Invariant and Group Theoretical Integrations over the Unitary Group

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January 2004

A thesis submitted to McGill University in partial fulfilment of the requirements of the degree of Master of Science.

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ISBN: 0-612-98590-3

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While looking for an opportunity to think for myself ...

Abstract

Integrals of products of unitary matrix elements over the unitary group will be discussed. A group theoretical formula is available to convert them into multiple sums, but unfortunately the latter are often tedious to compute. In this thesis, we develop an alternative method in which these sums are avoided, and group theory is rendered unnecessary. Only unitarity and the invariance of the Haar measure are required for the computation. Then, we return to the more traditional group theoretical approach, and compare its strengths and weaknesses with those of our *invariant method.* As a result, we are able to introduce another method that combines the respective strengths of the other two methods. There are many examples illustrating how each of the approaches works.

Résumé

Des intégrales comportant des monomes d'éléments de matrice unitaire seront examinées. Il existe une formule provenant de la théorie des groupes qui permet de convertir les intégrales en une somme multiple. Malheureusement, cette dernière requiert souvent un travail assidu. Nous développons dans cette thèse une méthode alternative dans laquelle la somme ci-mentionnée est absente et une connaissance de la théorie des groupes n'est pas requise. Cette nouvelle méthode ne dépend que de la définition du groupe unitaire ainsi que de l'invariance de la mesure de Haar. Par la suite, nous retournons à la méthode plus traditionelle de la théorie des groupes. Nous comparons alors les forces et faiblesses relatives aux deux méthodes, en vue d'élaborer une nouvelle méthode tirant avantage des deux autres. Il y a de nombreux exemples qui illustrent comment ces approches fonctionnent.

ACKNOWLEDGEMENTS

The present thesis would not have been possible without the excellent guidance of professor C. S. Lam. I feel priviledged to have had a supervisor of his talent. I am also grateful for his generosity and enlightening suggestions. The research was supported by the Natural Sciences and Engineering Research Council of Canada.

Contents

CHAPTER 1

INTRODUCTION

The integrals of monomials of $n \times n$ unitary matrix elements,

$$\int dU\, U^*_{i_1 j_1} \cdots U^*_{i_p j_p} U_{k_1 l_1} \cdots U_{k_q l_q}, \tag{1.1}$$

where dU is the invariant unitary group measure, and their generating functions are useful in many areas of physics. For instance, in a famous 1974 article, K. G. Wilson used the invariant group measure in his definition of lattice gauge theory [1]. Group integration was thus lent clear mathematical elegance. The integral (1.1) can be computed in several ways. There exist methods that rely on diagrammatic representations. Others rely on generating functionals. However, the more popular methods make principal use of the group theory of the symmetric group, S_p, and the unitary group, $U(n)$. In particular, group theoretical formulae for the computation of all $U(n)$ group integrals (1.1) have been derived in [2]. A similar formula, to be designated plainly the *group theoretical formula*, can be obtained from a result due to Frobenius and convenient group theoretical relations, namely, the orthogonality relations of group characters. The group theoretical formula can be deduced from another powerful integration technique, that is the Itzykson-Zuber formula [3]. The latter allows to compute a class of integrals with a $U(n)$ dependent exponential.

The group theoretical approach is effective as it offers a formula to calculate all integrals of the type (1.1) directly. Nevertheless, the calculation becomes rapidly cumbersome and prone to error as the number of U^* and U matrix elements increases. We first wish to develop an alternative integration technique that relies

only on the unitarity of the matrices U and the invariance of the group measure dU. No knowledge of group theory is then required. We shall refer to this approach as the *invariant method.* Then, we will study the group theoretical formula, and try to extract helpful information from it. The mentioned two approaches have different advantages and disadvantages, and in some way they complement each other. Thus, following an informative comparison of the methods, we shall attempt to combine them into a more efficient *hybrid* computational scheme. The present work will be oriented in a pure mathematical physics direction.

Let us now examine the content of the thesis more closely. The first chapter offers a review of group theoretical concepts that are important to what follows. The symmetric and unitary groups are particularly discussed. In a consideration of group representations, fundamental orthogonality relations are given. The Frobenius relation, which is essential to the derivation of the group theoretical formula, is also introduced in the last section of the chapter.

The second chapter is twofold. Some physical applications of the integration techniques are discussed initially. Lattice gauge theory as well as the topic that created our interest in the $U(n)$ integrals, that is the problem of parton saturation, are very briefly presented. Then, two important and known $U(n)$ group integration methods are given. They are the group theoretical and Itzykson-Zuber formulae. For consistency, it is shown how the latter formula reduces to the former.

The core of the work is contained in Chapter 4. First, the invariant method is developed. Before attacking the problem of the $U(n)$ group integrals, the reader is introduced to the parallel but simpler problem of integrating monomials over a hypersphere. Two useful conditions are then the invariance of the differential dx under rotation and $\sum_i x_i^2 = 1$. In the case of $U(n)$ group integrals, the invariance of the group measure dU produces various relations between the integrals. The unitarity of the members of $U(n)$ also yields useful relations. In particular, unitarity allows to compute the *values* of integrals recursively. Noticeable results of this first method are the fan and double-fan relations, as well as the Z-formula. Second,

the group theoretical formula is studied and extended, resulting into what will be known as the *group theoretical method.* The single-fan relation is also shown to be obtainable from this approach. Finally, after comparing the relative advantages of the two methods, a synthetic method, the *hybrid method*, is designed. The hybrid method takes advantages of the many relations inherent to the invariant method and of the universality of the group theoretical method.

Some appendices at the end contain details of various derivations. Integral results for $p = q = 4, 5$ in (1.1) are also included. Moreover, we have inserted in Appendix G some additional work that does not require a knowledge of symmetric group characters. The reader may want to play with and perhaps develop that theory. Finally, we have inserted a basic notation reminder at the very end.

CHAPTER 2

A Group Theoretical Review

In this chapter are presented the principal tools, though sometimes very briefly, to sufficiently judge or appreciate the remainder of the thesis. For the most part, the theory was taken from sources [4, 5, 6, 7], the last two being particularly useful.

2.1 Groups and Related Notions

As described by D. E. Littlewood in [4], a *group* is a system composed of a set of elements, $\mathcal{G} = \{a, b, c, \ldots\}$, and a rule for combining any two of them to obtain their 'product', such that:

1. for any $a, b \in \mathcal{G}$, their product $ab \in \mathcal{G}$;
2. the associative law holds, i.e. $a(bc) = (ab)c$;
3. there exists an *identity* element, $e \in \mathcal{G}$, which satisfies $ea = ae = a$ for every $a \in \mathcal{G}$;
4. for each element, $a \in \mathcal{G}$, there is an associated inverse element, $a^{-1} \in \mathcal{G}$, such that $aa^{-1} = a^{-1}a = e$.

A subset $\mathcal{H}$ of a group $\mathcal{G}$ is called a *subgroup* if it fulfills the above requirements, and the product in $\mathcal{H}$ is induced by $\mathcal{G}$.

From the definition of a group, we directly go to a notion that is primordial to the present work, that is the one of *conjugate classes*. Given that a, b, u are elements

of a group $\mathcal{G}$, it is said that b is *conjugate* to a if

$$uau^{-1} = b. \tag{2.1}$$

Clearly, every element of a group is conjugate to itself; choose $u = e$ in (2.1). Furthermore, if b conjugate to a, then the converse is also true; in fact, $u^{-1}bu = u^{-1}(uau^{-1})u = a$. Finally, where $c, v \in \mathcal{G}$, note that if a is conjugate to b, and b is conjugate to c, then a must be conjugate to c; for $a = ubu^{-1} = u(vcv^{-1})u^{-1} = (uv)c(uv)^{-1}$. We now have sufficient reasons to believe that the conjugate relation satisfies the requirements of an *equivalence relation*[1]. Using an equivalence relation, elements of a set can be organized into *classes.* In any case, we will classify conjugate group elements in the same *conjugate class.*

Groups come in two varieties; they may contain a finite number of elements (*finite groups*), or an infinite number of them (*infinite groups*). The number of elements contained in a group is termed the *order* of the group. An example of a finite group is the group of $p!$ permutations of p objects, which is discussed in section 2.3. An example of an infinite group is the $U(n)$ group, which is discussed in section 2.4.

The elements of a group may be considered as a set of "points" in some abstract space, thus forming what is called a *group manifold.* If some generalized definition of "nearness" or continuity can be imposed on the manifold, the group is said to be *continuous.* Many continuous groups, like $U(n)$, can be described by a finite set of continuously varying parameters. Assuming that m parameters α_i, $i = 1, \ldots, m$ are sufficient to label the elements of a group, the latter may be written symbolically as $r_\alpha = r(\alpha_1, \ldots, \alpha_m)$. For details as to how the elements r_α satisfy the requirements of forming a continuous group, refer to [7].

The elements of a group may be without any interpretation except as elements of the group. In such a case, the group is called an *abstract group.* An abstract group can be mapped homomorphically on a group of operators in a vector space. The related idea of *group representation* must now be discussed.

[1]Refer to [8] for a definition.

2.2 Group Representations

2.2.1 Definition

A clear idea of a group representation can be obtained from [7]. Consider a set of operators A, B, ... in a vector space L, and which satisfies the postulates given the previous section. If C is some operator of the set, then the composition of any two operators A and B is given by

$$Cx = A(Bx),$$

for all vectors x in L. The identity operator E leaves all vectors invariant. Hence if an arbitrary group $\mathcal{G}$ is mapped homomorphically on a group of operators $D(\mathcal{G})$ in the vector space L, $D(\mathcal{G})$ is said to form a *representation* of $\mathcal{G}$ in the *representation space* L. In this context, the *degree*, or *dimension*, of a representation refers to the dimensionality of L. For an element r of $\mathcal{G}$, the corresponding element in $D(\mathcal{G})$ is denoted by $D(r)$. Since the map $D : \mathcal{G} \to D(\mathcal{G})$ is a homomorphism,

$$\begin{aligned} D(rs) &= D(r)D(s), \\ D(r^{-1}) &= [D(r)]^{-1}, \text{ and} \\ D(e) &= E \end{aligned}$$

are sufficient conditions for D to furnish a representation.

The groups considered in this thesis can be described by groups of *linear* operators[2]. Given a basis in some n-dimensional space L, the operators may take the form of matrices. The homomorphism then becomes a homomorphism of $\mathcal{G}$ on the group of n-dimensional nonsingular matrices $GL(n)$, and the corresponding representation is referred to as a *matrix representation* of the group $\mathcal{G}$. We will denote the matrix element of $D(r)$ located in the ith row and jth column by $D_{ij}(r)$. In the case where different representations are under consideration, we here distinguish them with a superscript, e.g. $D^{(f)}_{ij}(r)$.

[2]We everywhere restrict ourselves to *linear* spaces and operators. Moreover, by 'operator' we understand 'group linear operator'.

A matrix representation is called *unitary* when all the mapped matrices A satisfy $AA^\dagger = A^\dagger A = E$. In other words, a unitary representation is a homomorphism $D : \mathcal{G} \to U(n)$ from $\mathcal{G}$ to the unitary group (see Section 2.4). An important fact concerning any finite group $\mathcal{G}$ is that every representation $H : \mathcal{G} \to GL(n)$ of such group is conjugate to a unitary representation (refer to [8] for a proof).

2.2.2 Reducibility of Representations

Consider a representation $D(r)$ in n dimensions. If there exists some subspace of dimension $m < n$ that is invariant under all transformations of the group, then the representation is said to be *reducible*; otherwise it is *irreducible.*

Both S_p, the symmetric group on p objects, and $U(n)$ are conjugate to a unitary representation. In fact, the former group is finite, and the latter is unitary by nature. We shall then restrict ourselves to representations of the unitary type in the remainder of the text.

An $n \times n$ unitary representation $D(r)$ is *fully reducible*, i.e. it can be expressed in terms of some k unitary irreducible representations $D^{(h_i)}(r), i = 1, \ldots, k$, in the form

$$D(r) = \begin{bmatrix} D^{(h_1)}(r) & 0 & 0 & \cdots & 0 \\ 0 & D^{(h_2)}(r) & 0 & \cdots & \vdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \vdots & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & D^{(h_k)}(r) \end{bmatrix}$$

The above may also be written in the simpler form $D^{(h_1)}(r) + D^{(h_2)}(r) + \cdots + D^{(h_k)}(r)$. We denote the dimension of an irreducible representation $D^{(h)}(r)$ by d_h. Therefore, $\sum_{i=1}^{k} d_{h_i} = n$. Unless otherwise stated, when mentioning a representation, it will be implied that it is irreducible.

2.2.3 Equivalent Representations and Characters

Consider a vector space L of dimensionality n which may be described by the basis $\{\mathfrak{e}\} = \{\mathfrak{e}_1, \ldots, \mathfrak{e}_n\}$. Transition to another coordinate system $\{\mathfrak{e}'\} = \{\mathfrak{e}'_1, \ldots, \mathfrak{e}'_n\}$ can be effected by a nonsingular matrix $M = [M_{ik}]$ according to $\mathfrak{e}_k \to \mathfrak{e}'_i = \sum_k M_{ki}\mathfrak{e}_k$. In this case, the coordinates of the vector $x = x_1\mathfrak{e}_1 + \cdots + x_n\mathfrak{e}_n = x'_1\mathfrak{e}'_1 + \cdots + x'_n\mathfrak{e}'_n$ are related by $x_i = \sum_k M_{ik}x'_k$, or simply $x = Mx'$. Accordingly, a *linear mapping* $A : x \to A(x)$ *of* L *upon itself* in terms of the fixed coordinate system $\{\mathfrak{e}\}$ is transformed into its corresponding A' in $\{\mathfrak{e}'\}$ in the following manner:

$$A \to A' = M^{-1}AM. \tag{2.2}$$

Two matrix representations $D(\mathcal{G})$ and $D'(\mathcal{G})$ may be related as above, i.e. for some $r \in \mathcal{G}$, let $A = D(r)$ and $A' = D'(r)$, in which case they are said to be *equivalent.* It is not hard to see that the structure, or multiplication table of the group remains unchanged vis-à-vis the transformation (2.2).

Given a representation, we have seen that different bases in L lead to various legitimate representations. A natural question to ask is whether or not there exists some intrinsic quantity related to $D(r)$ that is unaffected by a change of basis. An answer can easily found by summing the diagonal elements of $M^{-1}D(r)M$. We thus obtain

$$\begin{aligned}\sum_i (M^{-1}D(r)M)_{ii} &= \sum_{ijk} M^{-1}_{ik} D_{kl}(r) M_{li} \\ &= \sum_{kl} \delta_{kl} D_{kl}(r) = \sum_l D_{ll}(r),\end{aligned}$$

and realize that performing the trace of an operator is invariant under the transform operation, or a change of coordinates. The symbol δ_{ij} is the Kronecker delta, i.e. $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ otherwise. In group terminology, the trace of an operator is the *character* of the operator, and is denoted by

$$\chi(r) = \sum_l D_{ll}(r). \tag{2.3}$$

Equivalent representations have the same character. We will denote the character of $D(r)$ in the representation (f) by $\chi_f(r)$. A primordial fact should be realized from (2.3): the dimension of a representation (f) is given by $\chi_f(e)$.

Knowing that D is a homomorphism, using it to map both sides of equation (2.1) shows that $D(a)$ and $D(b)$ are conjugate in $D(\mathcal{G})$. Note that $D(u)$ is a possible change of coordinates matrix in the space L, or let $D(u) = M$ above. Thus, conjugate elements in $\mathcal{G}$ have the same character; or equivalently, elements in $\mathcal{G}$ belonging to the same class share the same character. If the class of an element r is denoted by c_r, then we may write $\chi_f(c_r)$ instead of $\chi_f(r)$. Let the ρ classes of $\mathcal{G}$ be given by $c_1, \ldots, c_\rho$, then each representation is associated with a set of characters for all classes, e.g. $\chi_h(c_1), \ldots, \chi_h(c_\rho)$ for representation (h). It is thus interesting to note that for any representation (h), such characters may be considered as components of a ρ-dimensional vector in character space, namely $\boldsymbol{\chi}_h = (\chi_h(c_1), \ldots, \chi_h(c_\rho))$.

2.2.4 Orthogonality Relations

We will here only state the relations; the proofs can be found in Chapter 3 of [7] for finite groups, and in Chapter 8 of [5] for continuous groups.

Consider all nonequivalent irreducible representations of a finite group $\mathcal{G}$ of order $n_\mathcal{G}$. With their row and column indices, as well as representation fixed, quantities like $D^{(f)}_{ij}(r)$ form vectors in an $n_\mathcal{G}$-*dimensional* space; there is a coordinate associated with each element of $\mathcal{G}$. Such vectors satisfy the orthogonality relation

$$\sum_{r\in\mathcal{G}} D^{(f)}_{il}(r) D^{(h)}_{mj}(r^{-1}) = \frac{n_\mathcal{G}}{d_f}\,\delta_{fh}\,\delta_{ij}\,\delta_{lm}, \tag{2.4}$$

or

$$\sum_{r\in\mathcal{G}} D^{(f)}_{il}(r) D^{(h)*}_{jm}(r) = \frac{n_\mathcal{G}}{d_f}\,\delta_{fh}\,\delta_{ij}\,\delta_{lm} \tag{2.5}$$

for unitary representations. Recall that d_f denotes the dimension of representation (f).

A similar relation for characters of unitary representations can be found from

(2.5). First set $l = i$ and $m = j$, then sum over i and j to obtain

$$\sum_{r\in\mathcal{G}} \chi_f(r)\chi_h^*(r) = n_{\mathcal{G}}\,\delta_{fh}. \tag{2.6}$$

A character is a class function (recall that $\chi_f(r) = \chi_f(c_r)$). By labelling the classes of $\mathcal{G}$ by $c_1, c_2, \ldots$, the above can be rewritten as

$$\sum_i n_i\,\chi_f(c_i)\chi_h^*(c_i) = n_{\mathcal{G}}\,\delta_{fh}, \tag{2.7}$$

where n_i is the number of elements in the class c_i, and the sum runs over all classes of $\mathcal{G}$. For a given (f), the numbers $n_i^{\frac{1}{2}}\chi_f(c_i)$ form a nonzero vector in a 'class' space. From (2.7), we can conclude that the vectors of nonequivalent representations are *orthogonal*. As a consequence, the number of nonequivalent irreducible representations must be smaller than or equal to the number of classes. More precisely, it can be shown that those two numbers are equal (refer to [7]).

It is worth mentioning that there are useful variants to relation (2.6). For instance,

$$\sum_s \chi_f(sq)\chi_h(rs^{-1}) = \frac{n_{\mathcal{G}}}{d_f}\chi_f(qr)\,\delta_{fh} \tag{2.8}$$

and

$$\sum_s \chi_f(qsrs^{-1}) = \frac{n_{\mathcal{G}}}{d_f}\chi_f(q)\chi_f(r), \tag{2.9}$$

where $s, q, r \in \mathcal{G}$ prove to be useful later. They can be demonstrated by first writing the characters explicitly in the form of sums of matrix elements, then by applying relation (2.4).

For a continuous group $\tilde{\mathcal{G}} = \{u_\alpha\}$, the discrete sums over finite group elements must be replaced by their integral equivalents over the continuous group manifold, i.e. $\sum_{u\in\mathcal{G}} \to \int_R du_\alpha$, where du_α denotes the element of volume, or *measure*, of $\tilde{\mathcal{G}}$ whose parametric region R is bounded and closed (more details to be found in section 2.4). For simplicity, the reference to parameters α on group elements may also be omitted where no confusion arises. One may thus expect continuous analogs to the previous relations. In fact, if the "volume" of the continuous group is given

by $V = \int_R du$, the relation

$$\int_R du\, \tilde{D}^{(f)}_{il}(u)\tilde{D}^{(h)}_{mj}(u^{-1}) = \frac{V}{\tilde{d}_f}\,\delta_{fh}\,\delta_{ij}\,\delta_{lm} \tag{2.10}$$

is the continuous version of (2.4). Note that V may be normalized to unity, i.e. $\int_R du = 1$, which is understood subsequently.

If the characters of $\tilde{\mathcal{G}}$ are written with a tilde, and $u, v, w \in \tilde{\mathcal{G}}$, then the relations

$$\int_R du\, \tilde{\chi}_f(u)\tilde{\chi}^*_h(u) = \delta_{fh},$$

$$\int_R du\, \tilde{\chi}_f(uv)\tilde{\chi}_h(wu^{-1}) = \frac{1}{\tilde{d}_f}\tilde{\chi}_f(vw)\,\delta_{fh}, \tag{2.11}$$

and

$$\int_R du\, \tilde{\chi}_f(vuwu^{-1}) = \frac{1}{\tilde{d}_f}\tilde{\chi}_f(v)\tilde{\chi}_f(w)$$

parallel (2.6), (2.8), and (2.9) for continuous groups.

2.3 The Symmetric Group

A *permutation* of a set S is a bijective map φ from S to itself, i.e. $\varphi : S \to S$. In more concrete terms, consider a set of p boxes (or points) labelled from 1 to p, each containing one object. The latter could be made discernible by having different shapes or colours, yet if one does not fear confusion, they may be labelled by the numbers from 1 to p. A permutation s of degree p is an operation that sends a standard arrangement of the objects in the boxes, say $(1, 2, \ldots, p)$, to any possibly other arrangement of them. A common notation for such a move is

$$s = \begin{pmatrix} 1 & 2 & \cdots & p \\ s_1 & s_2 & \cdots & s_p \end{pmatrix},$$

which means that s replaces 1 by s_1, 2 by s_2, and so on, where each of the s_i, $i = 1, \ldots, p$ assumes a distinct value in $(1, 2, \ldots, p)$. Note that the objects in the top row could be put in a different order, and as long as the order in the lower row is

changed accordingly, the meaning of the permutation is unaltered, e.g. the symbol $\begin{pmatrix} 3 & 1 & 2 \\ s_3 & s_1 & s_2 \end{pmatrix}$ is one of the 3! possible ways of writing $\begin{pmatrix} 1 & 2 & 3 \\ s_1 & s_2 & s_3 \end{pmatrix}$.

Another convenient way of expressing a permutation is provided by the cycle notation, which is most important in the rest of the thesis. Imagine m boxes being organized in a circle where each object is sent to its successor, i.e. the permutation sends s_i, $i = 1, \ldots, m-1$ to s_{i+1}, and s_m into s_1. Such a *cycle* on m objects, called an *m-cycle*, is denoted by the symbol $(s_1 s_2 \cdots s_m)$. A permutation may be composed of more than one cycle, e.g.

$$\begin{pmatrix} s_1 & \cdots & s_{q-1} & s_q & s_{q+1} & \cdots & s_{p-1} & s_p \\ s_2 & \cdots & s_q & s_1 & s_{q+2} & \cdots & s_p & s_{q+1} \end{pmatrix} = (s_1 \cdots s_q)(s_{q+1} \cdots s_p),$$

for a permutation of degree p expressible in terms of two cycles of lengths q and $p-q$. In the case where an object is sent to itself, a one-cycle is formed. Where clarity may not suffer, it is common practice to omit one-cycles when writing a permutation.

As was the case in the first notation, there is not a unique way of expressing a permutation in the cycle notation. First, for a permutation that contains several cycles, note that the order in which they are juxtaposed is irrelevant, for each of them is involved with different objects. Second, note that it is immaterial with which of the objects a cycle begins; what is of significance is the sequence in which the objects' labels are written.

The $p!$ permutations on p objects form a group, which is known as the *symmetric group* of degree p, and is denoted by S_p. The identity permutation is composed only of one-cycles, and is usually denoted by the italicized 'e'. For a given element, its inverse is simply obtained by reversing the order of the cycles, e.g. for $s = (s_1 s_2 \cdots s_m)(s_{m+1} s_{m+2} \cdots s_p)$, the corresponding inverse is given by $s^{-1} = (s_m s_{m-1} \cdots s_1)(s_p s_{p-1} \cdots s_{m+1})$. By the law of combination or product ts, we understand that the permutation s is first performed on the objects, then it is followed by t. In other words, permutations in a 'string' are performed sequentially from the right.

A permutation may be composed of cycles of various lengths. Denote the number

of cycles of length x by ν_x. Then, for any permutation s on p objects, the relation

$$\nu_1 + 2\,\nu_2 + \cdots + p\,\nu_p = p \tag{2.12}$$

must clearly be satisfied. After all, the total number of labels in the cycle(s) of a degree p permutation is p. A permutation with a given set of ν_i values that satisfy (2.12) is said to have the *cycle structure* $(\nu) \equiv (1^{\nu_1}, 2^{\nu_2}, \ldots, p^{\nu_p})$.

We shall now demonstrate that *elements belonging to the same class share the same cycle structure.* As seen earlier, two elements $r, s \in S_p$ contained in the same class are related through $qrq^{-1} = s$, where q is also an element of S_p. Let us write the elements r and q in the general forms

$$r = \begin{pmatrix} 1 & \cdots & p \\ r_1 & \cdots & r_p \end{pmatrix}$$

and

$$q = \begin{pmatrix} 1 & \cdots & p \\ q_1 & \cdots & q_p \end{pmatrix} = \begin{pmatrix} r_1 & \cdots & r_p \\ r_{q_1} & \cdots & r_{q_p} \end{pmatrix}.$$

Then

$$\begin{aligned} qrq^{-1} &= \begin{pmatrix} r_1 & \cdots & r_p \\ r_{q_1} & \cdots & r_{q_p} \end{pmatrix} \begin{pmatrix} 1 & \cdots & p \\ r_1 & \cdots & r_p \end{pmatrix} \begin{pmatrix} q_1 & \cdots & q_p \\ 1 & \cdots & p \end{pmatrix} \\ &= \begin{pmatrix} q_1 & \cdots & q_p \\ r_{q_1} & \cdots & r_{q_p} \end{pmatrix}, \end{aligned}$$

which implies that s is obtained from r by independently relabelling its bottom and top rows according to q. For example, if

$$r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}, \qquad q = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix},$$

then

$$s = \begin{pmatrix} 3 & 4 & 1 & 2 \\ 4 & 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}.$$

In the cycle notation, the above demonstration translates into the fact that conjugate elements differ only by a relabelling of the content of their cycle(s), thus leaving

unchanged the cycle structure. Elements of the same of class must then possess the same cycle structure and vice versa.

A few remarks are now in order. The identity element, being composed uniquely of one-cycles, is the only element of its class. Moreover, since the cycles of both an element and its inverse differ only by the order of their labels, such elements must have the same cycle structure, and thus they must belong to the same class.

Given a certain class $c_{(\nu)}$ in S_p, which has the cycle structure $(\nu) = (\nu_1, \nu_2, \ldots, \nu_p)$, one may deduce from combinatorial arguments the number of elements that are contained in the class. An element in $c_{(\nu)}$ has the structure

$$\underbrace{(\bullet)\cdots(\bullet)}_{\nu_1}\ \underbrace{(\bullet\bullet)\cdots(\bullet\bullet)}_{\nu_2}\ \underbrace{(\bullet\bullet\bullet)\cdots(\bullet\bullet\bullet)}_{\nu_3}\cdots, \tag{2.13}$$

where '$\bullet$' is a 'cipher' among p of them. The elements in $c_{(\nu)}$ may be obtained by organizing the labels $(1, \ldots, p)$ in all possible ways in (2.13), and then by identifying the distinct permutations that result. There are $p!$ ways of replacing the ciphers with p labels, i.e. p ways to insert the first label, $p-1$ ways to insert the second label, and so on until there is one label and one cipher remaining. However, this organization of the objects leads to a duplication of the elements. First, elements that differ by a permutation of their cycles, e.g. $(12)(34) = (34)(12)$, are the same; the factor $1/\prod_{i=1}^{p} \nu_i!$ cancels such duplication. Second, since it is irrelevant where a cycle begins, elements are the same up to cyclic moves of the labels in each of their cycles, e.g. $(123) = (231) = (312)$; the factor $1/\prod_{i=1}^{p} i^{\nu_i}$ removes this extra counting. Hence, the number of elements $n_{(\nu)}$ in the class $c_{(\nu)} \subset S_p$ is obtained by combining the above three factors, that is

$$n_{(\nu)} = \frac{p!}{\prod_{i=1}^{p} \nu_i! \cdot i^{\nu_i}}. \tag{2.14}$$

The number of distinct classes in S_p may be obtained by finding all allowed cycle structures for S_p, that is all nonnegative integral solutions (ν) of equation (2.12). The latter are the p partitions, $f_i, i = 1, \ldots, p$ of the integer p, and they may be

written as

$$\begin{aligned} \nu_1 + \nu_2 + \cdots + \nu_p &= f_1 \\ \nu_2 + \cdots + \nu_p &= f_2 \\ \vdots &= \vdots \\ \nu_p &= f_p, \end{aligned}$$

from which it is clear that $f_1 + f_2 + \cdots + f_p = p$, and $f_1 \geq f_2 \geq \cdots \geq f_p \geq 0$. A partition (f) is designated by $(f_1, f_2, \ldots, f_p)$. When x_{f_i} members f_i of (f) are the same, we may replace them by $f_i^{x_{f_i}}$. Moreover, it is customary to omit values that are zero in the partition. As an illustration, $(f) = (4, 2, 1, 1, 0, 0, 0, 0)$ is more conveniently written as $(4, 2, 1^2)$.

Due to Frobenius and Schur, an essential connection exists between partitions and irreducible representations. Hence, for every partition (f) of S_p as given above, there is an attached irreducible representation of S_p, and *all* irreducible representations may be identified in this way. A proof of this result can be found in [6].

2.4 The Unitary Group

2.4.1 Definition

Consider the form, as written by H. Weyl in [6],

$$G(x, y) \equiv \sum G_{ik} x_i^* y_k,$$

in an n-dimensional complex linear space. If the vectors x, y are transformed by the matrix A according to $x \to x' = Ax$ and $y \to y' = Ay$, the form undergoes the transformation

$$G(x, y) \to G'(x, y) \equiv A^\dagger G A(x, y) = G(Ax, Ay).$$

The form is characterized as *Hermitean* if

$$G(y, x) = [G(x, y)]^*, \text{ or } G_{ki} = G_{ik}^*.$$

The unit Hermitean form refers to the case where $G_{ki} = \delta_{ki}$, and it may be written as

$$(xy)_H = (yx)_H^* = x_1^* y_1 + \cdots + x_n^* y_n = x^\dagger y. \tag{2.15}$$

Note that y may be equal to x, in which case the form is a sum of real terms.

The *unitary group*, $U(n)$, is defined in terms of (2.15). It is the group of change of basis matrices A that leave the scalar product $(xy)_H$ invariant. More symbolically, the group is defined by

$$U(n) = \{A \mid A^\dagger A = AA^\dagger = E\}, \tag{2.16}$$

where $E = E_n$ is the $n \times n$ identity matrix. The matrix A may then be referred to as a *unitary transformation* in n dimensions. The definition (2.16) can be rewritten in terms of the matrix elements to give

$$\sum_{j=1}^{n} A_{ij} A_{lj}^* = \sum_{j=1}^{n} A_{ji} A_{jl}^* = \delta_{il}, \tag{2.17}$$

which we designate the *unitarity condition.*

For $A = [A_{ij}]$, equation (2.16) requires that $\sum_j |A_{ij}|^2 = 1$, such that $|A_{ij}|^2 \leq 1$ for all i and j. The parameters of $U(n)$ are thus restricted to vary over a finite range. The parametric region R that describes the unitary group manifold is thus closed and bounded, that is compact. Therefore, letting $\int dU = 1$ later is not problematic.

As alluded before, the $U(n)$ group is continuous. A unitary transformation has n^2 complex matrix elements, thus it contains $2n^2$ real yet not necessarily independent parameters. Since the unitary condition (2.16) imposes n^2 constraints on such parameters, every element of $U(n)$ can be described by a set of n^2 essential parameters. For example, let us look at the subgroup $U(1) \subset U(n)$, where every element $a = e^{i\theta}$ can be described by the single parameter θ. The group is infinite as there corresponds an element for every θ in the domain $[0, 2\pi]$. Let us now jump directly to aspects of the group measure on the continuous $U(n)$ group manifold.

2.4.2 The Unitary Group Measure

The *group measure*, or *Haar measure*, ds_α, of a continuous group $\tilde{\mathcal{G}} = \{s_\alpha\}$ is expressed in terms of the parameters that define every element of the group. For example,

$$dU_\theta = \frac{1}{2\pi}\, d\theta,$$

where the parameter θ satisfies $0 \leq \theta \leq 2\pi$, and

$$dU_\alpha = \frac{1}{32\pi^3} \sin(\alpha_1)\, d\alpha_1\, d\alpha_2\, d\alpha_3\, d\alpha_4,$$

for which $0 \leq \alpha_1 \leq \pi$, $0 \leq \alpha_2, \alpha_3 \leq 2\pi$, and $0 \leq \alpha_4 \leq 4\pi$, are group measures for $U(1)$ and $U(2)$ respectively. Observe that in the case of the $U(n)$ group, we choose to write the Haar measure ds as dU.

The measure dU possesses invariance properties that are of fundamental importance to the work of Chapter 4. Thus dU satisfies:

(1) *left-* and *right-invariance* as described respectively by $dU = d(VU) = d(UV)$, where $V \in U(n)$;

(2) invariance under transposition, i.e. $dU = dU^t$;

(3) invariance under complex conjugation, i.e. $dU = dU^*$.

It is worth mentioning that dU^{-1} defines another invariant measure, which by uniqueness, must equal dU. Such an equality could replace item (2) or (3) above, yet since it is not of direct use here, it has been omitted. The proofs of the invariance properties can be found and deduced from Chapter 8 of [5]. Items (1)–(3) immediately lead to the integral relations

$$\begin{aligned} \int dU\, f(U, U^*) &= \int dU\, f(VU, V^*U^*) = \int dU\, f(UV, U^*V^*) \\ &= \int dU\, f(U^t, U^{*t}) = \int dU\, f(U^*, U), \end{aligned} \tag{2.18}$$

where f is any function over the group.

2.5 The Frobenius Relation

A convenient relation, the Frobenius relation, exists between the characters of the $U(n)$ group and those of S_p. It can be arrived at by examining some of the results presented in [6]. In particular, by combining Theorems (7.1.C), (7.5.B), and (7.7.A), the Frobenius relation may be derived to give:

$$\sum_{i_1,\ldots,i_p} \prod_{a=1}^{p} A_{i_a\, i_{s(a)}} \Bigg|_{s\in S_p} = \sum_{(f)} \chi_f(s)\tilde{\chi}_f(A), \tag{2.19}$$

where A is an element of $U(n)$, and the sum on the right side ranges over all partitions (representations) (f) of p. In fact, as shown earlier, the S_p group representations may be labeled by the different partitions $(f) = (f_1, f_2, \ldots, f_p)$; in addition, the $U(n)$ irreducible representations contained in a pth rank tensor can be labeled in the same way, i.e. $(f) = (f_1, f_2, \ldots, f_n)$, where $f_1 \geq f_2 \geq \cdots \geq f_n \geq 0$, and $\sum_{i=1}^{n} f_i = p$, such that $f_i = 0$ for $i > p$. Note that (2.19) is class dependent, that is no new relation is obtained when changing s for another element of c_s. This can be seen with ease when the left side of (2.19) is given as

$$\prod_{i=1}^{p} (\mathrm{tr}(A^i))^{\nu_i},$$

the cycle structure attached to the class c_s of s being $(\nu_1, \nu_2, \ldots, \nu_p)$. In more compact form, the Frobenius relation can be rewritten as

$$(A)_s = \sum_{(f)} \chi_f(s)\tilde{\chi}_f(A), \tag{2.20}$$

by denoting the left side of (2.19) by $(A)_s$.

Sufficient tools have now been gathered to help us study and develop techniques for integrating over the $U(n)$ group. Yet before doing so, we shall mention some physical reasons for performing the group integrations. We also wish to introduce some fundamental work done to date on the subject.

CHAPTER 3

ADDITIONAL BACKGROUND

3.1 Applications in Physics

Integrals over the unitary group occur in several areas of physics, those include two-dimensional quantum gravity [9], matrix models [10], lattice gauge theory [1, 11, 12], and the problem of parton saturation at small Feynman x_F [13]. There are also applications in condensed matter physics, [14] is an example.

The development of techniques for integrating over the unitary group manifold has been largely encouraged by the primal role taken by $U(n)$ integrals in *lattice gauge theory*. We intend to briefly introduce how the integrals come about in the latter field. Furthermore, the original motivation for developing the work of Chapter 4 comes from the problem of *parton saturation at small Feynman* x_F. We thus wish to discuss the problem and its connection with the $U(n)$ integrals.

A word of warning is now proper. Whereas integrands of $U(n)$ integrals sometimes occur in physical problems with a $U(n)$ dependent exponential (refer to (3.3) for instance), which implies integrating a *series* of monomials of unitary matrix elements, the method of Chapter 4 is designed for integrating one such monomial at a time. Furthermore, when considering a lattice in some space, each point x of the lattice has a related $U(n)$ group. Where integrals contain elements associated with different points x, a great number of integrations must then be performed, i.e. one for each x. Our method deals with integrals of the form (1.1), which are concerned

with a specific point in space. It thus allows to compute the more general integral in a step by step process, treating one x at a time. One may also regard our work as a pure mathematical physics tool.

3.1.1 Lattice Gauge Theory

A derivation of the complete force between quarks is very hard to perform from the QCD Lagrangian. Analytic calculations have been done in the limits of weak and strong coupling. A strong coupling approximation to a quantum field theory leads to a problem, described by M. E. Peskin and D. V. Schroeder [15] as follows: "In a quantum field theory in which the coupling g^2 is very large, the elementary particles or their bound states typically acquire masses that grow with g^2. For $g^2 \to \infty$, the masses become comparable to the cutoff, and the field theory ceases to have a local continuum description."

A solution to the problem was proposed by K. G. Wilson [1], and it consists in quantizing a gauge field theory on a discrete lattice in Euclidean spacetime, where exact gauge invariance is preserved, and gauge fields are treated as angular variables. As required, this lattice gauge theory has a computable strong-coupling limit. Moreover, in this limit the binding mechanism for total confinement of quarks applies, and there are no free quarks. A pedagogical description of the work has been given by M. Creutz [16]. The principal ideas are in part discussed below.

One way of defining a gauge theory is as a theory of phases. The interaction of a particle with a gauge field introduces a phase factor related to any allowed world line for the particle. In a non-Abelian gauge theory, a path-dependent phase factor of this kind becomes a matrix in the gauge group. Hence the wave function of a material particle that traverses a contour in spacetime acquires a factor from electromagnetic interactions,

$$\psi \to \psi \exp\left[ig \int_P dx^\mu A_\mu^\alpha \lambda^\alpha\right] = U(P)\psi,$$

where the integral is along the path in question. The constant g is the coupling

constant of the gauge theory. The matrix potential $A^\alpha_\mu \lambda^\alpha$ is composed of the gauge potential A^α_μ, also known as *connection*, and the Hermitean matrices λ^α that generate the unitary gauge group.

Within a non-Abelian theory, an element of the gauge group is associated with any path. Given a path $x^\mu(s)$ described by the parameter $s \in [0,1]$, the differential equation,

$$\frac{d}{ds}U(s) = \frac{dx^\mu}{ds} igA^\alpha_\mu \lambda^\alpha U(s), \tag{3.1}$$

defines a group element for the portion of the path from $x^\mu(0)$ to $x^\mu(s)$. Taking $U(0) = 1$ as the initial condition, (3.1) can be solved to give

$$U(s) = P.O.\left(\exp\left[ig\int_0^s ds \frac{dx^\mu}{ds} A^\alpha_\mu \lambda^\alpha\right]\right), \tag{3.2}$$

where $P.O.$ is a path-ordering instruction for the non-commuting matrices $A^a_\mu \lambda^a$.

In order to describe a lattice model of the strong interactions, basic degrees of freedom on the discrete lattice must be defined. Such variables should correspond principally to non-Abelian gauge fields. Wilson suggested that the latter be the line elements (3.2) from a lattice vertex $y_{s=0}$ to another neighbouring one y_s; the element associated with an arbitrary path connecting a sequence of vertices is the group product of the elements describing the path. As such, the expression (3.2) is called a *Wilson line*[1].

The dynamics of the variables can be described by an action S, which is to be constructed. Since the discrete lattice should eventually be considered in the continuum limit, by diminishing the distance between lattice sites, the Lagrangian should then reduce to the classical Yang-Mills theory. Furthermore, because the field strength is a generalized curl of the potential, the use of integrals of $A^\alpha_\mu \lambda^\alpha$ around small closed contours appears to be proper. Thus motivated, Wilson proposed that the action be a sum over all elementary squares or *plaquettes* of the lattice,

$$S = \sum_{\square} S_{\square},$$

[1]It is worth mentioning that the path-dependent phase (3.2) was used before in Schwinger's early work on QED, and by Y. Aharonov and D. Böhm in Phys. Rev. **115**, 485 (1959).

where the action $S_\square = S_\square(U)$ for each plaquette is the trace of the product of the group elements surrounding the plaquette. Similarly, gauge-invariant observables can be constructed by taking a product of the U elements around a closed path.

Advance into the quantum theory is done by inserting the action into a path integral

$$Z = \int [dU]\, e^{-S(U)}, \tag{3.3}$$

thus leading to a mechanical statistical treatment of the interacting quarks. The symbol $[dU]$ denotes the invariant unitary group measures for all links or corners. The integration is done over all possible values for the gauge variables.

The partition function (3.3) is useful in the computation of expectation values or correlation functions. For instance, if H is some function of the field variables U, then its expectation is defined as

$$\langle H \rangle = Z^{-1} \int [dU]\, H(U) e^{-S(U)}. \tag{3.4}$$

In a quantum mechanical Hilbert space, the above gives the vacuum expectation value of the time-ordered operator of interest. Clearly, computation of either (3.3) or (3.4) calls for a method of integration in the unitary group manifold. Other details as to how $U(n)$ integrals enter lattice gauge theory can be found in [11].

3.1.2 Parton Saturation

A large momentum suddenly imparted on an energetic nucleon or nucleus causes it to radiate gluons, in much the same way that an electron radiates photons under similar circumstances. However, notice that as gluons can radiate other gluons, the same does not apply to photons. The softer the gluon, the more it is being radiated, until the gluon density reaches a point where further radiation is prevented. When do saturations occur, and how many gluons there are when that occurs are important questions to answer, as they affect the cross-sectional behaviour of various processes. There is presently intense theoretical activity in this area as saturation in nucleon and nucleus are beginning to be observed.

In an approach made by C. S. Lam et al [13], a distribution function $\overline{W}_\tau(v)$ is introduced to study the problem of gluon saturation at small Feynman x_F. The x_F value can be understood as the ratio of a parton momentum to the fixed momentum of the associated hadron. The distribution is given in terms of the unitary Wilson line v, and $\tau = \ln(1/x_F)$. Energetic partons interact with soft gluons via their Wilson lines. As explained in [13], it is this interaction that is responsible for the saturation process.

A distribution function is a probability function. It can thus be used for determining average values of various observables related to the system under consideration. In the context of [13], assuming $\overline{W}_\tau(v)$ to be normalized to unity, the average of any functional $B(v, v^\dagger)$ is given by

$$\int \mathcal{D}_H[v]\, B(v, v^\dagger)\overline{W}_\tau(v),$$

where $\mathcal{D}_H[v] = \prod_{\mathbf{x}} d_H[v(\mathbf{x})]$ is the product of the Haar measures associated with each transverse position $\mathbf{x}$ within a Lorentz-contracted nucleus. Note that the authors cover the transverse $\mathbf{x}$-plane with a lattice. Again, a necessity to compute $U(n)$ integrals is apparent, and the work to follow is justified.

3.2 Known Unitary Group Integration Methods

A variety of methods for computing $U(n)$ integrals have been developed in the context of lattice gauge theory [12]. Thus, techniques have been explored in the simplified case where $n \to \infty$. Moreover, it was shown that the integrals may be expressed in terms of totally antisymmetric tensors or Kronecker delta symbols, which allows for diagrammatic representations of the integrals. Another key element from [12] is that generating functionals can be used efficiently in the computation. An important contribution to the subject has also been made via [2], in which formulae for the evaluation of all $U(n)$ integrals of the type (1.1) are given. The article makes extensive use of the group theory of the symmetric group. As is the case with our work, integrals over a single link are examined in the mentioned papers.

A similar group theoretical formula to the ones found in [2] can be obtained from the Frobenius relation (2.20) and the orthogonality relations of Section 2.2.4. Such formula will be used extensively later. An important integral that involves a $U(n)$ dependent exponential function is the Itzykson-Zuber integral, which was first derived by Harish-Chandra [17], and later studied and fully exploited by C. Itzykson and J.-B. Zuber [3]. Stated differently, the Itzykson-Zuber integral is a special case of a more general result due to Harish-Chandra. The equation for the integral is powerful as it is concerned with an exponential series of $U(n)$ monomials. Moreover, this integral is well-known in the field of lattice QCD and in the context of matrix models. The two $U(n)$ integration methods should now be presented. In Section 3.2.2, we shall also demonstrate how the Itzykson-Zuber formula can be reduced to the simple group theoretical formula.

3.2.1 The Group Theoretical Formula

Using the Frobenius relation (2.20) and the orthogonality relations (2.8) and (2.11), the group theoretical formula[2],

$$\int dU\, U^*_{i_1 j_1} \cdots U^*_{i_p j_p} U_{k_1 l_1} \cdots U_{k_p l_p} = \sum_{r,s' \in S_p} \prod_{a=1}^{p} \delta_{k_a i_{r(a)}} \delta_{j_a l_{s'(a)}} \sum_{(f)} \frac{d_f^2}{p!^2 \tilde{d}_f} \chi_f(rs'), \quad (3.5)$$

can be derived. Refer to Appendix A for the derivation. The first two sums are over all elements r and s' of S_p, and the last one is over all inequivalent representations (f) of the same group. Recall that $d_f = \chi_f(e)$. The characters of S_p together are given by γ_p^2 numbers, where γ_p is either the total number of distinct classes or the number of inequivalent representations. Character tables are available for relatively high p values, get acquainted with the *Maple 7* mathematics software[3] or a more recent version for instance. The character tables for p up to 7 can also be found in [7]. The dimension of the representation (f) of the unitary group is given by

$$\tilde{d}_f = \frac{D(l_1, \ldots, l_n)}{D(n-1, \ldots, 0)}, \quad (3.6)$$

[2]We would like to thank A. D'Adda for having introduced us to the formula.

[3]For more information, refer to www.maplesoft.com.

$$l_i = f_i + (n - i), \qquad i = 1, \ldots, n,$$

where $D(x_1, \ldots, x_n)$ is the difference product of its arguments, also known as Vandermonde determinant, i.e. $D(x_1, \ldots, x_n) = \prod_{i<k}(x_i - x_k)$. For more details as to how (3.6) is obtained, refer to [6]. The group theoretical formula will be re-examined in Section 4.3.

3.2.2 The Itzykson-Zuber Formula

The expression,

$$I(M_1, M_2; \beta) = \int dU \, \exp[\beta \, \mathrm{tr}(M_1 U M_2 U^\dagger)],$$

where M_1 and M_2 are $n \times n$ Hermitean matrices, is studied in [3]. A series expansion in terms of the characters of the linear (or unitary) group is derived for $I(M_1, M_2; \beta)$, that is

$$I(M_1, M_2; \beta) = \sum_{\{f\}} \frac{\beta^{|f|}}{|f|!} \frac{d_f}{\tilde{d}_f} \tilde{\chi}_f(M_1) \tilde{\chi}_f(M_2), \tag{3.7}$$

where the sum is over all representations $\{f\}$, i.e. all representations (f) for all $|f| \equiv \sum_i f_i$ positive and integral. The other definitions are the same as before except that they apply to the representation $\{f\}$ as above. The derivation relies principally on: the orthogonality relation (2.10), the completeness relation

$$\sum_{\{f\},a,a'} \tilde{d}_f D^{\{f\}}_{aa'}(U) D^{\{f\}*}_{aa'}(U') = \delta(U, U'),$$

the continuous and inverted version of the Frobenius relation,

$$\int_{V \in U(n)} dV \, (\mathrm{tr}V)^q \tilde{\chi}^*_f(V) = \delta_{q,|f|} \, d_f,$$

and the invariance of the Haar measure under the adjoint action, i.e. $V \to UVU^\dagger$, for $U, V \in U(n)$, which is equivalent to the invariance property (1) encountered in Section 2.4.2.

The group theoretical formula (3.5) can be obtained from the Itzykson-Zuber formula (3.7). Using the orthogonality relation (2.6), Frobenius relation (2.20) can

be inverted such that (3.7) reads as

$$I(M_1, M_2; \beta) = \sum_{\{f\}} \frac{\beta^{|f|}}{|f|!^3} \frac{d_f}{\tilde{d}_f} \sum_{r,s} \chi_f(r)\chi_f(s)(M_1)_r(M_2)_s.$$

Next, use the orthogonality relation (2.9) to combine the two characters into one,

$$\begin{aligned} I(M_1, M_2; \beta) &= \sum_{\{f\}} \frac{\beta^{|f|}}{|f|!^4} \frac{d_f^2}{\tilde{d}_f} \sum_{r,s,t} \chi_f(rtst^{-1})(M_1)_r(M_2)_s \\ &= \sum_{\{f\}} \frac{\beta^{|f|}}{|f|!^4} \frac{d_f^2}{\tilde{d}_f} \sum_{r,s',t} \chi_f(rs')(M_1)_r(M_2)_{t^{-1}s't}. \end{aligned} \tag{3.8}$$

Since $(M_2)_{s'}$ depends only on the class in which s' lies, the last factor is independent of t. Therefore the sum over t yields only a factor $|f|!$. Let us then rewrite (3.8) as

$$\int dU \sum_{|f|} \frac{1}{|f|!} \beta^{|f|} (\mathrm{tr}(M_1 U M_2 U^\dagger))^{|f|} = \sum_{\{f\}} \frac{\beta^{|f|}}{|f|!^3} \frac{d_f^2}{\tilde{d}_f} \sum_{r,s'} \chi_f(rs')(M_1)_r(M_2)_{s'}.$$

Identify the terms that are proportional to β^p on both sides, and obtain

$$\int dU \, (\mathrm{tr}(M_1 U M_2 U^\dagger))^p = \sum_{(f)} \frac{1}{p!^2} \frac{d_f^2}{\tilde{d}_f} \sum_{r,s'} \chi_f(rs')(M_1)_r(M_2)_{s'}. \tag{3.9}$$

As M_1 and M_2 are arbitrary, the equation (3.5) is recovered by comparing the coefficients of $\prod_{a=1}^{p} (M_1)_{i_a k_a} (M_2)_{l_a j_a}$ on both sides of (3.9).

CHAPTER 4

INVARIANT AND GROUP THEORETICAL INTEGRATIONS

4.1 Notation

As done in Appendix A, let I_s denote the ordered index set[1] $(i_{s(1)}, \ldots, i_{s(p)})$ of arbitrary p indices, where $s \in S_p$. In the case where $s = e$, we may write I instead of I_e. An ordered index set with repeated indices may be written in such a way that the indices are raised to their number of occurences, e.g. $(a, a, b, c, c, c) = (a^2, b, c^3)$. Unless otherwise indicated, an index set should always be understood as ordered.

Let A_{IJ} stand for the juxtaposition $\prod_{a=1}^{p} A_{i_a j_a}$ of p elements of any matrix A. The symbol δ_{IJ} obeys a similar definition where $A_{i_a j_a}$ is replaced by $\delta_{i_a j_a}$. Note that because the matrix elements in A_{IJ} commute,

$$A_{IJ} = A_{I_s J_s}, \tag{4.1}$$

for any $s \in S_p$. The integral (1.1) is dependent only on the index sets I, J, K, and L, we may therefore denote it by $\langle IJ|KL\rangle$. Hence, in the case where $p = q$, another

[1] As opposed to unordered sets, which are enclosed with curly brackets, we enclose ordered sets with round brackets. Thus the ordered set (a, b) is considered different from the ordered set (b, a), whereas the unordered sets $\{a, b\}$ and $\{b, a\}$ are considered equal. Repetitions is allowed in an ordered set, e.g. (a, a, b) is different from (a, b).

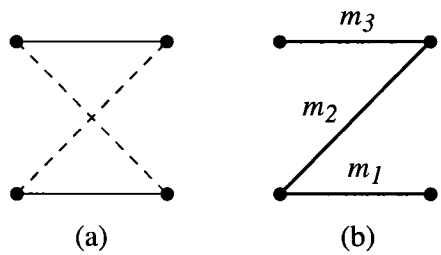


Figure 4.1: Examples of $U(n)$ integral diagrams. (a) The unique exchange integral for $p = 2$; (b) a Z-integral with arbitrary multiplicities m_1, m_2, and m_3.

way of writing (1.1) is

$$\langle IJ|KL\rangle = \int dU\, U^*_{IJ} U_{KL}. \tag{4.2}$$

The number p of U^* (or U) matrix elements is then referred to as the *degree* of the integral.

As will be shown later, any nonvanishing integral $\langle IJ|KL\rangle$ can be written as $\langle IJ|IJ_q\rangle$, for $q \in S_p$. If $J_q = J$, then the integral $\langle IJ|IJ\rangle \equiv \|IJ\|$ is positive definite. We shall refer to integrals of this type as *direct integrals.* If $J_q \neq J$, the sign is not guaranteed, and we shall refer to the corresponding integrals as *exchange integrals.*

It is convenient to represent the integral $\langle IJ|IJ_q\rangle$ graphically. Thus, consider two columns of dots where each dot on the left (L-dot) is assigned a distinct value in the index set I, and every dot on the right (R-dot) a distinct value in the index set J or J_q. In other words, no two dots in the same column may refer to the same index value, but no such restriction applies for two dots in different columns. The factor U^*_{ij} is shown as a thin solid line between the L-dot i and the R-dot j, and the factor U_{ij} is shown as a dotted line between the same two dots. The factor $U^*_{ij}U_{ij}$ is represented by a thick line, or more generally, the factor $U^{*m}_{ij}U^n_{ij}$ is represented by a thick line with a pair of numbers (m, n) written beside it. If $m = n$, then only a single number m is written. We designate the numbers (m, n) or m as the *multiplicities* of their attached line. See Fig. 4.1 for an illustration.

4.2 The Invariant Method (IM)

We have seen that (1.1) can be computed using the group theoretical formula (3.5). The method is powerful as it allows to compute any integral directly. However, as the degree p of the integral increases, the work becomes rapidly wearisome. In fact, the number $p!$ of elements of S_p increases rapidly with increasing p. The sums $\sum_{r,s' \in S_p}$ thus become generally more extensive, and there are potentially more products rs' to compute. The sum $\sum_{(f)}$ also becomes more important in the same conditions; for instance, the size $\gamma_p \times \gamma_p$ of a character table changes according to $\gamma_p = 2, 3, 5, 7, 11, 15, 22, 30$ for the respectively changing $p = 2, 3, 4, 5, 6, 7, 8, 9$.

In this section, we discuss an alternative method to calculate the monomial integral, using as input only the unitarity of the matrices U and the invariance of the Haar measure dU. No knowledge of group theory is required, and the multiple sums are avoided. We refer to this method as the *invariant method (IM)* [18].

As pointed out by C. S. Lam, the IM is also applicable to the much simpler case of a monomial integral over a hypersphere. This simpler case will be used as a testing ground for the method. The integration over a hypersphere will be discussed in Sec. 4.2.1, as a preparation for the computation of the monomial $U(n)$ group integrals in Sec. 4.2.2.

4.2.1 Integration over a Hypersphere

Let Ω_{n-1} be the unit sphere in n dimensions, defined by

$$\sum_{i=1}^{n} x_i^2 = 1, \tag{4.3}$$

and $d\omega$ be its rotationally symmetric volume element, normalized to $\int d\omega = 1$. We wish to calculate the integral $\int d\omega\, Y$ over Ω_{n-1}, where Y is a monomial of the coordinates x_j $(j = 1, 2, \ldots, n)$. The integral is zero unless the power of every x_j is even, in which case it can be written in the form

$$\langle J|J\rangle = \int d\omega\, X_J X_J, \tag{4.4}$$

where $X_J \equiv \prod_{a=1}^{p} x_{j_a}$ is a monomial of degree p, indexed by the set $J = (j_1, \ldots, j_p)$.

One can attempt to calculate the integral in several ways. Three of them having analogs with the $U(n)$ integrals will be singled out, because the simpler setting of a sphere should make their relative merits more transparent. The first two approaches are standard, both relying on spherical coordinates. The third, which we wish to develop here, is an *invariant approach*, requiring no coordinate system in its computation.

Spherical coordinates

The spherical coordinates of x_j on the unit sphere are:

$$\begin{aligned} x_n &= \cos\theta_1 \\ x_{n-1} &= \sin\theta_1 \cos\theta_2 \\ x_{n-2} &= \sin\theta_1 \sin\theta_2 \cos\theta_3 \\ &\vdots \\ x_2 &= \left(\prod_{i=1}^{n-2} \sin\theta_i\right) \cos\phi \\ x_1 &= \left(\prod_{i=1}^{n-2} \sin\theta_i\right) \sin\phi. \end{aligned} \tag{4.5}$$

The variables θ_i $(i = 1, \ldots, n-2)$ range between 0 and π, while ϕ between 0 and 2π. The volume element is

$$\begin{aligned} d\omega' &= \left(\prod_{i=1}^{n-2} (\sin\theta_i)^{n-i-1} d\theta_i\right) d\phi \\ d\omega &= \frac{d\omega'}{\int d\omega'}. \end{aligned}$$

Using the formula

$$\int_0^{\frac{\pi}{2}} d\theta \, \sin^{r-1}\theta \cos^{s-1}\theta = \frac{1}{2} \frac{\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{r+s}{2}\right)} \tag{4.6}$$

and (4.5), the integral $\langle J|J\rangle$ can be calculated for every index set J.

For example, if all the indices in J are equal to n, i.e. $J = (n \cdots n) = (n^p)$, then the integral is equal to

$$\begin{aligned} \langle J|J\rangle \equiv S(p) &= \frac{\int_0^{\pi/2} d\theta \cos^{2p}\theta \sin^{n-2}\theta}{\int_0^{\pi/2} d\theta \sin^{n-2}\theta} \\ &= \frac{1}{\sqrt{\pi}} \frac{\Gamma(p+\frac{1}{2})\Gamma(\frac{n}{2})}{\Gamma(p+\frac{n}{2})}. \end{aligned} \tag{4.7}$$

However, if we replace $J = (n^p)$ by $J = (1^p)$, or equivalently, x_n^{2p} in the integrand by x_1^{2p}, then the integral becomes much harder to calculate, because $(n-1)$ times more integrations must be performed. Yet, on account of the spherical symmetry, the result must come out to be the same as (4.7). This complication arises because a choice of axes breaks the spherical symmetry. Such a situation is avoided in the invariant approach discussed below.

The present method relies on an explicit parametrization of Ω_{n-1} via the spherical coordinates, as well as formula (4.6), which allows for the integrations to be carried out. A similar apparatus is much more difficult to instate in the $U(n)$ case, and the method is not quite useful there. There shall be no further discussion of this method.

Group theory

Alternatively, X_J can be expanded in terms of spherical harmonics, and the integral can be transformed into a sum using the orthonormality of the spherical harmonics. For $n = 3$, the expansion is

$$X_J = \sum_{l,m} a_{lm} Y_{lm}(\theta, \phi),$$

and the integral becomes

$$\langle J|J\rangle = \sum_{l,m} |a_{lm}|^2. \tag{4.8}$$

For $n > 3$, more sums are involved in (4.8). There are two non-trivial tasks in this approach: to find the coefficients a_{lm}, and to carry out the sum (4.8). Such tasks become quite difficult in practice for large n or p.

An analogous group theoretical technique to calculate a parallel $U(n)$ integral was presented in Sec. 3.2.1. As hinted already, formula (3.5) yields any integral directly, yet a multiple sum must be computed, and this could be tedious to perform. In practice, this expansion method may not be the best way to obtain a result. The invariant approach discussed below and in Sec. 4.3 is potentially simpler.

The invariant approach

The integral in (4.4) can be calculated directly, by using the condition (4.3) and the invariance of $d\omega$ under rotation. In particular, there is no need to employ the spherical coordinate system, and no need to know any integration formula, nor group theory.

The invariant approach will be used shortly to calculate integrals (of monomials of unitary matrix elements) over the unitary group. In that case, (4.3) is replaced by the unitarity condition (2.17), and $d\omega$ by the invariant Haar measure dU of the unitary group.

It is convenient to arrange the p indices in J according their distinct values (between 1 and n). If m_1 of these p indices take on a value, m_2 of them take on a second value, and so on, then the integral $\langle J|J\rangle$ will be denoted by $S(m_1, m_2, \ldots, m_t)$, where t is the number of distinct values in J, and $\sum_{i=1}^{t} m_i = p$. Spherical symmetry guarantees that the integral is independent of the specific values that the indices assume. As a result, S is symmetrical in all its arguments.

We calculate an arbitrary S by first relating it to $S(p)$, which is done by making use of the invariance of $d\omega$. Then, the value of $S(p)$ is calculated by using the sphere condition (4.3).

The invariance of $d\omega$ can be exploited in the following way. A rotation in the $x_i x_k$-plane by an angle ξ,

$$\begin{aligned} x_i &\rightarrow +\mathrm{c}x_i + \mathrm{s}x_k \\ x_k &\rightarrow -\mathrm{s}x_i + \mathrm{c}x_k, \end{aligned} \tag{4.9}$$

where $\mathrm{c} \equiv \cos\xi$ and $\mathrm{s} \equiv \sin\xi$, leaves $d\omega$ invariant. Similarly, if we subject the

integrand $X_J X_J$ in $\langle J|J\rangle$ to such a rotation, the integral $S(m_1,\ldots,m_k)$ remains unchanged.

Let us begin with the integral $S(p)$ whose X_J is equal to x_1^p. Under (4.9), with $(i,k)=(1,2)$, the integrand becomes

$$(x_1^2)^p \to (\mathrm{c}x_1+\mathrm{s}x_2)^{2p}=\sum_{e=0}^{p}\binom{2p}{2e}(\mathrm{c}x_1)^{2(p-e)}(\mathrm{s}x_2)^{2e}+\cdots .$$

The symbol $\binom{n}{k}$ denotes the binomial coefficient $n!/k!(n-k)!$. The ellipsis represents terms odd in x_1 and x_2, which can be dropped because they do not contribute to the integral. The invariance of the integral under this transformation yields the relation

$$S(p)=\sum_{e=0}^{p}\binom{2p}{2e}\mathrm{c}^{2(p-e)}\mathrm{s}^{2e}S(p-e,e).$$

Since the relation must be true for all ξ, its right side must be independent of this ξ, such that

$$S(p-e,e)=\frac{\binom{p}{e}}{\binom{2p}{2e}}S(p).$$

Similarly, we can apply (4.9) and the whole procedure to $(i,k)=(2,3)$ to get

$$S(p-e,e-f,f)=\frac{\binom{e}{f}}{\binom{2e}{2f}}S(p-e,e)=\frac{\binom{e}{f}}{\binom{2e}{2f}}\frac{\binom{p}{e}}{\binom{2p}{2e}}S(p).$$

Continuing thus, we finally obtain

$$S(m_1,\ldots,m_t)=\frac{(\sum_{i=1}^t m_i)!}{\prod_{j=1}^t m_j!}\frac{\prod_{j=1}^t(2m_j)!}{(\sum_{i=1}^t 2m_i)!}S(p). \tag{4.10}$$

Relations between arbitrary S integrals and $S(p)$ have now been established.

To complete the calculation we must calculate $S(p)$. This can be done by using condition (4.3). The integral $S(p-1,1)$ contains two distinct coordinates, say x_i and x_j $(i\neq j)$, such that its integrand may be written as $x_i^{2(p-1)}x_j^2$. Since the specific identity of the coordinates is irrelevant, the relevance lying only in the fact that two of them are distinct with $m_1=p-1$ and $m_2=1$, the sphere condition (4.3) applied to the coordinate x_j with $m_2=1$ can be translated to read

$$(n-1)S(p-1,1)+S(p)=S(p-1). \tag{4.11}$$

Using (4.10), we know that $S(p-1,1) = S(p)/(2p-1)$. Substituting this back into (4.11), we get the recursion relation

$$S(p) = \frac{2p-1}{n+2p-2} S(p-1). \tag{4.12}$$

With the initial value $S(0) = 1$, (4.12) can be solved to yield

$$S(p) = \frac{(2p-1)(2p-3)\cdots 1}{(n+2p-2)(n+2p-4)\cdots n} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(p+\frac{1}{2})\Gamma(\frac{n}{2})}{\Gamma(p+\frac{n}{2})}, \tag{4.13}$$

which agrees with the answer given by (4.7). The general result is obtained by substituting (4.13) into (4.10).

4.2.2 Integration over the Unitary Group

The invariant calculation of (1.1) relies only on the unitarity of the matrices in the integrand, expressed by (2.17), as well as the invariance of the Haar measure, which allows for (2.18). We shall initially apply (2.18) to the function $f'(U,U^*) = \prod_{a=1}^{p} U^*_{i_a j_a} \prod_{b=1}^{q} U_{k_b l_b}$, then applications to $f(U,U^*) = U^*_{IJ} U_{KL}$ will be more proper.

The calculation is very similar to that in the last section, though more complicated. The spherical condition (4.3) is now replaced by the unitarity condition (2.17), and the rotational invariance of $d\omega$ is now replaced by the group invariance of dU.

Relations from Invariance

The set of equalities in (2.18) is very powerful. Depending on the choice of V, many relations can be derived. Below are given some useful examples.

$V_{kl} = e^{i\phi}\delta_{kl}$ With this choice, $f'(UV, U^*V^*) = e^{i(q-p)\phi} f'(U,U^*)$. Hence the integral (1.1) is zero unless $q = p$. For this reason, we shall assume that $q = p$ from now on, and thus concentrate only on the integral (4.2).

$V_{kl} = e^{i\phi_k}\delta_{kl}$ With this choice, $f(UV, U^*V^*) = e^{i\zeta}f(U, U^*)$, where $\zeta = \sum_{a=1}^{p}(\phi_{l_a} - \phi_{j_a})$. The vanishing of (4.2) is avoided when the phase ζ is zero. If all the ϕ_k quantities are different, this happens only when L is the same as J up to a permutation. Similarly, one can show that K must differ from I by at most a permutation. For proper $r, s \in S_p$, the above can be rewritten symbolically as

$$K = I_r, \qquad L = J_s. \tag{4.14}$$

Thus, using (4.1) and denoting sr^{-1} by q, the nonzero integrals (4.2) reduce to

$$\langle IJ|IJ_q\rangle = \int dU\, U^*_{IJ}U_{IJ_q}.$$

As a consequence, every dot in a diagram for the integral is attached to an equal number of dotted lines and thin solid lines.

While the nonzero integral over a hypersphere encountered before is of the form $\langle J|J\rangle$, the nonzero integral over $U(n)$ appears as $\langle IJ|IJ_q\rangle$. The presence of an additional index set I and the possibility that $J_q \neq J$ both make it harder to calculate the $U(n)$ integral than the spherical integral.

The direct integral ($J_q = J$) looks deceptively similar to the spherical integral (4.4) in a complex n^2-dimensional space. By mapping U_{ij} to x_a, with $a = (i, j)$ running between 1 and n^2, the complex equivalent of (1.1), namely $\sum_{a=1}^{n^2}|x_a|^2 = n$, is guaranteed by the unitarity relation (2.17). One might therefore think that the direct integrals would turn out to be very similar to the spherical integrals, whose result is given by (4.10) and (4.13). Unfortunately that is not the case, because the measure dU is not rotational invariant in the n^2-dimensional complex vector space. As a result, even the direct integrals are more difficult to calculate than the spherical integrals.

V is a permutation matrix of n objects In that case, VU is obtained from U by permuting its rows, and UV is obtained from U by permuting its columns. With this V, equation (2.18) implies that

$$\langle IJ|KL\rangle = \langle I'J|K'L\rangle = \langle IJ'|KL'\rangle, \tag{4.15}$$

where I' is obtained from I by a reassignment of the values of its indices, and K' is obtained from K by the same reassignment. For instance, let us take $p = 6$ and $n = 8$. Suppose that $I = (1,1,5,5,5,2)$ and $K = (5,1,5,2,1,5)$. If we make the reassignment $1 \leftrightarrow 8$, $2 \leftrightarrow 5$, $4 \leftrightarrow 7$, then $I' = (8,8,2,2,2,5)$ and $K' = (2,8,2,5,8,2)$. In other words, the integral is affected by whether the indices take on the same or different values, but is independent of what these values are. For that reason, labels need not be affixed on the dots in the diagram representation of an integral.

Row-column index interchange Due to the invariance $dU = dU^t$ in (2.18), an integral is unaltered by an interchange of the row and column index sets, i.e.

$$\langle IJ|KL\rangle = \langle JI|LK\rangle.$$

In graphical terms, the above implies that two diagrams that differ by an interchange of the R-dots with the L-dots yield the same integral.

$U \leftrightarrow U^*$ interchange Due to the invariance $dU = dU^*$ in (2.18), an integral remains unchanged under the complex conjugation of its integrand, which implies

$$\langle IJ|KL\rangle = \langle KL|IJ\rangle.$$

In graphical terms, the above translates into the fact that two diagrams that differ by an interchange of the thin solid lines with the dotted lines yield the same integral.

V, a rotation matrix by angle θ in the ab-plane The matrix has values 1 along the main diagonal, except at the (a,a) and (b,b) positions, where the matrix element is $\mathrm{c} = \cos\theta$. The off-diagonal matrix elements are all zero, except at the positions (a,b) and (b,a), where the matrix elements are respectively $\mathrm{s} = \sin\theta$ and $-\mathrm{s}$.

The replacement $U \to UV$ causes the following change in the matrix elements:

$$\begin{aligned} U_{ia} &\to +\mathrm{c}U_{ia} - \mathrm{s}U_{ib}, \\ U_{ib} &\to +\mathrm{s}U_{ia} + \mathrm{c}U_{ib}, \\ U_{ij} &\to U_{ij}, \end{aligned}$$

provided that $j \neq a, b$. Similar replacements on U^*_{ia} and U^*_{ib} should also be made. The transformation changes $\langle IJ|IJ_q\rangle$ into a sum of terms of the form

$$M_e(\mathrm{c}^2)^{d-e}(\mathrm{s}^2)^e, \tag{4.16}$$

where d is the total number of column indices in U^*_{IJ} with value a or b, and e varies between 0 and d. Note that odd powers of cs never enter (4.16) because of the (4.15) requirement. The invariance condition (2.18) demands that

$$\langle IJ|IJ_q\rangle = \sum_{e=0}^{d} M_e(\mathrm{c}^2)^{d-e}(\mathrm{s}^2)^e.$$

In order for this to be true for all θ angles, the equality

$$M_e = \langle IJ|IJ_q\rangle \binom{d}{e} = M_0 \binom{d}{e} \tag{4.17}$$

must be satisfied. We may refer to the previous relation as the spin-off relation.

Let us see how M_e is computed in the graphical language. Take any two dots on the right column of $\langle IJ|IJ_q\rangle$ to be affixed the values a and b. One of the two dots should have some lines attached to it, but the other may or may not be free. The total number of thin solid (or dotted) lines attached to the two dots is d. Consider a move of e thin solid and e dotted lines between the two dots, subject to the constraint that at the end of the move, each dot must have an equal number of solid and dotted lines attached to it (otherwise the integral is zero). A weight of -1 must be assigned for a line moved from a to b, and a weight of $+1$ for each line moved from b to a. The quantity M_e in (4.17) is simply the sum of all the resulting integrals after the move, weighted by the product of the ± 1 associated with each move.

An important remark is that the above relations are local; they involve only the indices of J and J_q with values a and b. It does not matter what I is, and what the rest of the indices of J and J_q are. A similar relation applicable to L-dots (row indices) is also valid, and it can be derived similarly.

Let us illustrate the graphical application of the rotational relation with three examples.

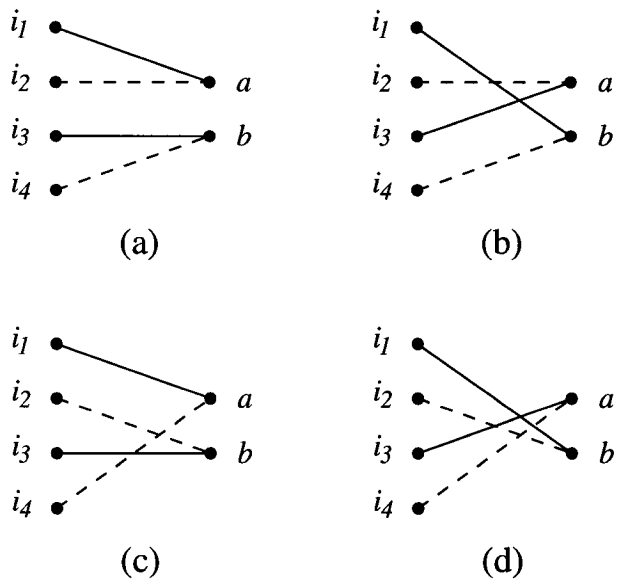


Figure 4.2: Four diagrams illustrating the use of equation (4.17), which is derived from the rotational invariance of the Haar measure.

Example 1. Fig. 4.2 shows partial diagrams. The whole diagram may have many more dots and lines, yet they are not drawn as they do not affect the relation to be derived shortly. The two right dots a and b are different, and no lines other than the ones shown are connected to them. The indices i_1, i_2, i_3, i_4 may or may not have the same values, and there may be many more lines attached to them than is shown, as long as these other lines do not connect with either a or b. If some of their values are the same, say $i_1 = i_2$, then graphically the two dots i_1, i_2 simply merge together into a single dot. If the values are different, then other lines must also come out of these dots in order to make the integral nonzero.

Let the integral corresponding to Fig. 4.2(a) be denoted by $I(4.2a)$, and similarly for the other partial diagrams of the same figure. Everywhere we are dealing with $d = 2$, because there are two pairs of (solid and dotted) lines ending on the right dots.

Applying rotation to Fig. 4.2(a) yields some useful information. There are two cases of interest, i.e. two and four lines being moved, which respectively correspond

to $e = 1, 2$. The situation in which all lines remain in position is trivial. In the case where two lines are moved, we get $I(4.2b)$ (two solid lines), $I(4.2c)$ (two dotted lines), and two other integrals, I_1 (one solid and one dotted lines from a to b) and I_2 (one solid and one dotted lines from b to a). The graphs for I_1 and I_2 not shown, but they can be obtained from $I(4.2a)$ by merging the a and b dots. In this way, we obtain $M_1 = -I(4.2b) - I(4.2c) + I_1 + I_2 = 2(-I(4.2b) + I_1)$, where (4.15) was used in the last step. Referring to (4.17), we find that $M_1 = 2M_0 = 2I(4.2a)$, such that the relation

$$I(4.2a) = -I(4.2b) + I_1$$

follows clearly. When four lines are moved, the only diagram obtained is the one found in Fig. 4.2(d), thus $M_2 = I(4.2d)$. The formula in (4.17) demands that $M_2 = M_0$, or $I(4.2d) = I(4.2a)$. We already know this to be true from (4.15).

Example 2: the fan relation. From the rotation result (4.17), a useful relation, the *fan relation*, emerges. It is given by

$$\int dU\, A\, |U_{ac_1}|^{2m_1} \cdots |U_{ac_t}|^{2m_t} = \frac{\prod_{j=1}^{t} m_j!}{(\sum_{j=1}^{t} m_j)!} \int dU\, A\, |U_{ac}|^{2m}, \qquad (4.18)$$

which relates integrals of the same $m = \sum_{j=1}^{t} m_j$, where A is an arbitrary product of matrix elements of U and U^* whose column indices are different from $c_1, c_2, \ldots, c_t$. The column index c on the right could be taken to be one of the c_i's. However, for the sake of the derivation to follow, and without loss of generality, let $c = c_1$.

Diagrammatically, the integral on the right of (4.18) is shown in Fig. 4.3(a), and the integral on the left is shown in Fig. 4.3(b). The additional lines and dots corresponding to the factor A are not shown, because they do not affect the fundamental result (4.17). We shall refer to Fig. 4.3(a) as a *closed fan*, and Fig. 4.3(b) as a *partially opened fan*. If every $m_i = 1$, then it will be said to be a *fully opened fan*, or simply an *opened fan*.

For convenience, let the $U(n)$ integral with integrand $A\, |U_{ac_1}|^{2m_1} \cdots |U_{ac_x}|^{2m_x}$ be denoted by $f(m_1, \ldots, m_x)$. For example, Fig. 4.3(a) could be written as $f(m)$, and

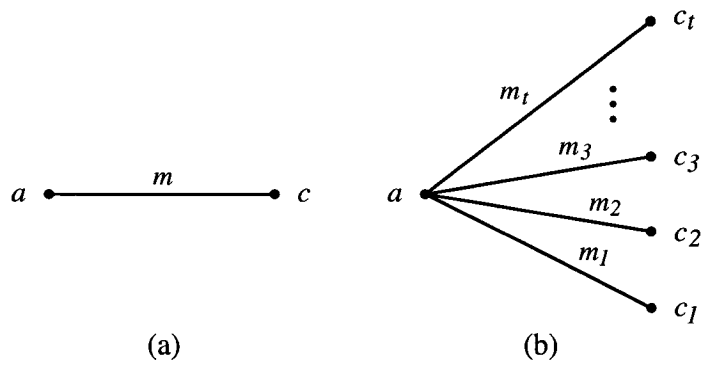


Figure 4.3: The fan diagrams shown here can be a part of a larger diagram. In that case, there may be more dots and lines in the complete diagram, provided none of the additional lines land on the R-dots shown. (a) A closed fan; (b) a partially opened fan. If all multiplicities $m_i = 1$, then it is said to be a fully opened fan, or just an opened fan.

Fig. 4.3(b) as $f(m_1, \ldots, m_t)$. We wish to start with Fig. 4.3(a), then reach Fig. 4.3(b) via a sequence of moves of dotted and solid lines. Thus, consider a rotation away of $e = \sum_{i=2}^{t} m_i$ solid and e dotted lines from $c = c_1$ to an empty dot c_2. There are $\binom{m}{e}$ ways of choosing the set of solid lines to move, and independently there are also $\binom{m}{e}$ ways to select the dotted lines. Hence $M_e = \binom{m}{e}^2 f(m-e, e)$. From (4.17), we know that $M_e = \binom{m}{e} M_0 = \binom{m}{e} f(m)$. As a result,

$$f(m-e, e) = \frac{1}{\binom{m}{e}} f(m).$$

The process can be repeated by moving $g = \sum_{i=3}^{t} m_i$ pairs of lines from c_2 to an empty dot c_3. Then

$$f(d-e, e-g, g) = \frac{1}{\binom{e}{g}} f(m-e, e) = \frac{1}{\binom{m}{e}} \frac{1}{\binom{e}{g}} f(m).$$

Following a repetition of this procedure, we arrive at the fan relation (4.18), which indicates how to fan out a thick line with sufficiently high multiplicity $m > 1$ into t different lines. In particular, a closed fan is $m!$ times an opened fan integral.

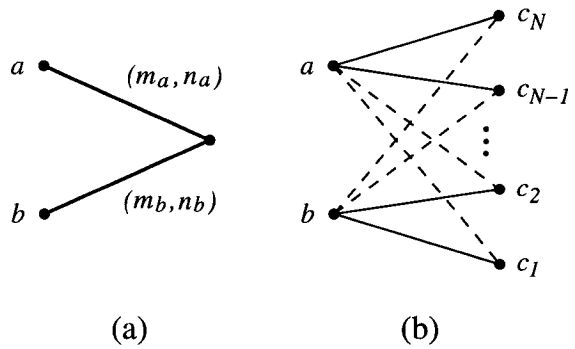


Figure 4.4: Double-fan diagrams. (a) A closed diagram; (b) a fully opened diagram.

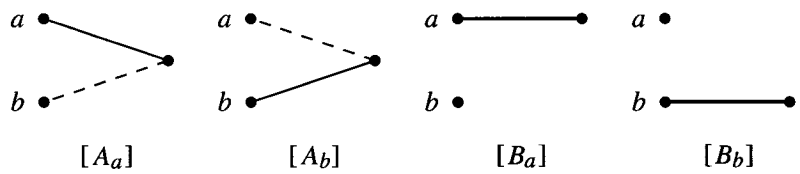


Figure 4.5: The four basic patterns for the R-dots of a fully opened double-fan graph.

Example 3: the double-fan relation. The fan relation can be generalized to a double-fan relation, connecting the closed 'double-fan' diagram of Fig. 4.4(a) with (fully) opened double-fan diagram(s) such as Fig. 4.4(b). As in Fig. 4.3, there may be additional dots and lines in the integral, but none of them may end up on the R-dots shown. In what follows we will always refer to the partial graphs of Fig. 4.4 or related ones. From the double-fan relation, it is possible to deduce relations between closed double-fans and partially opened double-fans; both types of double-fan can be related to fully opened double-fans.

The double-fan relation is considerably more complicated than the single-fan relation (4.18), because there are many more double-fan graphs. Each R-dot of a (fully) opened (double-fan) graph such as Fig. 4.4(b) falls into one of four *basic patterns*: $[A_a], [A_b], [B_a]$, and $[B_b]$, shown in Fig. 4.5. If the solid and dotted lines end up on the same L-dot, the pattern is a $[B]$; otherwise it is an $[A]$. The subscripts a and b tell us which L-dot the *solid* line emerges from.

Suppose there are α_i number of $[A_i]$ and β_i number of $[B_i]$ patterns in a (fully) opened (double-fan) graph. Then there are m_a solid and n_a dotted lines emerging from the L-dot a, and m_b solid and n_b dotted lines emerging from the L-dot b, where

$$\begin{aligned} m_a &= \alpha_a + \beta_a, \qquad & n_a &= \alpha_b + \beta_a, \\ m_b &= \alpha_b + \beta_b, \qquad & n_b &= \alpha_a + \beta_b. \end{aligned} \tag{4.19}$$

The total number of R-dots in the opened graph is $N = m_a + m_b = n_a + n_b$.

When the N R-dots are merged together, the closed (double-fan) graph depicted in Fig. 4.4(a) arises, and it will be denoted by $[(m_a, n_a)(m_b, n_b)]$. For the case $N = 1$, this is just one of the four basic patterns discussed before. If $N > 1$, we call the graph a *compound pattern.*

From (4.19), it can be seen that two graphs with α_i and β_i values related as $\alpha_i' = \alpha_i + \zeta$ and $\beta_i' = \beta_i - \zeta$, where ζ is an integer that keeps α_i' and β_i' nonnegative, yield the same closed graph when collapsed. Conversely, it is shown in Appendix B that the closed graph $[(m_a, n_a)(m_b, n_b)]$ can be spun off into a sum of several opened graphs, one for each (α_i', β_i'). The double-fan relation expressing this idea quantitatively is

$$\begin{aligned} [(m_a, n_a)(m_b, n_b)] &= \sum v(\alpha_a', \alpha_b', \beta_a', \beta_b')\, [A_a]^{\alpha_a'} [A_b]^{\alpha_b'} [B_a]^{\beta_a'} [B_b]^{\beta_b'} \\ &\equiv \sum [\alpha_a' A_a + \alpha_b' A_b + \beta_a' B_a + \beta_b' B_b], \end{aligned} \tag{4.20}$$

where

$$v(\alpha_a', \alpha_b', \beta_a', \beta_b') = \frac{m_a!\, n_a!\, m_b!\, n_b!}{\alpha_a'!\, \alpha_b'!\, \beta_a'!\, \beta_b'!}, \tag{4.21}$$

and the sum is over all solutions $(\alpha_a', \alpha_b', \beta_a', \beta_b')$ of (4.19).

The double fan becomes a single fan if the L-dot b is not connected, namely, if $m_b = n_b = 0$ and $m_a = n_a = m$. In that case, (4.20) becomes

$$[m\, B_a] = m!\, [B_a]^m,$$

which is just (4.18) (all multiplicities m_i being 1) in another notation.

Results from unitarity

We have discussed how to extract convenient tools from the invariance requirements (2.18). In particular, the 'rotation' invariance has led us to the useful single- and double-fan relations. To calculate the actual value of an integral, the unitarity condition (2.17) must be used.

The unitarity sum (2.17) for $i \neq l$ simply brings out more relations between different integrals. However, for $i = l$, a pair U^* and U matrix elements disappears on the right side of (2.17). As a result, the unitarity condition relates integrals of degree p to ones of degree $p-1$. Following a repeated application of (2.17), the integral of degree 0, which is known to be $\int dU = 1$, is eventually reached. In this way, the values of the integrals can be computed recursively. This recursive procedure will be illustrated with various examples to follow.

The fan integral The simplest direct integral of first degree ($p = 1$) is

$$\|(i)(j)\| = \frac{1}{n}. \tag{4.22}$$

The result follows from (4.15), i.e. $\|(i)(j)\|$ is independent of i and j, used conjointly with the unitarity relation. In fact,

$$\sum_{j=1}^{n} \|(i)(j)\| = n\|(i)(j)\| = 1, \tag{4.23}$$

yielding (4.22) in an instant.

The calculation can be generalized to the integral in Fig. 4.6(a) to give the closed *fan integral*

$$\|(i^m)(j^m)\| \equiv F(m) = \frac{m!(n-1)!}{(n+m-1)!}. \tag{4.24}$$

Note that Fig. 4.6(a) is just the diagram Fig. 4.3(a) without any additional dot and line.

The above result can be derived by first considering the integral $F(m-1)$, which is shown in Fig. 4.6(b). The integral is independent of the value of the index (dot)

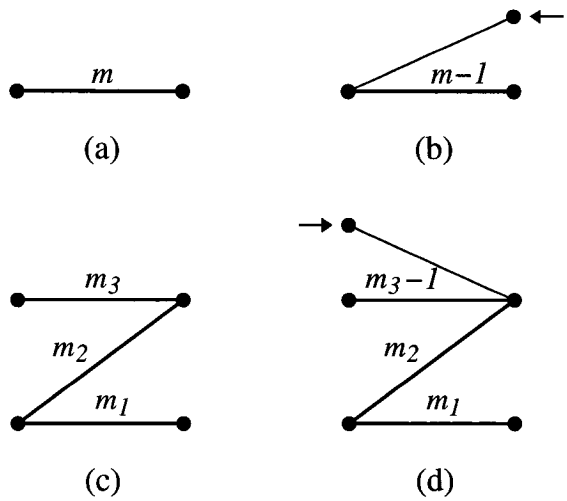


Figure 4.6: The diagrams in (a)–(c) are given special symbols, which are respectively $F(m)$, $F(m-1,1)$, and $Z(m_1,m_2,m_3)$. Diagram (a) can be obtained from (b) by summing over the index indicated by an arrow. Similarly, the 'Z-integral' in (c) can be obtained by summing over the indicated index in (d).

indicated by an arrow. If we sum from 1 to n over this indicated index, then in a similar fashion as in (4.23), unitarity and (4.15) imply

$$(n-1)F(m-1,1)+F(m)=F(m-1). \tag{4.25}$$

The fan relation (4.18) tells us that $F(m-1,1)=F(m)/m$, which when substituted in (4.25) gives the recursion relation between $F(m)$ and $F(m-1)$, that is

$$F(m)=\frac{m}{n+m-1}F(m-1,1).$$

Knowing that $F(0)=1$, this recursion relation can be solved to give (4.24).

Let the *fan integral*, $F(m_1,m_2,\ldots,m_t)$, be given by Fig. 4.3(b) without any extra dot and line. It follows from (4.24) and (4.18) that

$$F(m_1,m_2,\ldots,m_t)=\frac{(\prod_{i=1}^{t}m_i!)(n-1)!}{(n+\sum_{i=1}^{t}m_i-1)!}. \tag{4.26}$$

The Z-integral Consider the integral shown in Fig. 4.6(c), which we designate by $Z(m_1,m_2,m_3)$, where m_1, m_2, and m_3 are arbitrary multiplicities. Because of the shape of its diagram, the integral shall be referred to as a *Z-integral.*

We wish to demonstrate that

$$Z(m_1,m_2,m_3) = \int dU\,|U_{ij}|^{2m_1}|U_{il}|^{2m_2}|U_{kl}|^{2m_3}$$
$$= \frac{m_1!\,m_2!\,m_3!\,(n-2)!(n-1)!\,(n+m_1+m_3-2)!}{(n+m_1-2)!\,(n+m_3-2)!\,(n+m_1+m_2+m_3-1)!}. \quad (4.27)$$

Thus consider Fig. 4.6, with respective integral $I(4.6d)$. The sum in (2.17) applied to the index indicated by an arrow gives

$$(n-2)I(4.6d)+Z(m_1,m_2,m_3)+Z(m_1,m_2+1,m_3-1) = Z(m_1,m_2,m_3-1), \quad (4.28)$$

where as before (4.15) was used. The fan relation tells us that

$$I(4.6d) = Z(m_1,m_2,m_3)/m_3.$$

Substituting this into (4.28), we get the recursion relation in m_3:

$$Z(m_1,m_2,m_3) = \frac{m_3}{n+m_3-2}[Z(m_1,m_2,m_3-1) - Z(m_1,m_2+1,m_3-1)].$$

Using the limiting result

$$Z(m_1,m_2,0) = F(m_1,m_2) = (n-1)!m_1!m_2!/(n+m_1+m_2-1)!,$$

the recursion relation can be solved to yield (4.27). A remark is that relation (4.18) can also be used to fan out the two open ends in Fig. 4.6(c) for (4.27) to obtain the *fanned Z-integrals.*

As suggested by C. S. Lam, a recursion relation for 'X-integrals' can also be derived in a similar way, yet its solution is much more complicated. The recursion relation is worked out in Appendix C. A more powerful technique for computing such integrals will be presented later in Sec. 4.5.

Exchange integrals of second and third degree. We have so far dealt only with direct integrals. To illustrate how exchange integrals may be computed within the IM, all second and third degree exchange integrals will be computed. In addition, in the $p = 3$ case, there are direct integrals that are not of the fan type or Z type. We shall also obtain those from unitarity or invariance of the measure.

$p = 2$ All integrals of second degree fall into the $F(2) = Z(2,0,0)$, $F(1,1) = Z(1,1,0)$, $Z(1,0,1)$, or $E_2 \equiv \langle (i,k)(j,l)|(i,k)(l,j)\rangle$ types. The last integral, depicted in Fig. 4.1(a), is the only exchange integral for $p = 2$. It can be computed either by using the double-fan relation, or by unitarity. We will discuss both methods.

By the double-fan relation Consider the integral $F(1,1)$, which is the compound pattern $[(1,1)(1,1)]$ in the notation of (4.20). The double-fan relation gives

$$\begin{aligned} F(1,1) &= v(0,0,1,1)\, Z(1,0,1) + v(1,1,0,0)\, E_2 \\ &= Z(1,0,1) + E_2. \end{aligned}$$

Hence,

$$E_2 = F(1,1) - Z(1,0,1) = \frac{1}{n(n+1)} - \frac{1}{(n-1)(n+1)} = \frac{-1}{n(n^2-1)}.$$

By unitarity Summing over any index of E_2 from 1 to n gives $(n-1)E_2 + F(1,1) = 0$, hence

$$E_2 = \frac{-1}{(n-1)} F(1,1) = \frac{-1}{n(n^2-1)}.$$

$p = 3$ Fig. 4.7 shows the two direct integrals that do not fall into the fan or Z category and all exchange integrals of third degree. Remember that an integral is invariant under the exchange of its dotted lines with its thin solid lines, or of its R-dots with its L-dots.

The integral of Fig. 4.7(a), $I(4.7a)$, can be obtained by carrying out a unitarity sum on the indicated index. As a result, $(n-1)I(4.7a) + F(1,1,1) = F(1,1)$, hence

$$I(4.7a) = \frac{1}{(n-1)} \left(\frac{1}{n(n+1)} - \frac{1}{n(n+1)(n+2)} \right) = \frac{1}{(n-1)n(n+2)}.$$

Note that the same result follows by fanning out the bottom line of $Z(2,0,1)$.

To compute $I(4.7b)$, perform a unitary sum over any index. This results in $(n-2)I(4.7b) + 2I(4.7a) = Z(1,0,1)$. Hence

$$I(4.7b) = \frac{1}{(n-2)} \left(\frac{1}{n^2-1} - \frac{2}{(n-1)n(n+2)} \right) = \frac{n^2-2}{n(n^2-1)(n^2-4)}.$$

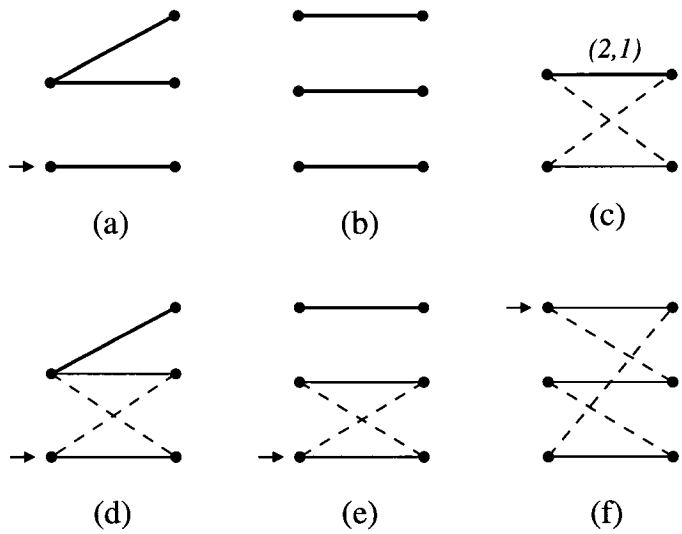


Figure 4.7: Diagrams for integrals of third degree. Arrows indicate the indices on which unitarity sums are to be performed.

The exchange integrals can be known by again taking unitarity sums on the indicated dots. Note that the right side of the sum is always zero in the case of exchange integrals. Thus the equations

$$\begin{aligned}
(n-1)I(4.7c) + F(1,2) &= 0, \\
(n-1)I(4.7d) + F(1,1,1) &= 0, \\
(n-2)I(4.7e) + I(4.7a) + I(4.7d) &= 0, \\
(n-2)I(4.7f) + 2I(4.7d) &= 0
\end{aligned}$$

are obtained. The solutions are:

$$\begin{aligned}
I(4.7c) &= \frac{-2}{n(n^2-1)(n+2)}, \\
I(4.7d) &= \frac{-1}{n(n^2-1)(n+2)}, \\
I(4.7e) &= \frac{-1}{(n^2-1)(n^2-4)}, \\
I(4.7f) &= \frac{2}{n(n^2-1)(n^2-4)}.
\end{aligned} \tag{4.29}$$

4.3 The Group Theoretical Method (GTM)

The formula (3.5) can be written in a more convenient form where no Kronecker symbol appears. Yet before we do so, we need to introduce a definition. By the *symmetry group* $\mathcal{G}_X$ of an ordered index set X, we understand

$$\mathcal{G}_X = \{s \in S_p | s(X) = X\}.$$

For example, the symmetry group $\mathcal{G}_I$ of $I = (9,4,4,4,2,2)$ is $S_1 \otimes S_3 \otimes S_2 \subset S_6$, where S_1 acts on the first element of $\mathcal{G}_I$, S_3 on the next three, and S_2 on the last two. In that sense, symmetry groups should also be taken as *ordered.*

Let us rewrite (3.5) in terms of the symmetry groups $\mathcal{G}_I$ and $\mathcal{G}_J$. The factors $\prod_{a=1}^{p} \delta_{k_a i_{r(a)}} \delta_{j_a l_{s'(a)}} = \delta_{KI_r} \delta_{JL_{s'}}$ in (3.5) imply that $\langle IJ|KL\rangle$ is nonvanishing only if the elements in K are the same as the ones in I, and similarly for J and L. Thus, from (4.1), a nonvanishing (3.5) is given by $\langle IJ|IJ_q\rangle$, where $q \in S_p$. Since $K = I$, the factor δ_{II_r} is nonzero if and only if $r \in \mathcal{G}_I$. Similarly, since $L = J_q$, the factor $\delta_{JL'_s}$ equals $\delta_{JJ_{qs'}}$, and $s \equiv qs'$ must be in the group $\mathcal{G}_J$. The argument of χ_f in (3.5) then becomes $rs' = rq^{-1}s$. Both r and s are summed over S_p, it is therefore legitimate to replace r by r^{-1} and s by s^{-1} in the summand. With such replacements, the argument of χ_f is $(sqr)^{-1}$. Because $\chi_f((sqr)^{-1}) = \chi_f(sqr)$, a convenient form for (3.5) is

$$\langle IJ|IJ_q\rangle = \sum_{r\in\mathcal{G}_I} \sum_{s\in\mathcal{G}_J} \sum_{(f)} \frac{d_f^2}{p!^2 \tilde{d}_f} \chi_f(sqr). \tag{4.30}$$

A character is class dependent. With this in mind, (4.30) can be rewritten as

$$\langle IJ|IJ_q\rangle = \sum_c N[c]\xi[c], \tag{4.31}$$

where the sum is taken over all classes c of S_p,

$$N[c] = \sum_{r\in\mathcal{G}_I} \sum_{s\in\mathcal{G}_J} \delta(sqr \in c) \tag{4.32}$$

is the number of elements of the type sqr in the class c, and

$$\xi[c] = \sum_{(f)} \frac{d_f^2}{p!^2 \tilde{d}_f} \chi_f(c) \tag{4.33}$$

are the simplest integrals to calculate. Note that since $\chi_f(c_r) = \chi_f(r)$, we may write $\xi[r]$ instead of $\xi[c_r]$ where judged more convenient.

For a general integral $\langle IJ|IJ_q\rangle$, it is not difficult to see that q is not unique, because $q' = sq = qt$ for any $s \in \mathcal{G}_J$ and any $t \in \mathcal{G}_{J_q}$ is another possible q. Different choices of q yield the same answer in (4.32). Therefore, another possible form for (4.32) is

$$N[c] = \sum_{r\in\mathcal{G}_I} \sum_{t\in\mathcal{G}_{J_q}} \delta(qtr \in c). \tag{4.34}$$

We will mostly use the above $N[c]$ in the rest of the work.

The computation of (4.34) is straightforward, yet it is generally tedious to perform. In fact, the products qtr for every $t \in \mathcal{G}_{J_q}$ and $r \in \mathcal{G}_I$ must be obtained, so that their corresponding classes be identified. Then, the number $N[c]$ can be known by counting the number of such products qtr that fall in the class c. However, the task becomes considerably more manageable if either $\mathcal{G}_I$ and $\mathcal{G}_{J_q}$ are disjoint, or if one is contained in the other. We shall refer to integrals with such properties as *orderly*. Further simplification occurs for direct integrals, because in that case q can always be chosen to be the identity e, such that the triple product qtr reduces to the double product tr.

The calculation of $\xi[c]$ in (4.33) is simpler than the calculation of $N[c]$, yet we know of no closed form for it that is valid for every class c and symmetric group S_p. The best that we can do is to compute them case by case. Results are given in Sec. 4.3.2. Each $\xi[c]$ is an orderly integral with $\mathcal{G}_I = \mathcal{G}_J = \{e\}$, to be referred to as *primitive integral*.

Other integrals can be computed in terms of the primitive integrals, if $N[c]$ is known. We shall discuss two orderly integrals for which $N[c]$ can easily be obtained. In Sec. 4.3.3, we discuss the *stack integrals*, which are direct integrals with $\mathcal{G}_I = \mathcal{G}_J$. In Sec. 4.3.4, we discuss the fully opened double-fan integrals of the type $[A_a]^\alpha[A_b]^\alpha$.

Relations between orderly integrals may be obtained without knowing the explicit values of the basic $\xi[c]$ integrals, if the corresponding $N[c]$ numbers are related in a simple way. This is the case for the single-fan relation, and the double-fan

relation with $n_a = m_b = 0$ and $m_a = n_b = m$. They will be discussed in Sec. 4.3.1. However, general double-fan integrals are not orderly, and we cannot obtain the general double-fan relation by the group theoretical method, at least not in the present way. We will also show that the closed (single-)fan *integral* can be computed without knowing explicitly what the $\xi[c]$ integrals are. This is a case where integrals can be obtained group theoretically without a need to know the primitive integrals $\xi[c]$.

There remain the non-orderly integrals, for which each term of the summand in (4.34) has to be calculated separately to get $N[c]$. The first non-orderly integral occurs in degree $p = 3$. In Sec. 4.3.5, we shall show how to calculate some of the $p = 3$ and $p = 4$ non-orderly integrals.

4.3.1 Single-Fan and Simple Double-Fan Relations

Let us write the index sets for the closed fan of Fig. 4.3(a) as

$$\begin{pmatrix} label \\ I \\ J \\ J_q \end{pmatrix} = \begin{pmatrix} 1 & \cdots & m & \cdots \\ a & \cdots & a & \cdots \\ c & \cdots & c & \\ c & \cdots & c & \end{pmatrix}. \tag{4.35}$$

The first row gives the index labels, and the next three rows give the relevant indices in the sets I, J, and J_q respectively. Different letters are understood to correspond to different values. Remember that additional dots and lines may be present in the graph, as long as none of the lines end up on the R-dots shown. These additional lines and dots are not drawn because they do not affect the fan relation in any way. The corresponding indices are also not shown in the index sets of (4.35) other than the ellipsis in the last column of the second row, which reminds us that there may be more lines connected to the L-dot of Fig. 4.3, i.e. the number of a indices may be greater than m. We have omitted such ellipses in the last two rows because no additional line is allowed to connect to the R-dots shown, i.e. the ignored indices in J and J_q are all different from c. Similarly, the index sets for the fully opened fan,

Fig. 4.3(b) with all $m_i = 1$, are

$$\begin{pmatrix} label \\ I \\ J \\ J_q \end{pmatrix} = \begin{pmatrix} 1 & \cdots & m & \cdots \\ a & \cdots & a & \cdots \\ c_1 & \cdots & c_m & \\ c_1 & \cdots & c_m & \end{pmatrix}, \tag{4.36}$$

differing from (4.35) only by the first m values of J and J_q.

Using S_m to denote the symmetric group on the first m labels, the symmetry groups for Fig. 4.3(a) can be read off from (4.35) to give $\mathcal{G}_I \supset S_m$ and $\mathcal{G}_J = \mathcal{G}_{J_q} = S_m$; indices with labels superior to m are ignored. Doing so is legitimate as the latter indices and their respective symmetry groups are common in both Fig. 4.3(a) and Fig. 4.3(b), which will have their results compared via (4.34). The symmetry groups for Fig. 4.3(b) can similarly be read off from (4.36) to give $\mathcal{G}_I \supset S_m$ and $\mathcal{G}_J = \mathcal{G}_{J_q} = e$. We may choose $q = e$ in both cases. Then for Fig. 4.3(a), $t\mathcal{G}_I = \mathcal{G}_I$ for every $t \in \mathcal{G}_{J_q}$, hence $N[c] = m! \sum_{r \in \mathcal{G}_I} \delta(r \in c)$. The last sum is simply the $N[c]$ for Fig. 4.3(b) or (4.36). Therefore it follows from (4.31) that the fan relation (with all $m_i = 1$) is true.

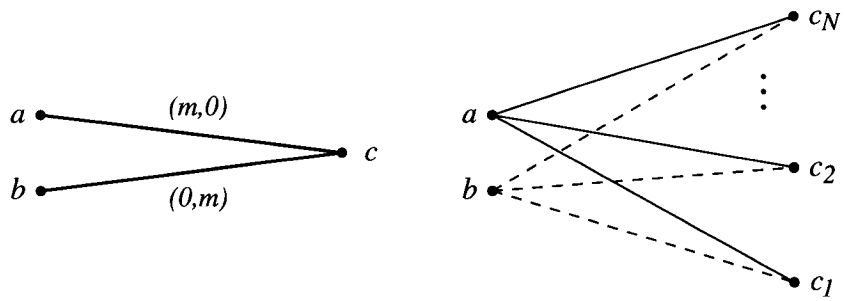
Let us now derive the double-fan relation (4.20) and (4.21) for the case $n_a = m_b = 0$ and $m_a = n_b$. The solution of (4.19) is in that case unique, giving $\alpha_a = m_a = n_b \equiv m$, and $\alpha_b = \beta_a = \beta_b = 0$. The double-fan relation (4.20) then becomes

$$[m\, A_a] = m!\, [A_a]^m. \tag{4.37}$$

We will now try to reach the same result from the (4.31) and (4.34).

The closed double-fan is shown in Fig. 4.8(a). Its index sets are

$$\begin{pmatrix} label \\ I \\ J \\ J_q \end{pmatrix} = \begin{pmatrix} 1 & \cdots & m & \cdots & n & \ldots & n+m & \cdots \\ a & \cdots & a & \cdots & b & \cdots & b & \cdots \\ c & \cdots & c & & & & & \\ & & & & c & \cdots & c & \end{pmatrix}. \tag{4.38}$$



(a) (b)

Figure 4.8: Double-fan diagrams with $n_a = m_b = 0$, and $m_a = n_b = m$. (a) A closed fan; (b) a fully opened fan.

The opened double-fan is shown in Fig. 4.8(b). Its index sets are

$$\begin{pmatrix} label \\ I \\ J \\ J_q \end{pmatrix} = \begin{pmatrix} 1 & \cdots & m & \cdots & n & \ldots & n+m & \cdots \\ a & \cdots & a & \cdots & b & \cdots & b & \cdots \\ c_1 & \cdots & c_m & & & & & \\ & & & & c_1 & \cdots & c_m & \end{pmatrix}. \tag{4.39}$$

We may choose $q = (1,n)(2,n+1)\cdots(m,n+m)$ in both cases. For (4.38), $\mathcal{G}_I \supset S_m \otimes S'_m$, $\mathcal{G}_J = S_m$, and $\mathcal{G}_{J_q} = S'_m$, where S_m is the permutation group acting on the elements labelled from 1 to m, and S'_m is the permutation group acting on the elements labelled from n to $n+m$. For (4.39), $\mathcal{G}_I \supset S_m \otimes S'_m$, but $\mathcal{G}_J = \mathcal{G}_{J_q} = e$. As before, we have ignored the column indices (R-dots) that are not drawn in writing the symmetry groups. With respect to Fig. 4.8(a), $t\mathcal{G}_I = \mathcal{G}_I$ for every $t \in \mathcal{G}_{J_q}$. As a consequence, $N[c] = m! \sum_{r \in \mathcal{G}_I} \delta(qr \in c)$. Yet the last sum is simply the $N[c]$ of Fig. 4.8(b). In this way (4.37) is verified by the GTM.

The single-fan integrals are orderly. Let us examine how the fan *integral* (4.24) can be obtained from the GTM. It is given by Fig. 4.3(a) without extra dots and lines, or (4.35) without the ellipses at the end. Then $\mathcal{G}_I = S_m$, and there are no

ignored dots in $\mathcal{G}_J = \mathcal{G}_{J_{q=e}} = S_m$. From (4.30) we get

$$\langle IJ|IJ_q\rangle \equiv F(m) = m! \sum_{r\in\mathcal{G}_I}\sum_{(f)} \frac{d_f^2}{(m!)^2\,\tilde{d}_f}\,\chi_f(r) \tag{4.40}$$

$$= \frac{1}{m!}\sum_{(f)}\frac{d_f^2}{\tilde{d}_f}\sum_{r\in S_m}\chi_f(r)\,\chi^*_{(m)}(r) = \frac{d^2_{(m)}}{\tilde{d}_{(m)}} \tag{4.41}$$

$$= \frac{m!(n-1)!}{(m+n-1)!}. \tag{4.42}$$

In order to get from (4.40) to (4.41), the character $\chi^*_{(m)}(r) = 1$ of the totally symmetric representation (m) of the S_m group is inserted, and the orthogonality relation (2.6) is used. To get to (4.42), $d_{(m)} = \chi_{(m)}(e) = 1$ as well as the formula (3.6) for $\tilde{d}_{(m)}$ are used.

The result in (4.42) agrees with the result (4.24). It is one of a few cases where the value of an integral can be obtained group theoretically without a knowledge of the individual $\xi[c]$ expressions.

4.3.2 Primitive Integrals

The integrals in which both symmetry groups $\mathcal{G}_I$ and $\mathcal{G}_J$ consist only of the identity e will be called *primitive*. This happens when all the indices i_a in the set I assume distinct values, and all the indices j_b in the set J are also different. The corresponding diagrams have p dots on each of both columns, and precisely one solid and one dotted lines connecting to any one of them. The primitive diagrams for $p \leq 3$ are shown in Fig. 4.9, and the ones for $p = 4, 5$ are contained in Appendix D.

Since $\mathcal{G}_I = \mathcal{G}_J = e$, it follows from (4.32) that $N[c] = \delta(q \in c)$, where q can be any element of S_p. The primitive integrals (4.31) are simply $\xi[c]$, one for each class c of S_p. For convenience, we may use an element of each cycle structure (class) to label the primitive integrals, as is done in Fig. 4.9. Diagrammatically, the cycle structure is translated into the loop structure of its diagram, as can be seen in Fig. 4.9. Using (4.33) along with (3.6) and the character tables found in Appendix E, the primitive integrals for $p \leq 3$ can easily be computed. The results are displayed in Table 4.1. We have also included the results for $p = 4, 5$ in Table D.1 of Appendix D.

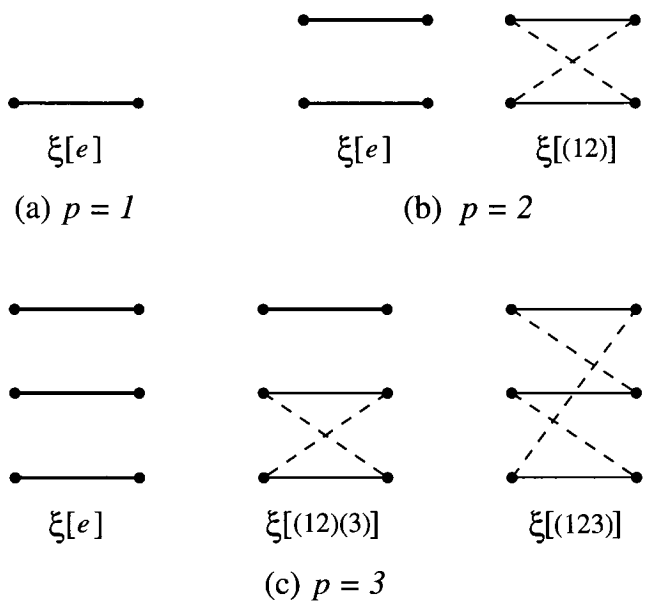


Figure 4.9: Primitive diagrams for (a) $p = 1$, (b) $p = 2$, and (c) $p = 3$. The identity element is everywhere denoted by e.

$\xi[c]$			
c	$p=1$	$p=2$	$p=3$
e	$\frac{1}{n}$	$\frac{1}{n^2-1}$	$\frac{n^2-2}{n(n^2-1)(n^2-4)}$
(12)		$\frac{-1}{n(n^2-1)}$	$\frac{-1}{(n^2-1)(n^2-4)}$
(123)			$\frac{2}{n(n^2-1)(n^2-4)}$

Table 4.1: Algebraic expressions for the primitive diagrams of $p = 1, 2, 3$.

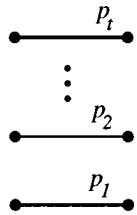


Figure 4.10: Arbitrary stack diagram, $\Xi(p_1, p_2, \ldots, p_t)$, of degree $p = \sum_{i=1}^{t} p_i$.

4.3.3 Stack Integrals

The stack diagrams (see Fig. 4.10) are direct integrals made up of disconnected lines of arbitrary multiplicities. As such, $q = e$, and J differs from I only by relabelling. Using (4.15), we may assume that $J = I$. Hence stack integrals are integrals of the form $\|II\|$.

Let $p_1, p_2, \ldots, p_t$ be the multiplicities of the disconnected lines in a stack diagram. Then $\mathcal{G}_I = \mathcal{G}_J \equiv \mathcal{G} = S_{p_1} \otimes S_{p_2} \otimes \cdots \otimes S_{p_t}$, and $N[c]$ is nonzero only when the class c is a direct product of the classes c_i of the groups S_{p_i}. In that case,

$$N[c] = \prod_{i=1}^{t} p_i!\, n_i(c_i), \tag{4.43}$$

where $n_i(c_i)$ is the number of elements of S_{p_i} in the class c_i. In other words,

$$n_i(c_i) = \frac{p_i!}{\prod_{j=1}^{p_i} j^{\alpha_j} \alpha_j!}\ , \tag{4.44}$$

where the class c_i consists of α_j cycles of length j. Denoting the stack integral $\|II\|$ by $\Xi(p_1, p_2, \ldots, p_t)$, we get

$$\Xi(p_1, p_2, \ldots, p_t) = \sum_{c_1, c_2, \ldots} \left(\prod_{i=1}^{t} p_i!\, n_i(c_i) \right) \xi(c_1 \otimes c_2 \otimes \cdots \otimes c_t). \tag{4.45}$$

All stack diagrams can be obtained by making the assignment $f_i \to p_i$ for every representation (f). In this way, we expect a same number of stack diagrams as of primitive diagrams, or classes. Using the ξ expressions obtained previously, the stack integrals for $p \leq 3$ can be computed to yield the expressions in Table 4.2.

$\Xi(p_1, \ldots, p_p)$	
$\Xi(1) = \frac{1}{n}$	$\Xi(3) = \frac{3!}{n(n+1)(n+2)}$
$\Xi(2) = \frac{2}{n(n+1)}$	$\Xi(2,1) = \frac{2}{(n-1)n(n+2)}$
$\Xi(1,1) = \frac{1}{n^2-1}$	$\Xi(1,1,1) = \frac{n^2-2}{n(n^2-1)(n^2-4)}$

Table 4.2: Algebraic expressions for the stack diagrams of $p = 1$, 2, and 3.

4.3.4 Special Double-Fan Integrals

The index sets for the fully opened double-fan integrals $[A_a]^\alpha[A_b]^\alpha$ (Fig. 4.4(b) with $N = 2\alpha$) are

$$\begin{pmatrix} label \\ I \\ J \\ J_q \end{pmatrix} = \begin{pmatrix} 1 & \cdots & \alpha & \alpha+1 & \ldots & 2\alpha \\ b & \cdots & b & a & \cdots & a \\ c_1 & \cdots & c_\alpha & c_{\alpha+1} & \cdots & c_{2\alpha} \\ c_{\alpha+1} & \cdots & c_{2\alpha} & c_1 & \cdots & c_\alpha \end{pmatrix}. \tag{4.46}$$

Hence both $\mathcal{G}_J$ and $\mathcal{G}_{J_q}$ consist only of the identity e. As for $\mathcal{G}_I$, it is given by $S_\alpha \otimes S_\alpha$, where the permutation groups S_α act respectively on the b and a indices in I. The element q maps J_q to J, i.e. $q = (1, \alpha+1)(2, \alpha+2)\cdots(\alpha, 2\alpha)$.

The fully opened integral can be computed using (4.31), with $N[c]$ given by (4.34). Thus,

$$\begin{aligned} N[c] &= \sum_{r\in\mathcal{G}_I}\sum_{t\in\mathcal{G}_{J_q}} \delta(qtr \in c) \\ &= \sum_{r\in\mathcal{G}_I} \delta(qr \in c) \;=\; \sum_{q'} \delta(q' \in c), \end{aligned} \tag{4.47}$$

where the last sum is over *every* permutation q' that sends *all* b indices in (4.46) to the positions labelled from $\alpha+1$ to 2α, and similarly *all* a indices to the positions labelled from 1 to α. As a consequence, the allowed cycles of q' must be of even length, and they can be specified by a sequence of nonnegative integers $(k) \equiv (k_1 k_2 \cdots k_\alpha)$, k_i being the number of cycles of length $2i$. The number of q' with the class structure (k)

α	$[A_a]^\alpha\,[A_b]^\alpha$
1	$\frac{-1}{n(n^2-1)}$
2	$\frac{2}{(n^2-1)n^2(n+2)(n+3)}$
3	$\frac{-6}{(n-1)n^2(n+1)^2(n+2)(n+3)(n+4)(n+5)}$

Table 4.3: Values of the monomial integrals $[A_a]^\alpha[A_b]^\alpha$ for $\alpha = 1, 2, 3$.

that is related to c is given by

$$N[c] = \frac{(\alpha!)^2}{\prod_{i=1}^{\alpha} i^{k_i} \cdot k_i!}. \tag{4.48}$$

In order to see how this is arrived at, consider the example where $k_1 = 2$, $k_2 = 2$, and all other k_i values are zero. Then q' is of the form $(ba)(ba)(baba)(baba)$, where the b and a letters take distinct index labels in $(1, \ldots, \alpha)$ and $(\alpha+1, \ldots, 2\alpha)$ respectively. Another q' with the same cycle structure can thus be obtained by permuting individually all the a and b labels. This accounts for the numerator in (4.48). However, such permutations do not necessarily give distinct q' elements. The cyclic nature of a cycle tells us that each cycle of length $2i$ will appear i times; this accounts for the i^{k_i} factor in the denominator. Moreover, no new q' is obtained if we permute cycles of the same length; that accounts for the other factor $k_i!$ in the denominator. Note that the above ideas are analogous to the ones used in determining the number of elements $n_{(\nu)}$ in a class $c_{(\nu)}$ of S_p, that is (2.14).

We may now return to (4.31) to calculate the integral $[A_a]^\alpha[A_b]^\alpha$ in terms of the primitive integrals $\xi[c]$. The result for the first few α values are listed in Table 4.3.

4.3.5 Non-Orderly Integrals

All integrals with degree $p < 3$ are orderly. The non-orderly integrals of $p = 3$ are the ones shown in Fig. 4.11, and those related to them by the fan relation (4.18). The calculation of the former integrals is discussed below. As done previously, the integrals will be labelled by their figure, e.g. the integral represented by Fig. 4.11(a)

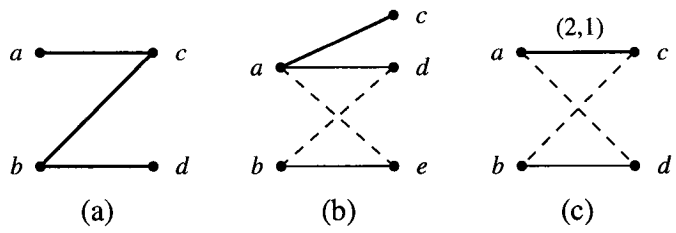


Figure 4.11: Non-orderly integrals of $p = 3$.

is denoted by $I(4.11a)$.

The index sets for Fig. 4.11(a) are

$$\begin{pmatrix} label \\ I \\ J \\ J_q \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ b & b & a \\ d & c & c \\ d & c & c \end{pmatrix}.$$

They give rise to the symmetry groups $\mathcal{G}_I = \{e, (12)\}$ and $\mathcal{G}_J = \mathcal{G}_{J_q} = \{e, (23)\}$. Moreover, the element q can be taken to be the identity element. In order to obtain the coefficients $N[c]$ of equation (4.31), we need to compute qtr for all $t \in \mathcal{G}_{J_q}$ and $r \in \mathcal{G}_I$. That triple product is $q\mathcal{G}_{J_q}\mathcal{G}_I = \{e, (12), (23), (132)\}$. As a result,

$$I(4.11a) = Z(1,1,1) = \xi[e] + 2\xi[(12)(3)] + \xi[(123)] = \frac{1}{(n^2-1)(n+2)}. \qquad (4.49)$$

In the same way, the index sets of Fig. 4.11(b) are

$$\begin{pmatrix} label \\ I \\ J \\ J_q \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ b & a & a \\ e & d & c \\ d & e & c \end{pmatrix},$$

such that the symmetry groups are $\mathcal{G}_{J_q} = \{e\}$ and $\mathcal{G}_I = \{e, (23)\}$, and the exchange element is $q = (12)$. We thus obtain $q\mathcal{G}_{J_q}\mathcal{G}_I = \{(12), (123)\}$, from which

$$I(4.11b) = \xi[(12)(3)] + \xi[(123)] = \frac{-1}{(n^2-1)n(n+2)}$$

follows.

Finally, for Fig. 4.11(c), the index sets are

$$\begin{pmatrix} label \\ I \\ J \\ J_q \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ b & a & a \\ d & c & c \\ c & d & c \end{pmatrix}.$$

The relevant symmetry groups are $\mathcal{G}_{J_q} = \{e, (13)\}$ and $\mathcal{G}_I = \{e, (23)\}$. With $q = (12)$, the products from $q\mathcal{G}_{J_q}\mathcal{G}_I$ result in $\{(12), (13), (123), (132)\}$, and formula (4.31) gives:

$$I(4.11c) = 2(\xi[(12)(3)] + \xi[(123)]) = \frac{-2}{(n^2-1)n(n+2)}. \tag{4.50}$$

The calculation of $q\mathcal{G}_{J_q}\mathcal{G}_I$ is not highly cumbersome for $p = 3$, but it gets worse rapidly as p increases. For example, let us look at some examples for $p = 4$. We wish to first calculate $Z(2,1,1)$ of Fig. 4.1(b), whose index sets can be written in the following way:

$$\begin{pmatrix} label \\ I \\ J \\ J_q \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ b & b & b & a \\ d & d & c & c \\ d & d & c & c \end{pmatrix}.$$

Then $q = e$, $\mathcal{G}_{J_q} = \{e, (12), (34), (12)(34)\}$ and $\mathcal{G}_I = \{e, (12), (13), (23), (123), (132)\}$. Thus $q\mathcal{G}_{J_q}\mathcal{G}_I = \{e, (12), (34), (12)(34), (12), e, (12)(34), (34), (13), (132), (143), (1432), (23), (123), (243), (1243), (123), (23), (1243), (243), (132), (13), (1432), (143)\}$, such that

$$\begin{aligned} Z(2,1,1) &= 2\xi[e] + 8\xi[(12)(3)(4)] + 8\xi[(123)(4)] + 2\xi[(12)(34)] + 4\xi[(1234)] \\ &= \frac{2}{(n-1)n(n+2)(n+3)}. \end{aligned} \tag{4.51}$$

Our last example is the Σ-integral, shown in Fig. 4.12. The index sets are

$$\begin{pmatrix} label \\ I \\ J \\ J_q \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ b & b & a & a \\ e & d & d & c \\ e & d & d & c \end{pmatrix}.$$

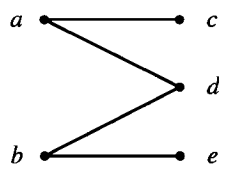


Figure 4.12: Sigma integral where all lines have multiplicity one.

Hence $q = e$, $\mathcal{G}_{J_q} = \{e, (23)\}$ and $\mathcal{G}_I = \{e, (12), (34), (12)(34)\}$. Multiplying such elements accordingly, the set $\{e, (12), (34), (12)(34), (23), (132), (234), (1342)\}$ is obtained for $q\mathcal{G}_{J_q}\mathcal{G}_I$. Hence

$$\begin{aligned}\Sigma &= \xi[e] + 3\xi[(12)(3)(4)] + 2\xi[(123)(4)] + \xi[(12)(34)] + \xi[(1234)] \\ &= \frac{n+1}{(n-1)n^2(n+2)(n+3)}.\end{aligned} \tag{4.52}$$

4.4 Comparison of the IM and the GTM

We have discussed the computation of $U(n)$ integrals (4.2) in two ways: the IM in Sec. 4.2, and the GTM in Sec. 4.3. Each of the two methods has its own merits and drawbacks, and in a way they complement each other. The purpose of this section is to compare their relative strong and weak points.

The IM is based solely on the unitarity condition (2.17) and the invariance of the Haar Measure (2.18). The method is simple because there is no need to know group theory. The conditions relate integrals of the same degree, and also integrals of degree p to integrals of degree $p-1$. Through these relations, specific integrals such as the fan integrals (4.26) and the Z-integrals (4.27) can be obtained, and general relations such as the single-fan relation (4.18) and the double-fan relation (4.20) can be worked out.

The GTM has the advantage of being general, in the sense that all integrals can be computed using the formula (3.5) or (4.31). The price to pay is that one has to know the characters of the irreducible representations of the appropriate symmetric

group, and a triple sum has to be carried out, which can prove to be very tedious for integrals of high degrees. Furthermore, unlike the IM, relations between integrals are hard to come by, so one must calculate the integrals one by one. There are however certain classes of integrals, the orderly integrals, for which relations can be developed, where the quantity $N[c]$ in (4.31) can be relatively easily computed. We then merely have to know the primitive integrals $\xi[c]$ in to get the value of the orderly integral of interest. The stack integrals (4.45) and the special double-fan integrals are examples of this kind. The primitive integrals $\xi[c]$ themselves must be calculated using (4.33).

To summarize, the IM gives a great number of relations, yet the value of any specific integral is not easily obtainable. The GTM allows us to calculate any specific integral, albeit rather tedious at times, but it is difficult to obtain relations between integrals. In the next section, we shall discuss a *hybrid method* which makes use of the advantages of both methods. We shall use the general GTM formula to calculate a specific set of integrals, then use the IM relations to obtain many other integrals.

In the rest of this section, we shall enlarge the given general remarks about the IM and GTM, by using specific examples from the last two sections as concrete illustrations.

The single-fan relation (4.18) can be obtained by both the IM and the GTM. However, the double-fan relation (4.20) in its general form can be obtained only by the IM, because most of the integrals involved are not orderly, thus making it hard to derive useful relations using the GTM. Nevertheless, in special cases involving only orderly integrals, (4.37) for instance, the GTM can also be used to derive the relation.

The Z-formula (4.27) is obtained using the IM, by a series of relations connecting it down to $\int dU = 1$. Since the Z-integrals (with nonvanishing m_1 or m_3) are not orderly, it is hard to compute them using the GTM, except at low degrees. The calculation of some of those by the GTM is shown in equations (4.49) and (4.51).

Since the values of the integrals in the IM are obtained only through chain of

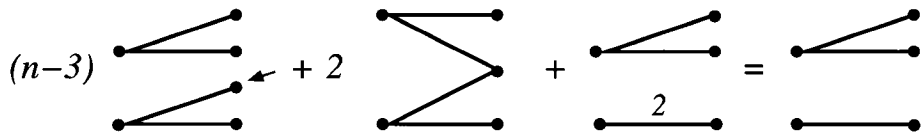


Figure 4.13: Unitarity sum relation involving the Σ diagram. The sum is performed on the index indicated by an arrow. Using the fan relation (4.18), the unitary sum can be written as: $(\frac{n-3}{4} + \frac{1}{2})Z(2,0,2) + 2\Sigma = \frac{1}{2}Z(2,0,1)$.

relations, it may be relatively complicated to calculate just one specific integral. This is where the GTM is superior, because of the general formula (3.5) or (4.30) valid for any one integral. For example, it is easy to obtain the Σ integral (4.52), assuming of course the $\xi[c]$ expressions to be already known. We can also obtain it using the IM, as we shall show below, but that involves a few steps as the integral must be got from relations. To see how that is done, look at Fig. 4.13, which illustrates the unitarity relation applied to the dot indicated by an arrow. The first and third diagrams can be related to $Z(2,0,2)$ by using the fan relation (4.18), and similarly the diagram on the right can be related to $Z(2,0,1)$. Using the Z-formula (4.27), we then obtain

$$\begin{aligned} \Sigma &= \frac{1}{4}\left[Z(2,0,1) - \left(\frac{n-3}{2}+1\right)Z(2,0,2)\right] \\ &= \frac{n+1}{(n-1)n^2(n+2)(n+3)}, \end{aligned}$$

which is the same as (4.52) reached via the GTM.

4.5 The Hybrid Method

Having considered the relative merits of the GTM and the IM, it is possible to combine them into a more efficient hybrid calculational scheme. The strategy is to start with one or more integrals that can be computed by the GTM with relative ease. Generally speaking, such integrals are ordered. Once they are obtained, the many relations of the IM can be used to calculate other integrals from them.

To illustrate the strategy, we will consider how the hybrid method can be used to calculate all double-fan integrals. By a double-fan integral, we mean any integral with two L-dots and any number of R-dots. Fig. 4.4(a) shows a closed double-fan integral (with the understanding that there are no extra dots or lines than those shown), and Fig. 4.4(b) shows a fully opened double-fan integral. We may also have partially opened double-fan integrals, in which every branch, that is every R-dot with its connecting lines, can be regarded as a closed integral. See Fig. 4.14 for an example of a partially opened integral.

As in Example 3 of Sec. 4.2.2, a fully opened double-fan integral is written in terms of the basic patterns as $[A_a]^{\alpha_a}[A_b]^{\alpha_b}[B_a]^{\beta_a}[B_b]^{\beta_b}$, and its corresponding closed integral is denoted by $[\alpha_a A_a + \alpha_b A_b + \beta_a B_a + \beta_b B_b]$. For a partially opened integral, we will denote it as a product of the closed integrals of each branch. See Fig. 4.14 for examples.

Using (4.20) and (4.21), all double-fan integrals can be expressed as sums of fully opened integrals. Integrals of the form $[A_a]^{\alpha}[A_b]^{\alpha}$ are given by (4.48), and some of them are contained in Table. 4.3, yet we still have to know how to calculate a fully opened integral when $\beta_i \neq 0$. As shown in Appendix F, the IM allows us to relate the latter to the ones with $\beta_i = 0$, by using the following formula

$$
\begin{aligned}
&[A_a]^{\alpha}[A_b]^{\alpha}[B_a]^{\beta_a}[B_b]^{\beta_b} = \\
&\sum_{e=0}^{\min(\beta_a,\beta_b)} \left\{ (-1)^e\, e! \binom{\beta_a}{e}\binom{\beta_b}{e} (n+2\alpha-1+2e)\cdot \right. \\
&\left. \frac{(n+2\alpha-2+e)!\,(n+2\alpha-1+2e)!}{(n+2\alpha+\beta_a-1+e)!\,(n+2\alpha+\beta_b-1+e)!}\,[A_a]^{\alpha+e}[A_b]^{\alpha+e} \right\}.
\end{aligned}
\tag{4.53}
$$

We close the section by showing how to use (4.20), (4.21) along with (4.53) to calculate the integrals in Fig. 4.14.

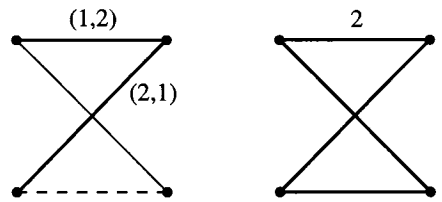


Figure 4.14: Partially opened double-fan integrals. (a) There are two equivalent forms for this graph: $[A_a + 2A_b][A_a]$and $[A_b + B_a + B_b][A_a]$. (b) There are four equivalent forms for this graph: $[A_a + A_b + B_a][A_a + A_b]$, $[2B_a + B_b][A_a + A_b]$, $[A_a + A_b + B_a][B_a + B_b]$, $[2B_a + B_b][B_a + B_b]$.

4.5.1 Fig. 4.14(a)

There are two equivalent forms for this diagram. One is

$$[A_a + 2A_b][A_a] = \left(2[A_a][A_b]^2\right)[A_a] = 2[A_a]^2[A_b]^2 \tag{4.54}$$

and the other is

$$[A_b + B_a + B_b][A_a] = 4\left([A_b][B_a][B_b]\right)[A_a] = 4[A_a][A_b][B_a][B_b]; \tag{4.55}$$

equations (4.20) and (4.21) have been used. The integral $I(4.14a)$ is obtained by adding up (4.54) and (4.55).

Using (4.53), we can express all fully opened integrals in terms of ones of the form $[A_a]^\alpha[A_b]^\alpha$. Applying the formula to (4.55), we get

$$[A_a][A_b][B_a][B_b] = \frac{1}{(n+2)^2}[A_a][A_b] - \frac{1}{(n+2)}[A_a]^2[A_b]^2.$$

We have seen that the integrals $[A_a]^\alpha[A_b]^\alpha$ are calculated with ease from (4.48). The expressions for $\alpha \leq 3$ have been inserted in Table 4.3, and the results for $\alpha = 1, 2$ can be used to yield

$$\begin{aligned} I(4.14a) &= 2[A_a]^2[A_b]^2 + 4[A_a][A_b][B_a][B_b] \\ &= \frac{2n}{(n+2)}[A_a]^2[A_b]^2 + \frac{4}{(n+2)^2}[A_a][A_b] \\ &= \frac{-4}{(n^2-1)n(n+2)(n+3)}. \end{aligned}$$

4.5.2 Fig. 4.14(b)

As shown in Fig. 4.14(b), $I(4.14b)$ has four equivalent forms. For one branch, the factors are:

$$\begin{aligned}[A_a + A_b + B_a] &= 4\,[A_a][A_b][B_a], \\ [2B_a + B_b] &= 2\,[B_a]^2[B_b];\end{aligned}$$

and for the other, they are:

$$\begin{aligned}[A_a + A_b] &= [A_a][A_b], \\ [B_a + B_b] &= [B_a][B_b].\end{aligned}$$

Hence by combining the above terms,

$$I(4.14b) = 4\,[A_a]^2[A_b]^2[B_a] + 6\,[A_a][A_b][B_a]^2[B_b] + 2\,[B_a]^3[B_b]^2. \tag{4.56}$$

We will now express each of the three fully opened integrals in (4.56) in terms of $[A_a]^\alpha[A_b]^\alpha$. First, with respect to (4.53), $[A_a]^2[A_b]^2[B_a]$ is characterized by $\alpha = 2$, $\beta_a = 1$, and $\beta_b = 0$. The vanishing of β_b causes (4.53) to consist of the single term:

$$[A_a]^2[A_b]^2[B_a] = \frac{1}{(n+4)}[A_a]^2[A_b]^2. \tag{4.57}$$

Second, $[A_a][A_b][B_a]^2[B_b]$ has $\alpha = 1$, $\beta_a = 2$, $\beta_b = 1$, and the sum in (4.53) gives:

$$[A_a][A_b][B_a]^2[B_b] = \frac{1}{(n+2)^2(n+3)}[A_a][A_b] - \frac{2}{(n+2)(n+4)}[A_a]^2[A_b]^2. \tag{4.58}$$

Finally, $[B_a]^3[B_b]^2$, having $\alpha = 0$, $\beta_a = 3$, $\beta_b = 2$, can be expressed as

$$\begin{aligned}[B_a]^3[B_b]^2 \;=\; & \frac{1}{n^2(n+1)^2(n+2)} - \frac{6}{n(n+2)^2(n+3)}[A_a][A_b] + \\ & \frac{6}{(n+1)(n+2)(n+4)}[A_a]^2[A_b]^2\end{aligned} \tag{4.59}$$

from equation (4.53). Using the fan relation, notice that $[B_a]^3[B_b]^2$ can also be reduced to $\frac{1}{3!}\frac{1}{2!}Z(3,0,2)$.

As done in Sec. 4.5.1, we fetch the expressions for $[A_a][A_b]$ and $[A_a]^2[A_b]^2$ in Table 4.3. The final answer is obtained by inserting (4.57)–(4.59) into (4.56). The result is:

$$\begin{aligned}[A_a + A_b + B_a][A_a + A_b] &= \frac{2}{n^2(n+1)^2(n+2)} + \frac{6(n-2)}{n(n+2)^2(n+3)}[A_a][A_b] + \\ &\quad \frac{4(n^2+2)}{(n+1)(n+2)(n+4)}[A_a]^2[A_b]^2 \\ &= \frac{2(n^2+2n+4)}{(n^2-1)n^2(n+2)(n+3)(n+4)},\end{aligned}$$

which can be verified using the plain group theoretical formula (3.5).

CHAPTER 5

CONCLUSION

Many areas of physics require techniques to integrate over the unitary group manifold, lattice gauge theory is an example. As a result, efforts have been made by many to solve the problem. The most powerful methods that have been developed are of group theoretical nature. One of those methods is efficient as it allows to compute any monomial integral (1.1) directly, via a formula (4.30) that involves a triple sum; two being over the symmetry groups of the index sets, and one over the symmetric group representations. The summand also involves a symmetric group character which has as argument a triple product of group elements. Due to the double sum, there may be many such triple products to be computed. As such, integrals of increasing degrees are in general tedious to compute from the group theoretical formula. There are exceptions however. In the case where one symmetry group is contained in the other, systematics arise and simplify the calculations. Such simpler integrals were characterized as orderly.

We have developed an alternative method, the invariant method (IM for short), to compute the monomial integrals. It relies solely on the invariance of the group measure and the unitarity of the $U(n)$ group elements. Hence no knowledge of group theory and group characters is needed. Within the method, we have derived useful relations such as the single-fan and double-fan relations, and the closed Z-formula. The Z-formula is an example of how a $U(n)$ integral can be computed recursively by using the unitarity and invariance conditions. The principal advantage of the IM

is the abundance of relations that it creates between different integrals.

Following a description of the invariant method, we have chosen to study and extend the already existing group theoretical formula. The related work was designated as the group theoretical method (GTM). As a consequence, we could better identify and compare the relative advantages of both the IM and GTM. We have then designed the hybrid method, which is a more efficient tool for the computation of non-orderly integrals. It uses the IM to relate the latter integrals to other integrals that can be calculated by the GTM with relative ease. The hybrid method was illustrated with an example on the calculation of all double-fan integrals.

As a final remark, consider (1.1) one last time, and note that the matrix elements are everywhere of integral power. Then, in a pure mathematical context, an interesting avenue for the daring reader may be to explore $U(n)$ integrals with elements of non-integral powers.

APPENDIX A

GROUP THEORETICAL FORMULA

In this appendix will be explained how the group theoretical formula,

$$\int dU\, U^*_{i_1 j_1} \cdots U^*_{i_p j_p} U_{k_1 l_1} \cdots U_{k_p l_p} = \sum_{r,s' \in S_p} \prod_{a=1}^{p} \delta_{k_a i_{r(a)}} \delta_{j_a l_{s'(a)}} \sum_{(f)} \frac{d_f^2}{p!^2 \tilde{d}_f} \chi_f(rs'), \quad \text{(A.1)}$$

is arrived at.

A crucial input to the computation of the integral is the Frobenius relation (2.20),

$$(U)_s = \sum_{(f)} \chi_f(s) \tilde{\chi}_f(U), \quad \text{(A.2)}$$

valid for any $U \in U(n)$ and $s \in S_p$. Let us also recall the orthogonality relation (2.11),

$$\int dU\, \tilde{\chi}_f(UV) \tilde{\chi}_h(WU^\dagger) = \frac{1}{\tilde{d}_f} \tilde{\chi}_f(VW)\, \delta_{fh}, \quad \text{(A.3)}$$

where $\tilde{\mathcal{G}} = U(n)$, and relation (2.6),

$$\frac{1}{p!} \sum_{r \in S_p} \chi_f(r) \chi_h(r) = \delta_{fh}, \quad \text{(A.4)}$$

where $\mathcal{G} = S_p$. Applying (A.2) to $(UV)_e$ and $(WU^\dagger)_e$, and using (A.3) to integrate, one arrives at the expression

$$\int dU\, (UV)_e (WU^\dagger)_e = \sum_{(f)} \frac{d_f^2}{\tilde{d}_f} \tilde{\chi}_f(VW), \quad \text{(A.5)}$$

where $d_f = \chi_f(e)$ has been used. The factor (A.4) can be introduced on the right side of (A.5) provided that it is accompanied with a sum over all representations (h).

In doing so, (A.5) turns into

$$\begin{aligned}\int dU\,(UV)_e(WU^\dagger)_e &= \sum_{r\in S_p}\sum_{(f)}\frac{d_f^2}{p!\tilde{d}_f}\chi_f(r)(VW)_r \\ &= \sum_{r,s'\in S_p}\sum_{(f)}\frac{d_f^2}{p!^2\tilde{d}_f}\chi_f(s'r)(VW)_{s'r}. \qquad \text{(A.6)}\end{aligned}$$

Let us introduce the shorthands

$$\delta_{IJ} = \prod_{a=1}^{p}\delta_{i_a j_a},$$

and

$$A_{IJ} = \prod_{a=1}^{p}A_{i_a j_a},$$

for any matrix A and *ordered* index sets $I = (i_1,\ldots,i_p)$, $J = (j_1,\ldots,j_p)$; an ordered index set with a subscript $s \in S_p$ means that its indices are organized according to this subscript s, e.g. $I_s = (i_{s(1)},\ldots,i_{s(p)})$. Thus, the sum over r and s' on the right side of (A.6) becomes

$$\begin{aligned}\sum_{r,s'}\chi_f(rs')\,V_{LK}W_{IJ}\,\delta_{KI}\delta_{JL_{s'r}} &= \sum_{r,s'}\chi_f(rs')\,V_{LK}W_{IJ_r}\,\delta_{KI}\delta_{JL_{s'}} \\ &= \sum_{r,s'}\chi_f(rs')\,V_{LK}W_{I_{r^{-1}}J}\,\delta_{KI}\delta_{JL_{s'}} \\ &= \sum_{r,s'}\chi_f(rs')\,V_{LK}W_{IJ}\,\delta_{K_{r^{-1}}I}\delta_{JL_{s'}} \\ &= \sum_{r,s'}\chi_f(rs')\,V_{LK}W_{IJ}\,\delta_{KI_r}\delta_{JL_{s'}},\end{aligned}$$

where sums over repeated indices are understood. Recall that a character is invariant under a cyclic move effected on its argument, e.g. $\chi_f(s'r) = \chi_f(rs')$. Since V and W are arbitrary, the coefficients of $V_{LK}W_{IJ}$ on both sides must be the same. Hence

$$\int dU\,U^*_{IJ}U_{KL} = \sum_{r,s'}\delta_{KI_r}\delta_{JL_{s'}}\sum_{(f)}\frac{d_f^2}{p!^2\tilde{d}_f}\chi_f(rs'),$$

which is (A.1) in a more compact form.

APPENDIX B

DERIVATION OF EQUATIONS (4.20) AND (4.21)

In order to demonstrate (4.20) and (4.21), we use the rotation technique discussed in Sec. 4.2.2, and particularly equation (4.17). Thus we spin off from the R-dot of Fig. 4.4(a) a new R-dot attached to a pair of solid-dotted lines. Depending on whether the basic pattern of this new R-dot is $[A_a]$, $[A_b]$, $[B_a]$, or $[B_b]$, we get the graphs shown in Fig. B.1(a), B.1(b), B.1(c), and B.1(d) respectively.

By repeating this operation of rotating away pairs of solid and dotted lines from the R-dot still containing a compound pattern, we eventually obtain a graph where every R-dot is given by a basic pattern. Let us call the corresponding graph a *basic graph.* All possible decompositions of the compound pattern generally yield distinct basic graphs occuring a number of $v(\alpha'_a, \alpha'_b, \beta'_a, \beta'_b)$ times. Equation (4.20) corresponds to this decomposition of the compound pattern into a sum of *all* possible fully opened integrals obtained by spinning off. The coefficient $v(\alpha'_a, \alpha'_b, \beta'_a, \beta'_b)$ of every distinct basic graph is derived from a combination of three factors:

1. Each time that we spin off a basic pattern from a compound pattern with d pairs of solid-dotted lines, there is a factor $1/d$ arising from (4.17), by taking $e = 1$. Since we start from a compound pattern with N pair of lines, by the time we come to a fully-opened pattern we have accumulated a factor $1/N!$.

2. The N R-dots in a final graph that is fully opened can be spinned off in a

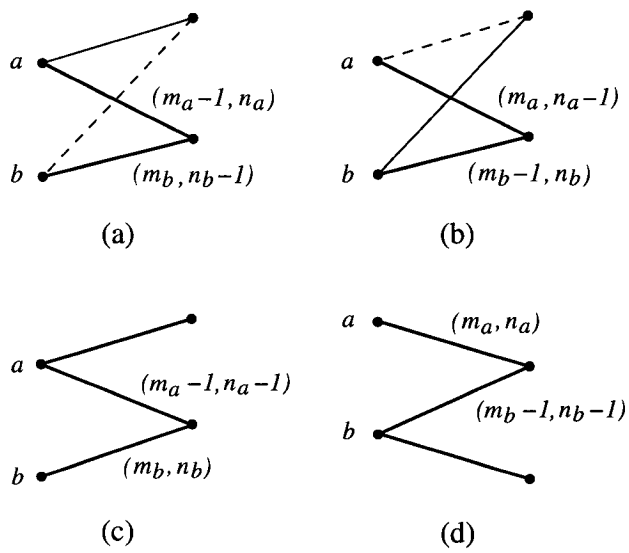


Figure B.1: Diagrams that result from rotating away the pairs of lines (a) $[A_a]$, (b) $[A_b]$, (c) $[B_a]$, and (d) $[B_b]$, from the compound pattern of Fig. 4.4.

different sequential order. According to (4.17), they must be summed over. This gives rise to a factor $N!/\alpha'_a!\alpha'_b!\beta'_a!\beta'_b!$.

3. At any time when we spin off a basic pattern, we can choose its single pair of solid-dotted lines in all possible ways. Equation (4.17) says that we must sum over all of these possibilities. The mulitipliciy factor is given by the number of permutations of these lines that lead back to the same basic integral. It is a factor of $m_a!\, m_b!\, n_a!\, n_b!$.

Assembling the three factors, we get

$$v(\alpha'_a, \alpha'_b, \beta'_a, \beta'_b) = \frac{m_a!\, m_b!\, n_a!\, n_b!}{\alpha'_a!\alpha'_b!\beta'_a!\beta'_b!}, \tag{B.1}$$

which is identical to (4.21).

APPENDIX C

RECURSION RELATION FOR X-INTEGRALS

To illustrate how direct and exchange integrals may be coupled in the recursion relation, let us look at the 1-loop *X-integral* depicted in Fig. C.1(a). The latter, specified by the multiplicities r, s, t, u of U^* elements, and r', s', t', u' of U elements, will be designated as $X(r, s, t, u|r', s', t', u') = I(C.1a)$. Since the number of dotted lines and solid lines that emerge from any dot must be equal, there are three relations for the eight parameters, i.e.

$$\begin{aligned} r' + s' &= r + s, \\ s' + t' &= s + t, \\ t' + u' &= t + u, \end{aligned}$$

so only five independent parameters are required to specify all the X-integrals.

It is fairly complicated to calculate all the X-integrals, so we will only derive the recursion relation here, and illustrate how it can be used in the simplest case. Let $\rho = r + r'$ be the total number of top lines in X, and $\sigma = s + s'$ the total number of lines on one side. The idea is to find a recursion relation in $\rho + \sigma$, each time reducing either ρ or σ by 1. Eventually one gets down to either $\rho = 0$ or $\sigma = 0$, which corresponds to a Z-integral. We have obtained a closed form, (4.27), for the Z-integrals before.

Let us derive the recursion relation. Rotate one solid and one dotted line from the $r - s$ junction of Fig. C.1(a) to an empty dot. The resulting diagrams are shown

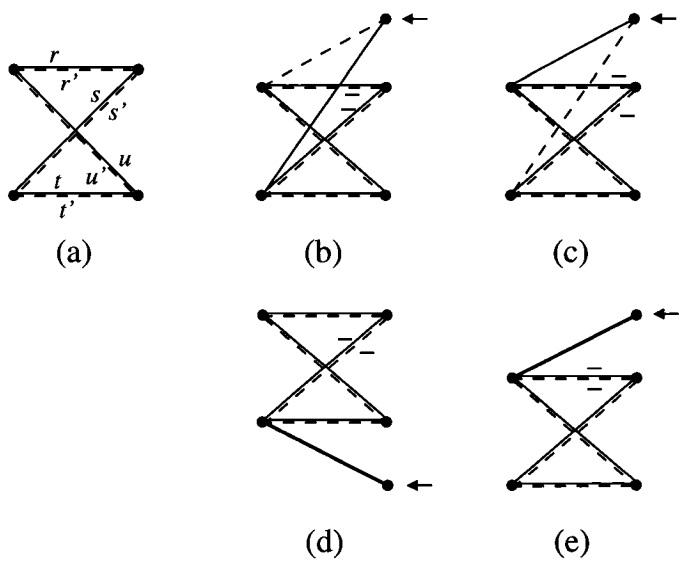


Figure C.1: (a) The X-integral and its multiplicities. (b)–(e) Diagrams obtained from (a) by rotating away a pair of lines from the $r - s$ junction. The multiplicities in these four diagrams are the same as those in (a), except for the ones with a '-' sign, in which case the corresponding multiplicity is decreased by 1. For simplicity, the thick line notation with a pair of numbers written beside it is not used.

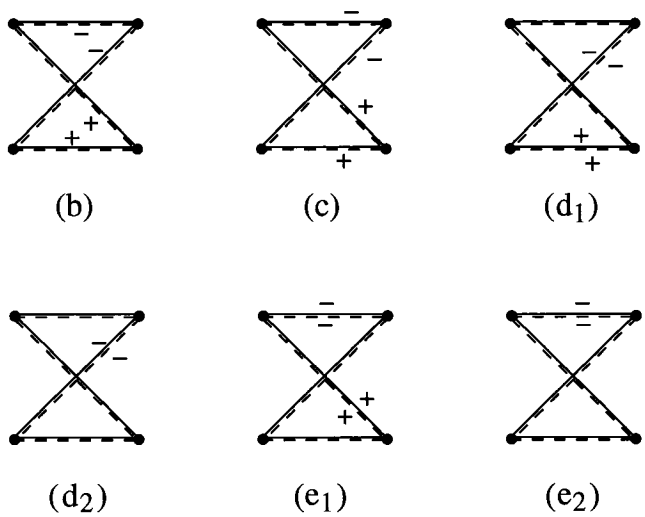


Figure C.2: Terms obtained from the unitarity sum of Figs. C.1(b) to C.1(e). The multiplicities are the same as in Fig. C.1(a), except for the lines indicated by a '-' or a '+', in which cases the multiplicities are decreased or increased by 1 respectively.

in Fig. C.1(b) to Fig. C.1(e). According to (4.17), we have

$$\begin{aligned} M_1 = r's I(C.1b) + rs' I(C.1c) + ss' I(C.1d) + rr' I(C.1e) &= (r+s) M_0 \\ &= (r+s) I(C.1a), \end{aligned} \quad \text{(C.1)}$$

where $I(C.1b)$ is the integral depicted in Fig. C.1(b), etc.

The parameters of Figs. C.1(b) to C.1(e) are those of Fig. C.1(a), except where a '-' sign occurs, in which case the corresponding parameter is reduced by 1. In Fig. C.2 we also use a '+' sign to indicate where the parameter is increased by 1.

The unitarity sum, applied to the indicated index in Figs. C.1(b) to C.1(e), results in the relations

$$\begin{aligned} (n-2) I(C.1b) + I(C.1a) + I(C.2b) &= 0, \\ (n-2) I(C.1c) + I(C.1a) + I(C.2c) &= 0, \\ (n-2) I(C.1d) + I(C.1a) + I(C.2d_1) &= I(C.2d_2), \\ (n-2) I(C.1e) + I(C.1a) + I(C.2e_1) &= I(C.2e_2). \end{aligned} \quad \text{(C.2)}$$

A substitution of (C.2) into (C.1) yields the desired recursion relation

$$I(C.1a) = \frac{-1}{(r+s)(r'+s'+n-2)}\,(\,r'sI(C.2b) + rs'I(C.2c) + \\ + ss'[I(C.2d_1) - I(C.2d_2)] + rr'[I(C.2e_1) - I(C.2e_2)]\,)\,. \quad \text{(C.3)}$$

Let us illustrate the recursion relation relation by computing the simplest cases, with $\rho + \sigma = 2$. There are four possibilities with $r + r' + s + s' = 2$. Two of them are the known Z-integrals, namely $X(0,1,t,u|0,1,t,u) = Z(1,t,u)$ and $X(1,0,t,u|1,0,t,u) = Z(1,u,t)$. The other two are exchange integrals, which can be obtained from (C.3):

$$\begin{aligned} X(1,0,t,u|0,1,t-1,u+1) &= -\frac{I(C.2c)}{n-1} \\ &= -\frac{F(t,u+1)}{n-1} \\ &= -\frac{t!(u+1)!(n-2)!}{(n+t+u)!}, \end{aligned} \quad \text{(C.4)}$$

$$\begin{aligned} X(0,1,t-1,u|1,0,t,u-1) &= -\frac{I(C.2b)}{n-1} \\ &= -\frac{F(t+1,u)}{n-1} \\ &= -\frac{(t+1)!u!(n-2)!}{(n+t+u)!}. \end{aligned}$$

In particular, (C.4) reduces to (4.29) for $I(4.7c)$ when $t = u = 1$.

APPENDIX D

PRIMITIVE DIAGRAMS FOR $p = 4$ AND $p = 5$

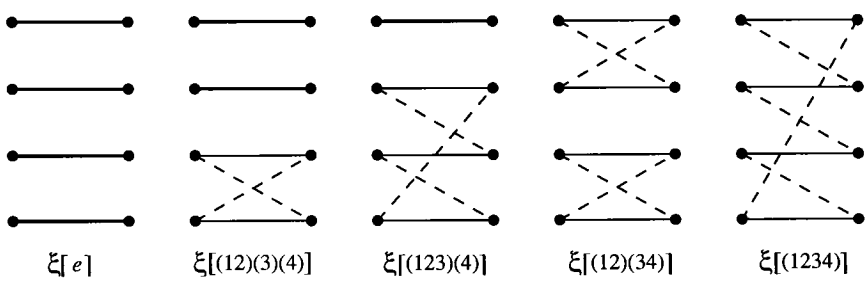


Figure D.1: The $p = 4$ primitive diagrams.

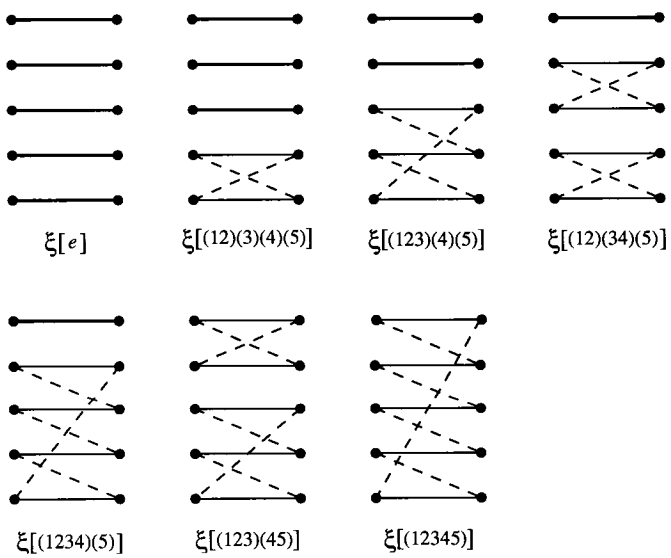


Figure D.2: The $p = 5$ primitive diagrams.

$\xi[c_q]$		
q	$p = 4$	$p = 5$
e	$\frac{n^4-8n^2+6}{n^2(n^2-1)(n^2-4)(n^2-9)}$	$\frac{n^4-20n^2+78}{n(n^2-1)(n^2-4)(n^2-9)(n^2-16)}$
$(12)(3)(4)$	$\frac{-1}{n(n^2-1)(n^2-9)}$	$\frac{-(n^2-2)(n^2-12)}{n^2(n^2-1)(n^2-4)(n^2-9)(n^2-16)}$
$(123)(4)$	$\frac{2n^2-3}{n^2(n^2-1)(n^2-4)(n^2-9)}$	$\frac{2}{n(n^2-1)(n^2-4)(n^2-16)}$
$(12)(34)$	$\frac{n^2+6}{n^2(n^2-1)(n^2-4)(n^2-9)}$	$\frac{n^2-2}{n(n^2-1)(n^2-4)(n^2-9)(n^2-16)}$
(1234)	$\frac{-5}{n(n^2-1)(n^2-4)(n^2-9)}$	$\frac{-5n^2+24}{n^2(n^2-1)(n^2-4)(n^2-9)(n^2-16)}$
$(123)(45)$		$\frac{-2(n^2+12)}{n^2(n^2-1)(n^2-4)(n^2-9)(n^2-16)}$
(12345)		$\frac{14}{n(n^2-1)(n^2-4)(n^2-9)(n^2-16)}$

Table D.1: Algebraic expressions for the primitive diagrams of fourth and fifth degrees. In the $p = 5$ case, the elements from row two to row five should be written with the additional (5) one-cycle.

Using (4.33) and the character tables for S_4 and S_5 [7], the algebraic expressions for the primitive diagrams of Fig. D.1 and Fig. D.2 can be obtained; they are given in Table D.1.

APPENDIX E

CHARACTER TABLES

The character tables for $p = 2, 3$ are given here in the form used by M. Hamermesh in [7]. The rows are labelled by the partitions that define the representations, and the columns are labelled by the cycle structures that define the classes. The number of elements in each class, $n(c)$, is written above the classes. The table for $p = 1$ is trivial as it consists of the sole value 1.

$p = 2$

	1	1
Part.\Class	(1^2)	(2)
(2)	1	1
(1^2)	1	-1

$p = 3$

	1	3	2
Part.\Class	(1^3)	$(1,2)$	(3)
(3)	1	1	1
$(2,1)$	2	0	-1
(1^3)	1	-1	1

APPENDIX F

DERIVATION OF EQUATION (4.53)

We would like to express the general fully opened integral $[A_a]^\alpha[A_b]^\alpha[B_a]^{\beta_a}[B_b]^{\beta_b}$ in terms of the special ones of the form $[A_a]^{\alpha'}[A_b]^{\alpha'}$. The idea is to apply a unitarity sum on the $[B_a]$ or $[B_b]$ basic patterns in order to get rid of them. To obtain the final result we also need to apply the fan relation (4.18) or the double-fan relation (4.20) and (4.21). Our approach is to first determine how can $[A_a]^\alpha[A_b]^\alpha[B_a]^{\beta_a}$ be reduced to fully opened integrals involving only the $[A_a]$ and $[A_b]$ patterns. With such information at hand, we will then attempt to reduce the more general $[A_a]^\alpha[A_b]^\alpha[B_a]^{\beta_a}[B_b]^{\beta_b}$ integrals into the $[A_a]^\alpha[A_b]^\alpha[B_a]^{\beta_a}$ integrals.

Let us apply a unitarity sum on one of the $[B_a]$ patterns in $[A_a]^\alpha[A_b]^\alpha[B_a]^{\beta_a}$, i.e.

$$\begin{aligned}
&(n-(2\alpha+\beta_a-1))\,[A_a]^\alpha[A_b]^\alpha[B_a]^{\beta_a}+(\beta_a-1)\,[A_a]^\alpha[A_b]^\alpha[B_a]^{\beta_a-2}[2B_a]\\
&\quad+\alpha\left([A_a]^\alpha[A_b]^{\alpha-1}[A_b+B_a][B_a]^{\beta_a-1}+[A_a]^{\alpha-1}[A_a+B_a][A_b]^\alpha[B_a]^{\beta_a-1}\right)\\
&\quad=[A_a]^\alpha[A_b]^\alpha[B_a]^{\beta_a-1}. \qquad \text{(F.1)}
\end{aligned}$$

Using (4.18), $[2B_a]$, in the second term above, can be rewritten as $2[B_a]^2$. Furthermore, equation (4.21) tells us that $[A_b+B_a]$ and $[A_a+B_a]$ can respectively be rewritten as $2[A_b][B_a]$ and $2[A_a][B_a]$, such that the term in parentheses above simplifies to $4[A_a]^\alpha[A_b]^\alpha[B_a]^{\beta_a}$. Therefore, relation (F.1) reduces to:

$$[A_a]^\alpha[A_b]^\alpha[B_a]^{\beta_a}=\frac{1}{n+2\alpha+\beta_a-1}\,[A_a]^\alpha[A_b]^\alpha[B_a]^{\beta_a-1}. \qquad \text{(F.2)}$$

Using the relation (F.2) recursively on its right side, until no $[B_a]$ remains, we obtain:

$$\begin{aligned}
&[A_a]^\alpha[A_b]^\alpha[B_a]^{\beta_a} \\
&= \frac{1}{(n+2\alpha+\beta_a-1)}\frac{1}{(n+2\alpha+\beta_a-2)}\cdots\frac{1}{(n+2\alpha)}[A_a]^\alpha[A_b]^\alpha \\
&= \frac{(n+2\alpha-1)!}{(n+2\alpha+\beta_a-1)!}[A_a]^\alpha[A_b]^\alpha, \qquad \text{(F.3)}
\end{aligned}$$

and the first step of the work is completed.

Assuming that $\beta_b \leq \beta_a$, let us perform a unitarity sum on a $[B_b]$ pattern in $[A_a]^\alpha[A_b]^\alpha[B_a]^{\beta_a}[B_b]^{\beta_b}$:

$$\begin{aligned}
&(n-(2\alpha+\beta_a+\beta_b-1))\,[A_a]^\alpha[A_b]^\alpha[B_a]^{\beta_a}[B_b]^{\beta_b} \\
&+\ (\beta_b-1)\,[A_a]^\alpha[A_b]^\alpha[B_a]^{\beta_a}[B_b]^{\beta_b-2}[2B_b] \\
&+\ \beta_a\,[A_a]^\alpha[A_b]^\alpha[B_a]^{\beta_a-1}[B_a+B_b][B_b]^{\beta_b-1} \\
&+\ \alpha\big([A_a]^\alpha[A_b]^{\alpha-1}[A_b+B_b][B_a]^{\beta_a}[B_b]^{\beta_b-1} \\
&\qquad +[A_a]^{\alpha-1}[A_a+B_b][A_b]^\alpha[B_a]^{\beta_a}[B_b]^{\beta_b-1}\big) \\
&= [A_a]^\alpha[A_b]^\alpha[B_a]^{\beta_a}[B_b]^{\beta_b-1}. \qquad \text{(F.4)}
\end{aligned}$$

Relation (4.18), or formula (4.21), again permit to make some simplifications, i.e. $[2B_b] = 2[B_b]^2$, $[B_a+B_b] = [A_a][A_b]+[B_a][B_b]$, $[A_b+B_b] = 2[A_b][B_b]$, $[A_a+B_b] = 2[A_a][B_b]$. By making the proper substitutions in (F.4), the recursion equation,

$$\begin{aligned}
&[A_a]^\alpha[A_b]^\alpha[B_a]^{\beta_a}[B_b]^{\beta_b} = \\
&\quad\frac{1}{(n+2\alpha+\beta_b-1)}\big\{[A_a]^\alpha[A_b]^\alpha[B_a]^{\beta_a}[B_b]^{\beta_b-1} \\
&\quad-\ \beta_a\,[A_a]^{\alpha+1}[A_b]^{\alpha+1}[B_a]^{\beta_a-1}[B_b]^{\beta_b-1}\big\},
\end{aligned}$$

results. The above can be solved to give

$$\begin{aligned}
&[A_a]^\alpha[A_b]^\alpha[B_a]^{\beta_a}[B_b]^{\beta_b} = \\
&\quad\sum_{e=0}^{\beta_b}\left\{(-1)^e\,e!\binom{\beta_a}{e}\binom{\beta_b}{e}(n+2\alpha-1+2e)\cdot\right. \\
&\quad\left.\frac{(n+2\alpha-2+e)!}{(n+2\alpha+\beta_b-1+e)!}[A_a]^{\alpha+e}[A_b]^{\alpha+e}[B_a]^{\beta_a-e}\right\},
\end{aligned}$$

which upon substitution of (F.3) yields the desired equation (4.53).

APPENDIX G

GAMES

G.1 The Primitive Integrals and the IM

We will present a method, though limited, to calculate the primitive integrals from the IM. As was seen in Sec. 4.3.3, if for a representation, $(f_1, f_2, \ldots, f_p)$, we associate the stack diagram, $\Xi(f_1, f_2, \ldots, f_p)$, then it is clear that there is a one-to-one correspondence between them, and that there must be as many different stack diagrams as there are conjugacy classes, or $\xi[c]$ integrals. In the case where all the stack integrals can be calculated via the IM, we wish to obtain a system of linear equations (refer to (4.45)) that can easily be solved for the various $\xi[c]$'s, hence rendering a knowledge of the symmetric group characters obsolete. For any p, it is not hard to convince oneself that all linear expansions of stack integrals into $\xi[c]$'s are linearly independent; examine the group structure of each $\Xi(f_1, f_2, \ldots, f_p)$'s symmetry group, $\mathcal{G} = S_{f_1} \otimes S_{f_2} \otimes \cdots \otimes S_{f_p}$, for instance. In fact, if the equations due to (4.45) are organized according to a dictionary order in the f_1, f_2, ... values, one can note that the various $\xi[c]$ coefficients gradually vanish when reading the equations in order (from $\Xi(p)$, with $\mathcal{G} = S_p$, and containing all classes of S_p, to $\Xi(1, \ldots, 1)$, with $\mathcal{G} = S_1 \otimes \cdots \otimes S_1$, which contains only c_e). As such, the equations can be arranged in an upper triangular form. For greater clarity, refer to the example below.

For a given degree p, knowing that an upper triangular system of equations exists for the primitive integrals in terms of the stack ones, the difficulty lies in

the determination of the latter integrals from unitarity, the invariance of the group measure, or directly the Z-formula (4.27). We leave it to the reader to verify up to what order can the stack diagrams be determined from the IM. Meanwhile, let us apply the procedure to the case of $p = 4$.

Using equation (4.45), the system of linear equations,

$$\begin{bmatrix} \Xi(4) \\ \Xi(3,1) \\ \Xi(2,2) \\ \Xi(2,1,1) \\ \Xi(1,1,1,1) \end{bmatrix} = \begin{bmatrix} 4! & 4!\cdot 6 & 4!\cdot 8 & 4!\cdot 3 & 4!\cdot 6 \\ 3! & 3!\cdot 3 & 3!\cdot 2 & 0 & 0 \\ 4 & 4\cdot 2 & 0 & 4 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi[c_e] \\ \xi[c_{(12)(3)(4)}] \\ \xi[c_{(123)(4)}] \\ \xi[c_{(12)(34)}] \\ \xi[c_{(1234)}] \end{bmatrix}, \tag{G.1}$$

can be built, and can easily be transformed into an upper triangular system of linear equations.

The first three stack integrals can be calculated directly from the Z-formula, and they are given as follows:

$$\begin{aligned} \Xi(4) &= Z(4,0,0) = \frac{4!}{n(n+1)(n+2)(n+3)}, \\ \Xi(3,1) &= Z(3,0,1) = \frac{3!}{(n^2-1)n(n+3)}, \\ \Xi(2,2) &= Z(2,0,2) = \frac{4}{(n-1)n^2(n+3)}. \end{aligned}$$

As for $\Xi(2,1,1)$, it can be obtained by performing a unitarity sum on a line with multiplicity one, i.e.

$$(n-2)\,\Xi(2,1,1) + \frac{1}{2}\,Z(2,0,2) + \frac{1}{3}\,Z(3,0,1) = Z(2,0,1),$$

where the fan relation was used to rewrite the second and third summands. The solution for $\Xi(2,1,1)$ is:

$$\Xi(2,1,1) = \frac{2\,(n^3+2n^2-2n-2)}{n^2(n^2-1)(n^2-4)(n+3)}.$$

With the obove result known, the last stack integral can also be obtained through a unitarity sum applied on any of its lines, i.e.

$$(n-3)\,\Xi(1,1,1,1) + 3\cdot\frac{1}{2}\,\Xi(2,1,1) = \Xi(1,1,1),$$

where the factor of $\frac{1}{2}$ is again due to the fan relation. The integral $\Xi(1,1,1)$ has been calculated previously (refer to Table 4.2). As a result,

$$\Xi(1,1,1,1) = \frac{n^4 - 8n^2 + 6}{n^2(n^2-1)(n^2-4)(n^2-9)}.$$

We are now in a position to solve (G.1). The solutions for all $\xi[c]$ integrals are shown in Table D.1.

G.2 Calculation of Symmetric Group Characters

We now wish to deduce the S_p group characters from the primitive integrals $\xi[c]$. Recall that the latter are given by equation (4.33),

$$\xi[q] = \xi[c_q] = \sum_{(f)} \frac{\chi_f^2(e)}{p!^2\, \tilde{d}_f}\, \chi_f(c_q), \tag{G.2}$$

where $\tilde{d}_f$ is given by (3.6). We also assume that all primitive integrals are known, and given in terms of n; the method of the preceding section could have been used for their obtainment for instance. Before attempting to determine the characters, we suggest to first write the equations for (G.2) in terms of n, that is to express the $\xi[c]$ and $\tilde{d}_f$ functions in terms of n. The reason for doing so is that, in a way very much similar to what is done in complex analysis, the pole structures in n may be used to obtain the various combinations $\chi_f^2(e)\,\chi_f(c_q)$. Hence, for $p = 1, \ldots, 4$, systems of equations with the mentioned combinations as unknowns can be written in an upper triangular form. For higher p values, the systems of equations can be simplified, yet not sufficiently for them to be triangular. For greater clarity, refer to the example below.

When the combinations $\chi_f^2(e)\,\chi_f(c_q)$ have been found, there remains to untangle the quantities $\chi_f(e)$ and $\chi_f(c_q)$. The task is facilitated if one notices that when working with the identity class, the combinations reduce to $\chi_f^3(e)$, thus allowing for a unique real solution for each of the $\chi_f(e)$ characters. As a check, $d_f = \chi_f(e)$ should be positive. Once more, we leave it to reader to verify up to what order can this

method be applied. In the remainer of the section, we will apply the above ideas to the case of $p = 3$.

From equations (G.2) and (3.6), the $\xi[c]$ functions can be expressed in terms of the characters as:

$$\begin{aligned}
\xi[e] &= \frac{n^2-2}{(n-2)(n-1)n(n+1)(n+2)} \\
&= \frac{1}{3!^2}\left[\begin{array}{ccc} \frac{6}{n(n+1)(n+2)} & \frac{3}{(n-1)n(n+1)} & \frac{6}{(n-2)(n-1)n} \end{array}\right]\left[\begin{array}{c} \chi^3_{(3)}(e) \\ \chi^3_{(2,1)}(e) \\ \chi^3_{(1^3)}(e) \end{array}\right] \\
&\equiv \frac{1}{3!^2}\frac{1}{\tilde{D}(n)}\left[\begin{array}{c} \chi^3_{(3)}(e) \\ \chi^3_{(2,1)}(e) \\ \chi^3_{(1^3)}(e) \end{array}\right], \qquad \text{(G.3)}
\end{aligned}$$

$$\begin{aligned}
\xi[c_{(12)(3)}] &= \frac{-1}{(n-2)(n-1)(n+1)(n+2)} \\
&= \frac{1}{3!^2}\frac{1}{\tilde{D}(n)}\left[\begin{array}{c} \chi^2_{(3)}(e)\,\chi_{(3)}(c_{(12)(3)}) \\ \chi^2_{(2,1)}(e)\,\chi_{(2,1)}(c_{(12)(3)}) \\ \chi^2_{(1^3)}(e)\,\chi_{(1^3)}(c_{(12)(3)}) \end{array}\right], \qquad \text{(G.4)}
\end{aligned}$$

$$\begin{aligned}
\xi[c_{(123)}] &= \frac{2}{(n-2)(n-1)n(n+1)(n+2)} \\
&= \frac{1}{3!^2}\frac{1}{\tilde{D}(n)}\left[\begin{array}{c} \chi^2_{(3)}(e)\,\chi_{(3)}(c_{(123)}) \\ \chi^2_{(2,1)}(e)\,\chi_{(2,1)}(c_{(123)}) \\ \chi^2_{(1^3)}(e)\,\chi_{(1^3)}(c_{(123)}) \end{array}\right]. \qquad \text{(G.5)}
\end{aligned}$$

Taking advantage of the pole structure, we shall first solve for $\chi_{(3)}(e)$, $\chi_{(2,1)}(e)$, and $\chi_{(1^3)}(e)$. Thus, by multiplying (G.3) by $(n+2)$ before setting $n = -2$, we obtain

$$\frac{1}{12} = \frac{1}{12}\chi^3_{(3)}(e),$$

or $\chi_{(3)}(e) = 1$. Similarly, let us multiply equation (G.3) by $(n-2)$, and then set $n = 2$. As a result,

$$\frac{1}{12} = \frac{1}{12}\chi^3_{(1^3)}(e),$$

which also yields $\chi_{(1^3)}(e) = 1$. If the same equation is multiplied by $(n-1)$, with n subsequently set to 1, then

$$\frac{1}{6} = \frac{1}{36}\left(\frac{3}{2}\chi^3_{(2,1)}(e) - 6\right),$$

such that $8 = \chi^3_{(2,1)}(e)$, or $\chi_{(2,1)}(e) = 2$.

Now that all the characters of the identity class are known, the exact same procedure as above can be applied to equations (G.4) and (G.5). The results have been verified to agree with the second table of Appendix E.

NOTATION

S_p	the symmetric group of degree p
$U(n)$	the unitary group in n dimensions
dU	the unitary group measure or unitary Haar measure
$*$	(superscript $*$) the *complex conjugate* of a number or matrix
t	(superscript t) the *transpose* of a matrix
$\dagger$	(superscript $\dagger$) the *conjugate-transpose* operation, i.e. if A is a complex matrix or vector, then $A^\dagger = A^{*t}$.
tr	the trace operator, i.e. if A is some $n \times n$ matrix, then $\mathrm{tr}A = \sum_i A_{ii}$.
δ_{ij}	the Kronecker delta, i.e. $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ otherwise.
$\binom{n}{k}$	the binomial coefficient $n!/k!(n-k)!$
$s \in \mathcal{S}$	s is an element of the set $\mathcal{S}$.
$\mathcal{S} \subset \mathcal{T}$	$\mathcal{S}$ is a subset of $\mathcal{T}$, or $\mathcal{S}$ is contained in $\mathcal{T}$, i.e. every element of $\mathcal{S}$ is also an element of $\mathcal{T}$.
e	in a group theoretical context, the identity element of any symmetric group
$\chi(r)$	the character of the element r of a symmetric group
$\chi(c)$	the character of the class c of a symmetric group
$\tilde{\chi}(u)$	the character of the element u of the unitary group $U(n)$
(ν)	the cycle structure $(1^{\nu_1}, 2^{\nu_2}, \ldots, p^{\nu_p})$ associated with an element or class of S_p, where ν_i is the number of cycles of length i.

(f) representation $(f_1, f_2, \ldots, f_x)$ of the symmetric group S_p $(x = p)$, or $U(n)$ group $(x = n)$. The conditions $f_1 \geq f_2 \geq \cdots \geq f_x \geq 0$ and $\sum_{i=1}^{x} f_i = p$ are satisfied everywhere in the work.

γ_p the number of distinct classes of S_p, or equivalently the number of inequivalent irreducible representations of the same group

d_f the dimension of the representation (f) of the symmetric group S_p

$\tilde{d}_f$ the dimension of the representation (f) of the unitary group $U(n)$

I_s the ordered index set $(i_{s(1)}, i_{s(2)}, \ldots, i_{s(p)})$, where $s \in S_p$. In the case where $s = e$, $I_e \equiv I$. The number of indices p should be clearly identifiable from the context.

A_{IJ} the juxtaposition $\prod_{a=1}^{p} A_{i_a j_a}$ of p elements of any matrix A. The number p should be clearly identifiable from the context.

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