EXCEPTIONAL SETS IN A PRODUCT OF HARMONIC SPACES AND APPLICATIONS

by

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A thesis submitted to the Faculty of Graduate Studies and Research, in partial fulfillment of the requirement for the degree of Doctor of Philosophy.

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March 1980

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ABSTRACT

A study of exceptional sets in a finite product of Brelot spaces is made. The principal results obtained are a convergence theorem for decreasing sequences of n-superharmonic functions and an extension theorem for positive n-superharmonic functions. Similar results are obtained for plurisuperharmonic functions.

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RESUME

Une étude des ensembles exceptionnels dans un produit fini des espaces harmoniques de Brelot est faite. Les résultats principaux obtenus sont un théorème pour une suite decroissante des fonctions n-surharmoniques et un théorème de prolongement pour les fonctions positifs n-surharmoniques. Les résultats similaires sont obtenus pour les fonctions plurisurharmoniques.

Département de Mathématiques

Université de McGill

Montreal

Mars 1980
Ph.D.

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Introduction

The convergence theorem of Cartan and Brelot which concerns the limit of a decreasing sequence of potentials (p_n) on a domain in $\mathbb{R}^n, n \geq 2$, has widespread applications in potential theory. See [0] for a detailed discussion. The theorem states that the lower envelope of (p_n) differs from its lower semicontinuous regularization at most on a set of outer capacity zero. Its principal use is in connection with the Dirichlet problem on a relatively compact open set. If the problem is solved by means of the P.W.B. method then the convergence theorem implies the set of irregular boundary points has outer capacity zero. The sets of outer capacity zero are precisely the subsets of sets on which superharmonic functions take the value ∞, the so called polar sets. These ideas can therefore be framed in the axiomatic setup of Brelot and as is well known the results go through with the additional assumption of the Axiom of Domination.

It is now natural to consider the convergence theorem on a finite product of n Brelot spaces with a decreasing sequence of n-superharmonic functions. The problem is to choose the appropriate associated exceptional set, the analogue of the polar set. In this thesis we investigate such analogues and survey the basic areas of potential theory where they arise, most notably the above mentioned convergence theorem. Until now such a systematic study has not been made. There are two obvious types of sets to consider. Firstly the subsets of sets on which n-superharmonic functions take the value ∞ (the n-polar sets) and secondly the sets "most" of whose sections are polar in each of the respective underlying Brelot spaces (the n-negligible sets). See Chapter 2.2 for a precise definition. We shall see that n-polar implies n-negligible and though at present we do not have a counterexample we suspect the converse is false. It appears that the n-negligible sets are the more useful of the two to look at in considering deeper results since the n-polar condition seems somewhat restrictive. The n-negligible sets were apparently first introduced with a significant application by K. Gowrisankaran in the special case of polydiscs in C^n where he studied the class of good inner functions [9].

In Chapter 1 a summary of all results needed of a single and a finite product of Brelot spaces is made, in most cases without proof. One exception is the discussion of the Cartan-Brelot topology where we prove Proposition 1.1.23 that a sequence of uniformly locally bounded positive superharmonic functions has a Cartan-Brelot convergent subsequence with the assumption of Axiom D instead of the usual assumption of a base of completely determining regular domains. We use this in 2.3 to solve the Dirichlet problem on a product of relatively compact domains. In Chapter 2 the n-polar and n-negligible sets are defined and a preliminary study is made. In Chapter 3 the two main results of the thesis are demonstrated, namely Theorem 3.1.7 where we show that the lower envelope of a family of locally uniformly lower bounded n-superharmonic functions differs from its lower semicontinuous regularization at most on an n-negligible set and Theorem 3.3.1 where we show locally lower bounded n-superharmonic functions can be extended across closed n-negligible sets to be n-superharmonic. In Chapter 4 we prove analogues of these two results in the more concrete setting of C^n and the plurisuperharmonic functions. We first do this with a type of exceptional set of our own invention, the n-P negligible set. We then consider these results with associated exceptional sets the so-called sets of zero Ronkin Γ-capacity, (See [16]). For this class both of these results are already known. The extension theorem was proved by U. Cegrell in [3]. The first results concerning the convergence of decreasing sequences of plurisuperharmonic functions were proved with the additional requirement that the regularized limit function be pluriharmonic. In this case the exceptional set is pluripolar. See [14]. This was generalized by Ronkin in [16] where he showed without additional assumption the exceptional set is of zero Γ-capacity. Following the work of Favorov [6] U. Cegrell also proved this result using a general theory of product capacities: See [4]. We present here alternative proofs of these theorems to show how easily they follow from the axiomatic framework.

I wish to thank Professor K. Gowrisankaran for suggesting the topic of this thesis to me, for his help in my preparation of it, and for the years of guidance he has given me in my mathematical development. I would also like to thank R. Jesuraj and Bernard Mair for many stimulating discussions of this work as well as Hilde Schroeder for typing the manuscript.

CHAPTER 1

Preliminaries

In this chapter we review the topics in axiomatic potential theory which we shall be using. For more details and proofs see [1],[7], and [8].

Section 1 Brelot Spaces

Let Ω be a locally compact, non-compact, connected, locally connected, Hausdorff space such that for every open set U of Ω there is a real vector space H(U) of real valued continuous functions on U called harmonic functions. Denote the non-negative harmonic functions on U by $H^+(U)$. We impose the following three axioms on Ω and the harmonic functions.

Axiom 1: i) If U and V are open with U contained in V then the restriction of any member of H(V) to U is in H(U).

ii) Let U be open and v a real valued function defined on U. If for each x in U there is a neighbourhood U_x of x such that v is in $H(U_x)$ then v is in H(U).

We say a relatively compact open set δ is regular if for every real valued continuous function f on $\partial\delta$ there exists a unique continuous function on $\bar{\delta}$ such that it is non-negative if f is non-negative and its restriction to δ is harmonic. This function is denoted by H_f^δ .

Axiom 2: There is a base of open sets consisting of regular domains.

Axiom 3: Given a pointwise increasing sequence in H(U) for U any domain, the limit function is either in H(U) or identically ∞ .

Constantinescu and Cornea have shown that in Axiom 3 no generality is lost by replacing the sequence with an increasing directed subset of $H(U)$. See [5].

With such a structure we call Ω a Brelot space. Note that any connected open subset of a Brelot space is a Brelot space as well.

The fundamental example of a Brelot space is Euclidean n-space, $n \geq 1$, where the harmonic functions are the twice continuously differentiable functions satisfying Laplace's equation $\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0$.

We shall be imposing three more restrictions on our Brelot spaces. Essentially all natural examples will satisfy these restrictions.

The first is that we assume each point of Ω has a countable base of neighbourhoods. It has been shown by Constantinescu and Cornea that this implies the existence of a countable base for all open sets. See [5]. As a consequence every open set can be written as a countable union of relatively compact open sets $(\omega_n)_{n\geq 1}$ with $\overline{\omega_n} \subset \omega_{n+1}$ for every n.

We postpone for the moment a description of the other two restrictions.

For each regular open set δ and x in δ the mapping $C_{\mathbb{R}}(\partial\delta) \to \mathbb{R}$, $f \to H_f^{\delta}(x)$ defines a positive linear functional, that is a Radon measure. We denote this Radon measure by ρ_x^{δ} and its value on f in $C_{\mathbb{R}}(\partial\delta)$ by $\int f \, d\rho_x^{\delta}$. Three properties of these measures are given in the first proposition.

<u>Proposition 1.1.1:</u> Let δ be a regular domain.

(a) The sets of outer ρ_x^{δ}-measure 0 are independent of x in δ.

(b) For any extended real valued function f on $\partial\delta$ the map $x \to \overline{\int} f \, d\rho_x^{\delta}$ is either identically ∞, identically $(-\infty)$, or in $H(\delta)$ and if f is ρ_x^{δ}-integrable for one x in δ it is integrable with respect to all such measures.

(c) Let δ_n be a sequence of regular domains such that for all n, $\overline{\delta_{n+1}} \subset \delta_n$ and $\bigcap_{n=1}^{\infty} \delta_n = \{x\}$. Then for all f in $C_{\mathbb{R}}(\overline{\delta}_1)$

$$\lim_{n\to\infty} \int f \, d\rho_x^{\delta_n} = f(x).$$

Definition 1.1.2: Let U be open. An extended real valued function v on U is said to be hyperharmonic in U if

i) $v(x) > -\infty$ for all x in U,

ii) v is lower semi-continuous,

iii) for every regular open set δ with $\overline{\delta} \subset U$ and x in δ

$$\int v \, d\rho_x^{\delta} \leq v(x).$$

If in addition v is finite at least at one point of every connected component of U then v is said to be superharmonic on U. We denote the set of all superharmonic and non-negative superharmonic functions on U by S(U) and $S^+(U)$ respectively.

Notice harmonic functions are superharmonic and a superharmonic function is harmonic if and only if its negative is also superharmonic. We summarize other important properties in the following.

Proposition 1.1.3: Let U be a domain.

(a) If v_1, v_2 are in S(U) and α, β are non-negative real numbers then $\alpha v_1 + \beta v_2$ and $\min(v_1, v_2)$ are in S(U).

(b) If v is hyperharmonic on U and $v(x) = \infty$ for all x in an open subset of U then v is identically ∞ on U.

(c) If $(v_i)_{i \in I}$ is a pointwise increasing directed family of hyperharmonic functions on U then the upper envelope is also hyperharmonic. As a consequence any v in $S^+(U)$ is either identically 0 or strictly positive. (Just consider $(n \cdot v)_{n \geq 1}$ and apply (b)).

(d) If v is in $S(\Omega)$ and δ is a regular open set, define E_v^δ on Ω by

$$E_v^\delta(x) = \begin{cases} v(x) & x \text{ in } \Omega - \delta \\ \int v \, d\rho_x^\delta & x \text{ in } \delta \end{cases} .$$

Then E_v^δ minorizes v pointwise, is in $S(\Omega)$, is harmonic on δ, and every point of continuity of v is a point of continuity of E_v^δ.

(e) (local property). Let v be an extended real valued lower semi-continuous function on U such that $v(x) > -\infty$ for all x. Suppose for each x in U and each neighbourhood ω of x there exists δ a regular neighbourhood of x contained in ω such that $\int v \, d\rho_x^\delta \leq v(x)$. Then v is hyperharmonic on U.

(f) (minimum principle) Suppose U is also relatively compact and v is in S(U). If for all x in $\partial\delta$

$$\liminf_{\substack{z \to x \\ z \in U}} v(z) \geq 0$$

then $v(z) \geq 0$ for all z in U.

<u>Remark 1.1.4:</u> If v is in S(U) for any open set U and δ is a regular open set such that $\overline{\delta} \subset U$ then part (b) above and Proposition 1.1.1(b) implies v is ρ_x^δ -integrable for all x in δ.

Definition 1.1.5: Let $\mathcal{B}$ be a base of regular domains. An extended real valued function v on Ω is said to be an $S_{\mathcal{B}}$ functions (and a nearly superharmonic function if $\mathcal{B}$ is the set of all regular domains) if

i) v is locally lower bounded and

ii) for all δ in $\mathcal{B}$ and x in δ, $\overline{\int} v \, d\rho_x^\delta \leq v(x)$.

It is easy to see the lower envelope of any uniformly locally lower bounded family of $S_{\mathcal{B}}$ functions is $S_{\mathcal{B}}$ and the upper envelope of any increasing directed family of $S_{\mathcal{B}}$ functions is $S_{\mathcal{B}}$. In particular if $(v_n)_{n\geq 1}$ is a pointwise decreasing sequence in $S^+(\Omega)$ the lower envelope is nearly superharmonic. More generally for any sequence $(w_n)_{n\geq 1}$ in $S^+(\Omega)$, $\liminf_{n\to\infty} w_n$ is nearly superharmonic.

Let v be $S_{\mathcal{B}}$. We denote the lower semi-continuous regularization of v by $\hat{v}$. That is $\hat{v}$ is pointwise the largest lower semi-continuous function minorizing v. Explicitly

$$\hat{v}(x) = \liminf_{z\to x} v(z)$$

(This of course holds for extended real valued functions on any Hausdorff space). The fundamental result for $S_{\mathcal{B}}$ functions is the following.

Proposition 1.1.6: Let v be $S_{\mathcal{B}}$. Then $\hat{v}$ is hyperharmonic. Hence a lower semi-continuous $S_{\mathcal{B}}$ function is hyperharmonic. For every x in Ω and every sequence $(\delta_n)_{n\geq 1}$ in $\mathcal{B}$ with $\overline{\delta}_{n+1} \subset \delta_n$ for all n and $\bigcap_{n\geq 1} \delta_n = \{x\}$, the sequence $(\overline{\int} v \, d\rho_x^{\delta_n})_{n\geq 1}$ is increasing and has limit $\hat{v}(x)$.

We consider now an important class of nearly superharmonic functions.

Definition 1.1.7: Let E be any subset of Ω and let v be in $S^+(\Omega)$. The reduced function R_v^E is defined on Ω by

$$R_v^E(x) = \inf \{w(x) : w \in S^+(\Omega), w(y) \geq v(y) \text{ for all } y \text{ in } E\} .$$

Proposition 1.1.8: R_v^E is nearly superharmonic, pointwise minorizes v, equals v on E, is harmonic on $\Omega - \overline{E}$, is pointwise monotone non-decreasing in E and v (subsets of Ω are here ordered by inclusion), and is subadditive in v.

Remark 1.1.9: If ω is a relatively compact open set it follows easily from the minimum principle that for u harmonic in Ω, $R_u^{\Omega-\omega}$ is identical to u.

Given any v in $S^+(U)$ where U is a domain there exists a unique function u on U which is harmonic and for the pointwise order is the greatest harmonic minorant of v on U. Explicitly if $(U_n)_{n\geq 1}$ is a sequence of relatively compact open sets with union U and for all n, $U_n \subset U_{n+1}$ then u is the lower envelope of the pointwise decreasing sequence of functions $(R_v^{U-U_n})_U$. (Here the subscript U means the reduced function is defined with respect to the Brelot space U). This follows easily from Remark 1.1.9. Thus v can be written uniquely as the sum of a function in $H^+(U)$ and one in $S^+(U)$ which pointwise majorizes no member of $H^+(U)$ other than 0. That is $v=u+(v-u)$. Classically this is the Riesz decomposition of v.

Definition 1.1.10: Let U be open and let p be in $S^+(U)$. Then p is said to be a potential on U if its greatest harmonic minorant on U is 0. If $U = \Omega$ we call p a potential

It is clear that the minimum of a potential and a positive superharmonic function as well as the product of a potential and a non-negative real number are both potentials. From our explicit construction and the subadditivity of the reduced function it is easy to see the sum of two potentials is a potential.

In a given Brelot space there is no need for positive potentials to exist. For example $\mathbb{R}^2$ has none. However in such a case it is immediate from the decomposition of positive superharmonic functions that they are all harmonic. Using this and Proposition 1.1.3(a) and (c) we see that of any two positive superharmonic functions one is everywhere greater than the other. Now multiplication by a suitable real number gives that any two positive superharmonic functions are proportional. In order to avoid the trivialities resulting from this situation we shall assume from now on the existence of a positive potential on Ω. In $\mathbb{R}^n$ for $n \geq 3$ these do indeed exist and for $n = 2$ any open set having a Green function has a positive potential, the potentials being precisely the convolution of the Green function with positive measures. (See [10]).

As a consequence of the existence of a positive potential the following can be deduced.

<u>Proposition 1.1.11:</u> Given an open set U there exists a positive potential which is real valued, continuous, harmonic on $\Omega-\overline{U}$, and not harmonic on U. Thus every domain of Ω is itself a Brelot space with positive potential.

The following <u>continuation theorem</u> can be proved: Let U,U' be open sets with $\overline{U} \subset U'$ and v in $S^+(U')$. Then there exist potentials p_1, p_2 such that on U, $p_1 = p_2 + V$. We demonstrate this as well as a generalization in Chapter 2.4.

<u>Definition 1.1.12:</u> Let U be open in Ω and E a subset of U. E is said to be polar in U if there exists v in $S^+(U)$ such that E is contained in $\{x \in U : v(x) = \infty\}$.

Using the continuation theorem the following <u>local property</u> can be shown: If for every x in E there is an open set U_x containing x such that $E \cap U_x$ is polar in U_x then E is polar in Ω. In particular E polar in U implies it is polar in Ω. Thus we may refer to a set as being simply polar without reference to an open set containing it.

The following proposition is a property of closed polar sets which we generalize in Theorem 3.3.1.

<u>Proposition 1.1.13:</u> Let E be a closed polar subset of Ω. Then every v in $S(\Omega - E)$ that is locally lower bounded on Ω has an extension to a function in $S(\Omega)$.

We consider now the Dirichlet problem on a relatively compact domain ω. Let f be an extended real valued function on $\partial\omega$. Put

$$U(f) = \{v\colon v \text{ hyperharmonic on } \omega,\ v \text{ lower bounded}, \liminf_{\substack{z \to x \\ z \in \omega}} v(z) \geq f(x) \text{ for all } x \text{ in } \partial\omega\}.$$

The upper solution $\overline{H}_f^\omega$ is defined pointwise on ω as the lower envelope of U(f) and the lower solution $\underline{H}_f^\omega$ is defined to be $-\overline{H}_{(-f)}^\omega$. It is easy to see if f is defined and superharmonic on Ω then $\overline{H}_f^\omega$ is nothing but $R_f^{\Omega-\omega}$. The minimum principle shows the upper solution is always greater than or equal to the lower solution. Both can be shown to be identically ∞, identically $(-\infty)$, or in $H(\omega)$ and hence if they are equal at one point they are identical. In case they are the same and in $H(\omega)$ we say f is resolutive and write for the common function H_f^ω. It can be shown that real valued continuous functions are resolutive and for each x in ω the mapping $C_{\mathbb{R}}(\partial\omega) \to \mathbb{R}$, $f \to H_f^\omega(x)$ is a Radon measure. We denote this Radon measure by μ_x^ω. It is also called the harmonic measure. In case ω is regular H^{ω}_{f} is precisely H^{ω}_{f} and hence μ^{ω}_{x} is just ρ^{ω}_{x} . The integrability of any f with respect to ρ^{ω}_{x} is independent of x in ω and it is integrable with respect to any (hence all) such measures if and only if it is resolutive. Note this shows functions superharmonic on a neighbourhood of ω are resolutive. The sets of μ^{ω}_{x} outer measure 0 can be shown to be independent of x in ω hence we can deduce polar subsets of $\partial\omega$ have 0 harmonic measure.

A point x in $\partial\omega$ is said to be regular if for each f in $C_{\mathbb{R}}(\partial\omega)$ the solution $H^{\omega}_{f}(z)$ tends to f(x) as z tends to x from ω . Otherwise a point is called irregular. In order to characterize the set of these irregular boundary points as well as deduce other deep results we need to introduce the following "axiom of domination".

Axiom D: Let ω be any relatively compact open set and v a locally bounded member of $S^{+}(\omega)$ which is harmonic on ω . Then any w in $S^{+}(\Omega)$ which pointwise majorizes it on Ω-ω majorizes it on Ω . That is $v \equiv R_{v}^{\Omega-\omega}$.
It can be shown this is equivalent to the following: For any positive locally bounded superharmonic function v on Ω and relatively compact open set ω the greatest harmonic minorant of v on ω is $R_{v}^{\Omega-\omega}(\cdot) = \int v d\mu^{\omega}_{\cdot}$.

It can also be shown that Axiom D holding on Ω implies it holds on any open subset of Ω .

Remark 1.1.14: Suppose in our statement of Axiom D, w only majorizes v on $(\Omega - \omega)-P$ where P is polar. We claim with Axiom D we can still deduce w majorizes v. Indeed let u be in $S^{+}(\Omega)$ with $u(x) = \infty$ if x is in P. For every positive integer n the function $w + (\frac{1}{n})u$ is in $S^{+}(\Omega)$ and majorizes v on Ω-ω . Thus Axiom D implies $w + (\frac{1}{n})u$ majorizes v on Ω . Since n is arbitrary we deduce w majorizes v on the set $\{x \in \Omega : u(x) < \infty\}$, that is everywhere except on the polar Borel set $\{x \in \Omega : u(x) = \infty\}$. (It is Borel because it is just $\bigcap_{n=1}^{\infty} \{x \in \Omega :\ u(x) > n\}$). Since such a set has 0 measure for any harmonic measure we apply Proposition 1.1.6 to deduce w majorizes v on Ω .

We shall assume in much of the thesis that Axiom D holds. As a consequence the following very important results can be demonstrated.

Theorem 1.1.15: (a) (Convergence Theorem) If $(v_n)_{n\geq 1}$ is a pointwise decreasing sequence in $S^+(\Omega)$ with limit function v then v is nearly superharmonic and equals $\hat{v}$ everywhere except on a polar set.

(b) The irregular boundary points of a relatively compact domain are polar.

In Chapter 3 we generalize the convergence theorem to a product of Brelot spaces and a sequence of "n-superharmonic" functions.

Remark 1.1.16: Recall the topological lemma of Choquet: Let X be a topological space with a countable base of open sets and $(f_i)_{i\in I}$ any collection of extended real valued functions on X. Then there exists I_o , a countable subset of I, such that

$$\widehat{\inf_{\alpha \in I} f_\alpha} = \widehat{\inf_{\alpha \in I_o} f_\alpha} .$$

It follows from this and the convergence theorem that for any v in $S^+(\Omega)$ and E contained in Ω, $\hat{R}_v^E$ and R_v^E differ at most on a polar set.

We now consider topologies on spaces of differences of positive harmonic and superharmonic functions. Both are defined without Axiom D though we introduce Axiom D later in order to prove Proposition 1.1.23.

Consider first the space $H^+ - H^+$ of differences of positive harmonic functions on Ω. With the topology of uniform convergence on compact sets and the obvious vector space structure $H^+ - H^+$ is a metrizable, locally convex, topological vector space. The following important result is due to Mokobodski and Brelot. See [15] and [1].

Theorem 1.1.17: For any real number M and any x in Ω, $\{u \in H^+(\Omega): u(x) \leq M\}$ is compact.

Thus each sequence in $H^+(\Omega)$ which is bounded at one point has a subsequence which converges locally uniformly to a function in $H^+(\Omega)$.

Define an equivalence relation on the set of pairs of functions in $S^+(\Omega)$. We say (u,v) is equivalent to (u_1,v_1) if for all x, $u(x) + v_1(x) = u_1(x) + v(x)$. The equivalence class containing (u,v) is denoted by $[(u,v)]$ and the set of all equivalence classes we call $S^+ - S^+ = S$. With the obvious operations S becomes a vector space. Notice S^+ can be identified with the set $\{[u,0]: u \in S^+(\Omega)\}$.

Now fix a countable base $\mathcal{B}$ of regular domains of Ω and let X be a countable dense subset of Ω. For ω in $\mathcal{B}$ and x in $\omega \cap X$ define the functional $\Pi_{\omega,x}$ on S by

$$\Pi_{\omega,x}[(u,v)] = \left|\int u\, d\rho_x^\omega - \int v\, d\rho_x^\omega\right| .$$

Clearly $\Pi_{\omega,x}$ is well defined, is a seminorm, and the countable family of all such seminorms defines a metrizable, locally convex, topological vector space structure on S. We call this topology the Cartan-Brelot topology.

Proposition 1.1.8: The Cartan-Brelot topology is Hausdorff and $S^+(\Omega)$ is closed.

Proof: To show the topology is Hausdorff it is enough to show that for (u,v) in $S^+(\Omega) \times S^+(\Omega)$, $\Pi_{\omega,x}[(u,v)] = 0$ for all ω in $\mathcal{B}$ and x in $X \cap \omega$ implies u and v are identical. Well fix ω in $\mathcal{B}$. Since the maps $\int u d\rho^{\omega}_{\cdot}$ and $\int v d\rho^{\omega}_{\cdot}$ are continuous on ω and equal on the dense subset $\omega \cap X$, they are equal on ω. This being true for all ω in $\mathcal{B}$ it follows from Proposition 1.1.6 that u and v are identical.

Suppose now $\{u_n\}_{n\geq 1}$ is a sequence in $S^+(\Omega)$ converging in the Cartan-Brelot topology to $[(v_1,v_2)]$. For all ω in $\mathcal{B}$ and x in $\omega \cap X$ the non-negative sequence $(\int u_n d\rho^{\omega}_x)_{n\geq 1}$ converges to $\int v_1 d\rho^{\omega}_x - \int v_2 d\rho^{\omega}_x$. Thus, since $\int v_1 d\rho^{\omega}_{\cdot}$ and $\int v_2 d\rho^{\omega}_{\cdot}$ are continuous on ω, we have $\int v_1 d\rho^{\omega}_y \geq \int v_2 d\rho^{\omega}_y$ for all y in ω. This being true for all ω we deduce from Proposition 1.1.6 that $v_1(y) \geq v_2(y)$ for all y in Ω.

Put $E = \{y \in \Omega : v_2(y) = \infty\}$. Define w on Ω by

$$w(y) = \begin{cases} v_1(y) - v_2(y) & y \text{ in } \Omega - E \\ \infty & y \text{ in } E \end{cases}.$$

As in Remark 1.1.14 E is Borel. Thus w is a Borel function. Now let x_o be in $\Omega - E$ and ω a neighbourhood of x_o in $\mathcal{B}$. There exists $(\omega_k)_{k\geq 1}$ in $\mathcal{B}$ such that for all k, $\overline{\omega}_{k+1} \subset \omega_k \subset \omega$ and $\bigcap_{k\geq 1} \omega_n = \{x_o\}$. Suppose x_o is in X. Since for all n and k, $\int u_n d\rho^{\omega_k}_{x_o} \geq \int u_n d\rho^{\omega}_{x_o}$ (Proposition 1.1.6), taking the limit as $n \to \infty$ gives

$$(1) \quad \int v_1 d\rho^{\omega_k}_{x_o} - \int v_2 d\rho^{\omega_k}_{x_o} \geq \int v_1 d\rho^{\omega}_{x_o} - \int v_2 d\rho^{\omega}_{x_o} .$$

Since the maps $y \to \int v_1 d\rho_y^{\omega_k} - \int v_2 d\rho_y^{\omega_k}$ and $y \to \int v_1 d\rho_y^{\omega} - \int v_2 d\rho_y^{\omega}$ are both continuous on ω_k and (1) holds for all x_o in the dense set $X \cap \omega_k$, we see (1) must hold even if x_o is not in X. Letting $k \to \infty$ in (1) and using Proposition 1.1.6 we get

$$v_1(x_o) - v_2(x_o) \geq \int v_1 d\rho_{x_o}^{\omega} - \int v_2 d\rho_{x_o}^{\omega} .$$

Since polar sets have 0 harmonic measure this says

(2) $\quad w(x_o) \geq \int w d\rho_{x_o}^{\omega}$.

Clearly (2) holds if x_o is in E. Thus w is an $S_{\mathcal{B}}$ functions. Since for all x in Ω

(3) $\quad v_1(x) = v_2(x) + w(x)$

it follows from Proposition 1.1.6 that (3) holds with w replaced by $\hat{w}$. Thus $\hat{w}$ is in $S^+(\Omega)$ and $[(\hat{w},0)] = [(v_1,v_2)]$. The proof is complete.

<u>Proposition 1.1.19:</u> Let $(v_n)_{n\geq 1}$, v be in $S^+(\Omega)$ and let v_n converge to v in the Cartan-Brelot topology. If δ is in $\mathcal{B}$ then for all x in δ the sequence $(\int v_n d\rho_x^{\delta})_{n\geq 1}$ converges to $\int v d\rho_x^{\delta}$.

<u>Proof:</u> Suppose there is a y in δ such that $\lim_{n\to\infty} \int v_n d\rho_y^{\delta}$ either does not exist or is not $\int v d\rho_y^{\delta}$. This implies there exists a positive number ε, a subsequence $(v_{n_j})_{j\geq 1}$ and an integer N such that

(1) $\quad \left| \int v_{n_j} d\rho_y^{\delta} - \int v d\rho_y^{\delta} \right| > \varepsilon \quad$ for $j > N$.

Now $(\int v_{n_j} d\rho_{\cdot}^{\delta})_{j\geq 1}$ is a sequence of positive harmonic functions on δ which converge pointwise on $X \cap \delta$ to a finite limit. Thus there is a subsequence $(v_{n_{j_k}} d\rho_{\cdot}^{\delta})_{k\geq 1}$ which converges locally uniformly to a function w harmonic on δ.

The function w agrees with $\int v d\rho^{\delta}_{\cdot}$ on a dense set $X \cap \delta$. By continuity they agree on all of δ, in particular at y. This contradicts (1). The proof is complete.

<u>Proposition 1.1.20</u>: Let $(v_n)_{n\geq 1}$ and v be as in the previous proposition. Then $v = \widehat{\liminf_{n\to\infty} v_n}$.

<u>Proof:</u> Let x be any point in Ω. Choose $(\delta_\ell)_{\ell\geq 1}$ contained in $\mathcal{B}$ such that $\overline{\delta}_{\ell+1} \subset \delta_\ell$ for all ℓ and $\bigcap_{i\geq 1} \delta_\ell = \{x\}$. Then from the previous proposition, for each ℓ the sequence $(\int v_n d\rho_x^{\delta_\ell})_{n\geq 1}$ converges to $\int v d\rho_x^{\delta_\ell}$. Thus

$$\int v d\rho_x^{\delta_\ell} = \lim_{n\to\infty} \int v_n \, d\rho_x^{\delta_\ell}$$

$$\geq \int \liminf_{n\to\infty} v_n \, d\rho_x^{\delta_\ell} \qquad \text{(Fatou lemma).}$$

We have seen $\liminf_{n\to\infty} v_n$ is nearly superharmonic. Letting $\ell \to \infty$ and applying Proposition 1.1.6 gives

$$\text{(1)} \qquad v(x) \geq \widehat{\liminf_{n\to\infty} v_n}(x).$$

Conversely, since for all n, ℓ and z in δ_ℓ

$$\int v_n d\rho_z^{\delta_\ell} \leq v_n(z),$$

taking the lower limit as $n \to \infty$ and applying Proposition 1.1.19 gives

$$\int v d\rho_z^{\delta_\ell} \leq \liminf_{n\to\infty} v_n(z).$$

But the left hand side is continuous in z. Therefore

$$\int v d\rho_x^{\delta_\ell} \leq \widehat{\liminf_{n\to\infty} v_n}(x).$$

Letting $\ell \to \infty$ gives

$$(2) \qquad v(x) \leq \widehat{\liminf_{n \to \infty} v_n(x)}$$

From (1) and (2) we see we are done.

<u>Corollary 1.1.21:</u> The mapping $f: S^+(\Omega) \times \Omega \to \mathbb{R} \cup \{\infty\}$, $(v,x) \to v(x)$ is lower semi-continous.

<u>Proof</u>: Let $(v_n)_{n\geq 1}$ and v_o be in $S^+(\Omega)$ with $(v_n)_{n\geq 1}$ converging in the Cartan-Brelot topology to v_o and let $(x_n)_{n\geq 1}$ and x_o be in Ω with $(x_n)_{n\geq 1}$ converging to x_o. Let α and ε be positive real numbers such that

$$\alpha < v_o(x_o) \quad \text{and} \quad \varepsilon < \min\left(\frac{\alpha}{4}, \frac{v_o(x_o) - \alpha}{4}\right) .$$

Let ω_1 be a relatively compact neighbourhood of x_o and h a positive harmonic function defined on a neighbourhood of ω_1. Put $U = \{x \in \Omega: \frac{v_o(x)}{h(x)} > \frac{\alpha}{h(x_o)}\}$. Then U is an open set containing x_o. Let δ, δ^1 be in $\mathcal{B}$ such that $x_o \in \delta \subset \overline{\delta} \subset \delta^1 \subset \overline{\delta^1} \subset U$. We have seen $(\int v_n d\rho_x^{\delta^1})_{n\geq 1}$ converges to $\int v_o d\rho_x^{\delta^1}$ for all x in δ^1. Since this is a sequence of positive harmonic functions on δ^1, there is a subsequence $(\int v_{n_j} d\rho_x^{\delta^1})_{j\geq 1}$ converging uniformly on $\overline{\delta}$ to $\int v_o d\rho_x^{\delta^1}$. Thus there is an integer N such that for $j > N$ and all x in $\overline{\delta}$

$$\begin{aligned} v_{n_j}(x) &\geq \int v_{n_j} d\rho_x^{\delta^1} \\ &\geq \int v_o d\rho_x^{\delta^1} - \frac{\varepsilon}{2} \\ &\geq \alpha \int \frac{h}{h(x_o)} d\rho_x^{\delta^1} - \frac{\varepsilon}{2} \\ &= \alpha \frac{h(x)}{h(x_o)} - \frac{\varepsilon}{2} . \end{aligned}$$

Put $U_1 = \{x \in \Omega : \alpha \frac{h(x)}{h(x_o)} > \alpha - \frac{\varepsilon}{2}\}$. Then U_1 is a neighbourhood of x_o. If x is in $U_1 \cap \bar{\delta}$ and $j > N$, $v_{n_j}(x) \geq \alpha - \varepsilon$. From this we easily deduce f is lower semicontinuous and we are done.

We proceed now with the assumption of Axiom D on Ω.

Lemma 1.1.22: Let $(v_n)_{n\geq 1}$ be a sequence in $S^+(\Omega)$ which is uniformly locally bounded on Ω and let δ be a regular open subset of Ω. Then there is a subsequence $(v_{n_j})_{j\geq 1}$ such that $(\int v_{n_j} d\rho_x^\delta)_{j\geq 1}$ converges for all x in δ and for any subsequence $(v_{n_{j_k}})_{k>1}$ and x in δ

$$\lim_{j\to\infty} \int v_{n_j} d\rho_x^\delta = \int \widehat{\liminf_{k\to\infty} v_{n_{j_k}}} d\rho_x^\delta .$$

Proof: The sequence $(\int v_n d\rho_x^\delta)_{n\geq 1}$ is contained in $H^+(\delta)$ and is pointwise bounded. Therefore there is a subsequence $(\int v_{n_j} d\rho_x^\delta)_{j\geq 1}$ converging locally uniformly on δ to a function in $H^+(\delta)$. Take any subsequence $(v_{n_{j_k}})_{k\geq 1}$. Put

$$v'_{n_{j_k}} = E^\delta_{v_{n_{j_k}}} .$$

From Proposition 1.1.3 (d) we see this is in $S^+(\Omega)$, it minorizes $v_{n_{j_k}}$, and it is harmonic on δ. Now define

$$v = \widehat{\liminf_{k\to\infty} v'_{n_{j_k}}} \text{ on } \Omega .$$

v is in $S^+(\Omega)$. Since a countable union of polar sets is polar (see Proposition 2.14) it follows from Proposition 1.1.3(c) and Theorem 1.1.15 (a) that for all x in $\Omega - \delta$ except a polar set

$$v(x) = \liminf_{k\to\infty} v_{n_{j_k}}(x).$$

(Note $v_{n_{j_k}} = v'_{n_{j_k}}$ on $\Omega - \delta$). Also v is locally bounded and harmonic on δ .

Define w on Ω by

$$w = \widehat{\liminf_{k\to\infty} v_{n_{j_k}}}$$

and put $w' = E_w^\delta$. Then w' is in $S^+(\Omega)$, it minorizes w hence is locally bounded on Ω, and is harmonic on δ. Since $w' = w$ on $\Omega - \delta$ we have w' and v equal everywhere on $\Omega-\omega$ except a polar set. It follows from Remark 1.1.14 that w' and v are identical. In particular for x in δ

$$\int \widehat{\liminf v_{n_{j_k}}}\, d\rho_x^\delta = \lim_{j\to\infty} \int v_{n_j}\, d\rho_x^\delta .$$

The proof is complete.

<u>Proposition 1.1.23:</u> If $(v_n)_{n\geq 1}$ is a uniformly locally bounded sequence in $S^+(\Omega)$ then there is a subsequence converging in the Cartan-Brelot topology.

<u>Proof:</u> Let $(\delta_\ell)_{\ell\geq 1} = \mathcal{B}$. Consider first δ_1 and $(v_n)_{n\geq 1}$. From the lemma there is a subsequence $(v_{n,1})_{n\geq 1}$ such that for any subsequence $(v_{n_{k,1}})_{k\geq 1}$ of this and all x in δ_1

$$\int \widehat{\liminf_{k\to\infty} v_{n_{k,1}}}\, d\rho_x^{\delta_1} = \lim_{n\to\infty} \int v_{n,1}\, d\rho_x^{\delta_1} .$$

Now we proceed inductively. Suppose the sequence $(v_{n,\ell})_{n\geq 1}$, $\ell \geq 1$, has been constructed. Consider this sequence and $\delta_{\ell+1}$. From the lemma there is a subsequence $(v_{n,\ell+1})_{n\geq 1}$ of $(v_{n,\ell})_{n\geq 1}$ such that for any subsequence $(v_{n_k,\ell+1})_{k\geq 1}$ of this and x in $\delta_{\ell+1}$

$$\int \widehat{\liminf_{k\to\infty} v_{n_{k,\ell+1}}}\, d\rho_x^{\delta_{\ell+1}} = \lim_{n\to\infty} \int v_{n,\ell+1}\, d\rho_x^{\delta_{\ell+1}} .$$

Choose the diagonal sequence $(w_n)_{n\geq 1} = (v_{n,n})_{n\geq 1}$. Put

$$w = \widehat{\liminf_{n\to\infty} v_n} .$$

Then w is in $S^+(\Omega)$. We claim $(w_n)_{n\geq 1}$ (which is a subsequence of $(v_n)_{n\geq 1}$) converges in the Cartan-Brelot topology to w. Indeed consider δ_ℓ in $\mathcal{B}$ and x in δ_ℓ . Since $(w_n)_{n\geq \ell}$ is a subsequence of $(v_{n,\ell})$ we have

$$\begin{aligned}\int w\, d\rho_x^{\delta_\ell} &= \int \widehat{\liminf_{n\to\infty} w_n}\, d\rho_x^{\delta_\ell} \\ &= \lim_{n\to\infty} \int v_{n,\ell}\, d\rho_x^{\delta_\ell} \\ &= \lim_{n\to\infty} \int w_n\, d\rho_x^{\delta_\ell}\end{aligned}$$

This completes the proof.

Section 2 Finite Products of Brelot Spaces

Let n be any positive integer and $\Omega_1, \ldots, \Omega_n$ Brelot spaces each having a positive potential and each having at every point a countable base of open sets. We mentioned earlier that this implies each one of these spaces has a countable base of all open sets. If U is an open subset of Ω_i then the harmonic, superharmonic, non-negative harmonic, and non-negative superharmonic functions on U are denoted respectively by $H_i(U)$, $S_i(U)$, $H_i^+(U)$ and $S_i^+(U)$.

Definition 1.2.1: Let U be an open subset of $\Omega_1 \times \ldots \times \Omega_n$. A real valued continuous function on U is said to be n-harmonic on U if for any n-1 fixed variables it is a harmonic function of the remaining variable. The set of all (respectively non-negative) n-harmonic functions on U is denoted by n-H(U) (respectively by n-H^+(U)).

It can be shown that continuity in Definition 1.2.1 can be omitted if u is non-negative. That is continuity is a consequence of the rest of the definition. See [7].

It is clear from the corresponding property for each $H_i(U)$ that n-H(U) is a real vector space. It is also clear that the n-harmonic functions satisfy the sheaf property. In other words if u is in n-H(U) then it is n-H(V) for all open sets V contained in U. Conversely if for every x in U there is a neighbourhood U_x of x such that u is in n-H(U_x) then u is in n-H(U). We summarize other properties of n-harmonic functions in the next proposition.

Proposition 1.2.2: (a) If $\omega_1, \ldots, \omega_n$ are regular domains in $\Omega_1, \ldots, \Omega_n$ respectively and f is a real valued continuous function defined on the distinguished boundary $\partial\omega_1 \times \ldots \times \partial\omega_n$ of $\omega_1 \times \ldots \times \omega_n$ then there exists a function Γ_f on $\bar{\omega}_1 \times \ldots \times \bar{\omega}_n$ such that

i) Γ_f is real valued continuous,

ii) $\Gamma_f = f$ on $\partial\omega_1 \times \ldots \times \partial\omega_n$

iii) $\Gamma_f(x) \geq 0$ for all x in $\bar{\omega}_1 \times \ldots \times \bar{\omega}_n$ if $f(x) \geq 0$ for all x in $\partial\omega_1 \times \ldots \times \partial\omega_n$,

iv) for each integer j from 1 to n and fixed point in $\prod_{\substack{i=1 \\ i\neq j}}^{n} \omega_{ij}$, Γ_f is a harmonic function of the remaining variable on ω_j. In particular Γ_f is in n-$H(\omega_1 \times \ldots \times \omega_n)$.

Furthermore Γ_f is uniquely determined by properties i,ii,iii,iv.

(b) If $(u_i)_{i\in I}$ is a pointwise increasing directed family in n-$H(U)$ where U is a domain then the upper envelope is either in n-$H(U)$ or identically ∞.

(c) Let $\omega_1, \ldots, \omega_n$ be as in (a) and g any extended real valued function on $\partial\omega_1 \times \ldots \times \partial\omega_n$. Then the $\rho_{x_1}^{\omega_1} \times \ldots \times \rho_{x_n}^{\omega_n}$ integrabity of g is independent of $(x_1, \ldots, x_n)$ in $\omega_1 \times \ldots \times \omega_n$ and if integrable with respect to one such measure the mapping

$$(x_1, \ldots x_n) \to \int g(z_1, \ldots, z_n)\ (\rho_{x_1}^{\delta_1} \times \ldots \times \rho_{x_n}^{\delta_n})(dz_1, \ldots, dz_n)$$

is in n-$H(\omega_1 \times \ldots \times \omega_n)$.

Remark 1.2.3: (a) We generalize parts (a) and (c) above in the next chapter by considering instead products of relatively compact domains.

(b) Proposition 1.2.2 (b) implies a non-negative n-harmonic function u defined on a domain is either strictly positive or identically 0. (Just consider the sequence $(\ell.u)_{\ell\geq 1}$).

Definition 1.2.4: Let U be an open subset of $\Omega_1 \times \ldots \times \Omega_n$. An extended real valued function v on U is said to be n-hyperharmonic on U if

i) $v(x) > -\infty$ for all x,

ii) v is lower semi-continuous,

iii) for any n-1 fixed variables it is hyperharmonic in the remaining variable.

If in addition v is finite at one point of each connected component of U then v is said to be n-superharmonic on U. The set of all n-superharmonic (respectively non-negative n-superharmonic) functions on U is denoted by n-S(U) (respectively $n\text{-}S^+(U)$).

It is easy to see that if v_1, v_2 are in n-S(U) and α_1, α_2 positive real numbers then $\alpha_1 v_1 + \alpha_2 v_2$ and $\min(v_1, v_2)$ are in n-S(U). It is also clear v is in n-H(U) if and only if v and (-v) are both in n-S(U). In addition the n-superharmonic functions satisfy the following.

Proposition 1.2.5: Let U be a domain in $\Omega_1 \times \ldots \times \Omega_n$.

(a) If v is in n-S(U) then it is finite on a dense subset of U.

(b) If $(v_i)_{i \in I}$ is a pointwise increasing directed subset of n-S(U) then the upper envelope is either in n-S(U) or identically ∞.

(c) If w is a locally lower bounded extended real valued function on U satisfying (iii) of Definition 1.2.4 then w is lower semicontinuous. Thus it is n-hyperharmonic.

Remark 1.2.6: (a) Let U be a domain in $\Omega_1 \times \ldots \times \Omega_n$, v in n-S(U), ω_i a regular domain in Ω_i for each i from 1 to n such that $\bar{\omega}_1 \times \ldots \times \bar{\omega}_n \subset U$, and $(x_1, \ldots, x_n)$ a point in $\omega_1 \times \ldots \times \omega_n$ with $v(x_1, \ldots, x_n) < \infty$. (From

Proposition 1.2.5 (a) we see this point exists). Since v is lower semi-continuous (hence locally lower bounded and Borel measurable) we can apply Fubini's theorem and deduce

$$\int v(z_1,\ldots,z_n)(\rho_{x_1}^{\omega_1} \times \ldots \times \rho_{x_n}^{\omega_n})(dz_1 \ldots dz_n) = \int \ldots \int v \, d\rho_{x_1}^{\omega_1} \ldots d\rho_{x_n}^{\omega_n}$$

$$\leq v(x_1,\ldots,x_n)$$

$$< \infty .$$

Thus Proposition 1.2.2 (c) implies v integrable with respect to all measures $\rho_{y_1}^{\omega_1} \times \ldots \times \rho_{y_n}^{\omega_n}$ for $(y_1,\ldots,y_n)$ in $\omega_1 \times \ldots \times \omega_n$ and

$$(y_1,\ldots,y_n) \to \int v(z_1,\ldots,z_n)(\rho_{y_1}^{\omega} \times \ldots \times \rho_{y_n}^{\omega})(dz_1,\ldots,dz_n)$$

is in n-$H(\omega_1 \times \ldots \times \omega_n)$.

(b) From Proposition 1.2.5 (a) and (b) we see v in n-$S^+(U)$ where U is a domain is everywhere on U positive or else identically 0. This follows since if for some x_o in U $v(x_o) > 0$ then $v(x) > 0$ on a neighbourhood of x_o and the increasing sequence $(\ell . v)_{\ell \geq 1}$ converges to ∞ on an open set hence everywhere on U. Thus v must be positive everywhere on U.

From this last remark we can prove a general minimum principle for n-superharmonic functions.

<u>Proposition 1.2.7</u>: <u>(Minimum Principle)</u> Let U be a relatively compact open subset of $\Omega_1 \times \ldots \times \Omega_n$ and let v be in n-S(U). Suppose for all x in ∂U

(1) $\liminf_{\substack{z \to x \\ z \in U}} v(z) \geq 0$.

Then $v(z) \geq 0$ on U.

Proof: Without loss of generality we may assume U is connected since (1) holds on each connected component of U.

Define w on $\overline{U}$ by

$$w(x) = \begin{cases} v(x) & x \text{ in } U \\ 0 & x \text{ in } \partial U . \end{cases}$$

Inequality (1) implies w is lower semicontinuous on $\overline{U}$. Let h be a positive n-harmonic function defined on a neighbourhood of U. Certainly this exists since $\overline{U}$ can be contained in a product $\omega_1 \times \ldots \times \omega_n$ of relatively compact open sets and if h_i is positive harmonic on ω_i then $h:(x_1,\ldots,x_n) \to h_1(x_1)\cdot h_2(x_2)\cdot \ldots \cdot h_n(x_n)$ is positive n-harmonic on $\omega_1 \times \ldots \times \omega_n$. Now w/h is lower semicontinuous on $\overline{U}$. Thus it attains a minimum value at a point x_o in $\overline{U}$. If v were negative somewhere on U we would have x_o in U. Now the function $v - (v(x_o)/h(x_o))h$ is in n-S(U), it is non-negative, and it equals 0 at x_o. Remark 1.2.6 (b) thus implies $v(x) = (v(x_o)/h(x_o))h(x)$ for all x in U. The continuity of h gives w negative at each boundary point of U. This is a contradiction, hence v is non-negative on U. The proof is complete.

Let k be an integer strictly between 1 and n, let v be in k-S($\Omega_1 \times \ldots \times \Omega_k$), and w in (n-k)-S($\Omega_{k+1} \times \ldots \times \Omega_n$). Put $u = v\cdot w$. That is for $(x_1,\ldots,x_n)$ in $\Omega_1 \times \ldots \times \Omega_n$ we put $u(x_1,\ldots,x_n) = v(x_1,\ldots,x_k)\cdot w(x_{k+1},\ldots,x_n)$. Then u is in n-S($\Omega_1 \times \ldots \times \Omega_n$). Indeed since v and w are locally lower bounded so is u. Clearly u is never $(-\infty)$ and u is finite at least at one point. It is also clear u is hyperharmonic in any variable if the other variables are fixed. Thus Propsotion 1.2.5(c) implies u is in n-S($\omega_1 \times \ldots \times \omega_n$). In the next proposition we show how to generate other n-superharmonic functions.

Proposition 1.2.8: Let v be a locally lower bounded Borel measurable function on $\Omega_1 \times \ldots \times \Omega_n$ such that for every $(x_1,\ldots,x_k)$ in $\Omega_1 \times \ldots \times \Omega_k$ (k a fixed integer between 1 and n-1) the mapping $y \to v(x_1,\ldots,x_k,y)$ is (n-k)-hyperharmonic (respectively (n-k)-harmonic) on $\Omega_{k+1} \times \ldots \times \Omega_n$. Let $\delta_1,\ldots,\delta_k$ be regular domains in $\Omega_1,\ldots,\Omega_k$ respectively. Then for each $x = (x_1,\ldots,x_n)$ in $\delta_1 \times \ldots \times \delta_k$ the mapping

$$g:(x_{k+1},\ldots,x_n) \to \int v(z_1,\ldots,z_k,x_{k+1},\ldots,x_n)(\rho_{x_1}^{\delta_1} \ldots \rho_{x_k}^{\delta_k})(dz_1,\ldots,dz_k)$$

is (n-k)-hyperharmonic (respectively (n-k)-harmonic) on $\Omega_{k+1} \times \ldots \times \Omega_n$.

Proof: Since v is locally lower bounded and the product of harmonic measures is totally finite, g never takes the value $-\infty$. Furthermore we may apply Fubini's Theorem to g. Let $(x^\ell)_{\ell\geq 1} = ((x_{k+1}^\ell,\ldots,x_n^\ell))_{\ell\geq 1}$ be a sequence converging to $(x_{k+1},\ldots,x_n)$ in $\Omega_{k+1} \times \ldots \times \Omega_n$.

$$\liminf_{\ell \to \infty} g(x^\ell) = \liminf_{\ell \to \infty} \int v(z_1,\ldots,z_k,x^\ell)(\rho_{x_1}^{\delta_1} \times \ldots \times \rho_{x_k}^{\delta_k})(dz_1,\ldots,dz_k)$$

$$\geq \int \liminf_{\ell \to \infty} v(z_1,\ldots,z_k,x^\ell)(\rho_{x_1}^{\delta_1} \times \ldots \times \rho_{x_k}^{\delta_k})(dz_1,\ldots,dz_k)$$

(Fatou lemma)

$$\geq \int v(z_1,\ldots z_k,x_{k+1},\ldots,x_n)(\rho_{x_1}^{\delta_1} \times \ldots \times \rho_{x_k}^{\delta_k})(dz_1,\ldots,dz_k)$$

(since v is lower semicontinuous in its last n-k variables)

$$= g(x_{k+1},\ldots,x_n) .$$

Thus g is lower semicontinuous. Finally let δ be a regular domain in Ω_{k+1}. For any $(x_{k+1},\ldots,x_n)$ in $\Omega_{k+1} \times \ldots \times \Omega_n$ with x_{k+1} in δ we have

$$\int g(z_{k+1},x_{k+2},\ldots,x_n)d\rho^{\delta}_{x_{k+1}}(z_{k+1})$$

$$= \int \rho^{\delta}_{x_{k+1}}(dz_{k+1}) \int v(z_1,\ldots,z_{k+1},x_{k+2},\ldots,x_n)(\rho^{\delta_1}_{x_1}\times\ldots\times \rho^{\delta_k}_{x_k})(dz_1,\ldots,dz_k)$$

$$= \int(\rho^{\delta_1}_{x_1}\times\ldots\times \rho^{\delta_k}_{x_k})(dz_1,\ldots,dz_k) \int v(z_1,\ldots,z_{k+1},x_{k+2},\ldots,x_n)\rho^{\delta}_{x_{k+1}}(dz_{k+1})$$

$$\leq \int v(z_1,\ldots,z_k,x_{k+1},\ldots,x_n)(\rho^{\delta_1}_{x_1}\times\ldots\times\rho^{\delta_k}_{x_k})(dz_1,\ldots,dz_k)$$

$$= g(x_{k+1},\ldots,x_n).$$

By symmetry we see g is separately (n-k)-hyperharmonic. Thus g is (n-k)-hyperharmonic on $\Omega_{k+1}\times\ldots\times\Omega_n$. By applying the same result to -v we see that if $y \to v(x_1,\ldots,x_k,y)$ is (n-k)-harmonic on $\Omega_{k+1}\times\ldots\times\Omega_n$ then g is (n-k)-harmonic.

We now consider the analogue of the $S_{\mathcal{B}}$ functions.

<u>Definition 1.2.9:</u> Let for each integer i from 1 to n $\mathcal{B}_i$ be a base of open sets of Ω_i consisting of regular domains. An extended real valued function f on $\Omega_1\times\ldots\times\Omega_n$ is said to be an n-S($\mathcal{B}_1,\ldots,\mathcal{B}_n$) function (and nearly n-superharmonic if $\mathcal{B}_i$ is the set of all regular domains for each i) if

i) f is locally lower bounded and

ii) for all δ_i in $\mathcal{B}_i$, i = 1,...,n, and for all $(x_1,\ldots,x_n)$ in $\delta_1\times\ldots\times\delta_n$

$$\overline{\int} f(z_1,\ldots,z_n)(\rho^{\delta_1}_{x_1}\times\ldots\times\rho^{\delta_n}_{x_n})(dz_1,\ldots,dz_n) \leq f(x_1,\ldots,x_n) .$$

Clearly any n-hyperharmonic function is nearly n-superharmonic. It is also simple to show the upper envelope of an increasing directed family of n-S($\mathcal{B}_1,\ldots,\mathcal{B}_n$) functions and the lower enevelope of any uniformly locally lower bounded family of n-S($\mathcal{B}_1,\ldots,\mathcal{B}_n$) functions is in n-S($\mathcal{B}_1,\ldots,\mathcal{B}_n$).

Thus in particular if $(v_\ell)_{\ell \geq 1}$ is a uniformly locally lower bounded sequence in $n\text{-}S(\Omega_1 \times \ldots \times \Omega_n)$, $\liminf_{\ell \to \infty} v_\ell$ is nearly n-superharmonic.

The fundamental result concerning these functions and one we use repeatedly is the following.

Proposition 1.2.10: Let v be an $n\text{-}S(\mathcal{B}_1, \ldots, \mathcal{B}_n)$ function which is not identically ∞. Then the lower semicontinuous regularization $\hat{v}$ of v is in $n\text{-}S(\Omega_1 \times \ldots \times \Omega_n)$. Furthermore for every $(x_1, \ldots, x_n)$ in $\Omega_1 \times \ldots \times \Omega_n$

$$\hat{v}(x_1, \ldots, x_n) = \sup \{\int v(z_1, \ldots, z_n)(\rho_{x_1}^{\delta_1} \times \ldots \times \rho_{x_n}^{\delta_n})(dz_1, \ldots, dz_n) :$$

$$(x_1, \ldots, x_n) \in \delta_1 \times \ldots \times \delta_n,\ \delta_i \in \mathcal{B}_i,\ i = 1, \ldots, n\}.$$

As a consequence a lower semicontinuous $n\text{-}S(\mathcal{B}_1, \ldots, \mathcal{B}_n)$ function is n-superharmonic.

In the course of the proof of Proposition 1.2.10 (see [8] page 81) Gowrisankaran actually deduces more if v is lower semicontinuous. Indeed he shows the following.

Proposition 1.2.11: (local property) Let v be an extended real valued lower semicontinuous function on a domain U in $\Omega_1 \times \ldots \times \Omega_n$ such that v is not identically ∞ and never $(-\infty)$. If for every $x = (x_1, \ldots, x_n)$ in U and neighbourhood V of x there exists $\delta_1, \ldots, \delta_n$ in $\mathcal{B}_1, \ldots, \mathcal{B}_n$ respectively such that $x \in \delta_1 \times \ldots \times \delta_n \subset \overline{\delta}_1 \times \ldots \times \overline{\delta}_n \subset V$ and

$$\int \ldots \int v \, d\rho_{x_1}^{\delta_1} \times \ldots \, d\rho_{x_n}^{\delta_n} \leq v(x),$$

then v is in $n\text{-}S(U)$.

Finally we consider the analogue of the reduced function.

Definition 1.2.12: Let v be in $n\text{-}S^+(\Omega_1 \times \ldots \times \Omega_n)$ and E any subset of $\Omega_1 \times \ldots \times \Omega_n$. The reduced function R_v^E is defined on Ω as follows:

$$R_v^E(x) = \inf \{w(x) : w \in n\text{-}S^+(\Omega_1 \times \ldots \times \Omega_n), w(y) \geq v(y) \text{ for all } y \text{ in } E\}.$$

As in the case n = 1 it is easy to show this is a monotone non-decreasing function of v and of E (if subsets of $\Omega_1 \times \ldots \times \Omega_n$ are ordered by inclusion). For E and F two subsets of $\Omega_1 \times \ldots \times \Omega_n$

$$R_v^{E \cup F}(x) \leq R_v^E(x) + R_v^F(x) \quad \text{on } \Omega_1 \times \ldots \times \Omega_n ,$$

and for λ a non-negative real number

$$R_{\lambda v}^E(x) = \lambda R_v^E(x) \quad \text{on } \Omega_1 \times \ldots \times \Omega_n .$$

Since the finite infimum of n-superharmonic functions is again n-superharmonic we see $R_v^E = v$ on E. Moreover R_v^E is the lower envelope of a family of functions in $n\text{-}S^+(\Omega_1 \times \ldots \times \Omega_n)$. Thus R_v^E is nearly n-superharmonic and hence $\hat{R}_v^E$ is in $n\text{-}S^+(\Omega_1 \times \ldots \times \Omega_n)$. Since v is lower semicontinuous, $\hat{R}_v^E = R_v^E$ on the interior of E. Thus in case E is open $\hat{R}_v^E$ equals v on all of E hence it majorizes R_v^E on $\Omega_1 \times \ldots \times \Omega_n$, and therefore in this case $\hat{R}_v^E$ and R_v^E are identical.

We remark that unlike in the case of n = 1 the reduced function is not necessarily n-harmonic on $\Omega_1 \times \ldots \times \Omega_n - \overline{E}$. It is this fact that produces the greatest difficulties in the theory.

CHAPTER 2

n-Polar and n-Negligible Sets

In this chapter we introduce and begin a preliminary study of the two principle types of exceptional sets which we shall be considering in this thesis; the n-polar and n-negligible sets. In Section 3 we solve the Dirichlet problem on a product of relatively compact domains and show the "irregular boundary points" form an n-polar set. In Section 4 we generalize the Continuation Theorem to functions n-superharmonic on an open set and bounded on the boundary of an open subset.

Before beginning we state once and for all that $\Omega_1, \ldots, \Omega_n, n \geq 1$, are Brelot spaces having a positive potential, a countable base of open sets for each point, and each satisfying Axiom D. We remark, however, Axiom D is actually used only in consideration of the Dirichlet problem, the Convergence Theorem, and thin sets.

Section 1 n-Polar Sets

Definition 2.1.1: Let U be an open subset of $\Omega_1 \times \ldots \times \Omega_n$. A subset E of U is said to be n-polar in U if there exists a v in n-$S^+(U)$ such that E is contained in $\{x \in U: v(x) = \infty\}$. If $U = \Omega_1 \times \ldots \times \Omega_n$ we simply call E an n-polar set. We shall say that any such v is associated to E in U.

It is obvious a subset of a set n-polar in U is n-polar in U. We also notice that a set E n-polar in U is contained in a G_δ (hence a Borel measurable) set n-polar in U. Indeed if v is associated to E in U,

$$E \subset \{x \in U: v(x) = \infty\}$$

$$= \bigcap_{\ell=1}^{\infty} \{x \in \Omega_1 \times \ldots \times \Omega_n : v(x) > \ell\} \cap U$$

which is G_δ since v is lower semicontinuous on U.

One way to generate n-polar sets is given in the following.

Proposition 2.1.2: Let P be k-polar in $\Omega_1 \times \ldots \times \Omega_k$, $1 \leq k < n$. Then $P \times \Omega_{k+1} \times \ldots \times \Omega_n$ is n-polar in $\Omega_1 \times \ldots \times \Omega_n$.

Proof: Let v be associated to P in $\Omega_1 \times \ldots \times \Omega_k$ and let w be any member of $(n-k)-S^+(\Omega_{k+1} \times \ldots \times \Omega_n)$. Then the mapping $(x_1,\ldots,x_n) \to v(x_1,\ldots,x_k) \cdot w(x_{k+1},\ldots,x_n)$ is in $n-S^+(\Omega_1 \times \ldots \times \Omega_n)$ and is associated to $P \times \Omega_{k+1} \times \ldots \times \Omega_n$.

We show now the extremely useful fact that a countable union of n-polar sets is n-polar.

Lemma 2.1.3: Let $\{U_k\}_{k \geq 1}$ be the connected components of the open set U in $\Omega_1 \times \ldots \times \Omega_n$. Then E is n-polar in U if and only if $E \cap U_k$ is polar in U_k for each k.

Proof: Suppose $E \cap U_k$ is polar in U_k for each k. Let v_k be associated to $E \cap U_k$ in U_k. We extend v_k to U by making it 0 on the other connected components. Then clearly $\sum_{k=1}^{\infty} v_k$ is in $n-S^+(U)$ and is associated to E in U.

Conversely if E is n-polar in U and v is associated to E in U then the restriction of v to any connected component U_k of U is in $n-S^+(U_k)$ and associated to $U_k \cap E$ in U_k. Thus $E \cap U_k$ is n-polar in U_k.

Proposition 2.1.4: A countable union of sets n-polar in U is n-polar in U.

Proof: By the lemma we may assume without loss of generality that U is connected. Let $(E_k)_{k \geq 1}$ be a sequence of sets n-polar in U and let v_k be associated to E_k. Choose $\delta_1,\ldots,\delta_n$ regular domains in $\Omega_1,\ldots,\Omega_n$ respectively with $\overline{\delta}_1 \times \ldots \times \overline{\delta}_n \subset U$. Let $(x_1,\ldots,x_n)$ be any point in $\delta_1 \times \ldots \times \delta_n$. Put for each k

$$\lambda_k = \int v_k(z_1,\ldots,z_n)(\rho_{x_1}^{\delta_1} \times \ldots \times \rho_{x_n}^{\delta_n})(dz_1,\ldots,dz_n) .$$

These exist and are finite since each v_k is integrable with respect to any such product of harmonic measures. (Remark 1.2.6(a)). Put

$$v(z) = \sum_{k=1}^{\infty} (v_k(z)/\lambda_k 2^k) \quad \text{for each } z \text{ in } U.$$

Then v is the limit of an increasing sequence of functions in $n\text{-}S^+(U)$ hence is in $n\text{-}S^+(U)$ if it is finite at least at one point of U (Proposition 1.2.5(b)). But this follows since

$$0 \leq \int v(z_1,\ldots,z_n)(\rho_{x_1}^{\delta_1} \times \ldots \times \rho_{x_n}^{\delta_n})(dz_1,\ldots,dz_n)$$

$$= \sum_{k=1}^{\infty} \int (v_k(z_1,\ldots,z_n)/2^k\lambda_k)(\rho_{x_1}^{\delta_1} \times \ldots \times \rho_{x_n}^{\delta_n}) \; (dz_1,\ldots,dz_n)$$

(Monotone Convergence Theorem)

$$= \sum_{k=1}^{\infty} 2^{-k}$$

$$< \infty$$

Since clearly $E = \bigcup_{k=1}^{\infty} E_k$ is contained in $\{z \in U : v(z) = \infty\}$, v is associated to E and we are done.

Remark 2.1.5: For n = 1 of course 1-polar means polar. In this case a set E which is polar in an open subset U of Ω_1 is polar in Ω_1. Indeed choose a sequence $(U_\ell)_{\ell \geq 1}$ of relatively compact open sets such that for each ℓ, $\overline{U}_\ell \subset U_{\ell+1}$ and $\bigcup_{\ell \geq 1} U_\ell = U$. Let v be associated to E in U. The Continuation Theorem implies for every ℓ there exist p_1, p_2 in $S_1^+(\Omega_1)$ such that $p_1 = p_2 + v$ on U_ℓ. Therefore

$$E \cap U_\ell \subset \{x \in \Omega_1 : P_1(x) = \infty\}$$

and $E \cap U_\ell$ is polar. The previous proposition now gives $E = \bigcup_{\ell \geq 1} (E \cap U_\ell)$ is polar thus proving the claim. We wish to note here that we do not know if this is true for n greater than 1. However for a special case of this see [13].

Definition 2.1.6: Let E be a subset of $\Omega_1 \times \ldots \times \Omega_n$, $n \geq 2$, i an integer from 1 to n-1, and $k_1,\ldots,k_i$ integers satisfying $1 \leq k_1 < k_2 < \ldots < k_i \leq n$. Let $x = (x_{\ell_1},\ldots,x_{\ell_j})$ be a point in $\Omega_{\ell_1} \times \ldots \times \Omega_{\ell_j}$ where $\{k_1,\ldots,k_i\} \subset \{\ell_1,\ldots,\ell_j\} \subset \{1,\ldots,n\}$. The $(k_1,\ldots,k_i)$-section of E through x is defined to be

$$\{z \in \prod_{\substack{r=1 \\ r \notin \{k_1,\ldots,k_i\}}}^{n} \Omega_r : (y_1,\ldots,y_n) \in E \text{ where } y_r = x_r \text{ for } r \in \{k_1,\ldots,k_i\} \text{ and } y_r = z_r \text{ for } r \notin \{k_1,\ldots,k_i\}\}.$$

It is denoted by $E_{k_1,\ldots,k_i}(x)$. The set of all $(k_1,\ldots,k_i)$-sections of E as $k_1,\ldots,k_i$ vary over all possible values are called the i-sections of E.

The $(k_1,\ldots,k_i)$-projection of E, denoted by $\Pi_{k_1,\ldots,k_i}(E)$, is defined to be

$$\{z \in \Omega_{k_1} \times \ldots \times \Omega_{k_i} : E_{k_1,\ldots,k_i}(z) \neq \emptyset\}.$$

Notice if E is open, closed, or compact any i-section or projection $\Pi_{k_1,\ldots,k_i}(E)$ has the corresponding property.

For a polar set P in Ω_1 it is true that for any point x of $\Omega_1 - P$ there is a function associated to P in Ω_1 that is finite at x. Indeed let $(\delta_\ell)_{\ell \geq 1}$ be a sequence of regular neighbourhoods of x such that for every ℓ, $\overline{\delta}_{\ell+1} \subset \delta_\ell$ and $\bigcap_{\ell \geq 1} \delta_\ell = \{x\}$. Then if v is associated to E in Ω_1,

$$w = \sum_{\ell \geq 1} (E_v^{\delta_\ell}/E_v^{\delta_\ell}(x)2^\ell)$$

is in $S_1^+(\Omega_1)$, $w(x) < \infty$, and $w(z) = \infty$ for z in P. In case $n > 1$ this is too much to hope for since (as we shall prove) it is necessary for all i-sections of P though x to be (n-i)-polar. We prove below in Theorem 2.1.9 that this condition is also sufficient.

We first prove two results both of which we return to in later sections.

Lemma 2.1.7: Let v be in $n\text{-}S(\Omega_1 \times \ldots \times \Omega_n)$ and δ_1 a regular domain in Ω_1. Define w on $\Omega_1 \times \ldots \times \Omega_n$ by

$$w(x_1,\ldots,x_n) = \begin{cases} v(x_1,\ldots,x_n) & x_1 \text{ in } \Omega_1 - \delta_1 , \\ \int v(z,x_2,\ldots,x_n)\, d\rho_{x_1}^{\delta_1}(z) , & x_1 \text{ in } \delta_1 . \end{cases}$$

Then w is in $n\text{-}S(\Omega_1 \times \ldots \times \Omega_n)$ and w minorizes v.

Proof: Clearly w minorizes v hence it is not identically ∞. Further since v is locally lower bounded and harmonic measure is uniformly totally bounded as x_1 varies over δ_1 we see w is also locally lower bounded. Thus to complete the proof it suffices to show w is separately hyperharmonic, the joint lower semicontinuity following from Proposition 1.2.5(c).

First fix $(x_2,\ldots,x_n)$ in $\Omega_2 \times \ldots \times \Omega_n$. Then the mapping $x_1 \to w(x_1,x_2,\ldots,x_n)$ is just

$$x_1 \to E^{\delta_1}_{v(\cdot,x_2,\ldots,x_n)}(x_1) ,$$

which is hyperharmonic by Proposition 1.1.3(d).

Next fix $(x_1,x_3,\ldots,x_n)$ in $\Omega_1 \times \Omega_3 \times \Omega_4 \times \ldots \times \Omega_n$. If x_1 is not in δ_1 there is nothing to prove. If x_1 is in δ_1, $x_2 \to w(x_1,\ldots,x_n)$ is

$$x_2 \to \int v(z_1,x_2,\ldots,x_n)d\rho_{x_1}^{\delta_1}(z_1)$$

which by Proposition 1.2.8 is hyperharmonic. We see by symmetry w is separately hyperharmonic and we are done.

Lemma 2.1.8: Let v be in $n\text{-}S(\Omega_1 \times \ldots \times \Omega_n)$ and $\delta_1,\ldots,\delta_n$ regular domains in $\Omega_1,\ldots,\Omega_n$ respectively. Then there exists a function in $n\text{-}S(\Omega_1 \times \ldots \times \Omega_n)$ which is positive if v is positive, minorizes v, equals v on $(\Omega_1-\delta_1)\times \ldots \times(\Omega_n-\delta_n)$, and is in $n\text{-}H(\delta_1 \times \ldots \times \delta_n)$.

Proof: Define $u_1,\ldots,u_n$ inductively as follows. Put $u_1 = w$, w as in Lemma 2.1.7. Assuming u_i is defined, $1 \le i \le n-1$, define u_{i+1} as

$$u_{i+1}(x_1,\ldots,x_n) = \begin{cases} u_i(x_1,\ldots,x_n) & x_i \text{ in } \Omega_i-\delta_i \\ \int u_i(x_1,\ldots,x_{i-1},z,x_{i+1},\ldots,x_n)d\rho_{x_i}^{\delta_i}(z) & x_i \text{ in } \delta_i \end{cases}$$

It is clear u_n satisfies all of the requirements but the last. There exists a point $(x_1,\ldots,x_n)$ such that $(x_1,\ldots,x_n)$ is in $\delta_1 \times \ldots \times \delta_n$ and $v(x_1,\ldots,x_n) < \infty$ (Proposition 1.2.5(a)). At this point

$$\begin{aligned} u_n(x_1,\ldots,x_n) &= \int \ldots \int v \, d\rho_{x_1}^{\delta_1} \ldots d\rho_{x_n}^{\delta_n} \\ &\le v(x_1,\ldots,x_n) \\ &< \infty . \end{aligned}$$

Thus Proposition 1.1.3(d) implies u_n is in n-H($\delta_1 \times \ldots \times \delta_n$). The proof is complete.

Theorem 2.1.9: Let E be an n-polar subset of $\Omega_1 \times \ldots \times \Omega_n$, $n \geq 1$, and $x = (x_1, \ldots, x_n)$ a point in $\Omega_1 \times \ldots \times \Omega_n - E$. Then there exists u in n-$S^+(\Omega_1 \times \ldots \times \Omega_n)$ with $u(x) < \infty$ and $u(z) = \infty$ for all z in E if and only if for each integer i from 1 to n-1 every i-section of E through x is (n-i)-polar. (Of course for n = 1 there are no i-sections hence the condition trvially holds.)

Proof: Suppose first such a u exists. Let $E_{k_1,\ldots,k_i}(x)$ be any i-section of E through x. Then the mapping

$$\prod_{\substack{j=1 \\ i \notin \{k_1,\ldots,k_i\}}}^{n} \Omega_j \to \overline{\mathbb{R}},\ z \to v(y_1,\ldots,y_n),$$

where $y_j = x_j$ for j in $\{k_1,\ldots,k_i\}$ and $y_j = z_j$ for j not in $\{k_1,\ldots,k_i\}$ is clearly (n-i)-hyperharmonic. It is thus (n-i)-superharmonic since it is finite at the point z where $z_j = x_j$ for j not in $\{k_1,\ldots,k_i\}$. It is also ∞ on $E_{k_1,\ldots,k_i}(x)$. This proves the proposition one way.

The converse is proved by induction on n. For n = 1 we have already remarked that the result holds. Assume then n > 1 and the implication holds for smaller integers. Choose for each integer j from 1 to n a sequence $(\delta_{j,k})_{k\geq 1}$ of regular domains in Ω_j each containing x_j such that for every k, $\overline{\delta}_{j,k+1} \subset \delta_{j,k}$ and $\bigcap_{k\geq 1} \delta_{j,k} = \{x_j\}$. Let v be in n-$S^+(\Omega_1 \times \ldots \times \Omega_n)$ with $v(z) = \infty$ for all z in E. By Lemma 2.1.8 there exists for each k a function v_k in n-$S^+(\Omega_1 \times \ldots \times \Omega_n)$ such that v_k minorizes v, v_k equals v on $(\Omega_1 - \delta_{1,k}) \times \ldots \times (\Omega_n - \delta_{n,k})$, and v_k is in n-$H^+(\delta_{1,k} \times \ldots \times \delta_{n,k})$ hence is finite at x. Put $\lambda_k = (2^k v_k(x))^{-1}$. Define w on $\Omega_1 \times \ldots \times \Omega_n$ by

$$w(z) = \sum_{k=1}^{\infty} \lambda_k v_k(z) .$$

Then w is in $n\text{-}S^+(\Omega_1 \times \ldots \times \Omega_n)$ since $w(x) < \infty$ and $w(z) = \infty$ for z in

$$F = E - \bigcup_{j=1}^{n} \{(z_1,\ldots,z_n) \in E : z_j = x_j\} .$$

For if $z = (z_1,\ldots,z_n)$ is in F then z is in E and $z_j \neq x_j$ for j from 1 to n hence there is a positive integer k such that z is in $(\Omega_1 - \delta_{1,k}) \times \ldots \times (\Omega_n - \delta_{n,k})$. This implies $w(z) \geq \lambda_k v_k(z) = \lambda_k v(z) = \infty$.

Consider now $E_1(x)$. It is a 1-section of E through x hence it is (n-1)-polar and every i-section of it through $(x_2,\ldots,x_n)$, $1 \leq i \leq n-2$, is an (i+1)-section of E through x. Furthermore $(x_2,\ldots,x_n)$ is not in $E_1(x)$. Hence by the inductive hypothesis there exists w_1 in $(n-1)\text{-}S^+(\Omega_2 \times \ldots \times \Omega_n)$ such that $w_1(x_2,\ldots,x_n) < \infty$ and $w_1(z) = \infty$ for z in $E_1(x)$. Let w_1^1 be any real valued member of $S_1^+(\Omega_1)$. Define u_1 on $\Omega_1 \times \ldots \times \Omega_n$ by

$$u_1(z_1,\ldots,z_n) = w_1'(z_1) \cdot w_1(z_2,\ldots,z_n) .$$

Then u_1 is in $n\text{-}S^+(\Omega_1 \times \ldots \times \Omega_n)$, $u_1(x) < \infty$, and $u_1(z) = \infty$ on $\{(z_1,\ldots,z_n) \in E : z_1 = x_1\}$. Similarly for $j = 2,\ldots,n$ there exists u_j in $n\text{-}S^+(\Omega_1 \times \ldots \times \Omega_n)$ such that $u_j(x) < \infty$ and $u_j(z) = \infty$ on $\{(z_1,\ldots,z_n) \in E : z_j = x_j\}$. Finally, define u on $\Omega_1 \times \ldots \times \Omega_n$ by

$$u(z) = w(z) + \sum_{j=1}^{n} u_j(z) .$$

Then u is in $n\text{-}S^+(\Omega_1 \times \ldots \times \Omega_n)$, $u(x) < \infty$, and $u(z) = \infty$ for z in E. The proof is complete.

We prove now some characteriztions of n-polar sets.

Proposition 2.1.10: Let E be a subset of $\Omega_1 \times \ldots \times \Omega_n$. Then the following are equivalent.

(a) E is n-polar

(b) For some v in $n\text{-}S^+(\Omega_1 \times \ldots \times \Omega_n)$ R_v^E is 0 at one point or $\hat{R}_v^E$ is identically 0.

(c) For every v in $n\text{-}S^+(\Omega_1 \times \ldots \times \Omega_n)$ R_v^E is 0 everywhere except on an n-polar set or $\hat{R}_v^E$ is identically 0.

Proof: (b) ⇒ (a): Suppose first for some v and point x we have $R_v^E(x) = 0$. For every positive integer k there exists w_k in $n\text{-}S^+(\Omega_1 \times \ldots \times \Omega_n)$ such that w_k majorizes v on E and $w_k(x) \leq 2^{-k}$. Then $w = \sum_{k\,1}^{\infty} w_k$ is in $n\text{-}S^+(\Omega_1 \times \ldots \times \Omega_n)$ (since the series converges at x) and $w(z) = \infty$ for z in E. Hence E is n-polar.

Next suppose only $\hat{R}_v^E$ is identically 0 for some v in $n\text{-}S^+(\Omega_1 \times \ldots \times \Omega_n)$. Choose $(U_k)_{k\geq 1}$ a sequence of relatively compact open sets in $\Omega_1 \times \ldots \times \Omega_n$ such that $\bigcup_{k\geq 1}^{\infty} U_k = \Omega_1 \times \ldots \times \Omega_n$. Fix any k. We have $\hat{R}_v^{E \cap U_k}$ identically 0 on $\Omega_1 \times \ldots \times \Omega_n$. Since $E \cap U_k$ is relatively compact there exist $\delta_1, \ldots, \delta_n$, regular domains in $\Omega_1, \ldots, \Omega_n$ respectively such that

$$\overline{\delta}_1 \times \ldots \times \overline{\delta}_n \subset \prod_{j=1}^{n} \Omega_j - \overline{\Pi_j(E \cap U_k)}\,.$$

It follows from Lemma 2.1.8 and Proposition 1.2.2(b) that $R_v^{E \cap U_k}$ is in $n\text{-}H^+(\delta_1 \times \ldots \times \delta_n)$. In particular it is continuous on $\delta_1 \times \ldots \times \delta_n$ and hence equals $\hat{R}_v^{E \cap U_k}$ there. Thus $R_v^{E \cap U_k}(z) = 0$ for z in $\delta_1 \times \ldots \times \delta_n$ and from the first part of the proof we see $E \cap U_k$ is n-polar. Proposition 2.1.4 now implies $E = \bigcup_{k\geq 1} (E \cap U_k)$ is n-polar and the assertion is proved.

(a) $\Rightarrow$ (c): Suppose E is n-polar in $\Omega_1 \times \ldots \times \Omega_n$. Let w be associated to E, v in n-$S^+(\Omega_1 \times \ldots \times \Omega_n)$, and x any point where w is finite. For every positive integer k, $k^{-1}w$ majorizes v on E hence it majorizes R_v^E on $\Omega_1 \times \ldots \times \Omega_n$. Since $w(x) < \infty$ and k is arbitrary this implies $R_v^E(x) = 0$. It follows $R_v^E(z) = 0$ for all z except in the n-polar set $\{z \in \Omega_1 \times \ldots \times \Omega_n : w(z) = \infty\}$. Finally $\hat{R}_v^E(x)$ must be 0 thus $\hat{R}_v^E$ is identically 0 (Remark 1.2.6 (b)).

The last assertion (c) $\Rightarrow$ (b) being obvious, the proof is complete.

<u>Corollary 2.1.11:</u> Let E be n-polar in $\Omega_1 \times \ldots \times \Omega_n$, F any subset of $\Omega_1 \times \ldots \times \Omega_n$, and v in n-$S^+(\Omega_1 \times \ldots \times \Omega_n)$. Then $R_v^{E \cup F}$ and R_v^F are equal everywhere except on an n-polar set.

<u>Proof:</u> Let x be any point such that $R_v^E(x) = 0$. Then

$$\begin{aligned} R_v^{E \cup F}(x) &\leq R_v^E(x) + R_v^F(x) \\ &= R_v^F(x). \end{aligned}$$

Since $R_v^{E \cup F}$ always majorizes R_v^F we see $R_v^{E \cup F}(x) = R_v^F(x)$. Proposition 2.1.10(c) implies this holds everywhere except on an n-polar set.

<u>Section 2</u> <u>n-Negligible Sets</u>

We define the n-negligible sets inductively (to include the earlier definition of polar) as follows.

<u>Definition 2.2.1:</u> Let U be an open subset of $\Omega_1 \times \ldots \times \Omega_n$. A subset E of U is said to be n-negligible in U in case n = 1 if E is polar and in case n > 1 if there exist sets $P_1, \ldots, P_n$ polar in $\Omega_1, \ldots, \Omega_n$ respectively

such that for each integer i from 1 to n, if x_i is in $\Pi_i(U)-P_i$, $E_i(x_i)$ is (n-1)-negligible in $U_i(x_i)$.

For n = 2, E 2-negligible in $\Omega_1 \times \Omega_2$ means that for all x_1 except in a polar subset of Ω_1 the section of E through x_1 is polar and for all x_2 except in a polar set in Ω_2 the section of E through x_2 is polar.

Notation: For each integer i from 1 to n we put

$$\Omega^i = \Pi\{\Omega_j : j \in \{1,\ldots,n\} - \{i\}\}.$$

Remark 2.2.2: Since subsets of polar sets are polar, E n-negligible in $\Omega_1 \times \ldots \times \Omega_n$ says precisely that for i any integer from 1 to n the set

$$\{x_i \in \Omega_i : E_i(x_i) \text{ is not (n-1)-negligible in } \Omega^i\}$$

is polar.

We observe now that in working with n-negligible sets, unlike with n-polar sets, we will not have to keep track of the open set U in Definition 2.2.2. This is shown in the following proposition.

Proposition 2.2.3: E is n-negligible in U if and only if it is n-negligible in $\Omega_1 \times \ldots \times \Omega_n$.

Proof: Suppose first E is n-negligible in U. We proceed by induction on n. If n = 1 this is just Remark 2.1.5. Next suppose n > 1 and the implication holds for integers 1 through n-1. There exist sets P_i, i = 1,...,n, each polar in Ω_i such that if x_i is in $\Pi_i(U) - P_i$, $E_i(x_i)$ is (n-1)-negligible in $U_i(x_i)$ hence by the inductive hypothesis (n-1)-negligible in Ω^i. If x_i is in $(\Omega_i - P_i) \cap (\Omega_i - \Pi_i(U))$, since E is contained in U, $E_i(x_i)$ is empty and hence (n-1)-negligible. This proves E is (n-1)-negligible in $\Omega_1 \times \ldots \times \Omega_n$.

The converse is proved by a simple induction and by noting that if a set P is polar in Ω_i, $P \cap V$ is polar in V for any open subset V of Ω_i. (Observe from Proposition 1.2.5(a) the restriction of an n-superharmonic function to V is in n-S(V)).

As a consequence we can unambiguously refer to such an exceptional set as being n-negligible without referring to any particular open set that contains it once it is clear which product of Brelot spaces it is in.

The next proposition gives a useful equivalent formulation of the definition of n-negligible set.

<u>Proposition 2.2.4:</u> Let E be a subset of $\Omega_1 \times \ldots \times \Omega_n, n \geq 2$. The following condition is equivalent to E being n-negligible For each integer i from 1 to n

$$N_i = \{(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n) \in \Omega^i : \{x_i \in \Omega_i : (x_1,\ldots,x_n) \in E\} \text{ not polar}\}$$

is (n-1)-negligible.

<u>Proof:</u> The proof in both directions is by induction on n. Note the proposition trivially holds for n = 2 since 1-negligible means polar.

Now suppose E is n-negligible, $n > 2$, and the condition holds for subsets of a product of any k of these Brelot spaces, k between 2 and n-1. By symmetry in the definition of $N_1,\ldots,N_n$ it is enough to show N_n is (n-1)-negligible. Since E is n-negligible there exist $P_1,\ldots,P_n$ polar subsets of $\Omega_1,\ldots,\Omega_n$ respectively such that if x_i is in $\Omega_i - P_i$, $E_i(x_i)$ is (n-1)-negligible in Ω^i. In particular consider any point x_1 in $\Omega_1 - P_1$ and the set $E_1(x_1)$. By the inductive hypothesis

$$\{(x_2,\ldots,x_{n-1}) \in \Omega_2 \times \ldots \times \Omega_{n-1} : \{x_n \in \Omega_n : (x_2,\ldots,x_n) \in E_1(x_1)\} \text{ not polar}\}$$

is (n-2)-negligible. But this set is precisely

$$\{(x_2,\ldots,x_{n-1}) \in \Omega_2 \times \ldots \times \Omega_{n-1} : (x_1,\ldots,x_{n-1}) \in N_n\}.$$

Similarly by considering P_i, $i = 2,\ldots,n-1$, we can show

$$\{(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{n-1}) \in \prod_j \{\Omega_j : j \in \{1,\ldots,n-1\}-\{i\}\} : (x_1,\ldots,x_{n-1}) \in N_n\}$$

is (n-2)-negligible. By the definition this shows N_n is (n-1)-negligible. The assertion is proved.

Conversely suppose $n > 2$, the condition holds for E, and if it holds for a subset of a product of k such Brelot spaces, $2 \leq k \leq n-1$, the subset is k-negligible. We will show E is n-negligible. Again by symmetry it is enough to construct P_1 in Definition 2.2.1. For each integer i from 2 to n, since N_i is (n-1)-negligible, there exists a polar set Q_i in Ω_1 such that if x_1 is in $\Omega_1 - Q_i$

$$R_{i,x_1} = \{(x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_n) \in \Pi\, \{\Omega_j : j \in \{2,\ldots,n\} - \{i\}\} : (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n) \in N_i\}$$

is (n-2)-negligible. Put $P_1 = \bigcup_{i=2}^{n} Q_i$. Then P_1 is polar in Ω_1 (Proposition 2.1.4) and if x_1 is in $\Omega_1 - P_1$ each set $R_{2,x_1},\ldots,R_{n,x_1}$ is (n-2)-negligible. Since

$$R_{i,x_1} = \{(x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_n) \in \Pi\, \{\Omega_j : j \in \{2,\ldots,n\} - \{i\}\} : \{x_i \in \Omega_i : (x_2,\ldots,x_n) \in E_1(x_1)\} \text{ not polar}\},$$

we have by the inductive hypothesis that $E_1(x_1)$ is (n-1)-negligible for such x_1. Thus P_1 satisfies the requirements of Definition 2.2.1 and we are done.

In future we will often have occasion to use both of these formulations of n-negligible sets and we will do so freely without referring back to Proposition 2.2.4.

We proceed now to demonstrate some of the most basic properties of n-negligible sets.

Proposition 2.2.5: A subset of an n-negligible set is n-negligible.

Proof: Let E be n-negligible and F any subset of E. We proceed by induction. The case n = 1 is trivial since clearly a subset of a polar set is polar. Now suppose n > 1 and the assertion holds for positive integers smaller than n. Since F is contained in E, $F_i(x_i)$ is contained in $E_i(x_i)$ for each i from 1 to n and x_i in Ω_i. It follows by the inductive hypothesis that

$$\{x_i \in \Omega_i : F_i(x_i) \text{ not } (n-1)\text{-negligible in } \Omega^i\}$$

$$\subset \{x_i \in \Omega_i : E_i(x_i) \text{ not } (n-1)\text{-negligible in } \Omega^i\} .$$

The latter set is polar since E is n-negligible hence the former set is also polar (Remark 2.2.2). This says precisely that F is n-negligible and we are done.

Proposition 2.2.6: A countable union of n-negligible sets is n-negligible.

Proof: The proof is by induction on n. If n = 1 this is just Proposition 2.1.4. Suppose then n > 1 and the result holds for smaller integers. Let $(E_k)_{k \geq 1}$ be a sequence of n-negligible sets and denote their union by E. For each k and integer i from 1 to n there exists a polar set $Q_{k,i}$ in Ω_i such that if x_i is in $\Omega_i - Q_{k,i}$, $(E_k)_i(x_i)$ is (n-1)-negligible in Ω^i. Put

$$Q_i = \bigcup_{k \geq 1} Q_{k,i} .$$

Then Q_i is polar (Proposition 2.1.4) and if x_i is in $\Omega_i - Q_i$, x_i is in $\Omega_i - Q_{k,i}$ for every k hence $(E_k)_i(x_i')$ is (n-1)-negligible and

$$E_i(x_i) = \bigcup_{k\geq 1} (E_k)_i(x_i)$$

is (n-1)-negligible by the induction hypothesis. The proves the proposition.

Proposition 2.2.7: Let E be (n-1)-negligible in $\Omega_2 \times \ldots \times \Omega_n, n\geq 2$. Then $\Omega_1 \times E$ is n-negligible.

Proof: The proof as before is by induction on n. If n = 2 then E is polar. If x_1 is any point in $\Omega_1, (\Omega_1 \times E)_1(x_1) = E$ which is polar. If x_2 is any point of Ω_2 outside the polar set $E, (\Omega_1 \times E)_2(x_2)$ is void, and hence polar. This proves the result for n = 2.

Now suppose n > 2 and the result holds for integers smaller than n. Clearly $E = N_1$ where

$$N_1 = \{x_2,\ldots,x_n) \varepsilon\ \Omega_2 \times \ldots \times \Omega_n : \{x_1 \varepsilon\ \Omega_1 : (x_1,\ldots,x_n) \varepsilon\ \Omega_1 \times E\} \text{ not polar}\}$$

since $\{x_1 \varepsilon\ \Omega_1 : (x_1,\ldots,x_n) \varepsilon\ \Omega_1 \times E\}$ is Ω_1 or ϕ depending on whether or not $(x_2,\ldots,x_n)$ is in E. Thus N_1 is (n-1)-negligible. For i = 2,...,n the set

$$N_i^1 = \{(x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_n) \varepsilon\ \Pi\{\Omega_j : j\varepsilon\ \{2,\ldots,n\} - \{i\}\} : \{x_i \varepsilon\ \Omega_i : (x_2,\ldots,x_n)\ \varepsilon\ E\} \text{ not polar}\}$$

is (n-2)-negligible by Proposition 2.2.4. Thus

$$N_i = \{(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n) \varepsilon\ \Omega^i : \{x_i \varepsilon\ \Omega_i : (x_1,\ldots,x_n)\ \varepsilon\ \Omega_1 \times E\} \text{ not polar}\}$$

$$= \Omega_1 \times N_i^1$$

is (n-1)-negligible by the induction hypothesis. Proposition 2.2.4 thus shows $\Omega_1 \times E$ is n-negligible and we are done.

Proposition 2.2.8: The complement of an n-negligible set is dense.

Proof: Let E be n-negligible. We show by induction that E contains no non-void open set. If n = 1, E is polar hence by Proposition 1.2.5 it contains no non-void open set. Suppose now $n > 1$ and $U \times V$ is contained in E where U is open in Ω_1 and V is open in $\Omega_2 \times \ldots \times \Omega_n$. Since E is n-negligible there exists a polar set P in Ω_1 such that if x_1 is in $\Omega_1 - P$, $E_1(x)$ is (n-1)-negligible. Since P does not contain U there exists x_1 in U - P. For this point $E_1(x_1)$ is (n-1)-negligible and it contains $(U \times V)_1(x_1) = V$. By the induction hypothesis V must be void. Therefore so is $U \times V$ and we are done.

Proposition 2.2.9: (local property) Let E be any subset of $\Omega_1 \times \ldots \times \Omega_n$ such that for every x in E there is an open set ω_x containing x with $E \cap \omega_x$ n-negligible in ω_x. Then E is negligible.

Proof: Note first that $E \cap \omega_x$ n-negligible in ω_x implies it is n-negligible in $\Omega_1 \times \ldots \times \Omega_n$ (Proposition 2.2.3). Now $\{\omega_x : x \in E\}$ is an open cover of E. Since $\Omega_1 \times \ldots \times \Omega_n$ has a countable base of open sets there exists a countable number of points $\{x_k\}_{k \geq 1}$ with x_k in E and $\{\omega_{x_k}\}_{k \geq 1}$ covering E (Lindelof property). Since for each k $E \cap \omega_{x_k}$ is n-negligible

$$E = \bigcup_{k \geq 1} E \cap \omega_{x_k}$$

is also n-negligible (Proposition 2.2.6). The proof is complete.

We wish now to show a set n-polar in an open set is n-negligible. This will follow from the next proposition.

Proposition 2.2.10: Let U be open in $\Omega_1 \times \ldots \times \Omega_n$, v in n-S(U), and k an integer between 1 and n-1. Then there is a k-negligible set N in $\Omega_1 \times \ldots \times \Omega_k$ such that if x is in $\Pi_{1,\ldots,k}(U)-N$, the mapping $y \to v(x,y)$ is in $(n-k)-S(U_{1,\ldots,k}(x))$.

Remark 2.2.11: If v is non-negative and $U = \Omega_1 \times \ldots \times \Omega_n$ it is easy to see we can choose N to be k-polar in $\Omega_1 \times \ldots \times \Omega_n$. Simply choose $y = (y_1,\ldots,y_n)$ any point at which v is finite and put

$$N = \{x \in \Omega_1 \times \ldots \times \Omega_k : v(x,z) = \infty \text{ for all } z \text{ in } \Omega_{k+1} \times \ldots \times \Omega_n\}.$$

Then since $v(y) < \infty$, $x \to v(x,y_{k+1},\ldots,y_n)$ is in $k\text{-}S^+(\Omega_1 \times \ldots \times \Omega_k)$ and equals ∞ on N.

The problem in general is that even if U is connected the $(1,\ldots,k)$-sections of U are not necessarily connected and so for some x in $\Pi_{1,\ldots,k}(U)$, $y \to v(x,y)$ might be (n-k)-superharmonic on one connected component of $U_{1,\ldots,k}(x)$ and identically ∞ on another.

Proof of the proposition: We prove this by induction on k. Suppose first k = 1. Put

$$E = \{x \in \Pi_1(U) : y \to v(x,y) \text{ is not in } (n-1)\text{-}S(U_1(x))\}$$

$$= \{x \in \Pi_1(U) : v(x,y) = \infty \text{ for all } y \text{ in a connected component of } U_1(x)\}.$$

By Proposition 1.2.5 there is a sequence $(x^\ell)_{\ell\geq 1} = ((x_1^\ell,\ldots,x_n^\ell))_{\ell\geq 1}$ which is dense in U such that for each ℓ, $v(x^\ell) < \infty$. Define G_ℓ for each ℓ to be the connected component of $U_{2,\ldots,k}(x_2^\ell,\ldots,x_n^\ell)$ containing x_1^ℓ. Put

$$N_\ell = \{x \in G_\ell : v(x,x_2^\ell,\ldots,x_n^\ell) = \infty\} \quad \text{and}$$

$$N = \bigcup_{\ell\geq 1} N_\ell .$$

Since $v(x_1^\ell,\ldots,x_n^\ell) < \infty$, $w:x \to v(x,x_2^\ell,\ldots,x_n^\ell)$ is in $S_1(G_\ell)$ and as we observed in Proposition 2.2.3 it is also in $S_1^+(G_\ell \cap \{x \in \Pi_1 U: w(x) > 0\})$. Thus N_ℓ is polar in $G_\ell \cap \{x \in \Pi_1 U: w(x) > 0\}$ hence also in Ω_1. Proposition 2.1.4 now implies N is also polar. We will be done if we show E is contained in N. Let x_o be in E. Let y_o be any point in the connected component of $U_1(x_o)$ in which $v(x_o,.)$ is identically ∞. Choose W_1, W_2 connected neighbourhoods of x_o, y_o respectively such that $W_1 \times W_2$ is contained in U. There exists an integer ℓ such that $(x_1^\ell,\ldots,x_n^\ell)$ is in $W_1 \times W_2$. Since W_1 and W_2 are connected it is easy to see x_o is in G_ℓ and $(x_2^\ell,\ldots,x_n^\ell)$ is in the same connected component of $U_1(x_o)$ as y_o. Therefore $v(x_o,x_2^\ell,\ldots,x_n^\ell) = \infty$ and hence x_o is in N_ℓ. This proves E is contained in N and the proof for $k = 1$ is complete.

Now suppose $1 < k \leq n-1$ and the proposition holds for smaller integers. Define

$$F = \{x \in \Pi_{1,\ldots,k}(U): y \to v(x,y) \text{ not in } (n-k)\text{-}S(U_{1,\ldots,k}(x))\} .$$

To show F is k-negligible in $\Omega_2 \times \ldots \times \Omega_k$, by symmetry, it is enough to show there exists P polar in Ω_1 such that for x_1 in $\Pi_1(F)-P$, $F_1(x_1)$ is $(k-1)$-negligible in $\Omega_2 \times \ldots \times \Omega_k$. Well from the first part of the proof there is a polar set P such that if x_1 is in $\Pi_1(U) - P$, $z \to v(x_1,z)$ is in $(n-1)\text{-}S(U_1(x_1))$. By the induction hypothesis, for such an x_1 there is a $(k-1)$-negligible set N in $\Omega_2 \times \ldots \times \Omega_k$ depending on x_1 such that if $(x_2,\ldots,x_k)$ is in $\Pi_{2,\ldots,k}(U_1(x_1)) - N$, $y \to v(x_1,\ldots,x_k,y)$ is in $(n-k)\text{-}S(U_{1,\ldots,k}(x_1,\ldots,x_k))$. Thus for x_1 in $\Pi_1(U) - P$

$$F_1(x_1) = \{(x_2,\ldots,x_k) \in \Pi_{2,\ldots,k}(U_1(x_1)): y \to v(x_1,\ldots,x_k,y) \text{ not in } (n-k)\text{-}S(U_{1,\ldots,k}(x_1,\ldots,x_k))\}$$

$$\subset N.$$

Therefore by Proposition 2.2.5 $F_1(x_1)$ is (k-1)-negligible in $\Omega_2 \times \ldots \times \Omega_k$ and we are done.

Proposition 2.2.12: Let U be an open subset of $\Omega_1 \times \ldots \times \Omega_n$ and E an n-polar subset of U. Then E is n-negligible.

Proof: Again the proof is by induction on n. If n = 1 there is nothing to show. Suppose then n > 1 and the assertion holds for smaller integers. Let v in $n\text{-}S^+(U)$ be associated to E in U. From the previous proposition there is a polar set P in Ω_1 such that if x is in $\Pi_1(U) - P, w: y \to v(x,y)$ is in $(n-1)\text{-}S^+(U_1(x_1))$. Now for any such x,

$$E_1(x) \subset \{z \in U_1(x): w(z) = \infty\}.$$

Thus $E_1(x)$ is (n-1)-polar in $U_1(x)$ hence by the induction hypothesis is (n-1)-negligible. It follows by symmetry E is n-negligible in U hence by Proposition 2.2.3 it is n-negligible. The proof is complete.

Remark 2.2.13: It is not known at this time whether or not the converse of Proposition 2.2.12 is true.

Remark 2.2.14: We saw in the beginning of Chapter 2.1 that every n-polar set is contained in a Borel n-polar set. Unfortunately we do not have a corresponding result for n-negligible sets. However the only use we now have for such a result is for performing integrations outside of n-negligible sets without changing the value of the integrals. We demonstrate how to sidestep this particular difficulty in Theorem 2.2.17.

Before we do this consider a Borel n-negligible set E. Let $\delta_1, \ldots, \delta_n$ be regular domains in $\Omega_1, \ldots, \Omega_n$ respectively and $(x_1, \ldots, x_n)$ any point of $\delta_1 \times \ldots \times \delta_n$. Recall while discussing the Dirichlet problem in Chapter 1.1

we remarked that polar sets have 0 harmonic measure. It follows from this fact, Fubini's Theorem, and a simple induction that $(\rho_{x_1}^{\delta_1} \times \ldots \times \rho_{x_n}^{\delta_n})(E) = 0$. Combining this with Proposition 1.2.10 we get the following result.

Proposition 2.2.15: If two n-$S(\mathcal{B}_1, \ldots, \mathcal{B}_n)$ functions are equal everywhere except on a Borel n-negligible set, their lower semicontinuous regularizations are equal everywhere.

Corollary 2.2.16: Let E be n-polar in $\Omega_1 \times \ldots \times \Omega_n$, F any subset of $\Omega_1 \times \ldots \times \Omega_n$, and v in n-$S^+(\Omega_1 \times \ldots \times \Omega_n)$. Then $\hat{R}_v^{E \cup F}$ and $\hat{R}_v^F$ are identical.

Proof: We saw in Corollary 2.1.1 that $R_v^{E \cup F}$ and R_v^F are equal everywhere except on an n-polar set, hence except on a Borel n-polar set, hence by Proposition 2.2.12 except on a Borel n-negligible set. But these two functions are nearly n-superharmonic. The result therefore follows from the Proposition.

Theorem 2.2.17: Let E be n-negligible, U open in $\Omega_1 \times \ldots \times \Omega_n$, $x = (x_1, \ldots, x_n)$ a fixed point of U, and v in n-S(U). Then

$$(1) \qquad v(x) = \liminf_{\substack{z \to x \\ z \in U - E}} v(z) .$$

Proof: Since E is n-negligible there exists a polar set P in Ω_1 such that for z_1 in $\Omega_1 - P$, $E_1(z_1)$ is (n-1)-negligible in $\Omega_2 \times \ldots \times \Omega_n$. We have seen that there is a Borel polar set Q_1 containing P. Let z_1 be any point in $\Omega_1 - Q_1$. Then a priori z_1 is in $\Omega_1 - P$ hence $E_1(z_1)$ is (n-1)-negligible. It follows similarly that there is a Borel set Q_{2,z_1} depending on z_1 such that for any z_2 in $\Omega_2 - Q_{2,z_1}$, $E_{1,2}(z_1, z_2)$ is (n-2)-negligible in $\Omega_3 \times \ldots \times \Omega_n$. Now choose any point z_2 in $\Omega_2 - Q_{2,z_1}$. We can form similarly the Borel polar set Q_{3,z_1,z_2}

in Ω_3 depending on z_1 and z_2 such that for z_3 in $\Omega_3 - Q_{3,z_1,z_2}$, $E_{1,2,3}(z_1,z_2,z_3)$ is (n-3)-negligible. Continuing in this way we get the sets $Q_1, Q_{2,z_1}, \ldots, Q_{n,z_1,\ldots,z_{n-1}}$ each polar in $\Omega_1, \ldots, \Omega_n$ respectively, each G_δ, each depending on the indicated points and the last containing $E_{1,\ldots,n-1}(z_1,\ldots,z_{n-1})$.

Now suppose if possible (1) fails. Since v is lower semicontinuous this says $v(x) < \infty$ and

$$\liminf_{\substack{z \to x \\ z \in U-E}} v(z) > v(x) .$$

Thus there exists a neighbourhood W of x and a positive real number ε such that

$$(2) \qquad v(z) \geq v(x) + \varepsilon , \quad z \text{ in } W \cap U-E .$$

For each integer i from 1 to n choose a sequence $(\delta_k^i)_{k\geq 1}$ of regular domains in Ω_i such that for every k, $\overline{\delta}_k^1 \times \ldots \times \overline{\delta}_k^n \subset W$, $\delta_k^i \supset \overline{\delta}_{k+1}^i$, and $\bigcap_{k\geq 1} \delta_k^i = \{x_i\}$. Now since v is integrable with respect to any product of harmonic measures we can apply Fubini's Theorem to obtain

$$\int v(z_1,\ldots,z_n)(\rho_{x_1}^{\delta_k^1} \times \ldots \times \rho_{x_n}^{\delta_k^n})(dz_1,\ldots,dz_n)$$

$$= \int d\rho_{x_1}^{\delta_k^1}(z_1) \int d\rho_{x_2}^{\delta_k^2}(z_2) \ldots \int v(z_1,\ldots,z_n) d\rho_{x_n}^{\delta_k^n}(z_n)$$

$$= \int_{z_1 \notin Q_1} d\rho_{x_1}^{\delta_k^1}(z_1) \int_{z_2 \notin Q_{2,z_1}} d\rho_{x_2}^{\delta_k^2}(z_2) \ldots \int_{z_n \notin Q_{n,z_1,\ldots,z_{n-1}}} v(z_1,\ldots,z_n) d\rho_{x_n}^{\delta_k^n}(z_n)$$

$$\geq \int_{z_1 \notin Q_1} d\rho_{x_1}^{\delta_k^1}(z_1) \ldots \int_{z_n \notin Q_{n,z_1,\ldots,z_{n-1}}} (v(x) + \varepsilon) d\rho_{x_n}^{\delta_k^n}(z_n)$$

(from (2) and the fact that $Q_{n,z_1,\ldots,z_{n-1}}$ contains $E_{1,\ldots,n-1}(z_1,\ldots,z_n)$)

$$= (v(x) + \varepsilon) \int d\rho_{x_1}^{\delta_k^1} \ldots \int d\rho_{x_n}^{\delta_k^n} .$$

Letting $k \to \infty$ and applying Proposition 1.1.1 (c) gives

$$v(x) \geq v(x) + \varepsilon .$$

This is impossible since $v(x) < \infty$. Thus (1) does indeed hold and the the proof is complete.

The following is an immediate consequence of Theorem 2.2.17.

Corollary 2.2.18: If two functions n-superharmonic on an open set are equal everywhere except on an n-negligible set then they are identical.

Section 3 The Dirichlet Problem

In this section, for convenience we state and prove results only for the special case $n = 2$. However, the obvious generalizations to $n > 2$ do hold and can be proved easily by induction.

Let ω_1, ω_2 be relatively compact domains in Ω_1, Ω_2 respectively. Using methods similar to Gowrisankaran in [7] we solve the Dirichlet problem on $\omega_1 \times \omega_2$ where the boundary values are specified only on the distinguished boundary, $\partial\omega_1 \times \partial\omega_2$. Then, by proving a minimum principle special to $\omega_1 \times \omega_2$ we arrive at the same solution by means of a P.W.B.type method.

Denote the irregular boundary points of ω_1 by P_1 and the irregular boundary points of ω_2 by P_2. Put

$$P = (P_1 \times P_2) \cup (\omega_1 \times P_2) \cup (P_1 \times \omega_2).$$

Then P is a subset of $\partial(\omega_1 \times \omega_2)$ which is 2-polar in $\Omega_1 \times \Omega_2$ (Proposition 2.1.2) and we shall see, in some sense, the irregular boundary points for the Dirichlet problem on $\omega_1 \times \omega_2$.

Proposition 2.3.1: Let ω_1, ω_2 and P be as above and f a real valued continuous function on $\partial\omega_1 \times \partial\omega_2$. Define φ_f on $\overline{\omega}_1 \times \overline{\omega}_2$ by

$$\varphi_f(x,y) = \begin{cases} f(x,y) & (x,y) \text{ in } \partial\omega_1 \times \partial\omega_2 \\ \int f(x,z^1)\,d\mu_y^{\omega_2}(z^1) & (x,y) \text{ in } \partial\omega_1 \times \omega_2 \\ \int f(z,y)\,d\mu_x^{\omega_1}(z) & (x,y) \text{ in } \omega_1 \times \partial\omega_2 \\ \iint f(z,z^1)\,d\mu_x^{\omega_1}(z)\,d\mu_y^{\omega_2}(z^1) & (x,y) \text{ in } \omega_1 \times \omega_2 \end{cases}$$

Then φ_f is non-negative if f is non-negative, it is in $2\text{-}H(\omega_1 \times \omega_2)$, it is continuous on $\overline{\omega}_1 \times \overline{\omega}_2 - P$, it equals f on $\partial\omega_1 \times \partial\omega_2$, and for every (x,y) in $\partial\omega_1 \times \partial\omega_2$ the mappings $z \to \varphi_f(z,y)$ and $z^1 \to \varphi_f(x,z^1)$ are in $H_1(\omega_1)$ and $H_2(\omega_2)$ respectively.

Proof: We first prove the continuity assertion. Suppose first f splits as $g_1 \cdot g_2$. That is for (x,y) in $\partial\omega_1 \times \partial\omega_2$, $f(x,y) = g_1(x)\cdot g_2(y)$ where g_1 and g_2 are in $C_{\mathbb{R}}(\partial\omega_1)$ and $C_{\mathbb{R}}(\partial\omega_2)$ respectively. Then φ_f splits as $G_1 \cdot G_2$, where G_1 solves the Dirichlet problem on ω_1 with boundary values g_1 and G_2 solves the Dirichlet problem on ω_2 with boundary values g_2. It is easy to see φ_f is continuous on $\overline{\omega}_1 \times \overline{\omega}_2 - P$. Clearly if f is a finite linear combination of continuous functions that split, the same conclusions holds. By the Stone-Weierstrass Theorem such functions are uniformly dense in $C_{\mathbb{R}}(\partial\omega_1 \times \partial\omega_2)$. Thus it suffices to show that if $(f_k)_{k\geq 1}$ is a sequence in $C_R(\partial\omega_1 \times \partial\omega_2)$ converging uniformly to f, and φ_{f_k} is for each k continuous on $\overline{\omega}_1 \times \overline{\omega}_2 - P$, then $(\varphi_{f_k})_{k\geq 1}$ converges uniformly on $\overline{\omega}_1 \times \overline{\omega}_2$ to φ_f. For if so, φ_f will be continuous at each point of continuity of every φ_{f_k}, that is on $\overline{\omega}_1 \times \overline{\omega}_2 - P$.

Let M be a bound for $\mathcal{H}_1^{\omega_1}$ on ω_1 and $\mathcal{H}_1^{\omega_2}$ on ω_2. Certainly this exists since there are functions harmonic on a neighbourhood of say ω_1, greater than 1 and by definition these majorize $\mathcal{H}_1^{\omega_1}$ on ω_1 and are themselves bounded on ω_1. Now given any positive real number ε there exists an integer K such that for k > K,

$$|f_k(x,y) - f(x,y)| < \varepsilon/M^2, \quad (x,y) \text{ in } \partial\omega_1 \times \partial\omega_2 .$$

It follows that for k > K,

$$|\varphi_{f_k}(x,y) - \varphi_f(x,y)| < \varepsilon/M^2 < \varepsilon \quad , \quad (x,y) \text{ in } \partial\omega_1 \times \partial\omega_2 ,$$

$$|\varphi_{f_k}(x,y) - \varphi_f(x,y)| < \varepsilon/M < \varepsilon , \quad (x,y) \text{ in } (\omega_1 \times \partial\omega_2) \cup (\partial\omega_1 \times \omega_2),$$

and $$|\varphi_{f_k}(x,y) - \varphi_f(x,y)| < \varepsilon ,(x,y) \text{ in } \omega_1 \times \omega_2 .$$

Thus $(\varphi_{f_k})_{k\geq 1}$ converges to φ_f uniformly on $\overline{\omega}_1 \times \overline{\omega}_2$ and the continuity assertion is proved.

From the solution of the Dirichlet problem for n = 1 it is clear that for each (x,y) in $\partial\omega_1 \times \partial\omega_2$, $z \to \varphi_f(z,y)$ and $z^1 \to \varphi_f(x,z^1)$ are harmonic on ω_1 and ω_2 respectively.

The only non-trivial assertion that remains to be proved is for fixed y in ω_2, $z \to \varphi_f(z,y)$ is in $H_1(\omega_1)$. By symmetry this will show φ_f is in $2\text{-}H(\omega_1 \times \omega_2)$. We show first $z \to \varphi_f(z,y)$ is continuous on $\partial\omega_1$. Let $(z_k)_{k\geq 1}$

be a sequence in $\partial\omega_1$ converging to z in $\partial\omega_1$. Since f is uniformly continuous on $\partial\omega_1 \times \partial\omega_2$, the sequence $(f(z_k,.))_{k\geq 1}$ converges uniformly to f(z,.). It follows from the Dominated Convergence Theorem that

$$\lim_{k\to\infty} \varphi_f(z_k,y) = \lim_{k\to\infty} \int f(z_k,z')d\mu_y^{\omega_2}(z')$$

$$= \int f(z,z')d\mu_y^{\omega_2}(z')$$

$$= \varphi_f(z,y) .$$

Thus the mapping is indeed continuous. Now let x be a point of ω_1 and δ a regular neighbourhood of x with $\overline{\delta} \subset \omega_1$.

$$\int \varphi_f(z,y)d\rho_x^\delta(z) = \int d\rho_x^\delta(z) \iint f(z'z'')d\mu_z^{\omega_1}(z')d\mu_y^{\omega_2}(z'')$$

$$= \int d\rho_x^\delta(z) \int \varphi_f(z',y)d\mu_z^{\omega_1}(z')$$

$$= \int \mathcal{H}^{\omega_1}_{\varphi_f(\cdot,y)}(z)\, d\rho_x^\delta(z)$$

(by the continuity of $z' \to \varphi_f(z',y)$ on $\partial\omega_1$ established at the beginning of this paragraph)

$$= \mathcal{H}^{\omega_1}_{\varphi_f(\cdot,y)}(x)$$

$$= \int \varphi_f(z,y)d\mu_x^{\omega_1}(z)$$

$$= \iint f(z,z')d\mu_y^{\omega_2}(z')d\mu_x^{\omega_1}(z)$$

$$= \varphi_f(x,y) .$$

This proves the harmonicity and we are done.

We will demonstrate the uniqueness of this solution below in Corollary 2.3.4.

We now consider the problem in a different way. We begin by deducing a minimum principle that is special to rectangles.

Theorem 2.3.2: Let ω_1, ω_2 be relatively compact domains in Ω_1, Ω_2 respectively and v in $2\text{-}S(\omega_1 \times \omega_2)$, v bounded below. Then, if for all (x,y) in $\partial\omega_1 \times \partial\omega_2$

$$(1) \qquad \liminf_{\substack{(z,z') \to (x,y) \\ (z,z') \in \omega_1 \times \omega_2}} v(z,z') \geq 0,$$

v is non-negative on $\omega_1 \times \omega_2$.

Proof: From Proposition 1.2.7 it suffices to show that (1) holds for all (x,y) in $\partial(\omega_1 \times \omega_2) = (\partial\omega_1 \times \partial\omega_2) \cup (\partial\omega_1 \times \omega_2) \cup (\omega_1 \times \partial\omega_2)$. We will show it for a fixed point (x_o, y_o) in $\omega_1 \times \partial\omega_2$. By symmetry this is enough.

Let us suppose first that v is also bounded above. If (1) fails at (x_o, y_o), there exists a positive real number ε and a sequence $((z_k, z'_k))_{k > 1}$ in $\omega_1 \times \omega_2$ converging to (x_o, y_o) with

$$(2) \qquad v(z_k, z'_k) < -\varepsilon \text{ for all } k.$$

Consider for each such k the mappings $v_k : z \to v(z, z'_k)$ defined on ω_1. This is a positive uniformly bounded sequence in $S_1(\omega_1)$. Since ω_1 is relatively compact there exists u, positive harmonic on a neighbourhood of $\bar{\omega}_1$, such that $v_k(z) + u(z) \geq 0$ for all k and z in ω_1 and $v_k + u$ is pointwise uniformly bounded above on ω_1. Proposition 1.1.23 implies there is a subsequence $(v_{k_j}+u)_{j\geq 1}$ converging in the Cartan-Brelot topology to a function w_1 in $S_1^+(\omega_1)$. It follows $(v_{k_j})_{j\geq 1}$ converges in the Cartan-Brelot topology to $w = w_1 - u$.

We claim w is non-negative on ω_1. Indeed let x_1 be any point in $\partial\omega_1$ and γ a positive real number. Then (x_1,y_o) is in $\partial\omega_1 \times \partial\omega_2$ and from (1) we deduce there exist relatively compact neighbourhoods U,V of x_1,y_o respectively such that

$$v(z,z') \geq -\gamma \text{ for all } (z,z') \text{ in } (U\times V)\cap(\omega_1 \times \omega_2).$$

Without loss of generality we may assume z'_{k_j} is in V for all j. Thus

$$v(z,z'_{k_j}) \geq -\gamma \text{ for every } z \text{ in } U\cap\omega_1 \text{ and every } j.$$

Now let x_2 be in $U\cap\omega_1$ and let $(\delta_\ell)_{\ell\geq 1}$ be a sequence of regular neighbourhoods of x_2 such that for each ℓ, $\overline{\delta}_{\ell+1} \subset \delta_\ell \subset \overline{\delta}_\ell \subset U\cap\omega_1$ and $\bigcap_{\ell\geq 1} \delta_\ell = \{x_2\}$. Then for every such ℓ,

$$\begin{aligned} \int w(z)d\rho_{x_2}^{\delta_\ell}(z) &= \lim_{j\to\infty} \int v_{k_j}(z)d\rho_{x_2}^{\delta_\ell}(z) \\ &= \lim_{j\to\infty} \int v(z,z'_{k_j})d\rho_{x_2}^{\delta_\ell}(z) \\ &\geq (-\gamma) \int d\rho_{x_2}^{\delta_\ell}(z) . \end{aligned}$$

Taking the limit as $\ell\to\infty$ gives $w(x_2)\geq -\gamma$ (Proposition 1.1.1(c) and Proposition 1.1.6). This being true for all x_2 in $U\cap\omega_1$ and γ being arbitrarily positive we get

$$\liminf_{\substack{z\to x\\ z\in\omega_1}} w(z) \geq 0.$$

This holds for all x_1 in $\partial\omega_1$ therefore it follows from the Minimum Principle on Ω_1 (Proposition 1.1.3(f)) that w is indeed non-negative on ω_1.

Now the mapping $f: S_1^+(\omega_1) \times \Omega_1 \to \overline{\mathbb{R}}, (s,x) \to s(x)$ is lower semi-continuous if $S_1^+(\omega_1)$ is provided with the Cartan-Brelot topology. (Corollary 1.1.21). Thus

$$\begin{aligned}
\liminf_{j\to\infty} v_{k_j}(z_{k_j}) + u(x_o) &= \liminf_{j\to\infty} [v_{k_j}(z_{k_j}) + u(z_{k_j})] \\
&= \liminf_{j\to\infty} f(v_{k_j} + u, z_{k_j}) \\
&\geq f(w_1, x_o) \\
&= w_1(x_o) \\
&= w(x_o) + u(x_o).
\end{aligned}$$

Therefore

$$\liminf_{j\to\infty} v(z_{k_j}, z'_{k_j}) \geq w(x_o) \geq 0.$$

But this contradicts (2). Therefore the theorem holds in case v is bounded above.

In general choose h positive 2.-harmonic on a neighbourhood of $\overline{\omega}_1 \times \overline{\omega}_2$ and define for each positive integer k the function w_k by

$$w_k(z,z') = \min\,(v(z,z'),\ k\cdot h(z,z')),\quad (z,z') \text{ in } \omega_1 \times \omega_2.$$

Then for each k, w_k is in $2\text{-}S(\omega_1 \times \omega_2)$, it is bounded, and it converges pointwise to v. In addition, for every (x,y) in $\partial\omega_1 \times \partial\omega_2$,

$$\liminf_{\substack{(z,z')\to(x,y) \\ (z,z')\in\omega_1\times\omega_2}} w_k(z,z') \geq 0\ .$$

By the special case this implies v_k is non-negative on ω_1. Letting $k \to \infty$ gives that v is also non-negative and the proof is complete.

Corollary 2.3.3: Let P be a subset of $\partial\omega_1 \times \partial\omega_2$ which is 2-polar in $\Omega_1 \times \Omega_2$ and suppose in Theorem 2.3.2,(1) is satisfied only for (x,y) in $\partial\omega_1 \times \partial\omega_2 - P$. Then we can still deduce v is non-negative.

Proof: Let (x_o, y_o) be in $\omega_1 \times \omega_2$. Since P is contained in $\partial\omega_1 \times \partial\omega_2$, $P_1(x_o) = P_2(y_o) = \emptyset$ hence from Theorem 2.1.9 there exists w in $2\text{-}S^+(\Omega_1 \times \Omega_2)$ such that $w(x_o, y_o) < \infty$ and $w(z,z') = \infty$ for all (z,z') in P. Therefore for every positive integer k, $v + k^{-1} \cdot w$ is in $2\text{-}S(\omega_1 \times \omega_2)$, is lower bounded, and

$$\liminf_{\substack{(z,z') \to (x,y) \\ (z,z') \in \omega_1 \times \omega_2}} (v(z,z') + k^{-1} \cdot w(z,z')) \geq 0$$

for all (x,y) in $\partial\omega_1 \times \partial\omega_2$. From the theorem we get $v + k^{-1} \cdot w$ is non-negative on $\omega_1 \times \omega_2$, in particular $v(x_o, y_o) + k^{-1} \cdot w(x_o, y_o) \geq 0$. Letting $k \to \infty$ gives $v(x_o, y_o) \geq 0$. The point (x_o, y_o) being arbitrary we have indeed v is non-negative on $\omega_1 \times \omega_2$.

Corollary 2.3.4: The function φ_f in Proposition 2.3.1 is the only function in $2\text{-}H(\omega_1 \times \omega_2)$ bounded on $\omega_1 \times \omega_2$ and tending to the same value as f at all points of $\partial\omega_1 \times \partial\omega_2$ except a subset 2-polar in $\Omega_1 \times \Omega_2$.

Proof: If u is another such function just apply Corollary 2.3.3 to $\varphi_f - u$ and $u - \varphi_f$.

Let ω_1 and ω_2 be as above and f any extended real valued function on $\partial\omega_1 \times \partial\omega_2$. Consider now the following families of functions.

$U(f) = \{v : v$ is lower bounded, 2-hyperharmonic on $\omega_1 \times \omega_2$, and for all (x,y) in $\partial\omega_1 \times \partial\omega_2$ except a set 2-polar in $\Omega_1 \times \Omega_2$

$$\liminf_{\substack{(z,z') \to (x,y) \\ (z,z') \in \omega_1 \times \omega_2}} v(z,z') \geq f(x,y) \}$$

$L(f) = \{w : (-w)$ is lower bounded, 2- hyperharmonic on $\omega_1 \times \omega_2$, and for all (x,y) in $\partial\omega_1 \times \partial\omega_2$ except a set 2-polar $\Omega_1 \times \Omega_2$

$$\limsup_{\substack{(z,z') \to (x,y) \\ (z,z') \in \omega_1 \times \omega_2}} w(z,z') \leq f(x,y) \} .$$

Define the upper and lower solution respectively by

$$\overline{\mathcal{H}}_f^{\omega_1 \times \omega_2}(x,y) = \inf \{v(x,y) : v \in U(f)\}$$

$$\underline{\mathcal{H}}_f^{\omega_1 \times \omega_2}(x,y) = \sup \{w(x,y) : w \in L(f)\}$$

for all (x,y) in $\omega_1 \times \omega_2$. Notice $\overline{\mathcal{H}}_f^{\omega_1 \times \omega_2} = -\underline{\mathcal{H}}_{-f}^{\omega_1 \times \omega_2}$. Also $\overline{\mathcal{H}}_f^{\omega_1 \times \omega_2}$ majorizes $\underline{\mathcal{H}}_f^{\omega_1 \times \omega_2}$. For if v is in U(f) and w is in $L(f)$, v-w is 2-hyperharmonic on $\omega_1 \times \omega_2$, lower bounded, and (since the union of two sets 2-polar in $\Omega_1 \times \Omega_2$ is 2-polar in $\Omega_1 \times \Omega_2$) we have for all (x,y) in $\partial\omega_1 \times \partial\omega_2$ except in a set 2-polar in $\Omega_1 \times \Omega_2$

$$(1) \quad \liminf_{\substack{(z,z') \to (x,y) \\ (z,z') \in \omega_1 \times \omega_2}} (v(z,z') - w(z,z')) \geq \liminf_{\substack{(z,z') \to (x,y) \\ (z,z') \in \omega_1 \times \omega_2}} v(z,z') - \limsup_{\substack{(z,z') \to (x,y) \\ (z,z') \in \omega_1 \times \omega_2}} w(z,z')$$

$$\geq 0 .$$

(Clearly (1) holds if $f(x,y)$ is finite. If $f(x,y) = \infty$ the first member of the right hand side of (1) is ∞ and since w is bounded above the second member is strictly smaller than ∞ thus (1) still holds. Similarly it holds if $f(x,y) = -\infty$). Therefore Corollary 2.3.3 implies v majorizes w hence $\overline{\mathcal{H}}_f^{\omega_1 \times \omega_2}$ does indeed majorizes $\underline{\mathcal{H}}_f^{\omega_1 \times \omega_2}$.

Proposition 2.3.5: $\overline{\mathcal{H}}_f^{\omega_1 \times \omega_2}$ is either identically ∞, identically $-\infty$, or it is in $2\text{-}H(\omega_1 \times \omega_2)$.

Proof: Suppose the first case does not occur. Then $U(f) \cap (2\text{-}S(\omega_1 \times \omega_2))$ is non-empty. Now let (x_o,y_o) be in $\omega_1 \times \omega_2$ and δ_1,δ_2 regular domains in Ω_1, Ω_2 respectively with (x_o,y_o) in $\delta_1 \times \delta_2$ and $\overline{\delta}_1 \times \overline{\delta}_2$ contained in $\omega_1 \times \omega_2$. From Lemma 2.1.8, every member of $U(f) \cap (2\text{-}S(\omega_1 \times \omega_2))$ can be replaced with a smaller member also in $2\text{-}H(\delta_1 \times \delta_2)$ with values unchanged on $\omega_1 \times \omega_2 - ((\delta_1 \times \Omega_2) \cup (\Omega_1 \times \delta_2))$. Clearly then such a function is still in $U(f)$ and since $U(f)$ is decreasing directed, it follows from Proposition 1.2.2 (b) that $\overline{\mathcal{H}}_f^{\omega_1 \times \omega_2}$ is in $2\text{-}H(\delta_1 \times \delta_2)$ or is identically $-\infty$ on $\delta_1 \times \delta_2$. This shows $\{(x,y) \in \omega_1 \times \omega_2 : \overline{\mathcal{H}}^{\omega_1 \times \omega_2}(x,y)$ 2-harmonic on a neighbourhood of $(x,y)\}$ and $\{(x,y) \in \omega_1 \times \omega_2 : \overline{\mathcal{H}}_f^{\omega_1 \times \omega_2}(x,y) = -\infty\}$ are two disjoint open subsets of $\omega_1 \times \omega_2$ hence, since $\omega_1 \times \omega_2$ is connected, one is empty and the other is $\omega_1 \times \omega_2$. This completes the proof.

Theorem 2.3.6. For every extended real valued function f on $\partial\omega_1 \times \partial\omega_2$ and every (x,y) in $\omega_1 \times \omega_2$

(1) $$\overline{\mathcal{H}}_f^{\omega_1 \times \omega_2}(x,y) = \overline{\int} f(z,z')(\mu_x^{\omega_1} \times \mu_y^{\omega_2})(dzdz')$$

and if f is $\mu_{x_1}^{\omega_1} \times \mu_{y_1}^{\omega_2}$ - integrable for one point (x_1,y_1) in $\omega_1 \times \omega_2$ it is integrable with respect to all such measures, $\overline{\mathcal{H}}_f^{\omega_1 \times \omega_2}$ and $\underline{\mathcal{H}}_f^{\omega_1 \times \omega_2}$ are identical, and are in $2\text{-}H(\omega_1 \times \omega_2)$. (We denote the common function by $\mathcal{H}_f^{\omega_1 \times \omega_2}$).

Proof: For any such f we let $\Lambda(f)$ denote the mapping $\omega_1 \times \omega_2 \to \overline{\mathbb{R}}$, $(x,y) \to \int f(z,z')(\mu_x^{\omega_1} \times \mu_y^{\omega_2})(dzdz')$.

Suppose first f is continuous and real valued. Then Proposition 2.3.1 says $\Lambda(f)$ is in $U(f)$, hence it majorizes $\overline{\mathcal{H}}_f^{\omega_1 \times \omega_2}$. Conversely, if v is in $U(f)$, $v - \Lambda(f)$ is (again from Proposition 2.3.1) lower bounded 2-hyper-harmonic on $\omega_1 \times \omega_2$ with lower limit greater than or equal to 0 at all points of $\partial\omega_1 \times \partial\omega_2$ except on a set 2-polar in $\Omega_1 \times \Omega_2$. Corollary 2.3.3 then implies v majorizes $\Lambda(f)$ and hence so does $\overline{\mathcal{H}}_f^{\omega_1 \times \omega_2}$. Thus the theorem holds for real valued continuous functions.

Next, if f is lower bounded lower semicontinuous, since $\partial\omega_1 \times \partial\omega_2$ is compact, there exists a sequence $(f_k)_{k\geq 1}$ in $C_{\mathbb{R}}(\partial\omega_1 \times \partial\omega_2)$ increasing pointwise to f. From the Monotone Convergence Theorem, $(\Lambda(f_k))_{k\geq 1}$ increases pointwise to $\Lambda(f)$, therefore we have $\Lambda(f)$ is either identically ∞ or in $2\text{-}H(\omega_1 \times \omega_2)$ (Proposition 1.2.2(b)). For any (x,y) in $\partial\omega_1 \times \partial\omega_2 - P$, Proposition 2.3.1 says $\mu_z^{\omega_1} \times \mu_{z'}^{\omega_2}$ converges weakly to the point mass at (x,y) as $(z,z') \to (x,y)$, (z,z') in $\omega_1 \times \omega_2$. For such (x,y) since f is lower semi-continuous, we have

$$\liminf_{\substack{(z,z') \to (x,y) \\ (z,z') \in \omega_1 \times \omega_2}} \int f(z_1,z_2)(\mu_z^{\omega_1} \times \mu_{z'}^{\omega_2})(dz_1dz_2) \geq f(x,y) .$$

Thus $\Lambda(f)$ is in $U(f)$ and it therefore majorizes $\overline{\mathcal{H}}_f^{\omega_1 \times \omega_2}$. On the other hand for (x,y) in $(\omega_1 \times \omega_2)$,

$$\overline{\mathcal{H}}_f^{\omega_1 \times \omega_2}(x,y) \geq \sup\{ \overline{\mathcal{H}}_g^{\omega_1 \times \omega_2}(x,y) : g \text{ real valued continuous on } \partial\omega_1 \times \partial\omega_2, g \text{ minorizing } f\} .$$

$$= \sup\{ \int g(z_1,z_2)(\mu_x^{\omega_1} \times \mu_y^{\omega_2})(dz_1dz_2): g \text{ real valued continuous on } \partial\omega_1 \times \partial\omega_2,\ g \text{ minorizing } f \}$$

$$= \int f(z_1,z_2)(\mu_x^{\omega_1} \times \mu_y^{\omega_2})(dz_1dz_2)$$

$$= \wedge(f)(x,y).$$

Therefore (1) holds for lower bounded lower semicontinuous functions.

In general, if (x,y) is in $\omega_1 \times \omega_2$

$$\wedge(f)(x_1,y_1) = \inf \{ \wedge(g)(x_1,y_1): g \text{ lower bounded lower semicontinuous on } \partial\omega_1 \times \partial\omega_2,\ g \text{ majorizing } f\} .$$

(This is precisely the definition of $\wedge(f)(x,y)$.)

$$= \inf\{ \overline{\mathcal{H}}_g^{\omega_1 \times \omega_2}(x_1,y_1): g \text{ lower bounded lower semicontinuous on } \partial\omega_1 \times \partial\omega_2,\ g \text{ majorizing } f\} .$$

Call the last member of this equality M. We show $M = \overline{\mathcal{H}}_f^{\omega_1 \times \omega_2}(x_1,y_1)$. If v is in $\mathcal{U}(f)$ we can extend v to $\partial(\omega_1 \times \omega_2)$ by

$$v(x,y) = \liminf_{\substack{(z,z') \to (x,y) \\ (z,z') \in \omega_1 \times \omega_2}} v(z,z'), \quad (x,y) \text{ in } \partial(\omega_1 \times \omega_2).$$

It is easy to prove v is lower bounded lower semicontinuous on $\overline{\omega}_1 \times \overline{\omega}_2$ and since it is in $\mathcal{U}(f)$ it majorizes f on $\partial\omega_1 \times \partial\omega_2 - Q$, where Q is a subset of $\partial\omega_1 \times \partial\omega_2$ which is 2-polar in $\Omega_1 \times \Omega_2$. Clearly $Q_1(x_1) = Q_2(y_1) = \phi$. Therefore Proposition 2.1.9 implies there is a w in $2\text{-}S^+(\Omega_1 \times \Omega_2)$ such that

$w(x_1,y_1) < \infty$ and $w(z,z') = \infty$ for all (z,z') in Q. Now for every positive integer k the function $v + k^{-1}\cdot w$ majorizes f on $\partial\omega_1 \times \partial\omega_2$. Therefore $v(x_1,y_1) + k^{-1}\cdot w(x_1,y_1) \geq M$. Letting $k \to \infty$ gives $v(x_1,y_1) \geq M$. This being true for every v in $\mathcal{U}(f)$ we have $\overline{\mathcal{H}}_f^{\omega_1 \times \omega_2}(x_1,y_1) \geq M$. The opposite inequality being obvious we have equality and (1) holds in general.

Finally, if f is integrable with respect to $\mu_{x_1}^{\omega_1} \times \mu_{y_1}^{\omega_2}$ for some (x_1,y_1) in $\omega_1 \times \omega_2$, the first part of the theorem says $\overline{\mathcal{H}}_f^{\omega_1 \times \omega_2}(x_1,y_1)$ is finite and hence by Proposition 2.3.5 $\overline{\mathcal{H}}_f^{\omega_1 \times \omega_2}$ is in 2-H$(\omega_1 \times \omega_2)$. We also have $\overline{\mathcal{H}}_f^{\omega_1 \times \omega_2}(x_1,y_1) = \underline{\mathcal{H}}_f^{\omega_1 \times \omega_2}(x_1,y_1)$. From Proposition 2.3.5 and Remark 1.2.3(b) we deduce

$$\overline{\mathcal{H}}_f^{\omega_1 \times \omega_2}(x,y) = \underline{\mathcal{H}}_f^{\omega_1 \times \omega_2}(x,y) = \wedge(f)(x,y)$$

for every (x,y) in $\omega_1 \times \omega_2$ and hence f is $\mu_x^{\omega_1} \times \mu_y^{\omega_2}$- integrable. This completes the proof.

<u>Remark 2.3.7:</u> Let v be an n-S$(\mathcal{B}_1,\ldots, \mathcal{B}_n)$ function on $\Omega_1 \times \ldots \times \Omega_n$ and for each integer i from 1 to n let δ_i and ω_i be regular domains in $\mathcal{B}_i$ with $\overline{\delta}_i \subset \omega_i$. We claim we can now deduce that for any $x = (x_1,\ldots,x_n)$ in $\delta_1 \times \ldots \times \delta_n$

$$\overline{\int} v(z_1,\ldots,z_n)(\rho_{x_1}^{\delta_1} \times \ldots \rho_{x_n}^{\delta_n})(dz_1,\ldots,dz_n) \geq \overline{\int} v(z_1,\ldots,z_n)(\rho_{x_1}^{\omega_1} \times \ldots \times \rho_{x_n}^{\omega_n})(dz_1,\ldots,dz_n).$$

Indeed if v were identically ∞ on $\delta_1 \times \ldots \times \delta_n$ there would be nothing to prove since then $\hat{v}$ would be in n-S$(\Omega_1 \times \ldots \times \Omega_n)$ (Proposition 1.2.10) and identically ∞ on $\delta_1 \times \ldots \times \delta_n$ hence identically ∞ on $\Omega_1 \times \ldots \times \Omega_n$ (Proposition 1.2.5). If v is finite at some point of $\delta_1 \times \ldots \times \delta_n$ we conclude from

Proposition 2.3.5 and Theorem 2.3.6 that $\overline{\mathcal{H}}_v^{\omega_1 \times \ldots \times \omega_n}$ is in $n\text{-}H(\omega_1 \times \ldots \times \omega_n)$. (Recall v is locally lower bounded). For all $y = (y_1, \ldots, y_n)$ in $\partial\delta_1 \times \ldots \times \partial\delta_n$

$$\liminf_{\substack{z \to y \\ z \in \delta_1 \times \ldots \times \delta_n}} \overline{\mathcal{H}}_v^{\omega_1 \times \ldots \times \omega_n}(z) = \overline{\mathcal{H}}_v^{\omega_1 \times \ldots \times \omega_n}(y)$$

$$= \overline{\int} v(z_1, \ldots, z_n)(\rho_{y_1}^{\omega_1} \times \ldots \times \rho_{y_n}^{\omega_n})(dz_1, \ldots, dz_n)$$

$$\leq v(y).$$

Thus $\overline{\mathcal{H}}_v^{\omega_1 \times \ldots \times \omega_n}$ is in $L(v)$ for the Dirichlet problem on $\delta_1 \times \ldots \times \delta_n$. It follows

$$\overline{\mathcal{H}}_v^{\omega_1 \times \ldots \times \omega_n}(x) \leq \underline{\mathcal{H}}_v^{\delta_1 \times \ldots \times \delta_n}(x)$$

$$\leq \overline{\mathcal{H}}_v^{\delta_1 \times \ldots \times \delta_n}(x) .$$

An application of Theorem 2.3.6 now establishes the claim.

Section 4 n-Potentials and the Continuation Theorem

Let v be in $n\text{-}S^+(\Omega_1 \times \ldots \times \Omega_n)$ and for each integer i from 1 to n let $(\omega_{k,i})_{k \geq 1}$ be a sequence of relatively compact domains in Ω_i such that $\overline{\omega}_{k,i} \subset \omega_{k+1,i}$ for every k and $\bigcup_{k \geq 1} \omega_{k,i} = \Omega_i$. For such k define v_k on $\omega_{k,1} \times \ldots \times \omega_{k,n}$ by

$$v_k(x_1, \ldots, x_n) = \int \ldots \int v \; d\mu_{x_1}^{\omega_{k,1}} \ldots d\mu_{x_n}^{\omega_{k,n}} .$$

It follows from Theorem 2.3.6, as in Remark 1.2.6, that an n-superharmonic function is integrable with respect to the product of any n harmonic measures. Proposition 2.3.5 then implies v_k is in n-$H^+(\omega_{k,1} \times \ldots \times \omega_{k,n})$ and clearly v_k minorizes v on $\omega_{k,1} \times \ldots \times \omega_{k,n}$. Furthermore for each k we have from Remark 2.3.7 that $(v_\ell)_{\ell \geq k}$ is decreasing pointwise on $\omega_{k,1} \times \ldots \times \omega_{k,n}$. Thus for any x in $\omega_{k,1} \times \ldots \times \omega_{k,n}$ we can define w(x) to be the limit of the sequence $(v_\ell(x))_{\ell \geq k}$. Since k is arbitrary here we see w is well defined on all of $\Omega_1 \times \ldots \times \Omega_n$. From the Sheaf property and Proposition 1.2.2(b) we have that w is in n-$H^+(\Omega_1 \times \ldots \times \Omega_n)$ and it minorizes v at every point of $\Omega_1 \times \ldots \times \Omega_n$. We claim it is pointwise the greatest n-harmonic minorant of v. Indeed, if h is in n-$H(\Omega_1 \times \ldots \times \Omega_n)$ and minorizes v, then for every $k, x = (x_1, \ldots, x_n)$ in $\omega_{k,1} \times \ldots \times \omega_{k,n}$, and $k' > k$,

$$
\begin{aligned}
v_{k'}(x) &= \int \ldots \int v \, d\mu_{x_1}^{\omega_{k',1}} \ldots d\mu_{x_n}^{\omega_{k',n}} \\
&\geq \int \ldots \int h \, d\mu_{x_1}^{\omega_{k',1}} \ldots d\mu_{x_n}^{\omega_{k',n}} \\
&= h(x) .
\end{aligned}
$$

Taking the limit as $k \to \infty$ gives $w(x) \geq h(x)$ thus proving the claim.

<u>Definition 2.4.1:</u> We say the function p on $\Omega_1 \times \ldots \times \Omega_n$ is an n-potential if it is in n-$S^+(\Omega_1 \times \ldots \times \Omega_n)$ and its greatest n-harmonic minorant is identically 0.

Denote the set of all n-potentials by n-P.

We note first that n-P has positive members. Indeed if p_1 is a positive potential on Ω_1 and v is any positive member of $(n-1)-S^+(\Omega_2 \times \ldots \times \Omega_n)$ we can show the mapping $w: (x_1,\ldots,x_n) \to p_1(x_1)v(x_2,\ldots,x_n)$ is in n-P. Well certainly it is in $n\text{-}S^+(\Omega_1 \times \ldots \times \Omega_n)$ (page 23). Call its greatest n-harmonic minorant h. Since 0 is an n-harmonic minorant of w, h is non-negative. If x is any point in $\Omega_2 \times \ldots \times \Omega_n$ such that $v(x) < \infty$ the potential $v(x)\cdot p_1(\cdot)$ majorizes the non-negative harmonic function $h(\cdot,x)$ on Ω_1. Thus $h(\cdot,x)$ is identically 0. Remark 1.2.3(b) then implies h is identically 0 and w is in n-P.

From our explicit construction of greatest n-harmonic minorants of n-superharmonic functions it is clear that n-P is closed for finite sums, multiplication by positive constants, and finite pointwise infimum.

The following proposition generalizes a result of R.M. Hervé and will be used to prove the Continuation Theorem.

Theorem 2.4.2: Let K be a compact subset of $\Omega_1 \times \ldots \times \Omega_n$, ε any positive real number, and f a real valued continuous function on K. Then there exist Q and Q' real valued continuous members of n-P such that for all x in K

$$|Q(x) - Q'(x) - f(x)| < \varepsilon \quad .$$

If f is non-negative Q and Q' can be chosen so that $Q(x) \geq Q'(x)$ for all x in $\Omega_1 \times \ldots \times \Omega_n$.

Proof: Fix $Q_1,\ldots,Q_n$ finite continuous positive potentials on $\Omega_1,\ldots,\Omega_n$ respectively. Put

$$V = \{(P-P')/Q_1\cdot\ldots\cdot Q_n : P,P' \text{ real valued continuous members of n-P}\},$$

where $Q_1 \cdot \ldots \cdot Q_n$ denotes the function on $\Omega_1 \times \ldots \times \Omega_n$, $(x_1,\ldots,x_n) \to Q_1(x_1)\cdot \ldots \cdot Q_n(x_n)$. From our previous disucssion it is clear V is a vector space of continuous functions containing the constants. We wish to apply the vector space version of the Stone-Weierstrass Theorem. If $(P-P')/Q_1 \cdot \ldots \cdot Q_n$ is in V,

$$|(P-P')/Q_1 \cdot \ldots \cdot Q_n| = (P+P' - 2\min(P,P'))/Q_1 \cdot \ldots \cdot Q_n$$

which is also in V. It remains only to show V separates points of $\Omega_1 \times \ldots \times \Omega_n$. Let $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n)$ be distinct points of $\Omega_1 \times \ldots \times \Omega_n$. Without loss of generality we may assume x_1 and y_1 are not the same. We show first there exist continuous potentials p_1 and p_1' on Ω_1 such that $p_1(x_1) > p_1'(x_1)$ and $p_1(y_1) = p_1'(y_1)$

Let ω be a regular domain of Ω_1 containing x_1 and not containing y_1. There exists a continuous psoitive potential p_1 on Ω_1 such that p_1 is not harmonic on ω(Proposition 1.1.11). Put $p_1' = E^{\omega}_{p_1}$. Then $p_1(y_1) = p_1'(y_1)$. Since p_1 is superharmonic, $p_1(z) - p_1'(z) \geq 0$ for all z in ω. Since p_1' is harmonic on ω we have from Remark 1.2.6(b) that $p_1 - p_1'$ is either identically 0 or strictly positive on ω. It is not identically 0 because p_1 is not in $H_1(\omega)$. In particular $p_1(x_1) > p_1'(x_1)$ hence p_1 and p_1' are the required functions.

Now put $p = p_1 \cdot Q_2 \cdot \ldots \cdot Q_n$ and $p' = p_1' \cdot Q_2 \cdot \ldots \cdot Q_n$. Then p and p' are continuous members of n-P. Furthermore

$$(p(x) - p'(x))/Q_1(x_1)\cdot \ldots \cdot Q_n(x_n) = (p_1(x_1) - p_1'(x_1))/Q_1(x_1) > 0, \text{ and}$$

$$(p(y) - p'(y))/Q_1(x_1)\cdot \ldots \cdot Q_n(x_n) = (p_1(y_1) - p_1'(y_1))/Q_1(y_1) = 0.$$

Thus V does indeed separate the points of $\Omega_1 \times \ldots \times \Omega_n$. We may now apply the Stone-Weierstrass Theorem. Put

$$M = \sup \{Q_1(x_1) \cdot \ldots \cdot Q_n(x_n) : (x_1, \ldots, x_n) \in K\} \quad .$$

Then there exist Q and Q' real valued continuous functions in n-P such that for all $x = (x_1, \ldots, x_n)$ in K,

$$|((Q(x) - Q'(x))/Q_1(x_1) \cdot \ldots \cdot Q_n(x_n)) - (f(x)/Q_1(x_1) \cdot \ldots \cdot Q_n(x_n))| < \varepsilon/M.$$

Therefore

$$|Q(x) - Q'(x) - f(x)| < \varepsilon \; Q_1(x_1) \cdot \ldots \cdot Q_n(x_n)/M$$

$$\leq \varepsilon$$

and this proves the first part of the proposition.

Finally, if f is non-negative, put $\overline{Q} = Q + Q'$ and $\overline{Q}' = 2 \cdot \min(Q, Q')$. Then $\overline{Q}$ majorizes $\overline{Q}'$ on $\Omega_1 \times \ldots \times \Omega_n$ and for any x in K,

$$|\overline{Q}(x) - \overline{Q}'(x) - f(x)| = ||Q(x) - Q'(x)| - |f(x)||$$

$$\leq |Q(x) - Q'(x) - f(x)|$$

$$< \varepsilon \quad .$$

The proof is complete.

Following closely the method of R.M. Hervé we can prove the following result. (See [12] Lemma 13.1)

Theorem 2.4.3 (Continuation Theorem). Let U_1, U_2, U_3 be relatively compact domains in $\Omega_1 \times \ldots \times \Omega_n$ such that $U_1 \subset \overline{U}_1 \subset U_2 \subset \overline{U}_2 \subset U_3$ and let v be in $n\text{-}S^+(U_3)$ with v bounded on ∂U_2. Then there exist p,p' in $n\text{-}S^+(\Omega_1 \times \ldots \times \Omega_n)$

such that p' is a continuous member of n-P and $p(x) = p'(x) + v(x)$ for all x in U_1.

Proof: Choose U'_2 open and connected with $\overline{U}_2 \subset U'_2 \subset \overline{U}'_2 \subset U_3$. Let M be an upper bound for v on ∂U_2 and m a positive lower bound for v on $\partial U'_2$. (Note m exists since v is positive lower semicontinuous and $\partial U'_2$ is compact). Let ε be any positive real number smaller than m. Define f on the compact set $K = \partial U_2 \cup \partial U'_2$ by

$$f(x) \begin{cases} M+\varepsilon & x \text{ in } \partial U_2 \\ m-\varepsilon & x \text{ in } \partial U'_2 \end{cases} .$$

Then f is clearly continuous. Hence from the previous proposition there exist Q and p' real valued continuous members of n-P such that for all x in K

$$|Q(x) - p'(x) - f(x)| < \varepsilon \quad .$$

Thus for x in $\partial U'_2$ we have $Q(x) - p'(x) < f(x) + \varepsilon = m \leq v(x)$. That is

(1) $Q(x) < v(x) + p'(x)$ for x in $\partial U'_2$

For x in ∂U_2 we have $Q(x) - p'(x) > f(x) - \varepsilon = M \geq v(x)$. That is

(2) $Q(x) > v(x) + p'(x)$ for x in ∂U_2.

Define p on $\Omega_1 \times \ldots \times \Omega_n$ by

$$p(x) = \begin{cases} p'(x) + v(x) & x \text{ in } U_2 \\ \inf\,(p' + v(x), Q(x)) & x \text{ in } U'_2 - U_2 \\ Q(x) & x \text{ in } \Omega_1 \times \ldots \times \Omega_n - U'_2 \end{cases} .$$

It is clear from inequality (1) that p is lower semicontinuous. Thus we may use the local property (Proposition 1.2.11) to show p is in $n\text{-}S^+(\Omega_1 \times \ldots \times \Omega_n)$. For this it is clearly enough to consider only points in K. If $x = (x_1,\ldots,x_n)$ is in ∂U_2, choose $\delta_1,\ldots, \delta_n$ regular domains in $\Omega_1,\ldots, \Omega_n$ respectively such that $x \in \delta_1 \times \ldots \times \delta_n \subset \overline{\delta}_1 \times \ldots \times \overline{\delta}_n \subset U'_2$. Then

$$\int \ldots \int p \, d\rho_{x_1}^{\delta_1} \ldots d\rho_{x_n}^{\delta_n} \leq \int \ldots \int (p' + v) d\rho_{x_1}^{\delta_1} \ldots d\rho_{x_n}^{\delta_n}$$

$$\leq p'(x) + v'(x)$$

$$= p(x) \quad \text{(from (2))}.$$

Finally, if $x = (x_1,\ldots,x_n)$ is in $\partial U'_2$, choose $\omega_1,\ldots, \omega_n$ regular domains in $\Omega_1,\ldots,\Omega_n$ respectively such that $x \in \omega_1 \times \ldots \times \omega_n \subset \overline{\omega}_1 \times \ldots \times \overline{\omega}_n \subset \Omega_1 \times \ldots \times \Omega_n - \overline{U}_2$. Then

$$\int \ldots \int p \, d\rho_{x_1}^{\omega_1} \ldots d\rho_{x_n}^{\omega_n} \leq \int \ldots \int Q \, d\rho_{x_1}^{\omega_1} \ldots d\rho_{x_n}^{\omega_n}$$

$$\leq Q(x)$$

$$= p(x) \quad \text{(from (1))}.$$

Thus the local property holds and p is in $n\text{-}S^+(\Omega_1 \times \ldots \times \Omega_n)$. Clearly p and p' are the required functions and we are done.

Remark 2.4.4: The similar continuation theorem proved by R.M. Hervé for functions of one variable does not impose the boundedness restriction on v. A continuation theorem without any boundedness restriction in several variables would enable us to prove that a set which is locally n-polar is also globally n-polar. However, at present we are unable to remove this restriction and the validity of the local property remains a conjecture.

CHAPTER 3

Principal Results

In this chapter we prove results in which the associated exceptional sets are n-negligible. In section 1 we prove the major result of the thesis, Theorem 3.1.1. It is a generalisation of the well known Cartan-Brelot convergence theorem for a sequence of potentials (cf. Theorem 1.1.15(a).). We hope that since the Dirichlet problem cannot be solved on a general open set in $\Omega_1 \times \ldots \times \Omega_n$ this theorem can serve as a useful substitute. In section 2 we present an application of this result to the study of thin sets. In the last section we present a generalization of Proposition 1.1.13.

Section 1 The Convergence Theorem

We will demonstrate the following result which we henceforth refer to as the Convergence Theorem. Recall that it is necessary to assume Axiom D which gives the result in one variable.

Theorem 3.1.1: Let U be an open subset of $\Omega_1 \times \ldots \times \Omega_n$ and $(v_k)_{k\geq 1}$ a decreasing sequence of uniformly locally lower bounded functions in n-S(U) with limit function v. Then $\hat{v}$ is in n-S(U) and equals v everywhere except on an an n-negligible set.

We remark first that since v is the pointwise limit of a sequence of Borel measurable functions it too is Borel measurable. Since it is locally lower bounded and bounded above by v_1 we see v is $\mu_{x_1}^{\omega_1} \times \ldots \times \mu_{x_n}^{\omega_n}$ - integrable for any choice of relatively compact open sets $\omega_1, \ldots, \omega_n$ in $\Omega_1, \ldots, \Omega_n$ respectively and $x_1, \ldots, x_n$ in $\omega_1, \ldots, \omega_n$. Thus we may apply Fubini's Theorem to v and such a measure.

At this point we introduce the following useful notation. If f is an extended real valued function on a set G in $\Omega_1 \times \ldots \times \Omega_n$, for any integer i from 1 to n, $\hat{f}^{i}$ and $\hat{f}^{\bar{i}}$ are defined in G by

$$\hat{f}^{i}(x_1,\ldots,x_n) = \liminf_{\substack{(z_1,\ldots,z_n)\to(x_1,\ldots,x_n)\\ z_i = x_i}} f(z_1,\ldots,z_n)$$

$$\hat{f}^{\bar{i}}(x_1,\ldots,x_n) = \liminf_{z_i \to x_i} f(x_1,\ldots,x_{i-1},z_i,x_{i+1},\ldots,x_n).$$

Before proving the theorem we consider several preliminary results.

Lemma 3.1.2: Let $U_1,\ldots,U_n$ be domains in $\Omega_1,\ldots,\Omega_n$ respectively, $n \geq 2$, and $(w_k)_{k\geq 1}$ a pointwise decreasing sequence of locally bounded functions in $n\text{-}S^+(U_1 \times \ldots \times U_n)$ with limit function w. Suppose that for every positive integer k and $(x_1,\ldots,x_{n-1})$ in $U_1 \times \ldots \times U_{n-1}$ the mapping $x_n \to w_k(x_1,\ldots,x_n)$ is harmonic on U_n. Assume further that the Convergence Theorem holds on $U_1 \times \ldots \times U_{n-1}$ for pointwise decreasing sequences $(v_k)_{k\geq 1}$ in $(n-1)\text{-}S^+(U_1 \times \ldots \times U_{n-1})$ with v_1 locally bounded. (This will later be an induction assumption.) Then w equals $\hat{w}$ everywhere except on an n-negligible set contained in a set $N \times U_n$ where N is (n-1)-negligible in $\Omega_1 \times \ldots \times \Omega_n$. In addition $\hat{w} = \hat{w}^n$ on $U_1 \times \ldots \times U_n$.

Proof: Let $(x_1,\ldots,x_n)$ be any point of $U_1 \times \ldots \times U_n$ and for each integer i from 1 to n let $(\omega^i_\ell)_{\ell\geq 1}$ be a sequence of regular domains in Ω_i such that for each ℓ, $\bar{\omega}^i_{\ell+1} \subset \omega^i_\ell$ and $\bigcap_{\ell\geq 1} \omega^i_\ell = \{x_i\}$. Since $\hat{w}$ is nearly n-superharmonic on $U_1 \times \ldots \times U_n$ it follows from Proposition 1.2.10 that

$$(1) \qquad \hat{w}(x_1,\ldots,x_n) = \sup_{\ell} \int \ldots \int w \, d\rho^{\omega^1_\ell}_{x_1} \ldots d\rho^{\omega^n_\ell}_{x_n} \quad .$$

Now by assumption we have for each k and ℓ

$$\int \cdots \int w_k \, d\rho_{x_1}^{\omega_\ell^1} \cdots d\rho_{x_n}^{\omega_\ell^n} = \int \cdots \int w_k(z_1,\ldots,z_{n-1},z_n)\, d\rho_{x_1}^{\omega_\ell^1}(z_1) \cdots d\rho_{x_{n-1}}^{\omega_\ell^{n-1}}(z_{n-1})$$

thus the Monotone Convergence Theorem implies it is also true for w. Hence by (1)

$$(2) \qquad \hat{w}(x_1,\ldots,x_n) = \sup_\ell \int \cdots \int w(z_1,\ldots,z_{n-1},x_n)\, d\rho_{x_1}^{\omega_\ell^1}(z_1) \cdots d\rho_{x_{n-1}}^{\omega_\ell^{n-1}}(z_{n-1})$$

$$(3) \qquad = \hat{w}^n(x_1,\ldots,x_n) \quad \text{(Proposition 1.2.10).}$$

This proves the last assertion.

From Remark 2.3.7 the sequence

$\left(\int \cdots \int w(z_1,\ldots,z_{n-1},x_n)\, d\rho_{x_1}^{\omega_\ell^1}(z_1) \cdots d\rho_{x_{n-1}}^{\omega_\ell^{n-1}}(z_{n-1})\right)_{\ell \geq 1}$ is non-decreasing. Since for every $(z_1,\ldots,z_{n-1})$ the mapping $x_n \to w(z_1,\ldots,z_{n-1},x_n)$ is harmonic on U_n (Axiom 3) it follows that for every $(x_1,\ldots,x_{n-1})$ in $U_1 \times \cdots \times U_{n-1}$ and positive integer ℓ, $x_n \to \int \cdots \int w(z_1,\ldots,z_{n-1},x_n)$ $d\rho_{x_1}^{\omega_\ell^1}(z_1) \cdots d\rho_{x_{n-1}}^{\omega_\ell^{n-1}}(z_{n-1})$ is harmonic on U_n (Proposition 1.2.8) and by Axiom 3 we deduce from (2) and (3) that $x_n \to \hat{w}^n(x_1,\ldots,x_n)$ is a harmonic function on U_n.

We show now that the set E defined by

$$E = \{(x_1,\ldots,x_n) \in U_1 \times \cdots \times U_n : \hat{w}(x_1,\ldots,x_n) < w(x_1,\ldots,x_n)\}$$

$$= \{(x_1,\ldots,x_n) \in U_1 \times \cdots \times U_n : \hat{w}^n(x_1,\ldots,x_n) < w(x_1,\ldots,x_n)\}$$

is contained in a set of the form $N \times U_n$ where N is (n-1)-negligible in $\Omega_1 \times \cdots \times \Omega_{n-1}$. For x_n in U_n define

$$E(x_n) = \{(x_1,\ldots,x_{n-1}) \in U_1 \;\cdots\; U_{n-1} : \hat{w}^n(x_1,\ldots,x_n) < w(x_1,\ldots,x_n)\}.$$

Clearly this set is $(n-1)$-negligible by the assumption concerning the Convergence Theorem on $U_1 \times \ldots \times U_{n-1}$. Now fix any x'_n in U_n. We claim E is contained in $E(x_n') \times U_n$. For if $(x_1,\ldots,x_n)$ is in E, since $z_n \to \hat{w}^n(x_1,\ldots,x_{n-1},z_n)$ and $z_n \to w(x_1,\ldots,x_{n-1},z_n)$ are both harmonic on U_n and $\hat{w}^n(x_1,\ldots,x_n) < w(x_1,\ldots,x_n)$ it follows from Remark 1.2.3(b) that

$$(4) \qquad \hat{w}^n(x_1,\ldots,x_{n-1},z_n) < w(x_1,\ldots,x_{n-1},z_n) \quad \text{for all } z_n \text{ in } U_n.$$

In particular (4) holds for $z_n = x'_n$. Thus $(x_1,\ldots,x_{n-1})$ is in $E(x'_n)$ and the claim is proved. It follows from Proposition 2.2.7 that E is n-negligible and we are done.

Lemma 3.1.3: Let $(v_k)_{k\geq 1}$, v, and U be as in the Convergence Theorem with v assumed to be non-negative and v_1 locally bounded. Then $\hat{v}^n$ is Borel measurable on U.

Proof: Let α be any real number. We must show that $E = \{(x_1,\ldots,x_n) \in U : \hat{v}^n(x_1,\ldots,x_n) \geq \alpha\}$ is a Borel set. Let $(\omega_\ell)_{\ell\geq 1}$ be a countable base of open sets of $\Pi_{1,\ldots,n-1}(U)$ consisting of relatively compact open sets. Then it is easy to see that

$$E = \bigcap_{m=1}^{\infty} \bigcup_{\ell=1}^{\infty} \{(x_1,\ldots,x_n) \in U : (x_1,\ldots,x_{n-1}) \in \omega_\ell, v(z,x_n) \geq \alpha - \frac{1}{m} \text{ for all } z \text{ in } \overline{\omega_\ell}\}.$$

Indeed if $x = (x_1,\ldots,x_n)$ is in E, then for all positive integers m $\hat{v}^n(x) > \alpha - (1/m)$ and hence there is an ℓ with $(x_1,\ldots,x_{n-1})$ in ω_ℓ and $v(z,x_n) \geq \alpha - (1/m)$ for all z in $\overline{\omega}_\ell$. Conversely if x is in the right hand side, then for all positive integers m there is an ℓ such that $(x_1,\ldots,x_{n-1})$ is in ω_ℓ and $v(z,x_n) \geq \alpha - (1/2m) > \alpha - (1/m)$ on $\overline{\omega}_\ell$. Thus $\hat{v}^n(x) \geq \alpha - (1/m)$.

Since m is arbitrary we have x is in E and the claim is proved. It thus suffices to show that for any relatively compact open set ω in $\Pi_{1,\ldots,n-1}(U)$ and any real number β,

$$\{(x_1,\ldots,x_n) \in U:(x_1,\ldots,x_{n-1}) \in \omega,\ v(z,x_n) \geq \beta \text{ for all } z \text{ in } \overline{\omega}\}$$

is Borel. But this set is just

$$\omega \times \{x_n \in \Pi_n(U):\ v(z,x_n) \geq \beta \text{ for all } z \text{ in } \overline{\omega}\}.$$

Thus we need consider only the last set in this product. For that set we have

$$\{x_n \in \Pi_n(U):v(z,x_n) \geq \beta \text{ for all } z \text{ in } \overline{\omega}\} = \bigcap_{k,\ell=1}^{\infty} \{x_n \in \Pi_n(U):v_k(z,x_n) > \beta - \frac{1}{\ell} \text{ for all } z \text{ in } \overline{\omega}\}.$$

Therefore we will be done once we show that for any k and any real number γ,

$$G = \{x_n \in \Pi_n(U):v_k(z,x_n) > \gamma \quad \text{for all } z \text{ in } \overline{\omega}\}$$

is open. Indeed let x_n' be a point of G. Since v_k is lower semicontinuous, for every z in $\overline{\omega}$ there exists δ_z a neighbourhood of z and δ_z' a neighbourhood of x_n' such that

(1) $v_k(z',z'') > \gamma$ for all z' in δ_z and z'' in δ_z'.

Now $\{\delta_z:z \in \overline{\omega}\}$ is an open cover of $\overline{\omega}$ thus there exists a finite subcover $\{\delta_{z_1},\ldots,\delta_{z_\ell}\}$. Put $\delta' = \delta'_{z_1} \cap \ldots \cap \delta'_{z_\ell}$. Then δ' is a neighbourhood of x_n' and for any x_n in δ' and z in $\overline{\omega}$, since (z,x_n) is in $\delta_{z_i} \times \delta'_{z_i}$ for some i, we have from (1) that $v_k(z,x_n) > \gamma$. Thus G is indeed open and we are done.

Lemma 3.1.4: Let $\omega_1, \ldots, \omega_n$ be relatively compact domains in $\Omega_1, \ldots, \Omega_n$ respectively and v a non-negative locally bounded n-superharmonic function defined on a neighbourhood of $\bar{\omega}_1 \times \ldots \times \bar{\omega}_n$. Then the mapping

$$w: (x_1, \ldots, x_n) \rightarrow \int \ldots \int v \, d\mu_{x_1}^{\omega_1} \ldots d\mu_{x_n}^{\omega_n}$$

is the greatest n-harmonic minorant of v on $\omega_1 \times \ldots \times \omega_n$.

Proof: The proof is by induction on n. If $n = 1$ this is Axiom D. Suppose then $n > 1$ and the lemma holds for smaller integers. We have seen that w is in $n\text{-}H^+(\omega_1 \times \ldots \times \omega_n)$ (Theorem 2.3.6) and since v is n-superharmonic on a neighbourhood of $\bar{\omega}_1 \times \ldots \times \bar{\omega}_n$, w minorizes v at each point of $\omega_1 \times \ldots \times \omega_n$. Now let u be an n-harmonic minorant of v on $\omega_1 \times \ldots \times \omega_n$. For any x_n in ω_n the induction hypothesis implies that the greatest (n-1)-harmonic minorant of $(x_1, \ldots, x_{n-1}) \rightarrow v(x_1, \ldots, x_n)$ on $\omega_1 \times \ldots \times \omega_{n-1}$ is $x_n \rightarrow \int \ldots \int v(z_1, \ldots, z_{n-1}, x_n) d\mu_{x_1}^{\omega_1}(z_1) \ldots d\mu_{x_{n-1}}^{\omega_{n-1}}(z_{n-1})$. Thus

$$u(x_1, \ldots, x_n) \leq \int \ldots \int v(z_1, \ldots, z_{n-1}, x_n) d\mu_{x_1}^{\omega_1}(z) \ldots d\mu_{x_{n-1}}^{\omega_{n-1}}(z_{n-1})$$

on $\omega_1 \times \ldots \times \omega_n$.

If $(\delta_k)_{k\geq 1}$ is a sequence of relatively compact subsets of ω_n with $\bar{\delta}_k \subset \delta_{k+1}$ for each k and $\bigcup_{k\geq 1} \delta_k = \omega_n$, then for all $(x_1, \ldots, x_n)$ in $\omega_1 \times \ldots \times \omega_n$ and all sufficiently large k,

$$\text{(1)} \quad u(x_1, \ldots, x_n) = \int u(x_1, \ldots, x_{n-1}, z_n) d\mu_{x_n}^{\delta_k}(z_n)$$
$$\leq \int \ldots \int v(z_1, \ldots, z_n) d\mu_{x_1}^{\omega_1}(z_1) \ldots d\mu_{x_{n-1}}^{\omega_{n-1}}(z_{n-1}) d\mu_{x_n}^{\delta_k}(z_n).$$

Let $k \rightarrow \infty$. Since for each $(x_1, \ldots, x_{n-1})$ in $\omega_1 \times \ldots \times \omega_{n-1}$ the mapping

$$g: z_n \to \int \dots \int v(z_1,\dots,z_n)\, d\mu_{x_1}^{\omega_1}(z_1)\dots d\mu_{x_{n-1}}^{\omega_{n-1}}(z_{n-1})$$

is locally bounded and in $S_n^+(\omega_n)$ (same proof as Proposition 1.2.8), the last expression in (1) converges to the greatest harmonic minorant of g on ω_n evaluated at x_n. Axiom D implies this is just the mapping $x_n \to \int g\, d\mu_{x_n}^{\omega} = \int \dots \int v\, d\mu_{x_1}^{\omega_1}\dots d\mu_{x_n}^{\omega_n}$. Thus (1) gives

$$u(x_1,\dots,x_n) \le \int \dots \int v\, d\mu_{x_1}^{\omega_1}\dots d\mu_{x_n}^{\omega_n} \quad \text{on } \omega_1 \times \dots \times \omega_n$$
$$= w(x_1,\dots,x_n).$$

This completes the proof.

<u>Corollary 3.1.5</u>: Let $\omega_1,\dots, \omega_n$ be relatively compact domains in $\Omega_1,\dots, \Omega_n$ respectively and $(v_k)_{k\geq 1}$, v, and U as in the Convergence Theorem with $\overline{\omega}_1 \times \dots \times \overline{\omega}_n \subset U$, v assumed to be non-negative, and v_1 assumed to be locally bounded. Then for all $(x_1,\dots,x_n)$ in $\omega_1 \times \dots \times \omega_n$,

$$(1) \qquad \int \dots \int v\, d\mu_{x_1}^{\omega_1} \dots d\mu_{x_n}^{\omega_n} = \int \dots \int \hat{v}\, d\mu_{x_1}^{\omega_1} \dots d\mu_{x_n}^{\omega_n}$$

and hence for each integer i from 1 to n,

$$(2) \qquad \int \dots \int v\, d\mu_{x_1}^{\omega_1} \dots d\mu_{x_n}^{\omega_n} = \int \dots \int \hat{v}^i\, d\mu_{x_1}^{\omega_1} \dots d\mu_{x_n}^{\omega_n}.$$

<u>Proof:</u> For every positive integer k and $(x_1,\dots,x_n)$ in $\omega_1 \times \dots \times \omega_n$ we have

$$\int \dots \int v_k\, d\mu_{x_1}^{\omega_1} \dots d\mu_{x_n}^{\omega_n} \le v_k(x_1,\dots,x_n).$$

Letting $k \to \infty$ and applying the Monotone Convergence Theorem we get a similar result for v. Furthermore Theorem 2.3.6 implies

$$g: (x_1,\dots,x_n) \to \int \dots \int v\, d\mu_{x_1}^{\omega_1} \dots d\mu_{x_n}^{\omega_n}$$

is in $n\text{-}H^+(\omega_1 \times \ldots \times \omega_n)$. In particular it is continuous and hence g minorizes $\hat{v}$ on $\omega_1 \times \ldots \times \omega_n$. Now $\hat{v}$ is locally bounded and n-superharmonic on U (Proposition 1.2.10 and Proposition 1.2.11). Thus g minorizes the greatest n-harmonic minorant of $\hat{v}$ on $\omega_1 \times \ldots \times \omega_n$. Lemma 3.1.4 gives us that for all $(x_1,\ldots,x_n)$ on $\omega_1 \times \ldots \times \omega_n$,

$$\int \ldots \int v \, d\mu_{x_1}^{\omega_1} \ldots d\mu_{x_n}^{\omega_n} \leq \int \ldots \int \hat{v} \, d\mu_{x_1}^{\omega_1} \ldots d\mu_{x_n}^{\omega_n}.$$

The reverse inequality being obvious we have equality and equation (1) does indeed hold. Finally (2) follows since for each i and x in $\omega_1 \times \ldots \times \omega_n$, $\hat{v}(x) \leq \hat{v}^i(x) \leq v(x)$.

Lemma 3.1.6: Let $(v_k)_{k\geq 1}$, v, and U be as in the Convergence Theorem with v assumed to be non-negative and v_1 assumed to be locally bounded. Let $U_1,\ldots,U_n$ be relatively compact domains in $\Omega_1,\ldots,\Omega_n$ respectively such that $\overline{U}_1 \times \ldots \times \overline{U}_n \subset U$. Then

$$\{(x_1,\ldots,x_n) \in U_1 \times \ldots \times U_n : \hat{v}(x_1,\ldots,x_n) < v(x_1,\ldots,x_n)\}$$

is n-negligible.

Proof: The proof is by induction on n. For $n = 1$ the result follows from Theorem 1.1.15. Now assume $n > 1$ and the lemma holds for positive integers smaller than n. Let δ_n be a regular domain in Ω_n with $\overline{\delta}_n \subset U_n$. Define

$$F = \{(x_1,\ldots,x_n) \in U_1 \times \ldots \times U_{n-1} \times \delta_n : \int \hat{v}^n (x_1,\ldots,x_{n-1},z_n) d\rho_{x_n}^{\delta_n}(z_n)$$

$$< \int v(x_1,\ldots,x_{n-1},z_n) d\rho_{x_n}^{\delta_n}(z_n)\} .$$

We first show F is n-negligible.

Define for each positive integer k and $(x_1,\ldots,x_n)$ in $U_1 \times \ldots \times U_{n-1} \times \delta_n$,

$$w_k(x_1,\ldots,x_n) = \int v_k(x_1,\ldots,x_{n-1},z_n) d\rho_{x_n}^{\delta_n}(z_n) \quad \text{and}$$

$$w(x_1,\ldots,x_n) = \int v(x_1,\ldots,x_{n-1},z_n) \; d\rho_{x_n}^{\delta_n}(z_n) .$$

Then $(w_k)_{k\geq 1}$ is a pointwise decreasing sequence of locally bounded functions in $n\text{-}S^+(U_1\times\ldots\times U_{n-1}\times\delta_n)$ (Proposition 1.2.8, Proposition 1.1.1(b), and Proposition 1.2.5(c)) with limit function w(Monotone Convergence Theorem) such that for every k and $(x_1,\ldots,x_{n-1})$ in $U_1\times\ldots\times U_{n-1}$ the mapping $x_n\to w_k(x_1,\ldots,x_n)$ is harmonic on δ_n (Proposition 1.1.1(b)). Furthermore our induction hypothesis implies the Convergence Theorem holds on $U_1\times\ldots\times U_{n-1}$ for pointwise decreasing sequences in $(n-1)\text{-}S^+(U_1\times\ldots\times U_{n-1})$ with first member locally bounded. Thus we may apply Lemma 3.1.2 and deduce $w=\hat{w}$ except on an n-negligible subset of $U_1\times\ldots\times U_{n-1}\times\delta_n$. In order to show F is n-negligible it therefore suffices to prove that on $U_1\times\ldots\times U_{n-1}\times\delta_n$

$$(1)\qquad \hat{w}(x_1,\ldots,x_n)=\int \hat{v}^n(x_1,\ldots,x_{n-1},z_n)\,d\rho_{x_n}^{\delta}(z_n).$$

Let $(x_1,\ldots,x_n)$ be any point in $U_1\times\ldots\times U_{n-1}\times\delta_n$ and for each integer i from 1 to n-1 let $(\omega_\ell^i)_{\ell\geq 1}$ be a sequence of regular domains with $\bar{\omega}_{\ell+1}^i\subset\omega_\ell^i\subset U_i$ for every ℓ and $\bigcap_{\ell\geq 1}\omega_\ell^i=\{x_i\}$. Again by Lemma 3.1.2

$$\hat{w}(x_1,\ldots,x_n)=\hat{w}^n(x_1,\ldots,x_n)$$

$$=\sup_\ell \int\ldots\int w(z_1,\ldots,z_{n-1},x_n)\,d\rho_{x_1}^{\omega_\ell^1}(z_1)\ldots d\rho_{x_{n-1}}^{\omega_\ell^{n-1}}(z_{n-1})$$

(Propositon 1.2.10)

$$=\sup_\ell \int\ldots\int v\,d\rho_{x_n}^{\delta_n}\,d\rho_{x_1}^{\omega_\ell^1}(z_1)\ldots d\rho_{x_{n-1}}^{\omega_\ell^{n-1}}(z_{n-1})$$

$$=\sup_\ell \int\ldots\int \hat{v}^n(z_1,\ldots,z_{n-1},z_n)\,d\rho_{x_n}^{\delta_n}(z_n)\,d\rho_{x_1}^{\omega_\ell^1}(z_1)\ldots d\rho_{x_n}^{\omega_\ell^{n-1}}(z_n)$$

(Corollary 3.1.5)

$$=\int \hat{v}^n(x_1,\ldots,x_{n-1},z_n)\,d\rho_{x_n}^{\delta_n}(z_n)\ .$$

(This last equality holds for the following reason: from Lemma 3.1.3, Proposition 1. 2. 8, and Proposition 1.2.10 we see the mapping $(u_1,\ldots,u_{n-1}) \to \int \hat{v}^n (u_1,\ldots,u_{n-1},z_n) d\rho^{\delta_n}_{x_n}(z_n)$ is in $(n-1)\text{-}S^+(U_1 \times \ldots \times U_{n-1})$. Now by again applying Proposition 1.2.10 the result follows). We therefore have verified (1) and have hence shown F is n-negligible.

Now for each integer i from 1 to n let $\mathcal{B}_i$ be a countable base of open sets of U_i consisting of regular domains. Define the sets G_i by

$$G_i = \bigcup_{\omega \in \mathcal{B}_i} \{ (x_1,\ldots,x_n) \in U_1 \times \ldots \times U_{i-1} \times \omega \times U_{i+1} \times \ldots \times U_n : \int \hat{v}^i (x_1,\ldots,x_{i-1},z_i,x_{i+1},\ldots,x_n) d\rho^{\omega}_{x_i}(z_i) < \int v(x_1,\ldots,x_{i-1},z_i,x_{i+1},\ldots,x_n) d\rho^{\omega}_{x_i}(z_i) \} .$$

From what we have just seen and the fact that a countable union of n-negligible sets is n-negligible, G_i is n-negligible.

Define the set K by

$$K = \bigcap_{i=1}^{\infty} \{(x_1,\ldots,x_n) \in U_1 \times \ldots \times U_n : \hat{v}^i (x_1,\ldots,x_n) < v(x_1,\ldots,x_n)\} .$$

We claim K is n-neglgible. Indeed by the symmetric nature of K it is enough to show that there is an (n-1)-negligible set N in $U_1 \times \ldots \times U_{n-1}$ such that if $(x_1,\ldots,x_{n-1})$ is in $U_1 \times \ldots \times U_{n-1} - N$, $P(x_1,\ldots,x_{n-1}) = \{x_n \times U_n : (x_1,\ldots,x_n) \in K\}$ is polar. Well just take N to be the empty set. For if $(x_1,\ldots,x_{n-1})$ is any point in $U_1 \times \ldots \times U_{n-1}$, $P(x_1,\ldots,x_{n-1})$ is polar by the Convergence Theorem on U_1.

Thus $E = K \cup \bigcup_{i=1}^{n} G_i$ is n-negligible and we will be done if we can show that for $(x_1,\ldots,x_n)$ in U-E, $\hat{v}(x_1,\ldots,x_n) = v(x_1,\ldots,x_n)$.

Indeed let $(x_1,\ldots,x_n)$ be a fixed point in U-E. Without loss of generality we may assume

$$(2) \qquad \hat{v}^{\bar{n}}(x_1,\ldots,x_n) = v(x_1,\ldots,x_n).$$

We can find, for each integer i from 1 to n, sequences $(\omega^i_{\ell_i})_{\ell_i \geq 1}$ in $\mathcal{B}_i$ such that for each ℓ_i, $\bar{\omega}^i_{\ell_i+1} \subset \omega^i_{\ell_i}$ and $\bigcap_{\ell_i \geq 1} \omega^i_{\ell_i} = \{x_i\}$. Now for each positive integer k the n-time indexed sequence $(\int\ldots\int v_k \, d\rho_{x_1}^{\omega^1_{\ell_1}} \ldots d\rho_{x_n}^{\omega^n_{\ell_n}})_{\ell_1,\ldots,\ell_n}$ increases in each ℓ_i if all other indices ℓ_j, $j \neq i$, are fixed (Proposition 1.1.6). The Monotone Convergence Theorem then gives the same result for $(\int\ldots\int v \, d\rho_{x_1}^{\omega_{\ell_1}} \ldots d\rho_{x_n}^{\omega_{\ell_n}})_{\ell_1,\ldots,\ell_n}$. Therefore

$$\hat{v}(x_1,\ldots,x_n) = \sup_{\ell_1,\ldots,\ell_n} \int\ldots\int v \, d\rho_{x_1}^{\omega^1_{\ell_1}} \ldots d\rho_{x_n}^{\omega^n_{\ell_n}} \qquad \text{(Proposition 1.2.10)}$$

$$= \sup_{\ell_n} \left(\sup_{\ell_1,\ldots,\ell_{n-1}} \int\ldots\int v \, d\rho_{x_1}^{\omega^1_{\ell_1}} \ldots d\rho_{x_n}^{\omega^n_{\ell_n}} \right)$$

$$= \sup_{\ell_n} \int d\rho_{x_n}^{\omega^n_{\ell_n}}(z_n) \left(\sup_{\ell_1,\ldots,\ell_{n-1}} \int\ldots\int v(z_1,\ldots,z_n) d\rho_{x_1}^{\omega^1_{\ell_1}}(z_1)\ldots d\rho_{x_{n-1}}^{\omega^{n-1}_{\ell_{n-1}}}(z_{n-1})\right)$$

(from the Monotone Convergence Theorem and Fubini's Theorem)

$$= \sup_{\ell_n} \int \hat{v}^n(x_1,\ldots,x_{n-1},z_n) d\rho_{x_n}^{\omega^n_{\ell_n}}(z_n) \qquad \text{(Proposition 1.2.10)}$$

$$\geq \sup_{\ell_n} \int v(x_1,\ldots,x_{n-1},z_n) d\rho_{x_n}^{\omega^n_{\ell_n}}(z_n)$$

(since $\omega^n_{\ell_n}$ is in $\mathcal{B}_n$ and $(x_1,\ldots,x_n)$ is not in G_n)

$$= \hat{v}^{\bar{n}}(x_1,\ldots,x_n) \qquad \text{(Proposition 1.1.6)}$$

$$= v(x_1,\ldots,x_n) \qquad \text{(equation (2))}.$$

This completes the proof.

Proof of the Convergence Theorem: Define E to be

$$E = \{x \in U : \hat{v}(x) < v(x)\}.$$

We must show E is n-negligible. Let us suppose first that v_1 is locally bounded above. Let $U_1,\ldots,U_n$ be relatively compact open sets in $\Omega_1,\ldots,\Omega_n$ respectively with $\overline{U}_1 \times \ldots \times \overline{U}_n \subset U$. Choose u to be a function n-harmonic on a neighbourhood of $\overline{U}_1 \times \ldots \times \overline{U}_n$ such that $v(x) + u(x)$ is positive for every x in $U_1 \times \ldots \times U_n$. Then we may apply Lemma 3.1.6 to the sequence $(v_k + u)_{k \geq 1}$ to deduce

$$E \cap U_1 \times \ldots \times U_n = \{(x_1,\ldots,x_n) \in U_1 \times \ldots \times U_n : \hat{v}(x_1,\ldots,x_n) < v(x_1,\ldots,x_n)\}$$

$$= \{(x_1,\ldots,x_n) \in U_1 \times \ldots \times U_n : \widehat{v+u}(x_1,\ldots,x_n) < v(x_1,\ldots,x_n) + u(x_1,\ldots,x_n)\}$$

is n-negligible. Now by applying the local property for n-negligible sets we see E is n-negligible. Thus the theorem holds if v_1 is locally bounded.

In general let v_o be a positive continuous member of $n\text{-}S^+(U)$. For each pair of positive integers k and ℓ we define $w_{k,\ell}$ on U to be

$$w_{k,\ell}(x) = \min(v_k(x), \ell\, v_o(x)).$$

Then for each ℓ the sequence $(w_{k,\ell})_{k\geq 1}$ is in n-S(U), it is uniformly locally lower bounded, it decreases pointwise to a function we call w_ℓ, and $w_{1,\ell}$ is locally bounded above. By the special case of the theorem which we have just verified, $w_\ell = \hat{w}_\ell$ except on an n-negligible subset E_ℓ of U. Put $E = \bigcup_{\ell\geq 1} E_\ell$. Then E is n-negligible (Proposition 2.2.6). It is clear that for all x in U, $w_{k,\ell+1}(x) \geq w_{k,\ell}(x)$ hence $w_{\ell+1}(x) \geq w_\ell(x)$, that is $(w_\ell)_{\ell\geq 1}$ is a pointwise increasing sequence of functions. Notice also that for each x in U, $v(x) = \lim_{\ell\to\infty} w_\ell(x)$. Indeed if $v(x) < \infty$, then for all ℓ sufficiently large $v(x) = w_\ell(x)$ hence $v(x) = \lim_{\ell\to\infty} w_\ell(x)$. If $v(x) = \infty$, $w_{k,\ell}(x) = \ell\, v_o(x)$ for all ℓ and k hence $w_\ell(x) = \ell\, v_o(x)$. Letting $\ell \to \infty$, gives $\lim_{\ell\to\infty} w_\ell(x) = \infty = v(x)$. Denote $\lim_{\ell\to\infty} \hat{w}_\ell$ by w. Then w is the limit of an increasing sequence in n-S(U), it is not identically ∞ on a connected component of U (since it is bounded by v_1), therefore it is in n-S(U). It minorizes v on U hence since it is lower semicontinuous it minorizes $\hat{v}$ on U. It follows that if x is in U-E,

$$\begin{aligned} v(x) &= \lim_{\ell\to\infty} w_\ell(x) \\ &= \lim_{\ell\to\infty} \hat{w}_\ell(x) \\ &= w(x) \\ &\leq \hat{v}(x) \\ &\leq v(x) \end{aligned}$$

This completes the proof.

Now by using the topological lemma of Choquet ([1] page 3), we deduce easily the more general form of the convergence result, Theorem 3.1.7: Let $(v_i)_{i \varepsilon I}$ be any family of locally uniformly lower bounded n-superharmonic functions on an open set and let v be the pointwise lower envelope of this family. Then $v = \hat{v}$ except on an n-negligible set.

Corollary 3.1.8: Let v be in $n\text{-}S^+(\Omega_1 \times \ldots \times \Omega_n)$ and let E be any subset of $\Omega_1 \times \ldots \times \Omega_n$. Then for all x except in an n-negligible set

$$R_v^E(x) = \hat{R}_v^E(x) .$$

Using the fact that a countable union of n-negligible sets is n-negligible we also deduce the following result.

Corollary 3.1.9: Let U be open in $\Omega_1 \times \ldots \times \Omega_n$ and $(v_k)_{k \geq 1}$ a sequence of uniformly locally lower bounded functions in n-S(U). Then for all x of U except an n-negligible subset

$$\liminf_{k \to \infty} v_k(x) = \widehat{\liminf_{k \to \infty}} \, v_k(x) .$$

Section 2 Thin Sets

Definition 3.2.1: Let E be contained in U an open subset of $\Omega_1 \times \ldots \times \Omega_n$, and let x be any point in U. We will say E is thin at x in U if one of the following three properties holds:

(1) x is not in $\overline{E}$,

(2) x is in $\overline{E} - E$ and there exists w in $n\text{-}S^+(U)$ with $\liminf_{\substack{y \to x \\ y \varepsilon E}} w(y) > w(x)$,

(3) x is in E, {x} is n-polar, and (1) or (2) holds for x and E-{x} .

In (3) above nothing is lost in taking x to be n-polar as opposed to n-negligible. We demonstrate this in the following simple result.

Proposition 3.2.2: A point set $\{x\} = \{(x_1,\ldots,x_n)\}$ is n-polar if and only if it is n-negligible.

Proof: Since n-polar sets are n-negligible (Proposition 2.2.12) there is only one assertion to prove and this we do by induction on n. If n = 1 there is nothing to show. Suppose then n > 1 and the proposition holds for integers smaller than n. Let $\{x\} = \{(x_1,\ldots,x_n)\}$ be n-negligible. Without loss of generality we may assume x_1 is not polar in Ω_1 for if it were, $\{x\} = \{x_1\}\times\{(x_2,\ldots,x_n)\}$ would be n-polar. (Proposition 2.1.2). Now since $\{x\}$ is n-negligible there is a polar set P in Ω_1 such that if y_1 is in $\Omega_1 - P$, $\{(y_2,\ldots,y_n)\} \in \Omega_2 \times\ldots\times\Omega_n : (y_1,\ldots,y_n) = (x_1,\ldots,x_n)\}$ is (n-1)-negligible. Since $\{x_1\}$ is not polar x_1 can not be in P. Hence $\{(y_2,\ldots,y_n)\} \in \Omega_2\times \ldots\times \Omega_n : (y_1,\ldots,y_n) = \{(x_1,\ldots,x_n)\} = \{(x_2,\ldots,x_n)\}$. Thus $(x_2,\ldots,x_n)$ is (n-1)-negligible and hence by the inductive hypothesis is n-polar. It follows $\{(x)\} = \{x_1,\ldots,x_n)\}$ is n-polar.

We wish now to apply the Continuation Theorem and deduce a local property of thinness.

Lemma 3.2.3: Let f and g be positive extended real valued functions defined on a Hausdorff space X, E a subset of X, and x a point in $\overline{E} - E$. Then if h is defined in X as $h(y) = \min(f(y),g(y))$, we have

$$\text{(1)} \qquad \liminf_{\substack{y\to x\\ y\in E}} h(y) = \min\Big(\liminf_{\substack{y\to x\\ y\in E}} f(y),\ \liminf_{\substack{y\to x\\ y\in E}} g(y) \Big).$$

Proof: Clearly the left hand side of (1) is smaller than or equal to the right hand side. If $\lambda_1,\lambda_2,\lambda_3$ are real numbers with $\lambda_1 > \lambda_2 > \lambda_3$ and the right hand side of (1) is bigger than λ_1, then there exist open sets W_1,W_2 such that $f(y) \geq \lambda_1$ for y in $W_1 \cap E$ and $g(y) \geq \lambda_1$ for y in $W_2 \cap E$. Thus $h(y) > \lambda_2$ on $W_1 \cap W_2 \cap E$ and hence

$$\liminf_{\substack{y \to x \\ y \in E}} h(y) \geq \lambda_2 > \lambda_3 .$$

This proves the result.

Lemma 3.2.4: Let E be thin at the point x in the open set U with x in $\overline{E}$. Then there exists w in $n\text{-}S^+(U)$ such that w is locally bounded and

$$\liminf_{\substack{y \to x \\ y \in E-\{x\}}} w(y) > w(x)$$

Proof: By definition there exists v in $n\text{-}S^+(U)$ such that

$$\liminf_{\substack{y \to x \\ y \in E-\{x\}}} v(y) > v(x).$$

Let u be a continuous member of $n\text{-}S^+(U)$ with u(x) lying strictly between these two numbers. Define w on U by w(y) = min(u(y),v(y)). Then w is in $n\text{-}S^+(U)$, it is locally bounded, and from Lemma 3.2.3

$$\begin{aligned}\liminf_{\substack{y \to x \\ y \in E-\{x\}}} w(y) &= \min\Big(\liminf_{\substack{y \to x \\ y \in E-\{x\}}} v(y), \liminf_{\substack{y \to x \\ y \in E-\{x\}}} u(y)\Big) \\ &= u(x) \\ &> v(x) \\ &= w(x)\end{aligned}$$

completing the proof.

Proposition 3.2.5 (local property of thinness) Let E be a subset of $\Omega_1 \times \ldots \times \Omega_n$ and x in $\overline{E}$. If for some neighbourhood ω of x we have $\omega \cap E$ thin at x in ω, then E is thin at x in $\Omega_1 \times \ldots \times \Omega_n$.

Proof: Let U_1, U_2 be relatively compact neighbourhoods of x with $\overline{U}_1 \subset U_2 \subset \overline{U}_2 \subset \omega$. From the previous lemma there exists w in $n\text{-}S^+(\omega)$ such that w is bounded on $\overline{U}_2$ and

$$\liminf_{\substack{y \to x \\ y \in E-\{x\}}} w(y) > w(x).$$

From Theorem 2.4.3 there exist p_1, p_2 in $n\text{-}S^+(\Omega_1 \times \dots \times \Omega_n)$ such that p_2 is continuous and $p_1(y) = p_2(y) + w(y)$ for all y in U_1. Therefore

$$\liminf_{\substack{y \to x \\ y \in E-\{x\}}} p_1(y) = \liminf_{\substack{y \to x \\ y \in E-\{x\}}} (p_2(y) + w(y))$$

$$= p_2(x) + \liminf_{\substack{y \to x \\ y \in E-\{x\}}} w(y)$$

$$> p_2(x) + w(x)$$

$$= p_1(x) .$$

This completes the proof.

From now on we will just refer to a set as being thin at a point without reference to any open set.

Using tools as in [1] and the Convergence Theorem we show below in Proposition 3.2.7 that the set of points in a given set at which the set is thin is n-negligible.

<u>Proposition 3.2.6:</u> Let E be a subset of $\Omega_1 \times \ldots \times \Omega_n$ and x a point in $\Omega_1 \times \ldots \times \Omega_n - E$. Then E is thin at x if and only if for each positive continuous v in $n\text{-}S^+(\Omega_1 \times \ldots \times \Omega_n)$ there exists a neighbourhood ω of x such that $R_v^{E \cap \omega}(x) < v(x)$.

<u>Proof:</u> Suppose first E is thin at x. If x is not in $\overline{E}$ we can find ω a neighbourhood of x such that $\omega \cap E = \phi$. Hence $R_v^{E \cap \omega}(x) = 0 < v(x)$. Thus we may assume x is in $\overline{E}$. There exists w in $n\text{-}S^+(\Omega_1 \times \ldots \times \Omega_n)$ such that

$$\liminf_{\substack{y \to x \\ y \in E}} w(x) > w(x) .$$

Choose a real number λ so that $\lambda v(x)$ lies strictly between both members of this inequality. Since v is continuous

$$\liminf_{\substack{y \to x \\ y \in E}} (w(y) - \lambda v(y)) = (\liminf_{\substack{y \to x \\ y \in E}} w(y)) - \lambda v(x)$$

$$> 0$$

Therefore there exists a neighbourhood ω of x such that for y in $\omega \cap E$, $w(y) - \lambda v(y) > 0$, hence for all y in $\Omega_1 \times \ldots \times \Omega_n$, $w(y) \geq R_{\lambda v}^{E \cap \omega}(x)$. It follows

$$\begin{aligned} \lambda v(x) &> w(x) \\ &\geq R_{\lambda v}^{E \cap \omega}(x) \\ &= \lambda R_v^{E \cap \omega}(x). \end{aligned}$$

Since λ is positive the result follows

Conversely let v be a positive continuous member of $n\text{-}S^+(\Omega_1 \times \ldots \times \Omega_n)$ and suppose there exists a neighbourhood ω of x such that $R_v^{E \cap \omega}(x) < v(x)$.

We may assume x is in $\overline{E}$ for otherwise there is nothing to show. There exists w in $n\text{-}S^+(\Omega_1 \times \ldots \times \Omega_n)$ such that w majorizes v on $E \cap \omega$ and $w(x) < v(x)$. It follows

$$\liminf_{\substack{y \to x \\ y \in E}} w(y) \geq \liminf_{\substack{y \to x \\ y \in E}} v(y)$$

$$\geq v(x)$$

$$> w(x) .$$

Thus E is thin at x and we are done.

<u>Proposition 3.2.7</u>: Let E be a subset of $\Omega_1 \times \ldots \times \Omega_n$. Then $T = \{x \in E:$ E is thin at $x\}$ is n-negligible.

<u>Proof:</u> Let $(\omega_i)_{i \geq 1}$ be a countable base of open sets of $\Omega_1 \times \ldots \times \Omega_n$, v a continuous member of $n\text{-}S^+(\Omega_1 \times \ldots \times \Omega_n)$ and x in T. From Proposition 3.2.6 there exists an integer i such that ω_i is a neighbourhood of x and $R_v^{(E-x) \cap \omega_i}(x) < v(x)$. Since $\{x\}$ is n-polar it follows

$$\hat{R}_v^{E \cap \omega_i}(x) = \hat{R}_v^{E \cap \omega_i - \{x\}}(x) \qquad \text{(Corollary 2.2.16)}$$

$$\leq R_v^{E \cap \omega_i - \{x\}}(x)$$

$$= R_v^{(E-\{x\}) \cap \omega_i}(x)$$

$$< v(x)$$

$$= R_v^{E \cap \omega_i} \qquad \text{(since x is in } E \cap \omega_i)$$

This shows that

$$T \subset \bigcup_{i \geq 1} \{x \in \Omega_1 \times \ldots \times \Omega_n : \hat{R}_v^{E \cap \omega_i}(x) < R_v^{E \cap \omega_i}(x)\} \quad .$$

This set is n-negligible by Corollary 3.1.8 and Proposition 2.2.6. The proof is complete.

Section 3 The Extension Theorem

Theorem 3.3.1: Let N be a closed n-negligible subset of an open set U in $\Omega_1 \times \ldots \times \Omega_n$ and v an n-superharmonic function on U-N that is locally lower bounded on U. (By this we mean for every compact set K, v is lower bounded on (U-N) $\cap$ K.) Then there exists a unique member of n-S(U) which equals v on U-N.

Notice that no generality is lost in having v in n-S(U-N) as opposed to n-hyperharmonic on U-N. For if v were identically ∞ on a connected component of U-N we could just extend it to be ∞ on the whole component.

Remark 3.3.2: Define h on U by

$$h(x) = \begin{cases} v(x) & x \text{ in } U-N \\ \liminf\limits_{\substack{z \to x \\ z \in U-N}} v(z) & x \text{ in } N \end{cases}$$

Notice h is Borel measurable. Theorem 2.2.17 implies that if the Extension Theorem holds then the extension must be h. (This gives the uniqueness assertion immediately). However, we cannot show directly that h is the required extension. The idea of the proof is to extend u(x,·) n-hyperharmonically for "most" x by an inductive procedure. (This procedure is of course extending by means of limit infimum in the last n-1 variables. Notice this gives a function which majorizes $h(x,\cdot)$). Then it is easy to extend it to all of N. We then show we get a nearly n-superharmonic function on U and the lower semicontinuous regularization of it gives the required solution.

<u>Proof of the theorem:</u> The proof is by induction on n. If $n = 1$ it is just Proposition 1.1.13. Now assume $n > 1$ and the theorem holds for positive integers smaller than n. Since N is n-negligible, there exists a G_δ polar set P such that if z_1 is in $\Pi_1(U)-P$, $N_1(z_1)$ is (n-1)-negligible in $\Omega_2 \times \ldots \times \Omega_n$ and closed. By the induction hypothesis, for each such z_1, the mapping $(z_2,\ldots,z_n) \to v(z_1,\ldots,z_n)$ can be extended to one that is (n-1)-hyperharmonic on $U_1(z_1)$. Thus, by a slight abuse of notation, we may assume v is defined everywhere on U except on M, where

$$M = N \cap \{(z_1,\ldots,z_n) \in U: z_1 \in P\},$$

and, for each z_1 in $\Pi_1(U)-P$, the mapping $(z_2,\ldots,z_n) \to v(z_1,\ldots,z_n)$ is (n-1)-hyperharmonic on $U_1(z_1)$. As we observed in Remark 3.3.2 this mapping majorizes $h(z_1,\cdot)$ for such z_1.

Observe that M is n-polar in $\Omega_1 \times \ldots \times \Omega_n$ since it is contained in $P \times \Omega_2 \times \ldots \times \Omega_n$ (Proposition 2.1.12). Therefore there exists u in $n\text{-}S^+(\Omega_1 \times \ldots \times \Omega_n)$ such that $u(z) = \infty$ for all z in M. Define for each positive integer k the function u_k on U by

$$u_k(z) = \begin{cases} v(z) + k^{-1}u(z) & z \text{ in } U-M \\ \infty & z \text{ in } M\,. \end{cases}$$

We claim u_k is a nearly n-superharmonic function on U. Clearly u_k is locally lower bounded. Let $\delta_1, \ldots, \delta_n$ be regular domains in $\Omega_1, \ldots, \Omega_n$ respectively with $\overline{\delta}_1 \times \ldots \times \overline{\delta}_n$ contained in U and $(x_1, \ldots, x_n)$ a point in $\delta_1 \times \ldots \times \delta_n$. We must show that

(1) $$\overline{\int} u_k(z_1, \ldots, z_n)(\rho_{x_1}^{\delta_1} \times \ldots \times \rho_{x_n}^{\delta_n})(dz \ldots dz_n) \leq u_k(x_1 \ldots x_n) .$$

Since N is a Borel measurable set with 0 product measure (Remark 2.2.14), u_k equals the locally lower bounded Borel measurable function $h + k^{-1} u$ on U-N (here they are both just $v + k^{-1}u$), and $\rho_{x_1}^{\delta_1} \times \ldots \times \rho_{x_n}^{\delta_n}$ is totally finite we may apply Fubini's Theorem and deduce

$$\overline{\int} u_k(z_1, \ldots, z_n)(\rho_{x_1}^{\delta_1} \times \ldots \times \rho_{x_n}^{\delta_n})(dz_1 \ldots dz_n) =$$

$$= \int (h + k^{-1}u)(z_1, \ldots, z_n)(\rho_{x_1}^{\delta_1} \times \ldots \times \rho_{x_n}^{\delta_n})(dz_1 \ldots, dz_n)$$

$$= \int_{z_1 \in \Omega_1 - P} d\rho_{x_1}^{\delta_1}(z_1) \int (h+k^{-1}u)(z_1, \ldots, z_n)(\rho_{x_2}^{\delta_2} \times \ldots \times \rho_{x_n}^{\delta_n})(dz_2 \ldots dz_n)$$

(since P has 0 $\rho_{x_1}^{\delta_1}$ measure)

$$\leq \overline{\int}_{z_1 \in \Omega_1 - P} d\rho_{x_1}^{\delta_1}(z_1) \int u_k(z_1, \ldots, z_n)(\rho_{x_2}^{\delta_2} \times \ldots \rho_{x_n}^{\delta_n})(dz_2, \ldots, dz_n)$$

$$\leq \overline{\int}_{z_1 \in \Omega_1 - P} u_k(z_1, x_2, \ldots, x_n) d\rho_{x_1}^{\delta_1}(z_1) .$$

(This last inequality follows from the fact that if z_1 is in $\Pi_1(U)-P$, the mapping $(z_2, \ldots, z_n) \to u_k(z_1, \ldots, z_n)$ is (n-1)-hyperharmonic on $U_1(z_1)$). Thus (1) will be proved if we can show

(2) $$\overline{\int} u_k(z_1,x_2,\ldots,x_n)\,d\rho_{x_1}^{\delta_1}(z_1) \leq u_k(x_1,\ldots,x_n).$$

Suppose first $(x_1,\ldots,x_n)$ is in U-N. Since N is n-negligible there exists an (n-1)-negligible set Q in $\Omega_2 \times \ldots \times \Omega_n$ such that if $(z_2,\ldots,z_n)$ is in $\Omega_2 \times \ldots \times \Omega_n - Q$, $N_{2,\ldots,n}(z_2,\ldots,z_n)$ is polar. If $(x_2,\ldots,x_n)$ is in $\Omega_2 \times \ldots \times \Omega_n - Q$ then (2) holds since by Proposition 1.1.13 there exists a function hyperharmonic on $U_{2,\ldots,n}(x_2,\ldots,x_n)$ which equals $z_1 \to u_k(z_1,\ldots,z_n)$ everywhere except on $N_{2,\ldots,n}(x_2,\ldots,x_n)$, a Borel (closed) set of $\rho_{x_1}^{\delta_1}$ measure 0. Now use the fact that x_1 is not in $N_{2,\ldots,n}(x_2,\ldots,x_n)$. If $(x_2,\ldots,x_n)$ is in Q, since u_k is n-hyperharmonic on U-N,N is closed, and Q is (n-1)-negligible, it follows from Theorem 2.2.17 that there is a sequence $(y^\ell)_{\ell\geq 1}$ in $\Omega_2 \times \ldots \times \Omega_n$ converging to $(x_2,\ldots,x_n)$ such that for every ℓ, y^ℓ is not in Q, (x_1,y^ℓ) is not in N, and

$$u_k(x_1,\ldots,x_n) = \liminf_{\ell \to \infty} u_k(x_1,y^\ell).$$

Thus from the previous case we have for each ℓ,

(3) $$u_k(x_1,y^\ell) \geq \overline{\int}_{z_1 \in \Omega_1 - P} u_k(z_1,y^\ell)\,d\rho_{x_1}^{\delta_1}(z_1).$$

Taking lower limits as $\ell \to \infty$ gives

(4) $$u_k(x_1,\ldots,x_n) = \liminf_{\ell\to\infty} u_k(x_1,y^\ell)$$

(5) $$\geq \liminf_{\ell\to\infty} \overline{\int}_{z_1 \in \Omega_1 - P} u_k(z_1,y^\ell)\,d\rho_{x_1}^{\delta_1}(z_1)$$

(6) $$\geq \overline{\int}_{z_1 \in \Omega_1 - P} \liminf_{\ell\to\infty} u_k(z_1,y^\ell)\,d\rho_{x_1}^{\delta_1}(z_1) \quad \text{(Fatou's Lemma)}$$

$$(7) \qquad \geq \overline{\int}_{z_1 \in \Omega-P} u_k(z_1,x_2,\ldots,x_n)\,d\rho_{x_1}^{\delta_1}(z_1)$$

since for z_1 in Ω_1-P the mapping $(z_2,\ldots,z_n) \to u_k(z_1,\ldots,z_n)$ is lower semicontinuous. Thus (2) holds if $(x_1,\ldots,x_n)$ is in U-N.

If $(x_1,\ldots,x_n)$ is in M, (2) clearly holds since in this case $u_k(x_1,\ldots,x_n) = \infty$.

Finally suppose $(x_1,\ldots,x_n)$ is in N-M. Then x_1 is not in P and therefore $N_1(x_1)$ is (n-1)-negligible. Since the mapping $(z_2,\ldots,z_n) \to u_k(z_1,\ldots,z_n)$ is (n-1)-superharmonic in $U_1(x_1)$ we can find a sequence $(y^\ell)_{\ell\geq 1}$ in $\Omega_2 \times \ldots \times \Omega_n$ such that for every ℓ, y^ℓ is not in $N_1(x_1)$, $(y^\ell)_{\ell\geq 1}$ converges to $(x_2,\ldots,x_n)$, and

$$u_k(x_1,\ldots,x_n) = \liminf_{\ell\to\infty} u_k(x_1,y^\ell)$$

(Theorem 2.2.17). Since (x_1,y^ℓ) is not in N we may apply a previous case and proceed exactly as in inequalities (3)′ - (7) to deduce (2) holds. Thus (2) holds for all $(x_1,\ldots,x_n)$ in U and therefore u_k is indeed nearly n-superharmonic.

Define w on U by

$$w(x) = \liminf_{k\to\infty} u_k(x).$$

Then w is nearly n-superharmonic and hence $\hat{w}$ is n-superharmonic on U. We claim $\hat{w} = v$ on U-N. Well clearly $\hat{w} = v$ on the subset of U-N where u is finite, that is everywhere on U-N except an n-polar set . Since n-polar sets are n-negligible it follows $\hat{w} = v$ on U-N (Corollary 2.2.18). Thus $\hat{w}$ is the required extension and we are done.

Remark 3.3.2: Let h be n-harmonic on U-N where U is open in $\Omega_1 \times \ldots \times \Omega_n$ and N is n-negligible. If h is locally bounded on U then we may apply the Extension Theorem to h and (-h) and deduce there is a unique n-harmonic extension of h to U.

CHAPTER 4

Applications to Plurisuperharmonic Functions

In this chapter we apply the results obtained so far to the study of plurisuperharmonic functions. In particular we consider two types of exceptional sets, the n-P negligible sets and the sets of Ronkin Γ-capacity zero, and prove theorems analagous to Theorem 3.1.1 and Theorem 3.3.1 in which the exceptional sets are either of these types.

Section 1 Introduction

Recall that $C(= R^2)$ is a Brelot space if the harmonic functions are the twice continuously differentiable functions satisfying Laplace's equation and the set of discs is the base of regular domains. However only the open sets having a Green function (for example the relatively compact open sets) have a positive potential. Thus to apply results obtained so far to C^n we must first of all make some additional definitions and check that certain fundamental properties go through. For more details see [10].

First of all the hyperharmonic, superharmonic, n-hyperharmonic, and n-superharmonic functions are defined exactly as before. They all satisfy a local property and therefore they have the same basic properties as the corresponding functions we have been studying. It is also true that the composition of a superharmonic and a holomorphic mapping is superharmonic.

We next consider the polar sets.

Definition 4.1.1: A subset E of C is said to be polar if there exists a v superharmonic on a neighbourhood of $\bar{E}$ such that $v(z) = \infty$ for all z in E.

In this definition we can assume v is non-negative since clearly, because of the lower semicontinuity of v, v is non-negative on a neighbourhood of E. It thus follows from the local property that our two notions of polar set agree if E is contained in an open set having a Green function. It can be shown a countable union of polar sets is polar. Polar sets also have the following global property: a set E is polar if and only if there exists a superharmonic function v on C such that $v(z) = \infty$ for all z in E. Furthermore it can be shown that the polar sets are precisely the sets of outer logarithmic capacity 0. Thus point sets are polar and line segments are not (since the logarithmic capacity of a line segment is 1/4 its length).

With this definition of polar we define the n-negligible sets just as before. It is immediate that if $\Omega_1, \ldots, \Omega_n$ are open subsets of C having a Green function, then a subset E of $\Omega_1 \times \ldots \times \Omega_n$ is n-negligible in this new sense if and only if it is n-negligible in $\Omega_1 \times \ldots \times \Omega_n$ in the sense of Definition 2.2.1. We can also show (exactly as in the proof of Proposition 2.2.6) a countable union of n-negligible sets is n-negligible. With this fact we can deduce all basic properties. In particular we have that a set E is n-negligible if and only if for all integers i from 1 to n,

$$N_i = \{(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n) \in C^{n-1} : \{x_i \in C : (x_1,\ldots,x_n) \in E\} \text{ not polar}\}$$

is (n-1)-negligible (see Proposition 2.2.4), for v in n-S(U) where U is an open subset of C^n, $n \geq 2$, and k an integer between 1 and n-1 there exists a k-negligible set N in C^k such that if x is in $\Pi_{1,\ldots,k}(U)$-N the mapping $y \to v(x,y)$ is in $(n-k)\text{-}S(U_{1,\ldots,k}(x))$ (see Proposition 2.2.10), and for w in n-S(U) and N an n-negligible set,

$$w(x) = \liminf_{\substack{z \to x \\ z \in U-N}} w(z)$$

for all x in U. (See Theorem 2.2.17).

Finally we deduce that the Convergence Theorem and Extension Theorem both hold on any open subset U of C^n. First suppose $(v_k)_{k\geq 1}$ is a pointwise decreasing and locally uniformly lower bounded sequence in n-Sup(U) with limit function v. Choose $(U_\ell)_{\ell\geq 1}$ a sequence of relatively compact open sets in C^n such that $\bigcup_{\ell\geq 1} U_\ell = U$. Theorem 3.1.1 implies $\{x \in U_\ell : \hat{v}(x) < v(x)\}$ is n-negligible for each ℓ. It follows

$$\{x \in U: \hat{v}(x) < v(x)\} = \bigcup_{\ell\geq 1} \{x \in U_\ell : \hat{v}(x) < v(x)\}$$

is n-negligible and hence the Convergence Theorem does indeed hold. Suppose now w is n-superharmonic on U-E where E is n-negligible and w is locally lower bounded on U. Define w' on U by

$$w'(x) = \begin{cases} w(x) & x \text{ in } U-E \\ \liminf\limits_{\substack{z \to x \\ z \in U-E}} w(z) & x \text{ in } E \end{cases}$$

Theorem 3.3.1 and Theorem 2.2.17 imply that w' is n-superharmonic in any relatively compact open subset of U. Thus from the local property w' is in n-S(U) and we see the Extension Theorem too is valid.

Section 2. n-P Negligible Sets

Definition 4.2.1: Let U be an open subset of C^n, $n \geq 1$. An extended real valued function v defined on U is said to be plurisuperharmonic on U if

(i) $v(z) > -\infty$ for all z in U,

(ii) v is not identically ∞ on a connected component of U,

(iii) v is lower semicontinuous,

(iv) for every z and w in C^n the mapping $\lambda \to v(\lambda z + w)$ is hyperharmonic on $\{\lambda \in C: \lambda z + w \in U\}$.

We denote the set of all plurisuperharmonic functions on U by P Sup (U).

Let v be in P Sup(U). If $\{e_1,\ldots,e_n\}$ is the canonical basis of C^n, by choosing in (iv) $z = e_i$ for any i between 1 and n and $w = \sum_{j=1}^{n} e_j - e_i$, we see that v is separately hyperharmonic on U. Since v in addition satisfies (i),(ii), and (iii) we see v is in n-Sup(U). Thus P Sup(U) is a subset of n-Sup(U). It is now almost immediate that a convergence theorem analagous to Theorem 3.1.1 holds for P Sup(U) in which the exceptional set is n-negligible. However, as we shall see shortly, we can say much more about this set. With this in mind we define the following class of sets.

Definition 4.2.2: A subset E of C^n is said to be n-P negligible if for every z in C^n

$$\{w \in C^n: \{\lambda \in C; \lambda z + w \in E\} \text{ not polar}\}$$

is n-negligible.

Proposition 4.2.3: An n-P negligible set is n-negligible.

Proof. Let E be n-P negligible. By symmetry it is enough to show that there exists an (n-1)-negligible set N such that if $(w_2,\ldots,w_n)$ is in $C^{n-1} - N$, $\{\lambda \in C: (\lambda,w_2,\ldots,w_n) \in E\}$ is polar. Choose z in Definition 4.2.2 to be $(1,0,\ldots,0)$. Then

$$M = \{w \in C^n: \{\lambda \in C: (\lambda + w_1, w_2,\ldots,w_n) \in E\} \text{ not polar}\}$$

is n-negligible. Note now that if for any $(w_1,..,w_n)$ the set $\{\lambda \in C:(\lambda+w_1,w_2,...,w_n) \in E\}$ is polar, so is $\{\lambda \in C:(\lambda,w_2,...,w_n) \in E\}$. (Indeed if v is superharmonic on C and equal to ∞ on the former set then the mapping $\lambda \to v(\lambda - w_1)$ is also superharmonic on C and equal to ∞ on the latter.) Since M is n-negligible, there is an (n-1)-negligible set N such that if $(w_2,...,w_n)$ is in $C^{n-1}-N$, $\{w_1 \in C:(w_1,...,w_n) \in M\}$ is polar. Thus for each $(w_2,...,w_n)$ in $C^{n-1}-N$ there is at least one w_1 such that $\{\lambda \in C:(\lambda + w_1,w_2,...,w_n) \in E\}$ is polar. By our previous remark then, for such $(w_2,...,w_n)$ $\{\lambda:(\lambda, w_2,...,w_n) \in E\}$ is also polar and we are done.

Definition 4.2.4: A subset ~~E of an open set U~~ in C^n is said to be pluripolar in U if there is a function v in P Sup(U) such that E is contained in $\{z \in U: v(z)= \infty\}$.

Proposition 4.2.5: A pluripolar subset of U is n-P negligible.

Proof. Let E be a pluripolar subset of U. There exists v in P Sup(U) such that $v(z) = \infty$ for all z in E. Now fix z in C^n. We must show there is an n-negligible set N such that for w in C^n-N, $\{\lambda \in C: \lambda z + w \in E\}$ is polar.

Put $W = \{(\lambda,w) \in C^{n+1}: \lambda z + w \in U\}$. On this open set the mapping $(\lambda,w) \to v(\lambda z + w)$ is (n+1)-superharmonic and hence there is an n-negligible set N such that if w is in $\{w \in C^n$: there exists λ in C with (λ,w) in $W\} - N$, the mapping $\lambda \to v(\lambda z + w)$ is superharmonic on $\{\lambda \in C:(\lambda,w) \in W\}$. It follows that for such a w, $\{\lambda \in C:\lambda z + w \in U$ and $v(\lambda z + w) = \infty\}$ is a polar set containing $\{\lambda \in C: \lambda z + w \in E\}$. This implies the latter set is polar and we are done.

The converse of Proposition 4.2.5 is false, as is illustrated in the following counterexample.

Example (Kiselman. See also [4].) Put $H = \{(z_1,z_2) \in C^2 : \text{Im } z_1 = \text{Re}(z_1+z_2) = 0\}$. For every z_1 in C with $\text{Im } z_1 = 0$ (that is for every z_1 on a line segment in C, a non-polar set)

$$\{z_2 \in C: (z_1,z_2)\} \in H\} = \{z_2 \in C: z_2 = -z_1 + ib, b \in R\}.$$

which is a line in C. It follows H is not 2-negligible and therefore not pluripolar. Let g be the biholomorphic mapping $(z_1,z_2) \to (z_1 - z_2^2, z_2)$. Then

$$g(H) = \{(z_1,z_2) \in C^2 : \text{Im}(z_1+z_2^2) = \text{Re}(z_1 + z_2 + z_2^2) = 0\}.$$

This set is 2-P negligible since for every z, w in C^2, $\{\lambda \in C: \lambda z + w \in g(H)\}$ contains at most 4 points. It is not however pluripolar since if it were there would be a v in P Sup(C^2) with $v(z) = \varphi$ for all z in g(H). But $v \circ g$ is also in P Sup (C^2) and equals ∞ on H. This is impossible. Thus g(H) is not pluripolar.

The following convergence theorem is a simple consequence of Theorem 3.1.1

Theorem 4.2.6: Let U be an open subset of C^n and v the lower limit of a pointwise decrease uniformly locally lower bounded sequence $(v_k)_{k \geq 1}$ in P Sup(U). Then $\hat{v}$ and v are equal everywhere except on an n-P negligible subset of U.

Proof: Fix z in C^n. Put $W = \{(\lambda,w) \in C^{n+1} : \lambda z + w \in U\}$. Define the sequence of functions $(g_k)_{k \geq 1}$ on W by $g_k(\lambda,w) = v_k(\lambda z + w)$. Then $(g_k)_{k \geq 1}$ is a pointwise decreasing sequence of uniformly locally lower bounded functions in (n+1)-Sup(U) with limit function g, where $g(\lambda,w) = v(\lambda z + w)$. Therefore by Theorem 3.1.1, $\hat{g}$ and g differ at most on an (n+1)-negligible subset of W. It is easy to see that for all (λ,w) in W $\hat{g}(\lambda,w) = \hat{v}(\lambda z + w)$. Thus there exists an n-negligible set N in C^n such that if w is in C^n-N,

$$\{\lambda \in C: \hat{v}(\lambda z + w) < v(\lambda z + w)\} = \{\lambda \in C: \hat{g}(\lambda, w) < g(\lambda, w)\}$$

is polar. This completes the proof.

We now prove an analogue of Theorem 3.3.1.

Theorem 4.2.7: Let U be an open subset of C^n, E a closed n-P negligible subset of U, and v plurisuperharmonic on U-E. If v is locally lower bounded on U there exists a unique v_1 in P Sup(U) such that $v_1 = v$ on U-W.

We first prove a lemma.

Lemma 4.2.8: Let E be a closed n-P negligible set. Then for every $u = (u_1, \ldots, u_n)$ in C^n, $\{w \in C^n: u + w \in E\}$ is n-negligible.

Proof: The proof is by induction on n. If n = 1, since E is polar, there exists a function v superharmonic on C such that $v(w) = \infty$ for all w in E. Consider now the mapping $w \to v(w-u)$ on C. This is superharmonic on C and equals ∞ on $\{w \in C: w + u \in E\}$. Therefore the lemma holds for n = 1. Suppose now n > 1 and it holds for smaller integers. By symmetry it is enough to show there is a polar set P such that if w_1 is in C-P, $\{(w_2, \ldots, w_n) \in C^{n-1}: (u_1 + w_1, \ldots, u_n + w_n) \in E\}$ is (n-1)-negligible. Well certainly it is true that

$$\{w_1 \in C: \{(w_2, \ldots, w_n) \in C^{n-1}: (w_1, \ldots, w_n) \in E\} \text{ not } (n-1)\text{-negligible}\}$$

is polar. Therefore by the induction hypothesis.

$$P = \{w_1 \in C: \{(w_2, \ldots, w_n) \in C^{n-1}: (w_1 + u_1, w_2, w_3, \ldots, w_n) \in E\} \text{ not } (n-1)\text{-negligible}\}$$

is polar. If w_1 is in C-P the set N defined by

$$N = \{(w_2, \ldots, w_n) \in C^{n-1}: (w_1 + u_1, w_2, w_3, \ldots, w_n) \in E\}$$

is (n-1)-negligible. Thus by the induction hypothesis $\{(w_2,\ldots,w_n) \in C^{n-1}:(u_2+w_2,\ldots,u_n+w_n) \in N\}$ is also (n-1)-negligible. This is precisely what we wished to show.

Proof of Theorem: The uniqueness follows immediately since any such extension is in particular in n-Sup(U) and functions in this set are determined if specified up to only an n-negligible set.

Now since v is in n-Sup(U-E) and locally lower bounded on U, by the Extension Theorem there exists v_1 in n-Sup(U) such that $v = v_1$ on U-E. We claim v_1 is the required extension.

To prove this it is clear we need only prove that for each z and w in C^n the mapping $\lambda \to v_1(\lambda z+w)$ is hyperharmonic on $\{\lambda \in C: \lambda z+w \in U\}$. Well since v_1 is lower semicontinuous so is this mapping. It remains to show that for λ_o a fixed complex number and δ a regular domain contained with its closure in $\{\lambda \in C: z+w \in U\}$,

$$(1) \qquad \int v_1(\lambda z_o+w_o)\,d\rho^{\delta}_{\lambda_o}(\lambda) \le v_1(\lambda_o z_o+w_o) .$$

We know, since E is n-P negligible, there exists an n-negligible set N (depending on z_o) such that if w is in C^n-N, $\{\lambda \in C:\lambda z_o+w \in E\}$ is polar. Put

$$M = N \cup \{w \in C^n: \lambda_o z_o + w \in E\} .$$

We show first (1) holds if w_o is in C^n-M. Indeed in this case the mapping $\lambda \to v(\lambda z_o+w_o)$ is defined everywhere on $\{\lambda \in C:\lambda z_o+w_o \in U\}$ except $\{\lambda \in C:\lambda z_o+w_o \in E\}$ and by our choice of w_o the latter set is polar and of course closed. Thus this mapping has an extension to a function hyperharmonic on $\{\lambda \in C:\lambda z_o+w_o \in U\}$. Call this extension u.

Since closed polar sets have zero harmonic measure,

$$\begin{aligned}\int v_1(\lambda z_o + w_o)\,d\rho^{\delta}_{\lambda_o}(\lambda) &= \int u(\lambda)\,d\rho^{\delta}_{\lambda_o}(\lambda)\\ &\leq u(\lambda_o)\\ &= v(\lambda_o z_o + w_o) \quad \text{(since } w_o \text{ is in } C^n - M\text{)}\\ &= v_1(\lambda_o z_o + w_o)\ .\end{aligned}$$

This proves that (1) holds if w_o is in C^n-M. Now by the lemma and Proposition 2.2.6 we see M is n-negligible. It follows that for w_o in general, since v_1 is in n-Sup(U), we can find a sequence $(w_k)_{k\geq 1}$ in C^n converging to w_o such that for each k, w_k is in C^n-M and

$$v_1(\lambda_o z_o + w_o) = \liminf_{k\to\infty} v_1(\lambda_o z_o + w_k)$$

(This too uses the lemma). For every positive integer k we have from the special case we have just proven that

$$\int v_1(\lambda z_o + w_k)\,d\rho^{\delta}_{\lambda_o}(\lambda) \leq v_1(\lambda_o z_o + w_k)\ .$$

Taking lower limits gives

$$\begin{aligned}v_1(\lambda_o z_o + w_o) &= \liminf_{k\to\infty} v_1(\lambda_o z_o + w_k)\\ &\geq \liminf_{k\to\infty} \int v_1(\lambda z_o + w_k)\,d\rho^{\delta}_{\lambda_o}(\lambda)\\ &\geq \int \liminf_{k\to\infty} v_1(\lambda z_o + w_k)\,d\rho^{\delta}_{\lambda_o}(\lambda) \quad \text{(Fatou lemma)}\\ &\geq \int v_1(\lambda z_o + w_o)\,d\rho^{\delta}_{\lambda_o}(\lambda)\end{aligned}$$

since v_1 is lower semicontinuous. Thus (1) holds and the theorem is proved.

Section 3 Sets of Rankin Γ-Capacity Zero

Let c denote the interior logarithmic capacity on C. For E a subset of $C^n, n \geq 1$, we define the quantity $\gamma_n(E)$ inductively as follows:

$$\gamma_1(E) = c(E),$$

$$\gamma_n(E) = c\{z_1 \in C: \gamma_{n-1}\{z_2 \in C^{n-1}: (z_1,z_2) \in E\} > 0\}.$$

Ronkin's Γ-capacity is defined on any subset E of C^n to be

$$\Gamma_n(E) = \sup\{\gamma_n(\alpha E): \alpha \text{ a complex unitary transformation of } C^n\}.$$

We shall need the following result of Cegrell, See [4]

Proposition 4.3.1: If E is universally capacitable then

$$\gamma_n(E) = \overline{\mathrm{cap}}_2\{z_1 \in C: \gamma_{n-1}\{z \in C^{n-1}: (z_1,z) \in E\} > 0\},$$

where $\overline{\mathrm{cap}_2}$ is the outer logarithmic capacity on C.

We now investigate the relationship between n-negligible sets and sets of zero Ronkin Γ-capacity.

Lemma 4.3.2: Let E be a Borel subset of C^n such that for every permutation σ of $\{1,2,\ldots,n\}$, $\gamma_n\ (z_{\sigma 1}, z_{\sigma 2}, \ldots, z_{\sigma n}): (z_1,\ldots,z_n) \in E\} = 0$. Then E is n-negligible.

Proof: The proof is by induction on n. If $n = 1$ then $c(E) = 0$. Since E is Borel it is universally capacitable and hence its outer logarithmic capacity equals 0. Thus E is polar and the result holds for $n = 1$. Now suppose $n > 1$ and the lemma holds for Borel subsets of $C^k, k < n$. We claim there exists a polar set P such that if z_1 is in C- P, $\{(z_2,\ldots,z_n) \in C^{n-1}: (z_1,\ldots,z_n) \in E\}$ is (n-1)-negligible. For each permutation σ of $\{1,2,\ldots,n\}$ which fixes 1 define

$$M_\sigma = \{(z_1,\ldots,z_n) \in C^n : (z_1, z_{\sigma^{-1}2}, \ldots, z_{\sigma^{-1}n}) \in E\} .$$

Since $\gamma_n(M_\sigma) = 0$ and M_σ is Borel Proposition 4.3.1 implies there is a polar set P_σ such that for all z_1 in $C-P_\sigma$,

$$0 = \gamma_{n-1}\{(z_2,\ldots,z_n) \in C^{n-1} : (z_1,\ldots,z_n) \in M_\sigma\}$$

$$= \gamma_{n-1}\{(z_2,\ldots,z_n) \in C^{n-1} : (z_1, z_{\sigma^{-1}2}, z_{\sigma^{-1}3}, \ldots, z_{\sigma^{-1}n}) \in E\}$$

$$= \gamma_{n-1}\{(z_{\sigma 2},\ldots,z_{\sigma n}) \in C^{n-1} : (z_1,\ldots,z_n) \in E\} .$$

Define P by

$$P = \cup\{P_\sigma : \sigma \text{ a permutation of } \{1,2,\ldots,n\} \text{ fixing } 1\} .$$

This is a finite union hence P is polar. Now if z_1 is any point in C-P the set $F = \{(z_2,\ldots,z_n) \in C^{n-1} : (z_1,\ldots,z_n) \in E\}$ is Borel and for any permutation τ of $\{2,\ldots,n\}$ we have shown

$$\gamma_{n-1}\{(z_{\tau 2},\ldots,z_{\tau n}) \in C^{n-1} : (z_2,\ldots,z_n) \in F\} = 0 .$$

By the induction hypothesis F must be (n-1)-negligible thus proving the claim.

In general, for any integer i from 1 to n, by considering permutations mapping 1 to i rather than those fixing 1, we can find a polar set Q such that if z_i is in C-Q, $\{(z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_n) \in C^{n-1} : (z_1,\ldots,z_n) \in E\}$ is (n-1)-negligible. Thus E is n-negligible and we are done.

<u>Theorem 4.3.3:</u> Let E be a Borel subset of C^n with $\Gamma_n(E) = 0$. Then for every complex unitary transformation α of C^n, $\alpha(E)$ is n-negligible.

<u>Proof:</u> If n = 1 it follows easily from the lemma. Suppose $n > 1$ and α is a

complex unitary transformation of C^n. Let σ be any permutation of $\{2,3,\ldots,n\}$. There exists a unique complex unitary transformation β on C such that $\beta(z_1,\ldots,z_n) = (z_1,z_{\sigma 2},\ldots,z_{\sigma n})$. Since $\beta^{-1}\circ\alpha$ is complex unitary $\gamma_n(\beta^{-1}\circ\alpha(E)) = 0$. Therefore, there exists a polar set P_σ such that if z_1 is in $C-P_\sigma$,

$$0 = \gamma_{n-1}\{(z_2,\ldots,z_n) \in C^{n-1}: (z_1,\ldots,z_n) \in \beta^{-1}\circ\alpha(E)\}$$

$$\therefore\quad 0 = \gamma_{n-1}\{(z_2,\ldots,z_n) \in C^{n-1}: (z_1,z_{\sigma 2},\ldots,z_{\sigma n}) \in \alpha(E)\}$$

$$(1)\qquad \therefore\quad 0 = \gamma_{n-1}\{(z_{\sigma^{-1}2},\ldots,z_{\sigma^{-1}n}) \in C^{n-1}: (z_1,\ldots,z_n) \in \alpha(E)\}\,.$$

Define Q_1 to be

$$Q_1 = \cup\,\{P_\sigma: \sigma \text{ a permutation of } \{2,\ldots,n\}\}\,.$$

Since thisis a finite union, Q_1 is polar. If z_1 is in C-Q, (1) holds for all permutations of $\{2,\ldots,n\}$ and by the lemma we conclude $\{(z_2,\ldots,z_n) \in C^{n-1}: (z_1,\ldots,z_n) \in \alpha(E)\}$ is(n-1)-negligible. Note that such a polar set exists for every complex unitary transformation α .

Now choose a complex unitary transformation β_1 of C^n such that $\beta_1(z_1,\ldots,z_n) = (z_2,z_1,z_3,z_4,\ldots,z_n)$. Since $\beta_1\circ\alpha$ is unitary, from what we have just seen, there exists a polar set Q_2 such that if z_1 is in $C-Q_2$, $\{(z_2,\ldots,z_n) \in C^{n-1}:(z_1,\ldots,z_n) \in (\beta_1\circ\alpha)(E)\}$ is (n-1)-negligible. But this set is just $\{(z_2,\ldots,z_n) \in C^{n-1}:(z_2,z_1,z_3,z_4,\ldots,z_n) \in \alpha(E)\}$. By just relabelling we therefore have that if z_2 is in $C-Q_2$, $\{(z_1,z_3,z_4,\ldots,z_n) \in C^{n-1}:(z_1,\ldots,z_n) \in \alpha(E)\}$ is (n-1)-negligible. Again note that Q_2 exists for any complex unitary transformation α .

If $n = 2$ we are done. If $n > 2$ we choose β_2 to be the complex unitary transformation $\beta(z_1,\ldots,z_n) = (z_1,z_3,z_2,z_4,\ldots,z_n)$. Since $\beta_2 \circ \alpha$ is unitary there exists Q_3 polar such that if z_2 is in $C-Q_3$, $\{(z_1, z_3, z_4,\ldots,z_n) \in C^{n-1}: (z_1,\ldots,z_n) \in \beta_2 \circ \alpha(E)\}$ is (n-1)-negligible. This is just $\{(z_1,z_3,\ldots,z_n) \in C^{n-1}: (z_1,z_3,z_2,z_4,\ldots,z_n) \in \alpha(E)\}$ and by relabelling we see that if z_3 is in $C-Q_3$, $\{(z_1,z_2,z_4,\ldots,z_n) \in C^{n-1}: (z_1,\ldots,z_n) \in \alpha(E)\}$ is (n-1)-negligible. Continuing in this way it is now clear $\alpha(E)$ is n-negligible. The proof is complete.

The converse of Theorem 4.3.3 is much easier and does not require E to be Borel.

Proposition 4.3.4: Let E be an arbitrary subset of C^n. If E is n-negligible then $\gamma_n(E) = 0$. Consequently, if for every complex unitary transformation α, $\alpha(E)$ is n-negligible then $\Gamma_n^*(E) = 0$.

Proof: The proof is by induction on n. If $n = 1$ then E is polar and hence it has zero outer logarithmic capacity. It follows $\gamma_1(E) = 0$. Suppose now $n > 1$ and the proposition holds for smaller integers. Since E is n-negligible there exists a polar set P such that if z_1 is in $C-P$

$$M = \{(z_2,\ldots,z_n) \in C^{n-1}: (z_1,\ldots,z_n) \in E\}$$

is (n-1)-negligible. By the induction hypothesis $\gamma_{n-1}(M) = 0$. Thus by definition $\gamma_n(E) = 0$ and we are done.

At this stage we do not know how the n-P negligible sets and the sets of zero Ronkin Γ-capacity compare. However we do still have analogues of Theorem 3.1.1 and Theorem 3.3.1. We remarked in the introduction that both of these results are knonw. We wish to include the proofs as an application of our methods.

Theroem 4.3.5: Let U be an open subset of C^n and $(v_k)_{k\geq 1}$ a pointwise decreasing sequence of plurisuperharmonic functions on U that are uniformly locally lower bounded. Then the limit function v differs from $\hat{v}$ on a set of zero Ronkin Γ-capacity.

Proof: Let $E = \{z \in U: \hat{v}(z) < v(z)\}$ and let α be any complex unitary transformation of C^n. We will show $\alpha^{-1}(E)$ is n-negligible and appeal to Proposition 4.3.4.

Consider first the sequence $(v_k \circ \alpha)_{k\geq 1}$ defined on the open set $\alpha^{-1}(U)$. By the linearity of α it is easy to prove this sequence is in P Sup($\alpha^{-1}(U)$), it is decreasing pointwise to $v \circ \alpha$, and it is uniformly locally lower bounded. Theorem 3.1.1 proves that $\{z \in \alpha^{-1}(U): \widehat{v \circ \alpha(z)} < v \circ \alpha(z)\}$ is n-negligible. We claim that $\widehat{v \circ \alpha} = \hat{v} \circ \alpha$. Indeed since $\hat{v}$ is lower semicontinuous and α is continuous, $\hat{v} \circ \alpha$ is lower semicontinuous and minorizes $v \circ \alpha$. It follows $\hat{v} \circ \alpha$ minorizes $\widehat{v \circ \alpha}$. Conversely if λ is a real number such that $\widehat{v \circ \alpha}(z_o) > \lambda$ for z_o in $\alpha^{-1}(U)$, there is a neighbourhood W of z_o such that $v \circ \alpha(z) > \lambda$ for all z in W. It follows v majorizes λ on the open set $\alpha^{-1}(W)$ and hence so does $\hat{v}$. This just says $\hat{v} \circ \alpha(z) \geq \lambda$ on W and this implies $\hat{v} \circ \alpha(z_o) \geq \widehat{v \circ \alpha}(z_o)$. The claim is proved. We thus have that $\{z \in \alpha^{-1}(U): \hat{v} \circ \alpha(z) < v \circ \alpha(z)\}$ is n-negligible. But this set is nothing but $\alpha^{-1}(E)$. The proof is complete.

Theorem 4.3.6: Let U be an open subset of C^n, E a closed subset of U with $\Gamma_n(E) = 0$, and v a member of P Sup(U-E) that is locally lower bounded on U. Then there exists a unique v_1 in P Sup(U) such that $v_1 = v$ on U-E.

Proof: From Theorem 4.3.3 E is n-negligible and hence the Extension Theorem 3.3.1 implies there exists a unique v_1 in n-Sup(U) with $v_1 = v$ on U-E. This immediately gives the uniqueness in the theorem. We will prove v_1 is the required extension.

We first show that for every complex unitary transformation α of $\mathbb{C}^n$, $v_1 \circ \alpha$ is in n-Sup($\alpha^{-1}(U)$). Since it is lower semicontinuous we can prove this by using Proposition 1.2.11. That is it is enough to show for every $u = (u_1,\ldots,u_n)$ in $\alpha^{-1}(U)$ and $\delta_1,\ldots,\delta_n$ regular neighbourhoods of $u_1,\ldots,u_n$ respectively with $\overline{\delta}_1 \times \ldots \times \overline{\delta}_n \subset \alpha^{-1}(U)$ that

$$(1) \qquad \int \ldots \int v_1 \circ \alpha \, d\rho_{u_1}^{\delta_1} \ldots d\rho_{u_n}^{\delta_n} \leq v_1 \circ \alpha(u).$$

Suppose first u is in $\alpha^{-1}(U) - \alpha^{-1}(E)$. Since $\alpha^{-1}(E)$ is n-negligible (Theorem 4.3.3), a closed subset of $\alpha^{-1}(U)$, $v \circ \alpha$ is in n-Sup($\alpha^{-1}(U) - \alpha^{-1}(E)$), and $v \circ \alpha$ is locally lower bounded on $\alpha^{-1}(U)$, there exists (by Theorem 3.3.1) a function w in n-Sup($\alpha^{-1}(U)$) which equals $v \circ \alpha$ on $\alpha^{-1}(U) - \alpha^{-1}(E)$. Now $\alpha^{-1}(E)$ is closed and hence it has 0 $\rho_{u_1}^{\delta_1} \times \ldots \times \rho_{u_n}^{\delta_n}$ measure (Remark 2.2.14). It follows

$$\begin{aligned}\int \ldots \int v_1 \circ \alpha \, d\rho_{u_1}^{\delta_1} \ldots d\rho_{u_n}^{\delta_n} &= \int \ldots \int w \, d\rho_{u_1}^{\delta_1} \ldots d\rho_{u_n}^{\delta_n} \\ &\leq w(u) \\ &= (v \circ \alpha)(u) \\ &= (v_1 \circ \alpha)(u).\end{aligned}$$

Now in general, since v_1 is in n-Sup(U), there exists a sequence $(x^k)_{k \geq 1} = ((x_1^k, \ldots, x_n^k))_{k \geq 1}$ in U converging to $\alpha(u)$ such that for every k x^k is in U-E and

$$v_1(\alpha(w)) = \liminf_{k \to \infty} v_1(x^k)$$

Put $u^k = \alpha^{-1}(x^k)$. Then u^k is in $\alpha^{-1}(U)-\alpha^{-1}(E)$ and the sequence $(u^k)_{k\geq 1}$ converges to u. By the case we just proved we have for each k

$$\int \cdots \int v_1 \circ \alpha \; d\rho_{u_1^k}^{\delta_1} \cdots d\rho_{u_n^k}^{\delta_n} \leq v_1 \circ \alpha(u^k) .$$

Taking lower limits on both sides gives

$$v_1 \circ \alpha(u) = \liminf_{k \to \infty} v_1(\alpha(u^k))$$

$$\geq \liminf_{k \to \infty} \int \cdots \int v_1 \circ \alpha \; d\rho_{u_1^k}^{\delta_1} \cdots d\rho_{u_n^k}^{\delta_n}$$

$$= \int \cdots \int v_1 \circ \alpha \; d\rho_{u_1}^{\delta_1} \cdots d\rho_{u_n}^{\delta_n} \qquad \text{(Remark 1.2.6(a))}.$$

Thus (1) holds. This completes the proof that $v_1 \circ \alpha$ is in $n\text{-Sup}(\alpha^{-1}(U))$.

It remains to show v_1 is in P Sup(U) and of this only (iv) of Definition 4.2.1 needs to be verified. Fix a nonzero z in C^n. We must show for every w in C^n that the mapping $\lambda \to v_1(\lambda z + w)$ is hyperharmonic on $\{\lambda \in C: \lambda z + w \in U\}$. Suppose first z and w are orthogonal and w is nonzero. In this case there exists a complex unitary transformation α such that $\alpha(1,0,\ldots,0) = z/|z|$ and $\alpha(0,1,0,\ldots,0) = w/|w|$. We have shown $v_1 \circ \alpha$ is in $n\text{-Sup}(\alpha^{-1}(U))$. It follows the mapping $\lambda \to v_1 \circ \alpha(\lambda, |w|, 0,\ldots,0) = v_1(\lambda z/|z| + w)$ is hyperharmonic and hence so is $\lambda \to v_1(\lambda z + w)$. If $w = (0,\ldots,0)$, instead choose α satisfying only $\alpha(1,0,\ldots,0) = z/|z|$. Then again $\lambda \to v_1 \circ \alpha\, (\lambda,0,\ldots,0) = v_1(\lambda z/|z|)$ is hyperharmonic.

In general w can be written as $w = w' + \beta z$ where w' is orthogonal to z and β is some complex number. Since $\lambda \to v(\lambda z + w')$ is hyperharmonic so is $\lambda \to v((\lambda + \beta)z + w') = v(\lambda z + w)$. The proof is complete.

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