CALL ADMISSION CONTROL AND ROUTING CONTROL IN INTEGRATED COMMUNICATION NETWORKS VIA DYNAMIC PROGRAMMING

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ABSTRACT

The problems of Connection-oriented Networking Call Admission Control (CAC) and Routing Control (RC) in Integrated Networks are formulated as finite and infinite horizon finite-state Stochastic Dynamic Programs. In particular, Poisson Markovian communication networks are analysed in detail. Because of the complexity of communication networks and of the operation of some kind of communication networks, such as the Internet, by multi-agents, it is effectively impossible to obtain the optimal solutions. Currently it is reasonable to use decentralized aggregation methods to obtain sub-optimal solutions for CAC and RC communication problems. In this thesis, stochastic dynamic programming methods for the optimal control of such network are studied. The notion of a doubly stochastic network, which possesses Markovian aggregated dynamics, is introduced, this is exploited in the hierarchical stochastic control of such hierarchical networks.

RÉSUMÉ

Les problèmes de Contrôle d'appels (CAC) des Réseaux du contrôle de cheminement (RC) dans les réseaux intégrés sont formulés en tant que programmes dynamiques stochastiques d'etats finis, finis et infinis. En particulier, les réseaux de transmission markoviens de Poisson sont analysés en détail. En raison de la complexité des réseaux de transmission, il est évidemment impossible d'obtenir les solutions optimales. En pratique, il est raisonnable d'employer des méthodes décentralisées d'agrégation pour obtenir les solutions suboptimales pour des problèmes de CAC et de transmission de RC. Dans cette thèse, on étudie des méthodes de programmation dynamiques stochastiques pour la commande optimale d'un tel réseau. En particulier, on présente la notion d'un réseau doublement stochastique possédant la dynamique agrégée Markovienne; cette notion étant exploiteé dans la commande stochastique hiérarchique de tels réseaux.

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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

Introduction

Due to the complexity of communication networks, the Call Admission Control(CAC) and Routing Control(RC) problems have been formulated as stochastic control problems for which a large number of suboptimal solutions have been proposed and partially analyzed ([3],[4],[8]). Indeed, because of the complexity of communication networks, it is effectively impossible to implement the solutions to the underlying optimal stochastic control problems. Among the recent analyses of interest, Dziong and Mason ([3]) obtained suboptimal solutions by assuming the statistical independence of each link in the communication network; Marbach *et al.* ([4]) obtained the approximation result by the method of neuro-dynamic programming where their method for multi-link networks is actually an extension of one for the single-link case.

In this thesis, we connect the fundamental optimal control theory of point processes, which dates back to 1970s ([5],[6],[7]), to stochastic communication network control problems. This work concerns centralized network control depending upon observations of the entire state of the system. In fact, in practical complex communication networks, this centralized optimal control method with full state observation cannot be implemented. Here, we provide a novel hierarchical control mechanism. The theoretical and computational consequences of this method are very significant; they lead to problems of global state estimation and approximation from local data, and, in particular, lead to the use of state aggregation methodologies ([4]). In this thesis, we introduce the doubly hierarchical stochastic networks where the local state processes are randomized at specific instants, *i.e.* high level event instants, and the consequential extended high level state processes are Markov processes. Furthermore, the sub-optimal solutions to the CAC and RC control problems for original networks are obtained by hierarchical control methods for approximate doubly hierarchical stochastic networks.

The thesis is organized as follows. In Chapter 2, we present a formal definition of a communication network and formulate the optimal CAC and RC problems; in Chapter 3, we present the construction of tractable Poisson (call request and connection departure processes) Markovian network models within the general framework of Chapter 2; in Chapter 4, the hierarchical CAC and RC problems in complex communication networks are established; Chapter 5 concerns the conclusions and future work.

SYMBOLS

$\mathbb{Z} \triangleq \{\cdots, -1, 0, 1, 2, \cdots\}$, i.e. the set of integers.

$\mathbb{Z}_+ \triangleq \{0, 1, 2, \cdots\}$, i.e. the set of positive integers.

$\mathbb{N} \triangleq \{1, 2, 3, \cdots\}$, i.e. the set of natural numbers.

$\mathbb{N}_- \triangleq \{-1, -2, \cdots\}$, i.e. the set of negative natural numbers.

CHAPTER 2

Formulation of CAC and RC Problems

2.1 Communication Networks

Definition 2.1 A *network*, or *graph*, $Net(\mathcal{N}, \mathcal{L})$ consists of a set of nodes $\mathcal{N} = \{n_1, \cdots, n_N\}$, $N \in \mathbb{N}$ and a set of links $\mathcal{L} = \{l_1, \cdots, l_L\}, L \in \mathbb{N}$, where each link $l \in \mathcal{L}$ is an ordered pair $(n, m) \in \mathcal{N} \times \mathcal{N}$ of distinct nodes.
A *network* $Net(\mathcal{N}, \mathcal{L})$ *with (link) capacities* $C = \{C_s : 1 \leq s \leq L, C_s \in \mathbb{Z}_+\}$, shall be denoted by $Net(\mathcal{N}, \mathcal{L}, C)$.

□

Definition 2.2 A *connection*, r in the network $Net(\mathcal{N}, \mathcal{L})$, connecting a node $x \in \mathcal{N}$ to a node $y \in \mathcal{N}$ is a finite sequence of nodes $r = (m_1, \cdots, m_k)$, such that

$$m_1 = x, \; m_k = y,$$

$$m_i \neq m_j, \; \text{for } i \neq j,$$

$$(m_i, m_{i+1}) \in \mathcal{L}, \; \text{for } i = 1, \cdots, k-1.$$

The *set of connections* in the network $Net(\mathcal{N}, \mathcal{L})$ is denoted by $\mathcal{R}$, and we set $R = |\mathcal{R}|$, the cardinality of $\mathcal{R}$.

□

Figure (2.1) is an illustration of connections in a communication network.

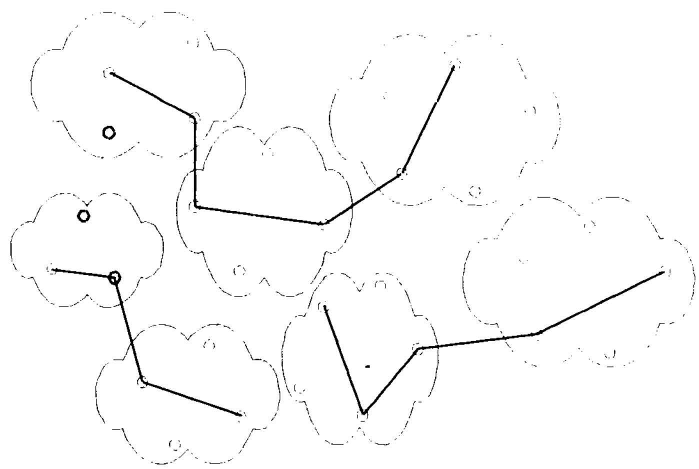

Figure 2.1: Three distinct connections in a communication network

Definition 2.3 The *state set* X (of *admissible sets* of *connections*) in $\mathcal{R}$ in the network with capacities $Net(\mathcal{N}, \mathcal{L}, C)$, is defined to be

$$X = \{x = (x_r) \in \mathbb{Z}_+^R : \sum_{r \in \mathcal{R}; l_s \in r} x_r \leq C_s, \ \forall s, 1 \leq s \leq L\}$$

□

We observe that in the definition of X, for each fixed l_s, the set of $r \in \mathcal{R}$ appearing in the sum is the set of connections each of which contains l_s as a link.
Since the connections in $\mathcal{R}$ are in one-to-one correspondence with the index of the components of a vector in $\mathbb{Z}_+^R \subset \mathbb{R}^R$, we shall by abuse of notation let $r \in \mathcal{R}$ also denote the integer indexing the corresponding vector component in $\mathbb{R}^R$.

Definition 2.4 For the network with capacities $Net(\mathcal{N}, \mathcal{L}, C)$, the *call (request) and (connection) departure event set*, E, is defined as:

$$E = \left\{\emptyset, e^+_{od}, e^-_r;\ \forall o, d \in \mathcal{N}, o \neq d, \forall r \in \mathcal{R}\right\},$$

where $e^+_{od} \in E$ corresponds to an ordered pair (o, d), origin/destination pair, of distinct nodes in $\mathcal{N} \times \mathcal{N}$, and $e^-_r \in E$ corresponds to a connection r in $\mathcal{R}$.

□

We interpret $e^+_{od} \in E$ as a call request from $o \in \mathcal{N}$ to $d \in \mathcal{N}$, i.e. a request for a connection in $Net(\mathcal{N}, \mathcal{L}, C)$ between o and d; $e^-_r \in E$ is interpreted as the departure of a previously established connection r between the initial and terminal nodes of r.

2.2 Stochastic Dynamics and Control

For a network with capacities $Net(\mathcal{N}, \mathcal{L}, C)$, a mapping $x : [0, T] \in \mathbb{R}_+ \to X$ constitutes a *state process trajectory* $x_t \in X$ for all $t, 0 \leq t \leq T \leq \infty$.

2.2.1 Control Set

Definition 2.5 For the network with capacities $Net(\mathcal{N}, \mathcal{L}, C)$, the *control set* U is specified by:

$$U = \{0\} \cup U^+ \cup U^- = \{0\} \cup \{1_r;\ \forall r \in \mathcal{R}\} \cup \{-1_r;\ \forall r \in \mathcal{R}\},$$

where, 1_r is a vector in the R dimensional space $\mathbb{R}^R$ with unit entry in the r-th position, and correspondingly, -1_r has an entry -1 in the r-th position.

□

For a call (request) event $e_{od}^{+} \in E$, a control action assigning the call to a connection in $\{\emptyset\} \cup \mathcal{R}$ is a mapping $u : E \rightarrow \{0\} \cup U^{+} \in \mathbb{Z}_{+}^{R}$, such that either $u \equiv u_{od}^{+} \equiv u(e_{od}^{+}) = 1_r$ for some $r \in \mathcal{R}$, where r has initial node o and terminal node d, or $u \equiv u_{\emptyset}^{+} \equiv u(e_{od}^{+}) = 0$; the former is termed as the *(controlled) acceptance* and *assignment* of the call request e_{od}^{+} to r, and the latter as a *(controlled) rejection* of the call request e_{od}^{+}.
Similarly at state $x \in X$, for a *(controlled) connection departure event* $e_r^{-} \in E$ occurence, a control mapping: $u : X \rightarrow U^{-} \in \mathbb{Z}_{-}^{R}$, such that $u = -1_r$ where $r \in x$.

We now introduce the stochastic framework for the analysis in this paper and henceforth we assume that all stochastic processes shall be defined on the underlying (filtered) probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{F}_s \subset \mathcal{F}_t, 0 \leq s \leq t, P)$.

2.2.2 Event Instants and Event Process

Definition 2.6 We term a sequence of (deterministic or stochastic) instants in $\mathbb{R}_{+}$

$$0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots$$

at which call and departure events occur as a *sequence of event instants*, $t : \mathbb{N}_{+} \rightarrow \mathbb{R}_{+}$.

□

Before giving the formal definition of the set of state processes x in $Net(\mathcal{N}, \mathcal{L}, C)$ we declare that a typical state process in X evolves in the following way: At any instant $t_k, k \geq 1$, at which some event e occurs a control action $u \equiv u_{t_k} \in U$ is instantaneously selected according to the pre-assigned control law (*i.e.* set of control responses) U. Then the state (value) $x_{t_k^-}$ is instantaneously transformed into the state (value) $x_{t_k^-} + u_{t_k}$. It then remains unchanged until the next event occurs at the instant t_{k+1}. In particular, this means the state equation is right continuous at the instant t.

Definition 2.7 We term a sequence of event instants $t(\omega)$ in $\mathbb{R}_{+}$

$$0 < t_1(\omega) < t_2(\omega) < \cdots < t_k(\omega) < t_{k+1}(\omega) < \cdots, (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P), \omega \in \Omega$$

at which random call request and connection departure events occur as a *sequence of random event instants* $t : \mathbb{N}_+ \times \Omega \longrightarrow \mathbb{R}_+$. The sequence $\tau : \mathbb{N}_+ \times \Omega \longrightarrow \mathbb{R}_+$, with $\tau \equiv t_k(\omega) - t_{k-1}(\omega)$, where $t_0(\omega) \triangleq 0$ is defined as the *sequence of event intervals* (associated to $t(\omega)$).

□

Evidently, we obtain that $t_k = \sum_{i=1}^{k} \tau_i$. We call an instant $t_k(\omega) \in \mathbb{R}_+$ at which a call request event occurs as a *random call request instant* t_k^+; and similarly, an instant $t_k(\omega) \in \mathbb{R}_+$ at which a (random) departure occurs a *random connection departure instant*, t_k^-.

Definition 2.8 We define the *event process* $e(t, \omega)$ as a stochastic process $e : \mathbb{R}_+ \times \Omega \rightarrow E$.

□

2.2.3 Control Laws

In this subsection, we will investigate a series of distinct Control Laws we are interested in. Information state is denoted as $w_t, t \in [s, T]$ without further analysis right now, and in the latter part of this section it will be analysed. Furthermore, $\{\mathcal{F}_t^w \subset \mathcal{F}_t; t \in [s, T]\}$ denote the sub-σ-field of $\mathcal{F}_t$ generated by the information state process $w_t, t \in \mathcal{R}_+$.

(1) control with full observations of past information;

(2) control with partial observations of past information;

(3) control with full observations of current information.

Definition 2.9 The set of *measurable control laws with full observations of past information* is denoted as $\mathcal{U}[s,T], s < T < \infty$,

$$\mathcal{U}[s,T] = \{u : [s,T] \times \Omega \to U;\ \text{ s.t. } u_t(.) \text{ is } \mathcal{F}^w_t \text{ measurable}, t \in [s,T]\}$$
$$\mathcal{U}[s,\infty) = \cup_{T \geq s} \mathcal{U}[s,T]$$

□

Definition 2.10 The set of *measurable control laws with partial observations of past information* is denoted as $\mathcal{U}^{\mathcal{G}}[s,T], s < T < \infty$, where $\mathcal{G}_r \subset \mathcal{G}_t \subset \mathcal{F}^w_t; s \leq r \leq t \leq T$,

$$\mathcal{U}^{\mathcal{G}}[s,T] = \{u : [s,T] \times \Omega \to U;\ u_t(\omega) \text{ s.t. } u_t(.) \text{ is } \mathcal{G}_t \text{ measurable}, t \in [s,T]\}$$
$$\mathcal{U}^{\mathcal{G}}[s,\infty) = \cup_{T \geq s} \mathcal{U}^{\mathcal{G}}[s,T]$$

□

Definition 2.11 The set of *measurable control laws with full observations of current information* also called *Markovian Control* denoted as $\mathcal{U}^M[s,T], s < T < \infty$,

$$\mathcal{U}^M[s,T] = \{u : [s,T] \times \Omega \to U;\ \text{ s.t. } u_t(.) \text{ is } \sigma(x^-_t, e_t) \text{ measurable}, t \in [s,T]\}$$
$$\mathcal{U}^M[s,\infty) = \cup_{T \geq s} \mathcal{U}^M[s,T]$$

□

We now specify the controlled stochastic dynamics of a state process x in $Net(\mathcal{N}, \mathcal{L}, C)$ subject to (random) call request events, (random) connection departure events, and subject to some specified control law just defined here, *i.e.* $u \in \mathcal{U}[s,T]$ (or $\mathcal{U}^{\mathcal{G}}[s,T]$, $\mathcal{U}^M[s,T]$).

2.2.4 State Transition Equation

Definition 2.12 We define the function $\mathbb{H}_{t_k} : \mathbb{R}_+ \to \{0,1\}$, where $t_k, k \in \mathbb{N}$ is the instant at which k-th event occurs, as

$$\mathbb{H}_{t_k}(t) = \begin{cases} 1, & t_k \leq t \\ 0, & t < t_k \end{cases} \tag{2.2.1}$$

Definition 2.13 The *state response* or *transition equation*, with the control law $u \in \mathcal{U}[s,T]$ (or $\mathcal{U}^{\mathcal{G}}[s,T]$, $\mathcal{U}^M[s,T]$), for the evolution of the state process $x : [s,T] \times \Omega \to X$ with initial state $x_s = \xi$, $0 \leq s < T < \infty$, where ξ is $\mathcal{F}_s^w$ measurable, is given by

$$x_t = \xi + \sum_{k=1}^{\max\{i;\, t_i \leq t\}} u_{t_k} \mathbb{H}_{t_k}(t) \tag{2.2.2}$$

where *the event instants satisfy* $s < t_1 < \cdots < t_{k-1} < t_k < t_{k+1} < \cdots < T, a.s.$ and where *the control law*

$u = u_{t_k}, k = 1, \cdots, \max\{i;\ t_i < t\}$ satisfy:

$$(1) \begin{cases} u \in \{0, 1_r : r = (m_1, \cdots, m_j) \in \mathcal{R}, m_1 = o, m_j = d\}, & \text{if at } t_k, e_{od}^+ \in E \text{ occurs} \\ u = -1_r, \text{ subject to } r \in x_{t_k^-}, & \text{if at } t_k, e_r^- \in E \text{ occurs} \end{cases}$$

(2) $x_{t_k} + u \in X$.

and where in general we say event processes e can be dependent on past information history, *i.e.* $e_t = e_t(x_s^{t-}, e_s^{t-})$

□

Definition 2.14 *We consider Network System with respect to* $Net(\mathcal{N}, \mathcal{L}, C)$ *is denoted as* $\mathcal{NS} \equiv \{Net(\mathcal{N}, \mathcal{L}, C); \mathcal{S}, \mathcal{U}\}$

□

We shall write (2.2.2) in *state space transition form* as $x_t = x(x_s, u_s^t), t \in [s, T]$, where $u \in \mathcal{U}[s,T]$ (or $\mathcal{U}^{\mathcal{G}}[s,T]$, $\mathcal{U}^M[s,T]$).

The following **(E1)**,**(E2)** and **(E3)** are equivalent to Definition (2.13)

(E1)

$$x_t = x_{t^-} + u_t, \qquad \forall\, t \in [s, T] \tag{2.2.3}$$

$$e_t = e_t(x_s^{t^-}, e_s^{t-}) \tag{2.2.4}$$

□

(E2)

$$x_{t_{k+1}} = x_{t_k} + u_{t_{k+1}}, \qquad x_{t_0} = x_s = \xi, \text{ and } t_{k+1} \in [s, T];\ \forall k \in N \tag{2.2.5}$$

$$x_t = x_{t_k}, \qquad t_k \leq t < t_{k+1} \tag{2.2.6}$$

□

(E3) We can express the state transition equation in term of stochastic differential equation (SDE) form with event process inputs $e_t; t \in [s, T]$,

$$dx_t = u_t(x_{t^-}, de_t), \qquad t \in [s, T] \tag{2.2.7}$$

with initial state $x_s = \xi$.
or, in integral form:

$$x_t = \xi + \int_s^t u_t(x_{t^-}, de_t), \qquad t \in [s, T] \tag{2.2.8}$$

□

With some admissible control $u \in \mathcal{U}[s,T]$ (or $\mathcal{U}^{\mathcal{G}}[s,T]$, $\mathcal{U}^M[s,T]$), from Definition (2.13), we get the sample path of state process $x_t; x_t \in X, 0 \leq s \leq t \leq \infty$, please see Figure (2.2).

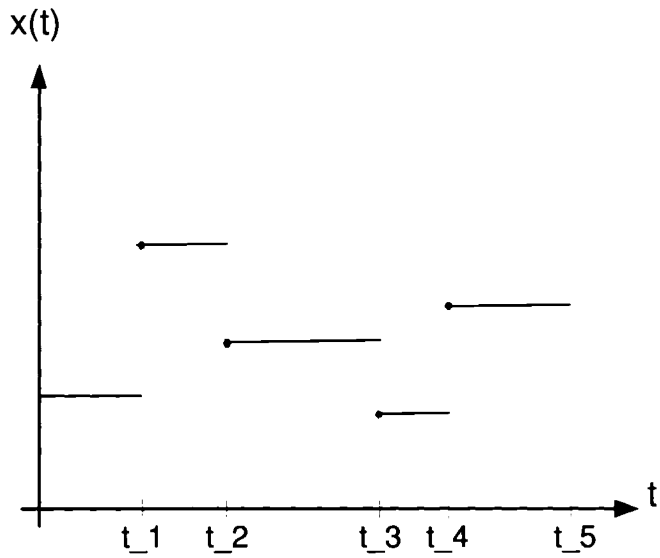

Figure 2.2: Sample path of state process

Remarks: The states in $X \equiv \{x_1, x_2, \cdots, x_{|X|}\}$ are in one-to-one correspondence with $\{1, 2, \cdots, |X|\}$, *i.e.* in Figure (2.2), $x(t) = n, n \in \{1, 2, \cdots, |X|\}$, just means $x(t) = x_n$.

Please see Figure (2.3), the state transition process.

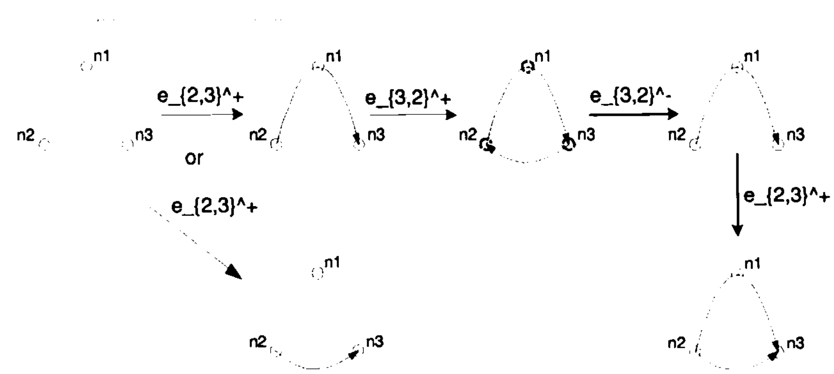

Figure 2.3: State transition process

From Definition (2.13), it may be seen that the state process x is everywhere right continuous, *i.e.* continuous from the right and is constant between the countable set of jump times ([5],[6]); hence x also has limits from the left everywhere. We shall be interested in the so-called *Markovian control* which is a function of the current state x_{t^-} and the event e_t.

Since we wish to have a Markovian description of the system in this case, consequently we introduce the notion of an information state transition form for the system.

2.2.5 Information State Process

Definition 2.15 The *information state transition equation* with information state $w : [s,T] \times \Omega \to \binom{X}{E}$, consists of the information state $w_t = \binom{x_{t^-}}{*}$, where $* = 0$ if t is not an event instant $\{t_k\}$ and $* = e_{t_k}$ at the kth event instant t_k. The system dynamics, with control law $u \in \mathcal{U}[s,T]$ (or $\mathcal{U}^{\mathcal{G}}[s,T], \mathcal{U}^M[s,T]$), have the representation:

$$w_t = w_{t_{k-1}^+}, \qquad t_{k-1} < t < t_k \tag{2.2.9}$$

$$w_{t_k} = \binom{x_{t_k^-}}{e_{t_k}} \tag{2.2.10}$$

$$w_{t_k^+} = \begin{pmatrix} I_R & 0 \\ 0 & 0 \end{pmatrix} w_{t_k} + \binom{I_R}{0} u_{t_k} \tag{2.2.11}$$

where the initial condition for the system is $w_s = \binom{\xi}{0}$.

□

Hence we see that:

$$w_t = w_{t_{k-1}^+}, \qquad t_{k-1} < t < t_k \tag{2.2.12}$$

$$w_{t_k} = \binom{x_{t_k^-}}{e_{t_k}} = \binom{x_{t_{k-1}^+}}{e_{t_k}} = \binom{x_{t_{k-1}}}{e_{t_k}} \tag{2.2.13}$$

$$w_{t_k^+} = \binom{x_{t_k^-} + u_{t_k}}{0} \tag{2.2.14}$$

Please see Figure (2.4) or Figure (2.5), sample path of information state process.

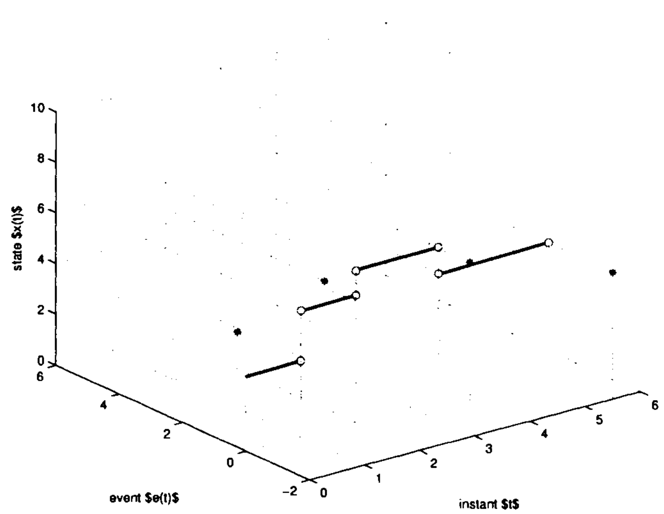

Figure 2.4: Sample path of information state process

2.3 Framework for Dynamic Programming

In this paper, we only consider the following situation:

(S1) the *(state conditionally) independent interval* and *event* case, that is to say, the case where x_s and $\left\{\binom{\tau_k}{e_{t_k}}\right\}_{k=1}^{\infty}$ are (state x_{t_k} conditionally) independent, and

$$\left\{x_s, \binom{\tau_0}{e_{t_0}}, \cdots, \binom{\tau_k}{e_{t_k}}\right\} \coprod{}_{x_{t_k}} \left\{\binom{\tau_{k+1}}{e_{t_{k+1}}}, \cdots\right\}, \qquad k \in \mathbb{N} \tag{2.3.1}$$

In other words,

$$P\Big(\binom{\tau_{k+1}}{e_{t_{k+1}}} \Big| \, \sigma(x_{t_k}), \sigma(x_s), \sigma\Big\{\binom{\tau_i}{e_{t_i}}_{i=1}^{k}\Big\}\Big) = P\Big(\binom{\tau_{k+1}}{e_{t_{k+1}}} \Big| \, \sigma(x_{t_k})\Big), \qquad k \in \mathbb{N} \tag{2.3.2}$$

$$P\Big(\binom{\tau_k}{e_{t_k}} \Big| \, \sigma(x_{t_{k+1}}), \sigma\Big\{\binom{\tau_i}{e_{t_i}}_{i=k+1}^{\infty}\Big\}\Big) = P\Big(\binom{\tau_k}{e_{t_k}} \Big| \sigma(x_{t_k+1})\Big), \qquad k \in \mathbb{N} \tag{2.3.3}$$

Lemma 2.1 With assumption **(S1)**, event process $\{e_t\}$ is state conditionally independent of past state process, *i.e.*

$$P\big(e_{t_k} | \{\sigma(x_r)\}_{r=s}^{t}\big) = P\big(e_{t_k} | \sigma(x_t)\big), \qquad t_{k-1} < t < t_k$$

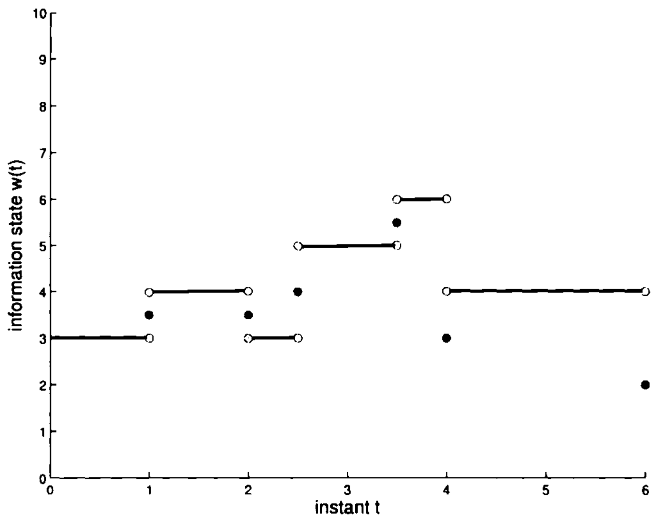

Figure 2.5: Another indication of sample path of information state process

Proof:

$$x_{t_1} = x_s + u_{t_1}(x_s, e_{t_1}) = x(\xi, e_{t_1});$$
$$x_{t_2} = x_{t_1} + u_{t_2}(x_{t_1}, e_{t_2}) = x(\xi, e_{t_1}, e_{t_2});$$
$$x_{t_3} = x_{t_2} + u_{t_3}(x_{t_2}, e_{t_3}) = x(\xi, e_{t_1}, e_{t_2}, e_{t_3});$$
$$\dots$$
$$x_{t_i} = x_{t_{i-1}} + u_{t_i}(x_{t_{i-1}}, e_{t_i}) = x(\xi, e_{t_1}, e_{t_2}, \cdots, e_{t_i});$$

$$\begin{aligned} & P\big(e_{t_k} | \{\sigma(x_r)\}_{r=s}^{t}\big), \qquad t_{k-1} < t < t_k \\ =& P\big(e_{t_k} | \sigma(x_t), \sigma(x_{t_{k-2}}), \cdots, \sigma(x_{t_2}), \sigma(x_{t_1})\big), \\ =& P\big(e_{t_k} | \sigma(x_t), \sigma(x(\xi, e_{t_1}, e_{t_2}, \cdots, e_{t_{k-2}})), \cdots, \sigma(x(\xi, e_{t_1}, e_{t_2})), \sigma(x(\xi, e_{t_1}))\big), \\ =& P\big(e_{t_k} | \sigma(x_t)\big), \qquad \text{by assumption } \textbf{(S1)} \end{aligned}$$

□

Theorem 2.1 With assumption **(S1)**, process $\{x_t\}$ and $\{w_t\}$ are Markovian processes.

Proof:

First we consider the control law $u \in \mathcal{U}^M[s,T]$, *i.e.* control $u_t = u_t(w_t)$ is the function of full observations of current information.

(1) With assumption **(S1)**, $\{x_t\}$ is a Markov process.

Consider any $s \leq t \leq t + \hat{s} \leq T$.

First, we suppose $t_{k-1} < t + \hat{s} < t_k$, for some $k \in \mathbb{N}$ and $t_{k-(m+1)} < t < t_{k-m}$, for some $m \in \{0, \mathbb{N}\}$,

$$
\begin{aligned}
x_{t+\hat{s}} &= x_{t_{k-1}} \\
&= x_{t_{k-2}} + u_{t_{k-1}}(x_{t_{k-2}}, e_{t_{k-1}}) \\
&= x_{t_{k-3}} + u_{t_{k-2}}(x_{t_{k-3}}, e_{t_{k-2}}) + u_{t_{k-1}}\big(x_{t_{k-3}} + u_{t_{k-2}}(x_{t_{k-3}}, e_{t_{k-2}}), e_{t_{k-1}}\big) \\
&= x_{t_{k-(m+1)}} + x(x_{t_{k-(m+1)}}, \{e_{t_{k-i}}\}_{i=1,\cdots,m}) \\
&= x_t + x(x_t, \{e_{t_{k-i}}\}_{i=1,\cdots,m}), \qquad x_t = x_{t_{k-(m+1)}}, \quad t_{k-(m+1)} < t < t_{k-m}
\end{aligned}
$$

We denote the space 2^X as $\mathcal{X}$ ($\mathcal{X}$ is a σ-field of set X), for any $\Gamma \in \mathcal{X}$,

$$
\begin{aligned}
&P\Big(x_{t+\hat{s}} \in \Gamma | \{\sigma(x_r)\}_{r=s}^{t}\Big) \\
=&P\Big(x_t + x(x_t, \{e_{t_{k-i}}\}_{i=1,\cdots,m}) \in \Gamma | \{\sigma(x_r)\}_{r=s}^{t}\Big) \\
=&P\Big(x_t + x(x_t, \{e_{t_{k-i}}\}_{i=1,\cdots,m}) \in \Gamma | \sigma(x_t)\Big), \qquad \text{by Lemma(2.1)} \\
=&P\Big(x_{t+\hat{s}} \in \Gamma | \sigma(x_t)\Big)
\end{aligned}
$$

In the same way, we can prove in the following cases, $t_{k-1} < t + \hat{s} < t_k$, for some $k \in \mathbb{N}$ and $t = t_{k-m}$, for some $m \in \{0, \mathbb{N}\}$

$t + \hat{s} = t_k$, for some $k \in \mathbb{N}$ and $t_{k-(m+1)} < t < t_{k-m}$, for some $m \in \{0, \mathbb{N}\}$

$t + \hat{s} = t_k$, for some $k \in \mathbb{N}$ and $t = t_{k-m}$, for some $m \in \{0, \mathbb{N}\}$

$$
P\Big(x_{t+\hat{s}} \in \Gamma | \{\sigma(x_r)\}_{r=s}^{t}\Big) = P\Big(x_{t+\hat{s}} \in \Gamma | \sigma(x_t)\Big)
$$

still holds. That is to say $\{x_t\}$ is Markov process.

(2) With assumption **(S1)**, $\{w_t\}$ is a Markov process.

Consider any $s \leq t \leq t + \hat{s} \leq T$.

First, we suppose $t_{k-1} < t + \hat{s} < t_k$, then for some $k \in \mathbb{N}$ and $t_{k-(m+1)} < t < t_{k-m}$, and for some $m \in \{0, \mathbb{N}\}$,

$$
\begin{aligned}
w_{t+\hat{s}} &= w_{t_{k-1}^+} \\
&= Aw_{t_{k-1}} + Bu_{t_{k-1}}(w_{t_{k-1}}) \\
&= A\Big(w_p + \big(\begin{smallmatrix} 0 \\ e_{t_{k-1}} \end{smallmatrix}\big)\Big) + Bu_{t_{k-1}}\Big(w_p + \big(\begin{smallmatrix} 0 \\ e_{t_{k-1}} \end{smallmatrix}\big)\Big) \\
&\qquad \Big(\text{since } w_{t_{k-1}} = w_p + \big(\begin{smallmatrix} 0 \\ e_{t_{k-1}} \end{smallmatrix}\big)\Big), \qquad t_{k-2} < p < t_{k-1} \\
&= Aw_p + Bu_{t_{k-1}}\Big(w_p + \big(\begin{smallmatrix} 0 \\ e_{t_{k-1}} \end{smallmatrix}\big)\Big) \\
&= Aw_r + Bu_{t_{k-2}}\Big(w_r + \big(\begin{smallmatrix} 0 \\ e_{t_{k-2}} \end{smallmatrix}\big)\Big) + Bu_{t_{k-1}}\Big(Aw_r + \big(\begin{smallmatrix} 0 \\ e_{t_{k-1}} \end{smallmatrix}\big) + Bu_{t_{k-2}}\big(w_r + \big(\begin{smallmatrix} 0 \\ e_{t_{k-2}} \end{smallmatrix}\big)\big)\Big), \\
&\qquad \Big(\text{since } w_{t_{k-2}} = w_r + \big(\begin{smallmatrix} 0 \\ e_{t_{k-2}} \end{smallmatrix}\big)\Big), \qquad t_{k-3} < r < t_{k-2} \\
&= Aw_q + Bu_{t_{k-3}}\Big(w_q + \big(\begin{smallmatrix} 0 \\ e_{t_{k-3}} \end{smallmatrix}\big)\Big) + Bu_{t_{k-2}}\Big(Aw_q + \big(\begin{smallmatrix} 0 \\ e_{t_{k-2}} \end{smallmatrix}\big) + Bu_{t_{k-3}}\big(w_q + \big(\begin{smallmatrix} 0 \\ e_{t_{k-3}} \end{smallmatrix}\big)\big)\Big) \\
&\qquad + Bu_{t_{k-1}}\Bigg(Aw_q + \big(\begin{smallmatrix} 0 \\ e_{t_{k-1}} \end{smallmatrix}\big) + Bu_{t_{k-3}}\big(w_q + \big(\begin{smallmatrix} 0 \\ e_{t_{k-3}} \end{smallmatrix}\big)\big) \\
&\qquad\qquad + Bu_{t_{k-2}}\Big(Aw_q + \big(\begin{smallmatrix} 0 \\ e_{t_{k-2}} \end{smallmatrix}\big) + Bu_{t_{k-3}}\big(w_q + \big(\begin{smallmatrix} 0 \\ e_{t_{k-3}} \end{smallmatrix}\big)\big)\Big)\Bigg) \\
&\qquad \Big(\text{since } w_{t_{k-3}} = w_q + \big(\begin{smallmatrix} 0 \\ e_{t_{k-3}} \end{smallmatrix}\big)\Big), \qquad t_{k-4} < q < t_{k-3} \\
&= Aw_t + w\big(w_t, \{e_{t_{k-j}}\}_{j=1,\cdots,m}\big), \\
&\qquad s < t_1 < \cdots < t_{k-(m+1)} < t < t_{k-m} < \cdots < t + \hat{s}, \hat{s} \geq 0 \text{ (by induction)}
\end{aligned}
\tag{2.3.4}
$$

We denote the space $2^{X\times E}$ as $\mathcal{W}$ ($\mathcal{W}$ is a σ-field of set $X \times E$), for any $\Gamma \in \mathcal{W}$,

$$
\begin{aligned}
&P\Big(w_{t+\hat{s}} \in \Gamma \big| \big(\sigma(w_r)\big)_{r=s}^{t}\Big)\\
=&P\Big(Aw_t + w(w_t, \{e_{t_{k-j}}\}_{j=1,\cdots,m}) \in \Gamma \big| \big(\sigma(w_r)\big)_{r=s}^{t}\Big)
\end{aligned}
$$

$$
\sigma(w_r) = \begin{cases} \sigma(x_s), & r = s \\ \sigma(x_r), & t_{i-1} < t < t_i, \quad i = 1,\cdots,k-(m+1) \\ \sigma(x_{t_{i-1}}) \cup \sigma(e_{t_i}), & r = t_i, \quad i = 1,\cdots,k-(m+1) \end{cases}
$$

$$
\begin{aligned}
=&P\Big(Aw_t + w(w_t, \{e_{t_{k-j}}\}_{j=1,\cdots,m}) \in \Gamma \big| \big(\sigma(x_s);\ \sigma(x_s)\cup\sigma(e_{t_1});\ \sigma(x_{t^1});\ \sigma(x_{t^1})\cup\sigma(e_{t_2});\ \cdots;\\
&\sigma(x_{t^{k-(m+2)}});\ \sigma(x_{t^{k-(m+2)}})\cup\sigma(e_{t_{k-(m+1)}});\ \sigma(x_{t^{k-(m+1)}});\ \sigma(x_{t^{k-(m+1)}})\cup\sigma(e_{t_{k-m}})\big)\Big)\\
&\text{where } t_i < t^i < t_{i+1}\\
=&P\Big(x_t + w(x_t, \{e_{t_{k-j}}\}_{j=1,\cdots,m}) \in \Gamma \big| \big(\sigma(x_s);\ \sigma(x_s)\cup\sigma(e_{t_1});\ \sigma(x_{t^1});\ \sigma(x_{t^1})\cup\sigma(e_{t_2});\ \cdots;\\
&\sigma(x_{t^{k-(m+2)}});\ \sigma(x_{t^{k-(m+2)}})\cup\sigma(e_{t_{k-(m+1)}});\ \sigma(x_t);\ \sigma(x_t)\cup\sigma(e_{t_{k-m}})\big)\Big)\\
&\sigma(x_{t^{k-(m+1)}}) = \sigma(x_t), \quad \text{if, } t_{k-(m+1)} < t^{k-(m+1)}, t < t_{k-m}\\
=&P\Big(x_t + w(x_t, \{e_{t_{k-j}}\}_{j=1,\cdots,m}) \in \Gamma | \sigma(x_t)\Big), \quad \text{by assumption } \textbf{(S1)} \text{ and by Lemma (2.1)}\\
=&P\Big(Aw_t + w(w_t, \{e_{t_{k-j}}\}_{j=1,\cdots,m}) \in \Gamma | \sigma(w_t)\Big)\\
=&P\Big(w_{t+\hat{s}} \in \Gamma | \sigma(w_t)\Big) \qquad (2.3.5)
\end{aligned}
$$

In the same way, we can prove in the following cases,

$t_{k-1} < t + \hat{s} < t_k$, for some $k \in \mathbb{N}$ and $t = t_{k-m}$, for some $m \in \{0, \mathbb{N}\}$

$t + \hat{s} = t_k$, for some $k \in \mathbb{N}$ and $t_{k-(m+1)} < t < t_{k-m}$, for some $m \in \{0, \mathbb{N}\}$

$t + \hat{s} = t_k$, for some $k \in \mathbb{N}$ and $t = t_{k-m}$, for some $m \in \{0, \mathbb{N}\}$

$$P\Big(w_{t+\hat{s}} \in \Gamma \big| \big(\sigma(w_r)\big)_s^t\Big) = P\Big(w_{t+\hat{s}} \in \Gamma | \sigma(w_t)\Big)$$

still holds.

The four equality relations above imply that $\{w_t = (x_{t^-}, e_t), s \le t \le T\}$ is a Markov process.

□

2.3.1 Value Functions and their Martingale Properties for Systems subject to Control Laws in $\mathcal{U}^{\mathcal{G}}[s,T]$

The CAC and RC problems in the network $Net(\mathcal{N}, \mathcal{L}, C)$ can be formulated as dynamic programming problems, which require the specification of **(i)** the state dynamics and then **(ii)** a system loss function covering a given interval $[s,T]$.

For any $s \in [0,T)$ and $\xi : \Omega \to X$, ξ is $\mathcal{F}_s$ measurable, state $\{x_t\}$ in the general space transition form:

$$x_t = x(\xi, u_s^t), \qquad s \le t \le T, u \in \mathcal{U}^{\mathcal{G}}[s,T], T \in \mathbb{R}_+ \tag{2.3.6a}$$

$$x_s = \xi, \qquad \text{a.s.} \tag{2.3.6b}$$

and consider the cost function

$$J(s,\xi;u) = E\{\int_s^T g(t, x_{t^-}, u_t)dt | \mathcal{G}_s\} \tag{2.3.7}$$

with control $u \in \mathcal{U}^{\mathcal{G}}[s,T]$, where $g : \mathbb{R}_+ \times X \times U \to \mathbb{R}_+$ is bounded and measurable $w.r.t.(t,x,u)$.

The *value function* $V^{\mathcal{G}}(s,\xi)$ is defined as

$$V^{\mathcal{G}}(s,\xi) = \inf_{u \in \mathcal{U}^{\mathcal{G}}[s,T]} J(s,\xi;u) \tag{2.3.8}$$

Let $u, v \in \mathcal{U}^{\mathcal{G}}[s,T]$ and $r \in [s,T]$, we define

$$\psi^{\mathcal{G}}(u,v,r) = E\{\int_r^T g(t, x_{t^-}, v_t)dt | G_r\}, \qquad \text{where, } x_t = x(x(\xi, u_s^r), v_r^t) \tag{2.3.9}$$

$$W^{\mathcal{G}}(u,r) = \inf_{v \in \mathcal{U}^{\mathcal{G}}[r,T]} \psi(u,v,r) \tag{2.3.10}$$

Lemma 2.2 With the initial condition as $x_s = \xi$, $\xi : \Omega \to X$ and ξ is $\mathcal{F}_s$ measurable, the following hold:

(1) $J(\hat{s}, x(\xi, u_s^{\hat{s}}); u) = \psi(u, u, \hat{s}), \qquad u \in \mathcal{U}^{\mathcal{G}}[s, T];$

(2) $V^{\mathcal{G}}(r, x(\xi, u_s^r)) = W^{\mathcal{G}}(u, r), \qquad r \in (s, T], u \in \mathcal{U}^{\mathcal{G}}[s, T];$

(3) $V^{\mathcal{G}}(s, \xi) = W^{\mathcal{G}}(u, s), \qquad u \in \mathcal{U}^{\mathcal{G}}[s, T].$

Proof:

(1)

$$J(\hat{s}, x(\xi, u_s^{\hat{s}}, e_s^{\hat{s}}); u) = E\{\int_{\hat{s}}^T g(t, x_{t^-}, u_t)dt|\mathcal{G}_{\hat{s}}\} = \psi(u, u, \hat{s}) \tag{2.3.11}$$

(2)

$$W^{\mathcal{G}}(u, r) = \inf_{v \in \mathcal{U}^{\mathcal{G}}[r,T]} \psi(u, v, r) = \inf_{v \in \mathcal{U}^{\mathcal{G}}[r,T]} E\{\int_r^T g(t, x_{t^-}, v_t)dt\big|G_r\} \equiv V^{\mathcal{G}}(r, x(\xi, u_s^r)) \tag{2.3.12}$$

(3) In the proof of **(ii)**, let $r = s$, we get

$$V^{\mathcal{G}}(s, \xi) = V^{\mathcal{G}}(s, x(\xi, u_s^s)) = W^{\mathcal{G}}(u, s) \tag{2.3.13}$$

□

Lemma 2.3 **(1)** For $s \leq t_1 \leq t_2 \leq T$ and $u \in \mathcal{U}^{\mathcal{G}}[s, T]$,

$$W^{\mathcal{G}}(u, t_1) \leq E\{\int_{t_1}^{t_2} g(t, x_{t^-}, u_t)dt\big|\mathcal{G}_{t_1}\} + E\{W^{\mathcal{G}}(u, t_2)\big|\mathcal{G}_{t_1}\} \tag{2.3.14}$$

(2) Furthermore, u is optimal on $[s, T]$ if and only if for all $s \leq t_1 \leq t_2 \leq T$, equality holds in (2.3.14).

Proof: (1)

For $\forall \epsilon > 0$, there exists a control $v_\epsilon \in \mathcal{U}^{\mathcal{G}}[t_2, T]$, such that

$$\begin{aligned} \psi^{\mathcal{G}}(u, v_\epsilon, t_2) &< \inf_{v \in \mathcal{U}^{\mathcal{G}}[t_2,T]} \psi^{\mathcal{G}}(u, v, t_2) + \epsilon \\ &= W^{\mathcal{G}}(u, t_2) + \epsilon \end{aligned}$$

The above inequality implies that

$$\begin{aligned}
\psi^{\mathcal{G}}(u,(u,v_\epsilon),t_1) &= E\Big\{\int_{t_1}^{t_2} g(t,x_{t^-},u_t)dt + \int_{t_2}^{T} g(t,x_{t^-},(v_\epsilon)_t)dt \,\big|\mathcal{G}_{t_1}\Big\} \\
&= E\Big\{\int_{t_1}^{t_2} g(t,x_{t^-},u_t)dt + E\Big\{\int_{t_2}^{T} g(t,x_{t^-},(v_\epsilon)_t)dt\big|\mathcal{G}_{t_2}\Big\} \,\big|\mathcal{G}_{t_1}\Big\} \\
&\qquad\qquad \big(\text{since, } E(E(X|\mathcal{G})|\mathcal{F}) = (E(X|\mathcal{F}), \text{ if } \mathcal{F}\subset\mathcal{G}\big) \\
&= E\Big\{\int_{t_1}^{t_2} g(t,x_{t^-},u_t)dt + \psi^{\mathcal{G}}(u,v_\epsilon,t_2) \,\big|\mathcal{G}_{t_1}\Big\} \\
&< E\Big\{\int_{t_1}^{t_2} g(t,x_{t^-},u_t)dt + W^{\mathcal{G}}(u,t_2) \,\big|\mathcal{G}_{t_1}\Big\} + \epsilon
\end{aligned}$$

So, we get, for all $\epsilon > 0$,

$$\begin{aligned}
W^{\mathcal{G}}(u,t_1) &= \inf_{v\in\mathcal{U}^{\mathcal{G}}[t_1,T]} \psi^{\mathcal{G}}(u,v,t_1) \\
&\le \psi^{\mathcal{G}}(u,(u,v_\epsilon),t_1) \\
&< E\Big\{\int_{t_1}^{t_2} g(t,x_{t^-},u_t)dt + W^{\mathcal{G}}(u,t_2)|\mathcal{G}_{t_1}\Big\} + \epsilon.
\end{aligned}$$

Hence,

$$W^{\mathcal{G}}(u,t_1) \le E\Big\{\int_{t_1}^{t_2} g(t,x_{t^-},u_t)dt\big|\mathcal{G}_{t_1}\Big\} + E\big\{W^{\mathcal{G}}(u,t_2)\big|\mathcal{G}_{t_1}\big\},$$

which yields (2.3.14).

(2) (i) Let u be optimal but suppose that, for some $t_1 \le t_2,\ t_1,t_2 \in [s,T]$,

$$W^{\mathcal{G}}(u,t_1) < E\Big\{\int_{t_1}^{t_2} g(t,x_{t^-},u_t)dt\big|\mathcal{G}_{t_1}\Big\} + E\big\{W^{\mathcal{G}}(u,t_2)\big|\mathcal{G}_{t_1}\big\}$$

Since u is optimal, by the definition of $W^{\mathcal{G}}(u,t_1)$, we have

$$E\Big\{\int_{t_1}^{t_2} g(t,x_{t^-},u_t)dt\big|\mathcal{G}_{t_1}\Big\} + E\Big\{\int_{t_2}^{T} g(t,x_{t^-},u_t)dt\big|\mathcal{G}_{t_1}\Big\} = W^{\mathcal{G}}(u,t_1)$$

But this implies

$$\int_{t_2}^{T} g(t,x_{t^-},u_t)dt < W^{\mathcal{G}}(u,t_2), \qquad \mathcal{G}_{t_1}\text{-a.s.}$$

which is a contradiction.

So, according to (2.3.14), we get, for u optimal and for any $t_1 \leq t_2,\ t_1, t_2 \in [s, T]$, the following equality holds,

$$W^{\mathcal{G}}(u, t_1) = E\Big\{\int_{t_1}^{t_2} g(t, x_{t^-}, u_t)dt \big| \mathcal{G}_{t_1}\Big\} + E\big\{W^{\mathcal{G}}(u, t_2)\big|\mathcal{G}_{t_1}\big\}$$

(ii) If for any $t_1 \leq t_2,\ t_1, t_2 \in [s, T]$,

$$W^{\mathcal{G}}(u, t_1) = E\Big\{\int_{t_1}^{t_2} g(t, x_{t^-}, u_t)dt \big| \mathcal{G}_{t_1}\Big\} + E\big\{W^{\mathcal{G}}(u, t_2)\big|\mathcal{G}_{t_1}\big\} \tag{2.3.15}$$

then for $t_1 = s$ and $t_2 = T$, (2.3.15) yields,

$$\begin{aligned} W^{\mathcal{G}}(u, s) &\equiv V^{\mathcal{G}}(s, \xi), \qquad \text{with the initial condition } x_s = \xi \\ &= E\Big\{\int_{s}^{T} g(t, x_{t^-}, u_t)dt \big| \mathcal{G}_s\Big\} + E\big\{W^{\mathcal{G}}(u, T)\big|\mathcal{G}_s\big\} \\ &= E\Big\{\int_{s}^{T} g(t, x_{t^-}, u_t)dt \big| \mathcal{G}_s\Big\}, \qquad \text{since } W^{\mathcal{G}}(u, T) = 0. \end{aligned}$$

That means u is optimal.

□

Theorem 2.2 For any $u \in \mathcal{U}^{\mathcal{G}}[0, t]$, the following process

$$\hat{J}_t^u = W^{\mathcal{G}}(u, t) + E\Big\{\int_{0}^{t} g(r, x_{r^-}, u_r)dr \big| \mathcal{G}_t\Big\} \tag{2.3.16}$$

is a $(\mathcal{G}_t, P)$ *sub-martingale*. Furthermore, $u \in \mathcal{U}^{\mathcal{G}}[0, t]$ is optimal if and only if this process is a $\mathcal{G}_t$-*martingale*, *i.e.* if and only if $\hat{J}_s^u = E\big\{\hat{J}_t^u | \mathcal{G}_s\big\}, \forall t > s \geq 0$.

Proof:

For $t_2 > t_1$, and any $u \in \mathcal{U}^{\mathcal{G}}[0,t]$,

$$\begin{aligned}
E\{\hat{J}^u_{t_2}|\mathcal{G}_{t_1}\} &= E\Big\{W^{\mathcal{G}}(u,t_2) + E\{\int_0^{t_2} g(r,x_{r-},u_r)dr|\mathcal{G}_{t_2}\}\ |\mathcal{G}_{t_1}\Big\} \\
&= E\Big\{W^{\mathcal{G}}(u,t_2) + E\{\int_{t_1}^{t_2} g(r,x_{r-},u_r)dr|\mathcal{G}_{t_2}\} + E\{\int_0^{t_1} g(r,x_{r-},u_r)dr|\mathcal{G}_{t_2}\}\ \Big|\mathcal{G}_{t_1}\Big\} \\
&= E\{\int_{t_1}^{t_2} g(r,x_{r-},u_r)dr|\mathcal{G}_{t_1}\} + E\{W^{\mathcal{G}}(u,t_2)|\mathcal{G}_{t_1}\} + E\{\int_0^{t_1} g(r,x_{r-},u_r)dr|\mathcal{G}_{t_1}\} \\
&\geq W^{\mathcal{G}}(u,t_1) + E\{\int_0^{t_1} g(r,x_{r-},u_r)dr|\mathcal{G}_{t_1}\}, \qquad \text{Lemma 2.3} \\
&= \hat{J}^u_{t_1}
\end{aligned}$$

Furthermore, by Lemma (2.3), if u is optimal,

$$\begin{aligned}
\hat{J}^u_{t_1} &= W^{\mathcal{G}}(u,t_1) + E\{\int_0^{t_1} g(r,x_{r-},u_r)dr|\mathcal{G}_{t_1}\} \\
&= E\{\int_{t_1}^{t_2} g(r,x_{r-},u_r)dr|\mathcal{G}_{t_1}\} + E\{W^{\mathcal{G}}(u,t_2)|\mathcal{G}_{t_1}\} + E\{\int_0^{t_1} g(r,x_{r-},u_r)dr|\mathcal{G}_{t_1}\} \\
&= E\Big\{W^{\mathcal{G}}(u,t_2) + \int_0^{t_2} g(r,x_{r-},u_r)dr\ |\mathcal{G}_{t_1}\Big\} \\
&= E\Big\{W^{\mathcal{G}}(u,t_2) + E\{\int_0^{t_2} g(r,x_{r-},u_r)dr|\mathcal{G}_{t_2}\}\ |\mathcal{G}_{t_1}\Big\} \\
&= E\{\hat{J}^u_{t_2}|\mathcal{G}_{t_1}\}
\end{aligned} \tag{2.3.17}$$

Since $\hat{J}^u_{t_1} = E\{\hat{J}^u_{t_2}|\mathcal{G}_{t_1}\}$ holds for all $0 \leq t_1 \leq t_2 \leq T$, the $\mathcal{G}_t$-adapted process $\{\hat{J}^u_t; 0 \leq t \leq T\}$ is a $\mathcal{G}_t$-martingale.

□

2.3.2 Value Functions and their Martingale Properties for Systems subject to Control Laws in $\mathcal{U}^M[s,T]$

This section is parallel with section (2.3.1). In this section, we consider the state dynamics with Markovian control laws $u \in \mathcal{U}^M[s,T]$.

The *cost function* $J^M(s,\xi;u)$ is defined as

$$J^M(s,\xi;u) = E\{\int_s^T g(t,x_{t^-},u_t)dt|\sigma(w_s)\}, \qquad u \in \mathcal{U}^M[s,T] \tag{2.3.18}$$

The *value function* $V^M(s,\xi)$ is defined as

$$V^M(s,\xi) = \inf_{u\in\mathcal{U}^M[s,T]} J^M(s,\xi;u) \tag{2.3.19}$$

Let $u, v \in \mathcal{U}^M[s,T]$ and $r \in [s,T]$, we define

$$\psi^M(u,v,r) = E\{\int_r^T g(t,x_{t^-},v_t)dt\big|\sigma(w_r)\}, \qquad \text{where, } x_t = x(x(\xi,u_s^r),v_r^t) \tag{2.3.20}$$

$$W^M(u,r) = \inf_{v\in\mathcal{U}^M[r,T]} \psi^M(u,v,r) \tag{2.3.21}$$

Lemma 2.4 With the initial condition as $x_s = \xi$, $\xi : \Omega \to X$ and ξ is $\mathcal{F}_s$ measurable, the following hold:

(1) $J^M(\hat{s},x(\xi,u_s^{\hat{s}});u) = \psi^M(u,u,\hat{s}), \qquad u \in \mathcal{U}^M[s,T]$;

(2) $V^M(r,x(\xi,u_s^r)) = W^M(u,r), \qquad r \in (s,T], u \in \mathcal{U}^M[s,T]$;

(3) $V^M(s,\xi) = W^M(u,s), \qquad u \in \mathcal{U}^M[s,T]$.

Proof:

(1)

$$J^M(\hat{s},x(\xi,u_s^{\hat{s}});u) = E\{\int_{\hat{s}}^T g(t,x_{t^-},u_t)dt|\sigma(w_{\hat{s}})\} = \psi^M(u,u,\hat{s}) \tag{2.3.22}$$

(2)

$$W^M(u,r) = \inf_{v\in\mathcal{U}^M[r,T]} \psi^M(u,v,r) = \inf_{v\in\mathcal{U}^M[r,T]} E\{\int_r^T g(t,x_{t^-},v_t)dt\big|\sigma(w_r)\} \equiv V^M(r,x(\xi,u_s^r)) \tag{2.3.23}$$

(3) In the proof of **(ii)**, let $r = s$, we get

$$V^M(s,\xi) = V^M(s,x(\xi,u_s^s)) = W^M(u,s) \tag{2.3.24}$$

□

Lemma 2.5 **(1)** For $s \le t_1 \le t_2 \le T$ and $u \in \mathcal{U}^M[s,T]$,

$$W^M(u,t_1) \le E\Big\{\int_{t_1}^{t_2} g(t,x_{t^-},u_t)dt\big|\sigma(w_{t_1})\Big\} + E\big\{W^M(u,t_2)\big|\sigma(w_{t_1})\big\} \tag{2.3.25}$$

(2) Furthermore, u is optimal if and only if equality holds in (2.3.25).

Proof: (1)

For $\forall\epsilon > 0$, there exists a control $v_\epsilon \in \mathcal{U}^M[t_2,T]$, such that

$$\begin{aligned}\psi^M(u,v_\epsilon,t_2) &< \inf_{v\in\mathcal{U}^M[t_2,T]} \psi^M(u,v,t_2) + \epsilon \\ &= W^M(u,t_2) + \epsilon\end{aligned}$$

The above inequality implies that

$$\begin{aligned}\psi^M(u,(u,v_\epsilon),t_1) &= E\Big\{\int_{t_1}^{t_2} g(t,x_{t^-},u_t)dt + \int_{t_2}^{T} g(t,x_{t^-},(v_\epsilon)_t)dt\ \big|\mathcal{F}^w_{t_1}\Big\} \\ &= E\Big\{\int_{t_1}^{t_2} g(t,x_{t^-},u_t)dt + E\Big\{\int_{t_2}^{T} g(t,x_{t^-},(v_\epsilon)_t)dt\big|\mathcal{F}^w_{t_2}\Big\}\ \big|\mathcal{F}^w_{t_1}\Big\} \\ &\qquad \big(\text{since, } E(E(X|\mathcal{G})|\mathcal{F}) = (E(X|\mathcal{F}),\ \text{if } \mathcal{F}\subset\mathcal{G}\big) \\ &= E\Big\{\int_{t_1}^{t_2} g(t,x_{t^-},u_t)dt + \psi^M(u,v_\epsilon,t_2)\ \big|\mathcal{F}^w_{t_1}\Big\} \\ &< E\Big\{\int_{t_1}^{t_2} g(t,x_{t^-},u_t)dt + W^M(u,t_2)\ \big|\mathcal{F}^w_{t_1}\Big\} + \epsilon\end{aligned}$$

So, we get for all $\epsilon > 0$

$$\begin{aligned}W^M(u,t_1) &= \inf_{v\in\mathcal{U}^M[t_1,T]} \psi^M(u,v,t_1) \\ &\le \psi^M(u,(u,v_\epsilon),t_1) \\ &< E\Big\{\int_{t_1}^{t_2} g(t,x_{t^-},u_t)dt + W^M(u,t_2)|\mathcal{F}^w_{t_1}\Big\} + \epsilon.\end{aligned}$$

Hence,

$$W^M(u,t_1) \le E\Big\{\int_{t_1}^{t_2} g(t,x_{t^-},u_t)dt\big|\sigma(w_{t_1})\Big\} + E\big\{W^M(u,t_2)\big|\sigma(w_{t_1})\big\}$$

which yields (2.3.25).

(2) **(i)** Let u be optimal but suppose that, for some $t_1 \leq t_2,\ t_1, t_2 \in [s, T]$,

$$W^M(u, t_1) < E\{\int_{t_1}^{t_2} g(t, x_{t^-}, u_t)dt|\sigma(w_{t_1})\} + E\{W^M(u, t_2)|\sigma(w_{t_1})\}$$

Since u is optimal, by the definition of $W^M(u, t_1)$, we have

$$E\{\int_{t_1}^{t_2} g(t, x_{t^-}, u_t)dt|\sigma(w_{t_1})\} + E\{\int_{t_2}^{T} g(t, x_{t^-}, u_t)dt|\sigma(w_{t_1})\} = W^M(u, t_1)$$

But this implies

$$\int_{t_2}^{T} g(t, x_{t^-}, u_t)dt < W^M(u, t_2), \quad \sigma(w_{t_1})\text{-a.s.}$$

which is a contradiction.

So, according to (2.3.25), we get, for u optimal and for any $t_1 \leq t_2,\ t_1, t_2 \in [s, T]$, the following equality holds,

$$W^M(u, t_1) = E\{\int_{t_1}^{t_2} g(t, x_{t^-}, u_t)dt|\sigma(w_{t_1})\} + E\{W^M(u, t_2)|\sigma(w_{t_1})\}$$

(ii) If for any $t_1 \leq t_2,\ t_1, t_2 \in [s, T]$,

$$W^M(u, t_1) = E\{\int_{t_1}^{t_2} g(t, x_{t^-}, u_t)dt|\sigma(w_{t_1})\} + E\{W^M(u, t_2)|\sigma(w_{t_1})\}, \tag{2.3.26}$$

then for $t_1 = s$ and $t_2 = T$, (2.3.26) yields,

$$\begin{aligned} W^M(u, s) &\equiv V^M(s, \xi), \qquad \text{with the initial condition } x_s = \xi \\ &= E\{\int_{s}^{T} g(t, x_{t^-}, u_t)dt|\sigma(w_s)\} + E\{W^M(u, T)|\sigma(w_s)\} \\ &= E\{\int_{s}^{T} g(t, x_{t^-}, u_t)dt|\sigma(w_s)\}, \qquad \text{since } W^M(u, T) = 0. \end{aligned}$$

That means u is optimal.

□

Theorem 2.3 For any $u \in \mathcal{U}^M[0, t]$, the following process

$$\hat{J}_t^u = W^M(u, t) + E\{\int_0^t g(r, x_{r^-}, u_r)dr|\mathcal{F}_t^w\} \tag{2.3.27}$$

is a $(\mathcal{F}_t^w, P)$ *sub-martingale.* $u \in \mathcal{U}^M[0,t]$ is optimal if and only if this process is a $\mathcal{F}_t^w$*-martingale, i.e.* if and only if $\hat{J}_s^u = E\{\hat{J}_t^u|\mathcal{F}_s^w\}, \forall t > s \geq 0$.

Proof:

For $t_2 > t_1$, and any $u \in \mathcal{U}^M[0,t]$,

$$\begin{aligned}
E\{\hat{J}_{t_2}^u|\mathcal{F}_{t_1}^w\} &= E\Big\{W^M(u,t_2) + E\{\int_0^{t_2} g(r,x_{r^-},u_r)dr|\mathcal{F}_{t_2}^w\}\ |\mathcal{F}_{t_1}^w\Big\} \\
&= E\Big\{W^M(u,t_2) + E\{\int_{t_1}^{t_2} g(r,x_{r^-},u_r)dr|\mathcal{F}_{t_2}^w\} + E\{\int_0^{t_1} g(r,x_{r^-},u_r)dr|\mathcal{F}_{t_2}^w\}\ \Big|\mathcal{F}_{t_1}^w\Big\} \\
&= E\{\int_{t_1}^{t_2} g(r,x_{r^-},u_r)dr|\mathcal{F}_{t_1}^w\} + E\{W^M(u,t_2)|\mathcal{F}_{t_1}^w\} + E\{\int_0^{t_1} g(r,x_{r^-},u_r)dr|\mathcal{F}_{t_1}^w\} \\
&\geq W^M(u,t_1) + E\{\int_0^{t_1} g(r,x_{r^-},u_r)dr|\mathcal{F}_{t_1}^w\} \\
&= \hat{J}_{t_1}^u
\end{aligned}$$

Furthermore, by Lemma (2.5), if u is optimal,

$$\begin{aligned}
\hat{J}_{t_1}^u &= W^M(u,t_1) + E\{\int_0^{t_1} g(r,x_{r^-},u_r)dr|\mathcal{F}_{t_1}^w\} \\
&= E\{\int_{t_1}^{t_2} g(r,x_{r^-},u_r)dr|\mathcal{F}_{t_1}^w\} + E\{W^M(u,t_2)|\mathcal{F}_{t_1}^w\} + E\{\int_0^{t_1} g(r,x_{r^-},u_r)dr|\mathcal{F}_{t_1}^w\} \\
&= E\Big\{W^M(u,t_2) + \int_0^{t_2} g(r,x_{r^-},u_r)dr\ |\mathcal{F}_{t_1}^w\Big\} \\
&= E\Big\{W^M(u,t_2) + E\{\int_0^{t_2} g(r,x_{r^-},u_r)dr|\mathcal{F}_{t_2}^w\}\ |\mathcal{F}_{t_1}^w\Big\} \\
&= E\{\hat{J}_{t_2}^u|\mathcal{F}_{t_1}^w\} \qquad (2.3.28)
\end{aligned}$$

Since $\hat{J}_{t_1}^u = E\{\hat{J}_{t_2}^u|\mathcal{F}_{t_1}^w\}$ holds for all $0 \leq t_1 \leq t_2 \leq T$, the $\mathcal{F}_t^w$-adapted process $\{\hat{J}_t^u; 0 \leq t \leq T\}$ is a $\mathcal{F}_t^w$-martingale.

□

2.3.3 Optimal Stochastic Control Problem

Definition 2.16 The *Optimal Stochastic Control (OSC) Problem* is defined as:

For any $s \in [0,T)$ and $\xi : \Omega \to X$, ξ is $\mathcal{F}_s$ measurable, we consider the system state

(2.13) in the space transition form with control law $u \in \mathcal{U}[s,T]$ (or $\mathcal{U}^{\mathcal{G}}[s,T], \mathcal{U}^M[s,T]$):

$$x_t = x(\xi, u_s^t), \quad s \le t \le T, \tag{2.3.29a}$$

$$x_s = \xi, \text{ a.s.} \tag{2.3.29b}$$

and consider the cost functions as

$$J^{\mathcal{G}}(s,\xi;u) = E\{\int_s^T g(t, x_{t^-}, u_t)dt \,|\mathcal{G}_s\}, \qquad u \in \mathcal{U}^{\mathcal{G}}[s,T] \tag{2.3.30}$$

$$J^{\mathcal{F}^w}(s,\xi;u) = E\{\int_s^T g(t, x_{t^-}, u_t)dt \,|\mathcal{F}_s^w\}, \qquad u \in \mathcal{U}^{\mathcal{F}^w}[s,T] \tag{2.3.31}$$

$$J^M(s,\xi;u) = E\{\int_s^T g(t, x_{t^-}, u_t)dt \,|\sigma(w_s)\}, \qquad u \in \mathcal{U}^M[s,T] \tag{2.3.32}$$

where general filtering space $\{\mathcal{G}_t; t \in [s,T], \mathcal{G}_t \subset \mathcal{F}_t\}$, $g : [s,T] \times X \times U \to \mathbb{R}_+$ is bounded and measurable *w.r.t.* (t,x,u).

The optimal stochastic control (OSC) Problem is given by the infimization:

$$V^{\mathcal{G}}(s,\xi) = \inf_{u \in \mathcal{U}^{\mathcal{G}}[s,T]} J^{\mathcal{G}}(s,\xi;u),$$
$$V^{\mathcal{F}^w}(s,\xi) = \inf_{u \in \mathcal{U}^{\mathcal{F}^w}[s,T]} J^{\mathcal{F}^w}(s,\xi;u),$$
$$V^M(s,\xi) = \inf_{u \in \mathcal{U}^M[s,T]} J^M(s,\xi;u),$$

where, when it exists, the function $V^{\mathcal{G}}(s,\xi)$ ($V^{\mathcal{F}^w}(s,\xi), V^M(s,\xi)$) : $\mathbb{R}_+ \times X \to \mathbb{R}_+$ called the *value functions of the OSC problem* and, when they exist, an infimizing function $\hat{u} \in \mathcal{U}^{\mathcal{G}}[s,T]$ or ($\mathcal{U}^{\mathcal{F}^w}[s,T], \mathcal{U}^M[s,T]$) shall be called an *optimal control for the OSC problem.*

Furthermore, when $\mathcal{G}_t = \mathcal{F}_t^w, t \in [s,T]$, then the *state-input* (x,e) *dependent optimal stochastic control (OSC) problem* is given by the infimization:

$$V(s,\xi) \equiv V^{\mathcal{F}^{x,e}}(s,\xi) = \inf_{u \in \mathcal{U}^{\mathcal{F}^{x,e}}[s,T]} J(s,\xi;u),$$

where, when it exists, the function $V(s,\xi) : \mathbb{R}_+ \times X \to \mathbb{R}_+$ called the *value function of the state-input dependent OSC problem* and, when it exists, an infimization function $\hat{u} \in \mathcal{U}^{\mathcal{F}^{x,e}}[s,T]$ shall be called an *optimal control for the state-input dependent OSC problem.*

□

The inequality parts of Lemmas (2.3, 2.5) and Theorems (2.2, 2.3) do not require the existence of an optimal control $u \in \mathcal{U}^{\mathcal{G}}[s,T]$ (or $\mathcal{U}^M[s,T]$) as appears in the second part of each result. However, because the only constraint on the class of control functions $\mathcal{U}^{\mathcal{G}}[s,T]$ (or $\mathcal{U}^M[s,T]$) is the specified adapted measurability with respect to $\{\mathcal{G}_t, s \leq t \leq T\}$, an OSC problem in fact always has a value function and a corresponding optimal control.

The first part of Theorem (2.4) below consists of a statement of the Dynamic Programming (DP) Principle for OSC also does not require the existence of an optimal control. Moreover, while the equality in the statement of the Dynamic Programming (DP) Problem in the Theorem (2.4) below, it is an important aspect of the state space system is that it has optimal controls within the class of state-input (adapted) measurable controls $\mathcal{U}^{\mathcal{F}^w}[s,T] \subset \mathcal{U}^{\mathcal{G}}[s,T]$ as stated in the second part of Theorem (2.4).

2.3.4 Optimality Principle

Theorem 2.4 For the SOCP (2.16),

(1) For $s \in [0,T)$, $\xi : \Omega \rightarrow X$, ξ is $\mathcal{F}_s$ measurable, and for all filtering spaces $\big\{\{\mathcal{G}_t\}_{s \leq t \leq T};\ \mathcal{G}_t \subset \mathcal{F}_t, \forall t \in [s,T]\big\}$, it is the case that

$$V^{\mathcal{G}}(s,\xi) = \inf_{u \in \mathcal{U}^{\mathcal{G}}[s,\hat{s}]} E\Big\{\int_s^{\hat{s}} g(t, x_{t-}, u_t)dt + V^{\mathcal{G}}(\hat{s}, x(\xi, u_s^{\hat{s}}, e_s^{\hat{s}}))\ \big|\mathcal{G}_s\Big\}, \tag{2.3.33}$$

(2) For $s \in [0,T)$, $\xi : \Omega \rightarrow X$, ξ is $\mathcal{F}_s$ measurable,

$$V^{M}(s,\xi) = \inf_{u \in \mathcal{U}^{M}[s,\hat{s}]} E\Big\{\int_s^{\hat{s}} g(t, x_{t-}, u_t)dt + V^{M}(\hat{s}, x(\xi, u_s^{\hat{s}}))\ \big|\sigma(w_s)\Big\}, \tag{2.3.34}$$

Proof: (1)

$$V^{\mathcal{G}}(s,\xi) = \inf_{u \in \mathcal{U}^{\mathcal{G}}[s,T]} J^{\mathcal{G}}(s,\xi;u) \equiv W^{\mathcal{G}}(u,s)$$

If u is optimal, then according to Lemma (2.3), we obtain

$$\begin{aligned}
V^{\mathcal{G}}(s,\xi) &\equiv W^{\mathcal{G}}(u,s) \\
&= E\Big\{\int_s^{\hat{s}} g(t,x_{t^-},u_t)dt\Big|\mathcal{G}_s\Big\} + E\Big\{W^{\mathcal{G}}(u,\hat{s})\Big|\mathcal{G}_s\Big\} \\
&= E\Big\{\int_s^{\hat{s}} g(t,x_{t^-},u_t)dt\Big|\mathcal{G}_s\Big\} + E\Big\{E\{W^{\mathcal{G}}(u,\hat{s})|\mathcal{G}_{\hat{s}}\}\Big|\mathcal{G}_s\Big\} \\
&\equiv E\Big\{\int_s^{\hat{s}} g(t,x_{t^-},u_t)dt + V^{\mathcal{G}}(\hat{s},x(\xi,u_s^{\hat{s}}))\ \Big|\mathcal{G}_s\Big\}
\end{aligned}$$

If u is not optimal control, then according to Lemma (2.3), we obtain

$$\begin{aligned}
V^{\mathcal{G}}(s,\xi) &\equiv W^{\mathcal{G}}(u,s) \\
&< E\Big\{\int_s^{\hat{s}} g(t,x_{t^-},u_t)dt\Big|\mathcal{G}_s\Big\} + E\Big\{W^{\mathcal{G}}(u,\hat{s})\Big|\mathcal{G}_s\Big\} \\
&\equiv E\Big\{\int_s^{\hat{s}} g(t,x_{t^-},u_t)dt + V^{\mathcal{G}}(\hat{s},x(\xi,u_s^{\hat{s}}))\ \Big|\mathcal{G}_s\Big\}
\end{aligned}$$

So,

$$V^{\mathcal{G}}(s,\xi) \leq \inf_{u\in\mathcal{U}^{\mathcal{G}}[s,\hat{s}]} E\Big\{\int_s^{\hat{s}} g(t,x_{t^-},u_t)dt + V^{\mathcal{G}}(\hat{s},x(\xi,u_s^{\hat{s}}))\ \Big|\mathcal{G}_s\Big\} \tag{2.3.35}$$

By definition of $V^{\mathcal{G}}(s,\xi)$, for any $\varepsilon > 0$, there exists a $u \in \mathcal{U}^{\mathcal{G}}[s,T]$, such that

$$\begin{aligned}
&V^{\mathcal{G}}(s,\xi) + \varepsilon > J^{\mathcal{G}}(s,\xi;u) \\
=&E\Big\{\int_s^{T} g(t,x_{t^-},u_t)dt|\mathcal{G}_s\Big\} \\
=&E\Big\{\int_s^{\hat{s}} g(t,x_{t^-},u_t)dt + E\Big\{\int_{\hat{s}}^{T} g(t,x_{t^-},u_t)dt\Big|\mathcal{G}_{\hat{s}}\Big\}\ \Big|\mathcal{G}_s\Big\} \\
=&E\Big\{\int_s^{\hat{s}} g(t,x_{t^-},u_t)dt + J(\hat{s},x(\xi,u_s^{\hat{s}});u_{\hat{s}}^T)\ \Big|\mathcal{G}_s\Big\} \\
\geq& \inf_{u\in\mathcal{U}^{\mathcal{G}}[s,\hat{s}]} E\Big\{\int_s^{\hat{s}} g(t,x_{t^-},u_t)dt + J(\hat{s},x(\xi,u_s^{\hat{s}});u_{\hat{s}}^T)\ \Big|\mathcal{G}_s\Big\} \\
\geq& \inf_{u\in\mathcal{U}^{\mathcal{G}}[s,\hat{s}]} E\Big\{\int_s^{\hat{s}} g(t,x_{t^-},u_t)dt + \inf_{u\in\mathcal{U}^{\mathcal{G}}[\hat{s},T]} J(\hat{s},x(\xi,u_s^{\hat{s}});u_{\hat{s}}^T)\ \Big|\mathcal{G}_s\Big\} \\
=& \inf_{u\in\mathcal{U}^{\mathcal{G}}[s,\hat{s}]} E\Big\{\int_s^{\hat{s}} g(t,x_{t^-},u_t)dt + V^{\mathcal{G}}(\hat{s},x(\xi,u_s^{\hat{s}}))\ \Big|\mathcal{G}_s\Big\}
\end{aligned}$$

Since, $\varepsilon > 0$ is arbitrary, we get

$$V^{\mathcal{G}}(s,\xi) \geq \inf_{u\in\mathcal{U}^{\mathcal{G}}[s,\hat{s}]} E\Big\{\int_s^{\hat{s}} g(t,x_{t^-},u_t)dt + V^{\mathcal{G}}(\hat{s},x(\xi,u_s^{\hat{s}}))\ \big|\mathcal{G}_s\Big\} \tag{2.3.36}$$

So, from (2.3.35) and (2.3.36), we get the conclusion.

Proof: (2)

$$V^M(s,\xi) = \inf_{u\in\mathcal{U}^M[s,T]} J^M(s,\xi;u) \equiv W^M(u,s)$$

If u is optimal, then according to Lemma (2.5), we obtain

$$\begin{aligned}
V^M(s,\xi) &\equiv W^M(u,s) \\
&= E\Big\{\int_s^{\hat{s}} g(t,x_{t^-},u_t)dt\big|\sigma(w_s)\Big\} + E\big\{W^M(u,\hat{s})\big|\sigma(w_s)\big\} \\
&= E\Big\{\int_s^{\hat{s}} g(t,x_{t^-},u_t)dt\big|\sigma(w_s)\Big\} + E\big\{W^M(u,\hat{s})\big|\sigma(w_s)\big\} \\
&\equiv E\Big\{\int_s^{\hat{s}} g(t,x_{t^-},u_t)dt + V^M(\hat{s},x(\xi,u_s^{\hat{s}}))\ \big|\sigma(w_s)\Big\}
\end{aligned}$$

If u is not optimal control, then according to Lemma (2.5), we obtain

$$\begin{aligned}
V^M(s,\xi) &\equiv W^M(u,s) \\
&< E\Big\{\int_s^{\hat{s}} g(t,x_{t^-},u_t)dt\big|\sigma(w_s)\Big\} + E\big\{W^M(u,\hat{s})\big|\sigma(w_s)\big\} \\
&\equiv E\Big\{\int_s^{\hat{s}} g(t,x_{t^-},u_t)dt + V^M(\hat{s},x(\xi,u_s^{\hat{s}}))\ \big|\sigma(w_s)\Big\}
\end{aligned}$$

So,

$$V^M(s,\xi) \leq \inf_{u\in\mathcal{U}^M[s,\hat{s}]} E\Big\{\int_s^{\hat{s}} g(t,x_{t^-},u_t)dt + V^M(\hat{s},x(\xi,u_s^{\hat{s}}))\ \big|\sigma(w_s)\Big\} \tag{2.3.37}$$

By definition of $V^{\mathcal{G}}(s,\xi)$, for any $\varepsilon > 0$, there exists a $u \in \mathcal{U}^{\mathcal{G}}[s,T]$, such that

$$\begin{aligned}
&V^M(s,\xi) + \varepsilon > J^M(s,\xi;u) \\
=&E\{\int_s^T g(t,x_{t^-},u_t)dt|\sigma(w_s)\} \\
=&E\{\int_s^T g(t,x_{t^-},u_t)dt|\mathcal{F}_s^w\} \\
=&E\Big\{\int_s^{\hat{s}} g(t,x_{t^-},u_t)dt + E\{\int_{\hat{s}}^T g(t,x_{t^-},u_t)dt|\mathcal{F}_{\hat{s}}^w\} \ \Big|\mathcal{F}_s^w\Big\} \\
=&E\Big\{\int_s^{\hat{s}} g(t,x_{t^-},u_t)dt + E\{\int_{\hat{s}}^T g(t,x(x_{\hat{s}},u_{\hat{s}}^t),u_t)dt|\mathcal{F}_{\hat{s}}^w\} \ \Big|\mathcal{F}_s^w\Big\} \\
=&E\Big\{\int_s^{\hat{s}} g(t,x_{t^-},u_t)dt + E\{\int_{\hat{s}}^T g(t,x(x_{\hat{s}},u_{\hat{s}}^t),u_t)dt|\sigma(w_{\hat{s}})\} \ \Big|\mathcal{F}_s^w\Big\} \\
=&E\{\int_s^{\hat{s}} g(t,x_{t^-},u_t)dt + J^M(\hat{s},x(\xi,u_s^{\hat{s}});u_{\hat{s}}^T) \ |\mathcal{F}_s^w\} \\
\geq& \inf_{u\in\mathcal{U}^M[s,\hat{s}]} E\{\int_s^{\hat{s}} g(t,x_{t^-},u_t)dt + J(\hat{s},x(\xi,u_s^{\hat{s}});u_{\hat{s}}^T) \ |\mathcal{F}_s^w\} \\
\geq& \inf_{u\in\mathcal{U}^M[s,\hat{s}]} E\{\int_s^{\hat{s}} g(t,x_{t^-},u_t)dt + \inf_{u\in\mathcal{U}^M[\hat{s},T]} J(\hat{s},x(\xi,u_s^{\hat{s}});u_{\hat{s}}^T) \ |\mathcal{F}_s^w\} \\
=& \inf_{u\in\mathcal{U}^M[s,\hat{s}]} E\{\int_s^{\hat{s}} g(t,x_{t^-},u_t)dt + V^M(\hat{s},x(\xi,u_s^{\hat{s}})) \ |\sigma(w_s)\}
\end{aligned}$$

Since, $\varepsilon > 0$ is arbitrary, we get

$$V^M(s,\xi) \geq \inf_{u\in\mathcal{U}^M[s,\hat{s}]} E\{\int_s^{\hat{s}} g(t,x_{t^-},u_t)dt + V^M(\hat{s},x(\xi,u_s^{\hat{s}})) \ |\sigma(w_s)\} \tag{2.3.38}$$

So, from (2.3.37) and (2.3.38), we get the conclusion.

□

Theorem 2.5 Information state dependent control(*i.e.* Markovian control) is optimal control, *i.e.*

$$\inf_{u\in\mathcal{U}} E\{\int_{t_0}^T g(t,x_{t^-},u_t(x_{t_0}^{t^-},e_{t_0}^t))dt|\mathcal{F}_{t_0}^w\} = \inf_{u\in\mathcal{U}^M} E\{\int_{t_0}^T g(t,x_{t^-},u_t(x_{t^-},e_t))dt|\mathcal{F}_{t_0}^w\} \tag{2.3.39}$$

Proof:

(1) Since $\mathcal{U}^M \subset \mathcal{U}$, we got that

$$\inf_{u\in\mathcal{U}} E\{\int_{t_0}^{T} g(t, x_{t^-}, u_t(x_{t_0}^{t^-}, e_{t_0}^{t}))dt|\mathcal{F}_{t_0}^{w}\} \leq \inf_{u\in\mathcal{U}^M} E\{\int_{t_0}^{T} g(t, x_{t^-}, u_t(x_{t^-}, e_t))dt|\mathcal{F}_{t_0}^{w}\} \tag{2.3.40}$$

(2) If for any $\epsilon > 0$, there exists some control $\hat{u} \in \mathcal{U}^M$, such that the following holds,

$$\inf_{u\in\mathcal{U}} E\{\int_{t_0}^{T} g(t, x_{t^-}, u_t(x_{t_0}^{t^-}, e_{t_0}^{t}))dt|\mathcal{F}_{t_0}^{w}\} + \epsilon \geq E\{\int_{t_0}^{T} g(t, x_{t^-}, \hat{u}_t(x_{t^-}, e_t))dt|\mathcal{F}_{t_0}^{w}\} \tag{2.3.41}$$

then we can say

$$\inf_{u\in\mathcal{U}} E\{\int_{t_0}^{T} g(t, x_{t^-}, u_t(x_{t_0}^{t^-}, e_{t_0}^{t}))dt|\mathcal{F}_{t_0}^{w}\} \geq \inf_{u\in\mathcal{U}^M} E\{\int_{t_0}^{T} g(t, x_{t^-}, \hat{u}_t(x_{t^-}, e_t))dt|\mathcal{F}_{t_0}^{w}\} \tag{2.3.42}$$

So, next we show (2.3.41) holds. First suppose there exists the positive finite integer K, such that $K = \min_{j\in\mathbb{N}}\{j; T < t_j, a.s.\}$.

(i) According to the Optimality Principle Theorem (2.4), for any $\epsilon > 0$, there exists some control $\widehat{u^0} \in \mathcal{U}^M$, such that the following holds,

$$\begin{aligned}
V(t_0, \xi) + \frac{\epsilon}{K+1} &= \inf_{u\in\mathcal{U}} E\{\int_{t_0}^{T} g(t, x_{t^-}, u_t(x_{t_0}^{t^-}, e_{t_0}^{t}))dt|\mathcal{F}_{t_0}^{w}\} + \frac{\epsilon}{K+1} \\
&= \inf_{u\in\mathcal{U}[t_0,t_1]} E\{\int_{t_0}^{t_1} g(t, x_{t^-}, u_t(x_{t_0}^{t^-}, e_{t_0}^{t}))dt + V(t_1, x_{t_1})|\mathcal{F}_0^{w}\} + \frac{\epsilon}{K+1} \\
&\geq E\{\int_{t_0}^{t_1} g(t, x_{t^-}, \widehat{u^0}_{t_0}(x_{t_0^-}, e_{t_0}))dt + V(t_1, x_{t_1})|\mathcal{F}_{t_0}^{w}\} \\
&= E\{\int_{t_0}^{t_1} g(t, x_{t_0}, \widehat{u^0}_{t_1}(x_{t_0^-}, \emptyset))dt + V(t_1, x_{t_1})|\mathcal{F}_{t_0}^{w}\} \\
&= E\{\int_{t_0}^{t_1} g(t, x_{t_0}, \emptyset)dt + V(t_1, x_{t_1})|\mathcal{F}_{t_0}^{w}\}
\end{aligned}$$

So, we see that $\widehat{u^0} \equiv \emptyset \in \mathcal{U}^M$.

(ii) Similarly, we can get that there exists some control $\widehat{u^1} \in \mathcal{U}^M$, such that the following holds,

$$
\begin{aligned}
V(t_1,x_{t_1})+\frac{\epsilon}{K+1} &= \inf_{u\in\mathcal{U}} E\{\int_{t_1}^{T} g(t,x_{t-},u_t(x_{t_0}^{t^-},e_{t_0}^{t}))dt|\mathcal{F}_{t_1}^{w}\}+\frac{\epsilon}{K+1}\\
&= \inf_{u\in\mathcal{U}[t_1,t_2]} E\{\int_{t_1}^{t_2} g(t,x_{t-},u_t(x_{t_0}^{t^-},e_{t_0}^{t}))dt+V(t_2,x_{t_2})|\mathcal{F}_{t_1}^{w}\}+\frac{\epsilon}{K+1}\\
&\geq E\{\int_{t_1}^{t_2} g(t,x_{t-},\widehat{u^1}_{t_1}(x_{t_0},x_{t_1},e_{t_0},e_{t_1}))dt+V(t_2,x_{t_2})|\mathcal{F}_{t_1}^{w}\}\\
&= E\{\int_{t_1}^{t_2} g(t,x_{t_1},\widehat{u^1}_{t_1}(x_{t_1}-,e_{t_1}))dt+V(t_2,x_{t_2})|\mathcal{F}_{t_1}^{w}\}
\end{aligned}
$$

(iii) In the similar way, we can got that there exists some control $\widehat{u^2}\in\mathcal{U}^M$, the following holds,

$$
\begin{aligned}
V(t_2,x_{t_2})+\frac{\epsilon}{K+1} &= \inf_{u\in\mathcal{U}} E\{\int_{t_2}^{T} g(t,x_{t-},u_t(x_{t_0}^{t^-},e_{t_0}^{t}))dt|\mathcal{F}_{t_1}^{w}\}+\frac{\epsilon}{K+1}\\
&= \inf_{u\in\mathcal{U}[t_2,t_3]} E\{\int_{t_2}^{t_3} g(t,x_{t-},u_t(x_{t_0}^{t^-},e_{t_0}^{t}))dt+V(t_3,x_{t_3})|\mathcal{F}_{t_1}^{w}\}+\frac{\epsilon}{K+1}\\
&\geq E\{\int_{t_2}^{t_3} g(t,x_{t-},\widehat{u^2}_{t_2}(x_{t_0},x_{t_1},e_{t_0},e_{t_1},e_{t_2}))dt+V(t_2,x_{t_2})|\mathcal{F}_{t_1}^{w}\}\\
&= E\{\int_{t_2}^{t_3} g(t,x_{t_1},\widehat{u^2}_{t_2}(x_{t_2}-,e_{t_2}))dt+V(t_3,x_{t_3})|\mathcal{F}_{t_2}^{w}\}
\end{aligned}
$$

Since the Markovian property of information state process, given state (x_{t_1},e_{t_2}), we observe that $t_3,\tau_2=t_3-t_2$ are independent of x_{t_0},e_{t_0},e_{t_1}. Furthermore

$$
\int_{t_2}^{t_3} g(t,x_{t_1},\widehat{u^2}_{t_2}(x_{t_0},x_{t_1},e_{t_0},e_{t_1},e_{t_2}))dt=\int_{t_2}^{t_3} g(t,x_{t_1})dt
$$

(iv) In the same way, we can got that there exists some control $\widehat{u^{K-1}}\in\mathcal{U}^M$, the following holds,

$$
\begin{aligned}
V(t_{K-1},x_{t_{K-1}})+\frac{\epsilon}{K+1} &= \inf_{u\in\mathcal{U}} E\{\int_{t_{K-1}}^{T} g(t,x_{t-},u_t(x_{t_0}^{t^-},e_{t_0}^{t}))dt|\mathcal{F}_{t_{K-1}}^{w}\}+\frac{\epsilon}{K+1}\\
&= \inf_{u\in\mathcal{U}[t_{K-1},T]} E\{\int_{t_{K-1}}^{T} g(t,x_{t-},u_t(x_{t_0}^{t^-},e_{t_0}^{t}))dt|\mathcal{F}_{t_{K-1}}^{w}\}+\frac{\epsilon}{K+1}\\
&\geq E\{\int_{t_{K-1}}^{T} g(t,x_{t-},\widehat{u^{K-1}}_{t_{K-1}}(x_{t_0},x_{t_1},\cdots,x_{t_{K-2}},e_{t_1},e_{t_2},\cdots,x_{t_{K-1}}))dt|\mathcal{F}_{t_1}^{w}\}\\
&= E\{\int_{t_{K-1}}^{T} g(t,x_{t_{K-1}},\widehat{u^{K-1}}_{t_{K-1}}(x_{t_{K-1}}-,e_{t_{K-1}}))dt|\mathcal{F}_{t_{K-1}}^{w}\}
\end{aligned}
$$

Since the Markovian property of information state process, given state $(x_{t_{K-2}}, e_{t_{K-1}})$, we observe that $\tau_{K-1} = t_K - t_{K-1}$ is independent of $x_{t_0}, x_{t_1}, \cdots, x_{t_{K-3}}, e_{t_1}, e_{t_2}, \cdots, e_{t_{K-2}}$. Furthermore

$$\int_{t_{K-1}}^{T} g(t, x_{t_{K-1}}, \widehat{u^{K-1}}_{t_{K-1}}(x_{t_0}, x_{t_1}, \cdots, x_{t_{K-2}}, e_{t_1}, e_{t_2}, \cdots, x_{t_{K-1}}))dt = \int_{t_{K-1}}^{T} g(t, x_{t_{K-1}})dt$$

So, from **(i)-(iv)**, we can get that

$$\begin{aligned}
V(t_0, x_{t_0}) + \frac{(K-1)\epsilon}{K+1} &\geq E\{J(t_0, \widehat{u^0}) + V(t_1, x_{t_1})|\mathcal{F}^w_{t_0}\} + \frac{(K-2)\epsilon}{K+1} \\
&\geq E\{J(t_0, \widehat{u^0}) + E\{J(t_1, \widehat{u^1}) + V(t_2, x_{t_2})|\mathcal{F}^w_{t_1}\}|\mathcal{F}^w_{t_0}\} + \frac{(K-3)\epsilon}{K+1} \\
&= E\{J(t_0, \widehat{u^0}) + J(t_1, \widehat{u^1}) + V(t_2, x_{t_2})|\mathcal{F}^w_{t_0}\} + \frac{(K-3)\epsilon}{K+1} \\
&\geq E\{\sum_{i=0}^{K-2} J(t_i, \widehat{u^i}) + V(t_{K-1}, x_{t_{K-1}})|\mathcal{F}^w_{t_0}\} + \frac{\epsilon}{K+1} \\
&\geq E\{\sum_{i=0}^{K-1} J(t_i, \widehat{u^i})|\mathcal{F}^w_{t_0}\}
\end{aligned}$$

Furthermore, we get

$$V(t_0, x_{t_0}) + \epsilon > E\{J(t_0, \widehat{u})|\mathcal{F}^w_{t_0}\}$$
$$\text{where, } \hat{u} = (\widehat{u^0}, \widehat{u^1}, \cdots, \widehat{u^{K-1}}) \in \mathcal{U}^M$$

So, according to **(1)** and **(2)** we get our conclusion.

□

Definition 2.17 The *Infinite Horizon Discounted OSC Problem* is defined as:
For any $s \in [0, \infty)$ and $\xi : \Omega \to X$, ξ is $\mathcal{F}_s$ measurable, we consider the system state (2.13) in the space transition form with control law $u \in \mathcal{U}[s, \infty)$ ($\mathcal{U}^{\mathcal{G}}[s, \infty), \mathcal{U}^M[s, \infty)$):

$$x_t = x(\xi, u_s^t), \quad s \leq t < \infty, \tag{2.3.43a}$$

$$x_s = \xi, \text{ a.s.} \tag{2.3.43b}$$

and consider the cost functions as

$$J^{\mathcal{G}}(s,\xi;u) = E\{\int_s^\infty e^{-\beta t} g(t, x_{t^-}, u_t)dt\, |\mathcal{G}_s\}, \qquad u \in \mathcal{U}^{\mathcal{G}}[s,\infty) \tag{2.3.44}$$

$$J^{\mathcal{F}^w}(s,\xi;u) = E\{\int_s^\infty e^{-\beta t} g(t, x_{t^-}, u_t)dt\, |\mathcal{F}_s^w\}, \qquad u \in \mathcal{U}[s,\infty) \tag{2.3.45}$$

$$J^{M}(s,\xi;u) = E\{\int_s^\infty e^{-\beta t} g(t, x_{t^-}, u_t)dt\, |\sigma(w_s)\}, \qquad u \in \mathcal{U}^{M}[s,\infty) \tag{2.3.46}$$

respectively, where discounted fact $\beta > 0$ and general filtering space $\{\mathcal{G}_t; t \in [s,\infty), \mathcal{G}_t \subset \mathcal{F}_t^w\}$, $g : [s,\infty) \times X \times U \rightarrow \mathbb{R}_+$ is bounded and measurable *w.r.t.* (t,x,u).
The infinite horizon discounted OSC Problems are given by the infimizations:

$$V^{\mathcal{G}}(s,\xi) = \inf_{u \in \mathcal{U}^{\mathcal{G}}[s,\infty)} J^{\mathcal{G}}(s,\xi;u) \tag{2.3.47}$$

$$V^{\mathcal{F}^w}(s,\xi) = \inf_{u \in \mathcal{U}^{\mathcal{F}^w}[s,\infty)} J^{\mathcal{F}^w}(s,\xi;u) \tag{2.3.48}$$

$$V^{M}(s,\xi) = \inf_{u \in \mathcal{U}^{M}[s,\infty)} J^{M}(s,\xi;u) \tag{2.3.49}$$

respectively, where, when they exist, the functions $V^{\mathcal{G}}(s,\xi)$ ($V^{\mathcal{F}^w}(s,\xi), V^M(s,\xi)$) : $\mathbb{R}_+ \times X \rightarrow \mathbb{R}_+$ called the *value functions of the infinite horizon discounted OSC problem* and, when it exists,an infimization function $\hat{u} \in \mathcal{U}^{\mathcal{G}}[s,\infty)$ ($\mathcal{U}^{\mathcal{F}^w}[s,\infty), \mathcal{U}^M[s,\infty)$) shall be called an *optimal control for the infinite horizon discounted OSC problem* with respect to control laws $\mathcal{U}^{\mathcal{G}}[s,\infty)$ ($\mathcal{U}^{\mathcal{F}^w}[s,\infty), \mathcal{U}^M[s,\infty)$), respectively.

□

CHAPTER 3

Poisson Markovian Network Systems

3.1 Call Request and Connection Departure Specifications

In this section, we consider a special but important class of the CAC and RC problems formulated for a network system $\mathcal{NS}$; this class will satisfy the following hypotheses.
(S2) We suppose the call request event process from node o to node d is a *Poisson process* with *parameter* λ_{od}, $\lambda_{od} > 0$, equivalently, the interval between call requests from node o to node d has the *exponential distribution* with parameter λ_{od}; furthermore for each distinct pair $(o, d) \in \mathcal{N} \times \mathcal{N}$, the associated Poisson processes are independent.
(S3) Any call holding time $\tau(t, r), t \in [s, T], r \in \mathcal{R}$, has an exponential distribution, with parameter $\tau, \tau > 0$, and the holding times $\tau(t_1, r_{(1)}); \tau(t_2, r_{(2)}); \cdots; \tau(t_n, r_{(n)}); \cdots$ on all connections initiated at any set of times $\{t_1, t_2, \cdots, t_n, \cdots\}$ are independent random variables.

According to the memoryless property of exponential distribution, the assumptions

(S2) and **(S3)** actually imply assumption **(S1)** (*i.e.* the (state conditionally) independent interval and event property) enunciated in the previous section.

Lemma 3.1 For $\mathcal{NS} \equiv \{Net(\mathcal{N}, \mathcal{L}, C); S, \mathcal{U}\}$ subject to assumptions **(S2)** and **(S3)**, it is the case that:

(1) The interval between call requests in $\mathcal{NS}$ has an exponential distribution with parameter $\sum_{o,d=1;o\neq d}^{N} \lambda_{od}$.

(2) The interval between the instant t at which the system state is $x_t = (x_t^1, \cdots, x_t^R), x_t \in \mathcal{R}, t \in [s, T]$, and the instant at which the first call connection departure occurs has an the exponential distribution with parameter $\mu \sum_{r=1}^{R} x_t^r$.

(3) At the instant t (with state $x_t = (x_t^1, \cdots, x_t^R), x_t \in \mathcal{R}, t \in [s, T]$), the interval between the instant t and the instant at which first event (call request or call connection departure) occurs has an exponential distribution with parameter $\sum_{o,d=1;o\neq d}^{N} \lambda_{od} + \mu \sum_{r=1}^{R} x_t^r$.

Proof: **(1)** We consider $\{\tau_{od}^+;\ o, d \in \mathcal{N} \times \mathcal{N}, o \neq d\}$ as the set of intervals between call requests from node o to node d , according to **(S2)**, they are independent exponential distributions with parameter $\lambda_{od} \in \mathbb{R}_+$ respectively. So, we see the interval between calls request in $Net(\mathcal{N}, \mathcal{L}, C)$ denoted by τ^+ is $\tau^+ = \min\{\tau_{od}^+; (o, d) \in \mathcal{N} \times \mathcal{N}, o \neq d\}$. For convenience, in the next part of this lemma, we arbitrarily order $\{\tau_{od}^+;\ o, d \in \mathcal{N} \times \mathcal{N}, o \neq d\}$, and denote the reordered sequence of random variables as $\{\tau_i^+;\ i = 1, \cdots, N(N-1)\}$; there have the independent exponential distributions with parameters $\{\lambda_i\}$ which correspond one-by-one to $\{\lambda_{od}\}$ respectively. Correspondingly, $\tau^+ = \min\{\tau_{od}^+; (o, d) \in \mathcal{N} \times \mathcal{N}, o \neq d\} \equiv \min\{\tau_i^+; i = 1, \cdots, N(N-1)\}$.

Firstly, we suppose $\tau^+ = \min\{\tau_1^+, \tau_2^+\}$, then,

$$\begin{aligned} P(\tau^+ < t) &= P\big(\min\{\tau_1^+, \tau_2^+\} < t\big) \\ &= P(\{\tau_1^+ < t\} \cup \{\tau_2^+ < t\}) \\ &= P(\tau_1^+ < t) + P(\tau_1^+ < t) - P(\{\tau_1^+ < t\} \cap \{\tau_2^+ < t\}) \\ &= P(\tau_1^+ < t) + P(\tau_1^+ < t) - P(\tau_1^+ < t)P(\tau_2^+ < t) \\ &\qquad \text{(since, } \tau_1^+ \text{ and } \tau_2^+ \text{ are independent)} \\ &= 1 - e^{-(\lambda_1+\lambda_2)t} \end{aligned} \tag{3.1.1}$$

So, for $\tau^+ \triangle \min\{\tau_1^+, \tau_2^+\}$, the lemma holds. Next, we suppose $\tau^+ = \min\{\tau_1^+, \cdots, \tau_K^+\}$ satisfies,

$$P(\tau^+ < t) = 1 - e^{-(\sum_{i=1}^{K} \lambda_i)t} \tag{3.1.2}$$

and establish the lemma by induction. Let us define $\tau^+ = \min\{\tau_1^+, \cdots, \tau_K^+, \tau_{K+1}^+\}$, then

$$\begin{aligned} P(\tau^+ < t) &= P\big(\min\{\tau_1^+, \cdots \tau_{K+1}^+\} < t\big) \\ &= P\big(\{\min\{\tau_1^+, \cdots \tau_K^+\} < t\} \cup \{\tau_{K+1}^+ < t\}\big) \\ &= P\big(\min\{\tau_1^+, \cdots \tau_K^+\} < t\big) + P\big(\tau_{K+1}^+ < t\big) \\ &\qquad - P\big(\{\min\{\tau_1^+, \cdots \tau_K^+\} < t\} \cap \{\tau_{K+1}^+ < t\}\big) \\ &= P\big(\min\{\tau_1^+, \cdots \tau_K^+\} < t\big) + P\big(\tau_{K+1}^+ < t\big) - P\big(\min\{\tau_1^+, \cdots \tau_K^+\} < t\big)P\big(\tau_{K+1}^+ < t\big) \\ &\qquad \text{(since, } \min\{\tau_1^+, \cdots \tau_K^+\} \text{ and } \tau_{K+1}^+ \text{ are independent)} \\ &= 1 - e^{-(\sum_{i=1}^{K+1} \lambda_i)t} \end{aligned} \tag{3.1.3}$$

So, we can get the conclusion that $\tau^+\big(= \min\{\tau_{od}^+; (o,d) \in \mathcal{N} \times \mathcal{N}, o \neq d\} \equiv \min\{\tau_i^+; i = 1, \cdots, N(N-1)\}\big)$ satisfies an exponential distribution with parameter $\sum_{o,d=1;o\neq d}^{N} \lambda_{od}$.

(2) Similarly, we get the interval, $\tau^-(x_t)$, between the instant t at which state $x_t = (x_t^1, \cdots, x_t^R), x_t \in \mathcal{R}, t \in [s,T]$ and the instant at which some connection departure occurs satisfies the exponential distribution with parameter $\mu \sum_{r=1}^{R} x_t^r$.

(3) Finally, since the random variables τ^+ and $\tau^-(x_t)$ are independent and exponentially distributed with parameters $\sum_{o,d=1;o\neq d}^{N} \lambda_{od}$ and $\mu \sum_{r=1}^{R} x_t^r$ respectively, the interval, $\tau(x_t) = \min\{\tau^+, \tau^-(x_t)\}$, between current instant t and the instant at which first event (call request or connection departure) occurs has an exponential distribution with parameter $\sum_{o,d=1;o\neq d}^{N} \lambda_{od} + \mu \sum_{r=1}^{R} x_t^r$.

□

So, from the Lemma (3.1), we observe that, with state x_t, the rate of next event occurrence is $\sum_{o,d=1;o\neq d}^{N} \lambda_{od} + \sum_{r=1}^{R} x_t^r \mu$. Now, we define a constant λ by

$$\lambda = \sum_{o,d=1;o\neq d}^{N} \lambda_{od} + \mu \sum_{l\in\mathcal{L}} C_l \tag{3.1.4}$$

and

$$\lambda(x_t) = \sum_{o,d=1;o\neq d}^{N} \lambda_{od} + \mu \sum_{r=1}^{R} x_t^r,\ x_t \in \mathcal{R} \tag{3.1.5}$$

So, we can see that since $0 \leq x_t \leq C_l$ when $x_t \in C_l$, it follows that

$$\lambda \geq \lambda(x_t),\ \ \forall x_t \in \mathcal{R} \tag{3.1.6}$$

We can establish the study of framework for CAC and RC problems for an underlying network system $\mathcal{NS}$ subject to assumptions **(S2)** and **(S3)**.

3.2 Dynamic Programming for Poisson Markovian Network Systems

In this section, we consider infinite horizon discounted OSC problems for Poisson Markovian network systems.

3.2.1 Uniform Transition Rate Case

We consider the Markovian network system with identical interval, *i.e.* any interval between any instant and the instant at which the first event (call request or connection departure) occurs has an identical exponential distribution.

Lemma 3.2

$$J(t',x;u) = J(t'',x;u), x \in X, 0 \le t',t'' < \infty \tag{3.2.1}$$

So, we can denote $J(t',x;u)$ as $J(x;u)$, *i.e.* discounted infinite horizon cost function $J(t,x;u)$ with initial state $x_t = x$ and control laws u is time invariant.

Proof:

$$\begin{aligned}
J(t',x;u) &= E\{\int_{t'}^{\infty} e^{-\beta(t-t')} g(x_{t^-}, u_t)dt\} \\
&= E\{\int_{0}^{\infty} e^{-\beta t} g(x_{(t+t')^-}, u_{t+t'})dt\} \\
&= E\{\int_{0}^{\infty} e^{-\beta t} g(x_{t^-}, u_t)dt\} \\
&\qquad \text{(since state trajectory is time invariant with the same control laws)} \\
&= J(0,x;u) \\
&= J(t'',x;u)
\end{aligned}$$

□

Definition 3.1 The *Infinite Horizon Discounted OSC Problem for Poisson Markovian Network Systems* is defined as:

In Definition (2.17), we consider the Poisson Markovian Network Systems with assumptions **(S2)** and **(S3)** and suppose system is time-invariant, then :

(1) *Controlled transition probability* from state $x_{t^-} = x \in X$ to state $x_t = y \in X$ with control $u_t(x_{t^-}) \in \mathcal{U}^M$ is denoted as $P(x_{t^-} = x, x_t = y; u) \equiv P(x,y;u)$;

(2) *Loss function* with control u, is $g(x_{t^-} = x, u) = g(x, u)$;

(3) *Cost function*, with control u, $J(s, x; u)$ in Definition (2.17) satisfies $J(x; u) = J(s, x; u)$ (Lemma (3.2));

(4) *Value function* is denoted as $J^0(x; u) = J^0(s, x; u) = \inf_{u \in \mathcal{U}^M} J^0(s, x; u)$, where, when it exists, an infimization function $\hat{u} \in \mathcal{U}^M$ shall be called an *optimal control* for the infinite discounted OSC problem for Poisson Markovian network system with respect to the Markovian control laws $\mathcal{U}^M$.

□

Theorem 3.1 For infinite horizon discounted OSC problems for Poisson Markovian Network Systems with identical interval distribution with parameter λ, for some $\lambda > 0$, there exists an equivalent discrete infinite horizon discounted control problems with discount factor $\alpha = \frac{\lambda}{\beta + \lambda}$, controlled transition probability $P(x, y; u)$, and loss function $\tilde{g}(x, u) = \frac{g(x,u)}{\beta + \lambda}$, for all $x \in X, u \in U$. Bellman's equation for discrete problems is

$$J^0(x) = \min_{u \in U} \{\tilde{g}(x, u) + \alpha \sum_{y \in X} P(x, y; u) J^0(y)\}, \qquad x \in X \tag{3.2.2}$$

Proof:

$$\begin{aligned}
J^0(s, y; u_s^\infty) &= E\{\int_s^\infty e^{-\beta t} g(t, x_{t^-}, u_t) dt\}, \qquad \beta > 0 \\
&= \sum_{k=0}^\infty E\{\int_{t_k}^{t_{k+1}} e^{-\beta t} g(t, x_{t^-}, u_t) dt\} \\
&= \sum_{k=0}^\infty E\{\int_{t_k}^{t_{k+1}} e^{-\beta t} g(x_k, u_k) dt\} \\
&\quad \big(\text{since, } x_t, u_t \text{ are constants for } t_k \le t < t_{k+1}, \\
&\quad \text{we denote them as } x_k, u_k, \text{ respectively.}\big) \\
&= \sum_{k=0}^\infty E\{\int_{t_k}^{t_{k+1}} e^{-\beta t} dt\} E\{g(x_k, u_k)\} \qquad (3.2.3) \\
&\quad \big(\text{since, } g(x_k, u_k) \text{ is constant for } t_k \le t < t_{k+1}, \\
&\quad \text{and also } g(x_k, u_k) \text{ is independent of } \int_{t_k}^{t_{k+1}} e^{-\beta t} dt\big).
\end{aligned}$$

Furthermore,

$$E\Big\{\int_{t_k}^{t_{k+1}} e^{-\beta t}dt\Big\} = \frac{E\{e^{-\beta t_k}\}(1-E\{e^{-\beta \tau_{k+1}}\})}{\beta}$$
$$= \frac{E\{e^{-\beta(\tau_1+\cdots+\tau_k)}\}(1-E\{e^{-\beta \tau_{k+1}}\})}{\beta}$$
$$= \frac{\prod_{i=1}^{k} E\{e^{-\beta \tau_k}\}(1-E\{e^{-\beta \tau_{k+1}}\})}{\beta}$$
$$= \frac{\big(E\{e^{-\beta \tau}\}\big)^k(1-E\{e^{-\beta \tau}\})}{\beta}$$

(since, $\tau_i, i = 1, \cdots, k$ are identical independent random variables as exponential distribution with parameter λ. We consider such random variable as τ)

$$= \frac{\alpha^k(1-\alpha)}{\beta}$$

Where, $\alpha \triangleq E\{e^{-\beta\tau}\} = E\int_0^\infty e^{-\beta\tau}\lambda e^{-\lambda\tau}d\tau = \frac{\lambda}{\lambda+\beta}$

Then, we get

$$E\Big\{\int_{t_k}^{t_{k+1}} e^{-\beta t}dt\Big\} = \frac{\alpha^k}{\beta+\lambda} \tag{3.2.4}$$

So, from (3.2.3) and (3.2.4), the cost function is transformed into the following form.

$$J^0(s,y;u) = E\Big\{\int_s^\infty e^{-\beta t}g(t, x_{t^-}, u_t)dt\Big\}$$
$$= \frac{1}{\beta+\lambda}\sum_{k=0}^{\infty}\alpha^k E\{g(x_k,u_k)\}, \qquad \text{with } x_0 = y$$
$$= \sum_{k=0}^{\infty}\alpha^k E\Big\{\frac{g(x_k,u_k)}{\beta+\lambda}\Big\}$$

So, Bellman's equation is obtained as:

$$
\begin{aligned}
&J^0(x_{t_0})\\
=&J^0(t_0, x_{t_0})\\
=&\inf_{u\in\mathcal{U}(t_0,t_1]} E\Big\{\int_{t_0}^{t_1} e^{-\beta(t-t_0)}g(x_{t^-},u_t)dt + \inf_{u\in\mathcal{U}(t_1,\infty)} E\{\int_{t_1}^{\infty} e^{-\beta(t-t_0)}g(x_{t^-},u_t)dt\}\Big\}\\
=&\inf_{u\in\mathcal{U}(t_0,t_1]} E\Big\{\int_{t_0}^{t_1} e^{-\beta(t-t_0)}g(x_{t^-},u_t)dt + \inf_{u\in\mathcal{U}(t_1,\infty)} E\{\int_{t_1}^{\infty} e^{-\beta(t-t_1)}e^{-\beta(t_1-t_0)}g(x_{t^-},u_t)dt\}\Big\}\\
=&\inf_{u\in\mathcal{U}(t_0,t_1]} E\Big\{g(x_s,u(x_s,e_{t_1}))\int_{t_0}^{t_1} e^{-\beta(t-t_0)}dt\\
&\qquad + E\{e^{-\beta(t_1-t_0)}\}\inf_{u\in\mathcal{U}(t_1,\infty)} E\{\int_{t_1}^{\infty} e^{-\beta(t-t_1)}g(x_{t^-},u_t)dt\}\Big\}\\
=&\inf_{u\in\mathcal{U}(t_0,t_1]} E\Big\{g(x_s,u(x_s,e_{t_1}))\int_{t_0}^{t_1} e^{-\beta(t-t_0)}dt + \alpha\inf_{u\in\mathcal{U}(t_1,\infty)} E\{\int_{t_1}^{\infty} e^{-\beta(t-t_1)}g(x_{t^-},u_t)dt\}\Big\}\\
=&\inf_{u\in U(x_{t_0})} E\Big\{g(x_s,u(x_s,e_{t_1}))\int_{t_0}^{t_1} e^{-\beta(t-t_0)}dt + \alpha\inf_{u\in\mathcal{U}(t_1,\infty)} EJ(t_1,x_{t_1};u)\Big\}\\
=&\inf_{u\in U(x_{t_0})} E\Big\{g(x_s,u(x_s,e_{t_1}))\int_{t_0}^{t_1} e^{-\beta(t-t_0)}dt + \alpha\sum_{x_{t_1}\in X}\inf_{u\in U(x_{t_1})} J(x_{t_1};u)P(x_{t_0};x_{t_1};u(x_{t_0}))\Big\}\\
=&\inf_{u\in U(x_{t_0})} E\Big\{g(x_s,u(x_s,e_{t_1}))\int_{t_0}^{t_1} e^{-\beta(t-t_0)}dt + \alpha\sum_{x_{t_1}\in X} J^0(x_{t_1})P(x_{t_0},x_{t_1};u(x_{t_0}))\Big\}\\
=&\inf_{u\in U(x)} E\Big\{g(x,u(x,e_{t_1}))\int_{t_0}^{t_1} e^{-\beta(t-t_0)}dt + \alpha\sum_{y\in X} J^0(y)P(x,y;u(x))\Big\}, \qquad \text{with } x_{t_0}=x, x_{t_1}=y\\
=&\inf_{u\in U(x)}\Big\{E\{\int_{t_0}^{t_1} e^{-\beta(t-t_0)}dt\}\,E\{g(x,u(x,e_{t_1}))\} + \alpha\sum_{y\in X} J^0(y)P(x,y;u(x))\Big\}\\
=&\inf_{u\in U(x)}\Big\{\frac{E\{g(x,u(x))\}}{\lambda+\beta} + \alpha\sum_{y\in X} J^0(y)P(x,y;u(x))\Big\} \qquad (3.2.5)
\end{aligned}
$$

□

With the stationary control law $u = (u(x_1), \cdots, u(x_{|X|})); u(x_i) \in U(x_i), i = 1, \cdots, |X|$:

(1) The vector form of loss function $\begin{pmatrix} g(x_1, u(x_1)) \\ \cdots \\ g(x_{|X|}, u(x_{|X|})) \end{pmatrix}$ is denoted as g_u;

(2) The vector form of cost function $\begin{pmatrix} J(x_1, u(x_1)) \\ \cdots \\ J(x_{|X|}, u(x_{|X|})) \end{pmatrix}$ is denoted as J_u;

(3) The controlled transition probability matrix

$$\begin{pmatrix} P(x_1, x_1; u(x_1)) & P(x_1, x_2; u(x_1)) & \cdots & P(x_1, x_{|X|}; u(x_1)) \\ \cdots & \cdots & \cdots & \cdots \\ P(x_{|X|}, x_1; u(x_{|X|})) & P(x_{|X|}, x_2; u(x_{|X|})) & \cdots & P(x_{|X|}, x_{|X|}; u(x_{|X|})) \end{pmatrix} \text{ is denoted as } P_u.$$

□

We implement the infinite horizon discounted OSC problems for Poisson Markovian network systems with identical interval distribution with parameter $\lambda > 0$ by *policy iteration algorithm* [9].

(**Step 1**) Take an initial control $u^0(x), x \in X$ *e.g.* a control with which the call request from node $n_i \in \mathcal{N}$ to node $n_j \in \mathcal{N}$ is accepted if and only if it can be allocated in the direct connection (n_i, n_j).

(**Step 2**) Implement the cost function J_{u^k} with control $u^k(x), x \in X, k \in \{0, \mathbb{N}\}$,

$$J_{u^k} = \frac{g_{u^k}}{\beta + \lambda}(I - \alpha P_{u^k})^{-1}, \qquad \text{where } \alpha = \frac{\lambda}{\lambda + \beta} \tag{3.2.6}$$

(**Step 3**) A new control u^{k+1} is obtained from

$$u^{k+1} = \text{argmin}\left\{u : \frac{g_u}{\beta + \lambda} + \alpha P_u J_{u^k}\right\} \tag{3.2.7}$$

if $u^{k+1} = u^k$, then u^{k+1} is the optimal control, otherwise return to **Step 2**.

□

3.2.2 Non-Uniform Transition Rate Case

In actual Poisson Markovian network system, call request and connection departure event (we have analysed in *Section* 3.1) has transition rate $\lambda(x_{t^-} = x) = \lambda(x); x \in X$

which is time invariant and depends on state x. Nonetheless, we can establish an equivalent uniform transition rate case problem which we have studied in Section (3.2.1) for non-uniform transition rate case [9].

Here we define a new controlled transition probability from state $x \in X$ to $y \in X$ with controlled $u \in U$ which is related to control transition probability $P(x, y; u)$ in uniform transition system,

$$\tilde{P}(x,y;u) = \begin{cases} \frac{\lambda(x)}{\lambda} P(x,y;u) & \text{if } x \neq y \\ \frac{\lambda(x)}{\lambda} P(x,x;u) + 1 - \frac{\lambda(x)}{\lambda} & \text{if } x = y \end{cases} \tag{3.2.8}$$

Theorem 3.2 For infinite horizon discounted OSC problems for Poisson Markovian network systems with interval distribution with parameter $\lambda(x) > 0$, and controlled transition probability $P(x, y; u)$, there exists an equivalent discrete uniform infinite horizon discounted control problems with discount factor $\alpha = \frac{\lambda}{\beta+\lambda}$, controlled transition probability $\tilde{P}(x, y; u)$ (3.2.8), and loss function $\tilde{g}(x, u) = \frac{g(x,u)}{\beta+\lambda}$, for all $x \in X, u \in U$. Furthermore, Bellman's equation is

$$J^0(x) = \min_{u\in U}\{\tilde{g}(x,u) + \alpha \sum_{y\in X} \tilde{P}(x,y;u)J^0(y)\}, \quad x \in X \tag{3.2.9}$$

□

From Theorem (3.2) we see implementation of general infinite horizon discounted OSC problems for Poisson Markovian network systems is just same as the corresponding uniform problems.

3.3 A Simple Example

We consider a simple Poisson Markovian network system $\mathcal{NS} = \{Net(\mathcal{N}, \mathcal{L}, C); S, \mathcal{U}\}$, Figure (3.1), where $\mathcal{N} = \{n_1, n_2, n_3\}$, $\mathcal{L} = \{l_1 = (n_1, n_2), l_2 = (n_2, n_1), l_3 = (n_1, n_3), l_4 = (n_3, n_1), l_5 = (n_2, n_3), l_6 = (n_3, n_2)\}$ and $C = \{C_s = 1 \ : \ 1 \le s \le L = 6\}$.

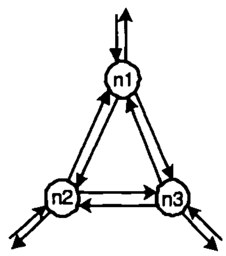

Figure 3.1: A fully connected 3-nodes network

We get the set of connections as

$$\mathcal{R} = \Big\{ r_1 = (n_1, n_2), r_2 = (n_1, n_3, n_2), r_3 = (n_2, n_1), r_4 = (n_2, n_3, n_1),$$
$$r_5 = (n_1, n_3), r_6 = (n_1, n_2, n_3), r_7 = (n_3, n_1), r_8 = (n_3, n_2, n_1),$$
$$r_9 = (n_2, n_3), r_{10} = (n_2, n_1, n_3), r_{11} = (n_3, n_2), r_{12} = (n_3, n_1, n_2) \Big\} \qquad (3.3.1)$$

and the state set is,

$$X = \left\{ x_1 = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 0 \\ \cdots \\ 0 \end{pmatrix}, x_3 = \begin{pmatrix} 1 \\ 1 \\ \cdots \\ 0 \end{pmatrix}, \cdots \right\} \qquad (3.3.2)$$

Even in this extremely simple network, the problem seems much more complex than we expected, *e.g.*, for state $x = (10 \cdots 0)$, the cardinality of admissible control set

$U(x)$ is 17,

$U(x) = \{$ connection departure allocated at connection $(n1, n2)$;

accept only one of following call requests and allocate it in the direct connection

$n2 \rightarrow n3, n3 \rightarrow n2, n2 \rightarrow n1, n1 \rightarrow n3, n3 \rightarrow n1$;

accept only one of following call requests and allocate it in the multi-link connection

$n2 \rightarrow n3, n2 \rightarrow n1, n3 \rightarrow n1$;

accept only two of following call requests and allocate it in the direct connection

$n2 \rightarrow n3$ and $n2 \rightarrow n1$, $n2 \rightarrow n3$ and $n1 \rightarrow n3$,

$n2 \rightarrow n3$ and $n3 \rightarrow n1$, $n3 \rightarrow n2$ and $n2 \rightarrow n1$,

$n3 \rightarrow n2$ and $n1 \rightarrow n3$, $n3 \rightarrow n2$ and $n3 \rightarrow n1$,

$n2 \rightarrow n1$ and $n1 \rightarrow n3$, $n2 \rightarrow n1$ and $n3 \rightarrow n1$ $\}$

In this section we have implemented an optimal CAC and RC problem for a very simple Poisson Markovian network. From this illustrated simple example we can see the complexity of the CAC and RC problems in communication networks and it is actually one of reasons to seek the sub-optimal control laws with so-called decentralized control methods.

CHAPTER 4

Hierarchical CAC and RC Problems in Communication Networks

Because of the great complexity of communication networks, we consider here hierarchical stochastic control methods and present the resulting sub-optimal control laws for the CAC and RC problems introduced in Chapter (2). The hierarchical formulation of the control problems will innovate state aggregation methods and we refer the readers to ([1], [2], [11], [24], [27]) for related state aggregation methods for routing problems.

See Figure (4.1) which is an example of this kind of communication networks.

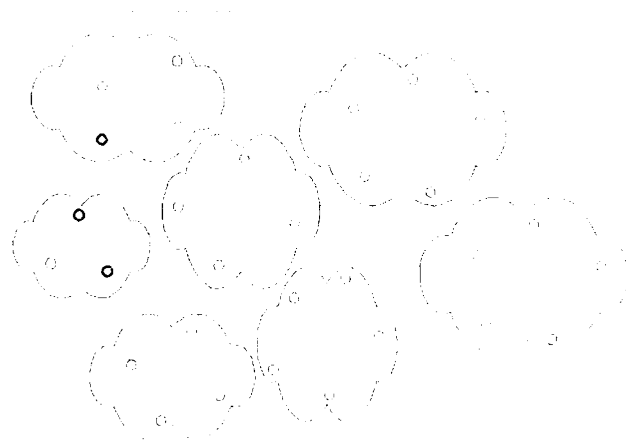

Figure 4.1: A communication network

4.1 Hierarchical Network

Definition 4.1 A *set of local networks with respect to a network* $Net(\mathcal{N}, \mathcal{L}, C)$, denoted $\{Net(\mathcal{N}_i, \mathcal{L}_i, C_i); i = 1, \cdots K\}$, is defined as:

$$\mathcal{N}_i \subset \mathcal{N},$$

$$\mathcal{N}_i \cap \mathcal{N}_j = \emptyset, \qquad \forall i \neq j; i, j \in \{1, \cdots, K\},$$

$$\cup_{i=1}^{K} \mathcal{N}_i = \mathcal{N}, \tag{4.1.1}$$

$$\mathcal{L}_i = \{l = (o, d) \in \mathcal{L}; o, d \in \mathcal{N}_i\}, \tag{4.1.2}$$

$$C_i = \{c_l \in C; l \in \mathcal{L}_i\}, \tag{4.1.3}$$

Each local network Net_i is a connected graph. (4.1.4)

□

Definition 4.2 *A High Level Network* $Net_H \triangleq Net(\mathcal{N}_H, \mathcal{L}_H, C_H)$ with respect to a network $Net(\mathcal{N}, \mathcal{L}, C)$ and its set of constituent local networks $\mathcal{N}_H = \{Net_i = Net(\mathcal{N}_i, \mathcal{L}_i, C_i); i = 1, \cdots K\}$ is defined as the directed multi-graph, where

$$\mathcal{L}_H = \{l = (o, d) \in \mathcal{L}; o \in \mathcal{N}_i, d \in \mathcal{N}_j; \forall i \neq j, i, j \in \{1, 2, \cdots, K\}\}, \tag{4.1.5}$$

$$C_H = \{c_l \in C; l \in \mathcal{L}_H\}. \tag{4.1.6}$$

□

So we see that the high level network Net_H consists of the collection of local networks of $Net(\mathcal{N}, \mathcal{L}, C)$, taken as the nodes of Net_H, together with the set of edges of $Net(\mathcal{N}, \mathcal{L}, C)$ connecting them and their associated capacities, now viewed as high level links and capacities.

See Figure (4.2) which is high level network with respect to network shown in Figure (4.1); Figure (4.3) where bold links indicate information of local network $Net(\mathcal{N}_i, \mathcal{L}_i, C_i)$.

We say a communication network has a *hierarchical networks structure*, if it is decomposed into a set of local networks as in Definition (4.1), in which case, we obtain the high level network given in Definition (4.2).

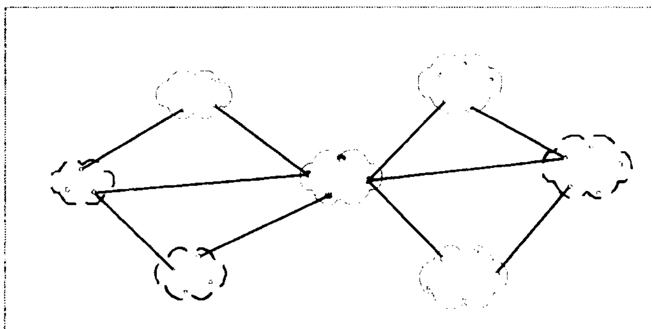

Figure 4.2: High level network

4.2 Stochastic Dynamics and Control for Local Networks of Hierarchical Networks

For any local network $Net_i = Net(\mathcal{N}_i, \mathcal{L}_i, C_i), i \in \{1, \cdots K\}$ of hierarchical network $Net(\mathcal{N}, \mathcal{L}, C)$, we denote set of connections, state set and event set as $\mathcal{R}^i, X^i$ and E^i respectively.

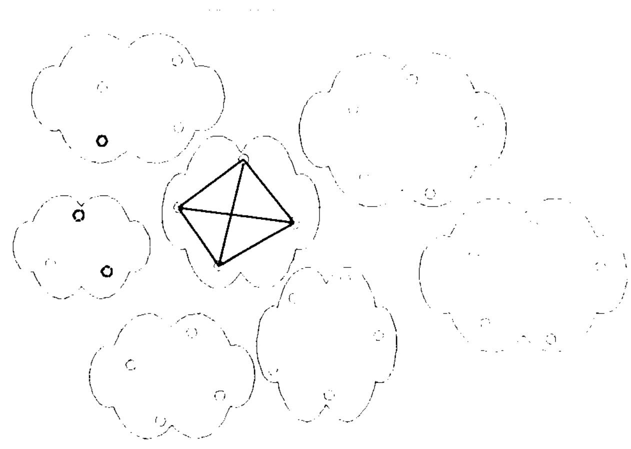

Figure 4.3: Information of a local network

Definition 4.3 For a local local network $Net(\mathcal{N}_i, \mathcal{L}_i, C_i), i \in \{1, \cdots, K\}$, the *set of boundary nodes*, $\mathcal{N}_i^b$, is defined as:

$$\mathcal{N}_i^b = \{n' \in \mathcal{N}_i;\ \exists\, n'' \in \mathcal{N}_j, j \neq i, j \in \{1, \cdots, K\}, (n', n'') \in \mathcal{L}\} \tag{4.2.1}$$

□

Definition 4.4 For a local network $Net_i, i \in \{1, \cdots, K\}$, the *call request and connection departure event set*, E^i, is defined as

$$E^i = E^{H,i} \cup E^{L,i},$$

$$\text{where, } E^{H,i} = \{\emptyset, e_r^-, e_{od}^+; \forall r \in \mathcal{R}^i, \forall o, d \in \mathcal{N}, o \neq d \text{ and } o \notin \mathcal{N}_i \text{ or } d \notin \mathcal{N}_i\},$$
$$E^{L,i} = \{\emptyset, e_r^-, e_{od}^+; \forall r \in \mathcal{R}^i, \forall o, d \in \mathcal{N}_i, o \neq d\} \tag{4.2.2}$$

□

The notion of (a) state (process) for a local network Net_i is defined in exact analogy with that of (a) state (process) for Net in Definition (2.13).

Definition 4.5 For a local network $Net(\mathcal{N}_i, \mathcal{L}_i, C_i)$, the *control set* is U^i, specified by:

$$U^i = \{0\} \cup \{1_r; \forall r \in \mathcal{R}^i\} \cup \{-1_r; \forall r \in \mathcal{R}^i\},$$

where, 1_r is a vector in the $|\mathcal{R}^i|$ dimensional space $\mathbb{R}^{|\mathcal{R}^i|}$ with unit entry in the r-th position; correspondingly, -1_r has an entry -1 in the r-th position.

□

Definition 4.6 We term a sequence of event instants $t^i(\omega)$ in $\mathbb{R}_+$

$$0 < t_1^i(\omega) < t_2^i(\omega) < \cdots < t_k^i(\omega) < t_{k+1}^i(\omega) < \cdots, (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P), \omega \in \Omega,$$

at which random call request and connection departure events occur as a *sequence of random event instants* $t^i : \mathbb{N}_+ \times \Omega \rightarrow \mathbb{R}_+$. The sequence $\tau^i : \mathbb{N}_+ \times \Omega \rightarrow \mathbb{R}_+$, with $\tau^i \equiv t_k^i(\omega) - t_{k-1}^i(\omega)$, where $t_0^i(\omega) \triangleq 0$ is defined as the *sequence of event intervals* (associated to $t^i(\omega)$).

□

Evidently, we obtain that $t_k^i = \sum_{j=1}^k \tau_k^i, \quad i \in \{1, \cdots, K\}, \; t \in \mathbb{R}_+$.

Definition 4.7 We define the *event process* $e^i(t,\omega)$ as a stochastic process $e^i : \mathbb{R}_+ \times \Omega \rightarrow E^i$.

□

Definition 4.8 The *set of measurable control laws with full observations of current information (Markovian control)* is denoted by $\mathcal{U}^{i,M}[s,T]$, where $i \in \{1, \cdots, K\}$,

$$\mathcal{U}^{i,M}[s,T] = \{u^i : [s,T] \times \Omega \rightarrow U; \text{ s.t. } u_t^i(.) \text{ is } \sigma(x_{t^-}^i, e_t^i) \text{ measurable}, t \in [s,T]\},$$

$$\mathcal{U}^{i,M}[s,\infty) = \cup_{T\geq s} \mathcal{U}^{i,M}[s,T].$$

□

We note that the control $u^i(.)$ for Net_i is a function of the high level event through the process $e^i_t \in E^i = E^{H,i} \cup E^{L,i}$, which depends upon both high level $E^{H,i}$ an local $E^{L,i}$ events as discussed below.

We now specify the controlled stochastic dynamics of a state process x^i in $Net(\mathcal{N}_i, \mathcal{L}_i, C_i)$ subject to call request, connection departure events and subject to some specified control law just defined here, *i.e.* $u \in \mathcal{U}^{i,M}[s,T]$.

Before giving the formal definition of the set of state processes x^i in $Net(\mathcal{N}_i, \mathcal{L}_i, C_i)$ we declare that a typical state process in X^i evolves in the following way:

- At a local event instant t^i_k, a local event $e \in E^i$ occurs:
  - if $e \in E^{H,i}_+$, then subject to current local capacity constraints, the call request is be instantaneously accepted and allocated by a pre-assigned local control law $u^i \equiv u^i_{t^i_k} \in U^{i,+}$, and the state (value) $x^i_{t^i_k-}$ is instantaneously transformed into the state (value) $x^i_{t^i_k} = x^i_{t^i_k-} + u^i_{t^i_k}$;
  - if $e \in E^{L,i}_+$, then the local call request can be accepted or refused by a pre-assigned local control law $u^i \equiv u^i_{t^i_k} \in U^i$, and the state (value) $x^i_{t^i_k-}$ is instantaneously transformed into the state (value) $x^i_{t^i_k} = x^i_{t^i_k-} + u^i_{t^i_k}$;
  - if $e = e^{i,-}_r \in \{E^{H,-}, E^{L,-}\}$, then the local control law $u^i = u^i_{t^i_k} = -1_r$ will be implemented.
- During $[t^i_k, t^i_{k+1})$, the local state will remain constant, *i.e.* the state trajectory is piece-wise constant and right continuous.
- At the instant t^i_{k+1}, the local state evolves in the same way as at instant t^i_k.

Definition 4.9 For the network $Net(\mathcal{N}_i, \mathcal{L}_i, C_i)$, the *state response* or *transition equation*, with the control law $u \in \mathcal{U}^{i,M}[s,T]$, for the evolution of the state process

$x^i : [s,T] \times \Omega \to X^i$ with initial state $x^i_s = \xi^i, 0 \le s \le T < \infty$, is given by

$$x^i_t = x^i_{t-} + u^i_t(x^i_{t-}, e^i_t),$$

$$(1)\begin{cases} u^i_t \in \{1_r : r = (m_1, \cdots, m_j) \in \mathcal{R}^i, m_1 = o, m_j = d\}, & \text{if at } t, e^i_t = e^{i,+}_{od} \in E^{H,i}_+ \text{ occurs} \\ u^i_t \in \{0, 1_r : r = (m_1, \cdots, m_j) \in \mathcal{R}^i, m_1 = o, m_j = d\}, & \text{if at } t, e^i_t = e^{i,+}_{od} \in E^{L,i}_+ \text{ occurs} \\ u^i_t = -1_r, \text{ subject to } r \in x^i_{t-}, & \text{if at } t, e^-_r \in E^i \text{ occurs} \end{cases}$$

$$(2)\ x^i_{t-} + u^i_t \in X^i.$$

□

Definition 4.10 For any $i \in \{1, \cdots, K\}$, $s \in [0,T)$ and $\xi^i : \Omega \to X^i, \xi^i$ is $\mathcal{F}_s$ measurable, we consider the system state (4.2.3a) in the space transition form with control law $u \in \mathcal{U}^{i,M}[s,T]$:

$$x^i_t = x(\xi^i, u^t_s), \quad s \le t \le T, \tag{4.2.3a}$$

$$x^i_s = \xi^i, \text{ a.s.} \tag{4.2.3b}$$

and consider the *cost function* as

$$J^{i,M}(s,\xi^i;u) = E\Big\{\int_s^T g^i(t, x^i_{t-}, u_t)dt \,\big|\sigma(x^i_s)\Big\}, \qquad u \in \mathcal{U}^{i,M}[s,T], \tag{4.2.4}$$

where, $g^i : [s,T] \times X^i \times U^i \to \mathbb{R}_+$ is bounded and measurable *w.r.t.* (t,x,u).

□

4.3 Stochastic Dynamics and Control for High Level Networks of Hierarchical Networks

For the high level network $Net(\mathcal{N}_H, \mathcal{L}_H, C_H)$, we denote set of connections, state set and event set as $\mathcal{R}^H, X^H$ and E^H respectively.

Definition 4.11 For the high level network $Net(\mathcal{N}_H, \mathcal{L}_H, C_H)$ of the hierarchical network $Net(\mathcal{N}, \mathcal{L}, C)$, the *call request and connection departure event set*, E^H, is defined as

$$E^H = \{\emptyset, e^+_{od}, e^-_r; \forall o, d \in \mathcal{N}_H, o \neq d, \forall r \in \mathcal{R}^H\}, \tag{4.3.1}$$

where, $e^+_{od} \in E^H$ corresponding to an ordered pair (o, d) of distinct nodes (local networks) in $\mathcal{N}_H \times \mathcal{N}_H$, and $e^-_r \in E^H$ corresponds to a connection r in $\mathcal{R}^H$.

□

For $Net(\mathcal{N}_H, \mathcal{L}_H, C_H)$, a mapping $x^H : [s, T] \in \mathbb{R}_+ \rightarrow X^H$ constitutes a state process trajectory $x^H_t \in X^H$ for all $t, s \leq t \leq T \leq \infty$.

Definition 4.12 For network $Net(\mathcal{N}_H, \mathcal{L}_H, C_H)$, the *control set* is U^H specified by:

$$U^H = \{0\} \cup \{1_r; \forall r \in \mathcal{R}^H\} \cup \{-1_r; \forall r \in \mathcal{R}^H\},$$

where, 1_r is a vector in the $|\mathcal{R}^H|$ dimensional space $\mathbb{R}^{|\mathcal{R}^H|}$ with unit entry in the r-th position, and correspondingly, -1_r has an entry -1 in the r-th position.

□

Definition 4.13 We term a sequence of event instants $t^H(\omega)$ in $\mathbb{R}_+$

$$0 < t^H_1(\omega) < t^H_2(\omega) < \cdots < t^H_k(\omega) < t^H_{k+1}(\omega) < \cdots, (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P), \omega \in \Omega$$

at which random call request and connection departure events occur as a *sequence of random event instants* $t^H : \mathbb{N}_+ \times \Omega \rightarrow \mathbb{R}_+$. The sequence $\tau^H : \mathbb{N}_+ \times \Omega \rightarrow \mathbb{R}_+$, with $\tau^H \equiv t^H_k(\omega) - t^H_{k-1}(\omega)$, where $t^H_0(\omega) \triangleq 0$ is defined as the *sequence of event intervals* (associated to $t^H(\omega)$).

□

Evidently, we obtain that $t_k^H = \sum_{j=1}^{k} \tau_k^H$.

Definition 4.14 We define the *event process* $e^H(t, \omega)$ as a stochastic process $e^H : \mathbb{R}_+ \times \Omega \rightarrow E^H$.

□

Definition 4.15 For the local network $Net(\mathcal{N}_i, \mathcal{L}_i, C_i), i \in \{1, \cdots, K\}$, at an instant t with local state as x_t^i, then the *feasible connection set* from $n' \in \mathcal{N}_i^b$ to $n'' \in \mathcal{N}_i^b$, denoted by $\mathcal{R}^{(n',n'')}(x_t^i)$, is defined as:

$$\begin{aligned}\mathcal{R}^{(n',n'')}(x_t^i) = \{r \in \mathcal{R}^i;\ r = (n^{(1)}, \cdots, n^{(k)}),\\ \text{where } n^{(1)} = n', n^{(k)} = n'' \text{and } c_l - \sum_{r' \in l} x_t^{i,r} > 0, \forall l \in r\}.\end{aligned} \tag{4.3.2}$$

□

Definition 4.16 For the local network $Net_i = Net(\mathcal{N}_i, \mathcal{L}_i, C_i), i \in \{1, \cdots, K\}$, suppose at an instant t, the local state is x_t^i, then we define *(feasible connection) capacities* as,

$$Q_t^{(n',n'')} = |\mathcal{R}^{(n',n'')}(x_t^i)|, \qquad \text{where, } n', n'' \in \mathcal{N}_i^b, n' \neq n'', \tag{4.3.3}$$

and

$$Q_t = (Q_t^1, \cdots, Q_t^{|Q|}), \qquad \text{where, } |Q| = |\mathcal{N}_i|[|\mathcal{N}_i| - 1]. \tag{4.3.4}$$

The set of Q corresponding to all feasible local network states is denoted by $\mathcal{Q}$.

□

EXAMPLE: We suppose that no local and high level connections are allocated at an instant t in Net_2 ,*i.e.* $x_t^2 = 0$, in the hierarchical communication network of Figure

(4.4). Then we get:

$$\mathcal{R}^{(n_{21},n_{24})}(x_t^i) = \{(n_{21}, n_{22}, n_{24}), (n_{21}, n_{23}, n_{24})\},$$
$$\mathcal{R}^{(n_{22},n_{24})}(x_t^i) = \{(n_{22}, n_{24})\},$$

and so

$$Q_t^{(n_{21},n_{24})} = |\mathcal{R}^{(n_{21},n_{24})}(x_t^2)| = 2,$$
$$Q_t^{(n_{22},n_{24})} = |\mathcal{R}^{(n_{22},n_{24})}(x_t^2)| = 1.$$

Clearly, assignments of connections through Net_2 are not independent, for instance if a (single) connection (which has capacity 1) is assigned to (n_{22}, n_{24}), then the feasible connection set becomes $\mathcal{R}^{(n_{22},n_{24})} = \{\emptyset\}$; $\mathcal{R}^{(n_{21},n_{24})} = \{(n_{21}, n_{23}, n_{24})\}$ (with $Q_t^{(n_{21},n_{24})} + Q_t^{(n_{22},n_{24})} = 1 < 3$).

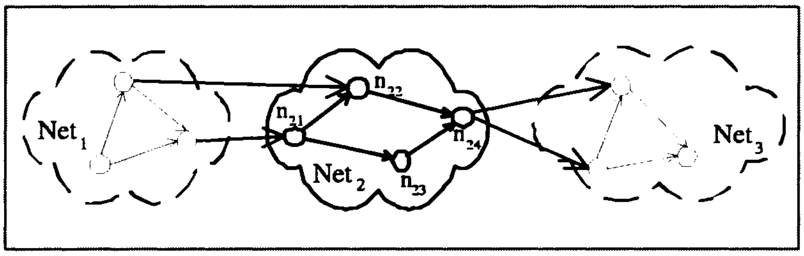

Figure 4.4: A hierarchical communication network

Definition 4.17 The *set of (measurable) control laws with (current) full observations (also called Markovian control laws)* denoted as $\mathcal{U}^{H,M}[s,T]$, is given by:

$$\mathcal{U}^{H,M}[s,T] = \{u^H : [s,T] \times \Omega \to U; \text{ s.t. } u_t^H(.) \text{ is } \sigma(x_{t^-}^H, Q_{t^-}, e_t^H) \text{ measurable}, t \in [s,T]\}$$
$$\mathcal{U}^{H,M}[s,\infty) = \cup_{T \geq s} \mathcal{U}^{H,M}[s,T]$$

□

We now specify the controlled stochastic dynamics of a state process x^H in $Net(\mathcal{N}_H, \mathcal{L}_H, C_H)$ subject to call request, connection departure events, and subject to some specified control law just defined here, *i.e.* $u \in \mathcal{U}^{H,M}[s,T]$.

Before giving the formal definition of the set of state processes x^H in $Net(\mathcal{N}_H, \mathcal{L}_H, C_H)$ we declare that a typical state process in X^H evolves in the following way:

- Assume that at a high level event instant t_k^H, a high level event $e \in E^H$ occurs:
  - in case $e \in E_+^H$, then it could be accepted or refused by a pre-assigned high level control law $u^H \equiv u_{t_k^H}^H \in U^H$, and the high level state (value) $x_{t_k^H-}^H$ is instantaneously transformed into the state (value) $x_{t_k^H}^H = x_{t_k^H-}^H + u_{t_k^H}^H$;
  - in case $e = e_r^{H,-} \in \{E_-^H\}$, then the high level control law $u^H = u_{t_k^H}^H = -1_r$ will be implemented.
- During $[t_k^H, t_{k+1}^H)$, the high level state will remain constant, *i.e.* the high level state equation is right continuous and pairwise constant at instant t.
- At instant t_{k+1}^H, the high level state evolves in the same way as at instant t_k^H.

Definition 4.18 For the high level network $Net(\mathcal{N}_H, \mathcal{L}_H, C_H)$, the *state response* or *transition equation*, with the control law $u \in \mathcal{U}^{H,M}[s,T]$, for the evolution of the state process $x^H : [s,T] \times \Omega \to X^H$ with initial state $x_s^H = \xi^H$, $x_s^i = \xi^i, i \in \{1, \cdots, K\}$ and $Q_s = Q(x_s^1, \cdots, x_s^K) = Q(\xi^1, \cdots, \xi^K)$, $0 \le s \le T < \infty$, is given by

$$x_t^H = x_{t-}^H + u_t^H(x_{t-}^H, Q_{t^-}, e_t^H)$$

$$(1)\begin{cases} u_t^H \in \{0, 1_r : \ r = (Net_{j_1}, \cdots, Net_{j_J}) \in \mathcal{R}^H, Net_{j_1} = o, Net_{j_J} = d; \\ \qquad \forall k > 1, \exists n', n'' \in \mathcal{N}_{j_k} \text{ and } n^{(1)} \in \mathcal{N}_{j_{k-1}}, n^{(2)} \in \mathcal{N}_{j_{k+1}}, \\ \qquad\qquad \text{such that } (n^{(1)}, n'), (n'', n^{(2)}) \in \mathcal{L} \text{ and } Q_{t-}^{(n',n'')}(x_{t-}^{j_k}) > 0, \\ \qquad\qquad\qquad\qquad \text{if at } t, e_t^H = e_{od}^{H,+} \in E_+^H \text{ occurs} \\ u_t^H = -1_r, \text{ subject to } r \in x_{t-}^H, \qquad \text{if at } t, e_r^- \in E_-^H \text{ occurs} \end{cases}$$

$$(2)\, x_{t-}^H + u_t^H \in X^H.$$

□

Definition 4.19 The *high level information state transition equation* with information state $w^H : [s,T] \times \Omega \to \begin{pmatrix} X^H \\ Q \\ E^H \end{pmatrix}$, consists of the information state $w_t^H = \begin{pmatrix} x_{t-}^H \\ Q_{t-} \\ * \end{pmatrix}$, where $* = 0$ if t is not a high level event instant $\{t_k^H\}$ and $* = e_{t_k^H}^H$ at the kth high level event instant t_k^H. The system dynamics, with high level control law $u^H \in \mathcal{U}^{H,M}[s,T]$, have the representation:

$$w_t^H = \begin{pmatrix} x_{t_{k-1}^H}^H \\ Q_{t-} \\ 0 \end{pmatrix}, \qquad t_{k-1}^H < t < t_k^H, \tag{4.3.5}$$

$$w_{t_k}^H = \begin{pmatrix} x_{t_{k-1}^H}^H \\ Q_{t-} \\ e_{t_k}^H \end{pmatrix}, \tag{4.3.6}$$

$$w_{t_k+}^H = \begin{pmatrix} x_{t_k^H}^H \\ Q_{t_k} \\ 0 \end{pmatrix}, \tag{4.3.7}$$

where the initial condition for the system is $w_s^H = \begin{pmatrix} \xi^H \\ Q_s \\ 0 \end{pmatrix}$.

□

Definition 4.20 For any $s \in [0,T)$ and $x_s^H = \xi^H$ and $x_s^i = \xi^i, i \in \{1, \cdots, K\}$, ξ^H and $\xi^i, i \in \{1, \cdots, K\}$ are $\mathcal{F}_s$ measurable, we consider the system state (4.4.1) in the space transition form with high level control law $u \in \mathcal{U}^{H,M}[s,T]$:

$$x_t^H = x(\xi^H, Q_s, {u^H}_s^t), \quad s \le t \le T, \tag{4.3.8a}$$

$$x_s^H = \xi^H, \text{ a.s.} \tag{4.3.8b}$$

$$Q_s = q(x_s^1, \cdots, x_s^K) = q(\xi^1, \cdots, \xi^K), \tag{4.3.8c}$$

and consider the *cost function* as

$$J^{H,M}(s, \xi^H, Q_s; u) = E\{\int_s^T g^H(t, x_{t-}^H, Q_{t-}, u_t^H)dt \,|\sigma(x_s^H, x_s^1, \cdots, x_s^K)\}, \quad u \in \mathcal{U}^{H,M}[s,T], \tag{4.3.9}$$

where, $g^H : [s,T] \times X^H \times U^H \to \mathbb{R}_+$ is bounded and measurable *w.r.t.* (t,x,u).

□

4.4 Doubly Stochastic Hierarchical Networks

The basic stochastic framework for the stochastic evolution of network connections developed in previous sections will now be generalized by a randomization phenomena; this will have the effect of retaining the overall features of the high level connection process in hierarchical networks, but will permit a crucial Markovian property to hold for the aggregation which, in turn, will permit a stochastic dynamic programming analysis in order to find optimal stochastic hierarchical controls for the hierarchical networks.

The extension of the previous formulation is defined as follows:
It is assumed that the underlying probability space $(\Omega, \mathcal{F}, P)$ is extended so as to carry all the new random phenomena introduced below; furthermore, parameter $\varepsilon > 0$ is introduced to index the extended process.

- **1** Let t_k^H be a high level event instant as defined in Definition (4.13), where this is a function of a predefined high level state dependent control $u^H_{t_k^H}(x^H_{t_k^{H-}}, Q_{t_k^{H-}}, e^H_{t_k^H})$ which has been defined for each feasible $(x^H, Q) \in X^H \times \mathcal{Q}$.

- **2** From t_k^H to the instant $(t_k^H + \varepsilon)^-$ no event processes are defined and hence no events occur.

- **3** At $(t_k^H + \varepsilon)$, the collection of internal connections of all local networks $\{Net_i, 1 \leq i \leq K\}$, are randomized within their respective local networks with respect to a *uniform distribution* over the set of internal connections, $S^i_{t_k^H+\varepsilon} \triangleq (x^H_{t_k^H}, Q_{t_k^H})^{-1}$, which are compatible with the high level information $(x^H_{t_k^H}, Q_{t_k^H})$. Necessarily only the information relevant to Net_i forms a non-trivial constraint for the randomization process within Net_i. The randomization process above results in a crucial Markovian property for the hierarchical network because, conditioned on $(x^H_{t_k^H}, Q_{t_k^H})_{Net_i}$, the randomization process within Net_i is (1) independent of the process of other local networks $\{Net_j; j \neq i\}$, and (2) (by virtue of the uniform distribution), independent of all network events at any t, $s \leq t \leq t_k^H + \varepsilon$.

- **4** From $(t_k^H + \varepsilon)$ onwards the global event process continous with respect to the global system state (*i.e.* complete set of global and internal local connections) with initial value $x_{t_k^H + \varepsilon}$. Hence in any given local network Net_i, local connections evolve only according to the $e_{od}^{i,+}, e_r^{i,-} \in E^i$.

- **5** At t_{k+1}^H, when a high level event $e_{od}^{H,+}, e_r^{H,-} \in E^H$ occurs, the high level state of the network Net_H makes an evolution as specified in Definition (4.18).

- **6** During the subsequent period $[t_{k+1}^H, t_{k+1}^H + \varepsilon)$, no events occur.

- **7** At $t_{k+1}^H + \varepsilon$, the randomization process re-occurs, etc.

Definition 4.21 For the hierarchical Markovian network $Net(\mathcal{N}, \mathcal{L}, C)$ with its local networks $Net(\mathcal{N}_i, \mathcal{L}_i, C_i), i = 1, \cdots, K$, and its high level network $Net(\mathcal{N}_H, \mathcal{L}_H, C_H)$, the *state response* or *transition equation*, with high level control law $u^H \in \mathcal{U}^{H,M}[s,T]$ and the local control laws $u^i \in \mathcal{U}^{i,M}[s,T]$, for the evolution of the state process $(x^H, x^1, \cdots, x^K) : [s,T] \times \Omega \to X^H \times X^i \times \cdots \times X^K$ with initial state $x_s^H = \xi^H$, $x_s^i = \xi^i, i \in \{1, \cdots, K\}$, $0 \leq s \leq T < \infty$, is given by

$$x_{t_k^H}^H = x_{t_k^H-}^H + u_{t_k^H}^H(x_{t_k^H-}^H, Q_{t_k^H-}, e_{t_k^H}^H), \qquad \text{(**1** high level dynamics and control)}$$

$$= x_{t_{k-1}^H}^H + u_{t_k^H}^H(x_{t_{k-1}^H}^H, Q_{t_k^H-}, e_{t_k^H}^H), \tag{4.4.1}$$

$$x_{t_k^H+\varepsilon}^i = x(x_{t_k^H}^H, Q_{t_k^H}), \qquad \text{(**3** randomization process over set } S_{t_k^H+\varepsilon}^i) \tag{4.4.2}$$

$$x_{t_{k_j}^i}^i = x_{t_k^H+\varepsilon}^i + u_{t_{k_j}^i}^i(x_{t_{k_j}^i-}^i, e_{t_{k_j}^i}^i), \qquad \forall j,\ t_{k_j}^i \in [t_k^H + \varepsilon, t_{k+1}^H),$$

$$\text{(**4** local network dynamics and control)} \tag{4.4.3}$$

□

Theorem 4.1 Given any high level event instant t^H, let $\varepsilon > 0$ be such that no event is defined in $[t^H, t^H + \varepsilon)$ and assume the randomization process occurs at $(t^H + \varepsilon)$, *i.e.* $x^i_{t^H+\varepsilon} = \xi^i(x^H_{t^H}, Q_{t^H})$ is a random variable which is uniformly distributed over the feasible local network states which satisfy the constraints of the high level state given by $(x^H_{t^H}, Q_{t^H})$. Then

$$(x^H_{t^H_k}, Q_{t^H_k}), \qquad s \leq \cdots < t^H_k < t^H_{k+1} < \cdots \leq T, \tag{4.4.4}$$

is a Markov process, *i.e.* for $s \leq t^H_k < \cdots < t^H_{k+m} \leq T$, the following holds:

$$P\big((x^H_{t^H_{k+m}}, Q_{t^H_{k+m}}) \in \Gamma |\, \sigma(x^H_{t^H_j}, Q_{t^H_j}), j = 1, \cdots, k\big) = P\big((x^H_{t^H_{k+m}}, Q_{t^H_{k+m}})) \in \Gamma |\, \sigma(x^H_{t^H_k}, Q_{t^H_k})\big) \tag{4.4.5}$$

Proof:

(i) First, we suppose $m = 1$, then by Definition (4.18),

$$x^H_{t^H_{k+1}} = x^H_{t^H_k} + u^H_{t^H_{k+1}}(x^H_{t^H_k}, Q_{t^H_{k+1}-}, e^H_{t^H_{k+1}}) \tag{4.4.6}$$

From Lemma (2.1), we can see that,

$$\{e^H_{t^H_{k+1}}\} \coprod_{x^H_{t^H_k}} \{x^H_{t^H_0}, x^H_{t^H_1}, \cdots, x^H_{t^H_{k-1}}\} \tag{4.4.7}$$

And from Definition (4.16) we get

$$Q_{t^H_{k+1}-} = Q(x^1_{t^H_{k+1}-}, \cdots, x^K_{t^H_{k+1}-}) \tag{4.4.8}$$

For high level event instants t^H_k and t^H_{k+1}, $s \leq t^H_k < t^H_{k+1} \leq T$, we suppose $t^H_k < t^i_{k_1} < \cdots < t^i_{k_{J_i}} < t^H_{k+1}$ for each local network Net_i (J_i could be zero, *i.e.* no local event occurs in Net_i in the period of $(t^H_k + \varepsilon, t^H_{k+1})$), then we get

$$\begin{aligned} x^i_{t^H_{k+1}-} &= x^i_{t^H_k+\varepsilon} + \sum_{j=1}^{J_i} u^i_{t^i_{k_j}}(x^i_{t^i_{k_{j-1}}}, e^i_{t^i_{k_j}}) \\ &= x(x^i_{t^H_k+\varepsilon}, e^i_{t^i_{k_1}}, \cdots, e^i_{t^i_{k_{J_i}}}), \end{aligned} \tag{4.4.9}$$

where, $x^i_{t^H_k+\varepsilon} = \xi^i$ is a random variable which is uniformly distributed over the feasible local network states which satisfy the constraints of the high level state information $(x^H_{t^H_k}, Q_{t^H_k})$, and is independent of past information of $(x^H_t, Q_t), t < t^H_k$. And, since, conditioned on state $x^i_{t^H_k+\varepsilon}, e^i_{t^i_{k_1}}, \cdots, e^i_{t^i_{k_{J_i}}}$ is independent of past local state information, we can see that $e^i_{t^i_{k_1}}, \cdots, e^i_{t^i_{k_{J_i}}}$ is independent of past information of $x^H_t, Q_t, t < t^H_k$.

From (4.4.7) and (4.4.8), we get

$$P\big((x^H_{t^H_{k+1}}, Q_{t^H_{k+1}}) \in \Gamma |\, \sigma(x^H_{t^H_j}, Q_{t^H_j}), j = 1, \cdots, k\big) = P\big((x^H_{t^H_{k+1}}, Q_{t^H_{k+1}})) \in \Gamma |\, \sigma(x^H_{t^H_k}, Q_{t^H_k})\big) \tag{4.4.10}$$

(ii) for $m = 2$,

$$\begin{aligned}
&P\big((x^H_{t^H_{k+2}}, Q_{t^H_{k+2}}) \in \Gamma |\, \sigma(x^H_{t^H_j}, Q_{t^H_j}), j = 1, \cdots, k\big)\\
=&\sum P\big((x^H_{t^H_{k+2}}, Q_{t^H_{k+2}}) \in \Gamma |\, \sigma(x^H_{t^H_j}, Q_{t^H_j}), j = 1, \cdots, k+1\big) P\big((x^H_{t^H_{k+1}}, Q_{t^H_{k+1}}) \in \Gamma' |\, \sigma(x^H_{t^H_k}, Q_{t^H_k})\big)\\
=&\sum P\big((x^H_{t^H_{k+2}}, Q_{t^H_{k+2}}) \in \Gamma |\, \sigma(x^H_{t^H_{k+1}}, Q_{t^H_{k+1}})\big) P\big((x^H_{t^H_{k+1}}, Q_{t^H_{k+1}}) \in \Gamma' |\, \sigma(x^H_{t^H_k}, Q_{t^H_k})\big)\\
=&P\big((x^H_{t^H_{k+2}}, Q_{t^H_{k+2}}) \in \Gamma |\, \sigma(x^H_{t^H_k}, Q_{t^H_k})\big)
\end{aligned} \tag{4.4.11}$$

(iii) Similarily, for any $m > 0, m \in \mathbb{N}, t^H_{k+m} \leq T$,

$$P\big((x^H_{t^H_{k+m}}, Q_{t^H_{k+m}}) \in \Gamma |\, \sigma(x^H_{t^H_j}, Q_{t^H_j}), j = 1, \cdots, k\big) = P\big((x^H_{t^H_{k+m}}, Q_{t^H_{k+m}})) \in \Gamma |\, \sigma(x^H_{t^H_k}, Q_{t^H_k})\big) \tag{4.4.12}$$

□

We conclude that Doubly Stochastic Hierarchical Networks establish Markovian high level processes and hence we shall sometimes refer to them as *Hierarchical Markovian Networks.*

See Figure (4.5) for an example of a hierarchical Markovian network.

We illustrate the state evolution of the doubly stochastic hierarchical networks with an example, see Figure (4.6):

- At $(t^H_k)^-$, suppose there exist two high level connections on high level connection (Net_2, Net_3, Net_5) in Figure (4.6), *i.e.* $x^{H,(Net_2,Net_3,Net_5)}_{t^H_k-} = 2$ and hence we get $Q^{(n',n'')}_{t^H_k-}(x^3_{t^H_k-}) = 0, \forall x^3_{t^H_k-} \in X^i$.

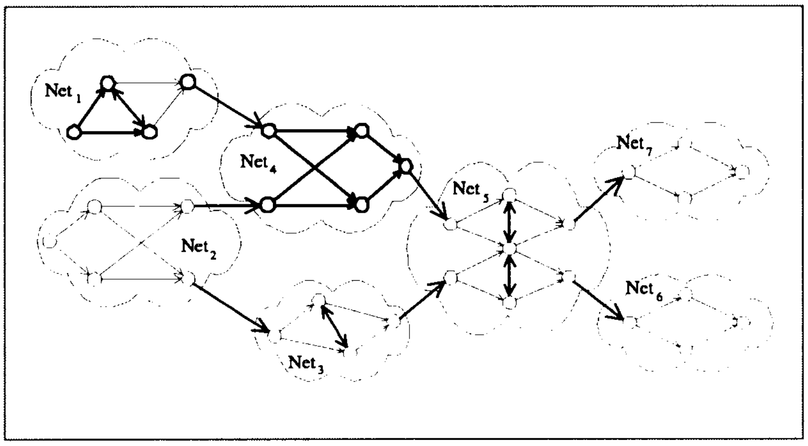

Figure 4.5: A hierarchical Markovian network

- At t_k^H, suppose one of two high level connections on (Net_2, Net_3, Net_5) is dropped, then the high level state evolves into $x_{t_k^H}^{H,(Net_2,Net_3,Net_5)} = 1$ and $Q_{t_k^H}^{(n',n'')}(x_{t_k^H}^3) = 1$.
- From t_k^H to $(t_k^H + \varepsilon)^-$, no event processes are defined and hence no events occur.
- At $(t_k^H + \varepsilon)$, all internal connections of local network Net_3 are re-assigned uniformly over the set of internal connections in Net_3 which is compatible with high level state information $(x_{t_k^H}^{H,(Net_2,Net_3,Net_5)}, Q_{t_k^H}^{(n',n'')}(x_{t_k^H}^3)) = (1,1)$ and hence independently of all previous events,
  i.e., conditional upon $(x_{t_k^H}^{H,(Net_2,Net_3,Net_5)}, Q_{t_k^H}^{(n',n'')}(x_{t_k^H}^3))$.

4.5 Hierarchical CAC and RC Control for Hierarchical Markovian Networks

In sections (4.2, 4.3), we have analysed the stochastic dynamics and control for local networks and high level network of hierarchical networks, respectively. Section 4.4 has established the hierarchical Markovian networks where the high level state processes with some feasible control are Markovian process. Here we define a novel hierarchical

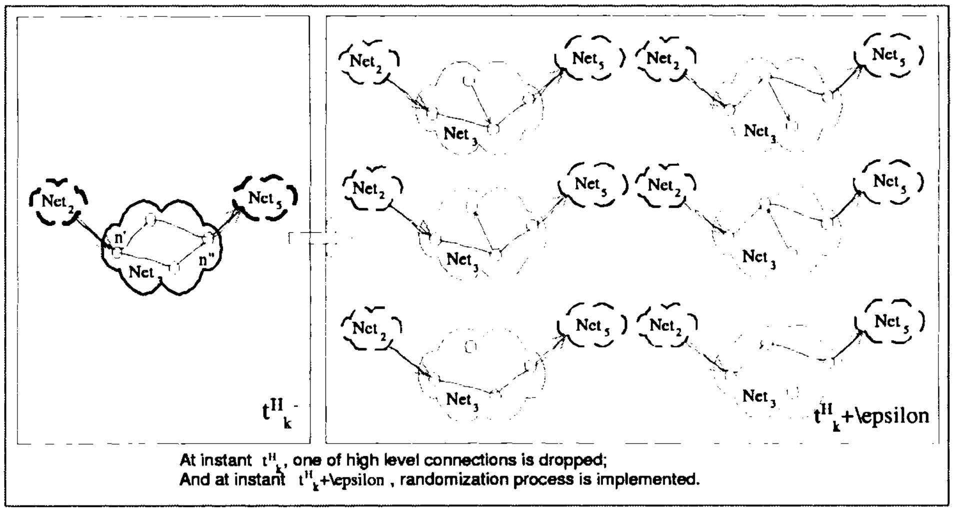

Figure 4.6: An example of randomization process in the local network Net_3

CAC and RC control method for hierarchical Markovian networks in this section.

Definition 4.22 For any $s \in [0,T)$ and $x_s^H = \xi^H$ and $x_s^i = \xi^i, i \in \{1, \cdots, K\}$, ξ^H and $\xi^i, i \in \{1, \cdots, K\}$ are $\mathcal{F}_s$ measurable, we consider the hierarchical state (4.21) in the space transition form with high level control law $u^H \in \mathcal{U}^{H,M}[s,T]$ and local control laws $u^i \in \mathcal{U}^{i,M}[s,T]$ respectively:

the *cost function* of hierarchical Markovian network system is defined as

$$\begin{aligned}
&J^M(s, \xi^H, \xi^1, \cdots, \xi^K; u^H, u^1, \cdots, u^K) \\
=&E\{\int_s^T g^H(t, x_{t-}^H, Q_{t-}, u_t^H)dt \,|\sigma(\xi^H, \xi^1, \cdots, \xi^K)\} \\
&\qquad + \sum_{i=1}^K E\{\int_s^T g^i(t, x_{t-}^i, u_t^i)dt \,|\sigma(\xi^i)\} \qquad (4.5.1)
\end{aligned}$$

where, $g^H : [s,T] \times X^H \times U^H \to \mathbb{R}_+$ and $g^i : [s,T] \times X^i \times U^i \to \mathbb{R}_+$ are bounded and measurable *w.r.t.* (t,x,u).

□

Definition 4.23 The *Optimal Hierarchical Stochastic Control (OHSC) Problem* is defined as:
For any $s \in [0, T)$ and $\xi^H : \Omega \to X^H$ and $\xi^i : \Omega \to X^i$, $\xi^H, \xi^1, \cdots, \xi^K$ are $\mathcal{F}_s$ measurable, consider the hierarchical state and cost function as in Definitions (4.21) and (4.22), respectively.

The optimal hierarchical stochastic control problem is given by the infimization:

$$V^M(s, \xi^H, \xi^1, \cdots, \xi^K) = \inf_{\substack{u^H \in \mathcal{U}^{H,M} \\ u^1 \in \mathcal{U}^{1,M} \\ \cdots \\ u^K \in \mathcal{U}^{K,M}}} J^{H,M}(s, \xi^H, \xi^1, \cdots, \xi^K; u^H) + \sum_{i=1}^{K} J^{i,M}(s, \xi^i; u^i) \tag{4.5.2}$$

□

Since, the hierarchical stochastic control methods introduced here use the doubly stochastic version of the network derived from the original model (by the randomization procedure) the control laws which are optimal for the hierarchical network will in general be sub-optimal for the original network. It is important to analyse the degree of suboptimality.

CHAPTER 5

Conclusion

5.1 Contributions of the Thesis

- Call admission control (*i.e.* CAC) and routing control (*i.e.* RC) problems in integrated communication networks have been formulated and analysed via stochastic dynamic programming.
- CAC and RC problems in Poisson communication networks have been formulated and analysed as discrete-time stochastic control problems.
- Doubly stochastic hierarchical networks have been defined, and CAC and RC control problems are then formulated as stochastic hierarchical control problems.

5.2 Future Work

- The properties of the doubly stochastic network models of Chapter (4) with respect to the standard models of Chapter (2) need to be thoroughly analysed.
- Even though a stochastic hierarchical control methodology has been provided its properties from a stochastic control and complexity viewpoint remain to be

analysed, as well as its performance in terms of implementation in realistic examples.

- Many complex networks, *e.g.* the Internet, are independently operated by many independent agents who are interested in their own revenues or costs and whose private information cannot be observed by others. For such communication networks, the CAC and RC control problems have been considered as game problems, see (*e.g.* [10], [12]-[23], [25], [26], [28], [29]). Based upon the models introduced in this thesis the following work should be undertaken:
  - The hierarchical stochastic control methods provided in this thesis should be generalized to multi-agent network environments and their potential benefits should be studied.
  - The game theoretic issue of simple networks with small group of agents need to be established; CAC and RC control (via dynamic programming) and pricing mechanisms for network resource allocation for multi-agent network need to be studied in detail, and Nash and other equilibria should be analyzed.

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