Improvement of Inertia Effects in Slender-Body Theory

by

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Dedicated to the Memory of

my Teacher, Supervisor and Mentor

Late Professor Raymond G. Cox

who passed away on July 29, 1995

after a brief illness.

Abstract

This research develops an analytical method for predicting the hydrodynamic force experienced by a long slender solid body of arbitrary cross-sectional shape and body centreline configuration, subjected to an unbounded uniform fluid flow. It is assumed the slenderness parameter, κ (the ratio of the body cross-sectional length scale to the body length) is small ($\ll 1$), the body centreline radius of curvature is everywhere large (of order body length), the cross-sectional shape varies slowly along the body length, and the Reynolds number R_e, based on the body length is of order unity.

The inner flow solution for an arbitrary cross-section is illustrated by applying the complex variable method for a body with an elliptical cross-section, which is extendable to any cross-sectional shape. The results agree with those obtained by Batchelor (1970) with the use of the elliptic cylinder coordinates.

By the method of matched asymptotic expansion, the force per unit length the fluid exerts on the body, is obtained as a solution of an integral equation which is correct to order κ. By neglecting the inertia effects, and for special case of body with a circular cross-section, the force integral equation reduces to that obtained by Johnson (1980) (for uniform flow) with a completely different approach.

The iterative solution of the force integral equation is illustrated by applying it to a long straight cylindrical body, with an arbitrary cross-sectional shape, at rest in a fluid with uniform velocity. The first two terms of the expansion of the force in a power series of ($1/ln\kappa$) is explicitly determined and found to be in accordance with previous work [Khayat & Cox (1989)].

The exact solution of the force integral equation for certain symmetric bodies is illustrated by applying it to a slender torus of arbitrary cross-sectional shape, which is constant along the body centreline, settling along its axis in an unbounded fluid at rest. The radial and axial components of the force per unit length are explicitly determined up to an error term of $O(\kappa)$. It is found that, for a particular case of a torus with a circular cross-section, as well as in the limit as $R_e \longrightarrow 0$, the axial component of the force (the only non-zero one) reduces to the results obtained by Johnson & Wu (1979) and Johnson (1980).

The novelty of this research is the improvement of the approximation of the force per unit length in *slender body theory* when inertia effects are not negligibly small.

Condensé

Cette recherche développe une méthode analytique pour prédire les forces hydrodynamiques causées par un long solide mince, ayant une forme de coupe transversale et une configuration arbitraire, soumis à un écoulement fluide uniforme sans bornes. Nous supposons que le paramétre de minceur, K (le ratio de la coupe transversale du corps à sa longueur) est petit ($\ll 1$), que le rayon de courbature de la ligne centrale du corps est grand, que la forme de la coupe transversale varie lentement sur la longueur du corps, et que le nombre de Reynolds Re, basé sur la longueur du corps est d'ordre unitaire.

La solution de lécoulement interne pour une coupe transversale arbitraire peut être démontrée en applicant la méthode de variable complexe pour un corps de coupe transversale elliptique. Cette dernière s'applique à n'importe quelle forme ou coupe transversale. Les résultats obtenus vont de paire avec ceux de Batchelor (1970) avec l'aide d'un système de coordonnées cylindrique elliptique.

Par la méthode d'association d'expansion asymptotique, la force par unité de longueur que le fluide exerce sur le corps est obtenu comme solution d'équation intégrale qui est correcte à l'ordre k. En négligeant les effets d'inertie, ainsi que les cas spéciaux de corps ayant une coupe transversale circulaire, l'équation intégrale de force se réduit à celle obtenue par Johnson (1980) (pour écoulement uniforme) et ce, en utilisant une approche complètement différente.

La solution itérative de l'équation intégrale de force est illustrée en l'appliquant à un long corps cylindrique droit, qui est stationnaire dans un fluide à vélocité uniforme. Les 2 premiers termes de l'expansion de la force dans une série de puissance ($1/lnk$) sont explicitement décidés et sont en accord avec les travaux précédents [Khayat et Cox, (1989)].

La solution exacte de l'équation de force intégrale pour certains corps symmétriques est illustrée en l'appliquant à un mince torus dont la forme de la coupe transversale est arbitraire, qui est constante le long de l'axe central du corps, dans un fluide aux délimitations très éloignées. Les composantes radiales et axiales de la force par unité de longueur sont explicitement determinées jusqu'à un terme d'erreur de $O(k)$. Nous trouvons, pour un cas particulier de torus à coupe transversale circulaire, de même que dans la limite $R_e \longrightarrow 0$, que la composante axiale de la force (la seule dans la littérature) se réduit aux résultats obtenus par Johnson et Wu (1979) et Johnson (1980).

La nouveauté de cette recherche est l'amélioration de l'approximation de la force par unité de longueur dans la *théorie de corps minces* quand les effets de l'inertie sont pris en considérations.

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CHAPTER 1

1.1 - Introduction

The hydrodynamics of low and intermediate Reynolds number, R_e, flows has numerous applications in physical and biological science, medicine, and engineering, including the study of pulp, asbestos, and wool fibre transport in streams, fibrous filter solid-liquid separation, micro-organism locomotion, rheology of blood circulation, wastewater treatment suspension sedimentation, and equipment lubrication.

The typical value of Reynolds number for these flow phenomenon based on the length of the body, kinematic viscosity of the liquid and the velocity of propulsion is of order 10^{-6} to 10^{-2} for small microscopic organisms (for example 10^{-3} for the size of spermatozoa) and of order unity for the larger ones of the size of nematodes, or for the sand-size particles settling in the water (as for a specified example, $Re = 1.17$ based on the length of a red cell in ascending aorta vessel). The common feature of the particles involved in these phenomena is that they possess irregular shapes and in many cases a slender body shape. So it is desirable to obtain the exact solution for this type of flows.

For low R_e flows, even though the standard Navier-Stokes equation is simplified to the Stokes equations as a first approximation, obtaining the solution for arbitrary body shapes is still difficult, and hence not many exact or even approximate solutions are known, except for the simplest of body shapes.

Non-zero Reynolds number flows, because of the difficulty in dealing with non-linear inertia terms in the Navier-Stokes equation, are usually studied by numerical methods. However, it is of interest to investigate the flow around a class of bodies of irregular shape for which one may solve analytically the equations of motion including inertia effects.

The present study develops an effective method by which a number of exact solutions can be determined for various long slender particles subject to uniform flow incident upon the body. Although, it can easily be extended to the any prescribed flow (steady or unsteady) for flexible slender bodies, far from the scope of this study, the exact solution for naturally occurring particles is desirable and hence further investigations are required.

However, a brief discussion of the background of the problem is presented under the title of history. The general problem is described in detail in Chapter 2. Chapter 3 deals with the outer expansion. By using the solution of the Greens function for Oseen's equation the asymptotic form of both pressure and velocity for a general point of the outer region as it approaches the body centreline (singularity line), is determined. The inner flow field is presented in Chapter 4. By applying the complex variable method, the inner flow field in the vicinity of a general point of the body centreline for a slender body of an elliptical cross-section is analyzed in detail in 4.1 and 4.2. The problem is then generalized by adopting Batchelor's solution for a general cross-sectional shape in 4.3 and 4.4. In both cases the asymptotic solution of the inner expansion of both pressure and velocity fields is expressed in terms of outer variables, and then the inner body conditions for the outer flow fields are matched onto those obtained, in Chapter 3, for the outer region near the line singularity. From results of matched asymptotic expansion the force integral equation is derived in Chapter 5. Chapter 5 also includes the iterative solution of the force integral equation together with a determination of its integrand. Chapters 6 and 7 contain examples of a long straight slender body and a slender torus, respectively. Consequently the forces per unit length experienced by the bodies are obtained with the error term of $O(ln\kappa)^{-3}$ and $O(\kappa)$ for the former and latter, respectively.

Throughout this research we tried to present the material in a relatively self-contained way with relatively simple mathematical procedures, which contain most details of the calculations, and whenever it was felt needed, a figure accompanies the material. Figures are located at the end of each Chapter. Finally, appendix A contains the solution of the biharmonic equation which is employed extensively in Chapter 4.

1.2 - History

The problem of determination of the force on fixed bodies in a slow uniform flow of viscous incompressible fluid is an old one. Stokes (1851) was the first one who paid attention to it. The problem originally considered by Stokes was flow past a sphere, for which, by neglecting completely the inertia of the fluid, he obtained a solution as $F = 6\pi a\mu U$ (where F is the drag force, U free-stream velocity at infinity, μ the dynamic viscosity of the fluid and a the radius of the sphere)[1] which is the well-known Stokes drag formula. However, in the case of a circular cylinder Stokes equations failed to give any solution. The non-existence of a Stokes solution for any two dimensional body fixed in unbounded flow is usually referred to as *Stokes' paradox*.

Oberbeck (1876) considered a spheroid with semi-axis a and b with a measured along the symmetry axis, by neglecting the inertia effects too, leading to a value for the force F (also along the symmetry axis) of magnitude

$$F = 16\pi\mu bU\left\{-\frac{2\left(\frac{a}{b}\right)}{\left(\frac{a}{b}\right)^2-1}+\frac{2\left(\frac{a}{b}\right)^2-1}{\left[\left(\frac{a}{b}\right)^2-1\right]^{\frac{3}{2}}}\ln\frac{\left(\frac{a}{b}\right)+\left[\left(\frac{a}{b}\right)^2-1\right]^{\frac{1}{2}}}{\left(\frac{a}{b}\right)-\left[\left(\frac{a}{b}\right)^2-1\right]^{\frac{1}{2}}}\right\}^{-1}, \qquad (1.2.1)$$

which for small b/a (i.e. for slender spheroid), reduces to

$$F = 4\pi\mu aU\left[\frac{1}{\ln\left(2\frac{a}{b}\right)-0.5}+O\left(\frac{b}{a}\right)\right] \quad as \quad \left(\frac{b}{a}\right)\longrightarrow 0. \qquad (1.2.2)$$

Whitehead (1889) attempted to develop the Stokes solution for a sphere by obtaining higher order approximations to the flow when the Reynolds number is not negligibly small. The method proposed by Whitehead was an iterative procedure to take

[1] Throughout this section F, U, μ and ν denote the drag force, uniform velocity at infinity, the dynamic and the kinematic viscosity of the fluid, respectively, unless otherwise stated.

the inertia effects into account. As noted by Proudman & Pearson (1957)" the particular difficulty encountered by Whitehead was that the second approximation to the velocity of flow past a sphere remains finite at infinity in a way which is incompatible with the uniform-stream condition. ... This mathematical phenomenon appears to be common to all problems of uniform streaming past a bodies of finite length scale and is sometimes referred to as *Whitehead's paradox.* "

The paradox was resolved by Oseen (1910). Oseen pointed out its physical origin and a mathematical device for overcoming the associated difficulties with Whitehead's paradox. Oseen showed the determination of a uniformly valid first approximation to the velocity and all its derivatives is itself a linear problem which can be solved analytically and hence resulted the famous Oseen's equation. In contrast to Stokes equations, Oseen's equation provided a uniformly valid approximation to the velocity and all its derivatives in the two dimensional flow past an infinite cylinder of finite cross-sectional length scale. The first such solution to be obtained was that for an infinite circular cylinder placed perpendicular to the uniform flow (Lamp 1911), for which Lamp retained some of the "inertia terms" but omitted the others in his solution. Oseen himself gave a solution for flow past a sphere of radius a and an infinite circular cylinder of radius b, respectively, as

$$F = 6\pi\mu a U\left(1 + \frac{3}{8}R_a\right) \qquad (1.2.3)$$

and

$$F = \frac{4\pi\mu U}{2\ln 2 - \ln R_b - \gamma + \frac{1}{2}}, \qquad (1.2.4)$$

where R_a and R_b are the Reynolds number based on the characteristic length of a and b respectively, defined by

$$R_a = \frac{aU}{\nu} \qquad \text{and} \qquad R_b = \frac{bU}{\nu}, \qquad (1.2.5)$$

and where ν is the kinematic viscosity defined by $\nu = \mu/\rho$ (ρ being the density of the fluid) and γ is Euler's constant the value of which is

$$\gamma = 0.5772\cdots \tag{1.2.6}$$

Equation (1.2.3,4) are well-known as Oseen drag formulas for a sphere and an infinite circular cylinder, respectively.

Burgers (1938) attempted to obtain directly the formula (1.2.2), obtained from Oberbeck's drag formula, for a long slender ellipsoid of revolution. He assumed the disturbance produced by the ellipsoid was like that which would be produced by a line of force of magnitude

$$\left.\begin{array}{ll} f(z) = A_0 + A_2(z/a)^2 + A_4(z/a)^4 & \text{if } |z| < a, \\ f(z) = 0 & \text{otherwise}, \end{array}\right\} \tag{1.2.7}$$

acting along the symmetry axis, where z is the distance along the symmetry axis measured from the centre of the ellipsoid and A_0, A_2, A_4 are constants. By minimizing the mean value of velocity on the body surface, he obtained the force on the ellipsoid exactly the same as equation (1.2.2). Burgers also applied his method to determine the force acting on a circular cylinder of finite length fixed in a uniform stream flowing in the direction of its symmetry axis. For this case, he obtained the total force acting on the cylinder as

$$F = \frac{4\pi\mu a U}{\ln\dfrac{2a}{b} - 0.72}, \tag{1.2.8}$$

where a and b are respectively the semi-length and the cross-sectional radius of the cylinder.

It seems the problem had remained unnoticed for many years after Burgers work, until a paper by Lagerstorm & Cole appeared in the literature in 1955. Lagerstorm & Cole (1955) introduced Oseen and Stokes variables and obtained Oseen and Stokes expansions which followed naturally from the limit processes they adopted.

A well-known paper by Proudman & Pearson (1957) considered the problem in more details giving an intensive theoretical study of the subject. Proudman & Pearson

(1957) and also Kaplun & Lagerstorm (1957) demonstrated that it is possible to obtain higher order approximations to the flow past a sphere and a circular cylinder by applying the method of Stokes and Oseen expansions, the so-called *matched asymptotic expansion* technique. Proundman & Pearson (1957)'s studies led to improving the approximation of the drag force, obtained by Oseen, acting on a sphere and on an infinite circular cylinder, respectively, as

$$F = 6\pi\mu a U\left[1 + \frac{3}{8}R_a + \frac{9}{40}R_a^2 \ln R_a + O(R_a^2)\right] \tag{1.2.9}$$

and

$$F = -4\pi\mu U\left\{\left[\frac{1}{\ln R_b} + \left(\frac{1}{\ln R_b}\right)^2\left(\gamma + \frac{1}{2} - 2\ln 2\right)\right] + O\left[\left(\frac{1}{\ln R_b}\right)^3\right]\right\}. \tag{1.2.10}$$

Although, by the aid of the binomial theorem, equation (1.2.10) leads to the Oseen drag formula given by (1.2.4), Proudman & Pearson's method is capable of yielding higher order approximations.

Broersma (1960) improved the method used by Burgers (1938). He took the disturbance produced by the cylindrical body as being that due to a line of force of magnitude

$$\left.\begin{array}{ll} f(z) = B_0 + B_2(z/a)^2 + B_4(z/a)^4 + \cdots & \text{if } |z| < a, \\ f(z) = 0 & \text{otherwise,} \end{array}\right\} \tag{1.2.11}$$

where B_0, B_2, B_4 ... are an infinite set of constants to be determined. Broersma computed the values of these constants numerically for the case of a circular cylinder of finite semi-length (a) and cross-sectional radius b being fixed in a fluid with uniform velocity U flowing in the direction of the symmetry axis, and obtained the force on the cylinder as

$$F = \frac{4\pi\mu a U}{\ln\frac{2a}{b} - 0.81}, \tag{1.2.12}$$

which is a rather different result than that obtained by Burgers given by (1.2.8).

However, the slender-body theory, in more general form, has been revived and considerably developed by Tuck (1964, 1970): Taylor (1967,1969); Cox (1970, 1971): Tillet (1970), and Batchelor (1970). All these authors neglect the inertia effects completely, i.e, they assumed that $R_a = 0$ and $R_b = 0$ [where R_a and R_b are the Reynolds number based on the characteristic length of the body length (a) and cross-section (b), respectively, defined by (1.2.5)].

Tuck (1964) investigated the translational resistance force on slender bodies by the use of spheroidal coordinates. Taylor (1967) illustrated that if the Reynolds number is very small, a slender body of revolution falls twice as fast axially as it does transversely, though, he had made a mistake in the sign of a term in his solution which was pointed out by Tillet (1970). Taylor (1969) also gave a theoretical study of the problem. Consequently, he formulated approximate integral equations for the stokeslet distributions (force density) in the two cases of axial and transverse flow.

Cox (1970) considered a curved slender body of a circular cross-section with, length l and with the characteristic dimension of the body cross-section b, expanding the solution directly in powers of $1/ln\kappa$ (where κ is the slenderness parameter defined by $\kappa = b/l$). He obtained a solution for force per unit length, on the body, correct up to the order $(1/ln\kappa)^3$ as

$$\frac{F(s)}{2\pi} = \left[\frac{U - U^*}{\ln\kappa} + \frac{J + (U - U^*)\ln\frac{2\varepsilon}{\lambda}}{(\ln\kappa)^2}\right]\cdot\left[\frac{dR}{ds}\frac{dR}{ds} - 2I\right] + \frac{\frac{1}{2}(U - U^*)}{(\ln\kappa)^2}\cdot\left[3\frac{dR}{ds}\frac{dR}{ds} - 2I\right] + O\left[\frac{1}{(\ln\kappa)^3}\right], \qquad (1.2.13)$$

where s $(0 \leq s \leq +1)$ is the dimensionless arch length of the body centreline measured from one end of the body; $\boldsymbol{R}$ is dimensionless vector function of s, representing the body centreline relative to a fixed dimensionless coordinate system; here $\boldsymbol{U}$ is the fluid velocity function of position; $\boldsymbol{U}^*$ is the velocity of the body centreline; $\boldsymbol{I}$ is the idemfactor; $\lambda(s)$ is dimensionless function of s, representing the radius of the cross-section at the point under consideration (s); ε is an arbitrary constant much smaller than unity and where $\boldsymbol{J}$

is a vector given by

$$J_i(s) = \frac{1}{2}\left[\int_0^{s-\epsilon} + \int_{s+\epsilon}^{+1}\right]\left\{\frac{\delta_{ij}}{|R-\hat{R}|} + \frac{[R-\hat{R}]_i[R-\hat{R}]_j}{|R-\hat{R}|^3}\right\} \times \left[\delta_{jk} - \frac{1}{2}\frac{d\hat{R}_j}{d\hat{s}}\frac{d\hat{R}_k}{d\hat{s}}\right]\left[U_k(\hat{R}) - U_k^{*}(\hat{s})\right]d\hat{s}, \quad (1.2.14)$$

where $\hat{s}$ is the integration variable; $\hat{R}$ is the value of R at point $s = \hat{s}$. Cox (1970) applied his theory (1.2.13-14) to examples involving bodies having a curved centreline. He obtained the force experienced by a long slender body, with its centreline bent in an arc of a circle of radius a, fixed in a fluid undergoing a uniform velocity U (U lying in the plane contains the body centerline) as

$$F_1 = 2\pi\mu U\left\{\frac{A}{\ln\kappa} + \frac{B}{(\ln\kappa)^2} + O\left[\frac{1}{(\ln\kappa)^3}\right]\right\}, \quad (1.2.15)$$

where F_l is the component of the force per unit length acting on the body in the direction of velocity U and where

$$A = \sin^2\theta - 2 \quad (1.2.16)$$

and

$$B = \frac{1}{4}(\sin^2\theta - 2)\{2\ln[\tan\tfrac{1}{4}(\theta-\theta_0)\tan\tfrac{1}{4}(\theta_1-\theta)] - \sin\theta_0\sin\tfrac{1}{2}(\theta+\theta_0) - \sin\theta_1\sin\tfrac{1}{2}(\theta+\theta_1) + 12\ln 2 + 2 + 2\sin^2\theta - 4\ln\lambda\} - \frac{1}{4}\sin\theta\cos\theta\{\sin\theta_0\cos\tfrac{1}{2}(\theta+\theta_0) + \sin\theta_1\cos\tfrac{1}{2}(\theta+\theta_1) - \sin 2\theta\} + \sin^2\theta. \quad (1.2.17)$$

where θ_0 is the angle between the radius passing from one end of the body and the direction of the velocity U and where $(\theta_1 - \theta_0)$ is the angle of the sector, the arch of which is the body centreline, hence $\theta_0 \leq \theta \leq \theta_1$ [see also Cox (1970) p. 807]. He also obtained the component of the total force by integration of (1.2.15-17). For the special case of a slender torus of constant circular cross-section (i.e. $\lambda = 1$), his studies results in a total force, experienced by the torus, of magnitude

$$F = \frac{6\pi^2 \mu a U}{\frac{1}{3} - \ln[(\frac{1}{4})\pi] - \ln\kappa} + O\left[\frac{1}{(\ln\kappa)^3}\right]. \qquad (1.2.18)$$

Tillet (1970) used the integral equations obtained by Tuck (1970) for axial flow and solved them iteratively by the aid of the method suggested by Tuck (1964). Consequently, he obtained a few terms of the expansion of the force in powers of $(1/ln\varepsilon)$ (where ε is the slenderness parameter), in addition to the recurrence relation for determination of the higher order terms. Whilst in the case of transverse flow, he only obtained the drag force correct up to order $(ln\varepsilon)^{-3}$. In the special case of a spheroid his results agree with those obtained by Lamb (1932) (solving the Stokes equations by the method of separation of variables).

Batchelor (1970) adopted the slender body theory to a straight non-axisymmetric body. From an investigation of the local inner flow field in the vicinity of a section of the body, and the condition that it should join smoothly with the outer flow which is determined by the body as a whole, Batchelor found that a given shape and size of the local cross-section is equivalent, in all cases of transverse relative motion, to an ellipse of certain dimensions and orientation, and, in all cases of longitudinal relative motion, to a circle of certain radius. As noted by Batchelor (1970) and is illustrated in the present study (Chapter 4), the equivalent circle and the equivalent ellipse (characteristic tensor) of the cross-sectional shape may be found from certain boundary-value problems by solving the harmonic and biharmonic equations, respectively.

Chwang & Wu presented a series of theoretical studies concerned with low Reynolds number flow in general. They (1974) neglected inertia effects and considered a viscous flow generated by pure rotation of an axisymmetric body having an arbitrary prolate form. Consequently, they obtained exact solution in closed form for a number of body shapes. In another paper [Chwang & Wu (1975)], they considered the singularity method for Stokes flows and obtained exact solution also in closed form for prolate spheroids, spheres and circular cylinders. Chwang & Wu (1976) took inertia effects into account and obtained the drag force on a spheroid (with semi-major and minor axis a and b, respectively) placed perpendicular to the flow for arbitrary R_a and small R_b [where R_a and R_b are the Reynolds number based on the length a and b, respectively, defined by (1.2.5)]. For a small R_a their result reduces to the Oberbeck drag formula given by (1.2.1). When a tends to infinity (i.e. for slender spheroid) and hence R_a tends to infinity, their result leads to the Oseen drag formula for an infinite circular cylinder given by (1.2.4-6). However, as pointed out by Khayat & Cox (1989) their result, for intermediate R_a (order unity); $b/a \ll 1$ and $R_b \ll 1$, is not valid because in their solution they made the incorrect assumption that the drag force is independent of the position along the body axis and consequently, they performed the matching at only one point on the body axis (i.e. the centre of the body).

The slender body theory also received attention and was further developed by Keller & Rubinow (1976), Geer (1976), Johnson & Wu (1979), Jhonson (1980) and Khayat & Cox (1989).

Keller & Rubinow (1976), by neglecting inertia effects, analyzed a curved slender body of circular cross-section capable of translating, twisting, stretching and dilating. Using the method of matched asymptotic expansion, they obtained an integral equation to determine the force per unit length experienced by the body, where the first approximation to its solution agrees with Cox's result given by (1.2.13,14).

Geer (1976) completed the method proposed by Tuck (1964) and Tillet (1970). He considered the disturbance flow due the presence of a slender body of revolution in Stokes flow, being that of produced by the superposition of three types of singularity, with unknown densities, distributed inside and along the body axis. By applying the no slip boundary condition, he obtained three pairs of linear integral equations for the density of the singularities, by solution of which he determined the drag, total force and torque experienced by the body. However, instead of applying the usual procedure of the inner and outer expansions, he obtained a uniform expansion for the asymptotic solution of the integral equations.

Johnson & Wu (1979) considered the Stokes flow passing a slender torus of circular cross-section. By the method of distribution of singularities (stokeslets, doublets, rotlets, sources, stresslets and quadruppoles) on the body centreline, they satisfied the no slip boundary condition on the body surface, in closed form, up to an error of $O(\varepsilon^2 ln\varepsilon)$ (ε is the semi-slenderness parameter), and hence they obtained force (correct up to order ε^2) and/or torque the torus experienced for the individual cases of the broadwise translation (motion along the longitudinal axis), translation normal to the longitudinal axis, rotation of a torus on its edge, spinning and expanding of a torus. For the case of axially translation of a torus with the cross-sectional radius b and the body centreline radius a, their studies results in

$$F = -\frac{4\pi\mu U}{\ln\frac{8}{\varepsilon} + \frac{1}{2}} + O(\varepsilon^2), \tag{1.2.19}$$

where $\varepsilon = b/a$, and F is the axial component of the force per unit length acting on the torus. For the case of transverse motion perpendicular to the torus axis, they obtained the total drag force as

$$F = 2\pi^2\mu a U\frac{\left(3\ln\frac{8}{\varepsilon} - \frac{17}{2}\right)}{\left(\ln\frac{8}{\varepsilon} - \frac{1}{2}\right)\left(\ln\frac{8}{\varepsilon} - 2\right) - 2} + O(\varepsilon^2), \tag{1.2.20}$$

where by neglecting terms of order $(1/ln\varepsilon)^3$ this leads to Cox's result given by (1.2.18).

Johnson (1980) extended the method used by Johnson & Wu (1979) (singularity method), for flow past slender bodies of finite centreline curvature. He considered a flexible slender body of circular cross-section with arbitrary prescribed motion (function of time and position). By neglecting inertia effects, he obtained an integral equation for a curved slender prolate spheroid and slender bodies of arbitrary longitudinal cross-sections (with prolate-spheroid ends) which may be written in dimensionless form [quantities made dimensionless by the characteristic length l (l being half length of the body function of time i.e., a body with extensible arc length) velocity U (U being an appropriate characteristic velocity) and μ (viscosity of the fluid)] as

$$V_\nu(s,t) = \frac{1}{8\pi}\alpha_\nu(s,t)L_\nu + \frac{1}{8\pi}\int_{-s_1}^{+s_1} K_\nu(R_0;\alpha)\,ds'; \qquad (\nu = s, n, b) \qquad (1.2.21)$$

where (there is no summation convention on the repeated index)

$$K_\nu(R_0;\alpha) = \frac{\alpha_\nu(s',t)}{R_0} + \frac{[\alpha(s',t)\cdot R_0]R_{0\nu}}{R_0^3} - \frac{D_\nu\alpha_\nu(s,t)}{|s-s'|}; \qquad (1.2.22)$$

s' ($-1 \le s' \le +1$) is the arch length along the body centreline measured from the midpoint of the centreline; $-s_1$ and $+s_1$ are the values of s' corresponding to the two ends of the body excluding the prolate-spheroidal ends; t is time; $\varepsilon = b/l$ (where b is the typical transverse cross-sectional radius); $V_\nu(s, t)$ and $\alpha_\nu(s, t)$ are, respectively, the dimensionless components of the prescribed velocity and force per unit length at the point under consideration ($s' = s$) [$\nu = (s, n, b)$ where s, n, b as subscripts of a variable denote the components of that variable in the direction tangent, normal and binormal to the body centreline at the point $s' = s$, respectively]; $L_s = 2(2L - 1)$; $L_n = L_b = 2L + 1$ [where L is defined by

$$L = \ln\frac{2}{\varepsilon} \text{ (for slender spheroid), or } L = \ln\frac{2(1-s^2)^{1/2}}{\varepsilon\eta} \text{ (for arbitrary longitudinal cross-section),} \qquad (1.2.23)$$

$\varepsilon\eta(s)$ being function of s, representing the body transverse cross-sectional radius at point $s' = s$]; $D_n = D_b = 1$; $D_s = 2$, and $-R_0$ is a vector, representing the position of a general point on the body centreline (s') with respect to the point s. The dimensionless force per unit length on the body, $\alpha(s, t)$, is determined by iterative solution of the integral equation (1.2.21-23), the recurrence relation of which is given by Johnson as

$$\alpha_\nu^{(k+1)}(s,t) = \frac{1}{L_\nu}\left\{8\pi V_\nu(s,t) - D_\nu \ln f(s)\alpha_\nu^{k}(s,t) - \int_{-s_1}^{+s_1} K_\nu(R_0;\alpha^{(k)})\,ds'\right\} \qquad (1.2.24)$$

where $\alpha_\nu^{(k)}$ is the kth iteration of α_ν; $f(s) = (1 - s^2)/\eta^2(s)$; and here $L = ln\ (2/\varepsilon)$ in L_ν. The first term of the force per unit length (α_ν^1) in a power series of ($1/L\nu$) is determined by letting $\alpha_r^{(0)} = 0$ in (1.2.24). Johnson applied his method (1.2.24) to examples such as translation of a toroidal ring along its symmetry axis, and translation of a partial spheroidal torus in its own plane. For the former he obtained an infinite number of iterations the summation of which is exact the same as that obtained by Johnson & Wu (1979) in a direct fashion given by (1.2.19). For the latter he obtained the first two terms of the expansion of the force in a power series of $(1/L_r)$, where by expanding $1/L_r$ in terms of $1/ln\varepsilon$ and neglecting terms of order $(1/ln\varepsilon)^3$, his results reduces to Cox's results given by (1.2.15-17).

Khayat & Cox (1989) took inertia effects into account and adopted the Batchelor results for a non-axisymmetric body [Batchelor (1970)] to Cox's theory [Cox (1970)]. They assumed the Reynolds number R_e based on the body length is arbitrary and obtained the force per unit length on a curved slender body of arbitrary transverse cross-section (at rest in unbounded fluid undergoing undisturbed uniform velocity U) in terms of slenderness parameter, κ, correct up to order $(1/ln\kappa)^3$. They applied the force equation to the uniform flow past a long straight slender body of arbitrary cross-section, and obtained the force per unit length experienced by the body, $f(s)$, as

$$
\begin{aligned}
\frac{f(s)}{2\pi\mu U} = {} & \left(\frac{1}{\ln\kappa}\right)(\cos\theta\,\boldsymbol{\beta} - 2\boldsymbol{e}) + \left(\frac{1}{\ln\kappa}\right)^2 \left\{\frac{1}{4}\left[2\cos\theta\,\boldsymbol{e} - (2 - \cos\theta + \cos^2\theta)\,\boldsymbol{\beta}\right]\right. \\
& \times \left[\frac{1 - e^{-\frac{1}{2}R_e(1-\cos\theta)(1+s)}}{\frac{1}{2}R_e(1-\cos\theta)(1+s)} - 1\right] - \frac{1}{4}\left[2\cos\theta\,\boldsymbol{e} - (2 + \cos\theta + \cos^2\theta)\,\boldsymbol{\beta}\right] \\
& \times \left[\frac{1 - e^{-\frac{1}{2}R_e(1+\cos\theta)(1-s)}}{\frac{1}{2}R_e(1+\cos\theta)(1-s)} - 1\right] - \frac{1}{2}(\cos\theta\,\boldsymbol{\beta} - 2\boldsymbol{e})\left\{E_1\left[\tfrac{1}{2}R_e(1-\cos\theta)(1+s)\right]\right. \\
& \left. + \ln(1-\cos\theta)\right\} - \frac{1}{2}(\cos\theta\,\boldsymbol{\beta} - 2\boldsymbol{e})\left\{E_1\left[\tfrac{1}{2}R_e(1+\cos\theta)(1-s)\right] + \ln(1+\cos\theta)\right\} \\
& \left. - (\cos\theta\,\boldsymbol{\beta} - 2\boldsymbol{e})\left[\gamma + \ln\left(\frac{1}{4}R_e R_s\right)\right] + \frac{3}{2}\cos\theta\,\boldsymbol{\beta} - \boldsymbol{e} + 2\boldsymbol{e}\cdot\boldsymbol{K} + \cos\theta\,\boldsymbol{\beta}\ln K\right\} + O\left(\frac{1}{\ln\kappa}\right)^3,
\end{aligned}
\tag{1.2.25}
$$

where $\kappa = b/a$ (b being the characteristic length of the cross-sectional shape and a is the half length of the body); β is a unit vector, representing the direction of the body centreline; $\boldsymbol{e}$ is the unit vector in the direction of velocity U; θ is the angle between the unit vectors e and β (i.e. $\theta = e.\beta$); γ is Euler's constant defined by (1.2.6); R_e is the Reynolds number based on the body half-length (a); R_s is the radius of the equivalent circle of the cross-section at the point under consideration, s [where s ($-1 \leq s \leq +1$) is the arch length along the body centreline measured from the midpoint of the centreline]; K is the characteristic tensor of the transverse cross-sectional shape; K is a dimensionless constant which depends on the cross-sectional shape, and where $E_1(x)$ is the exponential integral defined by

$$E_1(x) = \int_x^\infty \frac{e^{-\tau}}{\tau} d\tau . \tag{1.2.26}$$

They obtained the total force, by integration of (1.2.25), and also the torque experienced by the body. While they applied the theory to an infinite straight slender body, with large R_e, they realized that it fails to give a uniform valid solution, and hence a minor modification is needed. They gave a theoretical reason for this violation and pointed out that for an infinite slender body together with large R_e the force should be expanded in $(lnR_b)^{-1}$ (R_b being the Reynolds number based on the characteristic length of the cross-sectional shape) instead of $(ln\kappa)^{-1}$ and lengths, in the outer region, should be made dimensionless by (ν/U) (ν being the kinematic viscosity of the fluid) rather than a (half length of the body). By this modification, their research for special case of an infinite straight cylinder of constant circular cross-section, placed perpendicular to uniform flow, leads to the Proudman & Pearson's result given by (1.2.10).

CHAPTER 2

2 - The general problem

In this research we consider an isolated long slender body with arbitrary cross-section being at rest in an unbounded fluid undergoing a uniform velocity, U. We are interested in obtaining the hydrodynamic force per unit length which the fluid exerts on the body. The length of the body is l and the characteristic dimension of the cross-sectional shape is r_0, where r_0 is an arbitrary length chosen to be representative in some way of the value of the radius of the equivalent circle of the cross-section. For example, $2\pi r_0$ might be the perimeter of the cross-section at a typical point on the body centreline. It is assumed that the body centreline may be bent in any manner whatsoever so long as the radius of such a bending is at all points of order the body length (l). The arc length of the body centreline measured from one end is s' (see figure 2.1). A dimensionless quantity s is defined by

$$s = \frac{s'}{l} \tag{2.1}$$

so that, the value of s is bounded by zero and one, corresponding to two ends of the body. The body centreline itself is given by

$$r' = lR(s) \tag{2.2}$$

where r' is a position vector of a general point relative to a fixed set of rectangular

Cartesian axes (r'_1, r'_2, r'_3) with origin at O, as shown in figure 2.1, and $R(s)$ is a dimensionless vector function of s.

At a general point P on the body centreline we introduce a set of local Cartesian axes (x', y', z') with origin at P and the z' axis tangent to the body centreline. Therefore the relationship between the fixed Cartesian coordinate system (r'_1, r'_2, r'_3) and the local coordinate system (x', y', z') may be written as

$$X' = r' - lR_p \tag{2.3}$$

where X' is the position vector of a general point relative to the local Cartesian system (x', y', z') and R_p is the value of $R(s)$ at point P, the origin of the local Cartesian system. Associated with the rectangular Cartesian axes (x', y', z') is a set of local cylindrical polar coordinates (ρ', θ, z'), as shown in figure 2.2, so that the relationship between these two coordinates is given by

$$x' = \rho'\cos\theta, \qquad y' = \rho'\sin\theta \ . \tag{2.4}$$

The cross sectional shape might vary along the body centreline, hence it may be given in terms of local polar coordinates (ρ', θ) as

$$\rho' = r_0\lambda(s,\theta) \tag{2.5}$$

where λ is a dimensionless function of s and θ.

It is assumed that the slenderness parameter, κ, defined by

$$\kappa = \frac{r_0}{l} \tag{2.6}$$

is much smaller than unity; that is, the body is slender.

We assume the Reynolds number R_e based on the body length defined by

$$R_e = \frac{lU}{\nu} \tag{2.7}$$

is of order unity, where $U = |\,U\,|$ is the magnitude of the uniform velocity, U, and ν

is the kinematic viscosity of the fluid. Then as κ tends to zero the Reynolds number R based on the characteristic length of the body transverse cross-section (r_0) defined by

$$R = \frac{r_0 U}{\nu} = \frac{r_0}{l} \times \frac{lU}{\nu} = \kappa R_e \tag{2.8}$$

tends to zero.

It is in terms of the parameter κ that we make expansions of the velocity and pressure fields to obtain the force per unit length on the body in the limit as κ tends to zero. However, this type of expansion must be singular because the flow locally around the long slender body must be very nearly the flow around an infinite cylinder at zero Reynolds number R (see 2.8), and it is well known from the *Stokes' paradox* that it is impossible for such a flow field to satisfy the flow equations and simultaneously to satisfy both the no slip boundary condition on the body surface and uniform flow at infinity.

We use dimensionless quantities based on the body length, l, the fluid viscosity, μ, and the characteristic velocity U, hence the dimensionless position vectors $\boldsymbol{r}$ and $\boldsymbol{X}$, velocity $\boldsymbol{u}$ and pressure p may be written as

$$\boldsymbol{r} = \frac{\boldsymbol{r}'}{l}, \quad \boldsymbol{X} = \frac{\boldsymbol{X}'}{l}, \quad \boldsymbol{u} = \frac{\boldsymbol{u}'}{U}, \quad p = \frac{lp'}{\mu U}, \tag{2.9}$$

where the primed variables correspond to the dimensional forms of the unprimed variables.

It is assumed that the fluid is incompressible, hence the dimensionless governing equations of motion and continuity may be written as

$$R_e \boldsymbol{u}.\nabla \boldsymbol{u} = \nabla^2 \boldsymbol{u} - \nabla p \, ; \qquad \nabla.\boldsymbol{u} = 0 \tag{2.10}$$

The boundary conditions associated with equations (2.10) may be expressed as

$$\boldsymbol{u} \to \boldsymbol{e} \qquad as \qquad r = |\boldsymbol{r}| \to \infty \tag{2.11}$$

and

$$u \to 0 \quad \textit{on the body surface}, \qquad (2.12)$$

where $e = U/U$ is a unit vector in the direction of the uniform undisturbed flow.

In order to solve equations (2.10) together with boundary conditions (2.11,12), one should obtain a solution as an outer expansion in κ, valid in outer region [i.e., the uniform flow at infinity (2.11) being satisfied], where r is of order unity. Thus, in this region lengths are made dimensionless by l, and hence as κ tends to zero [see (2.6)], the body cross-sectional radius tends to zero. Therefore, in the limit as $\kappa \longrightarrow 0$, the body becomes very much like a line, as shown in figure 2.3. For reason later on apparent, we call it line singularity.

In the outer region, we write $X = (x, y, z) = (\rho, \theta, z)$, where the x, y, z and ρ are respectively x', y', z' and ρ' made dimensionless by l.

At each point P of the body centreline determined by $r = R_p$ we may introduce a local inner expansion in κ for which $X^{(i)}$ is used as the independent variable and $u^{(i)}$ and $p^{(i)}$ as dependent variables, where $X^{(i)}$, $u^{(i)}$ and $p^{(i)}$ are defined by

$$X^{(i)} = \frac{X'}{r_0}, \quad u^{(i)} = \frac{u'}{U}, \quad p^{(i)} = \frac{r_0 p'}{\mu U}. \qquad (2.13)$$

Here we use variables labelled by the superscript *(i)* to denote the inner variables. Hence, the dimensionless coordinates $(x^{(i)}, y^{(i)}, z^{(i)})$ and $(\rho^{(i)}, \theta, z^{(i)})$ are respectively the local coordinates (x', y', z') and (ρ', θ, z') made dimensionless by r_0. Thus, the relationship between the outer variables and inner variables may be written as [see (2.3,9,13)]

$$X^{(i)} = \frac{r - R_P}{\kappa}, \quad \rho^{(i)} = \frac{\rho}{\kappa}, \quad u^{(i)} = u, \quad p^{(i)} = \kappa p. \qquad (2.14)$$

In the inner expansion corresponding to each point P of the body centreline we made length dimensionless by r_0 so that, as κ tends to zero, the body becomes very much like an infinite cylinder with non circular cross-section. In fact there are an infinite number of local inner expansions corresponding to each point of the line singularity representing the body in the outer region. However, we use the same procedure as used by Cox and

Khayat [Cox (1970) and Khayat & Cox (1989)]. That is, we develop an inner expansion at a general point P of the line singularity in order to consider all such inner expansions simultaneously. Then the inner expansion obtained for such a point will be matched onto the asymptotic solution of the outer expansion near the line singularity for the same point.

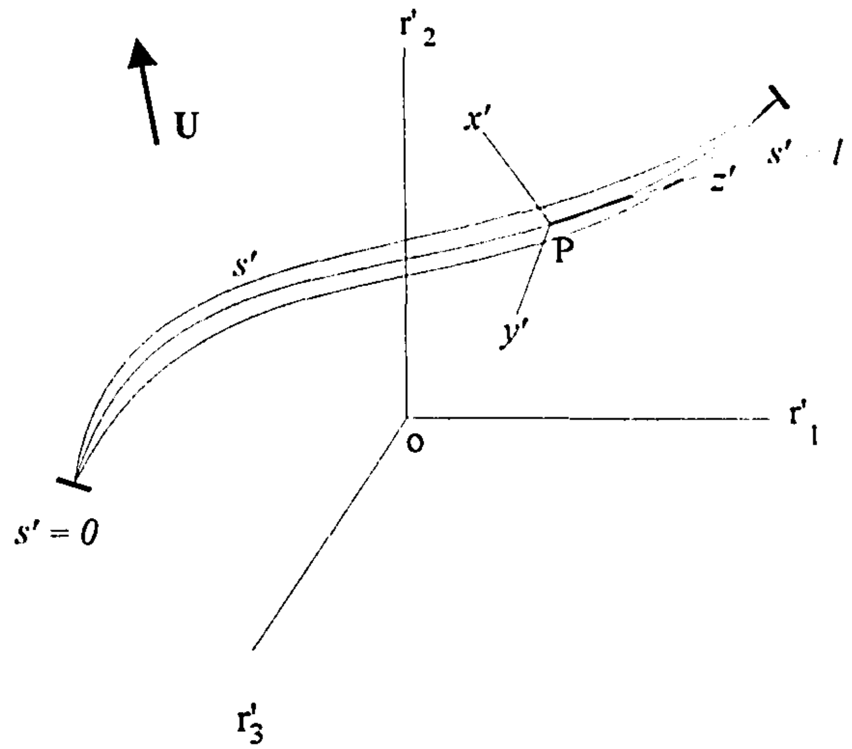

Figure 2.1 : Long slender body being at rest in the fluid undergoing an undisturbed uniform flow **U**.

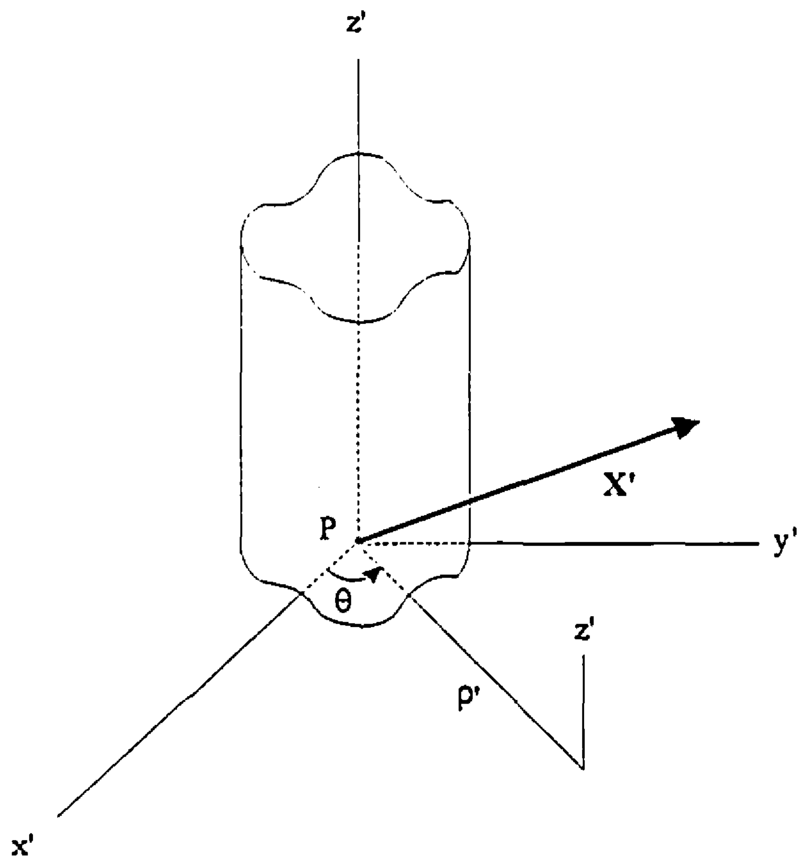

Figure 2.2 : Local cylindrical coordinate system (ρ' , θ , z') showing the local cross-sectional shape.

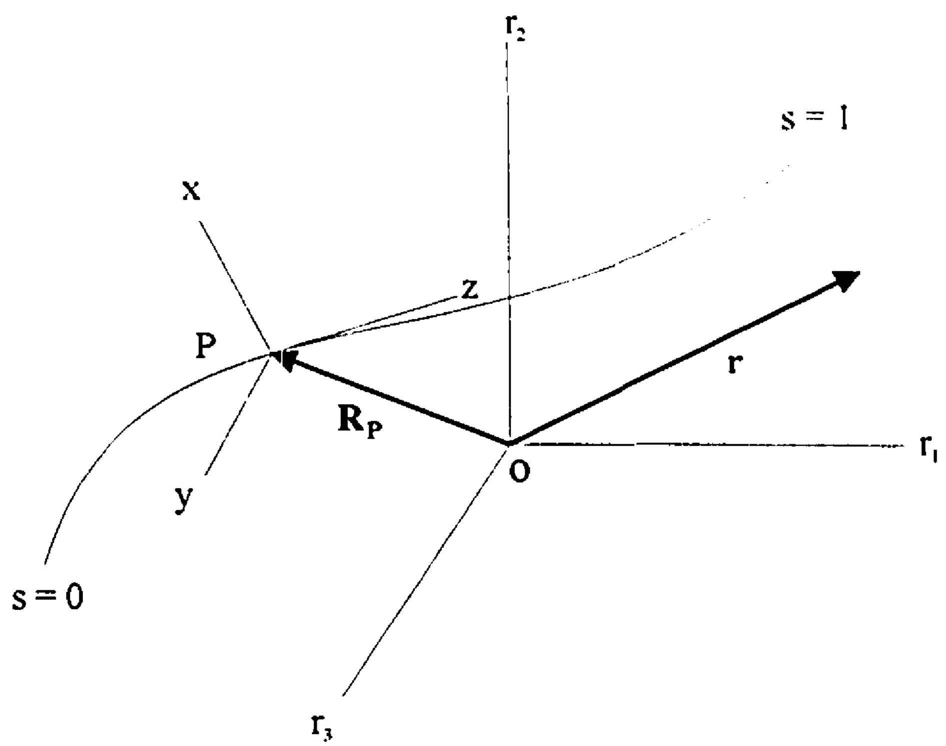

Figure 2.3 : The outer region in which lengths are made dimensionless by *l*.

In the limit as $\kappa \longrightarrow 0$ the body becomes a line singularity.

CHAPTER 3

3 - Outer expansion

In the outer region the velocity u and pressure p may be written as

$$u = e + u_1(\kappa) \tag{3.1}$$

and

$$p = p_1(\kappa), \tag{3.2}$$

where e represents the free stream velocity and $[u_1(\kappa), p_1(\kappa)]$ are the disturbance velocity and pressure due to the presence of the body S. In equation (3.2) the constant pressure associated with the free stream velocity e, without loss of generality, is taken to be zero. In the limit as κ tends to zero the body cross- sectional area shrinks to zero hence the effects of the disturbed velocity and pressure on the flow field (u, p) will diminish so we require

$$(u_1 , p_1) \rightarrow (0 , 0) , \quad as \quad \kappa \rightarrow 0. \tag{3.3}$$

In order to analyze these flow fields, at a general point P on the line singularity we take a set of rectangular Cartesian axes with unit base vectors $\underset{\sim}{i}_x$, $\underset{\sim}{i}_y$ and $\underset{\sim}{i}_z$ which lie in the same direction as the (x, y, z)-axes defined in chapter 2. Thus $\underset{\sim}{i}_z$ lies in the direction of the tangent to the body centreline, $r = R(s)$, at P.

From the form of u as one approaches the centreline we see that the flow is due to a force of magnitude

$$F'(s) = 4\pi B(\kappa,s)i_x + 4\pi D(\kappa,s)i_y - 2\pi A(\kappa,s)i_z. \tag{3.4}$$

This will be explicitly verified in the inner expansion. Since the x and y axes are arbitrary, it is convenient to take i_x to lie in the plane containing i_z and the velocity vector e, as shown in figure 3.1. Thus the unit vectors i_z, i_x and i_y may be determined by

$$i_z = t, \qquad i_x = \frac{e \cdot [I - tt)]}{[1 - |e\cdot t|^2]^{1/2}}, \qquad i_y = \frac{t \times e}{[1 - |e\cdot t|^2]^{1/2}}, \tag{3.5}$$

where

$$t(s) = \frac{dR(s)}{ds} \tag{3.6}$$

is a unit vector in the tangent direction of the line $r = R(s)$ at the general point P(s) and I is the idemfactor. Therefore $F'(s)$ may be expressed as

$$F'(s) = 2\pi\left[\frac{2B(\kappa,s)e\cdot(I - tt) + 2D(\kappa,s)t \times e}{(1 - |e\cdot t|^2)^{1/2}} - A(\kappa,s)t\right] \tag{3.7}$$

In order to make velocity u tend to be uniform at infinity we require

$$u_1 \to 0 \qquad as \qquad r \to \infty, \tag{3.8}$$

so that u_1 is the flow due to the distribution of the force density $F'(s)$ on the line $r = R(s)$.

The substitution of velocity u and pressure p (3.1,2) in the dimensionless forms of the Navier-Stokes and continuity equations given by (2.10) yields

$$R_e(e\cdot\nabla u_1 + u_1\cdot\nabla u_1) = \nabla^2 u_1 - \nabla p_1, \qquad \nabla \cdot u_1 = 0. \tag{3.9}$$

Since in the outer region the non-linear term $u_1 \cdot \nabla u_1$ is much smaller than $e \cdot \nabla u_1$, as a first approximation, we can neglect such a small term, and hence (3.9) may be written as

$$R_e e \cdot \nabla u_1 = \nabla^2 u_1 - \nabla p ; \qquad \nabla \cdot u_1 = 0 , \tag{3.10}$$

which is known as Oseen's equation for uniform flow in the direction e. It has a solution for u_1 and p_1 at location X due to the acting of a point force $f' = (f'_1, f'_2, f'_3)$ on the fluid at the origin given by [see Happel and Brenner (1970); Khayat & Cox (1989)]

$$(u_1)_i(X) = \frac{1}{8\pi}\left[g_{ij}(X) f_j'\right] \tag{3.11}$$

and

$$p_1(X) = \frac{1}{4\pi}\frac{X_i X_j}{X^3}, \tag{3.12}$$

where X is the radial distance from the origin defined by

$$X = |X| = (X_j X_j)^{1/2} \tag{3.13}$$

and g_{ij} is a symmetric tensor defined by

$$g_{ij}(X) = \delta_{ij}\nabla^2\Psi(X) - \Psi_{,ij}(X). \tag{3.14}$$

Here the summation convention is imposed on the repeated index, unless otherwise stated. In relationship (3.14) $\Psi(X)$ is given by

$$\Psi(X) = \frac{2}{R_e}\int_0^{\frac{1}{2}R_e(X - e\cdot X)}\left(\frac{1 - e^{-\alpha}}{\alpha}\right) d\alpha, \tag{3.15}$$

and the delta Kronecker δ_{ij} and $\Psi_{,ij}$ are respectively defined by

$$\delta_{ij} = \begin{cases} 1 , & for \ \ i = j \\ 0 , & for \ \ i \neq j \end{cases} \tag{3.16}$$

and

$$\Psi_{,ij} = \frac{\partial^2 \Psi}{\partial X_i \partial X_j} . \tag{3.17}$$

Thus, the velocity u_1 and pressure p_1 at location X due to action of a point force f' on the fluid at location X' may be written as

$$(u_1)_i(X) = \frac{1}{8\pi}[g_{ij}(X - X')f_j'] \tag{3.18}$$

and

$$p_1(X) = \frac{1}{4\pi}\frac{(X_j - X_j')f_j'}{|X - X'|^3}. \tag{3.19}$$

Therefore, the flow field (u_1, p_1) at a general point on the plane (x, y) with position vector $X = (x, y, 0)$, as shown in figure 3.2, produced by the whole line distribution of the force density $F'(\hat{s})$ may be determined by

$$(u_1)_i(X) = \frac{1}{8\pi}\int_0^1 g_{ij}[r - R(\hat{s})]F_j'(\hat{s})d\hat{s} \tag{3.20}$$

and

$$p_1(X) = \frac{1}{4\pi}\int_0^1 \frac{[r_j - R_j(\hat{s})]}{|r - R(\hat{s})|^3}F_j'(\hat{s})d\hat{s} . \tag{3.21}$$

In equations (3.20,21) the values of $(X - X')$ and f_j' of equations (3.18,19) are respectively replaced by their equivalent values $[r - R(\hat{s})]$ and $F'_j(\hat{s})d\hat{s}$ (see figure 3.2) and since equation (3.10) for u_1 and p_1 is linear we superimpose the velocity and pressure produced by all individual point forces $F'(\hat{s})d\hat{s}$ on the line singularity $r = R(\hat{s})$ by taking the integration over the line $0 \leq \hat{s} \leq 1$.

In order to match the value of velocity u and pressure p given by (3.1,2), respectively, onto those which will be obtained in the inner expansion we are required to determine the asymptotic behaviour of u_1 and p_1 near the line singularity $r = R(s)$. Since the integrands of u_1 in (3.20) and p_1 in (3.21) become singular on this line we divide u_1 and p_1 into two parts as

$$(u_1)_i = J_i + J_i' \tag{3.22}$$

and

$$p_1 = H + H', \tag{3.23}$$

where J'_i and H' are the integrals taken over the interval $(s - \varepsilon\ ,\ s + \varepsilon)$, whilst J_i and H are the integrals taken over the remaining interval. It is assumed that ε is an arbitrary constant, independent of κ and very much smaller than unity. Thus J_i , J'_i , H and H' may be expressed as

$$J_i(s) = \frac{1}{8\pi}\left\{\int_0^{s-\varepsilon} + \int_{s+\varepsilon}^{+1}\right\} g_{ij}[r - R(\hat{s})] F_j'(\hat{s}) d\hat{s}, \tag{3.24}$$

$$J_i'(s) = \frac{1}{8\pi}\int_{s-\varepsilon}^{s+\varepsilon} g_{ij}[r - R(\hat{s})] F_j'(\hat{s}) d\hat{s}, \tag{3.25}$$

$$H(s) = \frac{1}{4\pi}\left\{\int_0^{s-\varepsilon} + \int_{s+\varepsilon}^{+1}\right\} \frac{[r_j - R_j(\hat{s})]}{|r - R(\hat{s})|^3} F_j'(\hat{s}) d\hat{s} \tag{3.26}$$

and

$$H'(s) = \frac{1}{4\pi}\int_{s-\varepsilon}^{s+\varepsilon} \frac{[r_j - R_j(\hat{s})]}{|r - R(\hat{s})|^3} F_j'(\hat{s}) d\hat{s}\ . \tag{3.27}$$

We intend to obtain the asymptotic forms of u_1 and p_1 as X or its equivalent value ρ tends to zero; that is, as r tends to $R(s)$, the origin of local coordinate system $(x\ ,\ y,\ z)$ (see figure 3.3). Since the integrands in (3.24-27) only become singular at $\hat{s} = s$ if r lies on $R(s)$, it can be concluded that the integral J_i and H have integrands with no singularity , although their values will tend to infinity as ε tends to zero. Hence letting $r = R(s)$, the relationships (3.24) and (3.26) may be written as

$$J_i(s) = \frac{1}{8\pi}\left\{\int_0^{s-\varepsilon} + \int_{s+\varepsilon}^{+1}\right\} g_{ij}[R(s) - R(\hat{s})] F_j'(\hat{s}) d\hat{s} \tag{3.28}$$

and

$$H(s) = \frac{1}{4\pi}\left\{\int_0^{s-\varepsilon} + \int_{s+\varepsilon}^{+1}\right\} \frac{[R_j(s) - R_j(\hat{s})]}{|R(s) - R(\hat{s})|^3} F_j'(\hat{s}) d\hat{s}. \tag{3.29}$$

Since $\varepsilon \ll 1$, the integral J_i' and H' may be simplified if one notes that $F'(\hat{s}) \approx F'(s)$ in the range of the integration, that is, $s - \varepsilon \leq \hat{s} \leq s + \varepsilon$; so that we can write

$$J_i' \approx \frac{1}{8\pi} F_j'(s) I_{ij} \tag{3.30}$$

and

$$H' \approx \frac{1}{4\pi} F_j'(s) I_j, \tag{3.31}$$

where I_{ij} and I_j are respectively defined by

$$I_{ij} = \int_{s-\varepsilon}^{s+\varepsilon} g_{ij}[r - R(\hat{s})] d\hat{s} \tag{3.32}$$

and

$$I_j = \int_{s-\varepsilon}^{s+\varepsilon} \frac{[r_j - R_j(\hat{s})]}{|r - R(\hat{s})|^3} d\hat{s} . \tag{3.33}$$

For fixed but small ε, as one approaches the singularity line, $r = R(s)$, i.e. in the limit as ρ tends to zero, the flow approaches that due to a line of constant force $F'(s)$ acting on the z axis.

In order to obtain the asymptotic forms of I_{ij} and I_j for ρ tends to zero it is convenient to take the fixed rectangular Cartesian axes (r_1 , r_2 , r_3) to be parallel to the local coordinates set (z, x, y) at point $\hat{s} = s$; so that the r_1 , r_2 and r_3 axis respectively coincide with the z , x and y axis, respectively, as shown in figure 3.4. Thus the relationship between the position vector r and the vector $R(s)$ representing the body centreline at point $\hat{s} = s$ may be written as (see figure 3.4)

$$\begin{aligned} r_1 &= R_1(s) \\ r_2 &= R_2(s) + \rho\cos\theta \\ r_3 &= R_3(s) + \rho\sin\theta \end{aligned} \tag{3.34}$$

The expansion of the components of $R(\hat{s})$ around s by Taylor's series results in:

$$\begin{aligned} R_1(\hat{s}) &= R_1(s) + \frac{dR_1(s)}{ds}(\hat{s} - s) + \frac{1}{2}\frac{d^2R_1(s)}{ds^2}(\hat{s} - s)^2 + \dots \\ R_2(\hat{s}) &= R_2(s) + \frac{dR_2(s)}{ds}(\hat{s} - s) + \frac{1}{2}\frac{d^2R_2(s)}{ds^2}(\hat{s} - s)^2 + \dots \\ R_3(\hat{s}) &= R_3(s) + \frac{dR_3(s)}{ds}(\hat{s} - s) + \frac{1}{2}\frac{d^2R_3(s)}{ds^2}(\hat{s} - s)^2 + \dots \end{aligned} \tag{3.35}$$

Since $|\ \hat{s} - s\ | \leq \varepsilon \ll 1$ the errors in these relationships are of order ε^{+3}, so they tend to zero much faster than ε does. But, since the r_1 axis is parallel to the tangent direction of the centreline $R(\hat{s})$ at $\hat{s} = s$, $dR(s)/ds = t$ [see (3.6)], hence

$$\frac{dR_i(s)}{ds} = t_i = \delta_{1i} \tag{3.36}$$

which indicates that $dr_2/ds = dr_3/ds = 0$, therefore, (3.35) may be simplified as

$$\begin{aligned} R_1(\hat{s}) &= R_1(s) + (\hat{s} - s) + \frac{1}{2}\frac{d^2R_1(s)}{ds^2}(\hat{s} - s)^2 + \dots \\ R_2(\hat{s}) &= R_2(s) + \frac{1}{2}\frac{d^2R_2(s)}{ds^2}(\hat{s} - s)^2 + \dots \\ R_3(\hat{s}) &= R_3(s) + \frac{1}{2}\frac{d^2R_3(s)}{ds^2}(\hat{s} - s)^2 + \dots \end{aligned} \tag{3.37}$$

Thus, by the relationships (3.34) and (3.37), the components of the vector $[r - R(\hat{s})]$ may be expressed as

$$
\begin{aligned}
[r - R(\hat{s})]_1 &= -(\hat{s} - s) - \frac{1}{2}[\frac{d^2R_1}{ds^2}(s)](\hat{s} - s)^2 - \ldots \\
[r - R(\hat{s})]_2 &= + \rho\, Cos\theta - \frac{1}{2}[\frac{d^2R_2}{ds^2}(s)](\hat{s} - s)^2 - \ldots \\
[r - R(\hat{s})]_3 &= + \rho \cos\theta - \frac{1}{2}[\frac{d^2R_3}{ds^2}(s)](\hat{s} - s)^2 - \ldots
\end{aligned}
\tag{3.38}
$$

But $dR/ds = t$ is a unit vector, hence

$$
\frac{dR}{ds}\cdot\frac{dR}{ds} = 1\,. \tag{3.39}
$$

Differentiating (3.39) with respect to s results in :

$$
\frac{dR}{ds}\cdot\frac{d^2R}{ds^2} = 0 \qquad or \qquad \frac{dR_i}{ds}\frac{d^2R_i}{ds^2} = 0\,. \tag{3.40}
$$

Since $dr_i/ds = \delta_{1i}$, it follows that $d^2R_1/ds^2 = 0$, so that (3.38) may be written as

$$
\begin{aligned}
[r - R(\hat{s})]_1 &= - (\hat{s} - s) - O(\hat{s} - s)^3 - \ldots \\
[r - R(\hat{s})]_2 &= + \rho \cos\theta - \frac{1}{2}[\frac{d^2R_2}{ds^2}(s)](\hat{s} - s)^2 - \ldots \\
[r - R(\hat{s})]_3 &= + \rho \cos\theta - \frac{1}{2}[\frac{d^2R_3}{ds^2}(s)](\hat{s} - s)^2 - \ldots
\end{aligned}
\tag{3.41}
$$

Hence the square magnitude of vector $[r - R(\hat{s})]$ may be expressed as

$$
\begin{aligned}
|r - R(\hat{s})|^2 &= [(\hat{s} - s)^2 + \rho^2] + O[(\hat{s} - s)^4, \rho(\hat{s} - s)^2] \\
&= \left[(\hat{s} - s)^2 + \rho^2\right]\left\{1 + O\left[\frac{(\hat{s}-s)^4}{(\hat{s}-s)^2+\rho^2}\,,\ \frac{\rho(\hat{s} - s)^2}{(\hat{s}-s)^2+\rho^2}\right]\right\}.
\end{aligned}
\tag{3.42}
$$

Therefore by the aid of the binomial theorem the term in the dominator of I_j may be determined by

$$|r - R(\hat{s})|^{-3} = [(\hat{s} - s)^2 + \rho^2]^{-\frac{3}{2}} \times \left\{1 + O\left[\frac{(\hat{s} - s)^4}{(\hat{s} - s)^2 + \rho^2}, \frac{\rho(\hat{s} - s)^2}{(\hat{s} - s)^2 + \rho^2}\right]\right\} \tag{3.43}$$

The substitution of relationships (3.41) and (3.43) in I_j defined by (3.33) results in:

$$I_1 = \int_{s-\epsilon}^{s+\epsilon}\left\{\frac{-(\hat{s} - s)}{[(\hat{s} - s)^2 + \rho^2]^{3/2}} + O\left[\frac{(\hat{s} - s)^3}{[(\hat{s} - s)^2 + \rho^2]^{3/2}}, \frac{(\hat{s} - s)^5}{[(\hat{s} - s)^2 + \rho^2]^{5/2}}, \frac{\rho(\hat{s} - s)^3}{[(\hat{s} - s)^2 + \rho^2]^{5/2}}\right]\right\}d\hat{s}, \tag{3.44}$$

$$I_2 = \int_{s-\epsilon}^{s+\epsilon}\left\{\frac{\rho\cos\theta}{[(\hat{s} - s)^2 + \rho^2]^{3/2}} + O\left[\frac{\rho(\hat{s} - s)^4}{[(\hat{s} - s)^2 + \rho^2]^{5/2}}, \frac{\rho^2(\hat{s} - s)^2}{[(\hat{s} - s)^2 + \rho^2]^{5/2}}, \frac{(\hat{s} - s)^2}{[(\hat{s} - s)^2 + \rho^2]^{3/2}}\right]\right\}d\hat{s}, \tag{3.45}$$

and

$$I_3 = \int_{s-\epsilon}^{s+\epsilon}\left\{\frac{\rho\sin\theta}{[(\hat{s} - s)^2 + \rho^2]^{3/2}} + O\left[\frac{\rho(\hat{s} - s)^4}{[(\hat{s} - s)^2 + \rho^2]^{5/2}}, \frac{\rho^2(\hat{s} - s)^2}{[(\hat{s} - s)^2 + \rho^2]^{5/2}}, \frac{(\hat{s} - s)^2}{[(\hat{s} - s)^2 + \rho^2]^{3/2}}\right]\right\}d\hat{s}\ . \tag{3.46}$$

In order to evaluate I_j, it is convenient to introduce a new variable x defined by

$$\hat{s} - s = \rho x, \tag{3.47}$$

so that I_j in terms of variable x may be written as

$$I_1 = \int_{-\varepsilon/\rho}^{+\varepsilon/\rho}\left\{\frac{-x}{\rho(1+x^2)^{3/2}} + O\left[\frac{\rho x^3}{(1+x^2)^{3/2}}, \frac{\rho x^5}{(1+x^2)^{5/2}}, \frac{x^3}{(1+x^2)^{5/2}}\right]\right\}dx, \tag{3.48}$$

$$I_2 = \int_{-\varepsilon/\rho}^{+\varepsilon/\rho}\left\{\frac{\cos\theta}{\rho(1+x^2)^{3/2}} + O\left[\frac{\rho x^4}{(1+x^2)^{5/2}}, \frac{x^2}{(1+x^2)^{5/2}}, \frac{x^2}{(1+x^2)^{3/2}}\right]\right\}dx, \tag{3.49}$$

and

$$I_3 = \int_{-\varepsilon/\rho}^{+\varepsilon/\rho}\left\{\frac{\sin\theta}{\rho(1+x^2)^{3/2}} + O\left[\frac{\rho x^4}{(1+x^2)^{5/2}}, \frac{x^2}{(1+x^2)^{5/2}}, \frac{x^2}{(1+x^2)^{3/2}}\right]\right\}dx\ . \tag{3.50}$$

It can be seen that all the integrals in I_1 possess the property of symmetry, so that

$$I_1 \to 0 \quad as \quad \rho \to 0\ . \tag{3.51}$$

For fixed ε as ρ tends to zero the integrals in I_2 and I_3 may be evaluated as follows :

$$\int_{-\varepsilon/\rho}^{+\varepsilon/\rho}\frac{dx}{(x^2+1)^{3/2}} \sim \int_{-\infty}^{+\infty}\frac{dx}{(x^2+1)^{3/2}} = \int_{-\tan^{-1}\varepsilon/\rho}^{+\tan^{-1}\varepsilon/\rho}\cos\theta\, d\theta$$
$$= \left|\sin\theta\right|_{-\pi/2}^{+\pi/2} = +2\ , \tag{3.52}$$

$$\int_{-\varepsilon/\rho}^{+\varepsilon/\rho}\frac{x^2dx}{(x^2+1)^{3/2}} = \left|-x(x^2+1)^{-1/2}\right|_{-\varepsilon/\rho}^{+\varepsilon/\rho} + \left|\ln\left[x+(x^2+1)^{1/2}\right]\right|_{-\varepsilon/\rho}^{+\varepsilon/\rho}$$
$$= -2\frac{\varepsilon}{\rho}\left(\frac{\varepsilon^2}{\rho^2}+1\right)^{-1/2} + \ln\left[\frac{(\varepsilon^2+\rho^2)^{1/2}+\varepsilon}{(\varepsilon^2+\rho^2)^{1/2}-\varepsilon}\right]$$

or

$$\int_{-\varepsilon/\rho}^{+\varepsilon/\rho} \frac{x^2 dx}{(x^2+1)^{3/2}} \sim -\frac{2\dfrac{\varepsilon}{\rho}}{\dfrac{\varepsilon}{\rho} + \dfrac{1}{2}\left(\dfrac{\varepsilon^2}{\rho^2}\right)^{-1/2} + \dots} + \ln\left(\frac{\varepsilon + \dfrac{1}{2}(\varepsilon^2)^{-1/2}\rho^2 + \dots + \varepsilon}{\varepsilon + \dfrac{1}{2}\dfrac{\rho^2}{\varepsilon} + \dots - \varepsilon} \right) \tag{3.53}$$

$$\sim -2 + 2\ln\left(\frac{2\varepsilon}{\rho}\right),$$

$$\int_{-\varepsilon/\rho}^{+\varepsilon/\rho} \frac{x^2 dx}{(x^2+1)^{5/2}} = \left| -\frac{x}{3}(x^2+1)^{-3/2} \right|_{-\varepsilon/\rho}^{+\varepsilon/\rho} + \frac{1}{3}\int_{-\varepsilon/\rho}^{+\varepsilon/\rho} \frac{dx}{(x^2+1)^{3/2}} \tag{3.54}$$

$$\approx -\frac{2}{3}\frac{\rho^2}{\varepsilon^2} + \frac{2}{3}$$

and

$$\int_{-\varepsilon/\rho}^{+\varepsilon/\rho} \frac{x^4 dx}{(x^2+1)^{5/2}} = \left| -\frac{x^3}{3}(x^2+1)^{-3/2} \right|_{-\varepsilon/\rho}^{+\varepsilon/\rho} + \int_{-\varepsilon/\rho}^{+\varepsilon/\rho} \frac{x^2 dx}{(x^2+1)^{3/2}} \tag{3.55}$$

$$\approx -\frac{8}{3} + 2\ln\left(\frac{2\varepsilon}{\rho}\right).$$

The substitution of (3.52-55) in I_2 and I_3 given respectively by (3.49-50) gives

$$I_2 \sim 2\rho^{-1}\cos\theta + O\left[\rho, \rho\ln\left(\frac{\varepsilon}{\rho}\right), \left(\frac{\rho^2}{\varepsilon^2}\right), \rho^0, \ln\left(\frac{\varepsilon}{\rho}\right)\right] \tag{3.56}$$

and

$$I_3 \sim 2\rho^{-1}\sin\theta + O\left[\rho, \rho\ln\left(\frac{\varepsilon}{\rho}\right), \left(\frac{\rho^2}{\varepsilon^2}\right), \rho^0, \ln\left(\frac{\varepsilon}{\rho}\right)\right]. \tag{3.57}$$

As ε tends to zero the largest error in these relationships is of order ρ^0. Therefore H' defined by relationship (3.31) is obtained by

$$H' \sim \frac{1}{2\pi}\rho^{-1}[\cos\theta F_2'(s) + \sin\theta F_3'(s)] + O(\rho^0) \tag{3.58}$$

Thus, p_1 given by relationship (3.23) may be determined by

$$p_1 \sim H + \frac{1}{2\pi}\rho^{-1}[\cos\theta F_2'(s) + \sin\theta F_3'(s)] + O(1) \tag{3.59}$$

where H is defined by relationship (3.29). Therefore as ρ tends to zero, the asymptotic form of pressure p [see (3.2)] can be expressed as

$$\begin{aligned} p \sim \; & \frac{1}{2\pi}\rho^{-1}[\cos\theta F_2'(s) + \sin\theta F_3'(s)] \\ & + \frac{1}{4\pi}\left\{\int_0^{s-\epsilon} + \int_{s+\epsilon}^{+1}\right\}\frac{[R_j(s) - R_j(\hat{s})]}{|R(s) - R(\hat{s})|^3}F_j'(\hat{s})d\hat{s} + O(1)\,. \end{aligned} \tag{3.60}$$

In order to obtain the asymptotic behaviour of I_{ij} near the line singularity $r = R(\hat{s})$ one should obtain the asymptotic form of g_{ij}, defined by [see (3.14)]

$$g_{ij}(X'') = \delta_{ij}\nabla^2\Psi(X'') - \Psi_{,ij}(X''), \tag{3.61}$$

where $X'' = X - X' = r - R(\hat{s})$ (see figures 3.2,3), as both ρ and ε tend to zero. But $\rho = X$ and as ε tends to zero, X' the magnitude of the position vector of the point force, will tend to zero. Therefore as both ρ and ε tend to zero, X'' the magnitude of vector X'', will also tend to zero. Since the integral variable, α, in $\Psi(X'')$, defined by [see (3.15)]

$$\Psi(X'') = \frac{2}{R_e}\int_0^{\frac{1}{2}R_e(X''-e\cdot X'')}\left(\frac{1 - e^{-\alpha}}{\alpha}\right)d\alpha, \tag{3.62}$$

will tend to zero, if X'' tends to zero we require to determine the asymptotic behaviour of $\Psi(X'')$ for $\alpha \longrightarrow 0$, hence in this case the integrand of $\Psi(X'')$ may be simplified as

$$\begin{aligned} \frac{1 - e^{-\alpha}}{\alpha} \sim & \frac{1 - (1 - \alpha + \dfrac{\alpha^2}{2!} - \dfrac{\alpha^3}{3!} + \cdots)}{\alpha} \\ & = 1 - \frac{\alpha}{2!} + \frac{\alpha^2}{3!} - \cdots \end{aligned} \tag{3.63}$$

Thus as X'' tends to zero $\Psi(X'')$ is obtained by

$$\Psi(X'') \sim \frac{2}{R_e}\int_0^{\frac{1}{2}R_e(X''-e\cdot X'')}\left(1 - \frac{\alpha}{2!} + \dots\right)d\alpha$$
$$= \frac{2}{R_e}\left|\alpha - 4\alpha^2 + \dots\right|_0^{1/2R_e(x''-e\cdot X'')} \qquad (3.64)$$
$$= (X'' - e\cdot X'') - \frac{1}{8}R_e(X'' - e\cdot X'')^2 + \dots$$

The $\Psi,_{ij}$ and $\nabla^2\Psi$ in g_{ij} may be determined as follows :

$$\Psi,_i(X'') \sim \frac{X''_i}{X''} - e_i - \frac{1}{4}R_e(X'' - e_k X''_k)\left(\frac{X''_i}{X''} - e_i\right) + \dots \qquad (3.65)$$

Thus

$$\Psi,_{ij}(X'') \sim \left(\frac{\delta_{ij}}{X''} - \frac{X_i''X_j''}{X''^3}\right)$$
$$- \frac{1}{4}R_e\left[(X'' - e_k X_k'')\left(\frac{\delta_{ij}}{X''} - \frac{X_i''X_j''}{X''^3}\right) + \left(\frac{X_j''}{X''} - e_j\right)\left(\frac{X_i''}{X''} - e_i\right)\right] + \dots$$
$$(3.66)$$

Hence

$$\Psi,_{ii} \sim \left(\frac{3}{X''} - \frac{1}{X''}\right) - \frac{1}{4}R_e\left[(X'' - e_k X_k'')\left(\frac{3}{X''} - \frac{1}{X''}\right) + \left(\frac{X_i''}{X''} - e_i\right)\left(\frac{X_i''}{X''} - e_i\right)\right] + \dots$$
$$= \frac{2}{X''} - \frac{1}{4}R_e\left(2 - 2\frac{e_k X_k''}{X''} + 1 - 2\frac{x_i'' e_k}{X''} + 1\right) + \dots$$
$$(3.67)$$

Therefore

$$\nabla^2\Psi = \Psi,_{kk} \sim \frac{2}{X''} - R_e\left(1 - \frac{e_k X_k''}{X''}\right) + \dots \qquad (3.68)$$

Thus g_{ij} given by (3.61) may be determined by

$$
\begin{aligned}
g_{ij}(X'') &\sim \frac{2\delta_{ij}}{X''} - R_e\left(1 - \frac{e_k X_k''}{X''}\right)\delta_{ij} + \left(-\frac{\delta_{ij}}{X''} + \frac{X_i'' X_j''}{X''^3}\right) + \frac{1}{4}R_e\left(\delta_{ij} - \frac{e_k X_k'' \delta_{ij}}{X''}\right. \\
&\quad \left. - \frac{X_i'' X_j''}{X''^2} + \frac{e_k X_k'' X_i'' X_j''}{X''^3} + \frac{X_i'' X_j''}{X''^2} - \frac{e_j X_i''}{X''} - \frac{e_i X_j''}{X''} + e_i e_j\right) + \dots \\
&= \left(\frac{\delta_{ij}}{X''} + \frac{X_i'' X_j''}{X''^3}\right) + \frac{1}{4}R_e\left(-3\delta_{ij} + 3\frac{e_k X_k'' \delta_{ij}}{X''} = \frac{e_k X_k'' X_i'' X_j''}{X''^3}\right. \\
&\quad \left. - \frac{e_j X_i''}{X''} - \frac{e_i X_j''}{X''} + e_i e_j\right) + \dots
\end{aligned}
\tag{3.69}
$$

We see the term in R_e^{+1} is bounded [i.e. it is independent of $X'' = r - R(\hat{s})$], so it is independent of $\hat{s}$, hence it gives a contribution to I_{ij} [see (3.32)] of order ε^{+1} and so it tends to zero as ε does. After being substituted X'' by $[r\text{-}R(\hat{s})]$, the term in R_e^{0} gives

$$
g_{ij}[r - R(\hat{s})] \sim \frac{\delta_{ij}}{|r - R(\hat{s})|} + \frac{[r_i - R_i(\hat{s})][r_j - R_j(\hat{s})]}{|r - R(\hat{s})|^3} + \dots \tag{3.70}
$$

Substituting relationships (3.41) and (3.43) in g_{ij} results in :

$$
g_{11} \sim \frac{1}{[(\hat{s} - s)^2 + \rho^2]^{1/2}} + \frac{(\hat{s} - s)^2}{[(\hat{s} - s)^2 + \rho^2]^{3/2}} + \dots, \tag{3.71}
$$

$$
g_{22} \sim \frac{1}{[(\hat{s} - s)^2 + \rho^2]^{1/2}} + \frac{\rho^2\cos^2\theta}{[(\hat{s} - s)^2 + \rho^2]^{3/2}} + \dots, \tag{3.72}
$$

$$
g_{33} \sim \frac{1}{[(\hat{s} - s)^2 + \rho^2]^{1/2}} + \frac{\rho^2\sin^2\theta}{[(\hat{s} - s)^2 + \rho^2]^{3/2}} + \dots, \tag{3.73}
$$

$$
g_{12} = g_{21} \sim -\frac{(\hat{s} - s)\,\rho\cos\theta}{[(\hat{s} - s)^2 + \rho^2]^{3/2}} + \dots, \tag{3.74}
$$

$$
g_{13} = g_{31} \sim -\frac{(\hat{s} - s)\,\rho\sin\theta}{[(\hat{s} - s)^2 + \rho^2]^{3/2}} + \dots. \tag{3.75}
$$

and

$$g_{23} = g_{32} \sim + \frac{\rho^2 \sin\theta \cos\theta}{[(\hat{s} - s)^2 + \rho^2]^{3/2}} + \dots \,. \tag{3.76}$$

Therefore, I_{ij} defined by (3.32) may be determined by

$$I_{11} \sim \int_{s-\epsilon}^{s+\epsilon} \frac{2(\hat{s} - s)^2 + \rho^2}{[(\hat{s} - s)^2 + \rho^2]^{3/2}} \, d\hat{s} \,, \tag{3.77}$$

$$I_{22} \sim \int_{s-\epsilon}^{s+\epsilon} \frac{(\hat{s} - s)^2 + \rho^2 + \rho^2 \cos^2\theta}{[(\hat{s} - s)^2 + \rho^2]^{3/2}} \, d\hat{s} \,, \tag{3.78}$$

$$I_{33} \sim \int_{s-\epsilon}^{s+\epsilon} \frac{(\hat{s} - s)^2 + \rho^2 + \rho^2 \sin^2\theta}{[(\hat{s} - s)^2 + \rho^2]^{3/2}} \, d\hat{s} \,, \tag{3.79}$$

$$I_{12} = I_{21} \sim -\int_{s-\epsilon}^{s+\epsilon} \frac{(\hat{s} - s) \rho \cos\theta}{[(\hat{s} - s)^2 + \rho^2]^{3/2}} \, d\hat{s} \,, \tag{3.80}$$

$$I_{13} = I_{31} \sim -\int_{s-\epsilon}^{s+\epsilon} \frac{(\hat{s} - s) \rho \sin\theta}{[(\hat{s} - s)^2 + \rho^2]^{3/2}} \, d\hat{s} \,, \tag{3.81}$$

and

$$I_{23} = I_{32} \sim +\int_{s-\epsilon}^{s+\epsilon} \frac{\rho^2 \sin\theta \cos\theta}{[(\hat{s} - s)^2 + \rho^2]^{3/2}} \, d\hat{s} \,. \tag{3.82}$$

Or I_{ij} may be expressed in terms of the variable x defined by (3.47) as

$$I_{11} \sim \int_{-\epsilon/\rho}^{+\epsilon/\rho} \frac{2x^2}{(x^2 + 1)^{3/2}} dx + \int_{-\epsilon/\rho}^{+\epsilon/\rho} \frac{1}{(x^2 + 1)^{3/2}} dx \,, \tag{3.83}$$

$$I_{22} \sim \int_{-\epsilon/\rho}^{+\epsilon/\rho} \frac{x^2}{(x^2 + 1)^{3/2}} dx + \int_{-\epsilon/\rho}^{+\epsilon/\rho} \frac{1 + \cos^2\theta}{(x^2 + 1)^{3/2}} dx \,, \tag{3.84}$$

$$I_{33} \sim \int_{-\varepsilon/\rho}^{+\varepsilon/\rho} \frac{x^2}{(x^2+1)^{3/2}} dx + \int_{-\varepsilon/\rho}^{+\varepsilon/\rho} \frac{1+\sin^2\theta}{(x^2+1)^{3/2}} dx \; . \qquad (3.85)$$

$$I_{12} = I_{21} \sim -\int_{-\varepsilon/\rho}^{+\varepsilon/\rho} \frac{x\cos\theta}{(x^2+1)^{3/2}} dx \; , \qquad (3.86)$$

$$I_{13} = I_{31} \sim -\int_{-\varepsilon/\rho}^{+\varepsilon/\rho} \frac{x\sin\theta}{(x^2+1)^{3/2}} dx \; , \qquad (3.87)$$

and

$$I_{23} = I_{32} \sim \int_{-\varepsilon/\rho}^{+\varepsilon/\rho} \frac{\sin\theta\cos\theta}{(x^2+1)^{3/2}} dx \; . \qquad (3.88)$$

Therefore, as ρ tends to zero [i.e. $r \longrightarrow R(s)$ see figure 3.4], I_{ij} may be determined by [see (3.52-55)]

$$\begin{aligned} I_{11} &\sim 2[-2 + 2\ln(\frac{2\varepsilon}{\rho})] + 2 + \cdots \\ &\sim -4\ln\rho + \{4\ln\varepsilon - 2 + 4\ln 2\} + \cdots \; , \end{aligned} \qquad (3.89)$$

$$\begin{aligned} I_{22} &\sim -2 + 2\ln(\frac{2\varepsilon}{\rho}) + 2(1+\cos^2\theta) + \cdots \\ &\sim -2\ln\rho + \{2\ln\varepsilon + 2\ln 2 + 2\cos^2\theta\} + \cdots \; , \end{aligned} \qquad (3.90)$$

$$\begin{aligned} I_{33} &\sim -2 + 2\ln(\frac{2\varepsilon}{\rho}) + 2(1+\sin^2\theta) + \cdots \\ &\sim -2\ln\rho + \{2\ln\varepsilon + 2\ln 2 + 2\sin^2\theta\} + \cdots \; , \end{aligned} \qquad (3.91)$$

$$\begin{aligned} I_{12} = I_{21} &\sim \cos\theta \, | \, (x^2+1)^{-1/2} |_{-\varepsilon/\rho}^{+\varepsilon/\rho} \\ &\sim 0 \; , \end{aligned} \qquad (3.92)$$

similarly

$$I_{13} = I_{31} \sim 0 , \tag{3.93}$$

and

$$I_{23} = I_{32} \sim 2\sin\theta\cos\theta + \cdots . \tag{3.94}$$

The substitution of I_{ij} into J'_i defined by [see (3.30)]

$$J_i' \sim \frac{1}{8\pi}[I_{i1}F_1'(s) + I_{i2}F_2'(s) + I_{i3}F_3'(s)] \tag{3.95}$$

gives

$$J'_1 \sim \frac{1}{8\pi}\{ -4\ln\rho + [4\ln(2\varepsilon) - 2] + \cdots \}F'_1(s) , \tag{3.96}$$

$$J'_2 \sim \frac{1}{8\pi}\Big\{\{-2\ln\rho + \left[2\ln(2\varepsilon) + 2\cos^2\theta\right] + \cdots\}F'_2(s) + 2\sin\theta\cos\theta F'_3 + \cdots\Big\} , \tag{3.97}$$

and

$$J'_3 \sim \frac{1}{8\pi}\Big\{\{-2\ln\rho + [2\ln(2\varepsilon) + 2\sin^2\theta] + \cdots\}F'_3(s) + 2\sin\theta\cos\theta F'_2 + \cdots\Big\} . \tag{3.98}$$

Thus as ρ tends to zero $(u_l)_i$ defined by (3.22) may be determined by

$$(u_1)_1 \sim -\frac{1}{2\pi}F'_1(s)\ln\rho + \left\{ \frac{1}{4\pi}[2\ln(2\varepsilon) - 1]F'_1(s) + J_1(s) \right\} + \cdots , \tag{3.99}$$

$$\begin{aligned}(u_1)_2 \sim & -\frac{1}{4\pi}F'_2(s)\ln\rho \\ & + \left\{ \frac{1}{4\pi}[\ln(2\varepsilon) + \cos^2\theta]F'_2(s) + \frac{1}{4\pi}\sin\theta\cos\theta F'_3(s) + J_2(s) \right\} + \cdots \end{aligned} \tag{3.100}$$

and

$$(u_1)_3 \sim -\frac{1}{4\pi}F'_3(s)\ln\rho + \left\{ \frac{1}{4\pi}[\ \ln(2\varepsilon) + \sin^2\theta\ ]F'_3(s) + \frac{1}{4\pi}\sin\theta\cos\theta F'_2(s) + J_3(s) \right\} + \cdots . \tag{3.101}$$

Hence by (3.1) the asymptotic form of the velocity u_i when ρ tends to zero may be obtained by

$$u_1 \sim -\frac{1}{2\pi}F'_1(s)\ln\rho + \left\{ e_1 + \frac{1}{4\pi}[\ 2\ln(2\varepsilon) - 1\ ]F'_1(s) + J_1(s) \right\} + \cdots \tag{3.102}$$

$$u_2 \sim -\frac{1}{4\pi}F'_2(s)\ln\rho + \left\{ e_2 + \frac{1}{4\pi}[\ln(2\varepsilon) + \cos^2\theta]F'_2(s) + \frac{1}{4\pi}\sin\theta\cos\theta F'_3(s) + J_2(s) \right\} + \cdots \tag{3.103}$$

and

$$u_3 \sim -\frac{1}{4\pi}F'_3(s)\ln\rho + \left\{ e_3 + \frac{1}{4\pi}[\ln(2\varepsilon) + \sin^2\theta\ ]F'_3(s) + \frac{1}{4\pi}\sin\theta\cos\theta F'_2(s) + J_3(s) \right\} + \cdots . \tag{3.104}$$

where J_i is defined by (3.28) as

$$J_i(s) = \frac{1}{8\pi}\left\{\int_0^{s-\varepsilon} + \int_{s+\varepsilon}^{+1}\right\} g_{ij}[R(s) - R(\hat{s})]F_j'(\hat{s})d\hat{s} . \tag{3.105}$$

The relationships (3.103,104) may be combined and written in indices notation as

$$u_i \sim \frac{F'_j(s)}{4\pi}\left[(\delta_{ij})\ln\frac{2\varepsilon}{\rho} + \frac{X_i X_j}{\rho^2} \right] + e_i + J_i(s) + \cdots \tag{3.106}$$

where $(i, j) = 2, 3$; $X_2 = x$ and $X_3 = y$.

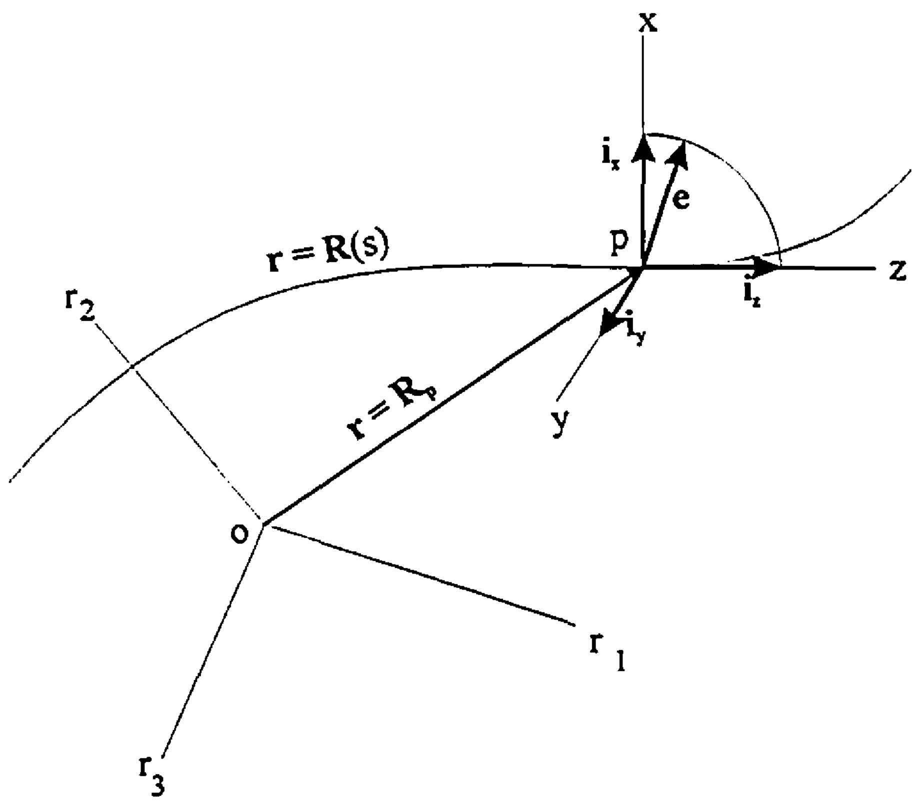

Figure 3.1 : The system of axes at the general point P on the line singularity with unit base vectors i_x, i_y, i_z showing the unit vector e lying in the (x, z)-plane.

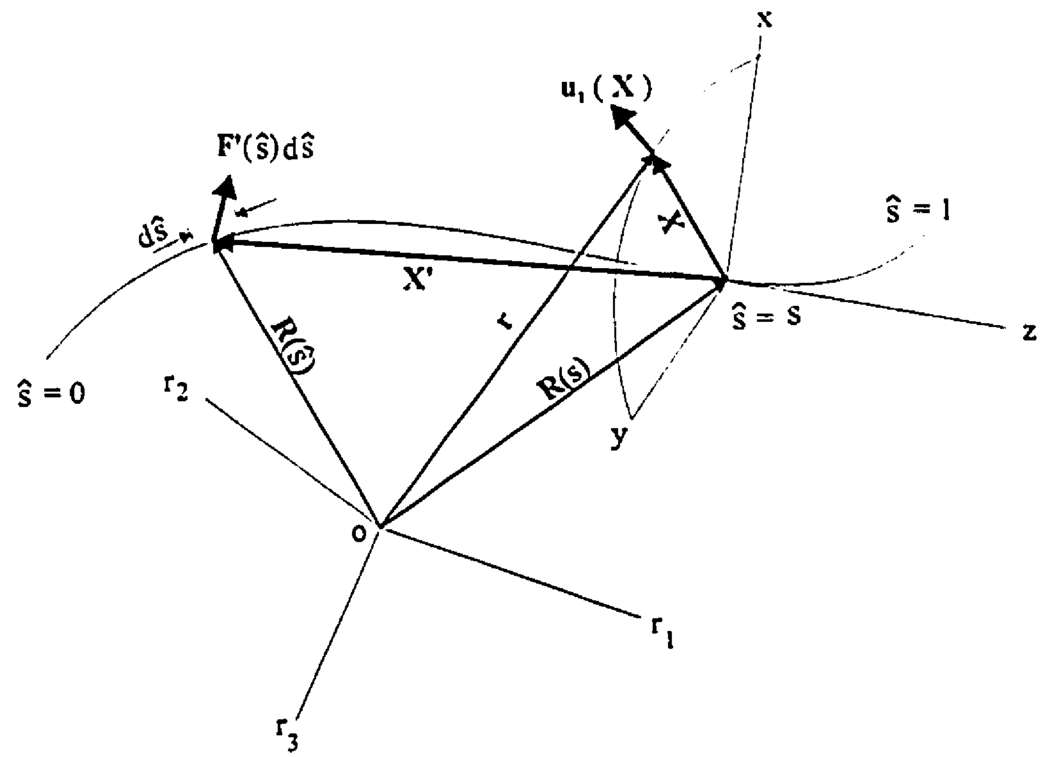

Figure 3.2 : The velocity $\mathbf{u}_t$ at position **X** in the (x-y)-plane due to the acting of the point force $\mathbf{F}'(\hat{s})d\hat{s}$ at position **X** ' on the line singularity $\mathbf{r} = \mathbf{R}(\hat{s})$.

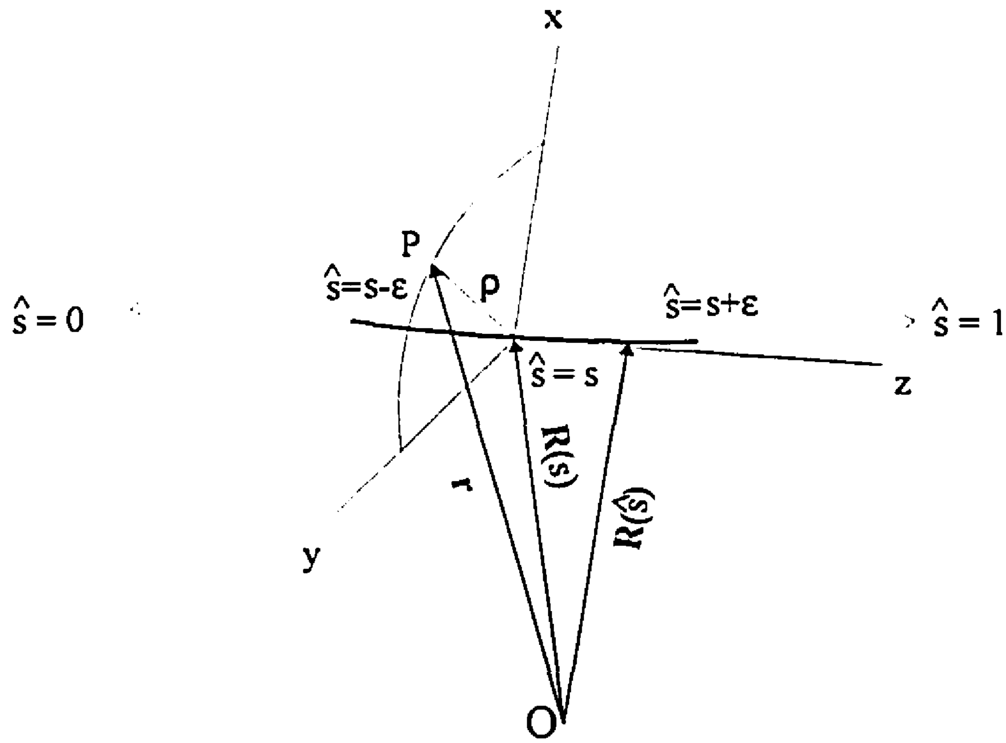

Figure 3.3 : The range of integration in which as $\rho \rightarrow 0$ the integrands in H' and J',
become singular.

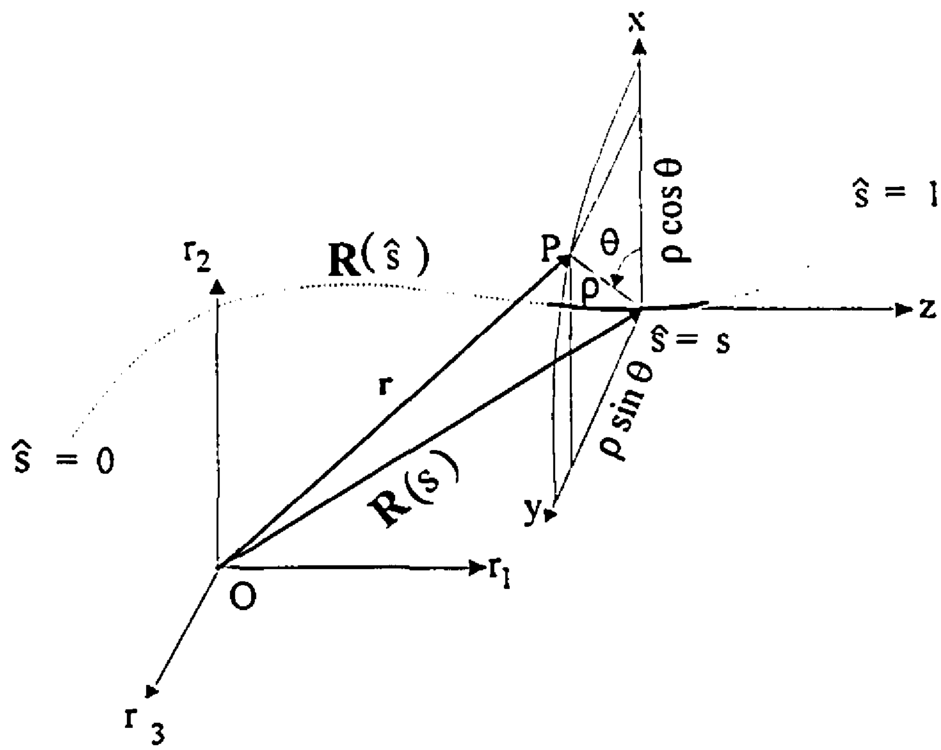

Figure 3.4 : The position of the general point P in the (x-y)-plane with respect to the (r_1, r_2, r_3)-axes which is taken to be parellel to the local coordinate system at point ŝ = s.

CHAPTER 4

4 - Inner flow field - Matched asymptotic expansion

In this chapter we consider the cylindrical flow in the neighbourhood of a general point on the body centreline (i.e. at P). The inner expansion of the flow fields ($u^{(i)}$, $p^{(i)}$) should be determined by solving the governing equations (2.10) expressed in inner variables as [see (2.8)]

$$\kappa R_e u^{(i)} \cdot \nabla^{(i)} u^{(i)} = \nabla^{(i)2} u^{(i)} - \nabla^{(i)} p^{(i)} \; ; \qquad \nabla^{(i)} \cdot u^{(i)} = 0 \; ; \tag{4.1}$$

subject to the boundary condition

$$u^{(i)} = 0 \qquad \textit{on the body surface} \; , \tag{4.2}$$

where $\nabla^{(i)}$ is the gradient operator with respect to the ($x^{(i)}$, $y^{(i)}$, $z^{(i)}$)-coordinates. Since the boundary condition at far distances from the body has been considered in the outer expansion, in inner flow field we only exert the no slip boundary condition on the body surface [i.e. (4.2)]. Thus, neglecting term of order κ, equations (4.1) may be written as

$$\nabla^{(i)2} u^{(i)} - \nabla^{(i)} p = 0 \; ; \qquad \nabla^{(i)} \cdot u^{(i)} = 0. \tag{4.3}$$

which is Stokes equations.

It is assumed the cross-sectional shape varies sufficiently slowly along the body centreline, and we recall that the curvature of the body centreline is assumed to be large everywhere, so we can neglect the dependence of the local flow fields on the z-axis. Thus, the problem is simplified to two dimensional flow fields in the plane-($x^{(i)}$, $y^{(i)}$).

Since terms, which have significant values in the region near the body surface, such as those of order $\rho^{(i)}$ to some negative power, when converted to the outer variable,[see (2.14) (i.e., $\rho = \kappa\rho^{(i)}$)], will turn into terms of order κ to some positive power, to the approximation being considered [i.e., $O(\kappa)$], we are not concerned with such terms. Therefore, it suffices to obtain the inner flow field valid at far distances from the body in the inner region.

To the approximation being considered the cases of transverse motion and longitudinal one involve only the transverse and longitudinal components of $\boldsymbol{F}'(s)$, respectively, and since the corresponding inner flow fields have slightly different characters we consider them separately. Thus, we decompose the flow fields ($\boldsymbol{u}^{(i)}$, $p^{(i)}$) in $\boldsymbol{u}^{(i)} = (u_1^{(i)}, 0, 0)$, which is parallel to the body centreline, together with constant pressure and in $\boldsymbol{u}^{(i)} = (0, u_2^{(i)}, u_3^{(i)})$, which is held in the cross-sectional plane ($x^{(i)}$, $y^{(i)}$), along with pressure $p^{(i)}$. This separation of the cylindrical flow will be explicitly verified to be in accordance with the exact solution of the problem.

First the flow field for a long cylindrical body with an elliptical cross-section is analyzed by the complex variable method and the problem is then generalized by the use of the inner flow field solution for a body with an arbitrary cross-section. In both cases the corresponding solutions will be expressed in terms of outer variables and then will be matched onto those obtained for the outer expansion at the same point.

4.1 - Longitudinal motion for elliptical cross-section

Now we may consider a cylindrical body with an elliptical cross-section, the semi-diameters of which are given by $r_0 a(s)$ and $r_0 b(s)$, where $a > b$. The dimensionless equation of the cross-sectional shape may be written as

$$\frac{x''^2}{a^2} + \frac{y''^2}{b^2} = 1, \tag{4.1.1}$$

where x'' and y'' are dimensionless axes, chosen to coincide with the larger semi-diameter a and the shorter one b, respectively. Associated with the rectangular coordinate system (x'', y'') is a dimensionless polar coordinate system ($\rho^{(i)}$, ϕ), as shown in figure 4.1, so that the relationship between these two coordinate systems is given by

$$x'' = \rho^{(i)}\cos\phi \qquad and \qquad y'' = \rho^{(i)}\sin\phi \, . \tag{4.1.2}$$

In the (x'', y'')-plane, which is called Z-plane, the dimensionless, equations of motion and boundary condition may be written as

$$\nabla''^2 u_1^{(i)} = 0; \qquad \nabla \cdot u = 0 \tag{4.1.3}$$

and

$$u_1^{(i)} = 0 \quad on \quad \frac{x''^2}{a^2} + \frac{y''^2}{b^2} = 1, \tag{4.1.4}$$

where $u_l^{(i)} = u_l^{(i)}(x'', y'')$ and ∇'' is the gradient operator with respect to (x'', y'')-coordinates. In equation (4.13) pressure p does not appear because it is constant and hence $\nabla''p = 0$. The continuity equation is automatically satisfied, that is, $\partial u_1/\partial z = 0$, which is one of the assumptions of the problem for the inner flow field. In order to impose the no slip boundary condition (4.1.4) on the body surface, it is convenient to transform the cross sectional shape onto a circle. Hence, the transformation

$$Z = w(\xi) = \left(\frac{a+b}{2}\right)\xi + \left(\frac{a-b}{2}\right)\xi^{-1} ; \tag{4.1.5}$$

transforms the ellipse defined by (4.1.1) in the Z-plane onto a unit circle, $e^{i\phi}$, in the ξ-plane as shown in figure 4.1. Solving the relationship (4.1.5) for ξ results in

$$\xi = f(Z) = \frac{Z + \sqrt{Z^2 - (a^2 - b^2)}}{a + b} . \tag{4.1.6}$$

In the derivation of the relationship (4.1.6) the root with positive sign is chosen in order not only to make a one to one[1] mapping, but also to map the exterior-region of the ellipse in the Z-plane onto the exterior-region of the unit circle in the ξ-plane. Since $f(Z)$ is analytic, it is conformal too, so that the harmonic function $u_l\,(x'', y'')$ remains

[1] Here we consider the principal value of the complex variable, unless otherwise stated.

harmonic under the change of variables arising from the conformal transformation. Therefore, in the ξ-plane, we have

$$\nabla''^2 u_1^{(i)} = 0, \qquad u_1^{(i)} = u_1^{(i)}(\,w_1'', w_2''\,) \qquad (4.1.7a,b)$$
$$u_1^{(i)} = 0 \qquad on \qquad r'' = 1,$$

where (w_1'', w_2'') and r'' are, respectively, the dimensionless coordinate system and radial distance from the origin in the ξ-plane, as shown in figure 4.1, and where the gradient operator ∇'' refers to (w_1'', w_2'')-coordinates. Letting $g(\xi) = P + i\,Q$ be analytic function of $\xi = w_1'' + i\,w_2'' = r''e^{i\phi}$ (i.e. $\nabla''^2 P = \nabla''^2 Q = 0$), $u_1^{(i)}$ may be written as

$$u_1^{(i)} = P = \Re[g(\xi)], \qquad (4.1.8)$$

where $\Re$ denotes the real part of the complex variable. The general solution of equation (4.1.7*a*) is

$$u_1^{(i)} = (\alpha\phi + \beta)(A'\ln r'' + B') + \sum_{n=1}^{\infty}\left(C'_n r''^{(+n)} + D'_n r''^{(-n)}\right)\left(E'_n \cos n\phi + F'_n \sin n\phi\right), (4.1.9a)$$

where all the coefficients are real constants and n is an integer number. But, the term $\alpha\phi$ can not appear in the solution since it is not periodic, neither can the terms of order $[\rho^{(i)}]^n$ (n being positive) since they would have to match onto the terms of the order κ to some negative power [see (2.14)] in the outer expansion, whereas no such term exists [see (3.102)]. In other words, at far distances from the body the velocity is finite or, strictly speaking, it is only logarithmic infinite. Therefore, (4.1.9*a*) may be written as

$$u_1^{(i)} = A\ln r'' + B + \sum_{n=1}^{\infty} r''^{(-n)}\left(E_n \cos n\phi + F_n \sin n\phi\right). \qquad (4.1.9b)$$

Imposing the boundary condition (4.1.7*b*) in (4.1.9*b*) results in:

$$0 = A\ln 1 + B + \sum_{n=1}^{\infty}\left(E_n \cos n\phi + F_n \sin n\phi\right). \qquad (4.1.9c)$$

Since the relationship (4.1.9*c*) must be held for any value of ϕ, it follows that, the only non-zero coefficient is A. Thus [see (4.1.8)]

$$u_1^{(i)} = A \ln r'' \\ = \Re[g(\xi)] = \Re(A \ln \xi). \tag{4.1.10}$$

In the derivation of the equation (4.1.10), it was noted that $ln\xi = ln(r'' e^{i\phi}) = lnr'' + i\phi$. However, by the aid (4.1.6), we have

$$\xi \longrightarrow [2/(a+b)]Z \quad as \quad r'' \longrightarrow \infty. \tag{4.1.11}$$

Therefore, at far distances from the boundary velocity $u^{(i)}{}_{l\infty}$ is determined by[1]

$$u_{1\infty}^{(i)} \sim \Re\left[A \ln\left(\frac{2Z}{a+b}\right)\right] \\ \sim A \ln\left(\frac{2\rho^{(i)}}{a+b}\right). \tag{4.1.12}$$

Since $\rho = \kappa\rho^{(i)}$ and $u_l = u_l^{(i)}$ [see (2.14)] the velocity $u_l^{(i)}$, in terms of outer variables, may be written as

$$u_1 \sim A \ln\left[\frac{2\rho}{\kappa(a+b)}\right] + O\left(\frac{\kappa^n}{\rho^n}\right) \tag{4.1.13}$$

where n is some positive integer. Because u_l is independent of ϕ, it possesses the same form under any rotation of the axes (x'', y''). Hence by neglecting the terms of order κ the inner body condition for the outer flow field also in terms of dimensionless polar coordinates (ρ, θ) is determined by

$$u_1 \sim A \ln \rho - A \ln[\kappa(a+b)/2]. \tag{4.1.14}$$

Matching the velocity u_l onto that obtained by (3.102) for the outer expansion requires

$$A = -[1/(2\pi)] F_1'(s) \tag{4.1.15}$$

and

[1]Here the dependent variables labelled by the sign ∞ denotes the values of those variables at far distances from the boundary.

$$-A\ln\left[\frac{\kappa(a+b)}{2}\right] = e_1 + \frac{1}{4\pi}[2\ln(2\varepsilon) - 1]F_1'(s) + J_1(s).$$

Substituting $A = -F_l'/(2\pi)$ results in

$$F_1'(s)\left\{2\ln\left[\frac{4\varepsilon}{\kappa(a+b)}\right] - 1\right\} = -4\pi[\,e_1 + J_1(s)\,]. \tag{4.1.16}$$

Next, let us examine the force which the fluid exerts on the body. Following Batchelor (1970) in the cylindrical flow field $F_l(s)$, the force per unit length which the fluid exerts on the body, may be determined by

$$F_1 = \oint\left(\frac{\partial u_1^{(i)}}{\partial n}\right)ds'', \tag{4.1.17}$$

where n and s'' represent distance normal and along to any closed curve in the cross-sectional plane at point s by which the body is surrounded. This equation can be easily derived by the use the relationships (A.35.36) (see appendix A) and (4.1.10). However, by choosing the curve as a circle with radius $\rho^{(i)}$ for which equation (4.1.12) holds, F_l may be obtained as [see (4.1.15)]

$$\begin{aligned} F_1(s) &= \int_0^{2\pi}\frac{\partial u_{1-}^{(i)}}{\partial\rho^{(i)}}\rho^{(i)}d\phi \\ &= \int_0^{2\pi}A\,d\phi \\ &= 2\pi A = -F_1'(s), \end{aligned} \tag{4.1.18}$$

which was expected.

4.2 - Transverse motion for elliptical cross-section

We may now consider the flow fields $u^{(i)} = (0,\, u^{(i)}_2,\, u^{(i)}_3)$ together with pressure $p^{(i)}$ in the plane (x'', y''). The governing dimensionless equations of motion and boundary condition, in the Z-plane, may be written as

$$\nabla''^2 u^{(i)} - \nabla'' p^{(i)} = 0; \qquad \nabla \cdot u = 0 \tag{4.2.1}$$

and

$$u^{(i)} = 0 \quad on \quad \frac{x''^2}{a^2} + \frac{y''^2}{b^2} = 1, \tag{4.2.2}$$

where $u^{(i)} = u^{(i)}(x'', y'')$. Equation (4.2.1) may be expressed in terms of the stream function Ψ as [see Appendix A, (A.12)]

$$\nabla''^4 \Psi = 0, \tag{4.2.3}$$

which is the biharmonic equation. It possesses the solution for Ψ in terms of the complex variables as (see appendix A)

$$\Psi = \Re(Z^* \Phi + \chi), \tag{A.27}$$

where Φ and χ are analytic functions of Z whose functions will be determined by using the boundary conditions; Z^* is the conjugate of Z, that is $Z^* = x - iy$.[1]

In transverse flow, as in longitudinal flow, the velocity at far distances from the boundary is logarithmic infinite. In fact, it was this singularity in space that didn't allow Stokes to satisfy the uniform flow condition at infinity, and hence caused the *Stokes' paradox*. Therefore, in the Z-plane, as $\rho^{(i)}$ tends to infinity, Ψ_∞ , corresponding to the velocities $u_2^{(i)}{}_\infty$ and $u_3^{(i)}{}_\infty$, has the following form:

$$\Psi_\infty - (C\rho^{(i)} \ln \rho^{(i)} + D\rho^{(i)}) \cos\phi + (E\rho^{(i)} \ln \rho^{(i)} + F\rho^{(i)}) \sin\phi . \tag{4.2.4}$$

letting

$$\Phi_\infty = (A' + iB') \ln Z$$

and

[1]As far as the complex variable is considered the variable labelled by an asterisk denotes the conjugate of that variable.

$$\chi_\infty = (C' + iD')Z\ln Z + (E' + iF')Z,$$

where A', B', C', D', E' and F' are real constants. Ψ_∞ may be btained by

$$\Psi_\infty = \Re[\rho^{(i)}e^{-i\phi}(A' + iB')(\ln\rho^{(i)} + i\phi)$$
$$+ (C' + iD')\rho^{(i)}e^{i\phi}(\ln\rho^{(i)} + i\phi) + (E' + iF')\rho^{(i)}e^{i\phi}]$$
$$= [(A' + C')\rho^{(i)}\ln\rho^{(i)} + E'\rho^{(i)}]\cos\phi$$
$$+ [(B' - D')\rho^{(i)}\ln\rho^{(i)} - F'\rho^{(i)}]\sin\phi - (B' + D')\rho^{(i)}\phi\cos\phi. \qquad (4.2.5)$$

Equating the equivalent terms of equations (4.2.4) and (4.2.5) to each other, results in:

$$A' = C' = \tfrac{1}{2}C,$$

$$B' = -D' = \tfrac{1}{2}E,$$

$$E' = D \qquad and \qquad F' = -F.$$

Hence, in Z-plane, Φ_∞ and χ_∞ may be written as

$$\Phi_\infty = (\frac{C}{2} + i\frac{E}{2})\ln Z$$

and

$$\chi_\infty = (\frac{C}{2} - i\frac{E}{2})Z\ln Z + (D - iF)Z.$$

But [see (4.1.3)] as $\rho \longrightarrow \infty$; $Z \longrightarrow (a + b)\xi/2$. Therefore, in the ξ-plane, Φ_∞ and χ_∞ may be expressed as

$$\Phi_\infty = \left(\frac{C}{2} + i\frac{E}{2}\right)\ln\left(\frac{a+b}{2}\xi\right)$$

and

$$\chi_\infty = \left(\frac{C}{2} - i\frac{E}{2}\right)\left(\frac{a+b}{2}\xi\right)\ln\left(\frac{a+b}{2}\xi\right) + (D - iF)\frac{a+b}{2}\xi .$$

Thus, one should take the general forms of Φ and χ as

$$\Phi = \left(\frac{C}{2} + i\frac{E}{2}\right)\ln\left(\frac{a+b}{2}\xi\right) + \sum_{n=1}^{\infty} G_n \xi^{-n} \tag{4.2.6}$$

and

$$\chi = \left(\frac{C}{2} - i\frac{E}{2}\right)\left(\frac{a+b}{2}\xi\right)\ln\left(\frac{a+b}{2}\xi\right) + (D - iF)\frac{a+b}{2}\xi + I\xi^{-1}\ln\xi + \sum_{n=1}^{\infty} H_n \xi^{-n}, \tag{4.2.7}$$

where G_n, I and H_n are complex constants; n is an integer.

The first term of relationship (4.2.6) and the first two terms of relationship (4.2.7) correspond to Φ_∞ and χ_∞, respectively, and the remainders are complementary terms, the values of which become significant in the region near the body surface and which functions are determined by imposing the no slip boundary condition on the body surface, that is, $u^{(i)} = 0$ on unit circle $\xi = e^{i\phi}$.

In the ξ-plane, velocity $u^{(i)}$ is given by (see appendix A)

$$i\left(u_x^{(i)} + i\,u_y^{(i)}\right) = \left[\frac{dw}{d\xi}\right]^{*-1}\left\{\left[\frac{dw}{d\xi}\right]^{*}\Phi + w\left[\frac{d\Phi}{d\xi}\right]^{*} + \left[\frac{d\chi}{d\xi}\right]^{*}\right\}, \tag{A.49}$$

where w is defined by (4.1.3) as

$$w(\xi) = \left(\frac{a+b}{2}\right)\xi + \left(\frac{a-b}{2}\right)\xi^{-1}.$$

Hence

$$\frac{dw}{d\xi} = \frac{a+b}{2} - \frac{a-b}{2}\xi^{-2}.$$

From relationships (4.2.6-7) $[d\Phi/\xi]^*$ and $[d\chi/\xi]^*$ are determined by

$$\left[\frac{d\Phi}{d\xi}\right]^{*} = \left(\frac{C}{2} - i\frac{E}{2}\right)\xi^{*(-1)} - \sum_{n=1}^{\infty} nG_n^{*}\xi^{*-(n+1)}$$

and

$$\left[\frac{d\chi}{d\xi}\right]^{*} = \left(\frac{C}{2} + i\frac{E}{2}\right)\frac{a+b}{2}\left[\ln\left(\frac{a+b}{2}\xi^{*}\right) + 1\right] + (D + iF)\frac{a+b}{2}$$
$$+ I^{*}(1 - \ln\xi^{*})\,\xi^{*(-2)} - \sum_{n=1}^{\infty} nH_n^{*}\xi^{*-(n+1)}.$$

We require $u_2^{(i)} = u_3^{(i)} = 0$ on $\xi = e^{i\phi}$, hence by substituting w, $[dw/d\chi]^{*}$, Φ, $[d\Phi/d\xi]^{*}$ and $[d\chi/d\xi]^{*}$ in equation (A.49) and letting $\xi = e^{i\phi}$, the no slip boundary condition can be expressed as

$$\left(\frac{a+b}{2} - \frac{a-b}{2}e^{2i\phi}\right)\left[\left(\frac{C}{2} + i\frac{E}{2}\right)\ln\left(\frac{a+b}{2}e^{i\phi}\right) + \sum_{n=1}^{\infty} G_n e^{-in\phi}\right]$$
$$+ \left(\frac{a+b}{2}e^{i\phi} + \frac{a-b}{2}e^{-i\phi}\right)\left[\left(\frac{C}{2} - i\frac{E}{2}\right)e^{i\phi} - \sum_{n=1}^{\infty} nG_n^{*}e^{i(n+1)\phi}\right]$$
$$+ \left(\frac{C}{2} + i\frac{E}{2}\right)\frac{a+b}{2}\left[\ln\left(\frac{a+b}{2}e^{-i\phi}\right) + 1\right] + (D + iF)\frac{a+b}{2}$$
$$+ I^{*}(1 + i\phi)e^{2i\phi} - \sum_{n=1}^{\infty} nH_n^{*}e^{i(n+1)\phi} = 0.$$

This relationship must hold for any value of ϕ, hence it must be independent of ϕ. Therefore, equating the coefficient of $(i\phi)^p e^{iq\phi}$ to zero results in:

$$i\phi e^{2i\phi}\,; \qquad -\frac{a-b}{2}\left(\frac{C}{2} + i\frac{E}{2}\right) + I^{*} = 0,$$

$$e^{2i\phi}\,; \quad -\frac{a-b}{2}\left(\frac{C}{2} + i\frac{E}{2}\right)\ln\frac{a+b}{2} + \frac{a+b}{2}\left(\frac{C}{2} - i\frac{E}{2}\right) + I^{*} - H_1^{*} = 0,$$

$$e^{0}\,; \qquad \frac{a+b}{2}\left(\frac{C}{2} + i\frac{E}{2}\right)\ln\frac{a+b}{2} + \frac{a-b}{2}\left(\frac{C}{2} - i\frac{E}{2}\right)$$
$$+ \left(\frac{C}{2} + i\frac{E}{2}\right)\frac{a+b}{2}\left(\ln\frac{a+b}{2} + 1\right) + (D + iF)\frac{a+b}{2} = 0,$$

and the other coefficients are equal to zero. Upon solving these equations simultaneously, the coefficients I, H_1, D and F are determined by

$$I = \frac{a-b}{2}\left(\frac{C}{2} - i\frac{E}{2}\right)$$

$$H_1 = a\frac{C}{2} + i\,b\frac{E}{2} - \frac{a-b}{2}\left(\frac{C}{2} - i\frac{E}{2}\right)\ln\frac{a+b}{2},$$

$$D = -C\left(\ln\frac{a+b}{2} + \frac{a}{a+b}\right),$$

and

$$F = -E\left(\ln\frac{a+b}{2} + \frac{b}{a+}\right).$$

Thus, $(\Phi,\ \chi)$ and Ψ_- given by (4.2.6,7) and (4.2.4), respectively, may be written as

$$\Phi \sim \left(\frac{C}{2} + i\frac{E}{2}\right)\ln\left(\frac{a+b}{2}\xi\right), \tag{4.2.8}$$

$$\begin{aligned}\chi \sim & \left(\frac{C}{2} - i\frac{E}{2}\right)\frac{a+b}{2}\xi\ln\left(\frac{a+b}{2}\xi\right) + \frac{a-b}{2}\left(\frac{C}{2} - i\frac{E}{2}\right)\xi^{-1}\ln\xi \\ & - \left[C\left(\ln\frac{a+b}{2} + \frac{a}{a+b}\right) - i\,E\left(\ln\frac{a+b}{2} + \frac{b}{a+b}\right)\right]\frac{a+b}{2}\xi \\ & + \left[\frac{C}{2}\left(a - \frac{a-b}{2}\ln\frac{a+b}{2}\right) + i\frac{E}{2}\left(b + \frac{a-b}{2}\ln\frac{a+b}{2}\right)\right]\xi^{-1}\end{aligned} \tag{4.2.9}$$

and

$$\Psi_- \sim -\rho^{(i)}(C\cos\phi + E\sin\phi)\ln\frac{a+b}{2\rho^{(i)}} - \rho^{(i)}\left(\frac{Ca\cos\phi + Eb\sin\phi}{a+b}\right). \tag{4.2.10}$$

We may now consider the elliptical cross-section in the general axes $(x^{(i)},\ y^{(i)})$ such that the direction of the larger principal of the ellipse $(2a)$ is given by the unit vector $\beta(s)$. For this case the x''-axis is obtained by the rotation of $x^{(i)}$-axis within the $(x^{(i)},\ y^{(i)})$-

plane by an angle λ in the counter-clockwise sense, as shown in figure 4.2, where

$$\beta_2 = \cos\lambda \qquad \text{and} \qquad \beta_3 = \sin\lambda .$$

Hence, the relationship between θ the polar angle associated with the $(x^{(i)}, y^{(i)})$-coordinate system and ϕ is determined by (see figure 4.2)

$$\phi = \theta - \lambda .$$

Thus, Ψ_∞ may also be expressed in terms of polar coordinates $(\rho^{(i)}, \theta)$ as

$$\Psi_\infty(\rho^{(i)},\theta) \sim -\rho^{(i)}[C\cos(\theta-\lambda) + E\sin(\theta-\lambda)]\ln\frac{a+b}{2\rho^{(i)}}$$
$$- \rho^{(i)}\left[\frac{Ca\cos(\theta-\lambda) + Eb\sin(\theta-\lambda)}{a+b}\right]$$

or Ψ_∞ in terms of $(x^{(i)}, y^{(i)})$-coordinates may be written as

$$\Psi_\infty(x^{(i)},y^{(i)}) \sim -[(C\beta_2 - E\beta_3)x^{(i)} + (C\beta_3 + E\beta_2)y^{(i)}]\ln\frac{a+b}{2\sqrt{x^{(i)^2} + y^{(i)^2}}}$$
$$- \frac{(Ca\beta_2 - Eb\beta_3)x^{(i)} + (Ca\beta_3 + Eb\beta_2)y^{(i)}}{a+b} .$$

At far distances from the boundary the velocities $u_{2\infty}^{(i)}$ and $u_{3\infty}^{(i)}$ may directly be obtained from the stream function Ψ_∞ as

$$u_{2\infty}^{(i)} = \frac{\partial\Psi_\infty}{\partial y^{(i)}} \sim -(C\beta_3 + E\beta_2)\ln\frac{a+b}{2\rho^{(i)}} - \frac{Ca\beta_3 + Eb\beta_2}{a+b}$$
$$+ [(C\beta_2 - E\beta_3)x^{(i)} + (C\beta_3 + E\beta_2)y^{(i)}]\frac{y^{(i)}}{\rho^{(i)^2}}$$

and

$$u_{3\sim}^{(i)} = -\frac{\partial \Psi_\sim}{\partial x^{(i)}} - (C\beta_2 - E\beta_3)\ln\frac{a+b}{2\rho^{(i)}} + \frac{Ca\beta_2 - Eb\beta_3}{a+b}$$

$$- [(C\beta_2 - E\beta_3)x^{(i)} + (C\beta_3 + E\beta_2)y^{(i)}]\frac{x^{(i)}}{\rho^{(i)^2}}.$$

Thus, the inner body conditions for the outer flow field are determined by [see (2.14)]

$$u_2 \sim -(C\beta_3 + E\beta_2)\ln(\frac{a+b}{2\rho}\kappa) - \frac{Ca\beta_3 + Eb\beta_2}{a+b}$$

$$+ (C\beta_2 - E\beta_3)\sin\theta\cos\theta + (C\beta_3 + E\beta_2)\sin^2\theta + O(\frac{\kappa^n}{p^n})$$

and

$$u_3 \sim + (C\beta_2 - E\beta_3)\ln\left(\frac{a+b}{2\rho}\kappa\right) + \frac{Ca\beta_2 - Eb\beta_3}{a+b}$$

$$- (C\beta_3 + E\beta_2)\sin\theta\cos\theta - (C\beta_2 - E\beta_3)\cos^2\theta + O(\frac{\kappa^n}{\rho^n}),$$

where n is some positive integer.

Matching the inner expansion onto the outer expansion obtained by (3.103,104) at order $ln\rho$ requires

$$C\beta_3 + E\beta_2 = -\frac{1}{4\pi}F_2'(s) \tag{4.2.12}$$

and

$$C\beta_2 - E\beta_3 = \frac{1}{4\pi}F_3'(s). \tag{4.2.13}$$

Solving these equations simultaneously for C and E results in:

$$C = \frac{\beta_2}{4\pi}F_3'(s) - \frac{\beta_3}{4\pi}F_2'(s)$$

and

$$E = -\frac{\beta_3}{4\pi} F_3'(s) - \frac{\beta_2}{4\pi} F_2'(s).$$

Thus u_2 and u_3 may be expressed as

$$u_2 \sim \frac{1}{4\pi}\left(\ln\left(\frac{a+b}{2\rho}\kappa\right) - \sin^2\theta + \frac{a\beta_3^2 + b\beta_2^2}{a+b}\right)F_2'(s) + \left(\sin\theta\cos\theta - \frac{\beta_2\beta_3(a-b)}{a+b}\right)F_3'(s)$$

and

$$u_3 \sim \frac{1}{4\pi}\left(\ln\left(\frac{a+b}{2\rho}\kappa\right) - \cos^2\theta + \frac{a\beta_2^2 + b\beta_3^2}{a+b}\right)F_3'(s) + \left(\sin\theta\cos\theta - \frac{\beta_2\beta_3(a-b)}{a+b}\right)F_2'(s).$$

Matching at order ρ^0 requires

$$4\pi\left[e_2 + J_2(s)\right] = \left(\ln\left(\frac{a+b}{4\varepsilon}\kappa\right) - 1 + \frac{a\beta_3^2 + b\beta_2^2}{a+b}\right)F_2'(s) - \frac{\beta_2\beta_3(a-b)}{a+b}F_3'(s)$$

and

$$4\pi\left[e_3 + J_3(s)\right] = \left(\ln\left(\frac{a+b}{4\varepsilon}\kappa\right) - 1 + \frac{a\beta_2^2 + b\beta_3^2}{a+b}\right)F_3'(s) - \frac{\beta_2\beta_3(a-b)}{a+b}F_2'(s).$$

These relationships may be combined and written in indices notation as

$$4\pi\,[\,c_i + J_i(s)\,] = \left\{\delta_{ij}\left[\ln\left(\frac{a+b}{4\varepsilon}\kappa\right) - \frac{1}{2} + \frac{a-b}{2(a+b)}\right] - \frac{a-b}{a+b}\beta_i\beta_j\right\}F_j'(s) \tag{4.2.14}$$

where $(i, j) = 2$ or 3.

In the Z-plane, pressure $p^{(i)}$ is given by (see appendix A)

$$p^{(i)} = -4\,Im\left(\frac{d\Phi}{dZ}\right) + p_0^{(i)}, \tag{A.30}$$

where Im denotes the imaginary part of the complex variable. But, in the Z-plane, Φ may be obtained by the aid of (4.2.8) and (4.1.6) as

$$\Phi = \left(\frac{C}{2} + i\frac{E}{2}\right)\ln\frac{Z + \sqrt{Z^2-(a^2-b^2)}}{2}.$$

Hence

$$\frac{d\Phi}{dZ} = \left(\frac{C}{2} + i\frac{E}{2}\right)\frac{1}{\sqrt{Z^2-(a^2-b^2)}}.$$

Thus

$$p^{(i)} = -4\,Im\left\{\left(\frac{C}{2} + i\frac{E}{2}\right)[Z^2 - (a^2-b^2)]^{-\frac{1}{2}}\right\} + p_0^{(i)}.$$

Upon expanding the bracket by the binomial theorem, the asymptotic form of the pressure may be determined as follows:

$$\begin{aligned} p^{(i)} &\sim -2\,Im\left\{(C+iE)\left[\frac{1}{Z} + \frac{1}{2Z^3}(a^2-b^2) + \cdots\right]\right\} + p_0^{(i)} \\ &\sim -2Im\left\{(C+iE)\left[\frac{e^{-i\phi}}{\rho^{(i)}} + \frac{e^{-3i\phi}}{2\rho^{(i)^3}}(a^2-b^2) + \cdots\right]\right\} + p_0^{(i)} \\ &\sim \frac{2C\sin\phi - 2E\cos\phi}{\rho^{(i)}} + \frac{C\sin(3\phi) - E\cos(3\phi)}{\rho^{(i)^3}}(a^2-b^2) + p_0^{(i)}, \end{aligned}$$

or $p^{(i)}$ can be expressed in terms of polar coordinates $(\rho^{(i)}, \theta)$ as

$$p^{(i)} \sim \frac{2C\sin(\theta-\lambda) - 2E\cos(\theta-\lambda)}{\rho^{(i)}} + \frac{C\sin(3\theta-3\lambda) - E\cos(3\theta-3\lambda)}{\rho^{(i)^3}}(a^2-b^2) + p_0^{(i)}.$$

Upon substitution $p^{(i)} = \kappa p$, $p_0^{(i)} = \kappa p_0$ and $\rho^{(i)} = \rho/\kappa$ [see (2.14)] the inner body condition for the outer flow field of pressure p is determined by

$$p \sim \frac{2}{\rho}[C(\sin\theta\cos\lambda - \cos\theta\sin\lambda) - E(\cos\theta\cos\lambda + \sin\theta\sin\lambda)] + p_0 + O(\frac{\kappa^2}{\rho^3})$$

or by (4.2.11) p may be written as

$$p \sim \frac{2}{\rho}[-(C\beta_3 + E\beta_2)\cos\theta + (C\beta_2 - E\beta_3)\sin\theta] + p_o + O(\frac{\kappa^2}{\rho^3}).$$

Matching inner expansion onto the outer expansion obtained by (3.60) at order ρ^{-1} requires

$$C\beta_3 + E\beta_2 = -\frac{1}{4\pi}F_2'(s) \qquad (4.2.15)$$

and

$$C\beta_2 - E\beta_3 = \frac{1}{4\pi}F_3'(s).$$

which agree with the matching obtained by the expansions of the velocity given by (4.2.12,13). Matching at order ρ^0 requires

$$p_0 = \frac{1}{4\pi}\left\{\int_0^{s-\epsilon} + \int_{s+\epsilon}^{1}\right\}\frac{[R_j(s) - R_j(\hat{s})]}{|R(s) - R(\hat{s})|^3}F_j'(\hat{s})d\hat{s} + O(1). \qquad (4.2.17)$$

The force per unit length which the fluid exerts on the body, $F(s)$, is determined by letting $\xi = e^{i\phi}$ in the following equation: (see appendix A)

$$\mathcal{F}x'' + i\mathcal{F}y'' = 2\left|\left[\frac{dw}{d\xi}\right]^{-1}\left\{\left[\frac{dw}{d\xi}\right]\Phi - w\left[\frac{d\Phi}{d\xi}\right] - \left[\frac{d\chi}{d1\xi}\right]^*\right\}\right|_{B'}^{A'} \qquad (A.50)$$

where $\mathcal{F}x''$ and $\mathcal{F}y''$ are respectively the components of $F(s)$ in the direction of the x'' and

y''-axis and where A' and B' correspond to $\phi = 0$ and $\phi = 2\pi$, respectively. From (4.1.3) and(4.2.8-9) $[dw/d\xi]^*$, $[d\Phi/d\xi]^*$ and $[d\chi/d\xi]^*$ are obtained as

$$\left[\frac{dw}{d\xi}\right]^* = \frac{a+b}{2} - \frac{a-b}{2}\xi^{*-2}; \qquad \left[\frac{d\Phi}{d\xi}\right]^* = \left(\frac{C}{2} - i\frac{E}{2}\right)\xi^{*-1}$$

and

$$\begin{aligned}\left[\frac{d\chi}{d\xi}\right]^* = &\left(\frac{C}{2} + i\frac{E}{2}\right)\frac{a+b}{2}\left[\ln\left(\frac{a+b}{2}\xi^*\right) + 1\right] - \left[C\left(\ln\frac{a+b}{2} + \frac{a}{a+b}\right)\right.\\ &\left.+ iE\left(\ln\frac{a+b}{2} + \frac{b}{a+b}\right)\right]\frac{a+b}{2} + \left(\frac{a-b}{2}\right)\left(\frac{C}{2} + i\frac{E}{2}\right)\xi^{*-2}(1 - \ln\xi^*)\\ &- \left[\frac{C}{2}\left(a - \frac{a-b}{2}\ln\frac{a+b}{2}\right) - i\frac{E}{2}\left(b + \frac{a-b}{2}\ln\frac{a+b}{2}\right)\right]\xi^{*-2}\end{aligned}$$

Thus $\mathcal{F}x'' + i\mathcal{F}y''$ on $\xi = e^{i\phi}$ is equal to

$$\begin{aligned}2\Bigg[&\left(\frac{a+b}{2} - \frac{a-b}{2}e^{2i\phi}\right)^{-1}\Bigg\{\left(\frac{a+b}{2} - \frac{a-b}{2}e^{2i\phi}\right)\left(\frac{C}{2} + i\frac{E}{2}\right)\left(\ln\frac{a+b}{2} + i\phi\right)\\ &- \left(\frac{a+b}{2}e^{i\phi} + \frac{a-b}{2}e^{-i\phi}\right)\left(\frac{C}{2} - i\frac{E}{2}\right)e^{i\phi} - \frac{a+b}{2}\left(\frac{C}{2} + i\frac{E}{2}\right)\\ &\times\left(\ln\frac{a+b}{2} + 1 - i\phi\right) + \frac{a+b}{2}\left[C\left(\ln\frac{a+b}{2} + \frac{a}{a+b}\right)\right.\\ &\left.+ iE\left(\ln\frac{a+b}{2} + \frac{b}{a+b}\right)\right] - \frac{a-b}{2}\left(\frac{C}{2} + i\frac{E}{2}\right)(1 + i\phi)e^{2i\phi}\\ &+ \left[a\frac{C}{2} - ib\frac{E}{2} - \frac{a-b}{2}\left(\frac{C}{2} + i\frac{E}{2}\right)\ln\frac{a+b}{2}\right]e^{2i\phi}\Bigg\}\Bigg|_{\phi=2\pi}^{\phi=0}\end{aligned}$$

or

$$\begin{aligned}\mathcal{F}x'' + i\mathcal{F}y'' = \frac{a}{b}\Bigg[&-\left(\frac{a+b}{2} - \frac{a-b}{2}\right)\left(\frac{C}{2} + i\frac{E}{2}\right)2\pi i - \frac{a+b}{2}\left(\frac{C}{2} + i\frac{E}{2}\right)2\pi i\\ &+ \frac{a-b}{2}\left(\frac{C}{2} + i\frac{E}{2}\right)2\pi i\Bigg] = 4\pi(E - iC)\end{aligned}$$

Therefore

$$\mathcal{F}x'' = 4\pi E \qquad \textit{and} \qquad \mathcal{F}y'' = -4\pi C.$$

But the components of $F(s)$ in the $(x^{(i)}, y^{(i)})$-coordinate system are determined by (see figure 4.3)

$$F_2(s) = \mathcal{T}x''\cos\lambda - \mathcal{T}y''\sin\lambda \quad and \quad F_3(s) = \mathcal{T}x''\sin\lambda + \mathcal{T}y''\cos\lambda .$$

Therefore, by the aid of (4.2.11-13) F_i may be written as

$$F_2(s) = 4\pi\,(\,C\beta_3 + E\beta_2\,) = -F_2'(s) \tag{4.2.18}$$

and

$$F_3(s) = 4\pi\,(\,E\beta_3 - C\beta_3\,) = -F_3'(s), \tag{4.2.19}$$

which relations verify the solution. It should be noted that F'_1 in (4.1.18) and (F'_2, F'_3) in (4.2.18,19) have come from the outer expansion, that is, they have satisfied the equations of motion (Oseen's equation) together with the uniform flow condition at infinity, whilst F_1, F_2 and F_3 has been derived in the inner expansion by the use of equations (4.1.12) and (A.50) (completely independent of the outer expansion), in other words, they have satisfied Stokes equations along with the no slip boundary condition on the body surface. Therefore, the solution may be verified to be in complete agreement with the exact solution of the problem {that is, ($F = -F'$) [obtained by the matched asymptotic expansion i.e., by the aid of (4.1.10) and (4.2.11-13)] simultaneously satisfies the equations of motion and both boundary conditions}, if and only if the assumption of the independence of the local flow field on the z-axis is satisfied (i.e., the curvature of the body centreline being large enough everywhere, as well as the cross-sectional shape varying sufficiently slowly). In addition, it has been assumed that the governing equations for the inner region are the Stokes equations, in other words, [see (4.1.3)] the term $\kappa R_e \boldsymbol{u}\cdot\nabla\boldsymbol{u}$ in the Navier-Stokes equation is assumed to be negligibly small. Therefore, (since R_e is of order unity) the slenderness parameter (κ) has to be small enough to satisfy the validity of this assumption.

4.3 - General cross-section - Longitudinal motion

We may now consider a body with arbitrary cross-section, cross-sectional shape of which is given by (2.5) as

$$\rho' = r_0\,\lambda\,(s, \theta\,),$$

where λ is a dimensionless function of s and θ (see figure 4.4).

The governing vorticity equation for longitudinal motion , $u' = (u_l', 0, 0)$, by neglecting the dependence on the z-axis is given by (see Appendix A)

$$\frac{\partial^2 \omega}{\partial x'^2} + \frac{\partial^2 \omega}{\partial y'^2} = 0\,, \tag{4.3.1}$$

where[see (A.8)] $\omega = (0, \partial u_l'/\partial y', - \partial u_l'/\partial x')$, is the fluid vorticity. Batchelor obtained a solution for the velocity valid in the region at far distances from the boundary as [see Batchelor (1970)]

$$u_{1\infty}' = \frac{f_1'(s)}{2\pi\mu}\left[\ln\left(\frac{r_0\lambda_s}{\rho'}\right) + Q + O\left(\frac{r_0\lambda_s}{\rho'}\right)\right],$$

where $u'_{l\infty}$ and f'_l are the corresponding dimensional form of $u_{l\infty}$ and F'_l, respectively, and where Q is a dimensionless constant whose value depends on cross-sectional shape, although it might vary along the body length, and $r_0\lambda_s$ is the radius of the circle of the equivalent cross-section i.e. $2\pi r_0\lambda_s$ is the perimeter of the local cross-section at point $\hat{s} = s$. Upon using quantities made dimensionless by the r_0, U and μ the velocity may be expressed in dimensionless form as

$$u_{1\infty}^{(i)} = \frac{F_1'(s)}{2\pi}\left[\ln\frac{\lambda_s}{\rho^{(i)}} + Q + O\left(\frac{\lambda_s}{\rho^{(i)}}\right)\right].$$

Thus, the inner body condition for the outer flow field may be determined by [see (2.14)]

$$u_1 = \frac{F_1'(s)}{2\pi}\left(\ln\frac{\kappa\lambda_s}{\rho} + Q\right) + O\left(\frac{\lambda F_1'\kappa}{\rho}\right).$$

Letting $Q = ln\, q$ and neglecting terms of order κ, u_l may be written as

$$u_1 = \frac{F_1'(s)}{2\pi}\ln\left(\frac{q\lambda_s}{\rho}\kappa\right).$$

Matching velocity u_l onto that obtained for the outer expansion given by (3.102) is straight forw... if we choose

$$F_1'(s)\left[2\ln\left(\frac{q\lambda_s}{2\varepsilon}\kappa\right)+1\right] = 4\pi\,[\,e_1 + J_1(s)\,]\,. \qquad (4.3.2)$$

Upon comparing relationship (4.3.2) with that obtained for a body with elliptical cross-section (4.1.16), we see that for an elliptical cross-section with semi-diameters a and b, the value of $q\lambda_s$ is determined by

$$q\lambda_s = \frac{1}{2}(a+b),$$

and on choosing $a = b = \lambda_s$ i.e. for a body with a circular cross-section, the value of q is equal to unity. Comparing the result (4.3.2) with that obtained for a circular cross-section, $qr_0\lambda_s$ can be regarded as the radius of a circle which is equivalent to this cross-section in the sense that a given total longitudinal force at the surface of the circular cylinder of this radius produces the same flow field in the region at far distances from the body surface [Batchelor (1970)]. This conclusion is consistent with that obtained in section 4.1 for an elliptical cross-section. It is clearly shown that the velocity is independent of any rotation of the axes - a property of the circular cross-section. In addition, since relationship (4.1.10) is valid for any cross-section, it follows that this conclusion is held for any cross-sectional shape.

4.4 - General cross-section - Transverse motion

The vorticity equation associated with the components of velocity $u' = (0, u_2', u_3')$ and force $f'(s) = (0, f_2', f_3')$ in the cross-sectional plane, (x', y'), is again of Laplacian form which is by neglecting the dependence on the z-axis is given by (4.3.1), although the vorticity vector is now $\omega = (\omega_1, 0, 0)$, where [see (A.9)]

$$\omega_1 = \frac{\partial u_3'}{\partial x'} - \frac{\partial u_2'}{\partial y'}.$$

Batchelor gave a solution for flow fields (u', p') at far distances from boundary as

$$u'_{i\infty} = \frac{f_j'}{4\pi\mu}\left[\delta_{ij}\left(\ln\frac{r_0\lambda_s}{\rho'} - \frac{1}{2}\right) + \frac{X_i'X_j'}{\rho'^2} + Q_{ij} + O\left(\frac{r_0\lambda_s}{\rho'}\right)\right]$$

and

$$p'_\infty = \frac{f_j'X_j'}{2\pi\rho'^2} + p'_0,$$

where $(i, j) = 2 \text{ or } 3$; $X_2' = x'$; $X_3' = y'$ and Q_{ij} is a constant dimensionless symmetric tensor whose magnitude depends on cross- sectional shape, although strictly speaking it, like q, might vary along the body length. Again using quantities maoe dimensionless by r_0, U, and μ, the dimensionless forms of the velocity and pressure may be written as

$$u_{i\infty}^{(i)} = \frac{F_j'}{4\pi}\left[\delta_{ij}\left(\ln\frac{\lambda_s}{\rho^{(i)}} - \frac{1}{2}\right) + \frac{X_i^{(i)}X_j^{(i)}}{\rho^{(i)2}} + Q_{ij} + O\left(\frac{\lambda_s}{\rho^{(i)}}\right)\right] \tag{4.4.1}$$

and

$$p_\infty^{(i)} = \frac{F_j'(s)\,X_j^{(i)}}{2\pi\,\rho^{(i)2}} + p_0^{(i)}. \tag{4.4.2}$$

Thus, the inner body conditions for the outer flow field are determined by [see (2.14)]

$$u_i = \frac{F_j'(s)}{4\pi}\left[\delta_{ij}\left(\ln\frac{\kappa\lambda_s}{\rho} - \frac{1}{2}\right) + \frac{X_iX_j}{\rho^2} + Q_{ij}\right] + O\left(\frac{\kappa\lambda_sF_j'}{\rho}\right),$$

and

$$p = \frac{F_2'\cos\theta + F_3'\sin\theta}{2\pi\rho} + p_0.$$

Matching the pressure onto that obtained for the outer expansion given by (3.60) is straightforward if we choose

$$p_0 = \frac{1}{4\pi}\left\{\int_0^{s-\varepsilon} + \int_{s+\varepsilon}^{1}\right\} \frac{[R_j(s) - R_j(\hat{s})]}{|R(s) - R(\hat{s})|^3} F_j'(\hat{s}) d\hat{s} + O(1),$$

which is the same as that obtained for the elliptical cross-section (4.2.17).

Matching the velocity onto that obtained for the outer expansion given by (3.106) results in :

$$F_j'(s)\left[\delta_{ij}\left(\ln\frac{\kappa\lambda_s}{2\varepsilon} - \frac{1}{2}\right) + Q_{ij}\right] = 4\pi\,[\,e_i + J_i(s)\,]. \tag{4.4.3}$$

Hence

$$F_2'(s)\left(2\ln\frac{\kappa\lambda_s}{2\varepsilon} - 1 + 2Q_{22}\right) + 2F_3'(s)Q_{23} = 8\pi[e_2 + J_2(s)] \tag{4.4.4}$$

and

$$F_3'(s)\left(2\ln\frac{\kappa\lambda_s}{2\varepsilon} - 1 + 2Q_{33}\right) + 2F_2'(s)Q_{23} = 8\pi\,[e_3 + J_3(s)]. \tag{4.4.5}$$

Upon comparing (4.4.3) with the results obtained for an elliptical cross-section given by (4.2.14) the value of Q_{ij} for an elliptical cross-section with semi-diameters a and b, whose direction of the larger diameter ($2a$) given by the unit vector β, is determined by

$$Q_{ij} = \delta_{ij}\left(\ln\frac{a+b}{2\lambda_s} + \frac{a-b}{2(a+b)}\right) - \beta_i\beta_j\frac{a-b}{a+b}, \tag{4.4.6}$$

where $2\pi r_0\,\lambda_s$ is the perimeter of the local cross-section. Hence λ_s is determined by

$$\lambda_s = \left(\frac{2a}{\pi}\right)E\left[\sqrt{1 - \frac{b^2}{a^2}}, \frac{\pi}{2}\right] \tag{4.4.7}$$

where $E(K,\pi/2)$ is the complete elliptic integral of the second kind defined by the numerical values of which for various values of $K = (1 - b^2/a^2)^{1/2}$ are available in tables. For a circular cross-section (i.e., $a = b$), Q_{ij} is equal to zero. The Q_{ij} obtained

$$E\left(K,\frac{\pi}{2}\right) = \int_0^{\frac{\pi}{2}}\sqrt{1 - K^2\sin^2\varphi}\, d\varphi\,, \qquad (4.4.8)$$

by (4.4.6) agrees with the result obtained by the use of elliptic cylinder coordinates [see Batchelor (1970)]. In general, Since Q_{ij} is a symmetric tensor and $Q_{ij} = 0$, it follows that it has only three independent components and four non-zero ones. And since the geometry of an ellipse is completely determined by only three scalars (i.e., a, b and λ), Batchelor concluded that any cross-sectional shape, for transverse motion, can be regarded as an equivalent ellipse with certain dimensions and orientation.

However, the advantage of the complex variable method, which is used in the present study is that it can easily be extended to any cross-sectional shape. That is, the values of q and Q_{ij} for a specified cross-sectional shape can be determined explicitly by applying the same procedure as that applied for the elliptical cross-section in 4.1-2. Moreover, for determination of the Q_{ij}, it is worth noting that, since the pressure is directly independent of the cross-sectional shape [see (4.4.2) and (3.60)] and is determined via the complex variable by the imaginary part of the derivation of the analytic function Φ with respect to Z [see (A.30)], it follows that $\Phi(Z)$ possesses the same function of Z, for any cross-sectional shape. And we recall that the relationship (4.1.10) is also held for any cross-sectional shape. Therefore, for the determination of q and Q_{ij}, it only remains to determine the function $\chi(Z)$, in addition to the determination of the transformation function.

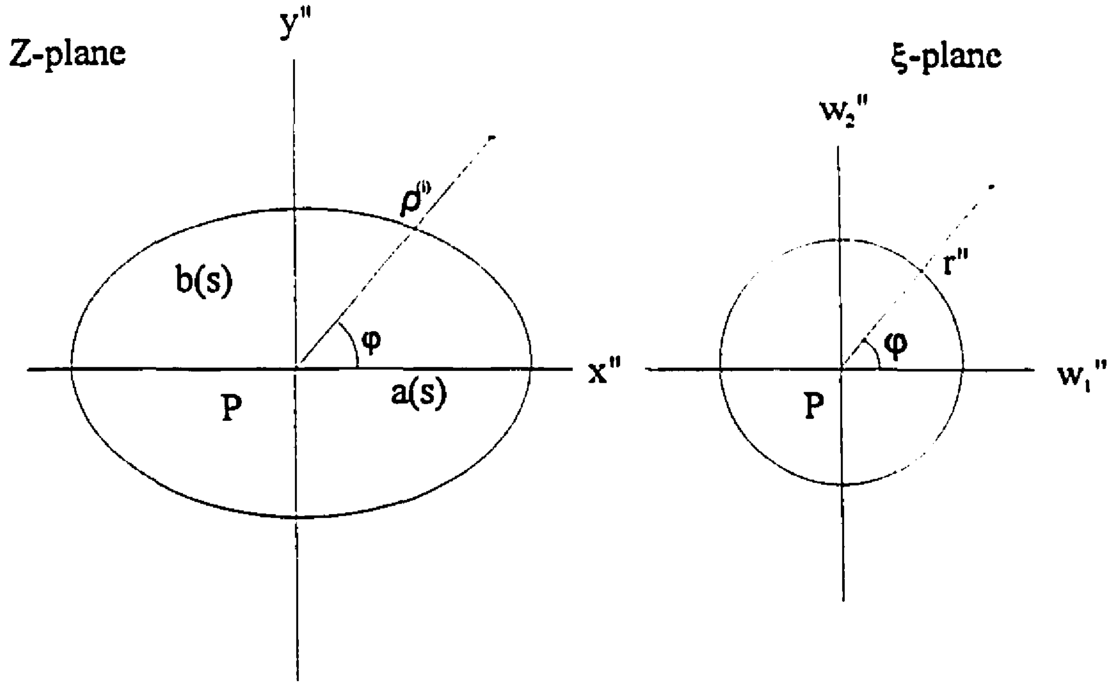

Figure 4.1 : Transformation of elliptical cross-section in Z-plane onto a unit circle in ξ-plane.

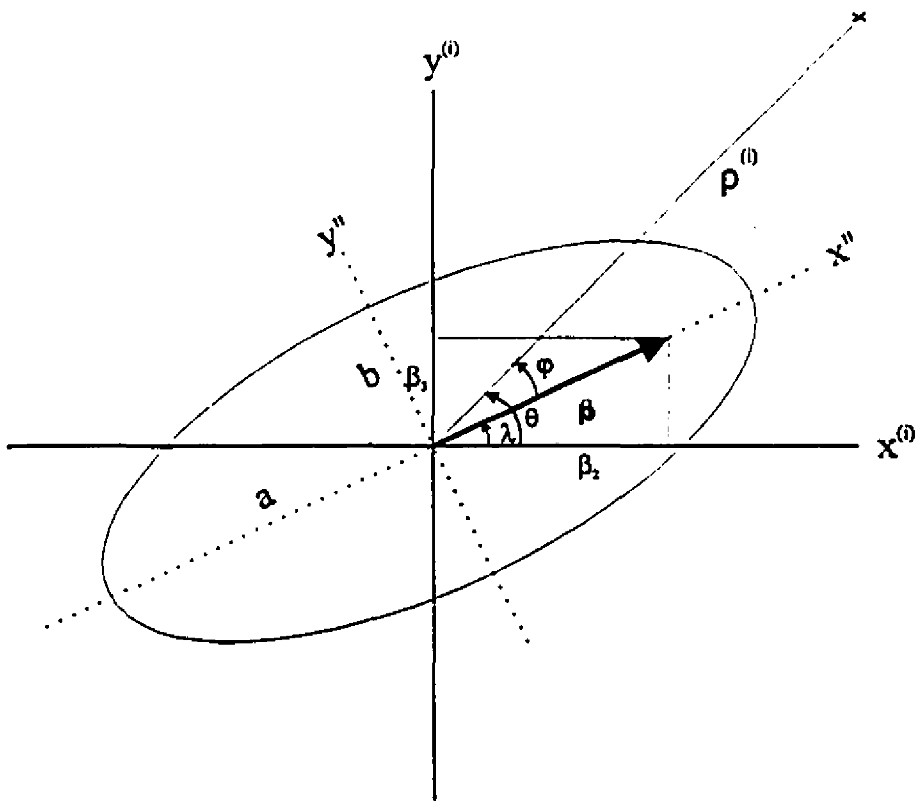

Figure 4.2 : Position of the elliptical cross-section in general axes ($x^{(i)}, y^{(i)}$) .

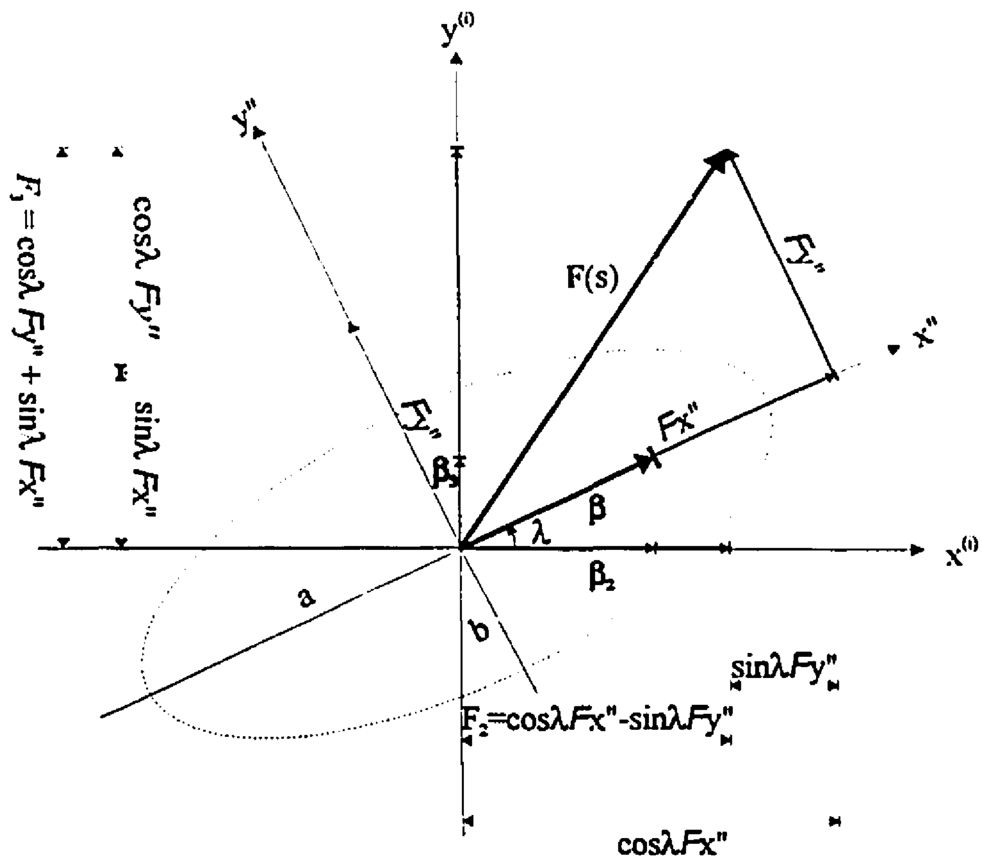

Figure 4.3 : The components of force F(s) in the $(x^{(i)}, y^{(i)})$-coordinate system.

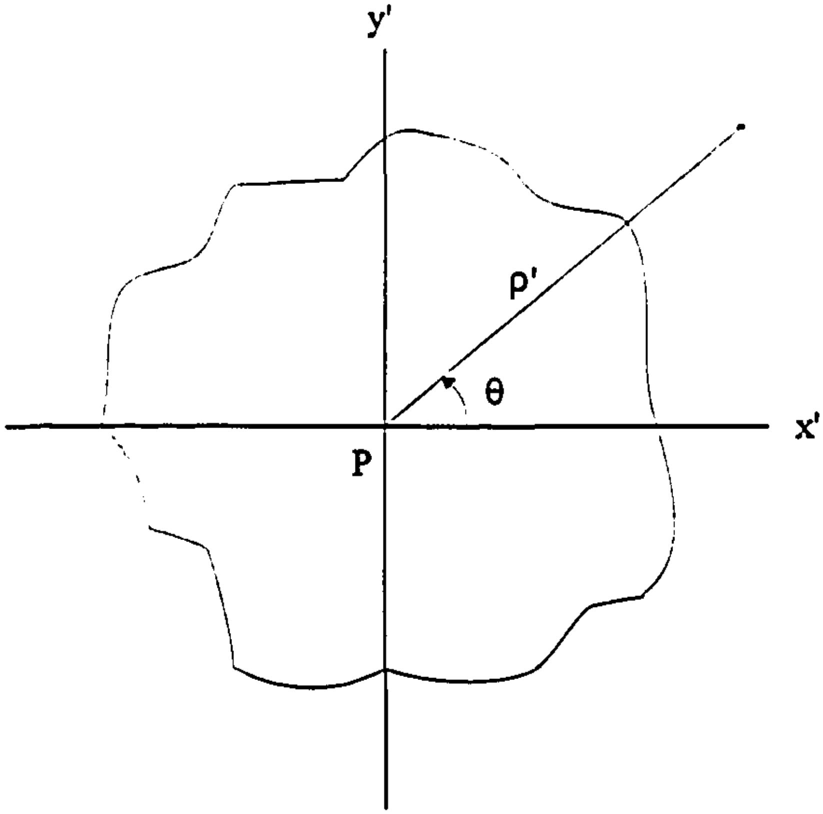

Figure 4.4 : General cross-sectional shape in the dimensional local polar Coordinate (ρ', Θ) at point P.

CHAPTER 5

5 - Force integral equation

We now return to the question of the determination of the function $F(s)$ representing the force density which the fluid exerts on the body. The results of the matching given by relationships (4.3.2) and (4.4.4.5) in the inner expansion provide a basis to determine the force integral equation. However, the force density which the fluid exerts on the body, $F(s)$, has the same magnitude as $F'(s)$ but with the opposite direction. Thus substituting $F_j' = -F_j$ into the relationships (4.3.2) and (4.4.4.5) results in :

$$F_1(s)\left(2\ln\frac{\kappa q\lambda_s}{2\varepsilon}+1\right) = -4\pi\,[\,e_1 + J_1(s)\,], \tag{5.1}$$

$$F_2(s)\left(2\ln\frac{\kappa\lambda_s}{2\varepsilon} - 1 + 2Q_{22}\right) + 2F_3(s)Q_{23} = -8\pi[e_2 + J_2(s)] \tag{5.2}$$

and

$$F_3(s)\left(2\ln\frac{\kappa\lambda_s}{2\varepsilon} - 1 + 2Q_{33}\right) + 2F_2(s)Q_{23} = -8\pi\,[e_3 + J_3(s)]\,. \tag{5.3}$$

where J_i defined by (3.105) may be written in terms of F_j as

$$J_i(s) = -\frac{1}{8\pi}\left\{\int_0^{s-\varepsilon} + \int_{s+\varepsilon}^{+1}\right\} g_{ij}[R - R(\hat{s})] F_j(\hat{s}) d\hat{s}. \tag{5.4}$$

Relationships (5.1-3) may be written as

$$F_1\left(2\ln\frac{q\lambda_s}{2\varepsilon} - 1\right) + 2F_1(1 + \ln q) = \frac{1}{2}\left(\int_0^{s-\varepsilon} + \int_{s+\varepsilon}^{+1}\right) g_{1j}(R - \hat{R})\hat{F}_j d\hat{s} - 4\pi e_1, \tag{5.5}$$

$$F_2\left(2\ln\frac{\kappa\lambda_s}{2\varepsilon} - 1\right) + 2(Q_{22}F_2 + Q_{23}F_3) = \left(\int_0^{s-\varepsilon} + \int_{s+\varepsilon}^{+1}\right) g_{2j}(R - \hat{R})\hat{F}_j d\hat{s} - 8\pi e_2 \tag{5.6}$$

and

$$F_3\left(2\ln\frac{\kappa\lambda_s}{2\varepsilon} - 1\right) + 2(Q_{33}F_3 + Q_{32}F_2) = \left(\int_0^{s-\varepsilon} + \int_{s+\varepsilon}^{+1}\right) g_{3j}(R - \hat{R})\hat{F}_j d\hat{s} - 8\pi e_3, \tag{5.7}$$

where $\hat{R}$ and $\hat{F}_j$ are respectively the values of R and F_j at point s = ŝ. But, since

$$\begin{aligned}\left(\int_0^{s-\varepsilon} + \int_{s+\varepsilon}^{+1}\right)\frac{d\hat{s}}{|s - \hat{s}|} &= \int_0^{s-\varepsilon}\frac{d\hat{s}}{s - \hat{s}} + \int_{s+\varepsilon}^{+1}\frac{d\hat{s}}{\hat{s} - s} \\ &= |-\ln(s - \hat{s})|_0^{s-\varepsilon} + |\ln(\hat{s} - s)|_{s+\varepsilon}^{+1} \\ &= -\ln\varepsilon + \ln s + \ln(1 - s) - \ln\varepsilon \\ &= -2\ln\varepsilon + \ln[s(1 - s)],\end{aligned} \tag{5.8}$$

lnε may be determined by

$$\ln\varepsilon = \frac{1}{2}\left\{\ln[s(1 - s)] - \left(\int_0^{s-\varepsilon} + \int_{s+\varepsilon}^{+1}\right)\frac{d\hat{s}}{|s - \hat{s}|}\right\}. \tag{5.9}$$

Thus, the relationships (5.5-7) may be written as

$$F_1\left(2\ln\frac{q\lambda_s}{2\sqrt{s(1-s)}}-1\right)+2F_1(1+\ln q) = \frac{1}{2}\left(\int_0^{s-\varepsilon}+\int_{s+\varepsilon}^{1}\right)\left[g_{1j}(R-\hat{R})\hat{F}_j-\frac{2F_1}{|s-\hat{s}|}\right]d\hat{s}$$
$$-4\pi e_1 .$$
(5.10)

$$F_2\left(2\ln\frac{\kappa\lambda_s}{2\sqrt{s(1-s)}}-1\right)+2(Q_{22}F_2+Q_{23}F_3) = \left(\int_0^{s-\varepsilon}+\int_{s+\varepsilon}^{1}\right)\left[g_{2j}(R-\hat{R})\hat{F}_j-\frac{F_2}{|s-\hat{s}|}\right]d\hat{s}$$
$$-8\pi e_2$$
(5.11)

and

$$F_3\left(2\ln\frac{\kappa\lambda_s}{2\sqrt{s(1-s)}}-1\right)+2(Q_{33}F_3+Q_{32}F_2) = \left(\int_0^{s-\varepsilon}+\int_{s+\varepsilon}^{1}\right)\left[g_{3j}(R-\hat{R})\hat{F}_j-\frac{F_3}{|s-\hat{s}|}\right]d\hat{s}$$
$$-8\pi e_3 .$$
(5.12)

However, from (3.71-76) for $\rho = 0$ and $\hat{s} \neq s$ the value of g_{ij} is determined by

$$g_{11} = \frac{2}{|s-\hat{s}|}\,;\; g_{22} = g_{33} = \frac{1}{|s-\hat{s}|}\,;\; g_{ij} = 0 \;\; for \;\; i \neq j. \qquad (5.13)$$

Thus, as $R(\hat{s}) \longrightarrow R(s)$, that is; $\rho = 0$ and $\hat{s} \longrightarrow s$ (see figure 3.3), the integrands in (5.10-12) may be evaluated as

$$\begin{aligned} g_{1j}(R-\hat{R})F_j(\hat{s}) - \frac{2F_1(s)}{|s-\hat{s}|} &= \frac{2F_1(\hat{s})}{|s-\hat{s}|} - \frac{2F_1(s)}{|s-\hat{s}|} \\ &= \frac{2\left\{\left[\frac{dF_1}{d\hat{s}}(s)\right](s-\hat{s}) + \cdots\right\}}{|s-\hat{s}|} \\ &= 2\frac{dF_1(s)}{d\hat{s}}sgn(s-\hat{s}) + O(\varepsilon), \end{aligned} \qquad (5.14)$$

$$g_{2j}(R-\hat{R})F_j(\hat{s}) - \frac{F_2(s)}{|s-\hat{s}|} = \frac{F_2(\hat{s})}{|s-\hat{s}|} - \frac{F_2(s)}{|s-\hat{s}|}$$

$$= \frac{\left\{\left[\frac{dF_2}{d\hat{s}}(s)\right](s-\hat{s}) + \dots\right\}}{|s-\hat{s}|} \tag{5.15}$$

$$= \frac{dF_2(s)}{d\hat{s}} sgn(s-\hat{s}) + O(\varepsilon),$$

and, similarly,

$$g_{3j}(R-\hat{R})F_j(\hat{s}) - \frac{F_3(s)}{|s-\hat{s}|} = \frac{dF_3(s)}{d\hat{s}} sgn(s-\hat{s}) + O(\varepsilon). \tag{5.16}$$

We see as $R(\hat{s}) \longrightarrow R(s)$, all integrals in (5.10-12) are finite, therefore, they converge as $\varepsilon \longrightarrow 0$. Thus (5.10-12) may be written as

$$F_1\left(2\ln\frac{q\lambda_s}{2\sqrt{s(1-s)}} - 1\right) + 2F_1(1+\ln q) = \frac{1}{2}\int_0^1\left[g_{1j}(R-\hat{R})\hat{F}_j - \frac{2F_1}{|s-\hat{s}|}\right]d\hat{s} - 4\pi e_1; \tag{5.17}$$

$$F_2\left(2\ln\frac{\kappa\lambda_s}{2\sqrt{s(1-s)}} - 1\right) + 2(Q_{22}F_2 + Q_{23}F_3) = \int_0^1\left[g_{2j}(R-\hat{R})\hat{F}_j - \frac{F_2}{|s-\hat{s}|}\right]d\hat{s} - 8\pi e_2; \tag{5.18}$$

$$F_3\left(2\ln\frac{\kappa\lambda_s}{2\sqrt{s(1-s)}} - 1\right) + 2(Q_{33}F_3 + Q_{32}F_2) = \int_0^1\left[g_{3j}(R-\hat{R})\hat{F}_j - \frac{F_3}{|s-\hat{s}|}\right]d\hat{s} - 8\pi e_3. \tag{5.19}$$

Letting i_1, i_2 and i_3 be unit vectors along the *1*, *2* and *3* directions, respectively, where

$$i_1 = \frac{dR}{ds} = t(s), \tag{5.20}$$

the relationships (5.17-19) may be combined and written in a unique equation as

$$(F_1 i_1 + F_2 i_2 + F_3 i_3)(2\ln\frac{\kappa\lambda_s}{2\sqrt{s(s-\hat{s})}} - 1) + 2(1+\ln q)F_1 i_1 + 2(Q_{22}F_2 + Q_{23}F_3)i_2$$
$$+ 2(Q_{33}F_3 + Q_{32}F_2)i_3 = \int_0^1 \Big\{ \{[g_{1j}(R-\hat{R})]i_1 + [g_{2j}(R-\hat{R})]i_2 + [g_{3j}(R-\hat{R})]i_3\}\hat{F}_j$$
$$- \frac{F_1 i_1 + F_2 i_2 + F_3 i_3}{|s-\hat{s}|}\Big\} d\hat{s} - \frac{1}{2}\int_0^1 [g_{1j}(R-\hat{R})]\hat{F}_j i_1 \, d\hat{s} - 8\pi(e_1 i_1 + e_2 i_2 + e_3 i_3) + 4\pi e_1 i_1 \tag{5.21}$$

But

$$g_{1j}F_j i_1 + g_{2j}F_j i_2 + g_{3j}F_j i_3 = g\cdot F = F\cdot g. \tag{5.22}$$

and, similarly,[1]

$$(Q_{22}F_2 + Q_{23}F_3)i_2 + (Q_{32}F_2 + Q_{33}F_3)i_3 = Q_{2j}F_j i_2 + Q_{3j}F_j i_3 = Q\cdot F = F\cdot Q. \tag{5.23}$$

Therefore, the force equation (5.21) may be written as

$$F(s)(2\ln\frac{\kappa\lambda_s}{2\sqrt{s(s-\hat{s})}} - 1) + 2(1+\ln q)F(s)\cdot t(s)t(s) + 2F(s)\cdot Q(s) = 4\pi e\cdot t(s)t(s)$$
$$-8\pi e + \int_0^1 \Big\{ g[R(s)-R(\hat{s})]\cdot F(\hat{s}) - \frac{F(s)}{|s-\hat{s}|}\Big\} d\hat{s} - \frac{1}{2}\int_0^1 t(s)\cdot g[R(s)-R(\hat{s})]\cdot F(\hat{s})t(s)d\hat{s} \tag{5.24}$$

or

$$F(s)\cdot\Big\{\Big(2\ln\frac{\kappa\lambda_s}{2\sqrt{s(s-\hat{s})}} - 1\Big)I + 2(1+\ln q)t(s)t(s) + 2Q(s)\Big\}$$
$$= -8\pi e\cdot\Big[I - \frac{1}{2}t(s)t(s)\Big] + \int_0^1 \Big\{\Big[I - \frac{1}{2}t(s)t(s)\Big]\cdot g[R(s)-R(\hat{s})]\cdot F(\hat{s}) - \frac{F(s)}{|s-\hat{s}|}\Big\} d\hat{s}, \tag{5.25}$$

where I is the idemfactor. Equation (5.25) may be expressed in dimensional form as

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It is noted that $Q_{1j} = 0$; g_{ij} and Q_{ij} are symmetric tensors, since, in general, the dot product of a tensor with a vector depends on which unit vector of the tensor being operated, that is, vector $g.F = g_{ij} f_j i_i$ would not be equal to vector $F.g = g_{jk} f_j i_k$.

$$\frac{F^{*}(s)}{\mu U}\cdot\left\{\left(2\ln\frac{\kappa\lambda_s}{2\sqrt{s(s-\hat{s})}}-1\right)I+2(1+\ln q)t(s)t(s)+2Q(s)\right\}=$$
$$-8\pi e\cdot\left[I-\frac{1}{2}t(s)t(s)\right]+\frac{1}{\mu U}\int_0^1\left\{\left[I-\frac{1}{2}t(s)t(s)\right]\cdot g[R(s)-R(\hat{s})]\cdot F^{*}(\hat{s})-\frac{F^{*}(s)}{|s-\hat{s}|}\right\}d\hat{s}$$

or

$$F^{*}(s)\cdot\left\{\left(2\ln\frac{\kappa\lambda_s}{2\sqrt{s(s-\hat{s})}}-1\right)I+2(1+\ln q)t(s)t(s)+2Q(s)\right\}=$$
$$-8\pi\mu U\cdot\left[I-\frac{1}{2}t(s)t(s)\right]+\int_0^1\left\{\left[I-\frac{1}{2}t(s)t(s)\right]\cdot g[R(s)-R(\hat{s})]\cdot F^{*}(\hat{s})-\frac{F^{*}(s)}{|s-\hat{s}|}\right\}d\hat{s},$$
(5.26)

where $F^{*}(s)$ is the corresponding dimensional form of the force density $F(s)$. Similarly (5.5-7) may be written as

$$F^{*}(s)\cdot\left\{\left(2\ln\frac{\kappa\lambda_s}{2\varepsilon}-1\right)I+2(1+\ln q)t(s)t(s)+2Q(s)\right\}=$$
$$-8\pi\mu U\cdot\left[I-\frac{1}{2}t(s)t(s)\right]+\left[I-\frac{1}{2}t(s)\,t(s)\right]\cdot\left(\int_0^{s-\varepsilon}+\int_{s+\varepsilon}^{1}\right)g[R(s)-R(\hat{s})]\cdot F^{*}(\hat{s})\,d\hat{s}.$$
(5.27)

The force integral equation (5.27) is a Fredholm equation of the third kind which reduces to an algebraic equation for certain symmetric bodies and as shown by the relationships (5.9-19), its weak singularity on $\ln\varepsilon$ cancels from both sides of the equation. Hence, its solution is independent of ε. It can also be solved for $F^{*}(s)$ reiteratively as a power series in $(1/\ln\kappa)$ correct to the order κ as follows :

Letting

$$F^{*}(s)=\frac{f_0^{*}(s)}{\ln\kappa}+\frac{f_1^{*}(s)}{(\ln\kappa)^2}\cdots+\frac{f_n^{*}(s)}{(\ln\kappa)^{n+1}}+\frac{f_{n+1}^{*}(s)}{(\ln\kappa)^{n+2}}\cdots,\qquad(5.28)$$

the force integral equation (5.27) may be expressed as

$$\left(\frac{f_0^*(s)}{\ln\kappa}+\frac{f_1^*(s)}{(\ln\kappa)^2}\cdots+\frac{f_n^*(s)}{(\ln\kappa)^{n+1}}+\frac{f_{n+1}^*(s)}{(\ln\kappa)^{n+2}}\cdots\right)\cdot\left[\left(\ln\kappa+\ln\frac{\lambda_s}{2\varepsilon}-\frac{1}{2}\right)I+(1+\ln q)t(s)t(s)+Q(s)\right]$$
$$= -4\pi\mu U\cdot\left[I-\frac{1}{2}t(s)t(s)\right]+\frac{1}{2}\left(\int_0^{s-\varepsilon}+\int_{s+\varepsilon}^1\right)\left\{\left[I-\frac{1}{2}t(s)t(s)\right]\cdot g\,[R(s)-R(\hat{s})]\cdot\right.$$
$$\left.\left(\frac{f_0^*(\hat{s})}{\ln\kappa}+\frac{f_1^*(\hat{s})}{(\ln\kappa)^2}\cdots+\frac{f_n^*(\hat{s})}{(\ln\kappa)^{n+1}}+\frac{f_{n+1}^*(\hat{s})}{(\ln\kappa)^{n+2}}\cdots\right)\right\}d\hat{s}. \tag{5.29}$$

Thus the term of order unity gives

$$f_0^*(s) = -4\pi\mu U + 2\pi\mu U\cdot t(s)t(s). \tag{5.30}$$

The term of order *1/lnκ* gives

$$f_1^*(s) + f_0^*(s)\cdot\left[\left(\ln\frac{\lambda_s}{2\varepsilon}-\frac{1}{2}\right)I+(1+\ln q)t(s)t(s)+Q(s)\right]$$
$$= +\frac{1}{2}\left(\int_0^{s-\varepsilon}+\int_{s+\varepsilon}^1\right)\left[I-\frac{1}{2}t(s)t(s)\right]\cdot g\,[R(s)-R(\hat{s})]\cdot f_0^*(\hat{s})\,d\hat{s}. \tag{5.31}$$

The recurrence formula for determination of the higher order terms is obtained by

$$f_{n+1}^*(s) + f_n^*(s)\cdot\left[\left(\ln\frac{\lambda_s}{2\varepsilon}-\frac{1}{2}\right)I+(1+\ln q)\,t(s)\,t(s)+Q(s)\right] = J_n(s)\,, \tag{5.32}$$

where J_n is defined by

$$J_n(s) = +\frac{1}{2}\left(\int_0^{s-\varepsilon}+\int_{s+\varepsilon}^1\right)\left[I-\frac{1}{2}t(s)t(s)\right]\cdot g\,[R(s)-R(\hat{s})]\cdot f_n(\hat{s})\,d\hat{s}. \tag{5.33}$$

It is readily seen that the leading term in the expansion of the force $F(s)$ is independent of the cross-sectional shape. But, since the unit vector t which is involved in the leading term [see (5.30)], is a function of the position of the point under consideration [see (5.20)], it follows that the leading term is a function of the shape of the body centreline. However, the force density depends not only on the uniform undisturbed flow, the viscosity of the fluid, the body centreline configuration and transverse cross-sectional shape and hence R_e, but also on κ, the slenderness parameter of the body S. That is, a more slender body causes less force per unit length on the body.

It remains to determine the value of g_{ij}, appears in the integrand of the force integral equation defined by (see 3.61,62)

$$g_{ij}(X) = \delta_{ij}\Psi,_{kk}(X) - \Psi,_{ij}(X) ,$$

where $\boldsymbol{X} = \boldsymbol{R} - \boldsymbol{R}(\hat{s})$ and

$$\Psi(X) = \frac{2}{R_e}\int_0^{\frac{1}{2}R_e(X - e\cdot X)}\left(\frac{1 - e^{-\alpha}}{\alpha}\right)d\alpha .$$

Differentiating Ψ with respect to X_i may be written as

$$\Psi,_i = \alpha,_i\Psi,_\alpha ,$$

where

$$\alpha,_i = \frac{\partial\alpha}{\partial X_i} \qquad and \qquad \Psi,_\alpha = \frac{\partial\Psi}{\partial\alpha} .$$

Thus

$$\Psi,_{ij} = \alpha,_{ij}\Psi,_\alpha + \alpha,_i\Psi,_{\alpha j} . \tag{5.34}$$

But

$$\alpha = \frac{1}{2}R_e(X - e\cdot X)$$

and

$$\Psi,_{\alpha} = \frac{2}{R_e}\left(\frac{1 - e^{-\alpha}}{\alpha}\right) = \frac{4}{R_e^2}\left(\frac{1 - e^{-\frac{1}{2}R_e(X - e\cdot X)}}{X - e\cdot X}\right).$$

Hence

$$\alpha,_i = \frac{1}{2}R_e\left(\frac{X_i}{X} - e_i\right),$$

$$\alpha,_{ij} = \frac{1}{2}R_e\left(\frac{\delta_{ij}X - \frac{X_iX_j}{X}}{X^2}\right)$$
$$= \frac{1}{2}R_e\left(\frac{\delta_{ij}}{X} - \frac{X_iX_j}{X^3}\right)$$

and

$$\Psi,_{\alpha j} = \alpha,_j \Psi,_{\alpha\alpha}$$
$$= \frac{R_e}{2}\left(\frac{X_j}{X} - e_j\right)\frac{2}{R_e}\left(\frac{\alpha e^{-\alpha} - 1 + e^{-\alpha}}{\alpha^2}\right)$$
$$= \left(\frac{X_j}{X} - e_j\right)\frac{\left[\frac{1}{2}R_e(X - e\cdot X) + 1\right]e^{-\frac{1}{2}R_e(X - e\cdot X)} - 1}{\frac{1}{4}R_e^2(X - e\cdot X)^2}.$$

Therefore, Ψ_{ij} given by (5.34) may be written as

$$\Psi,_{ij} = \frac{2}{R_e}\left(\frac{\delta_{ij}}{X} - \frac{X_iX_j}{X^3}\right)\left\{\frac{1 - e^{-\frac{1}{2}R_e(X - e\cdot X)}}{(X - e\cdot X)}\right\} + \frac{2}{R_e}\left(\frac{X_i}{X} - e_i\right)\left(\frac{X_j}{X} - e_j\right)\times$$
$$\left\{\frac{\left[\frac{1}{2}R_e(X - e\cdot X) + 1\right]e^{-\frac{1}{2}R_e(X - e\cdot X)} - 1}{(X - e\cdot X)^2}\right\}.$$

Letting $i = j$

$$\Psi_{,il} = \frac{2}{R_e}\left(\frac{\delta_{il}}{X} - \frac{X_i X_l}{X^3}\right)\left\{\frac{1 - e^{-\frac{1}{2}R_e(X - e\cdot X)}}{(X - e\cdot X)}\right\} + \frac{2}{R_e}\left(\frac{X_i X_l}{X^2} - 2\frac{e_i X_l}{X} + e_i e_l\right) \times$$

$$\left\{\frac{\left[\frac{1}{2}R_e(X - e\cdot X) + 1\right]e^{-\frac{1}{2}R_e(X - e\cdot X)} - 1}{R_e(X - e\cdot X)^2}\right\}$$

$$= \frac{4}{X}\left\{\frac{1 - e^{-\frac{1}{2}R_e(X - e\cdot X)}}{R_e(X - e\cdot X)}\right\} + \frac{4}{X}(X - e\cdot X)\left\{\frac{\left[\frac{1}{2}R_e(X - e\cdot X) + 1\right]e^{-\frac{1}{2}R_e(X - e\cdot X)} - 1}{R_e(X - e\cdot X)^2}\right\}.$$

Hence $\Psi_{,kk}$ is obtained by

$$\Psi_{,kk} = \frac{2}{X}e^{-\frac{1}{2}R_e(X - e\cdot X)}.$$

Therefore, $g_{ij}(X)$ is determined by

$$g_{ij}(X) = \frac{2\delta_{ij}}{X}e^{-\frac{1}{2}R_e(X - e\cdot X)} - 2\left(\frac{\delta_{ij}}{X} - \frac{X_i X_j}{X^3}\right)\left\{\frac{1 - e^{-\frac{1}{2}R_e(X - e\cdot X)}}{R_e(X - e\cdot X)^2}\right\} - 2\left(\frac{X_i}{X} - e_i\right)\left(\frac{X_j}{X} - e_j\right)$$

$$\times\left\{\frac{\left[\frac{1}{2}R_e(X - e\cdot X) + 1\right]e^{-\frac{1}{2}R_e(X - e\cdot X)} - 1}{R_e(X - e\cdot X)^2}\right\},$$

or

$$g_{ij}(X) = 2\left(\frac{1 - e^{-\frac{1}{2}R_e(X - e\cdot X)}}{R_e(X - e\cdot X)^2}\right)\left[\left(\frac{X_i}{X} - e_i\right)\left(\frac{X_j}{X} - e_j\right) - (X - e\cdot X)\left(\frac{\delta_{ij}}{X} - \frac{X_i X_j}{X^3}\right)\right]$$

$$+ \frac{e^{-\frac{1}{2}R_e(X - e\cdot X)}}{X - e\cdot X}\left[\frac{2(X - e\cdot X)\delta_{ij}}{X} - \left(\frac{X_i}{X} - e_i\right)\left(\frac{X_j}{X} - e_j\right)\right].$$

(5.35)

Thus the value of $g_{ij}[R(s) - R(\bar{s})]$ is obtained by

$$g_{ij}[R-\hat{R}] = 2\left(\frac{1-e^{-\frac{1}{2}R_e\{|R-\hat{R}|-e\cdot[R-\hat{R}]\}}}{R_e\{|R-\hat{R}|-e\cdot[R-\hat{R}]\}^2}\right)\left\{\left(\frac{[R-\hat{R}]_i}{|R-\hat{R}|}-e_i\right)\left(\frac{[R-\hat{R}]_j}{|R-\hat{R}|}-e_j\right)\right.$$

$$\left.-\{|R-\hat{R}|-e\cdot[R-\hat{R}]\}\left(\frac{\delta_{ij}}{|R-\hat{R}|}-\frac{[R-\hat{R}]_i[R-\hat{R}]_j}{|R-\hat{R}|^3}\right)\right\} \quad (5.36)$$

$$+\frac{e^{-\frac{1}{2}R_e(|R-\hat{R}|-e\cdot[R-\hat{R}])}}{|R-\hat{R}|-e\cdot[R-\hat{R}]}\left[\frac{2\{|R-\hat{R}|-e\cdot[R-\hat{R}]\}\delta_{ij}}{|R-\hat{R}|}\right.$$

$$\left.-\left(\frac{[R-\hat{R}]_i}{|R-\hat{R}|}-e_i\right)\left(\frac{[R-\hat{R}]_j}{|R-\hat{R}|}-e_j\right)\right],$$

where $\hat{R}$ is the value of R at point $s = \hat{s}$.

However, in the limit as $R_e \longrightarrow 0$ the value of g_{ij} may be determined by

$$g_{ij}[R-\hat{R}] = 2\left\{\frac{1-\left\{1-\frac{R_e}{2}\{|R-\hat{R}|-e\cdot[R-\hat{R}]\}+\cdots\right\}}{R_e\{|R-\hat{R}|-e\cdot[R-\hat{R}]\}^2}\right\}\left\{\left(\frac{[R-\hat{R}]_i}{|R-\hat{R}|}-e_i\right)\left(\frac{[R-\hat{R}]_j}{|R-\hat{R}|}-e_j\right)\right.$$

$$\left.-\{|R-\hat{R}|-e\cdot[R-\hat{R}]\}\left(\frac{\delta_{ij}}{|R-\hat{R}|}-\frac{[R-\hat{R}]_i[R-\hat{R}]_j}{|R-\hat{R}|^3}\right)\right\}$$

$$+\frac{1-\frac{R_e}{2}\{|R-\hat{R}|-e\cdot[R-\hat{R}]+\cdots\}}{|R-\hat{R}|-e\cdot[R-\hat{R}]}\left[\frac{2\{|R-\hat{R}|-e\cdot[R-\hat{R}]\}\delta_{ij}}{|R-\hat{R}|}\right.$$

$$\left.-\left(\frac{[R-\hat{R}]_i}{|R-\hat{R}|}-e_i\right)\left(\frac{[R-\hat{R}]_j}{|R-\hat{R}|}-e_j\right)\right]$$

or

$$g_{ij}[R-\hat{R}] = \frac{\delta_{ij}}{|R-\hat{R}|}+\frac{[R-\hat{R}]_i[R-\hat{R}]_j}{|R-\hat{R}|^3}. \quad (5.37)$$

It is not difficult to see that the force integral equation (5.27) or its equivalent dimensionless components given by (5.17-19) together with the relationship (5.37) (i.e. for $R_r = 0$) for a body with a circular cross-section (i.e. $q = 1$ and $Q_{ij} = 0$) reduces to that obtained by Johnson (for uniform flow) with a completely different approach, given by (1.2.21-23), excluding the prolate-spheroid ends.

CHAPTER 6

6 - Long straight cylinder

In this chapter we consider a long slender body with arbitrary cross section, but with a straight centreline being at rest in a fluid undergoing a uniform velocity, U. We intend to apply the force integral equation obtained in the previous chapter and solve it to determine explicitly the hydrodynamic force per unit length on the body.

It is convenient to use a set of fixed dimensionless rectangular axes (x_1, x_2, x_3) with origin at the midpoint of the body centreline, x_1 being parallel to the body centreline and $\boldsymbol{e}$, the unit vector in the direction of the velocity U, lying in the (x_1, x_2)-plane, as shown in figure 6.1.

Thus, $s = x_1 + 1/2$; $\boldsymbol{e} = (e_1, e_2, 0)$; and the body centerline may be written as $\boldsymbol{R}(x_1) = x_1 \boldsymbol{t}$, where $-1/2 \leq x_1 \leq 1/2$ and $\boldsymbol{t}$ is the unit base vector parallel to the body centreline. Thus, the body centreline in terms of the variable s is given by

$$\boldsymbol{R}(s) = (s - \frac{1}{2})\boldsymbol{t}. \tag{6.1}$$

The expansion of the force $\boldsymbol{F}^*(s)$ given by (5.28) may be written as

$$\boldsymbol{F}(s) = \frac{\boldsymbol{F}^*(s)}{\mu U} = \frac{\boldsymbol{f}_0(s)}{\ln\kappa} + \frac{\boldsymbol{f}_1(s)}{(\ln\kappa)^2} \cdots + \frac{\boldsymbol{f}_n(s)}{(\ln\kappa)^{n+1}} + \frac{\boldsymbol{f}_{n+1}(s)}{(\ln\kappa)^{n+2}} \cdots, \tag{6.2}$$

where the unlabelled variables correspond to the dimensionless forms of the variables labelled by superscript (*). Hence, the coefficient of the leading term, $\boldsymbol{f}_0$, may be readily determined by [see (5.30)]

$$f_0(s) = -4\pi e + 2\pi e \cdot t(s)t(s). \tag{6.3}$$

The coefficient of the second term, f_1, may be determined by [see (5.31)]

$$f_1(s) + f_0(s) \cdot \left[\left(\ln\frac{\lambda_s}{2\varepsilon} - \frac{1}{2}\right)I + (1 + \ln q)t(s)t(s) + Q(s)\right] = J_0, \tag{6.4a}$$

where J_0 is defined by

$$J_0 = + \frac{1}{2}\left(\int_0^{s-\varepsilon} + \int_{s+\varepsilon}^1\right)\left[I - \frac{1}{2}t(s)t(s)\right] \cdot g\,[R(s) - R(\hat{s})] \cdot f_0(\hat{s})\,d\hat{s}. \tag{6.4b}$$

Relationship (6.4b) may be expressed in index notation as

$$J_{0i} = + \frac{1}{2}\left(1 - \frac{1}{2}\delta_{1i}\right)\left(\int_0^{s-\varepsilon} + \int_{s+\varepsilon}^1\right)g_{ij}\,[R(s) - R(\hat{s})]f_{0j}(\hat{s})\,d(\hat{s}). \tag{6.5}$$

But, since f_{0j} is independent of $\hat{s}$, J_{0i} may be written as

$$J_{0i} = + \frac{1}{2}\left(1 - \frac{1}{2}\delta_{1i}\right)f_{0j}I_{ij}, \tag{6.6}$$

where I_{ij} is defined by

$$I_{ij} = \left(\int_0^{s-\varepsilon} + \int_{s+\varepsilon}^1\right)g_{ij}[R(s) - R(\hat{s})]\,d\hat{s}. \tag{6.7}$$

By the aid of the relationships (5.36) and (6.1), $g_{ij}\,[R(s) - R(\hat{s})]$ may be determined by

$$\begin{aligned} g_{ij}[R-\hat{R}] = 2&\left(\frac{1 - e^{-\frac{1}{2}R_e[|s-\hat{s}| - e_1(s-\hat{s})]}}{R_e[|s-\hat{s}| - e_1(s-\hat{s})]^2}\right)\left[\left(\frac{\delta_{1i}(s-\hat{s})}{|s-\hat{s}|} - e_i\right)\left(\frac{\delta_{1j}(s-\hat{s})}{|s-\hat{s}|} - e_j\right)\right. \\ &\left. - [|s-\hat{s}| - e_1(s-\hat{s})]\left(\frac{\delta_{ij}}{|s-\hat{s}|} - \frac{\delta_{1i}\delta_{1j}}{|s-\hat{s}|}\right)\right] + \frac{e^{-\frac{1}{2}R_e[|s-\hat{s}| - e_1(s-\hat{s})]}}{|s-\hat{s}| - e_1(s-\hat{s})} \\ &\times \left[\frac{2[|s-\hat{s}| - e_1(s-\hat{s})]\delta_{ij}}{|s-\hat{s}|} - \left(\frac{\delta_{1i}(s-\hat{s})}{|s-\hat{s}|} - e_i\right)\left(\frac{\delta_{1j}(s-\hat{s})}{|s-\hat{s}|} - e_j\right)\right]. \end{aligned} \tag{6.8}$$

Letting

$$s - \hat{s} = W \tag{6.9}$$

and noting that $e = (e_1, e_2, 0)$, the components of g_{ij} may be determined by

$$g_{11} = 2\left(\frac{1 - e^{-\frac{1}{2}R_e(|W| - e_1 W)}}{R_e(|W| - e_1 W)^2}\right)\left(\frac{W}{|W|} - e_1\right)^2 + \frac{e^{-\frac{1}{2}R_e(|W| - e_1 W)}}{|W| - e_1 W} \times \left[\frac{2(|W| - e_1 W)}{|W|} - \left(\frac{W}{|W|} - e_1\right)^2\right], \tag{6.10}$$

$$\begin{aligned} g_{12} = g_{21} &= 2\left(\frac{1 - e^{-\frac{1}{2}R_e(|W| - e_1 W)}}{R_e(|W| - e_1 W)^2}\right)\left(\frac{W}{|W|} - e_1\right)(-e_2) + \frac{e^{-\frac{1}{2}R_e(|W| - e_1 W)}}{|W| - e_1 W} \\ &\quad \times \left[-\left(\frac{W}{|W|} - e_1\right)(-e_2)\right] \\ &= e_2\left(\frac{W}{|W|} - e_1\right)\left[-2\left(\frac{1 - e^{-\frac{1}{2}R_e(|W| - e_1 W)}}{R_e(|W| - e_1 W)^2}\right) + \frac{e^{-\frac{1}{2}R_e(|W| - e_1 W)}}{|W| - e_1 W}\right], \end{aligned} \tag{6.11}$$

$$g_{13} = g_{31} = 0, \tag{6.12}$$

$$g_{22} = 2\left(\frac{1 - e^{-\frac{1}{2}R_e(|W| - e_1 W)}}{R_e(|W| - e_1 W)^2}\right)\left[(e_2)^2 - \frac{|W| - e_1 W}{|W|}\right] + \frac{e^{-\frac{1}{2}R_e(|W| - e_1 W)}}{|W| - e_1 W}\left[\frac{2(|W| - e_1 W)}{|W|} - (e_2)^2\right], \tag{6.13}$$

$$g_{23} = g_{32} = 0, \tag{6.14}$$

$$g_{33} = -2\left(\frac{1 - e^{-\frac{1}{2}R_e(|W| - e_1 W)}}{R_e(|W| - e_1 W)^2}\right)\frac{|W| - e_1 W}{|W|} + 2\left(\frac{e^{-\frac{1}{2}R_e(|W| - e_1 W)}}{|W| - e_1 W}\right)\frac{|W| - e_1 W}{|W|}. \tag{6.15}$$

By the aid of the relationship (6.9), I_{ij} defined by (6.7) may be written as

$$I_{ij} = -\left(\int_{s}^{+\varepsilon} + \int_{-\varepsilon}^{s-1}\right) g_{ij}(W)\,dW, \qquad (6.16)$$

where $W = R(s) - R(\hat{s}) = (s - \hat{s})t$. Noting that the limits of the first integral are positive and those of the second one are negative, I_{ij} may be determined by

$$\begin{aligned} I_{11} = &-\int_{s}^{\varepsilon}\left[2\left(\frac{1-e^{-\frac{1}{2}R_e(1-e_1)W}}{R_e w^2}\right) + (1+e_1)\left(\frac{e^{-\frac{1}{2}R_e(1-e_1)W}}{W}\right)\right]dW \\ &-\int_{-\varepsilon}^{1-s}\left[2\left(\frac{1-e^{\frac{1}{2}R_e(1+e_1)W}}{R_e w^2}\right) - (1-e_1)\left(\frac{e^{\frac{1}{2}R_e(1+e_1)W}}{W}\right)\right]dW, \end{aligned} \qquad (6.17)$$

$$\begin{aligned} I_{12} = I_{21} = &-\int_{s}^{\varepsilon} e_2\left[-2\left(\frac{1-e^{-\frac{1}{2}R_e(1-e_1)W}}{R_e(1-e_1)W^2}\right) + \frac{e^{-\frac{1}{2}R_e(1-e_1)W}}{W}\right]dW \\ &-\int_{-\varepsilon}^{1-s} e_2\left[2\left(\frac{1-e^{\frac{1}{2}R_e(1+e_1)W}}{R_e(1+e_1)W^2}\right) + \frac{e^{\frac{1}{2}R_e(1+e_1)W}}{W}\right]dW, \end{aligned} \qquad (6.18)$$

$$\begin{aligned} I_{22} = &-\int_{s}^{\varepsilon}\left[2\left(e_2^2-1+e_1\right)\left(\frac{1-e^{-\frac{1}{2}R_e(1-e_1)W}}{R_e(1-e_1)^2W^2}\right) + \left(2-2e_1-e_2^2\right)\left(\frac{e^{-\frac{1}{2}R_e(1-e_1)W}}{(1-e_1)W}\right)\right]dW \\ &-\int_{-\varepsilon}^{s-1}\left[2\left(e_2^2-1-e_1\right)\left(\frac{1-e^{\frac{1}{2}R_e(1+e_1)W}}{R_e(1+e_1)^2W^2}\right) - \left(2+2e_1-e_2^2\right)\left(\frac{e^{\frac{1}{2}R_e(1+e_1)W}}{(1+e_1)W}\right)\right]dW \\ = &-\int_{s}^{\varepsilon}\left[2e_1\left(\frac{1-e^{-\frac{1}{2}R_e(1-e_1)W}}{R_e(1-e_1)W^2}\right) + (1-e_1)\left(\frac{e^{-\frac{1}{2}R_e(1-e_1)W}}{W}\right)\right]dW \\ &-\int_{-\varepsilon}^{s-1}\left[-2e_1\left(\frac{1-e^{\frac{1}{2}R_e(1+e_1)W}}{R_e(1+e_1)W^2}\right) - (1+e_1)\left(\frac{e^{\frac{1}{2}R_e(1+e_1)W}}{W}\right)\right]dW, \end{aligned} \qquad (6.19)$$

and

$$I_{13} = I_{31} = I_{23} = I_{32} = 0. \tag{6.21}$$

But

$$\int_s^\varepsilon \frac{1 - e^{-\frac{R_e}{2}(1-e_1)W}}{W^2} dW = \left| \frac{-1 + e^{-\frac{R_e}{2}(1-e_1)w}}{W} \right|_s^\varepsilon + \frac{R_e}{2}(1 - e_1)\int_s^\varepsilon \frac{e^{-\frac{R_e}{2}(1-e_1)W}}{W} dw \tag{6.22}$$
$$= L + \frac{1}{2}R_e(1 - e_1)I$$

and

$$\int_{-\varepsilon}^{s-1} \frac{1 - e^{\frac{R_e}{2}(1+e_1)W}}{W^2} dW = \left| \frac{-1 + e^{\frac{R_e}{2}(1+e_1)W}}{W} \right|_{-\varepsilon}^{s-1} - \frac{R_e}{2}(1 + e_1)\int_{-\varepsilon}^{1-s} \frac{e^{\frac{R_e}{2}(1+e_1)W}}{W} dW \tag{6.23}$$
$$= L' - \frac{1}{2}R_e(1 + e_1)I',$$

where L, I, L' and I' are respectively defined by

$$L = \left| \frac{-1 + e^{-½R_e(1-e_1)w}}{W} \right|_s^\varepsilon, \tag{6.24}$$

$$I = \int_s^\varepsilon e^{-½R_e(1-e_1)W} \frac{dW}{W}, \tag{6.25}$$

$$L' = \left| \frac{-1 + e^{½R_e(1+e_1)W}}{W} \right|_{-\varepsilon}^{s-1}, \tag{6.26}$$

$$I' = \int_{-\varepsilon}^{s-1} e^{½R_e(1+e_1)W} \frac{dW}{W}. \tag{6.27}$$

Thus the components of I_{ij} given by (6.17- 21) may be simplified as

$$I_{11} = -\frac{2}{R_e}\left[L + \frac{1}{2}R_e(1 - e_1)I\right] - (1 + e_1)I - \frac{2}{R_e}\left[L' - \frac{1}{2}R_e(1 + e_1)I'\right] + (1 - e_1)I'$$
$$= -\frac{2}{R_e}(L + L') - 2(I - I'), \tag{6.28}$$

$$I_{12} = I_{21} = +\frac{2e_2L}{R_e(1 - e_1)} + e_2I - e_2I - \frac{2e_2L'}{R_e(1 + e_2)} + e_2I' - e_2I'$$
$$= \frac{2e_2}{R_e}\left(\frac{L}{1 - e_1} - \frac{L'}{1 + e_1}\right), \tag{6.29}$$

$$I_{22} = -\frac{2e_1L}{R_e(1 - e_1)} - e_1I - (1 - e_1)I + \frac{2e_1L'}{R_e(1 + e_1)} - e_1I' + (1 + e_1)I'$$
$$= -\frac{2e_1}{R_e}\left(\frac{L}{1 - e_1} - \frac{L'}{1 + e_1}\right) - (I - I'), \tag{6.30}$$

$$I_{33} = \frac{2L}{R_e(1 - e_1)} + I - 2I + \frac{2L'}{R_e(1 + e_1)} - I' + 2I'$$
$$= \frac{2}{R_e}\left(\frac{L}{1 - e_1} + \frac{L'}{1 + e_1}\right) - (I - I') \tag{6.31}$$

and the other components are equal to zero. Hence, by the aid of relationships (6.28-31), the components of J_α given by (6.6) may be determined by

$$J_{01} = \frac{1}{4}(f_{01}I_{11} + f_{02}I_{12} + f_{03}I_{13})$$
$$= \frac{\pi}{R_e}e_1(L + L') + \pi e_1(I - I') - \frac{2\pi}{R_e}e_2^2\left(\frac{L}{1 - e_1} - \frac{L'}{1 + e_1}\right)$$
$$= \frac{\pi}{R_e}\left\{\left[e_1 - 2(1 + e_1)\right]L + \left[e_1 + 2(1 - e_1)\right]L'\right\} + \pi e_1(I - I')$$
$$= \frac{\pi}{R_e}\left[-(2 + e_1)L + (2 - e_1)L'\right] + \pi e_1(I - I'), \tag{6.32}$$

$$J_{02} = \frac{1}{2}(f_{01}I_{21} + f_{02}I_{22} + f_{03}I_{23})$$

$$= -\frac{2\pi}{R_e}e_1e_2\left(\frac{L}{1-e_1} - \frac{L'}{1+e_2}\right) + \frac{4\pi}{R_e}e_1e_2\left(\frac{L}{1-e_1} - \frac{L'}{1+e_1}\right) + 2\pi(I - I') \qquad (6.33)$$

$$= \frac{2\pi}{R_e}e_1e_2\left(\frac{L}{1-e_1} - \frac{L'}{1+e_2}\right) + 2\pi(I - I')$$

and

$$J_{03} = \frac{1}{2}(f_{01}I_{31} + f_{02}I_{32} + f_{03}I_{33}) = 0. \qquad (6.34)$$

The integral I defined by (6.25) may be evaluated as

$$\begin{aligned} I &= \int_s^\varepsilon \frac{e^{-\frac{1}{2}R_e(1-e_1)W}}{W}\,dW \\ &= \left\{\int_s^\infty - \int_\varepsilon^\infty\right\}\frac{e^{-\frac{1}{2}R_e(1-e_1)W}}{W}\,dW \\ &= \left\{\int_{\frac{1}{2}R_e(1-e_1)s}^\infty - \int_{\frac{1}{2}R_e(1-e_1)\varepsilon}^\infty\right\}\frac{e^{-\tau}}{\tau}\,d\tau \\ &= E_1\left[\frac{1}{2}R_e(1-e_1)s\right] - E_1\left[\frac{1}{2}R_e(1-e_1)\varepsilon\right], \end{aligned} \qquad (6.35)$$

where $E_1(x)$ is the exponential integral defined by

$$E_1(x) = \int_x^\infty \frac{e^{-\tau}}{\tau}\,d\tau. \qquad (6.36)$$

But as x tends to zero [see Bender & Orszag (1978), p. 252],

$$E_1(x) \sim -\ln x - \gamma + x + O(x^2), \qquad (6.37)$$

where γ is the Euler constant. Hence as ε tends to zero, I may be expressed as

$$I = E_1\left[\frac{1}{2}R_e(1-e_1)s\right] + \ln\left[\frac{1}{2}R_e(1-e_1)\varepsilon\right] + \gamma + O(\varepsilon). \qquad (6.38)$$

Similarly I' defined by (6.27) may be written as

$$I' = \int_{-\varepsilon}^{s-1} \frac{e^{\frac{1}{2}R_e(1+e_1)W}}{W}\,dW$$

$$= \int_{\varepsilon}^{1-s} \frac{e^{-\frac{1}{2}R_e(1+e_1)W}}{W}\,dW$$

$$= -E_1\left[\frac{1}{2}R_e(1+e_1)(1-s)\right] - \ln\left[\frac{1}{2}R_e(1+e_1)\varepsilon\right] - \gamma + O(\varepsilon). \tag{6.39}$$

Thus, $I - I'$ is determined by

$$I - I' = E_1\left[\frac{1}{2}R_e(1-e_1)s\right] + E_1\left[\frac{1}{2}R_e(1+e_1)(1-s)\right] + \ln\left[\frac{1}{4}R_e^2(1-e_1^2)\varepsilon^2\right] + 2\gamma$$

$$= E_1\left[\frac{1}{2}R_e(1-e_1)s\right] + E_1\left[\frac{1}{2}R_e(1+e_1)(1-s)\right] + 2\ln\left(\frac{1}{2}R_e e_2 \varepsilon\right) + 2\gamma. \tag{6.40}$$

The value of L defined by (6.24) may be determined by

$$L = \left|\frac{-1 + e^{-\frac{1}{2}R_e(1-e_1)w}}{W}\right|_s^{\varepsilon}$$

$$= \frac{-1 + e^{-\frac{1}{2}R_e(1-e_1)\varepsilon}}{\varepsilon} - \frac{-1 + e^{-\frac{1}{2}R_e(1-e_1)s}}{s}$$

$$= \frac{-1 + 1 - \frac{1}{2}R_e(1-e_1)\varepsilon + O(\varepsilon^2)}{\varepsilon} - \frac{-1 + e^{-\frac{1}{2}R_e(1-e_1)s}}{s}$$

$$= -\frac{1}{2}R_e(1-e_1) + \frac{1 - e^{-\frac{1}{2}R_e(1-e_1)s}}{s} + O(\varepsilon). \tag{6.41}$$

Similarly L' defined by (6.26) may be obtained by

$$L' = \left|\frac{-1 + e^{-\frac{1}{2}R_e(1+e_1)w}}{W}\right|_{1-s}^{\varepsilon} = -\frac{1}{2}R_e(1+e_1) + \frac{1 - e^{-\frac{1}{2}R_e(1+e_1)(1-s)}}{1-s} + O(\varepsilon) \tag{6.42}$$

Therefore, as $\varepsilon \longrightarrow 0$, the components of $J_{\alpha i}$ given by (6.32-34) may be written as

$$
\begin{aligned}
J_{01} &= \frac{\pi}{R_e}\left\{ -(e_1+2)\left[-\frac{1}{2}R_e(1-e_1) + \frac{1-e^{-\frac{1}{2}R_e(1-e_1)s}}{s} \right] \right. \\
&\qquad \left. -(e_1-2)\left[-\frac{1}{2}R_e(1+e_1) + \frac{1-e^{-\frac{1}{2}R_e(1+e_1)(1-s)}}{1-s} \right] \right\} \\
&\quad + \pi e_1 \left\{ Ei\left[\frac{1}{2}R_e(1-e_1)s\right] + Ei\left[\frac{1}{2}R_e(1+e_1)(1-s)\right] + 2\ln\left(\frac{1}{2}R_e e_2 \varepsilon\right) - 2\gamma \right\} \\
&= \pi\left[-\frac{e_1^2+e_1-2}{2} - \frac{(2+e_1)\left(1-e^{-\frac{1}{2}R_e(1-e_1)s}\right)}{R_e s} + \frac{e_1^2-e_1-2}{2} \right. \\
&\qquad \left. + \frac{(2-e_1)\left(1-e^{-\frac{1}{2}R_e(1+e_1)(1-s)}\right)}{R_e(1-s)} \right] + \pi e_1 \left\{ Ei\left[\frac{1}{2}R_e(1-e_1)s\right] \right. \\
&\qquad \left. + Ei\left[\frac{1}{2}R_e(1+e_1)(1-s)\right] + 2\ln\left(\frac{1}{2}R_e e_2 \varepsilon\right) + 2\gamma \right\} \\
&= \pi e_1 \left\{ -1 - \frac{1-e^{-\frac{1}{2}R_e(1-e_1)s}}{R_e s} - \frac{1-e^{-\frac{1}{2}R_e(1+e_1)(1-s)}}{R_e(1-s)} + Ei\left[\frac{1}{2}R_e(1-e_1)s\right] \right. \\
&\qquad \left. + Ei\left[\frac{1}{2}R_e(1+e_1)(1-s)\right] + 2\ln\left(\frac{1}{2}R_e e_2 \varepsilon\right) + 2\gamma \right\} \\
&\quad + 2\pi \left\{ -\frac{1-e^{-\frac{1}{2}R_e(1-e_1)s}}{R_e s} + \frac{1-e^{-\frac{1}{2}R_e(1+e_1)(1-s)}}{R_e(1-s)} \right\},
\end{aligned}
\tag{6.43}
$$

$$
\begin{aligned}
J_{02} &= \frac{2\pi e_1 e_2}{R_e}\left(-\frac{1}{2}R_e + \frac{1-e^{-\frac{1}{2}R_e(1-e_1)s}}{(1-e_1)s} + \frac{1}{2}R_e - \frac{1-e^{-\frac{1}{2}R_e(1+e_1)(1-s)}}{(1+e_1)(1-s)} \right) \\
&\quad + 2\pi e_2 \left\{ Ei\left[\frac{R_e}{2}(1-e_1)s\right] + Ei\left[\frac{R_e}{2}(1+e_1)(1-s)\right] + 2\ln\left(\frac{R_e}{2}e_2\varepsilon\right) + 2\gamma \right\}
\end{aligned}
$$

or

$$J_{02} = \frac{2\pi e_1 e_2}{R_e}\left(\frac{1 - e^{-\frac{1}{2}R_e(1-e_1)s}}{(1-e_1)s} - \frac{1 - e^{-\frac{1}{2}R_e(1+e_1)(1-s)}}{(1+e_1)(1-s)}\right)$$
$$+ 2\pi e_2\left\{Ei\left[\frac{R_e}{2}(1-e_1)s\right] + Ei\left[\frac{R_e}{2}(1+e_1)(1-s)\right] + 2\ln\left(\frac{R_e}{2}e_2\varepsilon\right) + 2\gamma\right\} \quad (6.44)$$

$$J_{03} = 0 \,. \quad (6.45)$$

Hence, notin that $\mathbf{e} = e_1\mathbf{i}_1 + e_2\mathbf{i}_2$ (where $\mathbf{i}_1 = \mathbf{t}$ and $\mathbf{i}_2$ is the unit vector in the direction of x_2 axis), J_o may be written as

$$J_0 = \frac{\pi}{R_e}\left\{\frac{-e_1^2 + e_1 - 2}{1 - e_1}\left[-\frac{1}{2}R_e(1-e_1) + \frac{1 - e^{-\frac{1}{2}R_e(1-e_1)s}}{s}\right] + \frac{e_1^2 + e_1 + 2}{1 + e_1}\left[-\frac{1}{2}R_e(1+e_1)\right.\right.$$
$$\left.\left. + \frac{1 - e^{-\frac{1}{2}R_e(1+e_1)(1-s)}}{1-s}\right]\right\}t + \frac{2\pi}{R_e}\left\{\frac{e_1}{1-e_1}\left[-\frac{1}{2}R_e(1-e_1) + \frac{1 - e^{-\frac{1}{2}R_e(1-e_1)s}}{s}\right]\right.$$
$$\left. - \frac{e_1}{1+e_1}\left[-\frac{1}{2}R_e(1+e_1) + \frac{1 - e^{-\frac{1}{2}R_e(1+e_1)(1-s)}}{1-s}\right]\right\}e + 2\pi\left(e - \frac{1}{2}e_1 t\right)$$
$$\times\left\{E_1\left[\frac{1}{2}R_e(1-e_1)s\right] + E_1\left[\frac{1}{2}R_e(1+e_1)(1-s)\right] + \ln\left[\frac{1}{4}R_e^2(1-e_1^2)\varepsilon^2\right] + 2\gamma\right\}$$
$$= \frac{\pi}{R_e}\left[-e_1R_e + \left(\frac{-e_1^2 + e_1 - 2}{1 - e_1}\right)\frac{1 - e^{-\frac{1}{2}R_e(1-e_1)s}}{s} + \left(\frac{e_1^2 + e_1 + 2}{1 + e_1}\right)\frac{1 - e^{-\frac{1}{2}R_e(1+e_1)(1-s)}}{1-s}\right]t$$
$$+ \frac{2\pi e_1}{R_e}\left[\frac{1 - e^{-\frac{1}{2}R_e(1-e_1)s}}{(1-e_1)s} - \frac{1 - e^{-\frac{1}{2}R_e(1+e_1)(1-s)}}{(1+e_1)(1-s)}\right]e + 2\pi\left(e - \frac{1}{2}e_1 t\right)$$
$$\times\left\{E_1\left[\frac{1}{2}R_e(1-e_1)s\right] + E_1\left[\frac{1}{2}R_e(1+e_1)(1-s)\right] + \ln\left[\frac{1}{4}R_e^2(1-e_1^2)\varepsilon^2\right] + 2\gamma\right\}.$$

Substituting J_o and f_o (6.3) in relationship (6.4a) and noting that $Q_{it} = 0$ results in

$$
\begin{aligned}
f_1 &= 2\pi(2e - e_1 t)\cdot\left[\left(\ln\frac{\lambda_s}{2\varepsilon} - \frac{1}{2}\right)I + (1+\ln q)t(s)t(s) + Q(s)\right] \\
&\quad + \frac{\pi}{R_e}\left[-e_1 R_e + \left(\frac{-e_1^2 + e_1 - 2}{1 - e_1}\right)\frac{1 - e^{-\frac{1}{2}R_e(1-e_1)s}}{s} + \left(\frac{e_1^2 + e_1 + 2}{1 + e_1}\right)\frac{1 - e^{-\frac{1}{2}R_e(1+e_1)(1-s)}}{1-s}\right]t \\
&\quad + \frac{2\pi e_1}{R_e}\left[\frac{1 - e^{-\frac{1}{2}R_e(1-e_1)s}}{(1-e_1)s} - \frac{1 - e^{-\frac{1}{2}R_e(1+e_1)(1-s)}}{(1+e_1)(1-s)}\right]e + 2\pi\left(e - \frac{1}{2}e_1 t\right) \\
&\quad \times\left\{E_1\left[\frac{1}{2}R_e(1-e_1)s\right] + E_1\left[\frac{1}{2}R_e(1+e_1)(1-s)\right] + \ln\left[\frac{1}{4}R_e^2(1-e_1^2)\varepsilon^2\right] + 2\gamma\right\} \\
&= 2\pi\left\{(2e - e_1 t)\left\{\ln\frac{\lambda_s R_e}{4} + \frac{1}{2}\ln(1-e_1^2) + \frac{1}{2}E_1\left[\frac{1}{2}R_e(1-e_1)s\right] + \gamma - \frac{1}{2}\right.\right. \\
&\quad \left. + \frac{1}{2}E_1\left[\frac{1}{2}R_e(1+e_1)(1-s)\right]\right\} + \left(e_1 e + \frac{-e_1^2 + e_1 - 2}{2}t\right)\frac{1 - e^{-\frac{1}{2}R_e(1-e_1)s}}{R_e(1-e_1)s} \\
&\quad \left. + \left(-e_1 e + \frac{e_1^2 + e_1 + 2}{2}t\right)\frac{1 - e^{-\frac{1}{2}R_e(1+e_1)(1-s)}}{R_e(1+e_1)(1-s)} + \left(\frac{1}{2} + \ln q\right)e_1 t + 2e\cdot Q(s)\right\}.
\end{aligned} \tag{6.46}
$$

Upon substitution of f_0 given by (6.3) and f_1 into the expansion of the force density given by (6.2), the force per unit length on the straight long cylindrical body, correct to order $(1/\ln\kappa)^3$, may be determined by

$$
\begin{aligned}
\frac{F^*(s)}{2\pi\mu U} &= \left(\frac{1}{\ln\kappa}\right)(e_1 t - 2e) + \left(\frac{1}{\ln\kappa}\right)^2\left\{(2e - e_1 t)\left\{\ln\frac{\lambda_s R_e}{4} + \frac{1}{2}\ln(1-e_1^2) + \gamma - \frac{1}{2}\right.\right. \\
&\quad \left. + \frac{1}{2}E_1\left[\frac{1}{2}R_e(1-e_1)s\right] + \frac{1}{2}E_1\left[\frac{1}{2}R_e(1+e_1)(1-s)\right]\right\} + \frac{1 - e^{-\frac{1}{2}R_e(1-e_1)s}}{R_e(1-e_1)s} \\
&\quad \times\left[e_1 e - \frac{1}{2}(e_1^2 - e_1 + 2)t\right] - \frac{1 - e^{-\frac{1}{2}R_e(1+e_1)(1-s)}}{R_e(1+e_1)(1-s)}\left[e_1 e - \frac{1}{2}(e_1^2 + e_1 + 2)t\right] \\
&\quad \left. + \left(\frac{1}{2} + \ln q\right)e_1 t + 2e\cdot Q(s)\right\} + O\left(\frac{1}{\ln\kappa}\right)^3,
\end{aligned}
$$

or

$$\frac{F^*(s)}{2\pi\mu U} = \left(\frac{1}{\ln\kappa}\right)(\cos\theta\, t - 2e) + \left(\frac{1}{\ln\kappa}\right)^2 \left\{ (2e - \cos\theta\, t)\left\{\ln\frac{\lambda_s R_e}{4} + \ln\sin\theta + \gamma - \frac{1}{2} \right.\right.$$

$$\left. + \frac{1}{2}E_1\left[\frac{1}{2}R_e(1-\cos\theta)s\right] + \frac{1}{2}E_1\left[\frac{1}{2}R_e(1+\cos\theta)(1-s)\right]\right\} + \frac{1-e^{-\frac{1}{2}R_e(1-\cos\theta)s}}{R_e(1-\cos\theta)s}$$

$$\times\left[\cos\theta\, e - \frac{1}{2}(\cos^2\theta - \cos\theta + 2)\, t\right] - \frac{1-e^{-\frac{1}{2}R_e(1+\cos\theta)(1-s)}}{R_e(1+\cos\theta)(1-s)} \times$$

$$\left.\left[\cos\theta\, e - \frac{1}{2}(\cos^2\theta + \cos\theta + 2)\, t\right] + \left(\frac{1}{2} + \ln q\right)\cos\theta\, t + 2e\cdot Q(s)\right\} + O\left(\frac{1}{\ln\kappa}\right)^3. \tag{6.47}$$

where θ is the angle between the unit vector e and the unit base vector t, as shown in figure 6.1, and where γ is the Euler constant the value of which is given by (1.2.6) and $E_1(x)$ is the exponential integral given by (6.36) as

$$E_1(x) = \int_x^\infty \frac{e^{-\tau}}{\tau}\, d\tau .$$

The force equation (6.47) agrees with that obtained by Khayat & Cox (1989) given by (1.2.25). However, the recurrence formula given by (5.32,33) can be applied to obtaine the higher order terms.

It is worth noting that the only component of the force normal to the plane containing velocity U and the body centreline comes from the term $e.Q$. In other words, it is due to the effect of the cross-sectional shape on the flow field and hence for a body with a circular cross-section this component vanishes.

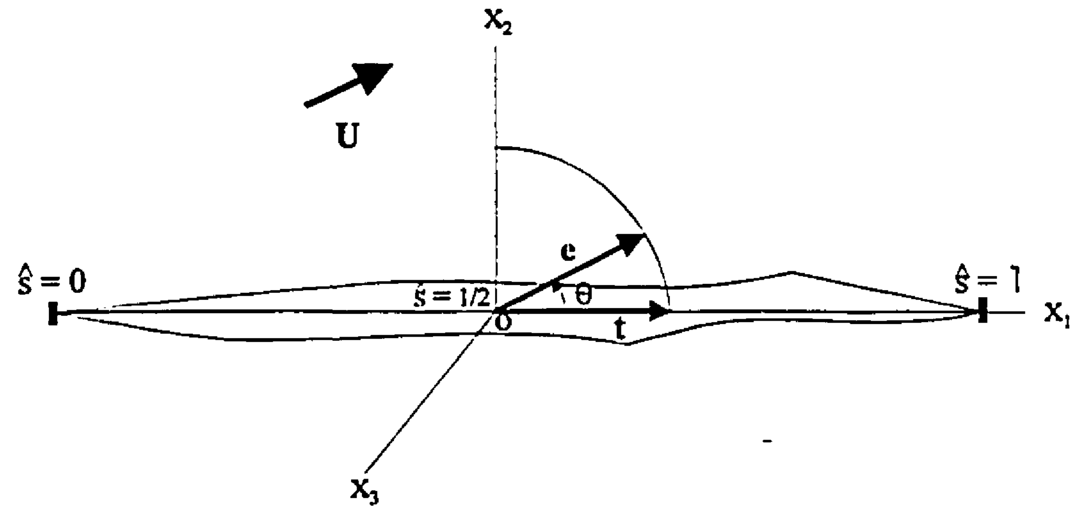

Figure 6.1 : Straight long slender body with arbitrary cross-section being at rest in fluid undergoing uniform velocity **U**.

CHAPTER 7

7 - Slender torus

In this chapter, as an example of a curved symmetric slender body, we consider a torus with an arbitrary cross-section but which is constant along the body centreline, settling along its axes in an unbounded fluid with constant velocity $-\boldsymbol{U}$. However, the problem is equivalent to that of a torus being at rest in a fluid undergoing a uniform velocity $\boldsymbol{U}$ in the direction normal to the plane containing the torus centreline (see figure 7.1).

It is assumed that the slenderness parameter κ defined by (2.6) is much smaller than unity, i.e, the radius of the torus is much larger than the characteristic length of the body cross-sectional shape (r_0) .

We intend to apply the force integral equation given by [see (5.27)]

$$\boldsymbol{F}^*(s)\cdot\left[\left(2\ln\frac{\kappa\lambda_s}{2\varepsilon}-1\right)\boldsymbol{I}+2(1+\ln q)\boldsymbol{t}(s)\boldsymbol{t}(s)+2\boldsymbol{Q}(s)\right]=-8\pi\mu\boldsymbol{U}\cdot\left[\boldsymbol{I}-\frac{1}{2}\boldsymbol{t}(s)\boldsymbol{t}(s)\right]+\boldsymbol{J}(s)\,, \tag{7.1}$$

where

$$\boldsymbol{J}(s)=\left[\boldsymbol{I}-\frac{1}{2}\boldsymbol{t}(s)\,\boldsymbol{t}(s)\right]\cdot\left(\int_0^{s-\varepsilon}+\int_{s+\varepsilon}^{1}\right)\boldsymbol{g}[\boldsymbol{R}(s)-\boldsymbol{R}(\hat{s})]\cdot\boldsymbol{F}^*(\hat{s})\,d\hat{s}\,, \tag{7.2}$$

to obtain the force per unit length the fluid exerts on the body. Since the torus is axisymmetric, in order to determine the vector J in (7.2), it is convenient to take a cylindrical polar coordinate system (ρ, θ, z) with the origin at the centre of the torus and the z-axis being parallel to the torus axis, as shown in figure 7.1. Associated with the cylindrical polar coordinate system is a set of rectangular Cartesian axes (x_1, x_2, x_3) with unit base vectors i_1, i_2 and i_3 which coincide with the x_1, x_2 and x_3 axis, respectively, and i_3 lying in the direction of velocity U (see figure 7.1). Thus, the relationship between these two coordinate systems may be written as (see figure 7.2)

$$x_1 = \rho\cos\theta\ , \qquad x_2 = \rho\sin\theta \ \textit{and} \qquad x_3 = z \tag{7.3}$$

and

$$i_1 = i_\rho\cos\theta - i_\theta\sin\theta\ ; \quad i_2 = i_\rho\sin\theta + i_\theta\cos\theta\ ; \quad i_3 = i_z \tag{7.4}$$

or

$$i_\rho = \cos\theta\, i_1 + \sin\theta\, i_2\ ; \quad i_\theta = -\sin\theta\, i_1 + \cos\theta\, i_2 \ \textit{and} \ i_z = i_3, \tag{7.5}$$

where i_ρ, i_θ and i_z are the cylindrical unit base vectors, corresponding to the ρ, θ and z coordinates, respectively, as shown in figure 7.1.

From the symmetric properties of the torus, it follows that the magnitude of the cylindrical components of the force density $F^*(s) = (F_\rho^*, F_\theta^*, F_z^*)$ are constant along the body length and since there is no variation on polar angle θ, F_θ^* is equal to zero. Therefore, it is sufficient to determine the cylindrical components of the force at a specified point on the body centreline and for convenience, we take it at the point $\theta = 0$. Thus, the relationship between the cylindrical and the Cartesian components of the force at the point under consideration ($\theta = 0$) may be written as (see figure 7.3)

$$F_1^*(0) = F_\rho^*\ ; \quad F_2^*(0) = F_\theta^* \ \textit{and} \ F_3^*(0) = F_z^*\ , \tag{7.6}$$

where $F_1^*(0)$, $F_2^*(0)$ and $F_3^*(0)$ are the Cartesian components of the force per unit length at the point $\theta = 0$, coinciding with the x_1, x_2 and x_3 axis, respectively.

The arc length of the body s is measured from the point $\theta = \pi$ in the counter-clockwise sense, as shown in figure 7.4. Hence, the relationship between the polar angle θ and s may be written as

$$s = (\pi + \theta)\, r, \tag{7.7}$$

where r is the radius of the torus. But the dimensionless length of the body centreline is equal to unity, therefore, the radius of the torus may be obtained by

$$2\pi r = 1 \qquad or \qquad r = \frac{1}{2\pi}.$$

Thus, (7.7) may be expressed as

$$s = \frac{1}{2\pi}(\theta + \pi), \tag{7.8}$$

so that, the point $\theta = 0$ on the body centreline corresponds to the point $s = 1/2$.

The body centreline may be written as (see figure 7.4)

$$\begin{aligned} \boldsymbol{R}(\theta) &= (r\cos\theta)\, \boldsymbol{i}_1 + (r\sin\theta)\, \boldsymbol{i}_2 \\ &= \frac{1}{2\pi}\left[(\cos\theta)\, \boldsymbol{i}_1 + (\sin\theta)\, \boldsymbol{i}_2\right]. \end{aligned} \tag{7.9}$$

Hence, vector $[\boldsymbol{R} - \hat{\boldsymbol{R}}]$ in (7.2) may be determined by

$$\boldsymbol{R}(\theta) - \boldsymbol{R}(\hat{\theta}) = \frac{1}{2\pi}\left[(\cos\theta - \cos\hat{\theta})\, \boldsymbol{i}_1 + (\sin\theta - \sin\hat{\theta})\, \boldsymbol{i}_2\right], \tag{7.10}$$

where $\hat{\theta}$ is the integration variable, or for the point $\theta = 0$, (7.10) may be written as

$$[R(0)-R(\hat\theta)] = \frac{1}{2\pi}\left[(1-\cos\hat\theta)\,i_1 - (\sin\hat\theta)\,i_2\right]$$
$$= \frac{1}{\pi}\left[(\sin^2\frac{\hat\theta}{2})\,i_1 - (\sin\frac{\hat\theta}{2}\cos\frac{\hat\theta}{2})\,i_2\right]. \qquad (7.11)$$

Thus

$$|R(0)-R(\hat\theta)| = \frac{1}{\pi}\left(\sin^4\frac{\hat\theta}{2} + \sin^2\frac{\hat\theta}{2}\cos^2\frac{\hat\theta}{2}\right)^{\frac{1}{2}}$$
$$= \frac{1}{\pi}\left[\sin^2\hat\theta\left(\sin^2\frac{\hat\theta}{2} + \cos^2\frac{\hat\theta}{2}\right)\right]^{\frac{1}{2}}$$
$$= \frac{1}{\pi}\left|\sin\frac{\hat\theta}{2}\right|. \qquad (7.12)$$

The force integral equation given by (7.1,2), for the point under consideration, may be expressed as

$$F^*(0)\cdot\left[\left(2\ln\frac{\kappa\lambda_s}{2\varepsilon}-1\right)I + 2(1+\ln q)\,t(0)t(0) + 2Q\right] = -8\pi\mu\,U\cdot\left[I-\frac{1}{2}t(0)t(0)\right] + J(0)\,, \qquad (7.13)$$

where, by the aid of (7.8,9), $t(0)$ (the unit vector parallel to the tangent of the body centreline at point $\theta = 0$) and $J(0)$ may respectively be obtained by

$$t(0) = \left.\frac{dR(s)}{ds}\right|_{s=\frac{1}{2}}$$
$$= \left.\frac{d\theta}{ds}\frac{dR}{d\theta}\right|_{\theta=0}$$
$$= 2\pi\frac{1}{2\pi}\left[(-\sin\theta)\,i_1 + (\cos\theta)\,i_2\right]_{\theta=0}$$
$$= +i_2 \qquad (7.14)$$

and

$$J(0) = \frac{1}{2\pi}\left[I - \frac{1}{2}t(0)\,t(0)\right]\cdot\left(\int_{-\pi}^{-2\pi\varepsilon} + \int_{2\pi\varepsilon}^{\pi}\right) g[R(0) - R(\hat{\theta})]\cdot F^{*}(\hat{\theta})\,d\hat{\theta}\,. \tag{7.15}$$

We intend to determine the vector $J(0)$ in cylindrical polar coordinates (ρ, θ, z), hence relationship (7.15) may be written as [see (7.5.14)]

$$J(0) = \left[I - \frac{1}{2}i_0(0)\,i_0(0)\right]\cdot\left(\int_{-\pi}^{-2\pi\varepsilon} + \int_{2\pi\varepsilon}^{\pi}\right) g[R(0) - R(\hat{\theta})]\cdot F^{*}(\hat{\theta})\,d\hat{\theta}\,, \tag{7.16}$$

where i_θ (0) is the value of the unit base vector i_θ at the point $\theta = 0$ [see (7.5.14)].

However, as $\varepsilon \longrightarrow 0$;

$$\left(\int_{-\pi}^{-2\pi\varepsilon} + \int_{2\pi\varepsilon}^{\pi}\right)\frac{d\hat{\theta}}{\left|\sin\frac{\hat{\theta}}{2}\right|} = \int_{-\pi}^{-2\pi\varepsilon}\frac{-d\hat{\theta}}{\sin\frac{\hat{\theta}}{2}} + \int_{2\pi\varepsilon}^{\pi}\frac{d\hat{\theta}}{\sin\frac{\hat{\theta}}{2}}$$

$$= \int_{2\pi\varepsilon}^{\pi}\frac{1 + \tan^2\frac{\hat{\theta}}{4}}{\tan\frac{\hat{\theta}}{4}}\,d\hat{\theta}$$

$$= 4\left|\ln\tan\frac{\hat{\theta}}{4}\right|_{2\pi\varepsilon}^{\pi}$$

$$= 4\left(\ln\sin\frac{\pi}{4} - \ln\cos\frac{\pi}{4} - \ln\sin\frac{\pi\varepsilon}{2} + \ln\cos\frac{\pi\varepsilon}{2}\right)$$

$$= 4\left[-\ln\left(\frac{\pi\varepsilon}{2} - \frac{(\pi\varepsilon/2)^3}{3!} + \ldots\right) + \ln\left(1 - \frac{(\pi\varepsilon/2)^2}{2!} + \ldots\right)\right]$$

$$\sim -4\ln\frac{\pi}{2} - 4\ln\varepsilon$$

or

$$-2F^{*}(0)\ln\varepsilon = 2F^{*}(0)\ln\frac{\pi}{2} + \frac{1}{2}F^{*}(0)\left(\int_{-\pi}^{-2\pi\varepsilon} + \int_{2\pi\varepsilon}^{\pi}\right)\left(\sin\frac{\hat{\theta}}{2}\right)^{-1} d\hat{\theta}\,. \tag{7.17}$$

Hence, as $\varepsilon \longrightarrow 0$; the force equation may be expressed as [see (7.13-15)]

$$F^*(0)\cdot\left[\left(2\ln\frac{\pi\kappa\lambda_s}{4}-1\right)I+2(1+\ln q)i_2i_2+2Q\right]=-8\pi\mu U\cdot\left[I-\frac{1}{2}i_2i_2\right]+J'(0),\quad (7.18)$$

where $J'(0)$ is defined by

$$J'(0)=\frac{1}{2\pi}\left(\int_{-\pi}^{-2\pi\epsilon}+\int_{2\pi\epsilon}^{\pi}\right)\left\{\left[I-\frac{1}{2}i_0(0)i_0(0)\right]\cdot g[R(0)-R(\hat\theta)]\cdot F^*(\hat\theta)-\frac{\pi F^*(0)}{|\sin\frac{\hat\theta}{2}|}\right\}d\hat\theta\,.\quad (7.19)$$

Relationship (7.19) may be written in index notation as

$$J'_i=\frac{1}{2\pi}\left(\int_{-\pi}^{-2\pi\epsilon}+\int_{2\pi\epsilon}^{\pi}\right)\left\{\left(1-\frac{1}{2}\delta_{i\theta}\right)g_{ij}F_j^*-\frac{\pi F_i^*(0)}{|\sin\frac{\hat\theta}{2}|}\right\}d\hat\theta\,.\qquad (7.20)$$

Noting that F_j^* is constant along the body centreline and $F_i^*(0)=\delta_{ij}F_j^*$ [see (7.6)], J'_i may be simplified as

$$J'_i=F_jI_{ij},\qquad (7.21)$$

where I_{ij} is defined by

$$I_{ij}=\frac{1}{2\pi}\left(\int_{-\pi}^{-2\pi\epsilon}+\int_{2\pi\epsilon}^{\pi}\right)\left[\left(1-\frac{1}{2}\delta_{i\theta}\right)g_{ij}-\frac{\pi\delta_{ij}}{|\sin\frac{\hat\theta}{2}|}\right]d\hat\theta\,.\qquad (7.22)$$

The components of $g[R(0)-R(\hat\theta)]$ in cylindrical polar coordinate system (ρ,θ,z) with the unit base vectors i_ρ, i_θ and i_z may be determined as follows :

$$\begin{aligned}g&=g_{\rho\rho}i_\rho i_\rho+g_{\rho\theta}i_\rho i_\theta+g_{\rho z}i_\rho i_z+g_{\theta\rho}i_\theta i_\rho+g_{\theta\theta}i_\theta i_\theta+g_{\theta z}i_\theta i_z+g_{z\rho}i_z i_\rho+g_{z\theta}i_z i_\theta+g_{zz}i_z i_z\\&=g_{11}i_1i_1+g_{12}i_1i_2+g_{13}i_1i_3+g_{21}i_2i_1+g_{22}i_2i_2+g_{23}i_2i_3+g_{31}i_3i_1+g_{32}i_3i_2+g_{33}i_3i_3,\end{aligned}\qquad (7.23)$$

where the term on the right hand side of the second equality contains the components of $g[R-\hat{R}]$ with respect to the rectangular Cartesian coordinate system (x_1, x_2, x_3). But [see (7.4)]

$$i_1 = i_\rho \cos\hat{\theta} - i_\theta \sin\hat{\theta}\,; \quad i_2 = i_\rho \sin\hat{\theta} + i_\theta \cos\hat{\theta}\,; \quad i_3 = i_z\,,$$

where $\hat{\theta}$ is the integration variable. Therefore, (7.23) may be written as

$$\begin{aligned}
& g_{\rho\rho} i_\rho i_\rho + g_{\rho\theta} i_\rho i_\theta + g_{\rho z} i_\rho i_z + g_{\theta\rho} i_\theta i_\rho + g_{\theta\theta} i_\theta i_\theta + g_{\theta z} i_\theta i_z + g_{z\rho} i_z i_\rho + g_{z\theta} i_z i_\theta + g_{zz} i_z i_z = \\
& g_{11}\left(i_\rho \cos\hat{\theta} - i_\theta \sin\hat{\theta}\right)\left(i_\rho \cos\hat{\theta} - i_\theta \sin\hat{\theta}\right) + g_{12}\left(i_\rho \cos\hat{\theta} - i_\theta \sin\hat{\theta}\right)\left(i_\rho \sin\hat{\theta} + i_\theta \cos\hat{\theta}\right) \\
& + g_{13}\left(i_\rho \cos\hat{\theta} - i_\theta \sin\hat{\theta}\right) i_z + g_{21}\left(i_\rho \sin\hat{\theta} + i_\theta \cos\hat{\theta}\right)\left(i_\rho \cos\hat{\theta} - i_\theta \sin\hat{\theta}\right) \\
& + g_{22}\left(i_\rho \sin\hat{\theta} + i_\theta \cos\hat{\theta}\right)\left(i_\rho \sin\hat{\theta} + i_\theta \cos\hat{\theta}\right) + g_{23}\left(i_\rho \sin\hat{\theta} + i_\theta \cos\hat{\theta}\right) i_z \\
& + g_{31} i_z\left(i_\rho \cos\hat{\theta} - i_\theta \sin\hat{\theta}\right) + g_{32} i_z\left(i_\rho \sin\hat{\theta} + i_\theta \cos\hat{\theta}\right) + g_{33} i_z i_z = \\
& g_{11}\left(\cos^2\hat{\theta}\, i_\rho i_\rho - \sin\hat{\theta}\cos\hat{\theta}\, i_\rho i_\theta - \sin\hat{\theta}\cos\hat{\theta}\, i_\theta i_\rho + \sin^2\hat{\theta}\, i_\theta i_\theta\right) + g_{12}\left(\sin\hat{\theta}\cos\hat{\theta}\, i_\rho i_\rho\right. \\
& \left. + \cos^2\hat{\theta}\, i_\rho i_\theta - \sin^2\hat{\theta}\, i_\theta i_\rho - \sin\theta\cos\theta\, i_\theta i_\theta\right) + g_{13}\left(\cos\hat{\theta}\, i_\rho i_z - \sin\hat{\theta}\, i_\theta i_z\right) \\
& + g_{21}\left(\sin\hat{\theta}\cos\hat{\theta}\, i_\rho i_\rho - \sin^2\hat{\theta}\, i_\rho i_\theta + \cos^2\hat{\theta}\, i_\theta i_\rho - \sin\hat{\theta}\cos\hat{\theta}\, i_\theta i_\theta\right) + g_{22}\left(\sin^2\hat{\theta}\, i_\rho i_\rho\right. \\
& \left. + \sin\hat{\theta}\cos\hat{\theta}\, i_\rho i_\theta + \sin\hat{\theta}\cos\hat{\theta}\, i_\theta i_\rho + \cos^2\hat{\theta}\, i_\theta i_\theta\right) + g_{23}\left(\sin\hat{\theta}\, i_\rho i_z + \cos\hat{\theta}\, i_\theta i_z\right) \\
& + g_{31}\left(\cos\hat{\theta}\, i_z i_\rho - \sin\hat{\theta}\, i_z i_\theta\right) + g_{23}\left(\sin\hat{\theta}\, i_z i_\rho + \cos\hat{\theta}\, i_z i_\theta\right) + g_{33} i_z i_z .
\end{aligned} \tag{7.24}$$

Hence, the cylindrical components of $g[R(0) - R(\hat{\theta})]$ may be obtained by

$$g_{\rho\rho} = g_{11}\cos^2\hat{\theta} + 2g_{12}\sin\hat{\theta}\cos\hat{\theta} + g_{22}\sin^2\hat{\theta}\,; \tag{7.25}$$

$$g_{\rho\theta} = g_{\theta\rho} = -g_{11}\sin\hat{\theta}\cos\hat{\theta} + g_{12}\left(\cos^2\hat{\theta} - \sin^2\hat{\theta}\right) + g_{22}\sin\hat{\theta}\cos\hat{\theta}\,; \tag{7.26}$$

$$g_{\rho z} = g_{z\rho} = g_{13}\cos\hat{\theta} + g_{23}\sin\hat{\theta}\,; \tag{7.27}$$

$$g_{\theta\theta} = g_{11}\sin^2\hat{\theta} - 2g_{12}\sin\hat{\theta}\cos\hat{\theta} + g_{22}\cos^2\hat{\theta}\,; \tag{7.28}$$

$$g_{\theta z} = g_{z\theta} = -g_{13}\sin\hat{\theta} + g_{23}\cos\hat{\theta} \tag{7.29}$$

and

$$g_{zz} = g_{33}\,. \tag{7.30}$$

The Cartesian components of g may be determined by the aid of relationship (5.36) given in Chapter 5 as

$$\begin{aligned} g_{ij}[R-\hat{R}] = 2\Bigg(&\frac{1-e^{-\frac{1}{2}R_e\{|R-\hat{R}|-e\cdot[R-\hat{R}]\}}}{R_e\{|R-\hat{R}|-e\cdot[R-\hat{R}]\}^2}\Bigg)\Bigg\{\Bigg(\frac{[R-\hat{R}]_i}{|R-\hat{R}|}-e_i\Bigg)\Bigg(\frac{[R-\hat{R}]_j}{|R-\hat{R}|}-e_j\Bigg) \\ &-\{|R-\hat{R}|-e\cdot[R-\hat{R}]\}\Bigg(\frac{\delta_{ij}}{|R-\hat{R}|}-\frac{[R-\hat{R}]_i[R-\hat{R}]_j}{|R-\hat{R}|^3}\Bigg)\Bigg\} \\ &+\frac{e^{-\frac{1}{2}R_e(|R-\hat{R}|-e\cdot[R-\hat{R}])}}{|R-\hat{R}|-e\cdot[R-\hat{R}]}\Bigg[\frac{2\{|R-\hat{R}|-e\cdot[R-\hat{R}]\}\delta_{ij}}{|R-\hat{R}|} \\ &-\Bigg(\frac{[R-\hat{R}]_i}{|R-\hat{R}|}-e_i\Bigg)\Bigg(\frac{[R-\hat{R}]_j}{|R-\hat{R}|}-e_j\Bigg)\Bigg]\,. \end{aligned}$$

where $[R(0)-R(\hat{\theta})]_i$ and $|R(0)-R(\hat{\theta})|$ are respectively given by [see (7.11,12)]

$$[R-\hat{R}]_1 = \frac{1}{\pi}\sin^2\frac{\hat{\theta}}{2};\quad [R-\hat{R}]_2 = -\frac{1}{\pi}\sin\frac{\hat{\theta}}{2}\cos\frac{\hat{\theta}}{2};\quad [R-\hat{R}]_3 = 0 \tag{7.31}$$

and

$$|R - \hat{R}| = \frac{1}{\pi}\left|\sin\frac{\hat{\theta}}{2}\right|. \tag{7.32}$$

Noting that $e = (0,\ 0,\ 1)$, g_{ij} may be determined by the following calculations

$$g_{11} = 2\left[\frac{1 - e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}}{R_e\left(\frac{1}{\pi}|\sin\frac{\hat{\theta}}{2}|\right)^2}\right]\left\{\left(\frac{\frac{1}{\pi}\sin^2\frac{\hat{\theta}}{2}}{\frac{1}{\pi}|\sin\frac{\hat{\theta}}{2}|}\right)^2 - \frac{1}{\pi}|\sin\frac{\hat{\theta}}{2}| \times \left[\frac{1}{\frac{1}{\pi}|\sin\frac{\hat{\theta}}{2}|} - \frac{\left(\frac{1}{\pi}\sin^2\frac{\hat{\theta}}{2}\right)^2}{\left(\frac{1}{\pi}|\sin\frac{\hat{\theta}}{2}|\right)^3}\right]\right\} + \left(\frac{e^{-\frac{R_e}{2\pi}|\sin\frac{\hat{\theta}}{2}|}}{\frac{1}{\pi}|\sin\frac{\hat{\theta}}{2}|}\right)\left[2 - \left(\frac{\frac{1}{\pi}\sin^2\frac{\hat{\theta}}{2}}{\frac{1}{\pi}|\sin\frac{\hat{\theta}}{2}|}\right)^2\right] \tag{6.33}$$

$$= \frac{-2\pi^2\cos\hat{\theta}}{\sin^2\frac{\hat{\theta}}{2}}\left(\frac{1 - e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}}{R_e}\right) + \frac{\pi\left(2 - \sin^2\frac{\hat{\theta}}{2}\right)}{|\sin\frac{\hat{\theta}}{2}|}e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|},$$

$$g_{12} = g_{21} = 2\left[\frac{1 - e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}}{R_e\left(\frac{1}{\pi}|\sin\frac{\hat{\theta}}{2}|\right)^2}\right]\left\{\frac{\left(\frac{1}{\pi}\sin^2\frac{\hat{\theta}}{2}\right)\left(-\frac{1}{\pi}\sin\frac{\hat{\theta}}{2}\cos\frac{\hat{\theta}}{2}\right)}{\left(\frac{1}{\pi}|\sin\frac{\hat{\theta}}{2}|\right)^2} - \frac{1}{\pi}|\sin\frac{\hat{\theta}}{2}| \times \left[-\frac{\left(\frac{1}{\pi}\sin^2\frac{\hat{\theta}}{2}\right)\left(-\frac{1}{\pi}\sin\frac{\hat{\theta}}{2}\cos\frac{\hat{\theta}}{2}\right)}{\left(\frac{1}{\pi}|\sin\frac{\hat{\theta}}{2}|\right)^3}\right]\right\} + \left(\frac{e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}}{\frac{1}{\pi}|\sin\frac{\hat{\theta}}{2}|}\right)\left[-\frac{\left(\frac{1}{\pi}\sin^2\frac{\hat{\theta}}{2}\right)\left(-\frac{1}{\pi}\sin\frac{\hat{\theta}}{2}\cos\frac{\hat{\theta}}{2}\right)}{\left(\frac{1}{\pi}|\sin\frac{\hat{\theta}}{2}|\right)^2}\right]$$

$$= -\frac{2\pi^2\sin\hat{\theta}}{\sin^2\frac{\hat{\theta}}{2}}\left(\frac{1 - e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}}{R_e}\right) + \frac{\pi\sin\hat{\theta}}{2|\sin\frac{\hat{\theta}}{2}|}e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}, \tag{7.34}$$

$$
\begin{aligned}
g_{13} = g_{31} &= 2\left[\frac{1-e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}}{R_e\left(\frac{1}{\pi}|\sin\frac{\hat{\theta}}{2}|\right)^2}\right]\left(\frac{\frac{1}{\pi}\sin^2\frac{\hat{\theta}}{2}}{\frac{1}{\pi}|\sin\frac{\hat{\theta}}{2}|}\right)(-1) + \left(\frac{e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}}{\frac{1}{\pi}|\sin\frac{\hat{\theta}}{2}|}\right)\left(-\frac{\frac{1}{\pi}\sin^2\frac{\hat{\theta}}{2}}{\frac{1}{\pi}|\sin\frac{\hat{\theta}}{2}|}\right)(-1) \\
&= -\frac{2\pi^2}{|\sin\frac{\hat{\theta}}{2}|}\left(\frac{1-e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}}{R_e}\right) + \pi e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|},
\end{aligned}
\tag{7.35}
$$

$$
\begin{aligned}
g_{22} &= 2\left[\frac{1-e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}}{R_e\left(\frac{1}{\pi}|\sin\frac{\hat{\theta}}{2}|\right)^2}\right]\left\{\left(\frac{-\frac{1}{\pi}\sin\frac{\hat{\theta}}{2}\cos\frac{\hat{\theta}}{2}}{\frac{1}{\pi}|\sin\frac{\hat{\theta}}{2}|}\right)^2 - \frac{1}{\pi}|\sin\frac{\hat{\theta}}{2}|\left[\frac{1}{\frac{1}{\pi}|\sin\frac{\hat{\theta}}{2}|}\right.\right. \\
&\quad \left.\left. - \frac{\left(-\frac{1}{\pi}\sin\frac{\hat{\theta}}{2}\cos\frac{\hat{\theta}}{2}\right)^2}{\left(\frac{1}{\pi}|\sin\frac{\hat{\theta}}{2}|\right)^3}\right]\right\} + \left(\frac{e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}}{\frac{1}{\pi}|\sin\frac{\hat{\theta}}{2}|}\right)\left[2 - \left(\frac{-\frac{1}{\pi}\sin\frac{\hat{\theta}}{2}\cos\frac{\hat{\theta}}{2}}{\frac{1}{\pi}|\sin\frac{\hat{\theta}}{2}|}\right)^2\right] \\
&= \frac{2\pi^2\cos\hat{\theta}}{\sin^2\frac{\hat{\theta}}{2}}\left(\frac{1-e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}}{R_e}\right) + \frac{\pi\left(1+\sin^2\frac{\hat{\theta}}{2}\right)}{|\sin\frac{\hat{\theta}}{2}|}e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|},
\end{aligned}
\tag{7.36}
$$

$$
\begin{aligned}
g_{23} = g_{32} &= 2\left[\frac{1-e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}}{R_e\left(\frac{1}{\pi}|\sin\frac{\hat{\theta}}{2}|\right)^2}\right]\left(\frac{\frac{1}{\pi}\sin\frac{\hat{\theta}}{2}\cos\frac{\hat{\theta}}{2}}{\frac{1}{\pi}|\sin\frac{\hat{\theta}}{2}|}\right) + \left(\frac{e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}}{\frac{1}{\pi}|\sin\frac{\hat{\theta}}{2}|}\right)\left(\frac{-\frac{1}{\pi}\sin\frac{\hat{\theta}}{2}\cos\frac{\hat{\theta}}{2}}{\frac{1}{\pi}|\sin\frac{\hat{\theta}}{2}|}\right) \\
&= \frac{2\pi^2\cos\frac{\hat{\theta}}{2}}{\sin\frac{\hat{\theta}}{2}|\sin\frac{\hat{\theta}}{2}|}\left(\frac{1-e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}}{R_e}\right) - \frac{\pi\cos\frac{\hat{\theta}}{2}}{\sin\frac{\hat{\theta}}{2}}e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}
\end{aligned}
\tag{7.37}
$$

and

$$g_{33} = 2\left[\frac{1 - e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat\theta}{2}|}}{R_e\left(\frac{1}{\pi}|\sin\frac{\hat\theta}{2}|\right)^2}\right]\left[1 - \frac{1}{\pi}|\sin\frac{\hat\theta}{2}|\left(\frac{1}{\frac{1}{\pi}|\sin\frac{\hat\theta}{2}|}\right)\right] + \left(\frac{e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat\theta}{2}|}}{\frac{1}{\pi}|\sin\frac{\hat\theta}{2}|}\right)$$

$$= + \frac{\pi}{|\sin\frac{\hat\theta}{2}|}e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat\theta}{2}|}. \tag{7.38}$$

Thus, the cylindrical components of g may be obtained by [see (7.25-30,33-38)]

$$g_{\rho\rho} = g_{11}\left(\frac{1+\cos2\hat\theta}{2}\right) + g_{12}\sin2\hat\theta + g_{22}\left(\frac{1-\cos2\hat\theta}{2}\right)$$

$$= \frac{2\pi^2}{\sin^2\frac{\hat\theta}{2}}\left(\frac{1 - e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat\theta}{2}|}}{R_e}\right)\left[-\cos\hat\theta\left(\frac{1+\cos2\hat\theta}{2}\right) - \sin\hat\theta\sin2\hat\theta\right.$$

$$\left. + \cos\hat\theta\left(\frac{1-\cos2\hat\theta}{2}\right)\right] + \frac{\pi e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat\theta}{2}|}}{|\sin\frac{\hat\theta}{2}|}\left[1 + \frac{1+\cos\hat\theta}{2}\left(\frac{1+\cos2\hat\theta}{2}\right)\right.$$

$$\left. + \frac{1}{2}\sin\hat\theta\sin2\hat\theta + \left(1 + \frac{1-\cos\hat\theta}{2}\right)\left(\frac{1-\cos2\hat\theta}{2}\right)\right]$$

$$= \frac{2\pi^2}{\sin^2\frac{\hat\theta}{2}}\left(\frac{1 - e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat\theta}{2}|}}{R_e}\right)\left(\frac{-2\cos\hat\theta\cos2\hat\theta - 2\sin\hat\theta\sin2\hat\theta}{2}\right)$$

$$+ \frac{\pi e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat\theta}{2}|}}{|\sin\frac{\hat\theta}{2}|}\left(1 + \frac{2 + 2\cos\hat\theta\cos2\hat\theta - 2\sin\hat\theta\sin2\hat\theta}{4}\right)$$

$$= \frac{-2\pi^2\cos\hat\theta}{\sin^2\frac{\hat\theta}{2}}\left(\frac{1 - e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat\theta}{2}|}}{R_e}\right) + \frac{\pi\left(2 - \sin^2\frac{\hat\theta}{2}\right)}{|\sin\frac{\hat\theta}{2}|}e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat\theta}{2}|}, \tag{7.39}$$

$$
\begin{aligned}
g_{\rho\theta} = g_{\theta\rho} &= \frac{1}{2}(g_{22} - g_{11})\sin 2\hat{\theta} + g_{12}\cos 2\hat{\theta} \\
&= \frac{2\pi^2}{\sin^2\frac{\hat{\theta}}{2}}\left(\frac{1 - e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}}{R_e}\right)(\cos\hat{\theta}\sin 2\hat{\theta} - \sin\hat{\theta}\cos 2\hat{\theta}) \\
&\quad + \frac{\pi}{|\sin\frac{\hat{\theta}}{2}|}e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}\left(-\frac{1}{2}\cos\hat{\theta}\sin 2\hat{\theta} + \frac{1}{2}\sin\hat{\theta}\cos 2\hat{\theta}\right) \\
&= \frac{2\pi^2\sin\hat{\theta}}{\sin^2\frac{\hat{\theta}}{2}}\left(\frac{1 - e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}}{R_e}\right) - \frac{\pi\sin\hat{\theta}}{2|\sin\frac{\hat{\theta}}{2}|}e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}; \qquad (7.40)
\end{aligned}
$$

$$
\begin{aligned}
g_{\rho z} = g_{z\rho} &= g_{13}\cos\hat{\theta} + g_{23}\sin\hat{\theta} \\
&= \frac{2\pi^2}{\sin\frac{\hat{\theta}}{2}|\sin\frac{\hat{\theta}}{2}|}\left(\frac{1 - e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}}{R_e}\right)\left(-\sin\frac{\hat{\theta}}{2}\cos\hat{\theta} + \cos\frac{\hat{\theta}}{2}\sin\hat{\theta}\right) \\
&\quad + \frac{\pi}{\sin\frac{\hat{\theta}}{2}}e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}\left(\sin\frac{\hat{\theta}}{2}\cos\hat{\theta} - \cos\frac{\hat{\theta}}{2}\sin\hat{\theta}\right) \\
&= \frac{2\pi^2}{|\sin\frac{\hat{\theta}}{2}|}\left(\frac{1 - e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}}{R_e}\right) - \pi e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}; \qquad (7.41)
\end{aligned}
$$

$$
\begin{aligned}
g_{\theta\theta} &= g_{11}\left(\frac{1-\cos 2\hat{\theta}}{2}\right) - g_{12}\sin 2\hat{\theta} + g_{22}\left(\frac{1+\cos 2\hat{\theta}}{2}\right) \\
&= \frac{2\pi^2}{\sin^2\frac{\hat{\theta}}{2}}\left(\frac{1-e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}}{R_e}\right)\left[-\cos\hat{\theta}\left(\frac{1-\cos 2\hat{\theta}}{2}\right) + \sin\hat{\theta}\sin 2\hat{\theta} + \cos\hat{\theta}\left(\frac{1+\cos 2\hat{\theta}}{2}\right)\right] \\
&\quad + \frac{\pi}{|\sin\frac{\hat{\theta}}{2}|}e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}\left[\left(1+\frac{1+\cos\hat{\theta}}{2}\right)\left(\frac{1-\cos 2\hat{\theta}}{2}\right) - \frac{1}{2}\sin\hat{\theta}\sin 2\hat{\theta}\right. \\
&\quad \left. + \left(1+\frac{1-\cos\hat{\theta}}{2}\right)\left(\frac{1+\cos 2\hat{\theta}}{2}\right)\right] \\
&= \frac{2\pi^2}{\sin^2\frac{\hat{\theta}}{2}}\left(\frac{1-e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}}{R_e}\right)\left(\frac{2\cos\hat{\theta}\cos 2\hat{\theta} + 2\sin\hat{\theta}\sin 2\hat{\theta}}{2}\right) \\
&\quad + \frac{\pi}{|\sin\frac{\hat{\theta}}{2}|}e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}\left(2 - \frac{1+\cos\hat{\theta}\cos 2\hat{\theta} + \sin\hat{\theta}\sin 2\hat{\theta}}{2}\right) \\
&= \frac{2\pi^2\cos\hat{\theta}}{\sin^2\frac{\hat{\theta}}{2}}\left(\frac{1-e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}}{R_e}\right) + \frac{\pi\left(2-\cos^2\frac{\hat{\theta}}{2}\right)}{|\sin\frac{\hat{\theta}}{2}|}e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|},
\end{aligned}
\tag{7.42}
$$

$$
\begin{aligned}
g_{\theta z} = g_{z\theta} &= -g_{13}\sin\hat{\theta} + g_{23}\cos\hat{\theta} \\
&= \frac{2\pi^2}{\sin\frac{\hat{\theta}}{2}|\sin\frac{\hat{\theta}}{2}|}\left(\frac{1-e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}}{R_e}\right)\left(\sin\frac{\hat{\theta}}{2}\sin\hat{\theta} + \cos\frac{\hat{\theta}}{2}\cos\hat{\theta}\right) \\
&\quad + \frac{\pi}{\sin\frac{\hat{\theta}}{2}}e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}\left(-\sin\frac{\hat{\theta}}{2}\sin\hat{\theta} - \cos\frac{\hat{\theta}}{2}\cos\hat{\theta}\right)
\end{aligned}
$$

or

$$g_{\theta z} = g_{z\theta} = \frac{2\pi^2 \cos\frac{\hat{\theta}}{2}}{\sin\frac{\hat{\theta}}{2}\,|\sin\frac{\hat{\theta}}{2}|}\left(\frac{1 - e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}}{R_e}\right) - \frac{\pi\cos\frac{\hat{\theta}}{2}}{\sin\frac{\hat{\theta}}{2}} e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}, \tag{7.43}$$

and

$$\begin{aligned} g_{zz} &= g_{33} \\ &= \frac{\pi}{|\sin\frac{\hat{\theta}}{2}|} e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}. \end{aligned} \tag{7.44}$$

Thus, by the aid of (7.39-41) $I_{\rho\rho}$ defined by (7.22) may be obtained as follows:

$$\begin{aligned} I_{\rho\rho} &= \frac{1}{2\pi}\left(\int_{-\pi}^{-2\pi\epsilon} + \int_{2\pi\epsilon}^{\pi}\right)\left[g_{ij} - \pi\left(|\sin\frac{\hat{\theta}}{2}|\right)^{-1}\right]d\hat{\theta} \\ &= \frac{1}{2}\int_{-\pi}^{-2\pi\epsilon}\left[\frac{-2\pi\left(1 - 2\sin^2\frac{\hat{\theta}}{2}\right)}{\sin^2\frac{\hat{\theta}}{2}}\left(\frac{1 - e^{\frac{1}{2\pi}R_e\sin\frac{\hat{\theta}}{2}}}{R_e}\right) - \frac{\left(2 - \sin^2\frac{\hat{\theta}}{2}\right)}{\sin\frac{\hat{\theta}}{2}} e^{\frac{1}{2\pi}R_e\sin\frac{\hat{\theta}}{2}} + \frac{1}{\sin\frac{\hat{\theta}}{2}}\right]d\hat{\theta} \\ &+ \frac{1}{2}\int_{2\pi\epsilon}^{\pi}\left[-\frac{2\pi\left(1 - 2\sin^2\frac{\hat{\theta}}{2}\right)}{\sin^2\frac{\hat{\theta}}{2}}\left(\frac{1 - e^{-\frac{1}{2\pi}R_e\sin\frac{\hat{\theta}}{2}}}{R_e}\right) + \frac{\left(2 - \sin^2\frac{\hat{\theta}}{2}\right)}{\sin\frac{\hat{\theta}}{2}} e^{-\frac{1}{2\pi}R_e\sin\frac{\hat{\theta}}{2}} - \frac{1}{\sin\frac{\hat{\theta}}{2}}\right]d\hat{\theta} \\ &= \int_{2\pi\epsilon}^{\pi}\left[-\frac{2\pi\left(1 - 2\sin^2\frac{\hat{\theta}}{2}\right)}{\sin^2\frac{\hat{\theta}}{2}}\left(\frac{1 - e^{-\frac{1}{2\pi}R_e\sin\frac{\hat{\theta}}{2}}}{R_e}\right) + \frac{\left(2 - \sin^2\frac{\hat{\theta}}{2}\right)}{\sin\frac{\hat{\theta}}{2}} e^{-\frac{1}{2\pi}R_e\sin\frac{\hat{\theta}}{2}} - \frac{1}{\sin\frac{\hat{\theta}}{2}}\right]d\hat{\theta} \end{aligned}$$

or

$$I_{pp} = -\frac{2\pi}{R_e}\left[\int_{2\pi\epsilon}^{\pi}\frac{1-e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}}{\sin^2\frac{\hat\theta}{2}}d\hat\theta - 2\int_{2\pi\epsilon}^{\pi}d\hat\theta + 2\int_{2\pi\epsilon}^{\pi}e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}d\hat\theta\right.$$
$$\left. + 2\int_{2\pi\epsilon}^{\pi}\frac{e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}}{\sin\frac{\hat\theta}{2}}d\hat\theta - \int_{2\pi\epsilon}^{\pi}\sin\frac{\hat\theta}{2}e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}d\hat\theta - \int_{2\pi\epsilon}^{\pi}\left(\sin\frac{\hat\theta}{2}\right)^{-1}d\hat\theta\right]. \tag{7.45}$$

But

$$\int_{2\pi\epsilon}^{\pi}\frac{1-e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}}{\sin^2\frac{\hat\theta}{2}}d\hat\theta = \int_{2\pi\epsilon}^{\pi}\frac{d\hat\theta}{\sin^2\frac{\hat\theta}{2}} - \int_{2\pi\epsilon}^{\pi}\frac{e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}}{\sin\frac{\hat\theta}{2}}d\hat\theta$$
$$= \left|-\frac{2\cos\frac{\hat\theta}{2}}{\sin\frac{\hat\theta}{2}}\right|_{2\pi\epsilon}^{\pi} - \left\{\left|-\frac{2\cos\frac{\hat\theta}{2}e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}}{\sin\frac{\hat\theta}{2}}\right|_{2\pi\epsilon}^{\pi} - \frac{R_e}{2\pi}\int_{2\pi\epsilon}^{\pi}\frac{\cos^2\frac{\hat\theta}{2}e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}}{\sin\frac{\hat\theta}{2}}d\hat\theta\right\}$$
$$= \frac{2\cos(\pi\epsilon)}{\sin(\pi\epsilon)} - \frac{2\cos(\pi\epsilon)e^{-\frac{1}{2\pi}R_e\sin(\pi\epsilon)}}{\sin(\pi\epsilon)} + \frac{R_e}{2\pi}\int_{2\pi\epsilon}^{\pi}\frac{\left(1-\sin^2\frac{\hat\theta}{2}\right)e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}}{\sin\frac{\hat\theta}{2}}d\hat\theta$$
$$= \frac{2\cos(\pi\epsilon)}{\sin(\pi\epsilon)} - \frac{[2\cos(\pi\epsilon)]\left[1-\frac{R_e}{2\pi}\sin(\pi\epsilon) + \left(\frac{R_e}{2\pi}\right)^2\frac{\sin^2(\pi\epsilon)}{2!} + \dots\right]}{\sin(\pi\epsilon)} +$$
$$\frac{R_e}{2\pi}\int_{2\pi\epsilon}^{\pi}\frac{\left(1-\sin^2\frac{\hat\theta}{2}\right)e^{-\frac{R_e}{2\pi}\sin\frac{\hat\theta}{2}}}{\sin\frac{\hat\theta}{2}}d\hat\theta = \frac{R_e}{2\pi}\left\{\int_{2\pi\epsilon}^{\pi}\frac{e^{-\frac{R_e}{2\pi}\sin\frac{\hat\theta}{2}}}{\sin\frac{\hat\theta}{2}}d\hat\theta - \int_{2\pi\epsilon}^{\pi}\sin\frac{\hat\theta}{2}e^{-\frac{R_e}{2\pi}\sin\frac{\hat\theta}{2}}d\hat\theta + 2\right\}, \tag{7.46}$$

so that.

$$I_{pp} = -\frac{2\pi}{R_e}\left[\frac{R_e}{2\pi}\left(\int_{2\pi\varepsilon}^{\pi}\frac{e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}}{\sin\frac{\hat\theta}{2}}d\hat\theta - \int_{2\pi\varepsilon}^{\pi}\sin\frac{\hat\theta}{2}e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}d\hat\theta + 2\right) + 2\int_{2\pi\varepsilon}^{\pi}e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}d\hat\theta\right.$$

$$\left. - 2\left|\hat\theta\right|_{2\pi\varepsilon}^{\pi}\right] + 2\int_{2\pi\varepsilon}^{\pi}\frac{e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}}{\sin\frac{\hat\theta}{2}}d\hat\theta - \int_{2\pi\varepsilon}^{\pi}\sin\frac{\hat\theta}{2}e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}d\hat\theta - \int_{2\pi\varepsilon}^{\pi}\left(\sin\frac{\hat\theta}{2}\right)^{-1}d\hat\theta\,. \tag{7.47}$$

However, as $\varepsilon \longrightarrow 0$

$$\int_{2\pi\varepsilon}^{\pi}\left(\frac{e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}}{\sin\frac{\hat\theta}{2}} - \frac{1}{\sin\frac{\hat\theta}{2}}\right)d\hat\theta = \int_{2\pi\varepsilon}^{\pi}\left[\frac{1 - \frac{R_e}{2\pi}\sin\frac{\hat\theta}{2} + \left(\frac{R_e}{2\pi}\right)^2\frac{\sin^2\frac{\hat\theta}{2}}{2!} + \cdots}{\sin\frac{\hat\theta}{2}} - \frac{1}{\sin\frac{\hat\theta}{2}}\right]d\hat\theta$$

$$= \int_{2\pi\varepsilon}^{\pi}\left[-\frac{R_e}{2\pi} + \frac{1}{2}\left(\frac{R_e}{2\pi}\right)^2\sin\frac{\hat\theta}{2} + \cdots\right]d\hat\theta\,; \tag{7.48}$$

hence, as expected, there is no singularity in (7.47). Therefore, in the limit as $\varepsilon \longrightarrow 0$, I_{pp} may be written as

$$I_{pp} = A_1 + \frac{4\pi^2}{R_e} - \frac{4\pi}{R_e}A_2 - 2\,, \tag{7.49}$$

where A_1 and A_2 are respectively defined by

$$A_1 = \int_0^{\pi}\left(\frac{e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}}{\sin\frac{\hat\theta}{2}} - \frac{1}{\sin\frac{\hat\theta}{2}}\right)d\hat\theta\,; \tag{7.50}$$

and

$$A_2 = \int_0^{\pi} e^{-\frac{1}{2\pi}R_e \sin\frac{\hat{\theta}}{2}} d\hat{\theta}. \tag{7.51}$$

By the aid of (7.40-44,50,51), the other cylindrical components of I_{ij}, in the limit as $\varepsilon \longrightarrow 0$, may be determined by the following calculations

$$\begin{aligned}
I_{\rho\theta} = I_{\theta\rho} &= \frac{1}{2\pi}\left(\int_{-\pi}^{-2\pi\varepsilon} + \int_{2\pi\varepsilon}^{\pi}\right) g_{\rho\theta} d\hat{\theta} \\
&= \frac{1}{2}\left(\int_{-\pi}^{-2\pi\varepsilon} + \int_{2\pi\varepsilon}^{\pi}\right)\left\{\frac{2\pi\sin\hat{\theta}}{\sin^2\frac{\hat{\theta}}{2}}\left(\frac{1 - e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}}{R_e}\right) - \frac{\sin\hat{\theta}}{2|\sin\frac{\hat{\theta}}{2}|} e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}\right\} d\hat{\theta} \\
&= \frac{1}{2}\int_{-\pi}^{-2\pi\varepsilon}\left[\frac{2\pi\sin\hat{\theta}}{\sin^2\frac{\hat{\theta}}{2}}\left(\frac{1 - e^{\frac{1}{2\pi}R_e\sin\frac{\hat{\theta}}{2}}}{R_e}\right) + \frac{\sin\hat{\theta}}{2\sin\frac{\hat{\theta}}{2}} e^{\frac{1}{2\pi}R_e\sin\frac{\hat{\theta}}{2}}\right] d\hat{\theta} \\
&\quad + \frac{1}{2}\int_{2\pi\varepsilon}^{\pi}\left[\frac{2\pi\sin\hat{\theta}}{\sin^2\frac{\hat{\theta}}{2}}\left(\frac{1 - e^{-\frac{1}{2\pi}R_e\sin\frac{\hat{\theta}}{2}}}{R_e}\right) - \frac{\sin\hat{\theta}}{2\sin\frac{\hat{\theta}}{2}} e^{-\frac{1}{2\pi}R_e|\sin\frac{\hat{\theta}}{2}|}\right] d\hat{\theta} \\
&= \frac{1}{2}\int_{2\pi\varepsilon}^{\pi}\left[-\frac{2\pi\sin\hat{\theta}}{\sin^2\frac{\hat{\theta}}{2}}\left(\frac{1 - e^{-\frac{1}{2\pi}R_e\sin\frac{\hat{\theta}}{2}}}{R_e}\right) + \frac{\sin\hat{\theta}}{2\sin\frac{\hat{\theta}}{2}} e^{-\frac{1}{2\pi}R_e\sin\frac{\hat{\theta}}{2}}\right] d\hat{\theta} \\
&\quad + \frac{1}{2}\int_{2\pi\varepsilon}^{\pi}\left[\frac{2\pi\sin\hat{\theta}}{\sin^2\frac{\hat{\theta}}{2}}\left(\frac{1 - e^{-\frac{1}{2\pi}R_e\sin\frac{\hat{\theta}}{2}}}{R_e}\right) - \frac{\sin\hat{\theta}}{2\sin\frac{\hat{\theta}}{2}} e^{-\frac{1}{2\pi}R_e\sin\frac{\hat{\theta}}{2}}\right] d\hat{\theta} \\
&= 0,
\end{aligned} \tag{7.52}$$

$$I_{\rho z} = I_{z\rho} = \frac{1}{2\pi}\left(\int_{-\pi}^{-2\pi\varepsilon} + \int_{2\pi\varepsilon}^{\pi}\right) g_{\rho z} d\hat{\theta},$$

hence

$$I_{\rho z} = I_{z\rho} = \frac{1}{2}\int_{-\pi}^{-2\pi\varepsilon}\left[-\frac{2\pi}{\sin\frac{\hat{\theta}}{2}}\left(\frac{1-e^{\frac{1}{2\pi}R_e\sin\frac{\hat{\theta}}{2}}}{R_e}\right)-e^{\frac{1}{2\pi}R_e\sin\frac{\hat{\theta}}{2}}\right]d\hat{\theta}$$

$$+\frac{1}{2}\int_{2\pi\varepsilon}^{\pi}\left[\frac{2\pi}{\sin\frac{\hat{\theta}}{2}}\left(\frac{1-e^{-\frac{1}{2\pi}R_e\sin\frac{\hat{\theta}}{2}}}{R_e}\right)-e^{-\frac{1}{2\pi}R_e\sin\frac{\hat{\theta}}{2}}\right]d\hat{\theta}$$

$$=\frac{1}{2}\int_{2\pi\varepsilon}^{\pi}\left[\frac{2\pi}{\sin\frac{\hat{\theta}}{2}}\left(\frac{1-e^{-\frac{1}{2\pi}R_e\sin\frac{\hat{\theta}}{2}}}{R_e}\right)-e^{-\frac{1}{2\pi}R_e\sin\frac{\hat{\theta}}{2}}\right]d\hat{\theta}$$

$$+\frac{1}{2}\int_{2\pi\varepsilon}^{\pi}\left[\frac{2\pi}{\sin\frac{\hat{\theta}}{2}}\left(\frac{1-e^{-\frac{1}{2\pi}R_e\sin\frac{\hat{\theta}}{2}}}{R_e}\right)-e^{-\frac{1}{2\pi}R_e\sin\frac{\hat{\theta}}{2}}\right]d\hat{\theta}$$

$$=\int_{2\pi\varepsilon}^{\pi}\left[\frac{2\pi}{\sin\frac{\hat{\theta}}{2}}\left(\frac{1-e^{-\frac{1}{2\pi}R_e\sin\frac{\hat{\theta}}{2}}}{R_e}\right)-e^{-\frac{1}{2\pi}R_e\sin\frac{\hat{\theta}}{2}}\right]d\hat{\theta}$$

$$=-\frac{2\pi}{R_e}A_1-A_2, \qquad (7.53)$$

$$I_{\theta\theta}=\frac{1}{2\pi}\left(\int_{-\pi}^{-2\pi\varepsilon}+\int_{2\pi\varepsilon}^{\pi}\right)\left[\frac{1}{2}g_{\theta\theta}-\pi\left(\left|\sin\frac{\hat{\theta}}{2}\right|\right)^{-1}\right]d\hat{\theta}$$

$$=\frac{1}{2}\int_{-\pi}^{-2\pi\varepsilon}\left[\frac{\pi\cos\hat{\theta}}{\sin^2\frac{\hat{\theta}}{2}}\left(\frac{1-e^{\frac{1}{2\pi}R_e\sin\frac{\hat{\theta}}{2}}}{R_e}\right)-\frac{\left(2-\cos^2\frac{\hat{\theta}}{2}\right)}{2\sin\frac{\hat{\theta}}{2}}e^{\frac{1}{2\pi}R_e\sin\frac{\hat{\theta}}{2}}+\frac{1}{\sin\frac{\hat{\theta}}{2}}\right]d\hat{\theta}$$

$$+\frac{1}{2}\int_{2\pi\varepsilon}^{\pi}\left[\frac{\pi\cos\hat{\theta}}{\sin^2\frac{\hat{\theta}}{2}}\left(\frac{1-e^{-\frac{1}{2\pi}R_e\sin\frac{\hat{\theta}}{2}}}{R_e}\right)+\frac{\left(2-\cos^2\frac{\hat{\theta}}{2}\right)}{2\sin\frac{\hat{\theta}}{2}}e^{-\frac{1}{2\pi}R_e\sin\frac{\hat{\theta}}{2}}-\frac{1}{\sin\frac{\hat{\theta}}{2}}\right]d\hat{\theta}$$

or

$$I_{\theta\theta} = \frac{1}{2}\int_{2\pi\varepsilon}^{\pi}\left[\frac{\pi\left(1-2\sin^2\frac{\hat\theta}{2}\right)}{\sin^2\frac{\hat\theta}{2}}\left(\frac{1-e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}}{R_e}\right)+\frac{\left(1+\sin^2\frac{\hat\theta}{2}\right)}{2\sin\frac{\hat\theta}{2}}e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}-\frac{1}{\sin\frac{\hat\theta}{2}}\right]d\hat\theta$$

$$+\frac{1}{2}\int_{2\pi\varepsilon}^{\pi}\left[\frac{2\pi\left(1-2\sin^2\frac{\hat\theta}{2}\right)}{\sin^2\frac{\hat\theta}{2}}\left(\frac{1-e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}}{R_e}\right)+\frac{\left(1+\sin^2\frac{\hat\theta}{2}\right)}{2\sin\frac{\hat\theta}{2}}e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}-\frac{1}{\sin\frac{\hat\theta}{2}}\right]d\hat\theta$$

$$=\frac{\pi}{R_e}\left[\frac{R_e}{2\pi}\left(\int_{2\pi\varepsilon}^{\pi}\frac{e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}}{\sin\frac{\hat\theta}{2}}d\hat\theta-\int_{2\pi\varepsilon}^{\pi}\sin\frac{\hat\theta}{2}e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}d\hat\theta+2\right)+2\int_{2\pi\varepsilon}^{\pi}e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}d\hat\theta\right.$$

$$\left.+2\int_{2\pi\varepsilon}^{\pi}d\hat\theta\right]+\frac{1}{2}\int_{2\pi\varepsilon}^{\pi}\frac{e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}}{\sin\frac{\hat\theta}{2}}d\hat\theta+\frac{1}{2}\int_{2\pi\varepsilon}^{\pi}\sin\frac{\hat\theta}{2}e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}d\hat\theta-\int_{2\pi\varepsilon}^{\pi}\frac{d\hat\theta}{\sin\frac{\hat\theta}{2}}$$

$$=A_1-\frac{2\pi^2}{R_e}+\frac{2\pi}{R_e}A_2+1\,, \tag{7.54}$$

$$I_{\theta z}=I_{z\theta}=\frac{1}{2\pi}\left(\int_{-\pi}^{-2\pi\varepsilon}+\int_{2\pi\varepsilon}^{\pi}\right)g_{\theta z}\,d\hat\theta$$

$$=\frac{1}{2}\int_{-\pi}^{-2\pi\varepsilon}\left[-\frac{2\pi\cos\frac{\hat\theta}{2}}{\sin^2\frac{\hat\theta}{2}}\left(\frac{1-e^{\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}}{R_e}\right)-\frac{\cos\frac{\hat\theta}{2}}{\sin\frac{\hat\theta}{2}}e^{\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}\right]d\hat\theta$$

$$+\frac{1}{2}\int_{2\pi\varepsilon}^{\pi}\left[\frac{2\pi\cos\frac{\hat\theta}{2}}{\sin^2\frac{\hat\theta}{2}}\left(\frac{1-e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}}{R_e}\right)-\frac{\cos\frac{\hat\theta}{2}}{\sin\frac{\hat\theta}{2}}e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}\right]d\hat\theta$$

or

$$
\begin{aligned}
I_{\theta z} = I_{z\theta} &= \frac{1}{2}\int_{2\pi\epsilon}^{\pi}\left[-\frac{2\pi\cos\frac{\hat\theta}{2}}{\sin^2\frac{\hat\theta}{2}}\left(\frac{1-e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}}{R_e}\right)+\frac{\cos\frac{\hat\theta}{2}}{\sin\frac{\hat\theta}{2}}e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}\right]d\hat\theta \\
&+\frac{1}{2}\int_{2\pi\epsilon}^{\pi}\left[\frac{2\pi\cos\frac{\hat\theta}{2}}{\sin^2\frac{\hat\theta}{2}}\left(\frac{1-e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}}{R_e}\right)-\frac{\cos\frac{\hat\theta}{2}}{\sin\frac{\hat\theta}{2}}e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}\right]d\hat\theta \\
&= 0 \qquad (7.55)
\end{aligned}
$$

and

$$
\begin{aligned}
I_{zz} &= \frac{1}{2\pi}\left(\int_{-\pi}^{-2\pi\epsilon}+\int_{2\pi\epsilon}^{\pi}\right)\left[g_{zz}-\pi\left(\left|\sin\frac{\hat\theta}{2}\right|\right)^{-1}\right]d\hat\theta \\
&= \frac{1}{2}\int_{-\pi}^{-2\pi\epsilon}\left[\frac{-1}{\sin\frac{\hat\theta}{2}}e^{\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}+\frac{1}{\sin\frac{\hat\theta}{2}}\right]d\hat\theta+\frac{1}{2}\int_{2\pi\epsilon}^{\pi}\left[\frac{1}{\sin\frac{\hat\theta}{2}}e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}-\frac{1}{\sin\frac{\hat\theta}{2}}\right]d\hat\theta \\
&= \frac{1}{2}\int_{2\pi\epsilon}^{\pi}\left[\frac{1}{\sin\frac{\hat\theta}{2}}e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}-\frac{1}{\sin\frac{\hat\theta}{2}}\right]d\hat\theta+\frac{1}{2}\int_{2\pi\epsilon}^{\pi}\left[\frac{1}{\sin\frac{\hat\theta}{2}}e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}-\frac{1}{\sin\frac{\hat\theta}{2}}\right]d\hat\theta \\
&= \int_{2\pi\epsilon}^{\pi}\left[\frac{1}{\sin\frac{\hat\theta}{2}}e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}-\frac{1}{\sin\frac{\hat\theta}{2}}\right]d\hat\theta \\
&= A_1 . \qquad (7.56)
\end{aligned}
$$

But, vector $J'(0)$ defined by (7.19-22) may be expressed as [see (7.4,6)]

$$
\begin{aligned}
J'(0) &= F_j^* I_{ij} i_i(0) = \left(F_\rho^* I_{\rho\rho}+F_\theta^* I_{\rho\theta}+F_z^* I_{\rho z}\right)i_\rho(0)+\left(F_\rho^* I_{\theta\rho}+F_\theta^* I_{\theta\theta}+F_z^* I_{\theta z}\right)i_\theta(0) \\
&\quad+\left(F_\rho^* I_{z\rho}+F_\theta^* I_{z\theta}+F_z^* I_{zz}\right)i_z(0) \\
&= \left(F_\rho^* I_{\rho\rho}+F_\theta^* I_{\rho\theta}+F_z^* I_{\rho z}\right)i_1+\left(F_\rho^* I_{\theta\rho}+F_\theta^* I_{\theta\theta}+F_z^* I_{\theta z}\right)i_2+\left(F_\rho^* I_{z\rho}+F_\theta^* I_{z\theta}+F_z^* I_{zz}\right)i_3 , \qquad (7.57)
\end{aligned}
$$

so that, by the aid of (7.49-56), the Cartesian components of J' may be obtained by

$$J'_1(0) = F_\rho^*\left(A_1 - \frac{4\pi}{R_e}A_2 + \frac{4\pi^2}{R_e} - 2\right) - F_z^*\left(\frac{2\pi}{R_e}A_1 + A_2\right), \tag{7.58}$$

$$J'_2(0) = F_\theta^*\left(A_1 + \frac{2\pi}{R_e}A_2 - \frac{2\pi^2}{R_e} + 1\right) \tag{7.59}$$

and

$$J'_3(0) = -F_\rho^*\left(\frac{2\pi}{R_e}A_1 + A_2 + 2\right) + F_z^* A_1 . \tag{7.60}$$

However, the force equation (7.18) may be written in indices notation as

$$F_i^*\left(2\ln\frac{\pi\kappa\lambda_s}{4} - 1\right) + 2F_i^*(1+\ln q)\,\delta_{i2} + 2F_jQ_{ij} = -8\pi\mu U_i\left(1 - \frac{1}{2}\delta_{i2}\right) + J'_i . \tag{7.61}$$

Thus, noting that $Q_{i2} = 0$ [see section 4.4] and $U = (0, 0, U)$, the Cartesian components of the force equation, for the point under consideration ($\theta = 0$), may be expressed as [see (7.6)]

$$F_\rho^*\left(2\ln\frac{\pi\kappa\lambda_s}{4} - 1\right) + 2\left(F_\rho^* Q_{11} + F_z^* Q_{13}\right) = F_\rho^*\left(A_1 - \frac{4\pi}{R_e}A_2 + \frac{4\pi^2}{R_e} - 2\right) - F_z^*\left(\frac{2\pi}{R_e}A_1 + A_2\right) \tag{7.62}$$

$$F_\theta^*\left(2\ln\frac{\pi\kappa\lambda_s}{4} - 1\right) + 2F_\theta^*(1+\ln q) = F_\theta^*\left(A_1 + \frac{2\pi}{R_e}A_2 - \frac{2\pi^2}{R_e} + 1\right) \tag{7.63}$$

and

$$F_z^*\left(2\ln\frac{\pi\kappa\lambda_s}{4} - 1\right) + 2\left(F_\rho^* Q_{13} + F_z^* Q_{33}\right) = -8\pi\mu U - F_\rho^*\left(\frac{2\pi}{R_e}A_1 + A_2\right) + F_z^* A_1 . \tag{3.64}$$

As expected, relationship (7.63) is consistent with F_θ^* being zero. Relationships (7.62,64) may be written as

$$F_z = \frac{A_1 - \frac{4\pi}{R_e}A_2 + \frac{4\pi^2}{R_e} - 1 - 2\ln\frac{\pi\kappa\lambda_s}{4} - 2Q_{11}}{\frac{2\pi}{R_e}A_1 + A_2 + 2Q_{13}} F_\rho^*$$

and

$$F_\rho\left(\frac{2\pi}{R_e}A_1 + A_2 + 2Q_{13}\right) + F_z\left(-A_1 + 2\ln\frac{\pi\kappa\lambda_s}{4} - 1 + 2Q_{33}\right) = -8\pi\mu U.$$

Therefore

$$F_\rho^* = \frac{-8\pi\mu U}{\frac{2\pi}{R_e}A_1 + A_2 + 2Q_{13} + \frac{A_1 - \frac{4\pi}{R_e}A_2 + \frac{4\pi^2}{R_e} - 1 - 2\ln\frac{\pi\kappa\lambda_s}{4} - 2Q_{11}}{\frac{2\pi}{R_e}A_1 + A_2 + 2Q_{13}}\left(-A_1 + 2\ln\frac{\pi\kappa\lambda_s}{4} - 1 + 2Q_{33}\right)}$$

$$= \frac{-8\pi\mu U\left(\frac{2\pi}{R_e}A_1 + A_2 + 2Q_{13}\right)}{\left(\frac{2\pi}{R_e}A_1 + A_2 + 2Q_{13}\right)^2 + \left(A_1 - \frac{4\pi}{R_e}A_2 + \frac{4\pi^2}{R_e} - 1 - 2\ln\frac{\pi\kappa\lambda_s}{4} - 2Q_{11}\right)\left(-A_1 - 1 + 2\ln\frac{\pi\kappa\lambda_s}{4} + 2Q_{33}\right)} \tag{7.65}$$

and

$$F_z^* = \frac{-8\pi\mu U\left(A_1 - \frac{4\pi}{R_e}A_2 + \frac{4\pi^2}{R_e} - 1 - 2\ln\frac{\pi\kappa\lambda_s}{4} - 2Q_{11}\right)}{\left(\frac{2\pi}{R_e}A_1 + A_2 + 2Q_{13}\right)^2 + \left(A_1 - \frac{4\pi}{R_e}A_2 + \frac{4\pi^2}{R_e} - 1 - 2\ln\frac{\pi\kappa\lambda_s}{4} - 2Q_{11}\right)\left(-A_1 - 1 + 2\ln\frac{\pi\kappa\lambda_s}{4} + 2Q_{33}\right)} \tag{7.66}$$

Relationships (7.65,66) are correct up to order κ which is the approximation under which the force integral equation (7.1,2) was derived. Thus, the force per unit length may be written as

$$F_\rho^* = \frac{-8\pi\mu U\left(\frac{2\pi}{R_e}A_1 + A_2 + 2Q_{13}\right)}{\left(\frac{2\pi}{R_e}A_1 + A_2 + 2Q_{13}\right)^2 + \left(A_1 - \frac{4\pi}{R_e}A_2 + \frac{4\pi^2}{R_e} - 1 - 2\ln\frac{\pi\kappa\lambda_s}{4} - 2Q_{11}\right)\left(-A_1 - 1 + 2\ln\frac{\pi\kappa\lambda_s}{4} + 2Q_{33}\right)}$$
$$+ O(\kappa) \tag{7.67}$$

and

$$F_z^* = \frac{-8\pi\mu U\left(A_1 - \frac{4\pi}{R_e}A_2 + \frac{4\pi^2}{R_e} - 1 - 2\ln\frac{\pi\kappa\lambda_s}{4} - 2Q_{11}\right)}{\left(\frac{2\pi}{R_e}A_1 + A_2 + 2Q_{13}\right)^2 + \left(A_1 - \frac{4\pi}{R_e}A_2 + \frac{4\pi^2}{R_e} - 1 - 2\ln\frac{\pi\kappa\lambda_s}{4} - 2Q_{11}\right)\left(-A_1 - 1 + 2\ln\frac{\pi\kappa\lambda_s}{4} + 2Q_{33}\right)}$$
$$+ O(\kappa), \tag{7.68}$$

where A_1 and A_2 are function of R_e defined by (7.50-51) as

$$A_1 = \int_0^\pi \left(\frac{e^{-\frac{1}{2\pi}R_e \sin\frac{\hat{\theta}}{2}}}{\sin\frac{\hat{\theta}}{2}} - \frac{1}{\sin\frac{\hat{\theta}}{2}} \right) d\hat{\theta}$$

and

$$A_2 = \int_0^\pi e^{-\frac{1}{2\pi}R_e \sin\frac{\hat{\theta}}{2}} d\hat{\theta}.$$

For a torus with circular cross-section the components of the characteristic tensor of the cross-sectional shape, Q_{ij}, in (7.67,68) vanish. For an elliptical cross-section with semi-diameters a and b (where $a > b$) with the direction of the larger principal axis ($2a$) given by unit vector β, (see figure 7.5), Q_{ij} is determined by [see (4.4.6)]

$$Q_{11} = \ln\frac{a+b}{2\lambda_s} + \frac{a-b}{2(a+b)}\cos 2\lambda, \tag{7.69}$$

$$Q_{13} = -\frac{a-b}{2(a+b)}\sin 2\lambda \tag{7.70}$$

and

$$Q_{33} = \ln\frac{a+b}{2\lambda_s} - \frac{a-b}{2(a+b)}\cos 2\lambda\,, \tag{7.71}$$

where, here, λ is the angle between the unit vector β and the unit base vector i_3, the unit base vector in the direction of uniform velocity U (i.e. $\cos\lambda = \beta.i_3$), as shown in figure 7.5, and where λ_s is the radius of its equivalent circle the value of which is given by (4.4.7,8).

However as $R_e \longrightarrow 0$ the value of A_1 and A_2 may be respectively determined by

$$\begin{aligned} A_1 &= \int_0^\pi \left(\frac{e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}}}{\sin\frac{\hat\theta}{2}} - \frac{1}{\sin\frac{\hat\theta}{2}} \right) d\hat\theta = \int_0^\pi \left[\frac{1 - \frac{R_e}{2\pi}\sin\frac{\hat\theta}{2} + \left(\frac{R_e}{2\pi}\right)^2 \frac{\sin^2\frac{\hat\theta}{2}}{2!} + \dots}{\sin\frac{\hat\theta}{2}} - \frac{1}{\sin\frac{\hat\theta}{2}} \right] \\ &= \int_0^\pi \left[-\frac{R_e}{2\pi} + \frac{1}{2}\left(\frac{R_e}{2\pi}\right)^2 \sin\frac{\hat\theta}{2} + \dots \right] d\hat\theta \\ &= -\frac{R_e}{2\pi}\int_0^\pi d\hat\theta + O\left(R_e^2\right) = -\frac{R_e}{2} + O\left(R_e^2\right); \end{aligned}$$

and

$$\begin{aligned} A_2 &= \int_0^\pi e^{-\frac{1}{2\pi}R_e\sin\frac{\hat\theta}{2}} d\hat\theta = \int_0^\pi \left[1 - \frac{R_e}{2\pi}\sin\frac{\hat\theta}{2} + \left(\frac{R_e}{2\pi}\right)^2 \frac{\sin^2\frac{\hat\theta}{2}}{2!} + \dots \right] d\hat\theta \\ &= \pi + O(R_e)\,. \end{aligned}$$

Thus the radial and axial components of the force in the limit as $R_e \longrightarrow 0$ may be obtained by

$$F_\rho^* = \frac{-8\pi\mu U R_e (2Q_{13})}{(2Q_{13})^2 - \left(1 + 2\ln\frac{\pi\kappa\lambda_s}{4} + 2Q_{11}\right)\left(-1 + 2\ln\frac{\pi\kappa\lambda_s}{4} + 2Q_{33}\right)} + O(\kappa)$$

and

$$F_z^* = \frac{+8\pi\mu U\left(1 + 2\ln\frac{\pi\kappa\lambda_s}{4} + 2Q_{11}\right)}{(2Q_{13})^2 - \left(1 + 2\ln\frac{\pi\kappa\lambda_s}{4} + 2Q_{11}\right)\left(-1 + 2\ln\frac{\pi\kappa\lambda_s}{4} + 2Q_{33}\right)} + O(\kappa)\,,$$

where for a torus of circular cross-section (i.e. $Q_{ij} = 0$; $\lambda_s = 1$) the only non-zero component (F_z) reduces to those obtained by Johnson & Wu (1979) and Johnson (1980) given by (1.2.19).

However, the error terms in the present results is of order κ (the slenderness parameter), whereas for those obtained by Johnson & Wu (1979) and Johnson (1980) the error was of order ε^2 (where $\varepsilon = 2\pi\kappa$ is the semi-slenderness parameter), which opens a new question " Why are exactly the same values obtained with a different level of accuracy ? "

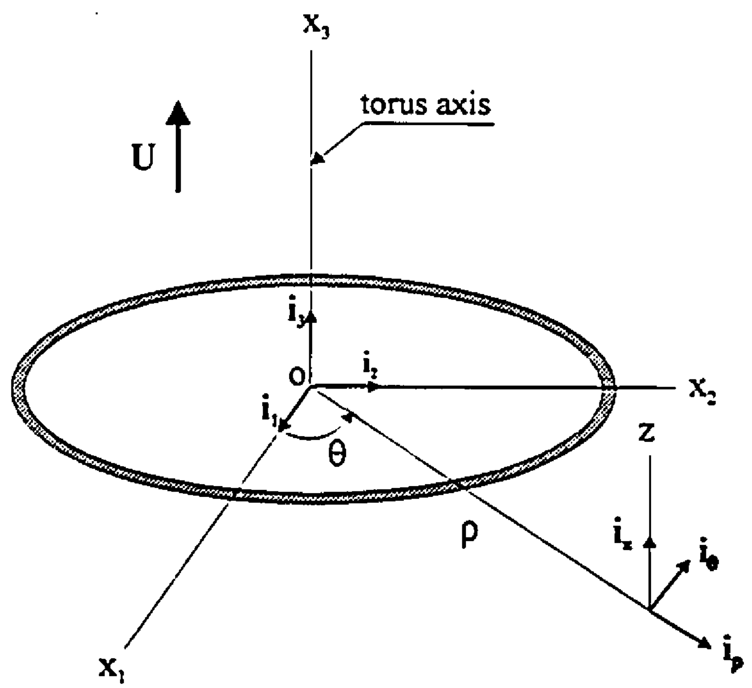

Figure 7.1 : Coordinate systems showing the torus settling along its axis with uniform velocity - **U.**

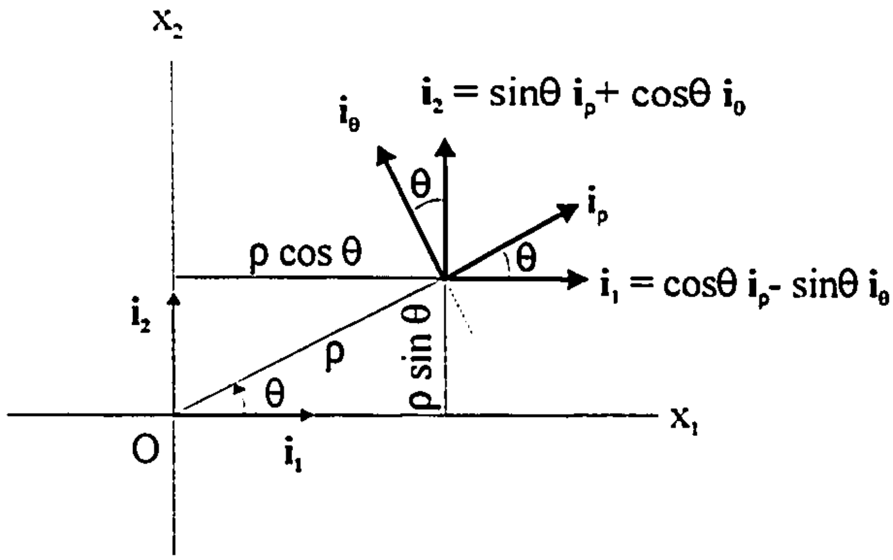

Figure 7.2 : Plane (x_1, x_2) showing the relationship between the rectangular Cartesian coordinate system and the cylidrical polar one.

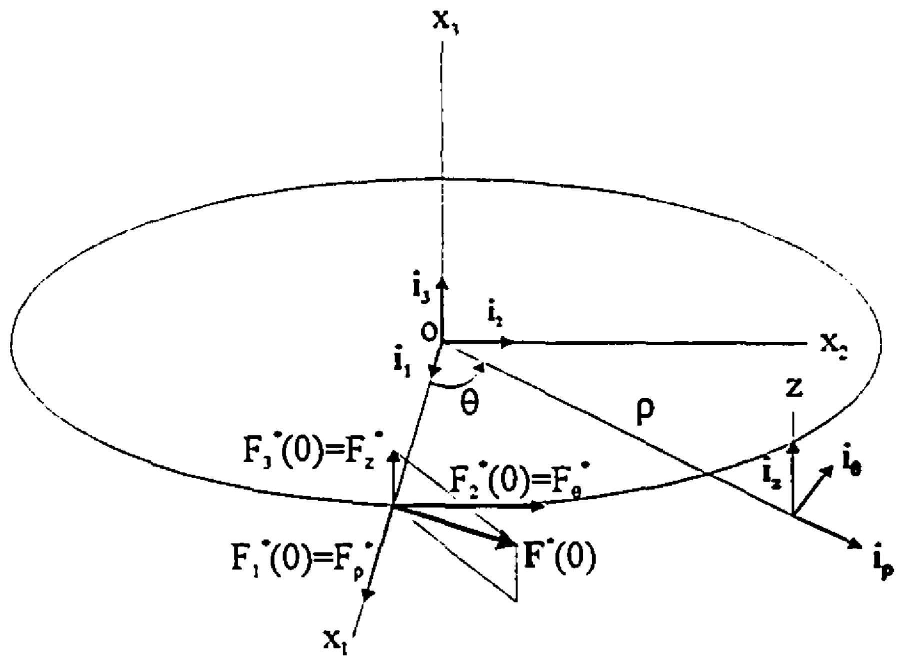

Figure 7.3 : Cartesian and cylindrical components of the force at the point under consideration ($\theta = 0$).

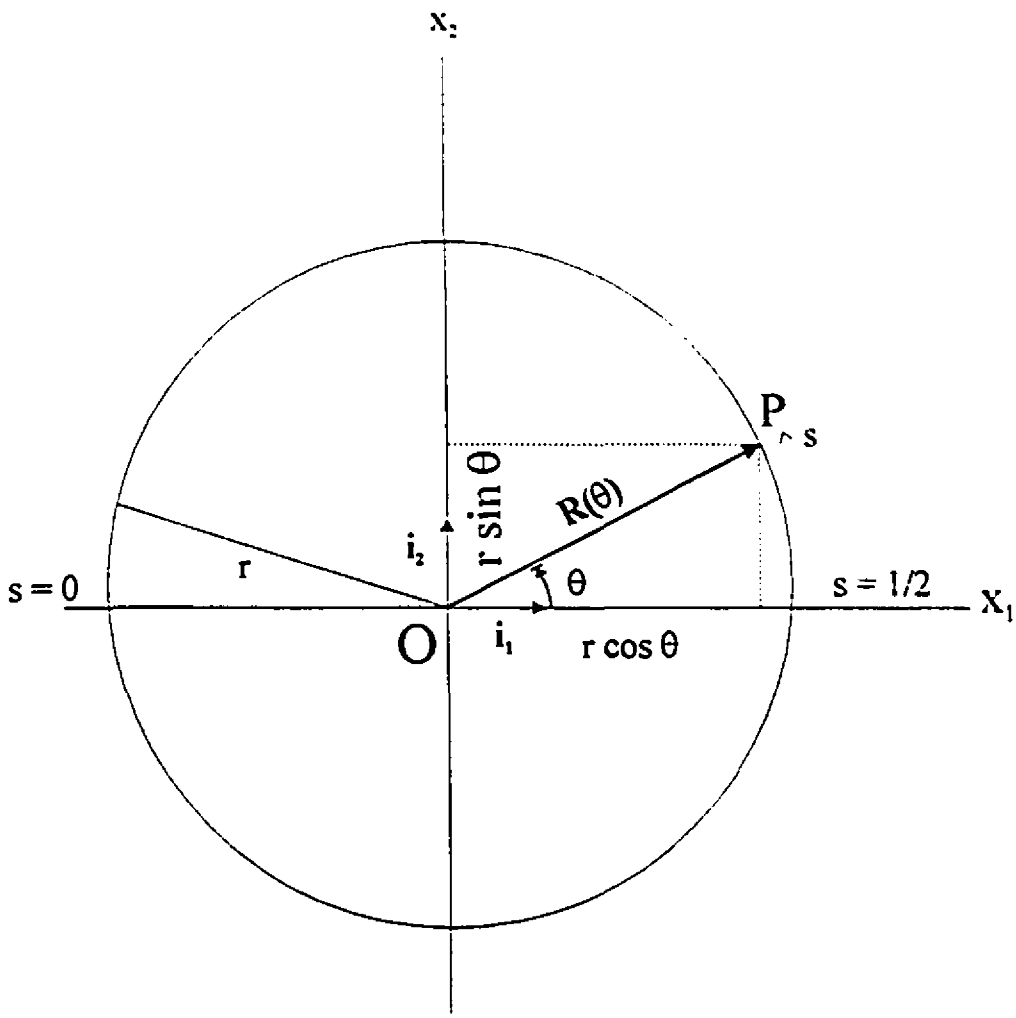

Figure 7.4 : Position of a general point P on the body centreline.

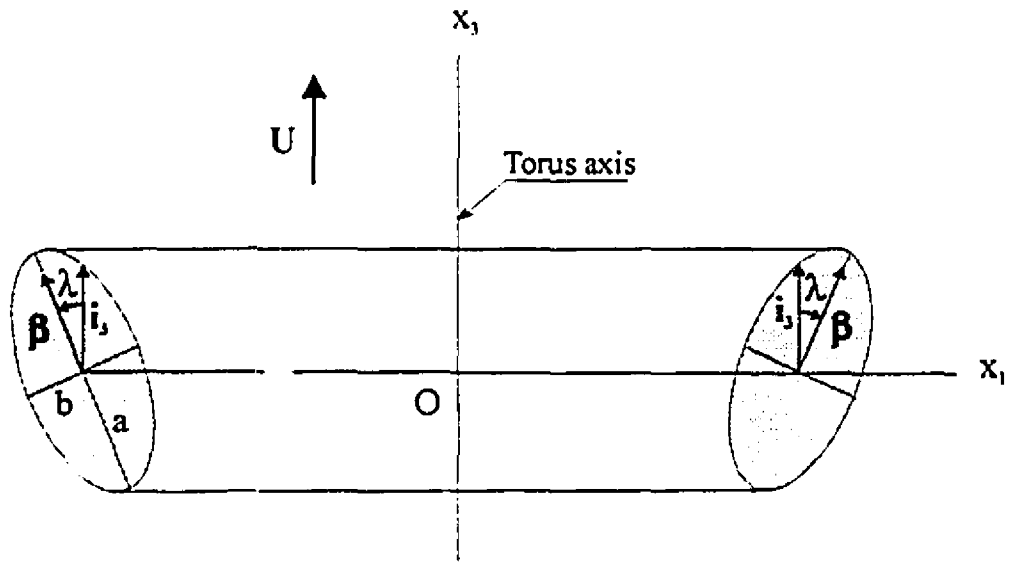

Figure 7.5 : Torus with elliptical Cross-section showing the position of the ellipse with respect to $\mathbf{i}_3$, the unit base vector in the direction of velocity **U**.

Appendix A

A - Solution of biharmonic equation

In this Appendix, we consider a planar (two dimensional) flow field in a dimensionless[1] plane (x, y), which is called the Z-plane (see figure A.1), applying the complex variable method to derive the equations used in Chapter 4.

In the absence of the body forces, and for zero Reynolds number the governing equations for an incompressible fluid may be written as

$$\nabla^2 \boldsymbol{u} - \nabla p = 0 \tag{A.1}$$

and

$$\nabla \cdot \boldsymbol{u} = 0, \tag{A.2}$$

where for two dimensional flow field ∇ and ∇^2 are respectively defined by

$$\nabla = i_x \frac{\partial}{\partial x} + i_y \frac{\partial}{\partial y} \tag{A.3}$$

and

[1]Throughout this appendix we use dimensionless quantities.

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \qquad \text{(A.4)}$$

Equations (A.1,2) are linear for u and p and are well-known as the creeping flow or Stokes equations. Taking the divergence of equation (A.1) results in

$$\nabla^2(\nabla \cdot \boldsymbol{u}) - \nabla^2 p = 0.$$

But $\nabla \cdot \boldsymbol{u} = 0$, hence

$$\nabla^2 p = 0, \qquad \text{(A.5)}$$

which is Laplace's equation. Thus, for creeping flow, the pressure is harmonic.

Taking the curl of equation (A.1) results in

$$\nabla^2 \boldsymbol{\omega} - \nabla(\nabla \times p) = \mathbf{0}. \qquad \text{(A.6)}$$

where ω is the fluid vorticity defined by

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u}.$$

But pressure p is a scalar, hence $\nabla \times \mathrm{p} = 0$. Thus, (A.6) may be written as

$$\nabla^2 \boldsymbol{\omega} = \mathbf{0}, \qquad \text{(A.7)}$$

which is the vorticity equation for the creeping flow (zero Reynolds number flow). The fluid vorticity for two dimensional flow fields $\boldsymbol{u}(x, y) = (u_z, 0, 0)$ and $\boldsymbol{u}(x, y) = (0, u_x, u_y)$ may respectively be determined by

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = \begin{vmatrix} \boldsymbol{i}_z & \boldsymbol{i}_x & \boldsymbol{i}_y \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ u_z & 0 & 0 \end{vmatrix}$$

$$= \boldsymbol{i}_x \frac{\partial u_z}{\partial y} - \boldsymbol{i}_y \frac{\partial u_z}{\partial x} \qquad \text{(A.8)}$$

and

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = \begin{vmatrix} \boldsymbol{i}_z & \boldsymbol{i}_x & \boldsymbol{i}_y \\ \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ 0 & u_x & u_y \end{vmatrix}$$

$$= \boldsymbol{i}_z \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right), \qquad \text{(A.9)}$$

where u_x and u_y are the components of velocity $\boldsymbol{u}$ in the x and y direction, respectively, as shown in figure A.1, and where u_z is the component of the velocity in the direction normal to the (x, y)-plane. But, for the flow field $\boldsymbol{u} = (0, u_x, u_y)$, the stream function, Ψ, is defined by

$$u_x = \frac{\partial \Psi}{\partial y} \quad \textit{and} \quad u_y = -\frac{\partial \Psi}{\partial x}, \qquad \text{(A.10)}$$

so that, by the aid of relationships (A.9,10) equation (A.7) may be expressed in terms of the stream function as

$$\begin{aligned} \nabla^2 \boldsymbol{\omega} &= \boldsymbol{i}_z \nabla^2 \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \\ &= \boldsymbol{i}_z \nabla^2 \left(-\frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi}{\partial y^2} \right) \\ &= \boldsymbol{i}_z \nabla^2 (-\nabla^2 \Psi) = \boldsymbol{0}. \end{aligned} \qquad \text{(A.11)}$$

Thus, the biharmonic equation

$$\nabla^4 \Psi = 0, \qquad \text{(A.12)}$$

where ∇^4 is defined by [see (A.4)]

$$\begin{aligned} \nabla^4 &= \nabla^2 \nabla^2 \\ &= \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}, \end{aligned}$$

satisfies the creeping flow equation. An immediate consequence of (A.11) is that $-\nabla^2\Psi$ is the component and the only component of the vorticity in the direction normal to the plane (x, y). Thus the vorticity equation may be written as

$$\nabla^2\omega = 0, \tag{A.13}$$

where

$$\omega = -\nabla^2\Psi. \tag{A.14}$$

Therefore, vorticity ω and pressure p in (A.5) are conjugate harmonic functions, and hence, they represent the real and imaginary parts of an analytic function of $Z = x + iy$.

Let us define

$$\Phi(Z) = f(x,y) + ig(x,y) = -\frac{1}{4}\int q(Z)dZ, \tag{A.15}$$

where $q(Z)$ is an analytic function of Z the real and imaginary parts of which are the fluid vorticity and pressure, respectively, i.e,

$$q(Z) = \omega + ip. \tag{A.16}$$

Since $q(Z)$ is analytic $\Phi(Z)$ is analytic too. Hence, f and g satisfy the Cauchy-Reiman equations, that is,

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y} \quad and \quad \frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x} \tag{A.17}$$

and, as a consequence of (A.17),

$$\nabla^2 f = \nabla^2 g = 0. \tag{A.18}$$

Further let

$$F(x,y) = \Psi(x,y) - xf(x,y) - yg(x,y). \tag{A.19}$$

Then, $\nabla^2 F$ may be determined by

$$\nabla^2 F = \nabla^2 \Psi - \nabla^2(xf) - \nabla^2(yg). \tag{A.20}$$

But [see (A.3,4)]

$$\begin{aligned}\nabla^2(UV) &= \nabla\cdot[\nabla(UV)] \\ &= \nabla\cdot(U\nabla V + V\nabla U) \\ &= U\nabla^2 V + 2\nabla U\cdot\nabla V + V\nabla^2 U.\end{aligned} \tag{A.21}$$

Hence [see (A.17-21)],

$$\begin{aligned}\nabla^2 F &= \nabla^2\Psi - x\nabla^2 f - 2\nabla x\cdot\nabla f + f\nabla^2 x - y\nabla^2 g - 2\nabla y\cdot\nabla g - g\nabla^2 y \\ &= \nabla^2\Psi - 2i_x\frac{\partial x}{\partial x}\cdot\left(i_x\frac{\partial f}{\partial x} + i_y\frac{\partial f}{\partial y}\right) - 2i_y\frac{\partial y}{\partial y}\cdot\left(i_x\frac{\partial g}{\partial y} + i_y\frac{\partial g}{\partial y}\right) \\ &= \nabla^2\Psi - 2\frac{\partial f}{\partial x} - 2\frac{\partial g}{\partial y} = \nabla^2\Psi - 4\frac{\partial f}{\partial x}.\end{aligned} \tag{A.22}$$

However, [see (A.14-16)]

$$\begin{aligned}\Phi'(Z) &= \frac{d\Phi(Z)}{dZ} = \frac{\partial f}{\partial x} + i\frac{\partial g}{\partial x} = -\frac{1}{4}q \\ &= -\frac{1}{4}\omega - \frac{1}{4}ip = \frac{1}{4}\nabla^2\Psi - \frac{1}{4}ip\end{aligned} \tag{A.23}$$

or

$$\frac{\partial f}{\partial x} = \frac{1}{4}\nabla^2\Psi,$$

so that [see (A.22)] $\nabla^2 F = 0$. Let $\chi(Z)$ be an analytic function of Z the real and the imaginary parts of which being $F(x, y)$ and $G(x, y)$, respectively, where G is the conjugate function of F, so that

$$\chi(Z) = F(x,y) + iG(x,y) \tag{A.24}$$

and

$$\nabla^2 F = \nabla^2 G = 0, \quad \frac{\partial F}{\partial x} = \frac{\partial G}{\partial y} \quad \textit{and} \quad \frac{\partial F}{\partial y} = -\frac{\partial G}{\partial x}. \tag{A.25}$$

Then, relationship (A.19) may be expressed as

$$\Psi(x,y) = xf(x,y) + yg(x,y) + \Re[\chi(Z)], \tag{A.26}$$

where $\Re$ denotes the real part of the complex variable.But[2]

$$\begin{aligned}\Re[Z^* \Phi(Z)] &= \Re[\,(x-iy)(f+ig)\,] \\ &= xf + yg,\end{aligned}$$

so that, the relationship (A.26) may be written as

$$\Psi(x,y) = \Re[Z^* \Phi(Z) + \chi(Z)]. \tag{A.27}$$

Therefore, we have $\Psi(x,y)$ expressed in terms of a pair of *complex potentials* $\Phi(Z)$ and $\chi(Z)$.

The components of the velocity $\boldsymbol{u}\,(x,\,y) = (0,\, u_x,\, u_y)$ may be expressed directly in terms of the stream function as [see (A.10)]

$$i(u_x + iu_y) = -u_y + iu_x = \frac{\partial \Psi}{\partial x} + i\frac{\partial \Psi}{\partial y}.$$

Thus by the aid of (A.26,24,17,25), the velocity may be expressed as

$$\begin{aligned} i(u_x + iu_y) &= f + x\frac{\partial f}{\partial x} + y\frac{\partial g}{\partial x} + \frac{\partial F}{\partial x} + i\left(g + x\frac{\partial f}{\partial y} + y\frac{\partial g}{\partial y} + \frac{\partial F}{\partial y}\right) \\ &= (f+ig) + x\left(\frac{\partial f}{\partial x} - i\frac{\partial g}{\partial x}\right) + iy\left(\frac{\partial f}{\partial x} - i\frac{\partial g}{\partial x}\right) + \frac{\partial F}{\partial x} - i\frac{\partial G}{\partial x} \\ &= (f+ig) + (x+iy)\left(\frac{\partial f}{\partial x} - i\frac{\partial g}{\partial x}\right) + \left(\frac{\partial G}{\partial x} - i\frac{\partial G}{\partial x}\right) \end{aligned} \tag{A.28}$$

or [see (A.15,23,24)]

[2]Here, the complex variable labelled by an asterisk denotes the conjugate of that variable.

$$i(u_x + iu_y) = \Phi(Z) + Z\left[\frac{d\Phi}{dZ}\right]^* + \left[\frac{d\chi}{dZ}\right]^*. \tag{A.29}$$

Pressure p may be obtained by relationships (A.23) as

$$p = -4\, Im\left[\frac{d\Phi}{dZ}\right],$$

where Im denotes the imaginary part of the complex variable. But, as usual, an additive constant in pressure is irrelevant, so that the pressure p may be expressed as

$$p = -4\, Im\left[\frac{\partial\Phi}{\partial Z}\right] + p_0, \tag{A.30}$$

where p_0 is a real constant.

Let us examine the pressure p and the velocity u obtained by (A.29,30), respectively. By the aid of the relationship (A.28), the components of the velocity may be written as

$$u_x = g - x\frac{\partial g}{\partial x} + y\frac{\partial f}{\partial x} - \frac{\partial G}{\partial x} \quad and \quad u_y = -f - x\frac{\partial f}{\partial x} - y\frac{\partial g}{\partial x} - \frac{\partial F}{\partial x}. \tag{A.31}$$

The substitution of u_x in the x- component of the creeping flow equation given by (A.1) results in [see (A.18,25,21)]

$$\begin{aligned}
\frac{\partial p}{\partial x} &= \nabla^2\left(g - x\frac{\partial g}{\partial x} + y\frac{\partial f}{\partial x} - \frac{\partial G}{\partial x}\right) \\
&= -\nabla^2\left(x\frac{\partial g}{\partial x}\right) + \nabla^2\left(y\frac{\partial f}{\partial x}\right) \\
&= -2\nabla x \cdot \nabla\frac{\partial g}{\partial x} + 2\nabla y \cdot \nabla\frac{\partial f}{\partial x} \\
&= -2\frac{\partial^2 g}{\partial x^2} + 2\frac{\partial^2 f}{\partial x \partial y} \\
&= -2\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)
\end{aligned}$$

or [see (A.17)]

$$p = -2\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) + k(y)$$
$$= -4\frac{\partial g}{\partial x} + k(y)\,. \qquad (A.32)$$

Thus,

$$\frac{\partial p}{\partial y} = -4\frac{\partial^2 g}{\partial x \partial y} + \frac{\partial k}{\partial y}. \qquad (A.33)$$

But from the y-component of the creeping flow $\partial p/\partial y$ may be determined by [see (A.31)]

$$\frac{\partial p}{\partial y} = -\nabla^2\left(f + x\frac{\partial f}{\partial x} + y\frac{\partial g}{\partial x} + \frac{\partial F}{\partial x}\right)$$
$$= -\nabla^2\left(x\frac{\partial f}{\partial x}\right) - \nabla^2\left(y\frac{\partial g}{\partial x}\right)$$
$$= -2\nabla x \cdot \nabla\frac{\partial f}{\partial x} - 2\nabla y \cdot \nabla\frac{\partial g}{\partial x}$$

or [see (A.17)]

$$\frac{\partial p}{\partial y} = -2\frac{\partial^2 f}{\partial x^2} - 2\frac{\partial^2 g}{\partial x \partial y}$$
$$= -2\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) \qquad (A.34)$$
$$= -4\frac{\partial^2 g}{\partial x \partial y}.$$

Hence, from (A.33,34), it follows that $k(y)$ is a constant, say p_0, so that, [see (A.32,23)]

$$p = -4\, Im\left[\frac{d\Phi}{dZ}\right] + p_0,$$

which is the same as the equation (A.30) obtained directly by the relationships (A.23).

Next we consider the force fluid exerts on a line element of length ds in the Z-plane, as shown in figure A.2. The components of the force per unit length, F_i, may be determined by

$$F_i = \sigma_{ij} n_j, \tag{A.35}$$

where n_j is the component of the unit vector normal to the element ds and where σ_{ij} is the stress tensor defined by

$$\sigma_{ij} = -p\delta_{ij} + \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}. \tag{A.36}$$

Thus the x-component of the force per unit length may be written as

$$\begin{aligned} F_x &= \sigma_{xx} n_x + \sigma_{xy} n_y \\ &= \left(-p + 2\frac{\partial u_x}{\partial x}\right) n_x + \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) n_y . \end{aligned} \tag{A.37}$$

But (see figure A.2)

$$n_x = -\sin\alpha = -\frac{dy}{ds} \quad and \quad n_y = \cos\alpha = \frac{dx}{ds}, \tag{A.38}$$

so that the x-component of the force on element ds may be determined by

$$\begin{aligned} F_x ds &= \left(\sigma_{xx} n_x + \sigma_{xy} n_y\right) ds \\ &= -\left(-p + 2\frac{\partial u_x}{\partial x}\right) dy + \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) dx . \end{aligned} \tag{A.39}$$

However, [see (A.30,10,23,31,17)]

$$\begin{aligned} -p + 2\frac{\partial u_x}{\partial x} &= 4\,Im\left(\frac{d\Phi}{dz}\right) - 2\frac{\partial u_y}{\partial y} \\ &= 4\frac{\partial g}{\partial x} + 2\frac{\partial}{\partial y}\left(f + x\frac{\partial f}{\partial x} + y\frac{\partial g}{\partial x} + \frac{\partial F}{\partial x}\right) \\ &= -4\frac{\partial f}{\partial y} + 2\frac{\partial}{\partial y}\left(f + x\frac{\partial f}{\partial x} + y\frac{\partial g}{\partial x} + \frac{\partial F}{\partial x}\right) \\ &= 2\frac{\partial}{\partial y}\left(-f + x\frac{\partial f}{\partial x} + y\frac{\partial g}{\partial x} + \frac{\partial F}{\partial x}\right), \end{aligned} \tag{A.40}$$

and [see (A.31,17,25)]

$$\begin{aligned}\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x} &= \frac{\partial}{\partial y}\left(g-x\frac{\partial g}{\partial x}+y\frac{\partial f}{\partial x}-\frac{\partial G}{\partial x}\right)+\frac{\partial}{\partial x}\left(-f-x\frac{\partial f}{\partial x}-y\frac{\partial g}{\partial x}-\frac{\partial F}{\partial x}\right)\\ &= \frac{\partial g}{\partial y}-x\frac{\partial^2 g}{\partial x\partial y}+\frac{\partial f}{\partial x}+y\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial^2 G}{\partial x\partial y}-\frac{\partial}{\partial x}\left(f+x\frac{\partial f}{\partial x}+y\frac{\partial g}{\partial x}+\frac{\partial F}{\partial x}\right)\\ &= \frac{\partial f}{\partial x}-x\frac{\partial^2 f}{\partial x^2}+\frac{\partial f}{\partial x}-y\frac{\partial^2 g}{\partial x^2}-\frac{\partial^2 F}{\partial x^2}-\frac{\partial}{\partial x}\left(f+x\frac{\partial f}{\partial x}+y\frac{\partial g}{\partial x}+\frac{\partial F}{\partial x}\right)\\ &= -2\frac{\partial}{\partial x}\left(-f+x\frac{\partial f}{\partial x}+y\frac{\partial g}{\partial x}+\frac{\partial F}{\partial x}\right).\end{aligned} \tag{A.41}$$

In relationship (A.40) the constant pressure p_0 does not appear since it acts equally in all orientations of the element and hence it doesn't contribute to the force on the immersed body in the fluid. Thus by the aid of (A.40,41), the x-component of the force may be expressed as

$$\begin{aligned}F_x ds &= -2\frac{\partial}{\partial y}\left(-f+x\frac{\partial f}{\partial x}+y\frac{\partial g}{\partial x}+\frac{\partial F}{\partial x}\right)dy-2\frac{\partial}{\partial x}\left(-f+x\frac{\partial f}{\partial x}+y\frac{\partial g}{\partial x}+\frac{\partial F}{\partial x}\right)dx\\ &= -2\,d\left(-f+x\frac{\partial f}{\partial x}+y\frac{\partial g}{\partial x}+\frac{\partial F}{\partial x}\right).\end{aligned} \tag{A.42}$$

Similarly, the y-component of the force may be obtained by

$$F_y ds = \left(-p+2\frac{\partial u_y}{\partial y}\right)dx-\left(\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}\right)dy. \tag{A.43}$$

But [see (A.30,10,23,31)]

$$\begin{aligned}-p+2\frac{\partial u_y}{\partial y} &= 4\,Im\left(\frac{d\Phi}{dz}\right)-2\frac{\partial u_x}{\partial x}\\ &= 4\frac{\partial g}{\partial x}-2\frac{\partial}{\partial x}\left(g-x\frac{\partial g}{\partial x}+y\frac{\partial f}{\partial x}-\frac{\partial G}{\partial x}\right)\\ &= 2\frac{\partial}{\partial x}\left(g+x\frac{\partial g}{\partial x}-y\frac{\partial f}{\partial x}+\frac{\partial G}{\partial x}\right)\end{aligned} \tag{A.44}$$

and [see (A.31,17,25)]

$$\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} = -\frac{\partial}{\partial x}\left(f + x\frac{\partial f}{\partial x} + y\frac{\partial g}{\partial x} + \frac{\partial F}{\partial x}\right) + \frac{\partial}{\partial y}\left(g - x\frac{\partial g}{\partial x} + y\frac{\partial f}{\partial x} - \frac{\partial G}{\partial x}\right)$$

$$= -\frac{\partial f}{\partial x} - x\frac{\partial^2 g}{\partial y \partial x} - \frac{\partial g}{\partial y} + y\frac{\partial^2 f}{\partial y \partial x} - \frac{\partial^2 G}{\partial y \partial x} + \frac{\partial}{\partial y}\left(g - x\frac{\partial g}{\partial x} + y\frac{\partial f}{\partial x} - \frac{\partial G}{\partial x}\right)$$

$$= -2\frac{\partial g}{\partial y} - x\frac{\partial^2 g}{\partial y \partial x} + \left(\frac{\partial f}{\partial x} + y\frac{\partial^2 f}{\partial y \partial x}\right) - \frac{\partial^2 G}{\partial y \partial x} + \frac{\partial}{\partial y}\left(-x\frac{\partial g}{\partial x} + y\frac{\partial f}{\partial x} - \frac{\partial G}{\partial x}\right)$$

$$= -2\frac{\partial}{\partial y}\left(g + x\frac{\partial g}{\partial x} - y\frac{\partial f}{\partial x} + \frac{\partial G}{\partial x}\right),$$

so that, the y-component of the force may be written as

$$F_y ds = 2\frac{\partial}{\partial x}\left(g + x\frac{\partial g}{\partial x} - y\frac{\partial f}{\partial x} + \frac{\partial G}{\partial x}\right)dx + 2\frac{\partial}{\partial y}\left(g + x\frac{\partial g}{\partial x} - y\frac{\partial f}{\partial x} + \frac{\partial G}{\partial x}\right)dy$$

$$= 2d\left(g + x\frac{\partial g}{\partial x} - y\frac{\partial f}{\partial x} + \frac{\partial G}{\partial x}\right). \qquad \text{(A.45)}$$

Relationships (A.42,45) may be expressed in complex variable form as

$$(F_x + iF_y)ds = -2d\left(-f + x\frac{\partial f}{\partial x} + y\frac{\partial g}{\partial x} + \frac{\partial F}{\partial x}\right) + 2id\left(g + x\frac{\partial g}{\partial x} - y\frac{\partial f}{\partial x} + \frac{\partial G}{\partial x}\right)$$

$$= 2d\left[(f + ig) - (x + iy)\left(\frac{\partial f}{\partial x} - i\frac{\partial g}{\partial x}\right) - \left(\frac{\partial F}{\partial x} - i\frac{\partial G}{\partial x}\right)\right]$$

or [see (A.15,23,24)]

$$(F_x + iF_y)ds = 2d\left\{\Phi(Z) - Z\left[\frac{d\Phi}{dZ}\right]^* - \left[\frac{d\chi}{dZ}\right]^*\right\}. \qquad \text{(A.46)}$$

Thus, the force on curve *AB*, as shown in figure A.3, may be determined by

$$\mathcal{F}x + i\,\mathcal{F}y = \int_B^A (F_x + iF_y)ds$$

$$= 2\int_B^A d\left\{\Phi(Z) - \left[Z\frac{d\Phi}{dZ}\right]^* - \left[\frac{d\chi}{dZ}\right]^*\right\}$$

$$= 2\left|\,\Phi(Z) - Z\left[\frac{d\Phi}{dZ}\right]^* - \left[\frac{d\chi}{dZ}\right]^*\right|_B^A, \qquad \text{(A.47)}$$

where $\mathcal{F}x$ and $\mathcal{F}y$ are the components of the force the fluid exerts on the curve *AB*, coinciding with the x and y axis, respectively.

Next consider the transformation function

$$Z = w(\xi) \tag{A.48}$$

mapping the Z-plane onto the ξ-plane (see, figure A.3). Thus the components of the velocity in (A.29) and the force on curve AB in (A.47), in the ξ-plane, may respectively be expressed as

$$\begin{aligned} i\left(u_x + iu_y\right) &= \Phi(Z) + Z\left[\frac{d\Phi}{dZ}\right]^* + \left[\frac{d\chi}{dZ}\right]^* \\ &= \Phi(w) + w\left[\frac{d\Phi}{d\xi}\right]^*\left[\frac{d\xi}{dw}\right]^* + \left[\frac{d\chi}{d\xi}\right]^*\left[\frac{d\xi}{dw}\right]^* \\ &= \left[\frac{dw}{d\xi}\right]^{*-1}\left\{\left[\frac{dw}{d\xi}\right]^*\Phi + w\left[\frac{d\Phi}{d\xi}\right]^* + \left[\frac{d\chi}{d\xi}\right]^*\right\} \end{aligned} \tag{A.49}$$

and

$$\begin{aligned} \mathcal{F}x + i\mathcal{F}y &= 2\left|\ \Phi(Z) - Z\left[\frac{d\Phi}{dZ}\right]^* - \left[\frac{d\chi}{dZ}\right]^*\ \right|_B^A \\ &= 2\left|\ \Phi(w) - w\left[\frac{d\Phi}{d\xi}\right]^*\left[\frac{d\xi}{dw}\right]^* - \left[\frac{d\chi}{d\xi}\right]^*\left[\frac{d\xi}{dw}\right]^*\ \right|_{B'}^{A'} \\ &= 2\left|\left[\frac{dw}{d\xi}\right]^{*-1}\left\{\left[\frac{dw}{d\xi}\right]^*\Phi - w\left[\frac{d\Phi}{d\xi}\right]^* - \left[\frac{d\chi}{d\xi}\right]^*\right\}\right|_{B'}^{A'}, \end{aligned} \tag{A.50}$$

where A' and B' are the two ends of the curve $A'B'$ which is obtained by the mapping of curve AB onto the ξ-plane, corresponding to the points A and B, respectively, as shown in figure A.3.

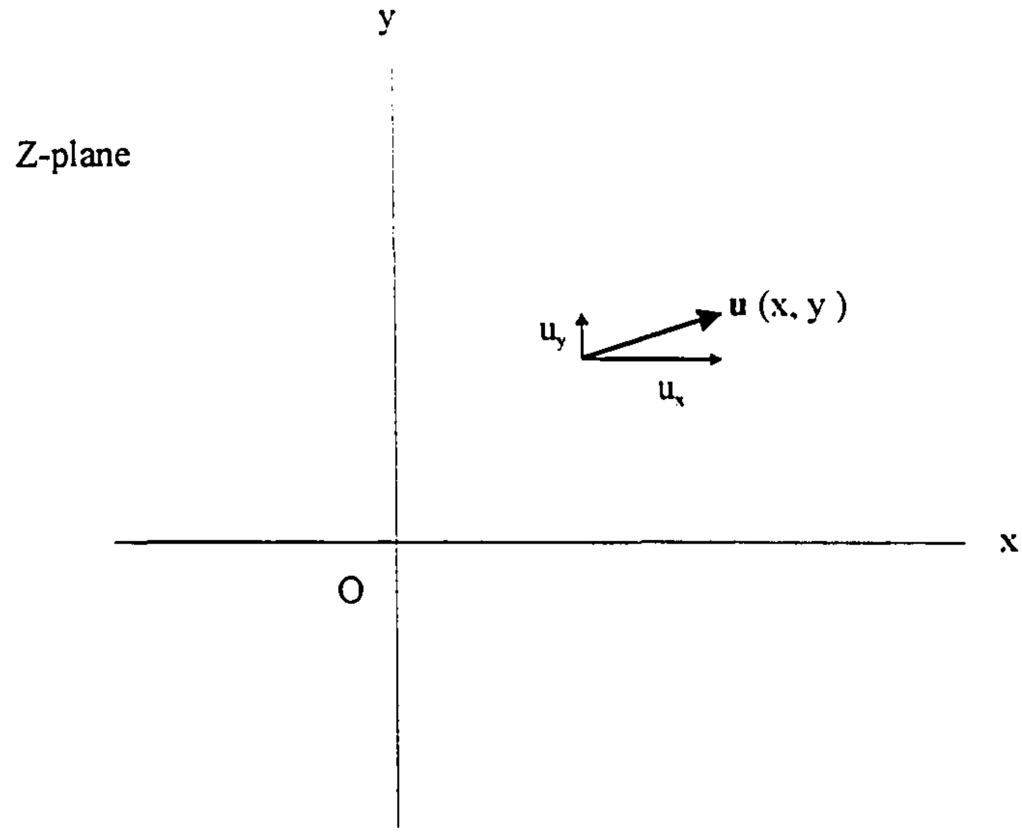

Figure A.1: (x, y)-plane showing the components of velocity $\boldsymbol{u}$.

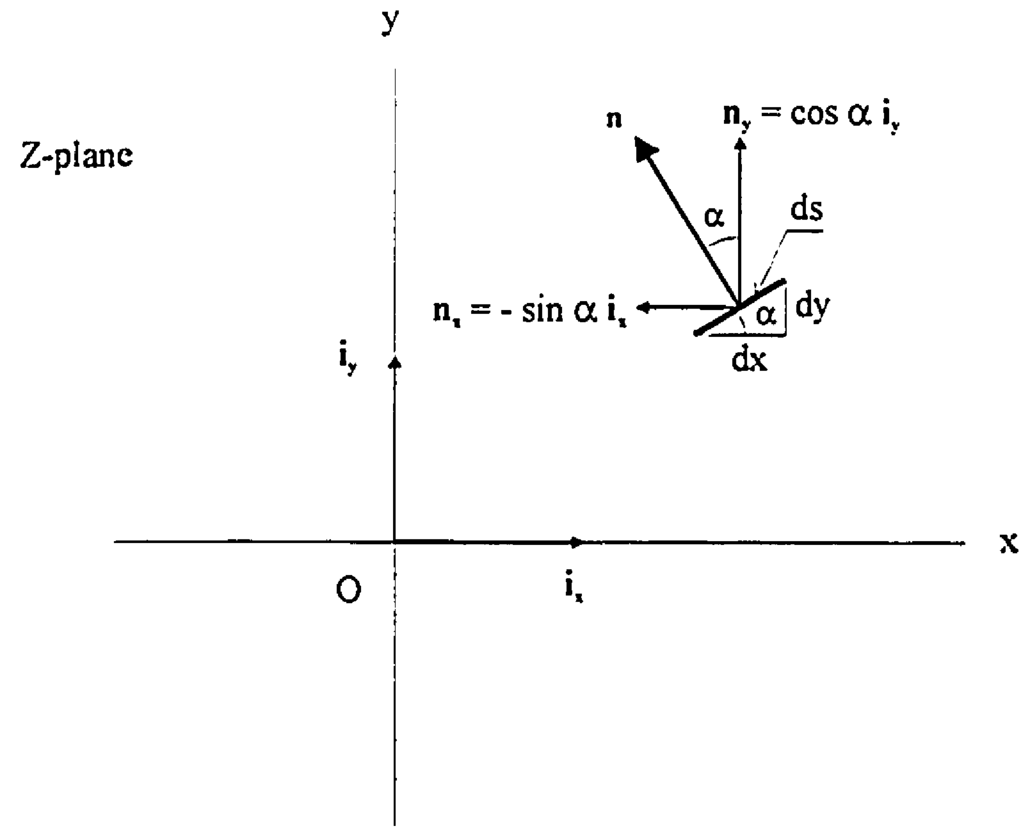

Figure A.2 : Z-plane showing the components of the unit vector normal to element ds.

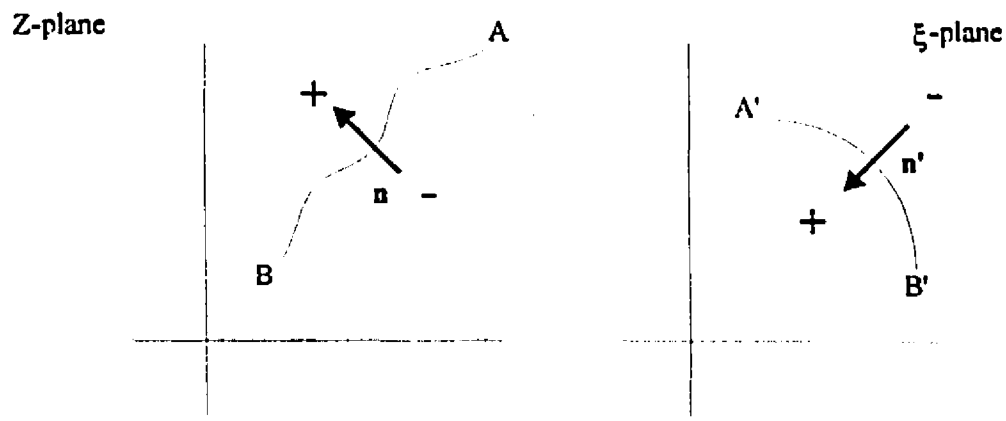

Figure A.3 : Z and ξ planes schematically showing the transformation of curve AB in Z–plane onto curve A'B' in the ξ–plane.

References

Batchelor, G. K. 1970 Slender-body theory for particles of arbitrary cross-section in Stokes flow. *J. Fluid Mech.* **44**, 419-440.

Bender, C. M. & Orszag, S. A. 1978 Advanced Mathematical Methods For Scientists and Engineers. McGRAW-HILL, New York.

Berry, A. & Swain, L. M. 1923 On the steady motion of a cylinder through infinite viscous fluid. *Proc. Roy. Soc. A* **102**, 766-778.

Chwang, A. T. & Wu, T. Y. 1974 Hydromechanics of Low-Reynolds-number flow. Part 1. Rotation of axisymmetric prolate bodies. *J. Fluid Mech.* **63**, 607-622.

Chwang, A. T. & Wu, T. Y. 1975 Hydromechanics of Low-Reynolds-number flow. Part 2. The singularity method for Stokes flow. *J. Fluid Mech.* **67**, 787-815.

Chwang, A. T. & Wu, T. Y. 1976 Hydromechanics of Low-Reynolds-number flow. Part 4. Translation of spheroids. *J. Fluid Mech.* **75**, 677-689

Cox, R. G. 1970 The motion of long slender bodies in a viscous fluid. part 1. General theory. *J. Fluid Mech.* **44**, 791-810.

Cox, R. G. 1971 The motion of long slender bodies in a viscous fluid. part 2. Shear flow. *J. Fluid Mech.* **45**, 625-657.

Geer, J. 1976 Stokes flow past a slender body of revolution. *J. Fluid Mech.* **78**, 577-600.

Hancouk, G. J. 1953 The self-propulsion of microscopic organisms through liquids. *Proc. Roy. Soc. A* **217**, 96-121.

Happle, J. & Brenner, H. 1975 Low Reynolds Number Hydrodynamics. Prentice-Hall.

Johnson, R. E. 1980 An improved slender-body theory for Stokes flow. *J. Fluid Mech.* **99**, 411-431

Johnson, R. E. & Wu, T. Y. 1979 Hydromechanics of Low-Reynolds-number flow. Part 5. Motion of a slender torus. *J. Fluid Mech.* **95**, 263-277.

Keller, J. B. & Rubinow, S. I. 1976 Slender-body theory for slow viscous flow. *J. Fluid Mech.* **75**, 705-714.

Khayat, R. E. & Cox, R.G. 1989 Inertia effects on the motion of long slender bodies. *J. Fluid Mech.* **209**, 435-462.

Kreyszig, E. 1983 ADVANCED ENGINEERING MATHEMATICS. John Willey and sons, New York.

Lamp, H. 1932 Hydrodynamics. Cambridge University Press.

Langlois, W. E. 1964 Slow Viscous Flow. Macmilian, New York.

Muskhelishvili, N. I. & Radok, J. R. M. 1963 SOME BASIC PROBLEMS OF THE MATHEMATICAL THEORY OF ELASTICITY. P. NOORDHOFF LTD, GRONINGGEN - THE NETHELANDSR.

Nayfeh, A. H. 1981 Introduction to Perturbation Techniques. John Willey and sons, New York.

Proudman, I. & Pearson , J. R. A. 1957 Expansions at small Reynolds numbers for the flow past a sphere and a circular cylinder. *J. Fluid Mech.* **2**, 237-262.

Richardson, S. 1968 Two-dimensional bubbles in slow viscous flows. *J. Fluid Mech.* **33**, 475-493.

Staff of Research and Education Association (director Fogiel, M.). 1992 Handbook of Mathematical, Scientific, and Engineering Formulas, Tables, Functions, Graphs, Transforms. REA, New York.

Tillet, J. P. K. 1970 Axial and transverse Stokes flow past slender axisymmetric bodies. *J. Fluid Mech.* **44**, 401-417.

Tuck, E. O. 1964 Some methods for flows past blunt slender bodies. *J. Fluid Mech.* **18**, 619-635.