Optimized Waveform Relaxation Methods for RC Type Circuits

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November 2002

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements of the degree of Master of Science

Your file Votre référence
ISBN: 0-612-88143-1
Our file Notre référence
ISBN: 0-612-88143-1

Abstract

The waveform relaxation method is a very efficient and reliable method, it has been widely used in several fields including circuit theory, for solving large systems of ordinary differential equations as well as partial differential equations. The convergence rate of the classical waveform relaxation approach is not uniform over the time interval for which the equations are integrated. A new approach called the optimized waveform relaxation approach was proposed with a remarkable and great improvement in the convergence behavior by introducing new transmission conditions. Here, we continue the work done on the optimized waveform relaxation by extending previous results and trying to get a better performance as well as a more general optimized waveform relaxation approach. We use two RC circuits to illustrate the theory and the performance obtained by improving the convergence behavior of the new algorithm.

Résumé

La méthode de relaxation d'ondes est une approche très efficace et fiable: elle est utilisée avec succès dans plusieurs domaines (comme en théorie des circuits par exemple) pour résoudre des systèmes d'équations différentielles ordinaires, ou aux dérivées partielles, de grandes dimensions. Le taux de convergence de la méthode de relaxation d'ondes dite classique n'est pas uniforme sur l'interval de temps sur lequel les équations sont intégrées. Une nouvelle approche, appelée relaxation d'ondes optimisée, a été proposée qui améliore de façon remarquable les propriétés de convergence en introduisant ce qu'on appelle des conditions de transmission. Ici, nous continuons le travail accomplit sur la relaxation d'ondes optimisée, en ayant pour but d'obtenir une meilleure convergence ainsi qu'un algorithme plus général. Nous utilisons deux exemples de circuits électriques RC pour illustrer la théorie et démontrer l'amélioration de la performance.

Acknowledgments

First I would like to thank my supervisor, Prof. Martin Gander, for his help, guidance and advice during the work on this thesis. Without his contribution and insight, this humble work would not have been done. I wish to express also my gratitude to Prof. Nilima Nigam and Paul Tupper who provided the encouragement and suggestions. I am thankful to all professors, staff and students in the department. Special thanks to my friends, Martin Caberlin, Charles Fortin, Olivier Dubois, Mohammad Al-Odat, Evridiki Efstathiou, Ronia Bitar and Juan-Manuel Perez-Abarca for their valuable discussions. Many thanks to my parents, brothers and sisters from whom I have been away during my study, for their support and encouragement.

Contents

Introduction

Usually in real life applications we have very large systems of Ordinary Differential Equations (ODEs) and Partial Differential Equations (PDEs), such as those large systems which we obtain from large circuits. Applying traditional numerical techniques to such systems can be quite time consuming. In the circuit domain, many circuit solver methods were introduced [6, 26] but the circuit simulation using these methods takes too much CPU time and too much storage to analyze a circuit. Picard discussed in a paper published in 1893, [23], iteration methods to study Initial Value Problems (IVPs) for systems of Ordinary Differential Equations, and Lindelöf showed in a paper that was published in 1894, [14], the super-linear convergence on all finite time intervals of the iteration methods that were discussed by Picard. In the quest for improving the efficiency of the numerical techniques and to speed up the solution of these large systems, the waveform relaxation methods (WR) were proposed to first use some continuous-time iterations (Picard-Lindelöf iterations) to decouple the large system and then discretize the resulting subsystems.

The Waveform Relaxation methods were first introduced by Lelarasmee [10] and Lelarasmee, Ruehli and Sangiovanni-Vincentelli [11] for time-domain analysis of non-linear dynamical systems, in particular, large-scale integrated circuits. The basic idea in these new methods is to apply relaxation such as the Gauss-Seidel and the Gauss-Jacobi relaxations [22] directly to the system of non-linear differential equations describing the circuit. As a consequence, the system is decomposed into decoupled subsystems of differential equations corresponding to decoupled dynamical sub-circuits. Each decoupled sub-circuit is then analyzed for the entire

simulation time interval by integration methods, like the backward Euler method, to obtain subsystems of non-linear algebraic equations and Newton-Raphson iterations to linearize the subsystems of the non-linear algebraic equations. The solutions to the sub-circuits are used to update the solutions of neighboring sub-circuits in an iterative fashion.

For instance, if we consider the initial value problem for systems of ordinary differential equations

$$\begin{cases} \dot{\boldsymbol{y}}(t) = \boldsymbol{f}(t, \boldsymbol{y}(t)), \quad t \geq t_0, \\ \boldsymbol{y}(t_0) = \boldsymbol{y}_0, \end{cases}$$

then the continuous-time Waveform Relaxation iterations using the Gauss-Jacobi relaxation is

$$\begin{cases} \dot{\boldsymbol{y}}_i^{k+1}(t) = \boldsymbol{f}_i(t, \boldsymbol{y}_1^k(t), \ldots, \boldsymbol{y}_{i-1}^k(t), \boldsymbol{y}_i^{k+1}(t), \boldsymbol{y}_{i+1}^k(t), \ldots, \boldsymbol{y}_m^k(t)), \\ \boldsymbol{y}_i^{k+1}(t_0) = \boldsymbol{y}_{0,i}, \\ \qquad i = 1, 2, \ldots, m, \quad t \in [t_0, T], \quad k = 0, 1, \ldots \quad . \end{cases}$$

A good study and survey of this technique with emphasis on simulation of large-scale electrical circuits was written by White, Sangiovanni-Vincentelli, Odeh and Ruehli [28]. Many circuit solvers and experimental solvers have been built based on the Waveform Relaxation technique e.g. [1, 27].

In practice one is interested in the best way of subdivisions which yields fast convergence for the iterations. Depending on the way of partitioning, most of the work on the Waveform Relaxation methods was concerned with convergence, error estimation, acceleration of the convergence of iterations and implementation advantages in a parallel computing environment.

The convergence of the Waveform Relaxation methods was analyzed in [11], it was shown that the Waveform Relaxation would converge in the continuous-time domain from an arbitrary initial guess if every node was connected by a capacitor to ground. Later, weaker requirements for convergence of the Waveform Relaxation methods were found by Zukowski [29] and by Desai [2]. The convergence theory of the Waveform Relaxation methods was founded on a mathematical basis by Nevan-

linna and his co-workers [15, 16, 17, 18, 19], and related results were also found by Lie and Skålin [13]. A new method for the analysis of the convergence properties of the Waveform Relaxation methods was derived by Gristede, Ruehli and Zukowski [5] with new sufficient conditions for the convergence which are less restrictive than those of [2, 11, 29].

In this work we will continue the work on the Waveform Relaxation methods in the circuit domain that was done by Gander and Ruehli [4] in which a new class of methods was introduced which improves the performance over the *classical* WR algorithm with little computational overhead. These methods are called *Optimized* Waveform Relaxation algorithms since they include an optimization process. The optimization concerns the *transmission* conditions from each subsystem to its neighbor subsystems. It was shown [3] that the convergence would be much faster if additional derivative information is exchanged in the transmission conditions. Since the classical Waveform Relaxation algorithm exchanges only nodal voltage values in the transmission conditions from each subsystem to its neighbor subsystems, the resulting convergence can be very slow and is non-uniform over the time interval for which the equations are integrated. Gander and Ruehli [4] proposed new transmission conditions which transmit a combination of voltages and currents. By these new transmission conditions the method becomes faster and the overall convergence becomes much more uniform in few iterations which means a faster and nicer convergence over the whole time interval.

Here, we follow the same approach as in [4] i.e., we exemplify the application of the new approach to the RC circuits but we extend the results by introducing first order transmission conditions to get a faster convergence than in [4]. Also, we prove the optimality of the constant transmission conditions proposed in [4]. The analysis of the small RC circuit shown in Figure 1.1 is the first part in this work and in the second part, we analyze the large circuit shown in Figure 2.1, which leads to larger subsystems corresponding to larger sub-circuits and shows that the size of the circuit does not affect the convergence of the optimized Waveform Relaxation

algorithm.

It is worth saying that, the convergence of the classical Waveform Relaxation methods for RC type circuits was shown by Nevanlinna [20] and this type of circuits has been investigated for the classical Waveform Relaxation algorithm by several researchers e.g. [5, 7, 8, 9].

This thesis is organized as follows. Chapter 1 describes the small RC circuit, contains some analysis and results for the convergence of the classical WR and the optimized WR as well as a detailed derivation of the exact conditions for the best convergence of the optimized WR and ends with a numerical experiment. Chapter 2, similar to the small circuit, describes the large RC circuit, includes some analysis and results for the convergence of the classical WR and the optimized WR as well as a detailed derivation of higher order transmission conditions for the optimized WR for a better convergence. We end this chapter also by a numerical experiment. Finally, we give the conclusions.

Chapter 1

A Small RC Circuit Model Problem

The equations for the small RC circuit, Figure 1.1, can be derived as follows [21]. The relationship between the current, I, and the voltage, v, through a resistance, R, is given by $I = \frac{v}{R}$, which is known as Ohm's law. The current through a capacitor, C, is given by $I = C\frac{dv}{dt}$ where $\frac{dv}{dt}$ is the derivative of the voltage with respect to the time, t. At each node in the circuit, the algebraic sum of all the currents equals zero, i.e., we have $\sum I_j = 0$ which is equivalent to $\sum(entering\ \ currents) = \sum(leaving\ \ currents)$, and this is known as Kirchhoff's current law, where the voltage is measured in volts, the current in amperes, the capacitor in farads, the resistance in ohms and the time in seconds. For example, at the first node x_1 we have

$$I_s = C_1\dot{x}_1 + \frac{x_1}{R_s} + \frac{(x_1 - x_2)}{R_1},$$

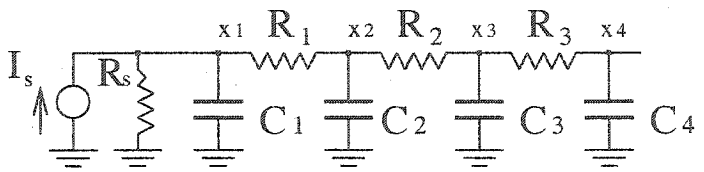

Figure 1.1: A small example RC circuit.

after some calculations

$$\dot{x}_1 = -(\frac{1}{R_s} + \frac{1}{R_1})\frac{1}{C_1}x_1 + \frac{1}{R_1 C_1}x_2 + \frac{I_s}{C_1},$$

and at the second node we have

$$\frac{(x_2 - x_1)}{R_1} + \frac{(x_2 - x_3)}{R_2} + C_2\dot{x}_2 = 0,$$

which implies

$$\dot{x}_2 = \frac{1}{C_2 R_1}x_1 - (\frac{1}{R_1} + \frac{1}{R_2})\frac{1}{C_2}x_2 + \frac{1}{C_2 R_2}x_3.$$

The equations at the other nodes can be found in a similar way, hence the circuit equations are of the form

$$\dot{\boldsymbol{x}} = \begin{bmatrix} b_1 & c_1 & & \\ a_1 & b_2 & c_2 & \\ & a_2 & b_3 & c_3 \\ & & a_3 & b_4 \end{bmatrix} \boldsymbol{x} + \boldsymbol{f}. \tag{1.1}$$

The entries in the tridiagonal matrix are given by

$$a_i = \frac{1}{R_i C_{i+1}}, \quad b_i = \begin{cases} -(\frac{1}{R_s} + \frac{1}{R_1})\frac{1}{C_1}, & i = 1 \\ -(\frac{1}{R_{i-1}} + \frac{1}{R_i})\frac{1}{C_i}, & i = 2, 3, \\ -\frac{1}{R_{i-1}C_i}, & i = 4 \end{cases} \quad c_i = \frac{1}{R_i C_i},$$

where the resistor values R_i and R_s and the capacitors C_i are strictly positive constants. The source term on the right hand side is given by $\boldsymbol{f} = (I_s(t)/C_1, 0, 0, 0)^T$ for some source function $I_s(t)$ and we are also given the initial voltage values $\boldsymbol{x}(0) = (v_1^0, v_2^0, v_3^0, v_4^0)^T$ at the time $t = 0$.

1.1 The Classical WR algorithm

We partition the circuit into two sub-circuits or subsystems and we call the unknown voltages in subsystem one $\boldsymbol{u}(t)$ and in subsystem two $\boldsymbol{w}(t)$. So the two subsystem

solutions satisfy

$$\begin{aligned} \dot{\boldsymbol{u}} &= \begin{bmatrix} b_1 & c_1 \\ a_1 & b_2 \end{bmatrix} \boldsymbol{u} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 u_3 \end{pmatrix}, \\ \dot{\boldsymbol{w}} &= \begin{bmatrix} b_3 & c_3 \\ a_3 & b_4 \end{bmatrix} \boldsymbol{w} + \begin{pmatrix} f_3 \\ f_4 \end{pmatrix} + \begin{pmatrix} a_2 w_0 \\ 0 \end{pmatrix}. \end{aligned} \tag{1.2}$$

The transmission conditions for the classical waveform relaxation are

$$u_3 \equiv w_1, \quad w_0 \equiv u_2.$$

By relaxing those transmission conditions we get the classical WR algorithm,

$$\begin{aligned} \dot{\boldsymbol{u}}^{k+1} &= \begin{bmatrix} b_1 & c_1 \\ a_1 & b_2 \end{bmatrix} \boldsymbol{u}^{k+1} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 w_1^k \end{pmatrix}, \\ \dot{\boldsymbol{w}}^{k+1} &= \begin{bmatrix} b_3 & c_3 \\ a_3 & b_4 \end{bmatrix} \boldsymbol{w}^{k+1} + \begin{pmatrix} f_3 \\ f_4 \end{pmatrix} + \begin{pmatrix} a_2 u_2^k \\ 0 \end{pmatrix}, \end{aligned} \tag{1.3}$$

with corresponding initial conditions $\boldsymbol{u}^{k+1}(0) = (v_1^0, v_2^0)^T$ and $\boldsymbol{w}^{k+1}(0) = (v_3^0, v_4^0)^T$. To start the WR iteration, we need to specify two initial waveforms $u_2^0(t))$ and $w_1^0(t)$ for $t \in [0, T]$, where T is the end of the transient analysis interval. The Laplace transform is used for the convergence study of the linear circuits. Actually, the Laplace transform plays a significant theoretical role in applied problems because of its versatility. It is often possible to find a desired information about a problem from the Laplace transform without finding the solution. The Laplace transform leads to a better understanding of a problem or an easier method of solution, it is used as a tool to solve (ODEs) as well as (PDEs). The Laplace transform for x will be denoted by $\hat{x} : \hat{x}(s) = \int_0^\infty e^{-st} x(t) dt$, $s \in \mathbb{C}$. The Laplace transform of $\dot{x}$ is given by $L(\dot{x}) = s\hat{x} - x_0$ where $x_0 = x(0)$, and more generally, the Laplace transform of $x^{(n)}$ is given by $L(x^{(n)}) = s^n \hat{x} - s^{n-1} x_0 - s^{n-2} \dot{x}_0 - \ldots - x_0^{(n-1)}$. The analysis of the Classical WR algorithm is discussed in [4], it was shown that $\hat{u}_2^{2k} = (\rho_{cla})^k \hat{u}_2^0$ and $\hat{w}_1^{2k} = (\rho_{cla})^k \hat{w}_1^0$ with the convergence rate ρ_{cla}

$$\rho_{cla}(s, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) = \frac{c_2(s - b_1)}{(s - b_1)(s - b_2) - a_1 c_1} \cdot \frac{a_2(s - b_4)}{(s - b_3)(s - b_4) - a_3 c_3}, \quad s = \sigma + i\omega. \tag{1.4}$$

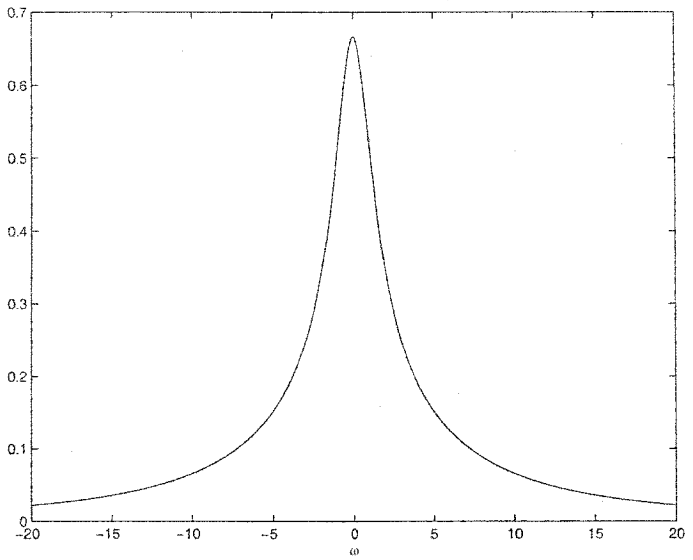

Figure 1.2: Convergence rate as a function of ω for the classical WR algorithm applied to the small RC circuit.

For convergence we need that $|\rho_{cla}(s, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})| < 1$ for $\Re(s) > 0$ and for fast convergence the modulus of ρ_{cla} should be much smaller than 1, $|\rho_{cla}| << 1$. But ρ_{cla} is a fixed function of the circuit parameters in the classical WR algorithm as is evident from (1.4). Thus the algorithm does not have any adjustable parameters like the optimized WR algorithm below. The convergence rate is different for different values of $s \in \mathbb{C}$ in the frequency domain, and the classical WR algorithm converges at different rates with different values of the time t. An example for the convergence rate as a function of ω is given in Figure 1.2.

1.2 The Optimal WR algorithm

For the optimized WR algorithm Gander and Ruehli proposed the transmission conditions

$$(u_3^{k+1} - u_2^{k+1}) + \alpha u_3^{k+1} = (w_1^k - w_0^k) + \alpha w_1^k, \quad (w_1^{k+1} - w_0^{k+1}) + \beta w_0^{k+1} = (u_3^k - u_2^k) + \beta u_2^k. \tag{1.5}$$

These new transmission conditions are more sophisticated since they attempt to match the interface voltages as well as the currents at the interface between the subsystems already during the iteration. The voltages u_3 and w_0 are exchanged and they are multiplied with weighting factors α and β, respectively, the voltage differences between the nodal voltages $(u_3 - u_2)$ and $(w_1 - w_0)$ insure that the currents are also taken into account in the transmission conditions since we could write the currents as $\alpha^{-1}(u_3 - u_2)$ and $\beta^{-1}(w_1 - w_0)$ where α and β can be viewed as resistors. It is shown in [4] that the new transmission conditions lead to the correct converged solution of the fundamental circuit equations. Together with the subsystems in (1.2) we get the optimized WR algorithm

$$\begin{aligned} \dot{\boldsymbol{u}}^{k+1} &= \begin{bmatrix} b_1 & c_1 \\ a_1 & b_2 + \frac{c_2}{\alpha+1} \end{bmatrix} \boldsymbol{u}^{k+1} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 w_1^k - \frac{c_2}{\alpha+1} w_0^k \end{pmatrix}, \\ \dot{\boldsymbol{w}}^{k+1} &= \begin{bmatrix} b_3 - \frac{a_2}{\beta-1} & c_3 \\ a_3 & b_4 \end{bmatrix} \boldsymbol{w}^{k+1} + \begin{pmatrix} f_3 \\ f_4 \end{pmatrix} + \begin{pmatrix} a_2 u_2^k + \frac{a_2}{\beta-1} u_3^k \\ 0 \end{pmatrix}, \end{aligned} \tag{1.6}$$

where the values u_3^k and w_0^k are determined by the transmission conditions (1.5). Hence, the parameters α and β enter the WR equations. It was shown in [4] as before that $\hat{u}_2^{2k} = (\rho_{opt})^k \hat{u}_2^0$ and $\hat{w}_1^{2k} = (\rho_{opt})^k \hat{w}_1^0$ where the convergence rate ρ_{opt} is given by

$$\begin{aligned} \rho_{opt}(s, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \alpha, \beta) &= -\frac{c_2(s-b_1)(\beta-1) + (s-b_1)(s-b_2) - a_1 c_1}{((s-b_3)(s-b_4) - a_3 c_3)(\beta-1) + a_2(s-b_4)} \\ &\quad \cdot \frac{-a_2(s-b_4)(\alpha+1) + (s-b_3)(s-b_4) - a_3 c_3}{((s-b_1)(s-b_2) - a_1 c_1)(\alpha+1) + c_2(b_1 - s)}. \end{aligned}$$

Further, it was shown that the best weighting factors α and β in the transmission conditions (1.5) are given by functions of s, namely

$$\alpha := \frac{-a_3 c_3}{(s-b_4)a_2} + \frac{s-b_3}{a_2} - 1, \quad \beta := \frac{a_1 c_1}{(s-b_1)c_2} - \frac{s-b_2}{c_2} + 1, \quad s \in \mathbb{C}. \tag{1.7}$$

In [4], an approximation of these best possible transmission conditions was proposed, because α and β are not polynomials in s and hence are symbols of non-local operators in time. An approximation of α and β by a constant was chosen, which

leads to a very practical algorithm with remarkable improvement in magnitude and uniformity for the convergence in comparison with the classical WR method. Here, we will use the optimal choice (1.7) for the parameters, and not an approximation, which will lead to the optimal WR algorithm for the small RC circuit that converges exactly in two iterations. The Laplace transform for $s \in \mathbb{C}$ of (1.5) is given by

$$(\hat{u}_3^{k+1}-\hat{u}_2^{k+1})+\alpha\hat{u}_3^{k+1} = (\hat{w}_1^k-\hat{w}_0^k)+\alpha\hat{w}_1^k, \quad (\hat{w}_1^{k+1}-\hat{w}_0^{k+1})+\beta\hat{w}_0^{k+1} = (\hat{u}_3^k-\hat{u}_2^k)+\beta\hat{u}_2^k. \tag{1.8}$$

If we substitute the values for α and β from (1.7) in (1.8) we get for the first transmission condition

$$(\hat{u}_3^{k+1}-\hat{u}_2^{k+1})+\left(\frac{-a_3c_3}{(s-b_4)a_2}+\frac{s-b_3}{a_2}-1\right)\hat{u}_3^{k+1} = (\hat{w}_1^k-\hat{w}_0^k)+\left(\frac{-a_3c_3}{(s-b_4)a_2}+\frac{s-b_3}{a_2}-1\right)\hat{w}_1^k, \tag{1.9}$$

and for the second transmission condition we have

$$(\hat{w}_1^{k+1}-\hat{w}_0^{k+1})+\left(\frac{a_1c_1}{(s-b_1)c_2}-\frac{s-b_2}{c_2}+1\right)\hat{w}_0^{k+1} = (\hat{u}_3^k-\hat{u}_2^k)+\left(\frac{a_1c_1}{(s-b_1)c_2}-\frac{s-b_2}{c_2}+1\right)\hat{u}_2^k. \tag{1.10}$$

By multiplying (1.9) by $a_2(s-b_4)$ and (1.10) by $-c_2(s-b_1)$, we find after some algebra,

$$\begin{aligned} &s^2\hat{u}_3^{k+1}-(b_3+b_4)s\hat{u}_3^{k+1}-a_2s\hat{u}_2^{k+1}+(b_4b_3-a_3c_3)\hat{u}_3^{k+1}+a_2b_4\hat{u}_2^{k+1} \\ &\quad = s^2\hat{w}_1^k-(b_3+b_4)s\hat{w}_1^k-a_2s\hat{w}_0^k+(b_4b_3-a_3c_3)\hat{w}_1^k+a_2b_4\hat{w}_0^k, \end{aligned} \tag{1.11}$$

$$\begin{aligned} &s^2\hat{w}_0^{k+1}-(b_1+b_2)s\hat{w}_0^{k+1}-c_2s\hat{w}_1^{k+1}-(a_1c_1-b_1b_2)\hat{w}_0^{k+1}+c_2b_1\hat{w}_1^{k+1} \\ &\quad = s^2\hat{u}_2^k-(b_1+b_2)s\hat{u}_2^k-c_2s\hat{u}_3^k-(a_1c_1-b_1b_2)\hat{u}_2^k+c_2b_1\hat{u}_3^k. \end{aligned} \tag{1.12}$$

Since a multiplication with s in the frequency domain corresponds to a time derivative, equations (1.11) and (1.12) become, respectively

$$\begin{aligned} &\ddot{u}_3^{k+1}-(b_3+b_4)\dot{u}_3^{k+1}-a_2\dot{u}_2^{k+1}+(b_4b_3-a_3c_3)u_3^{k+1}+a_2b_4u_2^{k+1} \\ &\quad = \ddot{w}_1^k-(b_3+b_4)\dot{w}_1^k-a_2\dot{w}_0^k+(b_4b_3-a_3c_3)w_1^k+a_2b_4w_0^k, \end{aligned} \tag{1.13}$$

$$\begin{aligned} &\ddot{w}_0^{k+1}-(b_1+b_2)\dot{w}_0^{k+1}-c_2\dot{w}_1^{k+1}-(a_1c_1-b_1b_2)w_0^{k+1}+c_2b_1w_1^{k+1} \\ &\quad = \ddot{u}_2^k-(b_1+b_2)\dot{u}_2^k-c_2\dot{u}_3^k-(a_1c_1-b_1b_2)u_2^k+c_2b_1u_3^k, \end{aligned} \tag{1.14}$$

and thus the non-local parameters in (1.7) can still lead to local transmission conditions in the optimal WR algorithm in this case. Substituting $\dot{u}_2^{k+1}$ and $\dot{w}_1^{k+1}$ from the first and second subsystems (1.2) in (1.13) and (1.14) respectively, and simplifying we get

$$\begin{aligned}\ddot{u}_3^{k+1} &= (b_3+b_4)\dot{u}_3^{k+1} - (b_4b_3 - a_3c_3 - a_2c_2)u_3^{k+1} - (a_2b_4 - a_2b_2)u_2^{k+1} + a_2a_1u_1^{k+1} \\ &\quad + a_2f_2 + \ddot{w}_1^k - (b_3+b_4)\dot{w}_1^k - a_2\dot{w}_0^k + (b_4b_3 - a_3c_3)w_1^k + a_2b_4w_0^k,\end{aligned} \tag{1.15}$$

$$\begin{aligned}\ddot{w}_0^{k+1} &= (b_1+b_2)\dot{w}_0^{k+1} + (a_1c_1 - b_1b_2 + a_2c_2)w_0^{k+1} - (c_2b_1 - c_2b_3)w_1^{k+1} + c_2c_3w_2^{k+1} \\ &\quad + c_2f_3 + \ddot{u}_2^k - (b_1+b_2)\dot{u}_2^k - c_2\dot{u}_3^k - (a_1c_1 - b_1b_2)u_2^k + c_2b_1u_3^k.\end{aligned} \tag{1.16}$$

Equations (1.15) and (1.16) are second order ordinary differential equations which can be written as two systems of two first order ordinary differential equations. We get for the first equation, equation (1.15),

$$\begin{aligned}\dot{u}_3^{k+1} &= \tilde{u}_3^{k+1}, \\ \dot{\tilde{u}}_3^{k+1} &= \ddot{u}_3^{k+1},\end{aligned} \tag{1.17}$$

and similarly for the second equation, equation (1.16),

$$\begin{aligned}\dot{w}_0^{k+1} &= \tilde{w}_0^{k+1}, \\ \dot{\tilde{w}}_0^{k+1} &= \ddot{w}_0^{k+1}.\end{aligned} \tag{1.18}$$

The system (1.17) which corresponds to the first transmission condition together with the first subsystem in (1.2), implies

$$\begin{pmatrix} \dot{u}_1^{k+1} \\ \dot{u}_2^{k+1} \\ \dot{u}_3^{k+1} \\ \dot{\tilde{u}}_3^{k+1} \end{pmatrix} = \begin{bmatrix} b_1 & c_1 & 0 & 0 \\ a_1 & b_2 & c_2 & 0 \\ 0 & 0 & 0 & 1 \\ a_2a_1 & a_2b_2 - a_2b_4 & a_3c_3 + a_2c_2 - b_4b_3 & b_3 + b_4 \end{bmatrix} \begin{pmatrix} u_1^{k+1} \\ u_2^{k+1} \\ u_3^{k+1} \\ \tilde{u}_3^{k+1} \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \\ 0 \\ a_2f_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \ddot{w}_1^k - (b_3+b_4)\dot{w}_1^k - a_2\dot{w}_0^k + (b_4b_3 - a_3c_3)w_1^k + a_2b_4w_0^k \end{pmatrix}, \tag{1.19}$$

with initial voltage values

$$(u_1^{k+1}(0), u_2^{k+1}(0), u_3^{k+1}(0), \tilde{u}_3^{k+1}(0))^T = (v_1^0, v_2^0, v_3^0, a_2v_2^0 + b_3v_3^0 + c_3v_4^0)^T,$$

and the system (1.18) which corresponds to the second transmission condition together with the second subsystem in (1.2), implies

$$\begin{pmatrix} \dot{\tilde{w}}_0^{k+1} \\ \dot{w}_0^{k+1} \\ \dot{w}_1^{k+1} \\ \dot{w}_2^{k+1} \end{pmatrix} = \begin{bmatrix} b_1+b_2 & a_1c_1-b_1b_2+a_2c_2 & c_2b_3-c_2b_1 & c_2c_3 \\ 1 & 0 & 0 & 0 \\ 0 & a_2 & b_3 & c_3 \\ 0 & 0 & a_3 & b_4 \end{bmatrix} \begin{pmatrix} \tilde{w}_0^{k+1} \\ w_0^{k+1} \\ w_1^{k+1} \\ w_2^{k+1} \end{pmatrix} + \begin{pmatrix} c_2f_3 \\ 0 \\ f_3 \\ f_4 \end{pmatrix} + \begin{pmatrix} \ddot{u}_2^k - (b_1+b_2)\dot{u}_2^k - c_2\dot{u}_3^k - (a_1c_1-b_1b_2)u_2^k + c_2b_1u_3^k \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{1.20}$$

with initial voltage values

$$(\tilde{w}_0^{k+1}(0), w_0^{k+1}(0), w_1^{k+1}(0), w_2^{k+1}(0))^T = (a_1v_1^0 + b_2v_2^0 + c_2v_3^0, v_2^0, v_3^0, v_4^0)^T.$$

Since at step $k \geq 1$,

$$\begin{aligned} &\dot{w}_0^k = \tilde{w}_0^k, \quad \dot{w}_1^k = a_2w_0^k + b_3w_1^k + c_3w_2^k + f_3, \quad \text{and} \\ &\ddot{w}_1^k = a_2\tilde{w}_0^k + b_3a_2w_0^k + (b_3^2 + c_3a_3)w_1^k + c_3(b_3+b_4)w_2^k + b_3f_3 + c_3f_4 + \dot{f}_3, \end{aligned}$$

we have

$$\ddot{w}_1^k - (b_3+b_4)\dot{w}_1^k - a_2\dot{w}_0^k + (b_4b_3 - a_3c_3)w_1^k + a_2b_4w_0^k = c_3f_4 - b_4f_3 + \dot{f}_3. \tag{1.21}$$

Similarly, at step $k \geq 1$, since

$$\begin{aligned} &\dot{u}_3^k = \tilde{u}_3^k, \quad \dot{u}_2^k = a_1u_1^k + b_2u_2^k + c_2u_3^k + f_2, \quad \text{and} \\ &\ddot{u}_2^k = a_1(b_1+b_2)u_1^k + (a_1c_1 + b_2^2)u_2^k + b_2c_2u_3^k + c_2\tilde{u}_3^k + b_2f_2 + a_1f_1 + \dot{f}_2, \end{aligned}$$

we have

$$\ddot{u}_2^k - (b_1+b_2)\dot{u}_2^k - c_2\dot{u}_3^k - (a_1c_1 - b_1b_2)u_2^k + c_2b_1u_3^k = a_1f_1 - b_1f_2 + \dot{f}_2. \tag{1.22}$$

By substituting from (1.21) and (1.22) in (1.19) and (1.20) respectively, we get the following two decoupled systems

$$\begin{pmatrix} \dot{u}_1^{k+1} \\ \dot{u}_2^{k+1} \\ \dot{u}_3^{k+1} \\ \dot{\tilde{u}}_3^{k+1} \end{pmatrix} = \begin{bmatrix} b_1 & c_1 & 0 & 0 \\ a_1 & b_2 & c_2 & 0 \\ 0 & 0 & 0 & 1 \\ a_2a_1 & a_2b_2 - a_2b_4 & a_3c_3 + a_2c_2 - b_4b_3 & b_3 + b_4 \end{bmatrix} \begin{pmatrix} u_1^{k+1} \\ u_2^{k+1} \\ u_3^{k+1} \\ \tilde{u}_3^{k+1} \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \\ 0 \\ a_2f_2 + c_3f_4 - b_4f_3 + \dot{f}_3 \end{pmatrix}, \tag{1.23}$$

and

$$\begin{pmatrix} \dot{\tilde{w}}_0^{k+1} \\ \dot{w}_0^{k+1} \\ \dot{w}_1^{k+1} \\ \dot{w}_2^{k+1} \end{pmatrix} = \begin{bmatrix} b_1 + b_2 & a_1c_1 - b_1b_2 + a_2c_2 & c_2b_3 - c_2b_1 & c_2c_3 \\ 1 & 0 & 0 & 0 \\ 0 & a_2 & b_3 & c_3 \\ 0 & 0 & a_3 & b_4 \end{bmatrix} \begin{pmatrix} \tilde{w}_0^{k+1} \\ w_0^{k+1} \\ w_1^{k+1} \\ w_2^{k+1} \end{pmatrix} + \begin{pmatrix} c_2f_3 + a_1f_1 - b_1f_2 + \dot{f}_2 \\ 0 \\ f_3 \\ f_4 \end{pmatrix}. \tag{1.24}$$

For the convergence study and analysis we do not want to find the solution for the non-homogeneous problem but we are looking for the convergence rate of the problem to see how fast and uniform the problem will converge to the solution. It is convenient and sufficient by linearity to consider the homogeneous problem or system of (ODEs) where the initial conditions, $\boldsymbol{u}^{k+1}(0) = \boldsymbol{w}^{k+1}(0) = (0,0,0,0)^T$, as well as the source terms, $\boldsymbol{f}(t) = (0,0,0,0)^T$, are zero for the convergence study, the reason why we can do so is that the homogeneous equations are the error equations themselves, for instance, studying the iterative method $\mathbf{M}\boldsymbol{x}^{k+1} = \mathbf{N}\boldsymbol{x}^k + \boldsymbol{b}$ for the

system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, where $\mathbf{A} = \mathbf{M} - \mathbf{N}$, we use the fact that $\mathbf{M}\boldsymbol{x} = \mathbf{N}\boldsymbol{x} + \boldsymbol{b}$, take the difference with the iteration and find $\mathbf{M}(\boldsymbol{x} - \boldsymbol{x}^{k+1}) = \mathbf{N}(\boldsymbol{x} - \boldsymbol{x}^k)$. Denoting by $\boldsymbol{e}^{k+1}$ the error, $\boldsymbol{e}^k = \boldsymbol{x} - \boldsymbol{x}^k$, one has the iteration $\mathbf{M}\boldsymbol{e}^{k+1} = \mathbf{N}\boldsymbol{e}^k$ and the vector $\boldsymbol{b}$ is not present any more. So instaed of introducing $\boldsymbol{e}$, we could have studied directly $\mathbf{M}\boldsymbol{x}^{k+1} = \mathbf{N}\boldsymbol{x}^k$, setting $\boldsymbol{b} = \mathbf{0}$ in the iteration. The homogeneous problem of (1.23) and (1.24) is given by

$$\begin{aligned}
\dot{\boldsymbol{u}}^{k+1} &= \begin{bmatrix} b_1 & c_1 & 0 & 0 \\ a_1 & b_2 & c_2 & 0 \\ 0 & 0 & 0 & 1 \\ a_2a_1 & a_2b_2 - a_2b_4 & a_3c_3 + a_2c_2 - b_4b_3 & b_3 + b_4 \end{bmatrix} \boldsymbol{u}^{k+1}, \\
\dot{\boldsymbol{w}}^{k+1} &= \begin{bmatrix} b_1 + b_2 & a_1c_1 - b_1b_2 + a_2c_2 & c_2b_3 - c_2b_1 & c_2c_3 \\ 1 & 0 & 0 & 0 \\ 0 & a_2 & b_3 & c_3 \\ 0 & 0 & a_3 & b_4 \end{bmatrix} \boldsymbol{w}^{k+1},
\end{aligned} \tag{1.25}$$

with initial conditions, $\boldsymbol{u}^{k+1}(0) = \mathbf{0}$, and $\boldsymbol{w}^{k+1}(0) = \mathbf{0}$. Since the coefficients in the above homogeneous linear systems are constant, by using the matrix exponential function $\exp(\mathbf{A}t)$, we can write the solution of the homogeneous problem,

$$\begin{aligned}
\dot{\boldsymbol{u}}^{k+1} &= \mathbf{A}\boldsymbol{u}^{k+1}, \quad \boldsymbol{u}^{k+1}(0) = \mathbf{0}, \\
\dot{\boldsymbol{w}}^{k+1} &= \mathbf{B}\boldsymbol{w}^{k+1}, \quad \boldsymbol{w}^{k+1}(0) = \mathbf{0},
\end{aligned}$$

where $\mathbf{A}$, and $\mathbf{B}$ are the matrices for the first and second equations in (1.25), respectively, in the form

$$\begin{aligned}
\boldsymbol{u}^{k+1} &= \boldsymbol{u}^{k+1}(0)\exp(\mathbf{A}t), \\
\boldsymbol{w}^{k+1} &= \boldsymbol{w}^{k+1}(0)\exp(\mathbf{B}t),
\end{aligned}$$

and since, $\boldsymbol{u}^{k+1}(0) = \mathbf{0}$, $\boldsymbol{w}^{k+1}(0) = \mathbf{0}$, we get

$$\boldsymbol{u}^{k+1} = \mathbf{0}, \quad \boldsymbol{w}^{k+1} = \mathbf{0}.$$

Lemma 1 *The optimized WR algorithm (1.19), (1.20) converges in two iterations if*

$$\alpha := \frac{-a_3c_3}{(s - b_4)a_2} + \frac{s - b_3}{a_2} - 1, \quad \beta := \frac{a_1c_1}{(s - b_1)c_2} - \frac{s - b_2}{c_2} + 1,$$

independently of the initial waveforms $\boldsymbol{u}^0$ *and* $\boldsymbol{w}^0$.

Proof The systems (1.23) and (1.24) which we get at step $k \geq 1$ do not depend on the guess for the initial waveforms, which means that from the second iteration the systems (1.19) and (1.20) will converge to the exact solution whatever the guess for the initial waveforms will be. Further, since $\boldsymbol{u}^{k+1} = \boldsymbol{0}$ and $\boldsymbol{w}^{k+1} = \boldsymbol{0}$, for $k \geq 1$, we have $\boldsymbol{u}^2$ and $\boldsymbol{w}^2$ are identically zero, independently of $\boldsymbol{u}^0$ and $\boldsymbol{w}^0$. ■

We ended up here with solving two systems of four unknowns each whereas we have originally only one system of four unknowns which could also have been solved directly, but we did the analysis for the small system to gain a better understanding and to show that in this case one can use the optimal transmission conditions using a simple transformation to make them local.

1.3 Numerical Experiments

In this section we give a numerical example that shows the results we found in the previous section. We assume the circuit parameters

$$R_s = \tfrac{1}{2}, \quad R_1 = \tfrac{1}{2}, \quad R_2 = \tfrac{1}{2}, \quad R_3 = \tfrac{1}{2},$$
$$C_1 = \tfrac{63}{100}, \quad C_2 = \tfrac{63}{100}, \quad C_3 = \tfrac{63}{100}, \quad C_4 = \tfrac{63}{100},$$

for the small circuit in Figure 1.1. We use for the numerical computations the forward Euler method with a time step $\Delta t = \frac{1}{100}$. We start with zero initial waveforms as well as zero initial conditions and use an input step function with an amplitude of $I_s = 1$ and a rise time of 1 time unit. We approximate the second order derivatives by a centered finite difference,

$$\ddot{w}^k_{1\,j} = \frac{w^k_{1\,j+1} - 2w^k_{1\,j} + w^k_{1\,j-1}}{(\Delta t)^2},$$
$$\ddot{u}^k_{2\,j} = \frac{u^k_{2\,j+1} - 2u^k_{2\,j} + u^k_{2\,j-1}}{(\Delta t)^2}, \quad j = 1, 2, \ldots, 1000,$$

and we use at $j = 0$, $\ddot{w}^k_1 = a_2\tilde{w}^k_0 + b_3a_2w^k_0 + (b_3^2 + c_3a_3)w^k_1 + c_3(b_3 + b_4)w^k_2 + b_3f_3 + c_3f_4 + \dot{f}_3$, and $\ddot{u}^k_2 = a_1(b_1 + b_2)u^k_1 + (a_1c_1 + b_2^2)u^k_2 + b_2c_2u^k_3 + c_2\tilde{u}^k_3 + b_2f_2 + a_1f_1 + \dot{f}_2$, since $\boldsymbol{u}^k(0)$

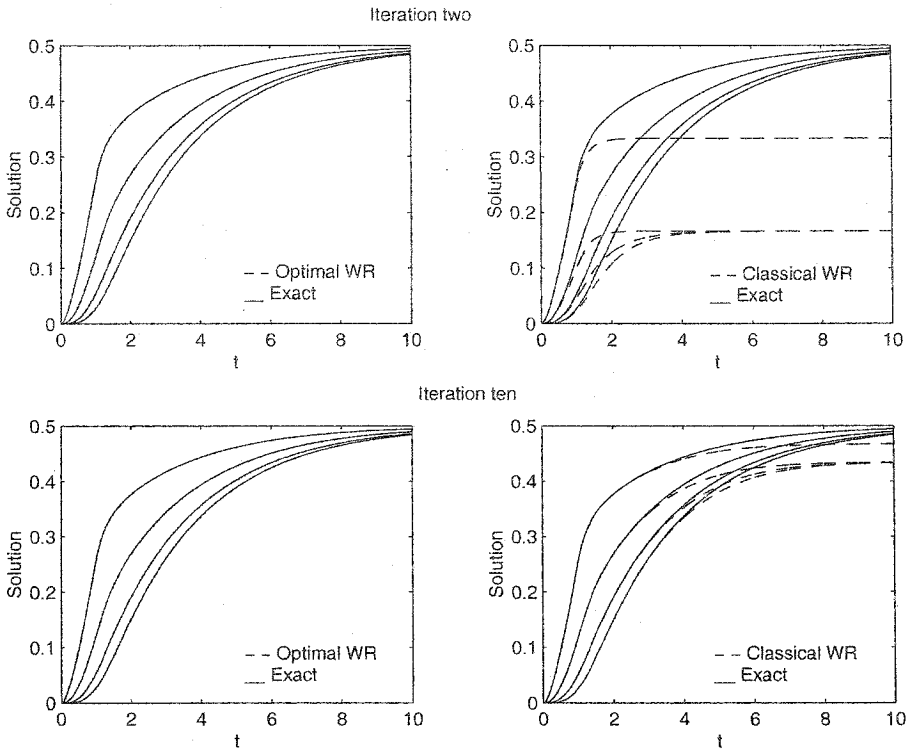

Figure 1.3: Results for Small Circuit.

and $\boldsymbol{w}^k(0)$ do not depend on k, they are given by the initial condoition. We assume at $j = 1000$, $w^k_{1j+1} = w^k_{1j-1}$, and $u^k_{2j+1} = u^k_{2j-1}$. Also, we approximate the first order derivatives by forward difference,

$$\begin{aligned}
\dot{w}^k_{0j} &= \frac{w^k_{0j+1} - w^k_{0j}}{\Delta t}, \\
\dot{w}^k_{1j} &= \frac{w^k_{1j+1} - w^k_{1j}}{\Delta t}, \\
\dot{u}^k_{2j} &= \frac{u^k_{2j+1} - u^k_{2j}}{\Delta t}, \\
\dot{u}^k_{3j} &= \frac{u^k_{3j+1} - u^k_{3j}}{\Delta t}, \quad j = 0, 1, \ldots, 1000,
\end{aligned}$$

and we again assume at $j = 1000$, $w^k_{0j+1} = w^k_{0j-1}$, $w^k_{1j+1} = w^k_{1j-1}$, $u^k_{2j+1} = u^k_{2j-1}$, and $u^k_{2j+1} = u^k_{2j-1}$. The above discretization depends on the time step, we assume symmetry around the right end point of the time interval, and it might not be the proper discretization to approximate the derivatives in the subsystems. There are four waveforms which we do not care about, $\tilde{u}_3$, $\tilde{w}_0$, u_3 and w_0. We need four waveforms to be our solution for the original system which are the first two of the

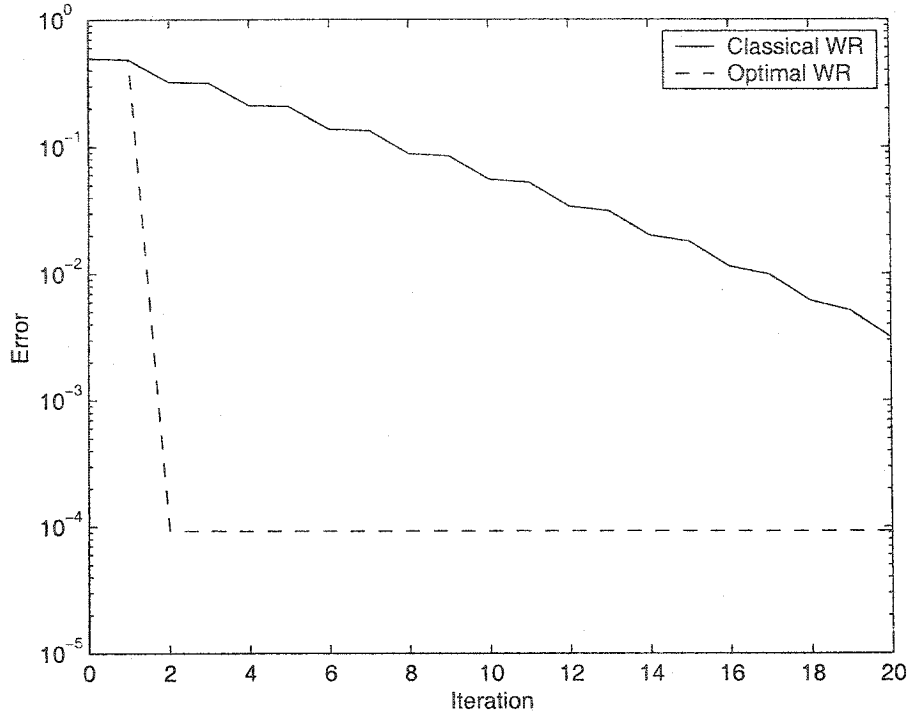

Figure 1.4: Convergence rates of classical versus optimal WR.

first subsystem and the last two of the second subsystem. We give the results for the classical WR and optimal WR algorithms together with the exact solution in Figure 1.3. The difference in convergence between the classical and the optimal WR algorithms is illustrated in Figure 1.4, which shows the error as a function of the iteration. The solution obtained by using the optimal WR algorithm is the same after the second iteration, and hence the error is the same which shows that the optimal WR algorithm converges in two iterations. In the optimal WR algorithm we use the second order transmission conditions which are the best possible ones that lead to the remarkable improvement of the optimized WR algorithm over the classical one and that is evident from the comparison in Figure 1.4. On the negative side, one has to think about the best discretization of the second order transmission conditions, which is not clear, to get a much better solution.

Chapter 2

The Large RC Circuit

We analyze in this chapter an infinitely long RC circuit and its infinite size system of equations, as is indicated in Figure 2.1, to show what the impact of the system size is on the convergence properties of the waveform relaxation algorithms. The equations for the infinitely large circuit matrix, Figure 2.1, are

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \ddots & \ddots & \ddots & & & \\ & a & b & c & & \\ & & a & b & c & \\ & & & \ddots & \ddots & \ddots \end{bmatrix} \boldsymbol{x} + \boldsymbol{f}. \tag{2.1}$$

The entries in the tridiagonal matrix are given by

$$a = \frac{1}{RC}; \quad b = -\left(\frac{2}{R}\right)\frac{1}{C}; \quad c = \frac{1}{RC} = a,$$

where the circuit elements, R and C are assumed to be strictly positive and constant. The source term on the right hand side is given by the vector of functions $\boldsymbol{f}(t) = (\ldots, f_{-1}(t), f_0(t), f_1(t), \ldots)^T$ and we need an initial condition $\boldsymbol{x}(0) =$

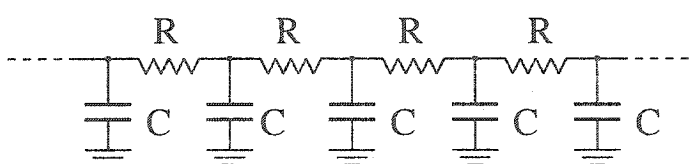

Figure 2.1: An infinitely long RC circuit chain.

$(\ldots, v_{-1}^0, v_0^0, v_1^0, \ldots)^T$. Since the circuit is infinitely large, we have to assume that all voltage values stay bounded as we move toward the infinite ends of the circuit to have a well posed problem.

2.1 The Classical WR Algorithm for the Large Circuit

We partition the system in (2.1) into the following two subsystems

$$\begin{aligned}
\dot{\boldsymbol{u}} &= \begin{bmatrix} \ddots & \ddots & \ddots & \\ & a & b & a \\ & & a & b \end{bmatrix} \boldsymbol{u} + \begin{pmatrix} \vdots \\ f_{-1} \\ f_0 \end{pmatrix} + \begin{pmatrix} \vdots \\ 0 \\ au_1 \end{pmatrix}, \\
\dot{\boldsymbol{w}} &= \begin{bmatrix} b & a & & \\ a & b & a & \\ & \ddots & \ddots & \ddots \end{bmatrix} \boldsymbol{w} + \begin{pmatrix} f_1 \\ f_2 \\ \vdots \end{pmatrix} + \begin{pmatrix} aw_0 \\ 0 \\ \vdots \end{pmatrix},
\end{aligned} \tag{2.2}$$

with the initial conditions $\boldsymbol{u}(0) = (\ldots, v_{-1}^0, v_0^0)^T$ and $\boldsymbol{w}(0) = (v_1^0, v_2^0, \ldots)^T$. The transmission conditions for the classical waveform relaxation are

$$u_1 \equiv w_1, \quad w_0 \equiv u_0.$$

Now relaxing those transmission conditions we get the waveform relaxation algorithm

$$\begin{aligned}
\dot{\boldsymbol{u}}^{k+1} &= \begin{bmatrix} \ddots & \ddots & \ddots & \\ & a & b & a \\ & & a & b \end{bmatrix} \boldsymbol{u}^{k+1} + \begin{pmatrix} \vdots \\ f_{-1} \\ f_0 \end{pmatrix} + \begin{pmatrix} \vdots \\ 0 \\ aw_1^k \end{pmatrix}, \\
\dot{\boldsymbol{w}}^{k+1} &= \begin{bmatrix} b & a & & \\ a & b & a & \\ & \ddots & \ddots & \ddots \end{bmatrix} \boldsymbol{w}^{k+1} + \begin{pmatrix} f_1 \\ f_2 \\ \vdots \end{pmatrix} + \begin{pmatrix} au_0^k \\ 0 \\ \vdots \end{pmatrix},
\end{aligned} \tag{2.3}$$

with the initial conditions $\boldsymbol{u}^{k+1}(0) = (\ldots, v_{-1}^0, v_0^0)^T$ and $\boldsymbol{w}^{k+1}(0) = (v_1^0, v_2^0, \ldots)^T$. To start the classical WR iteration, we use some initial waveforms $\boldsymbol{u}^0(t)$ and $\boldsymbol{w}^0(t)$.

The analysis for the classical WR is discussed in [4]. For the convergence study, as for the small circuit, it suffices to analyze the homogeneous problem, $\boldsymbol{f}(t) = \mathbf{0}$ with zero initial conditions $\boldsymbol{x}(0) = \mathbf{0}$. The Laplace transform yields in the $s \in \mathbb{C}$ domain

$$\begin{aligned} s\hat{\boldsymbol{u}}^{k+1} &= \begin{bmatrix} \ddots & \ddots & \ddots & \\ & a & b & a \\ & & a & b \end{bmatrix} \hat{\boldsymbol{u}}^{k+1} + \begin{pmatrix} \vdots \\ 0 \\ a\hat{w}_1^k \end{pmatrix}, \\ s\hat{\boldsymbol{w}}^{k+1} &= \begin{bmatrix} b & a & & \\ a & b & a & \\ & \ddots & \ddots & \ddots \end{bmatrix} \hat{\boldsymbol{w}}^{k+1} + \begin{pmatrix} a\hat{u}_0^k \\ 0 \\ \vdots \end{pmatrix}. \end{aligned} \tag{2.4}$$

Solving the first system of equations for $\hat{u}_j^{k+1}$ corresponds to solving the recurrence relation

$$a\hat{u}_{j-1}^{k+1} + (b-s)\hat{u}_j^{k+1} + a\hat{u}_{j+1}^{k+1} = 0, \quad j = 0, -1, -2, \ldots$$

which has the general solution

$$\hat{u}_j^{k+1} = A^{k+1}\lambda_+^j + B^{k+1}\lambda_-^j,$$

where $\lambda_\pm$ are the roots of the characteristic polynomial of the recurrence relation,

$$\lambda_\pm = \frac{s - b \pm \sqrt{(s-b)^2 - 4a^2}}{2a}. \tag{2.5}$$

It was shown in [4] that

$$\hat{u}_0^{2k} = (\rho_{cla})^k \, \hat{u}_0^0, \quad \hat{w}_1^{2k} = (\rho_{cla})^k \, \hat{w}_1^0,$$

where the convergence rate ρ_{cla} is given by

$$\rho_{cla}(s, a, b) = \frac{a^2}{(a\lambda_+^{-1} + b - s)(a\lambda_- + b - s)} = \frac{1}{\lambda_+^2}. \tag{2.6}$$

As in the small circuit case, the convergence rate depends on $s \in \mathbb{C}$, the parameter in the Laplace transform. Results may be obtained in the nodal formulation in the time domain, but they are obtained more easily by using Laplace transformation because of the power of the Laplace transform approach which is in its versatility. The classical WR, as is evident from (2.6), always converges for a large number of iterations since $|\lambda_+| > 1$, but convergence might be very slow.

2.2 The Optimal WR Algorithm

The two subsystems (2.2) with the new transmission conditions,

$$\begin{aligned}(u_1^{k+1} - u_0^{k+1}) + \alpha u_1^{k+1} &= (w_1^k - w_0^k) + \alpha w_1^k,\\ (w_1^{k+1} - w_0^{k+1}) + \beta w_0^{k+1} &= (u_1^k - u_0^k) + \beta u_0^k,\end{aligned} \tag{2.7}$$

which are similar to what we have used for the small circuit, will give the subsystems

$$\begin{aligned}\dot{\boldsymbol{u}}^{k+1} &= \begin{bmatrix} \ddots & \ddots & \ddots & \\ & a & b & a \\ & & a & b+\frac{a}{\alpha+1} \end{bmatrix} \boldsymbol{u}^{k+1} + \begin{pmatrix} \vdots \\ f_{-1} \\ f_0 \end{pmatrix} + \begin{pmatrix} \vdots \\ 0 \\ aw_1^k - \frac{a}{\alpha+1} w_0^k \end{pmatrix},\\ \dot{\boldsymbol{w}}^{k+1} &= \begin{bmatrix} b-\frac{a}{\beta-1} & a & & \\ a & b & a & \\ & \ddots & \ddots & \ddots \end{bmatrix} \boldsymbol{w}^{k+1} + \begin{pmatrix} f_1 \\ f_2 \\ \vdots \end{pmatrix} + \begin{pmatrix} au_0^k + \frac{a}{\beta-1} u_1^k \\ 0 \\ \vdots \end{pmatrix},\end{aligned} \tag{2.8}$$

together with the transmission conditions (2.7) which define the values u_1^k and w_0^k. In [4] it was shown that

$$\hat{u}_0^{2k} = (\rho_{opt1})^k \hat{u}_0^0, \quad \hat{w}_1^{2k} = (\rho_{opt1})^k \hat{w}_1^0,$$

where the convergence rate ρ_{opt1}, is given by

$$\rho_{opt1}(s,a,b,\alpha,\beta) = \frac{(\alpha+1)-\lambda_+}{(\alpha+1)\lambda_+ - 1} \cdot \frac{(\beta-1)+\lambda_+}{(\beta-1)\lambda_+ + 1}. \tag{2.9}$$

Theorem 1 (Optimal Convergence) *The optimized WR algorithm (2.8) converges in two iterations for the choice of parameters*

$$\alpha := \lambda_+ - 1, \quad \beta := -\lambda_+ + 1 \tag{2.10}$$

independently of the guess for the initial waveforms.

Proof The convergence rate vanishes if we insert (2.10) into ρ_{opt1} given by (2.9). Hence, $\hat{u}_0^2$ and $\hat{w}_1^2$ are identically zero, independent of $\hat{u}_0^0$ and $\hat{w}_1^0$. ■

For the large circuit, similar to the small circuit, the parameters in (2.10) are not

polynomials in s and hence are symbols of non-local operators in time. In the small circuit case, it is shown that the non-local parameters in (1.7) can lead to local transmission conditions in the optimal WR algorithm, by transforming them into polynomials, but in the large circuit case, the symbols contain square roots and hence there is no simple manipulation to transform them into polynomials. Again, in [4], an approximation of α and β by a constant was chosen. The best constant for the algorithm was found, which is not as good as the optimal conditions with the square root, but leads to a very practical algorithm which is already much better than the classical WR algorithm. In the next sections, we prove the optimality of the constants chosen for the transmission conditions in [4], and we introduce first order transmission conditions to get a faster convergence rate than in [4].

2.3 The Optimized WR Algorithm with Constant Transmission Conditions

The maximum principle for complex analytic functions is used in the optimization process for the large circuit.

Theorem 2 *Let R be the region consisting of C and its interior, and let $f(s)$ be regular and not identically constant in R. Then the maximum value of $|f(s)|$ in R occurs on the boundary C. If $f(s)$ has no zero in R, $|f(s)|$ also attains its minimum in R on boundary C.*

Proof See [12] for the proof. ■

The optimization process for the WR algorithm allows us to reduce the large $\rho_{cla}(\omega)$ of the classical WR and make it more uniform so that the overall convergence is improved. Mathematically, we want $|\rho_{opt1}| << 1$.

Lemma 2 *The convergence rate ρ_{opt1} in (2.9) is an analytic function in the right*

half of the complex plane, $s = \sigma + i\omega$ with $\sigma > 0$, if α, $\beta \in \mathbb{R}$,

$$b < 0, \quad a > 0, \quad 2a \leq -b, \tag{2.11}$$

$$\alpha > 0, \quad \beta < 0, \tag{2.12}$$

Proof See [4] for the proof. ■

The maximum principle can be used and the maximum of $|\rho_{opt1}(s)|$ for $s = \sigma + i\omega$, $\sigma > 0$ is attained on the boundary at $\sigma = 0$. That yields the optimization problem

$$\min_{\alpha>0,\beta<0} \left(\max_{\omega_{\min}<|\omega|<\omega_{\max}} |\rho_{opt1}(i\omega, a, b, \alpha, \beta)| \right) \tag{2.13}$$

where the frequency range was truncated by a minimal and maximal frequency relevant for the problem. The estimate for the lowest frequency occurring in the transient analysis depends on the length of the time interval $[0, T]$. The signal was expanded in a Fourier series $sin(k\pi t/T)$ for $k = 1, 2, \ldots$. Hence, the minimal relevant frequency can be estimated by $\omega_{min} = \pi/T$. The highest frequency depends again on the resolution of the discretization in time, and as before the maximal relevant frequency can be estimated by $\omega_{max} = \pi/\Delta t$ which is the highest possible oscillation on a grid with spacing Δt. The subsystems in the large circuit are behaving identically on both sides of the partition, so we assume, for simplicity, that $\beta = -\alpha$, although this might not give the best possible solution. We define a new parameter γ as $\gamma = \alpha + 1$. Also, λ_+ given in (2.5) with $s = i\omega$, $\omega \in [\omega_{min}, \omega_{max}]$, will be written as $\lambda_+ = x + iy$, where the real part, x, is given by

$$x = X(\omega) = \frac{-b}{2a} + \frac{\sqrt{2\sqrt{\omega^4 + 2\omega^2 b^2 + 8\omega^2 a^2 + b^4 - 8b^2 a^2 + 16a^4} - 2\omega^2 + 2b^2 - 8a^2}}{4a},$$

and the imaginary part, y, is given by

$$y = Y(\omega) = \frac{\omega}{2a} + \frac{\sqrt{2\sqrt{\omega^4 + 2\omega^2 b^2 + 8\omega^2 a^2 + b^4 - 8b^2 a^2 + 16a^4} + 2\omega^2 - 2b^2 + 8a^2}}{4a}.$$

The modulus of the convergence rate ρ_{opt1} in (2.9), will be

$$f(x, y; \gamma) = |\rho_{opt1}| = \left| \frac{(\gamma - x) - iy}{(\gamma x - 1) + i\gamma y} \right|^2 = \frac{(x - \gamma)^2 + y^2}{(\gamma x - 1)^2 + \gamma^2 y^2}, \tag{2.14}$$

where f is a function of two variables x, $y \in \mathbb{R}$, γ is our parameter. We know from before that $\alpha > 0$ which implies $\gamma > 1$, and $-b \geq 2a$ which gives $x > 1$. We set $a = 1$, and $b = -2c^2$, for $c \geq 1$, to eliminate one parameter, which is equivalent to a time scaling . The real and imaginary parts of the path, λ_+, become then

$$\begin{aligned} X(\omega, c) &= c^2 + \tfrac{1}{4}\sqrt{2\sqrt{\omega^4 + 8\omega^2 c^4 + 8\omega^2 + 16c^8 - 32c^4 + 16} - 2\omega^2 + 8c^4 - 8}, \\ Y(\omega, c) &= \tfrac{1}{2}\omega + \tfrac{1}{4}\sqrt{2\sqrt{\omega^4 + 8\omega^2 c^4 + 8\omega^2 + 16c^8 - 32c^4 + 16} + 2\omega^2 - 8c^4 + 8}. \end{aligned} \tag{2.15}$$

Solving $x = X(\omega, c)$ for $\omega(x, c)$ gives

$$\begin{aligned} \omega_1(x, c) &= -2\frac{\sqrt{-x(2c^2 - x)(2xc^2 - 1 - x^2)}(c^2 - x)}{2xc^2 - x^2}, \\ \omega_2(x, c) &= 2\frac{\sqrt{-x(2c^2 - x)(2xc^2 - 1 - x^2)}(c^2 - x)}{2xc^2 - x^2}. \end{aligned}$$

Since we have $\omega > 0$, we require

$$0 < 2xc^2 - x^2 < 1, \tag{2.16}$$

which implies

$$\omega(x, c) = -2\sqrt{\frac{-(2xc^2 - 1 - x^2)}{2xc^2 - x^2}}(c^2 - x) > 0. \tag{2.17}$$

With those conditions on c and x in (2.16), the second solution is ignored since it implies $\omega < 0$. Inserting the value for $\omega(x, c)$ from (2.17) into $Y(\omega, c)$, the second equation in (2.15), implies

$$\begin{aligned} Y(x, c) \;=\; & -\sqrt{\tfrac{-(2xc^2 - x^2 - 1)}{2xc^2 - x^2}}c^2 + \sqrt{\tfrac{-(2xc^2 - x^2 - 1)}{2xc^2 - x^2}}x \\ & + \tfrac{1}{2}\sqrt{2}\sqrt{\frac{(2xc^2 - x^2)\sqrt{\frac{(-c^4 + 4x^2c^4 - 4x^3c^2 + x^4)^2}{(2xc^2 - x^2)^2}} - 4xc^6 + 6x^2c^4 - 4x^3c^2 + c^4 + x^4}{2xc^2 - x^2}}. \end{aligned}$$

Since $(-c^4 + 4x^2c^4 - 4x^3c^2 + x^4) < 0$, for c and x satisfying the conditions (2.16), we have

$$\sqrt{(-c^4 + 4x^2c^4 - 4x^3c^2 + x^4)^2} = -(-c^4 + 4x^2c^4 - 4x^3c^2 + x^4).$$

This implies

$$Y(x, c) = -\sqrt{\frac{-(2xc^2 - x^2 - 1)}{2xc^2 - x^2}}c^2 + \sqrt{\frac{-(2xc^2 - x^2 - 1)}{2xc^2 - x^2}}x + \frac{1}{2}\sqrt{2}\sqrt{\frac{2c^4 + 2x^2c^4 - 4xc^6}{2xc^2 - x^2}}.$$

Simplifying more, we get

$$Y(x,c) = \sqrt{\frac{-(2xc^2 - x^2 - 1)}{2xc^2 - x^2}}x. \tag{2.18}$$

To find in what range x can vary, we find the limits in $x = X(\omega, c)$ as ω goes to zero and infinity,

$$x_{min} = \lim_{\omega \to 0} X(\omega, c) = c^2 + \sqrt{c^4 - 1},$$
$$x_{max} = \lim_{\omega \to \infty} X(\omega, c) = 2c^2.$$

Inserting $y = Y(x,c)$ from equation (2.18) into the function $f(x, y; \gamma)$ implies

$$F(x,c;\gamma) = \frac{2\gamma^2c^2 - \gamma^2x - 4\gamma xc^2 + 2\gamma x^2 + x}{-4\gamma xc^2 + 2\gamma x^2 + 2c^2 - x + \gamma^2 x}, \tag{2.19}$$

for which we minimize the maximum for $x \in [x_1, x_2]$, where $x_{min} < x_1 = X(\omega_{min}, c)$ and $x_2 = X(\omega_{max}, c) < x_{max}$. The derivative $F_x(x,c;\gamma)$ is

$$F_x(x,c;\gamma) = 2\frac{(2\gamma^3 - 2\gamma)x^2 + (4\gamma c^2 - 4\gamma^3c^2)x + c^2 - 4\gamma c^4 - \gamma^4c^2 + 4\gamma^3c^4}{(-2\gamma x^2 + (4\gamma c^2 + 1 - \gamma^2)x - 2c^2)^2}. \tag{2.20}$$

Solving $F_x = 0$ for x, we get two roots, namely,

$$r_\pm = c^2 \pm \frac{c}{2\gamma}\sqrt{-2\gamma(2\gamma c^2 - \gamma^2 - 1)}, \quad (2\gamma c^2 - \gamma^2 - 1) < 0. \tag{2.21}$$

Since $c^2 \geq 1$, we have $r_+ > 1$, and since $r_- < x_{min}$, and $x \geq x_1 > x_{min}$, r_- is ignored. For the study of the sign of $F_x(x,c;\gamma)$, we will use the fact that, if we have a quadratic polynomial, $p(x) = ax^2 + bx + c$, with two real zeros then the sign of the polynomial for those values of x lying between the zeros is different from the sign of the coefficient of x^2, i.e., a, and has the same sign as a everywhere else.

Lemma 3 *If $r_+ \leq x_1 < x_2$ then the function $F(x,c;\gamma)$ attains its maximum value at $x = x_2$, and minimum value at $x = x_1$. If $x_1 < r_+ < x_2$ then the function attains its maximum value either at $x = x_1$ or $x = x_2$, and minimum value at $x = r_+$. If $x_1 < x_2 \leq r_+$ then the maximum is attained at $x = x_1$, and the minimum is attained at $x = x_2$.*

Proof In the derivative $F_x(x, c; \gamma)$ given in (2.20), the coefficient of x^2 is $2(2\gamma^3 - 2\gamma) > 0$, and hence the sign of F_x for the values of x lying between the two roots of $F_x = 0$ is negative. The denominator does not affect the sign, since it is squared. By using basic calculus, if $r_+ \leq x_1 < x_2$, then $F_x(x, c; \gamma) > 0$ for $x \in (x_1, x_2)$, and hence the function $F(x, c; \gamma)$ is increasing in the whole interval and has a maximum value at $x = x_2$ and a minimum value at $x = x_1$. If $x_1 < r_+ < x_2$, then $F_x(x, c; \gamma) < 0$ for $x \in (x_1, r_+]$, and $F_x(x, c; \gamma) > 0$ for $x \in (r_+, x_2)$, so the function is decreasing for $x \in (x_1, r_+]$, and increasing for $x \in (r_+, x_2)$, hence there is a minimum value at $x = r_+$, and the function $F(x, c; \gamma)$ attains its maximum value either at $x = x_1$ or $x = x_2$. Finally, if $x_1 < x_2 \leq r_+$, then $F_x(x, c; \gamma) < 0$ for $x \in (x_1, x_2)$, and hence the function $F(x, c; \gamma)$ is decreasing in the whole interval and has a maximum value at $x = x_1$ and a minimum value at $x = x_2$. ■

Solving $r_+(c, \gamma) = x_1$, where $r_+(c, \gamma)$ is given in (2.21), for γ implies

$$\gamma_{min} = \frac{2c^4 + x_1^2 - 2x_1c^2 + \sqrt{(x_1^2 - c^2 - 2x_1c^2 + 2c^4)(x_1^2 + c^2 - 2x_1c^2 + 2c^4)}}{c^2}. \tag{2.22}$$

Since, by the conditions (2.16), $-1 < x_1^2 - 2x_1c^2 < 0$, and we know that $c^2 > 1$ which implies $(2c^2 - 1) > 1$, we have

$$(x_1^2 - c^2 - 2x_1c^2 + 2c^4) = (x_1^2 - 2x_1c^2) + c^2(2c^2 - 1) > 0,$$

and clearly, $0 < (x_1^2 - c^2 - 2x_1c^2 + 2c^4) < (x_1^2 + c^2 - 2x_1c^2 + 2c^4)$. Hence,

$$(x_1^2 - c^2 - 2x_1c^2 + 2c^4)(x_1^2 + c^2 - 2x_1c^2 + 2c^4) > 0.$$

Similarly, solving $r_+(c, \gamma) = x_2$, for γ gives

$$\gamma_{max} = \frac{2c^4 + x_2^2 - 2x_2c^2 + \sqrt{(x_2^2 - c^2 - 2x_2c^2 + 2c^4)(x_2^2 + c^2 - 2x_2c^2 + 2c^4)}}{c^2}, \tag{2.23}$$

where again, $(x_2^2 - c^2 - 2x_2c^2 + 2c^4)(x_2^2 + c^2 - 2x_2c^2 + 2c^4) > 0$.

Lemma 4 *The function $r_+(c, \gamma)$ in (2.21), is an increasing function of γ. Further, γ_{min} in (2.22) is strictly less than γ_{max} in (2.23).*

Proof The derivative of $r_+(c,\gamma)$ with respect to γ,

$$r'_+(c,\gamma) = \frac{c\sqrt{2}(\gamma^2-1)}{4\gamma\sqrt{-\gamma(2\gamma c^2-\gamma^2-1)}},$$

has two zeros, -1 and 1, the sign of the derivative for $\gamma > 1$ is positive, since the coefficient of γ^2 in the derivative is positive. Hence, the function $r_+(c,\gamma)$ is increasing for $\gamma > 1$. Moreover, since $x_1 < x_2$ and $r_+(c,\gamma)$ is increasing for $\gamma > 1$, we have $\gamma_{min} < \gamma_{max}$. ■

Lemma 5 *If $\gamma \in (1, \gamma_{min}]$, then the maximum of the function $F(x,c;\gamma)$ is attained at $x = x_2$. If $\gamma \in (\gamma_{min}, \gamma_{max})$, then the function attains its maximum value either at $x = x_1$ or $x = x_2$. If $\gamma \in [\gamma_{max}, \infty)$, then the maximum value is attained at $x = x_1$.*

Proof The proof follows directly from Lemma 3 and Lemma 4. ■

Lemma 6 *There exists $\gamma^* \in (\gamma_{min}, \gamma_{max})$, under the conditions (2.16), such that,*

$$F(x_1,c;\gamma^*) = F(x_2,c;\gamma^*).$$

Proof Solving the equation $F(x_1,c;\gamma) = F(x_2,c;\gamma)$ for γ implies

$$\gamma^* = \frac{x_1x_2+2c^4-x_2c^2-x_1c^2+\sqrt{(x_1x_2-x_1c^2-c^2+2c^4-x_2c^2)(x_1x_2-x_1c^2+c^2+2c^4-x_2c^2)}}{c^2}. \qquad (2.24)$$

We know that $x_1 < x_2$, and $\gamma_{min} < \gamma_{max}$. Since

$$\begin{aligned}(x_1x_2-x_1c^2-c^2+2c^4-x_2c^2)-(x_1^2-2x_1c^2-c^2+2c^4) &= x_1(x_2-x_1)-c^2(x_2-x_1)\\ &= (x_1-c^2)(x_2-x_1)\\ &> 0,\end{aligned}$$

we have $(x_1^2 - 2x_1c^2 - c^2 + 2c^4) < (x_1x_2 - x_1c^2 - c^2 + 2c^4 - x_2c^2)$ which implies $(x_1x_2 - x_1c^2 - c^2 + 2c^4 - x_2c^2) > 0$ and hence,

$$(x_1x_2 - x_1c^2 - c^2 + 2c^4 - x_2c^2)(x_1x_2 - x_1c^2 + c^2 + 2c^4 - x_2c^2) > 0.$$

This also implies $\gamma_{min} < \gamma^*$. Similarly, since

$$\begin{aligned}(x_2^2-2x_2c^2-c^2+2c^4)-(x_1x_2-x_1c^2-c^2+2c^4-x_2c^2) &= x_2(x_2-x_1)-c^2(x_2-x_1)\\ &= (x_2-c^2)(x_2-x_1)\\ &> 0,\end{aligned}$$

$\gamma^* < \gamma_{max}$. Hence, the existence of $\gamma^* \in (\gamma_{min}, \gamma_{max})$ has been assured, under the conditions (2.16). ■

Theorem 3 *Let $\rho_{opt1}(i\omega, a, b, \alpha, \beta)$ be the convergence rate of the optimized WR algorithm (2.8) with constant transmission conditions, then the global optimal solution of the min-max problem*

$$\min_{\alpha>0,\beta<0}\left(\max_{\omega_{\min}<|\omega|<\omega_{\max}}|\rho_{opt1}(i\omega, a, b, \alpha, \beta)|\right),$$

is obtained at $\alpha^* = \gamma^* - 1$ and $\beta^* = -\alpha^*$, where

$$\gamma^* = \frac{x_1x_2+2c^4-x_2c^2-x_1c^2+\sqrt{(x_1x_2-x_1c^2-c^2+2c^4-x_2c^2)(x_1x_2-x_1c^2+c^2+2c^4-x_2c^2)}}{c^2}.$$

Proof The proof follows directly from the above Lemmas. ■

For the numerical example in section 2.5, with $\omega_{min} = \pi/T$ and $\omega_{max} = \pi/\Delta t$ we find, by using the formula in (2.24), $\gamma^* = 1.73455$, and hence the optimal constant is $\alpha^* = 0.73455$, which is the same as what was found in [4] without a proof of optimality. The solution for the min-max problem given in (2.13), and the optimality of the constant transmission conditions proposed in [4] has now been proved, which is one of the key contributions of this thesis.

2.4 The Optimized WR Algorithm with First Order Transmission conditions

Since the optimal parameters, α and β in (2.10) are functions of $s \in \mathbb{C}$, we may approximate them as, $\alpha = \alpha_0 + \alpha_1 s$, and $\beta = -\alpha_0 - \alpha_1 s$, for some constants α_0 and

α_1. We assume $\beta = -\alpha$, for simplicity. When $\alpha_1 = 0$ we get the simplest expansion for the parameters and constant transmission conditions, which was done in [4] and we have analyzed the optimal choice of that constant in section 2.3. In this section we will use the first order expansion for α to get faster convergence. Taking the Laplace transform for the transmission conditions in (2.7), with $\beta = -\alpha$, and using the first order expansion for α, we get

$$\begin{aligned}(\hat{u}_1^{k+1} - \hat{u}_0^{k+1}) + \alpha_0\hat{u}_1^{k+1} + \alpha_1 s\hat{u}_1^{k+1} &= (\hat{w}_1^k - \hat{w}_0^k) + \alpha_0\hat{w}_1^k + \alpha_1 s\hat{w}_1^k,\\ (\hat{w}_1^{k+1} - \hat{w}_0^{k+1}) - \alpha_0\hat{w}_0^{k+1} - \alpha_1 s\hat{w}_0^{k+1} &= (\hat{u}_1^k - \hat{u}_0^k) - \alpha_0\hat{u}_0^k - \alpha_1 s\hat{u}_0^k.\end{aligned} \tag{2.25}$$

Since a multiplication by s in the frequency domain corresponds to a time derivative, by substituting

$$\begin{aligned}\dot{w}_1^k &= bw_1^k + aw_2^k + aw_0^k + f_1,\\ \dot{u}_0^k &= au_{-1}^k + bu_0^k + au_1^k + f_0,\end{aligned}$$

from the subsystems in (2.2), assuming $\alpha_1 \neq 0$, we have

$$\begin{aligned}\dot{u}_1^{k+1} &= \tfrac{1}{\alpha_1}u_0^{k+1} - \tfrac{(1+\alpha_0)}{\alpha_1}u_1^{k+1} + \tfrac{(1+\alpha_0+\alpha_1 b)}{\alpha_1}w_1^k + \tfrac{(\alpha_1 a-1)}{\alpha_1}w_0^k + aw_2^k + f_1,\\ \dot{w}_0^{k+1} &= \tfrac{1}{\alpha_1}w_1^{k+1} - \tfrac{(1+\alpha_0)}{\alpha_1}w_0^{k+1} + \tfrac{(1+\alpha_0+\alpha_1 b)}{\alpha_1}u_0^k + \tfrac{(\alpha_1 a-1)}{\alpha_1}u_1^k + au_{-1}^k + f_0.\end{aligned} \tag{2.26}$$

The ordinary differential equations which are found from the transmission conditions in (2.26) together with the subsystems in (2.2) will imply the following two decoupled subsystems,

$$\begin{pmatrix}\vdots\\ \dot{u}_{-1}^{k+1}\\ \dot{u}_0^{k+1}\\ \dot{u}_1^{k+1}\end{pmatrix} = \begin{bmatrix}\ddots & \ddots & \ddots & & \\ & a & b & a & \\ & & a & b & a\\ & & & \frac{1}{\alpha_1} & \frac{-(\alpha_0+1)}{\alpha_1}\end{bmatrix}\begin{pmatrix}\vdots\\ u_{-1}^{k+1}\\ u_0^{k+1}\\ u_1^{k+1}\end{pmatrix} + \begin{pmatrix}\vdots\\ f_{-1}\\ f_0\\ f_1\end{pmatrix} + \begin{pmatrix}0\\ \vdots\\ 0\\ \frac{(\alpha_1 a-1)}{\alpha_1}w_0^k + \frac{(1+\alpha_0+\alpha_1 b)}{\alpha_1}w_1^k + aw_2^k\end{pmatrix}, \tag{2.27}$$

and

$$\begin{pmatrix} \dot{w}_0^{k+1} \\ \dot{w}_1^{k+1} \\ \dot{w}_2^{k+1} \\ \vdots \end{pmatrix} = \begin{bmatrix} \frac{-(\alpha_0+1)}{\alpha_1} & \frac{1}{\alpha_1} & & \\ a & b & a & \\ & a & b & a \\ & & \ddots & \ddots & \ddots \end{bmatrix} \begin{pmatrix} w_0^{k+1} \\ w_1^{k+1} \\ w_2^{k+1} \\ \vdots \end{pmatrix} + \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \end{pmatrix} + \begin{pmatrix} au_{-1}^k + \frac{(1+\alpha_0+\alpha_1 b)}{\alpha_1} u_0^k + \frac{(\alpha_1 a-1)}{\alpha_1} u_1^k \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{2.28}$$

with the initial conditions $\boldsymbol{u}^{k+1}(0) = (\ldots, v_{-1}^0, v_0^0, v_1^0)^T$ and $\boldsymbol{w}^{k+1}(0) = (v_0^0, v_1^0, v_2^0, \ldots)^T$, respectively. To start the WR iteration, some initial waveforms $\boldsymbol{u}^0(t)$ and $\boldsymbol{w}^0(t)$ are used. As before, for the convergence study, it is sufficient to analyze the homogeneous problem, $\boldsymbol{f}(t) = \boldsymbol{0}$ with zero initial conditions $\boldsymbol{x}(0) = \boldsymbol{0}$. The Laplace transform yields in the $s \in \mathbb{C}$ domain

$$\begin{aligned} s\hat{\boldsymbol{u}}^{k+1} &= \begin{bmatrix} \ddots & \ddots & \ddots & & \\ & a & b & a & \\ & & a & b & a \\ & & & \frac{1}{\alpha_1} & \frac{-(\alpha_0+1)}{\alpha_1} \end{bmatrix} \hat{\boldsymbol{u}}^{k+1} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{(\alpha_1 a-1)}{\alpha_1}\hat{w}_0^k + \frac{(1+\alpha_0+\alpha_1 b)}{\alpha_1}\hat{w}_1^k + a\hat{w}_2^k \end{pmatrix}, \\ s\hat{\boldsymbol{w}}^{k+1} &= \begin{bmatrix} \frac{-(\alpha_0+1)}{\alpha_1} & \frac{1}{\alpha_1} & & \\ a & b & a & \\ & a & b & a \\ & & \ddots & \ddots & \ddots \end{bmatrix} \hat{\boldsymbol{w}}^{k+1} + \begin{pmatrix} a\hat{u}_{-1}^k + \frac{(1+\alpha_0+\alpha_1 b)}{\alpha_1}\hat{u}_0^k + \frac{(\alpha_1 a-1)}{\alpha_1}\hat{u}_1^k \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \end{aligned} \tag{2.29}$$

Solving the first subsystem of equations for $\hat{u}_j^{k+1}$, analogous to what is done for the classical WR, corresponds to solving the recurrence relation

$$a\hat{u}_{j-1}^{k+1} + (b-s)\hat{u}_j^{k+1} + a\hat{u}_{j+1}^{k+1} = 0, \quad j = 1, 0, -1, -2, \ldots$$

with the general solution

$$\hat{u}_j^{k+1} = A^{k+1}\lambda_+^j + B^{k+1}\lambda_-^j,$$

where $\lambda_\pm$ are the roots of the characteristic polynomial of the recurrence relation. To determine the constants A^{k+1} and B^{k+1} for the general solution, we will use that the solutions stay bounded at infinity as well as the last equation at the interface. We have

$$\lambda_+ = \frac{s-b+\sqrt{(s-b)^2-4a^2}}{2a},$$

which can be simplified to

$$\lambda_+ = \frac{-b}{2a} + \frac{s}{2a} + \frac{\sqrt{s^2-2sb+b^2-4a^2}}{2a}.$$

For $-b \geq 2a$ and $s = \sigma + i\omega$ with $\sigma > 0$, we get $|\lambda_+| > 1$, since $|s| > 0$. Since $\lambda_+\lambda_- = 1$, we have $|\lambda_+||\lambda_-| = 1$ which implies $|\lambda_-| < 1$, by knowing that $|\lambda_+| > 1$. By using the fact that $|\lambda_-| < 1$, $|\lambda_+| > 1$ and the boundedness condition we obtain $B^{k+1} = 0$. Further, we can determine A^{k+1} from the last equation at the interface,

$$(1+\alpha_0+\alpha_1 s)\hat{u}_1^{k+1} - \hat{u}_0^{k+1} = (a\alpha_1 - 1)\hat{w}_0^k + (1+\alpha_0+b\alpha_1)\hat{w}_1^k + a\alpha_1\hat{w}_2^k,$$

which implies

$$A^{k+1}(\lambda_+(1+\alpha_0+\alpha_1 s) - 1) = (a\alpha_1 - 1)\hat{w}_0^k + (1+\alpha_0+b\alpha_1)\hat{w}_1^k + a\alpha_1\hat{w}_2^k,$$

and hence

$$A^{k+1} = \frac{(a\alpha_1 - 1)\hat{w}_0^k + (1+\alpha_0+b\alpha_1)\hat{w}_1^k + a\alpha_1\hat{w}_2^k}{\lambda_+(1+\alpha_0+\alpha_1 s) - 1}. \tag{2.30}$$

Similarly, solving the second subsystem for $\hat{w}_j^{k+1}$ we obtain as a general solution

$$\hat{w}_j^{k+1} = B^{k+1}\lambda_-^{j-1}, \quad j = 0, 1, 2, 3, \ldots$$

and to determine B^{k+1} we again use the equation at the interface,

$$(1+\alpha_0+\alpha_1 s)\hat{w}_0^{k+1} - \hat{w}_1^{k+1} = (a\alpha_1 - 1)\hat{u}_1^k + (1+\alpha_0+b\alpha_1)\hat{u}_0^k + a\alpha_1\hat{u}_{-1}^k,$$

which gives

$$B^{k+1} = \frac{(a\alpha_1 - 1)\hat{u}_1^k + (1+\alpha_0+b\alpha_1)\hat{u}_0^k + a\alpha_1\hat{u}_{-1}^k}{\lambda_-^{-1}(1+\alpha_0+\alpha_1 s) - 1}. \tag{2.31}$$

By inserting the general solution for the second subsystem in equation (2.30) and the general solution for the first subsystem in equation (2.31) at step k, we obtain

$$A^{k+1} = \frac{(a\alpha_1 - 1)\lambda_-^{-1} + 1 + \alpha_0 + b\alpha_1 + a\alpha_1\lambda_-}{\lambda_+(1 + \alpha_0 + \alpha_1 s) - 1} B^k.$$

$$B^{k+1} = \frac{(a\alpha_1 - 1)\lambda_+ + 1 + \alpha_0 + b\alpha_1 + a\alpha_1\lambda_+^{-1}}{\lambda_-^{-1}(1 + \alpha_0 + \alpha_1 s) - 1} A^k.$$

Applying the second relation at step k to the first one, we find

$$\hat{u}_0^{k+1} = \rho_{opt2}(s, a, b, \alpha_0, \alpha_1)\hat{u}_0^{k-1},$$

where the convergence rate ρ_{opt2}, using that $\lambda_+ = \lambda_-^{-1}$, is given by

$$\rho_{opt2}(s, a, b, \alpha_0, \alpha_1) = \left(\frac{(a\alpha_1 - 1)\lambda_+ + 1 + \alpha_0 + b\alpha_1 + a\alpha_1\lambda_+^{-1}}{\lambda_+(1 + \alpha_0 + \alpha_1 s) - 1} \right)^2.$$

Since $\lambda_+ + \lambda_- = \frac{s-b}{a}$, we get

$$(a\alpha_1 - 1)\lambda_+ + 1 + \alpha_0 + b\alpha_1 + a\alpha_1\lambda_- = \alpha_0 + \alpha_1 s + 1 - \lambda_+.$$

Therefore, the convergence rate ρ_{opt2} is given by

$$\rho_{opt2}(s, a, b, \alpha_0, \alpha_1) = \left(\frac{(\alpha_0 + \alpha_1 s + 1) - \lambda_+}{\lambda_+(\alpha_0 + \alpha_1 s + 1) - 1} \right)^2. \tag{2.32}$$

The same relation also holds for the other subsystem and by induction we find $\hat{u}_0^{2k} = (\rho_{opt2})^k \hat{u}_0^0$ and $\hat{w}_1^{2k} = (\rho_{opt2})^k \hat{w}_1^0$. Similar to the optimized WR algorithm with constant transmission conditions, we discuss in this section an optimization process for the WR algorithm with first order transmission conditions, which we find by expanding α as $\alpha = \alpha_0 + \alpha_1 s$ and $\beta = -\alpha$ to get the best performance of the new WR algorithm. Mathematically, we again want $|\rho_{opt2}| << 1$.

Lemma 7 *The convergence rate ρ_{opt2} in (2.32) is, under the conditions*

$$b < 0, \quad a > 0, \quad 2a \leq -b, \tag{2.33}$$

$$\alpha_0 \geq 0, \quad \alpha_1 > 0, \tag{2.34}$$

an analytic function in the right half of the complex plane, $s = \sigma + i\omega$ with $\sigma > 0$.

Proof λ_+ is an analytic function in the right half plane, since the argument under the square root avoids the negative real axis under the conditions (2.33). Hence, it is sufficient to show that the denominator does not have zeros. Assume there is a zero, $\lambda_+(\alpha_0 + \alpha_1 s + 1) - 1 = 0$, this gives $\lambda_+ = \frac{1}{1+\alpha_0+\alpha_1 s}$, which implies $|\lambda_+|^2 = \frac{1}{(1+\alpha_0+\alpha_1\sigma)^2+\omega^2\alpha_1^2} < 1$, that is a contradiction to the fact that $|\lambda_+| > 1$. Hence, poles are excluded and the denominator has no zeros. ■

Since ρ_{opt2} is analytic we can again apply the maximum principle to find the maximum of $|\rho_{opt2}(s)|$ for $s = \sigma + i\omega$, $\sigma > 0$ on the boundary at $\sigma = 0$. This yields the optimization problem

$$\min_{\alpha_0 \geq 0, \alpha_1 > 0} \left(\max_{\omega_{\min} < |\omega| < \omega_{\max}} |\rho_{opt2}(i\omega, a, b, \alpha_0, \alpha_1)| \right), \tag{2.35}$$

where we again truncated the frequency range by a minimal and maximal frequency relevant for our problem. For our particular example in section 2.5 where we take $\omega_{min} = \frac{\pi}{20}$ and $\omega_{max} = 20\pi$, we find that the optimal solution for the min-max problem (2.35) is $\alpha_0^* = 0.1757$ and $\alpha_1^* = 0.6557$, by using a multidimensional unconstrained nonlinear minimization routine (Nelder-Mead). The Nelder-Mead algorithm was first published in 1965, it is a very popular direct search method for multidimensional unconstrained nonlinear minimization, it minimizes a scalar-valued nonlinear function of n real variables using only function values, without any derivative information. One can find more details about this method in many references, e.g. [24, 25]. We compare the convergence rates for the classical WR, optimized WR with $\alpha^* = 0.73455$ and optimized WR with $\alpha_0^* = 0.1757$, $\alpha_1^* = 0.6557$; we can see that the best method is the optimized WR with first order transmission conditions, i.e., with α_0 and α_1, Figure 2.2. The optimized convergence rate, $|\rho_{opt2}(\alpha_0^*, \alpha_1^*, \omega)|$, takes the smallest values which means a faster convergence in comparison with the classical WR and the optimized WR with α^* algorithms. In addition, the convergence rate is much more uniform.

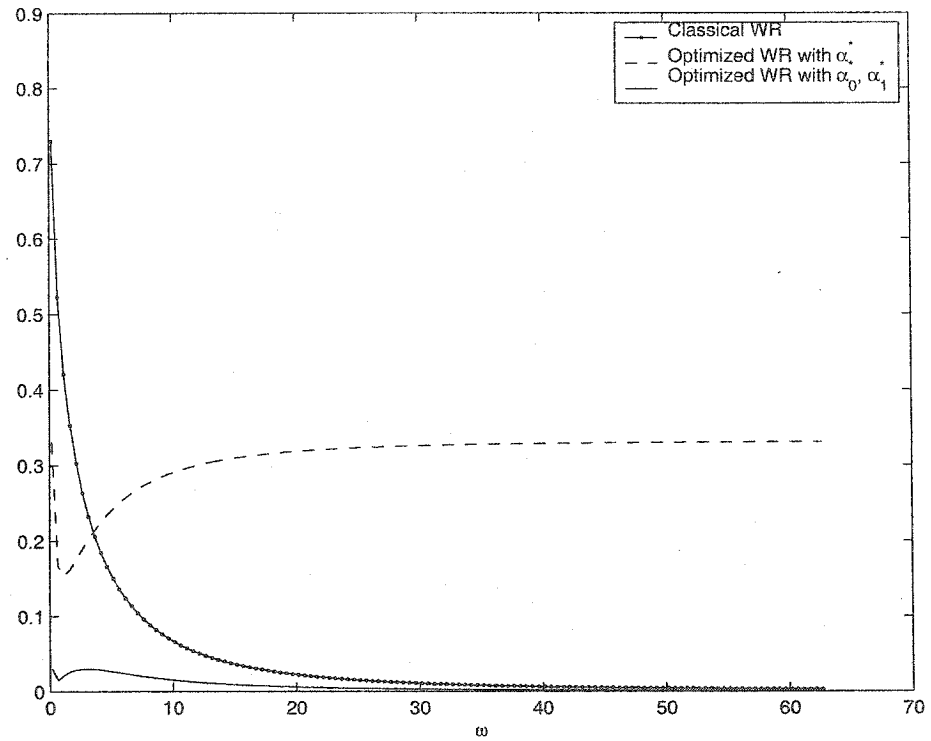

Figure 2.2: Classical rate $|\rho_{cla}(\omega)|$, Optimized rate $|\rho_{opt1}(\alpha^*, \omega)|$, and Optimized rate $|\rho_{opt2}(\alpha_0^*, \alpha_1^*, \omega)|$.

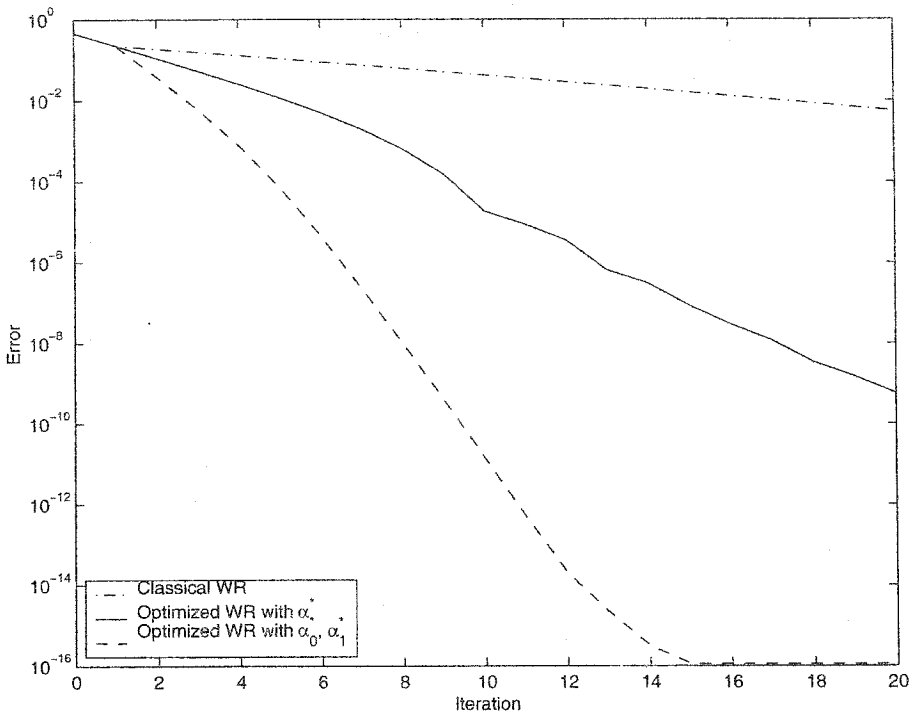

Figure 2.3: Convergence rates of classical versus optimized WR.

2.5 Numerical Experiments

For the large circuit we give a numerical example for a circuit with 10 nodes which can be extended to any number of nodes with the same parameters as we did for the small circuit,

$$R_s = \frac{1}{2}, \quad R_i = \frac{1}{2}, \quad i = 1, \ldots, 9; \qquad C_i = \frac{63}{100}, \quad i = 1, \ldots, 10.$$

We use again for the numerical computations the forward Euler method and our time is $t \in [0, 20]$ with a time step $\Delta t = \frac{1}{20}$. We start with zero initial waveforms as well as zero initial conditions and use an input step function with an amplitude of $I_s = 1$ and a rise time of 1 time unit. The difference in convergence between the classical, the optimized WR with $\alpha^* = 0.73455$, and the optimized WR with α_0^*, α_1^* algorithms is illustrated in Figure 2.3, which shows the error as a function of the iteration. In Figure 2.4 we varied α_0 and α_1 and computed the error after 4 iterations of the new optimized WR algorithm. We can see that the analytical

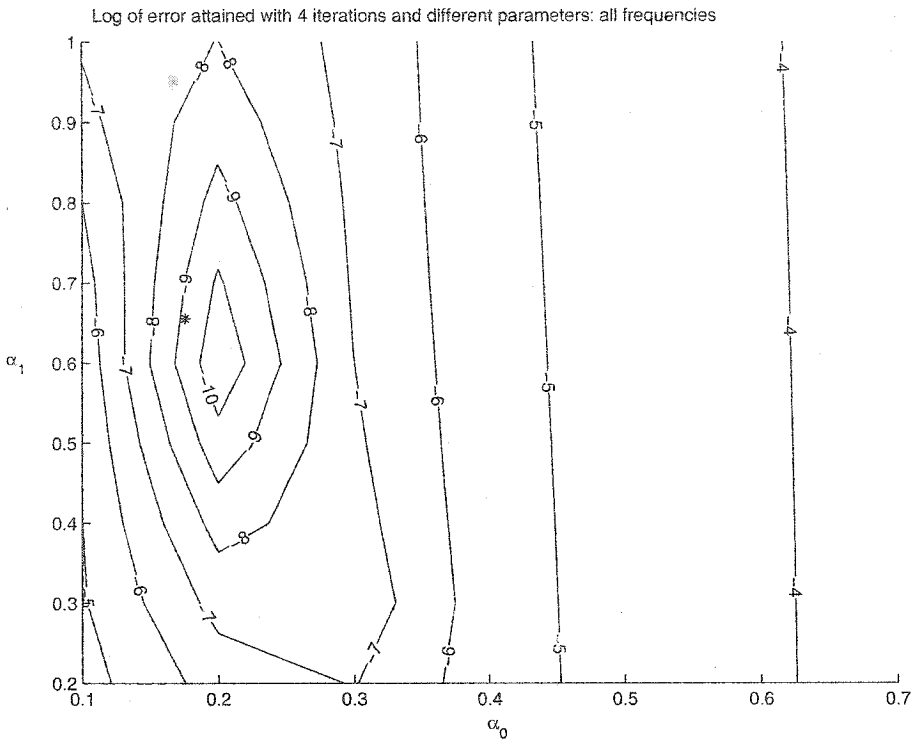

Figure 2.4: Numerical and analytical optima compared.

parameters obtained from the optimization process in section 2.4 are close to the optimal numerical ones.

Conclusions

The optimized waveform relaxation algorithm has a very uniform convergence in very few iterations, which is the key advantage of this new algorithm over the classical waveform relaxation algorithm which has a slow and non-uniform convergence over the time interval for which the equations are integrated. This remarkable improvement and great performance is achieved by new transmission conditions responsible for the exchange of both voltages and currents, or derivatives between the sub-circuits rather than just voltages as in the classical waveform relaxation algorithm. The optimized waveform relaxation algorithm is not complicated and can easily be implemented by only changing the few lines in the waveform relaxation code responsible for the transmission conditions. By considering the infinitely large circuit, we have shown that the size of the circuit does not have a major impact on the convergence of the optimized waveform relaxation method. In the small circuit case, the optimal waveform relaxation converges to the solution in two iterations, and in the large circuit case, a faster and more uniform convergence is obtained by taking a higher order expansion for the optimal parameters in the new transmission conditions. A numerical experiment for each case is given which confirms the theoretical results.

Future work will involve analyzing and proving the optimality of the constants, α_0 and α_1, in the first order transmission conditions, RC circuits connected in two dimensions, transmission line type circuits, and mixed circuits.

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