MATHEMATICS M.Sc.

SINGULAR PERTURBATION PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS

by

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ABSTRACT

The asymptotic solution of the boundary value problems for ordinary differential equation of the form

$$\sum_{j=0}^{k} a_j(x)\, y^{(j)}(x) + \sum_{r=1}^{\ell} \epsilon^r a_{k+r}(x) y^{(k+r)}(x) = 0$$

$$a_k(x) \neq 0 \text{ in } \quad 0 \leq x \leq 1,$$

which exhibits boundary layer behavior as ϵ tends to zero are studied. Sufficient conditions under which the solution of the full problem converges to the solution of the reduced problem (obtained by setting $\epsilon = 0$), except in the boundary layer, are stated.

The initial value problems for nonlinear ordinary differential equation of the forms

$$\frac{dx}{dt} = f(x, y, t),$$

$$\epsilon \frac{dy}{dt} = g(x, y, t),$$

where x and y are m- and M-dimensional vectors respectively are also considered. The thesis is primarily a compilation of some of the important works on singular perturbation problems by Visik and Lyusternik, and Vasil'eva.

Short Title of the Thesis:

PERTURBATION PROBLEMS OF ORDINARY

DIFFERENTIAL EQUATIONS

SINGULAR PERTURBATION PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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by

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INTRODUCTION

In the boundary value problems of mathematical physics the differential equations are often simplified by neglecting terms which are of higher order of differentiation than those taken into consideration. The best known example is the relationship between the theories of the flows of viscous and perfect fluids.

Now given a differential equation involving a small parameter ϵ, and of some boundary conditions (we call it the problem A_ϵ) in such a way that the reduced differential equation obtained by letting ϵ formally tend to zero (we call problem A_o to this reduced problem) is of lower order though positive. Then the solution of problem A_o, being the solution of a lower order differential equation, in general cannot be expected to satisfy all the original boundary conditions. This loss of boundary conditions in the passage to the limit means that the solution does not converge uniformly everywhere. This nonuniformity of the convergence is the most interesting aspect of this type of problem. In fluid dynamics it leads to the formation of boundary layers. These type of problems occur in the theory of viscous flow, in certain problems in the theory of elasticity and in other branches of Applied Mathematics.

A consideration of two very simple examples will show some of the features of such type of problems.

(1) A simple model that illustrates loss of the highest derivative in boundary layer theory is given by [1]* as

$$\epsilon \frac{d^2y}{dx^2} + \frac{dy}{dx} = a,$$

$$y(0) = 0 \text{ and } y(1) = 1$$

where 'a' is a constant.

The exact solution is

$$y(x, \epsilon) = (1 - a) \frac{1 - \exp(-x/\epsilon)}{1 - \exp(-1/\epsilon)} + ax,$$

and $$\lim_{\epsilon \to 0+} y(x, \epsilon) = (1 - a) + ax \qquad 0 < x \leq 1.$$

The limit value is the solution of the reduced equation that satisfies the boundary condition at $x = 1$, but not at $x = 0$ unless $a = 1$. The convergence is uniform in every closed interval $0 < \delta \leq x \leq 1$, but not in the whole interval $0 \leq x \leq 1$. In a narrow interval of width $O(\epsilon)$ the solution changes rapidly from $y(x, \epsilon) = 0$ at $x = 0$ to a value differing by a function that is $O(\epsilon)$ from the limit $y(x, 0) = (1 - a) + ax$. This is the interval of boundary layer.

(2) An example illustrating the additional difficulties which arise in the case of nonlinear differential equations is given by [2].

Consider the problem A_ϵ as

$$\epsilon \frac{d^2y}{dx^2} + \frac{dy}{dx} + \left(\frac{dy}{dx}\right)^3 = 0, \; 0 \leq x \leq 1, \; y(0) = y_0, \; y(1) = y_1 \qquad \text{...... (0.1)}$$

The limiting differential equation having the real solution is $\frac{dy}{dx} = 0$, and one might expect a behaviour somewhat resembling that of

* Numbers in square brackets refer to the Bibliography.

in example (1). Nevertheless, it will turnout that with the exception of the trivial case $y_o = y_1$ equation (0.1) fails to possess any solution at all when ϵ is sufficiently small. We consider the case when $\epsilon \to 0+$.

We know that $y(0) = y_o$. Let us set $\left.\frac{dy}{dx}\right|_{x=0} = \tan\gamma$, $-\pi/2 < \gamma < \pi/2$. Here γ may depend on ϵ and $\tan\gamma$, as a function of ϵ, may be unbounded. Now the differential equation in (0.1) may be integrated explictly in the form

$$y(x, \epsilon) = y_o + \epsilon\left[\gamma - \sin^{-1}(\exp.(-x/\epsilon \ \sin\gamma))\right] ,$$

where $\sin^{-1}$ denotes the principal value, between $-\pi/2$ and $\pi/2$, of the inverse sine function. It follows that $|y(x, \epsilon) - y_o| < \epsilon|\gamma|$, and for given y_o, y_1 there will be no solution of (0.1) when

$$0 < \epsilon < \frac{2(y_1 - y_o)}{\pi} .$$

The non-existence of the solution of (0.1) is generally attributed to the circumstance that the differential equation is nonlinear in $\frac{dy}{dx}$. For this reason some authors [2], [3] restrict themselves to the differential equations that are linear in $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ (not necessarily in y).

In this thesis we shall restrict ourselves to singular perturbation problems (defined in Theorem 1.1) involving ordinary linear and nonlinear differential equations assuming that the singular nature of the problem is carried entirely by a reduction of the order of the differential equation (and the consequent failure of some of the boundary conditions) as $\epsilon \to 0$. M. I. Visik and L. A. Lyusternik developed a powerful method for the solution of linear singular perturbation problems, which is independent of the general asymptotic theory of linear differential equations. Here the material is based on the work of M. I. Visik and L. A. Lyusternik [4] and A. B. Vasil'eva [5] .

The first three chapters carry a detailed investigation of the connection between the distribution of the signs of the roots of the characteristic equation of an arbitrary nth order linear differential equation and the nature of the supplementary conditions under which the passage to the limit leads to the solution of the equation. And the fourth chapter surveys the above problem for nonlinear equations.

In this paper we shall make use of the notations

$$(u, v) = \iint_Q u \cdot v \, dx, \qquad \|u\| = (u, u)^{1/2} ,$$

$W_2^k(Q)$ is the Hilbert space consisting of the functions $u(x_1, x_2 \ldots, x_n)$, which are in $\mathcal{L}_2$ together with all of their derivatives up to the k-th order, with the norm

$$\|u\|^2_{W_2^k} = \iint_Q \sum_{s=0}^{k} \sum_i \left| \frac{\partial^s u}{\partial x_{i_1} \ldots \partial x_{i_s}} \right|^2 dx.$$

$\| \|_r$ denotes the norm in a certain Banach space. Q is the region of n-dimensional space.

CHAPTER 1

ORDINARY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS.

We shall consider the asymptotic representation of the solution of problem A_ϵ as $\epsilon \to 0$.

Problem A_o consists in solving the differential equation of order k,

$$\sum_{j=0}^{k} a_j y^{(j)} = 0, \; a_k \neq 0 \tag{1.1}$$

with the k boundary conditions

$$y^{(i)}(0) = D_{io} \quad (i = 0, 1, \ldots, k_1 - 1) \tag{1.2'}$$

and $$y^{(j)}(1) = D_{j1} \quad (j = 0, 1, \ldots, k_2 - 1, \; k_1 + k_2 = k) \tag{1.2''}$$

The coefficients a_j are constant.

Problem A_ϵ consists in solving the equation of order $(k + \ell)$,

$$\sum_{j=0}^{k} a_j y^{(i)} + \sum_{r=1}^{\ell} \epsilon^{r} a_{k+r} y^{(k+r)} = 0 \tag{1.3}$$

where $\epsilon > 0$, with the $(k + \ell)$ boundary conditions

$$y^{(r)}(0) = D_{ro} \; (r = 0, 1, \ldots, k_1 + \ell_1 - 1) \tag{1.4'}$$

$$y^{(s)}(1) = D_{s1} \quad (s = 0, 1, \ldots, k_2 + \ell_2 - 1, \; \ell_1 + \ell_2 = 1) \tag{1.4''}$$

We shall first consider the case when all the $(k + \ell)$ boundary conditions are at the boundary point $x = 0$ and none at $x = 1$; i.e., when $k_2 = 0$, $\ell_2 = 0$, $\ell_1 \neq 0$ and $k_1 \neq 0$.

The characteristic equation corresponding to equation (1.1) is

$$P_o(\lambda) \equiv \sum_{j=0}^{k} a_j \lambda^j = 0. \tag{1.5}$$

For simplicity we assume that the roots $\mu_1, \mu_2, \ldots, \mu_k$ of (1.5) are all distinct. Therefore the general solution of (1.1) has the form

$$y(x) = \sum_{j=1}^{k} c_j \exp(\mu_j x)$$

and therefore using the boundary conditions at $x = 0$, we have

$$\sum_{j=1}^{k} c_j (\mu_j)^i = y^{(i)}(0) = D_{io} \quad (i = 0, 1, \ldots, k-1) \tag{1.6}$$

It is a system of linear equations in c_j, so its determinant $W(\mu_1, \mu_2, \ldots, \mu_k)$ is vandermonde determinant and since the numbers μ_i are distinct, we have

$$w(\mu_1, \mu_2, \ldots, \mu_k) \neq 0 \tag{1.7}$$

The auxiliary characteristic equation for (1.3) is

$$Q_o(\lambda) \equiv \sum_{r=0}^{\ell} a_{k+r} \lambda^r = 0 \tag{1.8}$$

Let the roots of this equation be $\nu_1, \nu_2, \ldots, \nu_\ell$.

Also the characteristic equation associated with (1.3) is

$$P_\epsilon(\lambda) \equiv \sum_{j=0}^{k} a_j \lambda^j + \sum_{r=1}^{\ell} \epsilon^r a_{k+r} \lambda^{k+r} = 0 \tag{1.9}$$

The following lemma holds.

Lemma 1.1 [4]: The roots of equation (1.9) have the form

$$\bar{\mu}_i = \mu_i + \epsilon_i \quad (i = 1, 2, \ldots, k)$$

and $$\frac{\bar{\nu}_r}{\epsilon} = \frac{\nu_r + \epsilon'_r}{\epsilon} \quad (r = 1, 2, \ldots, \ell),$$

where ϵ_i and ϵ'_r go to zero with ϵ, μ_i and ν_r are the roots of the equation (1.5) and (1.8).respectively.

Proof: see [4] page 262-3.

Let $v_\epsilon(x_1, x_2, \ldots, x_n) = v_\epsilon(x)$ be a function defined in a region

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<u>Proof</u>: see [4] page 262-3.

Let $v_\epsilon(x_1, x_2, \ldots, x_n) = v_\epsilon(x)$ be a function defined in a region

Q of n-dimensional space ($n \geqslant 1$) which is p times differentiable. We say that $v_\epsilon(x)$ is a function of boundary layer type of the k-th order ($k < p$) if:

1) the function $v_\epsilon(x)$ and its derivatives up to the p-th order ($p > k$) inclusive are concentrated near the boundary Γ of the region Q, i.e., these functions converge uniformly to zero as $\epsilon \to 0$ on an arbitrary closed subset of Q not containing points of Γ.

2) the k-th derivatives of the function $v_\epsilon(x)$ are bounded in Q as $\epsilon \to 0$, and at the same time among the (k + 1)-st derivatives of v_ϵ are functions converging to ∞ in the norm of the corresponding problem A_ϵ as $\epsilon \to 0$; the j-th derivatives of v_ϵ (for $j < k$) vanish on $\overline{Q}$ when $\epsilon \to 0$.

Theorem 1.1 [4] The solution y_ϵ of problem A_ϵ has the representation

$$y_\epsilon(x) = y_o(x) + v_\epsilon(x) + z_\epsilon(x), \tag{1.10}$$

where $y_o(x)$ is the solution of the reduced problem A_o; $v_\epsilon(x)$ is a function of boundary layer type of k-th order in a neighborhood of the point $x = 0$, and z_ϵ, together with all of its derivatives, converges uniformly to zero, as $\epsilon \to 0$, on an arbitrary interval $[a, b]$, $a < b$.

Proof: The general solution of equation (1.3) has the form

$$y_\epsilon = \sum_{j=1}^{k} \tilde{c}_j \exp(\bar{\mu}_j x) + \sum_{r=1}^{\ell} \tilde{c}_{k+r} \exp\left(\frac{\bar{\vartheta}_r x}{\epsilon}\right) \tag{1.11}$$

We call the perturbation of problem A_ϵ to problem A_o singular if the real parts of the roots of (1.8) are negative. Let the real parts of these roots be negative denoted by $-\lambda_r(-\lambda_r = \vartheta_r)$, $r = 1,\dots,\ell$. to these correspond the roots $-\frac{\bar{\lambda}_r}{\epsilon} = \frac{-(\lambda_r - \epsilon'_r)}{\epsilon}$ of the characteristic equation of (1.3), and, accordingly, we write the particular solutions of (1.3) in the form $\epsilon^k \exp(-\bar{\lambda}_r x/\epsilon)$.

Then the general solution of (1.3) is of the form:

$$y_\epsilon(x) = \sum_{j=1}^{k} \tilde{c}_j \exp(\bar{\mu}_j x) + \sum_{r=1}^{\ell} \tilde{c}_{k+r} \epsilon^k \exp\left(\frac{-\bar{\lambda}_r x}{\epsilon}\right). \quad (1.12)$$

Using the boundary conditions (1.4), we get the first k equations as

$$\sum_{j=1}^{k} \tilde{c}_j (\bar{\mu}_j)^i + \sum_{r=1}^{\ell} O(\epsilon) \quad \tilde{c}_{k+r} = D_{io} \quad (i = 0, 1, \ldots, k-1) \quad (1.13')$$

and the succeeding ℓ equations after multiplication by ϵ^α take on the form

$$\sum_{j=1}^{k} \epsilon^\alpha (\bar{\mu}_j)^{k+\alpha} \quad \tilde{c}_j + \sum_{r=1}^{\ell} (-\bar{\lambda}_r)^{k+\alpha} \tilde{c}_{k+r} = \epsilon^\alpha D_{k+\alpha 0} \quad (1.13'')$$

Where $\alpha = i - k$, $\alpha = 0, 1, \ldots, \ell - 1$.

When $\epsilon \to 0$ the system (1.13') goes to system (1.6) with determinant $W(\mu_1, \mu_2, \ldots, \mu_k)$ and the system (1.13") go into the equations

$$\left.\begin{aligned} &\sum_{j=1}^{k} (\mu_j)^k c_j + \sum_{r=1}^{\ell} (-\lambda_r)^k c_{k+r} = D_{ko} \text{ for } \alpha = 0 \\ &\sum_{r=1}^{\ell} (-\lambda_r)^{k+\alpha} \quad c_{k+r} = 0 \quad \text{for } \alpha = 1, 2, \ldots, \ell - 1. \end{aligned}\right\} \quad (1.14)$$

The determinat of system (1.6) and (1.14) is

$$B = W(\mu_1, \mu_2, \ldots, \mu_k) \begin{vmatrix} (-\lambda_1)^k & \ldots\ldots & (-\lambda_\ell)^k \\ \cdots & \cdots & \cdots \\ (-\lambda_1)^{k+\ell-1} & \ldots & (-\lambda_\ell)^{k+\ell-1} \end{vmatrix}$$

$$= W(\mu_1, \mu_2, \ldots \mu_k) W(-\lambda_1, \ldots, -\lambda_\ell)(\lambda_1, \ldots, \lambda_\ell)^k (-1)^{k-\ell} \neq 0,$$

since by assumption any of these factors is not zero.

The coefficients and the right members of the systems (1.13') and (1.13") approach the respective coefficients and right members of the limit system (1.6) and (1.14) as $\epsilon \to 0$. It means that B_ϵ (determinant of the system before the passage to the limit) approaches to B as $\epsilon \to 0$. Consequently for small ϵ, $B_\epsilon \neq 0$ and the systems (1.13') and (1.13") are solvable, their solutions $\tilde{c}_j (j = 1,\ldots,k + \ell)$ approaching the solutions c_j of the limit systems: $\tilde{c}_j = c_j + \eta_j, \eta_j \to 0$ as $\epsilon \to 0$.

Introducing the notations

$$v_\epsilon = \sum_{r=1}^{\ell} \epsilon^k \tilde{c}_{k+r} \exp\left[-\frac{\bar{\lambda}_r x}{\epsilon}\right]$$

and

$$z_\epsilon = y_\epsilon - y_o - v_\epsilon = \sum_{j=1}^{k} \tilde{c}_j \exp(\bar{\mu}_j x) - y_o(x)$$

$$= \sum_{j=1}^{k} (c_j + \eta_j) \exp(\bar{\mu}_j x) - \sum_{j=1}^{k} c_j \exp(\mu_j x)$$

$$= \sum_{j=1}^{k} \eta_j \exp\left((\mu_j + \epsilon_j)x\right) + \sum_{j=1}^{k} c_j \exp(\mu_j x)\exp(\epsilon_j x - 1)$$

in (1.12), we get the required solution

$$y_\epsilon(x) = y_o(x) + v_\epsilon(x) + z_\epsilon(x)$$

of the problem A_ϵ.

Now we consider the case when $k_2 = 0$, $\ell_1 = 0$, $k_1 \neq 0$, and $\ell_2 \neq 0$ in equations (1.1) to (1.4).

We write the general solution y_ϵ of (1.3) replacing $\tilde{c}_{k+r}$ by $\exp\left[-\frac{\nu_r}{\epsilon}\right]\tilde{c}_{k+1}$ in (1.11) as

$$y_\epsilon(x) = \sum_{j=1}^{k} \tilde{c}_j \exp(\bar{\mu}_j x) + \sum_{r=1}^{\ell} \tilde{c}_{k+r} \exp\left(\frac{\bar{\nu}_r(x-1)}{\epsilon}\right)$$

It is clear from the second term of this equation that if all $R_e \bar{\nu}_i > 0$, then for sufficiently small ϵ all such solutions have the character of a boundary layer near the point x = 1; otherwise the second term will tend to infinity as $\epsilon \to 0$. Therefore in this case the perturbation is singular provided all ℓ roots of equation (1.8) have positive real parts.

Now we consider the slightly more complicated problem when $k_2 = 0$, $k_1 \neq 0$, $\ell_1 \neq 0$ and $\ell_2 \neq 0$ in (1.1) to (1.4). Now the perturbation of problem A_ϵ to problem A_o is singular when $Q_o(\lambda) = 0$, has ℓ_1 roots $-\lambda_1, \ldots, -\lambda_{\ell_1}$, with negative real part and ℓ_2 roots $\nu_1, \nu_2, \ldots \nu_{\ell_2}$, with positive real part. The characteristic equation (1.9) has the roots

$$\bar{\mu}_i = \mu_i + \epsilon_i, \quad -\frac{\bar{\lambda}_r}{\epsilon} = \frac{-\lambda_r + \epsilon_r}{\epsilon}, \quad \frac{\bar{\nu}_s}{\epsilon} = \frac{\nu_s + \epsilon_s}{\epsilon}$$

$$(i = 1, \ldots, k; \; r = 1, \ldots, \ell_1; \; s = 1, \ldots, \ell_2)$$

Therefore the general solution of (1.3) can be written in the form

$$y_\epsilon(x) = \sum_{j=1}^{k} \tilde{c}_j \exp(\bar{\mu}_j x) + \sum_{r=1}^{\ell_1} \epsilon^{k_1} \tilde{c}_{k+r} \exp\left(-\frac{\bar{\lambda}_r x}{\epsilon}\right)$$

$$+ \sum_{s=1}^{\ell_2} \epsilon^{k_2} \tilde{c}_{k+\ell_1+s} \exp\left(\frac{\bar{\nu}_s (x-1)}{\epsilon}\right)$$

where the last two terms are the boundary layer of order k_1 and k_2 in the neighborhood of the points $x = 0$ and $x = 1$ respectively. This solution can be expressed in the form of (1.10).

Consider now the most general problem consisting of (1.1) to (1.4) where k_1, k_2, ℓ_1 and ℓ_2 are nonzero numbers.

Theorem 1.2. [4] If the reduced problem A_o is solvable and if the perturbation of problem A_ϵ to problem A_o is singular, then for sufficiently small ϵ problem A_ϵ is also solvable, and the solution $y_\epsilon(x)$ of the problem A_ϵ has the form

$$y_\epsilon(x) = y_o(x) + v_\epsilon(x) + z_\epsilon(x), \tag{1.10}$$

where $y_o(x)$ is the solution of the reduced problem A_o, $v_\epsilon(x)$ is a function of boundary layer type of the k_1 and k_2th order in the neighborhood of the point $x = 0$ and $x = 1$ respectively, which along with all of its derivatives vanishes uniformly with ϵ on an arbitrary interval interior to $[0, 1]$; the function $z_\epsilon(x)$ together with all of its derivatives vanishes uniformly with ϵ on the entire interval $[0,1]$.

Proof: Let $\mu_1, \mu_2, \ldots, \mu_k$ be the roots of the equation (1.5). The general solution of (1.1) has the form

$$y(x) = \sum_{j=1}^{k} c_j \exp(\mu_j x) = \sum_{j=1}^{k} c_j w_j$$

In order that boundary conditions (1.2) be satisfied, we have

$$\sum_{j=1}^{k} c_j(\mu_j)^i \exp(\mu_j) = D_{io} \quad (i = 0, \ldots, k_1 - 1)$$

and

$$\sum_{j=1}^{k} c_j(\mu_j)^i \exp(\mu_j) = D_{i1} (i = 0, 1, \ldots, k_2 - 1).$$

In case problem A_o is solvable the determinant B_1 of this system is not equivalent to zero.

Here problem A_ϵ perturbs singularly to problem A_o if $Q_o(\lambda) = 0$ has ℓ_1 roots $-\lambda_1, -\lambda_2, \ldots, -\lambda_{\ell_1}$ with negative real parts and $\ell_2 (= \ell - \ell_1)$ roots $\nu_1, \nu_2, \ldots, \nu_{\ell_2}$ with positive real parts, so in virtue of lemma 1.1, the characteristic equation (1.9) has roots

$$\bar{\mu}_i = \mu_i + \epsilon_i (i = 1, \ldots, k), \quad -\frac{\bar{\lambda} i}{\epsilon} = \frac{-(\lambda i + \epsilon' i)}{\epsilon}, \quad (i = 1, \ldots \ell_1),$$

$$\frac{\bar{\nu}}{\epsilon} j = \frac{(\nu_j + \epsilon''_j)}{\epsilon} \qquad (j = 1, \ldots, \ell_2).$$

The numbers ϵ_i, ϵ'_i, ϵ''_i vanish with ϵ.

The general solution $y_\epsilon(x)$ of (1.3) has the form

$$y_\epsilon(x) = \sum_{j=1}^{k} \tilde{c}_j \exp(\bar{\mu}_j x) + \sum_{j=1}^{\ell_1} \tilde{c}_{k+j}\, \epsilon^{k_1} \exp\left(-\frac{\bar{\lambda}_j x}{\epsilon}\right)$$

$$+ \sum_{j=1}^{\ell_2} \tilde{c}_{k+\ell_1+j}\, \epsilon^{k_2} \exp\left(\frac{\bar{\nu}_j (x-1)}{\epsilon}\right)$$

$$= \sum_{j=1}^{k+\ell} \tilde{c}_j \tilde{w}_j \tag{1.15}$$

It can be shown that the determinant B of the limit system obtained by the above problem is equal to

$$B = B_1 B_2 B_3$$

where B_1 is the determinant of problem A_o, which is not equal to zero by the assumption that the reduced problem A_o is solvable,

$$B_2 = \left| (-\lambda_r)^{k_1+\gamma} \right| \begin{matrix} r = 1, 2, \ldots, \ell_1 \\ \gamma = 0, 1, \ldots, \ell_1 - 1 \end{matrix} \neq 0,$$

$$B_3 = \left| \nu_s^{k_2+\delta} \right| \begin{matrix} s = 1, 2, \ldots, \ell_2 \\ \delta = 0, 1, \ldots, \ell_2 - 1 \end{matrix} \neq 0 .$$

Therefore $B \neq 0$.

Moreover by a similar argument as was given in theorem 1.2, the determinant B_ϵ of problem A_ϵ approaches to $B \neq 0$ as $\epsilon \to 0$. Consequently, for sufficiently small ϵ, $B_\epsilon \neq 0$, and so problem A_ϵ is solvable.

In the same fashion the constants $\tilde{c}_j (j = 1, \ldots, k + \ell)$ in the solution (1.15) tend to the constants $c_j (j = 1, \ldots, k + \ell)$ of the limit systems.

$$\tilde{c}_j = c_j + \eta_j, \quad \eta_j \to 0 \text{ as } \epsilon \to 0 \ (j = 1, 2, \ldots, k + \ell)$$

In (1.15) we introduce the notations

$$z_\epsilon(x) = \sum_{i=1} (c_i + \eta_i) \tilde{w}_i - y_o(x)$$

$$v_\epsilon(x) = \sum_{r=1}^{\ell} (c_{k+r} + \eta_{k+r}) \tilde{w}_{k+r}$$

then $y_\epsilon = y_o(x) + v_\epsilon(x) + z_\epsilon(x)$

and $z_\epsilon(x) = \sum_{1=1}^{k} (c_i + \eta_i) \exp(\mu_i + \epsilon_i)x - \sum_{1=1}^{k} c_i \exp(\mu_i x)$

$$= \sum_{1=1}^{k} \eta_i \exp(\mu_i + \epsilon_i)x + \sum c_i \exp(\mu_i x)\left(\exp(\epsilon_i x) - 1\right)$$

Since $\eta_i \to 0$ and $\epsilon_i \to 0$ when $\epsilon \to 0$, it follows that $\epsilon \to 0$, $z_\epsilon(x)$ vanishes uniformly on the segment $[0, 1]$ together with all of its derivatives.

$$v_\epsilon(x) = \sum_{r=1}^{\ell} (c_{k+r} + \eta_{k+r}) \tilde{w}_{k+r}$$

$$= \epsilon^{k_1} \sum_{r=1}^{\ell_1} (c_{k+r} + \eta_{k+r}) \exp\left(-\frac{\lambda_r x}{\epsilon}\right)$$

$$+ \epsilon^{k_2} \sum_{s=1}^{\ell_2} \left(c_{k+\ell_1+s} + \eta_{k+\ell_1+s}\right) \exp\left(\frac{\nu_s(x-1)}{\epsilon}\right)$$

The first sum on the right in (1.16) vanishes uniformly as $\epsilon \to 0$ together with all of its derivatives on $[0, 1]$ outside an arbitrary neighborhood of the point $x = 1$. Hence the proof of the theorem.

CHAPTER 2

ORDINARY DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

In the first chapter we calculated the asymptotic solution of the problem A_ϵ in the case of its singular perturbation. There the problem A_ϵ consists of the ordinary differential equation with constant coefficients. In this chapter we shall consider the ordinary differential equation with variable coefficients.

Problem A_o consists in solving the equation,

$$\sum_{j=0}^{k} a_j(x)\, y^{(j)} = f(x), \quad a_k(x) \neq 0 \text{ for } 0 \leq x \leq 1 \qquad (2.1)$$

with the boundary conditions

$$y^{(i)}(0) = 0 \quad (i = 0, 1, \ldots, k_1 - 1), \qquad (2.2')$$

$$y^{(j)}(1) = 0 \quad (j = 0, 1, \ldots, k_2 - 1,\ k_2 = k - k_1) \qquad (2.2'')$$

Under our hypothesis the coefficients $a_j(x)$ $(j = 1, 2, \ldots, k + \ell)$ are differential a sufficient number of times. In particular the coefficients in the neighborhood of the points $x = 0$ and $x = 1$ with remainder of order $N + 1$ can be expressed as

$$a_j(x) = a_{j_o} + \sum_{s=1}^{N} a_{j,s,o}\, x^s + a_{j,N+1,o}(x)\, x^{N+1} \qquad (2.3')$$

and, with the substitution $x_1 = 1 - x$,

$$a_j(x_1) = a_{j_1} + \sum_{s=1}^{N} a_{j,s,1}\, x_1^s + a_{j,N+1,1}(x_1)\, x_1^{N+1} \qquad (2.3'')$$

Here $a_{j_o} = a_j(x)|_{x=0}$, $a_{j_1} = a_j(x)|_{x=1}$.

Problem A_ϵ consists in solving the equation

$$L_\epsilon y = \sum_{j=0}^{k} a_j(x)\, y^{(j)} + \sum_{r=1}^{\ell} \epsilon^r a_{k+r}(x)\, y^{(k+r)} = f(x) \qquad (2.4)$$

$$a_{k+r}(x) \neq 0 \text{ for } 0 \leq x \leq 1$$

with the boundary conditions

$$y^{(k_1+r)}(0) = 0 \quad (r = 0, 1, \ldots, \ell_1 - 1) \tag{2.5'}$$

$$y^{(k_2+s)}(1) = 0 \quad (s = 0, 1, \ldots, \ell_2 - 1;\ \ell_2 = \ell - \ell_1) \tag{2.5''}$$

along with (2.2).

The auxiliary characteristic equations at the points $x = 0$ and $x = 1$ which appeared in the foregoing chapter also enter into our problem; namely, at $x = 0$ the equation is

$$Q_o(\lambda) = \sum_{r=0}^{\ell} a_{k+r,\,o}\,\lambda^r = 0, \tag{2.6'}$$

and at $x = 1$,

$$Q_1(\mu) = \sum_{s=0}^{\ell} (-1)^{k+s}\, a_{k+s,1}\,\mu^s = 0. \tag{2.6''}$$

Theorem 2.1 [4] The solution y of the problem A_ϵ has the representation

$$y = y_o + v_o + \epsilon\,\alpha_o$$

Where y_o is the solution of the problem A_o, v_o is the function of boundary layer type of order k_1 in a neighborhood of $x = 0$ and of order k_2 in a neighborhood of $x = 1$, and α_o is a bounded polynomial in ϵ and x.

Proof: The introduction of the polynomial $Q_o(\lambda)$ is motivated by means of the stretching transformation

$$t = \frac{x}{\epsilon}, \quad x = \epsilon t$$

In terms of new variable t we have from (2.4)

$$\epsilon^k L_\epsilon y = \sum_{j=0}^{k-1} \epsilon^k a_j(x)\, y^{(i)}(x) + \sum_{r=0}^{\ell} \epsilon^{k+r} a_{k+r}(x) y^{(k+r)}(x)$$
$$= \sum_{j=0}^{k-1} \epsilon^{k-j} a_j(x)\, y^{(i)}(t) + \sum_{r=0}^{\ell} a_{K+r}(x)\, y^{(k+r)}(t)$$

If the right member is expanded in powers of ϵ this becomes

$$\epsilon^k L_\epsilon y = M_o y + \sum_{j=1}^{N} \epsilon^j R_j y + \epsilon^{N+1} R_{N+1} y. \qquad (2.7')$$

Here $$M_o y = \sum_{r=0}^{\ell} a_{k+r,0}\, y^{(k+r)}(t) \qquad (2.7'')$$

is a linear differential operator with constant coefficients:

$$R_1 y = \sum_{r=0}^{\ell} t a_{k+r,1,0} y^{(k+r)}(t) + a_{k-1,0} y^{(k-1)}(t) \quad ,$$

and in general for $1 \leqslant i \leqslant N$, $R_i y$ is a linear differential operator, the coefficients of which are powers of t not greater than i; R_{N+1} is a linear differential operator, each of the coefficients of which is the product of a bounded function.

Similarly the stretching transformation $t_1 = \dfrac{x_1}{\epsilon} = \dfrac{1-x}{\epsilon}$ at the right end point produces, in analogous manner, the relation

$$\epsilon^k L_\epsilon y = M_1 y + \sum_{j=1}^{N+1} \epsilon^j R_{j1} y, \qquad (2.8')$$

$$M_1 y = \sum_{r=0}^{\ell} (-1)^{k+r} a_{k+r,1} y^{(k+r)}(t_1) \ , \qquad (2.8'')$$

in which R_{i1} is an operator like the operator R_i, $1 \leqslant i \leqslant N+1$.

Now for the differential equations

$$M_o y \equiv \sum_{r=0}^{\ell} a_{k+r,0}\, y^{(k+r)}(t) = 0, \qquad (2.9')$$

$$M_1 y \equiv (-1)^k \sum_{r=0}^{\ell} (-1)^r a_{k+r,1} \, y^{(k+r)}(t_1) = 0 \qquad (2.9'')$$

the respective characteristic equations can be written by virtue of (2.6) in the form

$$\lambda^k Q_o(\lambda) = 0, \qquad (2.10')$$

$$\mu^k Q_1(\mu) = 0. \qquad (2.10'')$$

The nonvanishing roots of these equations are identical with those of (2.6).

We assume that the perturbation of problem A_ϵ to problem A_o is singular i.e. equation (2.6') has ℓ_1 roots $-\lambda_1, -\lambda_2, \ldots, -\lambda_{\ell_1}$, and equation (2.6") has ℓ_2 roots $-\vartheta_1, -\vartheta_2, \ldots, -\vartheta_{\ell_2}$ with negative real parts. Also we shall assume that the problem A_o possesses a unique solution and that the same is true of the problem A_ϵ, provided $\epsilon \rightarrow 0$ is sufficiently small.

Let $y_o(x)$ be the solution of the problem A_o. In general it will fail to satisfy $\ell_1 + \ell_2$ conditions. In order to improve the approximation we add to $y_o(x)$ two correction terms which will yield an approximation to the boundary layer near the two end points, which will be obtained from stretched form of the differential operators (2.7') and (2.8'). In the first approximation the stretched equation

$$M_o y = \sum_{r=0}^{\ell} a_{k+r,0} \, y^{(k+r)}(t) = 0 \qquad (2.9)$$

is a differential equation in the independent variable t with constant coefficients. Now we shall construct a solution of (2.9) which remains bounded, as $\epsilon \rightarrow +0$ and which compensates for the discrepancy between the boundary conditions actually prescribed at x = 0 and those satisfied by $y_o(x)$ i.e., we require that,

$$y^{(i)}(x)\big|_{x=0} = \begin{cases} 0, & i = 0, 1, \ldots, k_1 - 1 \\ -y_o^{(i)}(x)\big|_{x=0}, & i = k_1, k_1 + 1, \ldots, k_1 + \ell_1 - 1 \end{cases} \tag{2.10}$$

or, equivalently,

$$y^{(i)}(t)\big|_{t=0} = \begin{cases} 0, & i = 0, 1, \ldots, k_1 - 1 \\ -\epsilon^i y_o^{(i)}(0), & i = k_1, k_1 + 1, \ldots, k_1 + \ell_1 - 1 \end{cases} \tag{2.11}$$

For simplicity we shall stipulate that the ℓ_1 roots (with negative real part) $-\lambda_1, -\lambda_2, \ldots, -\lambda_{\ell_1}$ of (2.6') are distinct, but this is not essential. The $k + \ell$ functions $t^j (j = 0, 1, \ldots, k - 1)$, $\exp(-\lambda_j t)$ $(j = 1, 2, \ldots, \ell_1)$ and $\exp(\lambda_j t)$ $(j = 1, 2, \ldots, \ell_2)$ constitute a fundamental system for the differential equation (2.9). We discard the $\ell - \ell_1 = \ell_2$ solutions $\exp(\lambda_j t)$ $j < \ell_1$, since $\exp(\lambda_j t) = \exp(\lambda_j \frac{x}{\epsilon})$ diverges for these j, as $\epsilon \to +0$. From the remaining $k + \ell_1$ particular solutions a linear combination can be formed that satisfies the $k_1 + \ell_1$ boundary conditions (2.11). To see this we first determine constants $\tilde{c}_j$, $j = 1, 2, \ldots, \ell_1$ so that

$$\left(\sum_{j=1}^{\ell_1} \tilde{c}_j \ \exp.(-\lambda_j t) \right)^{(i)} \Bigg|_{t=0} = -\epsilon^i y_o^{(i)}(x)\big|_{x=0}$$

$$i = k_1, k_1 + 1, \ldots, k_1 + \ell_1 - 1 \tag{2.12}$$

The coefficient matrix of the left member is of vandermonde type which is not zero by the assumption that roots λ_j, $j = 1, \ldots, \ell_1$ are different. Hence (2.12) have a unique solution, which is of the form

$$\tilde{c}_j(\epsilon) = \epsilon^{k_1} c_j^o(\epsilon) \quad j = 1, 2, \ldots, \ell_1 \tag{2.13}$$

where $c_j^o(\epsilon)$ are polynomials in ϵ. Therefore, the function

$$v_o^o = \sum_{j=1}^{\ell_1} \tilde{c}_j \exp(-\lambda_j t) = \epsilon^{k_1} \sum_{j=1}^{\ell_1} c_j^o \exp(-\lambda_j t) \tag{2.14'}$$

is the solution of (2.9) which satisfies the second group of conditions in (2.11). The first group of boundary conditions in (2.11) can be satisfied by subtracting the partial sum, up to terms of degrees $k_1 - 1$, of its Maclaurin series, from (2.14'). The partial sum,

$$\epsilon^{k_1} \sum_{j=1}^{\ell_1} c_j^o \sum_{k=0}^{k_1-1} (-\lambda_j)^k \frac{t^k}{k!} = -\epsilon \alpha_o^o ,$$

Therefore $y_o + v_o^o + \epsilon \alpha_o^o$ (2.15')

satisfies the $k_1 + \ell_1$ conditions at $x = 0$ and which is expected to approximate the solution of the problem A_ϵ better than y_o in $0 \leq x \leq 1$.

Similarly we get, by the same type of construction,

$$y_o + v_o^1 + \epsilon \alpha_o^1 , \tag{2.15''}$$

which satisfies all $k_2 + \ell_2$ conditions at $x = 1$.

where

$$\epsilon^{k_2} \sum_{j=1}^{\ell_2} c_j^1 \sum_{k=0}^{k_2-1} (-\mu_j)^k \frac{t_1^k}{k!} = -\epsilon \alpha_o^1$$

and

$$v_o^1 = \epsilon^{k_2} \sum_{j=1}^{\ell_2} c_j^1 e^{-\mu_j t_1} . \tag{2.14''}$$

Now we shall combine the two formulas (2.15') and (2.15") in one, which will give a uniformly solution in the interval $0 \leq x \leq 1$. Let

$\phi(x)$ be an infinitely differentiable function of x such that

$$\phi(x) = \begin{cases} 1, & 0 \leq x \leq \delta \\ 0, & 2\delta \leq x \leq 1, \end{cases}$$

where δ is small positive number. Set

$$\alpha_o = \phi(x)\,\alpha_o^o + \phi(1-x)\,\alpha_o^1 ,$$

$$v_o = \phi(x)\,v_o^o + \phi(1-x)\,v_o^1 . \tag{2.16}$$

Then

$$y_o + v_o + \epsilon\,\alpha_o \tag{2.17}$$

can be expected to be an improvement on the approximation y_o alone, in the whole interval $0 \leq x \leq 1$.

Now we illustrate the theorem by an example.

Let the problem A_o be

$$-\ y^{(1)}(x) = -\ 1, \qquad 0 \leq x \leq 1$$

under the condition $y(0) = 0$, and the problem A_ϵ be

$$L_\epsilon y = \epsilon^2 y^{(3)}(x) + \epsilon \sin\left(\frac{\pi x}{2}\right) y^{(2)}(x) - y^{(1)}(x) = -\ 1$$

with boundary conditions $y(0) = y^{(1)}(0) = 0$, $y(1) = 0$

clearly, here $a_o(x) = 0$, $a_1(x) = -1$, $a_2(x) = \sin\left(\frac{\pi x}{2}\right)$, $a_3(x) = 1$

$$k_1 = 1,\ k_2 = 0,\ \ell_1 = \ell_2 = 1.$$

The solution $y_o(x)$ of the problem A_o is $y_o(x) = x$. We find from (2.6) that,

$$Q_o(\lambda) = -1 + \lambda^2 = 0, \text{ with zeros } \lambda_1 = +\ 1,\ \lambda_2 = -\ 1,$$

$$Q_1(\mu) = 1 + \mu - \mu^2 = 0 \text{ with zeros } \mu_1 = \frac{1+\sqrt{5}}{2},\ \mu_2 = \frac{1-\sqrt{5}}{2} .$$

Furthermore from (2.9), we get

$$M_o y = -y^{(1)}(t) + y^{(3)}(t) = 0; \qquad t = x/\epsilon$$

$$M_1 y = y^{(1)}(t_1) + y^{(2)}(t_1) - y^{(3)}(t_1) = 0; \quad t_1 = \frac{1-x}{\epsilon} \quad .$$

Thus from (2.11) it follows that v_o^o must satisfy the conditions

$$M_o(v_o^o) = 0,$$

$$v_o^{o(1)}(t)\Big|_{t=0} = -\epsilon,$$

Thus, v_o^o, which must be a multiple of the particular solution $\exp(-t)$ of $M_o(v_o^o) = 0$, becomes

$$v_o^o = \epsilon \exp(-t) = \epsilon \exp\left(\frac{-x}{\epsilon}\right)$$

Moreover, we find that

$$\epsilon \alpha_o^o = -\epsilon \quad .$$

Similarly, v_o^1 must satisfy

$$M_1(v_o^1) = 0,$$

$$v_o^1(t_1)\Big|_{t_1=0} = -1$$

and must be a multiple of $\exp\left[\frac{1-\sqrt{5}}{2} t_1\right]$, i.e.,

$$v_o^1 = -\exp\left[\frac{1-\sqrt{5}}{2} \cdot \frac{1-x}{\epsilon}\right]$$

Finally, $\alpha_o^1 = 0$, since no other boundary condition need to be satisfied at $x = 1$.

The solution of problem A_ϵ becomes

$$y(x) = y_o(x) + \phi(x)\, v_o^o + \phi(1-x) v_o^1 + \epsilon\left[\phi(x)\, \alpha_o^o + \phi(1-x)\, \alpha_o^1\right]$$

$$= x + \epsilon\, \phi(x) \exp\left(\frac{-x}{\epsilon}\right) - \phi(1-x) \exp.\left[\frac{(1-\sqrt{5})(1-x)}{2\epsilon}\right] - \epsilon\phi(x).$$

We now aim at the following theorem which is a generalization of the formula (2.17) of the form

$$y(x) = \sum_{r=0}^{\infty} y_r \epsilon^r + \sum_{r=0}^{\infty} v_r \epsilon^r + \epsilon \sum_{r=0}^{\infty} \alpha_r \epsilon^r \qquad (2.18)$$

<u>Theorem 2.2.</u> [4] Under the conditions of solvability of problem A_o, of uniform solvability of problem A_ϵ, and of singular perturbation of problem A_ϵ to problem A_o, the solution $y(x)$ of problem A_ϵ for sufficiently small $\epsilon > 0$ admits the following representation:

$$y(x) = \sum_{r=0}^{N} \epsilon^r y_r + \sum_{r=0}^{N} \epsilon^r v_r + \epsilon \sum_{r=0}^{N} \epsilon^r \alpha_r \, .$$

Here y_o is the solution of problem A_o; y_r, $r = 1, 2, \ldots, N$ (together with its derivatives) are bounded with respect to ϵ on $0 \leq x \leq 1$; v_r is of boundary layer type of order k_1 in a neighborhood of $x = 0$, and of order k_2 in a neighborhood of $x = 1$ and α_r are bounded polynomials in x and ϵ.

<u>Proof</u>: Inserting (2.18) in (2.4), we get

$$L_\epsilon\left[\sum_{r=0}^{\infty} y_r \epsilon^r + \epsilon \sum_{r=0}^{\infty} \alpha_r \epsilon^r \right] + L_\epsilon\left[\sum_{r=0}^{\infty} v_r \epsilon^r \right] = f(x) \qquad (2.19)$$

The functions, y_r, α_r and v_r to be now constructed depend on x, as well as on ϵ. Expanding

$$L_\epsilon\left[\sum_{r=0}^{\infty} y_r \epsilon^r + \epsilon \sum_{r=0}^{\infty} \alpha_r \epsilon^r \right] - f(x)$$

in powers of ϵ and setting the coefficients of ϵ equal to zero, we get the differential equation of the form

$$\sum_{j=1}^{k} a_j(x) \; y_r^{(j)} = F_r \qquad r = 0, 1, \ldots . \tag{2.20}$$

where $F_o = f$ and the F_r, $r > 0$ are linear differential expressions in y_j, α_j, $j < r$, with coefficients holomorphic in $0 \leq x \leq 1$.

Let us assume that we have already determined $y_j(x, \epsilon)$ $j < r$, as infinitely differentiable function in $0 \leq x \leq 1$ and $\alpha_j(x, \epsilon)$, $j < r$ as polynomials in ϵ. Then the equation (2.20) has a unique solution satisfying the k boundary conditions of the problem A_o. This is our inductive definition of y_r. The function $y_r(x, \epsilon)$ is infinitely often differentiable if $y_j(x, \epsilon)$, $\alpha_j(x, \epsilon)$, $j < r$ are. Moreover, it is a polynomial in ϵ.

Now stretching the term $L_\epsilon\left[\sum_{r=0}^{\infty} v_r \epsilon^r\right]$ by independent variable $t = x/\epsilon$ and expressing it in the power of ϵ, we get after equating all terms to zero,

$$M_o v_r = G_r^o , \qquad r \geq 0, \quad G_o^o \equiv 0 \tag{2.21'}$$

Where G_r are linear differential expressions in the v_j, $j < r$, whose coefficients are polynomials in t. Similarly in terms of the independent variable $t_1 = \dfrac{1 - x}{\epsilon}$, we get the sequence of differential equations

$$M_1 v_r = G_r^1 \quad , \quad r \geq 0 \qquad G_o^1 \equiv 0 \tag{2.21''}$$

Now from (2.21'), we shall construct a sequence of particular solutions v_r^o, $r = 0, 1, \ldots$ of the form

$$v_r^o = \epsilon^{k_1} \sum_{i=1}^{\ell_1} c_{ir}^o \,(t, \epsilon)\, \exp(-\lambda_j t), \quad r = 0, 1, \ldots \tag{2.22}$$

where the c^o_{ir} are polynomials in both variables. For $r = 0$ we have already defined a solution v^o_o with this property. Under the inductive hypothesis that all v^o_j , $j < r$, are of the form (2.22) the functions G^o_r with $v_j = v^o_j$ also becomes a function of this type. Now by the method of undetermined coefficients the equation (2.21') possesses a particular solution of the type (2.22), say,

$$\tilde{v}^o_r = \epsilon^{k_1} \sum_{i=1}^{\ell_1} \tilde{c}_{ir}(t, \epsilon) \exp(-\lambda_j t), \qquad r = 0, 1, 2, \ldots$$

Now if we add to it a solution of the homogeneous equation of the form

$$\tilde{\tilde{v}}^o_r = \epsilon^{k_1} \sum_{i=1}^{\ell_1} \gamma_{ir} \exp(-\lambda_j t)$$

with constant coefficients γ_{ir}, $i = 1, 2, \ldots, \ell_1$ that may depend on ϵ, we can find that

$$v^{(o)}_r = \tilde{v}^o_r + \tilde{\tilde{v}}^o_r$$

satisfies the ℓ_1 conditions

$$v^{o\,(j)}_r(t)\big|_{t=0} = -\epsilon^j y^{(j)}(x)\big|_{x=0}, \quad j = k_1, k_1+1, \ldots k_1+\ell_1.$$

The function v^o_r so defined is a solution of (2.21'), and $y_r + v^o_r$ satisfies the ℓ_1 highest order boundary conditions at $x = 0$.

We also wish to satisfy the first k_1 boundary conditions (2.11). We expand v^o_r in powers of t about $t = 0$ and observe that the partial sum up to and including terms of degree $t^{k_1 - 1}$ of this series is annihilated by the operator M_o. We call this partial sum $-\epsilon\,\alpha^o_r$, which is polynomial in x and ϵ. The sum $y_r + v_r^o + \epsilon\,\alpha_r^o$ satisfies the all required boundary conditions (2.11) at x = 0. Similarly $y_r + v_r^1 + \epsilon\,\alpha_r^1$ can be constructed to satisfy the all prescribed boundary conditions at x = 1. Finally we define v_r and α_r by

$$v_r = \phi(x)\, v_r^o + \phi(1 - x)\, v_r^1$$

$$\alpha_r = \phi(x)\, \alpha_r^o + \phi\,(1 - x)\, \alpha_r^1$$

We wish to construct an approximate solution to the order N, for our singular perturbation problem. Here the ϵ^k term in (2.7') and (2.8') creates the trouble, which can be reduced by this correction modification: For $r \leq N$ we proceed as described. For $r > N$ we calculate a number of additional functions $\hat{v}_r^j$, $\hat{\alpha}_r^j$, j = 0, 1; in the same manner, except that we impose on $\hat{v}_r^j + \epsilon\,\hat{\alpha}_r^j$ the boundary conditions at x = 0 and x = 1, respectively, instead of doing it for $y_r + v_r^j + \epsilon\,\alpha_r^j$ as before. It is sufficient to go up to r = N + k in this manner. However, in view of the factor ϵ^{k_1} in (2.22) and similarly ϵ^{k_2} for the expression for v_r', it suffices to go up to

$$N_1 = N + k - \min.\,(k_1, k_2)$$

In this case the expression

$$Y_N = \sum_{r=0}^{N} y_r \epsilon^r + \sum_{r=0}^{N} v_r\, \epsilon^r + \epsilon \sum_{r=0}^{N} \alpha_r \epsilon^r$$

is the solution of the full problem A_ϵ. Here

$$L_\epsilon(Y_N) - f(x) = \epsilon^{N+1}\, g_N(x, \epsilon) \qquad 0 \leq \epsilon \leq \epsilon_o\,, \quad 0 \leq x \leq 1,$$

and $g_N(x, \epsilon)$ is bounded for $0 \leq x \leq 1$, $0 \leq \epsilon \leq \epsilon_o$.
(ϵ_o is constant).

The error $W_n = Y_N - y$ is the solution of the differential equation

$$L_\epsilon(W_n) = \epsilon^{N+1} g_N(x, \epsilon)$$

which satisfies the boundary conditions (2.5).

CHAPTER 3

ORDINARY DIFFERENTIAL EQUATIONS OF EVEN AND ODD ORDERS

This chapter deals with the conditions under which an ordinary differential operator L_ϵ (with parameters in the term with highest derivatives) has a uniformly positive symmetric part (3.7) and a uniform inverse, which guarantee the solvability of the problem A_ϵ. Furthermore we shall see under what conditions the problem A_ϵ perturbs singularly to problem A_o.

Let the differential operator

$$L_\epsilon y = \sum_{s=0}^{k} a_s(x) y^{(s)}(x) + \sum_{j=1}^{\ell} \epsilon^j a_{k+j}(x)\, y^{(k+j)}(x)$$

$$= L_o y + L_\epsilon^1 \; y \qquad (3.1)$$

$$(a_{k+1}(x) \neq 0 \text{ for } 0 \leq x \leq 1)$$

be given on the interval $0 \leq x \leq 1$. The characteristic forms $\pi_\epsilon(\xi \; ; x)$ and $\pi_\epsilon^1 \; (\xi \; , x)$ of the operators L_ϵ and L_ϵ^1 at the point x are defined respectively by,

$$\pi_\epsilon(\xi \; ; x) = \sum_{j=0}^{\ell} \epsilon^j a_{k+j}(x) \quad (i\xi)^{k+j} \qquad (3.2)$$

and

$$\pi_\epsilon^1 (\xi \; ; \; x) = \sum_{j=1}^{\ell} \epsilon^j a_{k+j}(x) \quad (i\xi)^{k+j} \qquad (3.3)$$

If $k + \ell = 2(k_1 + \ell_1)$ is an even number, then we consider the boundary conditions

$$y^{(s)}(0) = y^{(s)}(1) = 0, \qquad s = 0, 1, \ldots, k_1 + \ell_1 - 1 \qquad (3.4)$$

If $k + \ell = 2(k_1 + \ell_1) + 1$ is an odd number, then the number of boundary conditions at $x = 0$ and 1 depend on the sign of the coefficient $a_{k+\ell}(x)\,(-1)^{k_1+\ell_1}$ i.e., for $(-1)^{k_1+\ell_1} a_{k+\ell}(x) > 0$,

$$\left.\begin{aligned} y^{(r)}(0) &= 0, \quad r = 0, 1, \ldots, k_1 + \ell_1, \\ y^{(s)}(1) &= 0, \quad s = 0, 1, \ldots, k_1 + \ell_1 - 1 \end{aligned}\right\} \qquad (3.5)$$

and for $(-1)^{k_1+\ell_1} a_{k+\ell}(x) < 0$,

$$\left.\begin{aligned} y^{(r)}(0) &= 0, \quad r = 0, 1, \ldots, k_1 + \ell_1 - 1 \\ y^{(s)}(1) &= 0, \quad s = 0, 1, \ldots, k_1 + \ell_1 \end{aligned}\right\} \qquad (3.6)$$

We begin with the case in which $k = 2k_1$ and $\ell = 2\ell_1$ are even numbers. We shall derive below conditions under which the operator L_ϵ with the boundary conditions (3.4) are positive and moreover are uniformly positive, i.e.,

$$(L_\epsilon y, y) \geq \alpha^2 \left[\epsilon^{2\ell_1} \| y^{(k_1+\ell_1)}(x) \|^2 + \| y^{(k_1)}(x) \|^2 + \| y(x) \|^2 \right] \qquad (3.7)$$

where α^2 is independent of ϵ and y.

Theorem 3.1. [4] If the numbers $k = 2k_1$, $k + \ell = 2(k_1 + \ell_1)$ are even, if the characteristic form of the operator L_ϵ^1 has a positive real part, i.e.,

$$\text{Re. } \Pi_\epsilon^1(\xi\,; x) \equiv \sum_{j=1}^{\ell_1} \epsilon^{2j} (-1)^{k_1+j} a_{2(k_1+j)}(x)\, \xi^{2(k_1+j)} \geq \alpha^2 \sum^{\ell_1} \epsilon^{2j}\, \xi^{2(k_1+}$$

and moreover if the operator L_0 is positive:

$$(L_o y, y) \geqslant \gamma^2 \left(\| y^{(k_1)}(x) \| + \| y(x) \|^2 \right)$$

for arbitrary smooth function y satisfying conditions (3.4), then the operator L_ϵ under the boundary conditions (3.4) is uniformly positive for sufficiently small ϵ .

Proof of the Theorem will be given after quoting a few lemmas.

We denote by $\Lambda_\epsilon y$ the part of the operator $L_\epsilon y$ which is equal to the sum of the terms of L_ϵ with derivatives of even order and coefficients which contain ϵ .

Lemma 3.1. If $k = 2k_1$ and the characteristic form $\widetilde{\Pi}_\epsilon(\xi\ ;x) = \mathrm{Re.}\ \Pi^1_\epsilon(\xi\ ;x)$ of the operator Λ_ϵ is

$$\widetilde{\Pi}_\epsilon(\xi\ ;x) = \mathrm{Re.}\ \Pi^1_\epsilon(\xi\ ;x) = \sum_{j=1}^{\ell_1} (-1)^{k_1+j} \epsilon^{2j} a_{2(k_1+j)}(x)\, \xi^{2(k_1+j)}$$

$$\geqslant \alpha^2 \sum_{j=1}^{\ell_1} \epsilon^{2j} \xi^{2(k_1+j)}, \qquad (3.8)$$

Where α^2 is independent of x and ξ then for sufficiently small ϵ :

$$(\Lambda_\epsilon y, y) \geqslant \beta^2 \left[\sum_{j=1}^{\ell_1} \epsilon^{2j} \| y^{(k_1+j)}(x) \|^2 \right]$$

$$- M\epsilon \left[\| y^{(k_1)}(x) \|^2 + \| y(x) \|^2 \right] \qquad (3.9)$$

Where M is some constant, and y is an arbitrary smooth function which vanishes near the points 0 and 1 and outside of $[0, 1]$.

Proof: First we consider the case when the coefficients $a_i(x) = a_i$ are constants. Let the fourier transform of y(x) be

$$\tilde{y}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-i\xi x) \quad y(x) \quad dx$$

Therefore

$$(i\xi)^r \quad \tilde{y}(\xi) = 1/\sqrt{2\pi} \int_{-\infty}^{+\infty} \exp(-i\xi x) \quad y^{(r)}(x) \qquad dx.$$

Now making use of the Parseval's equation and (3.8), we get ,

$$(\Lambda_\epsilon y, y) = (\tilde{\Pi}_\epsilon(\xi)\tilde{y}(\xi), \tilde{y}(\xi)) \geqslant \alpha^2 \sum_{j=1}^{\ell_1} (\epsilon^{2j} \xi^{2(k_1+j)} \tilde{y}, \tilde{y})$$

$$= \alpha^2 \sum_{j=1}^{\ell_1} \epsilon^{2j} \left\| y^{(k_1+j)} \right\|^2 , \tag{3.9''}$$

and the estimate (3.9) is established, which holds for any finite functions $y(x)$.

Now we prove the inequality (3.9') when the coefficients $a_i(x)$ are variables. Let W is an arbitrary smooth function which vanishes near the points γ and $\gamma+\delta$ (and outside of $[\gamma, \gamma+\delta]$), $(\gamma, \gamma+\delta) \subset [0, 1]$. Denote by $\overset{\circ}{\Lambda}_\epsilon$ the operator Λ_ϵ when its coefficients are replaced with their values at the point x_0, $\gamma < x_0 < \gamma + \delta$. Applying (3.9'') to W, we get

$$(\Lambda_\epsilon W, W) = (\overset{\circ}{\Lambda}_\epsilon W, W) + ((\Lambda_\epsilon - \overset{\circ}{\Lambda}_\epsilon)W, W)$$

$$\geqslant \alpha^2 \left[\sum_{j=1}^{\ell_1} \epsilon^{2j} \left\| W^{(k_1+j)} \right\|^2 \right] + ((\Lambda_\epsilon - \overset{\circ}{\Lambda}_\epsilon) W, W) \tag{3.10}$$

Integrating by parts up to the order which is half the order of each component entering into Λ_ϵ, we get

$$\langle (\Lambda_\epsilon - \Lambda_\epsilon^{\circ}) W, W \rangle = \sum_{j=1}^{\ell_1} \epsilon^{2j} (-1)^{k_1+j} \left(W^{(k_1+j)}(x), \left(W(x) \eta_{k_1+j} \right)^{(k_1+j)} \right) \tag{3.11}$$

Where $\eta_i(x) = a_{2i}(x) - a_{2i}(x_o)$.

Also, $(\eta_s W^{(s)}) = \eta_s W^{(s)} + \sum_{i=1}^{s} {}^s c_i \, \eta_s^{(i)} \, W^{(s-i)}$

Estimating the scalar product in each term of (3.11), we get

$$\left| \left(W^{(s)}, (\eta_s W)^{(s)} \right) \right| \leqslant \left| \left(\eta_s W^{(s)}, W^{(s)} \right) \right| + \sum_{i=1}^{s} {}^s c_i \left| \left(W^{(s)}, \eta_s^{(i)} W^{(s-i)} \right) \right|$$

$$\leqslant \omega_\delta \| W^{(s)} \|^2 + c_o \left[\sum_{i=1}^{s} \left| \left(W^{(s)}, \eta_s^{(i)} W^{(s-i)} \right) \right| \right], \tag{3.12}$$

Where $\omega_\delta = \max. |\eta_s|$ for $x \in [\gamma, \gamma + \delta]$, $s = k_1 + 1, \ldots, k_1 + \ell_1$,
and c_o is constant.

Now after making use of the inequality

$$|(u, v)| \leqslant \frac{1}{2} \left(\omega^2 \|u\|^2 + \omega^{-2} \|v\|^2 \right)$$

for arbitrary $\omega > 0$, we get

$$\left| \left(W^{(s)}, \eta_s^{(i)} W^{(s-i)} \right) \right| \leqslant \frac{1}{2} \left[\omega^2 \| W^{(s)} \|^2 + \omega^{-2} \| \eta_s^{(i)} W^{(s-i)} \|^2 \right]$$

$$\leqslant \frac{1}{2} \left[\omega^2 \| W^{(s)} \|^2 + c^2 \omega^{-2} \| W^{(s-i)} \|^2 \right]$$

$$i = 1, 2, \ldots, s.$$

Where $c = \max. | a_{2j}^{(i)}(x) |$ for $0 \leqslant x \leqslant 1$

$$j = k_1 + 1, \ldots, k_1 + \ell_1; \; 1 \leqslant i \leqslant j$$

Taking $\omega^2 = \epsilon$, we have for $i \geqslant 1$,

$$\epsilon^{2(s-k_1)}\left|\left(W^{(s)}, \eta_s^{(i)} W^{(s-i)}\right)\right| \leqslant \epsilon\left(\frac{1}{2}\epsilon^{2(s-k_1)} \|W^{(s)}\|^2\right)$$

$$+ \frac{c^2}{2}\epsilon^{2(s-k_1)-1} \|W^{(s-i)}\|^2$$

For $s-i > k_1$, $i \geqslant 1$, $\epsilon^{2(s-k_1)-1} \leqslant \epsilon . \epsilon^{2(s-i-k_1)}$,

and for $s - i \leqslant k_1$, $\epsilon^{2(s-k_1)-1} \leqslant \epsilon$ (Since $s > k_1$, $\epsilon < 1$).

Therefore for $s > k_1$ (3.12) gives

$$\epsilon^{2(s-k_1)}\left|\left(W^{(s)}, (\eta_s W)^{(s)}\right)\right| \leqslant (\omega_\delta + c_1\epsilon)\,\epsilon^{2(s-k_1)}\|W^{(s)}\|^2$$

$$+ c_1\epsilon\left[\sum_{r=k_1+1}^{s-1}\epsilon^{2(r-k_1)}\|W^{(r)}\|^2 \sum_{\tau=0}^{k_1}\|W^{(\tau)}\|^2\right]$$

Substituting in (3.11) the estimate found and reducing like terms, we get

$$\left|((\Lambda_\epsilon - \Lambda_\epsilon^{\circ})W, W)\right| \leqslant (\omega_\delta + c_2\epsilon)\sum_{j=1}^{\ell_1}\epsilon^{2j}\|W^{(k_1+j)}\|^2 + c\,\epsilon\sum_{\tau=0}^{k_1}\|W^{(\tau)}\|^2$$

Choose ϵ and δ in such a way that $\omega_\delta + c_2\epsilon < \frac{\alpha^2}{2}$.

We get from (3.10)

$$(\Lambda_\epsilon W, W) \geqslant \frac{\alpha^2}{2}\sum_{j=1}^{\ell_1}\epsilon^{2j}\|W^{(k_1+j)}\|^2 - c_3\sum_{\tau=0}^{k_1}\|W^{(\tau)}\|^2 \quad (3.13)$$

and since $\|W^{(i)}\|^2 \leqslant M_i \|W^{(k_1)}\|^2$, $i \leqslant k$, (3.14) the estimate is established.

Let u be an arbitrary smooth function which vanishes near the points 0 and 1(and outside of $[0, 1]$) and $\zeta_i(x)$ be a smooth function not vanishing on $(\gamma_i, \gamma_i + \delta_i), \delta_i \leq \delta$ such that

$$1 \equiv \sum_{i=1}^{N} \zeta_i^2(x), \quad 0 \leq x \leq 1 \tag{3.15}$$

Therefore

$$(\Lambda_\epsilon y, y) = \left(\sum_{i=1}^{N} \zeta_i^2(x) \Lambda_\epsilon y, y \right)$$

$$= \sum_{i=1}^{N} \left[(\Lambda_\epsilon \zeta_i y, \zeta_i y) + B(y, \zeta_i) \right],$$

in which y appears in $B(y, \zeta_i)$ with derivatives of order lower at least by one than in the corresponding term of $\sum(\Lambda_\epsilon \zeta_i y, \zeta_i y)$. After integrating by parts and having the same estimate of scalar product as we have applied before, we find that

$$\left| B(y, \zeta_i) \right| \leq c\epsilon \left[\sum_{j=1}^{\ell_1} \epsilon^{2j} \| y^{(k_1+j)} \|^2 \right] + c\epsilon \left[\sum_{s=0}^{k_1} \| y^{(s)} \|^2 \right]$$

Choosing small enough ϵ such that $c\epsilon \leq \frac{\alpha^2}{4}$ and making use of (3.13) in each term $(\Lambda_\epsilon \zeta_i y, \zeta_i y)$, we get

$$(\Lambda_\epsilon y, y) \geq \frac{\alpha^2}{2} \sum_{j=1}^{\ell_1} \sum_{j=1}^{N} \epsilon^{2j} \| (\zeta_i y)^{(k_1+j)} \|^2 - c_3 \epsilon \sum_{\tau=0}^{k_1} \sum_{i=1}^{N} \| (\zeta_i y)^{(\tau)} \|^2$$

$$- c\epsilon \sum_{j=1}^{\ell_1} \epsilon^{2j} \| y^{(k_1+j)} \|^2 - c\epsilon \sum_{s=1}^{k_1} \| y^{(s)} \|^2 \tag{3.16}$$

Since

$$\|(\zeta_i y)^{(k_1+j)}\|^2 \geqslant \|\zeta_i y^{(k_1+j)}\|^2 - c_1 \sum_{r=1}^{k_1+j} \|y^{(k_1+j-r)}\|^2 ,$$

(3.16) becomes

$$(\Lambda_\epsilon y, y) \geqslant \frac{\alpha^2}{2} \sum_{j=1}^{\ell_1} \sum_{i=1}^{N} \epsilon^{2j} \|\zeta_i y^{(k_1+j)}\|^2 - c_2\, \epsilon \sum_{j=1}^{\ell_1} \epsilon^{2j} \|y^{(k_1+j)}\|^2$$

$$- c_4\, \epsilon \sum_{s=1}^{k_1} \|y^{(s)}\|^2 .$$

Again taking sufficient small ϵ so that $c_2 \leqslant \frac{\alpha^2}{4}$ and using (3.14) and (3.16), we get (3.9'), which was to be proved.

Now we shall see that Lemma 3.1 also holds for functions y in $W_2^{2(k_1+\ell_1)}$ satisfying the boundary conditions (3.4). To prove, in short, it is enough to add to the estimate (3.9') the expression

$$\sum_{j=1}^{\ell_1} (-1)^{k_1+j} \epsilon^{2j} \left(y^{(k_1+j)}, a_{2(k_1+j)} y\right) + M\epsilon \left(\|y^{(k_1)}\|^2 + \|y\|^2\right)$$

$$\geqslant \beta^2 \sum_{j=1}^{\ell_1} \epsilon^{2j} \|y^{(k_1+j)}\|^2$$

in which the sum in the left hand side results from integrating the form $(\Lambda_\epsilon y, y)$ by parts.

Let y belong to $W_2^{2(k_1+\ell_1)}$ and let y satisfy the boundary conditions (3.4). Then, we can construct a sequence of functions $\{y_n(x)\}$ $(= \{J_\eta(\zeta_n y)\}$, where ζ_n is a smooth function vanishing in $[0, \frac{1}{n}]$ and $[1 - \frac{1}{n}, 1]$; J_η is an averaging operator with radius $\eta = \frac{1}{2n}$), y_n is an arbitrary smooth function which vanishes near the points 0 and 1 (and outside of $[0, 1]$) and converges to y in $W_2^{2(k_1+l_1)}$. Substituting $y = y_n$ in the above written equation and passing to the limit as $n \to \infty$, we see the validity of this inequality for y of the above indicated type.

Denote by $M_\epsilon y$ the part of the $L_\epsilon y$ consisting of odd order which have ϵ in their coefficients:

$$M_\epsilon y = \sum_{j=0}^{\ell_1-1} \epsilon^{2j+1} a_{2(k_1+j)+1}(x) \, y^{2(k_1+j)+1}$$

Lemma 3.2. Let $y(x)$ belong to $W_2^{2(k_1+\ell_1)}$ and let it satisfy the boundary conditions (3.4). Then

$$(M_\epsilon y, y) \geq -c\epsilon \left[\sum_{j=1}^{\ell_1-1} \epsilon^{2j} \| y^{(k_1+j)} \|^2 + \| y^{(k_1)} \|^2 + \| y \|^2 \right]$$

Proof: Integrating by parts and using the boundary conditions (3.4), we get

$$\epsilon^{2\ell_1-1} \left(a_{2(k_1+\ell_1)-1} \, y^{(2(k_1+\ell_1)-1)} , \; y \right)$$

$$= \frac{1}{2}\,\epsilon^{2\ell_1 - 1}(-1)^{k_1 + \ell_1 - 1}\int_0^1 \frac{d}{dx}\left[a_{2(k_1+\ell_1)-1}\,y^{(k_1+\ell_1-1)}\right]^2 dx$$

$$+\ \epsilon^{2\ell_1 - 1}\sum_{j \leq k_1 + \ell_1 - 1}\left(b_j y^{(k_1+\ell_1-1)},\ y^{(j)}\right)$$

$$= 2\ell_1 - 1\sum_{j \leq k_1 + \ell_1 - 1}\left(b_j y^{(k_1+\ 1-1)},\ \ y^{(j)}\right)$$

Similarly as we have done in lemma 3.1, we get

$$\left|\epsilon^{2\ell_1 - 1}\sum_{j \leq k_1 + \ell_1 - 1}\left(b_j y^{(k_1+\ell_1-1)}, y^{(j)}\right)\right| \leq c_1\,\epsilon\left[\sum_{j=1}^{\ell_1 - 1}\epsilon^{2j}\|y^{(k_1+j)}\|^2 + \|y^{(k_1)}\|^2 + \|y\|^2\right].$$

We get the similar estimate for the remaining terms of M_ϵ also. Hence lemma is proved.

Proof of Theorem 4.1: For functions y in $W_2^{2(k_1+\ell_1)}$ which satisfy the boundary conditions (3.4), we have from the lemmae 3.1 and 3.2,

$$(L_\epsilon y, y) = (\Lambda_\epsilon y,\ y) + (M_\epsilon\, y,\ y) + (L_o y, y)$$

$$\geq (\beta^2 - c\epsilon)\sum_{j=1}^{\ell_1}\epsilon^{2j}\ \|y^{(k_1+j)}\|^2 + (\gamma^2 - M\epsilon - c\epsilon)\cdot(\|y\|^2 + \|y^{(k_1)}\|^2)$$

$$\geq \beta_1^2\left[\sum_{j=1}^{\ell_1}\epsilon^{2j}\|y^{(k_1+j)}\|^2 + \|y^{(k_1)}\|^2 + \|y\|^2\right]$$

Where ϵ is small such that $\beta^2 - c\epsilon \geqslant \beta_1^2 > 0$ and

$$\hat{\gamma}^2 - M\epsilon - c\epsilon \geqslant \beta_1^2 > 0$$

It follows from the uniform positiveness that the equation $L_\epsilon y = f(x)$ is solvable with the boundary conditions and arbitrary $f(x)$ in $\mathcal{L}_2$. In fact, the number of boundary conditions is equal to the order of the equation, and the positiveness then guarantees the uniqueness, and consequently the existence of the problem A_ϵ . Hence uniform positiveness implies uniform solvability of problem A_ϵ . Hence the theorem is proved.

Similar theorems on the uniform positiveness of L_ϵ hold also in the other cases, which can be stated as:

<u>Theorem 3.2.</u> If $k + \ell = 2(k_1 + \ell_1)$ is an even number, $k = 2k_1 + 1$ is odd, if the characteristic form $\overset{1}{\pi}_\epsilon$ of the operator $\overset{1}{L}_\epsilon$ has a positive real part

$$\text{Re. } \overset{1}{\pi}_\epsilon(\xi; x) = \sum_{j=1}^{\ell_1} \epsilon^{2j-1} (-1)^{k_1+j} a_{2(k_1+j)}(x)\, \xi^{2(j+k_1)}$$

$$\geqslant \alpha^2 \sum_{j=1}^{\ell_1} \epsilon^{2j-1} \xi^{2(j+k_1)} ,$$

and if the operator L_o is positive i.e.

$(L_o y, y) \geqslant \gamma^2 (\| y^{(k_1)} \|^2 + \| y \|^2)$, then the operator L_ϵ under the boundary conditions (3.4) are uniformly positive for sufficiently small ϵ .

Theorem 3.3. If the operator of L_ϵ is odd, i.e. $k + \ell = 2(k_1 + \ell_1) + 1$, if the characteristic form of the operator $\overset{1}{L}_\epsilon$ has a positive real part,

$$\mathrm{Re}\ \overset{1}{\pi}_\epsilon(\xi;x) \geqslant \alpha^2\left[\sum_{j=1}^{\ell_1} \epsilon^{2j}\ \xi^{2(j+k_1)}\right]$$

or

$$\mathrm{Re}\ \overset{1}{\pi}_\epsilon(\xi;x) \geqslant \alpha^2\left[\sum_{j=1}^{\ell_1} \epsilon^{2j-1}\ \xi^{2(j+k_1)}\right]$$

depending on the parity of k: $k = 2k_1$ or $k = 2k_1 + 1$, and if the operator L_o is positive i.e.

$$(L_o y, y) \geqslant \gamma^2\left(\| y^{(k_1)}\|^2 + \| y \|^2\right);$$

then the operator L_ϵ under the respective conditions (3.5) or (3.6) are uniformly positive for sufficiently small ϵ .

Now we shall find the number of the roots with negative real parts of the auxiliary characteristic equation

$$\overset{(k,\ell)}{Q}(t) = \sum_{j=0}^{\ell} a_{k+j}t^j = 0;\ a_k \neq 0,\ a_{k+\ell} \neq 0 \qquad (3.17)$$

We assume that the real part of the polynomial $t^k\overset{(k,\ell)}{Q(t)}$, t is imaginary, say $t = i\xi$ is positive i.e.

$$\mathrm{Re}\ (i\xi)^k\ \overset{(k,\ell)}{Q}(i\xi) \equiv \sum_{k \leqslant 2j \leqslant k+1} (-1)^j a_{2j}\ \xi^{2j}$$

$$\geqslant c^2\left(\xi^{2m} + \xi^{2M}\right) \qquad (3.18)$$

Where 2m and 2M are respectively the smallest and largest of the four numbers 2j for which $k \leq 2j \leq k + \ell$. If the free term a_k in (3.17) vanishes, then with the remaining coefficients $a_j (j > k)$ fixed and a_{k+1} assumed to be different from zero, only one of the roots λ_0 of (3.17) vanishes with a_k, and

$$\lambda_0 \approx \frac{-a_k}{a_{k+1}} \quad , \text{Re. } \lambda_0 \approx \frac{-a_k}{a_{k+1}} \quad .$$

Hence, for sufficiently small a_k,

$$\text{Sign Re. } \lambda_0 \approx \frac{-a_k}{a_{k+1}} \quad .$$

In the same fashion, if the leading coefficient $a_{k+\ell}$ in (3.17) vanishes, with all remaining coefficients a_j, $j < k+\ell$, fixed and $a_{k+\ell-1} \neq 0$, equation (3.17) for $a_{k+\ell-1} \neq 0$ has one root which becomes infinite as $k + \ell \to 0$, such that

$$\text{Sign Re } \mu = - \text{sign } \frac{a_{k+\ell-1}}{a_{k+\ell}} .$$

Now we consider the following lemmas.

<u>Lemma 3.3</u> Let $k = 2k_1 = 2m$ and $k + \ell = 2(k_1 + \ell_1) = 2M$ be even numbers. When condition (3.18) is satisfied, equation (3.17) has exactly ℓ_1 roots in the left half plane.

<u>Proof</u>: Let

$$Q_\tau^{(k,\ell)}(t) = \sum_{j=0}^{\ell_1} a_{2(k_1+j)} t^{2j} + \tau \sum_{j=0}^{\ell_1 - 1} a_{2(k_1+j)+1} t^{2j+1}$$

On account of (3.18) on the imaginary axis

$$(-1)^{k_1} \ \mathrm{Re.} \ Q_\tau^{(k,\ell)}(i\xi)$$

$$= (-1)^{k_1} \mathrm{Re.} \left[\sum_{j=0}^{\ell_1} a_{2(k_1+j)} (i\xi)^{2j} + \tau \sum_{j=0}^{\ell_1 - 1} a_{2(k_1+j)+1} (i\xi)^{2j+1} \right]$$

$$= \sum_{i=k_1}^{k_1+\ell_1} (-1)^i a_{2i} \xi^{2(i-k_1)} \geqslant c^2(1 + \xi^{2\ell_1}) > 0 \ , a_j\text{'s}$$

and τ are real. Thus for any real τ the equation $Q_\tau^{(k,\ell)}(t) = 0$ has no roots on the imaginary axis. Therefore if τ varies from 0 to 1, the roots of $Q_\tau^{(k,\ell)}(t) = 0$, which vary continuously, do not meet the imaginary axis. Now since the coefficient $a_{2(k_1+\ell_1)}$ in the term of the highest order is fixed and different from zero, the roots $Q_\tau^{(k,\ell)}(t) = 0$ do not become infinite when τ varies. It follows that the number of roots of $Q_\tau(t) = 0$ lying in the left half plane does not change with variation in τ. When $\tau = 0$, $Q_\tau^{(k,\ell)}(t) = 0$ becomes,

$$Q_0^{(k,\ell)}(t) = \sum_{j=0}^{\ell_1} a_{2(k_1+j)} t^{2j} = 0$$

and it has ℓ_1 pairs of roots $(\lambda_i, -\lambda_i)$. Therefore it has exactly ℓ_1 roots in the left half plane for arbitrary τ.

Hence if $\tau = 1$ then by [6] the lemma is proved.

In the same fashion the corresponding lemmas for the different value of k and ℓ can be carried out.

Lemma 3.4 If $k = 2k_1 + 1$ is odd, $k + \ell = 2(k_1 + \ell_1)$ is even and condition (3.18) is satisfied, then the equation (3.17) has in the left half-plane ℓ_1 roots when $(-1)^{k_1} a_{2k_1+1} < 0$ and $\ell_1 - 1$ roots when $(-1)^{k_1} a_{2k_1+1} > 0$.

Lemma 3.5 Let $k = 2k_1$ be even and $\ell = 2\ell_1 + 1$ be odd. When condition (3.18) is satisfied, equation (3.17) has in the left half-plane $\ell_1 + 1$ roots for

$(-1)^{k_1+\ell_1} a_{2(k_1+\ell_1)+1} > 0$ and ℓ_1 roots for $(-1)^{k_1+\ell_1} a_{2(k_1+\ell_1)+1} < 0$

Lemma 3.6 If $k = 2k_1 + 1$ and $k + \ell = 2(k_1 + \ell_1) + 1$ are odd and (3.18) is satisfied, then equation (3.17) has roots lying in the left half-plane equal in number to

(a) ℓ_1 for $(-1)^{k_1} a_{2k_1+1} > 0$, $(-1)^{k_1+\ell_1} a_{2(k_1+\ell_1)+1} > 0$,

(b) ℓ_1 for $(-1)^{k_1} a_{2k_1+1} < 0$, $(-1)^{k_1+\ell_1} a_{2(k_1+\ell_1)+1} < 0$,

(c) $\ell_1 - 1$ for $(-1)^{k_1} a_{2k_1+1} > 0$, $(-1)^{k_1+\ell_1} a_{2(k_1+\ell_1)+1} < 0$,

(d) $\ell_1 + 1$ for $(-1)^{k_1} a_{2k_1+1} < 0$, $(-1)^{k_1+\ell_1} a_{2(k_1+\ell_1)+1} > 0$,

In theorem (3.1) we have shown that the positiveness of the characteristic form of the operator L^1_ϵ and the positiveness of L_o guarantee the uniform positiveness of the operator L_ϵ and, consequently, the uniform solvability of problem A_ϵ .

Now we shall define the boundary conditions of the reduced problem in the same way as we have defined in the case of problem A_ϵ , to make the operator L_o positive.

When the order $k = 2k_1$ of the operator is even, then

$$y^{(s)}(0) = y^{(s)}(1) = 0, \qquad s = 0, 1, \ldots, k_1-1 \tag{3.19}$$

and for $k = 2k_1+1$ the boundary conditions at $x = 0$ and $x = 1$ depend on the sign of the coefficient $(-1)^{k_1} a_{2k_1+1}(x)$ i.e.,

for $(-1)^{k_1} a_{2k_1+1}(x) > 0$,

$$\left.\begin{aligned} &y^{(r)}(0) = 0, \qquad r = 0, 1, \ldots, k_1 \\ &y^{(s)}(1) = 0, \qquad s = 0, 1, \ldots, k_1-1 \end{aligned}\right\} \tag{3.20}$$

and for $(-1)^{k_1} a_{2k_1+1}(x) < 0$

$$\left.\begin{aligned} &y^{(r)}(0) = 0, \qquad r = 0, 1, \ldots, k_1-1 \\ &y^{(s)}(1) = 0, \qquad s = 0, 1, \ldots, k_1 \end{aligned}\right\} \tag{3.21}$$

Now we shall study the asymptotic behaviour of $y(x)$, the solution of problem A_ϵ . According to the chapter 2 if the perturbation of A_ϵ to A_o is singular then there is a representation in theorem 2.2 of the solution $y(x)$ containing a boundary layer. We shall prove now that the conditions of theorem 3.1 and above boundary conditions of problem A_o imply that the perturbation of problem A_ϵ to problem A_o is singular.

Theorem 3.4 [4]. Under the conditions of theorem 3.1 (3.2, 3.3), if for the operator L_o there are assigned the boundary conditions 3.19 (3.20, 3.21), then the problem A_ϵ is solvable, it singularly perturbs to problem A_o, and its solution $y(x)$ has the asymptotic behaviour given by formula in the theorem 2.2.

Proof: We shall prove it only for the case when L_ϵ and L_o are of even order, i.e., $k = 2k_1$, $k + \ell = 2(k_1 + \ell_1)$. On passage from problem A_ϵ to problem A_o, from conditions (3.4) to (3.19), exactly ℓ_1 conditions are lost at each end points $x = 0$ and $x = 1$. Now the theorem will be proved if the auxiliary characteristic equation has exactly ℓ_1 roots with negative real parts for both end points. The equations are

$$Q_o(\lambda) = \sum_{j=0}^{2\ell_1} a_{2k_1+j,0}\,\lambda^j = 0, \quad a_{s,0} = a_j(x)|_{x=0},$$

$$Q_1(\lambda) = \sum_{j=0}^{2\ell_1} (-1)^{2k_1+j} a_{2k_1+j,1}\,\lambda^j = 0, \quad a_{r,0} = a_r(x)|_{x=1}.$$

Since by assumption of theorem 3.1

$$\widetilde{\Pi}_\epsilon(\zeta\,;\,x) = \mathrm{Re.}\ \Pi^1_\epsilon(\xi\,;x) = \sum_{j=1}^{\ell_1} (-1)^{j+k_1} \epsilon^{2j} a_{2(k_1+j)}(x)\,\xi^{2(k_1+j)}$$

$$\geqslant c_1^2\,(\epsilon^{2\ell_1}\ \xi^{2(k_1+\ell_1)} + \epsilon^2\,\xi^{2(k_1+1)})$$

and by the fact that $(-1)^{k_1} a_{2k_1}(x) > 0$, we have

$$\mathrm{Re}\ \Pi_\epsilon(\xi\,;x) = (-1)^{k_1} a_{2k_1}(x)\ \xi^{2k_1} + \widetilde{\Pi}_\epsilon(\xi\,;x) \geqslant c^2(\xi^{2k_1} + \epsilon^{2\ell_1}\xi^{2(k_1+\ell_1)})$$

Multiplying by ϵ^{2k_1} and setting $\epsilon\,\xi = \eta$, we get

$$\sum_{r=0}^{\ell_1} (-1)^{k_1+r} a_{2(k_1+r)}(x)\,\eta^{2(k_1+r)} \geqslant c^2(\eta^{2k_1} + \eta^{2(k+\ell_1)})$$

which gives for x = 0 the inequality

$$\sum_{r=0}^{\ell_1} (-1)^{k_1+r} a_{2(k_1+r),0}\,\eta^{2(k_1+r)} \geqslant c^2(\eta^{2k_1} + \eta^{2(k_1+\ell_1)})$$

which proves the fact with the help of Lemma 3.3 that the auxiliary characteristic equation has for each of the ends exactly ℓ_1 roots with negative real parts. So we have the singular perturbation and hence the theorem is proved. Similarly it can be proved for the odd and even orders of L_ϵ and L_o operator.

CHAPTER 4

INITIAL VALUE PROBLEMS FOR NONLINEAR EQUATIONS

The Chapter is concerned with the asymptotic behaviour of the solution of nonlinear initial value problems. More precisely, we shall consider the system of differential equations of the type

$$\left.\begin{aligned} \frac{dx}{dt} &= f(x, y, t) \\ \epsilon \frac{dy}{dt} &= g(x, y, t) \end{aligned}\right\} \tag{4.1}$$

together with the initial conditions

$$z\big|_{t = t^{o}} = z^{o} \tag{4.2}$$

We shall denote the variables x and y by z.
ϵ is small positive parameter, x is an m-dimensional vector and y an M-dimensional vector. We shall denote by $z(t, \epsilon)$ the solution of (4.1) satisfying (4.2).

First we shall introduce some definitions.

<u>Reduced system of equations</u> Let $y = \phi(x, t)$ be one of the roots of the system of equations $g(x, y, t) = 0$ defined on a bounded closed set D. The system of equations

$$\left.\begin{aligned} &y = \phi(x, t), \\ \text{and} \quad & \\ &\frac{dx}{dt} = f(x,y, t) \end{aligned}\right\} \tag{4.3}$$

will be called the reduced system of equations corresponding to the root $y = \phi(x, t)$.

We shall denote the solution of this equation satisfying the initial condition $x|_{t = t^o} = x^o$ by $\bar{z}(t)$.

Isolated root We shall call the root $y = \phi(x, t)$ isolated on the set D if there exists an $\mu > 0$ such that the system $g(x, y, t) = 0$ has no solution other than $\phi(x, t)$ for $|y - \phi(x, t)| < \mu$.

Boundary layer system The system of differential equations

$$\frac{dy}{d\tau} = g(x, y, t) \tag{4.4}$$

in which x and t are parameters, will be called the boundary layer system of the equations (4.1).

Positively stable root The isolated root $y = \phi(x, t)$ will be called positively stable in D if, for all points (x^*, t^*) belonging to D, the points $y = \phi(x^*, t^*)$, are asymptotically stable stationary points, in the sense of Lyapunov[8], of the boundary layer system (4.4), as $\tau \to \infty$.

If the same situation holds as $\tau \to -\infty$, then the root will be called negatively stable.

Domain of Influence. The domain of influence of an isolated positively stable root $y = \phi(x, t)$ is the set of points (x^*, y^*, t^*) such that the solution of (4.4) with $y|_{\tau = 0} = y^*$ tends to the value $\phi(x^*, t^*)$, as $\tau \rightarrow \infty$. In the same way we may define the domain of influence of a negatively stable root.

<u>Uniform asymptotically stability</u> A stationary point $y = \phi(x^*, t^*)$ of the system (4.4) is called uniformly asymptotically stable with respect to the domain of variation D of x^*, t^* if, for any $\mu > 0$, there exists a $\delta(\mu)$ such that, for all (x^*, t^*) belonging to D, the inequality

$$|y(\tau) - \phi(x^*, t^*)| < \mu \tag{4.5}$$

holds, provided

$$|y(\tau_0) - \phi(x^*, t^*)| < \delta$$

and, moreover, the passage to the limit

$$\lim_{\tau \to \infty} y(\tau) = \phi(x^*, t^*)$$

is uniform in the set D.

<u>Theorem 4.1</u> [7] If some root $y = \phi(x, t)$ of the system $g(x, y, t) = 0$ is an isolated positively stable root in some bounded closed domain D, if the initial point (x^o, y^o, t^o) belongs to the domain of influence of this root, and if the solution $x = \bar{x}(t)$ of the reduced system (4.3) belongs to D for $t^o \leq t \leq T$, then the solution $z(t, \epsilon)$ of the original system (4.1) tends to the solution $\bar{z}(t)$ of the reduced system (4.3), as $\epsilon \rightarrow 0$, the passage to the limit

$$\lim_{\epsilon \to 0} y(t, \epsilon) = \bar{y}(t) = \phi(\bar{x}(t), t)$$

holding for $t^o < t \leq T^o < T$, and the passage to the limit

$$\lim_{\epsilon \to 0} x\,(t, \epsilon) = \bar{x}(t)$$

for $t^o \leq t \leq T^o < T$.

<u>Proof</u>: Let us fix an arbitrary small number $\mu_1 > 0$, then by the property of uniformly asymptotic stability, we may define $\delta(\frac{\mu_1}{2})$, allowing us to write $\frac{\mu_1}{2}$ on the right hand side of (4.5).

Consider the boundary layer system (4.4) reffered to the initial point, and its solution $y_o(\tau)$ defined by the condition $y_o \big|_{\tau = 0} = y^o$. Since by hypothesis the initial point belongs to the domain of influence of the stable root $y = \phi(x, t)$, therefore for any given μ, there exists a $\tau_o(\mu)$ such that, for $\tau \geqslant \tau_o$,

$$|y_o(\tau) - \phi(x^o, t^o)| < \mu \tag{4.6}$$

The stretching transformation

$$\tau = \frac{t - t^o}{\epsilon}$$

takes the system (4.1) and (4.2) into

$$\frac{dx}{d\tau} = \epsilon\, f(x, y, t^o + \tau\epsilon),$$

$$\frac{dy}{d\tau} = g(x, y, t^o + \tau\epsilon),$$

$$z \big|_{\tau = 0} = z^o$$

The system of equations contains the parameter ϵ on the right hand side; therefore by virtue of a standard theorem concerning the continuous dependence of the solution of a differential equation on the initial values and on parameters [9], there exists a $\epsilon^{0}(\mu)$ for any $\mu > 0$ such that, for $\epsilon \leq \epsilon^{0}$,

$$|y(t, \epsilon) - y_{0}(\tau)| < \mu, \tag{4.7}$$

$$|x(t, \epsilon) - x^{0}| < \mu, \tag{4.8}$$

if $0 \leq \tau \leq \tau_{0}$, where τ_0 is as large as we wish, but fixed. In addition, we may assume that, for $\epsilon \leq \epsilon^{0}$,

$$|t - t^{0}| < \mu \tag{4.9}$$

If we choose $\mu = \mu(\mu_1)$ appropriately in all these inequalities, then from (4.7), (4.6), (4.8) and (4.9)

$$|y(t, \epsilon) - \phi(x(t, \epsilon), t)| < \delta\left(\frac{\mu_1}{2}\right) \tag{4.10}$$

where $t = t_1 = t^{0} + \tau_0 \epsilon$, $\epsilon \leq \epsilon^{0}$.

Therefore at the point $t = t_1$, we may assert that

$$\Delta(t, \epsilon) = |y(t, \epsilon) - \phi(x(t, \epsilon), t)| < \mu_1 \quad \left(\delta\left(\frac{\mu_1}{2}\right) < \mu_1\right).$$

Now we shall show that this inequality is preserved when the projection of the integral curve under consideration into the space (x, t) belongs to D, i.e. $(x(t, \epsilon), t)$ belongs to D. Suppose that the equality occurs at the point $t_2 > t_1$; in general, t_2 is a function of ϵ, as is t_1. Let us take a sequence $\epsilon_n \to 0$.

So to every ϵ_n there corresponds a value t_{2n} for which

$$|y(t_{2n}, \epsilon_n) - \phi(x(t_{2n}, \epsilon_n), t_{2n})| = \mu_1$$

Also, by (4.10)

$$|y(t_{1n}, \epsilon_n) - \phi(x(t_{1n}, \epsilon_n), t_{1n})| < \frac{\mu_1}{2}.$$

Clearly we can find a value t_n such that $t_{1n} < t_n < t_{2n}$ and

$$|y(t_n, \epsilon_n) - \phi(x(t_n, \epsilon_n), t_n)| = \delta\left(\frac{\mu_1}{2}\right) \tag{4.11}$$

Let t_n be the last value of t beyond which the trajectory $y(t, \epsilon_n)$ no longer returns to the $\delta\left(\frac{\mu_1}{2}\right)$-neighborhood of ϕ, but falls on the boundary of the μ_1-neighborhood of ϕ, so that for $t_n < t < t_{2n}$, we have

$$\delta\left(\frac{\mu_1}{2}\right) < |y(t, \epsilon_n) - \phi(x(t, \epsilon_n), t)| < \mu_1 \tag{4.12}$$

The sequence of points

$$x(t_n, \epsilon_n), y(t_n, \epsilon_n), t_n$$

has a limit point $(x^{(0)}, y^{(0)}, t^{(0)})$ by virtue of (4.11) and the fact that $(x(t_n, \epsilon_n), t_n)$ belongs to D, so we can select a subsequence converging to this limit point. To avoid fresh notations let us suppose that $x(t_n, \epsilon_n), y(t_n, \epsilon_n), t_n$ be such a subsequence.

Consider the boundary layer system

$$\frac{dy}{d\tau} = g(x^{(0)}, y, t^{(0)})$$

$$y \big|_{\tau = 0} = y^{(0)}$$

since

$$\left| y(0) - \phi(x^{(0)}, t^{(0)}) \right| = \delta\left(\frac{\mu_1}{2}\right),$$

so for $\tau \geqslant 0$

$$\left| y(\tau) - \phi(x^{(0)}, t^{(0)}) \right| < \frac{\mu_1}{2}, \tag{4.13}$$

while for $\tau \geqslant \tau_o(\mu_2)$,

$$\left| y(\tau) - \phi(x^{(0)}, t^{(0)}) \right| < \mu_2 . \tag{4.14}$$

where μ_2 is any given positive small number.

Now stretching the system (4.1) by the transformation

$\tau = \dfrac{t - t_n}{\epsilon_n}$, we get

$$\frac{dx}{d\tau} = \epsilon_n \, f(x, y, t_n + \epsilon_n \tau)$$

$$\frac{dy}{d\tau} = g(x, y, t_n + \epsilon_n \tau)$$

$$z \big|_{\tau = 0} = z(t_n ; \epsilon_n)$$

Because of the continuous dependence of the solution of this system on ϵ for $0 \leqq \tau \leqslant \tau_o$, where τ_o is as large as we please but fixed. So by using (4.13), we see that for $n > N(\mu_1)$, i.e. for sufficiently small ϵ ,

$$|y(t, \epsilon_n) - \phi(x(t, \epsilon_n), t)| < \frac{3}{4}\mu_1 \qquad (4.15)$$

where $t_n \leqslant t \leqslant t^{(n)} = t_n + \tau_0 \epsilon_n$. Now using (4.14) and choosing $\mu_2 = \mu_2(\mu_1)$ appropriately, we find that, for $t = t^{(n)}$,

$$|y(t^{(n)}, \epsilon_n) - \phi(x(t^n, \epsilon_n), t^{(n)})| < \delta(\frac{\mu_1}{2}) \qquad (4.16)$$

The inequality (4.15) tells us that, in the interval from t_n to $t^{(n)}$, the trajectory we are considering has not yet reached the boundary of the μ_1-neighborhood of $\phi(x(t, \epsilon_n), t)$, i.e. $t^{(n)}$ satisfies the inequality $t_n < t^{(n)} < t_{2n}$ and, therefore, the inequality (4.12) must hold for $t^{(n)}$, which contradicts (4.16). So we have proved by contradiction that the inequality

$$\Delta(t, \epsilon) < \mu_1 \quad (\delta(\frac{u_1}{2}) < \mu_1)$$

is preserved.

Thus, we may assume that in the domain D

$$\frac{dx}{dt} = f(x(t, \epsilon), \phi(x(t, \epsilon), t), t) \qquad (4.17)$$

with $|\Delta(t, \epsilon)| < \mu_1$. In addition, by (4.8) and (4.9)

$$\left.\begin{array}{l} |x(t, \epsilon) - x^0| < \mu(\mu_1) \\ |t - t^0| < \mu(\mu_1) \end{array}\right\} \qquad (4.18)$$

Now we assume that the solution of the system (4.17) depends continuously on any variation in the right hand sides and in the initial point. Then for any μ_3, we can choose an $\mu_1(\mu_3)$ and so, in the last analysis, a $\epsilon_0(\mu_3)$ such that

$$|x(t, \epsilon) - \overline{x}(t)| < \mu_3, \quad (x(t, \epsilon), t) \in D, \quad (4.19)$$

Since the unperturbed system corresponding to (4.17) and (4.18) is just the system defining $\overline{x}(t)$. By hypothesis, the curve $x = \overline{x}(t)$ belongs to D for $t^0 \leq t \leq T$; but then, since μ_3 is arbitrarily small, we may deduce that $x = x(t, \epsilon)$ belongs to D in the interval $t^0 \leq t \leq T^0$, where T^0 is as close to T as we please, but fixed as $\epsilon \to 0$. If we choose $\mu_3 = \mu_3(\mu_4)$,

$$|y(t, \epsilon) - \phi(\overline{x}(t), t)| < \mu_4 \quad \text{for } t_1 \leq t \leq T^0$$

The last two inequalities prove the theorem.

Series expansion for the initial value problem

Now we shall construct the asymptotic expansion of the solution $z(t, \epsilon)$ of the system (4.1) in terms of the small parameter ϵ. Theorem 4.1 is only the first step in the asymptotic solution of initial value problems of the singular perturbation type. The actual approximate solution of such problems in series has been analyzed in a series of papers [10], [11], [12] and [13]. An entirely different approach is taken in [14] and [15]. Here we are giving an account of the important part of Miss Vasileva's theory. The nature of the asymptotic expansion depends essentially on whether or not the stationary point $\phi(x, t)$ is stable. Thus, we may assume that the real parts of the roots of the characteristic equation

$$\text{Det.} \left\| \frac{\partial g}{\partial y}(x, \phi(x, t), t) - \lambda I \right\| = 0$$

are negative in D.

Here $\frac{\partial g}{\partial y}$ is a matrix and I is an unit matrix.

We attempt a solution of (4.1) in the form of a series in powers of

$$z = \bar{z}_0(t) + \epsilon\, \bar{z}_1(t) + \cdots\cdots + \epsilon^n\, \bar{z}_n(t) + \cdots$$

$$= \sum_{r=0}^{\infty} \bar{z}_r(t)\, \epsilon^r \tag{4.20}$$

The system defining $\bar{z}_0(t)$ is of the form

$$\left.\begin{aligned} & g(\bar{x}_0, \bar{y}_0, t) = 0 \\ & \frac{d\bar{x}_0}{dt} = f(\bar{x}_0, \bar{y}_0, t) \end{aligned}\right\} \tag{4.21}$$

While the system defining $\bar{z}_1(t)$ is of the form

$$\left.\begin{aligned} \frac{d\bar{y}_0}{dt} &= \bar{x}_1 \frac{\partial \bar{g}}{\partial x_0} + \bar{y}_1 \frac{\partial \bar{g}}{\partial y_0} \\ \frac{d\bar{x}_1}{dt} &= \bar{x}_1 \frac{\partial \bar{f}}{\partial x_0} + \bar{y}_1 \frac{\partial \bar{f}}{\partial y_0} \end{aligned}\right\} \tag{4.22}$$

The subscript 0 and the - above mean that the function has arguements $\bar{x}_0$, $\bar{y}_0$, t; e.g.

$$\frac{\partial \bar{g}}{\partial x_0} = \frac{\partial g}{\partial x}(\bar{x}_0, \bar{y}_0, t)$$

The system defining the $z_k(t)$ are linear for all $k \geqslant 1$ with nonhomogeneous terms depending on the $z_i(t)$ $(i < k)$. To determine all these functions in succession we must specify the initial conditions, but since the first M equations in the systems (4.21), (4.22), etc. are not differential equations, we need only specify initial conditions for functions of type x.

For values of t that are small of order $O(\epsilon)$, the solution to our perturbation problem can be found by means of the stretching transformation

$$\tau = \frac{t - t^o}{\epsilon}$$

The stretched form of our problem is then

$$\frac{dx}{d\tau} = \epsilon f(x, y, t^o + \epsilon\tau)$$

$$\frac{dy}{d\tau} = g(x, y, t^o + \epsilon\tau)$$

If the functions f and g are analytic the solutions of this problem have convergent expansions in powers of ϵ, say,

$$z = \sum_{r=0}^{\infty} z_r(\tau)\,\epsilon^r \tag{4.23}$$

Here, also $z_o(\tau)$ and $z_1(\tau)$ are determined by the respective systems

$$\left.\begin{aligned} \frac{dx_o}{d\tau} &= 0 , \\ \frac{dy_o}{d\tau} &= g(x_o, y_o, t^o) \end{aligned}\right\} \tag{4.24}$$

and

$$\left.\begin{aligned} \frac{dx_1}{d\tau} &= f_o \\ \frac{dy_1}{d\tau} &= x_1 \frac{\partial g}{\partial x_o} + y_1 \frac{\partial g}{\partial y_o} + \tau \frac{\partial g}{\partial t^o} \end{aligned}\right\} \qquad (4.25)$$

In general, the $z_k(\tau)$ $(k \geqslant 1)$ satisfy a linear system of equations with non-homogeneous terms depending on the $z_i(\tau)$ $(i < k)$. To determine the solution of the systems (4.24), (4.25) etc. in succession we must specify initial conditions. Let the conditions be of this term

$$z_o \big|_{\tau = 0} = z^o ,$$

$$z_k \big|_{\tau = 0} = 0 \qquad (k > 0)$$

The fact that the initial values vanish for $k > 0$ is connected with the fact that z^o does not depend on ϵ.

Also we construct an expansion of (4.1) in the variables ϵ and $t - t^o$

$$z = z_{00} + (t - t^o)z_{1o} + \epsilon z_{01} + \dots\dots\dots$$
$$\dots + (t - t^o)^n z_{n0} + (t - t^o)^{n-1} \epsilon z_{n-1,1} + \dots + \epsilon^n z_{0n} + \dots \qquad (4.26)$$

which is obtained by expanding all coefficients in (4.20) in powers of $(t - t^o)$

$$\overline{z}(t) = z_{0k} + (t - t^o)z_{1k} + (t - t^o)^2 z_{2k} + \dots\dots$$

Here z_{ik} are constants.

We give the initial conditions defining $\bar{z}_k$ in the form

$$\bar{x}_0 \big|_{t = t^0} = x^0 , \tag{4.27'}$$

$$\bar{x}_k \big|_{t = t^0} = \int_0^{\infty} \left[f_{k-1}(\tau) - \tilde{f}_{k-1}(\tau) \right] d\tau , \tag{4.27''}$$

where $f_{k-1}(\tau)$ is the $(k - 1)$-st coefficient in the exposition of $f(x, y, t)$ of the type (4.23)

$$f(x_0 + \epsilon x_1 + \ldots, y_0 + \epsilon y_1 + \ldots, \epsilon\tau) = f_0 + \epsilon f_1 + \ldots .$$

while

$$\tilde{f}_{k-1}(\tau) = \sum_{i=0}^{k-1} \tau^i f_{i, k-1-i},$$

$f_{i,j}$ being the coefficient in the expansion of the same function $f(x, y, t)$ of the type (4.26).

It is clear that the data (4.27') determine the zero order terms in the expansion (4.20) and also the coefficients in (4.26) allow us to give the initial values $\bar{x}_1$ and, therefore, to determine $\bar{z}_1$, from which we find all the coefficients in the expansion (4.26) with 1 for their second index etc. Later we shall see that this choice of initial conditions for the determination of the $\bar{z}_k$ will make $\bar{z}_k$, the limiting values of the derivatives of the solution with respect to the parameter ϵ .

We shall show now that the required asymptotic formula [16] is

$$Z_n = (z)_n + (\bar{z})_n - (\tilde{z})_n \tag{4.28}$$

where $(z)_n$, $(\bar{z})_n$ and $(\tilde{z})_n$ are respectively the partial sums of the series (4.23), (4.20) and the double series (4.26) containing terms of order up to n.

If we suppose, in addition to the conditions of Theorem 4.1, that the right hand sides of (4.1) have continuous partial derivatives of order up to n + 2 inclusive in an arbitrary small neighborhood, fixed as $\epsilon \to 0$, of the limiting curve of the integral curve of the system (4.1), which consists of two parts

(a) $t = t^o$, $x = x^o$, $y = y_o(\tau)$ $\quad (0 \leq \tau < \infty)$,

(b) $t^o < t \leq T^o$, $x = \bar{x}(t)$, $y = \bar{y}(t) \equiv \phi(\bar{x}(t), t)$,

then the following theorem ensures the validity of the asymptotic formula (4.28).

<u>Theorem 4.2</u> The inequality $| z(t, \epsilon) - Z_n | < c \, \epsilon^{n+1}$ holds for the solution $z(t, \epsilon)$ of the system (4.1) which satisfies the initial conditions (4.2), where c is a constant independent of t and ϵ for sufficiently small ϵ ($\epsilon \leq \epsilon^o$) and for $t^o \leq t \leq T^o$.

First we state two propositions which we will use for the proof of Theorem 4.2.

Without loss of generality we may take $t^o = 0$.

<u>Proposition A</u>.

Consider the linear system of equations

$$\epsilon \frac{d\xi}{dt} = A(t, \epsilon)\, \xi + P,$$

where $A(t, \epsilon)$ is a matrix which is continuous in t and ϵ and for which the roots of the characteristic equation Det. $\|A - \lambda I\| = 0$ have negative real parts for $\epsilon \leq \epsilon^0$ and $0 \leq t \leq T^0$.

The solution $\xi(t, \epsilon)$ of this system satisfies the inequality

$$|\xi| < c\,|\xi^0| \exp. \left(-\frac{kt}{\epsilon}\right) + \int_0^t \frac{c}{\epsilon} \exp\left(k\left(\frac{t - t_1}{\epsilon}\right)\right) |P|\, dt_1,$$

where $\xi|_{t=0} = \xi^0$, k and c are sufficiently small and large constants, respectively, independent of t and ϵ for $\epsilon \leq \epsilon^0$ and $0 \leq t \leq T^0$.

<u>Proposition B</u>.

Consider the system of equations

$$\frac{d\xi}{dt} = A\,\xi + B\,\eta + P,$$

$$\frac{d\eta}{dt} = a\,\xi + b\,\eta + Q,$$

where $A(t, \epsilon)$ satisfies the above condition. The other coefficients may depend (continuously) on t and ϵ.

The solution $\xi(t, \epsilon)$, $\eta(t, \epsilon)$ of this system satisfies the inequality

$$|\eta| < |\eta^0| + c\,\epsilon\,\xi^0 + \int_0^t (\max.\ |P|, Q)\, dt,$$

where $\xi|_{t=0} = \xi^{o}$, $\eta|_{t=0} = \eta^{o}$, k and c are constants independent of t and ϵ for $\epsilon < \epsilon^{o}$ and $0 \leq t \leq T^{o}$.

Before passing to the proof of the theorem, we state the following lemmas, the proofs of which can be seen in [17] .

Lemma 4.1. If ϵ is sufficiently small ($\epsilon \leq \epsilon^{o}$), then the following inequalities hold:

$$| z_n(\tau) | < c \tau^{n} \qquad (0 \leq t \leq T^{o})$$

where c is a constant independent of t and ϵ .

Lemma 4.2. The following inequalities hold:

$$| \Pi_n(z) | < c \exp\left(\frac{-kt}{\epsilon}\right) \qquad (0 \leq t \leq T^{o})$$

$$\left| \frac{d}{d\tau} \Pi_n(z) \right| < c \exp. (-k\tau).$$

where $\Pi_n(z) = z_n(\tau) - \tilde{z}_n(\tau)$ and $\overline{z}_n(\tau) = \sum_{i=0}^{n} \tau^{i} z_{i,n-i}$.

Lemma 4.3. The following inequalities hold in the interval $0 \leq t \leq -A \epsilon \log \epsilon$, where A is some sufficiently large constant which is fixed as $\epsilon \to 0$:

$$| Z_n - (z)_{n+1} | < c \epsilon^{n+1}$$

$$\left| \frac{d}{d\tau} (Z_n - (z)_{n+1}) \right| < c \epsilon^{n}$$

Proof of Theorem 4.2.

We shall use the notations

$$D_{n+1} = z - Z_n (\Delta_{n+1} = y - Y_n, \quad \xi_{n+1} = x - X_n),$$

$$\tilde{D}_{n+1} = z - (\overline{z})_n (\tilde{\Delta}_{n+1} = y - (\overline{y})_n, \quad \tilde{\xi}_{n+1} = x - (\overline{x})_n)$$

We divide the interval $0 \leq t \leq T^{o}$ into two parts; one is $0 \leq t \leq t_o = -A \epsilon \log \epsilon$, where A is some sufficiently large constant which is fixed as $\epsilon \to 0$, and the other part is $t_o \leqslant t \leqslant T^o$. Using Lemma 4.2, on the interval (t_o, T^o), the assertion of Theorem 4.2 becomes

$$|\widetilde{D}_{n+1}| < c \; \epsilon^{n+1} \tag{4.29}$$

Since $D_{n+1} = \widetilde{D}_{n+1} + \sum_{k=0}^{n} \epsilon^k \, \Pi_k(x)$.

So, we shall prove this inequality for the interval (t_o, T^o), while the inequality

$$| D_{n+1} | < c \; \epsilon^{n+1} \tag{4.30}$$

will be proved on the interval $(0, t_o)$. First we prove (4.30) on $(0, t_o)$. The equations satisfied by D_n are of the form

$$\begin{aligned} \epsilon \frac{d}{dt} \Delta_{n+1} &= \overset{*}{g}_x \; \delta_{n+1} + \overset{*}{g}_y \Delta_{n+1} + g(X_n, Y_n, t) - \epsilon \frac{dY_n}{dt} \\ \frac{d}{dt} \delta_{n+1} &= \overset{*}{f}_x \; \delta_{n+1} + \overset{*}{f}_y \Delta_{n+1} + f(X_n, Y_n, t) - \frac{dX_n}{dt} \end{aligned} \tag{4.31}$$

where * means that the values of the functions are taken at some intermediate point between z and Z_n.

By definition

$$\begin{aligned} \frac{dx_n}{d\tau} &= f_{n-1} = \text{Coefficient of } \epsilon^n \text{ in } f(x_o + \epsilon x_1 + \dots, y_o + \epsilon y_1 + \dots, \epsilon\tau) \\ \frac{dy_n}{d\tau} &= g_n = \text{Coefficient of } \epsilon^n \text{ in } g(x_o + \epsilon x_1 + \dots, y_o + \epsilon y_1 + \dots, \epsilon\tau) \end{aligned} \tag{4.32}$$

and by Lemma 4.3

$$g(X_n, Y_n, t) - \epsilon \frac{d}{dt} Y_n$$
$$= g\big((x)_{n+1}, (y)_{n+1}, t\big) - \frac{d}{dt} (y)_{n+1} + \epsilon^{n+1},$$

While by Lemma 4.1

$$g\big((x)_{n+1}, (y)_{n+1}, t\big) = g_{n+1} + (t^{n+2}) = g_{n+1} + (\epsilon^{n+1}) \qquad (4.33)$$

Since on the interval $(0, -A\epsilon \log \epsilon)$, $t^{n+2} < c\,\epsilon^{n+1}$.

After using these last four equations, we obtain

$$g(X_n, Y_n, t) - \frac{d}{dt} Y_n = (\epsilon^{n+1}) \qquad (4.34')$$

Now using Lemmas 4.3 and 4.2, we get,

$$f(X_n, Y_n, t) - \frac{dX_n}{dt} = f\big((x)_{n+1}, (y)_{n+1}, t\big) - \frac{dX_{n+1}}{dt} + \epsilon^{n+1} \frac{d\bar{x}_{n+1}}{dt}$$
$$+ \epsilon^{n+1} \frac{d}{dt} \pi_{n+1}(x) + (\epsilon^{n+1}) = f\big((x)_{n+1}, (y)_{n+1}, t\big)$$
$$- \frac{d}{dt} (x)_{n+2} + \epsilon^{n+1} \frac{d}{dt} \pi_{n+1}(x) + (\epsilon^{n+1})$$
$$= f\big((x)_{n+1}, (y)_{n+1}, t\big)$$
$$- \frac{d}{dt} (x)_{n+2} + (\epsilon^{n+1}) + \epsilon^{n} \exp.\left(- \frac{kt}{\epsilon} \right).$$

Also by using the analogous result (4.33) for f and (4.32), we obtain

$$f(X_n, Y_n, t) - \frac{dX_n}{dt} = (\epsilon^{n+1}) + \epsilon^n \exp.\left(\frac{-kt}{\epsilon} \right) \qquad (4.34'')$$

By virtue of (4.34), equation (4.31) takes the form

$$\epsilon \frac{d}{dt} \Delta_{n+1} = \overset{*}{g}_x \delta_{n+1} + \overset{*}{g}_y \Delta_{n+1} + (\epsilon^{n+1}), \qquad (4.35')$$

$$\frac{d}{dt} \delta_{n+1} = \overset{*}{f}_x \delta_{n+1} + \overset{*}{f}_y \Delta_{n+1} + (\epsilon^{n+1}) + (\epsilon^{n} \exp.(-\frac{kt}{\epsilon})) \qquad (4.35'')$$

Here Δ_{n+1} and δ_{n+1} are zero at $t = 0$.

Also using the proposition B, we get for (4.35)

$$|\delta_{n+1}| < \int_0^t \left[(\epsilon^{n+1}) + \epsilon^{n} \exp.\left(-\frac{kt}{\epsilon} \right) \right] dt < c\, \epsilon^{n+1},$$

after which, using proposition A for (4.35'), we get

$$\left|\Delta_{n+1}\right| < \int_0^t \frac{c}{\epsilon} \exp.\left(\frac{-k(t-t_1)}{\epsilon} \right) \left(\overset{*}{g}_x \delta_{n+1} + \epsilon^{n+1} \right) dt_1 < c\epsilon^{n+1}.$$

Hence $|D_{n+1}| = |z - Z_n| < c\epsilon^{n+1}$.

Now we consider the part (t_o, T^o) and prove (4.29). We know that

$$\left.\begin{aligned} \epsilon \frac{d}{dt} \tilde{\Delta}_{n+1} &= \overset{**}{g}_x \tilde{\delta}_{n+1} + \overset{**}{g}_y \Delta_{n+1} + g\left((\bar{x})_n, (\bar{y})_n, t\right) - \epsilon \frac{d}{dt}(\bar{y})_n \\ \frac{d}{dt} \tilde{\delta}_{n+1} &= \overset{**}{f}_x \tilde{\delta}_{n+1} + \overset{**}{f}_y \tilde{\Delta}_{n+1} + f\left((\bar{x})_n, (\bar{y})_n, t\right) - \frac{d}{dt}(\bar{x})_n \end{aligned}\right\} \quad (4.36')$$

where ** means that the values of the functions are taken at some intermediate point between z and $(\bar{z})_n$.

It follows from the definition of the quantities $\bar{z}_k$ that

$$g\big((\bar{x})_n,\ (\bar{y})_n,\ t\big) - \epsilon \frac{d}{dt} (\bar{y}_n) = (\epsilon^{n+1}) \quad ,$$

$$f\big((\bar{x})_n,\ (\bar{y})_n,\ t\big) - \frac{d}{dt} (\bar{x})_n = (\epsilon^{n+1}),$$

and with these equations (4.36') becomes

$$\left.\begin{aligned} \epsilon \frac{d}{dt} \tilde{\Delta}_{n+1} &= \overset{**}{g}_x \tilde{\delta}_{n+1} + \overset{**}{g}_y \tilde{\Delta}_{n+1} + (\epsilon^{n+1}) \\ \frac{d}{dt} \tilde{\delta}_{n+1} &= \overset{**}{f}_x \tilde{\delta}_{n+1} + \overset{**}{f}_y \tilde{\Delta}_{n+1} + (\epsilon^{n+1}) \end{aligned}\right\} \qquad (4.36'')$$

Here $\tilde{\Delta}_{n+1}\big|_{t = t_o} = (\epsilon^{n+1}),\ \tilde{\delta}_{n+1}\big|_{t = t_o} = (\epsilon^{n+1})$.

Using the propositions A and B, we obtain (4.29) from (4.36) in the same way as we obtained (4.30) from (4.35).

This completes the proof of Theorem 4.2.

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