TEMPERATURE GRADIENT IN DE LAVAL STEAM NOZZLES, & FIVE OTHERS

BATHO

ON THE

TEMPERATURE GRADIENT IN DE LAVAL STEAM NOZZLES.

THE

TEMPERATURE GRADIENT IN DE LAVAL STEAM-NOZZLES.

BY

CYRIL BATHO, M.Sc., B.Eng.

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THE INSTITUTION OF CIVIL ENGINEERS.

SECT. II.—OTHER SELECTED PAPERS.

(*Paper No. 3713.*)

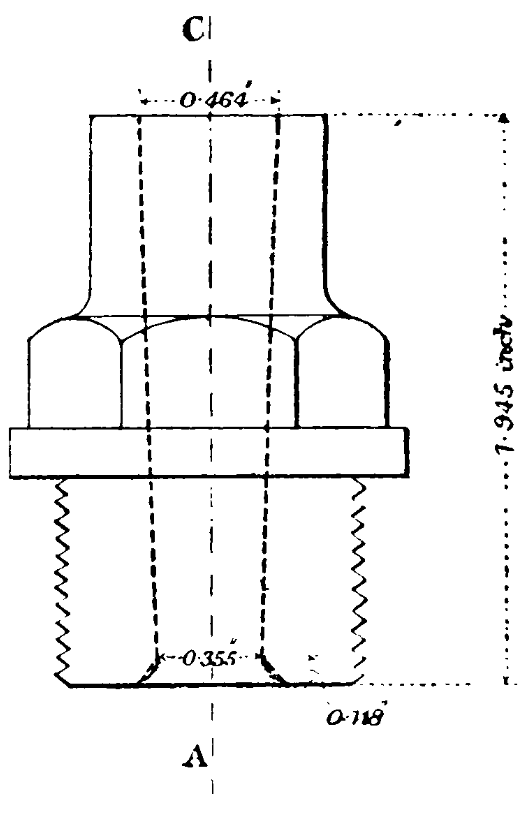
(*Abridged.*)

"The Temperature Gradient in De Laval Steam-Nozzles."

By CYRIL BATHO, M.Sc., B.Eng.

THE present Paper describes ~~some researches~~ an investigation carried out by the Author during the year 1906 to determine the temperature gradient along a steam-turbine nozzle. The experiments, under the direction of Professor W H. Watkinson, M.Eng., M. Inst. C.E., were commenced at the Walker Engineering Laboratories of the University of Liverpool, but some delay was caused during the summer by the failure of the steam supply at the University; through the courtesy of Mr. J. A. Brodie, M.Eng., M. Inst. C.E., (City Engineer), however, work was resumed at one of the Liverpool Corporation refuse-destructors.

Fig. 1.

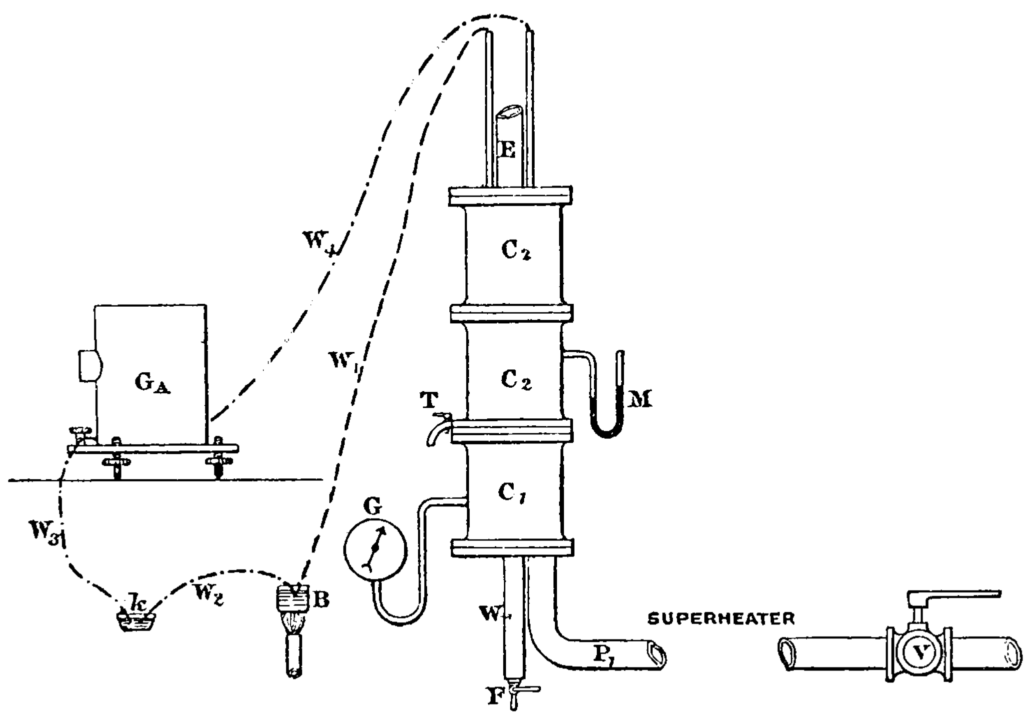


The De Laval nozzle is of the form shown in *Fig. 1.* Steam at a high pressure but low velocity enters the nozzle at A, and emerges from it at a high velocity but low pressure at C, a portion of its potential energy being converted into kinetic energy. In order to determine the action experimentally it is necessary to obtain the temperature, pressure and dryness of the steam at various points within the nozzle. Previous experimenters, for instance Dr. Stodola[1]

[1] A. Stodola, "Steam-Turbines." London, 1906.

of Zurich, have determined the pressure-distribution along the axis of the nozzle, by means of a small central tube having holes bored in its sides. This method is not very reliable, and the results obtained depend on the slope of the holes and the character of their edges; moreover the tube much increases the resistance to flow. It was suggested to the writer by Mr. J. H. Grindley, D.Sc., Assoc. Inst. C.E., then lecturer in Applied Mechanics at the University of Liverpool, that in the case of saturated steam the temperature, and hence the pressure, along the axis might be determined by means of a thermo-junction. Precautions had to be taken to prevent the stream of vapour from impinging against the junction, which would cause too high a temperature to be registered, and it appeared that the only

Fig. 2.

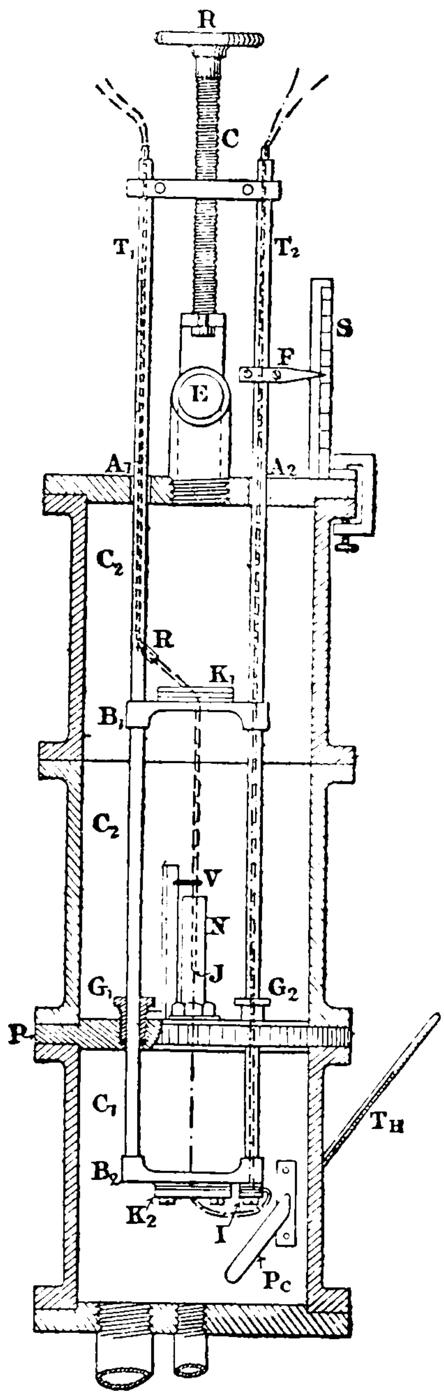


way was to form the junction in a wire stretched along the axis of the nozzle. The wire had to be of small diameter to minimize the resistance to the steam, and the junction had to be practically a point so as to obtain the temperature at one section only; the wires used in the experiments had a diameter of 0·008 inch, and a point-junction was obtained by a method which will be described later.

Fig. 2 shows the general arrangement of the apparatus, and *Fig. 3* the details of the cylinders, etc. Steam entered through the pipe P_1 to the cylinder C_1, its pressure being regulated by means of the valve V; from there it was discharged through the nozzle N into the exhaust-chamber C_2, and finally passed into the atmosphere by means of the exhaust-pipe E, which was $1\frac{1}{2}$ inch in diameter. The pressure in the supply-chamber C_1 was measured by means of a Bourdon pressure-gauge G, whilst a U-tube M, containing mercury, was used to measure the pressure in the exhaust-chamber C_2.

The cylinder C_1 was of cast-iron $\frac{3}{8}$ inch thick, and was $7\frac{1}{4}$ inches high and of $6\frac{1}{4}$ inches internal diameter. Any condensed water collected in a well W, and was allowed to escape through a cock F; this cock also served as an additional means of regulating the pressure in C_1. The exhaust-chamber consisted of two cast-iron cylinders, each $\frac{1}{4}$ inch thick and $7\frac{1}{4}$ inches high. The cast-iron plate P, spigoted into C_1 and C_2, carried the nozzle N under test, this being screwed into a central hole.

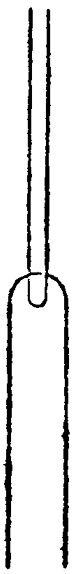
Fig. 3.



The measurement of the temperatures along the nozzle was made by means of a thermo-junction J (*Fig. 3*), the wires leading from this junction being stretched axially along the nozzle. Another junction was placed in a bath of cylinder-oil B (*Fig. 2*), and connected to the first by the wires W_1, W_2, W_3 and W_4. The circuit also included an Ayrton-Mather aperiodic galvanometer G and a mercury key *k*.

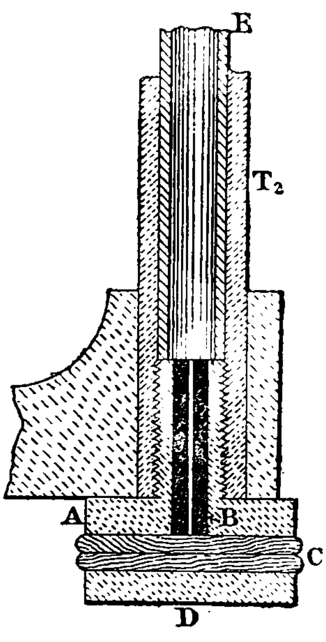
In the earlier experiments platinum and platinum-iridum wires were used (0·01 inch in diameter) welded together in an oxy-hydrogen flame, but no satisfactory point-junction, without increased size at the weld, could be so obtained. The form finally adopted was a loop-junction formed of iron and german-silver wires, 0·008 inch in diameter, as shown in *Fig. 4*; the loop was made as narrow as possible, and the wires were quite close together. A point-junction was thus obtained, offering very little resistance to the flow of steam. Considerable difficulty was experienced at first in obtaining a thermo-junction which would not break after a few minutes' exposure to steam, and the best seldom stood more than 5 or 6 hours' exposure. Much depended upon the manner of insertion, the wires had to be stretched fairly tightly, and precautions taken to ensure that an equal tension was given to the two limbs of each loop. It was noticed that breakage of the wires generally occurred at nearly the same point relatively to the nozzle, wherever the junction might be at the time. This point was in the nozzle and about $\frac{1}{8}$ inch below the outlet, which was singular, as it did not appear that the greatest amplitude of vibration would occur here. It was thought that a single steel wire might be used in place of the two loops, the thermo-junction being formed by annealing the steel up to a known point; it was not, however, found possible to anneal the steel uniformly, though by using a specially prepared material the method may yet be successful.

Fig. 4.



Several methods were tried for supporting the wire in the nozzle; that finally adopted was as follows: Two brass tubes, T_1 and T_2 (*Fig. 3*), $\frac{3}{8}$ inch in diameter and $\frac{1}{16}$ inch thick, passed through glands G_1 and G_2 in the nozzle-plate, and through easy-fitting holes in the top cover. They carried two bridge-pieces, B_1 and B_2, placed 14 inches apart, and carefully set in a direct line with the axis of the nozzle. The wires passed through $\frac{1}{8}$-inch diameter holes drilled in the centre of these pieces, and were held by the insulating clamps K_1 and K_2. The upper wire entered the tube T_1 through the rubber plug R, and the lower wire passed through an insulating gland I into the tube T_2.

Fig. 5.



A separate view of the insulating gland I is shown in *Fig. 5*. It consisted of a brass plug A screwed into the tube T_2, into the centre of which a hole was bored, and fitted with a piece of insulating fibre B. The wire passed through this hole, so as to prevent all contact with the metal, and was caught between two pieces of asbestos C, which were held by a brass plate D fastened to A by two screws—not shown in the figure. Both T_1 and T_2 were lined with glass tubing as shown at E. To reduce the vibration in the unsupported wire between B_1 and B_2 a guide V was used, carried by a brass standard S insulated at the foot. Very slight but quick vibrations still occurred, but they had the effect of keeping the junction clean.

To alter the position of the junction in the nozzle, the tubes T_1 and

T_2 were slid up and down, being operated by the screw R working in a collar C. The position of the junction in the nozzle was found as follows: two holes were bored in the exhaust-cylinder, one level with the top of the nozzle and the other a little higher. The junction was adjusted to be exactly on a level with the top of the nozzle, by sighting through the lower hole, and sliding the tubes T_1 and T_2 up or down until the right position was attained; the interior of the cylinder being illuminated by a light applied to the upper hole. A finger F, screwed to T_2 and sliding against a scale S, was then adjusted to the zero of the scale; the junction could be thus set to any required position in the nozzle. During a test the two holes in the cylinder were plugged up.

To ascertain whether unequal expansion of the wires and of the brass tubes affected the position of the junction relatively to the scale, steam was admitted to the bottom cylinder until the apparatus was thoroughly heated. It was then shut off, and the position of the junction was examined, but no alteration relative to the scale was found. This operation was repeated after each experiment.

The wires leaving the tubes T_1 and T_2 were connected with other wires of the same material to prevent exterior thermo-electric effects. Thus, the wires W_2, W_3 and W_4 (*Fig.* 2) were all of iron, whilst W_1 was of german-silver. All these wires were renewed each time a new junction was put in, and in the later experiments the wires leading from the tubes T_1 and T_2 were in one piece with the wires in the tubes. The second junction, i.e., the one between W_1 and W_2, was immersed in an iron cup B, filled with cylinder-oil, and heated by means of a blow-lamp.

The key *k* consisted of a porcelain crucible filled with mercury, into which the wires W_2 and W_3 dipped.

Method of Experimenting.—The thermo-junction having been adjusted, and the cocks T and F (*Fig.* 2) opened, steam was turned on and allowed to flow through the apparatus for some time, until a steady condition was attained, and water ceased to collect in the well W The cock F was then shut, and the pressure in C_1 was adjusted by means of the valve V The readings were then taken in the following manner:—The oil-bath B was heated until the temperature of the thermo-junction placed in it was higher than that of the junction J, which was observed by repeatedly closing the circuit at K. The bath was then allowed to cool, K remaining closed, until the galvanometer recorded zero deflection; at this moment the temperature of the junction in the bath B was equal to that of the junction J, and therefore to that of the steam in the nozzle at the point J. The thermometer in B was then read, as well as the back-pressure recorded by the mercury-gauge. The pressure and temperature in C_1 were of course kept constant during each experiment, and the pressure in C_1 had to be regulated very carefully, as even a slight change caused inaccuracy in the observed temperatures. It really would have been more satisfactory if a mercury-manometer instead of the Bourdon gauge had been used.

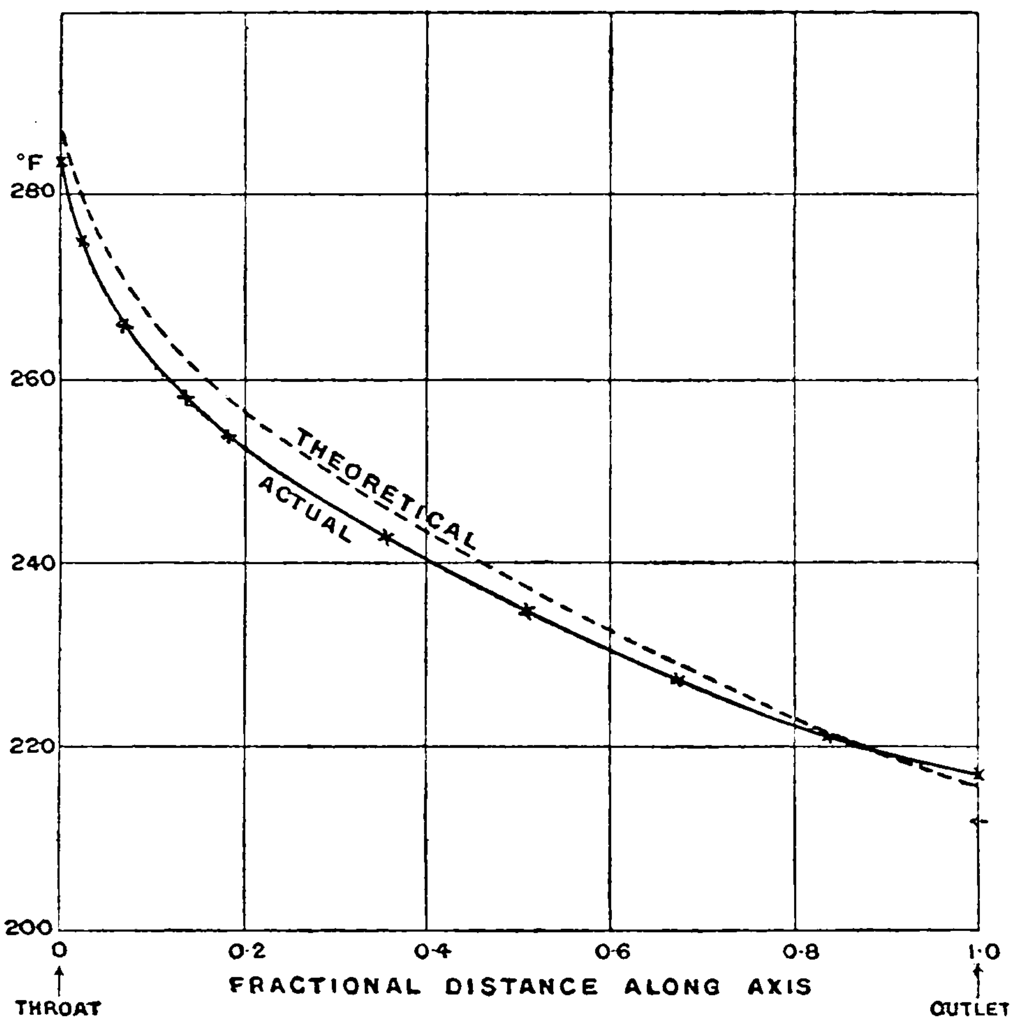
The first reading was taken when the junction J was in its lowest position, and subsequent readings as the junction was moved upwards towards the mouth. In this way it was possible, under favourable conditions, to obtain all the readings for one set of conditions with one heating of the bath B; usually however it was necessary to heat it several times. Readings could only be taken whilst the temperature of B was falling, because when rising the currents in the oil caused deceptive temperatures to be recorded by the bath thermometer, even though the junction was coiled round its bulb. Each pair of junctions was calibrated before being placed in the apparatus.

The first experiments were made with a brass nozzle of the ordinary De Laval type ~~; it was an actual turbine nozzle~~ made by Messrs. Greenwood and Batley. The diameter at the throat was 0·356 inch, and it was designed for an admission pressure of 94·7 lbs. per square inch absolute with a final pressure of 16 lbs. per square inch absolute. As it appeared that the back-pressure in the cylinder C_2 was practically 16 lbs. per square inch absolute, and as a condenser was not available, it was necessary with the low admission pressures used to have a nozzle designed for this back-pressure. Such a one was accordingly designed (*Fig. 1*) with a length of 1·945 inch, the throat 0·118 inch from the end was 0·355 inch in diameter, and the bore was a straight taper; the outlet diameter was 0·464 inch, and the nozzle was highly polished inside.

A large number of experiments were carried out with this nozzle under the conditions for which it was designed. The readings agreed in a striking manner, and occasional discrepancies (never amounting to more than about 2° F.) were explainable by slight changes in the back-pressure. The mean result of these readings is shown in *Fig. 6*.

For reasons which will appear later, it was thought desirable also to make experiments with a non-conducting nozzle, but considerable difficulty was experienced in obtaining the correct profile. A porcelain nozzle with a glazed bore was first used, but it was found that the glazing rendered the walls uneven, and conflicting results were obtained. It was, however, found possible to grind the inside of the nozzle by means of a steel lap, turned to the correct profile and rotated in a lathe; wet sand was used as a grinding material and the nozzle was pressed on by hand, several laps being used to obtain the proper taper. The entrance of the nozzle was ground by means of lead fed with wet sand, lead being used because it adapted itself to the profile and gave a rounded throat. The nozzle thus obtained gave fairly satisfactory results, but the friction

Fig. 6.

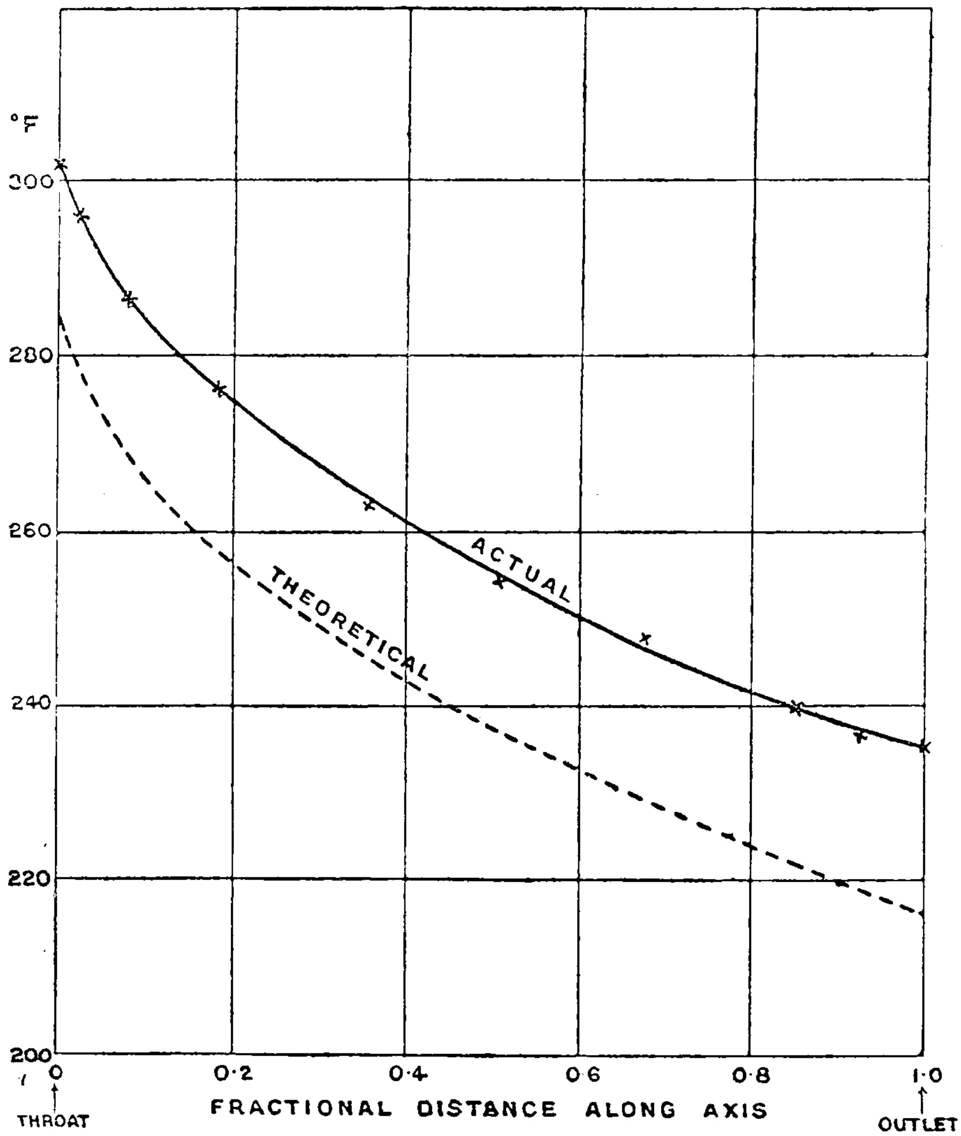


CURVES OBTAINED WITH THE METAL NOZZLE.

loss was high, although the walls were to all appearances quite smooth. The nozzle was 2·04 inches long, the throat being 0·187 inch from the entrance; it was 0·425 inch in diameter at the throat and 0·556 inch at the end, so that it was of the correct proportions for the same initial and final pressures as the metal nozzle, although the weight flow was greater (0·1918 lb. per second against 0·138 lb. per second with the metal nozzle). Several complete sets of experiments were made with this nozzle under the correct pressure conditions, and the results are shown in *Fig. 7*. The readings agreed in a satisfactory manner, the greatest discrepancy from the mean being only 1° F.

It was found that, with the wires and galvanometer used, temperatures could be read to about $\frac{1}{3}$ of a degree Fahrenheit, which was a

Fig. 7.



CURVES OBTAINED WITH THE PORCELAIN NOZZLE.

sufficient degree of accuracy for the method of pressure-regulation adopted.

Theoretical Considerations.—The temperature curves for frictionless adiabatic flow in the nozzles used were obtained as follows:—

Let p_1 be the initial pressure in lbs. per square inch absolute.
" p_2 " final " " " "
" p " pressure at any intermediate section having an area S.

Then the velocity, u, for frictionless adiabatic flow at the cross section of area S is given by the equation[1]

$$\frac{u^2}{2g} = \mathrm{J}\ \{\lambda_1 - (h + x\,l)\} \qquad (1)$$

λ_1 being the total heat of 1 lb. of dry saturated steam at the pressure p_1;

h the total heat of 1 lb. of water at temperature corresponding with p;

l the latent heat of evaporation of 1 lb. of steam at the pressure p;

and x the dryness fraction after adiabatic expansion from the pressure p_1 to the pressure p, the steam being initially dry.

But the weight-flow W is given by

$$\mathrm{W} = \frac{\mathrm{S}\,u}{x\,\mathrm{V}}$$

V being the volume of 1 lb. of dry saturated steam at the pressure p.

Hence
$$\mathrm{W} = \frac{\mathrm{S}\,u}{x\,\mathrm{V}} = \frac{\mathrm{S}_o\,u_o}{x_o\,\mathrm{V}_o}$$

the suffix o denoting the conditions at the throat,

therefore
$$\mathrm{S} = \mathrm{S}_o\,\frac{x\,\mathrm{V}\,u_o}{x_o\,\mathrm{V}_o\,u} \qquad (2)$$

The area of the section at which the pressure p exists is known in terms of the area of the throat, and x and x_o can be found from the energy chart. Thus a curve may be drawn showing the pressure or temperature distribution along the axis of the nozzle.

If the actual pressure at the section S is p' instead of p, and if it is assumed that the only losses in the nozzle are those due to friction,

then
$$\frac{u'^2}{2g} = \mathrm{J}\,\{\lambda_1 - (h' + x'\,l')\}$$

and
$$\mathrm{W} = \frac{\mathrm{A}\,u'}{x'\,\mathrm{V}'}$$

Therefore
$$\frac{u'^2}{2g} = \mathrm{J}\left\{\lambda_1 - \left(h' + \frac{\mathrm{A}\,u'}{\mathrm{W}\,\mathrm{V}'}\,l'\right)\right\} \qquad (3)$$

and W for the nozzle may be found from the theoretical equations,

[1] A. Stodola, "Steam-Turbines," p. 47.

which Rateau,[1] Mr. W. Rosenhain,[2] and others have proved to be practically correct. Equation (3) gives the actual velocity at the section S, and the efficiency of the flow up to this section is thus equal to

$$\frac{W\dfrac{u'^2}{2g}}{W\dfrac{u^2}{2g}} = \frac{u'^2}{u^2}.$$

There will be another loss at the end of the nozzle due to the sudden expansion of the steam to the back-pressure existing in the exhaust-chamber.

All these calculations assume that no heat-exchanges take place. If R units of heat are lost through the walls the velocity (u'') will be given by

$$\frac{u''^2}{2g} = J\left\{\lambda_1 - R - \left(h' + \frac{A\,u''}{W\,V'}\,l'\right)\right\} \quad . \quad (4)$$

whilst if R units are added to the stream,

$$\frac{u''^2}{2g} = J\left\{\lambda_1 + R - \left(h' + \frac{A\,u''}{W\,V'}\,l'\right)\right\} \quad . \quad (5)$$

equation (4) will give a lower and equation (5) a higher value than equation (1).

Discussion of the Results.—The results obtained with the metal nozzle are plotted as a temperature curve in *Fig. 6*, together with the theoretical frictionless adiabatic curve. The curves are only plotted from the throat to the end of the nozzle, since the entrance was too short to allow of temperature measurements being taken along it with any accuracy.

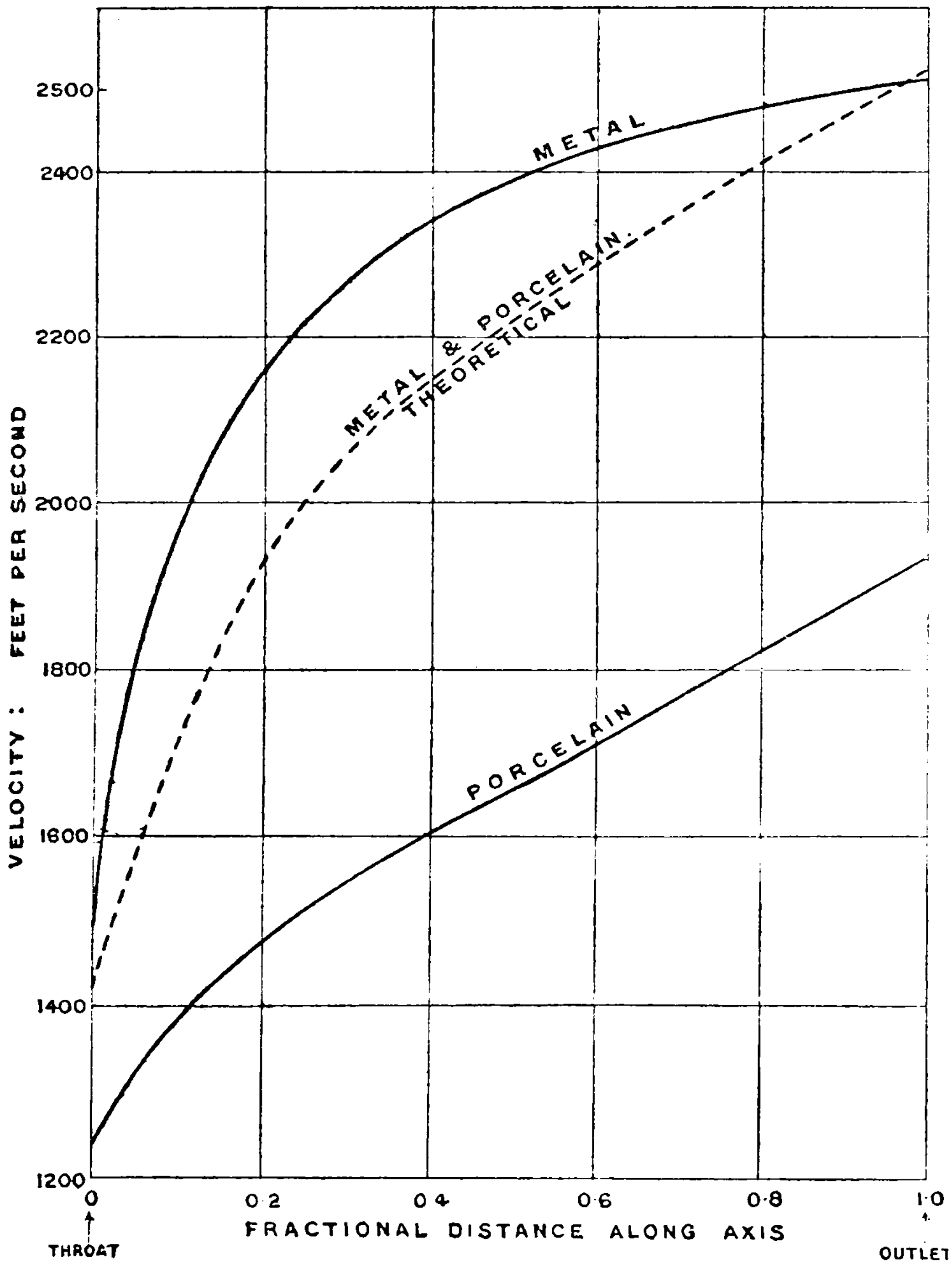
It will be noticed that the experimental temperature at the throat was only 284 1° F. instead of the theoretical 285·7° F. The actual temperature-curve drops more quickly than the adiabatic for about a third of the total length of the nozzle; it then begins to fall less quickly until, at 0·85 of the length of the nozzle from the throat, it cuts and rises above it; the actual temperature at the end of the nozzle being 217·5° F. instead of the theoretical 216° F. Now the velocity at the end of the nozzle calculated for adiabatic (not necessarily frictionless) flow from this final temperature of 217·5° F. was 2,520 feet per second, while the velocity for frictionless adiabatic flow in the nozzle was 2,532 feet per second. From this it would appear that the efficiency of the nozzle was about 99 per cent.—

[1] A. Rateau, "The Flow of Steam through Nozzles and Orifices." London, 1905.

[2] "Experiments on Steam-Jets." Minutes of Proceedings Inst. C.E., vol. cxl, p. 199.

a very high value. The matter requires closer consideration, however, and *Fig. 8* shows the velocity-curve obtained from the experimental temperatures by assuming no heat loss, together with the velocity-curve for frictionless adiabatic flow; it will be seen that the first curve rises above the second at the throat and remains

Fig. 8.



above it for some distance along the nozzle. But the actual velocity could not be greater than the velocity for frictionless adiabatic flow, and must therefore have been obtained from a wrong assumption; in short, the actual flow cannot have been adiabatic. Now if heat were lost from the steam, the velocity would be too high when calculated for adiabatic flow from the temperature at any section; therefore it would appear that the steam lost heat along the nozzle. Unfortunately, in this case, the efficiency could not be arrived at from the experimental results, as there was no means of calculating the rate of heat-loss. Part of this loss ~~at the throat~~ might be due to eddies at the entrance of the nozzle, but the main part must be due to conduction of heat through the walls.[1]

It was to obtain further evidence on this point that the experiments with the porcelain nozzle were carried out. Since porcelain is practically a non-conductor of heat, no heat exchange through the walls could take place. The experimental temperature curve for this nozzle is plotted together with the frictionless adiabatic temperature-curve in *Fig. 7*, and in *Fig. 8* the actual and theoretical velocity-curves are plotted. In the former figure it will be seen that the actual temperature-curve lies above the frictionless adiabatic curve throughout. At the throat, the actual temperature was 301·5° F. against 286·7° F. for frictionless flow, while at the mouth it was 234° F. as against 216° F.

If the efficiency at the end is calculated for adiabatic flow it will be found to be only 60 per cent., while at the throat it is about 67 per cent. It is to be feared that the frictional loss in this nozzle is higher than in the metal one, and that these efficiencies are unduly low; but although this would lessen the value of the comparison between the two, the experimental curve for the porcelain does not show any of the effects that have been attributed to heat-loss in the metal nozzle, as it is entirely above the adiabatic, and has nowhere a greater slope. Even supposing the friction in the porcelain nozzle to be three times that of the metal nozzle, the loss in the latter would be much greater than is apparently shown by the experimental curves.

Conduction of heat along the wires of the junction itself must be very small, because of the small diameter of the wires; besides, the friction of the steam against the junction would aid in equalising the temperature of the two if any difference existed. Again, if the conduction were appreciable, the experimental temperatures would be too high, and the real curve for the metal nozzle would be still further below the adiabatic.

The temperature registered by the junction might be rather higher than the mean temperature, owing to the friction of the

[1] Dr. Grindley has proved that there is such a heat-loss during flow through an orifice in a thin metal plate. See Proceedings Royal Society, vol. 66.

steam causing it to be superheated along the wire, but again this would also cause the real curve of temperature to be below the actual curve obtained.

Although the transmission of heat between a vapour and a surface probably varies directly as the velocity, the loss of heat will become less and less as the steam travels towards the end of the nozzle, because the temperature gradient between the outer wall and the core decreases. Some heat will travel along the walls of the nozzle, since the temperature of the walls must be higher at the throat than at the outlet, and possibly this may be given back to the steam at the end of the nozzle. But the friction of a vapour against a surface varies as some power (greater than 1) of the velocity, therefore the friction increases along the nozzle. It was thus probable that the friction-effect overcame the conduction-effect as the end of the nozzle was reached, causing the temperature curve to rise above the frictionless adiabatic curve, as shown in *Fig. 6.* It might be thought that the conduction of heat from the core of the stream to the walls of the nozzle would be inappreciable because of the very low conductivity of gases. But it must be remembered that the velocity of the steam in the nozzle was much above the critical velocity for steady motion of a gas, and that the rate of transmission was thereby greatly increased.

The experiments made by Dr. Stodola and others did not show this dip of the actual temperature curve below the adiabatic, but their method of experimenting was to measure the pressure gradient in the nozzle by means of a tube having a hole bored in the side, and communicating with a pressure-gauge. Besides greatly increasing the friction loss, this method could not be considered so reliable as the thermo-electric measurement of the temperature.

Fig. 9 shows the results of experiments made with the metal nozzle with an initial pressure of 64·7 lbs. per square inch absolute, i.e. less than the 94·7 lbs. per square inch absolute for which it was designed; the frictionless adiabatic curve is also given. It will be seen that the actual curve is entirely above the latter, and that it falls continuously along the axis, reaching a temperature lower than that corresponding with the back-pressure at the end of the nozzle.

Many experiments made with various initial pressures gave the same result, which is contrary to that obtained by Dr. Stodola.

Summary.—The chief conclusions to which these experiments lead are:—

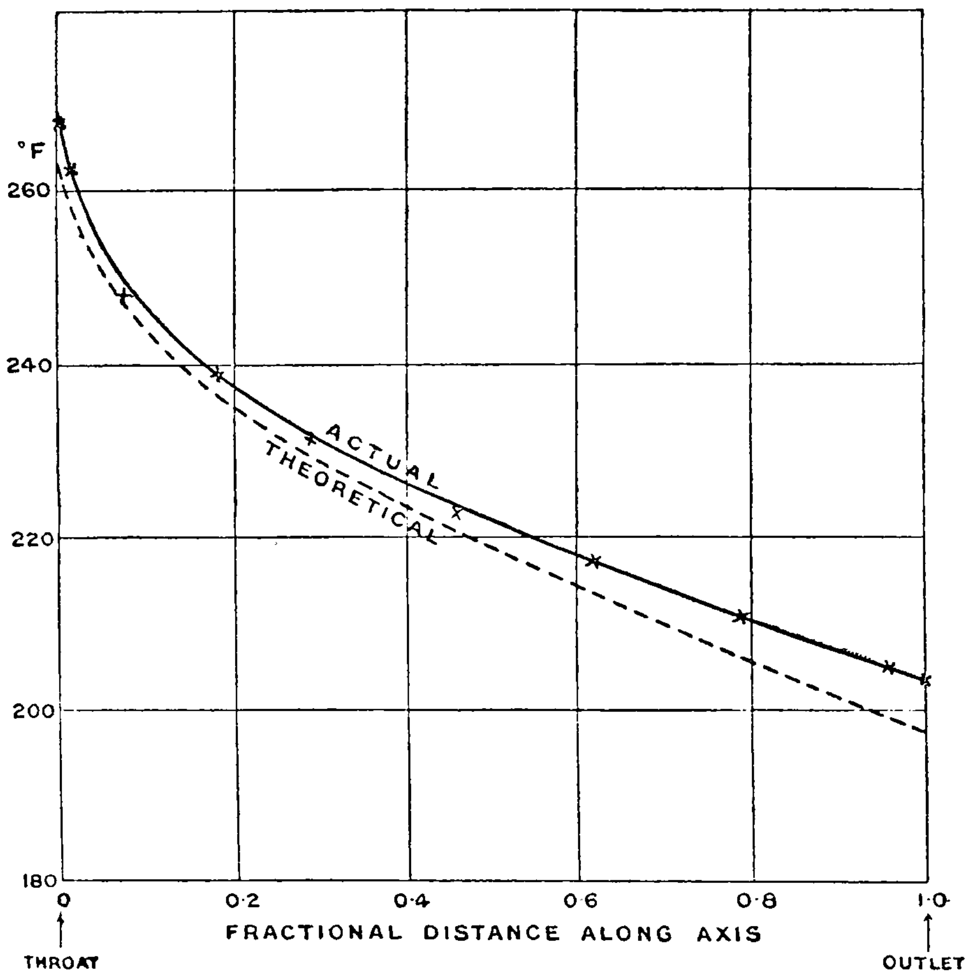
1. It is possible to measure accurately the temperature at various points in a nozzle by means of a thermo-junction.

2. It is shown by this method that there is a definite transfer of heat from the steam in the nozzle to the walls, part of which travels along the walls and may be given back to the steam at the end.

3. It is, therefore, incorrect to calculate the efficiency of the nozzle from the outlet-temperature without considering this heat loss.

4. With a nozzle of non-conducting material this heat transfer

Fig. 9.



Curves obtained with the Metal Nozzle at Reduced Pressure.

does not occur, and if the bore can be made smooth enough, such nozzles would probably prove more efficient in actual practice than those made of metal.

5. When the initial pressure is less than that for which a nozzle is designed, the temperature falls continuously along the nozzle and reaches a temperature lower than that corresponding with the back-pressure at the outlet.

In conclusion, the Author desires to express his indebtedness to Professor Watkinson and Dr. Grindley for their valuable suggestions during the progress of the experiments, and to Mr. Okill (Research Assistant at the Walker Engineering Laboratories), who was associated with him throughout the experimental part of the work.

The Paper is accompanied by nine drawings from which the Figures in the text have been prepared.

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The Distribution of Stress in Certain Tension Members.

Trans. Can. Soc. C. E. 191

MECHANICAL SECTION.

Chairman—H. H. Vaughan.

Vice-Chairman—W. J. Francis.

A meeting of the Mechanical Section was held on Thursday, 25th April, at 8.15 p.m., Mr. J. M. Shanly in the chair. A paper on "Distribution of Stress in Certain Tension Members," by Mr. Cyril Batho, A. M. Can. Soc. C.E., was read by the author.

PAPER No. 328

THE DISTRIBUTION OF STRESS IN CERTAIN TENSION MEMBERS.

By C. Batho, A. M. Can. Soc. C. E.

It is becoming generally recognized among engineers that a correct knowledge of the strength of structural members cannot be obtained by breaking tests alone. This is more especially the case with built up members in which it is probable that, as soon as some part reaches the elastic limit, the distribution of the load may change, so that the breaking load and the appearance of the specimen at fracture may not be any true guide to the action of the parts under working loads.

The most satisfactory way of obtaining a knowledge of the latter is by measuring the actual strain distribution under working loads, or, at any rate, at loads within the elastic limit of the parts, by means of some form of extensometer. Unfortunately, most forms of extensometers are open to many objections for this kind of work; some are inaccurate, others only measure the average strain over a long length, and nearly all are more or less complicated, take up a great deal of space and cannot be used in positions which are difficult of access, such as the interior of a built up column or between two angles. The writer knows of only one form of extensometer which, when proper precautions are taken, may be said to approach the ideal for this purpose. This is the Martens Extensometer, invented by Professor Martens, director of the Königliche Material Prüfungs Anstalt at Grosse Lichtefelde West, Berlin. This instrument is extremely simple in construction, easy to calibrate, and may be used in the most confined positions. (See Fig. 4.) It does not appear to have received the attention it deserves, possibly because of its simplicity, or because of inaccurate results obtained by lack of certain necessary precautions in its use. Under the conditions of the experiments described later, it was found to be capable of accurately estimating the strain over a length of 4″ to $\frac{1}{100,000}$″. The

Martens Extensometer was first used in the Testing Laboratory of McGill University, in 1906, for such work as is here described, but, owing to the fire of 1907, research work was considerably delayed, and has only lately been resumed.

The present paper gives an account of experiments made at McGill University to determine, by means of strain measurements with the Martens Extensometer, the distribution of stress in single and double angles with riveted end-plates loaded in tension, and to compare it with the theoretical distribution under different assumptions. Experiments are still in progress on similar members in compression and on built up members, and it is hoped that the present paper may be only a first contribution on the subject.

The experiments on built up members indicate that these do not, in general, act as one solid piece, but that the separate parts must be considered as eccentrically loaded members subject to constraints. From this it appears that the only way to build up a satisfactory theory of the action of such members is to commence with the problem, which is important in itself, of a uniform piece subjected to an eccentric load, and to work up gradually to more complicated members. This preliminary problem, with its application to the simplest form of compound member made from two angles placed back to back, is the subject of the present discussion.

Theoretical Considerations.

The method of finding the distribution of stress in a piece of uniform cross section, subjected to a load which is eccentrically applied, but which lies in an axis of symmetry of the cross-section, is well known and need not be considered in detail here. In this case the resultant stress at any point of the cross-section, the lateral deflection due to eccentricity being neglected, is given by the formula

$$f = \frac{N}{A} \pm \frac{Ney}{I} \qquad \ldots \ldots \ldots \ldots \ldots 1$$

Where N is the normal load, A the area of cross-section, I the moment of inertia of the cross-section about an axis in its plane through its centre of gravity and perpendicular to the line of symmetry on which the load axis lies, y the perpendicular distance of the given point from this axis, and e the eccentricity of the load, *i. e.* the distance of its point of application from the centre of gravity of the section. The + sign must be taken for points on that side of the centre of gravity on which the loading axis lies, and the — sign for points on the other side of the centre of gravity.

The equally, if not more, important case of a load applied eccentrically, and not in a line of symmetry of the cross-section (which includes, for example, the case of a single angle under tension riveted by one leg, and probably, as will be seen later, many cases of built up members where the load is apparently in a plane of symmetry) seems to be little known in this country, although it has been investigated thoroughly by many German writers. The only complete account in English, known to the writer, is in a paper by L. J. Johnson, Trans. Am. Soc. Civil Eng., Vol. 56, 1906*. The full development of the formulæ is considered in Appendix I, and only an outline of the method and the details of actual calculation will be given here.

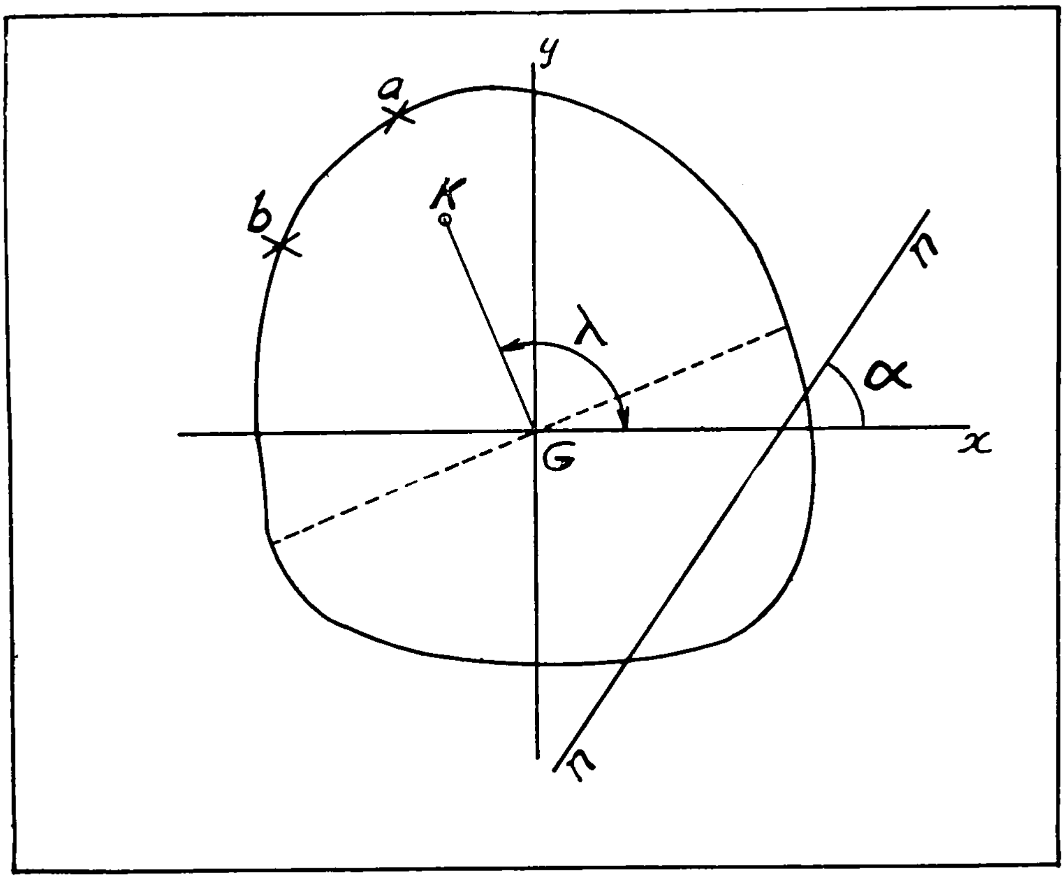

Fig. 1.

Consider a straight bar of uniform cross-section subjected to a load N, parallel to the axis of the bar, but which does not pass through the centre of gravity of the section. Let K (Fig. 1) be the loading point and G the centre of gravity of the cross-section. If KG is an axis of symmetry of the cross-section, the case will be that considered above, bending will take place about an axis in the plane perpendicular to KG and the maximum stress will be at a. If, however, K does not lie on an axis of symmetry, the neutral axis

*See Appendix I.

will be in some other direction such as *nn*, and the maximum stress will occur at *b*. Choose any convenient rectangular axes *Gx*, *Gy* through the centre of gravity (if the section is a standard one of which the moments of inertia are tabulated in the hand books, *Gx* and *Gy* should be the axes of the given moments of inertia) and indicate the angle *KGx* by λ Then the inclination, α, of the neutral axis to the axis *Gx* is given by the equation

$$tan\ \alpha = \frac{I_x - J\ tan\ \lambda}{J\ - I_y\ tan\ \lambda} \quad \ldots\ldots \quad 2$$

Where I_x is the moment of inertia of the cross-section about *Gx*, I_y the moment of inertia about *Gy* and *J*, the product of inertia about *Gx*, *Gy*. The only assumption made in deducing this is that the distribution of stress follows a linear law. Expressing this symbolically, and forming three equations expressing that the total normal internal force across the section is equal to *N*, and that the sums of the moments of the internal forces about *Gx* and *Gy* are equal to the moments of *N* about *Gx* and *Gy* respectively, equation (2) may be deduced. (See page 23.) In a similar way the equations

$$f = N \left[\frac{1}{A} + \frac{y - x\ \ tan\ \alpha}{J - I_y\ \ tan\ \alpha}\ x_k\right] \quad \ldots \quad 3$$

$$f = N \left[\frac{1}{A} + \frac{y\ - x\ tan\ \alpha}{I_x\ - J\ tan\ \alpha}\ y_k\right] \quad \ldots \quad 4$$

giving the stress, *f*, at any point (*x*, *y*) of the cross-section, may be found. In these equations *A* is the area of the cross-section and x_k and y_k are the co-ordinates of the load point *K*. In order to find the maximum stress, all that is necessary is to substitute for *x* and *y* in (3) or (4) the co-ordinates of the point *b* furthest away from the neutral axis. This may usually be determined readily by inspection. If *f* be made zero, either (3) or (4) will give the equations of the neutral axis and thus its position may be found.

The above equations become much simpler if *Gx* and *Gy* happen to be the principal axes of inertia of the cross-section, for in this case *J* is equal to zero. The moments of inertia given in the hand books for standard angle sections, etc., are not taken about the principal axes. For this and other reasons, it is better to take the axes for such sections parallel to the legs of the angle and to calculate *J*, which is

$$\iint xy\ dx\ dy$$

taken over the section. This is usually easy to evaluate, as will be seen from the example considered later.

A few points in the application of this theory to long members subjected to tension or compression must now be considered. In deducing the above equations it is, of course, assumed that the piece is free to bend in any direction. If it does so, the point K will be differently situated relatively to the cross-section at different sections, and this must be taken into account if correct values are to be obtained for the stresses, especially when near to the central section of a long member. In practice this will usually be a needless refinement, but in attempting to verify the theory by experiment, it must be considered. If the ends of the piece are constrained in any way, say for example, by the grips of the testing machine or the end connecting plate, or by riveted connections in actual structures, a constraining couple will be introduced, and this will have the effect of altering the position of the resultant force N. One of the deductions made from the experiments to be described is that the connecting plate in the case of riveted single angles does not introduce any considerable fixing couple, except in the plane of the plate, but, in attempting to build up a correct theory for the double angle, this constraint must be considered.

As an example of the method of calculation of the position of the neutral axis and the maximum stress in the cross-section, the case of a single angle 3″ x 3″ x ¼″ in cross-section, loaded at the middle point of one of its external faces, will now be worked out in full. This was the section of the angles used in experiments, and the results obtained from calculation will be necessary in the discussion of the experimental results.

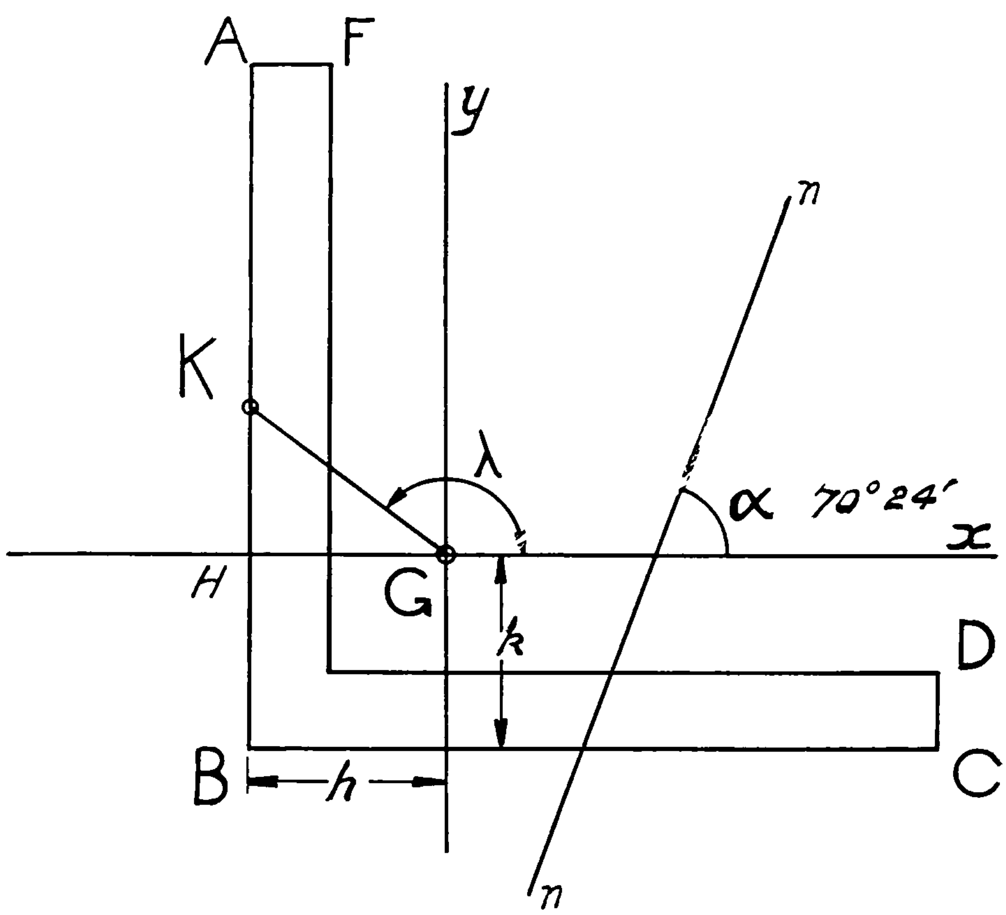

Fig. 2.

Figure 2 shows the cross-section. The axes Gx and Gy are taken parallel to the two legs of the angle. The following data are obtained from the Cambria Steel Handbook.

$A = 1.44$ square inches.

$I_x = I_y = 1.24$ (inch)^4 units.

Distance of G from the back of the leg $= 0.84''$. It is not very convenient to calculate J for the axes Gx and Gy, but as the calculation is very easy for the axes BC, BA, it will be made for these axes first, and then found for the axes through Gx, Gy by means of the formula

$$J_B = J_G + A\,h\,k$$

where J_B is the product of inertia about BC, BA.

J_G is the product of inertia about Gx, Gy and (h, k) are the coordinates of G referred to BC, BA. Now, using x' y' for coordinates referred to BC, BA,

$$J_B = \int\int x'\,y'\,dx'\,dy'$$

$$= \int_0^3\int_0^3 x'\,y'\,dx'\,dy' - \int_{0.25}^3\int_{0.25}^3 x'\,y'\,dx'\,dy'$$

$= 0.28$ (inch)^4 units,

the angle being considered as the difference between two squares. Hence

$$J_G = 0.28 - 1.44 \times (0.84)^2$$

$= -0.74$ (inch)^4 units.

This is correct to the second place of decimals, neglecting the rounding of the corners of the angles, etc., which is close enough for most purposes. It would save a great deal of calculation if the quantity J were tabulated in the handbooks on steel.*

The angle is supposed to be loaded at the point K. Thus $\tan\lambda$ is in this case equal to $-\dfrac{KH}{HG} = -\dfrac{1.5 - 0.84}{0.84}$

$$= -0.786$$

and the inclination of the neutral axis to the axis Gx is, from equation (2), given by

$$\tan\alpha = \frac{1.24 - 0.74 \times 0.786}{-0.74 + 1.24 \times 0.786}$$

$$= 2.81$$

Therefore $\alpha = 70^\circ\ 24'$.

The maximum stress obviously occurs at A and may be obtained

*See end of Appendix I for a simpler method for calculating J.

from equation (3) or (4). From 3, substituting $y = 2.16$, $x = -0.84$,

$$f_A = N\left[0.69 + \frac{2.16 + 0.84 \times 2.81}{-0.74 - 1.24 \times 2.81} \times (-0.84)\right]$$
$$= 1.59\ N$$

The ratio of the maximum to the mean stress is, therefore, $1.59 \times 1.44 = 2.29$, and thus the stress estimated on the not unusual assumption that the load is uniformly distributed gives only $43\frac{1}{2}\%$ of the correct amount.

The equation of the neutral axis may be obtained by giving f the value zero in equation (3) page 4.

$$0 = 0.69 + \frac{0.84}{4.71}(y - x \times 2.81)$$

or $\quad y = 2.81\,x - 3.87.$

It cuts Gx at the point $x = 1.22$ and is shown by the line nn in the figure.

It will be seen from the above that the calculation, using the correct theory, is simpler than that assuming a neutral axis perpendicular to KG and equation (1) for the stress distribution, because the latter, besides being entirely without rational basis, would involve the calculation of the moment of inertia of the cross-section about an axis perpendicular to KG. If bending were incorrectly assumed to take place about Gy the eccentricity of the load would be 0.84″ and the stress at A would be, from equation (1)

$$= \left[\frac{0.84 \times 0.84}{1.24} + 0.69\right] N$$
$$= 1.26\ N,$$

which is about 20% too small, whilst if it were assumed to take place about Gx the eccentricity would be 0.66″ and the stress A would be

$$= \left[\frac{0.64 \times 2.16}{1.24} + 0.69\right] N$$
$$= 1.84\ N,$$

which is approximately 16% too great, so that the correct value in the case of the given angle is approximately the mean of the values assuming bending about Gx and Gy respectively.

The Experiments.

All the experiments to be described were made in tension on specimens consisting of 3″ x 3″ x $\frac{1}{4}$″ angles having different forms of end connections. In the first experiments a simple angle was used, one leg being cut off shorter than the other, so that the specimens could be gripped in the machine by the other leg. It was tested in tension under different conditions, with the object of verifying the theory described above. It was found, however, that although the distribution of stress was planar, the positions of the line of pull varied with each placing in the machine, and the results are not thought sufficiently interesting to be published. Experiments were then made on the two single angle members shown in Fig. 3. The angles were 4′ 7½″ long and 3″ x 3″ x ¼″ cross section, and were riveted by means of four ¾″ rivets having a pitch of 2½″ to end plates ⅜″ and ¼″ thick respectively, different thicknesses of end plate being used with the object of determining the effect of the restraint to bending offered by end connections of different stiffnesses. The results of the test are given in Tables I, II, and IV. The remaining experiments were made on the double angle member shown in Fig. 3. This consisted of two angles placed back to back and connected at the ends to a loading plate ⅜″ thick, by four ¾″ rivets of 2½″ pitch.The results of the tests on this angle are given in Tables III and IV. The machine used was the Emery testing machine in the McGill University Testing Laboratory. This machine is of the vertical type and has a capacity of 150,000 lbs. The length of the specimens was governed by the limits of the machine. The Emery type is eminently adapted to this kind of work, because the line of pull is constant, the load may be very accurately estimated, and there is an entire absence of vibrations which would make the reading of the extensometer difficult.

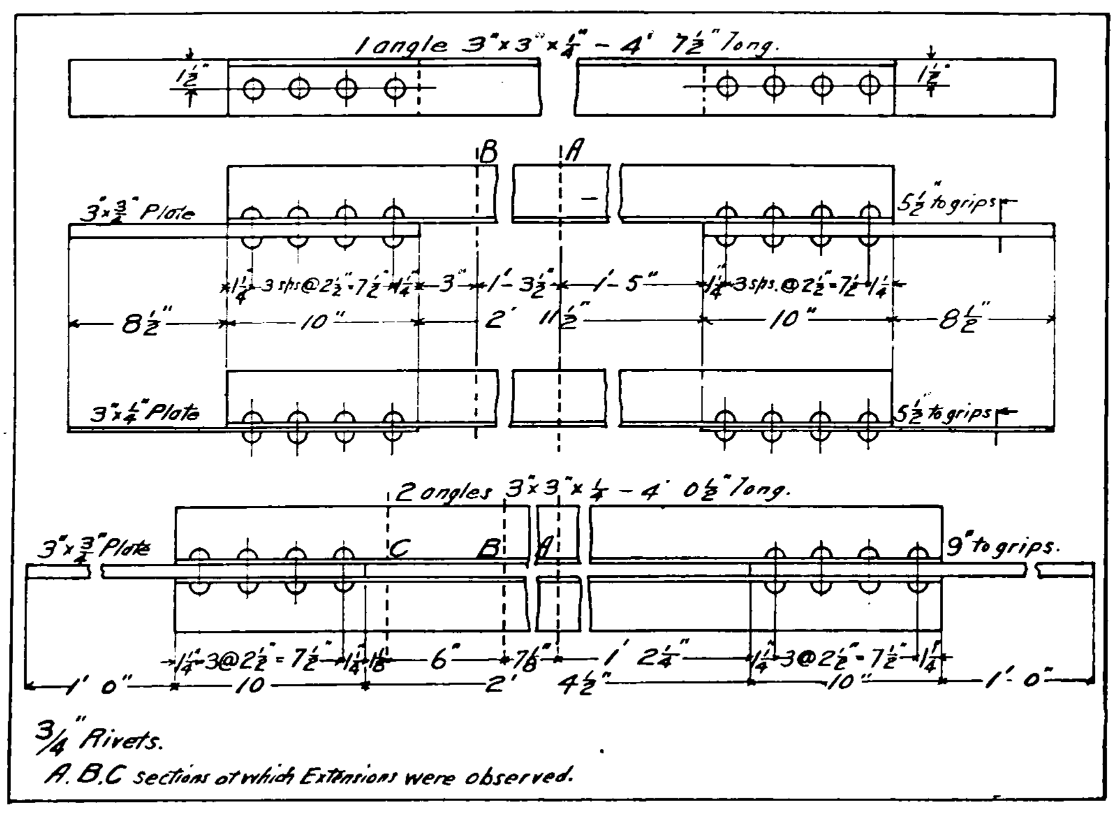

Fig. 3.

The Extensometers.

The extensometers used were a simplified form of the Martens type, designed and constructed in the McGill Testing Laboratory, where they have been in use since 1906, and have been proved capable of giving very accurate results. Figure 4 shows the prin-

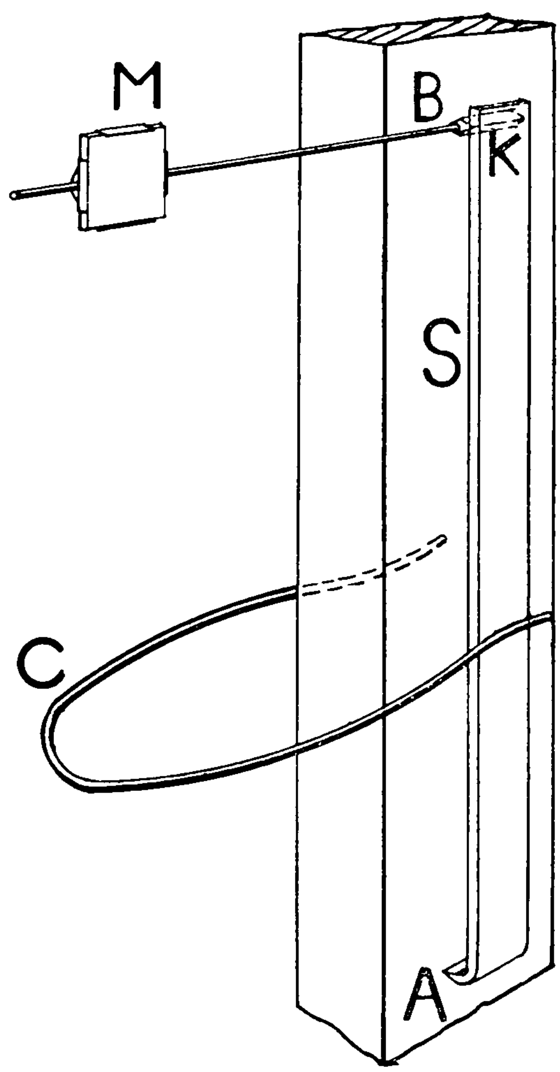

Fig. 4.

ciple of the instrument, and Figs. 5 and 6 show it in actual use on the specimens. It consists essentially of a double knife-edge, *K*, which fits between the specimen under test and a V groove in one end of a steel strip *S*, which is in contact with the specimen at *A*, and is pressed against it by means of a clip *C*. A change in the length of *AB* causes the knife edge to tilt and the tilt is measured by means of a telescope and scale, the scale being reflected in a mirror *M* attached to the knife edge. In the actual instrument the steel strip is $\frac{3}{8}''$ wide, $\frac{1}{8}''$ thick, and the length *A B* is 4″.

The end *A* is turned at right angles and brought to a sharp edge so that it may not slip on the specimen. The knife edge is of hardened steel about 0.18″ by 0.12″ by 0.45″, and the mirror is attached by means of a piece of steel knitting needle. The mirror is held in a clip of thin sheet steel which is arranged so that it can slide and rotate on the needle, a thin copper strip protecting its back from injury. This clip permits of a small amount of lateral adjustment. The mirror is about ½″ square and must be as truly plane as possible, as otherwise there will be an error introduced when the image of the scale moves to a different part of its surface, as it must do if the specimen deflects at all during test. In the original form of Martens' Extensometer there was a device for adjusting the mirror and also a balance weight at the opposite side of the knife-edge, but these refinements are not only unnecessary but cumbersome, and make the instrument less adapted to use in restricted positions.

The extensometer is calibrated in a Whitworth Measuring Machine and a calibrating rod is prepared for each instrument, giving the distance from the scale to the mirror, so that a definite distance on the scale may correspond to a given extension or compression on the specimen. In the case of the experiments described below, ½″ on the scale, subdivided into ten equal divisions, corresponded to $\frac{1}{1000}''$, so that the change of length of the specimens was easily read to $\frac{1}{100,000}''$ The length of the rod was about 4′, varying with different instruments. The angle turned by the mirror in any test is so small that there is no appreciable error in using a straight scale for the readings. This is verified by turning the mirror in the Whitworth measuring machine through much greater angles than those through which it turns in the tests. It was also found that different strips (S) did not affect the calibration, so that a knife-edge could be used with different lengths of strip without re-calibration. It is estimated that, under the conditions of test, the instrument reads accurately to $\frac{1}{100,000}''$

The kind of telescope used affects greatly the facility with which readings may be taken. The McGill Testing Laboratory telescopes were made at Charlottenburg, and are adjustable vertically and horizontally, besides moving bodily about a vertical axis (See Fig. 5). The extensometer must be carefully used in order to give correct results. The mirror should be, in its mean position, parallel to the scale and the telescope should be opposite to the mirror. The clip must be arranged so that the knife edge is held quite firmly, otherwise it will not tilt correctly. The best forms of clips are made from pieces of copper wire.

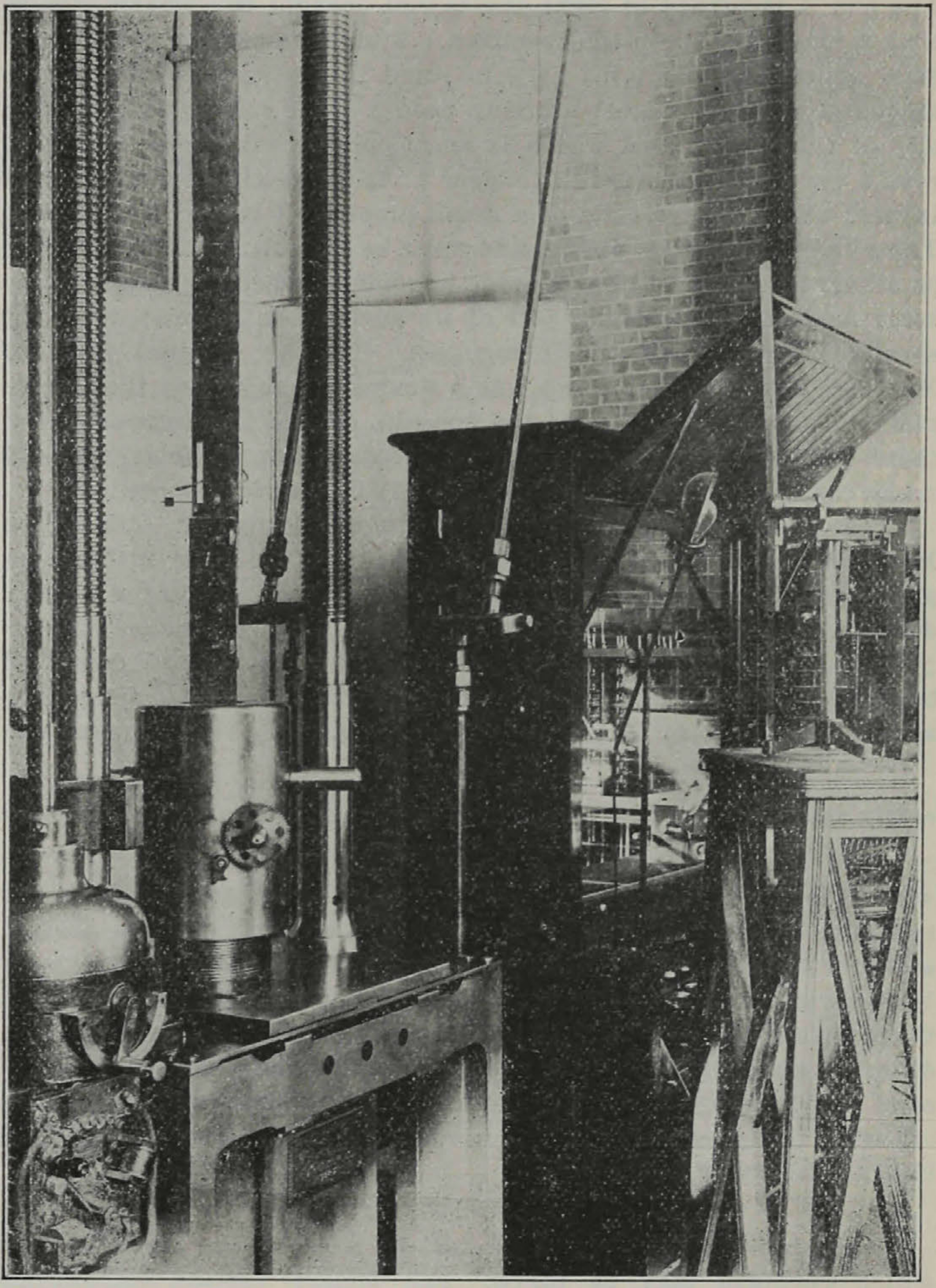

Fig. 5.

If the direction of *A B* (Fig. 4) remains unchanged during test, the difference of the scale reading between two loads will be an accurate measure of the strain of *A B* for the given load difference, but if *A B* alters in direction this will not be the case. If, however, two readings are taken, one with the extensometer in the position shown, and the other with the knife-edge at *A* and the sharp edge of the

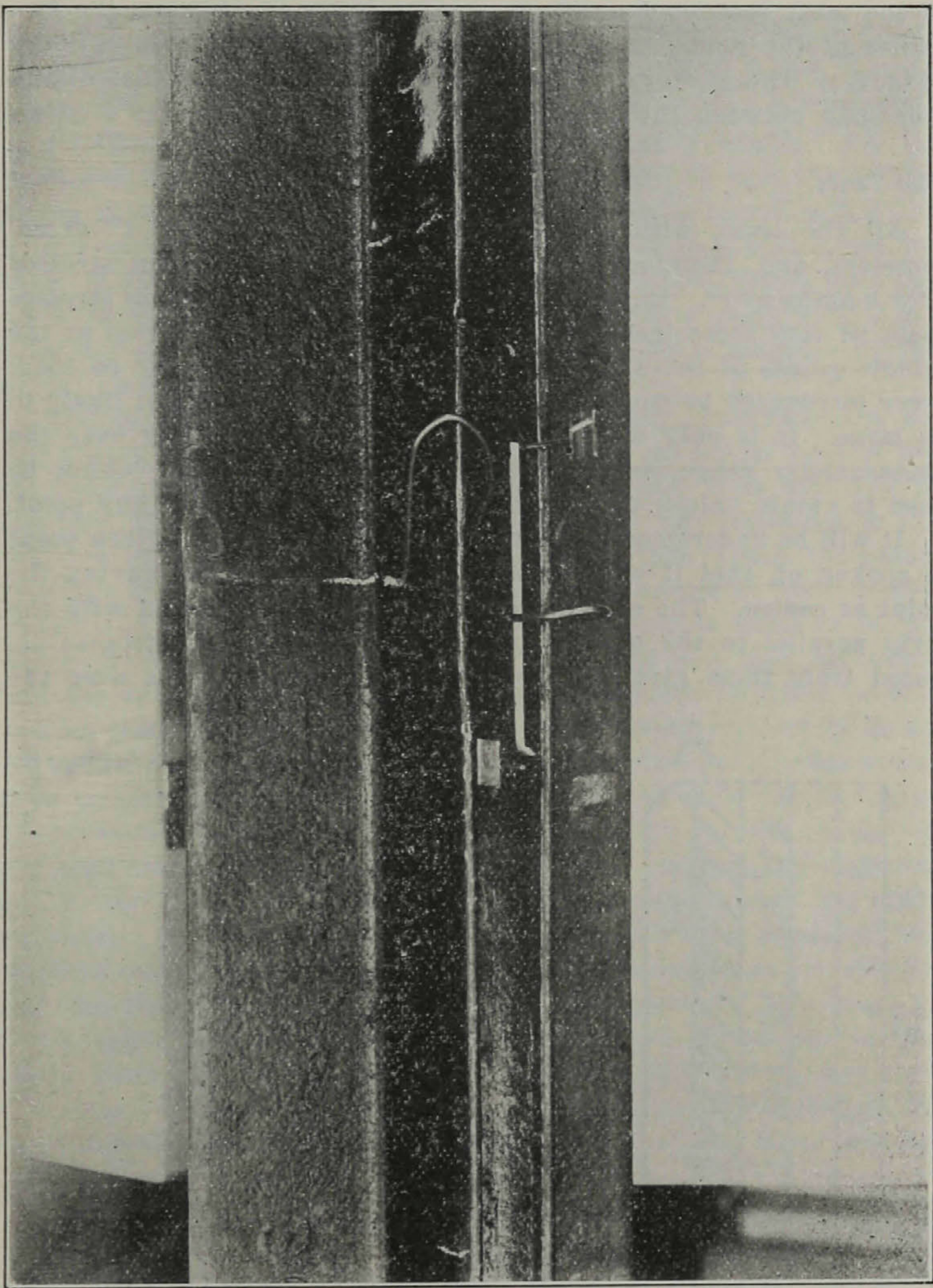

Fig. 6.

strip at *B*, the mean of the two will be correct. When any doubt exists it is always better to do this so as to eliminate possible error.

In the opinion of all who have used these instruments at McGill University, they are the most simple, practicable, and accurate extensometers in use. It will be seen that they may be readily used in the most restricted positions, as, for instance, between the two angles of the double angle members, where the width is only ⅜″. (Fig. 6.) The photograph shows two extensometers in use simultaneously between the angles.

The Tests.

All the tests, with one exception, were made with 4″ extensometers, and, therefore, the stresses tabulated are mean stresses over lengths of 4″ In the case of the central sections, these stresses must be very close indeed to the actual intensities of stress at the middle points of the 4″. For the end sections there may be some error introduced by considering them as such, but it is not likely to be large. It is only when the stress varies considerably over the extensometer range, as at the rivets, that the readings cannot be used to obtain values very close to the actual stresses at any point.

It will be understood then, wherever the reading at a given point is spoken of, that it was actually taken over 4″ range having the point as centre. The extensometers were always arranged with the strip parallel to the axis of pull, and, therefore, the stresses deduced from them give the distribution of normal stress over the

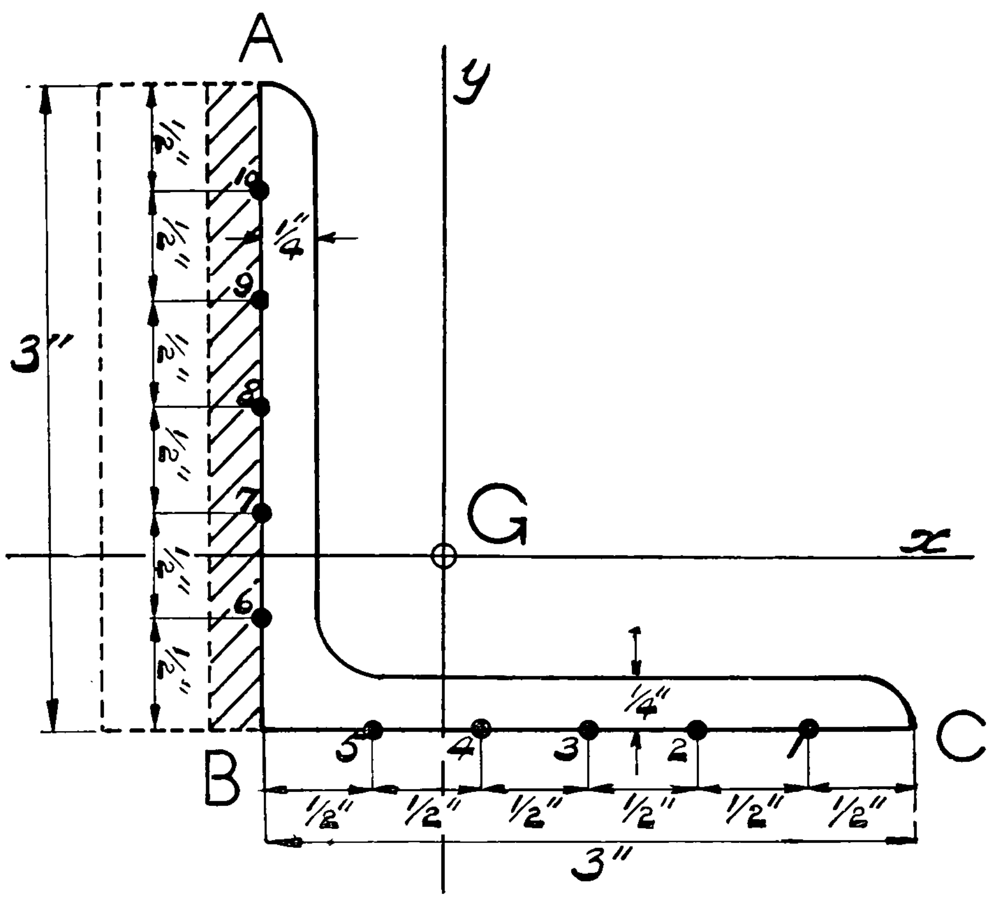

Fig. 7.

cross-section. All the stresses tabulated are for points on the outside faces of the angles. In the case of the single angles, the readings were taken at the central section and at a section 3" from the loading plate. The readings were taken across each section at intervals of ½" (See Fig. 7). For the double angle, 10 readings with the mirror at the lower end of the extensometer, and 10 with the mirror at the upper end were taken at the same intervals across each angle at the central section, and at two other sections, one *B*, 7⅜", and the other *C*, 1⅜" from the loading plate (See Table III). Other readings were taken at the rivets, but are not, at present, thought sufficiently interesting for publication, as they do not give a measure of the actual stress at the rivets.

The procedure of the tests was as follows. The specimen being placed between the grips of the machine, an initial load of 100 lbs. was applied. When two extensometers had been adjusted in position, and convenient zeros taken, the load was increased to the full amount, brought back to 100 lbs. and then again increased, readings being taken in the case of the single angles at 5,000, 10,000, 15,000, and 20,000 lbs., and in the case of the double angle at 10,000, 15,000, 20,000, 25,000, and 30,000 lbs. The load was then decreased and the zero checked. Usually the extensometers returned to zero and no readings were allowed to pass in which they failed to do so. All the readings were repeated at least once before the extensometers were moved to other points. It was determined early in the course of the experiments that the readings for all the riveted pieces did not alter when the piece was taken out of the machine and replaced, and so this was done whenever the machine was required for other purposes. Three complete sets of experiments were made at the sections tabulated, but there was very little variation in the results, and the Tables are compiled from one complete set. The value of *E* (Young's Modulus) for each specimen was found by cutting pieces from different parts of the actual sections and testing them in tension. The mean value of *E*, which did not differ greatly for the different specimens, was 28.6×10^6 lbs. per square inch, and this has been used in reducing all the results.

Careful measurements were also made of the lateral bending of the specimens at different points along them, by means of small scales graduated in $\frac{1}{100}$", and read through telescopes.

The scales were arranged so that the deflections of the points *A* and *B* (Fig. 7) at each cross-section, were obtained in the directions x and y, and thus the actual twist of *AB* was found. Table IV gives the principal results of these tests, which are used in determining the exact position of the load axis, as will be described later. Only the mean of the deflections at *A* and *B* is given in the table, as these were the values used in the reduction of the experimental results.

The Results.

In Tables I-III the stresses at the given points of the various cross-sections calculated from the actual extensometer readings are given. These were obtained by dividing the mean of the extensometer readings (with the mirror at upper end and with it at the lower end respectively) by 4", and multiplying by the mean modulus of elasticity for the piece, this being obtained by experiment, as described above. In Figures 8-13 (Plate 14) the actual mean extensometer readings are plotted, the mean straight lines being continued so as to give the maximum strain occurring at each section. The stresses corresponding to these estimated maximum strains are tabulated in Tables VI and VII, together with the ratio they bear to the average stresses over the sections.

It will be noticed, on examining Figs. 8-13 (Plate 14), how very closely the assumption of a linear distribution of stress over the cross-section is borne out by the experimental results. This is especially remarkable as the specimens were not elaborately prepared, but were ordinary shop products. The greater deviations from the mean occur in Figs. 8 and 10, which are for the unconnected limbs of the two single angle specimens at sections 3" from the loading plate. In these cases the deviations seem to follow definite curves, which are not only similar for the same place at different loads, but for the two different pieces. It is, therefore, probable that they are due to a real deviation from the linear law caused by the proximity of the sections to the rivets. This view is borne out by the results of experiments made with the object of determining the stress distribution near the rivets. Figs 9 and 12 also show rather large deviations, but these must be set down to irregularities of cross-section. The largest of these, in Fig. 9 (for specimen with ¾" plate), is at point 8, for the 20,000 lb. load, and amounts to about 6.6%, whilst the largest in Fig 12, for the left side of the double angle, is about 4.5%. In the other figures there is scarcely any deviation from the straight line. The stresses for the corner of the angle, obtained by producing the curves for the points 6-10 downwards, and those for 1-5 upwards, also agree in a very striking manner, as will be noticed from the figures, where the points surrounded by circles on the curves for 1-5 correspond with those found by producing 6-10.

These results show that the greatest confidence may be placed in the extensometers used, and that the assumption of a linear law for the stress distribution is justifiable.

The truth of this law having been established, the position of the neutral axis may be found for each load on a given section, and also the position of the load axis, according to the theory described above (page 4).

As the method of reduction is similar for all the experiments, one example will suffice to explain it. Consider the central section of the single angle member with $\frac{3}{4}''$ end plate, for which the stresses are given in Table 1, and the strains are plotted in Fig. 11 (Plate 14). The constants of the cross-section are given in Table 1. The line of stress for the points 1-5, at the 20,000 lb. load, intersects the base line at a point distant 1.88″ from the corner of the angle. This is, therefore, the point where the neutral axis cuts the leg *BC* of the angle for this load. Its distance from *B* is called *b* (See Table V). If the line of stress for the points 6-10 be produced until it reaches the base-line, as shown to a different scale by the small figure (Fig. 11), another point of zero stress may be found. Its distance from *B* is 7.5″, and is called *a* (Table V). The ratio of *a* to *b* gives the tangent of the angle of slope of the neutral axis to the axis *Gx*, which is called α in the analysis given above (page 4). In this case it is equal to 3.99 corresponding to an angle $\alpha = 75° \ 5'$. The neutral axis is thus determined and the loading point (x_k, y_k) may be found from equations 3 and 4, the axes being taken through the centre of gravity parallel to the legs. In order to simplify the calculations, the point of zero stress on the leg *BC* is taken. Thus *f* in equations 3 and 4 is equal to zero, whilst the co-ordinates (x, y) are $x = 1.88 - 0.85 = 1.03$, and $y = -0.85$, the distance of the centre of gravity from the back of the angle. (This is a little different from the distance for the standard angle, because the section was slightly heavy. See Table 1.) The values x_k and y_k, found in this way, are $x = -0.80$, $y = 0.59$. These are the co-ordinates of the point of action of the resultant load at these sections referred to axes through the centre of gravity of the section. In Table IV the deflections of this specimen at different cross-sections for different loads are given. Considering the central section, taking the mean of the deflections at these points *A* and *B* for a load of 20,000 lbs., and subtracting from these the deflections similarly taken at the middle of the riveting, a correction may be found for x_k, y_k, and, if this is applied, it will be found that the point of loading referred to the co-ordinates through the centre of gravity of the section, midway between the extreme rivets at the ends, is $x_k = -0.89$, $y_k = 0.63$. In a similar manner all the other figures in Table V have been obtained.

Discussion of the Results—Single Angles.

Consider Figs. 8-11 (Plate 14). If the point of application of the resultant force remained unchanged relatively to the section during loading, the stress lines in each of the figures would intersect at one point for all loads, *i. e.*, the distances a and b would be the same for different loads on the same section. This is not quite the case, as will be seen on inspection of the figures and tables. For example, at the central section of the angle with the ¾" plate, the point of application varies from (— 0.90, 0.65) at 5,000 lbs. load to (— 0.80, 0.59) at the 20,000 lb. load. This is largely due to the lateral bending of the members, and may be corrected from Table IV. In order to obtain a proper basis for comparison, the load point should be referred to the sections at which the load enters the angle. There is, of course, some uncertainty as to the exact position of this cross-section. It must be somewhere between the end of the angle and the end of the loading-plate, and it seems most nearly correct to take it at the mean section of the rivets, *i. e.*, between the two middle rivets. This has been done in the tables and the results must be close to the correct positions of the loads. It will be seen that this position is practically constant for the central section of the angle with the ¾" plate, and its mean is a point having co-ordinates (— 0.91", 0.64") (referred to the axes through the centre of gravity) which is $\frac{1''}{100}$ away from the centre of the connected limbs, and .06" within the load plate. For the angle with the ¼" plate the results are slightly more variable, their mean being a point having co-ordinates (0.91", 0.67") (referred to the axes through the centre of gravity) which is 0.02" from the centre of the connected leg, and 0.06" within the load plate. The mean angle of inclination of the neutral axis to the unconnected leg, for the angle with the ¾" plate, is 76°, and for the other angle 76° 50′. It appears from these results that there is a remarkable agreement between the action of the two angles, notwithstanding the great difference in the stiffness of their end connections. The results for the sections near to the ends give for the load points (— 1.01, 0.67) and (1.04, 0.71) for the specimens with the ¾" plate and ¼" plate respectively. These points are 0.16" and 0.20" respectively within the plate, and are .03" and .07" respectively from the centre of the connected leg. Here also the two different angles behave alike. The reason for the change in position of the load axis at this position is probably that some moment is caused here by proximity to the riveted joint.

Additional evidence that the heavy end plate does not appreciably restrain the bending of the angle is afforded by the deflections given in Table IV. It will be seen from these that the mean deflection of the central section of the connected leg in the direction of x, measured from the end of the angle, is 0.14" in the case of the ¾" end plate, and 0.15" in the case of the ¼" end plate, whilst in the direction of y, the values are 0.04″ and 0.06″ respectively. The difference between these values for the plates is small, especially considering that the first angle is slightly heavier than the other. In Appendix II a formula is developed for the central deflection of a piece subjected to an eccentric tensile force. It is shown that, when applied to a single angle of the dimensions of the specimens, the deflection of the centre of gravity arrived at is 0.15″. This is in a direction perpendicular to the neutral axis and assumes the load axis to be at the middle of the outside face of the connected leg. When this displacement is resolved parallel to Gx it gives 0.145″, and parallel to Gy 0.05″, which are close to the experimental values.

Now the constraint offered to bending by the ¾″ end plates is probably greater than that due to any end connections used in practice. Thus it will be evident, from the above, that in very few practical cases can the end of a single angle structural member be said to be fixed.

Careful measurements were made of the deflections of the plate and the angle near to the rivets, which showed that both bent together. The want of end rigidity must, therefore, be due to the stiffness of the angle being much greater than the stiffness of the plate, and not to any yielding of the rivets.

The next question which must be considered, is the position of the load axis. Evidently, from the above, it will not depend very much on the stiffness of the end connections. In Table VI the actual maximum stresses from measurements are given, together with those obtained from the theory, assuming the load axis as worked out from the experimental results. It will be seen that the agreement between the two is very close for the angle with the ¾″ plate. For the other angle, the calculated results are all 3% or 4% higher than the extensometer results, but a small variation in E would obviously bring them into agreement, and in any case the difference is small.

The truth of the theory may thus be said to be verified by the experimental values, and the stresses given in the second column of Table VI must be very close indeed to the actual maximum stresses. Considering the ratios of maximum to mean stress over the section, given in the last column of Table VI, it will be seen again that the two different angles behave very similarly, the ratio falling at the central sections from 2.23, at the lowest load, to about 2.10, at the highest. This change is, of course, due to the bending of the specimens. In the first column of Table VIII the stresses calculated from the mean position of the load axis, allowing for bending, are given, the ratio of maximum to mean stress being 2.16 for each angle. This may be taken as the mean experimental ratio for both of the sections. In this table the theoretical maximum stresses for different assumptions of the load axis, neglecting bending, are also given. It will be seen that the assumption which best fits the actual case is that the load axis is 1″ from the centre of gravity, corresponding to a point 0.15″ and 0.16″ respectively, within the load plate. (The values of a do not, of course, correspond exactly, because the deflection has not been considered.) The stresses at the ends of the piece are somewhat higher and correspond more closely to a load-axis at the junction of the plate and the angle, and it would seem that the best practical rule for obtaining the maximum stress of such a member would be to take the load-axis as along the line of rivets, and at the junction of the plate and angle, neglecting deflection. This would give results slightly on the safe side.

The Double Angle.

In figuring a section consisting of two angles placed back to back, connected by a plate to which the load is applied and riveted together at intervals, it is usually assumed that the section acts as one piece, *i. e.*, as a T section, thus bending about a neutral axis parallel to the unconnected legs of the angles. The load is thus assumed to act in an axis of symmetry of the cross-section, and the maximum stress in any given case may be easily calculated from equation 1 above. Applied to the experimental section, this method would give the ratio of maximum to mean stress as 2.65. A glance at Table VII will show how very far such calculated results are from the actual experimental values. In the actual specimen, the two angles did not take equal portions of the load, the angle L taking more than the angle R, but the greatest of the maximum stresses is only 2.28N at the lowest load, falling to 2.15N at the highest. The reason for this will be evident from Fig. 12, where the distribution of strain across the central cross-section is plotted for different loads in exactly the same way as in the case of the single angles. It will be seen from these figures that the two angles of the member bend each about its own neutral axis, and that they thus act like separate angles constrained at their ends. The results were, therefore, reduced to find the point of loading and the angle of inclination of the neutral axis, in the same way as for the single angles, and the results of the analysis are given in Table V. It will be seen from these that the angle of inclination is 20° 18′ for the right hand angle, and 18° 48′ for the left hand angle. The load axis for the right hand angle has a mean position (— 0.36, 0.46), and for the left hand (— 0.43, 0.55), and is constant for all the loads, except the lowest (10,000 lbs.). The results were not corrected for lateral bending, although deflections were measured (See Table IV), because the deflections were small, and it was recognized that these results could not, by reason of the unequal distribution of the load between the two angles, be so closely analysed as the results for a single angle. Assuming that the angles, if acting separately and unconstrained at the ends, behaved as in the experiments described above, the effect of the end constraint, caused by the riveting of the angles back to back, may be found from the shift of the axis of loading. This may be assumed at the centre of the connected leg for separate action, *i. e.*, at the point (— 0.84, 0.66). It has, therefore, shifted in the case of the angle *R* through a distance equal to $\sqrt{[(0.84-0.36)^2+(0.66-0.46)^2]}=0.52''$, and in the case of the left hand angle through a distance of 0.42''. This means that a restraining couple of moment 0.52 N inch lbs. acts on the right hand angle and a couple of 0.42 N inch lbs. on the left hand angle. Consider the adjoined figure (Fig. 14), which represents the two

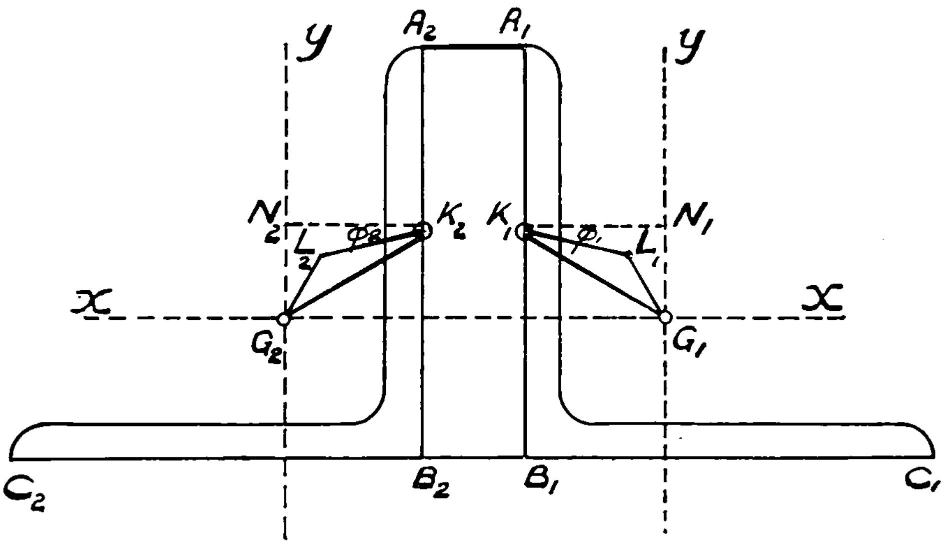

Fig. 14.

angles, G_1 being the centre of gravity of the right hand angle, G_2 that of the left hand angle. K_1 and K_2 represent the loading points for separate action, and L_1 and L_2 represent the actual axes of load found as above. The bending moment on the sections acting separately would be $N_1 \times K_1 G_1$ and $N_2 \times K_2 G_2$ respectively, where N_1 and N_2 are the loads carried by the angles. The actual moment for the right hand angle is $L_1 G_1 \times N_1$, and thus the constraining moment is $K_1 L_1 \times N_1 = 0.52\,N$ about an axis perpendicular to $K_1 L_1$. This may be resolved into moments

$$N_1 \times K_1 L_1 \cos \phi_1 = N_1 \times 0.52 \cos \phi_1$$
$$N_1 \times K_1 L_1 \sin \phi_1 = N_1 \times 0.52 \sin \phi_1$$

parallel to G_1y and G_1x respectively.

Now, $\tan\ \phi_1 = \dfrac{0.66 - 0.46}{0.84 - 0.36} = 0.417,$

and the constraining moments are thus 0.48 N_1 about an axis parallel to G_1y and 0.20 N_1 about an axis parallel to G_1x. Similar analysis for the left hand angle leads to the values tan $_2 = 0.27$, moment parallel to $G_2y = 0.41\,N_2$ and parallel to $G_2x = 0.11\,N_2$.

It is thus clear that the experimental angle is subject to imperfect constraints in directions parallel to the legs of the angles, the constraint parallel to the unconnected legs being roughly 50% of that required for perfect fixing, and the corresponding figure for the connected legs being 20%. If the load had been applied through pins in the end plates, the latter restraint would probably have been almost zero, since it is due to the stiffness of the end connections. In any actual members, however, there must be a certain fixing moment in this direction, which is probably never very much greater than the above experimental value. The length of unconnected angles in this case was 28.5″, which is not greater than that frequently used in practice, so that the restraining moment parallel to the connected leg is probably of the same order as that obtained in practice. It is hoped that other members with different lengths of unconnected angles, etc., may be tested in this way. With perfect constraint in both directions the stress would, of course, be uniformly distributed over the section, because the fixing moment would entirely counteract the eccentricity. With perfect fixing about the axis parallel to the connected leg and perfect freedom in a direction at right angles to it, the ordinary theory would be correct, because the line of pull would then be on G_1y at a distance $G_1\,N_1$ (Fig. 14). If, on the other hand, there were no constraint in either direction, the action would be like that of the single angle. In most practical cases there is probably imperfect restraint in both directions, as in the experimental member. It must not be assumed, however, that the greater the restraint the lower the maximum stress will be, because if, for example, the angles in a member of the section considered above acted separately, the ratio of maximum to mean stress would be 2.29, whilst with perfect constraint against bending of the unconnected limb, the ideal usually aimed at, it would be 2.65, about 16% higher. With perfect constraint in the direction at right angles, the ratio would be only 1.82, and with the actual imperfect restraints in both directions it is 2.15. From these results it will be seen that, for a member consisting of equal angles placed back to back, it is not desirable to stiffen the member so as to make it act as one single piece; and there must be many other cases of built up members in practice where extra stiffness given by distance pieces, diaphragms, etc., is a doubtful advantage. It must be remembered that the above figures only hold good for angles having equal legs. In the case of unequal legged angles connected by the longer legs, the stress may be much greater if they act separately, whilst if they are connected by the shorter legs, the reverse will be the case.

Remarks on Built Up Members.

A built up tension or compression member is one which is made of two or more simple sections, such as angles or channels, fastened together by rivets and by tie plates, lattice bars, or other connections, as in the case of a large column. Probably the simplest form is the double angle considered above. Such a built up member is usually considered as acting like one piece, and the forces in the tie plate or lattice connections are found on the assumption that, if any bending takes place, the whole member bends like a beam. The above experiments show that this is not true for the specimens tested, and it would probably be more nearly correct to consider such a member as an assemblage of simple members each trying to bend about its own neutral axis, but more or less constrained by the subsidiary latticing, etc. In the opinion of the writer, the only way to arrive at a correct theory of the action of such structures is to consider the simplest cases first and to approach gradually the more complex cases by introducing one constraint after the other, and finding their effect by experiment and analysis. This opens the way to a large field of research, to which it is hoped that the present paper may form a first contribution. An example will make this point clear. Consider a column in the form of a rectangle, built up of four angles, connected by tie-plates or lattice bars, and loaded through two loading plates riveted to the angles at the ends. The ordinary theory would assume that the whole member behaves like one piece, the tie-plates or lattice bars simply taking up the stress like the web of a girder. According to the theory advanced here, the four angles would be regarded as trying to bend about their own neutral axes in the way a single angle has been shown to behave above, and the tie-plates would constrain them against twisting, and so would themselves be under bending stresses, the whole action being, of course, somewhat complicated. It may be stated here that actual extensometer experiments on such a column, carried out under the direction of Professor H. M. Mackay, at McGill University, entirely bear out this view, the stresses in the tie-plates being found to be tensile on one side and compressive on the other. It is hoped that these results will be published shortly. The writer hopes to investigate the theory of this type of member by considering first the relatively simple case of two angles connected by tie-plates.

Summary and Conclusion.

As stated in the introduction, experiments of the kind described here are still in progress at McGill University. It is hoped to investigate in a similar manner single angle members in compression, double angle members with equal and unequal legs in tension and compression, as well as various forms of built up members. Experiments on some of these are in progress.

The chief conclusions to which the present paper leads are:

(1) That the form of extensometer described is very accurate and simple in operation, and that it is possible by its means to obtain very closely the distribution of stress in a piece of material under load;

(2) That experiments made with these extensometers on tension specimens of uniform cross-section subjected to eccentric axial loads not in an axis of symmetry of the cross-section, bear out very closely the general theory for such a case;

(3) That the point of application of the load for a single angle member loaded through a plate riveted to one of its legs may be taken as in the line of rivets and at the common face of the plate and angles;

(4) That the end plate, under ordinary conditions, offers no appreciable restraint to the bending of such a member;

(5) That a member consisting of two angles riveted together through a connecting plate does not act as one piece, but that each angle bends about its own neutral axis, and that it is not *always* an advantage to attempt to make it act as one piece by further constraints;

(6) That a built up member should not be regarded as a single piece bending as a beam, but as several pieces each trying to bend about its own neutral axis, but restrained from doing so by the subsidiary members, such as the tie-plates, or latticing.

In conclusion, the writer wishes to thank Professor H. M. Mackay (at whose suggestion the work was commenced), Professor E. Brown, and Mr. F. P Shearwood, of the Dominion Bridge Co., for their personal interest and advice; and Mr. S. D. Macnab, of the McGill University Testing Laboratory, who was associated with him throughout in the experimental parts of the work. He is indebted to the Dominion Bridge Co. for the specimens used in the tests.

APPENDIX I.

Theory of the Distribution of Stress in a Uniform Bar subjected to an eccentric force parallel to its axis, which does not lie in an axis of symmetry of the Cross-section.

This theory is to be found in the German text-books on Strength of Materials, but it seems to have been neglected by most English and American writers.* It was developed in one form by Mohr (See "Technische Mechanik," Otto Mohr, Berlin, 1906, P. 241). This form, however, although elegant, is not adapted to practical computations. C. Bach, in his work "Elasticität und Festigkeit" (p. 223, 4th edition), gives the results referred to the principal axes of inertia of the cross-section, and L. J. Johnson in Proc. Am. Soc. C. E., Vol. 56, 1906, works out the results in the form given here, which is that best suited for calculation.

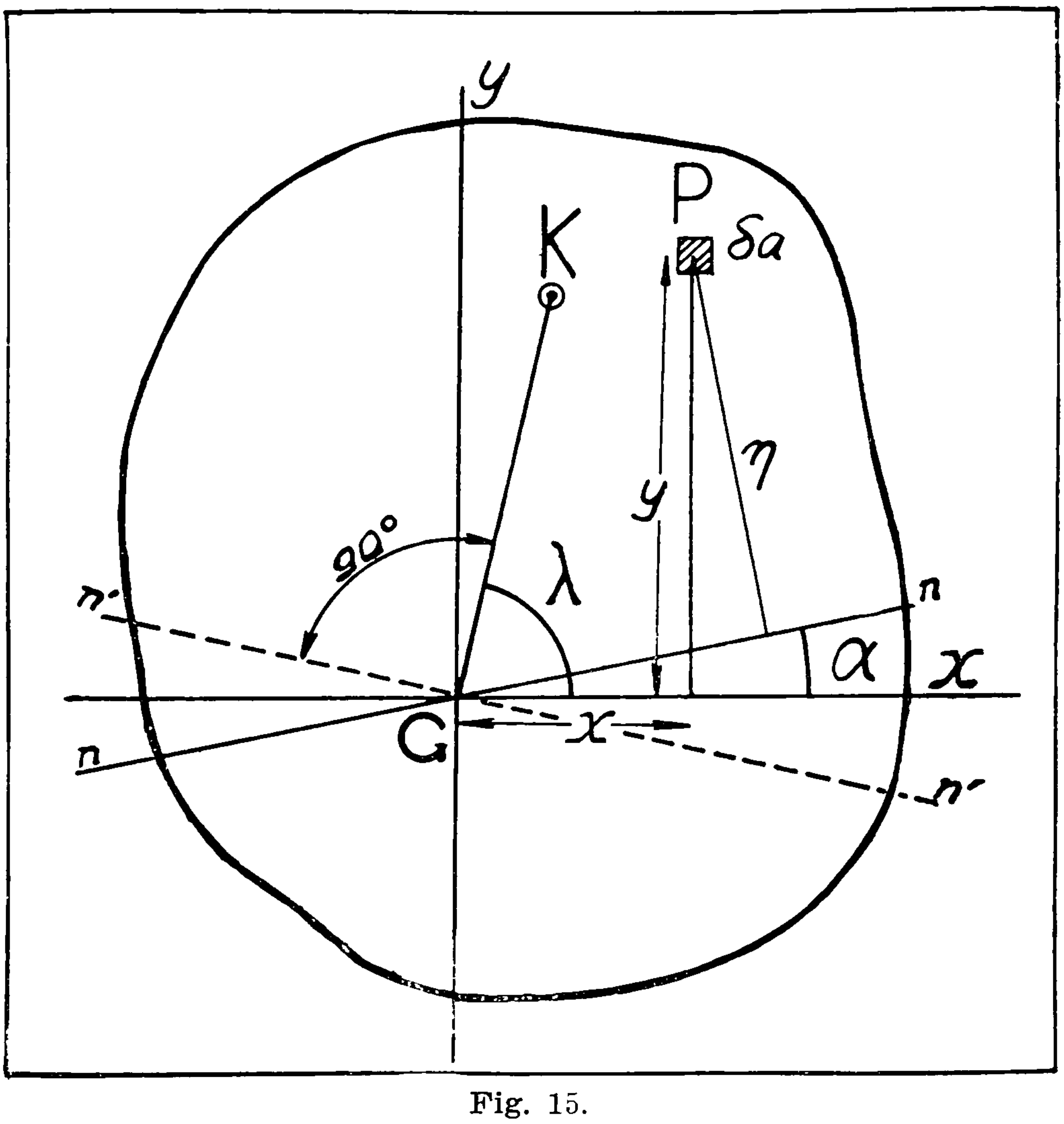

Fig. 15.

* Since writing this, a brief account of the theory has been published in the second edition of Morley's "Strength of Materials" (Longman's).

Let G (Fig. 15) be the centre of gravity of the cross-section, K the point of application of the normal load N, and Gx and Gy any rectangular axes through G. If the point K coincides with G, the stress over the cross-section will everywhere have the intensity $\frac{N}{A}$ where A is the area of the section. If K does not coincide with G, there will be in addition to this stress bending stresses caused by the moment $M = N.KG$, which has the axis GB perpendicular to GK. Consider the effect of this moment acting alone. It would cause the bar to bend about some neutral axis nn inclined at an angle a to the x— axis. Let η be the perpendicular distance of any element δa of the cross-section from nn and let (x, y) be its co-ordinates. By the ordinary laws of bending

$$\frac{f}{E\eta} = \frac{1}{R} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 1$$

where E is Young's Modulus, R the radius of curvature of the cross-section, and f the intensity of stress over δa. For equilibrium the sum of the moments of the stresses about Gx and Gy must be equal to the components of the bending moments about these axes,

Therefore $$M \sin \lambda = \Sigma f y \,\delta a \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 2$$
$$M \cos \lambda = \Sigma f x \,\delta a \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 3$$

But $$f = \frac{E\eta}{R} = (y \cos a - x \sin a) \frac{E}{R}$$

Therefore $$\frac{R}{E} M \sin \lambda = \Sigma y^2 \,\delta a \cos a - \Sigma x y \,\delta a \sin a$$
$$= I_x \cos a - J \sin a \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 4$$

and $$\frac{R}{E} M \cos \lambda = \Sigma x y \,\delta a.\cos a - \Sigma x^2 \,\delta a \sin a$$
$$= J \cos a - I_y \sin a \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 5$$

Where I_x and I_y are the moments of inertia of the section about Gy and Gx respectively, and J is the product of inertia about (Gx, Gy) Divide 4 by 5 and obtain

$$\tan \lambda = \frac{I_x \cos a - J \sin a}{J \cos a - I_y \sin a}$$

and on rearranging the terms

$$\tan a = \frac{I_x - J \tan \lambda}{J - I_y \tan \lambda}$$

which gives the angle of inclination of the neutral axis to Gx. (The effect of the direct stress $\frac{N}{A}$ will be to shift this axis parallel to itself to a position determined later).

From (1)
$$f = \frac{E\eta}{R}$$
$$= \frac{M \sin \lambda\ (y \cos \alpha\ - x \sin \alpha)}{I_x \cos \alpha\ - J \sin \alpha}$$
$$= \frac{N y_k\ (y\ - x \tan \alpha)}{I_x\ - J \tan \alpha} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ 7$$

and similarly from 1 and 5

$$f = \frac{N x_k\ (y\ - x \tan \alpha)}{J\ - I_y \tan \alpha} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ 8$$

Thus the actual stress at any point (xy) will be

$$f = \frac{N}{A} + \frac{N x_k\ (y\ - x \tan \alpha)}{J\ - I_y \tan \alpha} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ 9$$

the positive sign being taken because η was taken positive on the side of nn on which the point K lies. Putting $f = o$ in equation 9, the equation of the neutral axis may be obtained. Various graphical and semi-graphical methods have been devised by Mohr and others, but they do not appear to the writer to have any advantages over the above.

Note on the Calculation of J.

Let J be the product of inertia about any rectangular axes, and J_G that about parallel axes through the centre of gravity of the section. Then, if (x, y) are the co-ordinates of any point of the cross-section referred to the former axes, $(x^1 y^1)$ those referred to the parallel axes through the centre of gravity, and $(\bar{x},\ \bar{y})$ the co-ordinates of the centre of gravity referred to the first axes

$$J = \Sigma\, xy\ \delta a \qquad \text{over the section}$$

or

$$J = \Sigma\ (x' + \bar{x})\ (y' + \bar{y})\ \delta a$$
$$= \Sigma\ x'\ y'\ \delta a\ + \Sigma\ \bar{x}\ \bar{y}\ \delta a\ + \Sigma\ \bar{x}\ y'\ \delta a\ + \Sigma\ \bar{y}\ x'\ \delta a$$
$$= J_G + A\ \bar{x}\ \bar{y}$$

because $\Sigma\ y'\ \delta a\ = A\ \bar{y}' = o$

and $\Sigma x'\ \delta a\ = \quad \bar{x}'\ = o$

Prof. L. T. Johnson has called the attention of the author to a much neater method for calculating J for angle sections, originally due to Müller-Breslau.*

Suppose an angle is divided into two areas A_1 and A_2 by producing one of the inner faces until it cuts the outer, and let G_1 and G_2 respectively be the centres of gravity of these areas. Draw a line through G_1 parallel to the leg bounding the area A_2 and through G_2 parallel to the other leg and let these meet at O. Then J about axes through the centre of gravity of the angle parallel to the legs is given by

$$J = \frac{A_1\ A_2\ ab}{A_1 + A_2}$$

where $a = OG_1$, $b = OG_2$.

This is a particular case of a more general proposition which may be easily proved by means of equation 10 above.

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*Müller-Breslau—Graphische Statik der Ban-Constructionen I pp. 43, 44.
L. J. Johnson—"The General Case of Flexure," Journal of the Assoc. of Engineering Soc., Vol. XXVIII, 1902.

APPENDIX II.

The lateral deflection of a uniform bar under an eccentric tensile force parallel to the axis, but not in an axis of symmetry of the cross-section.

Let OA (figure 16) represent the axis of the bar and let N be the applied force of eccentricity d.

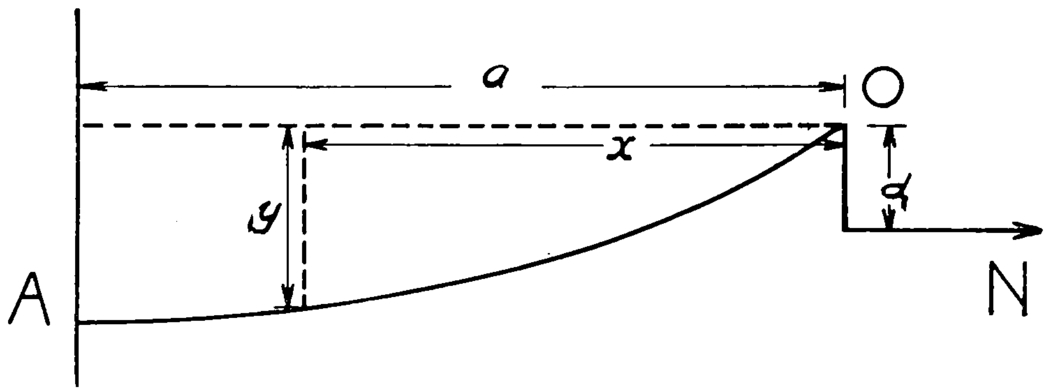

Fig. 16.

If the load were applied in an axis of symmetry the equation for bending would be

$$\mathrm{E\,I}\frac{d^2y}{dx^2} = N(y - d)$$

but since this is not the case, equations 4 and 5 of Appendix I must be used. Squaring and adding, these give—

$$\frac{d^2y}{dx^2} = \frac{1}{R} = \frac{M}{E}\,\frac{1}{\sqrt{[(I_x \cos\alpha - J\sin\alpha)^2 + (J\cos\alpha - I_y\sin\alpha)^2]}}$$

Now, at any section $M = N$ multiplied by the distance of the load point from the centre of gravity.

The bending will be perpendicular to the neutral axis, and thus if y be the distance of the centre of gravity from its position at the end o, the eccentricity of any section

$$= \sqrt{\left[d^2 + y^2 - 2\,dy\cos\left\{90^\circ - (\lambda + \alpha)\right\}\right]}$$

In many cases, including the angle section, the last term is practically equal to $2dy$, and the eccentricity then becomes.

$$(d - y)$$

Thus the differential equation of the axis is

$$\frac{d^2y}{dx^2} = k^2(y - \mathrm{d}) \dots\dots\dots\dots\dots \quad 10$$

where

$$k^2 = \frac{N}{E} \frac{1}{\sqrt{[(I_x \cos a - J \sin a)^2 + (J \cos a - I_y \sin a)^2]}}$$

or for the equal legged angle, since $I_x = I_y = I$

$$k^2 = \frac{N}{E \sqrt{[1^2 + J^2 - 4 I J \sin a \cos a]}} \qquad \text{11}$$

Solving (10)

$$y = d + A e^{kx} + B e^{-kx}$$

where A and B are constants.

Now, when $x = o$, $y = o$ and when $x = a, \frac{dy}{dx} = o$

Therefore $\quad -d = A + B$

and $\quad o = A e^{ka} - B e^{-ka}$

and (12) becomes

$$y = d \left[1 - \frac{e^{-ka}}{e^{ka} + e^{-ka}} e^{kx} - \frac{e^{ka}}{e^{ka} + e^{-ka}} e^{-kx} \right]$$

and the central deflection is given by

$$y = d \left[1 - \frac{2}{e^{ka} + e^{-ka}} \right] \qquad \text{13}$$

This result will now be applied to the 3″ x 3″ x ¼″ angle loaded at the ends at the mid-point of one of its sides as in the case considered above (page 228.). In order that the results may apply to the experiments, a has been taken 28.25″, which is the half length of the experimental angles, and $N = 20{,}000$ lbs.

The value of d is $\sqrt{[(0.84)^2 + (0.66)^2]} = 1.07''$

$$k^2 = \frac{N}{E \sqrt{[I^2 J^2 - 4 J I \sin a \cos a]}}$$

$$k = \sqrt{\frac{20{,}000}{28.5 \times 10^6 \times 1.8}}$$

$$= 0.02''$$

The deflection at the middle is, therefore, from equation 13

$$y = 1.07 \left[1 - \frac{2}{e^{0.566} + e^{-0.566}} \right]$$

$$= 0.15''$$

In the experiments, deflections were measured parallel to the legs of the angle. The components of the above in these directions are

$$0.15'' \cos a = 0.15'' \times 0.942 = -0.14''$$

$$0.15'' \sin a = 0.15'' \times 0.336 = 0.05''$$

which agree very closely with the experimental values. (See page (17).

TABLE I.—Stresses corresponding to the mean extensometer readings for 3″ x 3″ x ¼″ angle with ¾″ end-plate.

Area of cross-section 1.52 □″

Distance from c. g. to back of angle 0.85″

$I = 1.31$ (in)^4 units. $J = -.77$ (in)^4

$E = 28.6 \times 10^6$ lbs per □″

Stress s in lbs per □″ Tension + Compression −

	LOAD N LBS.	1	2	3	4	5	6	7	8	9	10
CENTRAL SECTION	5,000	− 2,360	− 780	710	2,290	3,570	5,710	6,000	6,360	6,790	7,150
	10,000	− 4,000	− 1,140	1,720	4,650	7,290	11,100	11,790	12,430	13,150	13,880
	15,000	− 5,360	− 1,140	2,790	7,000	11,000	16,400	17,500	18,200	19,300	20,450
	20,000	− 6,360	− 1,070	4,140	9,360	14,370	21,400	23,200	23,880	25,400	26,950
SECTION 3″ FROM END-PLATE	5,000	− 2,860	− 1,220	640	2,070	3,640	6,080	6,290	6,650	6,800	7,080
	10,000	− 5,290	− 2,430	1,500	4,290	7,360	11,900	12,300	12,950	13,300	13,880
	15,000	− 7,650	− 3,290	2,500	6,650	10,930	17,600	18,380	18,730	19,720	20,400
	20,000	− 9,650	− 4,080	3,500	8,930	14,980	23,200	23,950	25,000	25,900	26,500

TABLE II.—Stresses corresponding to the mean extensometer readings for 3" x 3" x ¼" angle with ¼" end-plate.

Area of cross-section 1.44 □"

Distance from c. g. to back of angle 0.85"

$I = 1.24\ (in)^4$ units. $J = -74\ (in)^4$ units.

$E = 28.6 \times 10^6$ lbs. per □"

	LOAD N (LBS.)	STRESSES IN LBS. PER □"—TENSION + COMPRESSION −									
		1	2	3	4	5	6	7	8	9	10
CENTRAL SECTION	5,000	− 2,430	− 720	640	2,290	3,500	5,790	6,080	6,430	6,790	7,510
	10,000	− 4,360	− 1,280	1,500	5,220	7,150	11,400	12,090	12,430	13,300	13,940
	15,000	− 6,000	− 1,280	2,500	7,430	10,670	16,680	17,800	17,930	19,600	20,600
	20,000	− 7,000	− 930	3,790	10,000	14,000	21,700	23,300	23,450	25,690	27,100
SECTION 3" FROM END-PLATE	5,000	− 3,280	− 1,360	640	2,430	4,000	6,210	6,430	6,790	7,150	7,440
	10,000	− 5,930	− 2,500	1,430	4,860	7,720	12,160	12,500	13,150	13,720	14,380
	15,000	− 8,290	− 3,360	2,430	7,440	11,430	17,930	18,500	19,300	20,350	21,300
	20,000	− 10,300	− 4,000	3,430	9,930	15,000	23,500	24,400	25,420	26,900	27,900

TABLE III.- -Stresses corresponding to the mean extensometer readings for 2—3" x 3" x $\frac{1}{4}$" angles with $\frac{3}{4}$" end-plate.

Constants for each angle are the same as in table No. II.

Section		Load N (lbs.)	Stresses in lbs. per □" 1	2	3	4	5	6	7	8	9	10
Central	Left	10,000	930	1,070	1,360	1,720	2,070	3,360	4,080	5,070	6,000	7,000
		15,000	1.280	1,720	2,150	2.640	3,290	5,150	6,150	7,580	8,860	10,580
		20,000	1.790	2,290	2,860	3.570	4,360	7,000	8,290	10,000	11,800	13,920
		25,000	2,220	2,930	3,860	4,570	5,650	8,790	10,380	12,500	14,650	17,420
		30,000	2,780	3,720	4,580	5,430	6,710	10,430	12,430	14,930	17,500	20,740
	Right	10,000	1,290	1,430	1,570	2,000	2,220	3,290	4,080	5,140	5,710	6,710
		15,000	1,860	2,220	2,720	2,930	3,430	5,210	6,150	7,500	8,500	9,650
		20,000	2,360	3,000	3,500	4,150	4,650	6,860	8.210	9,900	11,430	12,880
		25,000	3,070	3,790	4,360	5,070	5,710	8,510	10,380	12,360	14,000	15,920
		30,000	3,720	4,710	5,290	6,290	7,080	10,200	12,430	14,910	16,800	18,920
B 7⅛" from end	Left	20.000	2,220	2,790	3,220	4,000	4,500	6,860	8,150	9,500	10,720	12,220
	Right	20,000	3,220	3,500	3,930	4,430	4,710	6,720	8,090	9,440	10,720	12,160
Cx 3$\frac{1}{16}$" from end	Left	20,000	3,140	3,140	3,720	4,140	4,290	6,430	7,720	8,720	10,140	11,300
	Right	20,000	3,720	3,290	4,140	4,430	4,860	6,570	7,870	9,290	10,300	11,330

x For this section, a 2" extensometer was used.

TABLE IV. Mean lateral deflection of the specimens.

SPECIMEN	SECTION	N LOAD (LBS.)	MEAN DEFLECTION OF A B — With respect to middle of rivets X	Y	With respect to end of angle X	Y
			ALL MEASURED IN INCHES			
Single Angle with ⅜" end-plate	Central	5,000	−0.03	0.01	−0.04	0.01
		10,000	0.05	0.02	0.08	0.02
		15,000	0.09	0.03	0.11	0.03
		20,000	0.10	0.04	0.14	0.04
	3" from end-plate	5,000	0.02	0.01	0.03	0.01
		10,000	0.04	0.01	0.05	0.01
		15,000	0.05	0.01	0.08	0.02
		20,000	0.07	0.02	0.11	0.02
Single Angle with ¼" end-plate	Central	5,000	0.03	0.02	0.05	0.02
		10,000	0.06	0.03	0.08	0.03
		15,000	0.08	0.04	0.13	0.04
		20,000	0.10	0.05	0.15	0.06
	3" from end-plate	5,000	0.02	0.00	0.04	0.00
		10,000	0.03	0.01	0.07	0.01
		15,000	0.05	0.01	0.09	0.02
		20,000	0.08	0.03	0.12	0.03
Double Angle	Central Left	10,000	−0.00	0.01	−0.00	0.02
		20,000	0.01	0.02	0.01	0.03
		30,000	0.02	0.04	0.02	0.04
	Central Right	10,000	0.00	0.01	0.00	0.02
		20,000	0.01	0.02	0.01	0.03
		30,000	0.02	0.04	0.02	0.04

N.B.—All the deflections in the direction x are negative. The values given are the mean of readings taken at A and B.

TABLE V.—Reduction of experimental results to find load axis.

SPECIMEN	SECTION	LOAD (LBS)	Position of Neutral Line		Inclination of Neutral Line	Point of Zero Stress		Estimated loading axis referred to axes thro' c g of section		Estimated loading axis referred to axes thro' c g of mid rivet section	
			a	b	tan a	x	y	x_k	y_k	x_k	y_k
			INCHES		ALL MEASURED IN INCHES						
Single Angle with ¾" end-plate	Central	5,000	6.80	1.75	3.88	0.90	¯0.85	¯0.90	0.65	¯0.93	0.66
		10,000	7.40	1.80	4.11	0.95	¯0.85	¯0.86	0.63	¯0.91	0.65
		15,000	7.50	1.85	4.06	1.00	¯0.85	¯0.82	0.60	¯0.90	0.63
		20,000	7.50	1.88	3.99	1.03	¯0.85	¯0.80	0.59	¯0.89	0.64
		Mean values			4.01	$a=76°$		¯0.84	0.62	¯0.91	0.63
	3″ from end-plate	5,000	9.07	1.66	5.46	0.81	¯0.85	¯0.99	0.69	¯1.01	0.70
		10,000	11.86	1.70	6.98	0.85	¯0.85	¯0.96	0.65	¯1.00	0.66
		15,000	12.07	1.70	7.09	0.85	¯0.85	¯0.96	0.65	¯1.01	0.66
		20,000	12.45	1.73	7.20	0.88	¯0.85	¯0.94	0.66	¯1.01	0.68
		Mean values			6.68	$a=81°30'$		¯0.96	0.66	¯1.01	0.67
Single Angle with ¼″ end-plate	Central	5,000	6.00	1.70	3.53	0.86	¯0.84	¯0.84	0.64	¯0.87	0.66
		10,000	8.55	1.74	4.91	0.90	¯0.84	¯0.90	0.64	¯0.96	0.67
		15,000	8.10	1.80	4.50	0.96	¯0.84	¯0.85	0.62	¯0.93	0.66
		20,000	7.87	1.89	4.15	1.05	¯0.84	¯0.79	0.66	¯0.89	0.71
		Mean values			4.27	$a=76°50'$		¯0.84	0.64	¯0.91	0.67
	3″ from end-plate	5,000	9.67	1.60	6.04	0.76	¯0.84	¯1.05	0.73	¯1.07	0.73
		10,000	9.00	1.63	5.52	0.79	¯0.84	¯1.01	0.71	¯1.04	0.72
		15,000	9.67	1.67	5.79	0.83	¯0.84	¯0.97	0.68	¯1.02	0.70
		20,000	9.67	1.71	5.65	0.87	¯0.84	¯0.95	0.66	¯1.03	0.69
		Mean values			5.75	$a=80°10'$		¯0.99	0.69	¯1.04	0.71
Double Angle	Central Right	10 000	1.48	4.77	0.31	3.93	¯0.84				
		15,000	1.77	4.77	0.37	3.93	¯0.84	0.36	0.46		
		20,000	1.77	4.77	0.37	3.93	¯0.84				
		25,000	1.77	4.77	0.37	3.93	¯0.84				
		30,000	1.77	4.77	0.37	3.93	¯0.84				
		Mean values			0.37	$a=20°18'$					
	Central Left	10,000	1.47	3.97	0.37	3.13	¯0.84				
		15,000	1.42	3.87	0.37	3.03	¯0.84	0.43	0.55		
		20,000	1.41	3.87	0.37	3.03	¯0.84				
		25,000	1.24	3.87	0.32	3.03	¯0.84				
		30,000	1.15	3.87	0.30	3.03	¯0.84				
		Mean values			0.34	$a=18°48'$					
	7⅛″ from end-plate, Right	20,000	2.35	7.13	0.35	6.29	0.84	0.24	0.31		
	7⅛″ from end-plate, Left	20,000	2.46	4.95	0.49	4.11	0.84	0.33	0.39		
	3 9/16″ from end-plate, Right	20,000	2.70	8.95	0.30	8.11	0.84	0.24	0.31		
	3 9/16″ from end-plate, Left	20,000	2.62	8.65	0.30	7.81	0.84	0.24	0.32		

TABLE VI.—Maximum Stresses. Single Angles.

SPECIMEN	SECTION	LOAD (LBS)	Max. Stress over section from extensometer readings	RATIO Max./Mean	Max. Stress from calculated load axis	RATIO Max./Mean
Single Angle with ¾" end-plate	Central	5,000 10,000 15,000 20,000	7,500 14,420 21,400 28,000	2.28 2.20 2.17 2.13	7,350 14,500 21,200 28,000	2.23 2.20 2.15 2.13
	3" from end-plate	5,000 10,000 15,000 20,000	7,500 14,300 21,100 27,600	2.28 2.18 2.14 2.10	7,500 14,400 21.600 29,400	2.28 2.20 2.19 2.23
Single Angle with ¼" end-plate	Central	5,000 10,000 15,000 20,000	7,850 14.500 21,500 28,300	2.26 2.09 2.07 2.04	7,710 15,200 22,350 29,000	2.22 2.19 2.15 2.09
	3" from end-plate	5,000 10,000 15,000 20,000	7,730 15,000 22,150 29,000	2.23 2.16 2.13 2.09	8,100 15,900 23,200 30,600	2.33 2.29 2.23 2.21

N.B.—All stresses are measured in lbs per □"
$E = 28.6 \times 10^6$ lbs. per □".

TABLE VII.—Maximum stresses. Double Angle.
$E = 28.6 \times 10^6$ lbs. per □".

SECTION	LOAD (LBS)	Maximum stress from extensometer readings		RATIO Max./Mean	
		Left	Right	Left	Right
Central	10,000 15,000 20,000 25,000 30,000	7,930 11,650 15,300 18,930 22,450	6,720 9,600 12,880 15,900 18,930	2.28 2.24 2.21 2.18 2.15	1.94 1.84 1.85 1.83 1.82
7⅛" from end-plate	20,000	13,580	13,500	1.95	1.94
3$\frac{9}{16}$" from end-plate	20,000	12,500	12,600	1.80	1.81

TABLE VIII.—Maximum Stresses for different positions of Load Axis.

(a) ANGLE WITH ¾″ END-PLATE

LOAD (LBS)	Maximum Stress calculated from mean experimental load axis allowing for bending (—81, .60)	MAXIMUM STRESSES (THEORETICAL) NEGLECTING BENDING					
		Load axis having co-ordinates (−1.00, .65)	Load axis at outside face of plate (−1.60,. 65)	Load axis at middle of plate (−1.23, .65)	Load axis at junction of plate & angle (−0.85, .65)	Load axis at middle of angle (−0.72, .65)	Load axis at inner face of angle (−0.58, .65)
5,000	7,100	7,100	5,610	6,540	7,480	7,800	8,140
10,000	14,200	14,200	11,220	13,090	14,970	15,600	16,290
15,000	21,600	21,600	16,830	19,630	22,450	23,400	24,430
20,000	28,400	28,400	22,440	26,180	29,940	31,200	32,580
Max-Mean	2.16	2.16	1.70	1.99	2.28	2.37	2.48
a	74° 55′	84° 36′	−76° 12′	84° 54′	72° 25′	56° 24′	32° 36′

(b) ANGLE WITH ¼″ END-PLATE

LOAD (LBS)	Maximum Stress calculated from mean experimental load axis allowing for bending (−.81, .62)	MAXIMUM STRESSES (THEORETICAL) NEGLECTING BENDING					
		Load axis having co-ordinates (−1.00, .66)	Load axis at outside face of plate (−1.09, .66)	Load axis at middle of plate (−.96, .66)	Load axis at junction of plate & angle (−0.84, .66)	Load axis at middle of angle (−0.72, .66)	Load axis at inner face of angle (−0.59, .66)
5,000	7,490	7,490	7,250	7,600	7,950	8,300	8,650
10,000	14,980	14,980	14,500	15,200	15,900	16,600	17,300
15,000	22,470	22,470	21,750	22,800	23,850	24,900	25,950
20,000	29,960	29,960	29,000	30,400	31,800	33,200	34,600
Max - Mean	2.16	2.16	2.09	2.19	2.29	2.39	2.49
a	75° 10′	72° 42′	89° 16′	81° 18′	70° 24′	54° 48′	32° 21′

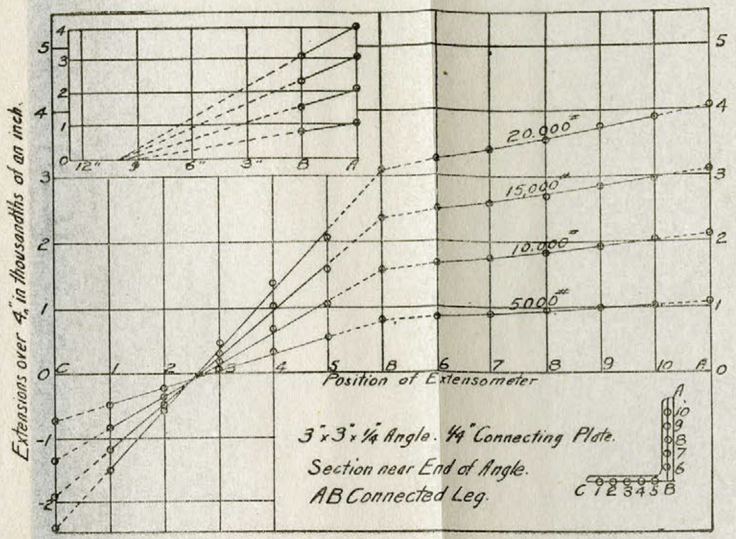

Fig. 8.

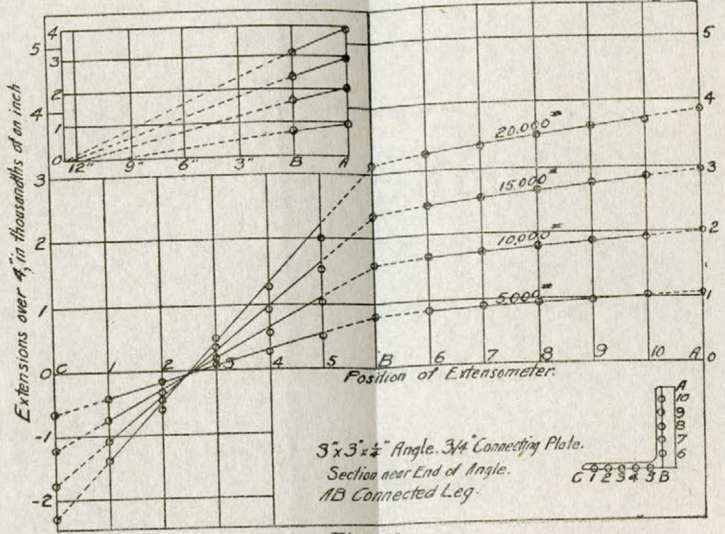

Fig. 10.

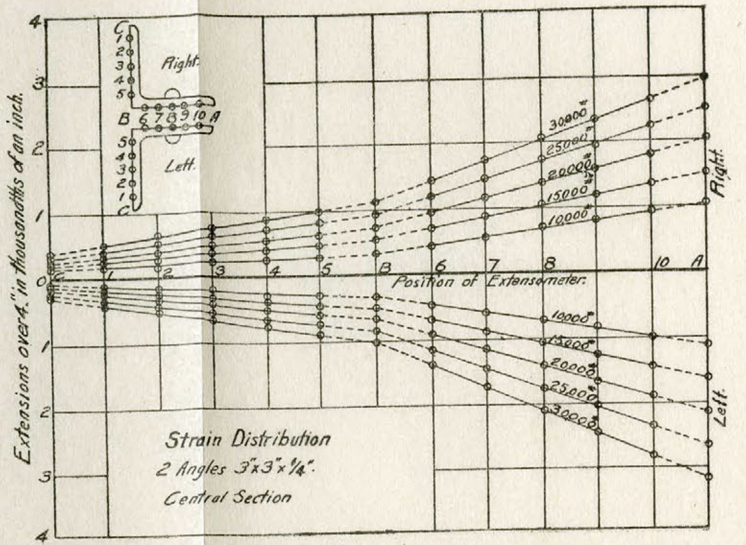

Fig. 12.

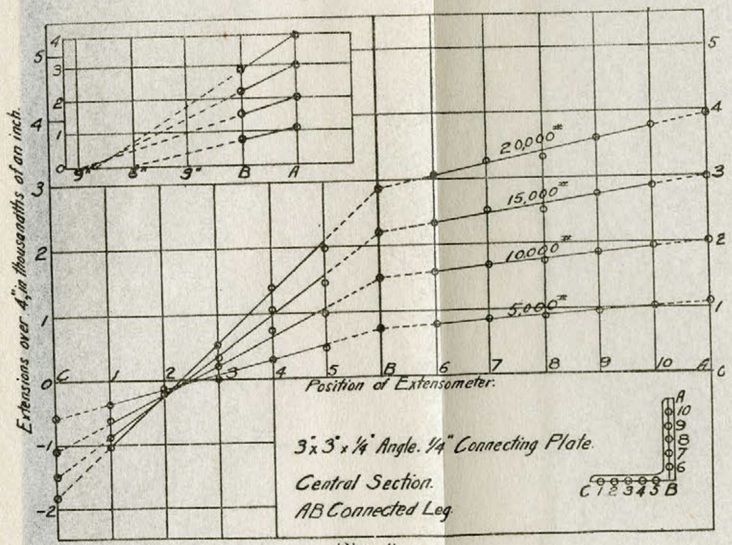

Fig. 9.

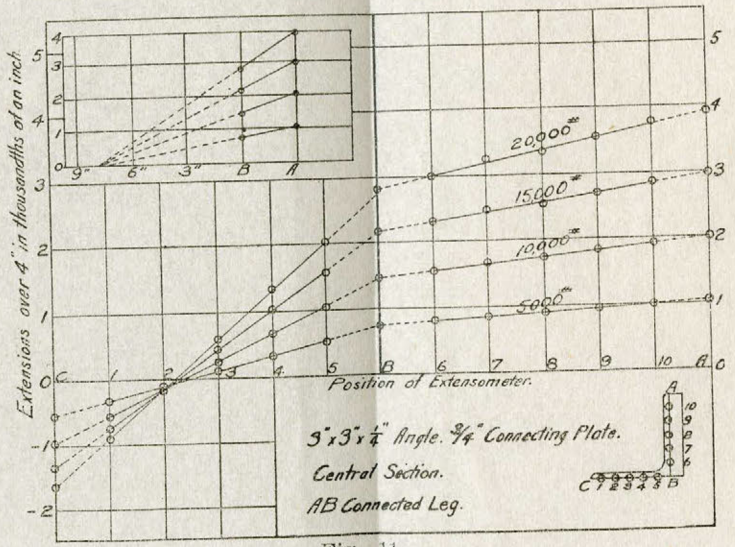

Fig. 11.

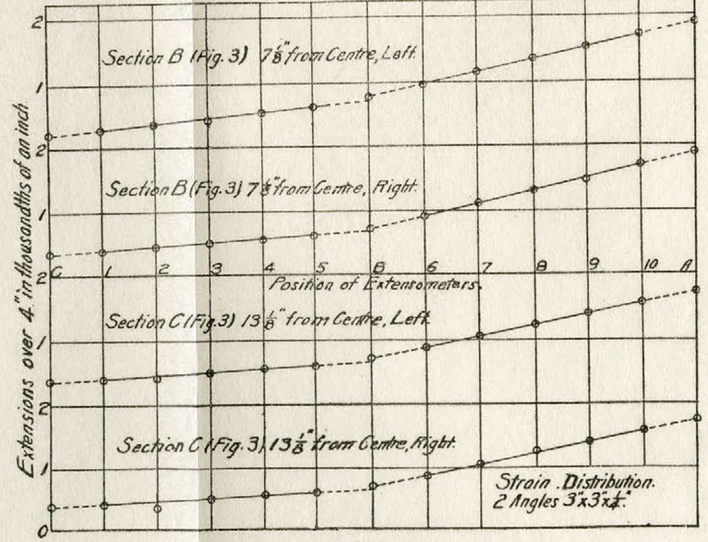

Fig. 13.

THE EFFECT OF THE END CONNECTIONS ON THE DISTRIBUTION OF STRESS IN CERTAIN TENSION MEMBERS

BY

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THE EFFECT OF THE END CONNECTIONS ON THE DISTRIBUTION OF STRESS IN CERTAIN TENSION MEMBERS.[1]

BY

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INTRODUCTION.

The experimental study of the distribution of stress in structural members has received considerable attention in recent years. The experiments of American investigators upon actual structures are well known. Most of these investigations, however, have been made with strain gauges, such as the Howard and Berry, which, while excellently adapted to the more or less rough determination of average strains in members of actual structures, are scarcely suited to refined measurement of the distribution of strain in a member subjected to stresses which vary considerably over the cross-section. Thus the analysis of the distribution of stress in single or built-up structural members and its modification due to different types of end connections has scarcely received the attention which it deserves from its practical importance. The optical determination of complex stress distribution in transparent materials by the use of polarized light as developed by Coker, Mesnager, Hönigsberg and others,[2] although giving accurate results within its range, does not appear to be

[1] Communicated by Professor H. M. Mackay.

[2] See British Association Report, 1914.

capable of extension to the investigation of built-up members, as the actual end conditions, connections between the parts, etc., cannot be correctly imitated in glass or xylonite models. The thermoelectric method would appear to be more promising, but has not yet been sufficiently investigated. For the present it would seem as if reliance must be placed upon measurement with some form of extensometer. Most extensometers, however, measure only the average stress over a considerable length, are bulky and inconvenient, and cannot be used in positions difficult of access, such as the inner portions of a built-up member.

Some time ago the writer read a paper before the Canadian Society of Civil Engineers [3] in which he described investigations of the distribution of stress in certain members made with a simplified form of Martens' mirror extensometer constructed in the laboratories of McGill University, Montreal. This instrument, extremely simple in construction and operation, was shown to be capable, when certain precautions are observed in its use, of measuring strains accurately to $\frac{1}{100,000}$ inch on a length as small as 2 inches, and of being used in the most confined positions, such, for example, as the space between two angles placed back to back and separated by as little as 3/8 inch. It was thus proved to be eminently suitable for the laboratory investigation of strain distribution in built-up members.

In the paper cited above, the distribution of stress in single- and double-angle tension members riveted to end plates was considered. It was shown that the assumption of planar distribution of stress over the cross-section was justified and that the actual distribution in a single-angle member was closely in accordance with that given by the theory of eccentric stresses developed by Bach, Müller-Breslau and others, and put into workable form by Professor L. J. Johnson, of Harvard.[4] It was also shown that a double-angle member did not act as one piece, but that each angle bent about its own neutral axis, and thus the correct way of considering a built-up member was not to regard it as a single piece bending as a beam, but as several pieces, each trying to bend about its own neutral axis, but restrained by the subsidiary members, such as tie plates or latticing.

In these experiments 3 inch × 3 inch × 1/4 inch angles were

[3] *Transactions Canadian Society of Civil Engineers*, vol. xxvi, p. 224.

[4] See *Trans. Am. Soc. Civ. Eng.*, 1906, p. 169.

used, riveted to end plates of the same width, that being the greatest which would fit into the grips of the testing machine. In this case the effect of the end constraints was found to be very little in the case of the single angle, but it was thought advisable to repeat the experiments with end plates more like the gusset plates to which such members are usually attached in practice. To this end special grips were made so that plates of considerable size could be used, and the results of the experiments to be described later show that there must be considerable end restraint under practical conditions. In the present paper the effect of this constraint is considered, and also the effect of changing the axis of pull on the gusset plate, but the chief object is the investigation of the effect of lock angles at the ends of the members. In practice the ends of single- and double-angle members are usually secured to the gusset plates directly by a row of rivets between the plate and the angle and indirectly through the medium of a small angle riveted both to the gusset plate and to the member as shown in Fig. 5, which represents the experimental specimens. The function of this small angle, called a "lock" or "lug" angle, is supposed to be two-fold. First, by transmitting a pull through both legs of the angle or angles forming the member, it is supposed to lessen the eccentricity of connection, so that often the stress is calculated as though uniformly distributed over the cross-section; secondly, by allowing more rivets to be used at the ends of the member, it is supposed to be equivalent to lengthening the end connections. The writer hopes to show that both of these claims are to a large extent unfounded; that lock angles really serve very little useful purpose, and that the slight effect which they have, helpful or otherwise, cannot be predetermined.

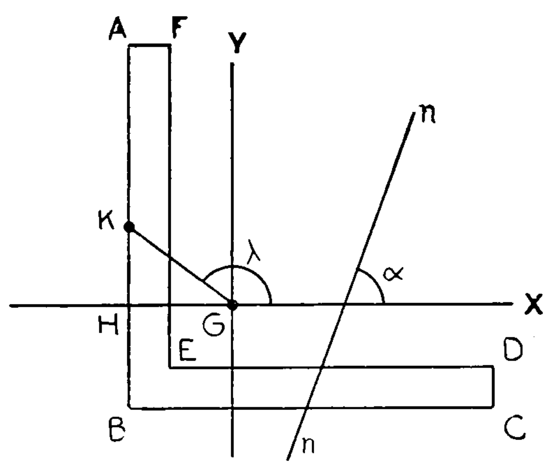
Before entering upon a discussion of the experiments a brief account of the theory upon which the analysis of the results is based will be given.

PART I.—THEORETICAL.

§1. *The Distribution of Stress in Eccentrically Loaded Members of Uniform Cross-section.*

A single angle such as is shown in Fig. 5, loaded in tension or compression through rivets in one leg, is an example of an eccentrically loaded member in which the load axis does not lie in an axis of symmetry of the cross-section. If the ends of the member are not appreciably restrained from bending by the end connections, the axis of loading may be said to lie along the line of rivets and at the common face of the end plate and the angle. This axis is represented by K in Fig. 1, which represents a cross-section of the member. It will readily be seen that the ordinary theory of eccentric loading, which is true only for loads applied in an axis of symmetry of the cross-section, cannot be applied to this case, and recourse must be had to the general theory already mentioned.[5] For a full account of this theory the reader is referred to the paper by Professor L. J. Johnson referred to above or to Appendix I of the paper by the present writer in the *Transactions of the Canadian Society of Civil Engineers*, vol. xxvi, p. 224. A brief *résumé* of the

FIG. 1.



theory will be given here, as it is essential to the understanding of the experimental results. Referring to Fig. 1, let G represent the centroid of the cross-section and Gx and Gy be a pair of coordinate axes parallel to the legs of the angle. Then, if N be the normal force applied at K, the stress at any point of the cross-section will be equal to $\frac{N}{A}$, where A is the area of the section, together with a stress arising from the bending moment $N.KG$ to which the angle is subjected. This moment will cause bending about a neutral axis nn, which will not, as in the case of loading in a plane of symmetry, be perpendicular to KG, but at an angle α to Gx, given by the equation

$$\tan \alpha = \frac{I_x - J \tan \lambda}{J - I_y \tan \lambda} \quad \ldots \ldots (1)$$

[5] L. J. Johnson, *loc. cit.*

where I_x and I_y are the moments of inertia of the section about Gx and Gy respectively and J is the product of inertia of the section with respect to the axes Gx and Gy,—*i.e.*, the $\int\int xydxdy$ with respect to these axes.[6] The above equation is deduced by equating the sum of the moments of the stresses over the section about Gx and Gy respectively to the components of the bending moment $N.KG$ about these axes, assuming that the distribution of stress over the section follows a linear law.[7] The stress at any point (xy) of the section may be shown to be given by either of the equations

$$f = N\left[\frac{1}{A} + \frac{y - x \tan a}{J - I_y \tan a}\,x_k\right] \qquad (2)$$

$$= N\left[\frac{1}{A} + \frac{y - x \tan a}{I_x - J \tan a}\,y_k\right] \qquad (3)$$

where (x_k, y_k) are the coördinates of the load axis K. In order to find the maximum stress, it is only necessary to substitute for (xy) the coördinates of the point most distant from the neutral axis. If f be equated to zero in either of the equations (2) or (3), the equation of the neutral axis nn will be obtained.

§2. *The S-Polygon.*

The S-polygon, a modification of the W-Fläche of Land, was first introduced by L. J. Johnson.[8] It is a figure which gives at a glance the point of maximum bending stress and the value of the latter for any given load axis, and, as its use will facilitate certain deductions to be made later, its construction will be considered briefly here.

For the bending of ordinary I sections the steel handbooks give a quantity called the "modulus" of the section which is equal to $\frac{M}{f} = \frac{I}{y}$ where M is the bending moment applied to the section, y the distance of the skin from the neutral axis, f the maximum stress and I the moment of inertia of the section about the neutral axis. It may be defined as that quantity by which the bending moment at any section must be divided in order to give the maximum

[6] I_x and I_y are given in the steel handbooks for all ordinary sections, but J has to be calculated. For the method of calculation see either of the papers cited above.

[7] This was shown to be the case experimentally by the writer, *loc. cit.*

[8] *Loc. cit.*

stress at the section. In the general case of flexure as considered above there will be a similar quantity which may be termed the "flexure modulus" (S) of the section. Thus $S = \frac{M}{f}$. The bending moment on the section is $N.KG$. The stress due to this at any point of the section is, from equation (2), given by

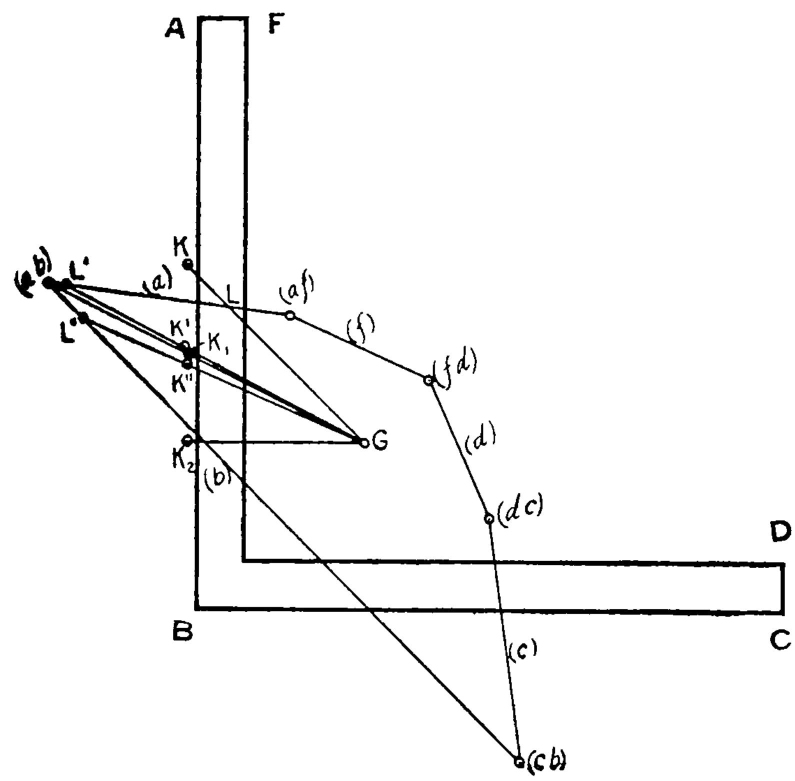
$$f_b = \frac{y - x \tan \alpha}{J - I_y \tan \alpha} \cdot NG \cos \lambda \qquad \ldots\ldots \quad (4)$$

Eliminating α between this equation and equation (1), and rearranging terms, the section modulus for (xy) is given by

$$S = \frac{M}{f} = \frac{I_x I_y - J^2}{(yI_y - xJ) \sin \lambda + (xI_x - yJ) \cos \lambda} \qquad \ldots\ldots \quad (5)$$

If the point (xy) remains fixed while λ changes, the above is readily seen to be the polar equation of a straight line, having

FIG. 2.



radius vector S and angle λ. This line may be termed the S-line for the point (xy). If S-lines be drawn for all points at which the maximum stress may occur, a polygon, called the S-polygon, will be obtained. Thus, for example, in the case of the single angle Fig. 2, the maximum stress may occur at A, B, C, D, or F. If G be chosen as the pole and Gx as the initial line, the figure (ab), (bc), (cd), etc., is the S-polygon drawn with S to the same scale as the linear dimensions of the figure. Thus for the load point K, GL represents S and the maximum stress occurs at B. But $S = \frac{M}{f_b}$ and $M = N.KG$, therefore f_b, the bending stress at B, is equal to

$$\frac{M}{S} = N \frac{KG}{LG}$$

Thus the total stress at B, which is the maximum stress at the section, is given by

$$f = \frac{N}{A} + N \cdot \frac{KG}{LG} \qquad \ldots\ldots\ldots\ldots (6)$$

In order that this may be true, it is, of course, necessary that the scale of the S-polygon should be the same as the linear scale of the figure and that the origin be G and the initial line Gx.

§3. *The Construction of the S-Polygon.*

The best method of constructing the S-polygon of an angle is by locating the apices. Each apex corresponds to a side. It may be shown by transforming equation (5) into rectilinear coördinates, substituting the coördinates of any two points $(x_a\ y_a)$, $(x_b\ y_b)$, and solving for the apex (ab) of the S-polygon, that

$$\left.\begin{aligned} x_{ab} &= \frac{(x_a - x_b)J - (y_a - y_b)I_y}{x_a y_b - x_b y_a}, \\ y_{ab} &= \frac{(x_a - x_b)I_x - (y_a - y_b)J}{x_a y_b - x_b y_a} \end{aligned}\right\} \qquad \ldots\ldots (7)$$

If the side considered be parallel to Gx, so that $y_a = y_b$, the above equation gives

$$x_{ab} = \frac{J}{y_a}, \quad y_{ab} = \frac{I_x}{y_a},$$

while, if the side be parallel to Gy,

$$x_{ab} = \frac{I_y}{x_a}, \quad y_{ab} = \frac{J}{x_a}$$

Considering, for example, the single-angle specimen I, used in the experiments to be described later and for which $I_x = I_y = 1.37$ $(\text{in.})^4$, $J = -0.81$, and the distance from the centroid to the back of the angle is 0.85″, the apex of the S-polygon corresponding to the side AB has the coördinates

$$x_{ab} = -\frac{1.37}{0.85} = -1.61, \qquad y_{ab} = \frac{0.81}{0.85} = 0.95.$$

The other apices may be determined in a similar manner, and the complete polygon is shown in Fig. 2.

The S-polygon furnishes the simplest method of determining the ratio of maximum to mean stress when a number of load axes have to be considered. It may also be used to establish a number of interesting results. For instance, as the point K moves down AB, starting from A, the radius vector GL terminates first on the f line, then on the a line, and finally on the b line, showing that the point of maximum stress changes from F to A and then to B. The ratio $\frac{GK}{GL}$ also changes considerably It will be shown later that the S-polygon is of material aid in analyzing the probable effect of a lock angle or a change in the restraining couple.

§4. *The Effect of Lateral Deflection and of End Restraints.*

The earlier experiments of the writer have shown that in the case of a single angle loaded in tension and not effectively restrained at the ends, the axis K is along the line of rivets and slightly within the loading plate. In a long member there will be a measurable lateral deflection as the load is applied, the centroid of each section trying to set itself in the load axis. This will cause a change in the position of K relative to the section and consequently affect the distribution of stress over the section. This effect is not usually great enough to be of much practical importance, but must be considered in reducing experimental results. Of much greater interest is the effect of the gusset plate and lock angle in affecting the position of the load axis. A stiff end connection will introduce constraining couples both in and at right angles to the plane of the end plate, and these may have important effects upon the distribution of stress. A lock angle is often supposed to so constrain the ends that the eccentricity of pull may be neglected, although how far this is from actually being the case will be shown later. The chief object of the present paper is the investigation of these different constraints in the case of single- and double-angle members.

§5. *The Working Loads upon Eccentrically Loaded Members.*

In a compression member it is, of course, always unsafe to allow the stress at any point to exceed the elastic limit of the material, even though the load which would cause this stress be well within the theoretical buckling load for a long column. Experiment has shown, however, that, in the case of a tension member, a redistribution of stress usually occurs under such conditions, and that the elastic limit may be considerably exceeded at certain parts of the section without danger to the structure as a whole. This would lead to the supposition that, when the stresses arising from eccentricity are taken into account, higher working stresses than usual might be used. It would, however, be unsafe to proceed on this assumption without more experimental evidence as to whether, in such case, the new distribution of stress persists without alteration after many reloadings. At present it would seem to be unwise to allow the stress over any considerable part of the structure to rise above the ordinary working stress of the material at any point of the cross-section, especially considering that there are always higher stresses locally near to the end connections. Thus it would appear that single-angle members, and to some extent double-angle members, should always be designed for the maximum stress, allowing for eccentricity,—*i.e.*, including what are often termed "secondary" stresses, although they are frequently quite as large as, or even larger than, the so-called "primary" or uniform stress, leaving what may be termed the "tertiary" or local stresses due to the end connections to be included in the factor of safety.

PART II.—THE EXPERIMENTS.

§1. *The Extensometers.*[9]

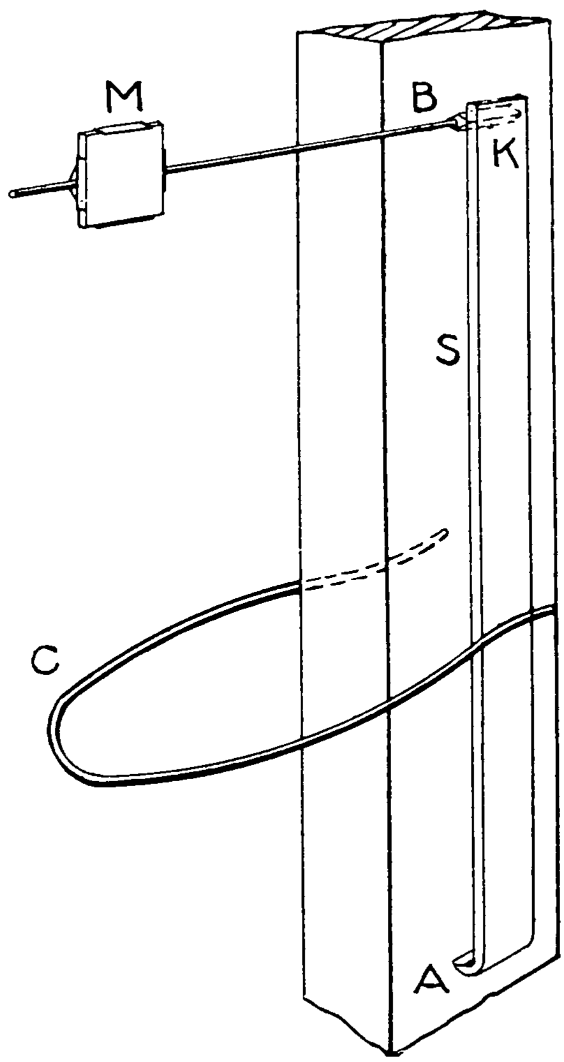
The extensometers used were a simplified form of the Martens' type, designed and constructed in the McGill Testing Laboratory, where they have been in use since 1906, and have been proved capable of giving very accurate results.

The principle of the instrument is shown in Fig. 3. It consists essentially of a double knife-edge, *K,* which fits between the specimen under test and a V groove in one end of a steel strip *S*, which is in contact with the specimen at *A,* and is pressed against it by means of a clip *C.* A change in the length of the specimen between *A* and *B* causes the knife-edge to tilt, and the tilt is measured by means of a telescope and scale, the scale being reflected in a mirror *M* attached to the knife-edge. In the actual instrument the steel strip is 3/8 inch wide, 1/8 inch thick, the length *A B* being chosen to suit requirements. The end *A* is is turned at right angles and brought to a sharp edge so that it may not slip on the specimen. The knife-edge is of hardened steel

[9] This section is reprinted from the writer's earlier paper, *loc. cit.*

about 0.18 inch × 0.12 inch × 0.45 inch, and the mirror is attached by means of a piece of steel knitting needle. The mirror is held in a clip of thin sheet steel which is arranged so that it can slide and rotate on the needle, a thin copper strip protecting its back from injury. This clip permits of a small amount of lateral adjustment. The mirror is about ½ inch square and must be as truly plane as possible, as otherwise there will be an error

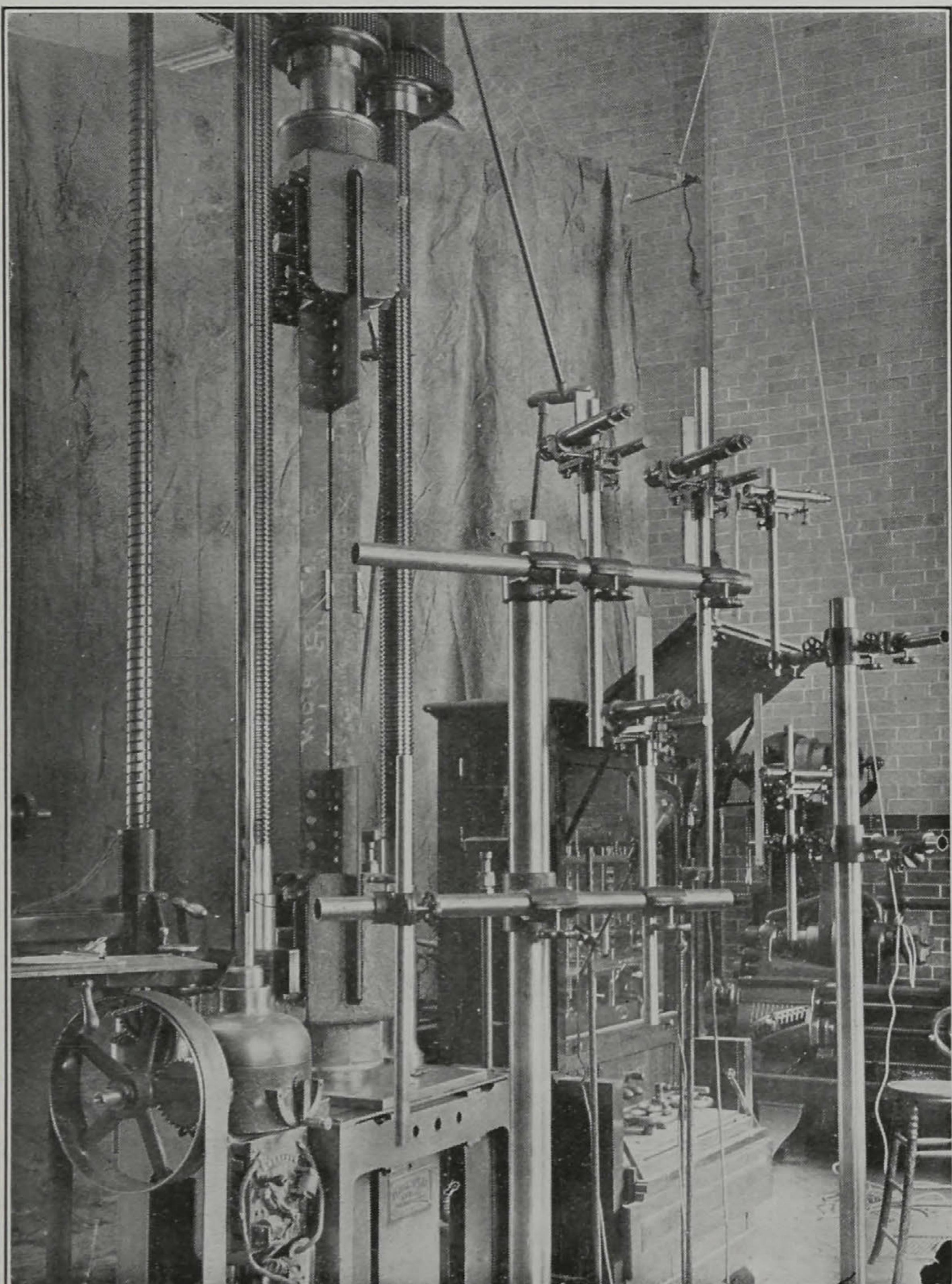
FIG. 3



introduced when the image of the scale moves to a different part of its surface, as it must do if the specimen deflects at all during test. In the original form of Martens' extensometer there was a device for adjusting the mirror and also a balance weight at the opposite side of the knife-edge, but these refinements are not only unnecessary but cumbersome, and make the instrument less adapted to use in restricted positions.

The extensometer is calibrated in a Whitworth measuring machine, and a calibrating rod is prepared for each instrument, giving the distance from the scale to the mirror, so that a definite

FIG. 4.



distance on the scale may correspond to a given extension or compression on the specimen. In the case of the experiments described below, ½ inch on the scale, subdivided into ten equal divisions, corresponded to $\frac{1}{1000}$ inch, so that the change of length of the specimens was easily read to $\frac{1}{100,000}$ inch. The length of the rod was about 4 feet, varying with different instruments. The angle turned through by the mirror in any test is so small that there is no appreciable error in using a straight scale for the readings. This was verified by turning the mirror in the Whitworth measuring machine through much greater angles than those through which it turns in the tests. It was also found that different strips (S) did not affect the calibration, so that a knife-edge could be used with different lengths of strip without recalibration. It was estimated that, under the conditions of test, the instrument reads accurately to $\frac{1}{100,000}$ inch.

The kind of telescope used affects greatly the facility with which readings may be taken. The McGill Testing Laboratory telescopes are adjustable vertically and horizontally, besides moving bodily about a vertical axis. They are carried on specially-made stands permitting any kind of adjustment to be made with ease (see Fig. 4). The extensometer must be carefully used in order to give correct results. The mirror should be, in its mean position, parallel to the scale and the telescope should be opposite to the mirror. The clip must be arranged so that the knife-edge is held quite firmly, otherwise it will not tilt correctly. The best clips are made from pieces of copper wire.

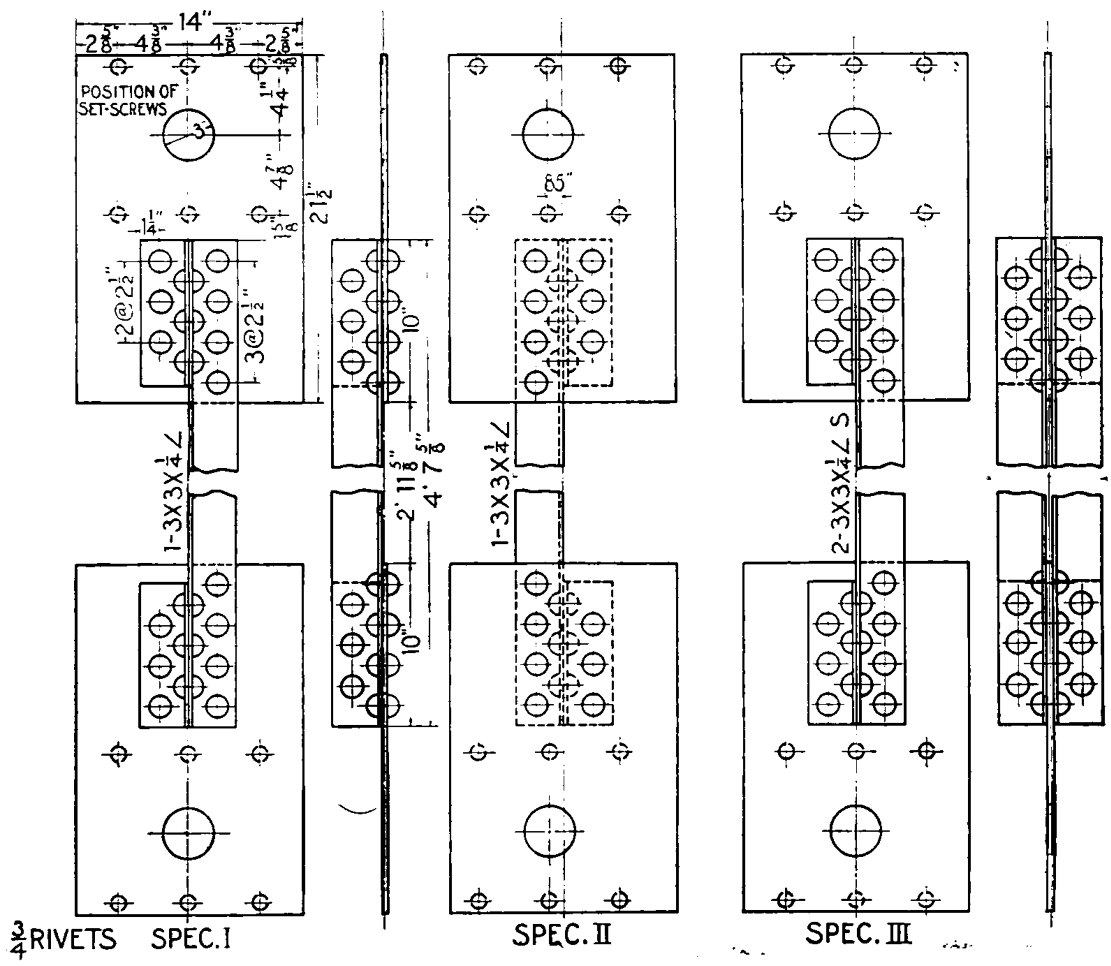
If the direction of $A\ B$ remains unchanged during test, the difference of the scale reading between two loads will be an accurate measure of the strain of $A\ B$ for the given load difference, but if $A\ B$ alters in direction this will not be the case. If, however, two readings are taken, one with the extensometer in the position shown, and the other with the knife-edge at A and the sharp edge of the strip at B, the mean of the two will be correct. When any doubt exists it is always better to do this so as to eliminate possible error.

In the opinion of all who have worked with these instruments at McGill University they are the most simple, practicable, and accurate extensometers in use. It will be seen that they may be readily used in the most restricted positions, as, for instance, between the two angles of the double-angle members, where the width is only ⅜ inch. Fig. 4 shows the general arrangement of the apparatus.

§2. *The Specimens and the Method of Holding Them.*

Experiments were made on the three specimens, two single-angle members and one double-angle, shown in Fig. 5. All were ordinary shop products made by the Dominion Bridge Company of Montreal, and the angles were of uniform section 3 inches × 3 inches × ¼ inch. Careful measurements showed that the section varied a little in the three specimens, and the actual areas, moments of inertia, etc., were computed and are given at the

FIG. 5.

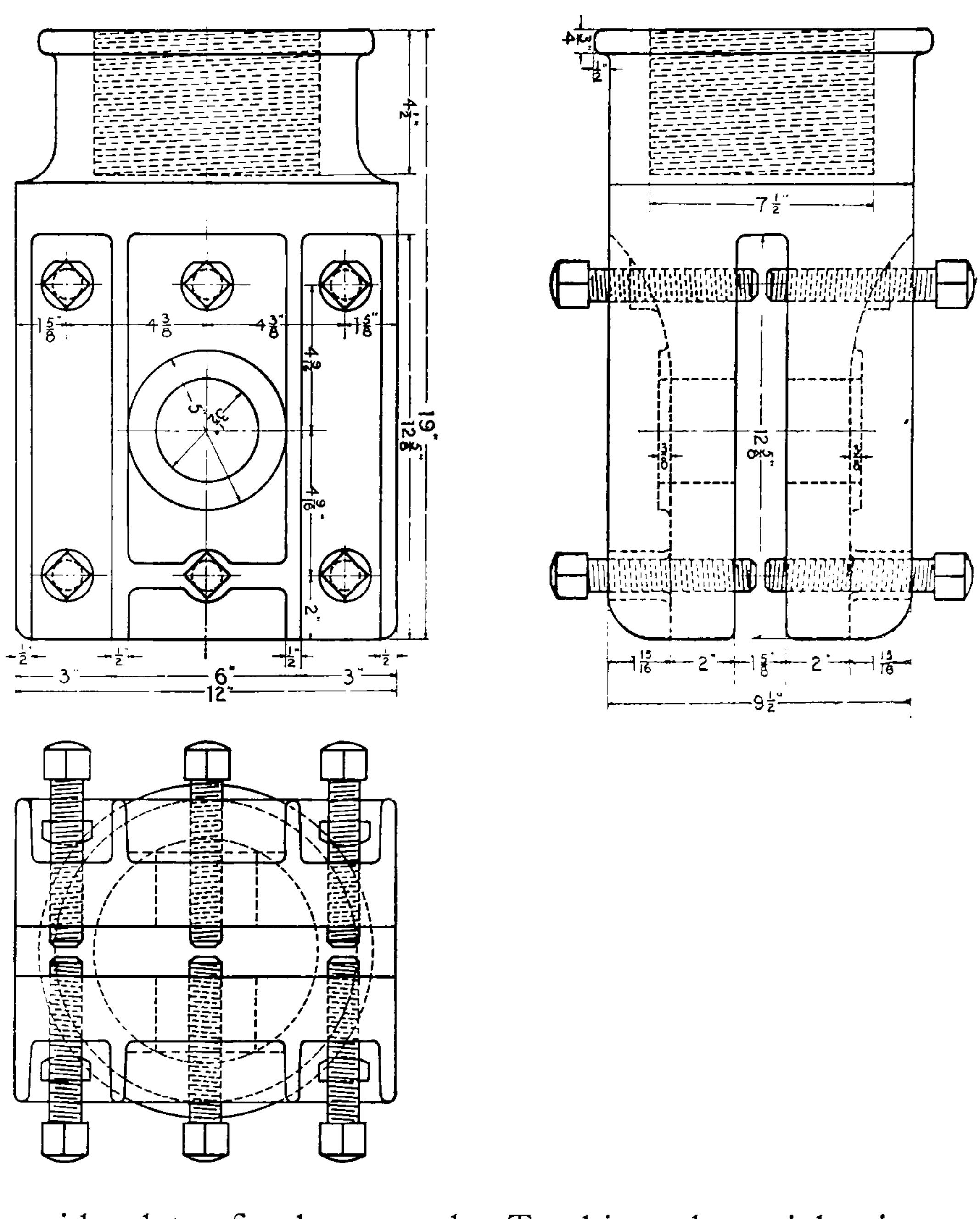


Specimens used in the experiments.

heads of Tables I, II, and III, pp. 153–155. The members were of a uniform length of 55⅝ inches over all and 35⅝ inches between the end plates, and were secured to the latter by means of four ¾-inch rivets having a pitch of 2½ inches and by lock angles riveted to the plate by three ¾-inch rivets and to the outstanding leg of the angle by a like number.

In the earlier experiments referred to above, similar angles were used, without lock angles, riveted to end plates of the same width, 3 inches, as the angles, secured directly in the grips of the testing machine, the distance from the end of the grips to the end of the angle being about 5½ inches. With these end plates it was found that the restraining couple upon the angle was very small, and it was thought desirable, in the present tests, to substitute end connections more nearly similar to those met with in practice,

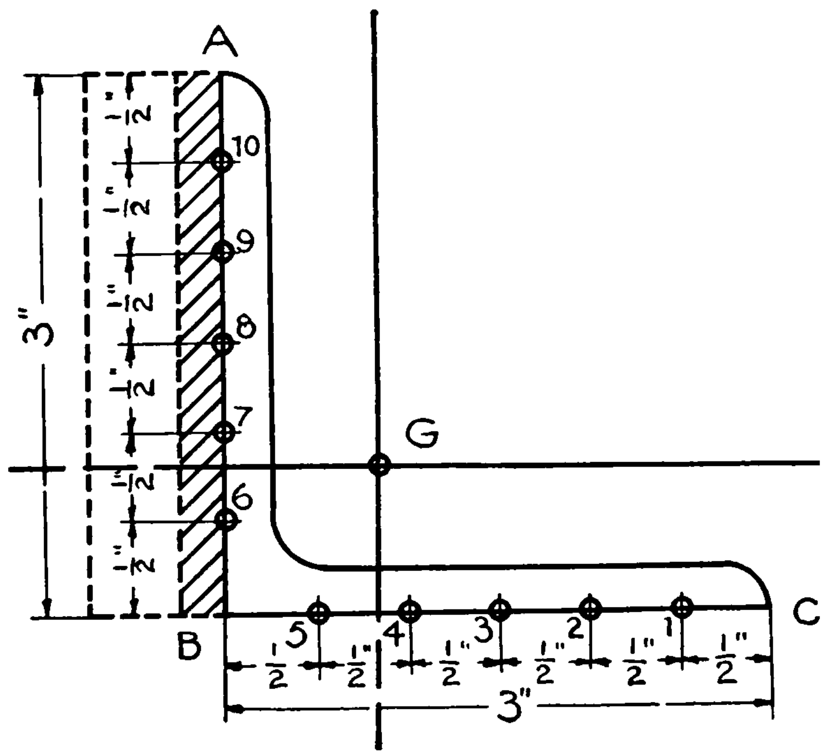
FIG. 6.



i.e., wide plates firmly secured. To this end special grips were designed to fit on the jaws of the machine. These are shown in Fig. 6. They were steel castings. The end plates, which were 1 foot 2 inches wide and ⅜ inch thick, were held by steel pins 3 inches in diameter fitting tightly into bushes in the castings, and were restrained from turning by six set screws on each side, as indicated in Fig. 6. The distance from the end of the angle to the set screws was approximately 2 inches, so that the ends of the members were quite as effectively held as in most practical cases.

The position of the pin with respect to the rivets was made different in the different specimens. In Specimens I and III it was in line with the back of the main angle, while in Specimen II it was in line with the centroid of the main angle. The object of

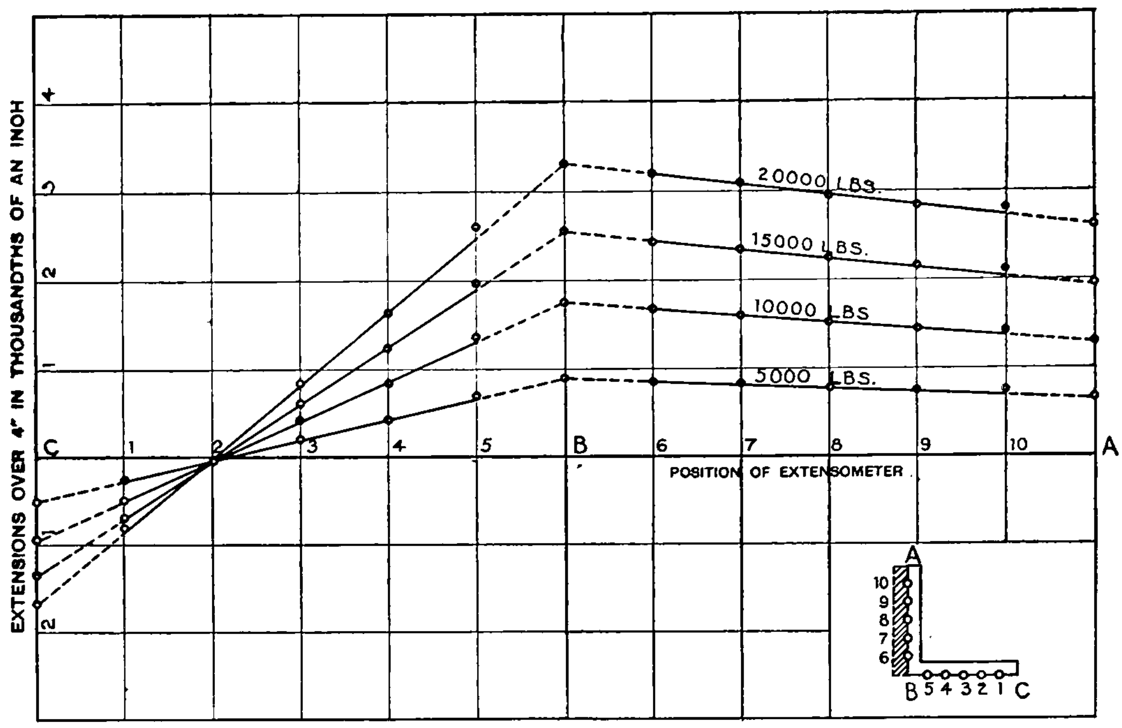
FIG. 7.



this was to find the effect of a change in the line of pull on the gusset plate,— *i.e.*, of different eccentricity of end connection. Further changes were also made during the tests, as will be described later. The specimens were also tested with the lock angles removed in order to study the action of the latter.

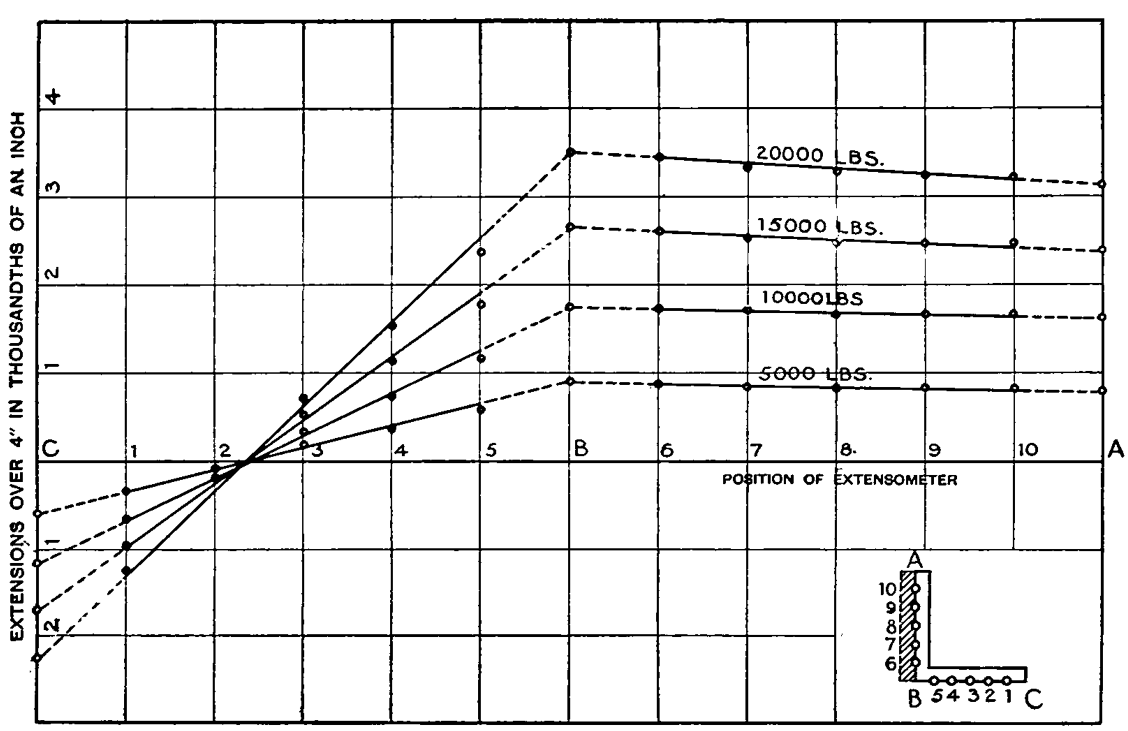
The machine used was the Emery testing machine in the Testing Laboratory of McGill University This machine is of the vertical type and has a capacity of 150,000 pounds. It is eminently suited to this kind of work as the line of pull, suitable means being taken to steady the straining head, remains constant, and there is an entire absence of vibration.

FIG. 8.



Specimen I with lock angle. Central section

FIG. 9.

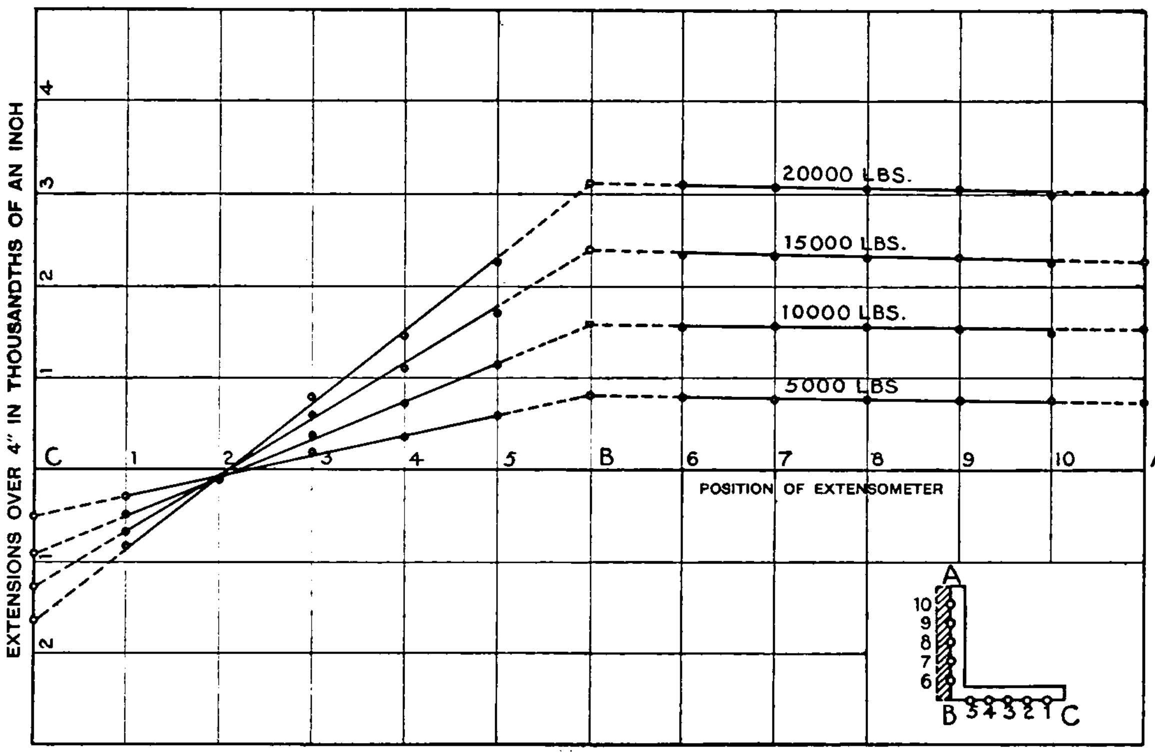


Specimen I with lock angle. End section.

§3. *The Tests.*

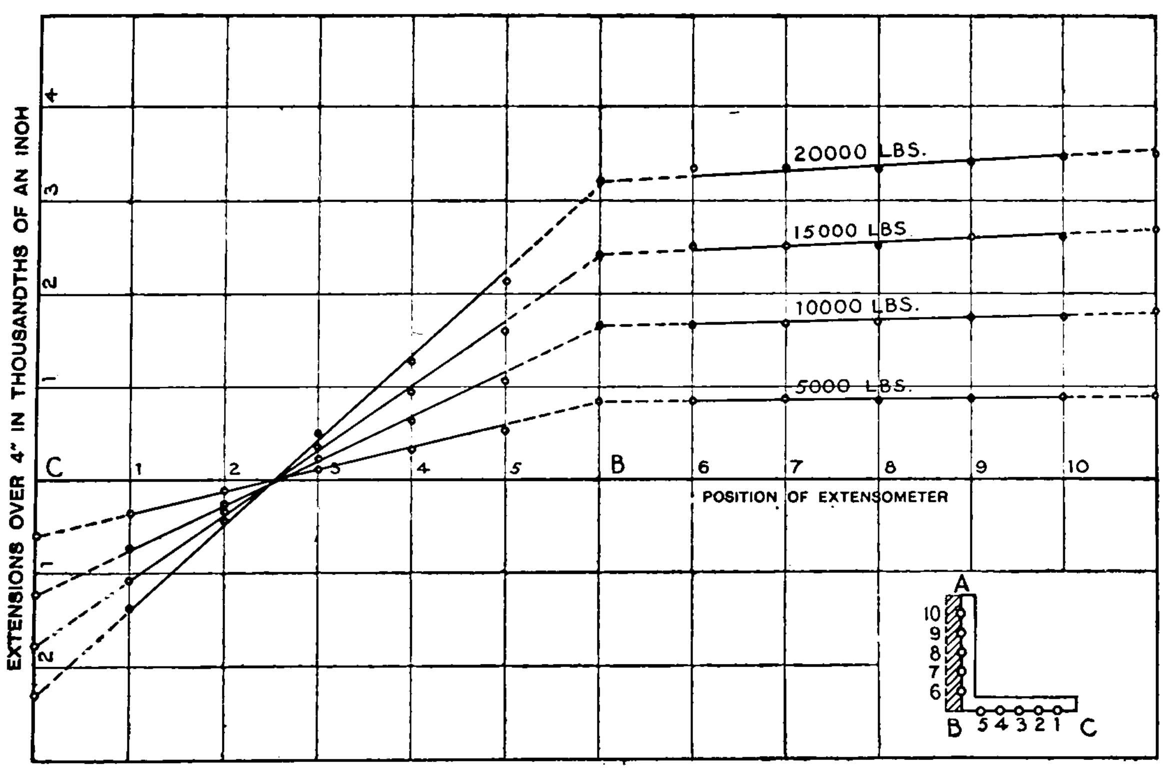
The experiments were directed to a determination of the distribution of stress under different loads at the central cross-sections of the specimens and also at sections near to the end plates.

FIG. 10.



Specimen I. Lock angle removed. Central section.

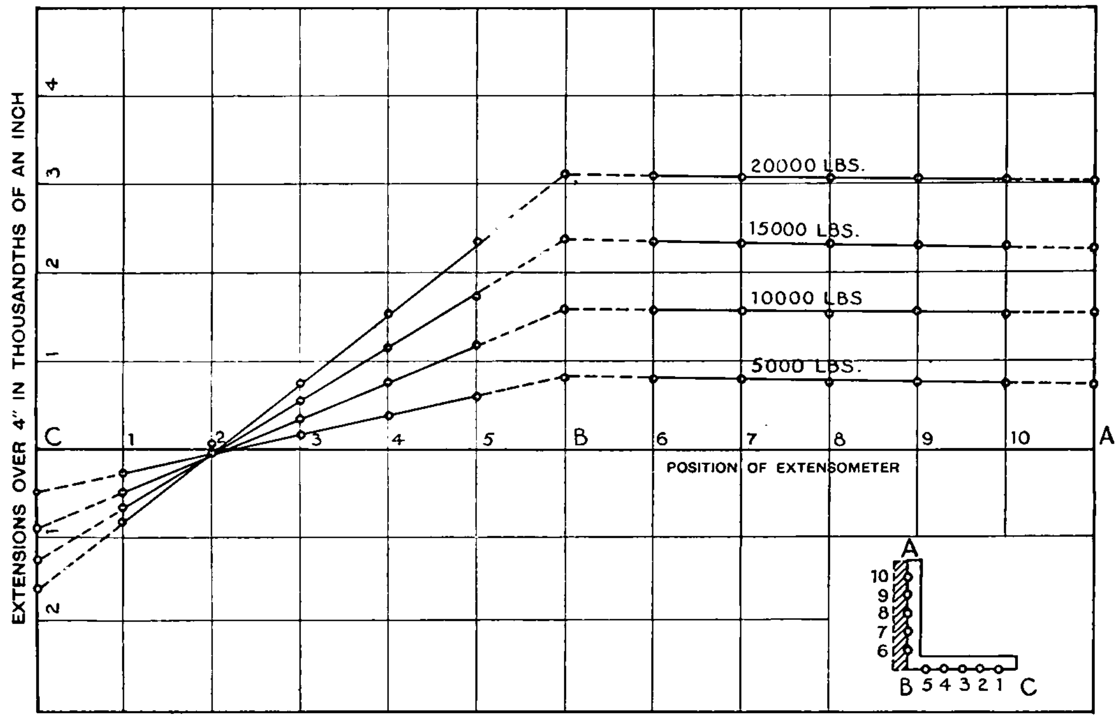
FIG. 11.



Specimen I. Lock angle removed. End section.

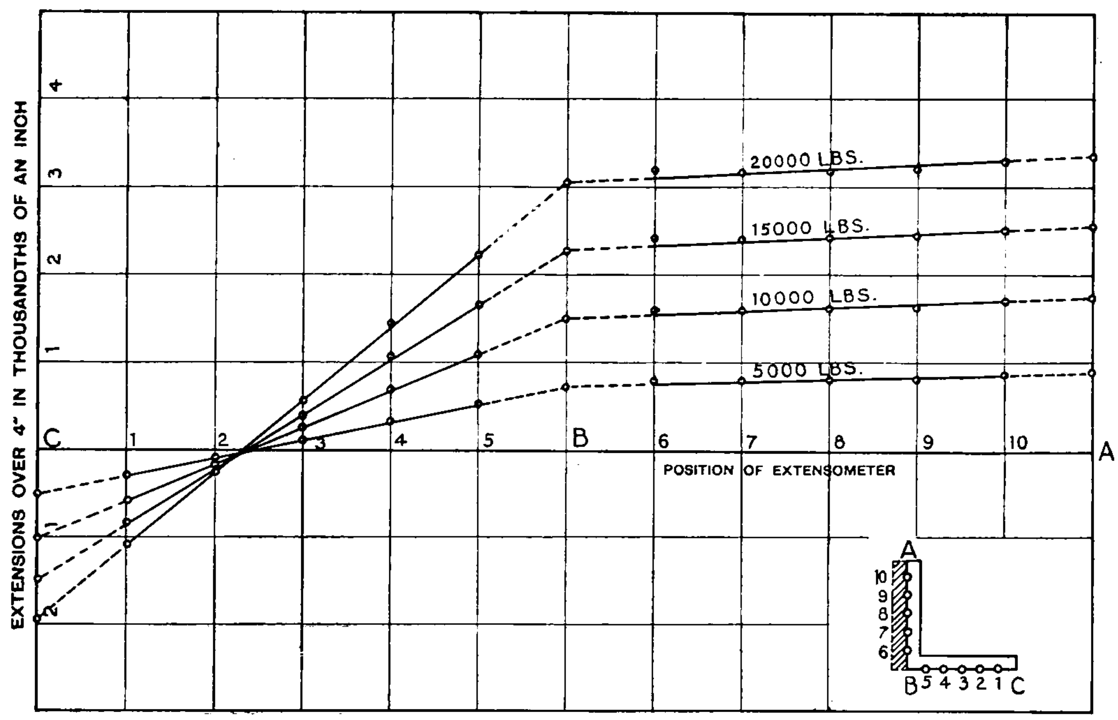
To this end extensometers were used at these sections with their distance pieces set parallel to the axis of pull. From the readings of these the longitudinal strains were found, and these were

FIG. 12.



Specimen II. With lock angle. ~~End~~ section.
Central

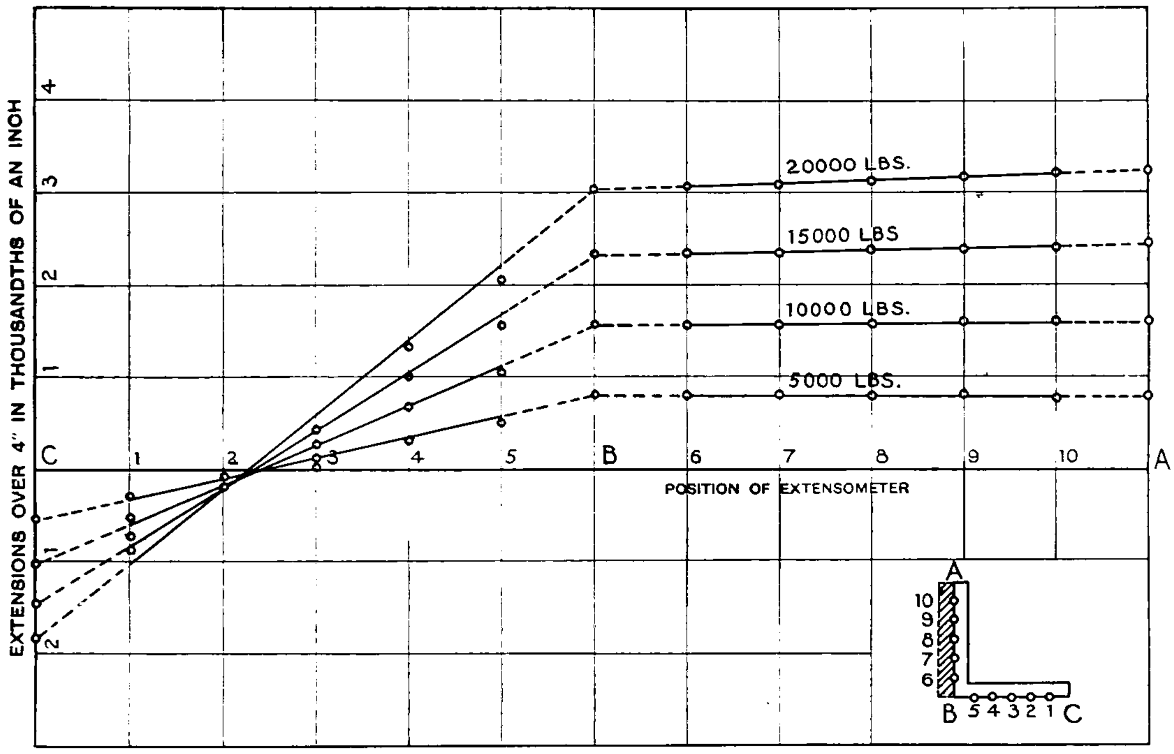
FIG. 13.



Specimen II with lock angle. End section.

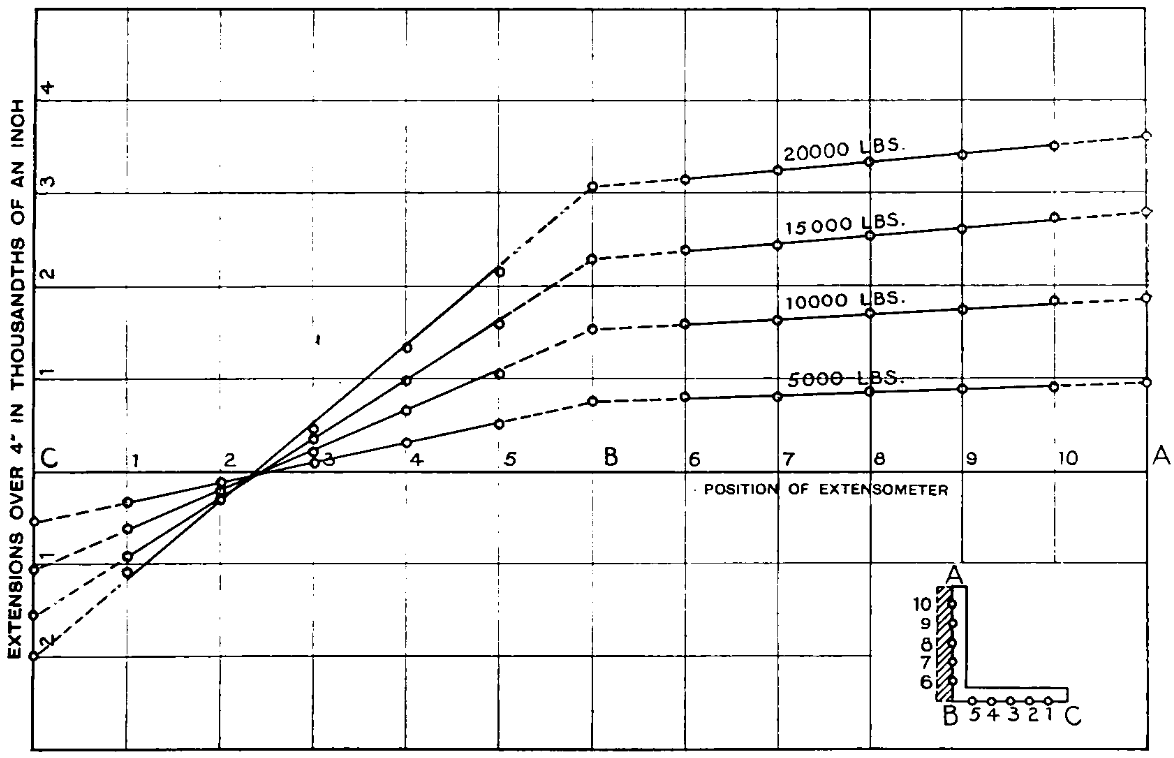
taken as being proportional to the longitudinal stresses, the effect of the strains perpendicular to the axis being so small as to be negligible. The distance pieces of the extensometers were 4

FIG. 14.



Specimen II. Lock angle removed. Central ~~station~~. section

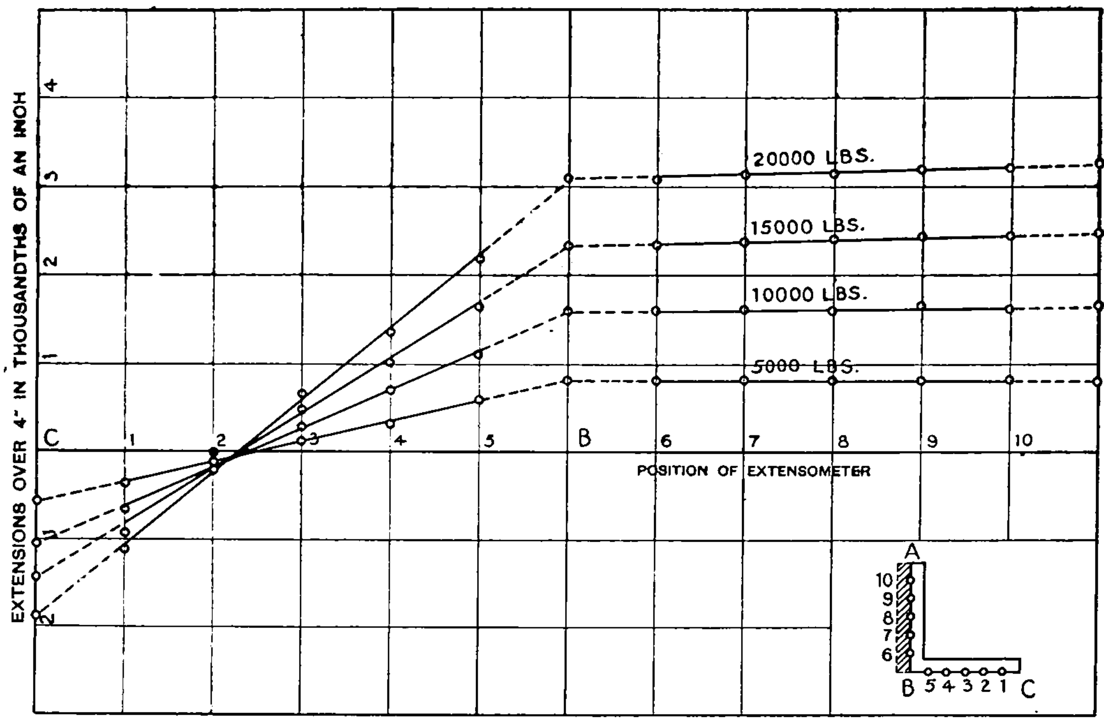
FIG. 15.



Specimen II. Lock angle removed. End section.

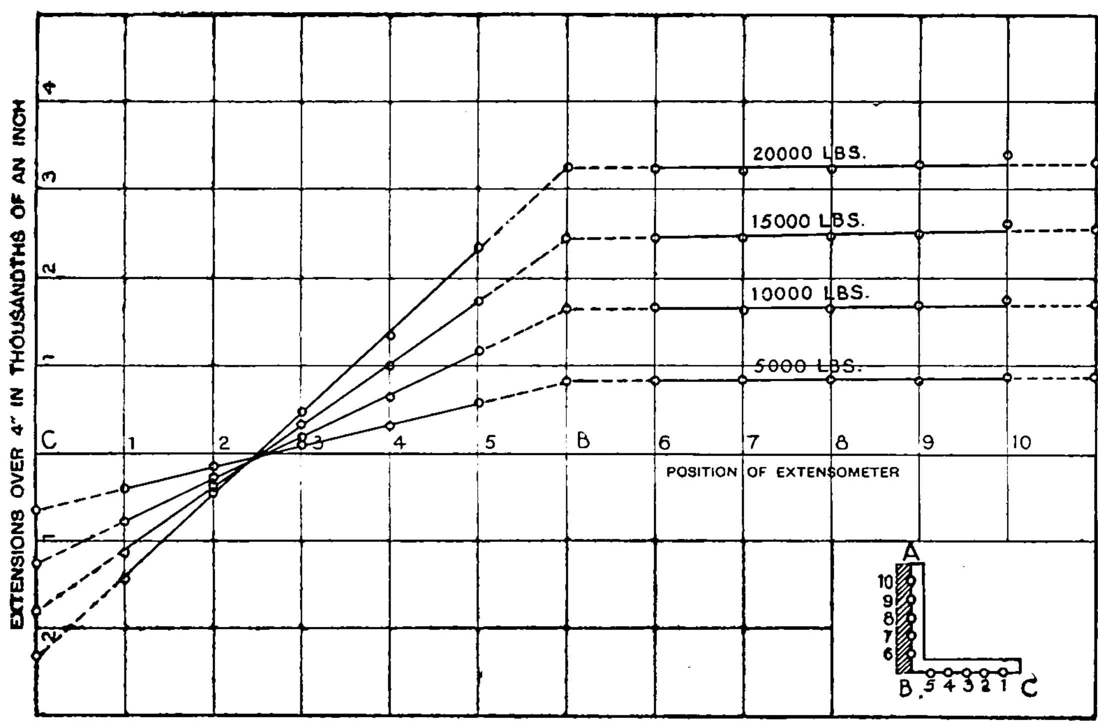
inches long, so that the strains measured were mean strains over that length. They could be considered as giving the distribution of stress over the central cross-section of the 4-inch range without

FIG. 16.



Specimen II. Line of pull in line of rivets. Central section.

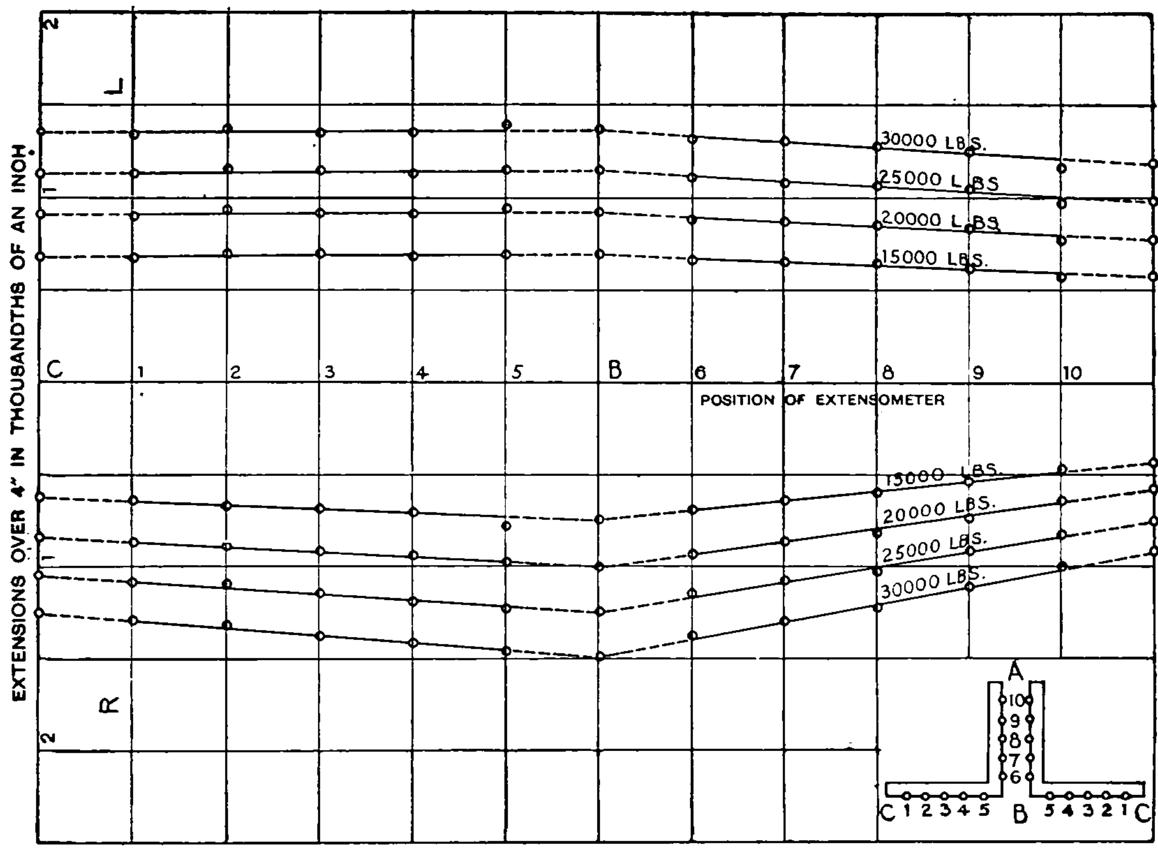
FIG. 17.



Specimen II. Line of pull in line of rivets. End section.

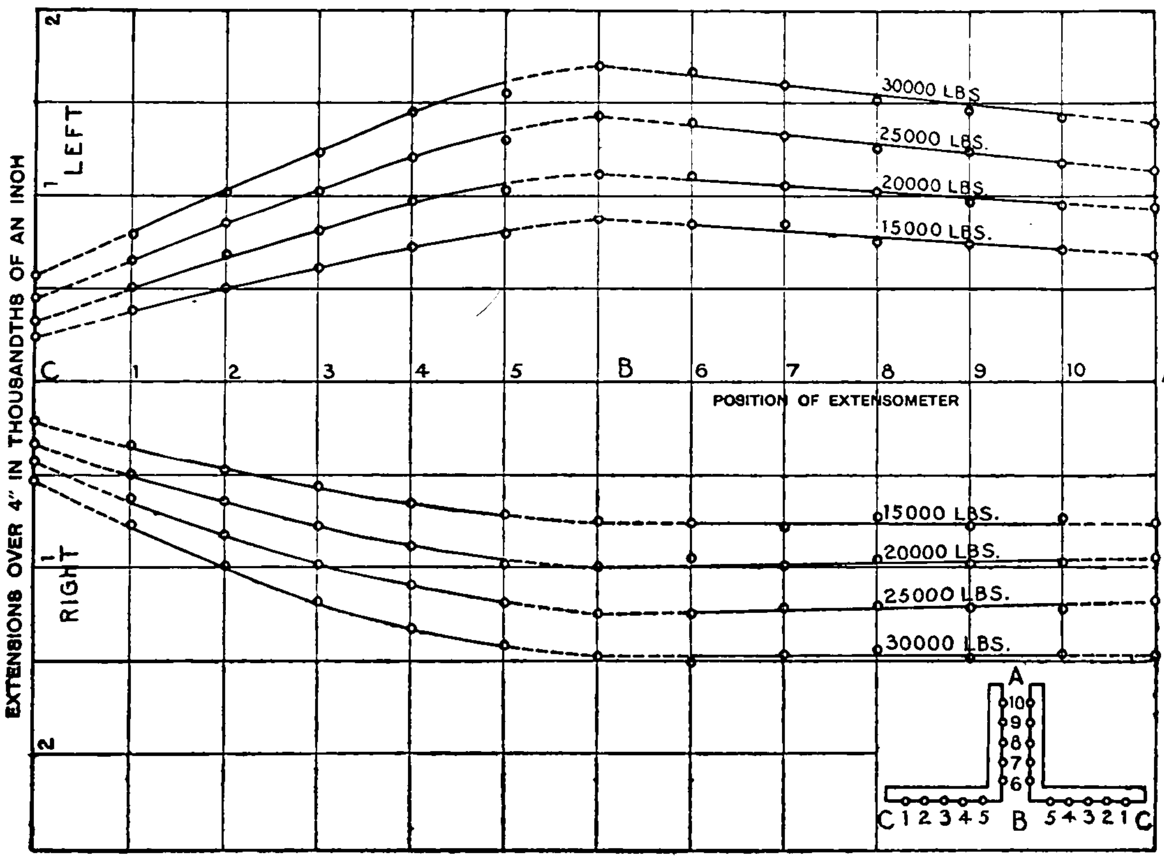
error at the central section, and to a fair degree of approximation at the ends. Readings of the extensometers were taken at half-inch intervals over the outside face of the angles, as shown in

FIG. 18.



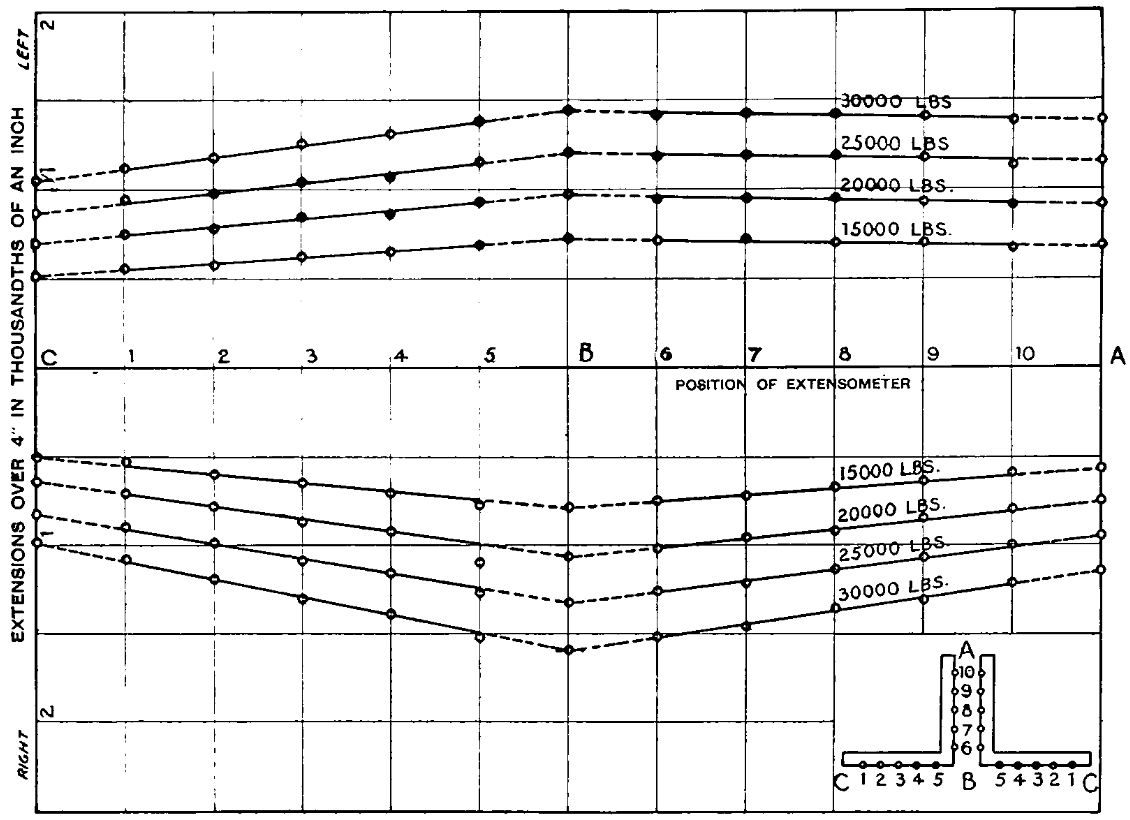
Double angle. Central ~~station~~. section

FIG. 19.



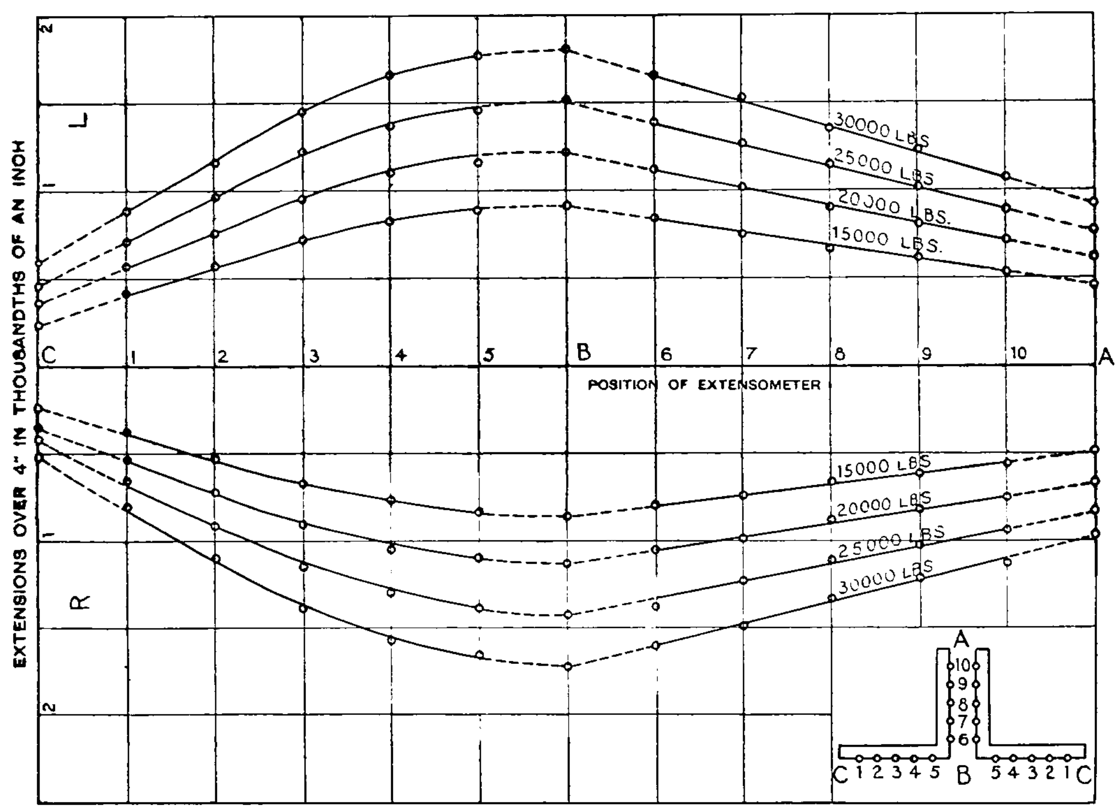
Double angle. End section.

FIG. 20.



Double angle without lock angles. Central section.

FIG. 21.



Double angle. Lock angles removed. End section.

Fig. 7, the extensometers being arranged first with the mirror at the lower end and then at the upper end and the mean of these readings taken, so that errors due to bending of the specimen were eliminated, as described above.

All the tests were carried out in a uniform manner. Two or more extensometers being set in position, an initial load of 100 pounds was applied to the specimen; the load was increased to its maximum value several times and then brought back to its initial value. The zeros of the extensometers were then set and readings taken at 5000, 10,000, 15,000, and 20,000 pounds respectively in the case of the single-angles and 15,000, 20,000, 25,000, and 30,000 pounds respectively in the case of the double-angle. The load was then brought back to its initial value and the zeros of the extensometers checked. No sets were allowed to pass in which the extensometers failed to return to their initial readings. All observations were repeated at least once before the extensometers were removed to other positions.

The uniformity of the results was remarkable. It was found that the specimen could be taken out of the machine and replaced without the extensometer readings for a given load being appreciably altered.

The lateral deflections of the specimens were carefully measured by means of telescopes and scales reading to $\frac{1}{100}$ inch. These deflections were allowed for in reducing the results, as will be explained later.

The Emery machine was calibrated by levers during the period of the tests and was found to be reading correctly to within 1 per cent. at all loads.

The sequence of tests was as follows:

(1) Specimen I.—Single, 3 inches × 3 inches × ¼ inch angle with lock angle. Line of pull on end plate in line with back of main angle.

(2) Same specimen, with lock angle removed.

(3) Specimen II.—Single, 3 inches × 3 inches × ¼ inch angle with lock angle. Line of pull on end plate in line with centroid of main section.

(4) Same specimen with lock angle removed.

(5) Same specimen with line of pull on end plate changed so as to be in line with rivets.

(6) Specimen III.—Double, 3 inches × 3 inches × 1/4 inch angle section, space of 3/8 inch between angles, with lock angles. Line of pull in line with unconnected legs of main angles.

(7) Same specimen with lock angles removed.

The stresses corresponding to the extensometer readings are given in Tables I to III, pp. 153–155, the values of E being calculated from the readings and the total load, as will be described later. The curves of stress distribution are shown in Figs. 8 to 21.

PART III.—ANALYSIS OF THE EXPERIMENTAL RESULTS.

§ 1. *The Planar Distribution of Stress.*

The theory described in Part I of this paper is based upon the assumption that the distribution of stress over a cross-section normal to the axis of load follows a linear law. That this is true for members of the type considered in this paper has already been shown by the writer.[10] It is also evident from the curves shown in Figs. 8 to 21. These represent the actual mean extensometer readings for the various cases, the mean straight line through the experimental points being drawn and continued so as to give the maximum strains, which occur at one or other of the corners of the angles. It will be noticed that in every case the mean straight lines for readings 1 to 5, on being produced to the corner of the angle, agree exactly with those for readings 6 to 10. The deviations of the experimental points from the straight lines are in nearly all cases so small as to be unimportant, especially considering that the specimens were ordinary shop products. Where deviations do occur they are usually regular and apparently denote a slight actual departure from planar distribution. They are most marked at the end sections and are probably due to warping produced by the end constraints. In one case only do they become important,—*i.e.*, at the end of the double-angle specimen (Figs. 19 and 21). For this section the mean straight lines could not be drawn and the results have not been reduced. These deviations, however, entirely vanish at the central section of the same specimen, and are thus probably due to the great end constraints in this case.

[10] *Loc. cit.*

§2. *The Method of Analysis.*

The planar distribution of stress having been established, the equations of Part I may be applied to determine the position of the load axis at each section. The details of this analysis are given in Table IV The method, being the same for all cases may be illustrated by a single example. Consider the central section of Specimen I with lock angle at 20,000 pounds load. The distribution of strain is shown in Fig. 8, and the constants for the section are given at the head of Table I. The results of the

TABLE I.

Stresses Corresponding to the Mean Extensometer Readings for Specimen I.

Constants.—Area of cross-section, 1.60 square inches.
Distance from centroid to back of angle, 0.85 inch.
$I = 1.37$ (inch)^4. $J = -0.81$ (inch)^4.
$E = 29.1 \times 10^6$ pounds per square inch.
Stresses in pounds per square inch. Tension +. Compression −.

	Load N. (lbs.)	1	2	3	4	5	6	7	8	9	10
(a) *With lock angle.*											
Central section	5,000	− 2,380	− 360	1,450	2,980	5,020	6,180	5,960	5,520	5,310	5,310
	10,000	− 3,420	− 440	3,060	6,040	9,810	12,140	11,560	10,980	10,450	10,310
	15,000	− 5,020	− 360	4,510	8,940	14,390	17,720	17,000	16,300	15,540	15,280
	20,000	− 5,960	− 70	6,250	11,910	18,810	23,280	22,400	21,440	20,580	20,280
Section 2½″ from end plate	5,000	− 2,470	− 510	1,310	2,620	4,220	6,400	6,250	6,110	6,110	6,110
	10,000	− 4,870	− 940	2,480	5,380	8,500	12,700	12,420	12,140	12,140	12,070
	15,000	− 7,060	−1,240	3,780	8,290	13,000	19,010	18,450	18,170	18,100	18,100
	20,000	− 9,090	−1,450	5,240	11,200	17,380	25,210	24,350	24,000	23,700	23,400
(b) *Without lock angle.*											
Central section	5,000	− 2,030	− 580	1,380	2,690	4,290	5,750	5,680	5,600	5,530	5,390
	10,000	− 3,640	− 730	2,760	5,390	8,440	11,490	11,490	11,330	11,200	10,830
	15,000	− 4,940	− 800	4,290	8,000	12,420	17,000	17,000	16,880	16,800	16,380
	20,000	− 6,030	− 730	5,810	10,690	16,420	22,470	22,400	22,240	22,240	21,780
Section 2½″ from end plate	5,000	− 2,690	− 870	650	2,260	4,290	5,750	5,670	5,600	5,520	5,380
	10,000	− 5,390	−1,960	1,530	4,510	8,440	12,160	12,220	12,380	12,720	12,720
	15,000	− 7,930	−2,620	2,470	6,840	12,420	18,180	18,270	18,320	18,920	18,990
	20,000	−10,100	−3,270	3,640	9,240	16,420	24,210	24,280	24,210	24,850	25,100

calculation are given in the first line of Table IV Lines through the centroid of the section parallel to the legs of the angle being taken as coördinate axes, the point of zero stress in the angle is seen to have coördinates $(1.15, -0.85)$. This is one point on the neutral axis. Another may be found by calculating where the $6-10$ line would cut the x-axis if produced. Thus the neutral axis is found to be inclined to the x-axis at an angle of $\tan^{-1} - 6.93$. This is $\tan\alpha$ in the notation of Part I. Substituting this value, the coördinates of the point of zero stress and $f = 0$ in equations (2) and (3), the coördinates (x_k, y_k) of the load axis are found to be $(-0.76, 0.37)$. All the values of (x_k, y_k) given in Table IV were calculated in a similar manner. The coördinates (x_k, y_k) give the position of the load axis relative to the cross-section considered. This depends, to some extent, upon the lateral deflection

TABLE II.

Stresses Corresponding to the Mean Extensometer Readings for Specimen II.

Constants.—Area of cross-section, 1.49 square inches.
Distance from centroid to back of angle, 0.85 inch.
$I = 1.29$ (inch)^4. $J = -0.76$ (inch)^4.
$E = 31.1 \times 10^6$ pounds per square inch.
Stresses in pounds per square inch. Tension + Compression −.

	Load N. (lbs.)	1	2	3	4	5	6	7	8	9	10
(a) *With lock angle.*											
Central section	5,000	− 2,100	− 310	1,250	3,040	4,590	6,300	6,230	5,990	5,990	5,920
	10,000	− 3,890	− 310	2,650	5,920	9,180	12,380	12,300	12,070	12,070	11,980
	15,000	− 5,130	− 150	4,200	8,950	13,610	18,350	18,290	18,120	17,980	17,810
	20,000	− 6,850	+ 540	5,760	11,980	18,130	24,200	24,150	23,900	23,810	23,500
Section 2½″ from end plate	5,000	− 2,340	− 780	780	2,490	4,050	6,150	6,150	6,300	6,300	6,780
	10,000	− 4,510	−1,400	1,870	5,370	8,480	12,450	12,370	12,510	12,770	13,210
	15,000	− 6,620	−1,870	2,960	8,250	12,830	18,750	18,500	18,750	18,990	19,600
	20,000	− 8,560	−2,100	4,440	11,280	17,280	24,900	24,650	24,750	24,970	25,680
(b) *Without lock angle.*											
Central section	5,000	− 2,250	− 700	930	2,410	4,040	6,230	6,150	6,230	6,230	6,230
	10,000	− 4,050	−1,090	2,020	5,140	8,100	12,210	12,210	12,380	12,430	12,430
	15,000	− 5,600	−1,320	3,270	7,860	12,100	18,220	18,290	18,430	18,600	18,680
	20,000	− 6,920	−1,320	4,590	10,500	15,880	23,800	24,100	24,280	24,680	24,820
Section 2½″ from end plate	5,000	− 2,410	− 860	700	2,410	3,970	6,150	6,230	6,610	6,840	7,000
	10,000	− 4,590	−1,550	1,550	5,060	8,160	12,380	12,780	13,080	13,540	14,220
	15,000	− 6,620	−2,100	2,570	7,780	12,420	18,600	18,980	19,600	20,500	21,170
	20,000	− 8,490	−2,410	3,660	10,420	16,720	24,600	25,100	25,900	26,440	27,700
(c) *Load axis on gusset plate changed to line of rivets.*											
Central section	5,000	− 2,880	− 780	1,010	2,490	4,360	6,300	6,300	6,300	6,300	6,300
	10,000	− 5,290	−1,400	2,260	5,290	8,640	12,450	12,480	12,480	12,680	12,480
	15,000	− 7,160	−1,630	3,660	7,860	12,820	18,290	18,420	18,590	18,810	18,810
	20,000	− 8,710	−1,550	5,130	10,670	16,950	23,800	24,250	24,420	24,800	24,980
Section 2½″ from end plate	5,000	− 3,190	−1,170	620	2,410	4,510	6,460	6,460	6,530	6,530	6,850
	10,000	− 6,230	−2,180	1,480	5,060	9,100	12,900	12,730	12,900	13,130	13,600
	15,000	− 8,950	−2,880	2,640	7,780	13,520	19,120	19,050	19,210	19,520	20,300
	20,000	−11,280	−3,500	3,680	10,420	18,280	25,200	25,120	25,280	25,750	26,420

of the section, so that, in order to find the position of the load axis where the load enters the angle, it is necessary to correct for this deflection. Column (4) of Table IV gives the mean deflections of the centroids of the sections under the different loads measured as described above, and column (5) gives the values of (x_k, y_k) referred to the end sections of the specimens taken as being the middle section of the riveted ends.

The value of E for each specimen may also be found from the experimental results. In the case considered above the ratio of maximum to mean stress for the calculated position of the load axis computed from equation (2) is 1.945[11] The elongation due

TABLE III.

Stresses Corresponding to the Mean Extensometer Readings for Specimen III (Double-angle).

Constants (Each angle).—Area of cross-section, 1.54 square inches.
Distance from centroid to back of angle, 0.85 inch.
$I = 1.33$ (inch).^{4} $J = -0.78$ (inch).^{4}
$E = 30.1 \times 10^6$ pounds per square inch.
Stresses in pounds per square inch. Tension +. Compression −.

	Load N. (lbs.)	1	2	3	4	5	6	7	8	9	10
(a) *With lock angle.*											
Central section, left.	15,000	5,040	5,190	5,190	5,110	5,190	4,960	4,890	4,810	4,590	4,210
	20,000	6,700	6,920	6,850	6,850	7,070	6,620	6,550	6,400	6,250	5,710
	25,000	8,440	8,660	8,580	8,500	8,660	8,360	8,130	7,980	7,830	7,220
	30,000	10,080	10,300	10,170	10,170	10,470	9,930	9,860	9,640	9,410	8,730
Central section, right.	15,000	4,890	5,040	5,190	5,270	5,490	5,190	4,820	4,510	4,060	3,610
	20,000	6,630	6,700	6,930	7,080	7,310	7,000	6,470	6,160	5,560	4890
	25,000	8,210	8,290	8,660	8,960	9,260	8,650	8,130	7,750	6,920	6,240
	30,000	9,800	9,940	10,380	10,690	10,990	10,380	9,770	9,250	8,350	7,520
Section 2½″ from end plate, left.	15,000	2,860	3,690	4,580	5,410	5,940	6,310	6,310	5,650	5,560	5,260
	20,000	3,760	5,110	6,090	7,300	7,680	8,280	7,900	7,670	7,230	7,070
	25,000	4,890	6,390	7,680	9,030	9,710	10,390	9,930	9,400	9,250	8,800
	30,000	5,940	7,600	9,250	10,900	11,580	12,490	11,970	11,370	10,900	10,600
Section 2½″ from end plate, right.	15,000	2,635	3,610	4,280	4,960	5,340	5,710	5,860	5,490	5,860	5,560
	20,000	3,760	4,890	5,860	6,690	7,300	7,150	7,450	7,220	7,370	7,300
	25,000	4,740	6,240	7,450	8,270	8,950	9,410	9,180	9,110	9,180	9,260
	30,000	5,860	7,520	8,960	10,000	10,680	11,340	11,030	10,890	11,200	11,030
(b) *Without lock angle.*											
Central section, left.	15,000	4,220	4,360	4,660	4,890	5,190	5,340	5,410	5,260	5,190	4,960
	20,000	5,650	5,850	6,320	6,470	7,000	7,070	7,150	7,070	7,000	6,760
	25,000	7,070	7,380	7,830	8,050	8,650	8,880	8,950	8,880	8,800	8,500
	30,000	8,430	8,880	9,410	9,860	10,390	10,600	10,680	10,600	10,530	10,380
Central section, right.	15,000	3,990	4,510	4,890	5,260	5,790	5,650	5,490	5,120	4,820	4,440
	20,000	5,340	6,020	6,550	6,920	8,280	7,670	7,230	6,920	6,400	6,020
	25,000	6,780	7,450	8,200	8,730	9,480	9,480	9,180	8,580	8,050	7,520
	30,000	8,130	8,950	9,790	10,430	11,410	11,420	10,990	10,220	9,860	9,180
Section 2½″ from end plate, left.	15,000	3,080	4,280	5,420	6,170	6,620	6,320	5,640	4,960	4,580	3,980
	20,000	4,290	5,720	7,150	8,280	8,730	8,420	7,680	6,770	6,090	5,340
	25,000	5,340	7,220	9,180	10,300	10,990	10,460	9,470	8,570	7,670	5,690
	30,000	6,620	8,740	10,900	12,490	13,220	12,500	11,500	10,150	9,250	7,960
Section 2½″ from end plate, right.	15,000	2,860	3,990	5,110	5,800	6,250	5,950	5,560	4,960	4,660	4,220
	20,000	3,990	5,490	6,850	7,900	8,280	7,900	7,370	6,620	6,170	5,650
	25,000	4,970	6,920	8,650	9,780	10,380	10,000	9,260	8,350	7,670	7,070
	30,000	6,100	8,280	10,440	11,890	12,490	12,030	11,200	10,090	9,180	8,500

[11]It would, of course, be incorrect to take the mean height of the plotted diagrams for a given section as being the mean extension over the section, since the curves only give the distribution of strain over the outside faces of the angles.

TABLE IV.

Reduction of Experimental Results.

Specimen and section	Load (lbs.)	1. Inclination of neutral line	2. Point of zero stress y= −0.85″	3. Load axis referred to axes through centroid of section		4. Lateral deflection of centroid of section referred to mid rivet section		5. Load axis referred to axes through mid rivet section		6. Max. stress lbs. per sq. in.	7. Ratio of maximum to mean stress
		tan α	x	x_k	y_k	x	y	x_k	y_k		
		(all measured in inches)									
Specimen I. With lock angle.											
				−	+			−	+		
Central	5,000	− 6.11	1.05	0.85	0.40	0.04	0.01	0.89	0.41	6,480	2.08
	10,000	− 5.73	1.10	0.81	0.37	0.08	0.02	0.89	0.40	12,720	2.05
	15,000	− 6.40	1.11	0.79	0.38	0.11	0.02	0.90	0.40	18,550	2.00
	20,000	− 6.93	1.15	0.76	0.37	0.15	0.03	0.91	0.40	24,080	1.94
					Mean values			0.90	0.40		
				−	+			−	+		
End	5,000	−15.00	0.95	0.92	0.50	0.03	0.00	0.95	0.50	6,550	2.04
	10,000	−20.95	0.95	0.92	0.51	0.05	0.01	0.97	0.52	12,720	1.98
	15,000	−16.26	0.97	0.90	0.49	0.07	0.01	0.97	0.50	19,420	1.98
	20,000	−14.81	0.97	0.90	0.49	0.09	0.02	0.99	0.51	25,540	2.00
					Mean values			0.97	0 51		
Specimen I. Without lock angle.											
				−	+			−	+		
Central	5,000	−16.5	0.99	0.88	0.48	0.04	0.01	0.92	0.49	5,900	1.96
	10,000	−31.6	1.05	0.82	0.47	0.08	0.01	0.90	0.48	11,640	1.86
	15,000	−28.4	1.10	0.78	0.44	0.12	0.01	0.90	0.46	18,030	1.82
	20,000	−52.9	1.11	0.77	0.45	0.15	0.02	0.92	0.47	22,620	1.79
					Mean values			0.91	0.47		
				−	+			−	+		
End	5,000	24.30	0.87	0.97	0.60	0.03	0.00	1.00	0.60	6,480	2.04
	10,000	19.15	0.87	0.97	0.60	0.05	0.00	1.02	0.61	13,100	2.05
	15,000	15.58	0.87	0.96	0.61	0.07	0.01	1.03	0.62	19,500	2.06
	20,000	14.45	0.88	0.95	0.60	0.09	0.01	1.04	0.61	25,820	2.07
					Mean values			1.02	0.61		
Specimen II. With lock angle.											
				−	+			−	+		
Central	5,000	−16.4	1.00	0.88	0.48	0.03	0.00	0.91	0.48	6,300	1.95
	10,000	−41,6	1.07	0.81	0.46	0.07	0.00	0.88	0.46	12,450	1.83
	15,000	−30.7	1.10	0.79	0.45	0.10	0.01	0.89	0.46	18,670	1.82
	20,000	−59.2	1.12	0.78	0.45	0.13	0.01	0.91	0.46	24,260	1.78
					Mean values			0.90	0.46		
				−	+			−	+		
End	5,000	10.6	0.90	0.93	0.61	0.02	0.00	0.95	0.60	6,920	2.08
	10,000	20.4	0.91	0.93	0.58	0.04	0.00	0.97	0.58	13,520	2.00
	15,000	23.0	0.95	0.90	0.55	0.06	0.01	0.96	0.56	19,820	1.96
	20,000	42.7	0.98	0.88	0.53	0.08	0.01	0.96	0.54	26,150	1.90
					Mean values			0.96	0.57		
Specimen II. Without lock angle.											
				−	+			−	+		
Central	5,000	∞	0.90	0.96	0.57	0.04	0.00	1.00	0.57	6,220	1.94
	10,000	87.60	0.93	0.93	0.55	0.08	0.00	1.01	0.55	12,370	1.93
	15,000	42,20	0.98	0.88	0.53	0.12	0.01	1.00	0.54	18,650	1.90
	20,000	23.30	1.00	0.86	0.53	0.14	0.01	1.00	0.54	25,100	1.83
					Mean values			1.00	0.55		
				−	+			−	+		
End	5,000	7.01	0.86	0.96	0.65	0.03	0.00	0.99	0.65	7,380	2,18
	10,000	8.20	0.91	0.92	0.61	0.04	0.00	0.96	0.61	14,470	2.09
	15,000	7.94	0.92	0.91	0.60	0.06	0.01	0.97	0.61	21,600	2.09
	20,000	9.40	0.96	0.88	0.57	0.08	0.01	0.96	0.58	28,000	2.03
					Mean values			0.97	0.61		

TABLE IV—*Continued.*
Reduction of Experimental Results.

Specimen and section	Load (lbs.)	1 Inclination of neutral line tan α	2 Point of zero stress y= −0.85″ x	3 Load axis referred to axes through centroid of section x_k	3 y_k	4 Lateral deflection of centroid of section referred to mid rivet section x	4 y	5 Load axis referred to axes through mid rivet section x_k	5 y_k	6 Max. stress lbs. per sq. in.	7 Ratio of maximum to mean stress
		(all measured in inches)									
Specimen II. Load axis changed to line of rivets.											
				−	+			−	+		
Central	5,000	∞	0.90	0.96	0.57	0.04	0.00	1.00	0.57	6,300	1.94
	10,000	53.30	0.95	0.91	0.55	0.09	0.00	1.00	0.54	12,820	1.92
	15,000	29.20	0.99	0.87	0.53	0.12	0.01	0.99	0.54	19,110	1.90
	20,000	27.35	1.02	0.84	0.52	0.15	0.01	0.99	0.52	25,260	1.89
						Mean values		1.00	0.54		
				−	+			−	+		
End	5,000	30.00	0.81	1.05	0.64	0.03	0.00	1.08	0.41	6,840	2.11
	10,000	58.80	0.83	1.03	0.62	0.05	0.00	1.08	0.40	13,220	2.05
	15,000	38.60	0.87	0.99	0.60	0.07	0.00	1.07	0.41	19,890	2.02
	20,000	111.30	0.90	0.96	0.57	0.10	0.00	1.06	0.40	25,740	1.96
						Mean values		1.07	0.40		
Specimen III. Double-angle with lock angles.											
				+	−						
Left angle central	15,000	− 0.15	115.0	0.02	0.05	Lateral deflections negligible				5,270	1.06
	20,000	− 0.12	153.3	0.02	0.05					6,960	1.06
	25,000	− 0.14	142.8	0.02	0.04					8,650	1.05
	30,000	− 0.15	136.6	0.02	0.04					10,340	1.05
		Mean values		0.02	0.04						
Left angle end	15,000	Non-planar distribution of stress								6,240	1.26
	20,000									8,320	1.26
	25,000									10,700	1.30
	30,000									12,720	1.29
				+	−						
Right angle central	15,000	− 0.39	17.90	0.03	0.11	Lateral deflections negligible				5,640	1.19
	20,000	− 0.36	19.15	0.03	0.11					7,520	1.19
	25,000	− 0.39	18.66	0.03	0.11					9,410	1.18
	30,000	− 0.40	18.75	0.02	0.10					11,200	1.18
		Mean values		0.03	0.11						
Right angle end	15,000	Non-planar distribution of stress								5,640	1.19
	20,000									7,520	1.19
	25,000									9,410	1.19
	30,000									11,200	1.14
Specimen III. Double-angle without lock angles.											
				−	+						
Left angle central	15,000	− 4.20	9.51	0.08	0.03	Lateral deflections negligible				5,460	1.11
	20,000	− 4.03	9.52	0.08	0.03					7,330	1.11
	25,000	− 6.61	10.05	0.08	0.04					9,030	1.10
	30,000	− 7.20	10.52	0.08	0.04					10,750	1.09
		Mean values		0.08	0.03						
Left angle end	15,000	Non-planar distribution of stress								6,810	1.39
	20,000									9,190	1.40
	25,000									11,360	1.38
	30,000									13,530	1.38
				−	−						
Right angle central	15,000	− 1.34	7.39	0.07	0.02	Lateral deflections negligible				5,950	1.22
	20,000	− 1.38	6.83	0.08	0.02					7,980	1.24
	25,000	− 1.33	7.13	0.07	0.02					9,970	1.23
	30,000	− 1.35	7.18	0.07	0.02					12,000	1.23
		Mean values		0.07	0.02						
Right angle end	15,000	Non-planar distribution of stress								6,440	1.32
	20,000									8,510	1.32
	25,000									10,710	1.33
	30,000									12,980	1.33

to the maximum stress is, from the curve in Fig. 8, 0.00331 inch. Thus the mean elongation over a length of 4 inches for a load of 20,000 pounds is $\frac{0.00331}{1.945} = 0.0017$ inch. This corresponds to a value of E given by $E = \frac{20000}{1.60} \times \frac{4}{0.0017} = 29.4 \times 10^6$ pounds per square inch. Proceeding in this manner, values of E were found for all the loadings at the central sections of the specimens, and the mean results are given at the heads of Tables I, II and III. The variations of E found in this way may be taken as a measure of the maximum over-all error in observation, measurement of the loads, calibration of the extensometers, plotting and calculation, and due to variations in the material. For Specimen I the maximum variation was 5 per cent., but this was in two cases only, the rest of the readings giving results which did not differ more than 2 per cent. The results for Specimen II were similar, while for the double angle, taking a mean of both angles, the deviation was in no case greater than 1 per cent.

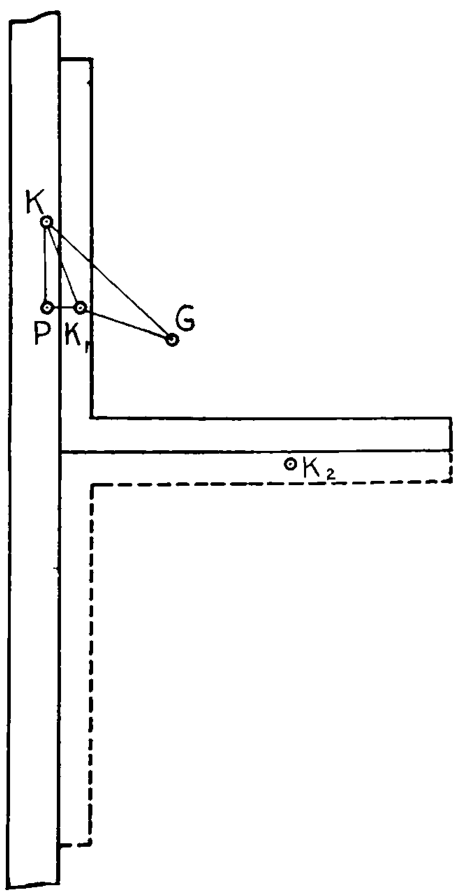
§3. *Discussion of the Results—General.*

If the load axis remained unchanged relatively to the section as the load increased, all the mean straight lines for any particular section would pass through the same points and the position of the neutral axis would be independent of the load. An inspection of the curves, Figs. 8 to 21, will show that this is not the case, and the discrepancy is due to the alteration of the load axis owing to lateral bending of the specimen. When this is allowed for by referring all load axes to the mean section through the end connections, the corrected axis remains practically constant as the load increases, as may be seen from the results given in Table IV, column 5. This is especially noticeable at the central sections. At the end sections there is in every case a slight shift, as the load increases, away from the centroid of the section perpendicular to the end plate and towards the centroid parallel to the end plate. This is probably due to a change in the fixing couple at the ends as the load increases.

§4. *The Effect of Constraints.*

In the case of single angles having long, narrow end plates the experimentally-determined load axis at the central section corrected for deflection never deviated more than 0.02 inch from the line of rivets. In the present tests y_k varies from 0.40 inch to 0.57 inch, while the y coördinate of the line of rivets is 0.90 inch. Thus the deviation from the line of rivets ranges from 0.33 inch to 0.50 inch. The reason for this is that the end plates in the present case are very much stiffer in their own plane than those used in the earlier experiments. The effect of such end constraints upon the position of the load axis and on the ratio of maximum to mean stress will first be considered in general. Let K in Fig. 22

FIG. 22.



represent the position of the load axis when there are no end constraints and K_1 its position as altered due to partial fixing. Assuming a normal load N, the bending couple in the former case would be $N.KG$ and in the latter $N.K_1G$, the restraining couple being thus $N.KK_1$. This couple may be resolved into the components $N.KP$ and $N.K_1P$ respectively parallel and perpendicular to the end plate. If the end connection consists of a gusset plate only, the former couple is the restraint due to its stiffness against bending in its own plane and the latter that due to its stiffness against bending perpendicular to its plane. It will readily be seen that the former must be much greater than the latter for an ordinary thin plate, and thus KK_1 will be nearly parallel to the connected leg. A lock angle may act in two ways. It may restrain the bending more or less and it may transfer part of the load to the line of rivets by which it is connected to the main angle (K_2). The latter would cause a movement of the line of pull along KK_2, and the former a somewhat similar motion, so that the two effects cannot be separated.

The effect of these changes of axis upon the ratio of maximum to mean stress may be studied by means of the S-polygon. Fig. 2 represents this polygon for the section of Specimen I (slightly heavier than the standard 3 inches × 3 inches × ¼ inch angle) calculated as described above. The polygon is drawn to the same linear scale as the angle, so that the maximum bending stress for a position K of the load axis is given by $N \cdot \frac{KG}{LG}$ as described in Part I. The load is applied through one row of rivets 1.75 inch from B, in the leg AB. Thus if there is no restraint due to the end plate, the position of the load axis will be approximately K. (The exact position of K perpendicular to the connected leg will depend to some extent upon the lateral bending, but may, for purposes of illustration, be taken on the outside face of the connected leg.) In this case the ratio of maximum to mean stress will be

$$A\left(\frac{1}{A}+\frac{KG}{LG}\right) = 1 + \frac{A \cdot KG}{LG} = 2.82,$$

and the maximum stress will occur at the corner A. If, however, a constraining couple be introduced parallel to AB, the point of loading will move from K along AB toward B. As the couple increases the ratio $\frac{KG}{LG}$ will obviously decrease until the point K_1 on the line joining the apex (ab) to G is reached. Thus the restraining couple will obviously decrease the ratio of maximum to mean stress. At the point K_1 this ratio is 1.89, a decrease of 33 per cent., and the stress is constant over AB. From K_1 to B the ratio $\frac{KG}{LG}$ again increases, and thus the ratio of maximum to mean stress increases. At the point K_2, level with the centroid of the angle, this ratio is 3.18, greater than without constraint. Thus the most favorable position of the load axis is on the line joining the apex (ab) to the centroid. From this it will be seen that an increase of constraint is not always an advantage.

With the load axis in any of the positions considered it will be noticed that a constraint perpendicular to the end plate, causing K to move inwards, always leads to a decrease of $\frac{KG}{LG}$, *i.e.*, a decrease in the ratio of maximum to mean stress. The only effective way of obtaining such restraint is by having another angle back to back with the first, *i.e.*, a double-angle section. In this case, as will be seen later, there is a very considerable restraining couple parallel to BC, causing the stress to be much more evenly distributed over the section. This is the correct way of considering such a section, and not as one piece bending about a neutral axis parallel to BC, as is often done. These points will be discussed when the experimental results upon the double angle are considered.

The effects of constraints upon the distribution of stress in other types of section, such as unequal-legged angles, Z-sections, etc., may be considered in a similar manner by use of the S-polygon.

§5. *Experimental Results.* (a) *The Restraining Effect of the End Plates.*

The results of the tests on the specimens with the lock angles removed will first be considered in order to find the restraining effects due solely to the end plate. The restraining couple perpendicular to the end plate,—*i.e.*, in the direction Ox,—is difficult to ascertain exactly, because it is neither certain where the load enters the angle nor exactly what is the position of the load axis through the thickness of the end plate. In Table IV, column 5, the position of x_k relative to the mid-section of the rivets is given, and it may be remarked that it remains practically constant as the load increases in each of the Specimens I and II, although it is not the same for each, the mean being $x_k = -0.91$ for Specimen I and $x_k = -1.00$ for Specimen II. In any case, however, the effect of this restraint is small and the specimen may be considered as practically free to bend perpendicularly to the end plate. This result is in agreement with the earlier experiments made with narrow end plates. In these it was found that $x_k = -0.91$ both with an end plate ¼ inch thick and with one ¾ inch thick.

The case of the double angle is quite different. The two angles, back to back, prevent each other from bending perpendicularly to the end plate, and the result is an almost complete fixing in that direction, considerably decreasing the ratio of maximum to mean stress over the section. Again, the effect is practically independent of the load and is, as might be expected, the same for each angle. The arms of the fixing couples parallel to Ox, measured from the back of the main angles, are given in the annexed table.

TABLE V.

Load	Left angle	Right angle
15,000.........	0.77	0.77
20,000.........	0.77	0.76
25,000.........	0.77	0.77
30,000....	0.77	0.76
Mean..........	0.77	0.77

The fixing parallel to the end plate is much more important for the single angles and of equal importance for the double angle. The annexed table shows the distance, measured parallel to Oy, of the actual axis from the line of rivets for each of the specimens at the central section:

TABLE VI.

Load, pounds	Specimen I, inch	Specimen II, inch	Double angle		
			Load, pounds	Left, inch	Right, inch
5,000...	0.41	0.33	15,000...	0.87	0.88
10,000...	0.42	0.35	20,000...	0.87	0.88
15,000...	0.44	0.36	25,000...	0.86	0.88
20,000...	0.43	0.36	30,000...	0.86	0.88
Mean...	0.42	0.35	Mean...	0.86	0.88

It will be noticed that there is fair agreement between Specimen I and Specimen II, although the line of pull on the end plate is different in the two cases. The fixing moment of the double angle, however, is more than twice that of the single angle.

In the earlier experiments on single angles connected to long, narrow gusset plates the line of pull, as mentioned above, never deviated more than 0.02 inch from the line of rivets. This shows the important effect of the type of gusset plate upon the position of the load axis and hence on the maximum stress in the member. The latter, of course, depends upon the lateral bending of the specimen, and in calculating the figures in column 7, Table IV, the axis of loading was not corrected for deflection. It will be noticed that, with a few exceptions, the ratio at the central sections decreases as the load increases. This is due to the alteration of the axis due to lateral bending. In every case the ratio is much less than for a load coinciding with the line of rivets, showing the value of a wide and firmly-connected gusset plate. The ratios of maximum to mean stress for a line of pull having the same x_k as that given by the experimental results, but y_k in line with the rivets, have been found by means of the S-polygon, as described above, and are compared with the actual ratios in the annexed table.

TABLE VII.

Load, pounds	Specimen I			Specimen II		
	Actual	Line of pull in line of rivets	Decrease due to fixing	Actual	Line of pull in line of rivets	Decrease due to fixing
			Per cent.			*Per cent.*
5,000.....	1.96	2.62	25.2	1.94	2.80	30.7
10,000.....	1.86	2.66	30.0	1.93	2.83	31.8
15,000.....	1.82	2.69	32.4	1.90	2.86	33.6
20,000.....	1.79	2.70	33.7	1.83	2.88	36.5

The average decrease of the ratio of maximum to mean stress due to the stiffness of the end plate in its own plane is thus about 35 per cent. at the highest load, the load axis (K^1, Fig. 2) being brought almost into its most favorable position as described in the last section.

At the end sections the ratio of maximum to mean stress is always greater than at the centre, the percentage increase being as shown in the annexed table.

TABLE VIII.

The Ratio of Maximum to Mean Stress at the End Section Compared with that at the Central Section.

Load, pounds	Specimen I			Specimen II		
	End	Central	Per cent. increase	End	Central	Per cent. increase
5,000.....	2.04	1.96	4.1	2.18	1.94	12.4
10,000.....	2.05	1.86	10.2	2.09	1.93	8.3
15,000.....	2.06	1.82	13.2	2.09	1.90	10.0
20,000.....	2.07	1.79	15.6	2.03	1.83	10.9
	Double angle					
	Left			Right		
	End	Central	Per cent. increase	End	Central	Per cent. increase
15,000.....	1.39	1.11	25.2	1.32	1.22	8.2
20,000.....	1.40	1.11	26.1	1.32	1.24	6.5
25,000.....	1.38	1.10	25.5	1.33	1.23	8.1
30,000.....	1.38	1.09	26.6	1.33	1.23	8.1

It will be noticed that the percentage increase varies widely, as might be expected. Tertiary stresses are indicated, ranging from 4 per cent. to 15 per cent. for the single angles and rising as high as 26 per cent. for the double angles. The curves for the latter show that the distribution of stress departed greatly from the planar. There is probably a readjustment of the longitudinal variations of stress at the different loads, depending upon the end connections, so that the results are not so reliable as at the central section and the exact tertiary stresses are thus indeterminate. They are local, however, and thus not so important, and may probably be neglected in designing if ordinary working stresses are used and the correct variation of stress over the main part of the member considered.

§5. (b) *The Effect of Lock Angles.*

It is often claimed, as stated in the introduction, that lock angles considerably reduce the ratio of maximum to mean stress in the member, some authorities going even so far as to say that when a lock angle is used the stress is practically uniformly distributed. A glance at the adjoined table will show how far this is from being the truth.

TABLE IX.

The Effect of Lock Angles on the Ratio of Maximum to Mean Stress at the Central Cross-sections.

Load, pounds	Specimen I: Ratio of maximum to mean stress, With lock	Specimen I: Ratio of maximum to mean stress, Without lock	Per cent. change due to lock	Specimen II: Ratio of maximum to mean stress, With lock	Specimen II: Ratio of maximum to mean stress, Without lock	Per cent. change due to lock
			Increase			*Decrease*
5,000....	2.08	1.96	6.1	1.95	1.95	0
10,000.....	2.05	1.86	10.2	1.83	1.93	5.2
15,000.....	2.00	1.82	9.9	1.82	1.90	4.2
20,000.....	1.94	1.79	8.4	1.78	1.83	2.7
	Double Angle, left.			Double angle, right.		
			Decrease			*Decrease*
15,000.....	1.06	1.11	4.5	1.19	1.22	2.5
20,000.....	1.06	1.11	4.5	1.19	1.24	4.0
25,000.....	1.05	1.10	4.5	1.18	1.23	3.2
30,000.....	1.05	1.10	4.5	1.18	1.23	3.2

This table compares the ratio of maximum to mean stress with the lock angles in position and with them removed. Considering the single angles first, it will be seen that in the case of Specimen II the ratio of maximum to mean stress is decreased about 3 per cent. at working load by the use of the lock angle, while in Specimen I it is actually greater by about 8.4 per cent. with the lock angle than without it. In the case of the double angle the stress in one leg is decreased about 4.5 per cent. and in the other 3.2 per cent. It is thus evident that the effect of the lock angle is very small, and that it is practically worthless for the purpose of distributing the stress more uniformly.

The variability and apparently paradoxical character of these results will be explained by a consideration of the shift of the load axis due to the lock angle as displayed by the adjoined table.

It will be seen that in all cases, as might be expected, the lock angle slightly increases the arms of the restraining couples both perpendicular to and parallel to the end plate. The change of y_k ranges from 0.07 inch to 0.09 inch, while that of x_k is 0.10 inch in all cases except Specimen I, in which it is only 0.01 inch. The agreement between the results in the different cases is remarkable, especially considering that no care was taken to get uniform work-

TABLE X.

Change of Load Axis at Central Section Due to Lock Angle.

Load, pounds	x_k With lock, inches	x_k Without lock, inches	Shift toward centroid produced by lock, inches	y_k With lock, inches	y_k Without lock inches	Shift toward centroid produced by lock, inches
	Specimen I					
	−	−		+	+	
5,000.....	0.89	0.92	0.03	0.41	0.49	0.08
10,000.....	0.89	0.90	0.01	0.40	0.48	0. 8
15,000.....	0.90	0.90	0.00	0.40	0.46	0.06
20,000.....	0.91	0.92	0.01	0.40	0.47	0.07
	Mean displacement...		0.01	Mean displacement.		0.07
	Specimen II					
	−	−		+	+	
5,000.....	0.91	1.00	0.09	0.48	0.57	0.09
10,000.....	0.88	1.01	0.13	0.46	0.55	0.09
15,000.....	0.89	1.00	0.11	0.46	0.54	0.08
20,000.....	0.91	1.00	0.09	0.46	0.54	0.08
	Mean displacement.		0.10	Mean displacement.		0.08
	Double angle	*Left*				
	+	−		−	+	
15,000.....	0.02	0.08	0.10	0.05	0.03	0.08
20,000.....	0.02	0.08	0.10	0.05	0.03	0.08
25,000.....	0.02	0.08	0.10	0.04	0.04	0.08
30,000.....	0.02	0.08	0.10	0.04	0.04	0.08
	Mean displacement...		0.10	Mean displacement.		0.08
	Double angle	*Right*				
	+	−		−	−	
15,000.....	0.03	0.07	0.10	0.11	0.02	0.09
20,000.....	0.03	0.08	0.11	0.11	0.02	0.09
25,000.....	0.03	0.07	0.10	0.11	0.02	0.09
30,000.....	0.02	0.07	0.09	0.10	0.02	0.08
	Mean displacement...		0.10	Mean displacement.		0.09

manship in the making of the specimens. It may be said that the lock angle produces an increase of, roughly, $^1/_{10}$ inch in the arms of each of the restraining couples. Now the effect of this depends upon the position of the load axis due to the stiffness of the end plate. In Specimen I the displacement of the axis brings the *GL* line of the S-polygon from *GL′* to *GL″* (Fig. 2). *L* thus passes over from the *a* line to the *b* line, and the maximum stress is actually increased by the additional restraint. In the other specimens the point L remains on the a line, and thus the restraint slightly diminishes the maximum stress.

The effect of the lock angle is no more marked at the end sections than at the central section, the change in the restraining arm due to its action varying from 0.02 inch to 0.10 inch.

TABLE XI.

The Effect of Lock Angles on the Ratio of Maximum to Mean Stress. End Sections.

Load, pounds	Specimen I Ratio of maximum to mean stress			Specimen II Ratio of maximum to mean stress		
	With lock	Without lock	Decrease due to lock	With lock	Without lock	Decrease due to lock
			Per cent.			*Per cent.*
5,000.....	2.04	2.04	0.0	2.08	2.18	4.6
10,000.....	1.98	2.05	3.4	2.00	2.09	4.3
15,000.....	1.98	2.06	3.9	1.96	2.09	6.2
20,000....	2.00	2.07	3.4	1.90	2.03	6.4
	Double angle					
	Left			Right		
15,000.....	1.26	1.39	9.3	1.19	1.32	9.9
20,000.....	1.26	1.40	10.0	1.19	1.32	9.9
25,000.....	1.30	1.38	5.8	1.19	1.33	10.5
30,000.....	1.29	1.38	6.5	1.14	1.33	14.3

TABLE XII.

Change of Load Axis Due to Lock Angles. End Sections.

Load, pounds	x_k (inches)			y_k (inches)		
	With lock	Without lock	Change due to lock	With lock	Without lock	change due to lock
Specimen I						
5,000...	0.95	1.00	0.05	0.51	0.61	0.10
10,000...	0.97	1.02	0.05	0.50	0.62	0.12
15,000..	0.97	1.03	0.06	0.50	0.61	0.09
20,000..	0.97	1.04	0.05	0.51	0.60	0.09
			0.05			0.10
Specimen II						
5,000..	0.95	0.99	0.04	0.60	0.65	0.05
10,000..	0.91	0.96	0.05	0.58	0.61	0.03
15,000...	0.96	0.97	0.01	0.56	0.61	0.03
20,000...	0.96	0.96	0.00	0.54	0.58	0.04
			0.02			0.04

It produces a slight decrease in the maximum stress, 3 per cent. to 7 per cent., for the single angles and a little greater, 6 per cent. to 10 per cent. and in one case 14 per cent., in the double angles. The results are more variable than at the centre, owing, no doubt, to the difference in the fixing couples near the rivets in the different specimens due to the different positions of the line of loading of the gusset plate.

To sum up, it may be said that neither in the single nor double angles does the lock angle cause a displacement of more than $^1/_{10}$ inch in the line of pull either parallel or perpendicular to the end plate, and that this produces only small changes in the maximum stress, insignificant in comparison with those arising from the stiffness of the gusset plate in its own plane. The effect of the lock angle is thus so small as to be practically negligible. It is also uncertain, since, depending upon the stiffness of the gusset plate, which is difficult to predict, it may increase or diminish the maximum stress. The contention that the lock angle increases the virtual length of attachment of the main angle to the gusset plate, allowing more rivets to be used, is also seen to be incorrect, since practically none of the stress is transmitted into the angle through the rivets in the lock angle, at any rate until the member is near to the breaking load, when it is of no importance. The tests of McKibben [12] have shown that the effect of lock angles upon the breaking load is uncertain and not great in any case.

It thus appears that lock angles are of very little practical value and are unnecessary, expensive and cumbersome additions to the end connections.

§5. (c) *The Effect of a Change in the Line of Loading of the Gusset Plate.*

In order to obtain some idea of the relative importance of this factor, the load axis on the end plate of Specimen II was altered by reboring the pin-hole and adding reinforcing plates, the change being from a line 0.85 inch from the corner of the angle,—*i.e.*, in line with the centroid of the section,—to one in line with the rivets, 1.75 inches from the corner. The effect of this upon the load axis in the specimen may be seen from the annexed table. There was practically no change at the central sec-

[12] *Proc. Am. Soc. Test. Mat.*, vol. 6, p. 267; vol. 8, p. 287.

tion, but at the end the arm of the fixing couple was decreased 0.10 inch perpendicular to the end plate and increased 0.21 inch parallel to the end plate. Thus only the tertiary stresses were affected by the change. A comparison of the load axes in Specimens I and II (Table XIV), which were practically similar (although one was a little heavier than the other), except that the load axis in Specimen I was in line with the back of the angle while in Specimen II it was in line with the centroid, as explained above, reveals a change of 0.08 inch in the arms of the restrain-

TABLE XIII.

Specimen II.—Effect of a Change in the Load Axis.

Load, pounds	x_k (inches)			y_k (inches)		
	Axis in line with centroid	Axis in line of rivets	Change	Axis in line with centroid	Axis in line of rivets	Change
Central Section.						
5,000..	1.00	1.00	0.00	0.57	0.57	0.00
10,000..	1.01	1.00	0.01	0.55	0.55	0.00
15,000...	1.00	0.99	0.01	0.54	0.54	0.00
20,000...	1.00	0.99	0.01	0.54	0.52	0.02
		Mean......	0.01		Mean...	0.00
End Section.						
5,000...	0.99	1.08	0.09	0.65	0.40	0.25
10,000...	0.96	1.08	0.12	0.61	0.41	0.20
15,000...	0.97	1.07	0.10	0.61	0.40	0.21
20,000...	0.96	1.06	0.10	0.58	0.41	0.17
		Mean.....	0.10		Mean....	0.21

ing couples at the central section and a smaller change at the end. In this case the differences may, of course, have been partly due to other causes than the difference in position of the axis, but in both cases they were so small as to make it appear that the exact position of the load axis on the end plate is not of very much importance, the distribution of stress being practically fixed by the line of rivets and the stiffness of the gusset plate. The effect on the tertiary stresses may be greater, but does not seem worthy of further investigation in view of its smallness in comparison with that due to the stiffness of the end plate in its own plane.

TABLE XIV.

Comparison of the Load Axes in Specimens I and II.

Load, pounds	x_k (inches)			y_k (inches)		
	Specimen I	Specimen II	Difference	Specimen I	Specimen II	Difference
Central Section.						
5,000...	0.92	1.00	0.08	0.49	0.57	0.08
10,000...	0.90	1.01	0.11	0.48	0.55	0.07
15,000...	0.90	1.00	0.10	0.46	0.54	0.08
20,000...	0.92	1.00	0.08	0.47	0.54	0.07
		Mean.....	0.09		Mean.....	0.08
End Section.						
5,000...	1.00	0.99	0.01	0.60	0.65	0.05
10,000...	1.02	0.96	0.06	0.61	0.61	0.00
15,000...	1.03	0.97	0.06	0.62	0.61	−0.01
20,000...	1.04	0.96	0.08	0.61	0.58	−0.03
		Mean.....	0.05		Mean....	0.00

§6. *Further Remarks on the Double Angle.*

It is the usual practice in designing a member, such as Specimen III, made up of two angles back to back, to consider that bending occurs about a neutral axis parallel to the unconnected legs, the whole member bending as one piece. It has already been shown by the writer [13] that this is incorrect and that the only true way in which to regard this and other similar built-up sections is to consider that each element of the section tends to bend about an individual neutral axis, being restrained more or less by the end or other connections. It is only necessary to remark here that the present experiments bear out this theory and also show that a stiff end plate produces such large constraining moments that almost complete fixing results, the greatest deviation of the load axes of the two angles from their centroids in Specimen III being about 0.1 inch. How far this fixing may be relied upon in any particular case it is difficult to say, and the subject is worthy of further investigation.

§7. *General Conclusions.*

The above analysis leads to the conclusion that, with a gusset plate connection of the usual type, wide and rigidly connected,

[13] *Loc cit.*

the chief factor influencing the distribution of stress over the cross-section of the member is the stiffness of the end plate in its own plane. In comparison with this the effect of lock angles is practically negligible, and a considerable change in the line of pull on the gusset plate may be made without appreciably altering the stress distribution.

The bearing of the above upon the design of single- and double-angle members is obvious, although it is difficult to formulate exact rules. It may, however, be definitely stated that lock angles are of little, if any, value, and this is perhaps the most important result of the investigation. As to the correct ratio of maximum to mean stress to be used in designing, the earlier experiments of the writer [14] have shown that with long, narrow gusset plates, unconnected at the sides, there is practically no end restraint, no matter what the thickness, within practical limits, of the plate may be, and that a fairly good approximation to the actual distribution of stress may be obtained by considering the load axis as coinciding with the line of rivets and lying slightly within the end plate. A broad, stiff connection, however, is generally advisable, and with this the ratio of maximum to mean stress is much, say 30 per cent., lower than would be given by the above rule. It is difficult to estimate exactly, however, and perhaps the same rule might be used taking a higher value for the working stress, since the tertiary stresses would also be covered by it. With a narrow plate, however, the ordinary working stress should be used, since the tertiary stresses are not included. There remains, of course, the possibility of exceeding the yield point at certain fibres without danger. This, however, it would not be safe to rely upon in the present state of experimental knowledge of the subject.

The experiments on double-angle members show that such a member with a stiff end plate is an excellent type in practice, the stress being almost uniformly distributed at the central section, but, unfortunately, it seems to be impossible to predict exactly what the distribution will be in any particular case.

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[14] In the earlier paper (*loc. cit.*) it was stated that the end plate had little constraining effect. This should be modified, in the light of the present experiments, to the statement given above.

§8. *Summary and Conclusion.*

The chief conclusions to which the present paper leads are:

1. That the only practicable experimental method at present available for investigating the distribution of stress in built-up members is by means of some form of extensometer, and that the simplified mirror extensometer used in the tests described is very suitable for this purpose.

2. That the assumption of a planar distribution of stress is justifiable in such members as are considered here, except perhaps close to the end connections, and that the ordinary theory may therefore be applied to an analysis of the distribution of stress in these members.

3. That in single- and double-angle tension members connected at their ends by means of rivets to wide and rigidly held gusset plates the stiffness of the gusset plate in its own plane has a considerable effect on the distribution of stress in the member, there being in every case a particular stiffness which will give the least maximum stress in the member for a given load.

4. That in such members lock angles are of very little, if any, value for the purpose either of giving a more equable distribution of stress in the member or of increasing the effective length of end connections.

5. That a slight change in the line of application of the load to the gusset plates does not materially affect the distribution of stress in the member, except possibly close to the end connections.

6. That the experiments on double angles bear out the theory that such members do not act as a single piece bending as a beam.

7. That the experiments lead to certain rules for design as formulated in Part III, §7.

In conclusion the writer wishes to thank Professors H. M. Mackay and E. Brown, of McGill University, for their personal interest and advice; and Mr. S. D. Macnab, of the McGill Testing Laboratory, who was associated with him throughout in the experimental work.

McGILL UNIVERSITY, Montreal,
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THE CALCULATION OF TORSION STRESSES IN FRAMED STRUCTURES AND THIN-WALLED PRISMS

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TORSION STRESSES IN FRAMED STRUCTURES.

*The Calculation of Torsion Stresses in Framed Structures and Thin-Walled Prisms.**

By CYRIL BATHO, M.Sc., B.Eng., Assistant Professor of Applied Mechanics, McGill University, Montreal.

§ 1. IN designing a double-track cantilever bridge with suspended span, it is necessary to consider the stresses arising in the suspended span due to unsymmetrical live loads on the cantilever and anchor-arms (see Example III.). It is also sometimes of importance to determine the stresses in an ordinary truss bridge, braced arch or other framed structure on four supports, due to unequal settlement of the supports. Similar problems arise in connection with erection travellers, etc. The stresses arising under such conditions may be termed "torsion stresses," since they correspond to those called into play by two equal and opposite couples in parallel planes acting at panel points of the structure. Methods for their calculation in the case of settlement of supports have been given by various authorities.† These methods are, however, long and tedious, and the work may be considerably shortened by the use of the following theorem, which is also of interest from a theoretical point of view. By its use the stresses in the lateral system, usually the most important, may be calculated in a few minutes, whilst the stresses in the main trusses may be determined by means of an ordinary reciprocal diagram.

§ 2. *Theorem.—If a framed structure consisting of two parallel trusses—$A\ B\ C$, $A^1\ B^1\ C^1$ (Fig. 1), similar in outline and connected by lateral bracing, be subjected to equal, opposite, and parallel couples consisting of unit forces at A, A^1, C and C^1 respectively, the shear S perpendicular to the plane of the trusses is constant throughout the lateral system and equal to the area of the base of the framework divided by twice the area of one of the trusses—i.e.*, using the notation of Fig. 1,

$$S = \frac{b\ l}{2 \times \text{area } A\ B\ C}$$

* Paper read before Section G of the British Association at Manchester, September, 1915.
† See Johnson, Bryan, and Turneaure, "Modern Framed Structures," vol. ii., page 375.

Consider the equilibrium of any panel-point m. Since there is no external force acting at this point, the force perpendicular to the plane of the truss A B C in the member $(m-1)^1$, m must be equal and opposite to that in the member $m\ m^1$. Similarly, considering the panel-point m^1 of the truss $A^1\ B^1\ C^1$, the force in $m\ m^1$ must be equal and opposite to the perpendicular component of the force in $m^1\ (m+1)$, etc. Thus the shear perpendicular to the plane of the trusses in the panel $(m-1)^1$, m of the lateral system is the same as that in the panel m^1, $(m+1)$, etc. In a similar manner it may be proved that for any type of lateral this shear is constant throughout the system. Let it be denoted by S.

The whole structure may now be considered as consisting of three parts joined together at the panel-points only, the trusses A B C, $A^1\ B^1\ C^1$ subjected to the unit forces at A, C, A^1, and C^1 respectively, and the lateral system transmitting the shear S. The chords of the main trusses must be regarded as belonging to both the truss and lateral systems. Each panel of the lateral system will transmit a force F to a panel-point of each of the main trusses, and these will form a system of forces parallel to the chord members of the trusses, as shown by the heavy arrows in Fig. 2. Thus at any panel-point m of the truss A B C a force F_m is acting from the panel m, $m+1$ of the lateral system. Considering the equilibrium of this panel, it will be seen that

$$F_m = S\frac{a}{b} \qquad (1)$$

where a is the length of the panel, and b the distance between the main trusses (Fig. 2). It is, of course, immaterial whether the force transmitted by the panel $m\ (m+1)$ of the laterals be considered as acting at m or $(m+1)$ of the main truss. In the former case the F forces would act as shown, in the latter there would be no horizontal force at C and no inclined force at A.

For equilibrium of the truss A B C the sum of moments of the forces shown in Fig. 2 about an axis through A perpendicular to the plane of the truss must be zero. Thus

$$1 \times l - \Sigma\,(F_m \cos\theta \;\; y_m - F_m \sin\theta \;\; x_m) = 0,$$

the summation extending over all the panel-points. Therefore, substituting values of sin θ and cos θ and using (1)—

$$l=\mathrm{S}\ \Sigma\left[\frac{a}{b}\left(\frac{x_{m+1}-x_m}{a}\right)y_m-\frac{a}{b}\left(\frac{y_{m+1}-y_m}{a}\right)x_m\right]$$

or

$$b\ l=\mathrm{S}\ \Sigma\ (x_{m+1}\cdot y_m-y_{m+1}\cdot x_m) \qquad (2)$$

Now twice the area A B C,

$$2\ \mathrm{P}=\Sigma\left[2\ y_m\ (x_{m+1}-x_m)+(y_{m+1}-y_m)\ (x_{m+1}-x_m)\right]$$
$$=\Sigma\ (x_{m+1}\ y_m-x_m\ y_{m+1})-\Sigma\ x_m\ y_m+\Sigma\ x_{m+1}\ y_{m+1}.$$

But

$$\Sigma\ x_m\ y_m=\Sigma\ x_{m+1}\ y_{m-1},$$

and therefore

$$2\ \mathrm{P}=\Sigma\ (x_{m+1}\ y_m-x_m\ y_{m+1}) \qquad (3)$$

Hence, substituting in (3) from (2),

$$\mathrm{S}=\frac{b\ l}{2\mathrm{P}} \qquad (4)$$

and the theorem is proved.

The stresses in the members of the lateral system may be found at once by considering the shear S as acting on each panel in turn. In order to find those in the main truss members, the stresses due to the F forces and the unit forces at A and C must be determined analytically or graphically. A short graphical method obviating the calculation of the F forces will be illustrated later (see Example III.). These stresses will be the correct ones for the web members, but to the chord members must be added the stresses arising from the lateral system, as in the calculation of wind stresses. These will be equal to or one half of the F forces, depending upon the type of lateral system.

§ 3. *Extensions of the Theorem:*

(*a*) *If the base of the structure be not plane*, the theorem still holds in the form given in equation (4), but $b\ l$ is not the area of the base. Thus equation (4) may be applied to braced arches, erection travellers, etc.; in fact, to any braced structure having similar parallel faces and subjected to any pair or pairs of equal, opposite, parallel couples in the planes of the two faces respectively.

(b) The theorem may be further extended to include any *thin-walled cylindrical or prismatic surface* having plane ends perpendicular to the surface.

Let Fig. 3 represent such a surface. b in this case represents the length of the cylinder, and the forces at A and C may be re-

Fig. 1.

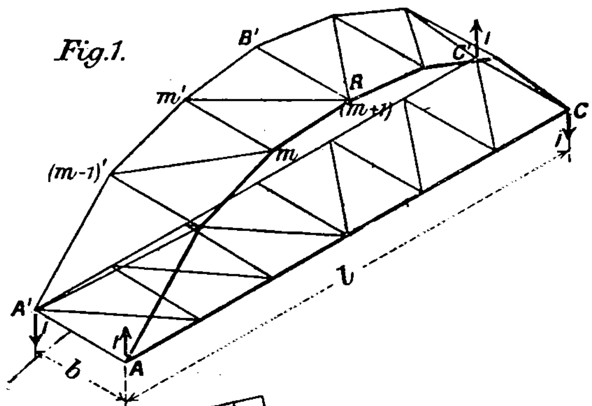


Fig. 3.

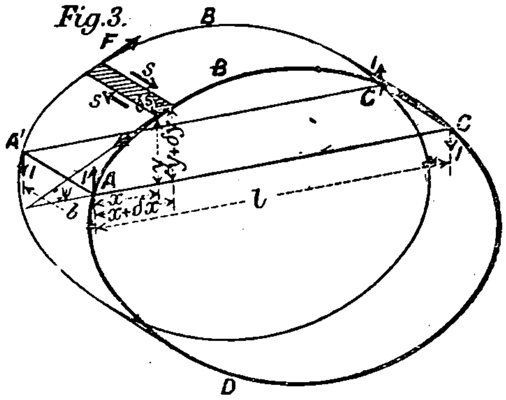
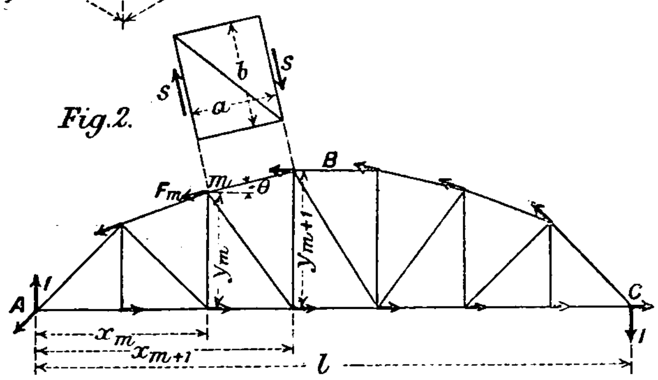


Fig. 2.

$F_{m,x}\,a = S\,s$



(4495 A.)

Fig. 4.

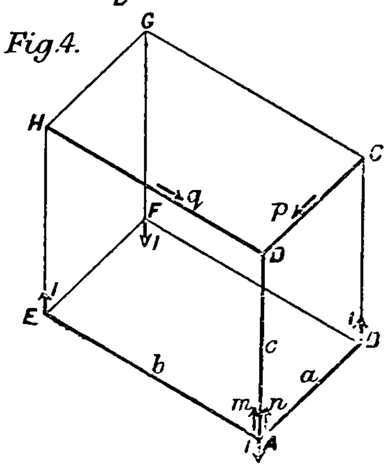
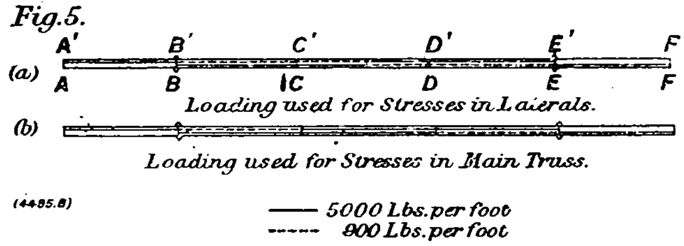


Fig. 5.



(4495. B)

garded as making up one couple, and those at A^1 and C^1 the other. Then, using the notation of Fig. 3,

$$F = \frac{S\ .\ \delta s}{b}$$

Taking moments about an axis through A parallel to the surface of the cylinder for equilibrium of the end A B C D,

$$1 \times l - \Sigma\ (\mathrm{F} \cos \psi \cdot y - \mathrm{F} \sin \psi \cdot x) = 0,$$

and therefore

$$b\, l = \mathrm{S}\ \Sigma\ (y\, \delta\, x - x\, \delta\, y),$$

the summation extending round the boundary A B C D. Thus:—

$$\mathrm{S} = \frac{b\, l}{2 \times \text{area A B C D}}$$

as before.

(*c*) In its most general form the theorem may be stated thus:—*If a hollow cylinder or prism, either continuous-walled or of framework, and having plane ends perpendicular to its length, be subjected to a twisting moment by couples in the planes of its ends, the total longitudinal shear is everywhere constant and equal to the twisting moment multiplied by the length of the cylinder, and divided by twice the area of one of its ends.*

§ 4. *Examples:*

(i.) *Bridge having Parallel Chords and Panels of Equal Length.*—Let there be n panels each of length d, and let the height of the trusses be h and their distance apart b. Full diagonal bracing is assumed. Then, if the end posts be vertical, as in a deck bridge,

$$\mathrm{S} = \frac{\text{area of base}}{2 \times \text{area of one truss}} = \frac{b\, n\, d}{2\, h\, n\, d} = \frac{b}{2h}$$

If the end posts be inclined, as in a through bridge,

$$\mathrm{S} = \frac{b\, n\, d}{2\, h\, \{ (n-2)\, d + d \}} = \frac{n}{n-1} \cdot \frac{b}{n} \cdot$$

These results are in agreement with those given by Johnson, Bryan, and Turneaure, "Modern Framed Structures," vol. ii., page 375.

(ii.) *Thin-Walled Rectangular Box.*—An interesting verification of the application of the theorem to a thin-walled prism is the case of a rectangular box, such as A B C D E F G H (Fig. 4), to which forces are applied as shown in the figure.

Considering A H and F C as the ends of the prism, the shear over the faces E B, A C, D G, and G E parallel to A B is given by:

$$S_1 = \frac{a\,b}{2\,b\,c} = \frac{a}{2\,c} \qquad (1)$$

Considering A C and E G as the ends, the shear over E D, D G, G B and B E parallel to A E is given by:

$$S_2 = \frac{a\,b}{2\,a\,c} = \frac{b}{2\,c} \qquad (2)$$

It will be noticed that $\frac{S_1}{S_2} = \frac{a}{b}$, which must be the case for equilibrium of the face H C.

Suppose the walls are of uniform thickness t throughout, then the intensity of shear over the faces A C and E G $= \frac{S_1}{a\,t} = \frac{1}{2ct}$, over D E and F C $= \frac{S_2}{b\,t} = \frac{1}{2c\,t}$ and over the faces E B and H C $= \frac{S_1}{a\,t} = \frac{1}{2\,c\,t}$. Thus the shear stress is constant throughout and of intensity $\frac{1}{2\,c\,t}$.

In this case the results may be arrived at quite simply from first principles as follows:—Of the force at A, let m be resisted by the face A H, and n by the face A C, as indicated in the figure. Then for equilibrium of the faces A H, A C, and B E, we have

$$m\;b = q\;c,$$
$$n\;a = p\;c,$$
$$p\;b = q\;a,$$

respectively. Thus

$$\frac{m}{n} = \frac{q}{p} \cdot \frac{a}{b} = 1,$$

or half of the force at A is taken by the face A H and half by the face A C, irrespective of the lengths of the sides of the figure. Also the intensity of shear over the face A C $= \frac{n}{c\,t} = \frac{1}{2c\,t}$, over A H $= \frac{m}{c\,t} = \frac{1}{2\,c\,t}$ and over B E $= \frac{q}{b\,t} = \frac{m\,b}{c\,b\,t} = \frac{1}{2\,c\,t}$, which correspond to the results obtained by the former method.

(iii.) *Torsion Stresses in the Suspended Span of a Cantilever Bridge Due to Unsymmetrical Live Loads.*—Fig. 5*a* is a diagrammatic plan of a double-track cantilever bridge having piers at B B^1 and E E^1. Thus A B and E F are the anchor-arms, B C and D E the cantilever arms, and C D is the suspended span. Let it be supposed that the position of the live load is as shown by the heavy lines. It will be seen at once that the suspended span is subjected to torsion stresses due to the unequal deflections at C and C^1 and at D and D^1 produced by the live load. The determination of these will be illustrated by an example taken from practice. Fig. 6 is an elevation of the suspended span of the new Quebec Bridge,

Fig. 6.

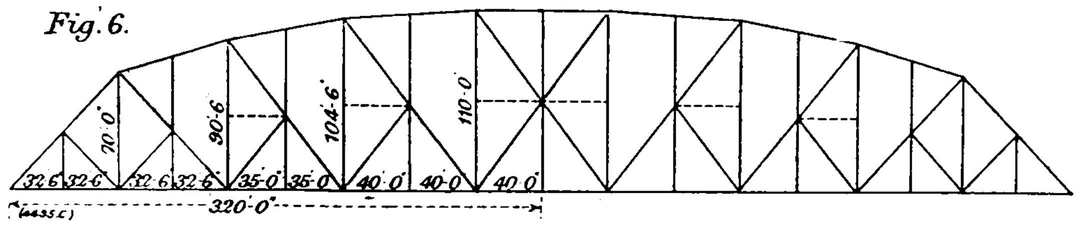
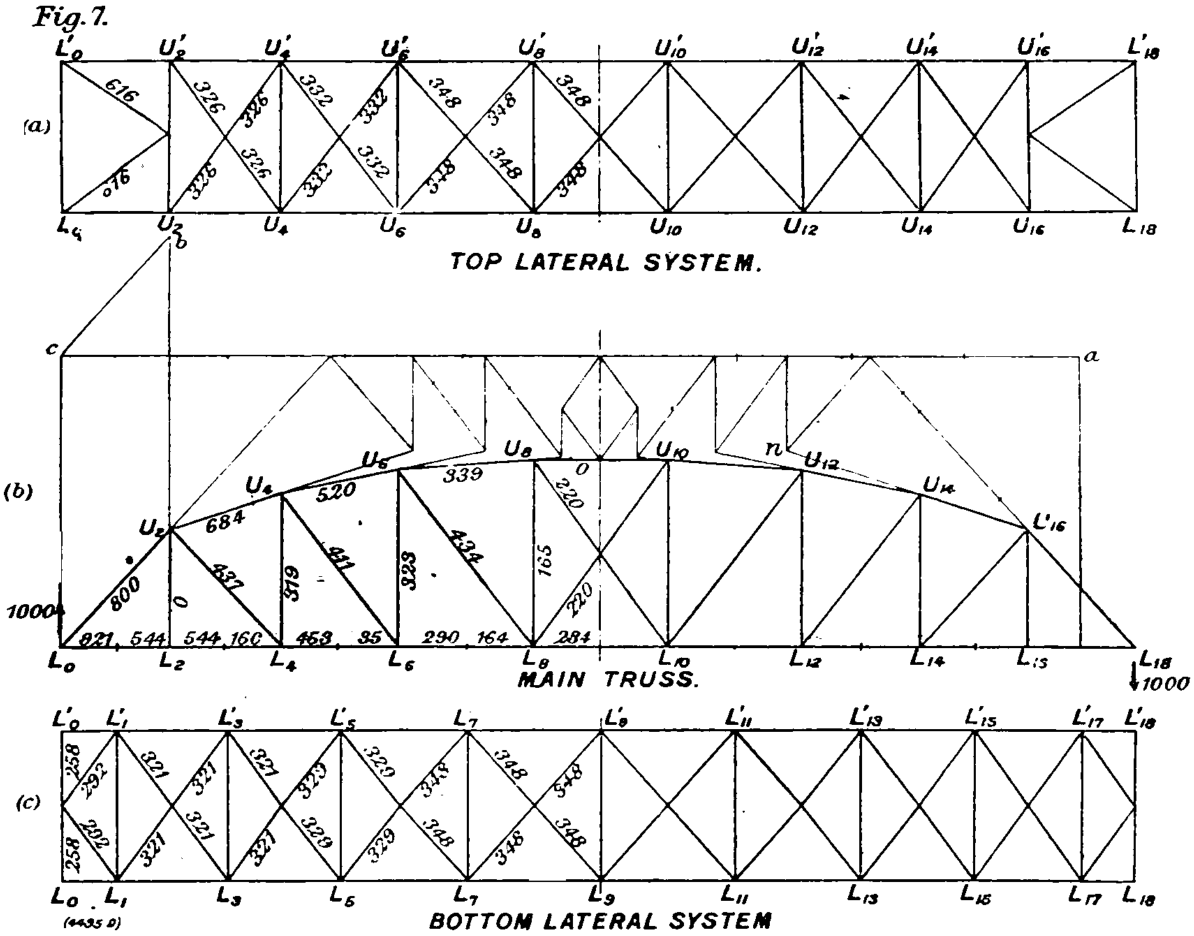


Fig. 7.



now in course of erection. The span is 640 ft. long, and the distance between the main trusses is 88 ft. The other dimensions are shown in the figure. The lateral system is shown in plan in Fig. 7, *a* and *c*. The diagonals of this system will be regarded as acting as both ties and struts, and the bending in the portals will be neglected. Each diagonal of a panel will be regarded as taking one half of the shear, in accordance with the usual assumption. This applies also to the central panels of the main trusses. The sub-members of the main trusses will receive no stresses from the torsional couples, and thus the only members which need be considered are those shown in Fig. 7*b*. The lengths of the cantilever and anchor-arms are shown in Fig. 5*a*, which is a plan of the whole bridge. The live load will be assumed as 5,000 lb. per lineal foot, together with empty cars weighing 900 lb. per lineal foot.

Any unsymmetrical loading of the cantilever and anchor-arms will give rise to torsion stresses in the suspended span, but only those which co-exist with maximum stresses from other causes will affect the design. In this bridge there is no top lateral system in the cantilever and anchor-arms, otherwise there would be torsion stresses in these as well as in the suspended span, but those in the latter would be somewhat reduced, owing to the greater stiffness of the cantilever and anchor-arms. Two cases of loading will be considered here, as shown in Figs. 5*a* and *b* respectively; *a* is the loading used in calculating the effective torsion stresses in the laterals, *b* that used for the main trusses. An exact calculation would require still other cases to be considered, but the magnitude of the torsion stresses in the main truss members is not such as to warrant the extra labour involved.

The first step is to calculate the stresses in the suspended span due to unit forces at L_0, L_{18}, L^1_0, and L^1_{18} respectively. The forces will be regarded as downward at C^1 and D, and upward at C and D^1. Then, from § 2, the transverse shear due to these is given by

$$S = \frac{\text{area of base}}{2 \times \text{area of face}} = \frac{88 \times 640}{2 \times 54{,}600} = 0.516.$$

Thus, for 1,000 lb. at the points L_0, L_{18}, L^1_0, and L^1_{18}, the transverse shear would be 516 lb.

From this the stresses in the laterals may be calculated immediately. For example, in the panel $U_8\,U^1_{10}$ of the upper laterals, each diagonal, being assumed to take one half the shear, will be subjected to a stress equal to

$$\frac{258 \times \text{length of the diagonal}}{\text{distance between the trusses}} = \frac{258 \times 118.8}{88}.$$
$$= 348 \text{ lb.}$$

The stresses in the other members may be calculated in a similar manner, and are shown in Figs. 7*a* and *c*.

In order to find the stresses in the main truss members, the whole system may be considered as made up of three parts—the lateral system and the two main trusses, joined together at the panel-points only. It will be seen from the discussion in § 2 above that the front main truss is then in equilibrium under two forces of 1,000 lb. each, at L_0 and L_{18} respectively, together with forces at each of the panel-points transferred from the lateral system. The latter forces will act parallel to the members of the top and bottom chords, as described in § 2 and indicated in Fig. 2. The force from the panel L_{18} U_{16} of the lateral system will be regarded as transferred at U_{16}, that due to U_{16} U_{14} at U_{14}, etc. It would, of course, be equally correct to regard the force from L_{18} U_{16} as transferred at L_{18}, and that due to U_{16} U_{14} at U_{16}, etc., and the final result would be the same. The stresses in the main members due to these forces may be found by drawing the reciprocal diagram in the usual manner; but, by a little manipulation, the labour may be materially lessened. The first step is to determine the F forces transferred from the lateral system. As an example, consider panel U_8 U^1_{10}. This is in equilibrium under the action of the shear S perpendicular to the main trusses, and the shear transmitted from the main trusses. Thus S × panel-length = F_{10} × distance between the trusses. Thus:—

$$F_{10} = \frac{S \times \overline{U_8 U_{10}}}{88} = \overline{U_8 U_{10}} \times 5.87.$$

It will thus be seen that the F forces are proportional to the lengths of the main chord members. The force diagram may, therefore, be constructed in the following manner, without actually calculating the magnitudes of the F forces:—Draw an outline of the truss as in Fig. 7*b* to any convenient scale. Find the scale to which the length of the chord members in this diagram represents the F forces by determining one of the latter. Erect the perpendicular $L_{17}a$ to represent a force of 1,000 lb. to this scale, draw *a c* parallel to the bottom chord, *c b* parallel to L_0 U_2, *b* U_2 perpendicular to the bottom chord. A little consideration will show that *a* L_{17}, L_{17} L_{18}, the top chord to U_2, U_2b, *b c* and *c a* form the force diagram. The reciprocal diagram may now be constructed and is shown in the figure. This diagram gives at once the stresses in the web members. The chord members, however, must be regarded as belonging to both the main truss and the lateral systems. Regarded as part of the lateral system each member, with the exception of the end posts, is subjected to one-half the F force in its panel, since the lateral system is stiff. This may be allowed for by measurement on the diagram. For example, $n\ U_{16}$ represents the stress in the member $U_{14}\ U_{16}$ regarded as part of the main truss system, $U_{14}\ U_{16}$ represents the F force, and, therefore, the actual force in $U_{14}\ U_{16}$ will be $n\ U_{16} - \frac{1}{2}\ U_{14}\ U_{16}$, which may be measured off on the figure. The other chord stresses may be found in a similar manner, and the final results for the whole system are given in Figs. 7*a*, *b* and *c*.

The next step is to calculate the relative deflections at the corners L_0, L^1_0, L_{18}, L^1_{18} due to the assumed couples. Knowing the sections of the members, this may be done in the usual manner by the use of the formula $\Sigma \frac{p\ U\ l}{A\ E}$ The details of the calculation need not be considered here. The result for the case of the Quebec Bridge is 0.0067 in. deflection of L_0 with respect to L^1_0 for 1,000 lb. load, as assumed above. Now the deflection of the point C with respect to C^1 (Fig. 5*a*) due to the loading shown, and assuming the suspended span to offer no resistance to twisting, was found to be 2.016 in., whilst the deflection at the point C due to a load of 1,000 lb. suspended at the point C under the same conditions is 0.00318 in. Hence, if X be the actual force transmitted from the cantilever arm to the suspended span at L, due to the loading shown in Fig. 5*a*,

$$0.0067\ X = 2.061 - 2 \times 0.00318\ X,$$

and

$$X = 157{,}800 \text{ lb.}$$

Thus the actual value of S for this loading

$$= 0.516 \times 157{,}800 \text{ lb.}$$
$$= 81{,}500 \text{ lb.}$$

For the second case, Fig. 5*b*, X = 99,400 lb. and S = 51,300 lb.

The actual torsion stresses may now be found by multiplying the stresses given in Fig. 7 by 157.8 and 99 4 respectively. For example, the actual torsion stress co-existing with maximum stresses from other causes in the top lateral member $U_8 U'_{10}$ is $157.8 \times 348 =$ 54,900 lb. = 3920 lb. per sq. in., the section of the member being 14 sq. in.

§ 5. *Conclusion.*—The above examples will be sufficient to show the application of the theorem. The case of stresses due to unequal settlement of supports is similar to Example III. but simpler, as it is not necessary to form an elastic equation in order to determine X. Further applications are to erection-travellers, three-hinged arches, etc., unsymmetrically loaded. The form of the theorem given in § 3 *c* may also be used to determine the angle of twist, etc., of any thin-walled prism subjected to a twisting moment by considering the work stored, and may also be extended to give a method of dealing with solid shafts of any cross-section.

THE TORSION OF SOLID AND HOLLOW PRISMS AND CYLINDERS.

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§1. *Introduction.*—In a paper read before the British Association in 1915 and published in ENGINEERING (October 15, 1915, page 392), the writer demonstrated that if any thin-walled cylinder or prism, either continuous-walled or of framework, is subjected to a twisting moment T, the total longitudinal shear S is given by the expression

$$S = \frac{T\,l}{2\,A}, \quad . \quad . \quad . \quad . \quad (1)$$

where l is the length of the cylinder and A the area bounded by the contour of its cross-section (not the area of actual cross-section). The application of this to a framed structure was considered in detail. It is the object of the present paper to consider more closely the case of the thin-walled cylinder or prism having a continuous boundary, to extend the method to solid and hollow sections, to obtain approximate formulæ for the torsion of rolled structural sections, and to criticise certain formulæ in general use for hollow sections.

§2. *Thin-walled Cylinders and Prisms.*—If the walls of a cylinder or prism subjected to torsion are thin, equation (1) may be used directly to find the distribution of shearing stress. Denoting the thickness of the walls by t, the average intensity of shearing stress q over a section of the walls normal to the contour at any point P, is given by

$$q = \frac{T}{2\,A\,t} \quad . \quad . \quad . \quad . \quad (2).$$

Thus, if the walls are of uniform thickness, the average shearing stress intensity is constant, and if of variable thickness, it varies inversely as the thickness. Since the walls are thin, the actual intensity of shear at any point is, in general, practically equal to the average intensity, and is thus given very approximately by equation (2). An exception to this occurs at the corners of a prismatic section. At any corner the stress is zero and thus the shear cannot be uniformly distributed in its immediate neighbourhood.*

Equation 2 may thus be applied to find the strength of any thin-walled cylinder or prism subjected to torsion. It appears from this that all sections of the same material, of the same thickness and having the same area A, have the same resistance under torsion. This does not mean, however, that thin tubes of all shapes are equally efficient when used as shafts. The amount of material for the same values of A and t obviously varies directly as the perimeter of the section, and thus the circle is the most efficient form. For example, a square tube having the same resistance under torsion as a circular tube of the same thickness will contain 1.128 times as much material.

The angle of twist may be found from a consideration of the work stored. This is equal to

$$\int \frac{q^2}{2\,G} \cdot d\,V,$$

where q is the stress at any element of the volume dV, G is the modulus of rigidity of the material and the integration is throughout the volume. It is also equal to $\frac{1}{2}$ T Θ, where T is the applied twisting moment and Θ the angle of twist. Thus

$$\Theta = \frac{2}{T} \int \frac{q^2}{2\,G} \cdot d\,V$$

But

$$q\,t = \frac{T}{2\,A}.$$

Therefore

$$\Theta = \frac{T}{4\,A^2\,G} \int \frac{d\,V}{t^2} \quad . \quad . \quad . \quad (3).$$

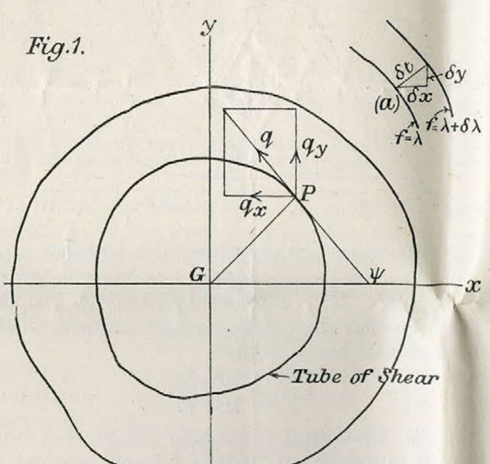
If t is uniform, this gives

$$\Theta = \frac{T\,P\,l}{4\,A^2\,t\,G} \quad . \quad . \quad . \quad . \quad (4)$$

where P is the perimeter of the section and l the length over which Θ is measured.

There seem to be few experimental data on the torsion of thin-walled tubes, but the above receives striking confirmation from the experiments of Mr. Ritchie which were published in ENGINEERING of January 21 and February 4, 1916. Mr. Ritchie used three hollow sections of the dimensions shown in Table I. It will be noticed that the ratio of width to thickness is not very great, so that the tubes can scarcely be called thin-walled. The angle of twist over a length of 5 ft. was measured for a twisting moment of 50 inch-lb. and the results are given in column 5. Column 6 gives the angles of twist calculated from equation (4).

Fig. 1.



The modulus of rigidity was not measured experimentally for these cases, so that its value is somewhat uncertain. Mr. Ritchie assumed it to be 11×10^6 lb. per square inch, and this value has been used in calculating column 6.

TABLE I.

1	2			3	4	5	6	7	8
Type of Section.	Dimensions (inches). Width.	Depth.	Thickness.	Ratio of Width to Thickness.	Area of Section of Metal.	θ (exp.) in Degrees.	θ from Equation 4.	Difference (per Cent.)	Value of G for which Experimental and Theoretical Results Agree.
									$\times 10^6$
Hollow rectangle	0.872	1.432	0.036	24.2	0.151	0.3245	0.3205	1.23	10.86
Hollow square ...	1.500	1.500	0.0502	33.4	0.296	0.0870	0.0923	6.09	11.68
Hollow oval ...	0.862	1.740	0.045	19.15	0.1788	0.3480	0.3668	5.40	11.60

It will be noticed that the agreement between columns 5 and 6 is within the range of experimental error and error in the evaluation of G. Column 8 gives the values of G for which the agreement would be exact. They are within the range of values given by Mr. Ritchie for those cases in which G was measured experimentally.

The above seems to show that reliance may be placed upon the method given for the calculation of stress and angle of twist even when the walls are not extremely thin, but further experimental evidence would be useful. It is interesting to note that Mr. Ritchie, applying the ordinary formula for a hollow elliptic section (Eq. 27 below) to the oval tube, obtains a result which is much too high. This he attributes to the actual section deviating from the elliptic. This may be so, but, even if the section were elliptic the formula would not give correct results, as it applies only to a section bounded by two similar ellipses and not to a tube of uniform thickness. This point will be considered later.

§3. *Extension of the Method to Solid Shafts.*—The shearing stress at any point within the cross-section of a solid shaft is tangential to one of a family of curves which may be called the lines of shear for the section,* the boundary of the section being itself a member of the family, since the stress at the contour must be everywhere tangential to the contour. Let Fig. 1 represent the cross-section, Gx and Gy being a pair of rectangular axes through its centroid G. The equation of the lines of shear may be taken as

$$f = \lambda,$$

where f is a function of x and y and λ a variable parameter, constant for any particular line of shear, the equation of the boundary being given by

$$f = 0.$$

The portion of the cylinder or prism bounded by two neighbouring surfaces of shear, of which the lines of shear are traces on the plane of the cross-section, may be called a *tube of shear*, and the whole shaft may be considered as made up of such tubes.

Now each tube of shear is under the same conditions of stress as a thin tube subjected to torsion, and thus equation (1) may be applied to it. Therefore,

$$q \,.\, \delta\,t = \frac{\delta\,T}{2\,a} \quad . \quad . \quad . \quad . \quad (5)$$

where δ T is the part of the twisting moment, T, carried by the tube under consideration, a is the area of the cross-section of the shaft bounded by the tube (not the sectional area of the tube), and $\delta\,t$ is the normal thickness of the wall of the tube at the point where the shearing stress is q, *i.e.*, the distance between the lines of shear bounding the tube at this point.

The resultant shear q at P (Fig. 1) may be resolved into q_x and q_y parallel to Gx and Gy respectively. Thus

$$q^2 = q^2{}_x + q^2{}_y\,,$$

or

$$q = \frac{q_x}{q} \,.\, q_x + \frac{q_y}{q} \,.\, q_y \,.$$

But

$$\frac{q_x}{q} = -\frac{\delta\,y}{\delta\,t} \text{ and } \frac{q_y}{q} = \frac{\delta\,x}{\delta\,t},$$

where $\delta\,x$ and $\delta\,y$ are the components, parallel to the axes, of the thickness δt (Fig. 1a).

Thus

$$q \,.\, \delta\,t = q_y \,.\, \delta\,x - q_x \,.\, \delta\,y \quad . \quad . \quad (6).$$

---

* For a fuller discussion of this point, see a letter in ENGINEERING, March 31, 1916, page 298.

* See Love, "Theory of Elasticity," page 309.

But, since q is tangential to the line of shear, $f = \lambda$, through P

$$\frac{q_y}{q_x} = \tan . \psi = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}},$$

where ψ is the angle of slope of the line of shear through P. This is satisfied by

$$q_x = -\mathrm{K}\frac{\partial f}{\partial y} \text{ and } q_y = \mathrm{K}\frac{\partial f}{\partial x} \quad . \quad . \ (7)$$

where K is a constant.

But, if ξ, η; and ζ are the displacements, parallel to Gx, Gy and the axis of the shaft respectively, of the point P

$$q_x = \mathrm{G}\left(\frac{\partial \zeta}{\partial x} + \frac{\partial \eta}{\partial z}\right) \quad . \quad . \quad . \ (8a)$$

and

$$q_y = \mathrm{G}\left(\frac{\partial \zeta}{\partial y} + \frac{\partial \xi}{\partial z}\right) \quad . \quad . \quad . \ (8b)$$

where G is the modulus of rigidity of the material.

Now under shear the radius GP (Fig. 1) turns through an angle $\delta\,\theta$ which is the same for all points, and $\frac{d\,\theta}{d\,l}$ is constant along the shaft.

Thus

$$\xi = -k\,x\,z \quad \text{and} \quad \eta = k\,y\,z \quad . \quad . \ (9)$$

where k is a constant and z is measured along the axis of the shaft. Differentiating $8a$ with respect to y and $8b$ with respect to x and subtracting

$$\frac{\partial q_x}{\partial y} - \frac{\partial q_y}{\partial x} = c,$$

where c is a constant. Substituting for q_x and q_y from equation (7), assuming K constant,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -\frac{c}{\mathrm{K}} = \text{a constant} \quad . \ (10).$$

Thus, if K be constant, f must satisfy (10). This equation, combined with the condition that $f = 0$ at the boundary, suffices for the determination of f. It is now only necessary to find K. Substituting the values of q_x and q_y from (7) in equation (6),

$$q\,.\,\delta\,t = \mathrm{K}\left(\frac{\partial f}{\partial x}\,.\,d\,x + \frac{\partial f}{\partial y}\,.\,d\,y\right),$$

or, using (5),

$$\frac{\delta\,\mathrm{T}}{2\,a} = \mathrm{K}\,.\,\delta\,f.$$

Integrating this over the whole cross-section,

$$\mathrm{T} = 2\,\mathrm{K}\int a\,.\,d f \quad . \quad . \quad . \ (11).$$

But a is zero at the centroid and f is zero at the boundary. Thus, integrating by parts,

$$\mathrm{T} = -2\,\mathrm{K}\int f\,.\,d\,a, \quad . \quad . \quad . \ (12)$$

the integral extending over the whole section. Thus, if f is obtained from equation (10), K may be found from equation (12), and the stress at any point of the section is then known from (7). The angle of twist Θ for a length l may be found, as in the case of the thin-walled tube, from

$$\Theta = \frac{l}{\mathrm{G\,T}}\int q^2\,.\,d\,a, \quad . \quad . \quad . \ (13)$$

the integration extending over the whole section.

§4. *Examples:* (*a*) *The Solid Elliptical Shaft.*—Let the semi-major axis be b and the semi-minor axis h. Then the equation of the boundary is

$$\frac{x^2}{b^2} + \frac{y^2}{h^2} - 1 = 0.$$

Thus a possible value for f is

$$f = \frac{x^2}{b^2} + \frac{y^2}{h^2} - 1.$$

This would give

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{2}{b^2} + \frac{2}{h^2},$$

which is a constant. Thus the lines of shear form a family of similar and similarly situated ellipses, for which

$$\frac{x^2}{b^2} + \frac{y^2}{h^2} - 1 = \lambda.$$

pplying equation (12) to this case

$$\mathrm{T} = -2\,\mathrm{K}\int\left(\frac{x^2}{b^2} + \frac{y^2}{h^2} - 1\right)d\,a.$$

But $\int x^2\,.\,d\,a$ and $\int y^2\,.\,d\,a$ are the moments of inertia of the section about the axes Gy and Gx respectively and $\int d\,a$ is the area of the section.

Substituting the values of these in the above equation,

$$\mathrm{K} = \frac{\mathrm{T}}{\pi\,b\,h}.$$

Thus the components of stress at any point of the section are

$$q_x = -\mathrm{K}\frac{\partial f}{\partial y} = -\frac{2\,\mathrm{T}\,y}{\pi\,b\,h^3},$$

and

$$q_y = \mathrm{K}\frac{\partial f}{\partial x} = \frac{2\,\mathrm{T}\,x}{\pi\,b^3\,h}.$$

The maximum shearing stress occurs at the end of the minor axis and is given by q_x, on substituting $y = h$. Thus

$$q_{\text{max.}} = \frac{2\,\mathrm{T}}{\pi\,b\,h^2} \quad . \quad . \quad . \ (14).$$

For a circular shaft of radius r, $b = h = r$, and

$$q_{\text{max.}} = \frac{2\,\mathrm{T}}{\pi\,r^3}$$

Thus the radius of a circular shaft of equivalent strength to the elliptical shaft is given by

$$r = \sqrt[3]{b\,h^2} \quad . \quad . \quad . \ (15).$$

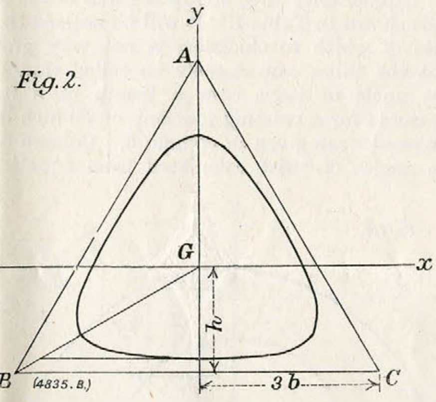
The angle of twist over a length l may be found from equation (13).

$$\Theta = \frac{l}{\mathrm{G\,T}}\int q^2\,.\,d\,a$$

$$= \frac{\mathrm{T}\,l}{\mathrm{G}\,\pi^2\,b^2\,h^2}\int\left(\frac{x^2}{b^4} + \frac{y^2}{h^4}\right)d\,a$$

$$= \frac{\mathrm{T}\,l}{\mathrm{G}\,\pi}\,.\,\frac{b^2 + h^2}{b^3\,h^3} \quad . \quad . \quad (16).$$

Fig. 2.



For a circular section of radius r this becomes

$$\Theta = \frac{2\,\mathrm{T}l}{\mathrm{G}\,\pi\,r^4}.$$

Thus a circular shaft of equivalent stiffness to the elliptical shaft has a radius*,

$$r = \sqrt[4]{\frac{2\,b^3\,h^3}{b^2 + h^2}} \quad . \quad . \quad . \ (17).$$

(*b*) *The Equilateral Triangular Shaft.*—Let ABC (Fig. 2) represent an isoceles triangle referred to axes Gx and Gy through the centroid, Gx being parallel to the base BC. If the length of BC is $6b$ and the height $3h$, the equation of the boundary is

$$\left\{\left(\frac{y}{2\,h} - 1\right)^2 - \frac{x^2}{4\,b^2}\right\}(y + h) = 0.$$

Thus the left-hand side of the above is a possible expression for f. This satisfies

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \text{a constant},$$

if $h = \sqrt{3}\,.\,b$, *i.e.*, if the triangle is equilateral. In this case the lines of shear will be given by

$$f = \left\{\left(\frac{y}{2\,h} - 1\right)^2 - \frac{x^2}{4\,b^2}\right\}(y + h) = \lambda,$$

and, applying equation (12),

$$\mathrm{T} = -2\,\mathrm{K}\int\left(\frac{y^3}{4\,h^2} - \frac{3}{4}\,.\,\frac{y^2}{h} - \frac{x^2\,y}{4\,b^2} + h - \frac{x^2\,h}{4\,b^2}\right)da.$$

Now

$$\int y^2\,d\,a = \mathrm{I}_{\mathrm{G}x} = 4.5\,b\,h^3,$$

$$\int h\,d\,a = 9\,b\,h^2,$$

$$\int x^2\,d\,a = \mathrm{I}_{\mathrm{G}y} = 13.5\,b^3\,h$$

$$\int x^2\,y\,d\,a = \int_{-3b}^{0}\int_{-h}^{\left(1 + \frac{x}{2b}\right)2h} x^2\,y\,.\,d\,x\,.\,d\,y + \int_{0}^{3b}\int_{-h}^{\left(1 - \frac{x}{2b}\right)2h} x^2\,y\,.\,d\,x\,.\,d\,y$$

$$= -5.4\,b^3\,h^2$$

and $\int y^3\,d\,a$ may be found by integrating in a similar manner and is equal to $1.8\,bh^4$.

Substituting these values in (18) and replacing h by $\sqrt{3}\,b$,

$$\mathrm{T} = -24.3\,\mathrm{K}\,b^3.$$

Thus

$$q_x = \frac{\mathrm{T}}{97.2\,b^5}\left(y^2 - 2\sqrt{3}\;y\,b - x^2\right).$$

This suffices for the determination of the distribution of shearing stress along BC, since $q_y = 0$ along this side. The maximum stress occurs at the mid-point of each side, *i.e.*, at the points of the contour nearest to the centroid, and is given by

$$q_{\text{max.}} = 20\,\frac{\mathrm{T}}{\mathrm{S}^3}, \quad . \quad . \quad . \ (19)$$

where S $= 6b =$ the length of the side.

Thus the radius of a circular shaft of equivalent strength is given by

$$r = 0.317\ \mathrm{S},$$

and the weight of a triangular shaft is 1.37 times the weight of a circular shaft of the same material, length and strength.

At the corner C, $x = 3b$, $y = -h$. Thus $q_x = 0$. The stress parallel to the side AC is also zero. Thus the stress at the corner is zero. The distribution of stress will, of course, be the same along each side.

The shape of the lines of shear is indicated in Fig. 2.

The angle of twist could be calculated as in example a, but the analysis, although not difficult, is too long to be given here.

The above examples will suffice to show the application of the method. In most of the other cases f takes a complicated form and the analysis is much more difficult. The results obtained are, of course, the same as those arising from the usual methods of analysis, but it appears to the writer that the above method is simpler, the mathematics involved more elementary, and the physical conditions kept more in evidence.

§5. *The Rectangular Shaft. Approximate Solutions.*—One of the most important cases is that of the rectangular shaft. Unfortunately the analysis in this case is complicated. The problem was first solved by St. Venant and has since been somewhat simplified by Goetzke.* Bach has shown that St. Venant's value for the maximum stress is given very approximately by the formula

$$q_{\text{max.}} = \left(3 + \frac{2\cdot 6}{0.45 + \frac{h}{b}}\right)\frac{\mathrm{T}}{b^2\,h} = \phi\,.\,\frac{\mathrm{T}}{b^2\,h}. \quad (20)$$

where b is the length of the shorter side and h that of the longer side. Bretschneider has verified this result experimentally.† Approximate solutions have been given by Bach and others leading to the formula

$$q_{\text{max.}} = 4\,.\,5\,\frac{\mathrm{T}}{b^2\,h} \quad . \quad . \quad . \ (21)$$

which is often used in practice for all ratios of b to h, although very far from the truth if h is much greater than b.

This formula may be obtained by the method of §3 and §4 by assuming

$$f = \left(x^2 - \frac{b^2}{4}\right)\left(y^2 - \frac{h^2}{4}\right).$$

This is the simplest function which satisfies the boundary conditions, but is not correct, since it does not satisfy equation (10), and so can only give an approximate solution. Proceeding as in §4, the

* It should be noted that shafts of different shapes having the same stiffness are not, in general, of the same strength, and thus the relative strengths of such shafts cannot be obtained from measurement of the angle of twist unless the law of variation of stress is known for each shaft. Mr. Ritchie seems to have fallen into this error in discussing the results of his experiments mentioned above. For example, the column of the table on page 50 (ENGINEERING, January 21, 1916), headed "Diameter of circular section of equivalent torsional strength," really gives the diameter of a circular section of equivalent torsional stiffness.

* Zeitschrift des Vereines Deutscher Ingenieure, 1909, page 935.

† Bach, "Elasticität und Festigkeit," 6th edition, page 342.

maximum stress will be that given by equation (21) and the stress at the corners will be found to be zero. The distribution of stress throughout agrees with that assumed by Bach.

A very approximate solution may be obtained simply if the ratio h/b is large,. In this case the lines of shear, except at the ends, must be parallel to the longer side, *i.e.*, $q_x = 0$ if the axis of y is parallel to the longer side. Thus equation (10) reduces to

$$\frac{\partial^2 f}{\partial x^2} = 2c$$

where c is a constant. Integrating this,

$$f = c\,x^2 + d,$$

where d is also a constant, q_y being equal to zero when $x = 0$. Thus the lines of shear, neglecting the effect of the ends, are given by

$$c\,x^2 + d = \lambda.$$

At the boundary,

$$x = \frac{b}{2}, \lambda = 0$$

and thus

$$f = c\left(x^2 - \frac{b^2}{4}\right).$$

This leads to

$$\mathrm{K}\,c = \frac{3\,\mathrm{T}}{h\,b^3}.$$

Thus

$$q_y = \frac{3\,\mathrm{T}}{\mathrm{A}\,t} \quad . \quad . \quad . \quad . \quad (22)$$

and is constant over the longer sides.

This agrees with St. Venant's result if $\frac{h}{b} = \infty$

6. *Structural Steel* I, ⊏, L *and* T *Sections.*—The result obtained for a thin rectangular section will be very approximately true for a thin section of any shape, as, for example, an ordinary angle section, provided that, as is usually the case in practice, the inner corners are rounded to prevent concentrations of stress near to them. This may be seen at once from the fact that the lines of shear in such a section must be parallel to the contour all over the section except in the immediate neighbourhood of the ends or corners. Thus for any of the rolled sections used in practice

$$q_{\max.} = \phi\,\frac{\mathrm{T}}{\mathrm{A}\,t} \quad . \quad . \quad . \quad (23)$$

where ϕ is a constant which will not differ greatly from 3.

The angle of twist is given by

$$\Theta = \frac{l}{\mathrm{T\,G}} \int q^2 \,.\, d\,a,$$

as in the cases considered above. At any point distant x from the centre line of the section,

$$q = \phi \,.\, \frac{\mathrm{T}}{\mathrm{A}\,t} \,.\, \frac{2\,x}{t}.$$

Thus

$$\Theta = \frac{\phi^2\,\mathrm{T}\,l}{3\,\mathrm{G\,A}\,t^2} \quad . \quad . \quad . \quad (24).$$

The above assumes that the thickness of the section is the same throughout. In I, channel and tee sections the thickness of the flanges is usually greater than the thickness of the web. Equation (23) is true for these if t be the least thickness, but in calculating θ it is necessary to take account of the difference in stress distribution in the web and flanges. If t_2 be the thickness of the flanges and t_1 of the web, the maximum stress in the flanges will be

$$\frac{t_1}{t_2} \cdot q_{\max.}$$

Thus, making the same assumptions as before,

$$\Theta = \frac{\phi^2\,\mathrm{T}\,l}{3\,\mathrm{G\,A}^2\,t_1^2}\left[\mathrm{A}_1 + \frac{t_1^2}{t_2^2}\mathrm{A}_2\right], \quad . \quad . \quad (25)$$

where A_1 is the area of the web, A_2 the total flange area, and $\mathrm{A} = \mathrm{A}_1 + \mathrm{A}_2$. If the web is thicker than the flanges, A_1 and A_2 and t_1 and t_2 must be interchanged.

The above may be extended to cases in which the flanges are of different thicknesses, &c., by adding terms similar to the last term in the bracket.

ϕ may be determined experimentally for any given section by measuring the angle of twist Θ for a given twisting moment T. Mr. Ritchie, in the paper referred to above, reports such measurements on a series of I, T, ⊏ and L sections. Table II gives the results of the tests and the values of ϕ calculated from them by using formula (25). It will be noticed that all the sections are stiffer than would be expected from the above theory, *i.e.*, ϕ is less than 3. For the channel it is only about 3 per cent. and for the angles 8.2 per cent. less than 3. For the T it is about 9.7 per cent., whilst for the I sections the results are very variable, the average being 19 per cent. lower than 3.

Thus the above theory seems to hold fairly well for channel and angle sections, but gives results much too high for I and T sections. This may be due partly to the conditions of stress at those portions of the flanges just above the junction with the web in the latter sections. A more complete series of tests on larger specimens, carefully graded as to relative width of flange, relative thickness of webs and flanges, &c., would probably throw more light on this and perhaps lead to a more complete theory or at least to an empirical formula for these cases. In the meantime equations (23) and (25) may be considered as giving results a little higher than the actual stress and angle of twist respectively.

TABLE II.

No.	Type of Section.	Dimensions in Inches. Width.	Depth.	Thickness of Flange.	Thickness of Web.	Area. (Sq. In.)	$G \div 10^6$ in Lb. per Sq. In.	Angle of Twist in Deg. over 5 ft. Span, for 50 In.-Lb. Torque.	ϕ.
1	I	5.01	8.02	0.605	0.30	8.02	11.43	0.0137	2.13
2	I	3.01	3.00	0.325	0.20	2.43	—	0.1445	2.38
3	I	1.75	4.78	0.324	0.19	1.91	—	0.2460	2.35
4	I	1.66	3.16	0.230	0.17	1.222	11.32	0.6450	2.50
5	I	0.99	1.95	0.165	0.22	0.6825	11.30	1.6210	2.76
6	I	0.76	1.50	0.165	0.14	0.4141	11.68	2.9470	2.42
7	Channel	0.97	2.00	0.227	0.22	0.7825	11.61	1.0650	2.91
8	Angle	1.175	1.175	0.250	—	0.5245	10.95	1.2260	2.76
9	Angle	1.00	1.00	0.185	—	0.3363	11.00	3.4300	2.76
10	Tee	1.58	1.58	0.231	0.21	0.650	11.50	1.0080	2.54
11	Tee	0.99	0.99	0.135	0.145	0.2573	11.41	8.3850	2.89

N.B.—Mr. Ritchie does not give the value of G for specimens 1 to 3. The value given here is the mean of those for specimens 4 to 6.

7. *Hollow Sections.*—Both the inner and outer boundaries of the cross-sections of a hollow shaft must be lines of shear. Thus the function f must vanish at the outer boundary and be equal to a constant at the inner boundary. The equation,

$$\mathrm{T} = 2\,\mathrm{K}\int a \,.\, d f,$$

will be true, but the integration will extend only from the inner boundary to the outer. Thus, integrating by parts,

$$\mathrm{T} = 2\,\mathrm{K}\,[a\,f] - 2\,\mathrm{K}\int f\,d\,a.$$

The product $a\,f$ vanishes at the outer boundary since $f = 0$, but is equal to a_1 at the inner boundary, λ_1, being the value of the parameter λ which gives the equation of the inner boundary and a_1 the area of the hollow.

Thus

$$\mathrm{T} = 2\,\mathrm{K}\left\{a_1\,\lambda_1 - \int f \,.\, d\,a\right\} \quad . \quad . \quad (26)$$

the integration extending from the inner to the outer boundary.

As an example of the method, a hollow elliptical cylinder bounded by two similar ellipses will be considered. Since the lines of shear of a solid elliptical shaft are similar ellipses to the boundary, the function f will be the same as in example a § 4, *i.e.*,

$$f = \frac{x^2}{b^2} + \frac{y^2}{h^2} - 1 = \lambda,$$

b and h being the semi-axes of the outer boundary.

Let the semi-axes of the inner boundary be $\gamma\,b$ and $\gamma\,h$ respectively. Then the equation of the inner boundary will be

$$\frac{x^2}{b^2} + \frac{y^2}{h^2} - 1 = \gamma^2 - 1.$$

Thus

$$a_1 = \pi\,\gamma^2\,b\,h$$

$$\lambda_1 = \gamma^2 - 1$$

and

$$\frac{\mathrm{T}}{2\,\mathrm{K}} = \gamma^2\,(\gamma^2 - 1)\,\pi\,b\,h - \int\left(\frac{x^2}{b^2} + \frac{y^2}{h^2} - 1\right)d\,a.$$

This leads to

$$\mathrm{K} = \frac{\mathrm{T}}{(1 - \gamma^4)\,\pi\,b\,h}.$$

Thus the maximum stress occurs, as in the case of the solid section, at the end of the minor axis and

$$q_{\max.} = \frac{2\,\mathrm{T}}{\pi\,b\,h^2\,(1 - \gamma^4)} \quad . \quad . \quad (27).$$

This formula is often given in engineering textbooks as applying to any hollow elliptical cylinder, but is, of course, true *only when the inner boundary is a similar ellipse to the outer.* No matter how thin the walls of the cylinder may be, this will not give a section having uniform thickness of walls, and so the formula cannot be applied to an ordinary elliptical tube. This was pointed out by the writer in a recent letter (ENGINEERING, March 31, 1916, page 298) in which it was shown that, for an elliptical tube of uniform thickness equal to $\frac{1}{100}$ of the semi-minor axis, the ratio of the axes being 3 to 1, the correct value of the maximum stress is about 21.2 per cent. lower than that given by equation (27).

In most hollow shafts which are used in practice the inner boundary is of the same shape as the outer, *i.e.*, the walls are of uniform thickness, and thus the inner contour is not, in general, a line of shear for the solid section. Thus f will not usually have the same form as in the solid shaft of the same shape and its determination becomes difficult. This is so, for example, in either a hollow triangular or hollow rectangular shaft of uniform thickness. The formula usually given for the latter is

$$q_{\max.} = \phi \,.\, \frac{\mathrm{T}\,b}{b^3\,h - b_0^3\,h_0},$$

where b_0 and h_0 are the sides of the inner boundary, the other symbols having the same meaning as in equation (20). This seems to have been arrived at, by analogy, from the formula for a hollow circular shaft:—

$$q_{\max.} = \frac{\mathrm{T}\,r}{\mathrm{J} - \mathrm{J}_0},$$

where r is the external radius and J and J_0 the polar moments of inertia for circles of radii equal respectively to the outer and inner radius of the section. It thus appears to be based upon two assumptions:—(1) that the shear stress varies as the distance from the centroid to the mid-point of the longer side; (2) that $b^3 h - b_0^3 h_0$ may be regarded as an equivalent J for the section. It has been shown by St. Venant that assumption (1) is incorrect. Assumption (2) is equivalent to stating that the inner rectangle is a line of shear for a solid rectangular shaft having the same boundary as the hollow shaft and that the lines of shear are the same as in the solid shaft. It is thus incorrect even when the walls are thin. An actual line of shear just within the contour of a solid rectangular shaft is a continuous curve practically parallel to the boundary near the centres of the sides but curving inwards quickly as it approaches the corners, somewhat as the line of shear for the triangular shaft shown in Fig. 2, page 2. Thus the formula cannot be expected to give correct results. In the letter already mentioned it was shown that for a thin hollow rectangular shaft having walls of thickness equal to $\frac{1}{200}$ of b, the error is probably about 21 per cent.

No satisfactory formulæ have, as far as the writer is aware, been given for thick-walled hollow shafts of the sections, other than circular, ordinarily used in practice. If the walls are thin or even of moderate thickness, the method of §2 may be used with some confidence.

In conclusion, the writer wishes to thank Professor E. Brown for his valuable suggestions and for checking the numerical work.

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THE PARTITION OF THE LOAD IN RIVETED JOINTS

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THE PARTITION OF THE LOAD IN RIVETED JOINTS.*

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INTRODUCTION.

RIVETED joints occur in many types of construction, and it is therefore of considerable practical importance to determine the exact manner in which they act, in order that a rational basis may be given for their design. The subject has attracted the attention of many experimenters, their investigations being mainly directed to a determination of the resistance of joints to rupture and of the frictional resistance to slip.[1] Attempts have also been made to determine the tension in the body of a rivet due to its contraction on cooling,[2] and Frémont has made an exhaustive study of the actual process of riveting and its effects upon the strength of the joint.[3] None of these experiments, however, have indicated very clearly the action of a riveted joint under working loads; *i.e.*, before permanent deformations of the plates or rivets have occurred. Very few attempts have been made either ex-

* Communicated by the author.

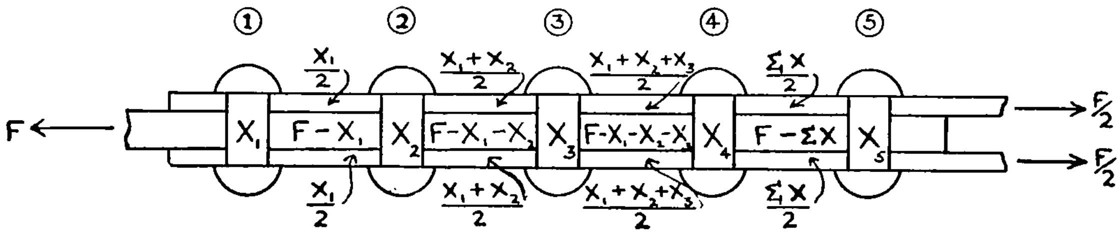
[1] A short bibliography of the subject up to the year 1909 is given in a paper by A. N. Talbot and H. F Moore—"Tests of Nickel-steel Riveted Joints." *University of Illinois Bulletin*, No. 49. See also Preuss, *Zeit. Ver. Deut. Ing.*, 1912, p. 404, and C. Bach and R. Baumann, *Ibid.*, p. 1890.

[2] R. Baumann.

[3] Étude Expérimentale du Rivetage, Paris, 1906.

each the same area of cross-section, and the connection being made by a single line of rivets of uniform diameter and pitch (Fig. 1).

FIG. 1.



Let a_p represent the cross-sectional area of the middle plate,
a_c represent the cross-sectional area of each cover plate,
l represent the pitch of the rivets,
n represent the number of rivets on each side of the junction of the main plates,
F represent the load, tensile or compressive, carried by the joint,
and X_1 represent the load carried by the 1st rivet, X_2, that carried by the second rivet, etc.

Then between the first and second rivets the load carried by the main plate is $F - X_1$, and the load carried by each cover plate is $\frac{X_1}{2}$; between the second and third rivets the load carried by the main plate is $F - X_1 - X_2$, and by each cover plate is $\frac{X_1+X_2}{2}$; and between the $(n-1)^{th}$ and n^{th} rivet the load carried by the main plate is $F - \sum_1^{n-1} X$ and by each cover plate is $\frac{\sum_1^{n-1} X}{2}$, where $\sum_1^{n-1} X = X_1 + X_2 + \quad X_{n-1}$. The distribution of the load for five rivets is shown in Fig. 1, which represents one-half of the joint.

Now assuming that the stress in any portion of a plate between two rivets is uniformly distributed (see § 1), the work stored in this portion, if the load carried by it is P, is $\frac{P^2 l}{2aE}$ where a is the cross-sectional area, l the pitch of the rivets,[8] and E is Young's Modulus for the material of the plate. It will be assumed that E is the same for both cover and middle plates.[9] The work stored in the rivets will be assumed to be of the form kX^2, as described above.

[8] l should really be taken a little less than the pitch of the rivets in order to allow for the portion of the plate cut away for the rivet hole.

[9] See § 3.

Then, if W represents the total work stored in one-half of the joint,

$$2EW = \frac{l}{a_p}\left[(F-X_1)^2+(F-X_1-X_2)^2+(F-X_1-X_2-X_3)^2+ \quad \ldots+(F-\Sigma X)^2\right]$$
$$+\frac{2l}{a_c}\left[\left(\frac{X_1}{2}\right)^2+\left(\frac{X_1+X_2}{2}\right)^2+\left(\frac{X_1+X_2+X_3}{2}\right)^2+ \quad \cdot +\left(\frac{\Sigma X}{2}\right)^2\right]$$
$$+k\left[X_1^2+X_2^2+X_3^2+ \quad \ldots +(F-\Sigma X)^2, \quad \right], \qquad \ldots (1)$$

where $\Sigma X = X_1+X_2+X_3+\ldots \quad +X_{n-1}$.

In accordance with the Principle of Least Work, the forces X_1, X_2, etc., in the above will take the values which will make W a minimum.

Thus

$$\frac{\partial W}{\partial X_1}=0, \frac{\partial W}{\partial X_2}=0, \frac{\partial W}{\partial X_3}=0 \quad \ldots \quad \frac{\partial W}{\partial X_{n-1}}=0$$

But $$\frac{\partial W}{\partial X_1} = -\frac{2l}{a_p}\left[(F-X_1)+(F-X_1-X_2)+ \quad +(F-\Sigma X)\right]$$
$$+\frac{l}{a_c}\left[X_1+(X_1+X_2)+\ldots \quad +\Sigma X\right]$$
$$+2k\left[X_1-(F-\Sigma X)\right].$$

Thus, equating this to zero and dividing through by $\frac{a_c}{l}$, ,

$$\left[(n-1)\left(1+\frac{2a_c}{a_p}\right)+\frac{4ka_c}{l}\right]X_1+\left[(n-2)\left(1+\frac{2a_c}{a_p}\right)+\frac{2ka_c}{l}\right]X_2$$
$$+ \quad \ldots +\left[\left(1+\frac{2a_c}{a_p}\right)+\frac{2ka_c}{l}\right]X_{n-1}=\left[(n-1)\frac{2a_c}{a_p}+\frac{2ka_c}{l}\right]F.$$

Writing $C=1+\frac{2a_c}{a_p}$ and $K=\frac{2ka_c}{l}$, and taking $F=1$ for convenience,

$$[(n-1)C+2K]X_1+[(n-2)C+K]X_2+ \quad \ldots+[C+K]X_{n-1}=(n-1)C+K$$

Differentiation with respect to X_2, X_3 etc., leads to the equations

$$[(n-2)C+K]X_1+[(n-2)C+2K]X_2+[(n-3)C+K]X_3+\ldots \quad +[C+K]X_{n-1}$$
$$=(n-2)C+K,$$
$$[(n-3)C+K]X_1+[(n-3)C+K]X_2+[(n-3)C+2K]X_3+\ldots \quad +[C+K]X_{n-1}$$
$$=(n-3)C+K,$$

$$[C+K]X_1+[C+K]X_2+[C+K]K_3+ \quad \ldots +[C+2K]X_{n-1}=C+K.$$

Thus a set of $(n-1)$ linear simultaneous equations has been established, from which $X_1, X_2 \ldots X_{n-1}$ may be found, while $X_n = 1 - \Sigma X$. For the sake of brevity in what follows, the above and all similar equations will be written in the abbreviated form

$$\left|\begin{array}{c|c|c|c|c|c}(n-1)C+2K & (n-2)C+K & (n-3)C+K & \ldots & C+K & (n-1)C+K \\ (n-2)C+K & (n-2)C+2K & (n-3)C+K & & C+K & (n-2)C+K \\ (n-3)C+K & (n-3)C+K & (n-3)C+2K & & C+K & (n-3)C+K \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ C+K & C+K & C+K & & C+2K & C+K\end{array}\right| \quad (2)$$

It will be noticed that the quantities C and K in the above have no dimensions, but are simply numbers. C is determined from the ratio of thickness of cover plate to thickness of main plate. Thus, if each of the cover plates has half the thickness of the middle plate $C = 2$, if each cover plate is of the same thickness as the middle plate $C = 3$, etc. K must be determined by experiment in the absence of an exact theoretical estimate of the work stored in a rivet carrying a given load. This matter will be discussed later.

It would serve no useful purpose to give a general solution of the above equations for any number of rivets, as the equations may be solved easily for any particular case and the results are similar in form, whatever n may be. In the experiments described later the number of rivets on each side of the junction was usually five, so that the solution for $n = 5$ will be considered in detail.

In this case equations (2) reduce to

$$\left|\begin{array}{c|c|c|c||c}4C+2K & 3C+K & 2C+K & C+K & 4(C-1)+K \\ 3C+K & 3C+2K & 2C+K & C+K & 3(C-1)+K \\ 2C+K & 2C+K & 2C+2K & C+K & 2(C-1)+K \\ C+K & C+K & C+K & C+2K & (C-1)+K\end{array}\right| \ldots\ldots (3)$$

Subtracting the second equation from the first, and the third from the second,

$$KX_2 = (C+K)X_1 - (C-1) \ldots\ldots (4)$$

$$KX_3 = CX_1 + (C+1)X_2 - (C-1) \ldots\ldots (5)$$

Eliminating X_4 from the third and fourth equations and substituting X_2 and X_3 from the above,

$$X_1 = \frac{K^4 + (C-1)10K^3 + (C^2-C)15K^2 + (C^3-C^2)7K + (C^4-C^3)}{5K^4 + 20CK^3 + 21C^2K^2 + 8C^3K + C^4} \ldots\ldots (6)$$

This expression may be used to find X_1 for any particular values of C and K. X_2 and X_3 may then be determined from equations (4) and (5), X_4 from the last equation of (3), and $X_5 = 1 - \Sigma X$.

It may readily be seen from equation (6) that the assumption ordinarily made in designing riveted joints, *i.e.*, that all the rivets take an equal proportion of the load, would only be true if $K = \infty$, in which case $X_1 = X_2 = X_3 = X_4 = X_5 = 1/5$. But this would mean that the rivets were quite flexible and offered no resistance to distortion, which would be a practical impossibility. Thus, in any joint of the type considered in this section, the load must be unequally distributed among the rivets.

If the rivets were absolutely rigid, *i.e.*, suffered no distortion when the joint was loaded, K would be equal to zero and

$$X_1 = \frac{C-1}{C},$$

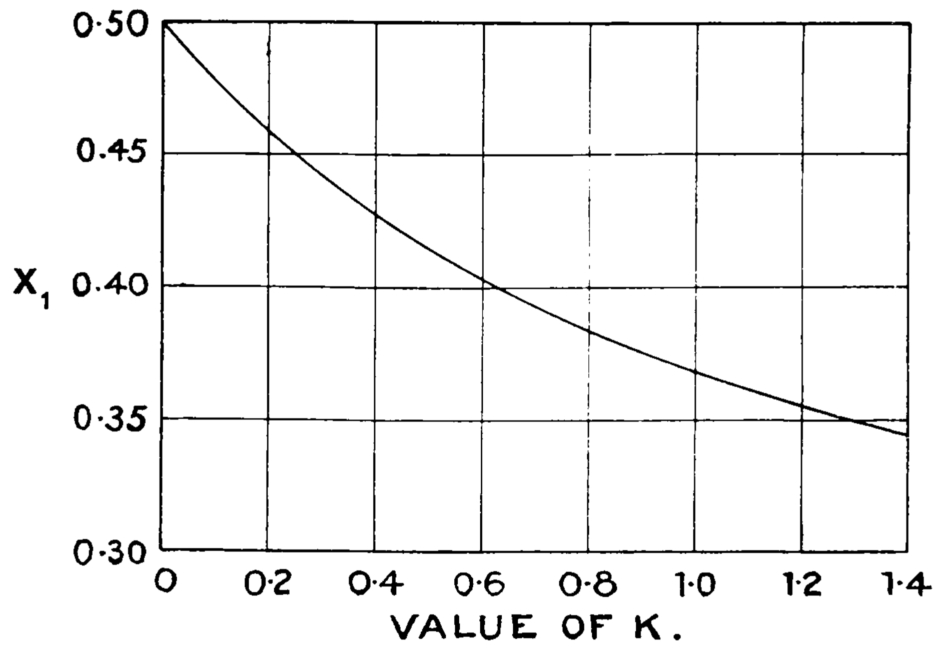
while $X_2 = X_3 = X_4 = 0$ and $X_5 = \frac{1}{C}$. Thus the first and last rivets would carry all the load.

If C were equal to 2, *i.e.*, if the cover plates were of the correct thickness, each one-half of the thickness of the main plate, these rivets would each take one-half of the load; if the covers were each equal in thickness to the main plate, the first rivet would carry two-thirds of the load and the last one-third; and if the covers were only one-quarter the thickness of the main plate, the first rivet would carry one-third of the load and the last two-thirds.

Actually the rivets are neither infinitely flexible nor infinitely rigid, but are elastic, and K has some finite value. Fig. 2 shows the value of X_1 for all values of K between 0 and 1.4 when there are five rivets. This curve will be used later in discussing the experimental results. The greater the value of K the more nearly uniform are the loads carried by the rivets, but, from the actual experimental values found from the specimens tested, it would appear that in most practical cases the two end rivets carry by far the greater part of the total load. The greatest experimental value of K was 1.30 for ½-inch rivets at about one and a half times the working load. At the working load it was approximately equal to unity In order to simplify the present discussion this value will be taken in most of the illustrative cases.

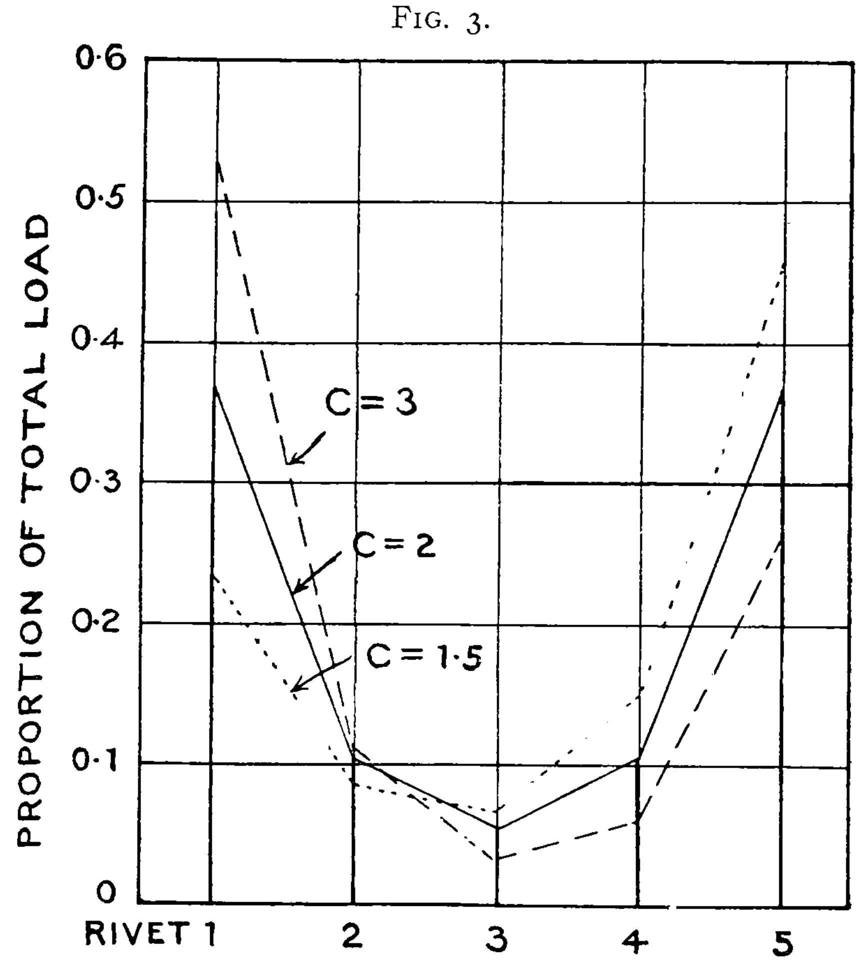
The proportion of the load carried by each rivet when $K = 1$ is tabulated in Table I for three values of C, while Fig. 3 shows the results graphically These three examples will serve to show the

FIG. 2.



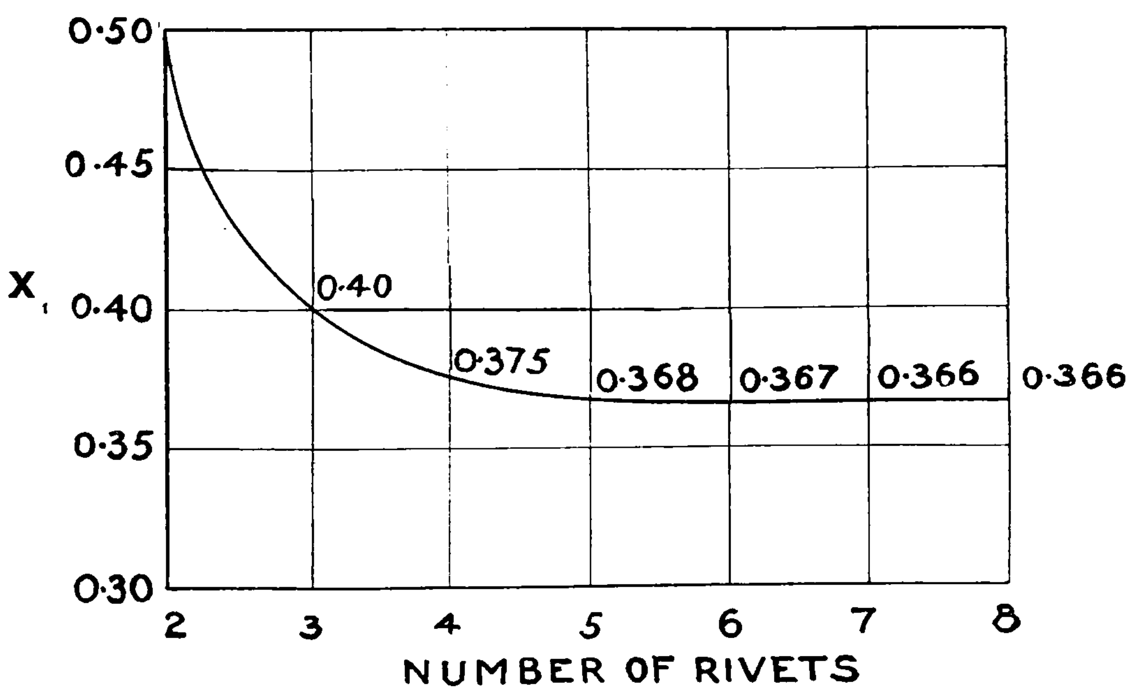
Curve connecting X_1 and K for five rivets.

FIG. 3.



general manner of distribution in most practical cases. In each case the end rivets take the greater part of the load, carrying 0.695, 0.736, and 0.793 of the total load when $C = 1.5$, 2, and 3.0 respectively. In no case does the middle rivet carry more than a very small fraction of the total load. It will be noticed that the distribution is symmetrical when the cover plates are of the correct thickness, *i.e.*, when $C = 2$. This is so no matter what value K may have, and, by taking account of this, the equations may be simplified by putting $X_2 = X_4$, thus reducing their number by one equation. When there is a large number of rivets, this shortens the solution considerably. For example, if $n = 10$, the number of

FIG. 4.



equations may be reduced from 9 to 6. When the thickness of each of the cover plates is more than one-half of the thickness of the main plate, the first rivet receives more than the last, as shown by the diagram for $C = 3$. On the other hand, when C is less than 2 this condition is reversed, the last rivet receiving more than the first.

Fig. 4 shows the proportion of the total load taken by the first rivet for $C = 2$, $K = 1$ for joints having 1–8 rivets on each side of the junction. It will be noticed that the curve becomes practically horizontal when the number 5 is reached, so that no matter how many more rivets may be added, the two end rivets get practically the same proportion of the total load as with five rivets. This means that as the number of rivets increases each of the other rivets receives a less and less proportion of the load, and those near to the middle of the joint are practically idle.

§3. *Effect of Factors Neglected in the Above Analysis.*

(*a*) Effect of Non-uniform Distribution of Stress in the Plates.

It was assumed, in estimating the work stored in the plates, that the stress is distributed uniformly between any pair of rivets. A glance at Fig. 9, page 583, will show that this is not so, the stress varying from a minimum along the centre line of the rivets to a maximum at the edges of the plate. In addition to this variation in the free portions of the plates between the rivets there must be considerable local variations of stress round the rivet holes.[10] It will be necessary to consider the effect of this non-uniform distribution on the equations (2) given above.

The effect will be equivalent to multiplying every term of the form $\frac{P^2l}{2aE}$ in the expression for the work stored, equation (1), by some coefficient α which will depend upon the manner of distribution of stress in the portion considered. If the variation of stress were the same in all portions of all the plates, *i.e.*, if α were the same for all the terms, it may easily be seen that the only effect of this would be to multiply K by the coefficient $\frac{1}{\alpha}$. If, however, α were different in different parts of the plates, as will usually be the case, the terms of equations (2) would be affected differently and there would be a modification of the distribution of load between the rivets. Fortunately the coefficient α cannot be very different from unity, as the following analysis will show.

Suppose that the stress in a portion of the cover plate between two rivets varies uniformly from f_1 at the centre line to f_2 at the edges. This is not far from the actual manner of distribution as shown in Fig. 9, page 583. Let the length of the portion considered be l, the breadth $2b$, and the thickness t, a_c being $2\ bt$. Then the mean stress in the plate is $\frac{f_1+f_2}{2}$ If this were uniformly

[10] This variation has been examined for a more or less analogous case by E. G. Coker, using his polarized light method. See *Trans. I. N. A.*, 1913.

distributed, as assumed in the analysis of §2, the work stored in the portion considered would be

$$W_1 = (f_1 + f_2)^2 \frac{btl}{4E}$$

The actual work stored, however, is

$$W_2 = 2\int_0^b \left\{ f_1 + \frac{f_2 - f_1}{b} x \right\}^2 \frac{tl}{2E} \cdot dx.$$

$$= \frac{btl}{3E}\left\{ f_1^2 + f_1 f_2 + f_2^2 \right\}$$

The ratio

$$r = \frac{W_2}{W_1} = \frac{4}{3} \frac{f_1^2 + f_1 f_2 + f_2^2}{(f_1 + f_2)^2} = \frac{4}{3}\left\{ 1 - \frac{f_1 f_2}{(f_1 + f_2)^2} \right\} \quad \ldots\ldots \quad (7)$$

Now the maximum variation of stress observed in the experiments was about 20 per cent.; *i.e.*, $\frac{f_2}{f_1} = \frac{5}{4}$. For this variation $r = \frac{4}{3}\left\{1 - \frac{20}{81}\right\} = 1.004$; *i.e.*, the work stored is about 0.4 per cent. greater than it would be if the stress were uniformly distributed. A longitudinal variation of stress of the same amount would give the same result. Thus no appreciable error is caused by neglecting this factor. The probably intense local variations of stress in the immediate neighborhood of the rivets are not easy to estimate, but they extend over a small area only and are probably similar in all the plates, so that they will be included in the experimental estimates of K, as will also be the error due to disregarding the parts of the plate cut away for the rivet holes in estimating the volume of the plate between a pair of rivets. Neither of these is likely to have any great effect upon the value of K.

(*b*) Effect of Unequal Partition of the Load Between the two Cover Plates.

It was assumed, in the analysis of §2, that the cover plates each received one-half of the load transmitted to them by the rivets. This was not so in most of the experiments described in Part II.

Suppose that one cover plate takes $\frac{1}{s}$ of the load, then the other takes $1 - \frac{1}{s}$ and equation (1) becomes

$$2EW=\frac{l}{a_p}\left[(F-X_1)^2+\ldots\right]+\frac{l}{s^2a_c}\left[X_1^2+(X_1+X_2)^2+\quad+(\Sigma X)^2\right]$$
$$+\frac{(s-1)^2}{s^2}\left[X_1^2+(X_1+X_2)^2+\ldots(\Sigma X)^2\right]+k\left[X_1^2+X_2^2+\quad\ldots+(F-\Sigma X)^2\right]$$

TABLE I.

C	X1	X2	X3	X4	X5
1½	0.235	0.086	0.068	0.151	0.460
2	0.368	0.105	0.054	0.105	0.368
3	0.528	0.112	0.034	0.061	0.265

On differentiating this with respect to X_1, X_2, etc., and forming the equations as before, it will be found that the equations take exactly the same form as in the last section if C be written for

$$\frac{2a_c}{a_p}+2\left(1-\frac{2}{s}+\frac{2}{s^2}\right)\ldots\ldots\ldots\ldots(8)$$

instead of for $\frac{2a_c}{a_p}+1$, as in equations (2).

Thus the effect of the non-equipartition of the load is equivalent to a change of C.

Suppose, for example, that $\frac{1}{s}=0.6$, the cover plates being of the correct thickness for which C would be equal to 2 if the load were equally divided. Then

$$C=2[0.5+1-1.2+0.72]$$
$$=2.04.$$

If $K=1$, the effect of this change of C may be seen from the annexed table. The load taken by the first rivet is increased about 2.44 per cent., and the load taken by the last rivet is decreased 1.36 per cent.

In practical cases, owing to want of straightness of the plates, etc., it will often happen that s is not the same throughout. This could be allowed for in a similar manner, but, of course, no general rule can be given. The effect is not likely to be great in any case, as may be seen from the above. It should, however, be allowed for in reducing experimental data to obtain exact values of K.

TABLE II.

C	X_1	X_2	X_3	X_4	X_5
2.00	0.368	0.105	0.054	0.105	0.368
2.04	0.377	0.106	0.052	0.102	0.363

(c) Effect of the Main Plate and Cover Plates Having Different Moduli of Elasticity.

It often happens that the cover plates, being thinner than the main plates, have a greater modulus of elasticity. Let E_c be the modulus of the cover plates, E_p that of the main plate, and let the ratio $\frac{E_c}{E_p}=r$ It may be shown from equation (1) that the effect of r being different from unity is to change C from $1+\frac{2a_c}{a_p}$ to

$$C=1+\frac{2a_c}{a_p}\cdot r \qquad \ldots \quad (9)$$

For example, if $E_c=31\times 10^6$ pounds per square inch and $E_r=29\times 10^6$ pounds per square inch, $r=\frac{31}{29}$, and, if the covers are each of half the thickness of the main plate,

$$C=2.069.$$

The effect is thus similar to that discussed in (b).

Thus factors (b) and (c) both tend to increase the load taken by the first rivet and decrease that taken by the last.

§4. *Joint in Which the Middle Plate is of Variable Width, as in the Connection of Members to a Gusset Plate.*

In this section an analysis will be made of a joint of the type shown in Fig. 6*a*, page 573, which consists of two similar plates of uniform width attached one on each side of a plate of variable width. In such a connection there will be no bending of the plates if the load carried by each of the outer plates is the same.

The analysis will be given for five rivets, but may readily be extended to any number. The same assumptions will be made as in §2.

Let a_c represent the cross-sectional areas of the outer plates, a_1, a_2, a_3 and a_4 the cross-sectional areas of the gusset plate mid-way between rivets 1 and 2, between rivets 2 and 3, etc., respectively Then, assuming uniform distribution of stress in the plates between each pair of rivets, the work stored in each outer plate between any pair of rivets is $\frac{P^2 l}{2a_c E}$, as in §2, but the work stored in the middle plate will take a more complicated form because of the variable width. Consider the portion of the middle plate between rivets 1 and 2. Then, if the load on this portion is P,

$$2EW = \Sigma \frac{P^2 dx}{A}$$

$$= \frac{P^2}{t} \int_o^l \frac{dx}{b_2 + \frac{b_1 - b_2}{2} . x} .$$

$$= \frac{P^2 l}{(b_1 - b_2)t} \cdot \log_e \frac{b_1}{b_2},$$

where b_1 and b_2 are the widths of the plate at rivet 1 and rivet 2 respectively and t is its thickness. If the plate were of uniform width, $\frac{b_1+b_2}{2}$,

$$2EW_1 = \frac{2P^2 l}{b_1 + b_2}$$

Thus the ratio

$$\eta_1 = \frac{W}{W_1} = \frac{b_1+b_2}{2(b_1-b_2)} \log_e \frac{b_1}{b_2} \ldots \qquad \ldots\ldots \quad (10)$$

Thus the work stored in the portions of the middle plate may be expressed as $\eta_1 \frac{P_1^2 l}{2a_1 E}$, $\eta_2 \frac{P_2^2 l}{2a_2 E}$, etc., where is a coefficient calculated from equation (10). On substituting these values in the work equation (equation (1), § 2), the first term becomes

$$l \left[\frac{\eta_1}{a_1} (F - X_1)^2 + \frac{\eta_2}{a_2} (F - X_1 - X_2)^2 + \qquad + \frac{\eta_4}{a_4} (F - \Sigma X)^2 \right]$$

the other terms remaining as before. The method of §2 then leads to the equations

$$\left|\begin{array}{cccc|c} 4+2a_c\sum_1^4\frac{\eta}{a}+2K & 3+2a_c\sum_2^4\frac{\eta}{a}+K & 2+2a_c\sum_3^4\frac{\eta}{a}+K & 1+2a_c\sum_4^4\frac{\eta}{a}+K & 2a_c\sum_1^4\frac{\eta}{a}+K \\ 3+2a_c\sum_2^4\frac{\eta}{a}+K & 3+2a_c\sum_2^4\frac{\eta}{a}+2K & 2+2a_c\sum_3^4\frac{\eta}{a}+K & 1+2a_c\sum_4^4\frac{\eta}{a}+K & 2a_c\sum_2^4\frac{\eta}{a}+K \\ 2+2a_c\sum_3^4\frac{\eta}{a}+K & 2+2a_c\sum_3^4\frac{\eta}{a}+K & 2+2a_c\sum_3^4\frac{\eta}{a}+2K & 1+2a_c\sum_4^4\frac{\eta}{a}+K & 2a_c\sum_3^4\frac{\eta}{a}+K \\ 1+2a_c\sum_4^4\frac{\eta}{a}+K & 1+2a_c\sum_4^4\frac{\eta}{a}+K & 1+2a_c\sum_4^4\frac{\eta}{a}+K & 1+2a_c\sum_4^4\frac{\eta}{a}+2K & 2a_c\sum_4^4\frac{\eta}{a}+K \end{array}\right|$$

where

$$\sum_1^4\frac{\eta}{a}=\frac{\eta_1}{a_1}+\frac{\eta_2}{a_2}+\frac{\eta_3}{a_3}+\frac{\eta_4}{a_4}$$

$$\sum_4^4\frac{\eta}{a}=\frac{\eta_4}{a_4}, \quad \text{etc.}$$

or writing

$$\alpha=4+2a_c\sum_1^4\frac{\eta}{a}+K,$$

$$\beta=3+2a_c\sum_2^4\frac{\eta}{a}+K,$$

$$\gamma=2+2a_c\sum_3^4\frac{\eta}{a}+K,$$

$$\delta=1+2a_c\sum_4^4\frac{\eta}{a}+K,$$

the equations become

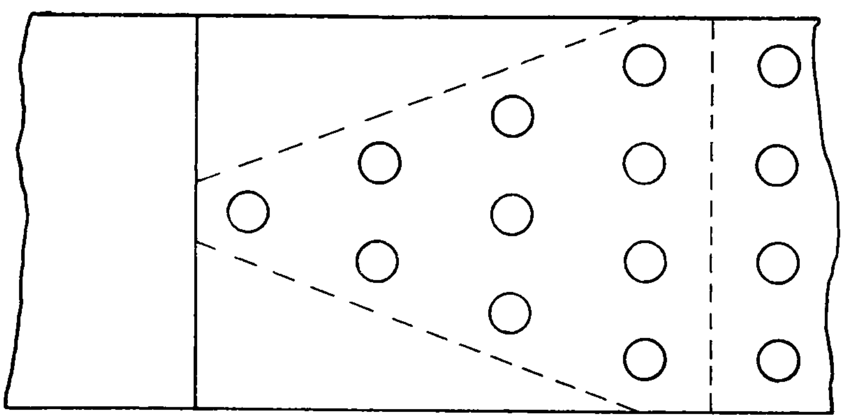
$$\left|\begin{array}{cccc|c} \alpha+K & \beta & \gamma & \delta & \alpha-4 \\ \beta & \beta+K & \gamma & \delta & \beta-3 \\ \gamma & \gamma & \gamma+K & \delta & \gamma-2 \\ \delta & \delta & \delta & \delta+K & \delta-1 \end{array}\right| \quad \ldots \ldots \quad (11)$$

These equations are similar in form to those obtained in §2 and the same method of solution may be used. An example is given in Part II, §5.

§5. *Splices with Various Groupings of Rivets.*

The joints considered in the former sections contained a single line of rivets only. When a large number of rivets is required for a connection, it is usual to group the rivets in several rows, a different number in each row, as, for example, in the splice shown in Fig. 5, which contains one rivet in the first row, two in the second, three in the third, etc. The distribution of the load between the various rivets of such a splice may be calculated by a similar method to that used in the cases considered above if the additional assumption is made that all the rivets in each row take the same load, which is probably nearly true in most

FIG. 5.



cases, although experimental data are not yet available on this point.

Let n_1 be the number of rivets in the first row, n_2 the number in the second row, etc., and, for simplicity, consider that there are only five rows on each side of the junction. Let X_1 be the total load taken by the first row of rivets, X_2 that taken by the second row, etc. Then each rivet of the first row will take the load $\frac{X_1}{n_1}$, each rivet of the second row $\frac{X_2}{n_2}$, etc. The work equation will be

$$2EW = \frac{l}{a_p}\left[(F-X_1)^2+(F-X_1-X_2)^2+\ \ldots\ +(F-\Sigma X)^2\right]$$
$$+\frac{2l}{a_c}\left[\left(\frac{X_1}{2}\right)^2+\left(\frac{X_1+X_2}{2}\right)^2+\ldots\ +\left(\frac{\Sigma X}{2}\right)^2\right]$$
$$+k\left[n_1\left(\frac{X_1}{n_1}\right)^2+n_2\left(\frac{X_2}{n_2}\right)^2+\ \ldots+n_5\left(\frac{F-\Sigma X}{n_5}\right)^2\right]\ldots\ldots\ldots(12)$$

This leads to the equations

$$\begin{vmatrix} 4C+K\left(\frac{1}{n_1}+\frac{1}{n_5}\right) & 3C+\frac{K}{n_5} & 2C+\frac{K}{n_5} & C+\frac{K}{n^5} & 4(C-1)+\frac{K}{n_5} \\ 3C+\frac{K}{n_5} & 3C+K\left(\frac{1}{n_2}+\frac{1}{n_5}\right) & 2C+\frac{K}{n_5} & C+\frac{K}{n_5} & 3(C-1)+\frac{K}{n_5} \\ 2C+\frac{K}{n_5} & 2C+\frac{K}{n_5} & 2C+K\left(\frac{1}{n_3}+\frac{1}{n_5}\right) & C+\frac{K}{n_5} & 2(C-1)+\frac{K}{n_5} \\ C+\frac{K}{n_5} & C+\frac{K}{n_5} & C+\frac{K}{n_5} & C+K\left(\frac{1}{n_4}+\frac{1}{n_5}\right) & (C-1)+\frac{K}{n_5} \end{vmatrix}$$

(13)

C and K having the same meanings as in §2. The method of solution is the same as for the equations of the preceding sections. Similar equations for any number of rows of rivets may be built up from a consideration of the above.

It will be interesting to consider one or two numerical examples in order to illustrate the method and to see how far the ordinary assumption of design—that each rivet carries an equal proportion of the load,—is justified. Take first the splice shown in Fig. 5, having four rows of rivets containing 1, 2, 3 and 4 rivets respectively. Let $C = 2, K = 1$ and $F = 1$. Equations (13) reduce to

$$\left| \begin{array}{c|c|c||c} 7.25 & 4.25 & 2.25 & 3.25 \\ 4.25 & 4.75 & 2.25 & 2.25 \\ 2.25 & 2.25 & 2.583 & 1.25 \end{array} \right|$$

TABLE III.

Specimen	Diameter of rivets	Thickness of middle plate	Thickness of each outer plate
A	½″	⅝″	5/16″
B	¾″	½″	¼″
C	⅞″	½″	¼″
D	¾″	¼″	⅜″
E	¾″	¼″	¼″
F and F′	¾″	¼″	¼″

These give

$$X_1 = 0.339,\ X_2 = 0.034,\ X_3 = 0.159,\ \text{and}\ X_4 = 0.468.$$

These are the total loads carried by each row. Thus

each rivet in the first row takes 0.339,
each rivet in the second row takes 0.017,
each rivet in the third row takes 0.053,
each rivet in the fourth row takes 0.117.

Thus more than one-third of the total force is carried by the first rivet, while the middle rows take very little load. The load carried by the first rivet is so great that it will probably fail. If this happens, or if the first rivet is removed, the values of X will become

$$X_2 = 0.416, \quad X_3 = 0.121, \quad X_4 = 0.463.$$

and

each rivet of the second row will take 0.208,
each rivet of the third row will take 0.040,
each rivet of the fourth row will take 0.116.

Thus the distribution is somewhat improved, but if the splice is in tension, the main plate is weakened by two rivet holes.

If the second row of rivets be removed,

$$X_3 = 0.441, \quad X_4 = 0.559.$$

Thus

each rivet in the third row takes 0.147,
each rivet in the fourth row takes 0.140,

and the distribution is much more uniform.

Usually in a splice of the type considered the cover plates are cut away as shown by the dotted lines in Fig. 5. This will alter the distribution of stress. Suppose, for example, that the widths of the cover plates at the first, second, third, and fourth rows of rivets are in the ratio 1 : 2 : 3 : 4. The equations will take the form

$$\left|\begin{array}{ccc||c} 3+a_c\sum_1^3\frac{\eta}{a}+K\left(\frac{1}{n_1}+\frac{1}{n_4}\right) & 2+a_c\sum_2^3\frac{\eta}{a}+\frac{K}{n_4} & 1+a_c\sum_3^3\frac{\eta}{a}+\frac{K}{n_4} & 3+\frac{K}{n_4} \\ 2+a_c\sum_2^3\frac{\eta}{a}+\frac{K}{n_4} & 2+a_c\sum_2^3\frac{\eta}{a}+K\left(\frac{1}{n_2}+\frac{1}{n_4}\right) & 1+a_c\sum_3^3\frac{\eta}{a}+\frac{K}{n_4} & 2+\frac{K}{n_4} \\ 1+a_c\sum_3^3\frac{\eta}{a}+\frac{K}{n_4} & 1+a_c\sum_3^3\frac{\eta}{a}+\frac{K}{n_4} & 1+a_c\sum_3^3\frac{\eta}{a}+K\left(\frac{1}{n_3}+\frac{1}{n_4}\right) & 1+\frac{K}{n_4} \end{array}\right| \tag{14}$$

where a_c is the area of cross-section of the cover plates when of the same width as the middle plate.

Substituting numerical values.

$$\eta_1 = \tfrac{3}{2}\log_\epsilon 2 = 1.0397,$$
$$\eta_2 = \tfrac{5}{2}\log_\epsilon 1.5 = 1.0136,$$
$$\eta_3 = \tfrac{7}{2}\log_\epsilon \tfrac{4}{3} = 1.007,$$

and the equations become

$$\left|\begin{array}{ccc||c} 9.795 & 5.022 & 2.401 & 3.25 \\ 5.022 & 5.522 & 2.401 & 2.25 \\ 2.401 & 2.401 & 2.734 & 1.25 \end{array}\right|$$

From these

$$X_1 = 0.225,\quad X_2 = 0.145,\quad X_3 = 0.132,\quad X_4 = 0.498.$$

Thus

each rivet in the first row takes 0.225,
each rivet in the second row takes 0.072,
each rivet in the third row takes 0.044,
each rivet in the fourth row takes 0.125.

Thus shaping the cover plate decreases the load taken by the first rivet. If the first rivet be omitted

$$X_2 = 0.338,\quad X_3 = 0.160,\ X_4 = 0.502$$

and

each rivet in the second row takes 0.169,
each rivet in the third row takes 0.053,
each rivet in the fourth row takes 0.125.

If the second row be also omitted

$$X_3 = 0.458,\quad X_4 = 0.542$$

and

each rivet of the third row takes 0.153,
each rivet of the third row takes 0.135.

These illustrations are sufficient to show how the method may be used to determine the partition of load in any form of splice. The problem of the best arrangements of rivets in splices will be deferred until further experiments have been made.

§6. *Joints Having Rivets of Different Sizes or for Which the Values of K are Different.*

Consider a joint having a single line of rivets, five on each side of the junction. Let the values of K for the rivets be K_1, K_2, K_3, K_4 and K_5 for the first, second, third, fourth, and fifth rivets respectively. Then the work equation will be similar to that given in §2, equation (1), but the last term will be

$$K_1X_1^2 + K_2X_2^2 + \quad \ldots \quad + K_5(F - \Sigma X)^2.$$

Thus the equations for this case will be

$$\left|\begin{array}{l|l|l|l||l} 4C+K_{15}+K_5 & 3C+K_5 & 2C+K_5 & C+K_5 & 4(C-1)+K_5 \\ 3C+K_5 & 3C+K_2+K_5 & 2C+K_5 & C+K_5 & 3(C-1)+K_5 \\ 2C+K_5 & 2C+K_5 & 2C+K_3+K_5 & C+K_5 & 2(C-1)+K_5 \\ C+K_5 & C+K_5 & C+K_5 & C+K_4+K_5 & (C-1)+K_5 \end{array}\right| \quad \ldots \quad (15)$$

§7. *Lap Joints.*

In lap joints the loads on the two plates are not in the same straight line. This causes bending of the plates, and the distribution of stress may be considerably modified by this action. The rivets are in single shear, and this will affect the value of K. In view of these factors and in the absence of experimental data, no attempt will be made in the present contribution to give a complete theory of such joints. If the bending of the plates is neglected, the work equation for a lap joint having a single line of rivets and connecting two similar plates will be

$$\begin{aligned} 2EW = & \frac{l}{a_p}\left[(F-X_1)^2+(F-X_1-X_2)^2+ \dots +(F-\Sigma X)^2\right] \\ & + \frac{l}{a_p}\left[X_1^2+(X_1+X_2)^2+ \dots +(\Sigma X)^2\right] \\ & + k\left[X_1^2+X_2^2+ \dots +(F-\Sigma X)^2\right] \end{aligned}$$

where a_p is the cross-sectional area of the plates.

Differentiating with respect to X_1, and equating the result to zero,

$$\begin{aligned} & -\frac{2l}{a_p}\left[(F-X_1)+(F-X_1-X_2)+ \dots +(F-\Sigma X)\right] \\ & + \frac{2l}{a_p}\left[X_1+(X_1+X_2)+ \dots +(\Sigma X)\right] \\ & + 2k\left[X_1-(F-\Sigma X)\right]=0 \end{aligned}$$

Thus, if $K=\frac{k \cdot a_p}{l}$, the equations are

$$\begin{vmatrix} (n-1)+2K & (n-2)+K & (n-3)+K & \dots & 1+K & (n-1)+K \\ (n-2)+K & (n-2)+2K & (n-3)+K & \dots & 1+K & (n-2)+K \\ (n-3)+K & (n-3)+K & (n-3)+2K & & 1+K & (n-3)+K \\ \dots & \dots & \dots & & \dots & \dots \\ 1+K & 1+K & 1+K & & 1+2K & 1+K \end{vmatrix} \quad (16)$$

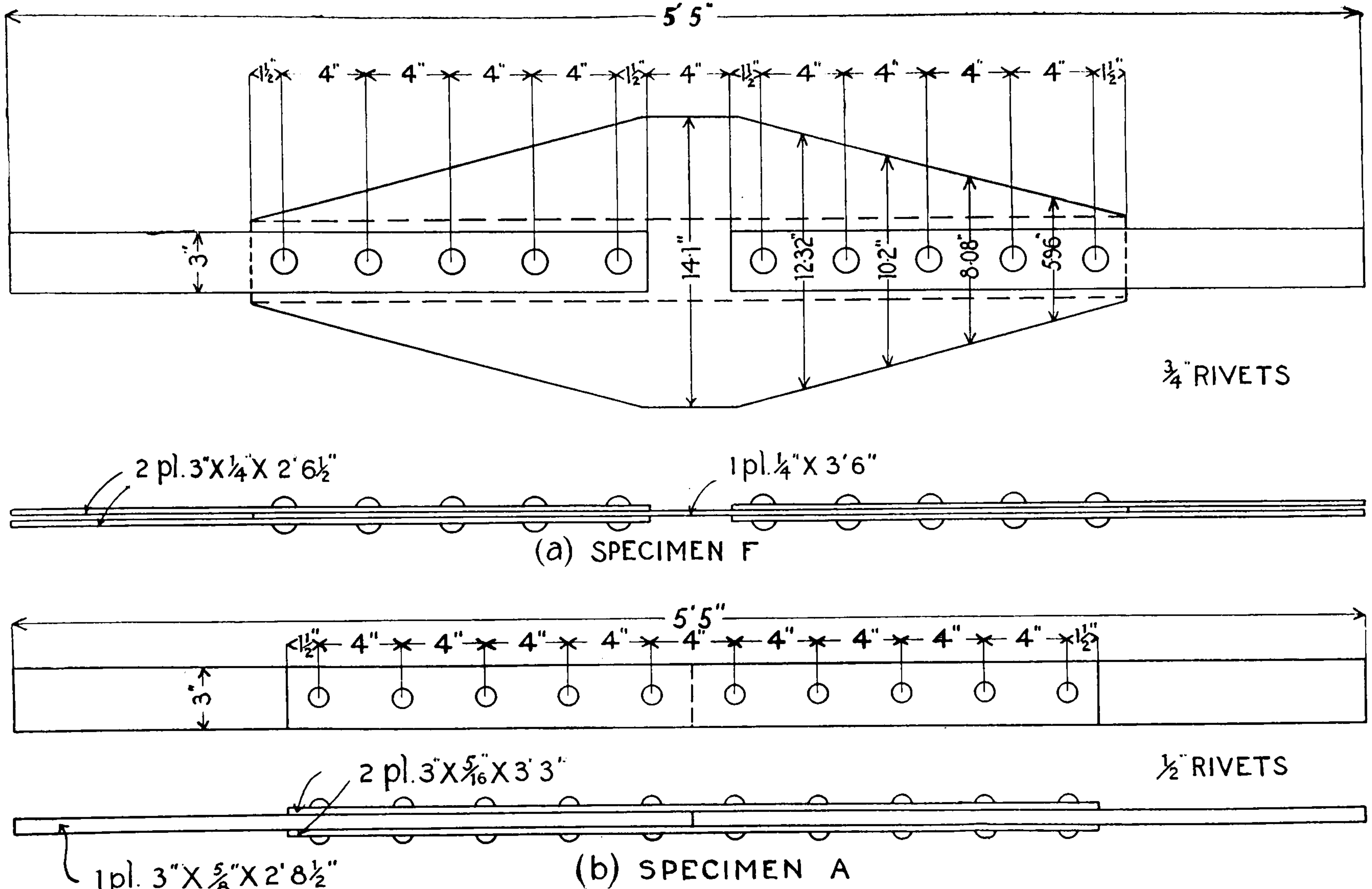
Thus the distribution of load is the same as for a butt joint having $C=2$, as considered in §2, if the value of K is the same. The above equations must be considered only as a first approximation.

PART II.—EXPERIMENTAL.

§1. *Specimens Used and Method of Experiment.*

The experiments described in the following pages were made with the object of determining the distribution of stress in the cover plates of a series of riveted butt joints having a single

FIG. 6.



Typical specimens.

line of rivets, and of thus deducing the load transferred from the main plate to the cover plates by each rivet. In order to establish the validity of the method, tests were also made on a specimen of the form shown in Fig. 6*a*, in which it was possible to measure the distribution of stress over a great part of the middle plate in addition to the distribution over the cover plates (see § 2). The butt joints were all of the form shown in Fig. 6*b*. The plates were of varying thicknesses and the rivets of various sizes, but the width of the plate was in every case three inches, the pitch of the rivets four inches, and the total number of rivets was ten; *i.e.*, five on each side of the junction. The annexed table shows the remaining dimensions and the designation of the specimens.

Specimen A was also tested with the first rivet removed, leaving four rivets on one side of the junction, and then with the fifth rivet removed, leaving three rivets.

Specimen F was also tested after the middle plate had been cut down to a uniform width of 4.09 inches, as shown by the dotted lines in Fig. 6*a*. This specimen will be designated by F^1

All the specimens were made by the Dominion Bridge Company, Montreal. The holes were drilled and the riveting was done by machine. Care was taken to keep the specimens as free from local bending as possible, but otherwise the joints were ordinary shop products, and as such were subject to minor irregularities in the position of the holes, etc.

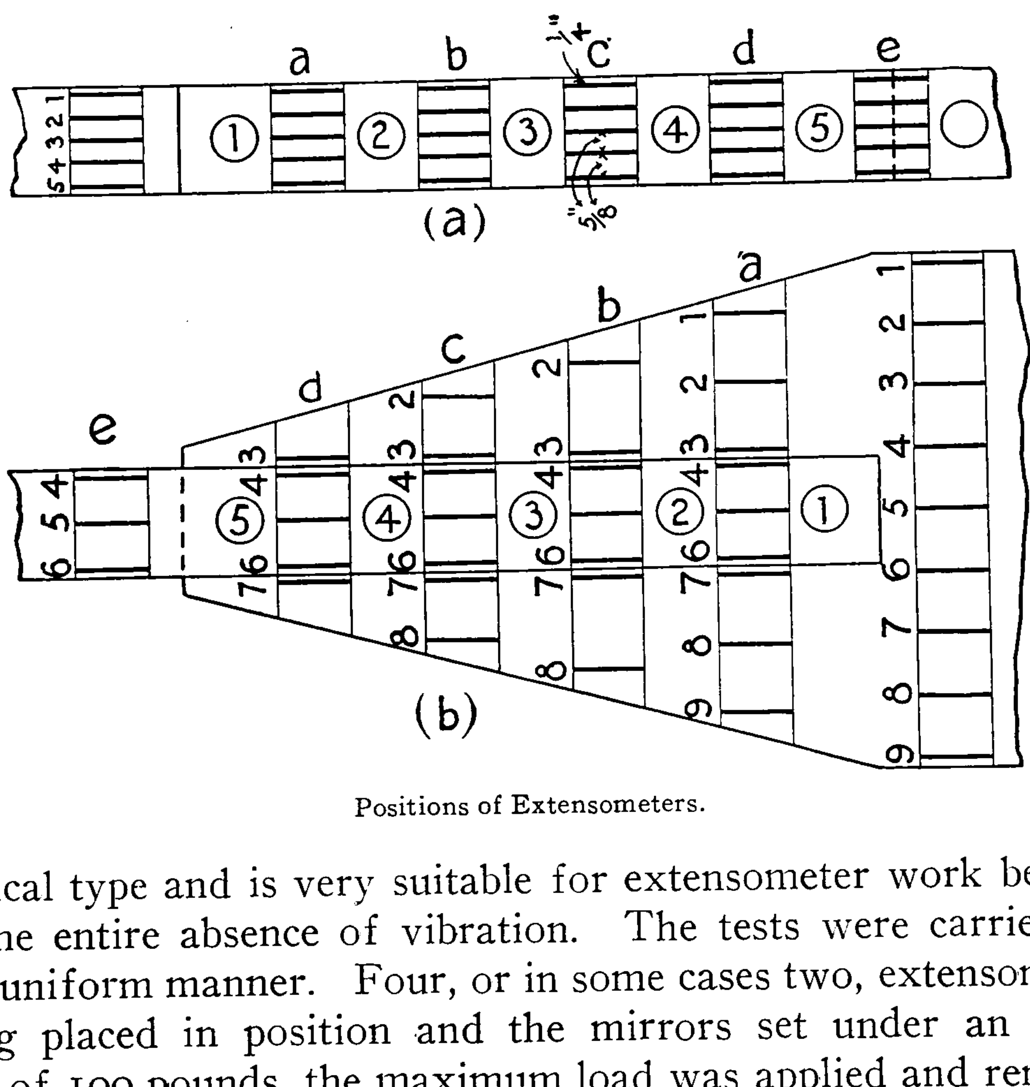
The strains were measured with the simplified form of the Marten's Mirror Extensometer developed in the McGill University Testing Laboratory. This instrument was described fully in a paper by the present writer which appeared in the JOURNAL OF THE FRANKLIN INSTITUTE, August, 1915.[11] The gauge length was 2 inches and the instrument read accurately to $\frac{1}{100{,}000}$ inch. All the instruments were carefully calibrated. The extensometers were set between each pair of rivets in positions such as are indicated in Fig. 7, their length being parallel to the axis of load. In specimens A and C, readings in five positions were taken between each pair of rivets; in B, D, E and F, in three positions only. In every case the instruments were read with the

[11] C. Batho, "The Effect of the End Connections on the Distribution of Stress in Certain Tension Members," J. F I., August, 1915, p. 129.

knife-edge at each end in turn and the mean of these readings taken in order to eliminate errors due to bending. The strains measured in this way may be regarded as proportional to the stresses, since the strains perpendicular to the axis of load are so small as to be negligible. Readings taken on the middle plate below the joint and on the cover plates at the central section enabled the value of the modulus of elasticity for each plate to be determined.

The specimens were loaded in the 150,000-pound Emery testing machine at McGill University This machine is of the

FIG. 7.



Positions of Extensometers.

vertical type and is very suitable for extensometer work because of the entire absence of vibration. The tests were carried out in a uniform manner. Four, or in some cases two, extensometers being placed in position and the mirrors set under an initial load of 100 pounds, the maximum load was applied and removed several times, the mirrors reset if necessary, and the load was then run up gradually, readings being taken at the required loads. The load was then reduced to its initial value. In nearly all cases the extensometer readings returned to zero. If not, the process was repeated until they returned satisfactorily. This latter precaution was, however, seldom necessary

It will be seen that the success of the method depends upon the distribution of stress remaining the same after many loadings. This point was very carefully tested by repeating readings at intervals. No differences were found that were not within the range of experimental error; *i.e.*, the readings checked to $\frac{1}{100,000}$ inch. An attempt was made in specimen A to obtain readings on the first loading of the piece. This was found rather difficult as the ends of the specimen always slip a little in the grips when the load is applied for the first time, disturbing the extensometers, so that the loading has to be carried out in stages, returning to the initial load and resetting the extensometers every time such a motion occurs. The results appeared to show some minor differences between the distribution of stress on the first and on subsequent loadings, but these may have been due to experimental errors. A closer examination of this matter would be interesting, but outside the scope of the present investigation, which deals with the distribution of stress in joints when a stable condition has been reached, and, as stated above, this distribution remains exactly the same, no matter how many times the piece has been loaded.

The specimens were loaded in tension. Experiments in compression would be more difficult because of the tendency for bending to occur, but would be necessary in order to determine the value of K for joints in compression.

§2. *Test of the Validity of the Method.*

The object of the experiments, as stated above, was to obtain the proportion of the load transmitted from the middle plate to the cover plates by each rivet. Since extensometer measurements could be taken only on the outer surfaces of the cover plates, it will be seen that the validity of the method depends upon whether or no the strains in the cover plates deduced from these measurements were a true estimate of the mean strains in the plates. If, for example, the plates were held together mainly by friction between the plates, the stresses at the inner surfaces of the plates would probably be much greater than at the outer. In order to obtain information on this point, experiments were made on specimen F, in which the middle plate was much wider than the outer plates, so that measurements could be taken both on the cover plates and on the middle plate. The extensometer positions are shown in Fig. 7, and Table IV gives the sum of the four extensometer readings on the two sides of the specimen at each position, one-hundred-thousandth of an inch being taken

TABLE IV.

Section	Sum of the extensometer readings on the two faces of the specimen								
	1	2	3	4	5	6	7	8	9
Central section of middle plate	76	100	113	167	179	162	130	101	70
a	103	110	112	76	67	77	126	118	107
b		122	107	110	90	109	117	111	
c		119	121	140	114	142	124	126	
d			123	162	150	166	128		
e				279	281	281			

as unit, for a tensile load of 16,000 pounds. The extensions were measured over a length of 2 inches. Thus, since the readings are correct to $\frac{1}{100,000}$ inch, an error of about 150 pounds, *i.e.*, about 1 per cent. of the load, is possible in the estimate of the stress. The error of the sum is probably much less than this.

TABLE V.

$E = 28.4 \times 10^6$ pounds per square inch for middle plate,
$= 30.4 \times 10^6$ pounds per square inch for outer plates.

Section	Mean strain in middle plate $\times 4 \left(\frac{1}{100,000}\right)''$	Mean strain in outer plate $\times 4 \left(\frac{1}{100,000}\right)''$	Load carried by middle plate Lbs.	Load carried by outer plates Lbs.	Total load from extensometer reading Lbs.	Error
						Per cent.
Central	127.7	....	16000	....	16000	.....
a	112.7	71.7	12320	4100	16420	+2.62
b	114.2	99.7	10340	5690	16030	+0.19
c	122.5	127.5	8800	7260	16060	+0.38
d	125.5	157.0	6650	8940	15590	−2.56
e	...	280.5		16000	16000	...

The values of E for the plates were determined from the readings at the central section for the middle plate, and at section *e*, below the rivets, for the outer plates. The results are given at the head of Table V.

Columns 1 and 2 in Table V give the mean strains at each section determined from the figures in Table IV To obtain the mean strains in the outer plates, twice the middle reading was added to the two outer readings and the result divided by four. Thus at section a the mean strain

$$71.7 = \frac{76+2\times67+77.}{4}$$

The reason for this will be explained in §3. The mean strains in the inner plate were obtained by taking the average of the figures given in Table IV The loads carried by the outer and inner plates respectively were calculated by using the values of E obtained as explained above and are given in columns 3 and 4 of Table V

At each section the sum of these, if the estimated strains are the mean strains in the plates, should be 16,000 pounds, the total load. The actual sums are given in column 5, and it will be noticed how close they are to this value. The percentage errors are given in column 6. They are quite negligible at sections b and c and only amount to about 2.6 per cent. at sections a and d. Thus the validity of the method may be regarded as fully established. The agreement of the results will be seen to be remarkably close when it is considered that the experimental error may amount to about 1 per cent. and that readings could not be taken over the parts of the middle plates covered by the outer plates.

It may occur to the reader that there is also an error due to the varying width of the middle plate not being taken into account in reducing the strains to stresses. This error, however, was estimated and found to be too small to be considered.

To sum up, the above experiment shows that extensometer measurements on the outside of the outer plates of a riveted joint are sufficient for the determination of the proportion of the load carried by each rivet, since they give accurate values for the mean strains in the plates. It also appears to prove that friction between the plates cannot play any part in transferring the load, except possibly in the parts of the plates close to the rivets.

§3. *Experimental Results for Specimens A, B and C in Which the Thickness of Each of the Cover Plates was One-half the Thickness of the Middle Plate.*

The results of the tests on specimens A, B, and C are given in Table VI, together with their reduction to find the load taken by each rivet. It has not been considered necessary to tabulate each individual extensometer reading. The figures in Table VI are the sums of the four readings at the corresponding positions on the two cover plates, the sum being given, for convenience, instead of the mean. Readings were also taken on the middle plates below the joint for the purpose of determining E for these plates, but are not tabulated.

The earlier experiments were made on specimens B, D, E and F, and in these readings were taken in three positions only at each cross-section, corresponding to 1, 3 and 5, Fig. 7. The results (see Table VI) showed clearly that the distribution of strain across the width of the plates was not uniform, the readings at the central position being always less than at the outer positions. It was therefore thought to be advisable to take readings at the intermediate positions, 2 and 4, Fig. 7, in order to study more closely the actual distribution of strain. This was done in specimens A and C.

The results for these specimens given in Table VI show that the distribution of strain is of substantially the same character at all loads and for each specimen. The strain is always least at the centre, rising to a maximum at each outer boundary. The readings for specimen A at the cross-section, between the fourth and fifth rivets, where the variations are most marked, are plotted in Fig. 8. It will be noticed that the shape of the curves is the same at all loads. The ratio of the mean of the readings at 1 and 5 to the reading at 3 has the values

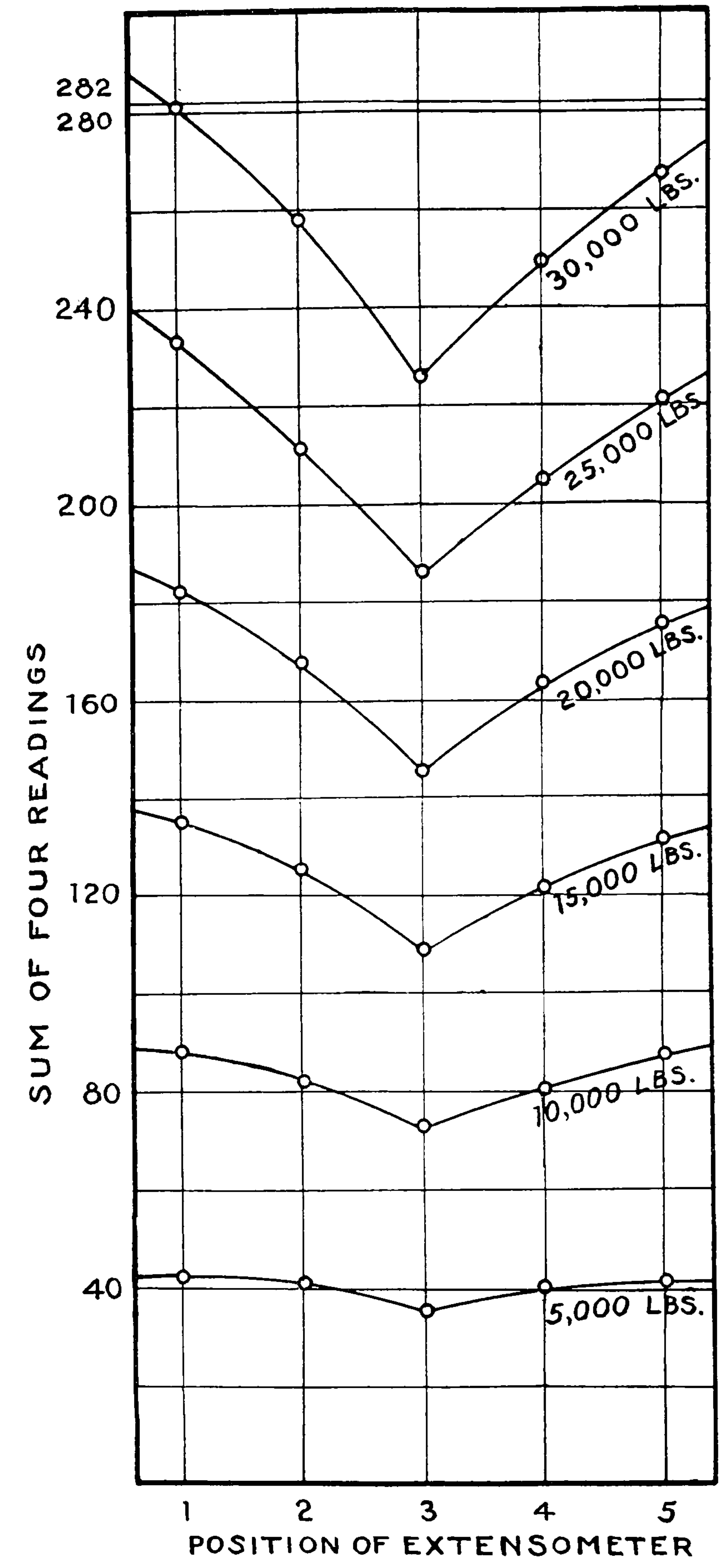
1.15, 1.20, 1.22, 1.23, 1.22, 1.21

at the loads

5000, 10000, 15000, 20000, 25000, 30000 pounds

respectively. It thus remains practically constant for all loads from 10,000 pounds to 30,000 pounds. It is somewhat less at the 5,000 pound load, but the readings for this load are too small to be relied upon. Curves drawn for the other sections give similar results. Now the maximum load on this specimen, 30,000 pounds, corresponds to an average stress of about 15,280 pounds per square inch of rivet, well above the working stress. Thus, at any rate when the joint has come to a stable condition

FIG. 8.



Distribution of strain at different loads between the fourth and fifth rivets of specimen A.

TABLE VI

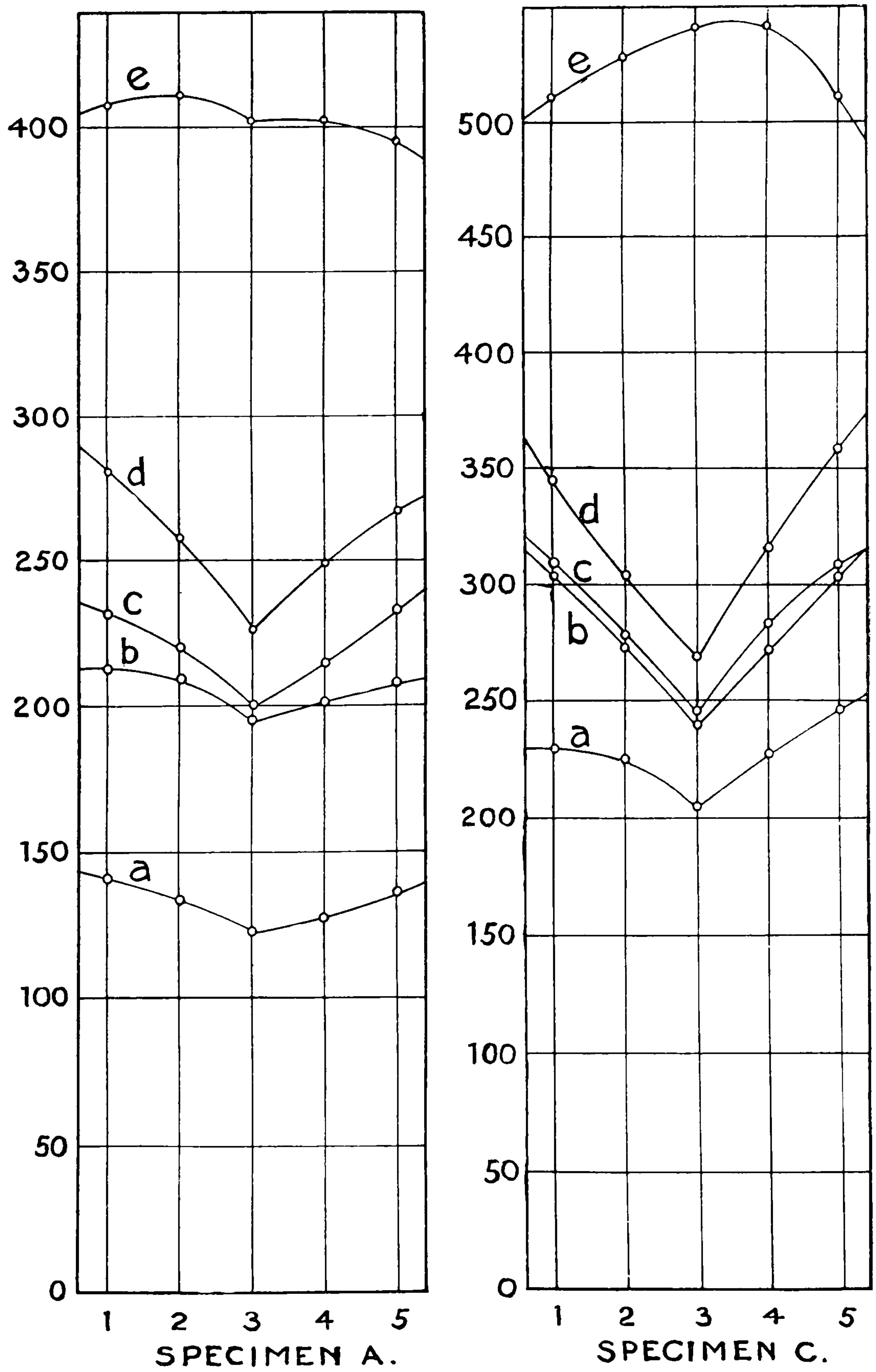
1	2																									3					4					5	6
Load	Sum of the four extensometer readings in corresponding positions on the two faces of the specimen																									Mean of the readings at each section (obtained as described on page 584 ~~587~~)					Percentage of the total load taken by each rivet					Mean of 1 and 5.	K
	a					b					c					d					e																
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	a	b	c	d	e	1	2	3	4	5		
A	E for middle plates = 30.4 × 10⁶ lbs. per sq. in., E for cover plates = 31.7 × 10⁶ lbs. per sq. in.																																				
5000	29	28	28	28	31	36	36	33	35	36	37	36	34	36	38	42	41	36	40	41	66	68	67	63	61	28	35	36	40	65	43.2	10.8	1.5	6.2	38.3	40.7	....
10000	60	60	55	57	61	73	70	65	69	73	75	72	66	70	75	88	82	73	80	87	134	137	134	135	126	58	69	70	80	133	43.6	8.3	0.8	7.5	39.9	41.7	0.485
15000	85	83	77	80	85	108	107	102	114	111	113	109	99	108	115	135	125	109	121	131	200	202	200	196	192	81	108	108	122	198	40.9	13.6	0.0	7.1	38.4	39.6	0.673
20000	105	102	95	100	103	147	142	133	142	145	151	148	131	145	154	182	168	145	163	175	272	274	266	263	258	100	141	147	164	267	37.5	15.4	2.2	6.4	38.6	38.1	0.837
25000	124	120	112	113	116	181	175	162	170	176	192	185	163	180	194	233	211	186	205	221	343	344	336	337	329	116	172	181	207	338	34.4	16.5	2.7	7.7	38.8	36.6	1.027
30000	141	134	123	127	136	214	209	195	201	208	232	222	200	214	233	281	258	226	249	267	408	411	402	402	395	131	204	217	252	404	32.4	18.0	3.2	8.7	37.6	35.0	1.300
A with 5th rivet removed.																																					
5000	36	32	32	32	35	40	37	36	35	38	45	39	37	39	43	...	...	...	...	...	...	...	...	...	...	33	36	40	..	...	50.8	4.6	6.1	38.5	....	44.6	0.273
10000	62	60	59	59	62	78	71	67	69	73	97	80	73	78	94	...	...	...	...	...	...	...	...	...	...	60	70	81	..	...	45.0	7.5	8.3	39.1	....	42.0	0.475
15000	89	83	83	86	94	115	105	104	105	111	141	122	111	120	138	...	...	...	...	...	...	...	...	...	...	86	106	122	..	...	43.5	10.1	8.1	38.4	....	40.9	0.573
20000	109	104	101	104	109	151	142	137	140	149	191	166	155	163	184	...	...	...	...	...	...	...	...	...	...	104	142	167	...	...	39.0	14.2	9.3	37.5	....	38.2	0.900
25000	126	119	120	125	129	199	177	172	174	184	241	209	195	205	235	...	...	...	...	...	...	...	...	...	...	123	177	210	...	...	36.4	16.0	9.8	37.9	....	37.1	1.066
30000	143	135	132	143	153	228	212	208	211	220	294	257	239	252	281	...	...	...	...	...	...	...	...	...	...	140	213	257	...	...	34.6	18.0	10.9	36.4	....	35.5	1.380
A with 1st and 5th rivets removed.																																					
5000	...	...	...	...	...	37	34	34	35	35	45	39	36	39	42	...	...	...	...	...	...	...	...	...	...	..	35	39	...	...	...	53.9	6.1	40.0	....	47.0	0.150
10000	...	...	...	...	...	64	62	62	66	65	96	83	69	79	90	...	...	...	...	...	...	...	...	...	...	...	63	81	...	...	....	47.4	13.5	39.1	....	43.3	0.450
15000	...	...	...	...	...	95	92	91	92	92	139	127	105	120	135	...	...	...	...	...	...	...	...	...	...	...	92	122	...	...	....	46.5	15.1	38.4	....	42.5	0.545
20000	...	...	...	...	...	116	112	112	119	119	184	168	143	166	177	...	...	...	...	...	...	...	...	...	...	..	115	165	..	..	....	43.1	18.7	38.2	....	40.7	0.825
25000	...	...	...	...	...	143	141	137	143	140	238	211	183	208	221	...	...	...	...	...	...	...	...	...	...	...	141	208	..	..	....	41.8	19.8	38.5	....	40.1	0.975
30000	...	...	...	...	...	164	160	165	169	164	286	259	229	251	269	...	...	...	...	...	...	...	...	...	...	...	164	254	...	...	....	40.6	22.3	37.2	....	38.9	1.320
C	E for middle plate = 29.4 × 10⁶ lbs. per sq. in., E for cover-plates = 30.4 × 10⁶ lbs. per sq. in.																																				
16000	135	129	119	131	148	161	146	127	147	161	161	149	132	152	161	175	155	135	156	182	264	282	299	292	265	130	146	149	156	280	46.5	5.7	1.1	2.5	44.3	45.4	0.225
20000	167	160	148	163	180	204	180	160	182	203	202	186	164	190	204	220	197	168	199	230	335	347	371	358	340	161	181	187	198	350	46.0	5.7	1.7	3.1	43.5	44.7	0.269
25000	199	192	180	201	215	253	227	200	229	254	255	232	205	235	259	278	246	215	250	288	420	439	457	450	423	195	227	232	248	438	44.5	7.3	1.1	3.6	43.5	44.0	0.314
30000	229	224	204	227	246	304	273	239	272	304	309	278	245	284	308	345	304	269	316	359	511	529	542	542	511	223	272	280	310	527	42.4	9.3	1.5	5.7	41.2	41.8	0.477
35000	258	249	235	261	280	354	319	277	319	356	360	326	285	335	365	406	361	320	368	425	595	617	629	635	605	254	318	328	365	616	41.2	10.4	1.6	6.0	40.7	41.0	0.545
B	E for middle plate = 30.1 × 10⁶ lbs. per sq. in., E for cover plates = 31.3 × 10⁶ lbs. per sq. in.																																				
16000	142	...	128	...	146	151	...	136	...	151	161	...	139	...	163	188	...	147	...	193	271	...	277	...	280	136	143	150	169	276	49.3	2.5	2.5	6.9	38.8	44.1	0.306
20000	173	...	153	...	179	199	...	169	...	200	199	...	167	...	197	234	...	184	...	246	343	...	347	...	345	164	184	182	212	345	47.6	5.8	−0.6	8.7	38.6	43.1	0.369
22000	189	...	169	...	194	217	...	182	...	222	219	...	182	...	219	265	...	204	...	276	378	...	382	...	379	180	201	200	237	380	47.5	5.5	−0.3	9.7	37.6	42.5	0.423
27000	201	...	205	...	247	244	...	220	...	279	257	...	223	...	280	322	...	250	...	337	462	...	468	...	464	214	241	246	290	465	46.0	5.8	1.1	9.4	37.7	41.8	0.477
30000	221	...	224	...	276	269	...	243	...	313	287	...	253	...	310	365	...	277	...	380	514	...	522	...	519	236	267	276	325	518	45.6	5.9	1.7	9.5	37.3	41.5	0.500
33000	241	...	244	...	295	299	...	269	...	344	320	...	273	...	344	406	...	307	...	425	563	...	573	...	572	256	295	302	361	569	45.0	6.9	1.2	10.3	36.6	40.8	0.564

after a few loadings, the manner of distribution of stress in the plates remains the same at all loads. This does not mean that the rivets carry the same proportions of the total load, but that there is no marked change in the way in which the load is transferred to the plates by the rivets.

It would require too much space to give all the curves for specimens A and C, and it is, fortunately, unnecessary, since they are, as shown above, similar for the same sections at different loads. Fig. 9 shows the readings at all the sections of the cover plates in specimens A and C for a load of 30,000 pounds. It will be remembered that the rivets in specimen A are ½ inch diameter and in specimen C are ⅞ inch diameter, the thickness of the cover plates being ¼ inch in specimen A and 5/16 inch in specimen C. In order to make direct comparison possible the ordinates of the curves for specimen A have been drawn to 5/4 the scale of the ordinates for specimen C, thus allowing for the difference in thickness of the plates. The curve at the central section, *c*, is higher in specimen C than in A, indicating that the value of E is lower in specimen C than in A. The actual values are 31.7×10^6 pounds per square inch for A and 30.4×10^6 pounds per square inch for C. Thus the ordinates do not represent the *stresses* in the two specimens to quite the same scale. The curves for sections *a, b, c* and *d* are very similar in form in the two specimens, although the load, 30,000 pounds, corresponds to an average load of about 15,280 pounds per square inch of nominal rivet section in A and of only ~~15,280~~ 5000 pounds per square inch in C, showing that the manner of distribution of stress at a cross-section is substantially the same not only at different loads, as shown above, but also in different specimens. All the experimental results show this in an equally striking manner. The curves are not quite symmetrical about the centre line. This is probably due to slight irregularities in the construction of the specimens. Most of the curves seem to show a discontinuity at the centre, position 3. Readings taken close to the centre might show that the curve is really continuous instead of coming to a sharp point.

The curves referred to above represent the sum of the readings in corresponding positions on the two cover plates and thus give (as shown in §2) the mean strain in the cover plates multiplied

FIG. 9.



Distribution of strains in specimens A and C at a load of 30,000 lbs.

by four. Fig. 10 shows, to the same scale as Fig. 9, the readings on the two cover plates for specimen A at 30,000 pounds load. Owing to bending in the plates, and slight irregularities in the placing and action of the rivets, the strain in one cover plate is at each section higher than in the other, the ratio varying somewhat at different sections. This has a slight effect upon the partition of load between the rivets, as was proved in Part I, § 3. The difference in the value of E for the cover plates and the middle plates has a similar effect.

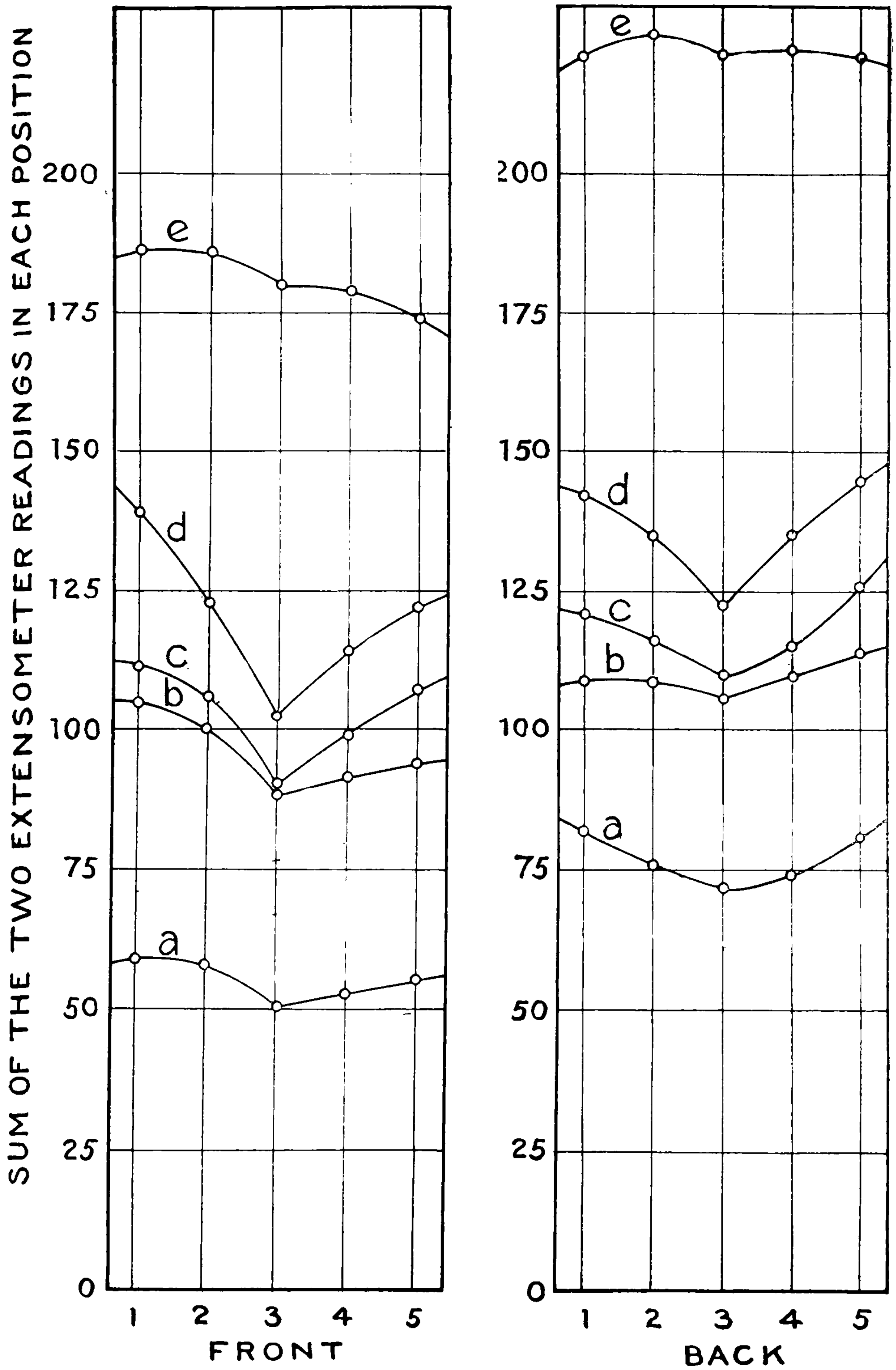
The curves given in Fig. 9 show that the ratio of maximum to minimum stress, or strain, across the section is least between the first and second rivets and increases to a maximum between the fourth and fifth rivets. This fact and the general form of the curves lend strong support to the view that the rivets are acting in shear, because, if the plates were, in the neighborhood of the rivets, held together by friction, it would seem that the pull transferred by each rivet would give a maximum stress at the centre line and a minimum at the outer edges. Also, it is very unlikely that in such a case the strains on the outer faces of the cover plates would give true values for the mean strains in the plates.

The area included between the curve for section c, the ordinates at its ends, and the horizontal axis is a measure, to some scale, of the total load on the specimen. The areas under curves a, b, c and d represent to the same scale the loads at the corresponding sections of the cover plates. Thus the area under a represents the load transferred to the cover plates by the first rivet, the area between a and b the load transferred by the second rivet, etc. It is at once evident that the first and last rivets transfer the major portion of the load and that the middle rivet transfers practically nothing. This is in accordance with the theory given in Part I. In order to obtain the exact loads taken by each rivet, all that is necessary is to obtain the mean heights of the curves. This has been done, and the results are given in Table VI, column 3. It was found that the mean corresponding to the curves could best be determined by using Simpson's rule on each side of the centre. Thus, for example, the mean for specimen A at section d for a load of 30,000 pounds is given by

$$\frac{281+4\times258+2\times226+4\times249+267}{12}=252,$$

the result being taken to the nearest integer.

FIG. 10.



Extensometer measurements on the two cover plates of A, at a load of 30,000 lbs.

For specimens B, D, E and F, since three readings only were taken, the mean was considered as the sum of the readings at 1 and 5, together with twice the reading at 3, the whole being divided by 4. Applied to the section of specimen A considered above, this would give 250 instead of 252, a difference of about 0.8 per cent. Thus the error from using three readings only is unimportant.

The mean strains are proportional to the loads in the cover plates at the different sections, and the proportion of the load taken by each rivet may readily be found from these. For example, for specimen A at 30,000 pounds load, the mean at section c is 404, and at section a is 131. Thus the first rivet takes

$$\frac{131}{404} \times 100 = 32.4 \text{ per cent.,}$$

the second rivet takes

$$\frac{204-131}{404} \times 100 = 18 \text{ per cent.,}$$

of the total load, etc.

Column 4, Table VI, gives the percentage of the total load taken by each rivet for the specimens A, B and C at different loads, and also for specimen A with the fifth rivet removed and with the first and fifth rivets removed. It will be at once apparent that the results are in general agreement with the theory given in Part I, the end rivets taking by far the greater portion of the load, the second and fourth rivets much less, and the centre rivet practically none.

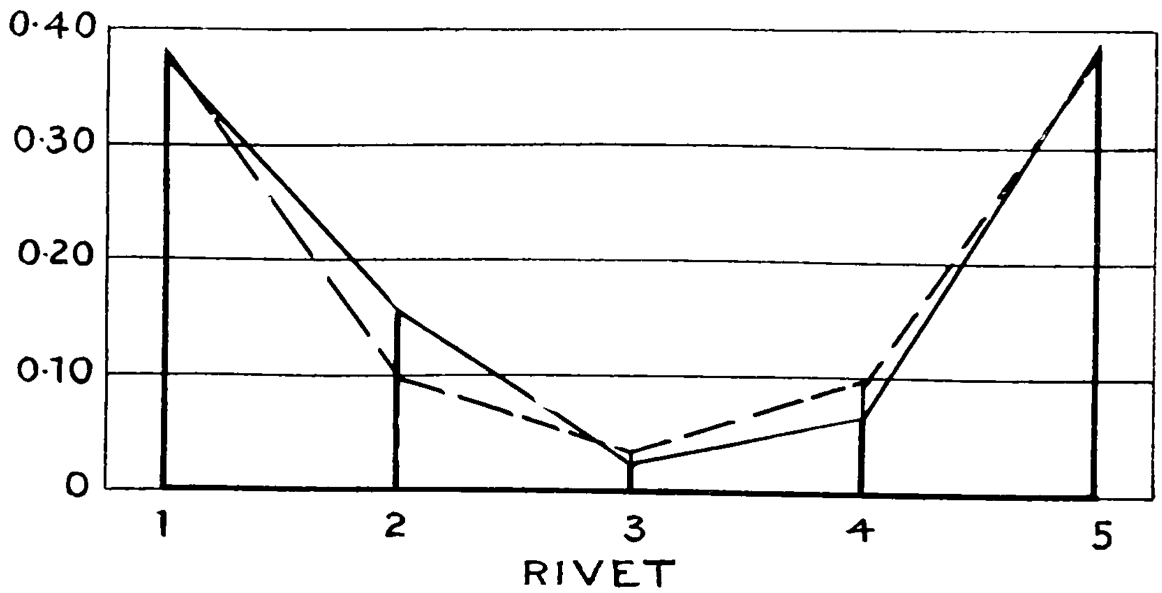
Column 5 gives the mean of the percentages of the load taken by the two end rivets, and it will be noticed that in every case this decreases as the load increases, which means that the value of K increases with the load. This point will be discussed in the next section. The general distribution is, however, similar at different loads, and Figs. 11 to 15 show typical results graphically

Figs. 11, 12 and 13 show the partition of the load between the rivets for specimen A at a load of 20,000 pounds with five, four, and three rivets respectively, and Figs. 14 and 15 show the same for specimens B and C respectively at a load of 30,000 pounds.

The heavy ordinates represent the proportions of the total load taken by each rivet. The tops are joined by lines in order to display the results more clearly These lines do not, of course, mean that interpolations may be made for a different number of rivets.

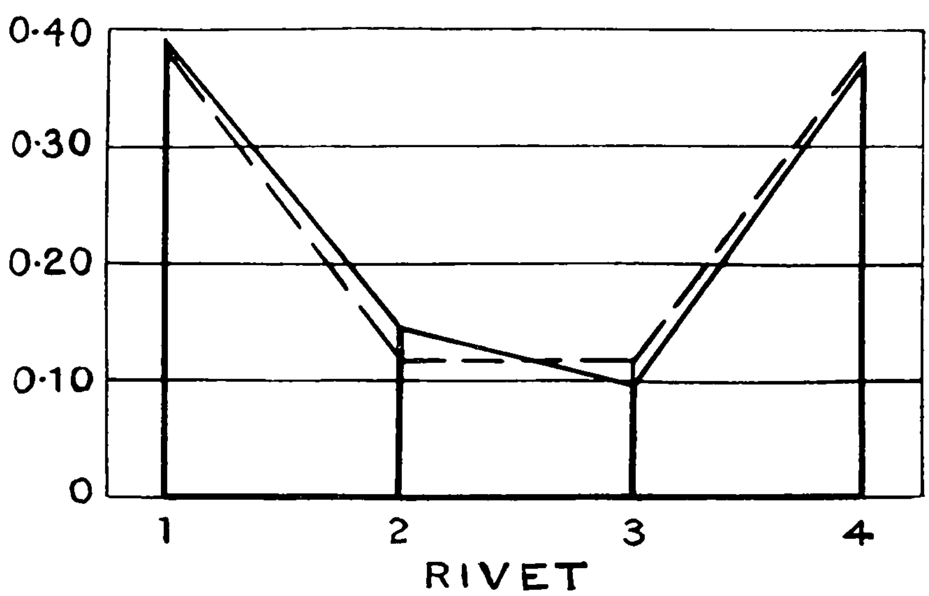
The distribution is not quite symmetrical in any of the

FIG. 11.



Specimen A. Load 20,000 lbs.

FIG. 12.

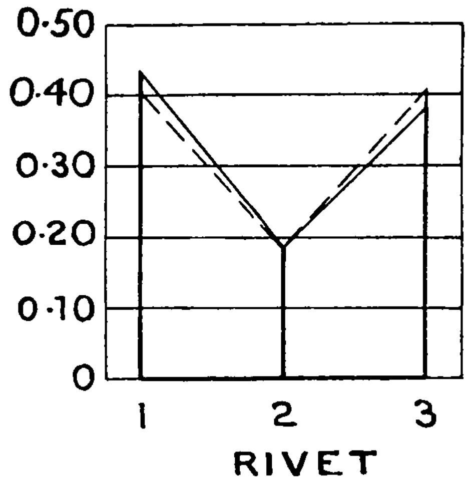


Specimen A (4 rivets). Load 20,000 lbs.

specimens, the last rivet, except for the three highest loads on specimen A and the two highest on the same specimen with the fifth rivet removed, taking a little less of the load than the first. This is, probably, mainly due to the causes discussed in Part I,

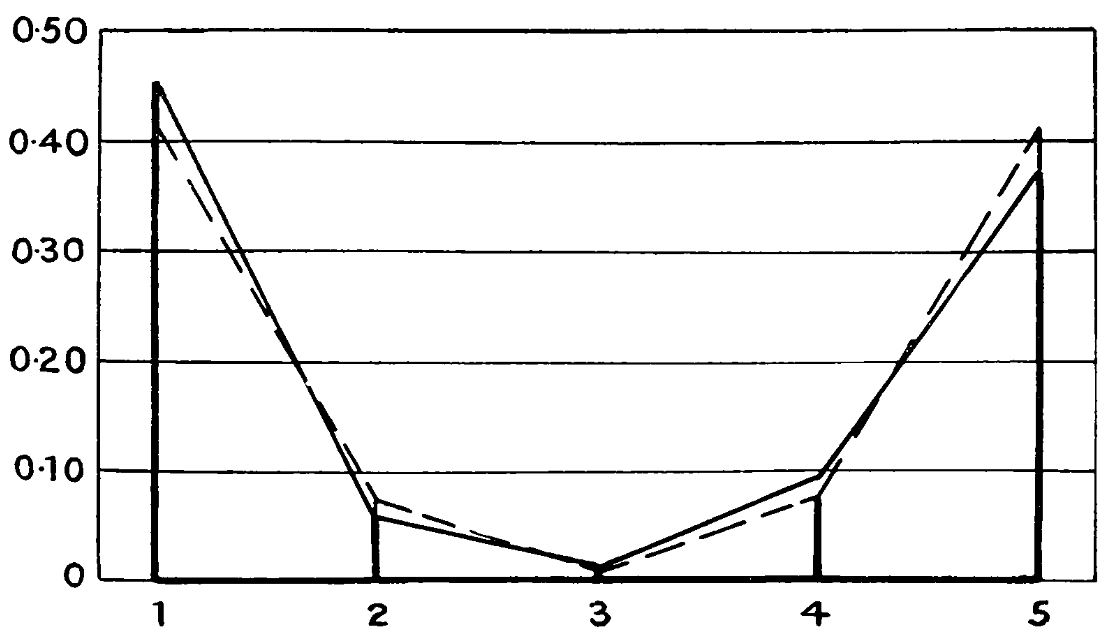
§ 3, *i.e.*, the difference in the values of E for the main plate and the cover plates and the unequal loads taken by the two cover plates: but want of straightness and minor irregularities in the specimens may also play some part. The difference is most

FIG. 13.



Specimen A (3 rivets). Load of 20,000 lbs.

FIG. 14.

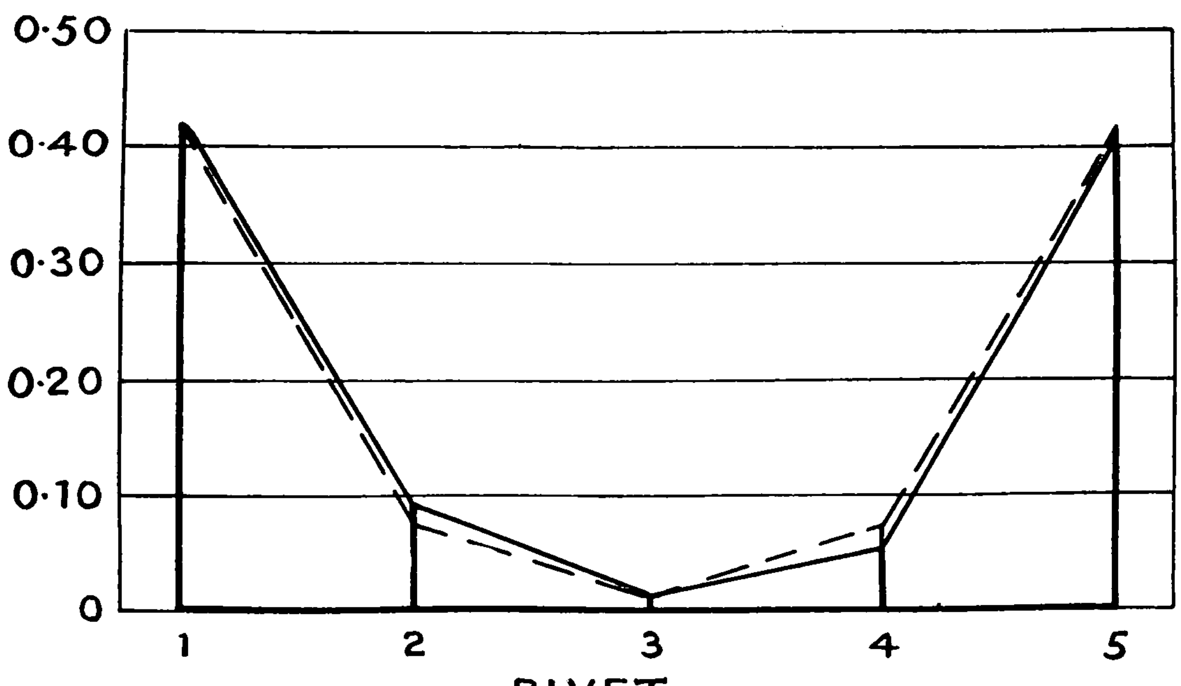


Specimen B. Load 30,000 lbs.

marked in specimen B, and practically vanishes in specimen C. In order to determine the value of K best fitting the experimental results, the mean of the loads taken by the first and last rivets was used, and from this K was determined, as described in Part I, by the use of Fig. 2, Part I. The values thus found are given in the last column of Table VI.

The dotted lines in Figs. 11 to 15 show the theoretical percentages of the load taken by each rivet, obtained by using these values of K. The agreement between the theoretical and experimental results is very striking, especially if it be remembered that the specimens were by no means ideal, but ordinary shop products. The experimental results for the rivets 2, 3 and 4 are somewhat irregular. This may be partly owing to experimental errors, since the results depend upon the comparatively small differences between the means of the readings at consecutive cross-sections of the plates, but it may also arise from taking the value of K as constant for all rivets. In any case, these rivets take such small loads that the differences between the theoretical and experimental results may be regarded as unimportant.

FIG. 15.

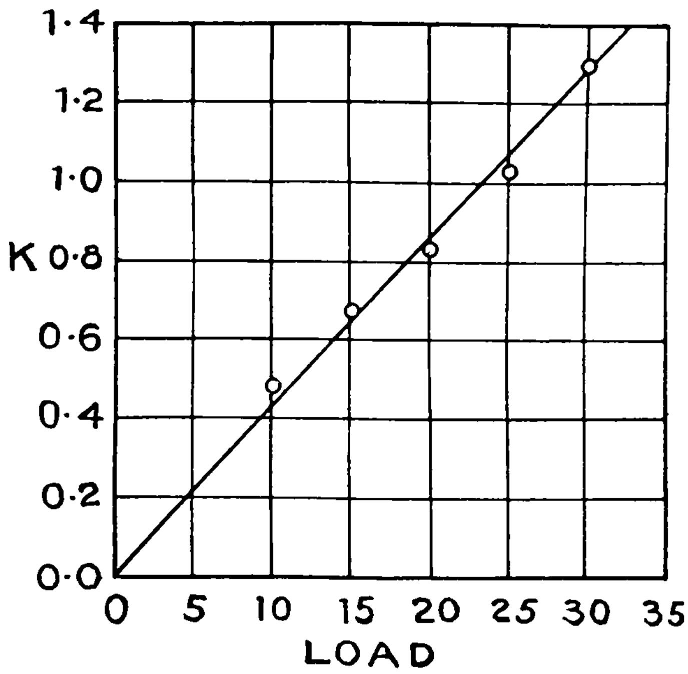


Specimen C. Load 30,000 lbs.

§ 4. *The Value of K.*

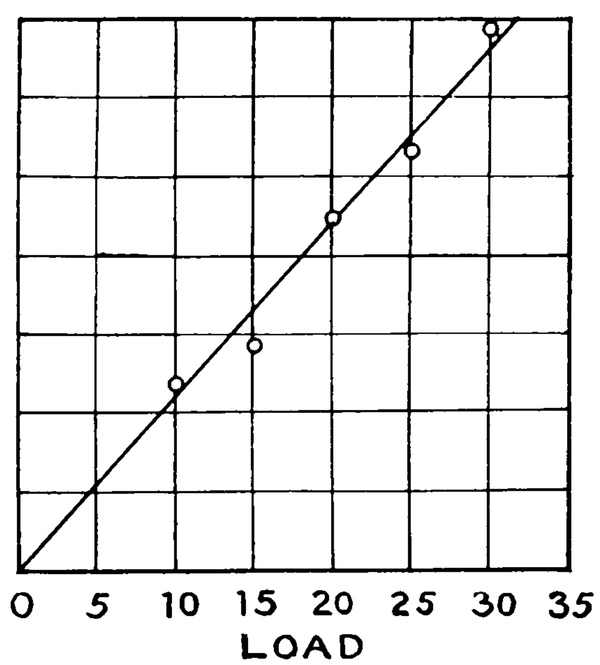
The values of K given in Table VI, column 6, are plotted in Figs. 16 to 20. Since a small variation in the percentage of the load carried by the end rivets causes a fairly large variation of K, the results cannot be considered as accurate to the third place of decimals, and the range of error is roughly indicated by the circles marking the experimental points in the figures. The variation of K with the load is somewhat irregular, but the points lie fairly well on straight lines except for specimen C. Probably

FIG. 16.



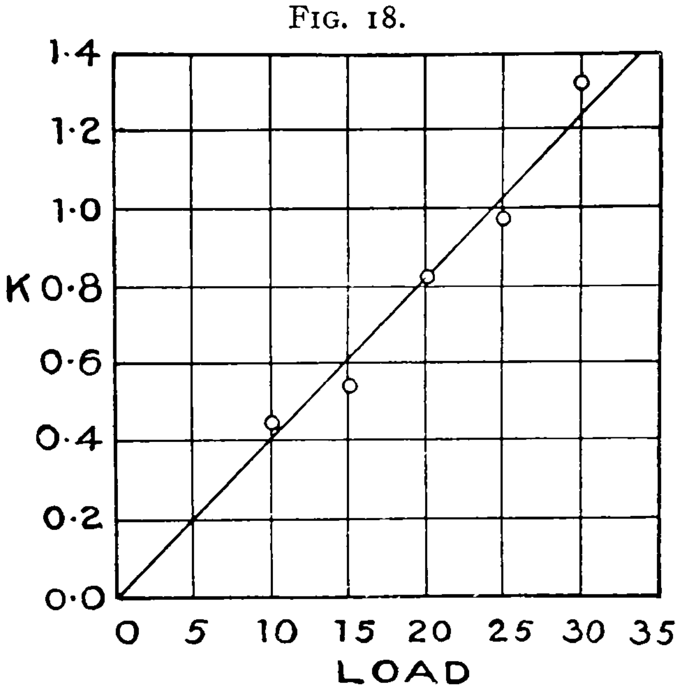
Specimen A.

FIG. 17.



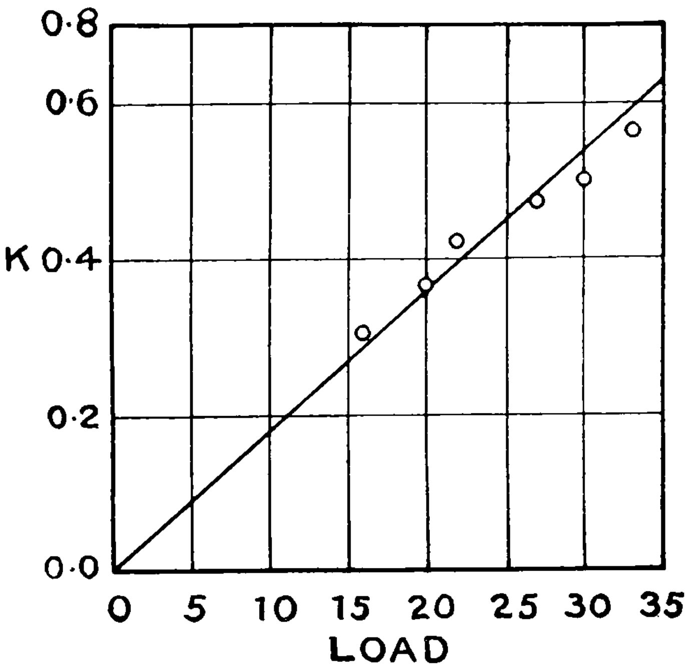
Specimen A, (4 rivets)

FIG. 18.



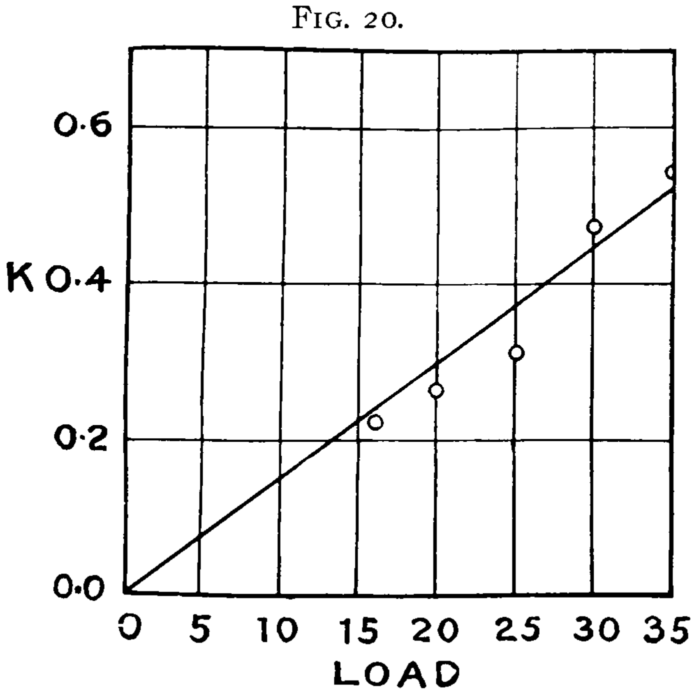
Specimen A, (3 rivets)

FIG. 19.



Specimen B.

FIG. 20.



Specimen C.

FIGS. 16–20. Variation of K with load. Unit of load–1000 lbs.

with more carefully made specimens the irregularities would disappear. In any case the law of variation may be regarded, for practical purposes, as linear. In order to obtain the mean straight lines, the method of least squares was used; *i.e.*, if F_n represent the abscissa (load), K_n the ordinate (K) of any point, the law of variation was taken as

$$K = \frac{\Sigma F_n K_n}{\Sigma F_n^2} \cdot F.$$

The results are given, correct to two figures, in the annexed Table.

TABLE VII.

Specimen	Diameter of rivets	Law of variation of K with load
C	⅞″	$K=0.15\times 10^{-4}$ F
B	¾″	$K=0.18\times 10^{-4}$ F
A	½″	$K=0.43\times 10^{-4}$ F
A (4 rivets)	½″	$K=0.44\times 10^{-4}$ F
A (3 rivets)	½″	$K=0.41\times 10^{-4}$ F

The results differ but little for the three specimens with ½-inch rivets, although the number of rivets was different. On the other hand, there is a considerable difference between the values for the specimens A, B and C, the ratio of *K*'s for any given load being

$$K_A : K_B : K_C = 2.87 : 1.2 : 1.$$

Now the nominal diameters of the rivets in A, B and C are ½ inch, ¾ inch, and ⅞ inch respectively, and the inverse ratio of their areas is as

$$3.07 : 1.34 : 1$$

Thus the values of *K* are roughly in inverse proportion to the nominal areas. A number of rivets were removed from these specimens, and it was found that the rivets fitted tightly in the holes, while the holes were somewhat irregular but always greater in diameter than the nominal diameter of the rivet, the mean diameters being 0.55 inch for specimen A, 0.83 inch for specimen

B, and 0.925 inch for specimen C. Thus the inverse ratio of the actual areas is as

$$2.83 : 1.24 : 1,$$

which is very close to the ratio of K's. It therefore appears that the values of K vary as $\frac{F}{A}$ where F is the total load on the specimen and A the area of cross-section of the rivets. This being so, an empirical formula for K is

$$K = \frac{a F}{10^{+4} A},$$

where a is a constant. The values of a given by the experimental results are 0.102, 0.098, and 0.101 for the specimens with ½-inch, ¾-inch, and ⅞-inch rivets respectively Thus the mean value is 0.100 and

$$K = \frac{F}{100{,}000 A} \quad \ldots\ldots \quad \ldots \ldots\ldots\ldots \quad \ldots (17)$$

where A is the actual area of cross-section of the rivets. This result indicates that in joints similar to the specimens A, B and C the value of K is the same for the same average load per square inch of total cross-section of the rivets. For example, at a load of 10,000 pounds per square inch of total cross-section of the rivets $K = 1.0$. On the other hand, the results for specimen A with the fifth rivet and with the first and fifth rivets removed show very little change in the value of K with the number of rivets. The results of these tests, however, cannot be taken as conclusive, because the rivets had been previously under stress in the complete specimen.

It will be shown in the next section that the value of K given by equation (17) gives results in accordance with the experimental results for the remaining specimens. This says much for the truth of the theory given in Part I, and shows that equation (17) is correct for specimens such as have been tested. It will be necessary, however, to make many more experiments on specimens in both tension and compression having different sizes and arrangements of rivets and different ratios of width of cover plate to rivet pitch before general rules can be given for the determination of K in any type of joint.

Equation (17) is entirely empirical. It will be interesting to examine the results theoretically. K, as explained in Part I, § 2, is a coefficient given by

$$K = \frac{2a_c}{l} \cdot k$$

where k is that quantity which, when multiplied by the square of the load transferred by a rivet and divided by $2E$, gives the work stored in the rivet or its equivalent. It is not easy to determine in exactly what manner work is stored in the rivets. Possibly the rivets act by giving a frictional hold between the plates. This does not seem probable, since K follows the same laws below and above loads at which slip has been shown to occur by other investigators, and because of other reasons already given. If, however, the rivets do hold by friction, K must be some function depending upon the work stored in the portions of the plate held together without slip by the rivets. If, on the other hand, the rivets are in shear, $\frac{kP^2}{2E}$ is the work stored in them when transmitting a load P. Now the rivets are so short and so rigidly held that the work stored in bending must be very small and the major portion of $\frac{kP^2}{2E}$ must be the work stored in shear. This will depend upon the exact manner in which the load comes upon the rivets. As the load increases the contact between plates and rivets will become more intimate and thus the value of K will increase. This is precisely what was found in the experiments. It is, of course, impossible to say theoretically how the load will be distributed at any particular stage. Assuming, however, that a load has been reached which gives a uniform load distribution over the length of the rivets, the shearing force on the rivet will increase uniformly from zero at the head to a maximum of $\frac{P}{2}$ at the junction of the cover and middle plates and will then decrease uniformly to zero at the centre of the middle plate. The intensity of shear, q, at any point of a cross-section over which the shearing force is S will be given by

$$q = \frac{4S}{3\pi R^2}\left(1 - \frac{y^2}{R^2}\right)$$

where R is the radius of the rivet and y the distance of the point from an axis through the centre in the plane of the cross-section and perpendicular to the load.[12] The work stored by shear will be given by

$$W_s = \Sigma \frac{q^2}{2G} \cdot dV,$$

where G is the modulus of rigidity of the material and dV an element of the volume. This, omitting the analysis, gives

$$W_s = \frac{5}{54G} \cdot \frac{t}{A} \cdot P^2,$$

where t is the thickness of the middle plate, or twice the thickness of the cover plates.

Thus

$$K = \frac{5}{27} \frac{E}{G} \cdot \frac{t}{A}$$

and

$$K = \frac{2a_c}{l} \cdot \frac{5}{27} \cdot \frac{E}{G} \cdot \frac{t}{A} \cdot$$

If b represents the breadth of the cover plate $a_c = \frac{bt}{2}$, and

$$K = \frac{5}{27} \cdot \frac{E}{G} \cdot \frac{b}{l} \cdot \frac{t^2}{A} \qquad \dots \quad \dots \qquad \dots\dots (18)$$

For specimen A, having a middle plate 3 inches wide and ⅝-inch thick and ½-inch rivets of 4 inches pitch, this would give, taking the nominal area of the rivets and $\frac{E}{G} = 2.5$,

$$K = 0.692.$$

This is of the same order as the experimental values; in fact, it is the actual value of K for a load of 16,000 pounds or an average load of 8150 pounds per square inch of rivet. If the work stored in bending were considered, the value would be raised somewhat. The value of K given by equation (18) varies inversely as A, and this would be true no matter how the load was distributed over the length of the rivet. This is in accordance with the experimental results. On the other hand, it varies as t^2 while the experimental results do not show such a variation. This would appear to indicate that at a given load per square inch of rivet

[12] See Morley, "Strength of Materials," p. 132. This assumes that the shear intensity is constant over an elementary slice of the cross-section perpendicular to the load.

the load is distributed in the same way and over the same length of the rivet, no matter what the total length of the rivet may be. This seems probable, at any rate for fairly short rivets. In this case K would vary with the ratio $\frac{b}{l}$. Further theorizing on this point, however, would be futile on the experimental evidence at present available.

To sum up, if the rivets act by clamping the plates together by their initial tension, it is evident from the experimental results of §2 that this action is local, and in this case K will depend on the way in which work is stored in the parts of the plates thus held. If, on the other hand, the rivets are in shear, K will depend principally upon the work stored in shear in the rivets and will vary at different loads, because the manner of distribution of the

TABLE VIII.

Specimen	Section	Sum of extensometer readings on the two faces			Mean	Specimen	Section	Sum of extensometer readings on the two faces			Mean
		1	2	3				1	2	3	
D	a	145	119	143	132	F	a	76	67	77	72
	b	164	138	163	151		b	110	90	109	100
	c	173	144	166	157		c	140	114	142	127
	d	176	144	170	158		d	162	150	166	157
	e	195	170	197	187		e	279	281	281	280
E	a	187	170	192	180	F′	a	144	134	150	140
	b	216	183	212	198		b	175	156	179	166
	c	221	184	223	203		c	197	157	196	177
	d	228	191	226	209		d	215	178	227	199
	e	280	284	281	282		e	..			280

load on a rivet depends upon the intensity of the load. Theoretical considerations and experimental results both appear to show that the second hypothesis is the correct one, but further experiments are needed. It is, of course, quite possible that the value of K is not the same for all the rivets in a joint and that the values given above are only equivalent ones for the whole joint.

§5. *The Specimens D, E, F and F′*

The experimental results for the specimens D, E, F and F′ under a load of 16,000 pounds are shown in Tables VIII and IX. The specimen F′ was, as mentioned above, obtained by cutting down the middle plate of specimen F to a uniform thickness of 4.096 inches. This, of course, changed the partition of the load among the rivets, as may be seen by comparing Figs. 23 and 24.

The specimens D, E, and F′ will be considered first. The cover plates in specimens D and E were 3/8 inch and 1/4 inch thick respectively, the middle plate being 1/4 inch thick, and all plates 3 inches wide. All the plates of F′ were 1/4 inch thick, but the outer plates were 3 inches wide and the inner plate 4.096 inches. Thus the values of C in the specimens D, E and F′ were 4, 3 and 2.468 respectively Now it will be remembered that the theory given in Part I indicated that, if C were greater than 2, the first rivet would take a greater percentage of the load than the last,

TABLE IX.

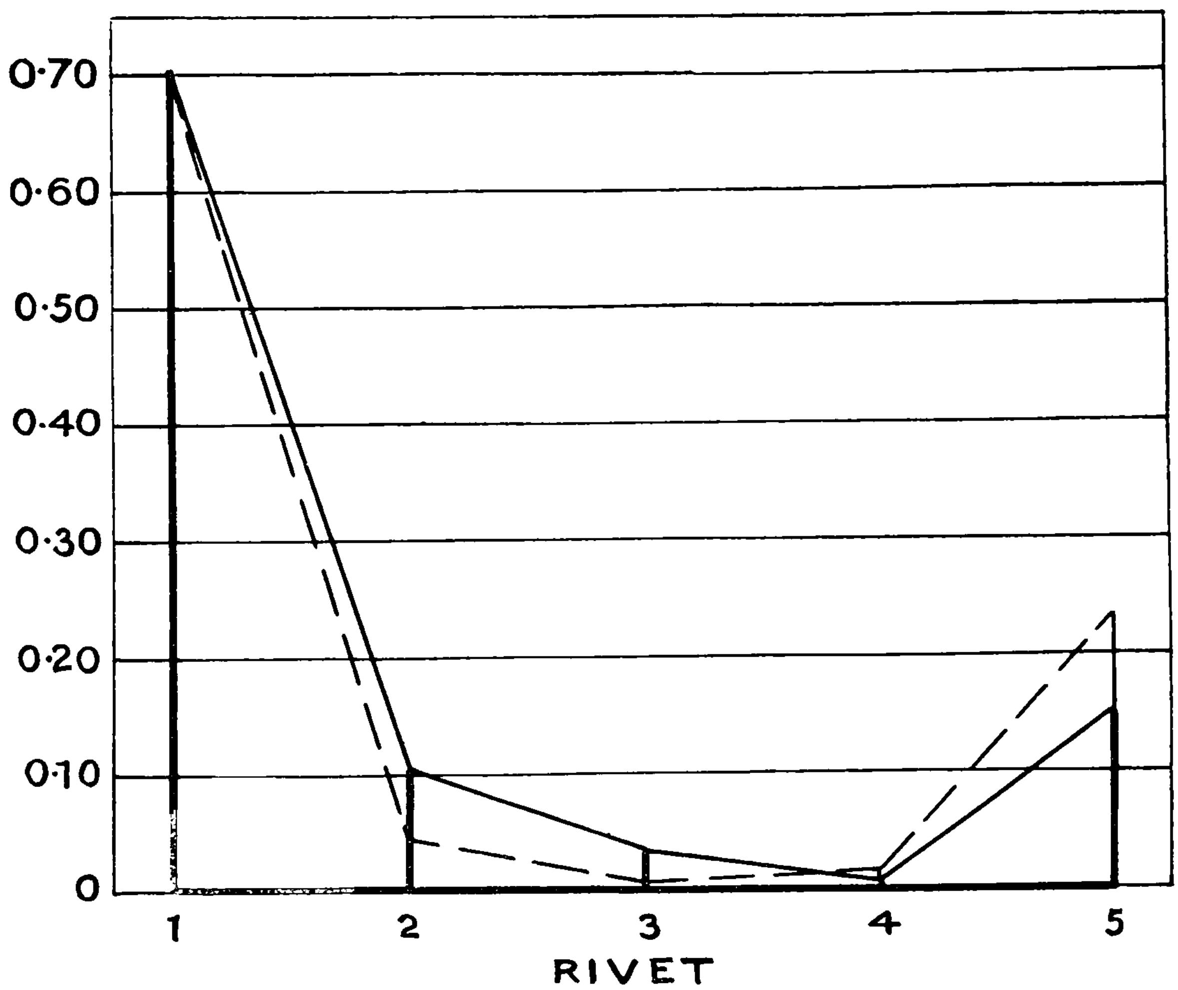
Specimen	C	K		Percentage of total load taken by each rivet				
				1	2	3	4	5
D	...	...	e	70.7	10.1	3.2	0.5	15.5
	4.04	0.29	t	70.5	4.4	0.4	1.5	23.2
E	...	...	e	63.8	6.4	1.8	2.1	25.9
	3.02	0.29	t	61.5	5.0	0.2	2.9	30.4
F′	....	...	e	50.0	9.3	3.9	7.9	28.9
	2.468	0.29	t	53.8	5.2	0.6	3.8	36.6
F	.....	...	e	25.7	10.0	9.6	10.7	44.0
	2.468	0.29	t	28.7	8.0	6.6	11.6	45.0

e = Experimental.
t = Theoretical.

the difference increasing with the value of C. This is entirely borne out by the experimental results as shown in Table IX, the percentage taken by the first rivets being respectively 70.7, 63.8, and 50.0 for the three specimens. The loads taken by the other rivets also agree in general with the theoretical results.

It was shown in the last section that K for 3/4-inch rivets is equal to 0.18×10^{-4} F For a load of 16,000 pounds this gives $K = 0.29$. If, as surmised above, a small difference in the length of the rivets does not alter K, the above value ought to give figures which agree with the experimental results. The two cover plates, however, did not in any of the specimens receive exactly the same loads, and allowance must be made for this, as explained in §3, Part I. The actual ratio of the loads taken by the two cover plates was not constant at each section. This could be allowed for if necessary, but the effect of the correction is, in any case, so small that it will be quite sufficient to take mean values. The mean values of $\frac{1}{s}$ in D, E, and F were 0.595, 0.574, and 0.510 respectively Thus, allowing for these, the values of C become 4.04 and 3.02 for D and E respectively, remaining practically unchanged for F' By substituting these values in the equations of Part I and taking $K = 0.29$, the figures in the rows marked t in Table IX were obtained, and the results are shown with the experimental results in Figs. 21 to 23. It will be seen that the agreement is fairly close. It is almost exact for the first rivets,

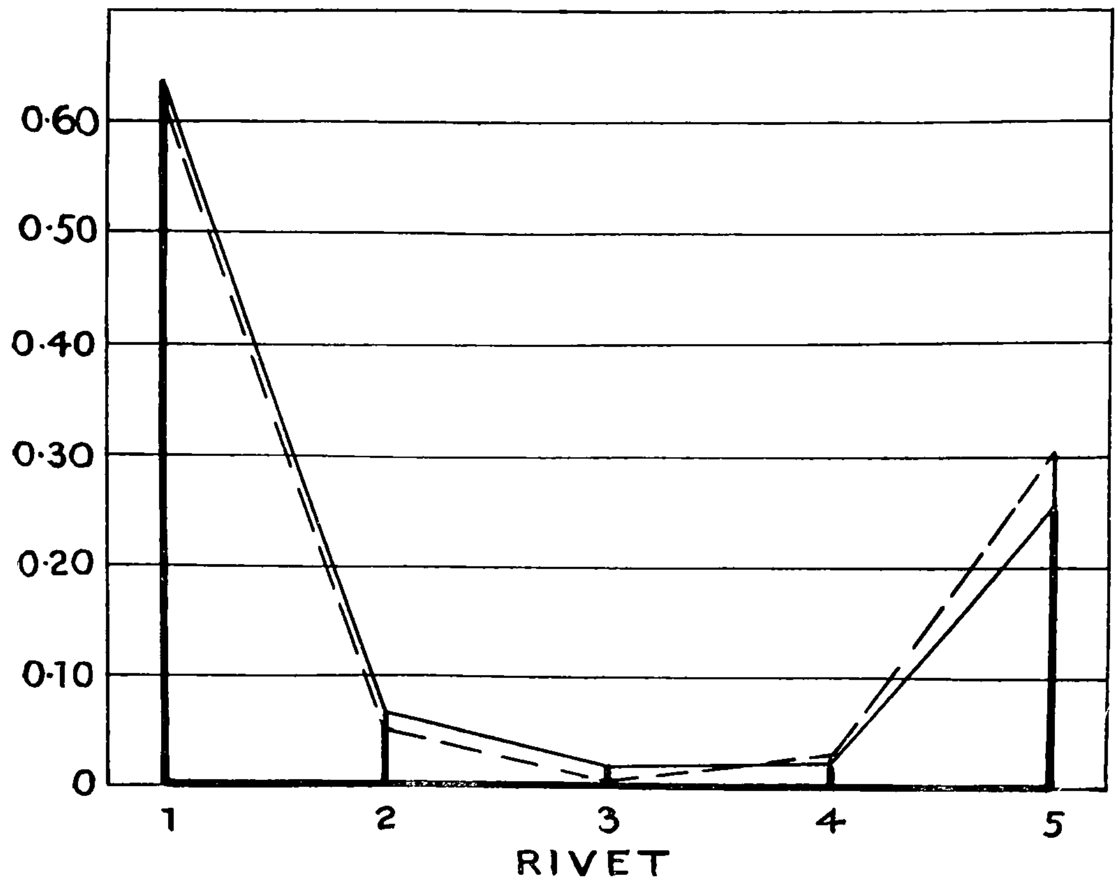
FIG. 21.



Specimen D.

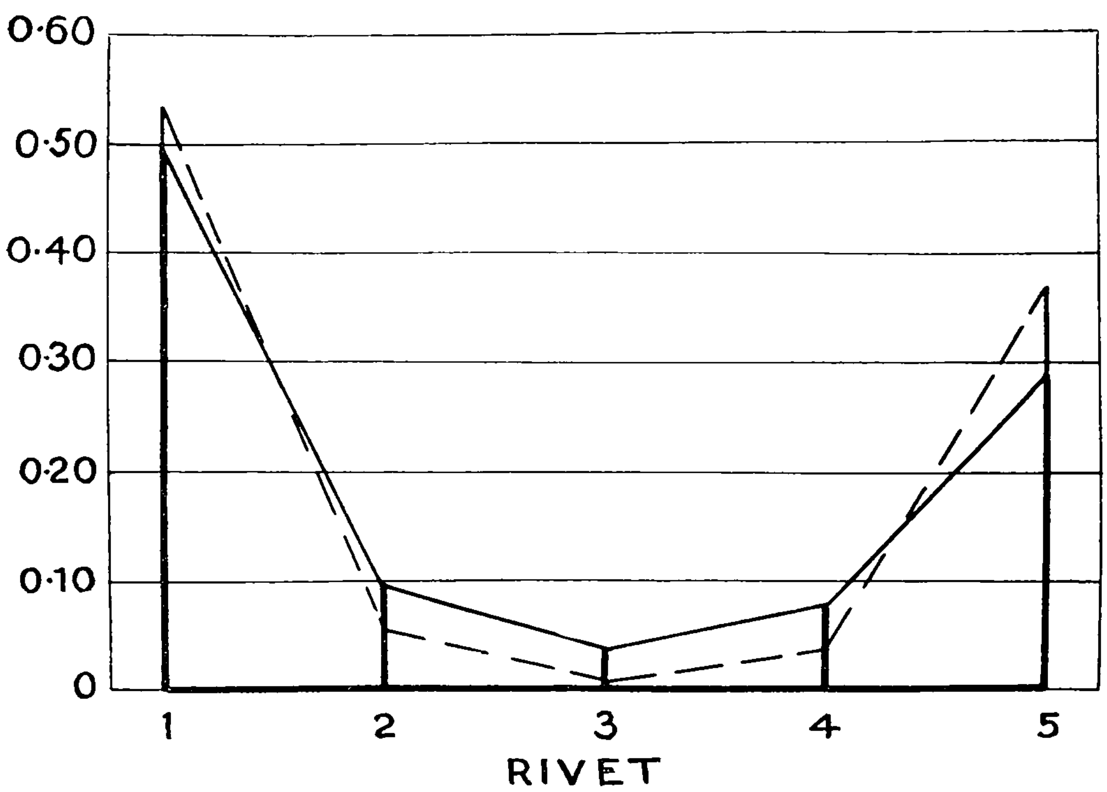
which are the most important. The fifth rivets, however, take less than the theoretical loads in each case, the difference being distributed among the middle rivets. This may arise from actual differences in K for the different rivets or from bending, but, considering the nature of the specimens and the probable irregularities in setting of the rivets, the agreement may be regarded as satisfactory, especially when it is remembered that the value of K used

FIG. 22.



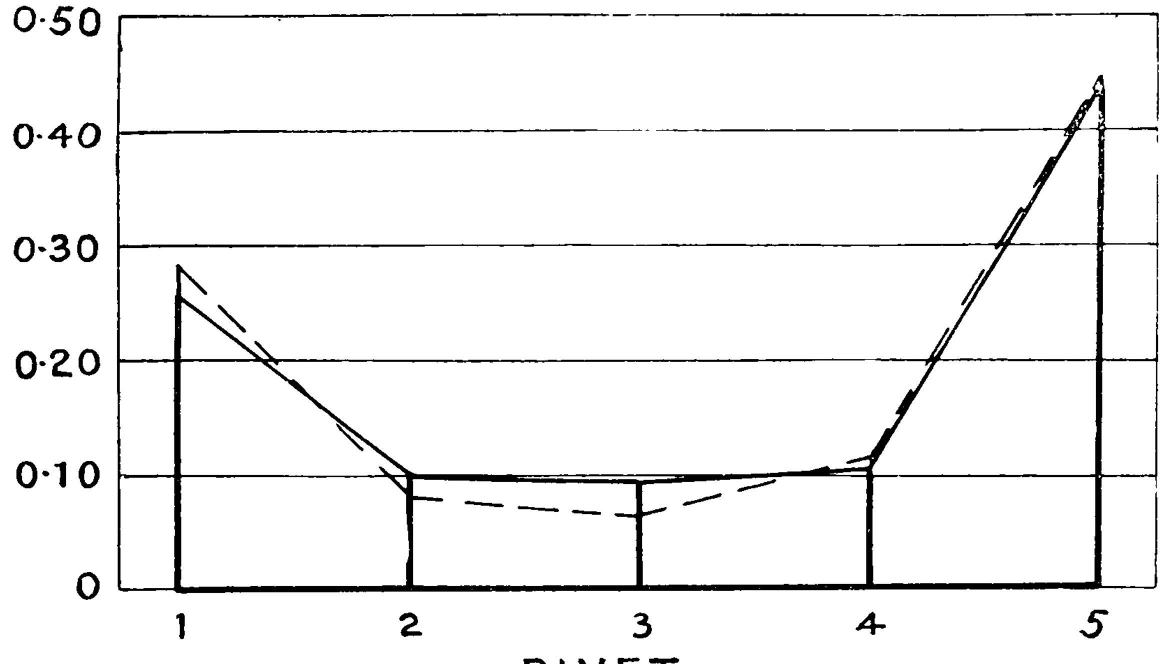
Specimen E.

FIG. 23.



Specimen F'.

FIG. 24.



Specimen F.

was obtained from a single set of experiments on another specimen.

F was the only specimen tested in which the width of the middle plate was variable. The theoretical equations for this specimen are of the type given in Part I, §4. η, the coefficient depending upon the variable width of the plates between each pair of rivets, is so near to unity that it may be neglected. The values of the terms in $2\Sigma\frac{a_c}{a_1}$ are

$$\frac{2a_c}{a_1} = \frac{2\times3}{12.32} = 0.487,$$
$$\frac{2a_c}{a_2} = \frac{2\times3}{10.2} = 0.588,$$
$$\frac{2a_c}{a_3} = \frac{2\times3}{8.08} = 0.743,$$
$$\frac{2a_c}{a_4} = \frac{2\times3}{5.96} = 1.007.$$

Thus

$$2\sum_{1}^{4}\frac{a_c}{a} = 2.825,\ 2\sum_{2}^{4}\frac{a_c}{a} = 2.338,\ 2\sum_{3}^{4}\frac{a_c}{a} = 1.750$$

and $2\sum_{4}^{4}\frac{a_c}{a} = 1.007$, and the equations for the loads taken by the various rivets, Part I, §4, p. 589, taking $K = 0.29$, become

$$\left|\begin{array}{cccc||c} 7.405 & 5.628 & 4.040 & 2.297 & 3.115 \\ 5.628 & 5.918 & 4.040 & 2.297 & 2.628 \\ 4.040 & 4.040 & 4.330 & 2.297 & 2.040 \\ 2.297 & 2.297 & 2.297 & 2.587 & 1.297 \end{array}\right|$$

The solution of these equations gives the results displayed in Table IX and shown by the dotted lines in Fig. 24. There is again a remarkable agreement between the theoretical and experimental results.

The two specimens F and F′ are similar to the ordinary types of end connections of riveted bridge members, and the results show that the widening of the gusset plate results in far less load being carried by the first rivet and also increases the proportion of the load taken by the middle rivets from 21.1 per cent. to 30.3 per cent. of the total load, giving a more even partition of the load.

§6. *General Conclusions.*

The results of the experiments carried out up to the present have now been given and analyzed, and it only remains to see what general conclusions can be drawn from them. In the first place, all the experiments are in remarkable agreement with the theory advanced in Part I, especially when it is taken into consideration that the specimens were ordinary shop products, and show that it is possible to predict, in general, the way in which the load will be divided among the rivets in any form of joint.

Only the specimen with ½-inch rivets was carried beyond the working load, but the regularity of its action showed that the partition of load obeyed the same laws at all loads up to that causing permanent deformation of the plates or rivets. In every specimen and at all loads the first and fifth rivets took by far the greater part of the total load, the actual proportion decreasing gradually as the load increased. For example, in the specimen with ½-inch rivets, the first and fifth rivets carried 83.5 per cent. of the total load at a load of 10,000 pounds, and this decreased to 70.0 per cent. at a load of 30,000 pounds. The latter load corresponds to an average stress of about 12,650 pounds per square inch of actual rivet section, or 15,280 pounds per square inch of nominal rivet section. This would usually be taken as the shearing stress on all the rivets, but actually the end rivets, if, as there seems little doubt, the rivets were in shear and not holding by friction, were each under an average shear stress of 22,150 pounds per square inch, while the third rivet at the same load took only 3.2 per cent., corresponding to an average shear stress of only 2020 pounds per square inch. Thus in joints having several rows each containing an equal number of rivets and designed in the usual manner, *i.e.*, allowing the average load per square inch of rivet section to be equal to the working stress in shear of the rivet material, the rivets in the end rows must carry stresses far above the allowable working stress. That this will not be remedied by increasing the number of rows of rivets was shown in Part I, §2.

The above refers to the specimens in which the cover plates were of the correct thickness; *i.e.*, each half as thick as the middle plate. If they are thicker, the first rivet takes an even greater proportion of the load, the proportion increasing with increased thickness. This was shown theoretically in Part I and experimentally on the specimens D, E and F′ Specimens B, D, E and F′ all had the same diameter of rivet. In specimen B, in which the cover plates were of correct thickness, the first and fifth rivets took 88.1 per cent. of the load at a load of 16,000 pounds. In specimens D, E and F at the same load they carried 86.2 per cent., 89.7 per cent. and 78.9 per cent. of the total load respectively, but of these the first rivets carried respectively 70.7 per cent., 63.8 per cent. and 50 per cent. of the total load. Specimen F, in which the middle plate was of varying width, illustrated the action in members riveted to a gusset plate, and it was found that the varying width of plate resulted in a rather more even distribution of stress, the first and fifth rivets carrying only 69.7 per cent. of the load, as compared with 78.9 per cent. when the middle plate was cut down to uniform width.

It must be noted that in all the specimens tested the ratio of width of cover plate to pitch of rivets was the same, ¾. Now it was shown in §4 that K probably varies as the width of cover plate divided by the pitch of the rivets; thus, with a smaller pitch or wider plates, K would be increased, and the effect of this would be to make the partition of the load rather more uniform. But, as stated above, a large variation of K only causes a comparatively small alteration in the percentage of the load carried by the end rivets. For example, in the specimen A a change of K from 0.485 to 1.3 only altered the load carried by each of these rivets from 40.7 per cent. to 35 per cent., and the alteration for a given change becomes less and less as the values of K increase. Thus the effect of change of pitch or breadth of cover is not likely to be very great, except possibly in splices containing a number of rivets in each row. However, further experiments are needed in order that a general law may be found for the value of K. When this is determined it will be possible to predetermine the exact partition of load in any proposed joint, and this will enable the joint to be designed in the most efficient manner. A very good approximation, sufficient for most purposes, may, however, be obtained from the data already given, since the general manner of partition of load is the same for all values of K. It would require too much space to illustrate this further here, but the examples given in Part I show clearly the method of procedure.

The writer has already in hand further experiments on the variation of K in different types of joints and also experiments designed to show the part, if any, played by friction in riveted joints.

§7. *Summary and Conclusion.*

The following is a summary of the principal contents of the present paper:

1. It is shown that a riveted joint may be considered as a statically indeterminate structure, and that a series of equations may be obtained for any joint by means of the Principle of Least Work, giving the loads carried by each of the rivets in terms of a quantity K, which depends upon the manner in which work is stored in, or by the action of, the rivets.

2. This theory is applied to various types of joints, and the modifying effects of non-uniform distribution of stress in the plates, unequal partition of the load between the two cover plates, and a difference in the modulus of elasticity of the middle plate and the cover plates are also considered.

3. It is shown experimentally that extensometer measurements on the outer surfaces of the cover plates of a riveted joint are sufficient for the determination of the mean stresses in the plates, and that the partition of the load among the rivets may be determined from such measurements. It is also shown that, at any rate after the first few loadings, the distribution of strain in the plates of a joint is not altered by repeated loadings.

4. It appears from 3 that if there is any frictional hold between the plates, it acts only over those portions in the immediate neighborhood of the rivets. All the experiments tend to show that friction does not play an important part, but further experiments are necessary on this point.

5. Experiments made on a number of specimens having a single line of rivets and loaded in tension give results in close agreement with the theoretical considerations. They also show that the longitudinal stresses in a portion of the cover plate between two consecutive rivets are a minimum along the line of rivets, rising to a maximum at the edges of the plates.

6. The experiments show that the value of K for a joint having a given ratio of width of cover plate to rivet pitch and a given number of rivets varies approximately directly as the load and inversely as the area of the rivets. An empirical rule is given for its value in joints similar to the experimental specimens, but a more general rule cannot be given until further experiments have been made. A theoretical estimate is made of the value of K for a rivet acting in shear, and the result is shown to be within the range of the experimental values.

7 Both the experimental results and the theoretical deductions show that:—

(*a*) in a double-cover butt joint having a single line of rivets, the two end rivets and the two rivets on each side of the junction of the middle plates take by far the greater part of the load at all loads within that causing permanent deformation of the plates or rivets, the actual proportion decreasing slowly as the load increases;

(*b*) if, in such joints, the total area of cross-section of the cover plates is equal to that of the middle plates, these four rivets take equal loads, but if it is greater the end rivets take greater loads than the others, the difference increasing as the area of the cross-section of the cover plates increases;

(*c*) if two plates of uniform width and equal thickness are connected by a single line of rivets to opposite sides of a gusset plate of uniform width, the first and last rivets take the greater part of the load, but if the gusset plate increases in width from the first to the last rivets, the partition of load is more uniform.

The results already obtained allow the general manner of partition of load in any riveted joint, in which there is no eccentricity of connection, to be estimated, and it is hoped that, when further experiments have given general laws for the value of K, it will be possible to predetermine the exact load that will be carried by each rivet. The practical value of this is obvious.

In conclusion, the writer wishes to thank Prof. H. M. Mackay and Prof. E. Brown for their personal interest and advice, and Mr. S. D. Macnab for his valuable assistance in the experimental part of the work.

McGill University,
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