The Use of Fuzzy Set Theory in Economics: Applications in Micro-Economics and Finance

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Abstract

This paper attempts to show how fuzzy set theory can be used to weaken some of the stringent rationality assumptions used in classical micro-economics. The objective of the paper is to see whether by introducing fuzziness we arrive to new results or just only generalizations of classical micro-economic results. We discover that the axiom of completeness is not needed anymore. Using fuzziness will also allow us to better explain the existing gap between delimiting possible choices and making the actual choice. We also introduce the notions of a fuzzy indifference set with a measurable area. The fuzzy utility surface is also discussed. The demand curve is now 'thick'.

In the producer area, the classical hypothesis that maximum profit entails maximum utility of profit is now substantially weakened when introducing fuzziness.

Finally, we consider revealed preference within a fuzzy context.

Abrégé

Par le présent travail, nous tenterons de démontrer comment l'application de la théorie des ensembles flous peut attenuer certaines des hypothèses rigoureuses sur la rationalité proposées par la micro-économie classique. Ce travail a pour objectif de rechercher si de nouveaux résultats sont obtenus par l'application des ensembles flous ou si nous n'arrivons qu'à une généralisation des résultats obtenus en micro-économie classique. Nous découvrirons que l'axiome de complétude n'a plus de sense. L'application des ensembles flous nous permettra également de mieux expliquer l'écart qui existe entre la délimitation des choix possibles et l'acte en soi de faire un choix. Nous introduirons également la notion d'un ensemble d'indifférence flou avec surface mesurable. La surface d'utilité floue sera également traitée. La courbe de demande est alors "épaisse".

Du point de vue du producteur, l'hypothèse classique selon laquelle le profit maximal engendre l'utilité maximale du profit perd de sa force par l'introduction des ensembles flous. Finalement, nous considérons les préférences révélés dans un contexte flou.

Introduction

This paper is concerned with seeing how fuzzy mathematics could contribute to an enriched vision of economics. We are specifically interested in seeing how the result of the traditional problem of maximizing a utility function subject to a budget constraint can be extended when utility and/or budget constraint are fuzzy. The idea behind the fuzzification of this type of problem is quite intuitive. As consumers our preferences are vague and basically the resulting demand function we derive from successive optimizations of utility functions subject to budget constraints should perhaps not have to lead to the 'ultra-thin' demand curve but rather to a 'thick' demand curve. The suggestion by the French economist Marchal is essential here. The main objective of the paper thus consists in seeing where exactly fuzzy sets may contribute in the area of micro-economics. We will try to pinpoint the advantage of using fuzzy sets as a procedure which allows us to uncover the way by which we express preferences. Furthermore we will try to argue in favor of getting rid of the completeness axiom which is used in classical choice theory.

Fuzzy set theory at this present stage is knowing a tremendous boost in research. Research in the axiomatization of fuzzy set theory is overwhelming. This paper uses only the very basics of fuzzy set theory and shuns as much as possible fuzzy set axiomatization. Therefore we suggest that, as expected, this paper can be at most a very small eye-opener to a newcomer in the field.

Our paper contains six different parts which are meant to form an integrated whole. Part I deals with the philosophical underpinnings of fuzzy set theory. It looks at the roots of multivalued logic through authors such as Black and Carnap. The main protagonists in formalizing multivalued logic are Lukasiewicz and Bochvar. We analyze some of Lukasiewicz's ideas. The law of the excluded middle, central to the discipline of fuzzy set theory is also analyzed.

We look also at some of the 'hot' questions which are currently 'hanging in the air' so to speak. In 1994 Elkan claimed that multivalued logic can be reduced to bi-valued logic. We analyze his argument and try to refute it.

Part II deals with some of the building blocks of fuzzy theory proper. The survey is simple, non-exhaustive and certainly not rigorous. We briefly survey Goguen's extension of the membership value set $[0,1]$ to a lattice. The extension is quite useful as it renders the task of defining certain classical concepts into a fuzzy set environment easier. An example is the issue of pseudo-complementation. We have also a look at the concept of a fuzzy binary relation, a concept which is most appropriate for defining fuzzy preferences. We start rounding off part II with a discussion on fuzzy numbers and possible algebraic operations on fuzzy numbers. The L-R fuzzy number is also discussed. Finally we propose a measure which may be used to indicate the fuzziness of a fuzzy set.

Part III deals with a crucially important topic which is the membership function. All too many applied papers have often assumed a membership function to be given from the outset. If we want to understand how we could possibly construct a membership function for a particular problem we must inquire about the meaning of a fuzzy sentence. Two views are presented the syntactic and the semantic approach. We survey several propositions mainly all belonging to the semantical approach. Dombi proposes a 'better' membership function based on a survey he conducted over a period of three years. Hisdal tries to model membership functions using a probabilistic approach. However, her approach is not clear-cut. Giles uses mainly a Bayesian approach where evidence does not have to back up beliefs. Shafer calls this the 'personalist view'. Smets and Magrez are using a more syntactical approach. This part of the paper also tries to distinguish between probabilities and possibilities; in view of separating the notions of respectively quantity of information from meaning of information.

Part IV deals with another essential topic which are the operators. We survey the axiomatic proposition of Bellman and Giertz who propose that the max and min operators may respectively correspond to fuzzy union and intersection. However, Bellman and Giertz mathematical justification of the two main operators is not sufficient to deal with real world problems. The argument by Zimmerman and Zysno tries to alter the max-min proposition into a weighted connective.

Part V finally gets to the economics subject which after all is the objective of this paper. Our goal is to depart somewhat from the all too much restrictive assumption of rationality which is so widely used in economics today. Whether we are in an environment where there is certainty, risk or uncertainty the information on the set of options we have is assumed to be perfect. Fuzziness is 'invited' in when we would reasonably assume that the options known in advance may only be partially known. Furthermore the choice the agent is supposed to make may be much less clear-cut than assumed. Therefore we do introduce the notion of fuzzy preference. This notion has a weak equivalent in the economic literature with concepts such as bounded rationality. Of course, the critique may be that in departing from the assumption of rationality we resume into a merely descriptive rather than a normative model. This is a difficult issue. The fine tread separating the two positions is indeed very hard to trace. The scope of probabilities is shown to be of little value in an imprecise environment. We introduce the notion of a fuzzy relation which we discussed in part II. We look at the all important definition of transitivity and see how fuzzy transitivity may indeed weaken the rationality assumption. We want to argue that when introducing fuzzy preferences we do uncover the procedure on *how* an agent arrives to a preference rather than with the result of the preference per se. Finally it is shown that the assumption of completeness is not at all needed within a fuzzy context. Incomparability is avoided in classical theory by imposing the axiom of completeness. Because we can use degrees of preference we can give a true expression to incomparability.

Part VI deals with applications in micro-economics and finance. We look at four applications in the micro-economics field. The optimization of a crisp utility function subject to a fuzzy budget constraint. The optimization of a fuzzy utility function subject to a fuzzy budget constraint. We also look at a fuzzy producer's equilibrium. Finally we consider revealed preference within a fuzzy setting. Part VI starts first with an expansion on the idea of fuzzy preferences. The tick demand function is an outgrowth of this assumption. Then we progress into the notion of a fuzzy indifference set. This set carries a membership function which is a fuzzy number. We then go into the notion of a weak and strong preference set. Finally we round off with an expectable proposition which refers to the convexity of a weak preference set.

The optimization problems are then tackled. First, we argue about the dangers of fuzzy optimization when simple linear membership functions are taken into account. Then we enter the problem of optimizing a crisp utility function subject to a fuzzy budget constraint. Then we look at optimizing a fuzzy utility function subject to a fuzzy budget constraint. We wonder whether a solution is possible using Brouwer's fixed point theorem. We try to argue whether we do indeed have an optimal solution. We also maximize the utility the producer gets from the profit he realizes when subjected to technological constraints. This is not a straightforward problem in a fuzzy environment as the objective of maximum profit does not necessarily coincide with maximum utility, as it would be in the classical case. Finally we make a brief discussion on Basu's paper which deals with fuzzy revealed preference. We round off part VI with some simple applications on basic finance concepts. We look at a fuzzification of future values and the net present value. Finally it is shown that the internal rate of return has no specific fuzzy equivalent.

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Part I: Fuzzy Logic

This first part of the paper deals with some of the philosophical issues surrounding the issue of fuzzy sets. This part will be subdivided into four chapters. Chapter I deals with a brief and simple survey on some of the reactions the academic community has been uttering against (or in favor) of fuzzy set theory. This chapter has no pretense at all. It just wants to shed some light on the popular stance vis-à-vis fuzzy set theory. Chapter II wants to enlighten us a little on the roots of fuzzy set theory. Chapter III deals with a formalization of what Black had to say on vagueness. Elements of Lukasiewicz logic are taken up. Finally chapter IV deals with some 'hot' questions on fuzzy sets.

Chapter I

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I.1.The academic community and some popular reactions

Professor Bart Kosko in his popular book on fuzzy thinking wanted to show to his class what a fuzzy set is all about. Says Kosko: "How many of you are male? Raise your hands. Males hands go up and female hands stay down. This gives us a non-fuzzy set. Now a harder question. How many of you are satisfied with your jobs? The hands bob up and down and soon come to rest with most elbows bent. A confident few point their arms straight up or do not raise them at all. Most persons are in between. That defines a fuzzy set...'([43]; p.13) A simple experiment but with big ramifications. To start with in the applied field of engineering for instance, numerous applications have been made using fuzzy set theory. To name but a few. Hitachi invented an air-conditioner which adjusts temperature in the most optimal way possible. Sony developed a palmtop computer using fuzzy sets so that it can recognize handwritten Kanji characters. Washing machines as developed through numerous Japanese and South Korean firms such as Daewoo, Samsung or Sanyo can now adjust their washing strategy based on sensing dirt level; fabric type; load size and water level. Finally the Sendai Subway System is ran using fuzzy logic. Fuzzy logic is truly engraved in the Japanese industrial landscape.

The Japanese Ministry of International Trade and Industry (MITI) launched the Laboratory for International Fuzzy Engineering Research (LIFE) in 1989. The chairman of LIFE is the president of Hitachi Corporation. Among the directors on LIFE are Joichi Aoi, president of Toshiba Corporation but also Yutaka Kume; president of Nissan Motors. The cream of the world largest and finest technology corporations are member of LIFE. This is to say that the practical implications of fuzzy set theory as they are applied in engineering can not be washed away.

At the non-applied front there is opposition however. I have remarked while writing up this thesis that most writings on the fundamentals of fuzzy set theory are coming from Western and Eastern Europe. Western Europe is heavily represented through French, Belgian and German Universities. Eastern Europe mainly through Polish and Chekoslovakian universities. North America is foremostly represented by the main inventor of fuzzy sets Lotfi Zadeh of Berkeley. Some Canadian universities are also involved with authors such as Giles and Gupta.

It is interesting however to see how fuzzy sets has been accepted in North America. One interesting story is the one which opposes Rudolf Kalman to Zadeh. Kalman was the inventor of the Kalman filter which is basically an optimal estimator. Says Kosko 'It gives the 'best' guess where the plane went when it flew behind a cloud'. I am not an engineer and can thus not grasp the beauty of Kalman's invention. One thing is sure would there have been a Nobel price in engineering Kalman would have won it 'hands down' as Kosko says. Kosko however argues that it is Zadeh's basic work in this field which helped lay the foundations of the Kalman filter. As Kosko says '..Zadeh missed the price and Kalman found it. That is why we call it the Kalman filter and not the Zadeh filter.' This is what Kalman had to say at the Man and Computer conference in Bordeaux of 1972: 'No doubt professor Zadeh's enthusiasm for fuzziness has been reinforced by the prevailing climate in the US- one of unprecedented permissiveness. 'Fuzzification' is a kind of scientific permissiveness; it tends to result in socially appealing slogans unaccompanied by the discipline of hard scientific work and patient observation.' I could go on for hours citing other big names who have opposed fuzzy set theory. However the big names are on both sides! Richard Bellman; the noted mathematician is certainly one of them; not to forget

Zadeh himself; chair of the department of electrical engineering at Berkeley. On the philosophy side there are great names such as Black and Russell.

It is truly extremely difficult to make an assessment of some depth on what fuzzy set theory has to offer. I do think; from my very limited exposure and my very limited capacities that it holds promise. It has shown to be useful in some areas of technology. From the proliferation of material written on the subject one may at least have a good 'gut feeling' about the future of this field. Finally little or nothing of this area has been used in economics. Some French economists such as Ponsard and Billot have been writing on the subject but little else is to be noted.

Chapter II

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II.1. Max Black and the roots to fuzzy set theory

Black's article is certainly one of the first articles which deals with the problem in depth. Philosophers of ancient Greece had posed the problem of vagueness; but little of a precise argument followed from their questioning. Black's article which appeared in the late thirties joins however a series of other papers on the same problem. The fundamental papers of Lukacsiewicz; Bochvar and Kleene came in 1938. They are quite more sophisticated in argument that Black's paper. It seems that the connection with fuzzy sets lies in Lukasiewicz 3-valued logic. We will inquire some of the elements of this multivalued logic and the possible connection with fuzzy sets in the next section. For now let us have a closer look at what Black has to say.

Black cites the well known philosopher Peirce who defines a 'vague' proposition as follows: 'a proposition is vague when there are possible states of things concerning which it is *intrinsically uncertain* whether, had they been contemplated by the speaker, he would have regarded them as excluded or allowed by the proposition. By intrinsically uncertain we mean not uncertain in consequence of any ignorance of the interpreter, but because the speaker's habits of language were indeterminate.'([6]; p.431) Peirce also invokes the idea of 'indeterminacy of habits' by which he means the hypothetical variation by the speaker in the application of the proposition; 'so that one day he would regard the proposition as excluding, another as admitting, those states of things'. The knowledge of such variation *could only* be deduced from a perfect knowledge of his state of mind...

Black's discussion on the location of the fringe gets us right to the point where vagueness may be defined. Says Black: 'The presupposition of the existence of a class of 'doubtful' objects will involve the assumption either of an exact boundary or of a doubtful region between the fringe and the class of unproblematic objects'. ([6];p.435) Either assumption will be shown to invalidate the concept of negation which is used in the classical logical principles. Black's example which purports to the above quote is as follows. Says the philosopher: 'Let *L* be a typical example of a vague symbol. The vagueness of *L* consists in the impossibility of applying *L* to certain numbers of a series. Let the series S be composed of a finite number; say 10 of terms x; and let the rank of each term in the series be used as its name. Let the region of doubtful application (or fringe) be supposed to consist of the terms whose numbers are 5 and 6 respectively. The choosing of those fringe terms is arbitrary. In the usual notation of prepositional calculus *Lx* will mean *L* applied to *x* and *~Lx; Lx* is false. Suppose now that L_1, L_2, L_3, L_4 *are true, but* L_5, L_6 are doubtful. The question which then arises is that of what is the range of *Lx* in that case?' We know the *L*'s which are true though we do not know for sure about the *L's* which refer to position 5 and 6. To exclude positions 5 and 6 may be right or it may be wrong; we do not know. We definitely exclude positions 7 to 10, though. The problem comes in when looking at *~Lx*. We should be positively excluding 1 to 4 including; not really knowing what to do with 5 and 6; though we certainly include positions 7 to 10 now. Assume now we include the fringe (i.e positions 5 and 6) in both cases. What can we say of *Lx* versus *~Lx*? We know that *~Lx* is only true when *Lx* is false. *~Lx* is true when excluding positions 1 to 4; or including positions 5 to 10. *Lx* is true only when we exclude positions 7 to 10; so the negation of this true *Lx* is to *include* positions 7 to 10! There is overlap between these two negations and this makes no sense according to the formal properties of negation in which a domain and its complement can not overlap. Here they clearly overlap!

The problem spot is certainly positions 5 and 6; which we called the fringe. We see clearly that because there is no clear boundary between *Lx* and *~Lx* we get to such a contradiction. However on the other hand and this leads us into quite some trouble, well defined boundaries to the fringe also lead to problems. The following is one among the many of such examples which illustrate the latter problem:

'A measure of corn when thrown out makes a sound. Each grain and each smallest part of a grain must therefore have made a sound yet no sound is made by a single grain.' ([6]; p.438). Black's paper does not offer any solutions to the circular problem he posed above. But he posed the problem. Though we are far still from a formalization of multivalued logic or even remoter from fuzziness altogether this paper has the merit to have shown in a simple way the problem of the 'excluded middle' in some sense. Goguen comes to a very similar conclusion as Black does. Says Goguen:' ...representing concepts by sets and deduction by methods of traditional logic does not yield an adequate model of our customary use of inexact concepts and deduction; for we have shown this representation leads to paradoxical conclusions.'([26]) Goguen's proposal for a resolution of this problem is to use fuzzy sets. We now go to chapter III which wants to look at some of the formalizations of multivalued logic. We then make the connection between this kind of logic with fuzziness.

Chapter III

III.1.Some Elements of multivalued logic

III.1.1. Early History

It is said that the founding fathers of many valued logic are the Scotsman Hugh MacColl (1837-1909) the American Charles Peirce (1839-1914); and the Russian Nikolai Vasil'ev (1880-1940). MacColl developed a system of prepositional logic in which three values could appear. The traditional true and false values and a 'variable' value. An example is this: '2=2';'3=2' and 'x=2'; which respectively would form a *certain* proposition (which is always true); an *impossible* proposition which is always false and the *variable* proposition which is sometimes false and sometimes true. (i.e if we attribute x to be 3 then the variable proposition is false)

Peirce also invented a similar kind of logic he called 'triadic' logic. Together with Frege he became the inventor of the truth tables for 2-valued logic. It is claimed that by 1909 Peirce had been extending this truth table to a three-valued logic truth table. Peirce was also instrumental in developing connectives specially geared towards three valued logic. Some of those connectives were taken over later by mathematical philosophers such as Lukasiewicz. Finally, Vasil'ev also developed a similar three valued logic. He proposed a world in which some objects have the predicate A; others its negation predicate not-A and still others which simultaneously have both A and non-A.

III.1.2. Breakthrough

The real breakthrough after some groundwork had been laid by MacColl; Peirce and Vasil'ev came in the early 20th century; mainly in the period 1920 to 1932. Instrumental authors were Lukasiewicz and Post. Lukasiewicz published the first systematization on a 3-valued system of logic in a lecture before the Polish Philosophical Society in Lwow in 1920. Axiomatization of Lukasiewicz logic was achieved by Mordchaj Wajsberg in 1931.

The interesting part of Lukasiewicz development of 3-valued logic is that it went through several stages. He distinguished about 3-valued and n-valued logic. While in the beginning of his development he held that 3-valued logic was of philosophical interest later on he pointed out that it would be 4-valued logic. Post himself discovered also this 3-valued logic and systematized it also. His discovery was independent of Lukasiewicz.

The difference between Post and the Polish philosopher was that Post started out right away with a formal development of n-valued logic while, as we said above Lukasiewicz progress went from a 3 valued to 4 valued and n valued logic. He also contemplated infinitely valued logic.

III.1.3. More recent

Following Resher the more recent period would cover 1932 to 1965. Tarski and Turquette carried further the work of mainly Lukasiewicz. Furthermore Lotfi Zadeh in fact also worked his fuzzy sets out based on this logic. There are however others systems of many-valued logic which have been worked out independently of what Lukasiewicz had to say. Prominent authors are Kleene, Godel and Bochvar. I would dare to classify in the applied field of multivalued logic authors such as Zadeh but also Shannon and Birkhoff, von Neumann and the Dutch mathematician Brouwer.

III.1.4. Elements of two valued logic

I do follow the notation introduced by Rescher. The following is at use then:

	Two – valued Systems	*Many – valued Systems*
Negation	$\sim$	$\neg$
Conjunction	$\&$	$\wedge$
Disjunction	V	$\vee$
Implication	$\supset$	$\rightarrow$
Equivalence	$\equiv$	$\leftrightarrow, \equiv$

The truth tables in two valued logic are well known. We re-iterate the following:

p	$\sim p$
T	F
F	T

where *T*=true and *F*=false

Using the other connectives of the table above we get then:

p	q	$p\&q$	pVq	$p\supset q$	$p\equiv q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Using the truth table we can find the truth value for what Rescher calls 'well formed formulas' (or wff).

An example of such wff is for instance this:

$$\alpha \supset (\beta \& [(\alpha V \beta) \supset (\gamma \equiv \alpha)])$$

where the truth values of the constituent elements are given as:

$$|\alpha| = T; |\beta| = F \; and \; |\gamma| = T$$

simply replacing in the wff we get:

$$T \supset (F \& [(TVF) \supset (T \equiv T)])$$

using the truth table from above we get:

$$T \supset (F \& [(T) \supset (T)]) = T \supset (F \& T) = T \supset F = F$$

The connectives are called *truth functional* as the truth value of the resulting wff, given that the constituent part's truth values are known, is always uniquely defined. This is the characteristic of the classical two-valued prepositional calculus designated as C_2.

III.1.5. Elements of 3-valued logic

With Lukasiewicz we introduce a third 'intermediate' or also 'neutral' truthvalue *I*. Lukasiewicz defends intuitively the introduction of *I* with the following example:

'I can assume without contradiction that my presence in Warsaw at a certain moment of next year; e.g. at noon on 21 December; is at the present time determined neither positively nor negatively. Hence it is *possible*; but not *necessary*; that I shall be present in Warsaw at the given time. On this assumption the proposition 'I shall be in Warsaw at noon on December 21 of next year', can at the present time be neither true nor false. For if it were true now; my future presence in Warsaw would have to be necessary; which is contradictory to the assumption. If it were false now ..my future presence in Warsaw would have to be impossible which is also contradictory to the assumption. Therefore the proposition considered is at the moment *neither true nor false* and must possess a third value; different from '0' or '1'. We can designate this value by 1/2. It represents the 'possible' and joins the 'true' and the 'false' as a third value....'[50]

Thus in other words Lukasiewicz points out thus that propositions regarding 'future-contingent' matters have a truth status that does not correspond to either of the orthodox truth-values of truth and falsity.

The truth tables based on this 3-valued logic, as proposed by Lukasiewicz himself looks then as follows:

p	$\neg P$
T	F
I	I
F	T

p	q	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
T	I	I	T	I	I
I	T	I	T	T	I
I	F	F	I	I	I
I	I	I	I	T	T
F	T	F	T	T	F
F	F	F	F	T	T
F	I	F	I	T	I

One can replace *T=1;I=1/2 and F=0.* Note that the results are somewhat less intuitive than in the case of C_2. This 3-valued logic we can abbreviate as L_3.

The guiding principles, following Rescher, for 3-valued logic would then be:

1) Obviously three truth values

2) The truth value of a conjunction is the *falsest*; and of a disjunction the *truest* of the truth values of its components. This can be seen easily in the above table. To take the disjunction of *T and I ; then the truest of both is T.*

III.1.6.The law of the excluded middle

We can not of course go into any serious details about the L_3 logic. The law of the excluded middle however for our purposes is worth to be mentioned.

Let us first for this explain the meaning of a two-valued tautology. This is a formula which always takes the truth value T regardless of what truth values may be assigned to the component prepositional variables. An example is for instance $pV \sim p$. The 3-valued truth table corresponds to the 2-valued truth table when only T's and F's are involved. From this follows then that any 3-valued tautology must also be a two valued tautology. The reason for this is that a tautology must take on the truth value T no matter the assignment of truth values to the constituent parts of the wff. This is easily illustrated. Take the case of $pV \sim p$ in C_2 and this is a tautology. But look now at a two valued tautology in L_3. Then $p\vee \sim p$ (where '∨ ; is the disjunctive operator in L_3) can yield easily *I* if *p* adopts the truth value *I*. $p \vee (or\ V) \sim p$ is exactly the law of the excluded middle; and it really fails to obtain in L_3 as a tautology. This could have been avoided would Lukasiewicz have taken for the disjunctive operator the following truth table:

p	q	$p\check{\vee}q$
T	T	T
T	F	T
T	I	T
I	T	T
I	F	I
I	I	T
F	T	T
F	F	F
F	I	I

One sees clearly that this new disjunctive operator on *I* yields now a true value *T*. The tautology can thus be maintained and the law of the excluded middle would remain valid in L_3. However this new disjunctive form has drawbacks. This paper is not the place in which we should further extend other avenues concerning the law of the excluded middle. At least what we have shown here is that it is not so obvious that in multivalued logic the law of the excluded middle would by definition not hold. *It can hold under certain circumstances.*

III.1.7. Many valued generalizations of 3-valued logic

3-valued logic can be extended into n-valued logic. Working with our truth values on [0,1] we can see the following quite easily:

n	$Division\ points$
2	(1,0)
3	(1,1/2,0)
.4	(1,2/3,1/3,0)
.	
.	
n	$1=(n-1/n-1);(n-2)/(n-1);\ldots 2/n-1;1/n-1;0/n-1=0$

Where for $n=2$ we worked in C_2; and for $n=3$ we worked in L_3.

Furthermore an extension can be made to infinite-valued logic in two cases:

1) $L_{\aleph_0}$ which symbolizes the case that if we take 0 and 1 together with all the rational numbers between 0 and 1 as truth values.

2) $L_{\aleph_1}$ when we take real numbers from [0,1] as truth values.

We now get into two propositions which look at the number of tautologies we may find in each system. This has relevance to our law of the excluded middle.

Proposition I: Every tautology of L_n is a tautology of C_2

References has been made already to this idea. The proposition is quite intuitive. The truth tables for L_n will agree with the two-valued system when only 0 or 1 is involved. This will be the overlap with C_2. The reverse is not true however as we have been remarking already above. Considering the law of the excluded middle; which is a tautology in C_2; it will not be a tautology in L_4 for instance. Take for instance from (1;2/3;1/3;0) for $|p| = 1/3;\ then\ |p \vee \neg p| = 1/3 \vee 2/3 = 2/3 \neq 1$; and thus the law of the excluded middle does not hold. Recall that the value of a conjunction is the *falsest*; and of a disjunction the *truest* of the truth values of its components; as we saw for L_3.

This is also valid in extension on L_3; say L_n. One can now also get a quite serious 'foretaste' of the use of max-min operators in fuzzy union and intersection. As the membership values of elements of fuzzy sets lie in [0,1] ,at least in the simplest case.

We now better understand why fuzzy union would imply maximum and intersection; minimum. Some of the logical connectives are then defined as follows:

(set *truth value equal to p(x))(please refer to part III for a discussion on probability logics.)*

$$\begin{cases} 0 \le p(x) \le 1 \\ p(\bar{x}) = 1 - p(x) \\ p(x \wedge y) = \min(p(x), p(y)) \\ p(x \vee y) = \max(p(x), p(y)) \end{cases}$$

The above development gives us an idea on how the fuzzy set theory draws its origins from multivalued logic.The math part and subsequent parts of this paper will, I hope make this clearer. Giles once said that "..Lukasiewicz logic is exactly appropriate for the formulation of the 'fuzzy set theory'; first described by Zadeh'; indeed it is not too much to claim that L_∞ is related to fuzzy set theory exactly as classical logic is related to ordinary set theory." (note that $L_\infty = L_{\aleph_i}$ *(i=0,1)* as we noted it above.). Giles argument requires much more of a development than it seems. Giles paper is an example of making this connection. We will however not take up a discussion on this subject. ([22])

Chapter IV

========

IV.1. Some 'hot' questions before the start

As I hope the reader will notice, after having read the subsequent parts of this paper; it is not clear whether fuzzy set theory is still as fuzzy as before or just not. There are still quite a lot of unresolved high caliber questions in this theory. I deliberately mention 'high caliber' because of the fundamental nature the questions involved. A short list of some of the problem areas may then be the following.

1) Are grades of membership to have a probabilistic or possibilistic interpretation?

2) How do we define grades of memberships?

3) What about ensuing operators?

4) Is fuzzy set theory a unified theory?

1) We will not define yet precisely what a grade of membership is. An example is the easiest to handle at this stage. Consider the set of golfers of Canada. This set contains professional golfers; golfers with 1 year of golfing experience etc...Each element of this basically fuzzy set belongs to some degree to the set. One is inclined to give a membership value of '1' to the element 'professional golfer'. Less of a value would go to the element which exemplifies the golfer with 2 years of free-time golf experience. We know from our discussion on multivalued logics grades of memberships may have to take values in [0,1]. The confusion which ensues out of that is serious. Toth says it in somewhat of a superficial way as follows: '..using membership functions of the form $\mu: X \rightarrow [0,1]$.. is a very appealing presentation, because in a simple way it generalizes both the logical truth functions and the characteristic function of ordinary set theory. But it is just that last property and the fact that for probabilities we have $\pi: Y \rightarrow [0,1]$; which have proved to be drawbacks because of the two following reasons: [1] the equality of the ranges of μ *and* π has misled many people to believe the underlying concepts to be the same too...[2]...'.[71] It was Goguen; by introducing a lattice structure which perhaps avoided further confusion with the [0,1] problem. We deal briefly with Goguen's extension in the math chapter of this paper. The above passage is to set the tune. The probabilistic versus possibilistic interpretation remains a problem. We will discuss it further in part III of this paper.

2) Related to the first question is the definition of grades of membership. Now, let us be clear the membership function and its grades is certainly the basic template upon which fuzzy set theory is to be build. I think part III of this paper gives us an idea of this. As we will see in part III defining and finding a workable and acceptable way to make grades of membership functional is far from easy.

3) The problem of operators is all important for the obvious reason that it will give us a tool kit by which we can optimize for instance. The mathematical branch of fuzzy set theory has opted widely for t-norms and t-conorms. As we also will see in part III the traditional max-min operators have been discarded somewhat; at least in the practice field.

4) Finally we may wonder if at all fuzzy set theory is unified at this present stage. This is again a very hard question.

Hisdal claims that there would be two orientations in fuzzy sets; the syntactical and semantical branch. Part III discusses the differences. From our readings however it looks as if there is still a lot of work to perform before to call fuzzy set theory unified.
Unified in the sense of a common base (such as membership functions and ensuing operators) upon which everybody agrees. On the other hand it may be asked whether a unified theory is that important; or even worse whether it is by definition possible in an area such as fuzzy sets.

IV.2. The hottest question of all

Finally there are even hotter questions! Elkan claims in [18] that fuzzy logic can basically be reduced to binary logic. This is an attack of the utmost serious sort! We need to expand on Elkan's argument. Elkan defines the the 'degree of truth' as follows:
Definition 1: Let A and B be arbitrary assertions. Then:

$$\begin{cases} 1) t(A \wedge B) = \min\{t(A), t(B)\} \\ 2) t(A \vee B) = \max\{t(A), t(B)\} \\ 3) t(\neg A) = 1 - t(A) \\ 4) t(A) = t(B) \; if \; A \; and \; B \; are \; \text{log}ically \; equivalent. \end{cases}$$

(with *t(.)* the truth value)
Then given this definition Elkan proposes the following 'revolutionary' theorem:
if $\neg(A \wedge \neg B)$ *and* $B \vee (\neg A \wedge \neg B)$ are logically equivalent then for any two assertions A and B; either $t(B)=t(A)$ or $t(B)=1-t(A)$. The proposition derived from this theorem takes then the following form:
Elkan's Proposition: Let P be a finite Boolean algebra of propositions and let a be a truth assignment function $P \rightarrow [0,1]$; then for all $p \in P, a(p) \in \{0,1\}$.
This is indeed an incredible claim given that all values in a multivalued setting; when following the four elements of definition 1, must collapse to a two-valued setting. Elkan gives the proof of his proposition; which we will not re-iterate here.
This looks like an enormous blow to all what has been constructed in the fuzzy literature. Would Elkan's statement have validity fuzzy sets would be just relegated as a mental exercise of no value; just as studying latin in high school.

Is Elkan's proposition refutable? The 'super' trio Didier Dubois-Henri Prade-Philippe Smets and also Lotfi Zadeh refute this statement completely. We want to look in their argumentation and start out first with the refutation of Dubois-Prade-Smets.

DPS do totally refute Elkan's use of the logical equivalence. They claim that this equivalence is true in a Boolean algebra setting but not at all in fuzzy logic. It is the fourth property in his definition 1 which is very bothering. Let us see why this is so. When looking at ONLY the first three statements we can, following DPS derive some classical logical equivalencies. They are for instance:

$$A \wedge (\vee) A \equiv A \; (idempotence)$$
$$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$$
$$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C) \; (distributivity)$$

For instance $A \wedge A \equiv A$, derives immediately from the first statement; i.e. that $\min(t(A), t(A)) = t(A)$.Much more important is that other Boolean equivalencies do NOT hold using the first three statements of definition 1. As an example, it is not true that: $A \wedge \neg A \equiv \mathrm{K} \; with \; t(K) = 0$ always holds. The reason follows immediately from statement 1 in definition 1. Namely that $t(A \wedge \neg A) = \min(t(A), 1 - t(A)) \leq 1/2$. Similarly $t(A \vee \neg A) = \max(t(A), 1 - t(A)) \geq 1/2$; meaning thus that $A \vee \neg A \equiv \mathrm{E}; with \; t(E) = 1$ does not always hold. Thus without statement 4 in definition 1 using thus the first three statements the *law of the excluded middle* is still refuted, as $A \vee \neg A \equiv \mathrm{E}; with \; t(E) = 1$ does not always hold. Zadeh gives us a very nice insight in what is wrong in Elkan's equivalence. We know that $\neg(A \wedge \neg B) = \neg A \vee B$; this just uses the fact that $\neg(A \wedge B) = \neg A \vee \neg B$ (law of de Morgan). Then extending the RHS with $(\neg A \vee B) \wedge (B \vee \neg B)$; where the new appendix is the law of the excluded middle. The latter expression is claimed to be equivalent to $: B \vee (\neg A \wedge \neg B)$. Hence $\neg(A \wedge \neg B) \equiv B \vee (\neg A \wedge \neg B)$; and this is precisely the equivalence Elkan uses in his theorem 1. No doubt the law of the excluded middle is 'disguised' to use Zadeh's words $(\neg A \vee B) \wedge (B \vee \neg B)$; through the latter term; i.e. $(B \vee \neg B)$. The non-equivalence of $\neg(A \wedge \neg B) \; and \; B \vee (\neg A \wedge \neg B)$ is also shown in Klir and Yuan.

A sharp note is by Ruspini who wonders when equivalence in a binary setting involves the equality of the truth values of two formulas; it looks definitely awkward to impose an axiom then; i.e. condition 4 of definition 1; to establish this truth.

An important point is this: the equivalence in the sense of $\alpha \leftrightarrow \beta$ is not the same as the equivalence in the sense of the equality of truth values *in multivalued logic*. It all depends what *type* of multivalued logic is considered. In part I we looked briefly at Lukasiewicz logic. We also mentioned the multivalued logic of Bochvar.

Part II: Fuzzy Mathematics - Building Blocks

Chapter I: Introduction

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Zadeh in 1965 formulated his ideas on fuzzy sets, which basically expressed the idea of introducing a degree of belonging of an element to a subset. Zadeh's creation is in some sense 'natural', but it needs a genius to create it! As Hans Zimmerman once said it is difficult to find 'a one-to-one mapping between the richness of our thoughts and the expression of it into daily language'. Formal language is even poorer in meaning , as opposed to daily language. Therefore Zimmerman concludes that a one-to-one mapping between rich human thoughts and formal logic is inexistant. In the social sciences the lack of truly crisp phenomena is omnipresent. Economics tries certainly to abstract from heavily complex situations so to provide approximate solutions to those real problems. Zadeh in 1973 pointed out that if the complexity of a system increases, our ability to make precise statements about its behavior diminishes until a threshold is reached beyond which precision and significance become almost mutually exclusive characteristics. Perhaps the most englobing definition of fuzzy mathematics was given by Goguen who said that :' the theory of fuzzy sets studies formal properties of *ill-posed problems and ill-defined sets...'* ([25]; p146) From the outset I think we can already clarify one main divergence of fuzzy math from the traditional bi-valued math and that is the *law of the excluded middle.* We know in probability theory that $p(A \cap \overline{A}) = 0$. This is precisely not universally true in a fuzzy context. Natural examples abound: 'yellow and yellowish' 'little tall, tall and very tall', all examples which we use with high frequency in our daily language. No clear boundaries as such and therefore no law of the excluded middle. We have been discussing the law of the excluded middle in part I.

If E is a 'universe of discourse' then I.1 expresses 'belonging' according to classical set theory.

I.1. Let E be a set and $A \subset E; x \in A$ *can be written as* $U_A(x) = 1; U_A(x) = 0$ *for* $x \notin A$.

1.2. Define $\{x:U_A(x)\};\forall x \in A;\, where\ U_A(x)$ indicates the DEGREE of membership of x in $\widetilde{A}$. In other terms let E be defined as above and let $x \in E; \widetilde{A} \subset E$ is a fuzzy subset such that $\{x|U_{\widetilde{A}}(x)\};\forall x \in A;$ where E is defined as above.

A fuzzy set on X (X being finite) can also be expressed as:

$$\widetilde{A} = U_{\widetilde{A}}(x_1)/x_1 + \ldots U_{\widetilde{A}}(x_n)/n = \sum_{i=1}^{n} U_{\widetilde{A}}(x_i)/x_i$$

Example 1: For X=N (positive integers) let $\widetilde{A}$.=.1/7+.5/8+.8/9+1/10+.8/11+.5/12+.1/13; where the fuzzy set A does indicate to be a fuzzy subset of integers of approximately 10.

Example 2: Let E={a,b,c,d,e,f} and M={0,1/2,1}(M=Membership set or *valuation set*) then: $\widetilde{A} = \{(a/0),(b/1),(c/1/2),(d/0),(e,1/2),(f/0)\}\ where\ \widetilde{A} \subset E$. Remark it is from M that the set $\widetilde{A}$ takes its membership values.

Example 3: If E={a,b,c} and M=[0,1] then A={(a/0),(b/1),(c/1)} is an ordinary set.

Example 4: An interesting example by Miaymoto is this:

Say E is the set of non-negative integers. Say E is the set including the ages of people. Can we define a subset B denoting 'young ages'? Clearly, there is no well defined objective criterion by which we can separate 'young' from 'old' ages. Define now a function $U_{\widetilde{B}}$ corresponding to the concept of 'young ages'.

So for instance $U_{\widetilde{B}}(x) = 1\ for\ x = 10\ and\ U_{\widetilde{B}}(x) = 0\ for\ x = 60$; where 'x' denotes age. Thus the fuzzy subset B is precisely 'fuzzy' because there is no clear defined boundary. ([52]; p.7) The membership function can also be drawn. In (age,membership value) space the function may be horizontal at 1 up to x=20 and start declining from x=20 onwards. The membership function is sometimes called *preference function*..(see [45] for instance) Note that in fuzzy set theory we could also define a *possibility function*. We then invoke the possibility that a certain event will happen and establish boundaries. Lai and Hwang attempt to make the difference clear between the two concepts: 'the grade of a membership function indicates a subjective degree of satisfaction within given tolerances.' and 'the grade of possibility indicates the subjective or objective degree of the occurrence of an event.' ([45]; p3) The above definitions also seem to show us that we explicitly deal with possibility and not probability; as in the latter case we refer solely to the *objective* degree of the occurrence of an event. Zadeh claimed in his original 1965 paper that the 'notion of a fuzzy set is completely non-statistical in nature' ([77]; p.340) Zadeh however does not offer, in this paper, any deep explanation on why this could be so. To make our stance clear on the differential between probability and possibility we can re-state Zadeh's intuition: 'what is possible may not be probable and what is improbable need not be impossible' ([37]; p.24) This is also known as the consistency principle. Part III deals with the problem more extensively.

I.3. Support and height:

The common set suppA= $\{x \in X, U_{\tilde{A}}(x) > 0\}$; where X is the universe of discourse set. 'suppA' is called the support of the fuzzy set A. We emphasize the fact that suppA only includes membership values which are *strictly greater than zero.* The least upper bound of $U_{\tilde{A}}(x)$ is called the height of A: hgt(A)='sup$_{x \in X}$ $U_{\tilde{A}}(x)$ ([1]; p.10) (see fig.1 - appendix)

A fuzzy set with height '=1' is normalized. Linked to the concept of 'hgt' is the concept of cardinality. If X is a finite set the cardinality, cardA is defined as:

$cardA = \sum_{x \in X} U_{\tilde{A}}(x)$ ([1]; p.11 and [52]; p.18) The definition makes some sense, when we think in analogy with the crisp set cardinality. A crisp set containing 5 elements will have cardinality 5; i.e the sum = 5 times 1. The membership value of elements in the set is 1. By analogy we do the same for a fuzzy set. Defining cardinality on X not being finite is however more tedious. The *relative cardinality* is defined as: $card_{rel}\tilde{A} = card\tilde{A} / cardX$. Cardinality and height are special cases of *energy-measures*. ([1]; p.12)

Toth [72] remarks that fuzzy cardinality as defined above is quite meaningless. For one thing cardinality could be defined as two sets having the same cardinality if there is a bijection between them. Of course this definition can not at all be followed when using $cardA = \sum_{x \in X} U_{\tilde{A}}(x)$. The reason is simply that in the crisp case the cardinality is formed by the binary value '1' of elements in the set. If one considers two fuzzy sets with same cardinality there need not at all to be a bijection between those two sets as we do not have to work with integers. I.e. say $card\widetilde{A} = card\widetilde{B} = 5$; then working with non integers there are tons of different ways how to reach this equality; and no bijection is at all guaranteed.

One of the great extensions in fuzzy set theory is the extension which can be made from the [0,1] to a more general structure which is a lattice.
At this point it may be useful to have a quick look at some of the peculiar lattices we can encounter.

I.4 Lattices

Let L be an ordered set; suppose for any ordinary subset $\{x_i, x_j\}$ of L there exists ONE AND ONLY ONE element of L constituting an inferior limit of the subset and likewise there exists one and only one element of L constituting a superior limit of the subset. If so then L is called a *lattice*. In symbols this is written as:

$$\{\ \forall x_i, \forall x_j (x_i \in L, x_j \in L) : \exists! x_k = x_i \Delta x_j ; x_k \in L\ \}$$

$$\{\ \forall x_i, \forall x_j (x_i \in L, x_j \in L) : \exists! x_l = x_i \nabla x_j ; x_l \in L\ \}$$

Four properties can be invoked: (A,B,C are elements of L.)

$A\Delta B = B\Delta A$ *and* $A\nabla B = B\nabla A$
(*commutativity*)

$A\Delta(B\Delta C) = (A\Delta B)\Delta C$
(*idem for* ∇)
(*associativity*)

$A\Delta A = A$
(*idem for* ∇)
(*idempotence*)

$A\nabla(A\Delta B) = A$
$A\Delta(A\nabla B) = A$
(*absorption*)

Some examples will clarify some of the lattices we may consider.

A Hasse diagram (figure 2) gives the following interesting results:

Setting $x_i < x_j$ *for* x_i *preceding* x_j; *we assume this is an ordered set, then:*

1) $A < B < C < F$
2) $A < B < E < F$
3) $A < B < D < F$

Then we can verify our four rules:

For instance $A\nabla(A\Delta B) = A$? *We know that:* $A\Delta B = A$ *and* $A\nabla A = A$; *hence* $A\nabla(A\Delta B) = A$

Note that *maximal chains* of a lattice are defined as chains which are non-overlapping with other chains. The lattice of figure 2 gives maximal chains which are as follows:
A<B<C<F or A<B<E<F. Counterexamples are useful as they show us immediately the important components of the definition of a lattice. Figure 3 shows us a first example of a non-lattice. C is inferior limit of both D and E. But B is also the inferior limit of D and E. Hence this violates the definition of a lattice. We only can have one inferior limit for a same given ordered set. In Figure 4 we may wonder what is the superior limit of D and E. This also violates the definition of a lattice as we are required to have a superior limit.

I.5. Important types of lattices

1. Modular Lattice

A lattice L is modular when for 3 arbitrary elements $x_1, x_2, x_3 \in L: x_1 \leq x_3 \Rightarrow x_1 \nabla (x_2 \Delta x_3) = (x_1 \nabla x_2) \Delta x_3$

Figure 5 provides an example.

2.Distributive Lattice

A lattice L is distributive when:

$$\forall x_1, x_2, x_3 \in L: x_1 \nabla (x_2 \Delta x_3) = (x_1 \nabla x_2) \Delta (x_1 \nabla x_3) \quad \textit{and} \quad x_1 \Delta (x_2 \nabla x_3) = (x_1 \Delta x_2) \nabla (x_1 \Delta x_3)$$

Figure 6 provides also an example.

Property: Any Distributive Lattice is Modular.

3.Sublattice

A lattice L and $A \subset L$ *if* $\forall x \in A, \forall y \in A: x \Delta y \in A$ *and* $x \nabla y \in A$ then A is a sublattice of L.

Property: A Sublattice L' of a distributive lattice L is itself distributive.

4. Complemented Lattice

Suppose a lattice L possesses an element denoted '0' which is the inferior limit of the entire lattice L; suppose L also possesses a superior limit for the entire lattice; say 'U'.

Then x_j *is a complement of* x_i $(x_j \in L, x_i \in L)$ *if*:
$x_i \Delta x_j = 0$ *and* $x_i \nabla x_j = U$

Figure 7 is an example.

A lattice L is complemented when:

a) it possesses a unique element '0'= inferior limit and U= a superior limit

b) each $x_i \in L$ possesses at least one complement in L.

Property: In a distributive lattice; the complement of an element x_i is always unique.

5. Boolean Lattice

A distributive and complemented lattice

a) for each $\overline{\overline{x_i}} := x_i$

b) $\overline{x_i \Delta x_j} = \overline{x_i} \nabla \overline{x_j}$

c) $\overline{x_i \nabla x_j} = \overline{x_i} \Delta \overline{x_j}$

6. Vector Lattice

Let A,B,...S be n sets; each totally ordered by '<'; the product set is an ordered set and forms a vector lattice. To create an order in this product set we use the dominance relation:

Ex.: $\vec{v} = (x_1, x_2, x_3)$ dominates $v^T = (x'_1, x'_2, x'_3) \Leftrightarrow x_1 \geq x'_1 ; x_2 \geq x'_2 ; x_3 \geq x'_3$

A vector lattice is distributive but not complemented.

7. Product of Lattices

Let L_1 *and* L_2 be two lattices. The product of these two lattices gives again a lattice.

An example:

Let $L_1 = \{A, B, C, D, E, F\}$ *and* $L_2 = \{\alpha, \beta, \gamma, \delta, \varepsilon\}$. Figure 8 shows the first lattice is not totally ordered; i.e. is B<C<D?

No! In fact a lattice may be constructed from partially ordered sets as long as the *maximal chains* indicate a total order and that there is a superior and inferior limit. The first lattice has maximal chains which are totally ordered, as one can see. Consider now this:

$(x_i, y_j)(x'_i, y'_j) \in L_1 \times L_2$ *if* $(x'_i, y'_j) > (x_i, y_j)$ *then* $x'_i > x_i$ *and* $y'_j > y_j$; *then* $L_1 \times L_2$ will be ordered; and the associativity etc.. rules can be verified; confirming $L_1 \times L_2$ to be a lattice. Thus in our example all possible combinations must be verified. I.e. $(F, \varepsilon) < (F, \gamma) < (F, \beta) < (F, \alpha)$ etc..

The figure 9 gives an idea of a partially ordered set not forming a lattice. D,E,F have no inferior limit.

8. Sup and Inf Semilattices

When only the superior limit (or only the inferior limit) belongs to the lattice we have respectively a sup-semilattice and inf-semilattice. The figure 10 is self-explanatory. A lattice is at the same time sup and inf semi lattice.

Chapter II. Generalization of the notion of a fuzzy subset

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So far we have been looking at fuzzy subsets taking membership values in M=[0,1].

We want to extend the membership set to L, which is more general than M.

We will then talk about L-fuzzy subsets; L being a lattice. The extension was worked out by Goguen. Up to this point fuzzy sets where subsets of an ordinary set E. Therefore we called them fuzzy subsets. One could perform operations on fuzzy subsets having the same reference set. We want to look specifically at the ordinary sets E_i.

II.1. Set of Mappings of E_1 into E_2

The set of functional mappings of E_1 into E_2 is denoted as: $E_2^{E_1}$.

Those mappings are functions; a function is a relation.

An example:

Let $E_1 = \{A, B\}$ *and* $E_2 = \{\alpha, \beta, \gamma\}$. All the possible combinations are totaling: $\#[E_2^{E_1}] = \# E_2^{\# E_1}$; i.e. $3^2 = 9$ combinations.

Two properties are:

1) $(E_1 x E_2)^{E_3} = E_1^{E_3} x E_2^{E_3}$

2) $(E_1^{E_2})^{E_3} = E_1^{E_2 x E_3}$

II.2. Fundamental Properties of a set of mappings of one set into another.

The set of mappings of E into L (lattice) is : L^E. The following property is extremely important:

Any internal law '*' defined on L induces a corresponding internal law $\otimes$ on L^E.

The law of internal composition says: to each ordered pair (x, y) $\in ExE$ corresponds one and only one element $z \in E$. Example: $E_1 = E_2 = \Re^+$ *if* $x - y = z; x, y \in \Re^+$ then it certainly can be $z \notin \Re^+$; one obtains thus then an external law.

Another example makes the concept even more clearer:

Consider a finite set: E=$\{x_1, x_2, \ldots x_n\}$ *and* $L = \{0,1\}$ Then:

$L^E = \{\{(x_1,0),(x_2,0),\ldots(x_n,0)\}\{(x_1,1),(x_2,0)..(x_n,0)\}\ldots\{(x_1,1),(x_2,1)\ldots(x_n,1)\}\}$

If L indicates the membership values of the elements then $L^E = \wp(E)$ where the latter indicates the power set of the set E. For the case of for instance E={ x_1, x_2 } and L={0,1}.

We can define the product operator on L as:

.	0	1
0	0	0
1	0	1

The law '.' is internal. An important question to know is whether '.' is remaining internal on L^E? Take for instance : $\{(x_1,o)(x_2,1)\} \otimes \{(x_1,1)(x_2,1)\} = \{(x_1,(o.1))(x_2,(1.1))\} = \{(x_1,o)(x_2,1)\} \in L^E$. Where the symbol '$\otimes$' stands for '.' on L^E. (multiplication)

We get in fact two interesting cases for L^E:

1) E=$\{x_1,\ldots x_n\}$ *and* $L = \{0,1\}$ yielding $\wp(E) = L^E$, L^E now being the set of subsets.

2) E=$\{x_1,\ldots x_n\}$ *and* $L = [0,1]$ yielding $\tilde{\wp}(E) = \tilde{L}^E$; where the latter is the set of fuzzy subsets. Some other interesting properties are forthcoming. To see for instance intuitively how for the case E=$\{x_1,x_2\}$ *and* $L = \{0,1\}$ the internal operator '*' where '*'='.' induces $\cap$ *on* L^E. Forming $\wp(E)$ is immediate. Take for instance $\{(x_2,1)(x_1,1)\}.\{(x_2,1)(x_1,o)\} = \{(x_2,1.1)(x_1,1.o)\} = \{(x_2,1)(x_1,o)\} = \{x_2,x_1\} \cap \{x_2\} = \{x_2\}$

For the case where E=$\{x_1,\ldots x_n\}$ *and* $L = \{0,1\}$

1) An internal operator '*' where *='.' induces $\cap$ *on* L^E.

2) An internal operator '*' where *= $\hat{+}$ induces $\cup$ *on* L^E. (where $A \hat{+} B = A + B - AB$)

3) A complement on L induces a complement on L^E.

For the case where E=$\{x_1,\ldots.x_n\}$ *and* $L = [0,1]$

1) An internal operator '*' where *=$\wedge$ on L induces $\cap$ on L^E.

2) An internal operator '*' where *=$\vee$ on L induces $\cup$ on L^E.

If E=Z and L=[0,1]; then L^E is the set of fuzzy integers.

If E= Z and L={0,1}; then L^E is the set of integers.

II.3. Proposal

a) If * is associative in L$\Rightarrow \otimes$ is associative in L^E

b) If * is commutative in L$\Rightarrow \otimes$ is commutative in L^E

c) If * is idempotent in L $\Rightarrow \otimes$ is idempotent in L^E

The inducement of operators and characteristics on L^E through operators and characteristics in L is absolutely crucial. The approach by Michel Prévôt starts from this line. Furthermore if there exists an operating structure (Monoid, Group etc..) we must check whether the structure is kept on L^E. A 'prelude' example could be the following: Two fuzzy subsets $\tilde{A}$ *and* $\tilde{B}$, the fuzzy subset known as the intersection of those two fuzzy subsets is $\tilde{A} \cap \tilde{B}$; *where* $U_{\tilde{A}\cap\tilde{B}}(x) = \min\{U_{\tilde{A}}(x), U_{\tilde{B}}(x)\}$. *The way we arrive to this, at first arbitrary definition; is to consider that if an internal operator '*' where '*'*$=\wedge$, *it induces* $\cap$ *on* L^E. Another example is the issue of pseudo-complementation. In the sense of Zadeh $\overline{\tilde{B}} = \tilde{A} \Leftrightarrow \forall x_i \in E: U_{\tilde{B}}(x_i) = 1 - U_{\tilde{A}}(x_i)$; but this result may not at all be obtained if we consider lattices other than the Boolean lattice!

Chapter III. Fuzzy Subsets Proper

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III.1. Generalities

Given a non-empty set and a lattice L containing at least two elements:

E---f---L

x-------u(x)

where the number of mappings possible is : $(\# L)^{\# E}$

A fuzzy subset $\widetilde{A}$ is an element of *f*(E,L). Thus $\widetilde{A}$ is a fuzzy subset: $\widetilde{A} \Leftrightarrow \widetilde{A} \in f(E,L)$ *where* $\widetilde{A} = \{x, U_{\widetilde{A}}: \forall x \in E: U_{\widetilde{A}}(x) \in L\}$. Thus if E is non empty ; a fuzzy subset $\widetilde{A}$ defined on E is a set of ordered couples such that: $\forall x \in \widetilde{A}: \widetilde{A} = \{x, U_{\widetilde{A}}(x)\}$; where L is the membership set and where $U_{\widetilde{A}}(x)$ is the degree of membership of x to $\widetilde{A}$.

III.2. Basic Operations on Fuzzy subsets

Consider a set E={A,B,C} with the following lattice L (a Hasse Diagram)(figure 11) and L={a,b,c,d,e,f}. Then we can look at:

a1) Inclusion:

Let $\leq$ be the order relation of the lattice; $\widetilde{A} \subseteq \widetilde{B} \Leftrightarrow \forall x_i \in E: \lambda_{\widetilde{A}}(x_i) \leq \lambda_{\widetilde{B}}(x_i)$.Two fuzzy subsets are comparable if *the respective values by the membership function in the lattice L are comparable.* As an example let $\widetilde{A} = \{(A|b)(B|a)(C|c)\}$ *and* $\widetilde{B} = \{(A|d)(B|e)(C|c)\}$. Are those two fuzzy subsets comparable? Is $b \leq d; a \leq e, c \leq c$? From the Hasse diagram we can see this is true. So we can conclude in this example that $\widetilde{A} \subset \widetilde{B}$. But remark that $\widetilde{C} = \{(A/f)(B/b)(C/d)\}$ is not comparable to fuzzy subset $\widetilde{A}$. Can we say for instance that $c \leq d$? We can not! Furthermore we need a relation of dominance to be able to compare two sets. Considering $\widetilde{D} = \{(A/d)(B/e)(C/b)\}$, this fuzzy subset is not comparable with the fuzzy subset $\widetilde{A}$ as $c \geq b$. So comparability and dominance are key.

Note that if L={0,1} we get ordinary inclusion; i.e $A \subset B \Leftrightarrow \forall x \in E: U_A(x) \le U_B(x)\, so\, \forall x \in A \Rightarrow U_A(x) = 1 \Rightarrow U_B(x) = 1 \Rightarrow x \in B$ which fits the standard definition of inclusion. Note that one could think about a *degree of containment.*

a2) Equality:

$$\tilde{A} = \tilde{B} \Leftrightarrow \forall x_i \in E: \lambda_{\tilde{A}}(x_i) = \lambda_{\tilde{B}}(x_i)$$

Similarly as in a1) one may think of a degree of equality of two fuzzy subsets.

a3) Complementation:

$$\overline{\tilde{B}} = \tilde{A} \Leftrightarrow \forall x_i \in E: U_{\tilde{B}}(x_i) = 1 - U_{\tilde{A}}(x_i)$$

The complementation issue is quite interesting, as highlighted already above. Complementation in lattices requires that $x_i \Delta x_j = 0\, and\, x_i \nabla x_j = U$ (U=Upper); furthermore the complement must be unique. Unique complements require thus Boolean lattices. Vector lattices are not complemented. Thus if L=[0,1] we will have a vector lattice and no unique complement. We must thus require L^E to be a Boolean lattice, in order to get a unique complement. So we can re-write the above definition of a complement as:

$$\overline{\tilde{B}} = \tilde{A} \Leftrightarrow \forall x_i \in E: U_{\tilde{A}}(x_i) \Delta U_{\tilde{B}}(x_i) = 0\, and\, U_{\tilde{A}}(x_i) \nabla U_{\tilde{B}}(x_i) = Upper$$

Therefore we could call Zadeh's complementation; *pseudo-complementation.* Pseudo - complementation and complementation coincide when L={0,1}. If A is a fuzzy set, the pair (A, A^c) is called a fuzzy partition of X, since $U_{\tilde{A}}(x) + U_{\tilde{A}^c}(x) = 1$. This definition of a partition requires obviously a Boolean lattice.

a4) Intersection:

$$\tilde{A} \cap \tilde{B}: \forall x \in E: \lambda_{\tilde{A} \cap \tilde{B}}(x) = \lambda_{\tilde{A}}(x) \Delta \lambda_{\tilde{B}}(x)$$

The above is of course not valid for sup-semilattices. The 'min' operator is also called 'aggregator'. As $U_{\sim}(.)$ belongs to a lattice; we know that every pair of elements possesses a greatest and a smallest element. Thus for every pair $\{U_{\tilde{A}}(x), U_{\tilde{B}}(x)\}$ part of a lattice we can define a greatest element:

$$\max[U_{\tilde{A}}(x), U_{\tilde{B}}(x)] = U_{\tilde{A}}(x) \vee U_{\tilde{B}}(x)$$

And a smallest element:

$$\min[U_{\tilde{A}}(x), U_{\tilde{B}}(x)] = U_{\tilde{A}}(x) \wedge U_{\tilde{B}}(x)$$

Note that we can of course picture such an intersection. Just take the two membership functions and take the maximum of both. (figure 12)

a5) Union:

$$\tilde{A} \cup \tilde{B} : \forall x \in E : \lambda_{\tilde{A} \cup \tilde{B}}(x) = \lambda_{\tilde{A}}(x) \nabla \lambda_{\tilde{B}}(x)$$

The above is not valid for inf-semilattices.

a6) Ordinary subset of level α :($\alpha - cuts$)

$$A_{\alpha} = \{x \xi x \in E : U_{A}(x) \geq \alpha\}$$

Example:

$\tilde{A} = \{(a;0.2)(b;0.3)(c;0.6)(d;0.55)(e;0.78)\}$ *where* $L = [0,1]$ *and* $A = \{a,b,c,d,e\}$ *Then* $A_{0.5} = \{c,d,e\}$

It is easy to see that if $\alpha_1 \subset \alpha_2 \Rightarrow A_{\alpha_2} \subset A_{\alpha_1}$.(monotonicity)

The $\alpha - cuts$ is an important concept. Such a cut induces an ordinary set derived from a fuzzy set. Hence for different levels of $\alpha \in [0,1]$ we get different ordinary sets referring to the same fuzzy subset. Hence the fuzzy subset can be defined as *a family of* α *-cuts*. The *resolution identity* is the following: $\tilde{A} = \bigcup_{\alpha} \alpha C(\alpha) A$ where C(.)A is just another notation for an alpha-cut (using an operator C(.)). Note however that the crisp membership grades of C(.)A are to be multiplied by α ; and so for all values of α . (see also part VI)

Distinction is also sometimes made in the literature between strong and weak $\alpha - cuts$; in the latter a strong inequality is used. ([52]; p.11). One may wonder whether there is not an operator transforming fuzzy subsets into other fuzzy subsets. L(α) is such an operator and it is defined as: $U_{L(\alpha)[\tilde{A}]}(x) = U_{\tilde{A}}(x)$ *if* $U_{\tilde{A}}(x) \geq \alpha$ *and* 0 *if* $U_{\tilde{A}}(x) < \alpha$. This is interesting and very different from the $\alpha - cuts$ definition, in that the membership value is accorded if U(.) is bigger than or equal to alpha. L(α)[A] is called a level fuzzy set. The figure 13 shows the alpha cut. It takes zero for all levels below alpha and 1 for all membership values above alpha. The concave line is the membership function. The alpha level set follows the membership function for a membership value above alpha.

III.3. Some operations involving union and intersection.

Because a lattice has the following properties: (see above)

For ∇ *and* Δ we get:

1)commutativity

2)associativity

3)idempotence

4)absorption

We can then derive the following:

a) $\forall(\widetilde{A},\widetilde{B},\widetilde{C}) \in f^3(E,L):(\widetilde{A}\cap\widetilde{B})\cap\widetilde{C}=\widetilde{A}\cap(\widetilde{B}\cap\widetilde{C})$; and similarly for union. The proof is immediate; use of the definition plus the associativity property of lattices gives the result.

b) $\forall(\widetilde{A},\widetilde{B}) \in f^2(E,L):\widetilde{A}\cap\widetilde{B}=\widetilde{B}\cap\widetilde{A}$; here the proof is also straightforward we use the commutativity of Δ *and* ∇ and the definition of union and intersection of fuzzy subsets.

c) $\widetilde{A}\cap(\widetilde{B}\cup\widetilde{A})=\widetilde{A}$ which uses the property of absorption. (also for union Vs intersection)

d) $\widetilde{A}\cap\widetilde{A}=\widetilde{A}$, which uses property 3. (also for union)

Theorem I:

$\forall(\widetilde{A},\widetilde{B}) \in f^2(E,L):\widetilde{A}\cap\widetilde{B}=\widetilde{B} \Leftrightarrow \widetilde{A}\cup\widetilde{B}=\widetilde{A}$

The proof uses commutativity and absorption.

Property:

$\forall(\widetilde{A},\widetilde{B},\widetilde{C}) \in f^3(E,L):\widetilde{A}\cup(\widetilde{B}\cap\widetilde{C})=(\widetilde{A}\cup\widetilde{B})\cap(\widetilde{A}\cup\widetilde{C})$, and similarly for intersection Vs. union. This property requires the lattice to be Boolean, as we explicitly need distributivity.

III.4. Convexity of fuzzy sets

A fuzzy set A is called convex if for any x, y $\in S$, and for any parameter $\lambda \in [0,1]$:

$$U_{\widetilde{A}}(\lambda x+(1-\lambda)y) \geq \min[U_{\widetilde{A}}(x),U_{\widetilde{A}}(y)] \text{ holds.}$$

See figure 14. A concave function f:S->[0,1] defines a membership of a convex fuzzy set $\tilde{A}$ by taking $U_{\tilde{A}}(x) = f(x)$. The converse is not true ([52]; p.16); a membership function of a convex fuzzy set is not necessarily a concave function of x.

Figure 15 shows that for A which is a convex set the membership function is not concave (at one point A is convex). As Miyamoto says the necessary and sufficient condition in order that a fuzzy set A be convex is that an arbitrary alpha-cut of A be a crisp convex set. This can be shown quite easily; i.e. the projection on the X-axis must be a closed interval.

If the fuzzy set A is a convex set and B is a fuzzy set which is also convex then the intersection will also be convex. (as we use 'min')

Chapter IV. Fuzzy Relations

Let us reiterate the case of an ordinary binary relation. For any relation R, there exists a crisp subset $G_R \subseteq S \times T$ *such that* $xRy \Leftrightarrow (x,y) \in G_R, x\overline{R}y \Leftrightarrow (x,y) \notin G_R$. Where S and T are two sets. The converse is also possible. For any subset $G \subseteq S \times T$ there exists a relation R_G *such that* $(x,y) \in G \Leftrightarrow xR_Gy, (x,y) \notin G \Leftrightarrow x\overline{R}_Gy$. ([52]; p.21)

IV.1. Cartesian Product

A crisp relation on ExF is a set of ExF. Similarly for a fuzzy relation $\widetilde{R}$. Consider two sets E and F; the set of ordered couples (x,y); $x \in E$ *and* $y \in F$ defines the product set $E \times F$. We get then $\widetilde{A} = \{(x,y), U_{\widetilde{A}}: \forall x \in E; \forall y \in F: U_{\widetilde{A}}(x,y) \in L\}$; L being a lattice. Thus x is in 'relation' with y to some degree. We can also say we have a binary relation between elements of E and F noted Ψ.

So we can define the fuzzy subset $\widetilde{A}$ then as:

$$\widetilde{A} = \widetilde{\Psi}(X,Y) = \{(x,y); U_{\widetilde{\Psi}}; \forall x \in E; \forall y \in F: U_{\widetilde{\Psi}}(x,y) \in L\}$$

Examples of fuzzy relations abound. For instance 'Car X is better than car Y' is an example.

1) Reciprocal Relation: Given a fuzzy binary relation $\widetilde{\Psi}$ of E to F; there is a reciprocal relation (dual) noted : $\widetilde{\Psi}^{-1}$ defined on F to E. We get: $\forall x,y \in E \times F: \widetilde{\Psi}^{-1}(y,x) \Leftrightarrow \widetilde{\Psi}(x,y)$

2) Complementary Relation: if we have a uniquely complemented lattice then we can define a new relation Ψ' of E to F which is the complementary relation.

Note: We can instead of only defining *binary* relations also define *n-ary* relations. I.e. a fuzzy relation R on $S_1 \times S_2 \times S_3 \ldots S_n$ which is a set of the Cartesian product of those sets.

IV.2. Operations on Fuzzy Relations

1. Inclusion:

A fuzzy relation is included in another fuzzy relation iff:

$$\forall (x,y) \in E \times F: U_{\tilde{\Psi}}(x,y) \leq U_{\tilde{\Psi}'}(x,y), \text{ i.e } \tilde{\Psi} \subset \tilde{\Psi}'.$$

2. Union:

Let $\tilde{\Psi}$ *and* $\tilde{\Psi}'$ be two fuzzy relations. Then $U_{\tilde{\Psi} \cup \tilde{\Psi}'}(x,y) = \max\left[U_{\tilde{\Psi}}(x,y), U_{\tilde{\Psi}'}(x,y)\right]$.

This follows entirely from the definition of the union of two fuzzy subsets, given L is a lattice.

3. Intersection:

$$U_{\tilde{\Psi} \cap \tilde{\Psi}'}(x,y) = \min\left[U_{\tilde{\Psi}}(x,y), U_{\tilde{\Psi}'}(x,y)\right]$$

4. Property: let $\tilde{\Psi}$ *and* $\tilde{\Omega}$ be two fuzzy relations of E into F. then:

$$\tilde{\Psi} \subset \tilde{\Omega} \Rightarrow \tilde{\Psi}^{-1} \subset \tilde{\Omega}^{-1}$$

The proof uses the concept of inclusion of a fuzzy relation by another relation.

5. Composing of fuzzy relations:

Given a fuzzy relation $\tilde{\Psi}$ of E to F and a fuzzy relation $\tilde{\Omega}$ of F to G then the composed relation $\tilde{\Omega} \circ \tilde{\Psi}$ is a relation of E to G such that: $\forall (x,z) \in E \times G: U_{(\tilde{\Omega} \circ \tilde{\Psi})}(x,z) = \max_y \left[\min(U_{\tilde{\Psi}}(x,y), U_{\tilde{\Omega}}(y,z)\right]$

IV.3. Binary Relations in a set E

We have been looking at fuzzy relations defined on a product set of E and F. We particularize somewhat now and look to the case where E=F.

1.The equality relation noted Π is the fuzzy relation of E to E such that:

$$\Pi = \left\{(x,y), U_{\Pi}; x \in E, y \in E: U_{\Pi}(x,x) = 1 \text{ and } U_{\Pi}(x,y) = 0 \text{ for } x \neq y\right\}$$

It looks intuitive that:

$\tilde{\Psi} \circ \Pi = \tilde{\Psi} = \Pi \circ \tilde{\Psi}$ *and* $\Pi = \Pi^{-1}$

2. Reflexive relation:

$$\forall x \in E: U_{\tilde{\Psi}}(x,x) = 1$$

We remark this property may be too strong in a fuzzy context.

We can therefore also define: $\alpha - reflexivity: \alpha \in]0,1[\Leftrightarrow \forall x \in X: U_{\tilde{\Psi}}(x,x) \geq \alpha$ ([1]; p.35)

3. Irreflexive relation $\Leftrightarrow \forall x \in E: U_{\tilde{\Psi}}(x,x) \neq 1$

4. Antireflexive relation $\Leftrightarrow \forall x \in E: U_{\tilde{\Psi}}(x,x) = 0$

5. Transitive relation $\Leftrightarrow \forall (x,z) \in E^2 : \max_y \left[\min(U_{\tilde{\Psi}}(x,y), U_{\tilde{\Psi}}(y,z)\right] \leq U_{\tilde{\Psi}}(x,z)$

One sees that the left part of this equation is nothing else than the definition of the composition of two fuzzy relations. The inclusion is embodied in the '$\leq$'. In general a fuzzy binary relation is not transitive but the $\tilde{\Psi}$ can be made transitive by using the notion of transitive closure. The transitive closure is defined as follows:

Given a binary relation defined on E; the transitive closure is defined as:

$$\hat{\tilde{\Psi}} = \tilde{\Psi} \cup \tilde{\Psi}^2 \cup \tilde{\Psi}^3 \ldots \cup \tilde{\Psi}^k \cup \ldots$$

Note: Transitive closure can be defined in other ways too.

Theorem I: **The transitive closure of a fuzzy binary relation is transitive.**

Proof:

We must show that $\tilde{\Psi} \circ \tilde{\Psi} \subset \tilde{\Psi}$. This follows directly from the definition of transitivity. I.e. the composition and inclusion are embodied in that definition as we remarked above.

Form:

$$\hat{\tilde{\Psi}}^2 = \hat{\tilde{\Psi}} \circ \hat{\tilde{\Psi}} = (\tilde{\Psi} \cup \tilde{\Psi}^2 \cup \ldots) \circ (\tilde{\Psi} \cup \tilde{\Psi}^2 \cup \ldots) = (\tilde{\Psi}^2 \cup \tilde{\Psi}^3 \cup \tilde{\Psi}^4 \cup \ldots) \subset (\tilde{\Psi} \cup \tilde{\Psi}^2 \cup \tilde{\Psi}^3 \cup \ldots)$$

This means that $\hat{\tilde{\Psi}}^2 \subset \hat{\tilde{\Psi}}$ *i.e* $\tilde{\Psi} \circ \tilde{\Psi} \subset \tilde{\Psi}$

Theorem II: Let us have a fuzzy binary relation. If for a certain 'k':

$$\tilde{\Psi}^{k+1} = \tilde{\Psi}^k \; then \; \hat{\tilde{\Psi}} = \tilde{\Psi} \cup \tilde{\Psi}^2 \cup \ldots$$

Proof: $\forall n = k+1 > k: \tilde{\Psi}^n = \tilde{\Psi}^k \; then \; \hat{\tilde{\Psi}} = \tilde{\Psi} \cup \tilde{\Psi}^2 \cup \ldots \tilde{\Psi}^k \cup \tilde{\Psi}^k = \tilde{\Psi} \cup \tilde{\Psi}^2 \cup \ldots \tilde{\Psi}^k$

(where use was made of associativity and idempotence)

6. Symmetric Relation:

$$\forall (x,y) \in E^2 : U_{\tilde{\Psi}}(x,y) = U_{\tilde{\Psi}}(y,x)$$

7. Antisymmetric relation:

$$\forall (x,y) \in E^2 : U_{\tilde{\Psi}}(x,y) \; and \; U_{\tilde{\Psi}}(y,x) \Rightarrow x = y$$

IV.4. Structures of Binary Relations

1. Resemblance relation

A fuzzy binary relation is a resemblance relation $\Leftrightarrow \tilde{\Psi}$ is symmetrical and reflexive i.e.:

$$\forall x \in E: U_{\tilde{\Psi}}(x,x) = 1 \; and \; \forall (x,y) \in E^2: U_{\tilde{\Psi}}(x,y) = U_{\tilde{\Psi}}(y,x)$$

2. Dissemblance relation

This comes forth when:

$$\forall x \in E: U_{\tilde{\Psi}}(x,x) = 0 \; and \; \forall (x,y) \in E^2: U_{\tilde{\Psi}}(x,y) = U_{\tilde{\Psi}}(y,x)$$

3. Fuzzy Pre-order relation

This relation is reflexive and transitive i.e.:

$$\forall x \in E: U_{\tilde{\Psi}}(x,x) = 1 \; and \; \forall (x,y,z) \in E^3: \max_y \left[\min(U_{\tilde{\Psi}}(x,y), U_{\tilde{\Psi}}(y,z))\right] \le U_{\tilde{\Psi}}(x,z)$$

We know when the fuzzy relation is reflexive that : $\tilde{\Psi} \subset \tilde{\Psi} \circ \tilde{\Psi}$

When the fuzzy relation is transitive then: $\tilde{\Psi} \circ \tilde{\Psi} \subset \tilde{\Psi}$

So for a pre-order we get: $\tilde{\Psi} \circ \tilde{\Psi} = \tilde{\Psi}$

4. Order relation

If the pre-order is anti-symmetric we have an order relation.

5. Similitude relation

If the pre-order is symmetric we have a similitude relation. The similitude relation is also called equivalence relation.

6. Subrelation of a similitude relation in a fuzzy pre-order

Let $\tilde{\Psi} \subset E \times E$ be a fuzzy pre-order relation. If there exists an ordinary subset $E_1 \subset E$ such that $\forall x,y \in E_1: U_{\tilde{\Psi}}(x,y) = U_{\tilde{\Psi}}(y,x)$; then there exists among E_1 a similitude relation which is called *a similitude sub-relation in the pre-order*. The similitude sub-relation is maximal if there is no other similitude relation of the same nature in the relation. An example makes this clearer.

$\widetilde{\Psi}$	A	B	C	D	E	F	G
A	1	.2	.2	.2	.2	.3	.4
B	.2	1	.5	.2	.2	.3	.5
C	.2	.5	1	.2	.2	.3	.5
D	.2	.2	.2	1	.8	.3	.5
E	.2	.2	.2	.8	1	.3	.5
F	.2	.2	.2	.2	.2	1	.4
G	.2	.2	.2	.2	.2	.2	1

a)We first want to ensure that the fuzzy relation is a pre-order (reflexive and transitive)

b)We do not obtain symmetry; as an example $U_{\widetilde{\Psi}}(F,D)=.2\ but\ U_{\widetilde{\Psi}}(D,F)=.3!$

c)We can however find subsets of $\widetilde{\Psi}$ which make similitude relations.

As an example the subset K_1={A,BC,D,E} verifies a pre-order and is symmetric; f.i. $U_{\widetilde{\Psi}}(A,C)=.2=U_{\widetilde{\Psi}}(C,A)$. The subset K_1 ={A,B,C} would also verify a similitude sub-relation but it would not be maximal as we can extend this subset into K_1. Two other subsets are also maximal i.e. $K_2=\{F\}\ and\ K_3=\{G\}$ are similitude sub-relations. All K_1,K_2,K_3 are disjoint from each other as one can easily verify. Thus the fuzzy relation $\widetilde{\Psi}$ is decomposable into maximal disjoint similitude sub-relations. K_1,K_2,K_3 form then similitude classes.

Note that:

a) a maximal subrelation does not have to be disjoint.

b) finding maximal subrelations is not at all easy. There is a general method to perform the quest for sub-relations, called the Malgrange algorithm as per Y.Malgrange a Belgian engineer. ([38]; p.387)

Chapter V. The Extension Principle

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This is an essential concept in fuzzy set theory. The principle allows us to extend non-fuzzy concepts in order for us to deal with fuzzy quantities. The concept basically asks: 'if there is some relationship between non-fuzzy entities; what is its equivalent between fuzzy entities?'([37]; p.19)

V.1. Definition

Let f be a mapping from X to a universe V such that: $y=f(x), x \in X, y \in V$.

We now assume that instead of having x being an element of X; only the fuzzy quantity $\widetilde{A}$ on X is given; f.i. 'approximately x'. We may now wonder what the *fuzzy* image by f is of the fuzzy argument $\widetilde{A}$. So to write $\widetilde{B} = f(\widetilde{A})$ is to say what exactly?

Figure 16 provides us with some insights:

Take an element x on the X-axis on the graph. This point has a precise mapping on the Y-axis. The problem to solve is what is the image of the fuzzy variable 'approximately x'? The answer is logical: the membership function associated to the fuzzy set $\widetilde{A}$ must be mapped *through* f on the Y-axis yielding the mapping of the membership function of the fuzzy set $\widetilde{A}$. For a one-to-one mapping (and assuming that membership values must be non-negative):

$$U_{\widetilde{B}}(y) = U_{\widetilde{A}}(f^{-1}(y)) = U_{\widetilde{A}}(x)$$

The case of non-one-to-one mappings is a little more tricky:

Considering again the same figure; we clearly see that y is the mapping of x and x'. The mapping of the membership function through f is now dubious; i.e. should we consider A or B? The definition Zadeh then proposes is:

$$U_{\widetilde{B}}(y) = \sup_{x \in X: f(x)=y} U_{\widetilde{A}}(x)$$

V.2.Fuzzy Numbers

Let the universe U be the real line. A fuzzy set $\tilde{A}$ on $\Re$ is called a fuzzy number iff $\tilde{A}$ is convex and there exists exactly one point, say $M \in \Re \; with \; U_{\tilde{A}}(M) = 1$. As Nather ([1]; p.20) says the linguistic expression for such number is 'approximately M'. A fuzzy interval is a straightforward extension of a fuzzy number. See figure 17. Zimmerman adds to the definition of Nather that the membership function should be piecewise continuous. This makes sense. In general thus a fuzzy number is a *restricted* case of a fuzzy set in that it has to be convex (i.e. the alpha cut sets must be convex) and it must be normalized. Fuzzy sets in general do not have to carry those properties.

Kaufmann and Gupta take a more intuitive approach to defining a fuzzy number. It basically involves a coupling of an interval of confidence with a level of presumption. The higher the level of presumption the smaller the interval of confidence gets. Very large intervals of confidence such as in the extreme case of the real line gives very low levels of presumption. The membership function which then couples both interval of confidence and level of presumption can either be smooth or flattened out. The interval of confidence should be a closed interval; this is precisely requiring A to be convex. The level of presumption called alpha is nothing else than the alpha we used in the alpha cuts definition.

V.2.1.The L-R Fuzzy Number

This is a particular fuzzy number and it is defined as:

$$\forall x' \in \Re; \Phi(.) \in [0,1]$$

$$\Phi(x') = F_L(x') \; for \; x' \in]-\infty,0[$$

$$\Phi(x') = 1 \; for \; x' = 0$$

$$\Phi(x') = F_R(x') \; for \; x' \in]0,\infty[$$

Increasing monotonicity is imposed on $F_L(x')$ and decreasing monotonicity on $F_R(x')$.

The function $\Phi(x')$ is a concave function. Left and right 'leg' of the function do not necessarily make up a symmetric function. We get a little more precise now on right and left leg of the function:

$$\forall x \in \Re: U(x) = F_L(x - M_A)/T_{\tilde{A}} \; for \; x \in]-\infty, M_A[$$

$$U(x) = 1 \; for \; x = M_A$$

$$U(x) = F_R(x - M_A)/V_{\tilde{A}} \; for \; x \in \left]M_A, \infty\right[$$

where T and V are both positive.

One sees that if x/T decreases when T increases; i.e. the slope gets flatter; we speak then of a dilatation. In the case where T decreases we speak about a contraction. M is the central value and T measures the left spread of the function and similarly for V.(right spread)

The usual notation is:

$$\tilde{A} = (M, T, V)_{LR}$$

An example of an L-R fuzzy number is:

$F_L(x') = 0, x' \leq -1$

$F_L(x') = \sqrt{1+x'}, x' \in [-1, 0]$

$F_R(x') = 1 - x'^2, x' \in [0, 1]$

$F_R(x') = 0, x' > 1$

We also may have L-R's with a flat; i.e. the $\Phi(.)$ function has a flat portion. We also can have (semi)-symmetric L-R's. Dubois and Prade's definition concerns a semi-symmetrical L-R. ([16]; p.53) We also have other special types of fuzzy numbers such as TFN's (Triangular Fuzzy Numbers (triangular shape of the function); or also T_r FN; Trapezoidal Fuzzy numbers. Kaufmann and Gupta also come forth with the concept of a hybrid number, which is a blend of fuzzy and random numbers. We do not go in detail on the latter however. At this point it is also interesting to note that a fuzzy number is a *subjective valuation.* Two human operators may assign different membership values for a same fuzzy number; say 'approximately five'. Observations can either be precise or statistically measurable or they are not measurable at all which puts them in the class of fuzziness.

V.2.2. Extended Real Operations

At this point it is interesting to investigate somewhat deeper the different operations we can perform on fuzzy numbers whatever their type may be. In this section we also want to look at the extended operations on fuzzy sets. This will give us the background to look at fuzzy functions and at non-fuzzy functions with fuzzy arguments.

We follow the set up of Kaufmann and Gupta. A fuzzy number derives from a fuzzy set. The fuzzy set derives from its membership function. As seen as above the fuzzy number is a coupling of interval of confidence with level of presumption which is nothing else than α. Therefore it is logical to see what the traditional operations on numbers will yield when applied on intervals of confidence.

1. Operations on Intervals of Confidence

We first define uncertain values; i.e. values belonging to an interval of confidence. Those uncertain values are not yet fuzzy numbers however!

Then :

1)Let A=[a,b] and B=[c,d] and [illegible]re:A(+)B=[a+c,b+d]

2)A(-)B=[a-c,b-d]

3)A(+)A^-=[a,b](+)[-b,-a]=[a-b,b-a] which is not equal to zero!

'(+)' is commutative and associative. There is also a neutral element i.e. [0,0]. However there is no inverse element.

4)A(.)B=[a,b](.)[c,d]=[a.b,c.d] ($a,b,c,d \in \Re^+$)

5)A(/)B=[a/d,b/c]

6)A^{-1}=[1/b,1/a]

For (.) there is no inverse but there is commutativity and associativity; there is also a neutral element; i.e [1,1].

7)$A(\wedge)B=\left[a \wedge c, b \wedge d\right]$

8)similarly for minimum

2. Levels of Presumption

$$\forall \alpha_1, \alpha_2 \in [0,1]: \alpha_1 < \alpha_2 \Rightarrow [a^{\alpha_2}, b^{\alpha_2}] \subset [a^{\alpha_1}, b^{\alpha_1}]$$

Thus the higher the level of presumption the smaller the interval of confidence gets. This makes sense.

3. Addition and subtraction of Fuzzy Numbers

Let A_α be the interval of confidence of the fuzzy set $\widetilde{A}$. Let B_α be the interval of confidence of the fuzzy set $\widetilde{B}$. Remark the subscript 'alpha' which indicates the level of presumption. Let the fuzzy number associated to the fuzzy set $\widetilde{A}$ be 'nearly 5' and the fuzzy number associated with the fuzzy set $\widetilde{B}$ be nearly 'nearly 8'. Both fuzzy sets are convex and reach at one point $\alpha = 1$. Furthermore the membership functions are piecewise continuous. Adding both fuzzy numbers will give:

$$A_\alpha (+) B_\alpha = [a_\alpha, b_\alpha] + [c_\alpha, d_\alpha] = [a_\alpha + c_\alpha, b_\alpha + d_\alpha], \forall \alpha \in [0,1]$$

For instance at $\alpha = 1$ we will get [5,5]+[8,8]=[13,13]

There is another method which is as follows:

$$U_{\widetilde{A}(+)\widetilde{B}}(z) = \vee_{x+y=z} (U_{\widetilde{A}}(x) \wedge U_{\widetilde{B}}(x))$$

An example can clarify this definition somewhat.

Consider the fuzzy set $\widetilde{A}$={(0,0),(1;.1),(2;.3),(3,.8),(4;1),(5,.7),(6,.3),(7,0),(8,0)...}

Consider the fuzzy set $\widetilde{B}$={(0,0),(1,.3),(2,.6),(3,1),(4,.7),(5,.2),(6,.1),(7,0),(8,0)...}

The membership functions sprouting out of those givens are for both more or less symmetrical concave functions. Using our new definition:

1) taking '1' for instance: we know 1=1+0 and 0+1 we must look for the membership values for both '1' in $\widetilde{A}$ and '0' in $\widetilde{B}$; we also must look at the membership values of '0' in $\widetilde{A}$ and '1' in $\widetilde{B}$. Applying our definition we get then:

$$\vee[.1 \wedge 0; 0 \wedge .3] = \vee[0,0] = 0$$

This gets substantially more convoluted when taking higher numbers such as 3. Because '3' can be the result of four different additions of 2 positive numbers.

It can be shown that (+) is an internal law. This shows that when using (+) we obtain again a fuzzy number. We get then the following properties for (+):

1) (+) is commutative (here for fuzzy numbers)

2) (+) is associative

3) Has a neutral element

4) Has no inverse.

Subtraction can be seen as adding the negative to the first argument. This resumes as expected in the interval of confidence case. Neither commutatitivity nor associativity is present. Subtraction is then defined as:

$$U_{\tilde{A}(-)\tilde{B}}(z) = \vee_{z=x-y}\left[U_{\tilde{A}}(x) \wedge U_{\tilde{B}}(y)\right]$$

4. Multiplication and division of fuzzy numbers

The result for the fuzzy number case is exactly as in the confidence interval case.

The properties are as in (+). Division will not be commutative nor associative.

The definition is as above except that we have that x.y=z figuring as the new subscript under the max sign. Furthermore the operation one must carry out to find the maximum (or more generally the supremum) is complicated.

Finally it can be shown that (.) is distributive vis a vis (+) but not vice versa.

Division is similar and we use x/y=z as subscript.

5. Maximum and Minimum of fuzzy numbers

The fuzzy minimum is defined as follows:

$$\forall \alpha \in [0,1]: A_\alpha (\wedge) B_\alpha$$

The fuzzy maximum is defined similarly.

Equivalently we can use:

$$U_{\tilde{A}(\wedge)\tilde{B}} = \vee_{z=x\wedge y}(U_{\tilde{A}}(x) \wedge U_{\tilde{B}}(y))$$

6. Convolution of Fuzzy Numbers

The definition we have been looking at so far are called by Kaufmann-Gupta *max-min convolutions*. It is as this point important to see those max-min convolutions as direct applications of the *extension principle* which we have treated above. We can also consider *min-max convolutions*. One can calculate a sum (+) for two fuzzy numbers according to min-max convolution without restriction. *What remains important however is that the fuzzy number which is yielded must be associated to a convex set and be normalized.* Min-max convolutions may not keep convexity and normality. This is however guaranteed with max-min convolutions.

7. Convolution of L-R Fuzzy Numbers

The max-min convolution for (+) on L-R numbers is proven by Dubois and Prade and we will not re-iterate the proof here. ([16]) The result for (+) is:

Given $\tilde{A} = [M_A, T_A, V_A]_{LR}$ *and* $\tilde{B} = [M_B, T_B, V_B]_{LR}$, two L-R fuzzy numbers. Then:

$$\tilde{A}(+)\tilde{B} = [M_A + M_B, T_A + T_B, V_A + V_B]_{LR}$$

(.) for L-R's is a little more complicated. It turns out we can have 2 different cases:

-a) when the right spreads of both L-R's are small with respect to their central values

-b) when the right spreads of both L-R's are not small with respect to their central values

For a) this yields:

$$M_1 > 0 \text{ and } M_2 > 0$$

$$\tilde{A}_1(.)\tilde{A}_2 \approx (M_1M_2, M_1T_2 + M_2T_1, M_1V_2 + M_2V_1)_{LR}$$

For b) this yields:

$$\tilde{A}_1(.)\tilde{A}_2 \approx (M_1M_2, M_1T_2 + M_2T_1 - T_1T_2, M_1V_2 + M_2V_1 - V_1V_2)_{LR}$$

For scalar multiplication we get:

$$a\tilde{A} = (aM, |a|T, |a|V)_{LR}$$

8. Deconvolution of fuzzy numbers

Let A,B and C be fuzzy numbers. Let C and B be given in C=A(+)B. How can we find A? Is A=C(-)B? The answer is no! We defined a max-min convolution for (-) as:

$$U_{\tilde{C}(-)\tilde{B}}(x) = \vee_{z=x-y}(U_{\tilde{C}}(x) \wedge U_{\tilde{B}}(y))$$

That C(-)B does not yield the solution can be easily seen:

$$A_\alpha = [a^\alpha, b^\alpha] \; and \; B_\alpha = [c^\alpha, d^\alpha] \; and \; C_\alpha = [a^\alpha + c^\alpha, b^\alpha + d^\alpha]$$

$$C_\alpha(-)B_\alpha = [a^\alpha + c^\alpha, b^\alpha + d^\alpha](-)[c^\alpha, d^\alpha] = [a^\alpha + c^\alpha - d^\alpha, b^\alpha + d^\alpha - c^\alpha] \neq A_\alpha$$

There is a way to determine whether a solution exists in this simple equation. We do not go in detail however on this issue. Similar theorems exist for deconvolutions of (-),(.) and (/).

Chapter VI: Fuzzy Functions

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We may consider two types of fuzzy functions. A non fuzzy function with a fuzzy argument. And a fuzzy function with a fuzzy or non-fuzzy argument.

The first type is what Nather calls *a fuzzy extension of non-fuzzy functions.*

An example makes the first type clearer.

Consider a straight line equation $f(x) = ax+b$ and there is a fuzzy argument which is: $(x, T=V)_L$ which is nothing else than a perfectly symmetric L-R fuzzy number. Left and right spread are equal. Now we want to find the image, following the extension principle and using the definitions on convolutions on L-R fuzzy numbers, we get:

$$f(x) = (ax+b, |a|T = V)_L$$

where we apply the definition of the multiplication of an L-R with a scalar. The fuzzy number L-R is mapped through a non-fuzzy function. We see that the central value of the L-R is changed through the mapping from '*x*' to '*ax+b*'. Furthermore the spread has changed also from T=V to T=V multiplied by a scalar. Figure 18 relates this non-fuzzy function with a fuzzy argument. It clearly shows us the fuzzy L-R which is symmetric and which is mapped through the ordinary linear function is reduced in spread and has changed it's central value on the Y-axis. Mind the height of this fuzzy number depiction is still the same as the level of presumption alpha is 1. Furthermore in this case we can say that the domain of the function is fuzzy, as the argument of the non-fuzzy function is a fuzzy L-R variable.

The second class is the fuzzy function which is also called *fuzzifying function.* ([1]; p.40) The fuzzifying function maps a crisp point $x \in X$ into a fuzzy set $\widetilde{B} = f(x) \in \widetilde{\wp}(V)$; where the latter shows the fuzzy subset $\widetilde{B}$ belongs to the set of fuzzy sets; i.e. the power set . The fuzzy function of the straight line we considered above is now written:

$$\widetilde{f}(x) = A\,(.)\,x\,(+)\,b$$

The situation is here somewhat different. If the climb parameter $\widetilde{A} = [a, T = V]_L$ which is again a symmetrical fuzzy L-R then the image of the fuzzifying function is:

$$\widetilde{f}(x) = [ax+b, T = V|x|]_L$$

Here the domain of the function is non-fuzzy, consisting of arguments which are for instance elements of the real numbers. As the ascent parameter is a fuzzy L-R (symmetrical) different versions of the straight line are possible. Figure 19 pictures the story. The images of the fuzzifying function are fuzzy as one can see from the graph.
For different values of crisp x variables we will have different L-R fuzzy numbers, with all equal height of 1.

VI.1. Fuzzy Extremum

In the case of an extremum of a fuzzy function on a non-fuzzy domain we may consider the fuzzy function and apply 'alpha cuts' on it. Recall that the alpha cut on a fuzzy set yielded an ordinary set. By applying an alpha cut on a fuzzy function we get an ordinary function. The fuzzy function corresponds to the different associated ordinary functions generated through the alpha cut for all alpha. So we could in maximizing the fuzzy function f, maximize $f_\alpha, \forall \alpha \in]0,1]$. Nather says that the maximum of $\tilde{f}$ could then be defined by a fuzzy set $\tilde{M}$ which contains all the maxima of f_α, $\forall \alpha \in]0,1]$.([1]; p.42)
We can then define y= $\sup_{x \in D} f_\alpha(x)$; the degree of membership of y belonging to the fuzzy set $\tilde{M}$ could then be defined as being equal to the largest alpha. Note that D stands for the domain of the function. This is written as:

$$U_{\tilde{M}}(y) = \sup\{\alpha : \sup_{x \in D} f_\alpha(x) = y\}$$

This is somewhat of an ad hoc definition which may fit its purpose in some circumstances but is of course not a general proposition.

Chapter VII. Flou, Disorder, and Entropy

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VII.1. Fuzzy Index:

a) Generalized Hamming Distance: $d(\tilde{A}, \dot{\tilde{A}}) = \sum_{i=1}^{n} \left| U_{\tilde{A}}(x) - U_{\dot{\tilde{A}}}(x) \right|$

Where $U_{\underset{\sim}{A}} = 0 \; if \; U_{\tilde{A}}(x) < .5 \; and \; U_{\underset{\sim}{A}} = 1 \; if \; U_{\tilde{A}}(x) > .5 \; and \; U_{\underset{\sim}{A}}(x) = 0 \vee 1 \; if \; U_{\tilde{A}}(x) = .5$.

This description is the one of *an ordinary subset nearest to a fuzzy subset.*

From this distance formulation we use the so called *linear fuzzy index:*

$v(\tilde{A}) = 2/n.d(\tilde{A}, \dot{\tilde{A}})$ This linear fuzzy index, which is one among many fuzzy indexes, can give us an idea of the gap between a fuzzy information and a binary information.

VII.2. Probabilistic Entropy

Fuzziness can also be investigated through entropy. The entropy measures the degree of disorder of components with respect to the probabilities of state. Define fuzzy entropy as:

$$H(\Pi_{\tilde{A}}(x_1), \ldots \Pi_{\tilde{A}}(x_n)) = -1/\ln(N).\sum_{i=1}^{N} \Pi_{\tilde{A}}(x_i).\ln(\Pi_{\tilde{A}}(x_i))$$

$$\text{where } \Pi_{\tilde{A}}(x_i) = U_{\tilde{A}}(x_i) / \sum_{i=1}^{N} U_{\tilde{A}}(x_i)$$

There is severe problem with such definition. Take the case where $\tilde{A} = (.1,.1,.1,.1,.1,.1) \; and \; \tilde{B} = (.8,.8,.8,.8,.8,.8)$. Then in both cases H will be the same (about .89) Why? Because through $\Pi_{\tilde{A}}(x_i) = U_{\tilde{A}}(x_i) / \sum_{i=1}^{N} U_{\tilde{A}}(x_i)$, $\Pi_{\tilde{A}}(x_1) = .1/.6 = .16 \; and \; \Pi_{\tilde{B}}(x_1) = .8/4.8 = .16$. A better example is to consider the fuzzy sets $\tilde{A}$ and $\tilde{B}$ with the ordinary set A=(1,1,1,1,1,1) being an ordinary set, and $\tilde{B}$= (.1,.1,.1,.1,.1,.1). Both cases yield H=1! This makes no sense! The ordinary set A would have the same entropy as the fuzzy set!

VII.3. The Fuzzy Index of de Luca and Termini

The serious problem which occurred with the probabilistic entropy approach is that it does not take into account the effective values of $U_{\tilde{A}}(.)$ but only looks at relative values.

De Luca and Termini propose a much more serious approach to indexing fuzziness.

The set up is as follows:

a) define $\beta(\tilde{A})$ as a degree of fuzziness of the fuzzy set $\tilde{A}$. Three properties should be satisfied:

i) $\beta(\widetilde{A})$=0 iff $U_{\widetilde{A}}(.)$=0 or $U_{\widetilde{A}}(.)$=1; this makes sense as we are in an ordinary set here and the fuzziness should therefore be 0.

ii) $\beta(\widetilde{A})$ is maximal iff $U_{\widetilde{A}}(.)$=1/2; this is also clear; the fuzziness becomes highest for membership values which becomes closest to 1/2. A membership values of .9 is less fuzzy than .49.

iii) $\forall x \in E: U_{\widetilde{A}}(x) \geq 1/2 \; and \; U_{\widetilde{B}}(x) \geq U_{\widetilde{A}}(x) \Rightarrow \beta(\widetilde{A}) > \beta(\widetilde{B})$
$\forall x \in E: U_{\widetilde{A}}(x) \leq 1/2 \; and \; U_{\widetilde{B}}(x) \leq U_{\widetilde{A}}(x) \Rightarrow \beta(\widetilde{A}) > \beta(\widetilde{B})$

This property makes again plain sense. If the membership value of an element of the fuzzy set $\widetilde{A}$ is close to .5 for instance then if the membership value of an element of the fuzzy set $\widetilde{B}$ is higher than .5 (and thus further removed from .5) then the fuzziness is higher relative to the fuzzy set A than to the fuzzy set B. The second line in iii) casts the same intuition.

iv) Let $\psi(\widetilde{A}) = -k.\sum_{i=1}^{N} U_{\widetilde{A}}(x_i).\ln(U_{\widetilde{A}}(x_i)), k > 0$; this is a formulation which is reminiscent of the probabilistic entropy definition we saw above. Formulation iv) creates very severe problems. In analogy with the probability content of the probabilistic entropy measure we encounter the problem that for instance $\sum_{i=1}^{N} U_{\widetilde{A}}(.)$ does not necessarily add up to '1', as it should be the case with probabilities. (as in the probabilistic entropy definition)

The goal is to keep the formulation of the probabilistic entropy but by avoiding the relative measuring. We of course must also solve the problem of non-adding to '1'.

Therefore de Luca and Termini introduce some additional properties on $\psi(.)$.

i1) $\forall \widetilde{A} \subset E: \psi(\widetilde{A}) \geq 0$, which looks acceptable out of the definition. We know the membership values are taken here in the simplest membership set; i.e. [0,1]; hence the logarithm must be smaller or equal to 0. Membership values are positive. The negative sign in $\psi(.)$ makes thus the whole expression positive.

i2) $\forall \widetilde{A} \subset E, \forall \widetilde{B} \subset E: \psi(\widetilde{A} \cup \widetilde{B}) + \psi(\widetilde{A} \cap \widetilde{B}) = \psi(\widetilde{A}) + \psi(\widetilde{B})$; the latter is proven in ([39]; p.43).

i3) $\psi(\widetilde{A}.\widetilde{B}) = card\widetilde{B}.\psi(\widetilde{A}) + card\widetilde{A}.\psi(\widetilde{B})$

We know the measure of cardinality of a fuzzy set, as defined before: card($\widetilde{A}$) $= \sum U_{\widetilde{A}}(x)$. It is clear that if $card(\widetilde{A}) = 1 = card(\widetilde{B})$ then $\psi(\widetilde{A}.\widetilde{B}) = \psi(\widetilde{A}) + \psi(\widetilde{B})$. Furthermore, and this is important, when $card(\widetilde{A}) = 1$ then $\psi(\widetilde{A}) = H(\widetilde{A})$ as then $\sum U_{\widetilde{A}}(.) = 1$ and we recuperate the definition of probabilistic entropy.

What we now want to show is whether $\psi(.)$ satisfies i)ii)iii) of above. Recall that conditions i)ii)iii) are very reasonable indeed.

i-) recall the definition of $\psi(.)$ which is: $\psi(\widetilde{A}) = -k.\sum_{i=1}^{N} U_{\widetilde{A}}(x_i).\ln(U_{\widetilde{A}}(x_i)), k > 0$; then $\psi(\widetilde{A}) = 0$ as through i) membership values are either 0 or 1. We must omit '0' however as the logarithm of '0' does not exist. This is a problem however as the logarithm of 0 can only be approximated by a limit which tends to $-\infty$. We conclude that for ordinary sets $\psi(\widetilde{A}) = 0$.

ii-) From the definition of $\psi(.)$ we could calculate the maximum, and it occurs at $U_{\widetilde{A}}(x) = 1/e$ which is definitely not 1/2 as prescribed through ii)!

The way to circumvent this problem is as follows:

a)define $\rho(\widetilde{A}) = \psi(\widetilde{A}) + \psi(\overline{\widetilde{A}})$ *and* $U_{\overline{\widetilde{A}}}(x_i) = 1 - U_{\widetilde{A}}(x_i)$; where we must remark that a Boolean lattice would be a necessity to provide for a unique complement.

Next de Luca and Termini introduce the so called *Shannon Function* which is a function which is monotonically increasing in [0,1/2] and monotonically decreasing in [1/2,1]; having a maximum at 1/2 (which we need dearly to satisfy ii))

The Shannon function is defined as:

$$S(x) = (-x)\ln(x) - (1-x)\ln(1-x)$$

Knowing how $\rho(\widetilde{A})$ is defined we can write it up:

$$\rho(\widetilde{A}) = -k\sum_{i=1}^{N} U_{\widetilde{A}}(x_i)\ln U_{\widetilde{A}}(x_i) + U_{\overline{\widetilde{A}}}(x_i)\ln U_{\overline{\widetilde{A}}}(x_i)$$

$$\rho(\widetilde{A}) = -k\sum_{i=1}^{N} U_{\widetilde{A}}(x_i)\ln U_{\widetilde{A}}(x_i) + (1 - U_{\widetilde{A}}(x_i))(\ln(1 - U_{\widetilde{A}}(x_i))$$

Using the definition of the Shannon function: $S(x)=(-x)\ln(x)-(1-x)\ln(1-x)$ and replacing 'x' by $U_{\tilde{A}}(.)$ we get then:

$$\rho(\tilde{A})=k.\sum_{i=1}^{N}S(U_{\tilde{A}}(x_i))$$

This result satisfies ii) and iii) and hence de Luca and Termini's fuzzy index guards the form of the probabilistic entropy without reverting to relative values, i.e. we use the full membership function. Furthermore the problem of non-adding to '1' is now also solved as $U_{\tilde{A}}(.)$ *and* $1-U_{\tilde{A}}(.)$ add up to '1'.

Note : k in $\rho(\tilde{A})=k.\sum_{i=1}^{N}S(U_{\tilde{A}}(x_i))$ is a factor which normalizes such that $0\le\rho(\tilde{A})\le 1$.

The necessary formulation for k is then $k=1/N.\ln 2$

Example:

Fuzzy set A:[.1,.1,.1,.1,.1,.1]

Fuzzy set B:[.8,.8,.8,.8,.8,.8]

Fuzzy set C:[1,1,1,1,1,1]

Using $H(\Pi_{\tilde{A}}(x_1),\ldots\Pi_{\tilde{A}}(x_n))=-1/\ln(N).\sum_{i=1}^{N}\Pi_{\tilde{A}}(x_i).\ln(\Pi_{\tilde{A}}(x_i))$ we obtain H=1 in all cases which makes no sense. de Luca and Termini's fuzzy index yields the following:

Our intuition should show us that fuzzy set $\tilde{A}$ is less fuzzy than fuzzy set $\tilde{B}$ as it is further removed from .5. Clearly fuzzy set C which is an ordinary set should not be fuzzy at all.

Using $\rho(\tilde{A})=k.\sum_{i=1}^{N}S(U_{\tilde{A}}(x_i))$ we obtain:

for fuzzy set $\tilde{A}$: 1.95 and normalized yields:.47

for fuzzy set $\tilde{B}$: 3 and normalized yields .72.

for ordinary set C: 0

If $\nu(\tilde{A})=2/n.d(\tilde{A},\dot{\tilde{A}})$ is used, which we called a linear fuzzy index we get:

for fuzzy set $\tilde{A}$= .2 =(.1(6).(2/6))

for fuzzy set $\tilde{B}$= .4

for ordinary set C: 0

The values are substantially different; the value for the ordinary set remains zero.

To make a true assessment of both approaches is difficult given that we do not have really a bench mark to which we can compare the obtained values. The highest value is of course for x=0.5 and is 0.69. The lowest is for 0 and 1; which yields 0. Those are however non normalized values. For instance for [.5,.5,.5,.5,.5,.5] a normalized value of 0.99 is yielded.

Part III: The Membership Function

Eliciting Membership functions and classes of membership functions: Introduction

Fuzzy sets has as main goal to provide for a more richer approach to modeling 'common' thinking patterns. Binary logic has been the basic tenet for the formalization of thinking patterns. In 1982 Robin Giles, wrote an interesting statement concerning the dangers practitioners and/or theorists of fuzzy set theory may face. Everything boils down in finding the *meaning* of a grade of membership. Giles advances the example of 'an unbreakable glass to a degree of 0.9' An agent asserting such statement expresses information about a belief he holds. But as Giles says, it is by no means clear what the information is behind that belief. This is where Giles point comes in about the dangers of fuzzy set theory. Says Giles 'it is common to avoid this question by saying the matter is not important....(this) provides a measure of freedom to the theorist, who can suggest rules (axioms) for manipulating grades of membership and for introducing new fuzzy sets almost without constraint, and certainly without fear of refutation...in turn this offers to the practitioner the possibility of choosing from a variety of such theoretical procedures each in general yielding different results. The conscientious investigator may be worried by this , but others can benefit from the freedom by selecting a procedure that yields the conclusions they prefer.' ([23]) This statement can not be clearer and truer.

When entering the field of eliciting membership functions we automatically have to analyze the meaning of a fuzzy sentence. Closely related to the problem of electing a suitable membership function is the discussion of suitable operators. This important subject will be treated in part IV.

The subject of electing membership functions can be approached, as in many fuzzy theory subjects, from two viewpoints: either a pragmatic more intuitive point of view or else a purely mathematical approach. The former tries to deduce immediate conclusions from a practical concept. This act of deduction brings us by repeated choosing of different subjects to build up gradually a theory. The latter however lies down first a mathematical structure and then will find possible connections between this mathematical structure and practice. This is the 'hard' way in some sense. Whether it is the best way is debatable.

Some authors will claim that by using the *syntactic* approach, as it is called by Ellen Hisdal, one looses track of the practical implication of the theories so derived. This is avoided in the *semantic* approach. This paper will stick as much as possible to the semantic approach for immediate reasons such as the author not being a mathematician.

As said above already the subject of electing appropriate membership functions is an important one. However it is often omitted! Dombi for instance remarks that many applied papers in the fuzzy set area have been written without a proper *ex-ante specification* of the membership function. It is as if in production theory we would not specify the specific production function we are about to use! Jain comes to the same conclusion. Says Jain 'Most papers in the field start with a given ..membership function; without any mention of how and why they were chosen...'.Chanduri and Majumder say that '..it is easy to find a function that takes on values in [0,1] and is monotonic over a range, but the compatibility of a sample to the set may not be reflected by the function....'

We will survey several viewpoints. For instance Dombi's historical survey of membership functions and his mathematical model of a membership function, and Giles' semantic approach. It is somewhat difficult to classify in a clear-cut way where Dombi's approach stands. We leave this up to the reader to decide. Giles point of view is clearly semantic; so is Hisdal's. Smets and Magrez are looked briefly at and their approach is more of a syntactic nature. At this point we also want to make the distinction between a measure of information approach versus a meaning of information approach. Bouchon and Kaufmann are in the measurement class; while Giles, Hisdal and Smets would be in the meaning of information class. Those two classes are in principle disjoint. I believe however that they can be used *together* to form a better picture. We will not expand on the work by Bouchon and Kaufmann. Bouchon is essentially concerned with a method to finding the best combination of fuzzy answers which reduces to a minimum the loss of information when fuzzy answers are attached to crisp answers. Her treatment is quite interesting but has the drawback she does not define a fuzzy event. Kaufmann's approach on different levels of uncertainty using essentially the probability of a fuzzy event and the Shannon function is also highly interesting. However though we may be capable of measuring a total entropy as he defines it, we do not have any benchmark value upon which we can refer.

Kaufmann's method defines explicitly a fuzzy event; something Bouchon does not do. For Bouchon see [37]; for Kaufmann see [39].

Chapter I. *Meaning of Information 1*: Dombi's Rough classification of Membership functions.

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An example of a membership function is for instance: Zadeh's Unimodal Function:

$$U_{young\ age} = \begin{cases} 1/\{1+[(x-25)/5]^2\} \ if\ x > 25 \\ 1\ if\ x \leq 25 \end{cases}$$

Where of course '25' is a perfectly arbitrary cut off point. The membership value for a value less than '25' is thus 1. We can formulate the 'old age' membership function similarly. Dubois and Prade's L-R fuzzy number is another example of a membership function.

I.1. Dombi's Common Traits

Based on a historical survey Dombi performed on fuzzy set articles for the period of 1987 to 1990, Dombi comes to the following common traits or properties for membership functions:

1. they are continuous functions
2. they map [a,b] into [0,1]
3. they are either monotonically increasing or decreasing or could be divided into monotonically increasing and decreasing parts.
4. the functions may be either concave or convex or both, in the latter case they are called S-shaped.
5. some membership functions have meaningless parameters while other membership functions are really too general in nature. Finally in some membership functions the parameters are very hard to calculate.

I.2. The 'better' membership function-Dombi's mathematical model of a membership function.

To create a 'better' membership function Dombi tries to use his 'common traits' as derived from the survey of membership functions and avoids the pitfalls mentioned under (5) above. The 'better' membership function according to Dombi should then have the following characteristics:

1. The membership function must be continuously increasing from $[0,1] \to [0,1]$. This is an expectable requirement; the more we advance on [0,1] the higher the membership value is supposed to be.

2. $U_{\tilde{A}}(0) = 0 \; and \; U_{\tilde{A}}(1) = 1$ which are the boundary conditions. This also is a natural requirement. I.e. the membership value at x=1 must be '1'.

3. S-shaped character; i.e. $U'_{\tilde{A}}(0) = 0 \; and \; U'_{\tilde{A}}(1) = 0$.

4. $U_{\tilde{A}}(x)$ is a rational function of polynomials. This is quite less obvious.

5. Find such membership function such that n+m is minimal; where 'n' stands for the degree of the nominator polynomial and 'm' as the degree of the denominator polynomial.

Properties '3','4' and '5' look less intuitive. There is clearly no apparent reason, assuming we would start from scratch, to require that the membership function should be S-shaped. But many membership functions surveyed by Dombi were S-shaped! There is no direct apparent reason why we should assume a rational function of polynomials, but again many surveyed membership functions were of that form. This is of course not enough of an argument to propose all five conditions. Below we will see whether Dombi comes forth with more solid arguments for proposing his five properties of the 'better' membership function.

I.2.1.Theorem 1:

There are no membership functions fulfilling the properties 1-5 if $n+m \leq 3$.

Proof:

Say we define *U(x)*=ax+b / (Ax+B); therefore m+n=2 $\leq$ 3.

Then as *U(0)*=0 (property 2) we have b=0 which is straightforward.

But if $U(1)=1$ and b=0 then a=A+B. Therefore substituting we obtain that: $U(x)=(A+B)x / (Ax+B)$. Taking the derivative of this and using $U'(0)=0$ (property 3) it must be that (A+B)B=0 inducing either A+B=0 or B=0. If B=0 then $U(x)\equiv 1$, this is obvious as in $U(x)=(A+B)x / (Ax+B)$, putting B=0 we get Ax/Ax which makes 1 for all x. Similarly for A+B=0, making a=0 and yielding $U(x)\equiv 0$. This contradicts property 2. Say we now define $U(x)=ax+b/(Ax^2+Bx+C)$; which gives for m+n=3. After calculation of the first derivative this again leads to a contradiction. Similarly if we want to define $U(x)=ax^2+bx+c/(Ax+B)$; giving again n+m=3; we will again encounter a contradiction. The argumentation is the same.

We can see that use is made of the imposed property of the S shape of the function.

I.2.2. Theorem 2:

The minimum of n+m is 4 and the membership function is:

$$U(x)=\frac{(1-v)x^2}{(1-v)x^2+v(1-x)^2}$$

where 'v' is the intersection value of $y=U(x)$ and the 45 degree line, v is called the characteristic value of the shape.

Proof:

Let $U(x)=\dfrac{ax^2+bx+c}{Ax^2+Bx+C}$. We know that $U(0)=0$ therefore c=0 is immediate. Calculating the first derivative we get:

$$U'(x)=(Ax^2+Bx+C)(2ax+b)-(2Ax+B)(ax^2+bx+c)/(Ax^2+Bx+C)^2$$

The derivative simplifies slightly because of the fact c=0. Because of property 3 we know that $U'(0)=0$ so we have in the above derivative that C.b=0 (remark that already c=0); but as theorem 1 asserts that n+m >3, to satisfy the five properties then C can not be 0, therefore b=0 in C.b=0. C can not be zero plainly because if $U(0)=0$ (property 2) then given that c=0 we would have 0/0 which is not defined.

Thus $U(x)$ becomes: $U(x)=\dfrac{ax^2}{Ax^2+Bx+C}$, (1); and as $U(1)=1$ and $U'(1)=0$ we get directly from the 'new' U(x), (1) that a=A+B+C and also taking the first derivative of the 'new' U(x),

(1) and using $U'(1)=0$ we obtain B=-2C. Substituting those newly obtained equalities in (1) yields:

$$U(x)=\frac{(A-C)x^2}{Ax^2-2Cx+C}$$

where it is required that A $\neq$ C.

The form can be re-written as:

$$U(x)=\frac{x^2}{x^2+\alpha(1-x)^2},(2)\ where\ \alpha=C/(A-C)$$

To show that $\alpha=v/(1-v)$ where $U(v)=v$ simply derive alpha from $v=\dfrac{v^2}{v^2+\alpha(1-v)^2}$

1)$U(v)=v$ takes care of y=$U(x)$ and y=x where v is the intersection value.

2)Substituting $\alpha=v/(1-v)$ into (2) we get the form $U(x)=\dfrac{(1-v)x^2}{(1-v)x^2+v(1-x)^2}$

The latter can be re-written by dividing by $(1-v)x^2$, we obtain then:

$U(x)=\dfrac{1}{1+\dfrac{v}{1-v}\dfrac{(1-x)^2}{x^2}},(3)$. As one can see from the above, the membership function depends on the parameter v. Dombi claims, and rightly so, that this membership function has the same form as the Zimmerman-Zysno function.

This function is defined as: $U(x)=1/2+(1/d)[1/(1+e^{-a(x-b)})-c]$; where 'd' indicates 'distance' between the given object and the ideal object. 'c' and 'd' have particular formulas attached for purposes of estimation.

Defining membership as a function of distance (i.e. the distance between the given object and an ideal object) Zimmerman-Zysno used: $U(x)=\dfrac{1}{1+d(x)}$ where in our general derivation using (3); $d(x)$ becomes $d(x)=\frac{v}{1-v}\frac{(1-x)^2}{x^2}$, (4). Zimmerman and Zysno used thus a 'distance' approach where the distance is '0' when the given object we are trying to evaluate has all ideal features and the distance is infinite when there is total dissimilarity between given and ideal object. We can easily see that if *d (x)* = 0 the membership value is '1' while if it is infinite the membership value is '0'. This is quite informative as it shows Zimmerman-Zysno respect the conditions put forth by Dombi for a 'better' membership function. Of course we must define the distance concept itself and then compare again to what Zimmerman and Zysno did, before we can arrive towards a final judgment. It turns out that *d(x)* has already a form as exemplified in (4). Dombi generalizes this distance function as follows: $d(x)=\frac{(v)^{\lambda-1}}{(1-v)^{\lambda-1}}\frac{(1-x)^{\lambda}}{x^{\lambda}}$, (5) where the generalization is straightforward by looking at (4). *v*'s exponential is one less than *(1-x)*; and similarly for the denominator. Plugging (5) in (3) and re-writing we get for the new more general membership function:

$$U(x)=\frac{(1-v)^{\lambda-1}x^{\lambda}}{(1-v)^{\lambda-1}x^{\lambda}+v^{\lambda-1}(1-x)^{\lambda}},\lambda>1,(6)$$

(6) is valid for $x\in[0,1]$, if we want to transform (6) to be valid on [a,b] we get:

$$U(x)=\frac{(1-v)^{\lambda-1}(x-a)^{\lambda}}{(1-v)^{\lambda-1}(x-a)^{\lambda}+v^{\lambda-1}(b-x)^{\lambda}},(7)$$

As one can see this takes the same form as (6); but adaptation is made to work within the interval [a,b]. The transformation on [a,b] is important because most of the time we will not work with x-values which are in [0,1]. Height is an example. Remark also that *U(x)* $\in[0,1]$. Finally remark that (7) exemplifies the case of a monotonically increasing S-shaped function. The distance function is now also modified: as:

$$d(x)=\frac{v^{\lambda-1}}{(1-v)^{\lambda-1}}\frac{(b-x)^{\lambda}}{(x-a)^{\lambda}},(8)\text{ (relative to (7))}$$

A brief intuitive discussion on the parameters is now important:

1) The larger the [a,b] we are considering the larger *d(.)* will be. (c.p)

2) The higher the λ factor is the larger *d(.)* will be.(c.p)

3) The lower v is the lower the $v/1-v$ fraction will be and the lower *d(.)* will be.(c.p)

4) The λ factor has no influence on *(b-x)/(x-a)* when choosing a+b/2 in [a,b].

We remark that v(x) is nothing else than a fixed point. We have a continuous function on a compact set, i.e [a,b]. v(.) can be seen as an indicator of variability of the membership function. This is also an indicator of ambiguity. While λ can be seen as an indicator of sharpness or also vagueness. The higher λ is the less vagueness we encounter. (figure 20)

We remark that (7) satisfies Dombi's properties for a 'better' membership function. We think Dombi does not make a solid argument why a rational form of polynomials should be followed for a 'better' membership function. The S shape is also not solidly argued. Finally we remark that property 5 is a property which we think has to do with a simplification of membership functions. Requiring m+n to be minimal implies finding the 'better' form in its most simplified form.

I.2.3. The parameters in action

Zysno did an experiment in which 64 subjects from age 21 to 25 were asked to rate 52 different statements concerning age with reference to four groups: very young/ young/ very old/ old. The four groups are four fuzzy sets. It is straightforward to show that; by using (7) and having $d(x)=\dfrac{v^{\lambda-1}}{(1-v)^{\lambda-1}}\dfrac{(b-x)^{\lambda}}{(x-a)^{\lambda}}$,(8); that $\dfrac{1-U}{U}=d(x)$. Taking logarithms on both sides we obtain then:

$$\ln(\frac{1-U}{U})=\lambda\ln(\frac{b-x}{x-a})+(\lambda-1)\ln(\frac{v}{1-v})$$

where we can simplify the notation somewhat:

y= $\ln(\dfrac{1-U}{U})$ and x= $\ln(\dfrac{b-x}{x-a})$ and c= $\ln(\dfrac{v}{1-v})$, so the membership function becomes now in shorthand notation: $y=\lambda x+(\lambda-1)c=\lambda x+d$.In short because of the fact we use logarithms, we established a linear relationship between x and y and therefore the straight line can be estimated through the least squares of deviation. (figure 21)

The results Dombi obtains yield optimal λ and ν. Those 'optimal' values are the result of minimizing the least square error when fitting the data against the regression.
The λ values are quite low for 'very young' and are higher for old, young and very old. This may indicate there is less 'vagueness' in the latter three fuzzy statements than in the first one. The slopes of the membership functions are also steeper the higher the λ values. What is less clear are the ν values. Comparing the ν values for very young and young we see a difference of approximately 0.17; while comparing old and very old we see a difference of about 0.1. This may indicate that the variability of the membership function is higher in the young/very young case than in the old and very old case. This is of course applicable to the sample which was selected to answer questions relating to the four fuzzy sets.

I.3. Discussion

Are we wiser after this small exposition? I do not think Dombi argues solidly why he proposes his five properties other than the fact they are based on common traits Dombi derived from surveying membership functions over a three year period. I do have some trouble seeing why the membership function should be necessarily S-shaped. Dombi shows that under his properties his 'better' membership function is very close to the Zimmerman-Zysno membership function. We could appreciate the Zimmerman function to a fuller extent would the properties Dombi proposes have been more solidly argued. Why assume that the membership function should be a rational function of polynomials? To assume an S-shape membership function is certainly not unreasonable, far from that but why only assume such a shape? Deriving common traits of membership functions based on a historical survey is not enough as an argument, I would say.

Chapter II. *Meaning of Information 2:* Smets and Magrez

The approach by Smets and Magrez tries to link degree of truth with grade of membership. This 'linking' is crucial and will be made explicit under the Giles approach. The basic idea is to consider an object which has a continuum of values attached. We then decide whether the proposition relating to this object is true or false for extreme values of this continuum. For the case of 'John is tall' when John is 100 cm in height the proposition would be false but true when John is 190 cm. Thus when the height moves from 100 cm towards 190cm the proposition of John being tall increases in value. We can then find a cut off value at which 'tallness' starts. Underlying this logic with a multivalued truth domain is a set of axioms which relate to the implication operator. This is where Smets and Magrez are merely syntactic than semantic in their approach. Smets and Magrez's method seems to have as objective to be precise in that it wants to attach significance to a membership value of say .2 as being equivalent to a truth value of .2. But we would omit a crucial assumption in Smets and Magrez's approach and that is that the reference scale is *strictly personal.* This means there is no assumption whatsoever by the authors there would be a universal definition on what for instance *tallness* would mean just as the 'meter' would be defined. The workability of the method however is quite questionable.

Chapter III. *Meaning of Information 3:* Giles

III.1. The classical sentence versus the fuzzy sentence

Giles starts out wondering what the use is of a 'sentence'. It is there to communicate information. The information, most of the time, will not consist of facts but of beliefs and even *degrees of belief.* The classical sentence apart from conveying facts conveys a belief but no degree of belief. The fuzzy sentence will convey a degree of belief. Furthermore, as Giles says, the degree of belief has a tight relation with a subjective degree of truth.

III.2. The notion of meaning-meaning of the classical sentence

In the case of a classical sentence it is sometimes claimed that the meaning of a sentence is given by its truth value. Thus if John is effectively over 180 cm tall then the meaning of 'John is over 180 cm tall' is the same as '1+1=2'.

As an example Carnap said that 'The connection between meaning and confirmation has sometimes been formulated by the thesis that a sentence is meaningful iff if its verifiable; and that its meaning is its method of verification' ([9];p.421) But one may wonder whether the concept of 'meaning' is to be restricted that much. In fact, as Giles points out, we can easily sense the meaning of the sentence 'John is over 180 cm tall' WITHOUT having to know whether this is actually true, and thus meaning may exist without an explicit immediate truth value. Furthermore, it is not very useful to engage in communication on sentences which are universally true. This is the case of '1+1=2'. Classical sentences will have the same meaning if their truth values would be the same under all conditions, or in all states of the world. Thus the meaning of a *classical* sentence is to be identified through its truth function which generates a map $\phi:\Omega \rightarrow \{0,1\}, \Omega$ being the set of all world states. And $\phi(\omega) = 1$ if the sentence is true in world state ω, and '=0' if the sentence is false. If I am confident that 'John is over 180cm tall' but I do not have measured his height; i.e. thus I am not knowledgeable *of the actual world state.* Say it is my *belief* that this current world state belongs to a subset of all possible world states, Ω. Giles is explicit in that here I have a belief, and not a degree of belief. The belief is that the current world state (i.e. John's actual height) , which I do not know , is part of a subset of Ω. When will I assert? In pure terms I will assert the sentence if $\phi(\omega)$ expresses the truth for every $\omega \; in \, \Omega_o \subset \Omega$. Mind that my *belief* refers to my assumption the actual world state belongs to Ω_o. Let us be careful however that the truth function in this case only refers to a subset of all world states. Hence truth is only partial. The reason for this is twofold:

a) we work with a belief the actual world state belongs to a subset of all world states

b) the truth function refers to the subset and the not the full set of all world states.

Thus Giles defines the *meaning of a classical sentence with its truth function as:*

Proposition I: The meaning of a classical sentence is that information that is necessary and sufficient , in conjunction with an agent's beliefs about the world state, to allow the agent to decide whether or not to assert the sentence.

For the example of 'John is over 180cm' the necessary and sufficient information will have to purport to who John really is. Once we know who John is, we must envision all the *possible* worlds in which John can be over 180cm. This can be for instance the interval [180,300].

III.3. The meaning of the fuzzy sentence

For a fuzzy sentence things are different. Now, I have various degrees of willingness to assert a fuzzy sentence ; my degree of willingness may be related to my degree of belief in the sentence. The reason for that is quite simple. Because I have a 'fuzzy' statement in my sentence I have degrees of beliefs as to which the possible world could be. For instance saying that 'John is tall' is a fuzzy sentence. I must find out who is John, this will yield me the necessary information; but this information will not yield sufficiency! I also must find information relevant to what tallness is all about. I then have to think about the possible world states and the height of John will be critical here.(f.i $\Omega = [130,200]$) But my belief in the world states which are possible is fuzzy because of the fuzzy qualifier 'tall' in the sentence. Thus, I must have degrees of belief, to the opposite of the case where 'John is over 180cm'.

Proposition 2: Two fuzzy sentences have the same meaning to me *iff*, under the same conditions, I am always exactly as willing to assert one as to assert the other.

III.4. Choosing to assert or not to assert

Assuming that a normal person wants to tell as much as possible the truth it is quite easy to imagine the agent may have trouble deciding what assertion to make.

Would he know all the outcomes of every assertion he makes he would have to choose this assertion which he *most prefers*. It is expectable that this is extremely rare. For the case of 'John is over 180cm' we have a belief about the possible states and this implies we are not sure of the outcome of our assertion. It is assumed that the possible outcome is a known function $f(A,\omega)$, where *A stands for 'act' (an assertion is an act)*; and ω stands for the world state. Of course the agent has partial knowledge of the actual world state. We have been introducing an important argument and that is the one of 'most preferred'; which implies utilities. Here some trouble arises with Giles' argument.

Proposition 3: Assuming that the agent's preferences between the various outcomes and between lotteries which have these outcomes as prizes are known; it is possible to represent these preferences by assigning to each outcome a numerical utility *u* which indicates its value to the agent; in such a way that also the value of any lottery is given by its corresponding weighted mean of the utilities of the outcomes involved.

The utility scale is ordinal and only affine transformations can be considered; as expected. We must imagine the full set of all possible outcomes. Different lotteries may yield different outcomes; i.e. some outcomes of the full set may be realized under certain lotteries with zero probability. The value of a lottery will be given by the weighted mean of utilities of the various outcomes permitted by the specific lottery in question. The latter is important because it refers to the fact the agent can distinguish between lotteries. He will choose that lottery which brings him the highest weighted average in utility. The issue of the lottery per se is appropriate also. The agent before asserting or *acting* is in an uncertain state in that he does not know what the actual world state is. Does the actual world state belong to $\Omega_o \subset \Omega$? Either I belief that it does belong or I belief it does not. If I belief it does belong then probable outcomes are considered following a certain lottery. The problem which occurs here is that we explicitly handle probabilities rather than possibilities. We must consider all possible worlds, rather than only probable worlds. Furthermore proposition 3 would be hard to maintain within a fuzzy context, as we are confronted with *degrees of beliefs*. In the case of the classical sentence things are quite simple; either you belief 'belonging' occurs either you do not. Then you consider all possible outcomes which follow different lotteries. In the fuzzy case, from the outset you have degrees of belief. Once the degree is determined you must follow a lottery of outcomes *given* that degree of belief. How do we determine this degree of belief? Does it also follow a probability distribution? This is unlikely!

What Giles is introducing here is Bayesianism. For belief functions the Bayesian approach may be awkward. This issue will be discussed further up.

III.5. Pay-off function

In simple terms the agent whether he is dealing with a classical or fuzzy sentence derives utility from either asserting the sentence or not. It may effectively be that not asserting yields higher utility than asserting. The problem to act is rendered more difficult within a fuzzy context. The utility scale will yield a *pay-off function.* We now must be more specific how this pay off function can be determined. When I assert a sentence, given I have decided upon my belief, given also I know the lottery outcomes; I will receive a certain utility, or pay-off. When I do not assert I basically receive the opposite of what I would have received by asserting. Say by asserting I obtain negative utility then by not asserting I will get zero utility. It looks acceptable to assume non-assertion will yield zero utility. The objective is to find a payoff of asserting for a state ω; given I assume zero utility for non-asserting. Giles calls this function $v(\omega)$; which gives in the state ω the additional utility of making the assertion as opposed to not making it; Giles calls this also *the pay-off function of the assertion.* It is this pay-off function which will form the basis of making a decision to assert or not to assert.

Proposition 4: The meaning of an assertion is given by its payoff function.

The pay off function is equally valid for a fuzzy sentence. We must be reminded however that the way of arriving to the payoff function in the fuzzy context is more difficult given that we are confronted with degrees of beliefs rather than beliefs.

III.6. 'John is Tall

We must envision a normal society, as Giles terms it. We gain prestige when we assert something which reveals to be true, and loose prestige when asserting something which turns out to be untrue. For the fuzzy sentence of 'John is tall' the truth of the sentence will certainly be a function of the height of John. Hence the pay-off function will certainly have as variable 'height'. The utility of the assertion of 'John is tall' will be *approximately* a function *f(h)* for a given state. *f(h)* is only a rough approximation of the pay off. Our degrees of belief influence the probable outcomes and it is 'tallness' being fuzzy which induces those degrees of beliefs. Thus the pay-off function can not be claimed to be *f(h)* but it is only *approximated* by this function. Finally we are explicitly assuming 'normal' truth seeking individuals. **The meaning of fuzziness lies in the realm of reasonably rational people, *and not in the marginal cases. (jokers, irrational behavior etc.)*** The height function, according to Giles, could also be viewed as an indicator for the *degree of willingness* of the agent to assert 'John is tall'. This is a strong statement because of the fact it is based solely on *f(h)*. We can resort into the 'protection' of approximation but we do not answer how well we approximate. This is a drawback of the serious sort. The *f(h)* function could be determined by offering bribes. Say the agent knows John's height is 170cm. We flip a fair coin and offer the following proposition to the agent: 'if the coin shows heads you'll get the reward of 5 units (figure 22) if not you must assert 'John is tall''. The latter outcome is diminishing the prestige of the agent in some world state. If the agent accepts we know that *f(170)* >5. Penalties would be used for the positive ordinates. The problems with the pay off function as based on *f(h)* in this example are multiple:

1. Height is not the only criterion to consider when considering the payoff function for 'John is tall'

2. We must assume rational individuals

3. Point 2 does not preclude possibilities such as 'stretching the facts'. Therefore Giles proposes the concept of 'average meaning of the assertion in society'. This is clearly vague and hard to describe in precise terms.

Chapter IV. *Meaning of Information 4:* Hisdal

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Hisdal is certainly worth to be mentioned. She presents a unified theory which tries to better explain where membership functions could really come from. She is detailed also on how shapes of membership functions can be explained.

IV.1. Set up of the TEE model

Zadeh's motive to develop a theory of fuzzy sets was basically to show that because individuals communicate with linguistic concepts such as 'tall', 'small' etc..there must exist some underlying procedure for this communication to work. Therefore Zadeh's invented the concept of degree of membership in a set. Hisdal's TEE model makes use of probabilities rather than possibilities. TEE stands for 'Threshold, Error, assumption of Equivalence'. Hisdal's starting point is to claim that in most of fuzzy theory, at least in the semantic realm, a mystic agent is at work making its fuzzy decisions according to some undefined procedure. This is a very important claim. It is crucial because if we want to apply fuzzy sets in a economics context particularly the one of consumer behavior it is useful to have a *serious* grasp of the workings of such procedures which lead us to model preferences for instance. Hisdal's TEE model is instrumental in that.

IV.1.1. Label sets and types of experiments

There are three types of experiments:

1)LB experiments; called labeling experiments

2)YN experiments; called yes or no experiments

3)MU experiments; called grade of membership experiments

The LB experiment corresponds to a situation in which an object is described as being 'very tall' and the YN experiment would refer to a situation in which a person is asked whether 'John is tall?' and he is supposed to answer with yes or no. Finally in the MU experiment a subject is asked to assign a membership value $\mu \in [0,1]$ to an object concerning the label λ. It is clear that the MU experiment can refer either to an LB or YN experiment. The latter two experiments being the ones we encounter in real life.

An important requirement is that the experiment must refer to a label set $\Theta = \{\lambda_1, \lambda_2, .. \lambda_L\}$. An example is for instance $\Theta = \{small, medium, tall\}$

IV.1.2. Sources of fuzziness

Hisdal's approach with the TEE model is to avoid any ad-hoc kind of definitions. I mean, definitions which are re-formulated or plainly changed to fit the situation. The objective of the TEE model is to be able *to derive correct formulas and operators.* We only mention two sources of fuzziness here.

1)Fuzziness #1a

This source of fuzziness is defined by Hisdal as follows:

Anticipation by the subject of errors of observation under nonexact conditions. Even if the subject performs an exact experiment in which he measures μ^{ex}; he is aware of the fact that under non-exact conditions of observations his estimate μ of the attribute value may assume varying values according to some probability distribution or 'error function' $P(\mu|\mu^{ex})$.' Some small explanation is needed on some of Hisdal's wording. The 'exact conditions of an experiment' refers to a semantic experiment in which the subject measures (or is told) the exact attribute value of each object. So $\mu = \mu^{ex}$.

2)Fuzziness #3a

Intersubject fuzziness. It is for instance the subject's awareness that different persons may have somewhat different ideas as to the threshold height value in an LB or YN experiment.

IV.2. Discussion

Hisdal's collection of papers makes a very detailed account on her TEE model. Space is in this thesis is very limited and therefore we must omit important details. For the sake of the argument we deem that what follows below is a reasonable summary of Hisdal's argument.

In *an exact labeling experiment* (where thus μ^{ex} is known) the subject will set a threshold and a 2-valued threshold function will emerge. This will yield a step-function. As an example the sentence 'tall men' may have a threshold of 170cm if the object has been measured at 170cm. The agent will therefore set an upper and lower limit for each label in her label set. Thus the agent may set 170cm as the threshold for tall. When the labeling experiment is *not exact* she may make an error in the cut off value , i.e where the value zero changes into one. Therefore the threshold function is now a rounded off version of the two valued step function as obtained under exact conditions. In the case of an MU experiment the connection between MU and LB is basically modulated through fuzziness #1a. In an MU experiment under exact conditions for instance ,the TEE assumes the agent sees a connection between the membership value and the labeling experiment. Through fuzziness #1a she knows that she can make an error in her estimate of the height value of the object. Thus even though we are in an exact experiment where the height is known the agent assumes that in everyday life this will not be so. This looks a little weird at first sight. However; the MU experiment is more general than YN or LB experiments; because of this generality we impose every day conditions through fuzziness #1a. This generality can even be more extended to fuzziness of others sorts such as #3a for instance. In our same example the agent will ask himself the following question in relation to the MU experiment: 'given I am confronted with this object (i.e. a man of exact height 170cm) what is my estimate of the fraction of times I would estimate the height of the object to be above 170 in an *everyday* situation?' Thus a reference is used and this is the exact height but an estimation is put relative to this exact value. Thus the membership curve, even in *exact* conditions is a rounded off version of the non-fuzzy threshold curve. We remark however that this membership function is the result of the 'easiest scenario' in the TEE model; i.e. fuzziness of #1a and exact conditions. Remark that if we would consider an MU under *non-exact* conditions that further rounding off would occur. First the MU experiment yields a rounded off version even in exact conditions. Consider now the case where a YN and MU are performed under the *same non-exact conditions* First the MU will be a rounded off version of the non-fuzzy step threshold function because of fuzziness #1a, as explained above. But further rounding off occurs because both experiments are conducted under the same non-exact conditions. The rounding off is the same because the experimental conditions are the same. Thus the MU membership function will be 'wider' than the rounded off curve in the YN (we round off twice in MU but only once in YN). Remark that for a YN experiment the situation is very similar to an LB experiment; the non-fuzzy threshold curves remain. As Hisdal says, based on $\sum_{l=1}^{L} p(\lambda_l|\mu^{ex})=1$ we get then Zadeh's negation such that $\mu_{not\lambda}(\mu^{ex})=1-\mu_{yes\lambda}(\mu^{ex})$.

IV.3. The problem of probabilities Vs possibilities

IV.3.1. Introduction

In 1978 Gaines wrote that '...there are significant differences between fuzzy logics and probability logics, in their motivations, applications and axioms. However, there are also close relationships between the two forms of logic which are themselves significant.

We must define a probability logic. Probability logic as defined in relation to many valued logic gives a probability to *statements*. This logic assumes the postulates of mathematical probability as we see in any statistics course. From part I's discussion on multivalued logic we said that a truth value of a proposition *p*, was denoted $|p|$. In similar terms will probability logic in a multivalued setting $|p|$ now adopt $pr(p); pr(p)=p$. A fuller enumeration of the links between the sorts of logics is as follows:

$$\begin{cases} |p| = pr(p) \\ |\neg p| = 1-pr(p) \\ |p\vee q| = pr(p)+pr(q)\ (if\ p,q\ are\ mutually\ exclusive) \\ |p\wedge q| = pr(p)+pr(q)-pr(p\vee q) \\ |p\equiv q| = |(p\rightarrow q)\wedge(q\rightarrow p)| \end{cases}$$

Note however that the system obtained is not truth functional anymore. The concept of truth functionality has been introduced in part I. It is easy to show there is no truth functionality anymore: example:

$$1/2\wedge 1/2 = |p\wedge p| = pr(p\wedge p) = pr(p)+pr(p)-pr(p\vee p) = 1-1/2 = 1/2$$

Vs.

$$1/2\wedge 1/2 = |p\wedge\neg(p)| = pr(p\wedge\neg(p)) = pr(p)+pr(\neg p)-pr(p\vee\neg p) = 1/2+1/2-(1/2+1/2) = 0$$

So a unique value is not guaranteed. Rescher however points out that the non-truth functionality makes a probability logic system not different from other systems (such as the one presented in part I). A provable fact (in Rescher) is this :

The tautologies of PL are exactly the same as the tautologies of C_2

I do want to make a remark in function of part IV which deals with operators. It is, I think, clear that the operators behavior will be extremely dependent on the kind of underlying multivalued logic which is being used. We introduced the connectives for the 'truth-value' approach in part I and those are the operators we find back in what Zadeh has been writing. Where is Hisdal here?

IV.3.2. Hisdal's argument

After all Hisdal may be working with such probability logic. This is the reason of our small introduction. The argumentation Hisdal gives for using probabilities in her TEE model can be briefly summarized through the following proposal. The sum of grade of membership values for *a given label*, such as tall f.i. over all μ^{ex} values may not add up to 1. But with the formulation that $p(\lambda|\mu^{ex})$ *is a probability distribution* over the different elements of Θ,as Hisdal defines it; the sum of this $p(\lambda|\mu^{ex})$ must add up to 1. Some clarification is needed here. Looking back at the figure 23 one can easily see that for a given μ^{ex} value and *different labels* the medium non-fuzzy threshold curve and the tall non-fuzzy threshold curves each determine a cut-off area under the error function. We know that the area under the probability density function (which is the error function here) must add up to 1. Considering all the different labels from the label set over a *given value of* μ^{ex} it is then totally expectable that $\sum_{l=1}^{L} p(\lambda_l|\mu^{ex}) = 1$; where '$l$' stands for the different labels in the label set Θ. Remark that for a YN experiment the situation is very similar to an LB experiment; the non-fuzzy threshold curves remain. As Hisdal says, based on $\sum_{l=1}^{L} p(\lambda_l|\mu^{ex}) = 1$ we get then Zadeh's negation such that $\mu_{not\lambda}(\mu^{ex}) = 1 - \mu_{yes\lambda}(\mu^{ex})$.

This is the crux of the argument Hisdal uses to defend the probabilistic interpretation of grades of memberships. A little more expansion is needed here too. The sum of grade of membership values *for a given label*, such as 'tall' f.i, over all μ^{cr} values may not add up to 1. This may be the reason, according to Hisdal, that the max-min approach was used. Mind here the label is given and we run over all attribute values. The 'summing up to 1' however is in a context where we vary the labels over a given μ^{cr} value. Or in 'non-Hisdal' terms, 'use is made of the summing up to 1 formula of the grades of membership of a point of the attribute universe in all clusters; i.e. in all labels.' In this crucial argument which tries to defend the use of probabilities it is important to give two important remarks. The first one is that Zadeh has claimed $p(\lambda_l|\mu^{cr}) = p(\mu^{cr}|\lambda_l)$. If this is true then Hisdal's argument can not possibly hold. We remark however that Zadeh sees this equality within a possibility context. The second argument is that as an alternative to max-min operators *t-norms and t-conorms* have been used. We have no idea what the implication is of the use of such norms. Certainly a min operator is an example of a t-norm; while a max operator would be an example of an s-norm (or t-conorm).(see [36])

IV.3.3. The probability approach

The probability approach opposes itself to the possibility approach and the related max-min approach. We must however be careful to assign the adjective 'subjective' to probability! A question is this: Is Hisdal using subjective probability; in that she looks at an individual assessing his own estimate of a probability of the occurrence of an event?

The issue at stake here is to briefly examine what the arguments may be for a possibilistic versus subjective probability framework. It is out of this discussion that we hope, we will be capable of better appreciating Hisdal's approach. Zadeh in 1978 wrote:' The possibility π_x is defined to be numerically equal to the membership function of F when we are given the proposition 'X is F'.' Grades of membership would thus be the same as possibilities out of this proposition. We hinted already between Zadeh's proposed equality between $p(\lambda_l|\mu^{cr}) = p(\mu^{cr}|\lambda_l)$; using possibilities however. In an example: the *possibility* that an object which is tall has a height of 175 cm= the *possibility* that an object with height 175cm is tall. It is not at all obvious why there may be an equality between these two propositions. Following Hisdal $p(\lambda_i|\mu^{n})$ is associated with the grade of membership. $p(\mu^{n}|\lambda_i)$ however is claimed by Hisdal to be equal to $p(\lambda_i|\mu^{n}).p(\mu^{n})$. Some examples, as presented still by Hisdal may clarify the debate in a more precise manner.

1. The tall man from the company

Suppose professor X expects a man from a company who is called Y. X is told on the phone that Y is tall and that he is the passenger on a certain plane. X goes to the airport to meet Y and waits at the exit from the plane. Three men come out of the plane; they are m_1, m_2, m_3. X assigns the grades of membership of respectively .7; .6; .4 for 'tall'. We ask: what is the (possibility) or (probability) that either man is Y? If we use the set up of the math chapter then all we need to do is to look for the fuzzy union; which uses max.

(as per Zadeh; $Poss\{X\ is\ A\} = \pi(A) = \sup$ *of the membership function)*

Therefore the result would be .7! This obviously makes no sense as it is absolutely certain that one of them is Y. We obtained a possibility value of .7 while there is certainty! This is indeed weird! This may be a first reason for which we may discard possibilities (as going through min-max)) Let us now modify the situation somewhat. Call case (a) the case above; i.e. let m_o be the person with membership value .7. Case (b) is a little different from (a) as now it is assumed that 750 people leave the plane.

We also assume that X assigns a grade of membership of 1 to 100 of those passengers. Let m_o be one of the hundredth man with membership of 1. m_o in case (a) is the person with membership value of .7. We ask again what is now the probability(or possibility)that Y is m_o? Using max-min the answer would be 1! This again makes no sense! m_o is one of the hundredth men having membership of 1; out of the 750 men in total. Still the max-min accords a higher value to the (b) case than to the (a) case. Some comments are in order here. Those are carefully searched out examples which would show us that using the possibilistic approach is erroneous. However, from the possibilistic camp examples may be found showing that the probabilistic camp may be wrong also in some instances. We do not have an example at first sight; but it should not be ruled out.

2. $p(\lambda_l|\mu^{ex}) = p(\mu^{ex}|\lambda_l)$; within possibilities.

Suppose a person has a height of 2.5m. We assign a grade of membership of 1; to tall; thus the possibility that this person is tall is 1. Because of the proposed equality (Zadeh), we get: the possibility that an object which is tall has a height of 2.5m = the possibility that an object with height 2.5m is tall. Intuitively if the object is a person the equality does not make much sense. The RHS of the equality has certainly possibility 1. For objects which are persons the LHS can not possibly be 1. Following Hisdal's proposition that $p(\mu^{ex}|\lambda_l) = p(\lambda_l|\mu^{ex}).p(\mu^{ex})$; which would come out to be very small given that $p(\mu^{ex})$ is extremely small, confirming our intuition.The equality would thus not hold if Hisdal's proposition is acceptable. It certainly is from an intuitive point of view. It is almost common sense that if we would accord to the possibility that an object which is tall has a height of 2.5m a value of 1 we would indeed be in a strange world!

IV.3.4. Subjective probability or probability?

1. Introduction

It is at this point useful to wonder whether Hisdal's use of probabilities is to be found within probabilities (and attached repeated experiments) or subjective probabilities. The issue is not really clear. Recall that $p(\lambda_l|\mu^{ex})$ was defined by Hisdal as the *probability* that an object with given attribute value μ^{ex} will be labeled λ in a YN or LB experiment. Furthermore we also introduced the bell-shaped error curve $p(\mu|\mu^{ex})$. The only way I can see this as a probability is that the experiment has been conducted over several subjects forming a large enough sample so that such probability distribution can be formed. We know that $p(\lambda_l|\mu^{ex})$ is cut off area under the error function when it is superimposed on the non-fuzzy threshold curve. The question then becomes how the error curve is derived. Hisdal does give very little clues to that. There is an estimated error function as determined by the subject. This can easily involve subjective probabilities. The estimate could also be based on a derived form of the real error function.

2. Bayesian Theory

We must make a halt however here. Confusion and non-sense may start building up rather quickly...The issue of subjective probability is not at all an easy concept. Following Shafer Bayesian theory has played a predominant role in the development of subjective probability. We may pause a little on the Bayesian stance. Before starting let us remark that in a great deal of economic literature Bayesianism is often equated with rationalism. Harsanyi for instance has maintained that every rational individual would be also be a Bayesian. ([53]; p.11) The Bayesian stance says that a *belief function* should obey three rules:

$$\begin{cases} Bel(\varnothing)=0 \\ Bel(\Theta)=1 \\ if\ A\cap B=\varnothing \Rightarrow Bel(A\cup B)=Bel(A)+Bel(B) \end{cases}$$

The set Θ is a finite set and $2^{\Theta}=\#(\wp(\Theta))$. Then we suppose the function

$$Bel{:}2^{\Theta}\to[0,1]\, satisfies{:} \begin{cases} Bel(\varnothing)=0 \\ Bel(\Theta)=1 \\ Sum\ of\ the\ Bel-functions\ attached\ to\ each\ subset\ of\ \wp(\Theta)\leq 1 \end{cases}$$

For instance if we would set a belief function on Hisdal's error function; instead of this very awkward probability function which happens to be nicely bell-shaped; then we would get the possibility we make an error at a certain height versus the possibility we do not make an error at a certain height. Define for instance for a particular height $t;\Theta=\{\theta_1,\theta_2\}$; where the individual thetas refer to making an error or not. The only very serious problem which occurs here is that evidence will have to be the main factor in assessing θ_1 *and* θ_2. Where will I get as an agent evidence on whether I, myself will make a mistake or not? This is important. If we assume evidence exists to this purpose we still have to look in what form there is a link between this evidence and degrees of support or belief. It is here where the Bayesian stance is introduced. There seem to be two orientations in Bayesian theory either the older stream which Shafer calls the 'logical view' or else the 'personalist view'.The first option insists that numerical degrees of support are indeed objectively determined by given evidence; the second option analyzes the degrees of belief as psychological facts; facts which can be discovered by observing an individual preferences among bets or risks but which may not bear any particular relation to any particular evidence as Shafer says. Giles proposition on the analysis of a fuzzy sentence at one point uses this personalistic point of view. It is important that this point of view does not look at the relation between degrees of belief and evidence.

Perhaps this is the correct stance to take. As we just remarked finding evidence in the case of the error function expressed as a belief function is a difficult issue to resolve. However the error function is instrumental in finding the membership function; at least in Hisdal's work. Giles does not need an error function. Giles' membership function uses the personalistic view and therefore does not care about evidence as such.

If one wants to model a degree of membership function as a degree of belief by which an agent attributes a label then if evidence is available in some form the Bayesian approach looks not promising; unless one wants to follow the personalist view. The additivity rule $Bel(A \cup B) = Bel(A) + Bel(B)(A \cap B = \varnothing)$ is the problem. Shafer gives an excellent example on where this additivity rule may go totally wrong. ([64]-p.24) In fact in Hisdal one could interpret the error function and the non-fuzzy threshold function as evidence to the membership function. Do we need to bother about evidence?

If we follow a non-Bayesian approach it will become harder to argue for subjective probabilities as we would think they are used in Hisdal.

Whether we are here in a pure subjective probabilistic approach is to question. The issue is important because if subjective probabilities are used in Hisdal's TEE model a problem of aggregation of *individual* membership functions imposes itself. This kind of aggregation is certainly very difficult to handle. Hisdal's stance can perhaps best be expressed through the following. Laviolette and Seaman in a critique on Hisdal's TEE model say that for events not reproducible; like the result of an upcoming election; the relative frequency approach clearly does not apply. Hisdal responds to this by saying that a subject could carry out a subjective analysis of errors *using the frequency-probabilistic approach.* She could estimate a probability distribution for the number of voters who change their mind at the last minute. Says Hisdal 'the subject's subjective uncertainties are thus given a frequency probabilistic approach. 'In my mind this says nothing else than deriving a subjective probability distribution. The point remains unclear. The key issue is to know , in the TEE setup where the estimated error function comes from. Is it truly a probability distribution? Could it be a belief function using evidence as in Shafer? Is it a personalistic-Bayesian approach? Do we need evidence or can we dissociate evidence from degrees of belief? If Hisdal does use subjective probability we will have to aggregate from individual membership functions; so to obtain the membership function for a specific label. In short the aggregation would have to run according to the following procedure. For the construction of the membership function 'tall' and using only fuzziness #1a, the estimated error function for a given attribute value is to be superimposed on the non-fuzzy threshold curve of the individual.

This will yield a membership value given this attribute value for the specific label 'tall'. The same procedure is to be repeated for all other attribute values ; and it is almost sure that the estimated error curves will change when the attribute value is changed. The membership function for the label 'tall' is then constructed. In very superficial terms aggregation to a *general membership* function for 'tall' will involve taking into account all the membership functions for the different individuals in the sample and tested on the label 'tall'. On the other hand as we pointed out in the introduction to this section Hisdal may be using a probability logic. The reader may follow the definition of probability logic as introduced above and then decide. The matter remains a hard sell however. As a final word Herbert Toth presents somewhat of another view on Hisdal's probability approach. Toth basically agrees on Hisdal's TEE model but finds that the dependency between conditional probabilities of a certain kind and membership degrees should be less strong. He proposes a more general format such as $\mu_\lambda(\mu^{ex}) = f(p(\lambda|\mu^{ex})$. Toth however does not go into any detail on how f may be typified.

IV.3.5. Hans ate X eggs for breakfast

At this point it still may not be clear what the differences are between possibility and probability. Zadeh provides about the best I came across showing the difference between those two concepts. We may associate a possibility distribution with X taking values in

U={1,2,3,...} by interpreting $\pi_x(u)$ as *the degree of ease* with which Hans can eat u eggs. Also, we can associate a probability distribution with X by interpreting $p_x(u)$ as the *probability* of Hans eating u eggs for breakfast. The values of $\pi_x(u)$ and $p_x(u)$ could then be the following:

u	1	2	3	4	5	6	7	8
$\pi_x(u)$	1	1	1	1	.8	.6	.4	.2
$p_x(u)$	.1	.8	.1	0	0	0	0	0

Thus Hans possibility; or degree of ease; to eat 3 eggs for breakfast is 1; while his probability of doing so may be much lower. His degree of ease of eating 7 eggs is .4 while his probability is plainly 0. The following conclusions follow directly:

1) a high degree of possibility does not imply a high degree of probability

2) a low degree of probability does not imply a low degree of possibility

3) if an event is impossible it can not be probable

One way to express a *degree of consistency* between probability and possibility distributions is simply as follows:

Let $\Pi = (\pi_1, \pi_2, ... \pi_n)$ *and* $P = (p_1, p_2, ... p_n)$, respectively for possibilities and probabilities. Then γ expresses the degree of consistency as follows:

$$\gamma = \pi_1 . p_1 + \pi_2 . p_2 + ... \pi_n . p_n$$

For the above $\gamma = 1$. It is intuitive that the higher the consistency the higher the level of γ. Let us remark however that in this example 'degree of ease' is associated to a possibility and not a probability as would be the case in Hisdal.The above example may give us also a very superficial but intuitive reason why perhaps the degree of a label belonging to a certain object may have to go through possibility rather than probability. How else could we define possibility?

Part IV: A Discussion On Fuzzy Operators

A discussion on operators or connectives imposes itself if we want to treat the subject of fuzzy sets in some depth. This part surveys some of the proposals in the area. We do not want to go into too great detail given that the main issue of this paper is the economic applicability of fuzzy sets.

As we have seen in the math part of this paper the definition of union and intersection following Zadeh was that:

$$\widetilde{A} \cap \widetilde{B} = \{x: \min\{U_{\widetilde{A}}(x), U_{\widetilde{B}}(x)\}\}$$
$$\widetilde{A} \cup \widetilde{B} = \{x: \max\{U_{\widetilde{A}}(x), U_{\widetilde{B}}(x)\}\}$$

We first look at two authors who are arguing in favor of this definition. Then we will confront this argument with opposing views. Bellman and Yager are in favor of Zadeh's definition. Thole, Zimmerman and Zysno are against it. From the outset we note that Zimmerman's argument is within the experimental realm. Yager and Bellman do argue within a theoretical context.

Chapter I.

I.1.Bellman and Giertz

The problem of operators is not an easy one to solve. As Bellmann says 'if an object is accepted to 60% as a member of fuzzy set $\widetilde{A}$; and to 40% as a member in $\widetilde{B}$ how willing should we be to accept it as a member in both fuzzy sets $\widetilde{A}$ and $\widetilde{B}$?' ([3];p.150)

Very interesting is Bellman's observation that though we are quite free to attach a subjective evaluation to a degree of membership in a fuzzy set, we are more constrained if we consider compound statements.

As an example say we attach a subjective valuation to x being a member to some degree of the fuzzy set $\widetilde{A}$; and y being a member to some degree of the fuzzy set $\widetilde{A}$. Can we be so

'free' in attaching a membership value to for instance ' *both x and y being members of the fuzzy set* $\widetilde{A}$ '? Certainly not! For instance assigning a larger membership to x *and* y members of $\widetilde{A}$ than to x member of $\widetilde{A}$; is inconsistent. The general problem is then:
Given two statements S_1 *and* S_2 with given truth values; what are the restrictions which are to be observed when a truth value to a compound statements such as S_1 *or* S_2 or also S_1 *and* S_2?

I.1.1. Assumptions

Bellman now proposes a list of quite straightforward assumptions which should be respected when using connectives. The assumptions are quite natural though we wonder whether they are exhaustive. Based on those assumptions an axiomatic structure is then developed. The goal is here to have a brief look at those two components.

a) Assumptions

Define $F = \{[S, U_S]\}$ be a fuzzy set of statements. Note that the statements in F are un-related to each other. We consider compound statements such as S_1 *and* S_2 etc...

A1: The truth value of a compound statement depends on the truth values of the statements in F. For S_1 *and* S_2 (sub-statements of S) for instance we specify a real-valued function *f(x,y)* with $x \in [0,1]$ *and* $y \in [0,1]$; so that: $U_{S_1 and\, S_2} = f(U_{S_1}, U_{S_2})$; $U_{S_1 or\, S_2} = g(U_{S_1}, U_{S_2})$

A2: If truth values have been assigned to arbitrary statements S or T (which are thus unrelated) then the same functions *f* and *g* provide us also with truth values for S *and* T and also S *or* T:

$$U_{S\, and\, T} = f(U_S, U_T)$$
$$U_{S\, or\, T} = g(U_S, U_T)$$

A3: *f* and *g* are non-decreasing and continuous in both variables. This makes sense. If the willingness to accept *S* or *T* increases there is no reason to assume *f* and *g* would not increase. (*S and T* are arbitrary statements)

A4: *f* and *g* are symmetric; i.e. *f(x,y)=f(y,x) or also g(x,y)=g(y,x)*. There is no reason to assign different truth values to S and T than to T and S.

A5: *f(x,x) and g(x,x)* are strictly increasing in x. This also makes sense. Say $U_S = U_T$ *and* $U_V = U_W$ *and* $U_s > U_V$ then we are more willing to accept *S and T than V and W*.(similarly for the 'or' connective)

A6: $f(x,y) \leq \min\{x,y\}$ *and* $g(x,y) \geq \max\{x,y\}$; this is quite intuitive. It says that the membership value of *S and T* will be lower (or equal) than the membership value of *S or T*. It is intuitive to claim that the membership value of *S and T* will be less than when *S* or *T* is considered alone. Clearly, the membership value of *S or T* is however higher (or equal) than when considering *S* or *T* alone.

A7: *f(1,1)=1 and g(0,0)=0*. If *S and T* are both completely accepted (individually) then '*S and T' must be completely accepted.* Similarly for the 'or' connective and rejection. We may wonder what happens to *g(1,1) or f(0,0) or even also g(1,0) for instance.*

A8: Logically equivalent statements have equal truth values.

I.1.2. Showing that *f(x,y)*=min{x,y} and *g(x,y)*=max{x,y}

Bellman introduces the notation *f(x,y)*=$x \wedge y$ and for $g(x,y) = x \vee y$. The following conditions are then emerging (based upon the assumptions):

$x \wedge y = y \wedge x$ *and* $x \vee y = y \vee x$ (1)(*from* A4)

$(x \wedge y) \wedge z = x \wedge (y \wedge z)$ (2)(*also for 'or'*)(*from* A8)

$(x \wedge y) \vee z = (x \wedge z) \vee (y \wedge z)$ (3)(*also for* $\vee$ *Vs.* $\wedge$)(*from* A8)

$x \wedge y$ is non-decreasing and continuous in x. (4)*(also for 'or')(from A3)*

$x \wedge x$ is strictly increasing in x. (5)*(also for 'or')(from A5)*

$x \wedge y \leq \min\{x,y\}$ *and* $x \vee y \geq \max\{x,y\}$ (6)*(from A6)*

$1 \wedge 1 = 1$ *and* $0 \vee 0 = 0$ (7) *(from A7)*

Bellman shows that out of those seven conditions which have been taken over from his initial assumptions that:

$$x \wedge y = \min\{x,y\}$$
$$x \vee y = \max\{x,y\}$$

The proof can be found in ([3],p.154)

The question we may raise is whether the list of assumptions is exhaustive enough.

I.2. Ronald Yager

Yager's treatment is somewhat less formal than what Bellman presented. Though I do think it goes somewhat in greater depth.

I.2.1. The choice of membership functions and their influence on union and intersection of fu..zy sets.

Yager makes a distinction between the *absolute* membership function and a non-absolute membership function. I find this distinction artificial and not of great use. A true fuzzy context will rarely generate an absolute membership function. We saw this in extensio in part III. The problem however to know whether union and intersection's validity and use is dependent on how we choose such membership function remains a crucial point.

Assume thus that we have two membership functions; call them $f_{\tilde{A}}$ *and* $f_{\tilde{B}}$. Now Yager says that he wants to define $\tilde{A} \cap \tilde{B} = \tilde{C}$; *where* $f_{\tilde{A}}(x) * f_{\tilde{B}}(x) = f_{\tilde{C}}(x)$.

Two properties must be imposed to the '*' operator:

1) $f_{\tilde{C}}(x)$ is *indifferent* to the particular selection of the membership function

2) reduces to the usual intersection of the sets if the memberships are binary.

The same criteria are needed for union. All what is being assumed here is that the individual has some idea of ranking the membership values, though there is NO known precise relationship between values. Let us therefore remark that the treatment we viewed under Hisdal or also Dombi came to the conclusion membership functions were definable in some quite precise form. Thus here we are at a quite more general level. Yager shows that under certain conditions there will be only one way to define intersection and union between two fuzzy subsets. His development shows that union and intersection basically reduce to Zadeh's min and max operators.The foregoing developments (Bellman and Yager) are mathematically justified. However such justification is insufficient when modeling real world phenomena. This is where Zimmerman et al. comes in.

Chapter II

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II.1. Thole, Zimmerman and Zysno

Zimmerman starts his analysis really at the bottom. We have been talking in part III about possible approaches to electing membership functions. Our discussion was quite general and compared several propositions. Zimmerman asks himself again this question:' how can grades of membership be determined in practice'? He distinguishes two different streams i.e. the direct and indirect approach. The direct approach assumes human beings work as good measuring devices. Hisdal's approach was based on this direct stream. I.e. we had a step function which was rounded off through our error function and bingo the membership function came forth! Yager's treatment on operators also contains a section on what he calls the 'cardinal approach'. This approach is equivalent to Zimmerman's 'direct stream'. Zimmerman also provides us with some more new insights. Says Zimmerman there are quite a serious amount of response biases and a very important bias is the 'end effect'. This effect says Zimmerman has 'subjects to shift stimuli towards the ends of the rating scale'.([70], p.169) There are other biasing effects. Then there is the indirect approach which is in fact the essence of Yager's argument as presented above. In this stream only *ordinal* judgments are being used. Preciseness is not the issue here. No doubt that this indirect stream does put much less weight on the human being as a measuring device as compared to the direct stream. Zimmerman proposes the use of both streams.

In the next section (II.1.1) we will look at a detailed set up of how a fuzzy experiment could be conducted. In part III we have often used the example of 'John is tall' to generate a membership function. Never have we gone into sufficient detail however on the issue of possible biases. Basically we always have been assuming that the human being would be a good measurer. This may not be so. The now following section provides us with some detailed background on how we may want to construct a membership function using all the details possible. The latter is more than useful when considering example such as 'John is modest' rather than 'John is tall'.

II.1.1. Set up

The steps as proposed by Zimmerman are as follows:

A) A sample of objects is to be taken which should represent membership grades rather evenly on the [0,1] interval. At least two objects should be included which translate in a full '0' and full '1' grade. This is what Zimmerman calls the 'condition of undisputed extremes'. I.e. all subjects assign the same number to the extremes. A pre-test is to be instated so that this can be verified.

B) Objects are now rated on a percentage scale by the subjects; the conversion to [0,1] is immediate.

C) The scale position of each object is now estimated by using the median of the distribution of ratings given by the subjects on that object. We are here thus at the aggregate level. This scale; call it D; may be distorted by several biases which belong to Zimmerman's direct stream. The end effect is an example. D is now compared with a scale S which is obtained through the indirect stream. One will check whether the order between the two streams is still preserved. If so D can be transformed into D' which is an approximation of S.

II.1.2. A concrete experiment

Consider three fuzzy sets: 'Metallic object'; 'container' and finally 'metallic container'.

The following hypothesis is formulated:

Set:

1) $U_{\tilde{M}}(x)$ as the grade of membership of some object x in the set 'metallic object'.

2) $U_{\tilde{C}}(x)$ as the grade of membership of x in the set 'container'

The grade of membership of x in the set 'metallic container' is hypothesized to be then:

$$H_1 : U_{\tilde{M}\cap\tilde{C}}(x) = \min[U_{\tilde{M}}(x), U_{\tilde{C}}(x)]$$
$$H_2 : U_{\tilde{M}\cap\tilde{C}}(x) = U_{\tilde{M}}(x).U_{\tilde{C}}(x)$$

At this point we may also clarify somewhat the difference between a max-min and product operator. In some cases the product operator may really be more appropriate. An example may clarify this. Consider two fuzzy statements A and B. Fuzzy statement A says 'a one-day old infant is young' and fuzzy statement B says 'living one more day does not alter one's youth'. Clearly statement B is definitely fuzzier than statement A. Now, intuitively statements B is difficult to agree with. However using the min operator it makes perfect sense! This is indeed awkward. Using the product operator however will show the statement not to be true. If we multiply the same membership value successively the product of all those same numbers will be smaller. ([66])

1. A pre-test

A pre-test is to be performed on two items:

-the intersection of two fuzzy subsets labels 'metallic object' and 'container' is in fact represented by the subset 'metallic container'.

-is D' an approximation of S?

1) For the first problem Zimmerman performed a test on a sample of (only) 5 subjects. He looked at whether objects which were referred to as metallic and containers were indeed also rated as metallic containers.

2) Zimmerman then started selecting objects which would fit in each of the three classes; i.e. 'metallic objects'; 'container' and finally 'metallic container'. Each class would have to contain an object which gets a full rating of '0'; and a full rating of '1'.

3) Furthermore the objects had to be so chosen as to enable an even 'spacing' of stimuli. For so doing Zimmerman took a sample of 20 subjects to rate 50 *provisionally* selected objects. From there the objects which best approximate conditions 2 and 3 would be chosen. There are other details to be mentionea but we limit ourselves to the above.

2. A real experiment.

Omitting the details; the results out of the experiment now based upon a sample of 60 subjects gave the following results.The table below incorporates a re-arrangement of scale as set out above. Both direct and indirect streams (as explained above) are used. The fuzzy set M stands for 'metallic objects'; and the fuzzy set C stands for 'containers'. The intersection as *tested* gave the following results:

Stimulus x	$U_{\tilde{M}\cap\tilde{C}}(x)[T]$	$U_{\tilde{M}\cap\tilde{C}}(x)[Min]$	$U_{\tilde{M}\cap\tilde{C}}(x)[Prod.]$
bag	0.007	0.000	0.000
baking - tin	0.517	0.419	0.380
ball - point	0.170	0.149	0.032
bathing - tub	0.674	0.552	0.444
book wrapper	0.007	0.023	0.010
car	0.493	0.437	0.219
cash register	0.537	0.400	0.252
container	1.000	0.847	0.847
fridge	0.460	0.424	0.264
hollywood - swing	0.142	0.212	0.067
kerosene - lamp	0.401	0.310	0.149
nail	0.000	0.000	0.000
parkometer	0.437	0.335	0.222
pram	0.239	0.283	0.127
press	0.101	0.130	0.067
shovel	0.301	0.239	0.078
silver - spoon	0.330	0.256	0.248
sledge - hammer	0.023	0.012	0.006
water - bottle	0.714	0.546	0.525
wine - barrel	0.185	0.127	0.124

The letters 'T' and 'Min' and 'Prod' refer to the 3 possibilities we have been alluding to. The latter two correspond to the two hypotheses we have set out above. 'T' stands for test and the results of the two hypotheses should be compared with the test results.

The issue here is to know what kind of criteria one should be using to either accept or discard some values. Zimmerman's proposal on that issue is then as follows:

1) the mean difference between observed and predicted values is not different from zero.

2) the correlation between observed and predicted values is higher than 0.95.

If 1) and 2) are observed then the connective operator should be accepted. It turns out after some statistical manipulation that both hypotheses can not be accepted. Hence in this experiment product and min-operator are not acceptable. A small word of caution is in order here. There is no explicit reason why the results of the experiment should be accepted. Where is there a generally accepted standard procedure which could act as a benchmark so to validate or also in-validate Zimmerman's procedure? Granted the pre-test he uses makes lot of sense. But the adaptation of the direct scale to the indirect scale may contain errors. When looking at the table above we see there is in a majority of cases an excess of the observed rating over the min operator. Zimmerman argues that this may be due to the fact that human beings 'compensate' when they combine fuzzy sets in the sense of 'and'. Says Zimmerman: '...in rating objects with respect to a composite attribute they do not process the relevant information as if they were choosing the smaller of two grades of membership, but proceed internally as if they were using the smaller one only as an orientation and then modifying it in the direction of the higher value.'([70], p.179) Interestingly enough the compensation seems to work also in the max-operator or union of fuzzy sets. An experiment carried out by Hersch and Caramazza showed that the observed values were mostly *below* the predicted result as going through the max-operator.

Assume now for a moment that there is nothing wrong with Zimmerman's experiment then we may wonder how workable the max-min operators really are. They may be theoretically justified but from a practice point of view they should still be operational. If Zimmerman's experiment is correct then there is a problem. The urgent question is then: what sort of operator should one use? Is there a connective for each situation?

II.2. A general connective

Zimmerman makes a proposal where he defines a grade of compensation. This leads to a more general form of operator which lies between the 'and' and 'or' as we know it. Whether this is a solution is to be seen. It is certainly useful to develop somewhat this idea. We have been defining the operator for the intersection either by

$$H_1 : U_{\tilde{M}\cap\tilde{C}}(x) = \min[U_{\tilde{M}}(x), U_{\tilde{C}}(x)]$$
$$H_2 : U_{\tilde{M}\cap\tilde{C}}(x) = U_{\tilde{M}}(x).U_{\tilde{C}}(x)$$

For the union we get:
$$H_1' = \max[U_{\tilde{M}}(x), U_{\tilde{C}}(x)]$$
$$H_2' = U_{\tilde{M}}(x) + U_{\tilde{C}}(x) - U_{\tilde{M}}(x).U_{\tilde{C}}(x)$$

Zimmerman's idea is this. Introduce a compensation parameter γ. This parameter takes the value zero when it is on the connective 'and' and '1' when it is on the connective 'or'.It is Zimmerman's goal to establish a weighted combination of both connectives. The 'end-points' ; thus either 'or' and 'and' can be expressed either in product or max-min form. Zimmerman opts for the product form by arguing that he wants an *interaction* of membership values. Let us remark that Yager has serious objections to use H_2' as a union connective. (see [74], p.198) The 'interaction' resides in the algebraic operation which has to be carried out and which links membership values to each other in a much more explicit way than if we would be using max or min.

Intuitively the weighted combination reposes on the following idea:

[(and)-no compensation] 0________________1 [(or)-full compensation]

The idea is intuitively appealing from what we said above when we dealt with Zimmerman's experiment. Out of his experiment it was shown that 'and' from a theoretical point of view had lower values that the 'experimental' 'and'. The theoretical 'or' through work by Hersch and Caramazza lied in value above the experimental 'or'. Using the interactive version of the connectives as defined above through H_2 *and* H_2 the following version of a weighted combination of 'and' and 'or' is seen: $U_{\tilde{A}\theta\tilde{B}} = U_{\tilde{A}\cap\tilde{B}}^{\,1-\gamma} . U_{\tilde{A}\cup\tilde{B}}^{\,\gamma}$. Using the interactive connectives (H_2 *and* H_2) the general form becomes then: $U_\theta = (\prod_{i=1}^{m} U_i)^{1-\gamma} (1-\prod_{i=1}^{m}(1-U_i))^{\gamma} ; U \in [0,1]\ and\ \gamma \in [0,1]$.

and i=1,2,....m; m = number of sets to be connected. The first term corresponds to H_2 and the second term corresponds to H'_2.This general form is thus easily checked. Taking the case for m=2 we see it immediately that: $U_1 + U_2 - U_1 . U_2 = 1-[1-U_1][1-U_2]$ which corresponds to the second part of the general form. The L.H.S of this equality simply corresponds to H'_2. One can also see quite easily that if $\gamma = 0\ and\ \gamma = 1$ respectively that: $U_\cap = \prod_{i=1}^{m} U_i\ (for\ \gamma = 0)\ and\ U_\cup = 1-\prod_{i=1}^{m}(1-U_i)\ (for\ \gamma = 1)$ which thus reduces to the interactive definitions $H_2\ or\ H_2$. Isolating γ from $U_\theta = (\prod_{i=1}^{m} U_i)^{1-\gamma} (1-\prod_{i=1}^{m}(1-U_i))^{\gamma} ; U \in [0,1]\ and\ \gamma \in [0,1]$ we get then:

$$\gamma = \frac{\log U_{ii} - \log \prod_{i=1}^{m} U_i}{\log(1-\prod_{i=1}^{m}(1-U_i)) - \log \prod_{i=1}^{m} U_i}$$; out of there one can find a value for γ and set this value in the connective. Finally one can match the obtained result of the experiment with the predicted value using the obtained value for γ.

One should remark that $U_{\tilde{A}\theta\tilde{B}} = U_{\tilde{A}\cap\tilde{B}}{}^{1-\gamma} . U_{\tilde{A}\cup\tilde{B}}{}^{\gamma}$ is not the only possibility for a weighted connective. There are other possibilities.

II.3.Conclusion

In some sense there is no single form of a general operator which can be used in any practical situation. This is worrisome to say the least. The 'applications' area is tremendously demanding especially when a descriptive rather than a prescriptive attitude is taken. The theoretical justification of max-min or also the interactive product operators are certainly too simple as to mimic the decision processes which go on in our brains. Zimmerman's proposal for a more general connective is perhaps a 'step in the right direction' when considering the highly demanding descriptive applications. Before switching to part V of this paper we may effectively wonder how tolerant one may be in accepting gaps between theory and practice. It remains the objective of fuzzy sets to formalize our everyday speech to some degree. Both practice and theory must be married and one may perhaps say the practice is a test on the validity of the theory. To develop a consistent and thus non-contradictory theory of fuzzy sets on itself is not a very praiseworthy objective with which practitioners will be contented with. Choosing the right assumptions and working out postulates on those assumptions in a consistent way is basically all what matters in the theoretical field. The consistency gets harder and harder to be obtained when the assumptions become more and more convoluted. The fuzzy set discipline looks to be like a great effort in optimization. Says Richard Bellman: 'We must balance the needs for exactness and simplicity, and reduce complexity without oversimplification in order to match the level of detail at each step with the problem we face'.([3],p.149) Zimmerman in [82] proposes some salient properties a suitable operator should have. Among those properties are:

1) Axiomatic Strength: An operator with less axiomatic restrictions is better

2) Empirical fit: An operator must be an appropriate model of real system behavior which can normally only be proven by empirical testing.

3)Adaptability: An operator should be dependent on the context and the semantic interpretation.

4) Numerical Efficiency: An operator should be computationally efficient.

Part V: Fuzzy logic and Economics

Part V will survey some of the possible contributions of fuzzy set theory in economics.

V.1 Introduction

In 1988 Michael Smithson asked the following question :' Why have so few researchers in these fields utilized fuzzy set theory, and why has the dialogue between them and fuzzy set theorists been so underdeveloped?' ([67], p.1) The words 'these fields' refer to the social sciences in general. Says Smithson: '...the human sciences tend to be methodologically conservative when mathematically sophisticated and mathematically ignorant when methodologically innovative.' ([67], p.2) From our literature survey there is exceedingly little to remark in the area of fuzzy sets and economics. Smithson puts fuzzy sets in a very new daylight. Says Smithson 'Qualitatively oriented researchers are fond of castigating quantitative researchers for their inability to convincingly translate sophisticated theories of human behavior into mathematical form...while quantitative proponents berate 'anti-positivists' for the vagueness of their concepts and techniques.' ([67], p.12) Herein may lie the great value of fuzzy sets in that it brings the two warring camps together.

Claude Ponsard makes an excellent statement on where exactly fuzzy set theory could enter economics. He distinguishes the three frameworks in which traditional micro-economics is performed. Either we are in a framework of *certainty* where the agent is perfectly knowledgeable of the consequences attached to the choices he makes.

Either we are in the framework of *risk* in which the consequences of a decision are still well known *but* now randomness is attached to the outcome of the decision(s). Finally we also may be in the framework of *uncertainty* where the probability law we knew in the risk framework is now unknown. The three frameworks have however some very salient features in common. The information on the set of options is perfect; the possible results are clearly known by the agent. Following Mongin the distinction Ponsard presents may actually be borrowed from Knight. ([53], p.11)

The question Ponsard asks himself is what will fuzzy set theory input be in this set up of the three frameworks mentioned above? Basically the options known in advance may be known in only an imperfect way; this of course then 'begs' for the introduction of fuzziness. Furthermore the choice the agent is supposed to make may be much less clear-cut than presented in the classical theory. The agent may have fuzzy preferences a concept which is intuitively clear. It is all too much of an oversimplification to prone that we have neat preferences over an allocation. This idea has been indirectly captured in the economic literature notably in fields such as industrial organization where use is made of the concept of *bounded rationality*. Perhaps Mongin says it in the most general way:' ...il faut clairement distinguer le principe de rationalité lui-même, comme modèle générique , et les modèles divers, *spécifiques*, qui peuvent se réclamer de lui, mais non prétendre en épuiser le contenu..[..]...rationnel $\Rightarrow$ économique; (mais) elle est neutre pour l'implication inverse'. ([53], p.12) This means basically that concepts such as bounded rationality or even expected utility can draw from a general template of rationality but each can NOT pretend to be the sole representative of it. Ponsard very aptly remarks the traditional argument we may expect as a critique on the 'new' stance; and that is that taking into account a more 'human' preference behavior of an agent will without question lead to positing a *descriptive* problem rather than a *normative* one. The answer to such a an important critique may not be conclusive. However, the purpose of fuzzy set theory is exactly to provide for a more softly-oriented decision making approach. To claim that the introduction of fuzzy set theory in the preference behavior of an agent will reduce the problem to a descriptive problem is to say, in some sense that fuzzy set theory has only descriptive power. This is incorrect as one could see by reading through the different parts of this paper. But such a loose refutation however should also have to include the possibility of the orientation we introduced in part III; i.e. either syntactic or semantic. And we do *not* come full circle with our argument. Perhaps we should use Ponsard's proposition which says that 'a descriptive model is also a normative model at the optimum' ([5], p.14) That says it all but without any great solid argumentation. A whole paper can be filled with the pro's and con's to the problem posited above.

Intuitively fuzzy set theory is not inherently descriptive. Would it be, then we would work on a case by case basis and there would not be any scope whatsoever for a theory as such. This is a debate which is also ruminating in artificial intelligence circles. One must continue to see the fine tread which separates a theory which has a certain level of generality from a fully context dependent construct. This tread is sometimes difficult to distinguish and certainly the fuzzy set theory is going in the direction of increasing difficulty. I however refuse the argument that fuzzy set theory in fuzzifyng the preferences of an agent yields a descriptive model of the consumer; which therefore is totally separated from normativity. This is too easy an argument which clears the debate in favor of classical micro-economic theory. The results obtained must show that this is not so.

Claude Ponsard writes : 'Un évènement imprécis est celui qui peut se réaliser incomplètement.' ([5]) This is exactly the problem. By recognizing that an imprecise event can not be completely realized we recognize indeed that our preferences are fuzzy. Using instead a probability distribution will severe us from Ponsard's statement because in that case an event will know a complete realization.

Or to cite Luhandjula ' situations where doubt arises about the exactness of concepts; correctness of statements and judgments; degree of credibility, have little to do with the occurrence of events, the backbone of probability.' ([49], p.257)

V.2. Imprecision-uncertainty

We are perhaps arriving again at a point of high confusion. Our task is to disentangle the ingredients which lead to this confusion. The main issue here at stake is to distinguish clearly between the two concepts stated in the title; i.e. uncertainty and imprecision. I do think that imprecision and uncertainty have a common link; in that they purport to the *meaning* of information. We may have to represent three elements: error, uncertainty and fuzziness (or imprecision). ([57], p.20) Uncertainty is linked to future events which may or may not realize. But this is essentially *not* talking about fuzziness.

Probability and fuzziness are not the same and a simple example should make this clear. The statement 'element x belongs to a fuzzy subset with degree of membership 0.5' is not the same as saying that x would belong with probability 50% to this set. If it turns out that x belongs to this set then the membership value would be 100%! Thus the probability measure, as a would-be equivalent to fuzziness takes away the notion of fuzziness altogether; i.e. it continues to operate within a binary setting. Following Bellman and

Zadeh in ([4], p.B141) the authors claim that fuzziness yields a type of imprecision which yields 'classes in which there is no sharp transition from membership to non-membership.' Zadeh in [76] claims that information is 'intrinsically statistical in nature' and therefore probability theory is to be used. However this statement has relevance to *the quantity* of information rather than to the *meaning* of information. Decision making under uncertainty is a typical example which refers to *meaning* of information. The tool used in the area of meaning of information is possibilities rather than probabilities. Following this line of thought, imprecision and uncertainty should be put in the possibility area rather than in the probability area. We have been talking already about the differences between possibility and probability. One major difference may be that there is very explicit link between the probability of an event and its opposite. However this may not at all be the case when talking about possibilities.

Assume that effectively this may all be true, i.e. that we choose to model uncertainty and imprecision through possibilities rather than through probabilities. We may then wonder as Chandrasekaran does whether fuzzy sets; which then plays the role of a possibility distribution can handle this calculus of uncertainty. Chandrasekaran makes the interesting comment whether we should view fuzzy sets as either a psychological or either a mathematical theory. If the former is chosen as Chandrasekaran says then 'we would need certain kinds of evidence about human behavior in uncertainty handling.'([10], p.11) If the latter is chosen then an abstract world would emerge whose constituents parts are uncertainties of certain types. As Chandrasekaran aptly remarks such an abstract world would first have to exist and also the fuzzy set axioms would then also have to show they capture 'the operations of this world'. ([10], p.12) Of course, this is open for debate but it certainly raises an important point. On the other hand Chandrasekaran may be taking too much of a purist stance. Christian Freksa tells us that Zadeh realized that it was 'much more important to have a good model of the semantics of human concepts and perform reasonable operations than to have a bad model and perform verifiably correct operations' ([19], p.21)

V.3. Shackle's theory

A very nice application to the treatment of uncertainty in a possibilistic framework rather than in a probabilistic framework is the work by the English economist Shackle. The central idea in Shackle's theory is the concept of *potential surprise*. Potential surprise is defined as follows. Says Shackle : 'we decide on a particular course of action out of a number of rival courses because this one gives us, as an immediately present experience, the most enjoyment (or distress) by anticipation of its outcome.' ([63],p.10) The entity which gives enjoyment by anticipation has, according to Shackle, two characteristics. The first characteristic says Shackle 'describes the situation or sequence of situations, saying what it would be like if it were to happen (without saying anything as to whether it will happen)'.([63],p.10) The other characteristic 'consists in our degree of belief that this picture will become true'.

The interesting question Shackle now proposes is what is this degree of belief referring to. Basically it refers to the fact that the more we are sure something will happen the higher our *potential surprise* if it does not happen. The reason why Shackle talks about surprise as the image of belief is that he considers 'degree of belief ' not to be a sensation or an emotion in itself; though feeling of surprise is. This is what Shackle says 'The concrete mental experience which corresponds to any given degree of belief in some particular hypothesis is , I think, the degree of surprise to which this belief exposes us...'([63],p.11) Shackle also makes the important remark that surprise felt at the *actual* occurrence can not serve as an *uncertainty* variable. The reason for this is quite simple; uncertainty will be linked to something which may happen in the future and we do have incomplete information as to whether it will happen or not. ([62],p.68) If the event has happened the question about uncertainty is of course irrelevant. Therefore it is important to talk about *potential* surprise. Imagine now that we have several, mutually exclusive hypotheses, concerning the same question. How are we now supposed to assign a potential surprise? Shackle's suggestion is this :' ..an individual degrees of belief in a hypothesis can be easily...expressed by means of the potential surprise he assigns to the least potentially surprising rival hypothesis.' ([62],p.71)

This looks nice but is troublesome. It really means that one can certainly NOT have positive beliefs in different rival hypothesizes at the same time. The reason is simple; as we just use the minimum of all the degrees attached to the potential surprises of all hypotheses we effectively rule out consideration of all other rival hypothesizes which have higher degrees of surprise. I do not find Shackle gives a reasonable explanation to this problem. Billot however in ([5],p.29) shows however how tight the relationship is between Shackle's theory and possibilities. There is much more to say on Shackle's theory than those very few words. The great achievement of Shackle's theory is certainly that it emulates human thinking behavior to some extent. In this sense it is close to fuzzy set theory.

V.4. Fuzzy Probabilities?

If fuzziness is not probability how could we then ever talk about a fuzzy probability? This looks like to be confusing! In fact it is not. Basically to talk about a fuzzy probability one must first accept the fuzziness which goes into the proposed subject. To put a probability on this subject is then a fuzzy probability. Nothing new here! Ponsard in Billot ([5], p.38) gives an example. Say a woman likes more or less fur coats. Say that her membership to the set 'coats' is for instance 0.2. Then good friends of this lady may construct a probability distribution centered around 0.2. This will then be a fuzzy probability distribution.

V.5. A new turning point in economics?

The goal in this section is to provide a taste of some of the changes which will occur; specifically in the area of preferences of the consumer, when the assumption of *perfect rationality* of the individual is not upheld. This assumption of a rational agent is crucial in economics. It looks as if this requirement of rationality, which is so far away from reality, is a necessity in the build up of a coherent and consistent economic theory. Of course, this is a very flamboyant statement. Gut feeling however would command that it may be quite useful to have assumptions which are a little closer to reality so to speak. Herein comes fuzziness.

V.5.1. The fuzzy relation

In part II of this paper under the heading of 'structure of binary relations' we surveyed some of the different order relations. The pre-order was defined to be a transitive and reflexive relation. In an economy where there are l goods and m individuals and where total resources to be distributed is $\vec{\Omega}$, any allocation will be a vector of $\Re_+^{lm}$.

The allocation in order to be feasable must satisfy the condition that: $X = \sum_{i=1}^{m} x^i \le \vec{\Omega}$;

where x^i represents a basket of goods allocated to agent i.

The set of all the allocations S={X,Y,Z,...} is non-fuzzy. An agent will classify allocations by using his *preferences.* This preference (non-strict) is a pre-order. We obtain a utility function out of this pre-order when also imposing completeness and continuity. Following Billot ([5], p.45) the claim is that when entering the fuzzy arena the structure as presented here remains valid; however the meaning will be altered very seriously towards 'a more closer to reality' setting.

We have been looking in part II at the concept of a fuzzy relation. Fuzzy reflexivity and transitivity have also been defined. We re-iterate the definitions here:

1) Fuzzy Relation:

A crisp relation on ExF is a set of ExF. Similarly for a fuzzy relation R. Consider two sets E and F; the set of ordered couples (x,y); $x \in E$ *and* $y \in F$ defines the product set $E \times F$. We get then $\tilde{A} = \{(x,y), U_{\tilde{A}}: \forall x \in E; \forall y \in F: U_{\tilde{A}}(x,y) \in L\}$. Where is for simplicity [0,1]. Thus x is in 'relation' with y to some degree. We can also say we have a binary relation between elements of E and F noted $\tilde{\Psi}$. So we can define the fuzzy subset $\tilde{A}$ then as:

$$\tilde{A} = \tilde{\Psi}(X,Y) = \{(x,y); U_{\tilde{\Psi}}; \forall x \in E; \forall y \in F: U_{\tilde{\Psi}}(x,y) \in L\}$$

Examples of fuzzy relations abound. For instance 'Car X is better than car Y' is an example.

2) Reflexive relation;

$$\forall x \in E : U_{\tilde{\Psi}}(x,x) = 1$$

We remark this property may be too strong in a fuzzy context.

We can therefore also define: $\alpha - \textit{reflexivity}: \alpha \in]0,1[\Leftrightarrow \forall x \in X : U_{\tilde{\Psi}}(x,x) \geq \alpha$ ([1],p.35)

3) Transitive relation;

The transitive relation is a key relation in decision making. As classical theory does acknowledge there are two classes of individuals i.e. rational and irrational ones, it must guarantee that for the group in which it is interested; rational consumers; there is consistency. The transitivity relation plays a crucial role in this. A rational consumer should reveal transitive preferences. The max-min transitivity definition is a formal statement which tries to weaken the all too rigid transitivity requirement of classical theory.

$$\forall (x,z) \in E^2 : \max_y \left[\min(U_{\tilde{\Psi}}(x,y), U_{\tilde{\Psi}}(y,z))\right] \leq U_{\tilde{\Psi}}(x,z)$$

See part II for the explanation of the form of this definition. There is somewhat of a problem with this definition. Kaushik Basu remarks that if for instance $U_{\tilde{\Psi}}(x,y) = 0.5 \textit{ and } U_{\tilde{\Psi}}(y,z) = 0.5$ then the min on this will give 0.5. However using for $U_{\tilde{\Psi}}(x,y) = 0.5 \textit{ and } U_{\tilde{\Psi}}(y,z) = 1$ we get again a minimum of 0.5, which makes not much intuitive sense. He therefore presents a definition of the following form: $\forall (x,y,z) \in E^3 : U_{\tilde{\Psi}}(x,z) \geq 1/2\, U_{\tilde{\Psi}}(x,y) + 1/2\, U_{\tilde{\Psi}}(y,z)$. ([2],p.215)Clearly the problem with such definition is cleared away. However, Billot in ([5], p.47) remarks that the first definition is not really a problem. In fact the 'weakening' of the definition of transitivity is just what is sought for; so as to weaken the rationality assumption. In some sense this is not so surprising given that we definitely want to get rid of a 'homo economicus' who has been given hyper-rational powers. Such super beings can distinguish one basket of goods from another. Those powers express themselves in a 'super-sensitivity' when comparing allocations. It is assumed that the finest details are not overlooked when comparing....This is the reason why we should be contented with the traditional definition of fuzzy transitivity.

Blin et al. ([7],p.19) provide a discussion in which they propose to create a fuzzy set of transitive preference patterns. The membership function of such fuzzy set would then give us an indication how close the pattern is to the rigid transitivity pattern as proposed in classical theory. Consumer preferences will be distinguished as to their closeness with the classical transitivity definition. The idea is interesting but lacks practical application.

We could so to speak replace the max-min definition with a general membership function on transitive preferences.

V.5.2. The preference relation

We said above that the preferences of the agent will yield a classification of the different allocations in *S*. The behavior of an agent will be determined by the structure $(S,\tilde{\Psi})$ *where* $\tilde{\Psi}$ is a fuzzy binary relation between the elements of the Cartesian product on $S \times S$. We get then the following expression: $x_j \tilde{\Psi} x_k = \{(x_j, x_k), U_{\tilde{\Psi}}; \forall x_j \in S, \forall x_k \in S: U_{\tilde{\Psi}}(x_j, x_k) \in M\}$

This needs a little explanation. x_k, x_j are the quantities of goods *j* and *k;* and $U_{\tilde{\Psi}}(.,.)$ expresses the *degree* of preference between the two goods. The set *M* contains the membership values; and is usually [0,1]. We can consider the two main cases; i.e. preference and indifference. We now separate the structure $(S,\tilde{\Psi})$ into $(S,\succ)$ *and* $(S,\sim)$.

1. $(S,\succ)$

The most expected property is that the preference relation is anti-symmetric. We defined this in part III. We re-iterate it here: $\forall (x_j, x_k) \in S^2 : U_{\tilde{\Psi}}(x_j, x_k)$ *and* $U_{\tilde{\Psi}}(x_k, x_j) \Rightarrow x_j = x_k$. This means thus that we can not find $x_j \neq x_k$; *such that* $U_{\tilde{\Psi}}(x_j, x_k) = U_{\tilde{\Psi}}(x_k, x_j)$. Furthermore we also obtain fuzzy transitivity. Recall that the max-min form of transitivity is weaker than what the classical binary form will yield.

2. $(S,\sim)$

In classical theory indifference will be reflexive and symmetric. We need some notation before introducing fuzzy symmetry. The following notation makes sense:

$$U_{\tilde{\Psi}}(x_j,x_k) > U_{\tilde{\Psi}}(x_k,x_j)$$

where $U_{\tilde{\Psi}}(x_j,x_k)$ expresses a strong preference; while $U_{\tilde{\Psi}}(x_k,x_j)$ expresses a weak preference. The degrees in the first case is bigger than the degree in the second case. In his decision making an individual will always compare his strong with weak preference. It is expected that if the weak and strong preferences are equal to each other then there will be indifference. Using the notation introduced above this yields then:

$$\forall(x_j,x_k) \in S \times S{:}\, U_{\tilde{\Psi}}(x_j,x_k) = U_{\tilde{\Psi}}(x_k,x_j)$$

where the R.H.S stands for a weak preference and the L.H.S for a strong preference. Remark that this equality also implies the fuzzy symmetry property.

Finally one needs also to look at fuzzy reflexivity. Two possible definitions are offered either: $\forall x \in E{:}\, U_{\tilde{\Psi}}(x,x) = 1$ or also $\alpha - \textit{reflexivity}{:}\, \alpha \in]0,1[\Leftrightarrow \forall x \in X{:}\, U_{\tilde{\Psi}}(x,x) \geq \alpha$ ([1],p.35). The latter form is less strong than the former form.

As preferences are fuzzy it is an overstatement to assume that the individual knows *exactly* what satisfaction to derive from being indifferent towards the identical allocation. In that case his preferences would not be fuzzy. When engaging into preferences on an allocation one will look at the intrinsic qualities of this allocation but also at the relative qualities; i.e. in relation with other allocations. In the reflexivity cases all what we are doing is looking at the intrinsic qualities of the allocation. However as preferences are fuzzy we are not that sure about the level of this kind of quality for a given allocation. In the classical definition of fuzzy reflexivity; when following this train of thought, we would have reached the highest satisfaction possible attached solely to intrinsic quality; as then $\forall x \in E{:}\, U_{\tilde{\Psi}}(x,x) = 1$; i.e. takes the value 1. The definition of $\alpha -$ reflexivity is appropriate in the set up we just exposed above.

We can thus summarize now that $(S,\succ,\sim)$ is a pre-order; i.e. fuzzy reflexive and fuzzy transitive.

V.5.3. A New Interpretation?

1) A very important point is that with the use of fuzziness in preferences we are much more concerned with how an agent arrives to a preference than with the result of the preference as would be the case in classical theory. For instance the exposition on reflexivity showed us this quite clearly. When using the traditional reflexivity we could not come to a reflection on the degree of satisfaction we had when contemplating the intrinsic qualities of an allocation. Another example of this concern; which is more indirect is the use of the max-min fuzzy transitivity which relaxes the rigid and all too precise classical definition.

2) Because we can use degrees of preference we can give significance to strong and weak preference, through using $U_{\varphi}(x_j,x_k) > U_{\varphi}(x_k,x_j)$. We also could use this inequality for giving true meaning to indifference ; i.e. the equality of strong and weak preference. Here again, we stress that with the introduction of fuzziness it is shown what the underlying 'ingredients' are before coming to the result. Only the result is of importance in the classical case.

3) The issue of comparability of two allocations is also very important. In classical theory incomparability is avoided by imposing the axiom of completeness which requires that all goods are supposed to be comparable. Here, because of the possibility of using degrees of preference we can declare incomparability! I.e. the degrees of preference of one allocation to the other is plainly 0. I.e. we get $\forall(x_j,x_k) \in S \times S{:}U_{\varphi}(x_j,x_k) = U_{\varphi}(x_k,x_j)$=0. Weak and strong preference are equal to each other and express thus indifference. A level of indifference of '0' is equivalent to incomparability.

In brief the introduction of fuzziness brings us the **possibility of the evaluation of a comparison.**

V.5.4. Degree of comparability

To set up degrees of comparability we can define the following:

$$x_j \widetilde{H} x_k = \{(x_j, x_k), U_{\widetilde{H}} : \forall (x_j, x_k) \in S \times S, U_{\widetilde{H}}(x_j, x_k) \in [0,1]\}$$

Where $\widetilde{H}$ stands for the relation expressing this degree of comparability; this is a fuzzy relation. This relation is said to be reflexive, symmetric; which following part II, we also called a *resemblance relation.* Out of this relation we can define a fuzzy set C such that: $\widetilde{C} = \{(x_j, x_k) : U_{\widetilde{C}} ; \forall (x_j, x_k) \in S \times S : U_{\widetilde{C}}(x_j, x_k) \in [0,1]\}$ Now we can define a level $\alpha \in [0,1]$. Below the level set there is no comparability. Depending on the level set we will have comparability in some case and non-comparability in other cases. This is then written as:

$$C_\alpha = \{(x_j, x_k), U_{C_\alpha} ; \forall (x_j, x_k) \in C; U_{C_\alpha}(x_j, x_k) = 1 \Leftrightarrow U_{\widetilde{C}}(x_j, x_k) \geq \alpha \text{ and } U_{C_\alpha}(x_j, x_k) = 0 \Leftrightarrow U_{\widetilde{C}}(x_j, x_k) < \alpha\}$$

Remark this alpha-cut is a non-fuzzy set. Integrating in the definition of non-comparability we get then: *If* $(x_j, x_k) \notin C_\alpha \Rightarrow \forall (x_j, x_k) \in S \times S : U_{\widetilde{\Psi}}(x_j, x_k) = U_{\widetilde{\Psi}}(x_k, x_j) = 0$. Furthermore $C_\alpha \cup \overline{C}_\alpha = S \times S$; where $\overline{C}_\alpha$ is the *complement* of C_α; i.e. this takes the union of all comparable and non-comparable allocation; this forms the total set of allocations.

V.5.5. Similitude Sub-Relations and the formalization how to arrive to indifference/preference

We have covered the concept of similitude sub-relation in a fuzzy pre-order in part II of this paper. Consider a fuzzy preference relation $\widetilde{\Psi} \subset S \times S$. We know the preference relation is a fuzzy pre-order relation. I.e. it is fuzzy reflexive and transitive. Now our goal is to find sub-relations which belong to $\widetilde{\Psi}$ and which are transitive, reflexive and symmetrical. Because the relation $\widetilde{\Psi}$ is already a pre-order the subrelation will also be a pre-order. Now, as we add symmetry the sub-relations or sub-classes will form what is called a similitude sub-relation in a fuzzy pre-order. The classes in question express *indifference* as now we have also symmetry. Furthermore the elements in those classes also express *degrees of indifference*. We need to go further however. It is claimed that the relation *among the different classes is an order relation*. This means that we now have reflexivity, anti-symmetry and transitivity. Consider the same example we had when looking at sub relations in part II.

$\tilde{\Psi}$	A	B	C	D	E	F	G
A	1	.2	.2	.2	.2	.3	.4
B	.2	1	.5	.2	.2	.3	.5
C	.2	.5	1	.2	.2	.3	.5
D	.2	.2	.2	1	.8	.3	.5
E	.2	.2	.2	.8	1	.3	.5
F	.2	.2	.2	.2	.2	1	.4
G	.2	.2	.2	.2	.2	.2	1

Is the relation $\tilde{\Psi}$ a pre-order relation?

1) We have to check fuzzy reflexivity and transitivity. Using the max-min definition of transitivity we get then the following. The couple (A,F) in the above matrix when interpreted as the result of a transitivity operation has as underlying couples (A,.) and (.,F); where '.' stands for either A,B,C,D,E or F. Taking the minimum with respect to the membership degree of each two possible couples; we will obtain 6 different minimae values from which we are supposed to take the maximum. The results are then as follows:

(A,A)<->(A,F):--------min[1;0.3]=0.3

(A,B)<->(B,F):-------- min[0.2;0.3]=0.2

(A,C)<->(C,F):-------- min[0.2;0.3]=0.2

(A,D)<->(D,F):--------min[0.2;0.3]=0.2

(A,E)<->(E,F):---------min[0.2;0.3]=0.2

(A,F)<->(F,F):---------min[0.3,1]=0.3

From this we need to take the maximum: 0.3. Does the membership value of 0.3 correspond to the couple (A,F)? It does.

One must perform the same operation for all other possible couples. Fuzzy reflexivity is immediate. Here the classical definition of fuzzy reflexivity has been taken.

2) The pre-order is not symmetrical: as an example $U_{\tilde{\Psi}}(F,D) = .2$ *but* $U_{\tilde{\Psi}}(D,F) = .3$!

3) We can however find subsets of $\widetilde{\Psi}$ which make similitude relations.

As an example the subset K_1={A,BC,D,E} verifies a pre-order and is symmetric; f.i. $U_{\widetilde{\Psi}}(A,C)=.2=U_{\widetilde{\Psi}}(C,A)$. The subset K_1 ={A,B,C} would also verify a similitude sub-relation but it would not be maximal as we can extend this subset into K_1. Two other subsets are also maximal i.e. $K_2=\{F\}$ *and* $K_3=\{G\}$ are similitude sub-relations. All K_1,K_2,K_3 are disjoint from each other as one can easily verify. Thus the fuzzy relation $\widetilde{\Psi}$ is decomposable into maximal disjoint similitude sub-relations. K_1,K_2,K_3 form then similitude classes.

4)We can observe that the levels of indifference in between the classes does vary. For instance the degree of indifference between (A,B) and (B,A) is not the same as the indifference between B and C in the similitude class K_1={A,BC,D,E}.

We now want to look at the idea behind an order relation *among* similitude classes. We continue our example. Take the case of the couples (B,F) and (F,B). One sees that we work among classes here as F belongs to $K_2=\{F\}$ and B belongs to K_1={A,B,C,D,E}. The degrees of membership are certainly not equal and this shows the anti-symmetric property necessary for an order relation. We can also compare the degrees of *preference* of B versus F. The degree of membership for (B,F)=0.3 and of (F,B) is 0.2.; so we strongly prefer B over F. Remark again that we work here *among classes.* Furthermore with the use of the order relation and the similitude sub-relations in a fuzzy pre-order we have been formalizing the set up of how we arrive to indifference or preference.

There is at least one problem however. It is claimed that the relation among classes is an order relation and thus reflexivity is implied. However it looks impossible to claim reflexivity among classes! For instance reflexivity of $\{F\}$ is possible as $U_{\widetilde{\Psi}}(F,F)=1$ but this is not a statement which links elements among classes! So we doubt the fact reflexivity can ever be present amongst classes. For a strong preference relation we can obviously not have reflexivity.

V.5.6. Discussion

Georgescu-Roegen tells us that when an individual makes a choice on the given set of allocations two steps are usually taken. The first one being the delimitation of possible allocations which will be considered by that individual. The second step consists in emitting the choice the individual made.([21],p.137) Classical theory does not treat what happens *in-between* those two steps. To put this in the traditional analogy with our 'homo economicus'; he just is again the robot, and the robot program is inaccessible for earthly beings. The kernel of bringing fuzziness into economics lies in the fact that now we have *degrees of preference*. We can now totally agree that to prefer object A to object B may have a different degree of preference as preferring object C to D. In classical theory we do not have this differential weight of preference. The same is valid for indifference where now we have *levels of indifference*. This led also to the important claim that the completeness axiom is of no use anymore; as now we can really pinpoint incomparability; we do not have to hide incomparability behind indifference. This is also confirmed in Basu.([2], p.225) There are more achievements which are worth mentioning. But we satisfy ourselves with the above. The above is more impressive than one may think. The root of the problem with economics, to my idea is that it treats human beings in the same way as physics would treat objects. Physics does a good job on this; but how possibly can economics do a good job? If it is assumed that we are all the same ;if it assumed there is a well known blueprint of rationality which we all carry in us then we are far removed from the real world. This reminds me of what Popper once said concerning rationality. Says Popper:'....c'est la méthode qui consiste à élaborer un modèle à partir de l'hypothèse de rationalité complète ... de la part de tout les individus concernés, puis à estimer l'écart entre le comportement effectif et le comportement postulé par le modèle...' ([53], p.56) This is of course a very sterile proposition.

However, it is a very honest proposition which basically asks not to ONLY theorize but also to experiment on the proposed theories.

The problem we have to seriously wonder about is the trade off between generalization and specificity. The more we generalize the less concrete we become and vice versa. It looks however as if we are now too far removed on the side of generalization.

There is a critique however also. If individuals can now express their degree of preference why is it so that they can be that precise? This is definitely a very reasonable critique. Instead of concentrating on such cardinal measures some theorists will prefer a much more weaker ordinal form. The notion of 'soft sets' derives from this approach. See [1.1] and [2.1].

Part VI: Applications

This is then the last part of this paper which deals with applications. It is certainly an exceedingly important part of this paper as, I hope, it will set out somewhat what the potential of fuzzy sets may be in the social science sphere; especially in the areas of economics and finance. We are specifically interested in the following applications:

1) The construction of a fuzzy utility function

2) Fuzziness and the Producer's Equilibrium

3) Applications in Finance.

We will of course draw on some of the concepts which have been introduced in the former parts of this paper.

VI.1. The Construction of a fuzzy utility function

Part V of this paper introduced already some of the possible newities which may be expected when using fuzzy sets in micro-economics.

Though the work by Chen et al. seems to lie down some intuitive groundwork; I do think the article is faulty in many respects. Therefore we follow *some of the points* Chen et al. in ([12]) propose and develop also our own arguments.

VI.1.1. Introduction

The utility function is traditionally defined as a mapping from an n-ary commodity space to a utility space. By fuzzifying this we will not use numbers (of course in an ordinal framework) attached to indifference curves but fuzzy sets in the form of fuzzy numbers.

VI.1.2. Mathematical recap and extensions

Some notions Chen et al. introduce in their set up have not been included in part II of this paper. Therefore we introduce the new concepts here. Furthermore we recap briefly some notions which have been covered in part II already.

1. Two dimensional fuzzy sets

This is a straightforward extension of the original definition of a fuzzy set. The only difference comes in here where the variable which is supposed to belong to *some degree* to the fuzzy set is now a vector with two coordinates. Hence the membership function will not be anymore a simple bi-dimensional graph but a graph in 3 dimensions. I.e. the coordinates take up two axis's and the membership value takes up one axis. Of course we can encounter membership functions of any dimension.

Related to point 1) is the concept of *cylindrical extension*. The basic idea of this is that the variable in its original form; whether a vector or not; will be transformed into a vector of dimension: dim(original)+1. Thus if the variable is originally a two dimensional vector; cylindrical extension will make it a three dimensional vector.

2. Projection of a fuzzy relation

A fuzzy relation was defined as: $\widetilde{A} = \widetilde{\Psi}(X,Y) = \{(x,y); U_{\widetilde{\Psi}}; \forall x \in E; \forall y \in F: U_{\widetilde{\Psi}}(x,y) \in L\}$; where L for the membership set, which is usually [0,1]. The projection of a fuzzy relation is a little less intuitive. If we consider a relation in product space $E \times F \times G$ then the ensuing relation could be represented as a three dimensional figure. Each point making up this figure will carry a membership value referring thus to the fuzzy relation in question. The projection of this fuzzy relation will be defined as the supremum of subsequences of membership values of this fuzzy relation. As an example consider the relation $\Psi(X,Y,Z)$; then the projection of this relation on the plane XY would be defined as: $\Psi'(X,Y) = P_{X \times Y}\Psi(X,Y,Z)$; with P standing for 'projection'.

3. Composition of fuzzy relations

The composition of a fuzzy relation was defined in part II as: given a fuzzy relation $\widetilde{\Psi}$ of E to F and a fuzzy relation $\widetilde{\Omega}$ of F to G then the composed relation $\widetilde{\Omega} \circ \widetilde{\Psi}$ is a relation of E to G such that: $\forall (x,z) \in E \times G: U_{(\widetilde{\Omega} \circ \widetilde{\Psi})}(x,z) = \max_y \left[\min(U_{\widetilde{\Psi}}(x,y), U_{\widetilde{\Omega}}(y,z)\right]$. There is an equivalent to this definition. Using the concept of projection the above can be stated alternatively as: $\tilde{\Omega} \circ \tilde{\Psi} = P_{E\times G}(\overline{\tilde{\Omega}} \cap \overline{\tilde{\Psi}})$. It is important to see first why this is an alternative definition. Furthermore remark that the relations are extended cylindrically. (noted with a bar); the reason for this will become clear in a moment. The equivalence sprouts out as follows:

as $U_{(\tilde{\Omega}\circ\tilde{\Psi})}(x,z) = \max_y \left[\min(U_{\tilde{\Psi}}(x,y), U_{\tilde{\Omega}}(y,z)\right]$ then we can re-write this as: $U_{(\tilde{\Omega}\circ\tilde{\Psi})}(x,z) = \vee_y \left[(U_{\tilde{\Psi}}(x,y) \wedge U_{\tilde{\Omega}}(y,z)\right]$; using the definition of projection which says that we use a supremum then: $P_{E\times G}[(U_{\tilde{\Psi}}(x,y) \wedge U_{\tilde{\Omega}}(y,z)] = P_{E\times G}(\overline{\tilde{\Omega}} \cap \overline{\tilde{\Psi}})$; where the equality is legal given the definition of fuzzy intersection; as defined by Zadeh. Hence the alternative way. We need still to come to terms with the fact that cylindrical extensions have been used. This is in fact not that hard to see. The extension is a necessity as we compose relations on $E \times F$ *and* $F \times G$; and the composite relation will thus have to lie in $E \times G$. But composing the two relations in $E \times F$ *and* $F \times G$ spaces; reduces to composing those relationships in $E \times F \times G$ space. Hence the need for an extension on both $\tilde{\Omega}$ *and* $\tilde{\Psi}$.

4. Fuzzy Numbers

Let the universe U be the real line. A fuzzy set A on $\Re$ is called a fuzzy number iff A is convex and there exists exactly one point, say $M \in \Re \; with \; U_{\tilde{A}}(M) = 1$. The reason for convexity has been explained in part II. The reason for the normalization is that for the case of f.i approximately 50; the membership value must be '1' at x=50. Note also that there are different types of fuzzy numbers. One special case is the L-R fuzzy number of Dubois and Prade. (see part II) Again note that a fuzzy number is a special case of a fuzzy set in that it is restricted through the normalization and convexity condition.

5. Ranking of fuzzy numbers

Chen et al. define a fuzzy set *K* to be greater (or equal) than a fuzzy set *L* if all of the alpha-cuts of *K* are greater than or equal than the alpha cuts of *L*. ([12], p.290) One of course compares at the same level of alpha for both fuzzy sets. A simple example of this is in the case of nicely shaped bell curves with a membership for x at 1 (in case of a fuzzy number); and 'one bell shaped curve sits on the other'.

VI.1.3. Basic Concepts in Fuzzy Economics

1. Fuzzy Preferences

We have been talking already about the rationale behind the idea of fuzzy preferences in part V of this paper. The utility level attached to the preference will be fuzzy. Chen et al. propose that a survey could be made based on different combinations of commodity bundles and consumers could be indicating the level of fuzzy utility in terms of whether they are very satisfied; or merely satisfied etc. with a particular selection.

A sufficient number of such indicators will lead us to derive a crude fuzzy utility function for the entire commodity space. The intuition here should now be clear. Consider the case of a commodity bundle consisting of two goods. In classical theory the utility function will be a 3-dimensional figure which has on its two axis's the goods of the commodity bundle in question and an axis indicating the level of utility. The surface traced out will then be called a utility surface. The very interesting thing here as compared with the fuzzy counterpart of this is that because utility would now be a fuzzy set; i.e. utility levels would now be fuzzy numbers the utility surface would have a 'skin' of varying thickness to use imagery. The 'tickness' of the 'skin' of the classical utility surface will be uniform and be as thin as a point basically.

Clearly, when using the ordinal concept of utility one does not bother about the exactness of the numbers obtained; levels of utility have only a ranking purpose. However because there is a precise number attributed to each combination of quantity of the goods in a given commodity bundle we assume that we have crystal clear preferences. The case of the thick utility surface does assume exactly this away. However, as a unique argument the above is not really revolutionary. Basically, because of the ordinal character of utilities the exactness of a number attached to a certain commodity bundle is clearly not relevant. In the same vein can one argue for the tick utility surface. The implication of a 'tick' utility surface however comes through the demand curve. This is what we want to survey now before continuing. The concept of 'thick demand' curves was invented by the French economist Marchal. The basic idea of such demand curve is that it expresses a variability of rationality. The ticker the curve becomes the less rational we are. Hence, in this framework the regular demand curves as we know them would express *ultra-rationality.* We know the versions of Hicks and Marshall on demand curves. The former introduced substitution and income effects; while the latter only bothered about substitution effects. In any case the resulting demand curve was assuming a *totally rational individual.* The interesting issue in Marchal's book is that he also considers the demand of an individual he calls 'l'individu partiellement rationnel et partiellement conditionné par le milieu.' ([51],p.543) This is of course closer to reality than the perfectly rational consumer as treated by Hicks and Marshall.

The idea is simple, the individual when facing a price for a good will not know what *exact* quantity he would be willing to buy. Let us remark first that more leeway is given to the true nature of the average consumer when also considering the income effect. However the 'exactness of quantity' problem is never solved under such framework. Says Marchal 'pour chaque prix, il y a, en fait, non plus une certaine quantité qui sera automatiquement demandée; mais une *quantité minima qui sera certainement atteinte et une quantité maxima qui ne sera pas depassé. Entre l'une et l'autre; la demande se fixera d'après l'intensité des stimulants que présentera le milieu externe.*' ([51], p.544) This looks straightforward and makes sense. But we should carry the analysis a little further however. Let us consider the indifference curve which can be viewed as a horizontal cut; through the utility surface at a given level of utility. The major implication of such indifference curve forthcoming thus from an ultra-thin utility surface is that we are now *able to compare any super small amount of one good versus an amount of the other good always respecting the level of indifference. We are capable of comparing any possible combination of two goods under the constraint of indifference.* This is of course gross exaggeration. The human being is not born with five Pentium® chips in his brain! Let us go a little further now. The points on the ultra-thin demand function are found successively by maximizing a utility function subject to a budget constraint. Hence the point found for a particular maximization derives from the fact that we are able to compare an infinite amount of quantities no matter how small or big they are, all respecting indifference! This is erroneous. Thus it must now be clear that if we leave some leeway to the individual, *as we know him*; then we will not obtain a precise quantity as the result of the optimization but a fuzzy quantity! Let us go still a little further. *In order to make human sense out of the ultra-thin indifference curve we are truly obliged to accept that some quantities of goods will be incomparable. I.e. we can not compare one millionth of a unit of good X to 10 units of good Y. This is then incomparability. But incomparability is lumped together with indifference here given that we face an ultra-thin indifference curve. Hence the necessity of the completeness axiom! The problem of lumping incomparability into indifference ; which is clearly faulty but necessary in order to make human sense out of an ultra-thin indifference curve is absent when considering fuzzy utility functions.*

The reason for that is quite simple. The fuzzy utility function now carries fuzzy numbers as levels of utility. The fuzzy utility surface will have a 'skin' of varying thickness! Now consider a horizontal cut through this fuzzy utility surface. Say that fuzzy utility is measured on the Z-axis (vertical) and the non-fuzzy quantities of both goods on X and Y axis. The projection of the cut on the XY plane will yield an indifference curve which will be fuzzy and will now be non-uniform. The non-uniformity of such projection could be roughly defined as indicating that we can denote an *area* to the indifference curve. There is no such *area* in the classical case. When optimizing a fuzzy utility function with a non-fuzzy budget constraint we obtain a quantity on our fuzzy demand function which precisely corresponds to the quantities found on Marchal's thick demand curves. For a fixed quantity of good Y; the indifference will purport to a certain interval of quantities for good X. This makes much more sense now. Remark that the utility level for the fuzzy indifference curve is also fuzzy. Another important question arises and that is the one of *levels of indifference*. The concept makes plain sense as we will see below. In effect the fuzzy indifference 'curve' has also a membership function; and hence *levels of indifference* can be contemplated.

2. Normalization of the fuzzy indifference set.

Point 1 dealt with an intuitive outline on the fuzzy utility function. We now want to formalize our thoughts a little more. We begin with the fuzzy indifference function. The fuzzy utility function is defined as a mapping from X in $\tilde{F}(U)$ thus: $\tilde{f}: X \rightarrow \tilde{F}(U)$. X is the commodity space ; i.e. the set of all commodity bundles. $\tilde{F}(U)$ is the collection of all fuzzy numbers which indicate the level of fuzzy utility. The locus of $\tilde{f}$ in the product space $X \times U$; which Chen et al. call the consumption space is then a fuzzy subset of $X \times U$. Equivalently one may also construct a fuzzy relation $\tilde{\Psi}_f$ between X and U.

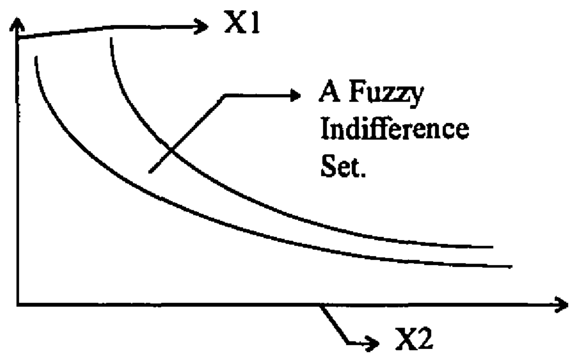
Now we could view in the case of the utility function that the level of fuzzy utility is the dependent variable. In the case of indifference curves we could view the commodity bundle as the dependent variable. Hence we could define for fuzzy utility functions the relation $\tilde{\Psi}_f(x,u)$; where x is a commodity bundle and u the level of *non-fuzzy* utility; this represents a relation from X to U. The inverse relation $\tilde{\Psi}_f^{-1}$ goes then from U to X. Following Chen et al. *the indifference set* of commodity bundles is defined as: $I_m = P_X(\tilde{\tilde{M}} \cap \tilde{\Psi}_f^{-1})$. This needs some explanation. First of all $\tilde{M} \in \tilde{F}(U)$; and it has to be *extended* to $\tilde{\tilde{M}}$ because $\tilde{\Psi}_f^{-1}$ operates in $U \times X$; hence as we know the projection to be equivalently defined to a composition of relations (see 3) under VI.1.2.) then as $\tilde{M}$ is defined in U; the composition $\tilde{M} \circ \tilde{\Psi}_f^{-1}$ will be defined in $U \times X$; so $\tilde{M}$ is to be extended in $U \times X$. The question arises why there is a need to compose with $\tilde{M}$. The only way I can see this is that u takes non-fuzzy values and all those non-fuzzy values linked to the different non-fuzzy commodity bundles some of them will fit within the fuzzy set $\tilde{M}$. The intersection of the inverse fuzzy relation and the extended $\tilde{M}$ will yield the indifference area; i.e. given a certain level of fuzzy utility what are the consumption bundles; which are non fuzzy here; which carry this same level of fuzzy utility? This 'intersected area' is then projected on the commodity space. Remark that in fact $\tilde{\Psi}_f(x,u)$ is not a fuzzy relation as neither x *nor* u is fuzzy; but $\tilde{M} \in \tilde{F}(U)$ is fuzzy. We are also concerned in finding a form for the membership function belonging to the fuzzy indifference set. That the indifference set is fuzzy is an immediate consequence of the fact that we are confronted with a fuzzy utility. Therefore we now have a look at this membership function. As seen in 3) under VI.1.2. the composition of two relations may also be expressed through max-min. This yields then the following:

$\mu_{\tilde{I}_M}(\chi) = \max_u \min[\mu_{\tilde{M}}(u), \mu_{\tilde{f}(\chi)}(u)]$. An explanation of the symbols used is immediately needed. $\mu = \textit{membership grade}$ and $\chi =$ non-fuzzy commodity bundle. Now it is important to correctly read this definition. $\tilde{M}$ *and* $\tilde{f}(\chi)$ are fuzzy sets. u is not a fuzzy number but is an ordinary number.

$\tilde{M}$ is a fuzzy number and $\tilde{f}(\chi)$ yields a fuzzy utility which is also a fuzzy number. By denoting $\mu_{\tilde{M}}(.)\ or\ \mu_{\tilde{f}(\chi)}(.)$ we look at how well the non-fuzzy utilities are members of the fuzzy set of utilities. How can we describe the membership grade of the non-fuzzy commodity bundle χ in the indifference set $\tilde{I}_M$? Basically we need to look at how close the fuzzy utilities through $\tilde{f}(\chi)$ match the given level of fuzzy utility $\tilde{M}$. A perfect match for instance will generate a perfect membership of '1' in $\tilde{I}_M$. Thus following Chen et al. 'this 'closeness' of two fuzzy sets $\tilde{f}(\chi)$ and $\tilde{M}$ is measured by the maximum of membership grades of the intersected set of $\tilde{f}(\chi)$ and $\tilde{M}$'.([12], p.288). Remark that $I_m = P_X(\tilde{\tilde{M}} \cap \tilde{\Psi}_f^{-1})$ indicates the set of commodity bundles eligible to be claimed indifferent. Using $\mu_{\tilde{I}_M}(\chi) = \max_u \min[\mu_{\tilde{M}}(u), \mu_{\tilde{f}(\chi)}(u)]$ we look at how well those quantities are members of the fuzzy indifference set; i.e. it traces out the membership function. Both formulations show us that the indifference set is clearly a fuzzy set. Remark finally that in the case of a horizontal cut (still with two commodities) the membership function will be three-dimensional.

3. Discussion

The fuzzy indifference set was defined through : $I_m = P_X(\tilde{\tilde{M}} \cap \tilde{\Psi}_f^{-1})$. The meaning of this formulation is crucially important. Let us go a little deeper. Assume commodity bundles of two goods. We can basically express the fuzzy indifference set in a one dimensional setting or a two dimensional setting. In the one dimensional setting a *vertical* cut is performed on the fuzzy utility surface. The intersection of the cylindrical extension of the fuzzy utility and the vertical cut of the fuzzy utility surface will give us an area which is to be projected on the commodity space. The result is a fuzzy indifference set. For our purposes it is , in line with the classical case, more interesting to perform a horizontal cut on the fuzzy utility surface. For this we take again the cylindrical extension of the fuzzy utility which is now horizontal of course and let it intersect with the fuzzy utility surface. This intersection is now projected on the commodity space. The fuzzy indifference set thus obtained looks like in the figure below. Remark that the fuzzy indifference set has an 'area'; to the contrary of the classical case where there is no area. A three-dimensional membership function is put on this fuzzy indifference set.



VI.1.4. Fuzzy Weak Preference Set

In classical theory the upper contour set of the convex indifference curve would show us the weak preference set; if of course the lower boundary is included. This upper contour set we know is a convex set. How would the fuzzy weak preference set be defined? In the classical case an upper contour set given a certain level of utility can be defined as the union of all indifference curves carrying a level of utility greater than or equal to the given utility. Says Chen et al. 'The weak preference set $\tilde{R}_{\tilde{M}}$ is the union of all indifference sets with fuzzy utility indicators greater or equal to $\tilde{M}$. ' ([12]; p.290)

An immediate question arises: how can one define one fuzzy number to be greater than another? This was looked at under VII.1.2. The definition as proposed under that heading makes sense as long as the membership functions of the fuzzy number are similar in shape. If membership functions can assume different shapes it will be difficult to start comparing the alpha-cuts for *all* alpha before we can conclude that a fuzzy number is really greater than another. The operability of the proposed definition is certainly questionable to some extend. One way to write $\widetilde{R}_{\widetilde{M}}$ (the weak preference set) is then as follows: $\bigcup_{t}\{\widetilde{I}_t = \widetilde{M}_t \circ \widetilde{R}_f^{-1} \xi\, \widetilde{M}_t \geq \widetilde{M}\}$.

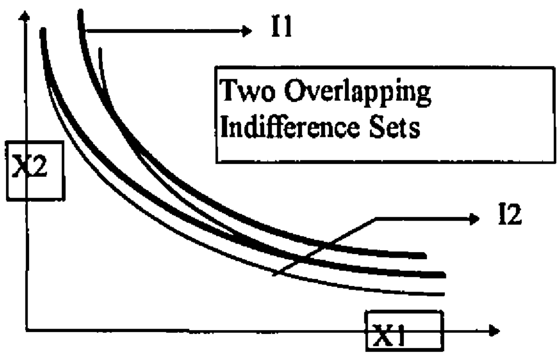
VI.1.5. The fuzzy step function

Let $\widetilde{M}$ be a fuzzy number; the fuzzy step function is defined as a fuzzy set obtained from $\widetilde{M}$ by letting $\mu_{\widetilde{M}}(x) = 1; \textit{for all}\ \chi > \chi_0$. The quantity at which the bundle assigns a membership value of '1' is denoted χ_0. For all quantities beyond χ_0 a membership value of 1 is counted. Remark that χ_0, χ are thus vectors. This proposition is extremely debatable, however! We still assume that we work with two commodities.

As the fuzzy indifference set is indeed carrying a membership function all couples in the given set will carry a degree of membership in that set. Clearly if a specific commodity vector has a low membership value in the given set, a set, which we recall is of course totally conditioned upon the given level of fuzzy utility, then we may either conclude, in rough terms, that the level of fuzzy utility is either too high or too low for this specific commodity vector. Chen et al. propose however that if a particular vector can be localized which has degree of membership '1' then all vectors greater than this vector would also have the same membership degree in the fuzzy indifference set. First a vector greater than another vector, in two-dimensional space, means that at least one of the coordinates has to be strictly bigger than the corresponding coordinate of the other vector. There is at least one problem to this proposition. By augmenting the quantity of at least one of the coordinates it is not appropriate to think the membership degree would have to remain at 100%. If a commodity vector is higher in quantity than another for which the membership was 100%, it is erroneous to think that the higher quantity couples would have also this membership grade! Instead they would have lower membership grades as they would fit less well in the given level of fuzzy utility; i.e. for the higher quantities the given level of fuzzy utility would be too low! Hence we can not subscribe to Chen et al. proposition of a fuzzy step function. Recall that the union of fuzzy indifference sets which have a fuzzy utility higher than or equal to an imposed threshold level of fuzzy utility would form the fuzzy preference set. Using the Zadeh operator for union which is 'max' we would then take the maximum of all membership grades of all the fuzzy indifference sets constituting the weak fuzzy preference set and we would then have found the membership function, in three dimensions, of the fuzzy weak preference set.

VI.1.6. Analysis of the strong preference set

We now need to think a little deeper on how a fuzzy preference set would be constructed in detail; when we are confronted with a *strong preference set.* We agree that $\bigcup_{i}\{\widetilde{I}_i = \widetilde{M}_i \circ \widetilde{R}_j^{-1} \xi \widetilde{M}_i > \widetilde{M}\}$ now expresses the strong preference set instead of the weak preference set. There is a serious difference as to the construction approach when looking at the two types of preference sets. In the strong fuzzy preference set we need *first* to disentangle the problem of knowing which coordinates belong to the fuzzy threshold utility level and which coordinates would not belong. In the weak fuzzy preference case there is no such problem as we have a weak inequality between fuzzy numbers. So in the weak preference case we can satisfy ourselves with just constructing the membership function of the preference set. The problem we want to discuss now is related to the fact that we do not really know how to delimit the strong fuzzy preference set. Under the condition of a concave fuzzy utility surface (see VI.1.7 below) we will in most circumstances be confronted to an overlap of fuzzy indifference sets. The situation is pictured in the figure below:



The fat-lined curves form one indifference set (I1) and the thin-lined another indifference set.(I2). Clearly the fuzzy indifference sets do overlap. The commodity vectors which lie in the overlapping area can not have a membership degree of 1. This is expectable as a 100% membership in both sets would imply a contradiction against ordinary set theory. We will exclude the commodity vectors which belong to the fuzzy indifference set carrying the benchmark fuzzy utility. The other 100% commodity vectors will fatally belong to the next higher level of fuzzy utility and should thus be included in the strong fuzzy preference set. The problem is of course not solved yet because we have to decide what to do with all the commodity vectors which have membership degrees which are less than '1'. The vectors in the overlapping area will normally have two membership values; i.e. in relation to each of the membership functions. The issue is clear if for a certain vector the membership value is higher in one set than versus another. If the membership of a vector is higher in the benchmark fuzzy utility indifference set then this commodity vector will be excluded. If the reverse occurs the vector should be included in the strong preference set. We are still not finished. We still have two remaining cases. The vectors which are in the non-overlapping areas but do not have membership values of '1' and the vectors which are in the overlapping areas but do not have two membership values. For the case of non-overlap and membership degrees inferior to 1 a proposition could consists in creating a neighborhood of some radius around the commodity vector in question and to find out whether membership increases if at least one of the coordinates is increased. If membership increases then the low original membership value is due to the fact that the fuzzy utility is too high; rather than too low. If the membership decreases while increasing at least one of the coordinates then we can conclude the fuzzy utility is in fact too low for the particular commodity vector. In the latter case for instance the commodity vector will be included in the preference set; while he will not be included in the former case. We need of course to take a radius which is small enough so that one remains in the same non-overlapping area. The same procedure could be followed for the case of overlap but no double membership. This is only a sketch of the possibilities. There may be more possibilities and things may get quite more complicated if the membership functions take peculiar forms.

There is some 'discipline' however in what possible forms the membership functions of fuzzy indifference sets can adopt. This discipline is imposed from the fact that the membership functions of the fuzzy utilities are fuzzy numbers and must thus be convex and normalized.

VI.1.7 Is the weak preference set a convex set?

In classical theory this is an important issue. A unique optimal point depends on the convexity of the preferences set. The same goal has to be pursued in a fuzzy set environment.; though this may well be more difficult. One of the crucial requirements for a convex indifference set is the requirement that the utility surface is concave; i.e. that it indicates diminishing marginal utility. This will yield in classical theory nice convex preferences and consequently a preference set which will be a convex set. We assume the commodity bundles are 'good' goods, i.e. not 'bads'. The problem when introducing fuzziness is that we are confronted with a non-uniform utility surface, as said already before. Hence we need a stringent requirement so to be able to claim that the fuzzy utility surface is indeed concave. Imagine that the 'top layer' of this surface is indeed concave; this does certainly not imply that the layers below that 'top layer' will be concave; they may have convex parts. So to be able to claim that a fuzzy utility surface is concave we must tear apart this 'tick' surface into several layers and examine each of the layers separately as to whether they are concave or not. Chen et al. ([12];p.294) call such layers *iso-membership grade surfaces.*(IMGS). Each layer has a varying *fuzzy* utility but carries the same membership grade in the different membership functions associated to the varying fuzzy utility. If each of those layers is indeed concave then we can declare the entire fuzzy utility surface to be concave. All this leads us to a possibly important theorem which may be the base of finding fuzzy equilibriae:

Theorem: *If every fuzzy utility indicator of a fuzzy utility relation is convex and normalized, and if there is a diminishing marginal utility for every IMGS then the weak preference set is convex.* ([12],p.298)

Let us briefly discuss the conditions of the theorem.

1)The fuzzy utility indicator has to be convex and normalized. In other words we must use fuzzy numbers as utility indicators. The requirement is important because it allows us to claim that the fuzzy utility surface will be uniform. Of course the uniformity of the fuzzy utility surface does not guarantee there will be no convex parts in this surface. The requirement of using fuzzy numbers only lead us to conclude that the ISMG surfaces will never intersect when the fuzzy sets (here utility indicators) are convex and normalized.

2) There is also a diminishing marginal utility needed for the fuzzy utility surface. In other words we must impose that every ISMG surface is to be subjected to diminishing marginal utility. Requirement 2) is expected, given of course that from 1) we only could conclude the fuzzy utility surface would be uniform. If every ISMG surface is indeed concave then the fuzzy utility surface will be concave.

VI.1.8. The optimization problem

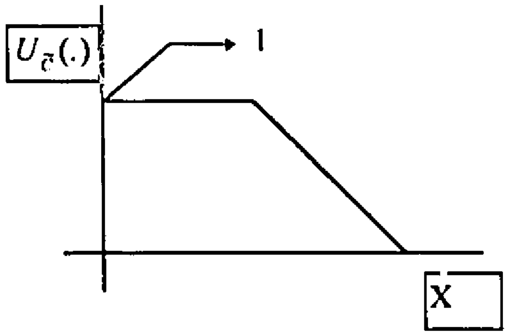
We now will consider the simple case of optimizing a utility function subject to a fuzzy budget constraint. The result out of this optimization should yield us Marchal's thick demand curve as discussed before.

This section has four objectives. First, to give a taste of the dangers of fuzzy optimization and second to show the dependency of fuzzy optimization on underlying membership functions. The third objective has as mission to spell out a possible approach to optimizing a crisp utility function subject to a fuzzy budget constraint. The fourth objective concerns the optimizing of a fuzzy utility function subject to a fuzzy budget constraint. Let us remark however we will not use any fuzzy optimization technique proper as this would require too much sophistication and space is not provided in this paper to expand upon a possible fuzzy optimization with sophisticated membership functions. This relates to the second objective we mentioned above.

1.Objective I and II: Dangers of fuzzy optimization. Dependency of fuzzy optimization on membership functions.

In most of fuzzy optimization max-min operators are used. Lai and Hwang are an example. ([45]). Part IV showed us that max-min operators have theoretical validity but are quite less useful in a practical descriptive context. The max-min operator within the programming context derives from a paper by Bellman and Zadeh in which the authors propose that the decision should be based upon the conjunction of objective and constraint and therefore the 'and' operator (or min) would be used on objective and constraint. A decision based upon those two ingredients is un-debatably a necessity however whether the min operator is to be used for that is to be discussed. The word 'conjunction' is somewhat debatable however. Following Bellman and Zadeh in ([4],p.B149) this conjunction refers to the 'hard' 'and'. However other operators may be looked at notably the product operator (i.e. compensatory) as seen in part IV of this paper. Note that Bellman and Zadeh only give an intuitive type argument for this 'and; operator in [4].

Many fuzzy optimization techniques which I came across with, mainly through Lai and Hwang ([46]) do only cover an extremely limited form of membership function: linear and of a particular shape. For more sophisticated shapes as the one which we will be looking at in objectives three and four; i.e. when optimizing (fuzzy) utility subject to a (fuzzy) budget the brief survey following here below will be of no use whatsoever. The particular shape of the membership functions considered in ([46]) would be of the following type:



Consider this basic problem in fuzzy optimization

$$\max z = cx$$
$$s.t.\ (Ax)_i \le \tilde{b}_i, \forall i; x \ge 0; \tilde{b}_i \in [b_i, b_i + p_i]$$

and the linear assumed membership function is defined as:

$$U_i(x) = \begin{cases} 1\ if\ (Ax)_i \le b_i \\ 1 - [(Ax)_i - b_i]/p_i\ if\ b_i < (Ax)_i < b_i + p_i \\ 0\ if\ (Ax)_i \ge b_i + p_i \end{cases}$$

The following points should be raised:

1) the membership functions are often assumed to be nicely linear. Lai and Hwang in ([46]) is an example. From part III we have been stressing enough that the way to get to membership functions is extremely dependent on the problem at hand. Also important are the derivations which are presented using this type of membership function. More complicated membership functions will also yield in general more complex solution. We can not go in detail on this however. Different approaches for specific more sophisticated sets of membership functions do exist (for instance piece-wise linear continuous functions). However the solution procedures are really quite limited to a quite simple set of membership functions. We did not come across specific solution procedures when membership functions adopt a much more 'non-classified' character. Remark however that

if fuzzy numbers are present we need to consider a normalized and convex membership function. Hence there is quite less scope to consider more convoluted membership functions in that case.

2) related to 1) is the problem of the assumed operators. Clearly max-min has been resolutely shown in part IV to be of doubtful value in practical cases. The Zimmerman-Zysno study showed this conclusively. Recall Zimmerman's attempt to define an operator with a weighted compensation build in. Here also the solution procedures when using those more sophisticated operators becomes clearly more difficult to handle.

3) An explicit form of the Operations Research approach is stochastic programming where randomness is key. One argument to leave this approach aside is by pretending that fuzzy programming is less concerned with the issue of quantity of information than with the problem of meaning of information. The quantity of information approach comes in there where randomness is introduced. I.e. it purports to the transmission of information. What has reference to meaning of information should be represented through possibility rather than probability. This is the stance Zadeh took in ([76]). We have left the possibilistic distribution approach, as an alternative to membership functions totally aside in this paper. As an aside it is however interesting to look at the distinction where fuzzy sets uses membership function and where it does uses possibility distributions. In ([17],p.15) Dubois and Prade tell us that the first area (i.e. membership function) purports really to a state of complete information; i.e. there is no uncertainty. Remark immediately that membership functions refer thus to imprecision, but not to uncertainty. The scope of fuzziness in economics remains valid however. The second area; i.e. of possibilistic distribution would be the area which relates to uncertainty; or also incomplete information. The important point is that the propositions in the second case are explicitly *Boolean;* i.e. true or false. Say Dubois and Prade ' degrees of uncertainty apply to all or nothing propositions, and do not model truth values but express the fact that the truth value (true or false) is unkown. ([17], p.16).

4) One should perhaps show a little more nuance towards the kind of black and white distinction of randomness and fuzziness. It well may be that there are instances in

programming problems where *both* randomness and fuzziness appear. What to do in such a case is a mystery.....

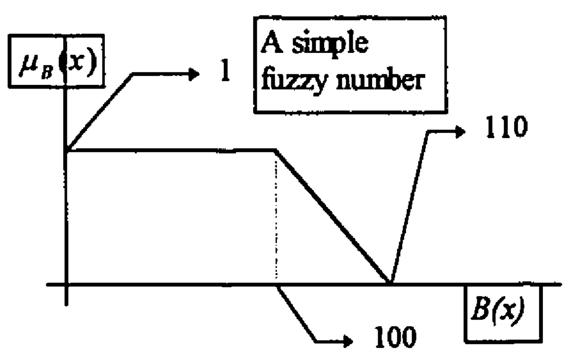
5) Last but not least let us show the case where with the assumed membership function (i.e. linear) we optimize a crisp utility function subject to a fuzzy budget constraint. Many of the methods used in Lai and Hwang for instance are in fact iterations on optimal solutions obtained in a crisp setting. Many optimal solutions do not make use of all input resources which are available in for instance a production problem. By allowing a producer to be 'fuzzy' on some constraint variables this input use can be re-modeled so that better optimal solutions are found. As such this is not really new at all! A simple example can confirm this. A specific utility function is as follows: $\begin{array}{l}\max x^{0.2}y^{0.8} = U(x,y) \\ s.t.\ 2x+y \leq 100\end{array}$; which is a classical problem. We assume non-satiation. The Lagrangian is as follows: $\ell = x^{0.2}y^{0.8} + \lambda[100-2x-y]$; solving this yields us for x=10 and y=80. Now consider the optimization problem when we are confronted with a budget which is not crisp. Say that the constraint of the budget is now a fuzzy number. We get then the following problem:

$$\begin{array}{l}\max x^{0.2}y^{0.8} = U(x,y) \\ s.t.\ 2x+y \leq 10\tilde{0}\end{array}$$

Assume the level of tolerance is 10. Name the budget function *B(x)*. The membership function, for the fuzzy number $10\tilde{0}$ would then be in its simplest form:

$$\mu_B(x) = \begin{cases} 1\ if\ B(x) < 100 \\ 1-[(B(x)-100)/10]\ if\ 100 \leq B(x) \leq 110 \\ 0\ if\ B(x) > 110 \end{cases}$$

So the membership function of this



fuzzy number is then just as in the figure. Remark that the membership at x=100 is 1; while it is '0' at 110. The point we want to make is that if we follow the approach of the basic fuzzy optimization problem as set out above, then the maximization problem could be re-written as follows:

$$\max x^{0.2}y^{0.8}$$
$$s.t.\, 2x+y\leq 100+10(1-\alpha)$$

The Lagrangian is then: $\ell = x^{0.2}y^{0.8} + \lambda[100+10(\theta)-2x-y];\, where\, \theta = 1-\alpha;\theta \in [0,1]$. One sees immediately that if one takes first derivatives towards x and y that the obtained result will not be very different. We would simply obtain that $x = 10+\theta; y = 80+8\theta$. Setting for instance $\theta = 1$ we would indeed find the budget of 110. So we could see this as just a separate optimization for different budget values lying between 100 and 110. There is effectively nothing new here! For more sophisticated membership function results may well be very different as said above. The solution method proposed here would not be usable.

2. Objective III: Crisp utility and fuzzy budget constraint

The set up of objective III is as follows. First we will make a general proposal for the case of optimizing a crisp utility subject to a fuzzy budget constraint. The proposed result will then be translated into the economic setting. The development follows here Billot and Ponsard.

2.1. Basic Set up

Define a set E of all possible commodity bundles. The agent has a crisp utility function on E but has to take into account that he is confronted with a *fuzzy* budget constraint. The objective function results can be captured in a crisp set F; subset of the set E. The objective function is defined as : $f\colon E \rightarrow [0,\infty[$; wherefrom we derive immediately $f(x) \in [0,\infty[$; where $f(x)$ measures thus utility. The fuzzy constraint is a *fuzzy subset of E*; which can be denoted as $\tilde{C}$; and which is defined as : $U_{\tilde{C}}\colon E \rightarrow [0,1]$; thus the membership values are taken in [0,1]. Those membership values signify the degree of membership to the constraint.

Following the proposal by Bellman and Zadeh ([4]) that optimization is a basically a max-min operation; we can write our problem then as: $\sup_{x\in\tilde{C}} f(x) = \sup_{x\in E}[\min(f(x), U_{\tilde{C}}(x))]$. This means thus that the best allocation possible is *the maximal element of the intersection of F and* $\tilde{C}$. The 'optimum' formulation can be re-written however. Use is made of the definition of an alpha-cut which we covered in part II of this paper. The following definition was presented: $\tilde{A} = \bigcup_{\alpha} \alpha.C(\alpha)A$. Let us take a simple example. Consider the following fuzzy set: $\tilde{A} = 0/1 + 0.5/2 + 0.8/3 + 1/4 + 0.2/5$. Taking $C(0.5)A$ of this yields: 0/1+1/2+1/3+1/4+0/5. The fuzzy set A is the union for *all levels of* α of $\alpha.C(\alpha)A$. This yields then for $\alpha = 0.5$: 0/1+0.5/2+0.5/3+0.5/4+0/5. This procedure is to be repeated for all levels of $\alpha \in [0,1]$; taking then the fuzzy union or what is equivalent of all the membership values for the *respective* variables we must obtain the original fuzzy set $\tilde{A}$. It is straightforward to see that $\alpha.C(\alpha)A = \min(\alpha, C(\alpha)A)$. This should be clear as we know that $C(\alpha)A$ adopts only binary values of '0' and '1'. The above example is an easy check on this. Following the definition; $\tilde{A} = \bigcup_{\alpha} \alpha.C(\alpha)A$ we can thus write that $U_{\tilde{A}}(x) = \max_{\alpha}[\alpha.C(\alpha)A]$; which is now equivalent given $\alpha.C(\alpha)A = \min(\alpha, C(\alpha)A)$, to: $U_{\tilde{A}}(x) = \max_{\alpha}[\min(\alpha, C(\alpha)A)]$. Given this last formulation then $\sup_{x\in\tilde{C}} f(x) = \sup_{x\in E}[\min(f(x), U_{\tilde{C}}(x))]$; can be re-written as:

$$\sup_{x\in\tilde{C}} f(x) = \sup_{x\in E}[\min[(f(x), \max_{\alpha}[\min(\alpha, U_{C(\alpha)}(x))]]].$$

Using this last formulation and applying several operations of distributivity, associativity and commutativity on max-min operators (see in part II) the above form can be reduced to the following formulation:

$$\sup_{x\in\tilde{C}} f(x) = \sup_{\alpha} \min[\alpha, \sup_{x\in C(\alpha)} f(x)]$$

which can easily be re-written as:

$$\sup_{x\in\tilde{C}} f(x) = \sup_{\alpha}[\alpha \wedge \sup_{x\in C(\alpha)} f(x)]$$

2.2. Sugeno Measures

The solution will not be fuzzy, though of course the obtained commodity bundle will belong to the fuzzy budget set. Would the solution be fuzzy then we would obtain a solution set with a membership function and every feasable solution would belong to some degree to this membership function. It is of great importance however to accept that the solution obtained is truly a function of the membership function which is proposed. My view is therefore that the Sugeno-integral gives us a clue at how the membership function may influence the probability that the optimal couple will belong to the fuzzy budget constraint we have been imposing. ([1],p.48) The solution statement $\sup_{x \in \tilde{C}} f(x) = \sup_{\alpha} [\alpha \wedge \sup_{x \in C(\alpha)} f(x)]$ can be reduced to a Sugeno integral following Billot in ([5], p.66). Before we come to a rudimentary explanation of this integral we first must briefly consider the fuzzy measures on *ordinary sets*. We will see that the fuzzy Sugeno measure on ordinary sets is in fact a very general approach to fuzzy measures.
Special cases from this general approach are for instance the belief function of Shafer ([64]) which we covered in a former part in this paper. Our objective is not to stress the fuzzy measures on ordinary sets but to look at the Sugeno integral. Both approaches have a common set up however.

2.2.1. Fuzzy measures on ordinary sets

We follow the development by Dubois and Prade. ([16],p.126-134)
Let g be a function from $\wp(X) \to [0,1]$; where $\wp(.)$ is the set of subsets of X. *g()* is said to be a fuzzy measure iff:

1) $g(\emptyset) = 0; g(X) = 1$
2) $\forall A, B \in \wp(X), \textit{if } A \subseteq B \Rightarrow g(A) \leq g(B)$ (*monotonicity*)
3) $\textit{if } \forall i \in N, A_i \in \wp(X); (A_i)_i$ *is monotonic:* $A_1 \subseteq A_2 \ldots \subseteq A_n$ *or also* $A_1 \supseteq A_2 \supseteq \ldots A_n$
$\Rightarrow \lim_{i \to \infty} g(A_i) = g(\lim_{i \to \infty} A_i)$ (*continuity*)

$g(A)$ is called by Sugeno a 'grade of fuzziness' of A.

To just establish a brief overview of the connection of a fuzzy Sugeno measure with for instance probabilities or also Shafer belief functions we need to define Sugeno's λ – *fuzzy measures*. Those measures relax the additivity property of probabilities in the following sense: $\forall A,B \in \wp(X); A \cap B = \varnothing: g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A) g_\lambda(B); \lambda > -1$. Note the added term '$\lambda g_\lambda(A) g_\lambda(B)$'. Would g_λ be a probability we would *not* need this additional term; as $A \cap B = \varnothing$. Dubois and Prade give a short proof which explains why $\lambda > -1$. ([16],p.127) The proof mainly uses the property 2 of monotonicity. We do not re-iterate the proof here. It is interesting to note that if $\lambda = 0$ the above formulation reduces to a probability statement with $A \cap B = \varnothing$. As long as $\lambda \geq 0$ the fuzzy measure will be a Shafer belief function. Thus a belief function is thus a special case of λ – fuzzy measures.

2.2.2. The Sugeno Integral

Billot claims that $\sup_{x \in \tilde{C}} f(x) = \sup_\alpha [\alpha \wedge \sup_{x \in C(\alpha)} f(x)]$ can be reduced to a Sugeno integral. For this the following is necessary:

1) $\mu_f(C(\alpha)) = \sup_{x \in C(\alpha)} f(x)$

2) $\sup_\varnothing f(x) = 0$

3) $\sup_{x \in E} f(x) = 1$

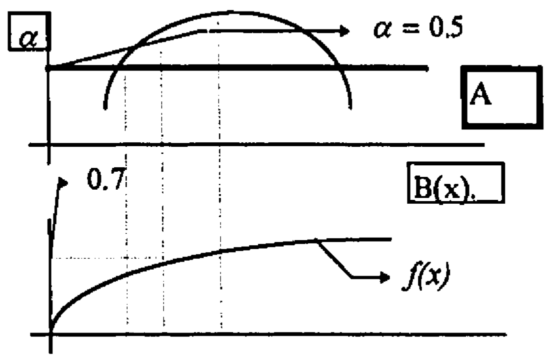
Note that $\mu_f(.)$ is a fuzzy measure on an ordinary set. (see below) *f(.)* is our utility function which is crisp here. $\mu_f(.)$ is found by running through the values found through an alpha-cut, i.e. $x \in C(\alpha)$. So we can truly speak about a fuzzy measure on an ordinary set though the budget constraint is fuzzy. The 'ordinariness' of the set comes thus through the alpha-cut.

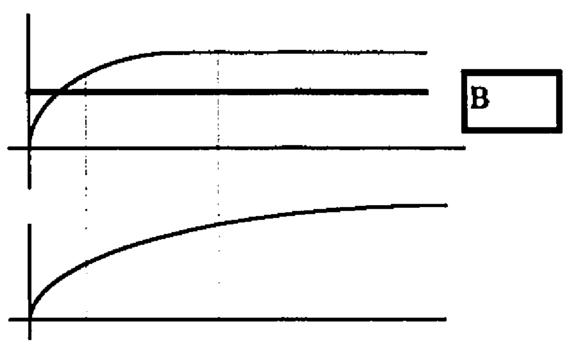
Of course the fuzzy budget has a membership function and the results obtained after different alpha-cuts reflect the membership function in some sense. Using $\tilde{A} = \bigcup_\alpha \alpha C(\alpha) A$

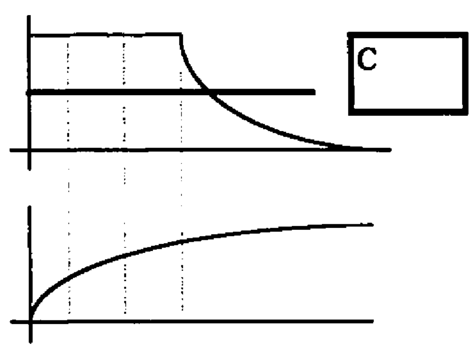
we can see immediately that such proposal makes sense. The crisp set E is the set of all commodity bundles. Note also that the utility value through *f(.)* will tend to 1; when all commodity bundles are considered. This is a necessity if we want to respect the conditions of a fuzzy measure on an ordinary set. It is such that the fuzzy measure on the entire set is 1. (condition 1; under 2.1) The intuition behind $\mu_f(C(\alpha)) = \sup_{x \in C(\alpha)} f(x)$ as a fuzzy measure on an ordinary set could be as follows. Considering that a fuzzy measure in our context would indicate a 'degree' of certainty of a commodity bundle belonging to E *also will belong to the budget set AND POTENTIALLY SOLVE THE OPTIMIZATION PROBLEM.* As an example if a commodity bundle is way above the fuzzy budget constraint the 'degree' of certainty will effectively be zero; plainly because this bundle will not be included in the *ordinary set* $C(\alpha)$. Hence if bundles are in $C(\alpha)$(thus for a certain level of α) which provoke a high singular level of utility through $\sup_{x \in C(\alpha)} f(x)$ then effectively the 'fuzzy measure' or equivalently the 'degree of certainty' the bundles belonging to E will belong to the budget set and potentially solve the optimization problem will be high. Note that for bundles which belong to $C(\alpha)$ one can not obtain a fuzzy measure of zero. This should be clear as the fuzzy measure is the result of a supremum and hence as values obtained through the utility function after an alpha-cut has been performed should be necessarily zero. This is weird indeed as we would need negative utilities! So we should conclude being a member of $C(\alpha)$ confers already a positive fuzzy measure. The higher this fuzzy measure however the better. The reason why this is so is that our maximization problem still consists in finding the highest utility possible subject to a constraint. Note also that if $\alpha = 0$ there is no reason to believe that *automatically* the fuzzy measure on $C(\alpha)$ will be 1. We only have that $\sup_{x \in E} f(x) = 1$; i.e. thus for *all commodity bundles.* We also remark that an alpha -cut of 0 is in fact a senseless statement. Consider a fuzzy set $\widetilde{A} = 1/0 + 2/0.4 + 3/0.6 + 4/1$. Taking an alpha cut for $\alpha = 0$ would lead to a contradiction if we work with a *weak alpha cut.* The reason is simple as then the element '1' would now belong to the ordinary set while it neither belongs to the fuzzy set nor to the ordinary set associated to the fuzzy set. We can set $\alpha = 0$ but then we should consider a strong alpha-cut. Another issue is that the fuzzy measure $\sup_{x \in C(\alpha)} f(x)$ is very dependent on the membership function which is attached to the budget constraint under consideration. In fact this specific fuzzy measure is dependent on three variables: α; the membership function and finally the utility function *f*.

Let us first see the dependence on the membership function with different cases of membership functions.

The figures represent three possible membership functions for the fuzzy budget constraint. Note that the possible budget quantities are on the X-axis (denoted B(x)). *f(x)* is the utility function.

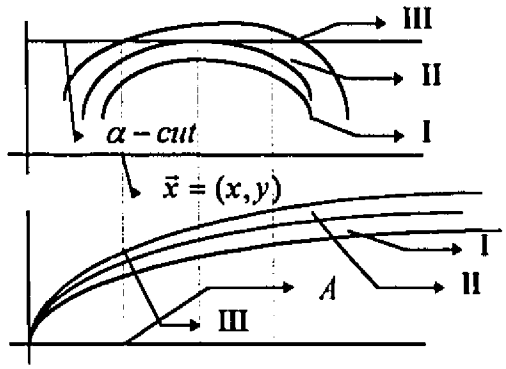






The three figures are A, B and C. Let us look at the results for $\sup_{x \in C(\alpha)} f(x)$. One sees that for the alpha cut of $\alpha = 0.5$ the eligible budget quantities are drawn horizontally down to the quantities used as input for the utility function. The figures should be three dimensional but that is impossible to draw here. Taking the supremum of the utility values on the Y-axis in the bottom graph of A we obtain a value of 0.7. Imagine the same alpha value for figure B. The eligible quantities will yield when the supremum is applied a value of '1'. *No matter what alpha value is considered in figure* B *the supremum value will always be '1'.* In case C we see that for lower values of alpha higher supremum values are obtained. Thus the relationship between the fuzzy measure and the level of alpha is very dependent on the kind of membership function under consideration.

Also the value of $\sup_{x \in C(\alpha)} f(x)$ is also dependent on the level of alpha under consideration. Finally it is also dependent on the shape of the utility function. Let us assume that the utility function has the concave shape as presented in the three figures. The above figures are caricatures. Considering our optimization problem it looks appealing to assume that allocations which are 'closer' to the richness of the individual will also be allocations which will be having a higher membership value in the fuzzy budget constraint. On the contrary the allocations which are far from exhausting the richness of the individual will have much lower membership values. Figures B and C are especially caricatures. It is unappealing indeed to assume that there is a whole set of allocations which would have a 100% membership value in the fuzzy budget constraint. It may indeed be exaggeration to assume that the membership set should be normalized. In the case of non-normalization the fuzzy budget set can not be a fuzzy number anymore however. Figure B does pose some serious trouble, apart from the flat part, in that no matter the alpha value the fuzzy measure will always yield a value of 1. This is truly senseless. Let us recall that a low alpha value as applied in an alpha cut implies that the fuzzy set is very fuzzy indeed. A high value will mean the opposite. Take the alpha cut with alpha=1 and we obtain the *same crisp set as the crisp set associated with the fuzzy set.* This is clearly non-fuzzy. Our proposition for a reasonable membership function would resemble figure A and would of course have to be drawn in three dimensions; given that we have commodity bundles with at least two goods. We could draw this in two dimensions as follows:



The membership function has now several contours. In a two good commodity case the membership function would be of the form $\alpha = g(x,y)$; where x and y are two goods. The function would best be captured in a three dimensional setting. The above membership function has several contours which come from cutting the membership 'surface' vertically. Using the fact that the more of each good is present the higher the utility would be up to a point of saturation; i.e. we assume non-satiation. The utility function has contours. The contours provide from a *vertical* cut on the crisp utility surface. We transit from a lower contour to a higher contour (i.e. for instance from I to II) by augmenting one of the fixed coordinates. Say the vertical cut was obtained by varying the quantities of good X but leaving quantities of good Y fixed then we get a higher contour line if we higher the fixed quantity of Y. Now let us look at the contoured membership function of the fuzzy budget constraint.

Following the proposition that the more we exhaust the budget the higher the membership value will be then it is logical to assume that for instance the membership values at vector $\vec{x}$ will be higher if we let positively vary the coordinate y; and keep the x-coordinate fixed. Note that the intercept A on the X-axis is a set which contains the different vectors where each vector has a fixed coordinate while the other coordinate is increased with a constant value. Thus if there are for instance three contours then the set A will contain three vectors.

We need to argue in favor of why the specific form of the membership function was chosen. Assume we would know our precise wealth then the membership values would increase to the commodity bundle which exhausts 100% our wealth; at that point the membership value would be one. Beyond that point the membership value would have to be 0. As we do not know our wealth precisely we could argue that considering some specific set of allocations will have successively higher and higher membership values up to where membership is highest and then decreases. The left part of the membership function would then 'mirror' the right part of the graph. Note however that it would be erroneous to assume that we could have a 100% membership value in this case. Merely the contours maximae *tend to* a 100% membership degree but never reach it. Would a commodity bundle (x=fixed and y=variable) reach 100% membership there would be no scope to accord positive membership grades for higher quantities beyond that point. Thus *we explicitly assume that the membership function can not be a fuzzy number; as it is not normalized.*

What is to be done now is to look at the evolution of α *Vs.* $\mu_f(C(\alpha))$. We can clearly see that for a given membership contour there is an inverse relation between the two variables. The lower alpha the higher the supremum value and the higher alpha the lower the supremum value. This is not a good relationship however! If a specific number of quantity vectors belonging to a certain very low alpha cut yield thus high utility then they are effectively good candidates for our optimization. The only *serious* problem is that those vectors have been found with very low alpha values and thus the set from which those vectors are drawn is a highly fuzzy set! The best result is to obtain quantity vectors coming from a high alpha-cut and yielding a high utility. The proposed membership function does not allow this. We do want to find an *overall* measure which would take into account the trade-off between a level of alpha and the fuzzy measure. The form could be as follows: $\sup_{\alpha\in[0,1]}[\alpha \wedge \mu_f(C(\alpha))]$. This form does totally mimic the solution form we have found under the section in which we dealt with the basic set up. After re-working we found for our problem the following solution form: $\sup_{x\in\tilde{C}} f(x) = \sup_\alpha[\alpha \wedge \sup_{x\in C(\alpha)} f(x)]$. Following the definition of a fuzzy measure all what is to be done is to replace the supremum in this statement by the fuzzy measure and we arrive to the form $\sup_{\alpha\in[0,1]}[\alpha \wedge \mu_f(C(\alpha))]$. This latter form, we propose is the *Sugeno Integral.* The Sugeno integral has the same definition in general. The interesting issue about this integral is that it is a *fuzzy measure of a FUZZY set.* The fuzzy measure we considered so far were fuzzy measures on *ORDINARY* sets. Let us look in our case what the result would be of the required operation. Of course the result depends on the shape of the utility function and the assumed shape of the membership function. Note however that $\alpha \in [0,1]$ *and* $f(x) \in [0,1]$ as defined above. So it would be impossible for instance to have consistently higher utility values than values of alpha.

So both are comparable. The min operator on alpha and the fuzzy measure does make a lot of sense. It takes care of the problem of high alpha and low fuzzy measures or the opposite. What is *NOT* reprimanded is high fuzzy measures and high alpha's; as we only take the minimum. Taking the minimum, instead of the maximum does take care of the problem that high (low) alpha and low (high) fuzzy measures is not a praiseworthy situation. This is of course not taken care of with a maximum. The supremum is taken for all values which have been obtained at each level of alpha. Note that we need a supremum as the results obtained may have decimal places of the same digits. We do not consider working only with integer values of alpha. The Sugeno integral gives us a global measure of the fuzziness of a fuzzy set. The fuzzy set in our problem is the set from which the solution will be drawn. Clearly the higher the supremum value the better as it means that the optimal result will have a lesser degree of fuzziness. There are certainly fuzzy optimization techniques which take the quite sophisticated shape of the membership function of the budget constraint into account.

As a final comment on this section we would like to mention two points. First, we choose commodity bundles which are in the fuzzy budget. Clearly we do not know the limit of this constraint as it is fuzzy but the limits do not go as far as the set of *ALL* commodity bundles. If we are not sure of our resources but we know we are not a millionaire then to consider the whole commodity space is not relevant. If we want to mimic as closely as possible human behavior then we must have an idea of a commodity space in which commodity bundles are members of our budget constraint. This commodity space is certainly a subset of the set of all commodity bundles. Second, the 'Sugeno integral' we have been using above will give us an indication of basically 'how well we are doing' in having our solution belonging to the *chosen* fuzzy budget constraint. Looking back at the figure above we can readily see, keeping the utility function in its same shape, that if we widen the tails of our membership function of the fuzzy budget constraint that the gap between α *and* $\mu_f C((\alpha))$ would effectively become wider. As we are taking the minimum of α *and* $\mu_f C((\alpha))$ it should not be precluded that the resulting value of the supremum is closer to '1' or even '1'. This is in fact to be expected given the fact that a membership function such as the one in the last figure but with much wider tails will have more chance to contain the solution which maximizes the problem. Nothing really new here, but at least we may get a better grasp of the meaning of the result of $\sup_{\alpha\in[0,1]}[\alpha \wedge \mu_f(C(\alpha))]$. Basically we do want to come to a result which is as close to '1' as possible. If this is so we do know that a solution is therefore findable and we have well positioned our fuzzy budget constraint.

3. Objective IV: Fuzzy Utility and fuzzy budget constraint.

The problem is now increased in difficulty as we are now also considering a fuzzy utility function. We will be brief on this as basically the set up is very similar to the first problem. Though solving the problem as such may be quite more difficult. We definitely do not attempt any solving. Neither did we in the first problem. Recall that the solution statement for the first problem was: $\sup_{x \in \tilde{C}} f(x) = \sup_{x \in E} [\min(f(x), U_{\tilde{C}}(x))]$; where E would stand for the set of all commodity bundles. Remark our comments on this set in the last pages of the former section. The problem with a fuzzy utility and fuzzy constraint is then having a very similar form: $\sup_{x \in E} U_{\tilde{F}}(x) = \sup_{x \in E} [\min(U_{\tilde{H}}(x), U_{\tilde{C}}(x))]$. The following fuzzy sets mentioned in the solution statement have to be defined:

1) fuzzy set $\tilde{H}$ which refers to the fuzzy utility function.

2) $\tilde{C}$ which is the fuzzy budget constraint

3) $\tilde{F}$ which stands for the fuzzy decision set; this means thus that the solution will clearly be fuzzy.

We can do the same re-working through mainly using $\tilde{A} = \bigcup_{\alpha} \alpha . C(\alpha) A$ as we did with the first problem and the result of the second problem becomes then: $\sup_{x \in E} U_{\tilde{F}}(x) = \sup_{x \in E} [\min(\alpha, \sup_{x \in C(\alpha)} U_{\tilde{H}}(x))]$. Note that $C(\alpha)$ refers to an alpha cut on the budget constraint, and NOT on the fuzzy utility.

3.1. Is a solution possible ?

It is of great interest to expand on a possible approach to see if a solution may exist to this problem. No optimization technique is presented to practically solve the problem but a formal way to see whether feasable solutions really can exist. The framework which is followed here goes by Billot ([5],p.72).

Before looking at Billot's very interesting set up relative to 'the' problem let us quickly re-iterate some of the ideas behind the fuzzy indifference set which we covered in the beginning of part V. The fuzzy indifference set can be the result of a horizontal cut on the fuzzy utility surface. It contains commodity vectors which belong to the same fuzzy utility. Recall that the fuzzy utility is a fuzzy number, thus a normalized and convex fuzzy set. The requirement of fuzzy utility as a fuzzy number is important for reasons of being able to rank the fuzzy utilities. I.e. the membership functions should be alike. Some discipline is thus imposed here. One of the issues with the indifference set notion was that they could overlap. Thus certain commodity vectors may indeed have for instance two membership values; as thus the indifference set itself is fuzzy and also has a membership function. Note however that the problem of 'sorting out' suitable commodity vectors in view of deciding whether they should or should not belong to the preference set is only problematic when we are confronted with a strong preference set.

Recall also that a weak fuzzy preference set is convex if the iso-membership grade surfaces (IMGS) are themselves concave. Finally recall that if every fuzzy utility indicator of a fuzzy utility relation is convex and normalized, and if there is a diminishing marginal utility for every IMGS then the weak preference set is convex. ([12],p.298)

Let us now with the argumentation. First we will re-write the solution expression in somewhat of a different way. We obtained that: $\sup_{x\in E} U_{F}(x) = \sup_{x\in E}[\min(\alpha, \sup_{x\in C(\alpha)} U_{\Pi}(x))]$. Define now the following functions:

1) $\varphi:[0,1]\rightarrow[0,1]$: $\alpha \rightarrow \varphi(\alpha) = \sup_{x\in C(\alpha)} U_{\Pi}(x)$

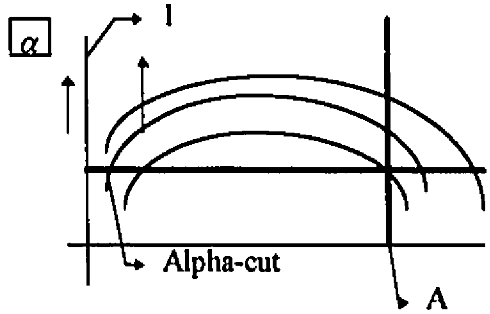
2) $\psi:[0,1]\rightarrow[0,1]$: $\alpha \rightarrow \psi(\alpha) = \alpha \wedge \varphi(\alpha)$

Given 1) and 2) we can thus re-write the solution statement as: $\sup_{x\in E} U_{F}(x) = \sup_{\alpha\in[0,1]} \psi(\alpha)$. Where thus 'min' comes thus from definition 2.

Some interpretation is needed as to definition 1. Definition 2 follows straight out of 1.

We are thus supposed, following definition 1, to take commodity vectors through an alpha cut on the fuzzy budget and subject those vectors to the degree of belongingness they have in each level of fuzzy utility. The supremum is to be taken on all those values. An example may better clarify this requirement.

Consider the figure below which follows the same idea that the higher the quantities of a commodity involved the higher the membership value will be to the fuzzy budget.



The alpha cut is thus the horizontal fat line. Remark that the possible maxima of the contour lines tend to '1'. The set A is a collection of commodity vectors where on coordinate is fixed and the other coordinate is variable. So the set A for instance would contain couples A={(5,2),(5,3),(5,4)...} No further explanation is needed on this graph as we discussed this graph already before. Say now that the elements of set A are to be subjected to membership grades in the fuzzy utility function. Then for instance (5,2) may belong with degree 0.3 to a utility level say $\tilde{U}_1$. But (5,2) may belong also to $\tilde{U}_2$ with for instance a lesser degree of say 0.25. Assume for instance $\tilde{U}_2 > \tilde{U}_1$. The issue here is that we take thus the supremum over all those membership degrees in the fuzzy utility over all eligible couples which are the result of an alpha-cut on the fuzzy budget constraint. Thus a certain value (i.e. the supremum) is obtained for a given level of α. Definition 2 only asks to take the minimum of this supremum at a given level of alpha and the level of alpha.

At first sight a problem occurs here and that is that it well may be that some of the couples do have 100% membership in a given level of fuzzy utility. This is entirely possible given that our levels of fuzzy utility are normalized and convex sets; i.e. fuzzy numbers. Taking the supremum at a certain level may indeed yield the value of '1'. As such however this is not really a problem as a minimum is guaranteed through $\psi(\alpha)$. It may also occur that when $\alpha = 1$ and it occurs that $\varphi(\alpha) = 1$ then $\psi(\alpha) = 1$ and hence the $\sup_{x \in E} U_{\tilde{F}}(x) = \sup_{\alpha \in [0,1]} \psi(\alpha) = 1$.

This means thus that the fuzzy solution would belong for a 100% to the membership function of the fuzzy decision set; and the solution would thus be crisp. There is little else to explain on what is 'behind' the functions. The reason for that is simple the functions fit in the definition of the decision problem.

The properties of those functions are however very important. We look at them now:

1) $\varphi(0) = \sup_{x \in E} U_{\Pi}(x)$

2) $\alpha \leq \beta \Rightarrow \varphi(\alpha) \geq \varphi(\beta)$

The first property is quite straightforward. If we take an alpha cut at $\alpha = 0$; we obtain the total set of all possible commodity vectors; hence the notation *E.* Let us make the comment that couples which have a 0 membership degree should still have to be ruled out after this 0-alpha cut has been performed otherwise there would be contradiction.

The second property has a simple proof. As $\alpha \leq \beta \Rightarrow C(\beta) \subseteq C(\alpha)$; *and as* $\varphi(\beta) = \sup_{x \in C(\beta)} U_{\Pi}(x) \leq \sup_{x \in C(\alpha)} U_{\Pi}(x) = \varphi(\alpha)$.

Remark that the implication is of course dependent on the shape of the membership function. Would the membership function be convex-shaped for instance then the implication would of course not hold. We have been discussing however the shape of the fuzzy budget constraint membership function, and the shape is appropriate for property 2. Out of property 2 we can readily map a relationship between $\varphi(\alpha)$ *and* α. The relationship is immediate: a decreasing function in α. The important factor in showing that a solution to our fuzzy problem is possible consists in introducing the notion of a fixed point. If that is possible some re-working is needed to show that effectively the supremum on the membership values of the fuzzy decision set has effectively a value. In other words if it can be shown that $\sup_{x \in E} U_{F}(x) = \overline{\alpha}$ then it will have been proven that the fuzzy optimization problem has indeed a solution. Now what still has then to be done is to show that $\overline{\alpha}$ is indeed the optimal solution. We proceed in the following order:

1) show that $\overline{\alpha} = \sup_{x \in C(\overline{\alpha})} U_{\Pi}(x)$

2) show that $\overline{\alpha} = \sup_{x \in E} U_{F}(x)$

3) show that $\overline{\alpha}$ is indeed optimal

1) For this, we need Brouwer's fixed point theorem basically. We need to show only that a fixed point exists. Any continuous function mapping of the unit interval into itself must

cross the 45 degree line at least once; this function has at least one fixed point.([68], p.516). The problem of the uniqueness of the fixed point is not a problem here given that we are confronted with a continuous *and decreasing function*. The proof is standard and follows Varian. ([73],p.320) We define $\varphi(\alpha)-\alpha = f(\alpha)$. This should be a continuous function as long as it can be shown that $\varphi(\alpha)$ is continuous. This is much less straightforward however. There is a theorem by Tanaka and Asai which says the following: If the fuzzy subset $\tilde{C}$ is strictly convex then $\varphi(\alpha)$ is continuous. ([69]) To have the fuzzy set $\tilde{C}$ to be convex we need to show that all alpha-cuts of this fuzzy set are indeed convex.. The proof of this theorem is convoluted and we leave it aside. Note however that the continuity of $\varphi(\alpha)$is a necessity when applying Brouwers's theorem. $f(\alpha)$ in the above measures the vertical gap between $\varphi(\alpha)$ and the 45 degree line. For a fixed point we need to have that $f(\alpha)$=0. We also know that $f(0)=\varphi(0) > 0$. The strict inequality comes forth from property 1. Obviously a supremum on membership values of all commodity vectors in the fuzzy utility can not possibly be 0. Would it be zero then all commodity vectors would have zero membership in the fuzzy utility which is senseless. Also we know that $f(1) = \varphi(1) - 1 \leq 0$. This is also expected, as the maximum value of the supremum can not possibly exceed '1'. Now the *intermediate* value theorem can be used. This theorem says that if f is continuous on [a,b] and C is a number between f(a) and f(b); then there is at least one number c between a and b for which f(c)=C. ([27],p.98)

Here as $\varphi:[0,1] \to [0,1]$ and the function is continuous then using the theorem we can effectively conclude that there is some $\alpha \in [0,1]$ such that $f(\alpha) = \varphi(\alpha) - \alpha = 0$. That shows thus that we have a fixed point.

2) This is a little more convoluted as an argument and we follow Billot here. ([5],p.72)

We know that $\sup_{x \in E} U_{\tilde{F}}(x) = \sup_{x \in E}[\min(\alpha, \sup_{x \in C(\alpha)} U_{\tilde{F}}(x))] = \sup_{\alpha \in [0,1]} \psi(\alpha)$. We also know that $\psi(\overline{\alpha}) = \overline{\alpha} \wedge \varphi(\overline{\alpha}) = \overline{\alpha} \wedge \overline{\alpha} = \overline{\alpha}$.

Hence there is a fixed point on the $\varphi(.)$ *and* $\psi(.)$ functions. In order to show that $\overline{\alpha} = \sup_{\alpha \in [0,1]} \psi(\alpha)$ we have to literally show that $\overline{\alpha}$ is the supremum indeed. This is quite easy to understand. There was no problem in case one as $\varphi(\overline{\alpha}) = \overline{\alpha} = \sup_{x \in C(\alpha)} U_{\beta}(x)$ by definition. We can however *NOT* define $\psi(\overline{\alpha}) = \overline{\alpha} = \sup_{x \in E} U_{\beta}(x)$; as $\psi(\overline{\alpha})$ is only defined as : $\psi(\overline{\alpha}) = \varphi(\overline{\alpha}) \wedge \overline{\alpha}$. This is the reason why we have to show that $\overline{\alpha}$ is thus the supremum of the $\psi(.)$.

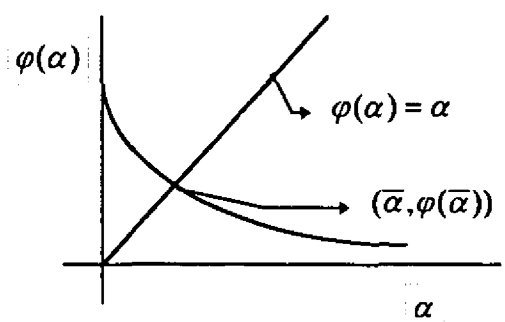
What has thus to be proven is:

a) can we say that if $\alpha < \overline{\alpha}$ then $\psi(\overline{\alpha}) \geq \psi(\alpha)$?

b) can we say that if $\alpha \geq \overline{\alpha}$ then $\psi(\overline{\alpha}) \geq \psi(\alpha)$ is *still* valid?

If both a) and b) are valid then effectively $\psi(\overline{\alpha}) = \overline{\alpha}$ will be the supremum; and therefore thus $\overline{\alpha} = \psi(\overline{\alpha}) = \sup_{x \in E} U_{\beta}(x)$ is true statement.

Let us tackle case a). Consider the graph below which shows the function $\varphi(\alpha)$:



The function $\varphi(\alpha)$ is indeed downward sloping, as we have remarked already under property 2.

a1) $\alpha < \overline{\alpha} \Rightarrow \varphi(\alpha) \geq \varphi(\overline{\alpha}) = \overline{\alpha} > \alpha$. Where use is made thus of the property of the $\varphi(\alpha)$function and also of the fact that it has a fixed point.

a2) $\psi(\alpha) = \varphi(\alpha) \wedge \alpha[I]; and\ \varphi(\alpha) \wedge \alpha = \alpha\ [II] < (\overline{\alpha} = \psi(\overline{\alpha}))[III]$. We have put roman numbers to separate the parts upon which we want to comment in the formulation. Part [I] obviously refers to the definition of the $\psi(.)$ function. The result obtained in part [II] refers to the implicit result obtained in a1.

In a1 we can read almost immediately that $\varphi(\alpha) > \alpha$; and hence the minimum is obviously α. Finally part [III] refers to the fixed point we obtain *also* through the $\psi(.)$ function, we have mentioned this already. Out of [I],[II],[III] one can readily see that effectively $\psi(\alpha) \leq \psi(\overline{\alpha})$.

b1) $\alpha \geq \overline{\alpha} \Rightarrow \varphi(\alpha) \leq \varphi(\overline{\alpha}) = \overline{\alpha} \leq \alpha$. This is as in a1.

b2) $\psi(\alpha) = \alpha \wedge \varphi(\alpha) = \varphi(\alpha)[I]; and\ \varphi(\alpha) \leq (\overline{\alpha} = \psi(\overline{\alpha}))[II]$. Part [I] obtains the minimum which follows straight out of b1. In b1 we read that $\varphi(\alpha) < \alpha$; and the minimum out of those two components is thus immediate. Part [II] follows also straight out of b1. Use is also made of the fixed point property of the $\psi(.)$.

Thus out of [I],[II] we conclude that $\psi(\alpha) \leq \psi(\overline{\alpha})$.

Thus no matter whether $\alpha < \overline{\alpha}\ or\ \alpha \geq \overline{\alpha}$ we obtain that $\psi(\alpha) \leq \psi(\overline{\alpha})$. Thus this means that $\psi(\overline{\alpha}) = \overline{\alpha} = \sup_{x \in E} U_{\tilde{F}}(x)$.

3) What remains to be done is to show that $\overline{\alpha}$ is indeed optimal.

We know so far that $\sup_{x \in E} U_{\tilde{F}}(x) = \overline{\alpha}\ and\ \sup_{x \in C(\overline{\alpha})} U_{\tilde{\Pi}}(x) = \overline{\alpha}$. Remark that the alpha-cut on the fuzzy budget set is taken at the specific level $\overline{\alpha}$. The conclusion is then immediate i.e.; that $\sup_{x \in E} U_{\tilde{F}}(x) = \sup_{x \in C(\overline{\alpha})} U_{\tilde{\Pi}}(x)$. Billot provides for an extensive proof on finding a *sharper* formulation to which $\sup_{x \in E} U_{\tilde{F}}(x)$ maybe equal to. The form he finds is the following: $\sup_{x \in E} U_{\tilde{F}}(x) = \sup_{x \in A} U_{\tilde{\Pi}}(x)$; where the ordinary set A is defined as: $A = \{\forall x \in E, U_{\tilde{C}}(x) \geq U_{\tilde{\Pi}}(x)\}$. This is indeed a sharper result than the specific alpha-cut result. The interpretation of this is that the optimal decision as exemplified through $\sup_{x \in E} U_{\tilde{F}}(x)$ will be a decision which maximizes a fuzzy objective on the set A or also which maximizes a fuzzy objective on elements of E which satisfy at least as well the constraint as the objective. All other elements are ruled out.

Nowhere has mention been made that the fuzzy utility surface would have to be concave in order to have a convex weak preference set. This is indeed a little strange to say the least. The solution form does only work with membership functions and the concavity of the fuzzy utility surface as such is not registered into those membership functions.

We may make a point here. From the classical background we know that the convexity of the weak preference set is a necessity. Using the set up here this convexity seems to be taking quite less importance. This is indeed bothering.

VI.2. The case of the producer

So far we have been concerned in looking at a framework in which it would be possible to find possible fuzzy optimae. This has been performed for the case of a fuzzy utility function which is subjected to a fuzzy budget constraint. It is certainly interesting to see what the conditions would be would we enter the producer area. This is what will be looked at now.

VI.2.1. Refutation of basic hypotheses

Ponsard in ([56], p.302) puts the producer in a spatial framework. This is indeed a highly interesting approach. The following hypotheses are commonly made, according to Ponsard:

A) all inputs and outputs are located in a single space where the producer is located and where the production is carried out.

B) the producer has *complete* information concerning the conditions of his producer's activity; i.e. he perfectly commands inputs and outputs and realizes a maximum profit given the constraints of technology and available price.

Assumption A, we know from real life, does of course not always hold. An input which is totally non-transportable will basically command the location of the production unit. As there are many combinations possible depending on the transportability of inputs and the necessity of those inputs and the location where outputs can be possibly sold we could easily envisage that the producer has a utility function which is linked here to profit of course and which is dependent on the possible combinations of location elements we have set out. This is the interest of considering a spatial model. It is much richer and closer to real life than non-spatial models. Condition B is definitely more cumbersome. The assumption of complete information again, is a very idealistic assumption. Inputs are fuzzy to some extent and maximum profit can be modulated based on tolerance levels one accords to the inputs. The ultimate goal of this section is to have an idea of what an *optimal fuzzy supply* may be.

Necessary assumptions as set forth by Ponsard are as follows:

1) The producer's space is characterized by the location of his production unit, by the inputs supply space and the outputs demand space.

2) Production capacity is fixed. We are in the short term.

3) The input supply space is denoted by Y_i^r; $r=1,...p$ and $i=1,..n$ Where r stands for the number of places where the input is available. i stands for the number of inputs needed in the production process. So when noting for instance Y_2^3 this means the 2 needed inputs for the production process can be found at three places. Beware however: 2 inputs can be found at the 3 places; or only one input can be found at 2 places; while the other input is available at only one place. etc... all combinations are thus possible.

4) The output demand space is defined identically as in 3); however some explicit dependencies are created between output and input space. There is a dependency between the places where inputs can be bought and the places where outputs are demanded. Similarly there also will be a dependency between the number of inputs and outputs considering of course a specific product. Ponsard is somewhat arbitrary in those dependencies and we do not want to re-iterate his proposition.

What is however very important to consider is that the space Y of productions has two important dependent variables i.e. inputs and places. Thus $Y = \{Y_i^r\}$ is the set of all productions. 'i' stands for the number of inputs used and 'r' stands for the number of places. Take the case of Y_2^3. This means here that 2 output goods can be sold in 3 different places. Any combination is possible as in the input case. I.e. we may be able to sell the 2 output goods only in one place or in two places etc.. but every option is possible. To the contrary in the case $Y^1{}_2$ the 2 goods can ONLY be sold in one place. The set of productions does contain all possible elements but not all possible elements are technically feasable for the producer. Take the case of Y_2^3 it may well be that for the producer none of the available inputs can produce two different outputs. Or it may be impossible to sell two different outputs at three different locations. Only the technologically feasable productions are a base from which we can draw a supply. Let $Z \subset Y$ and call Z the feasable production set. An element y of Z is called a producer's supply. Note that such an element denotes a quantity. A production y is said to be efficient in Z iff $\forall y' \in \Re^k : y' \geq y \Rightarrow y' \notin Z$. Remark that y is a multidimensional vector where the amount of the variable 'location' and the amount of the variable 'different outputs' determine the dimension.

We compare units of y' with y for the *same number of different types of output and for the same number of different locations.* The definition of efficiency is thus clear then.

All this however is classical and the point Ponsard wants to make is an important one. Says Ponsard: ' ...the result of a production process is by nature imprecise. It follows that a technically possible production is more or less efficient. It is not advisable to partition the set of all possible productions into two classes: the efficient productions and the inefficient productions.' ([56], p.304) Ponsard wants to show us that basically 1)the efficiency of an input is a relative and not an absolute concept and 2)the inputs even if their technical efficiency is maximal will not be the sole factors which will determine the output. We need to expand somewhat on this claim.That the efficiency of the input is in most of the cases relative rather than absolute is not really new. The prevailing technology for instance is in fact a constraint on attaining an absolute level of efficiency.

The absoluteness would basically mean that technology related to that input could not possibly be improved. This is theoretically achievable but not practically. The second point also makes sense. Even if such theoretical absoluteness would be achievable there are a lot of 'imponderabilia'; i.e. factors which can not be controlled for and which may negatively influence the optimal result. Argument two has the most weight we would say. In all, inputs should in fact *have degrees of efficiency.* The *limiting case* is the classical case which assumes 100% membership and which yields the unique optimal quantities of output.

Thus following this train of thought we can thus define a fuzzy set $\widetilde{H}$ which is a fuzzy subset of the ordinary reference set Z. Obviously a membership function is defined on this subset and the membership values are, for simplicity lying in [0,1]. The case where $U_{\widetilde{H}}(x) = 1$ resumes to the classical case of 100% technical efficiency; which equates thus an absolute efficiency and/or total control over all imponderabilia. Would the membership value be '0' then we would be working with inputs having no technical efficiency whatsoever and this would thus equate to waste. Clearly, the classical case resumes when the membership values are in $\{0,1\}$. Ponsard remarks in ([56], p.304) that the membership functions will here be determined by 'purely technical reasons'. This, says Ponsard makes 'that the fuzziness is objective'. This is a quite interesting point. There is no 'human judgment' so to speak which is to be reflected by the membership function; only the technology will determine its shape. We recall our discussion on finding membership functions for semantic problems such as 'John is tall' which involved quite some subjectivity. This is the first time in this paper that we can claim that the membership function is *entirely* dependent on the technology in question. In the section which treated with consumer behavior the membership function over the budget constraint was quite less clear-cut. Remark also that the membership function will not take into account imponderabilia. Furthermore we recall that probability states are not relevant . This 'eternal problem' was discussed in former parts of this paper.

VI.2.2. Fuzzy Profit

We need prices and quantities of output so to be able to use the notion of profit. The price system must be constructed to reflect the spatial nature of the production. Ponsard introduces FOB and CIF prices to account for the transport cost of bringing the input to the production location. The notation ${}^{f}p_{i}^{r}$ with $i=1,...n$ *and* $r=1,...p$ would be referring to an input price. The input price of course will be higher the higher i will be. It also may be expected that the higher is r the lower relatively speaking this price will be. The more locations at which inputs may be bought the lower, given competition, the input price should be. The output price will also have dependency on i and r; but the relationship is less clear when looking at r. As we said, the advantage of treating the profit maximization problem in a spatial context is that it provides for a richer context which is closer to reality. The maximization of profit in this more realistic set up is indeed quite more challenging than in an a non-spatial context. The problem which immediately occurs is that the producer must in fact not only maximize profit but also must think about how well he can reconcile possibly conflicting strategies in order to attain such maximum profit. In the classical case with one output and say two inputs the relativity of the technological efficiency of the inputs is an important factor in diminishing the absolute notion of maximum of profit. Also the imponderabilia also play a role. But the problem stops there. In a spatial context one must be looking at the implications of selling 3 outputs on 5 locations. The links with this decision and the fact that 10 inputs can be found at 20 locations are existent. The problem can become exceedingly complex. Just imagine you make the initial decision to sell three different outputs at 4 locations. The 10 inputs needed for the 3 different outputs go into each output at differing degrees of units of input. 3 inputs may be sold at 3 different locations while 2 other inputs at 2 different locations and the five remaining inputs may be sold at 15 different locations. Each location has different transport costs. This becomes a quite complex problem. To find a maximum profit given the efficiency of technology which would not be absolute given the imponderabilia and given the complex interaction between number of locations and number of outputs/inputs it would be pretentious to think that we may find a maximum profit output combination which at the same time gives highest utility to the producer. A very high profit may give less utility to the producer because it does not as well blend the conflicting strategies as a lower profit may do. Recall again that in the non-spatial and classical case it has always been assumed that there is absolute efficiency, no imponderabilia (i.e. a fixed environment); and inputs and outputs are always found in the same single space. No wonder of course that in such a context maximum profit and highest utility to the producer have a straightforward relationship. The classical hypothesis is that *the maximum profit entails a maximum utility of profit.* Thus we need to define, what Ponsard calls an *imprecise profit utility function.* ([56],p.306)

A degree of membership will be accorded in function of how well the conflicting strategies are blended together in relation to the level of profit.

Clearly the membership function exemplifying fuzzy utility of profit is *subjective* as opposed to the objective membership function of technological efficiency.

VI.2.3. The fuzzy optimization problem

It is the objective of the producer to maximize the utility he gets from the profit realized. This is a straightforward problem in the classical case as this objective will coincide with maximum profit. In our fuzzy context coincidence is not guaranteed at all. Following Ponsard we assume that the spatial price system is given. The fuzzy technological constraint is given by the fuzzy set $\widetilde{H}$. The fuzzy objective is the maximization of the fuzzy utility of profit. Define therefore the fuzzy set $\widetilde{P}$ *as* $\widetilde{P} = \{y, U_{\widetilde{P}}; \forall y \in Z, U_{\widetilde{P}}(y) \in [0,1]\}$ Then the solution will be a fuzzy set; following Bellman and Zadeh ([4]) we must marry objective an constraint together following max-min. So we can write that $\widetilde{S} = \widetilde{P} \cap \widetilde{H}$. Using max-min we can then write the solution statement, exactly in the same form as in the consumer's problem as:

$\sup_{y \in Z} U_{\widetilde{S}}(y) = \sup_{y \in Z} [U_{\widetilde{P}}(y) \wedge U_{\widetilde{H}}(y)]$. Using the notion of $\widetilde{A} = \bigcup_{\alpha} \alpha . C(\alpha) A$; we obtain a more summarized form, as in the consumer's case: $\sup_{y \in Z} U_{\widetilde{S}}(y) = \sup_{\alpha \in [0,1]} [\alpha \wedge \sup_{y \in H_\alpha} U_{\widetilde{P}}(y)]$.

The way we continue to proceed is exactly as in the consumers case. We will define two new functions and then try to find fixed points. We do not repeat the steps involved as they are identical to the consumer's case.

Define the following functions:

1) $\varphi: [0,1] \to [0,1]: \alpha \to \varphi(\alpha) = \sup_{y \in H_\alpha} U_{\widetilde{P}}(y)$

2) $\psi: [0,1] \to [0,1]$: $\alpha \to \psi(\alpha) = \alpha \wedge \varphi(\alpha)$

Then $\sup_{y \in Z} U_{\widetilde{S}}(y) = \sup_{\alpha \in [0,1]} \psi(\alpha)$.

The $\varphi(.)$ has the same properties as in the consumer's case. It also will yield a fixed point if it is continuous and decreasing over α. Recall however that it is not straightforward at all to find the conditions under which $\varphi(\alpha)$ is continuous. The following results are totally mimicking the consumer's case:

1) $\overline{\alpha} = \sup_{y \in H_\alpha} U_{\widetilde{P}}(y) = \sup_{y \in Z} U_{\widetilde{S}}(y)$; i.e. a solution exists to the problem.

2) $\sup_{y \in A} U_{\widetilde{P}}(y) = \sup_{y \in Z} U_{\widetilde{S}}(y)$; $A = \{y, y \in Z, U_{\widetilde{H}}(y) \geq U_{\widetilde{P}}(y)\}$; which provides thus for a sharper solution to the problem. This is totally similar to the consumer's case.

The interpretation of this is that the optimal decision as exemplified through $\sup_{y \in Z} U_{\tilde{s}}(y)$ will be a decision which maximizes a fuzzy objective on the set A or also which maximizes a fuzzy objective on elements of Z which satisfy at least as well the constraint as the objective. All other elements are ruled out.
Interestingly enough Ponsard proposes conditions under which a unique solution can be found. ([56],p.311)This unique solution would thus be non-fuzzy. The conditions however are very stringent. We do not pause on this however.

VI.2.4. Conclusion

The consumer and producer models have been solved in a fuzzy context. Basically what one may claim now is that the classical case is in fact a special case of the much more general set up proposed here. This may be termed as rather being generalization than innovation.
The innovation may come more in the uncovering of a choice behavior which is definitely more human. Clearly in the set up above we could render the problem more sophisticated by introducing *fuzzy* expected utility. We stress 'fuzzy' to clearly distinguish fuzziness from probability.

VI.3. Fuzzy Revealed Preference

Another interesting application where fuzziness may be of interesting use is revealed preference. We observe that revealed preference has as underpinnings very strong rationality assumptions. Richter Rationality or for instance also regular rationality are examples of that. The use of fuzzy sets may weaken again this tight level of rationality.

VI.3.1. Set up

The development here follows mainly Basu. ([2])

Basu uses the *Generalized Hamming Distance* which is defined as follows: $d(\widetilde{A},\widetilde{B}) = \sum |U_{\widetilde{A}}(x) - U_{\widetilde{B}}(x)|$. Where $\sim$ represents a fuzzy set. $U_{\sim}(.)$ represents the membership value of x in the fuzzy set. It is explicitly assumed that X which is the set of alternatives is a crisp set. We do have a fuzzy binary relation (FBR) which is then generally defined as: $\widetilde{R}: X \times X \to [0,1]; \widetilde{R}$ is a fuzzy subset of $X \times X$. The membership values of this binary relation are taken in [0,1]. But we could generalize this interval to a lattice. As an example $U_{\widetilde{R}}(x,y)$ measures the strongness of the relation between x and y. This FBR is defined by Basu as being a fuzzy order; i.e. it is fuzzy reflexive, fuzzy transitive and complete.

Here a first problem occurs. It hints towards Basu's definition of fuzzy reflexivity and fuzzy transitivity. Basu defines fuzzy reflexivity as: $\forall x \in X: U_{\widetilde{R}}(x,x) = 1$. This is too strong as a definition of fuzzy reflexivity, as a matter of fact it totally corresponds to the classical definition. Bandemer ([1]) for instance has been defining a weaker form as: $\forall x \in X: U_{\widetilde{R}}(x,x) \geq \alpha; \alpha \in]0,1[$. There is indeed no reason to believe; given fuzzy relations that the strongness of x with itself has to be necessarily 100%! The fuzzy order by Basu is defined then as follows:

1) reflexive

2) $\forall x,y \in X; x \neq y: U_{\widetilde{R}}(x,y) + U_{\widetilde{R}}(y,x) \geq 1$; which is the completeness property

3) $\forall x,y \in X: U_{\widetilde{R}}(x,y) \geq 1/2.U_{\widetilde{R}}(x,z) + 1/2.U_{\widetilde{R}}(z,y); \forall z \in X \setminus \{x,y\}; U_{\widetilde{R}}(x,z) \neq 0$
$U_{\widetilde{R}}(z,y) \neq 0$

The transitive property has already been discussed under V.5.1.

VI.3.2. The Greatest Set

Let X be the set of all alternatives. Define S as a non-empty subset of X. The greatest set in S is denoted $G(S,\tilde{R}):S \to [0,1]: \forall x \in S, G(S,\tilde{R})(x) = \min_{y \in S} U_{\tilde{R}}(x,y)$. An example clarifies the definition.

Example:

Let $U_{\tilde{R}}(x_1,y_1) = 0.4 // U_{\tilde{R}}(x_1,y_2) = 0.3 // U_{\tilde{R}}(x_1,y_3) = 0.2...$ Then the definition tells us that we have to take the minimum over all those membership values. Say the minimum is 0.1 then $G(S,\tilde{R})(x_1) = 0.1$. I must repeat the same procedure for all x. The significance of $G(S,\tilde{R})(x_1) = 0.1$ is thus that x_1 belongs to the greatest set with a membership value of 0.1. Thus $G(S,\tilde{R})$ is clearly a fuzzy set so defined. We can easily imagine that $G(S,\tilde{R})$ has a membership function. The individual expresses his preferences over pairs of alternatives. He assigns a value to denote the strengness of his preference. The immediate objection is that of course assigning such value is somewhat equivalent to relapsing in crisp preferences so to say. This is an important point. However we can use some of the theories which have been invoked for eliciting membership functions so to give a better grounding to this problem. Hisdal's approach may be an example.

VI..3.3. Choice Function and Rationality

Define the set of all alternatives X such that $3 \leq \# X \leq \infty$. So the set of alternatives must contain at least 3 elements. Now define K as the set of all subsets on X BUT each subset is to contain two or more elements. So $K \neq \wp(X)$. Then the choice function is defined as: $C: K \to K \text{ and } \forall S \in K: C(S) \subset S$.

The idea of course is to know whether the choice is indeed rational.

VI.3.4. Crisp Binary relations: Richter Rationality and Regular Rationality

Basu proposes two types of rationality definitions. Richter rationality and regular rationality. Richter's idea says that a choice function is rational if it has been generated as the outcome of preference maximization. *C(.)* is Richter rational iff there exists an exact binary relation R on C such that $\forall S \in K; C(S) = G(S,R)$. This is for an exact binary relation and there is no order requirement.

The greatest set is clearly a crisp set now. There is with a crisp relation no doubt about the fact that x may prefer y to some degree and vice versa that y may prefer x to some degree. If x strictly prefers y with degree 0; then either y strictly prefers x or is indifferent. Furthermore if x strictly prefers x to y with degree '1'; then there is no possibility y may prefer x or x to be indifferent to y. Recall that the definition of greatest set uses min. Hence the Richter idea of preference maximization says thus that if x_i is strictly preferred over any $y_i \in S$ then $x_1 \in G(S,R)$, and only then.

Thus the notion of greatest set coincides with the choice function if the greatest set is crisp. A second type of crisp binary relations refers to regular rationality. *C(.)* is regular rational iff there exists an exact binary ORDERING R on X: $\forall S \in K: G(S,R) = C(.)$

The definition is almost identical to Richter's rationality with the difference the crisp binary relation has now to be an order; i.e. be reflexive, complete and transitive.

VI.3.5. FBR's: Unfuzzy Dominance and D-rationality

It is clear that regular rationality and Richter rationality are of no great use within a fuzzy context. The unfuzzy dominance and D-rationality do explicitly work with FBR's but explicitly keep the greatest set to be crisp. There is an obvious reason to that. Having a fuzzy greatest set will not be very helpful to define greatest elements.

Unfuzzy dominance is defined as follows: Given a FBR, $\tilde{R}$ on X the *unfuzzy dominant set* is denoted $D(S,\tilde{R}) = \{x \in S: G(S,\tilde{R})(x) = 1\}$ There is no doubt that this is a extremely restrictive definition. All membership values for a fixed x must be minimally '1'. We do not see a choice function. This is what D-rationality will do. *C(.)* is *D-rational* iff there exists a fuzzy ordering $\widetilde{R}$ on X such that $\forall S \in K: C(S) = D(S, \widetilde{R})$. So D-rationality is like regular rationality except that it accepts FBR's.

All this leads to deceiving little newity. Basu's theorem 1 which says an individual is D-rational iff he is regular rational confirms the intuition we just had. We omit the proof.

The theorem is clear: if a person's behavior can be rationalized using a fuzzy ordering then there must exist an exact ordering. So fuzzy ordering is of no use.

VI.3.6. Ouf! There may be a way out!

The way out needs however first a new concept which is the idea of *nearest exact set.* Given a fuzzy set $\widetilde{A}$ in X, a nearest set of this fuzzy set is $N(\widetilde{A})$; it is an exact set which is nearest to $\widetilde{A}$ in terms of the Hamming Distance. The closer to zero this Hamming distance is the better. There is a useful property which is the following: $\{x \in X: U_{\widetilde{A}}(x) > 0.5\} \subset N(\widetilde{A}) \subset \{x \in X: U_{\widetilde{A}}(x) \geq 0.5\}$. An example may clarify this relation. Consider $\widetilde{A} = \{x_1 / 0.5; x_2 / 0.6; x_3 / 0.7; x_4 / 0.1\}$. Then $N(\widetilde{A}) = \{x_2, x_3\}$ and this yields using Hamming distance 0.5+0.4+0.3+0.1=1.3. For instance would I have taken for $N(\widetilde{A}) = \{x_1, x_2, x_3, x_4\}$ then the Hamming distance would have been 1.3+0.9=2.2

Note also that if $N(\widetilde{A}) = \{x_1, x_2, x_3\}$ then the Hamming distance would still have been 1.3. This is because the membership value of x_1 is 0.5. Hence one can now better see the intuition behind the relation above i.e. that $\{x_1, x_2\} \subset \{x_1, x_2, x_3\}$; i.e. that $\{N(\widetilde{A}) \subset \{x \in X: U_{\widetilde{A}}(x) \geq 0.5\}$. Now consider another example. Let $\widetilde{B} = \{x_1 / 0.1; x_2 / 0.2\}$ The nearest exact set is in fact the empty set. The Hamming distance would then be 0.1+0.2. There is no non-empty exact set which can beat this distance. For instance the singleton $\{x_1\}$ yields a Hamming distance of 0.9+0.2=1.1. **If all membership values are strictly lower than 0.5 the nearest exact set will be the empty set.**

VI.3.7. N-Rationality

C(.) is N-rational iff there exists a fuzzy ordering such that $\forall S \in K: C(S) = N[G(S, \tilde{R})]$.

This definition is quite different from D-rationality and thus from regular rationality. The relation is fuzzy and an order; so far for the overlap with D-rationality. The newity is now that the greatest set can be fuzzy. We approximate this greatest fuzzy set with a nearest exact set. The idea of nearest set is useful because we knew that a fuzzy greatest set is very hard to interpret. By introducing the nearest exact set we make this decision making simpler though WE AVOID to impose that the greatest set is to be crisp.

A theorem which we now must consider is Basu's theorem 2: *All individuals are N-rational.* We must discuss the proof of theorem 2.

Basu first defines a completely fuzzy binary relation as follows:

$$\forall x, y \in X: U_{\tilde{R}}(x, y) = \begin{cases} 0.5 \text{ if } x \neq y \\ 1 \text{ if } x = y \end{cases}$$

This relation so defined is indeed the fuzziest we can have. Remark also that this fuzziest relation is also an order. We can for instance immediately sense the fuzzy reflexivity property. As all membership values for x different of y are 0.5. From a fuzziness point of view such set is the most fuzzy.

The definition of such most fuzziest relation however is key. As $G(S, \tilde{R})(x) = 0.5$; $\forall x$; (as we use a minimum)then from the example we have seen above $N(G(S, \tilde{R}))$ must exist. It can not be an empty set as the membership values are 0.5. We said that if all membership values are strictly smaller than 0.5 then effectively $N(G(S, \tilde{R})) = \varnothing$, as seen at the end of section IX. This is the reason why the FBR is so defined by Basu. As $N(G(S, \tilde{R}))$ exists then so must the choice function. The fuzzy relation is also an ordering. So if the FBR is defined as above we can rationalize any choice function.

A question arises whether we need to restrict that much the FBR as being most fuzzy. We could easily assume the following definition:

$$\forall x, y \in X: U_{\tilde{R}}(x, y) = \begin{cases} \in [0.5, 1] \text{ if } x \neq y \\ \geq \alpha \text{ if } x = y; \alpha \in [0.5; 1] \end{cases}$$

This would relax Basu's definition and would also relax somewhat the too binary character of fuzzy reflexivity. We may seriously wonder if theorem 2 is acceptable. There is no a-priori reason to believe that all fuzzy order relation should be of the type as defined above. Therefore we may well wonder whether theorem 2 is at all realistic.

It is easy to make a proof when a fuzziest order relation is presented. You will not be able to proof that all individuals are N-rational when the fuzzy order relation is neither fuzziest or of the type right above. This is simply because the greatest set may have membership values which fall below 0.5 and therefore a nearest set may be impossible to find. The main problem with theorem 2 is that it does not make room for fuzzy orderings which are not necessarily of the type defined above. The problem could be solved if instead of defining the greatest set being the result of a minimum we define it as a maximum; i.e. $G(S,\widetilde{R}) = \max_{y \in S} U_{\widetilde{R}}(x,y)$. Would we be able to define the greatest set as a maximum then all what we would have to require in order to find the nearest crisp set of the greatest set is that there exists *at least one* fuzzy relation which has a membership value which is greater than 0.5. Then we know that $G(S,\widetilde{R})$ defined as a maximum will have a value of 0.5 or higher and therefore the nearest exact set of the greatest set could be found. We would think there is not necessarily an argument against using max of the greatest set. If so, then theorem 2 makes much more sense because the restriction on the fuzzy orderings is much less stringent. There is however one problem spot and that is that we must be able to guarantee we still have a fuzzy order. With low membership values for instance the completeness condition may for instance be violated. Thus a possible extension on Basu's theorem 2 could be: If $G(S,\widetilde{R}) = \max_{y \in S} U_{\widetilde{R}}(x,y)$, and if at least one fuzzy relation has a membership value greater than 0.5 and if the fuzzy ordering conditions can still be respected then *more* individuals can be declared N-rational. This does contradict somewhat theorem 2 which says that all individuals would be N-rational. The correct version of theorem 2, we think should be that *some* individuals will be N-rational if the fuzzy ordering is of the fuzziest type.

VI.3.8. Extent of Rationality

We want to know how fuzzy a binary relation really is. This will be useful to define levels of rationality and irrationality. The classical distinction is restricted to irrationality and rationality. There are several measures for fuzziness indexes. For instance in chapter VII of part II we develop the fuzzy index of de Luca and Termini. Basu uses another fuzzy index. The consequences of using a specific kind of fuzzy index are important. One flaw in Basu's paper is that he really does not give a serious argument on why he uses a specific fuzzy index. For any FBR (thus not necessarily an order) the index of fuzziness $\upsilon(\tilde{R}) = \frac{2.d(\tilde{R}, N(\tilde{R}))}{\#(X \times X)}$. It is obvious that $N(.)$ is non-unique. For instance the nearest set for $\tilde{A} = \{x/0.5; y/0.5\}$ is in fact any element of the $\wp(A)$. However the $d(.,.)$ will be unique and so $\upsilon(.)$ is well defined. For a completely fuzzy binary relation the index of fuzziness must be '1'. This is very easy to see. Consider thus the fuzziest relation defined as: $\forall(x,y) \in X \times X{:}U_{\tilde{R}}(x,y) = 0.5$. Then say $X = \{x,y\}; then\ X \times X{:}\{(x,x),(y,y),(x,y),(y,x)\}$. Given the definition of the fuzziest binary relation we use $\upsilon(\tilde{R}) = [2.(0.5.4)]/4 = 1$. It is obvious for an exact set that $d(\tilde{R}, N(\tilde{R})) = 0$ and hence $\upsilon(\tilde{R}) = 0$

The definitions here developed relate to a fuzzy binary relation which does not have to be an order. If we impose an order then there are some small differences to be taken into account. We use the same definition for a complete fuzzy ordering; i.e.: $\forall x,y \in X{:}U_{\tilde{R}}(x,y) = \begin{cases} 0.5\, if\ x \neq y \\ 1\, if\ x = y \end{cases}$. This is a fuzziest ordering. Consider the same example as above then given this fuzziest ordering the membership values will be somewhat different: $U_{\tilde{R}}(x,y) = U_{\tilde{R}}(y,x) = 0.5$ but $U_{\tilde{R}}(x,x) = U_{\tilde{R}}(y,y) = 1$. Using this in our definition of a fuzzy index we would get: We need to find a nearest exact set to the fuzzy set created. We need take all couples into account for a nearest set in this case and we obtain that $d(\tilde{R}, N(\tilde{R})) = 1$. Plugging this in our fuzzy index formula we obtain a value of 0.5. This is not a very nice result as the FBR here is still fuzziest but is now an ordering.

Basu wants a '1' measure for a fuzziest ordering and therefore modifies the fuzzy index formula as follows: $\delta(\tilde{R}) = \frac{2.d(\tilde{R}, N(\tilde{R}))}{\#(X \times X) - \#(X)}$. Plugging the numbers of our example shows then that $\delta(\tilde{R}) = 2.(0.50)/4 - 2 = 1$. The exact ordering gives also an index of zero.

VI.3.9. Degree of Fuzzy rationality

Basu calls $\Re(C(.))$ the set of fuzzy orderings which N-rationalize *C(.)*. N-rationalizing *C(.)* means that $N(G(S,\tilde{R})) = C(.)$. Of course, we do not need fuzziest orderings, any ordering is acceptable as long the membership values of the fuzzy relation is higher than or equal to 0.5. This is quite important because we need to define a fuzzy ordering for which the minimal values must be 0.5. The reason for that is that $G(S,\tilde{R}) = \min_{y \in S} U_{\tilde{R}}(x,y)$. In order to find the nearest crisp set to the fuzzy $G(S,\tilde{R})$; the minimal membership value for $G(S,\tilde{R})$ must be 0.5. So for instance an acceptable proposal for a fuzzy order relation would then be: $\forall x,y \in X: U_{\tilde{R}}(x,y) = \begin{cases} \in [0.5,1] \text{ } if \text{ } x \neq y \\ \geq \alpha \text{ } if \text{ } x = y; \alpha \in [0.5;1] \end{cases}$ which we discussed already above. **Thus $\Re(C(.))$ is the set of all fuzzy order relations which are defined as above.** This is important.

Basu defines *C(.)* to be fuzzy rational of degree $\Omega[C(.)]$. This degree would be defined as: $\Omega[C(.)] = 1 - \min_{\tilde{R} \in \Re(C(.))} \delta(\tilde{R})$. The formulation is self-explanatory. Remark that $\Omega[C(.)]$=0 when the minimum of $\delta(\tilde{R})$ (over all fuzzy orderings which N-rationalize *C(.)*) is effectively equal to unity. But as we use the minimum it MUST be that all fuzzy orderings which have been N-rationalizing *C(.)* must be complete fuzzy ordering or fuzziest orderings with definition: $\forall x,y \in X: U_{\tilde{R}}(x,y) = \begin{cases} 0.5 \text{ } if \text{ } x \neq y \\ 1 \text{ } if \text{ } x = y \end{cases}$ This makes then intuitive sense why we can claim that the degree of rationality in that case would be zero. Furthermore it also makes more sense to talk about the minimum here. Imagine for a moment we would have maximum then it is simple to show that if there is just one fuzziest ordering then $\delta(\tilde{R})$ will be '1' and the degree of rationality would be 0. This would indeed make little sense if the other fuzzy order relations in $\Re(C(.))$ would not be of the fuzziest type. Hence the use of the minimum. $\Re(C(.))$ can not be empty given Basu's theorem 2. Hence $\Omega[C(.)] = 1 - \min_{\tilde{R} \in \Re(C(.))} \delta(\tilde{R})$ is well defined. Remark however our discussion of having the greatest set defined over a maximum rather than a minimum. Then we do not need such pure type of fuzzy order and we still may have that $\Re(C(.))$ will not be empty.

VI.3.10. WARP

Basu defines WARP as follows:

If *C(.)* satisfies WARP iff there is not $S_1, S_2 \in K; \textit{for some } x, y \in X: x \in C(S_1), y \in S_1 \textit{ and } y \in C(S_2), x \in S_2 \setminus C(S_2)$.

This definition implies in the classical setting that if there is $S_1, S_2 \in K$ then the person will be called irrational. Otherwise if the above definition is satisfied then he is declared rational. When introducing fuzziness we will observe that this extreme situation of rational or irrational can be weakened. This is the object of Basu's third theorem.

Theorem 3 says this:

If *C(.)* satisfies WARP then $\Omega[C(.)] = 1$ and if *C(.)* everywhere violates WARP then $\Omega[C(.)] = 0$. Furthermore if $\Omega[C(.)] = 1$ then *C(.)* satisfies WARP. We omit the proof.

Theorem 3 is a crucial theorem as it shows that with fuzziness introduced we can weaken the extremes of rational/irrational spectrum. Basu provides for a nice example in which the concept of degrees of rationality is well shown.

The following is given:

$X = \{x,y,z\} \textit{ and } C(X) = \{x,y\}, C(\{x,y\}) = \{y\}; C(\{y,z\}) = \{y\}; C(\{x,z\}) = \{x\}$.

This choice violates WARP and this is checkable as follows. Set $S_1 = \{x,y,z\} \textit{ and } S_2 = \{x,y\}. x \in C(S_1)(i.e. x \in \{x,y\}); y \in S_1 (i.e. y \in \{x,y,z\})$. We do want to find $y \in C(S_2) \textit{ and } x \in S_2 \setminus C(S_2)$? It is possible. As $C\{x,y\} = \{y\}; so\ y \in C(S_2).\ x \in \{x,y\} \setminus \{y\} = \{x\} \Leftrightarrow x \in S_2 \setminus C(S_2)$. Hence we violated

WARP. As he violates WARP he can thus not be regular rational. If he is not regular rational he is irrational. The use of fuzzy theory will now show us that the individual is rational but of a certain degree.

We want a fuzzy order relation belonging to $\Re(C(.))$. This means thus that the fuzzy ordering can be used to N-rationalize *C()*. This means thus that $N(G(S,\widetilde{R}) = C(.)$ Therefore given the defined choice functions we obtain immediately that:

$$N[G(\{x,y,z\}),\widetilde{R}] = \{x,y\}; N[G(\{x,y\}),\widetilde{R}] = \{y\}; N[G(\{y,z\}),\widetilde{R}] = \{y\}$$
$$N[C(\{x,z\}),\widetilde{R}] = \{x\}$$

We get then:

$$U_{\widetilde{R}}(x,x) = 1; U_{\widetilde{R}}(y,y) = 1; U_{\widetilde{R}}(z,z) = 1; U_{\widetilde{R}}(x,y) \geq 0.5; U_{\widetilde{R}}(x,z) \geq 0.5; U_{\widetilde{R}}(y,x) \geq 0.5$$
$$U_{\widetilde{R}}(y,z) \geq 0.5; U_{\widetilde{R}}(z,x) \leq 0.5; U_{\widetilde{R}}(z,y) \leq 0.5; U_{\widetilde{R}}(x,y) \leq 0.5.$$

Remark that most of this derivation comes right out of the set $N[G(\{x,y,z\}),\widetilde{R}]$.

The next step is to find the nearest crisp relation to the relation with membership values as set out above. We also must keep in mind we still need a fuzzy order. Basu's proposition is this:

$$U_{\widetilde{R}^*}(x,y) = 0.5; U_{\widetilde{R}^*}(y,x) = U_{\widetilde{R}^*}(y,z) = U_{\widetilde{R}^*}(x,z) = U_{\widetilde{R}^*}(x,x) = U_{\widetilde{R}^*}(y,y) = U_{\widetilde{R}^*}(z,z) = 1$$
$$U_{\widetilde{R}^*}(z,y) = U_{\widetilde{R}^*}(z,x) = 0$$

That $U_{\widetilde{R}^*}(x,y) = 0.5$ follows immediately from the fact that $U_{\widetilde{R}^*}(x,y) \leq 0.5 \wedge U_{\widetilde{R}^*}(y,x) \geq 0.5$. The cases with the membership values of '1' derive directly from the fact that the fuzzy values can be greater or equal to 0.5. Also we choose them to be equal to '1' because the crisp case (i.e. the original given) accorded a membership value (i.1 '1' in the crisp case) to the same elements. We can give '0' membership value to the last two cases as in the crisp case they also have '0' membership value. Hence the $d(\widetilde{R}^*, N(\widetilde{R}^*)) = 1/2$. Says for instance we would give for the latter two cases $U_{\widetilde{R}^*}(z,y) = U_{\widetilde{R}^*}(z,x) = 0.2$ then this would not be very wise from a nearest set point of view as the Hamming distance would then be 0.9.

The rest of the calculations are immediate. $\#(X \times X) = 9; \#(X) = 3$, so as $\delta(\widetilde{R}) = \dfrac{2.d(\widetilde{R}, N(\widetilde{R}))}{\#(X \times X) - \#(X)} = \dfrac{2.(1/2)}{6} = \dfrac{1}{6}$. Hence $\Omega[C(.)] = 5/6$. So the degree of rationality is in fact 5/6. This is true even though WARP has been violated.

VI.3.11. Final Comments

There are some problems in Basu's paper. The first problem relates of course to Basu's definition of a fuzzy binary order. We may have to relax the fuzzy reflexivity relation. Furthermore Kaufmann's definition, following Billot's argument may have to be used. A second problem relates to theorem 2. Using a fuzziest order relation to proof this theorem seems to be insufficient. Problem 3 refers to the use of the fuzzy index Basu proposes. To violate WARP and to be still declared rational up to a certain degree is definitely an interesting idea. However the choice of the right fuzzy index is of paramount importance and influences the interpretation for a great part of the above mentioned idea.

VI.4. Simple applications on basic finance concepts

The theory of fuzzy sets has relevance also to the finance discipline. The following development deals with applying fuzzy numbers to some basic finance concepts. Recall that fuzzy numbers have normalized and convex membership functions. The basic idea of using fuzzy numbers is again related to the vagueness of estimates relative to possible invested amounts or estimated interest rates.

Applications in finance using fuzzy sets is picking up. Two recent publications by Refenes and De Boeck use several insights of fuzzy sets to re-model certain finance models. Some other applications; especially in the 80's have concerned the extension of the CAPM model with fuzzy policy constraints. Instead of having crisp policy constraints the constraints are fuzzy and a fuzzy mathematical programming method as covered in this paper may be used to solve such problem. The paper by Ostermark does treat the problem, in a very superficial way. Ostermark's paper does not explicitly indicate what sort of enrichment the CAPM will enjoy when policy constraints would be fuzzy. The argument that the CAPM rejection on empirical grounds may be resting on the fact that imprecision has never been introduced in the model is debatable and is certainly not proven by Ostermark. An honest statement may be that if one wants to fuzzify the CAPM one should be fuzzifying some of the assumptions underlying this model. This, however would certainly not be an easy task. The assumption of having rational investors with homogeneous expectations as to the minimum variance opportunity set is convertible to a an assumption of heterogeneous expectations but makes the model somewhat more complicated. The pre-requisite that CAPM is an equilibrium model which assumes that the market portfolio is efficient is however a very solid assumption which is not alterable. In fact the model is amenable to changes on assumptions such as normally distributed returns or homogeneous expectations. Even without riskless assets the model is feasable. But fuzzifying the fact that the CAPM is an equilibrium model with an efficient market portfolio is impossible. All this, just to say that the attempt to fuzzify the CAPM should not be limited to fuzzifying the policy constraints. If the CAPM proper should at all be fuzzified a judicious choice of what assumption should be fuzzified may be a first step.

It is not at all sure that empirical results will be better as compared to the ATP after having fuzzified.

Other papers have been written in the finance area and most of them have appeared in the 'Fuzzy Sets and Systems' collection. The main article to which we dedicate this last section of the paper is an article by J.J Buckley. Li Calzi in a more recent paper makes a follow up of Buckley's treatment of fuzzifying simple finance concepts. Li Calzi is a more sophisticated paper which deals with the fuzzification of discount and accumulation models. We omit this here.

VI.4.1. Future values

The applications treated here make solely use of fuzzy numbers. Please refer to part II on the section which deals with fuzzy numbers; for more extensive information on performing operations on fuzzy numbers. Remark that when applying fuzzy sets to social sciences we are not always obliged to use fuzzy numbers.. The section which dealt with the optimization of a fuzzy utility function subject to a fuzzy budget constraint showed us that the budget constraint being fuzzy could have a membership function which does not have to be normalized for instance.

If an amount A is invested now at rate of r (per period) for n periods then $S_n = (1+r)^n A$.

In most cases however the amount may be *more or less* an amount A and most important the interest rate will always be a 'more or less' figure. No matter what econometrics are being used the interest rate can not be predicted in a totally stable manner. We know therefore that there is some variation around the interest rate but we do not know how much variation. Then we get the expression in the fuzzy context as: $\tilde{S}_n = \tilde{A} \otimes (1 \oplus \tilde{r})^n$.

Recall that an operation such as addition and multiplication remains internal if the result of this operation remains a fuzzy number; i.e is a convex and normalized membership function. Also we assume that max-min (and not min-max) is being used for the execution of the operation; so to guarantee that the operation remains internal.

The point here consists in showing whether the fuzzy expression is a 'legal' one. From part II we know that multiplication is distributive vis a vis addition and also that multiplication is associative. This shows immediately that the above expression is indeed 'legal'.

$\tilde{S}_1 = \tilde{A} \otimes (1 \oplus \tilde{r})$ for the first period and $\tilde{S}_2 = \tilde{S}_1 \oplus (\tilde{S}_1 \otimes \tilde{r}) = \tilde{A} \otimes (1+\tilde{r})^2$; using the properties. What is interesting is to obtain the membership function for $\tilde{S}_n$. This must be a membership function associated to a fuzzy number as for the addition and multiplication we have been assuming that max-min was used and thus the operations remain internal. Obviously the membership function of $\tilde{S}_n$ will be strongly dependent on the membership functions of the independent variables i.e $\tilde{A}$ *and* $\tilde{r}$. Both being fuzzy numbers and thus convex and normalized membership functions. Using Buckley's notation we get as form for the membership function of $\tilde{S}_n$ the following $f_{ni}(y/\tilde{S}_n) = f_i(y/\tilde{A}).(1+f_i(y/\tilde{r}))^n$; for $i=1,2$ and $f_{n1}(0/\tilde{S}_n) = s_{n1}; f_{n1}(1/\tilde{S}_n) = s_{n2}; f_{n2}(0/\tilde{S}_n) = s_{n4}; f_{n2}(1/\tilde{S}_n) = s_{n3}$. This symbolism only wants us to tell that the membership function for $\tilde{S}_n$ when the membership value is 0 the point s_{n1} is obtained. The membership function attains again a value of '0' at point s_{n4}. Remark however that the membership function is flat leveled between the points s_{n2} *and* s_{n3} where the membership value is then 1. The membership function so obtained would thus give us different values all belonging with different degrees to the fuzzy number $\tilde{S}_n$. We could also contemplate having the number of periods to be fuzzy. This is however a little more complex and does not add much to the finding.

VI.4.2. Fuzzy Cash Flows

A more interesting application is the one which relates to the concepts which are used in comparing investment alternatives. The NPV or net present value method is well known. Another method is the IRR or the internal rate of return method.

1. A fuzzy NPV approach

Consider a sequence of cash flows over n-periods call it $\Delta = A_0, A_1, \ldots A_n$. To find the net present value of those projected cash flows (after deducting for initial cash outlays) the traditional formulation is used : $NPV(\Delta, n) = \sum_{i=0}^{n} A_i (1+r_0)^{-i}$ .Say now that different investment proposals are put forth $\Delta, \mathrm{B}, \mathrm{X}$...Following the value of the obtained NPV's projects will be selected. The important issue here is that of course r_0 (cost of capital to the firm) and the projected cash values are all estimated. So the scope for fuzzy numbers is more than appropriate. The way to go about it here is not that complicated. A fuzzy cash flow is now defined as $\widetilde{\Delta} = \widetilde{A}_1, \widetilde{A}_2, \ldots \widetilde{A}_n$. Each of those fuzzy cash flows are thus fuzzy numbers. We will assume that $\widetilde{A}_0$ is a negative fuzzy number. This means that the membership function of this fuzzy number lies *entirely* to the left of the Y-axis; i.e. the axis which registers the membership values. A positive fuzzy number will lie *entirely* to the right of the Y-axis. The $\text{NPV}(\widetilde{\Delta}, n) = \widetilde{A}_0 \oplus \sum_{i=1}^{n} PV(\widetilde{A}_i, i)$ where the summation is fuzzy. The membership function for each fuzzy cash flow will follow the crisp definition as was the case in the future value calculation. Buckley presents an interesting example where the method is exemplified. There some small problems which have to be tackled first:

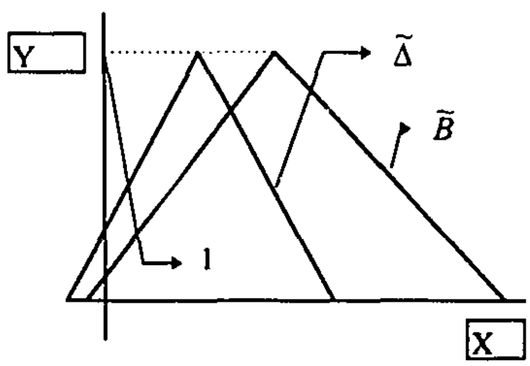
1. Fuzzy NPV must be greater than fuzzy 0; in order to be considered
2. Comparing fuzzy numbers

Problem 1 indicates a threshold level which is fuzzy 0. Buckley proposes for instance a membership function which is having membership value of '1' at x=0 to be a good candidate for fuzzy 0. To have a criterion which assesses whether the obtained fuzzy NPV is greater than fuzzy 0 we need to use a possible definition of ranking fuzzy numbers. One possibility is to require that one fuzzy number is greater than the other if *all* of the alpha cuts of the first fuzzy number are bigger than *all* alpha-cuts of the other fuzzy number. ([12]) The 'all' quantifier may be leading us in some trouble however. Problem 2 is already tackled through the definition we just gave.

Consider the following example in Buckley ([8]; p.270) Two projects are proposed in which the projected fuzzy cash flows are as follows:

Pr.$\widetilde{A}$	Pr.$\widetilde{B}$
$\widetilde{A}_0 = (-1100/-1000,-1000/-900)$	$\widetilde{B}_0 = (-1100/-1000,-1000/-900)$
$\widetilde{A}_1 = (450/500,500/550)$	$\widetilde{B}_1 = (50/100,100/150)$
$\widetilde{A}_2 = (350/400,400/450)$	$\widetilde{B}_2 = (150/200,200/250)$
$\widetilde{A}_3 = (250,300,300/350)$	$\widetilde{B}_3 = (250/300,300/350)$
$\widetilde{A}_4 = (150/200,200/250)$	$\widetilde{B}_4 = (350/400,400/450)$
$\widetilde{A}_5 = (50/100,100/150)$	$\widetilde{B}_5 = (450/500,500/550)$
	$\widetilde{B}_6 = (550/600,600/650)$

Assume also that the fuzzy cost of capital to the firm is : $\widetilde{r}_0 = (0.08/0.1,0.1/0.12)$ The table entries read simply for instance $\widetilde{A}_5$ as membership value of '0' for x=50 and at x=150; and membership value of '1' at x=100. Those are thus all triangular fuzzy numbers. The membership function for each project can be found using the same procedure we used for finding the membership function of the future value. The membership functions ([8];p.270) with some minor alterations, of the two projects is:



Remark that the two projects are thus triangular fuzzy numbers. We may prefer project B over the other as it looks as if the alpha cuts for all alpha would be greater than for the other project. There are more sophisticated definitions in ranking fuzzy numbers however. We do not go in detail on this issue however. Remark also that the number of periods considered here could also be fuzzy. Termination dates of the projects may indeed be imprecise. This is indeed a senseful idea. We do not go in detail on this issue however.

2. A fuzzy IRR

The internal rate of return is defined as the solution r>-1 which solves: $\sum_{t=0}^{n} A_t(1+r)^{-t} = 0$.

This equation may have no solutions , unique or several solutions. The projects if a unique solution exists will be ranked on the basis on how well $r > r_0$. We can translate the above homogeneous equation in a fuzzy setting as: $\widetilde{A}_0 \oplus \sum_{t=1}^{n} PV(\widetilde{A}_t, i) = \widetilde{0}$. Solving such a fuzzy equation resumes to using deconvolution as we have seen in part II. However it is clear that a clear-cut solution is far from easy. How do we define the fuzzy number 0? This makes thus that the internal rate of return has in fact no feasable fuzzy equivalent when the problem is written in this way. This may be an interesting observation and we may expect that in other areas of finance or economics such extensions may thus be missing at some times.

Conclusion

This paper has tried to show how fuzzy set theory could weaken the stringent rationality assumptions used in classical micro-economics. Fuzzy set theory in its simple form is intuitively appealing. This property has advantages but also disadvantages. As a main disadvantage we can mention that the theory as such is often relegated to the area of pseudo-theories. Theories thus, which have no intrinsic rigor and which only exist because nobody cares to get rid of them. Lotfi Zadeh who is the main protagonist in fuzzy theory has had to endure many cheap and not so cheap criticisms on his ideas. The strength of a theory however is also dependent on its maker. Zadeh is not a nobody, we all know that. As we have said in part I of this paper the practice of fuzzy set theory at this point in time draws people from both strongly scientific backgrounds as well as from much weaker backgrounds. Research in the field is extremely dynamic. Applications in whatever field makes it to be a theory which is gaining respect day by day. The discussion lists on comp.ai.fuzzy on the internet for instance show very clearly that all creatures 'great and small' are drawn to this new science, so to speak. I mention 'great and small' because 'champions of the brain' such as Marvin Minsky of M.I.T do post positive messages on this discussion list too.

I have proposed in this thesis that fuzzy set theory, in the very simple form presented here, can make a contribution to a discipline which is condemned to work within a highly imprecise environment. Thus, by definition an economist in his strong desire to model must make abstraction from the real world intricacies. The economist, it seems to an aspirant economist such as this author, must discover this magic fine thread which separates descriptivity from normativity. This is a very difficult task indeed. The question is whether we do fall into the realm of descriptivity when using fuzzy set theory in economics. A more 'acid test' oriented question is the one which asks whether fuzzy sets enriches or just only generalizes our propositions in economics. This thesis can not possibly be conclusive on the latter question.

The problem of descriptivity is indeed a very fuzzy problem and we leave it to the appreciation of the reader to decide on it.
The results obtained are mainly that we can get rid of the completeness axiom when working with fuzzy preferences. Furthermore fuzzy sets in micro-economics brings the added benefit of being able to uncover what happens between the stage where the individual delimits the possible allocations and the final choice he makes. From the optimization problems we considered we wondered whether a unique solution was at all possible and Ponsard does propose an argument which favors this possibility.

It must be stressed however that at best this thesis can be at the stage of the earliest of beginnings of much more extensive research of fuzzy set theory in economics. The drive which brought us to the field of fuzzy sets in economics is the one which derives from a truly uncomfortable feeling as to the classical assumptions which are used in micro-economics. As an example the property of negative transitivity has little relevance to observed choice behavior. It is however a property which is crucial in defining a utility function. We must always keep in mind the words of Karl Popper who reminds us that theory and testing have to be in a perpetual relationship. I do think, from my limited exposure to economics, that the testability of the models as proposed in classical choice theory brings quite bad news. This is not an argument in favor of using fuzzy set theory in economics, but it certainly opens somewhat the 'gates of opposition' which are against introducing it. This thesis, unfortunately comes to deceiving little formal conclusions. We know that fuzziness may relax rationality assumptions which we know are too farfetched. The only objective is to see that if we modulate the strongness of those assumptions by introducing fuzziness we may come to richer results. Either they are further generalizations or either they are truly new. The former is not of high interest. The latter is. In addition to the findings of the completeness axiom , we also looked at preference sets with measurable areas and fuzzy utility surfaces. Those are direct consequences of assuming levels of fuzzy utility. Another newity which is of interest is that the demand function is 'tick' according to Marchal.

In the producer area the relaxation of the equivalence between maximizing a utility function subject to a technological constraint and the maximization of profit subject to that same constraint may be another newity. Finally fuzzy revealed preference may hold some promise in that it shows that an individual violating WARP may still be declared rational to some degree.

So far for the meager accomplishments. The other side of the coin has also to be investigated. The biggest problem with fuzzy sets in economics is the precision by which we express a degree of membership. This precision is as far-fetched as assuming that we are all hyper-rational. However to leave this statement as is, is not that fair either. The part in this paper which dealt with membership functions showed us possible approaches of how we could attain such membership function. Hisdal's approach mainly probabilistic and therefore somewhat awkward (unless we really think about subjective probabilities) proposed an estimated error function. This may indeed refute the critique that membership grades are assumed to be all too precise. Finally there is the classical critique which equates fuzzy set theory with a theory of 'hidden probabilities'. I think we have been quite extensive on this issue. The most important point here is that fuzziness is imprecision and because of this it refers to events which can never realize completely, to re-iterate the words of Claude Ponsard. We recall also the words of Luhandjula who says that situations in which there is doubt about the exactness of concepts; correctness of statements have little to do with the occurrence of events which is the backbone of probability.

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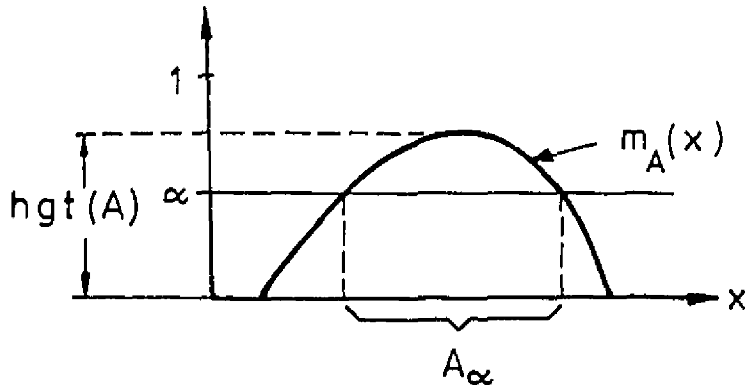
ILLUSTRATIONS



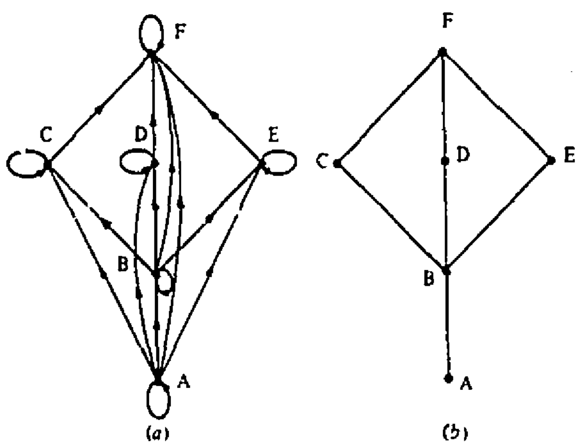
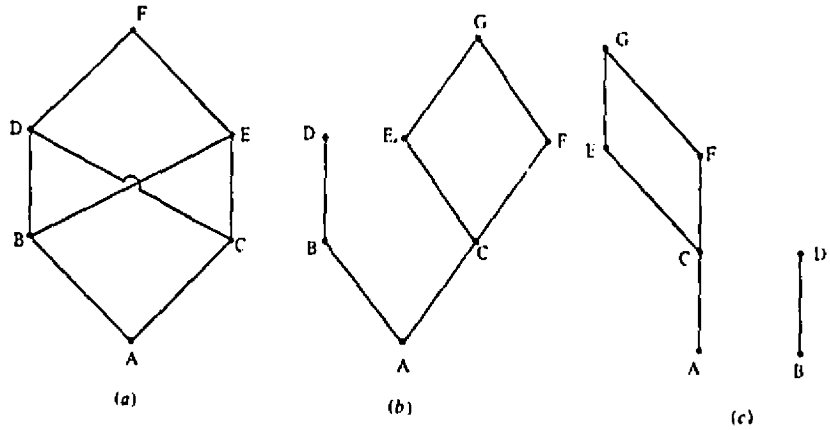
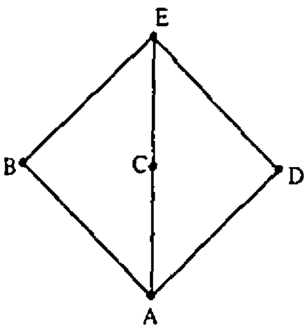
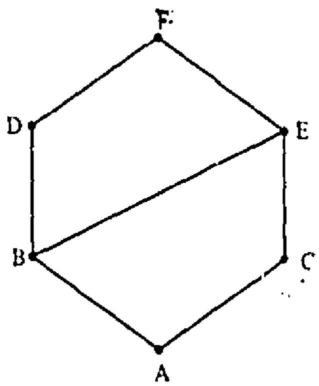


Figure 1-[1]:Top Figure 2-[38]:Bottom







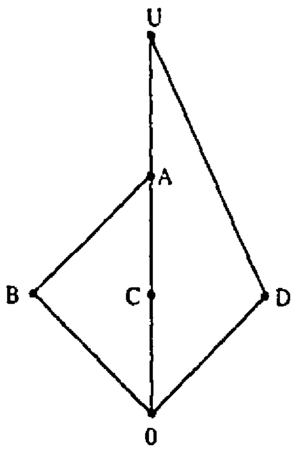
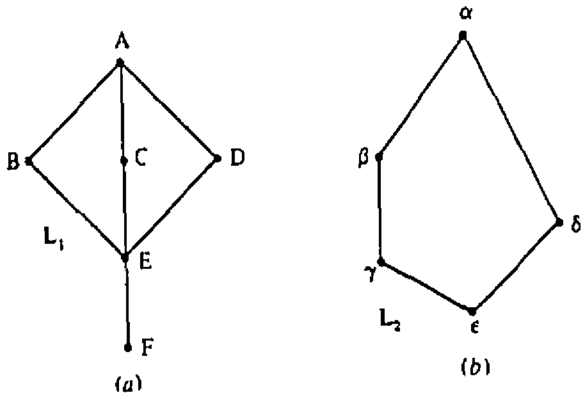
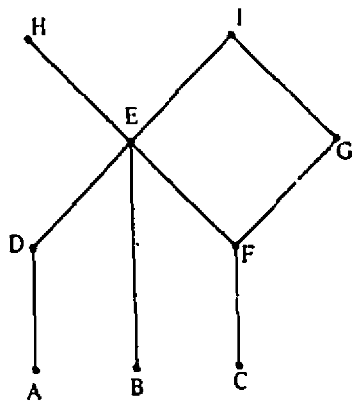


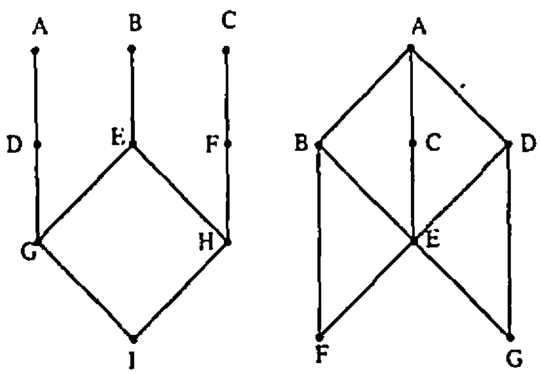
Fig.3/4-[38]:Top Fig. 5-[38]:Left-middle Fig. 6-[38]:Right Middle Fig. 7-[38]: Bottom

(54.28) $$L_1 = \{A, B, C, D, E, F\},$$

(54.29) $$L_2 = \{\alpha, \beta, \gamma, \delta, \epsilon\}$$







inf-semilattice sup-semilattice

Fig.8-[38]:Top Fig. 9-[38]:Middle Fig. 10-[38]:Bottom

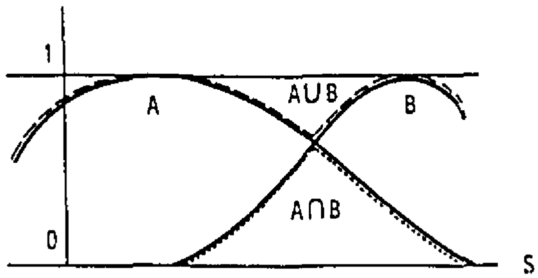
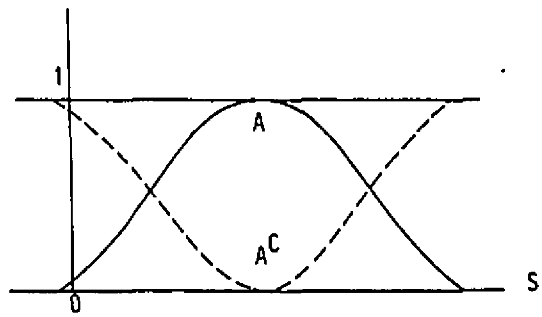


Figure 2.3 Union and intersection of two fuzzy sets.



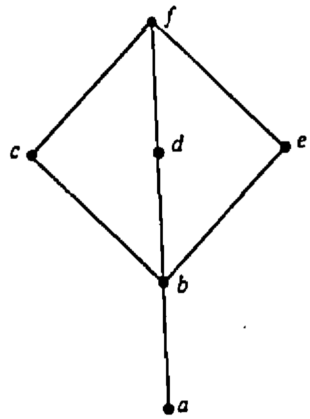


Fig. 11-[38]:Bottom Fig. 12-[52]:Top

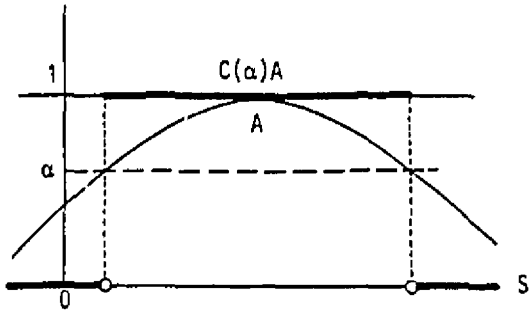


Figure 2.5 An α-cut of a fuzzy set.

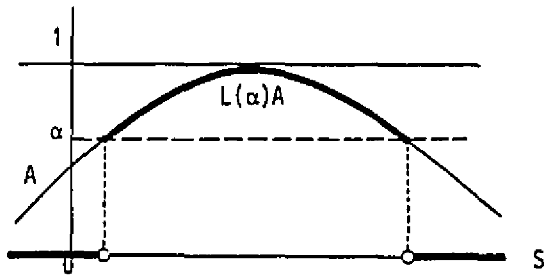


Figure 2.6 An α-level fuzzy set.

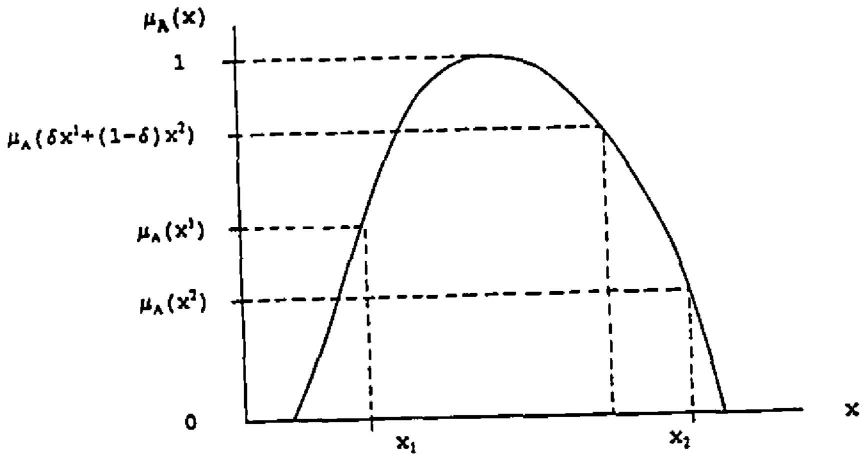
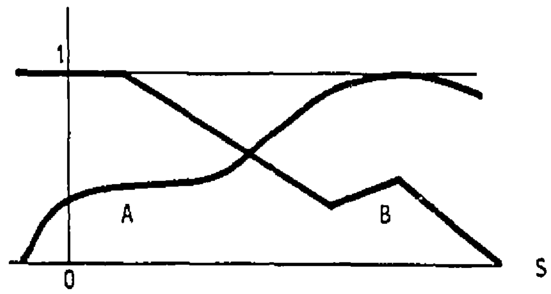


Figure 2.5 A convex fuzzy set

Fig. 13-[52]:Top Fig. 14-[45]:Bottom



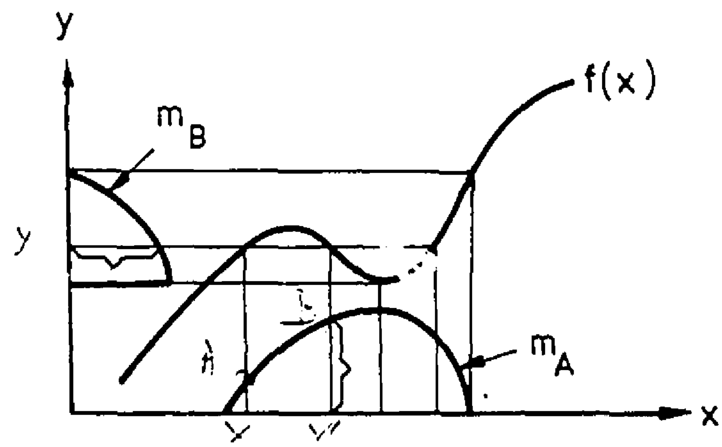


Figure 2.7: Extension principle

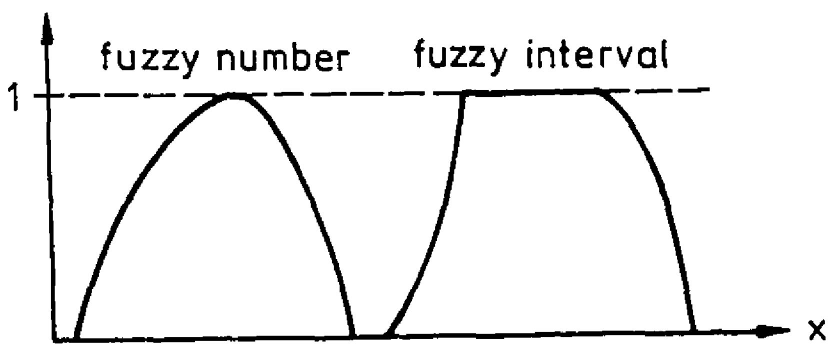
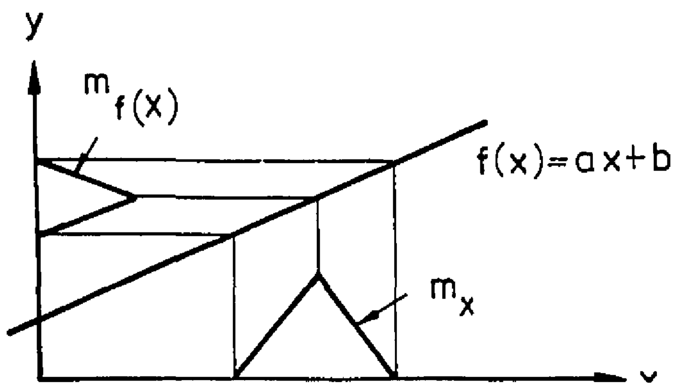


Figure 2.3: Fuzzy number, fuzzy interval

Fig.15-[52]:Top Fi g. 16-[1]:Middle Fig.17-[1]:Bottom



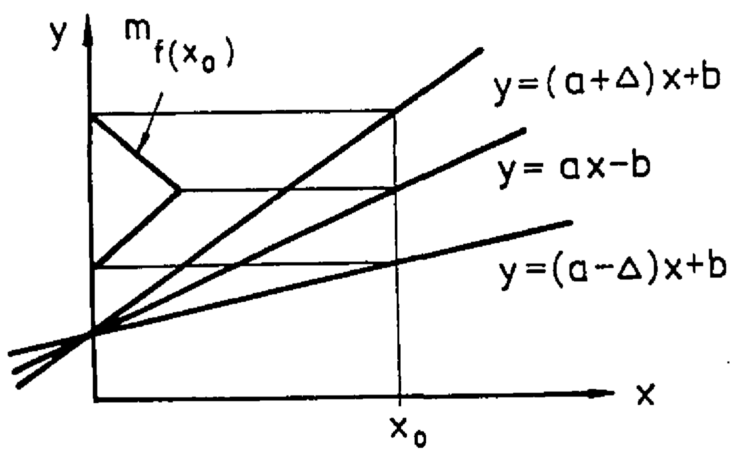


Fig. 18-[1]:Top Fig. 19-[1]:Bottom

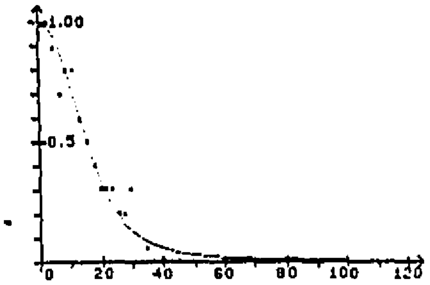


Fig. [illegible] Function name: *very young* (data stream number = 8, optimal λ = 2.255, optimal ν = 0.235, least squares error (LSE) = 0.132041).

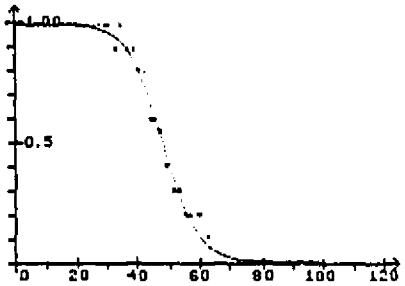


Fig. 5. Function name: *young* (data stream number = 18, optimal λ = 4.445, optimal ν = 0.43, LSE = 0.05103).

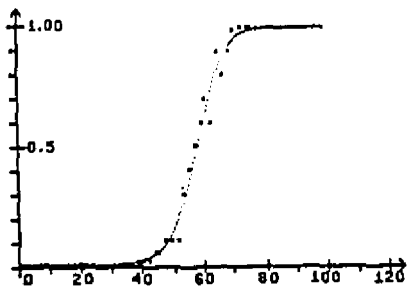


Fig. 6. Function name: *old* (data stream number = 44, optimal λ = 5.49, optimal ν = 0.605, LSE = 0.05284).

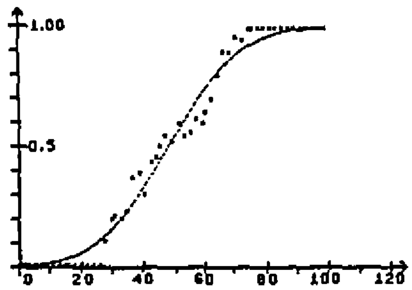


Fig. 7. Function name: *old* (data stream number = 35, optimal λ = 2.165, optimal ν = 0.475, LSE = 0.14841).

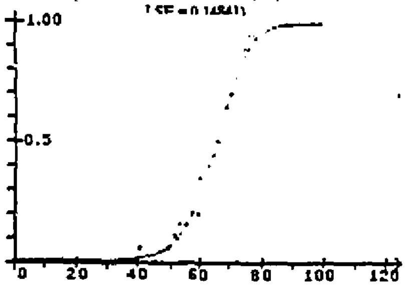


Fig. 8. Function name: *very old* (data stream number = 58, optimal λ = 4.215, optimal ν = 0.705, LSE = 0.03027).

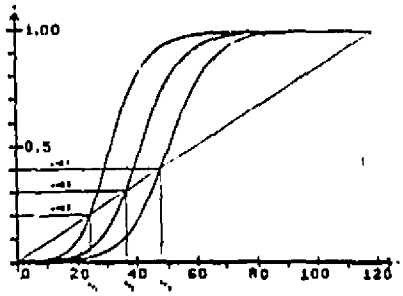


Fig. 1. Membership function for λ = 4 and ν = 0.2, 0.3, 0.4.

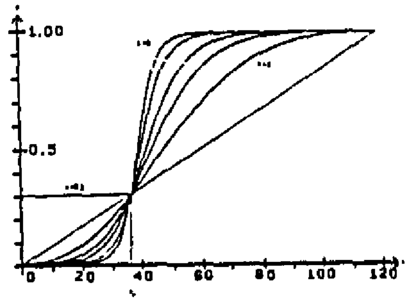


Fig. 2. Membership function for ν = 0.3 and λ = 2,3,4,6,8.

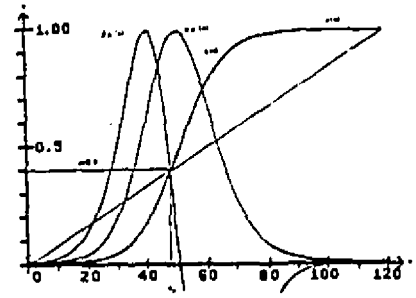


Fig. 3. Membership function and its derivatives for λ = 4, ν = 0.4.

Fig.20-[15]:Left Fig. 21-[15]:Right

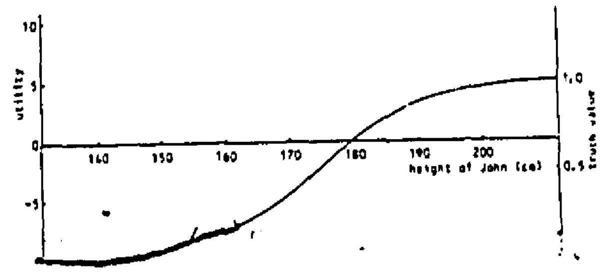
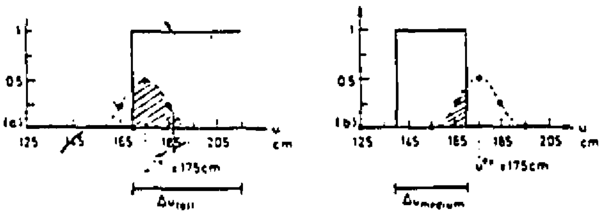


Fig. 1. Payoff function of the asse.tion John



ig. 4. Derivation of the $u^{**} = 175$ cm ordinate of the grade of membership curves for 'tall' and 'medium' respectively. It is equal to the area (actually sum of ordinates) cut-off by $\varepsilon_i(u)$, the threshold curve for λ, from the estimated error curve of Fig. 3(a) displaced to $u^{**} = 175$ cm (see Sect. 2.6, 2.8). u is the subject's estimate of the object's attribute value u^{**}.

Fig.22-[24]:Top Fig. 23-[29]:Bottom