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The problem of a hot two-dimensional laminar jet issuing vertically into an otherwise quiescent fluid of a lower temperature is studied by means of two coordinate-type expansions. Detailed calculations are carried out for a Prandtl number of two. Solutions for the temperature and velocity distributions obtained by the series expansions are compared with those obtained by a simple integral approach. It is found that the velocity along the vertical axis of symmetry initially decreases with distance from the virtual origin, reaches a minimum and thereafter steadily increases. The centreline temperature decreases monotonically from the virtual origin.

1. INTRODUCTION

This paper concerns the flow developed by a hot vertical two-dimensional laminar jet in an otherwise quiescent, uniform and colder fluid. Recently Brand and Lahey (1967) treated this problem, as well as the corresponding axisymmetric case, using the usual boundary layer equations for natural convection flows. For the two-dimensional case they found similarity solutions in which the maximum (centreline) velocity was proportional to $x^{1/5}$ where x is the vertical coordinate along the axis of symmetry of the jet. these solutions predict that the centreline velocity vanishes at the virtual origin. However, if we assume the flow to be generated by a virtual source of both heat and momentum, the velocity would increase without limit as $x \rightarrow 0$. Thus the analysis of Brand and Lahey (1967) is an asymptotic solution valid at large x such that the initial conditions of momentum flux have been "lost". From a somewhat different standpoint their solution corresponds to the flow generated by a virtual source of heat but no momentum. Such laminar natural convection plumes have been studied by Schuh (1948), Yih (1952), Bendor (1956), Mahony (1957), Serruck (1958), Crane (1959), Spalding and Cruddace (1961), and Fujii (1963). Some of these references contain most of the solutions presented by Brand and Lahey (1967). Experimental measurements of the velocity and temperature fields have been carried out for the plume by Bendor (1956), Brodowicz and Kierkus (1966), and Forstrom and Sparrow (1967).

The present work considers the problem of steady laminar two-dimensional flow developed by a virtual line source of both heat and momentum. A solution is obtained by means of direct and inverse coordinate expansions valid at small and large distances respectively from the virtual origin. The analysis is similar in certain respects to that of Van Dyke (1964a) for the laminar boundary layer on a parabola in a uniform free stream, and that of Wygnanski (1967) for a laminar jet in a streaming flow. The distributions of centreline temperature and velocity obtained from the series expansions are compared with those obtained by a simple integral method which is expected to be quite accurate for a Prandtl number, $\sigma = 2$. Although detailed numerical calculations are presented only for the case of $\sigma = 2$, the series expansions can be applied for arbitrary Prandtl numbers.

2. ANALYSIS

The usual form of the steady, two-dimensional, laminar boundary layer equations for the case of thermal buoyancy with small temperature differences (cf. ex. Ostrach 1964) may be expressed non-dimensionally as

$$\left.\begin{aligned} \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} &= 0 \quad , \\ U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} &= \frac{g\beta T^{*}L}{U^{*2}}\, T + \frac{\partial^2 U}{\partial y^2} \quad , \\ U\frac{\partial T}{\partial x} + V\frac{\partial T}{\partial y} &= \frac{1}{\sigma}\frac{\partial^2 T}{\partial y^2} \quad . \end{aligned}\right\} \qquad (1)$$

where

$$x = \frac{\tilde{x}}{L} \quad , \quad y = \frac{\tilde{y}}{L}\sqrt{\frac{U^{*}L}{\nu}} \quad ,$$

$$U = \frac{\tilde{U}}{U^*}\ , \qquad V = \frac{\tilde{V}}{U^*}\sqrt{\frac{U^* L}{\nu}}\ ,$$
$$T = \frac{\tilde{T} - \tilde{T}_\infty}{T^*} \quad \text{and} \quad \sigma = \frac{\nu}{\alpha}\ . \qquad (2)$$

Here $\tilde{x}$ and $\tilde{y}$ are Cartesian coordinates (the x-axis is directed vertically upwards and coincides with the axis of symmetry of the jet), $\tilde{U}$ and $\tilde{V}$ are the velocity components in the $\tilde{x}$ and $\tilde{y}$ directions respectively, $\tilde{T}$ is the local fluid temperature, $\tilde{T}_\infty$ is the uniform temperature of the ambient fluid, ν is the kinematic viscosity, α is the thermal diffusivity, σ is the Prandtl member, β is the volumetric expansion coefficient, g is the gravitational acceleration, and U^*, T^* and L are reference velocity, temperature and length. The Boussinesq approximation has been adopted and thus all fluid properties are assumed constant except for the essential density variations.

A virtual source of heat and momentum is assumed to be located at the origin of the coordinate system. The boundary conditions to be satisfied by the velocity and temperature profiles are:

$$\text{for} \quad x > 0\ , \ y = 0 \qquad \frac{\partial U}{\partial y} = \frac{\partial T}{\partial y} = V = 0\ ,$$
$$\text{for} \quad x \geq 0\ , \ y = \infty \qquad U = T = 0\ . \qquad (3)$$

Considering the flow in the region close to the virtual source where the flow velocity is high we anticipate that the buoyancy forces are negligible compared with the inertial and viscous forces. The velocity field will be approximated by the classical solution of Schlichting (1933) for a momentum jet in a uniform quiescent ambient fluid. Thus we are lead to attempt a perturbation solution for the velocity and temperature fields close to the origin by expanding in terms of direct (fractional) powers of the distance x from the virtual source. Since the governing equations are parabolic and x is time-like, subsequent terms in the expansion can be calculated without difficulty in terms of the initial conditions at the virtual source. Although the resulting series will diverge for some value of x, hopefully we can extend its radius of convergence by use of, for example, the Euler transformation (Van Dyke 1964b).

At large values of x the buoyancy forces will be of the same order as the other terms in the momentum equation and the flow will asymptotically approach the similarity solutions given by Brand and Lahey (1967), Schuh (1948), Bendor (1956), etc., that were mentioned earlier. Here it is natural to seek an approximate solution by means of an inverse coordinate expansion. There occur eigen-solutions, the undetermined multiplicative constants of which are related to the initial conditions at the virtual source. These constants may be determined by a suitable joining or numerical patching procedure.

2.1 Direct Expansion For Small x

It will be convenient to define the coefficient of T in the momentum equation of (1) as

$$g\frac{\beta T_s^* L}{U_s^{*2}} = 1 \quad , \tag{4}$$

where T_s^* and U_s^* are the reference temperature and velocity for small x. Unique definitions for T_s^*, U_s^* and L may be obtained by making use of equation (4) and two integral conditions. The initial jet momentum per unit length of source in terms of the physical variables is defined by

$$\tilde{J} = \lim_{\tilde{x}\to 0} \int_{-\infty}^{\infty} \tilde{\rho}\,\tilde{U}^2 dy \quad , \tag{5}$$

where $\tilde{\rho}$ is chosen as the fluid density emerging from the slit and is a convenient reference density. We note that the integral in equation (5) is not independent of x as in the classical solution of Schlichting (1933) for a constant temperature jet. Rewriting equation (5) in terms of the dimensionless variables yields

$$\lim_{x\to 0} \int_{-\infty}^{\infty} U^2 dy = k_1 = \frac{\tilde{J}\sqrt{\frac{U_s^* L}{\nu}}}{\tilde{\rho}\, U_s^{*2} L} \quad , \tag{6}$$

where k_1 is simply a numerical constant which may be determined from subsequent solutions.

From the original physical form of the last of equations (1) (the energy equation) it is found that

$$\int_{-\infty}^{\infty} \tilde{U}(\tilde{T}-\tilde{T}_\infty)\,dy = \text{const.} = \frac{\tilde{Q}}{\tilde{\rho}\,C_p} \;, \tag{7}$$

where $\tilde{Q}$ is the heat flow per unit length of line source, Cp is the specific heat at constant pressure.

Expressing equation (7) in dimensionless form gives

$$\int_{-\infty}^{\infty} UT\,dy = k_2 = \frac{\tilde{Q}\sqrt{\dfrac{U_s^* L}{\nu}}}{\tilde{\rho}\,C_p\,U_s^*\,T_s^*\,L} \;, \tag{8}$$

where k_2 is a numerical constant.

Equations (4), (6) and (8) provide three equations to determine the reference temperature, velocity and length as follows:

$$\begin{aligned} T_s^* &= \sqrt{\frac{1}{\tilde{\rho}}}\left(\frac{g\beta}{\nu}\right)^{1/4}\left(\frac{\tilde{Q}}{C_p k_2}\right)^{5/4}\left(\frac{k_1}{\tilde{J}}\right)^{3/4} \;, \\ U_s^* &= \sqrt{\frac{1}{\tilde{\rho}}}\left(\frac{\tilde{J}\tilde{Q}\,g\beta}{\nu\,C_p\,k_1 k_2}\right)^{1/4} \;, \\ L &= \sqrt{\frac{1}{\tilde{\rho}}}\,\frac{1}{\nu^{1/4}}\left(\frac{\tilde{J}}{k_1}\right)^{5/4}\left(\frac{C_p k_2}{\tilde{Q}\,g\beta}\right)^{3/4} \;. \end{aligned} \tag{9}$$

A stream function $\Psi(x,y)$ is now defined such that

$$U = \frac{\partial \Psi}{\partial y} \;, \quad V = -\frac{\partial \Psi}{\partial x} \;. \tag{10}$$

It is now assumed that the stream function and the temperature can be expanded in the region close to the origin as follows:

$$\Psi(x,\eta) = 2x^{1/3}\left[F_0(\eta) + \varepsilon F_1(\eta) + \varepsilon^2 F_2(\eta) + \cdots\cdots\right],$$
$$T(x,\eta) = \frac{2}{27x^{1/3}}\left[H_0(\eta) + \varepsilon H_1(\eta) + \varepsilon^2 H_2(\eta) + \cdots\cdots\right]. \tag{11}$$

where
$$\eta = \frac{y}{3x^{2/3}} .$$

The similarity variable η and the leading term of the stream function expansion are of the same form as those in the classical jet solution of Schlichting (1933). The form of the first term of the expansion for the temperature is determined by the integral condition (8). Substitution of equations (10) and (11) into the last two of equations (1) suggests that it is natural to take

$$\varepsilon = x^{4/3} \tag{12}$$

In fact, the choice of other fractional powers of x such as $x^{1/3}$, $x^{2/3}$, etc., for ε is found to produce identically zero for the related functions $F_n(\eta)$ and $H_n(\eta)$. We are thus lead to the following sequence of ordinary differential equations for the functions $F_n(\eta)$ and $H_n(\eta)$:

$$F_0''' + 2F_0F_0'' + 2(F_0')^2 = 0 ,$$
$$H_0'' + 2\sigma(F_0H_0)' = 0 ,$$

$$F_1''' + 2F_0F_1'' - 4F_0'F_1' + 10F_0''F_1 = -H_0 ,$$
$$\frac{1}{\sigma}H_1'' + 2F_0H_1' - 6F_0'H_1 = -10F_1H_0' - 2F_1'H_0 ,$$

$$F_2''' + 2F_0F_2'' - 12F_0'F_2' + 18F_0''F_2 = 6(F_1')^2 - 10F_1F_1'' - H_1 \quad ,$$

$$\frac{1}{\sigma}H_2'' + 2F_0H_2' - 14F_0'H_2 = 6F_1'H_1 - 10F_1H_1' - 18F_2H_0' - 2F_2'H_0 \, ,$$

$$F_3''' + 2F_0F_3'' - 20F_3'F_0' + 26F_3F_0'' = 20F_2'F_1' - 10F_1F_2''$$
$$-18F_2F_1'' - H_2 \, , \qquad (13)$$

$$H_3'' + 2F_0H_3' - 22F_0'H_3 = 14F_1'H_2 - 10F_1H_2' + 6F_2'H_1$$
$$-18F_2H_1' - 2F_3'H_0 - 26F_3H_0' \, ,$$

etc.,

where the primes denote differentiation with respect to η. The boundary conditions (3) reduce to

$$F_n(0) = F_n''(0) = F_n'(\infty) = H_n'(0) = H_n(\infty) = 0 \qquad (14)$$

The first of equations (13) is easily integrated to yield the classical solution of Schlichting (1933) for the constant temperature free jet

$$F_0 = \tanh\eta \, . \qquad (15)$$

The second of equations (13) yields

$$H_0 = (\operatorname{sech}\eta)^{2\sigma} \, , \qquad (15)$$

where the integration constant has been chosen as unity for convenience. The solution for the two dimensional heated jet corresponding to equations (15) and (16) has been obtained previously by Yih (1950). The remaining equations of (13) were integrated numerically by means of a Runge-Kutta technique.

For the particular case of Prandtl number equal to two, the shapes of the velocity and temperature profiles for $x \rightarrow 0$ (obtained from equations (15) and (16)) are identical to the shapes associated with the asymptotic solution for $x \rightarrow \infty$ (cf. ex. Brand and Lahey 1967). It would appear reasonable to assume that the profile shapes do not change significantly for intermediate values of x. Thus we expect that an integral method using fixed temperature and velocity profiles (of the proper asymptotic form) should yield accurate solutions of the flow development for the particular Prandtl number of 2. Such an integral approach is outlined in the Appendix and it will be compared subsequently with the series expansion solutions. For Prandtl numbers other than 2, a more general integral method which allows for variations in the profile shapes with axial distance would be desirable.

The numerical solutions for the functions F_0' to F_3' and H_0 to H_3 for the case of $\sigma = 2$ are shown in Figures 1 and 2. The numerical constants in equations (6) and (8) are found to be

$$k_1 = \frac{16}{9} ,$$

$$k_2 = \frac{64}{405} \qquad \text{for } \sigma = 2 \tag{17}$$

where the contributions to the integrals in (6) and (8) come only from the zeroth order terms F_0' and H_0.

The distributions of the centreline velocity and temperature obtained from the direct series expansions and from the integral method (cf. Appendix) are shown in Figures 3 and 4. In its present form the series diverges at some downstream distance. We shall now proceed to a discussion of the solution valid at large distances, after which we shall return to a further discussion of the preceding work.

2.2 Inverse Expansions For Large x

We shall make use of the equations of motion in the form of equations (1), subject to boundary conditions of the form of equations (3). For large x the non-dimensional velocity components and temperature will be defined by the lower case symbols u, v and t. The variables are non-dimensionalized in a manner similar to equations (2) where the large x reference velocity U_L^*, temperature T_L^* and length L are chosen in the following way. The reference length L is taken to be the same as that determined for the small x case (cf. equation (9)). It is convenient to define

$$\frac{g\beta T_L^* L}{U_L^{*2}} = 1 \quad . \tag{18}$$

Analogous to equation (8) we may obtain

$$\int_{-\infty}^{\infty} ut\, dy = k_3 = \frac{\tilde{Q}\sqrt{\frac{U_L^* L}{\nu}}}{\tilde{\rho}\, c_p\, U_L^* T_L^* L} \quad , \tag{19}$$

where k_3 is a numerical constant.

From equations (18) and (19) and the previous definition for the reference length L, the large x reference velocity U_L^* and temperature T_L may be uniquely determined. They are related to the small x reference velocity and temperature by

$$U_L^* = \left(\frac{k_2}{k_3}\right)^{2/5} U_s^* \quad ,$$

$$T_L^* = \left(\frac{k_2}{k_3}\right)^{4/5} T_s^* \quad . \tag{20}$$

We define a stream function $\psi(x,y)$ such that

$$u = \frac{\partial\psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial\psi}{\partial x} \quad . \tag{21}$$

By introducing the similarity variable ζ, the momentum and energy equations can be expressed as

$$\psi_\zeta \psi_{x\zeta} - \frac{2}{5x}\psi_\zeta^2 - \psi_x \psi_{\zeta\zeta} = x^{4/5} t + \frac{1}{x^{2/5}}\psi_{\zeta\zeta\zeta} \quad ,$$

$$\psi_\zeta t_x - \psi_x t_\zeta = \frac{1}{\sigma x^{2/5}} t_{\zeta\zeta} \quad , \tag{22}$$

where $\zeta = \dfrac{y}{x^{2/5}}$

and the subscripts denote partial differentiation.

We now seek a solution which is close to the asymptotic solution for large x. Thus assume that

$$\left.\begin{aligned}\psi(x,\xi) &= \psi_0 + \psi_1 + \cdots\cdots , \\ t(x,\xi) &= t_0 + t_1 + \cdots\cdots , \\ \text{where} \quad \psi_0 &= x^{3/5} f_0(\xi) , \\ t_0 &= \frac{1}{x^{3/5}} h_0(\xi) ,\end{aligned}\right\} \tag{23}$$

and $\quad \psi_{i+1} \ll \psi_i \;, \quad t_{i+1} \ll t_i \;.$

After substituting equations (23) into (22) the zeroth order problem is found to be

$$\begin{aligned} f_0''' + \tfrac{3}{5} f_0 f_0'' - \tfrac{1}{5}(f_0')^2 &= -h_0 , \\ h_0'' + \tfrac{3}{5}\sigma (f_0 h_0)' &= 0 \end{aligned} \tag{24}$$

subject to the boundary conditions

$$f_0(0) = f_0''(0) = f_0'(\infty) = h_0'(0) = h_0(\infty) = 0 . \tag{25}$$

These zeroth order equations describe the laminar plumes studied by Fujii (1963), Spalding and Cruddace (1961), Crane (1959) and others. The solution for $\sigma = 2$ may be obtained in closed form as

$$\begin{aligned} f_0 &= \tanh \tfrac{3}{10}\xi , \\ h_0 &= \frac{9}{125}\,\mathrm{sech}^4\left(\frac{3\xi}{10}\right) . \end{aligned} \tag{26}$$

The first order equations are

$$+\frac{3}{5} f_0 \psi_{1\xi\xi} + \frac{1}{5} f_0' \psi_{1\xi} - x\left[f_0' \psi_{1x\xi} - f_0'' \psi_{1x}\right] = -x^{6/5} t_1 \ ,$$

$$t_{1\xi\xi} + \frac{3}{5} f_0 t_{1\xi} - x f_0' t_{1x} = -\frac{3}{5} \frac{h_0}{x^{6/5}} \psi_{1\xi} - \frac{h_0'}{x^{1/5}} \psi_{1x} \ , \tag{27}$$

subject to the boundary conditions

$$(x,0) = \psi_{1\xi\xi}(x,0) = \psi_{1\xi}(x,\infty) = t_{1\xi}(x,0) = t_1(x,\infty) = 0 \ . \tag{28}$$

Separation of variables is now assumed, whereby we write

$$\psi_1(x,\xi) = Z_1(\xi)\, N_1(x) \ ,$$

$$t_1(x,\xi) = Y_1(\xi)\, M_1(x) \ . \tag{29}$$

Substitution of equations (29) into (27) yields

$$N_1 = C_1 x^{-\lambda_1} \ ,$$

$$M_1 = C_1 x^{-(\lambda_1 + \frac{6}{5})} \ ,$$

$$Z_1''' + \frac{3}{5} f_0 Z_1'' + \left(\lambda_1 + \frac{1}{5}\right) f_0' Z_1' - \lambda_1 f_0'' Z_1 = -Y_1 \ , \tag{30}$$

$$\frac{1}{\sigma} Y_1'' + \frac{3}{5} f_0 Y_1' + \left(\lambda_1 + \frac{6}{5}\right) f_0' Y_1 = -\frac{3}{5} h_0 Z_1' + \lambda_1 h_0' Z_1 \ .$$

The boundary conditions on Z_1 and Y_1 are

$$Z_1(0) = Z_1''(0) = Z_1'(\infty) = Y_1'(0) = Y_1(\infty) = 0 \ . \tag{31}$$

Equations (30) and (31) constitute an eigenvalue problem somewhat reminiscent of the investigation of the perturbed Blasius solutions carried out by Libby and Fox (1963). Only for discrete values of $\lambda_1 = \lambda_{1,k}$ can all of the boundary conditions (31) be satisfied. Thus the solutions for ψ_1 and t_1 are

$$\psi_1(x,y) = \sum_{k=1}^{\infty} C_{1,k}\, x^{-\lambda_{1,k}}\, Z_{1,k}(y) \quad ,$$
$$t_1(x,y) = \sum_{k=1}^{\infty} C_{1,k}\, x^{-(\lambda_{1,k}+\frac{6}{5})}\, Y_{1,k}(y) \quad . \qquad (32)$$

Our purposes are quite different from those of Libby and Fox (1963) and we need not compute a large number of eigenvalues and eigenfunctions.

Solutions to the last two of equations (30) were obtained numerically by a Runge-Kutta method. For convenience $Z'_{1,k}(0)$ may be chosen as unity. Using trial values for $\lambda_{1,k}$ and $Y_{1,k}(0)$ the equations were integrated from $y = 0$ to a suitably large value of y. The proper values of $\lambda_{1,k}$ and $Y_{1,k}(0)$ were selected when the boundary conditions for $y \to \infty$ were satisfied. The constants $C_{1,k}$ are related to the initial conditions at the virtual origin and can be determined by some joining or matching procedure. For a Prandtl number of 2, the first eigenvalue was found to $\lambda_{1,1} = 2/5$. The corresponding eigenfunctions $Z'_{1,1}(y)$ and $Y_{1,1}(y)$ are shown in Figures 5 and 6 along with $f'_0(y)$ and $h_0(y)$.*

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* The second eigenvalue for $\sigma = 2$ is found to be $\lambda_{1,2} = 1.800$, but because of the difficulties involved in joining the "large x" and "small x" expansions, only the first eigenfunctions are shown here.

2.3 Transformation of the Direct Expansion

From Figures 3 and 4 it is evident that the direct expansions for velocity and temperature obtained from equations (11) have utility only for small values of x. It is suspected that convergence of these series is restricted by singularities on the negative axis of the complex plane (cf. Van Dyke 1964b). Accordingly we shall attempt to extend the radius of convergence of each series to infinity by applying the Euler transformation. Attention will be restricted solely to the centreline velocity and temperature. However, the Euler transformation may be applied to the complete velocity and temperature profiles similar to the way in which Van Dyke (1964a) has applied it to the complete stream function in the boundary layer on a parabola.

It will be convenient to rearrange the expressions for the centreline velocity and temperature in the following form:

$$\varepsilon^{-3/20}\,\frac{\tilde{U}(0)}{U_s^*} = \frac{2}{3}\,\varepsilon^{-2/5}\left[F_0'(0) + \varepsilon F_1'(0) + \varepsilon^2 F_2'(0) + \varepsilon^3 F_3'(0) + \dots\right],$$

$$\varepsilon^{-9/20}\left(\frac{\tilde{T}(0) - \tilde{T}_\infty}{T_s^*}\right) = \frac{2}{27}\,\varepsilon^{1/5}\left[H_0(0) + \varepsilon H_1(0) + \varepsilon^2 H_2(0) + \varepsilon^3 H_3(0) + \dots\right]. \qquad (33)$$

Recasting the above series in terms of a new variable

$$\bar{\varepsilon} = \frac{\varepsilon}{1+\varepsilon} \qquad (34)$$

and making use of the numerical solutions for $F_k'(0)$ and $H_k(0)$ we obtain

$$\frac{\tilde{U}(0)}{U_s^*} = \frac{2}{3}(\bar{\varepsilon})^{-2/5}\left[1 - 0.29286\,\bar{\varepsilon} - 0.06340(\bar{\varepsilon})^2 - 0.02391(\bar{\varepsilon})^3 + \dots\right]$$

(35)

$$\left(\frac{\tilde{T}(0) - \tilde{T}_\infty}{T_s^*}\right) = \frac{2}{27}(\bar{\varepsilon})^{1/5}\left[1 + 0.18214\,\bar{\varepsilon} + 0.09982(\bar{\varepsilon})^2 + 0.06705(\bar{\varepsilon})^3 + \dots\right]$$

Hopefully the above transformed series converge to the exact solution for all x. It is straightforward to compare equations (35) for $\bar{\varepsilon} \to 1$ (i.e., $x \to \infty$) with the asymptotic solution for large x. From equations (20), (21), (23) and (26) we obtain

$$\varepsilon^{-3/20}\,\frac{\tilde{U}(0)}{U_s^*} = \left(\frac{k_2}{k_3}\right)^{3/5} f_0'(0) = 0.4004\,, \qquad (36)$$

$$\varepsilon^{9/20}\left(\frac{\tilde{T}(0) - \tilde{T}_\infty}{T_s^*}\right) = \left(\frac{k_2}{k_3}\right)^{4/5} h_0(0) = 0.1283$$

for the asymptotic large x solution.

The successive partial sums of first of equations (35) for $\bar{\varepsilon} \to 1$ are

$$\varepsilon^{-3/20}\,\frac{\tilde{U}(0)}{U_s^*} = 0.6667\,,\ 0.4714\,,\ 0.4292\,,\ 0.4132\,,\ \dots\dots \qquad (37)$$

These appear very likely to be converging quite rapidly to the required value of 0.4004. Figure 7 shows the variation of the centreline velocity as obtained from the successive partial sums of the transformed direct series. Four terms of the transformed series are seen to agree very well with the integral solution (cf. Appendix) which merges into the asymptotic solutions valid for both small and large values of x.

The successive partial sums of the second of equations (35) for $\bar{\varepsilon} \to 1$ are

$$\varepsilon^{9/20}\left(\frac{\tilde{T}(0)-\tilde{T}_\infty}{T_s^*}\right) = 0.07407,\ 0.08757,\ 0.09496,\ 0.1000,\ \ldots \tag{38}$$

Here, the series is converging more slowly, but it seems possible that the partial sums are approaching the required value of 0.1283. Figure 8 compares the centreline temperature distribution obtained from the transformed direct series with that from the integral solution.

2.4 Determination of Constants in Inverse Expansion

It was originally anticipated that the direct and inverse coordinate expansions could be joined in a manner similar to that followed by Van Dyke (1964a). However, it is not obvious how this may be done for the present case and we have had to resort to a crude numerical patching. The constant $C_{1,1}$ for the first eigen-solution was chosen to be approximately 0.18 by patching with the integral solution at large finite values of x. The two term inverse expansions for the centreline velocity and temperature using this value of $C_{1,1}$ are shown in Figures 9 and 10. The constants associated with the higher eigensolutions can be determined, but because of the rather arbitrary manner of the numerical patching, we shall not bother to show them here.

3. CONCLUDING REMARKS

The flow developed by a buoyant two-dimensional vertical laminar jet has been analysed by means of direct and inverse coordinate expansions valid at small and large distances from the virtual origin. After applying the Euler transformation to the direct expansions, it is found that for a Prandtl number of 2 the partial sum of the first form terms for the centreline velocity agrees to within 3% of the asymptotic solution for large x. The agreement for the temperature distribution is not as good and here the four term direct series underestimates the centreline temperature by some 22% as $x \to \infty$. Comparisons have also been made with a simple integral method which is expected to be valid for a Prandtl number of 2. The centreline velocity was found to decrease with distance from the virtual origin, reach a minimum and thereafter increase, whereas the centreline temperature was found to decrease monotonically. From a physical standpoint this type of behaviour is to be expected. Close to the orifice where bouyancy effects are negligible the flow develops like a momentum jet. Far from the orifice where the initial conditions of momentum flux have been "forgotten" and buoyancy effects are essential, the flow behaves like a buoyant laminar plume.

The series expansions could, of course, be carried out for other values of Prandtl number. However, it should be quite straightforward to work out an integral method which is more flexible than the present one in order to produce reasonable accuracy for Prandtl numbers other than 2. The asymptotic solutions for large and small x could be used as a guide for the choice of velocity and temperature profiles.

With some suitable assumption regarding eddy viscosity, series expansions could be carried out for the turbulent case. Because of the rather arbitrary nature of such assumptions, a simpler integral approach is undoubtedly more appropriate at this time.

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APPENDIX. INTEGRAL METHOD

Consider equations (1), (2) and (3) in which the reference temperature, velocity and length are the "small-x" values given by equation (9). By integrating the momentum equation between y = 0 and y = ∞ and making use of the continuity equation we obtain

$$\frac{d}{dx}\int_0^\infty U^2 dy = \int_0^\infty T dy \quad . \tag{A1}$$

The following expressions are assumed for the velocity and temperature profiles :

$$U(x,y) = a_1(x) \operatorname{sech}^2 \frac{y}{b(x)} \quad , \tag{A2}$$

$$T(x,y) = a_2(x) \operatorname{sech}^4 \frac{y}{b(x)} \quad . \tag{A3}$$

The above profile shapes should be reasonable for a Prandtl number, σ = 2, since they are of the same form as the asymptotic solutions for both small and large x. For other Prandtl numbers it would be more precise to adopt a general integral approach which permitted the profile shapes to vary rather than remain fixed as above.

Substituting equations (A2) and (A3) in (A1) we obtain

$$2a_1 b \frac{da_1}{dx} + a_1^2 \frac{db}{dx} = a_2 b \tag{A4}$$

Applying the momentum equation of (1) at y = 0 gives

$$\frac{da_1}{dx} = \frac{a_2}{a_1} - \frac{2}{b^2} \tag{A5}$$

Substituting (A2) and (A3) in equation (8) yields

$$\frac{16}{15} a_1 a_2 b = k_2 \tag{A6}$$

where $k_2 = \frac{64}{405}$ for $\sigma = 2$ as obtained from equation (17).

Equations (A4), (A5) and (A6) can be reduced to two first order simultaneous ordinary differential equations of the form

$$\begin{aligned} \frac{da_1}{dx} &= \frac{4}{27 a_1^2 b} - \frac{2}{b^2} \\ \frac{db}{dx} &= \frac{4}{a_1 b} - \frac{4}{27 a_1^3} \end{aligned} \tag{A7}$$

Equations (A7) were integrated by means of a Runge-Kutta method using the "small x" zeroth order solution to provide starting values. The solutions to (A7) combined with equation (A6) then provide complete information concerning the flow field.

REFERENCES

BENDOR, E. 1956 Some Studies in Natural Convection. Ph.D. Thesis, University of London, London.

BRAND, R.S. and LAHEY, F.J. 1967 J. Fluid Mech. 29, 305.

BRODOWICZ, K. and KIERKUS, W.T. 1966 Int. J. Heat Mass Transfer 9, 81.

CRANE, L.J. 1959 Z. Angew. Math. Phys. 10, 453.

FORSTROM, R.J. and SPARROW, E.N. 1967 Int. J. Heat Mass Transfer 10, 321.

FUJII, T. 1963 Int. J. Heat Mass Transfer 6, 597.

LIBBY, P.A. and FOX, H. 1963 J. Fluid Mech. 17, 433.

MAHONY, J.J. 1957 Proc. Roy. Soc. A 238, 412.

OSTRACH, S. 1964 Laminar Flows with Body Forces. Theory of Laminar Flows, (ed. F.K. Moore) Princeton: Princeton University Press.

SCHLICHTING, H. 1933 Z. Angew. Math. Mech. 13, 260.

SCHUH, H. 1948 Boundary Layers of Temperature. Boundary Layers (ed. W. Tollmien), Brit. Ministry of Supply, German Document Center Ref. 3220T.

SERRUK, I.G. 1958 J. Appl. Math. Mech. 22, 807.

SPALDING, D.B. and CRUDDACE, R.G. 1961 Int. J. Heat Mass Transfer 3, 55.

VAN DYKE, M. 1964a J. Fluid Mech. 19, 145.

VAN DYKE, M. 1964b Perturbation Methods in Fluid Mechanics. New York; Academic Press.

WYGNANSKI, I. 1967 J. Fluid Mech. 27, 431.

YIH, C.S. 1950 J. Appl. Mech. 17, 381.

YIH, C.S. 1952 Free Convection Due to a Point Source of Heat. Proc. First U.S. Nat. Cong. Appl. Mech., Chicago 1951, p. 941.

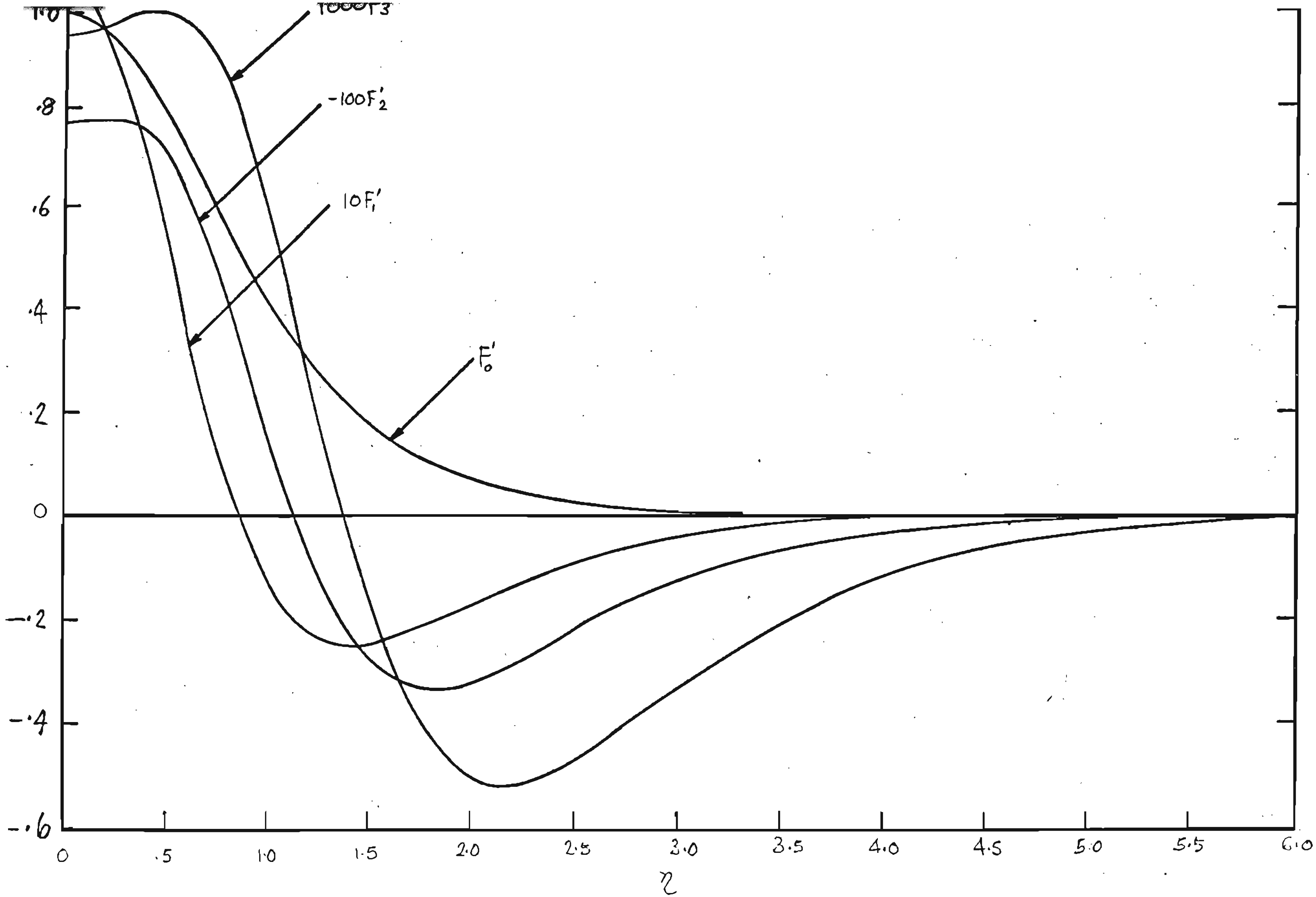

FIGURE 1. Velocity functions, F_0' to F_3'

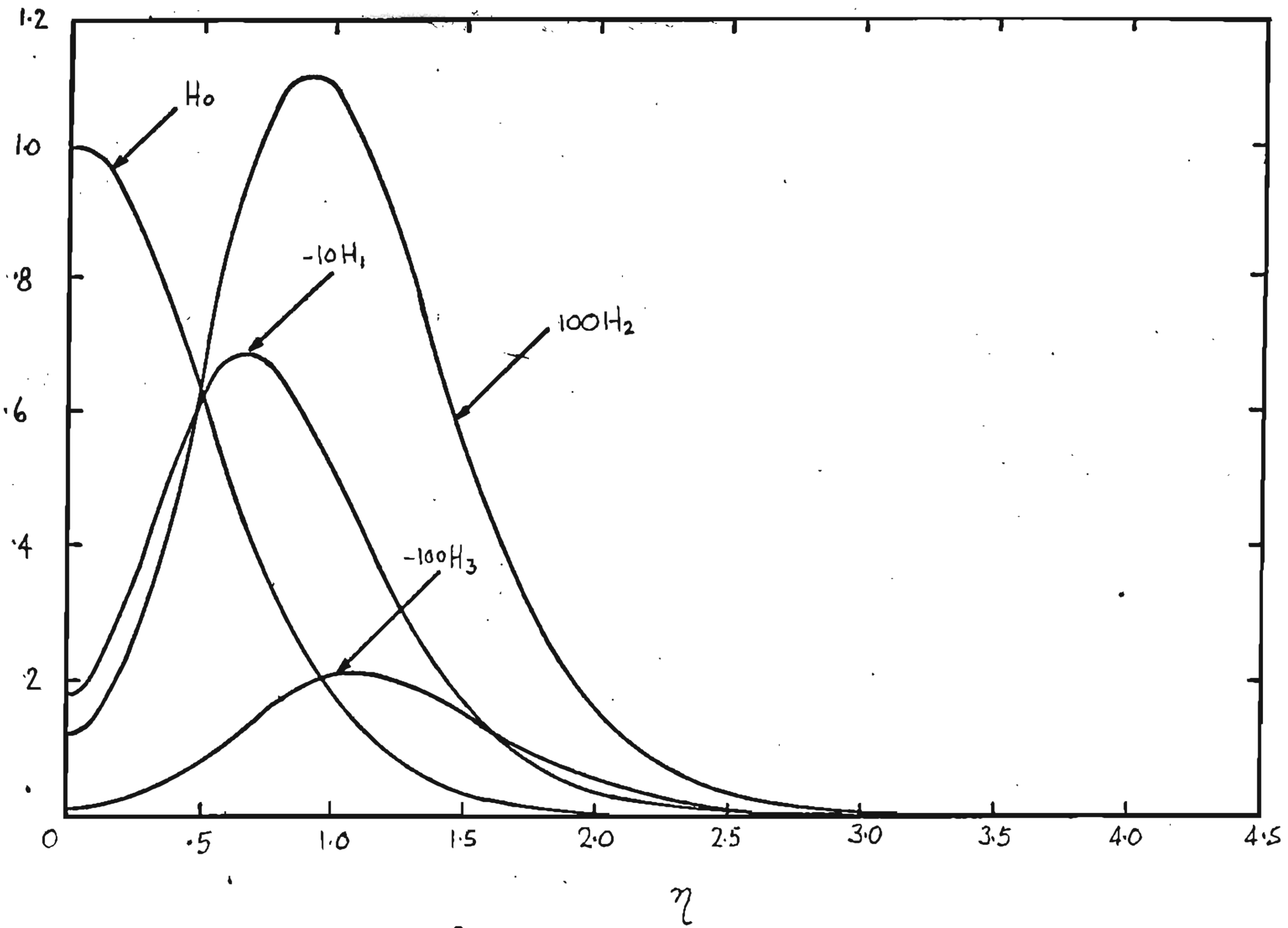

FIGURE 2: Temperature functions, H_0 to H_3.

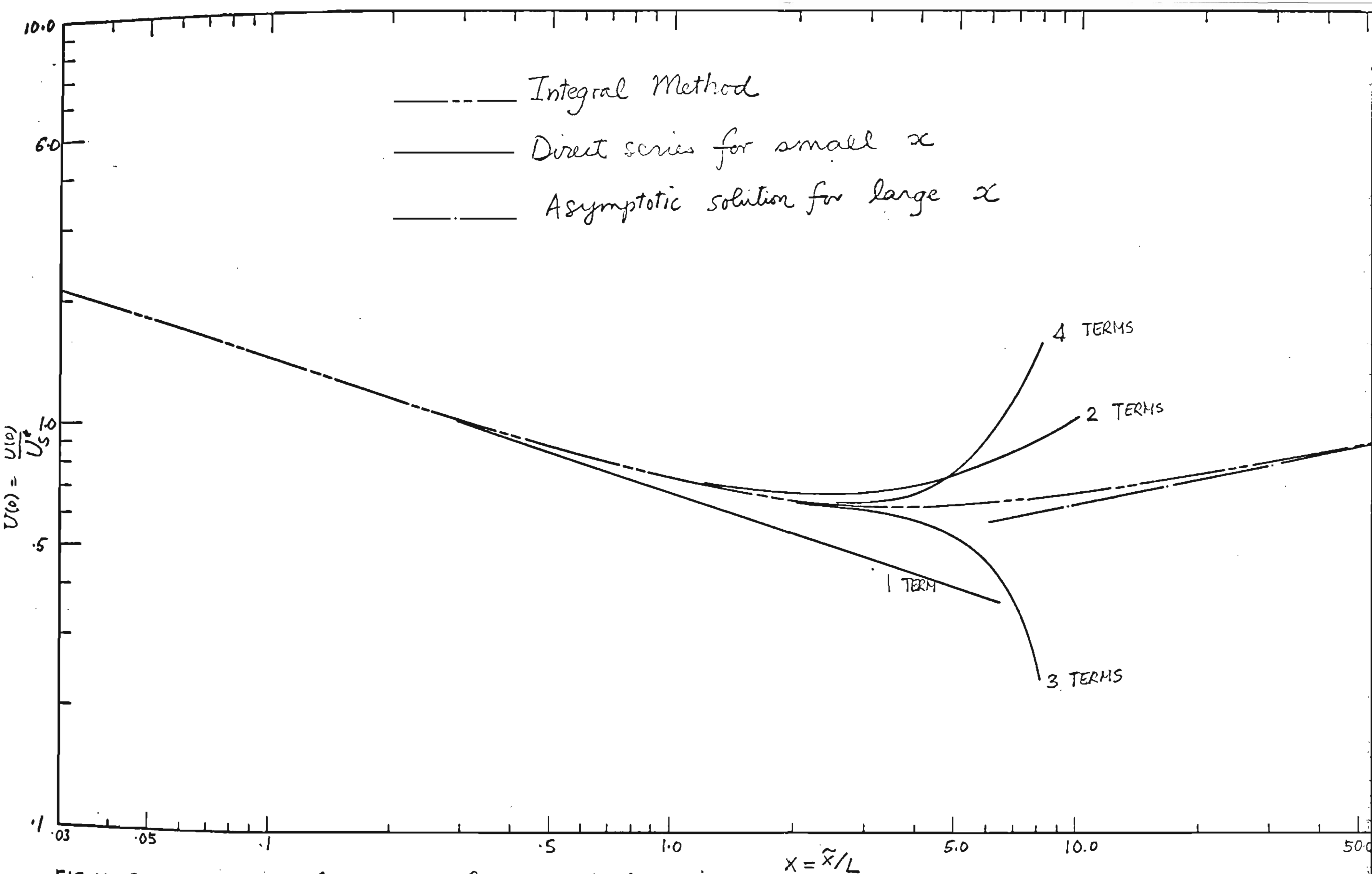

FIGURE 3. Variation of velocity along vertical axis of symmetry.

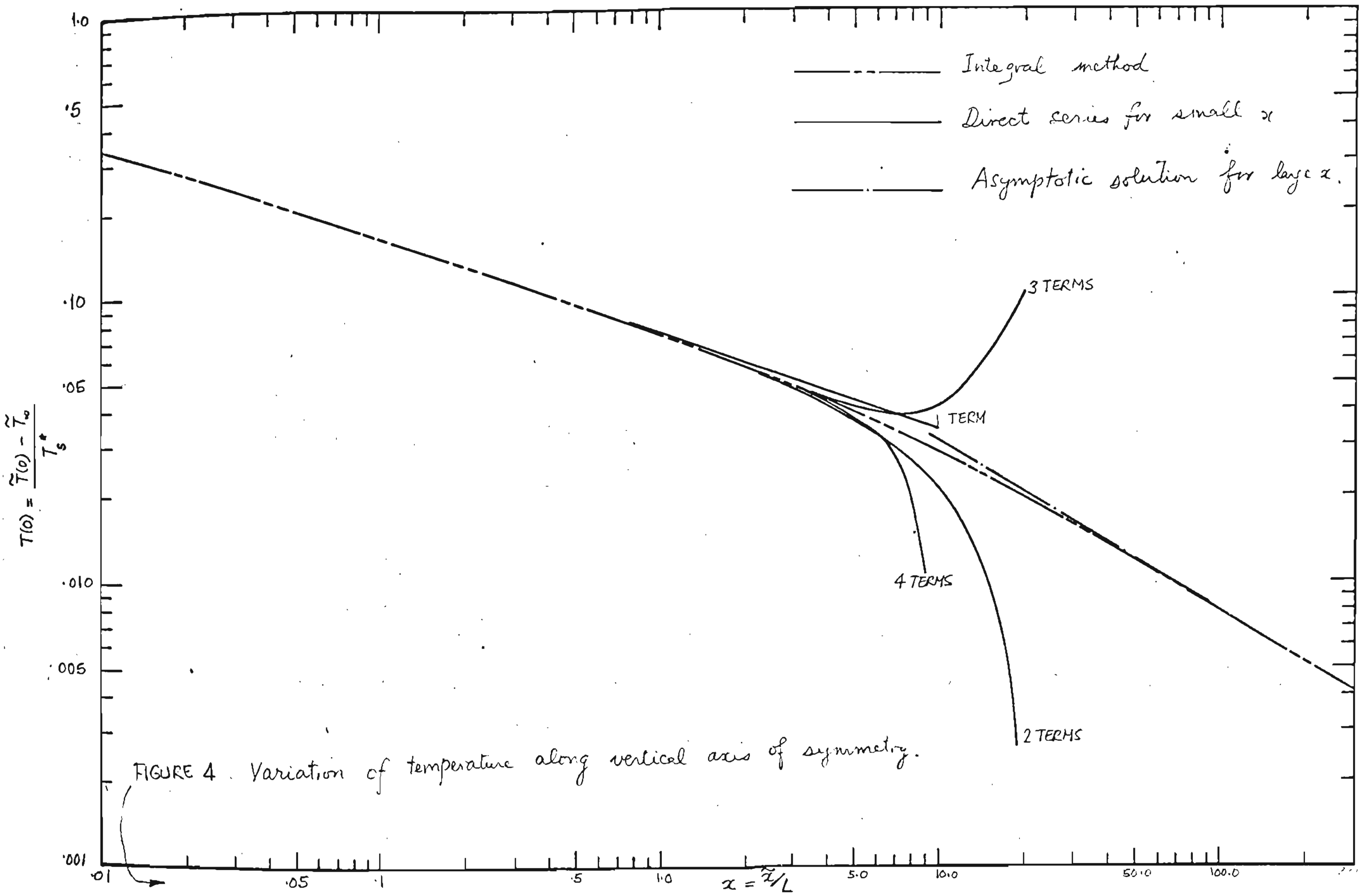

FIGURE 4. Variation of temperature along vertical axis of symmetry.

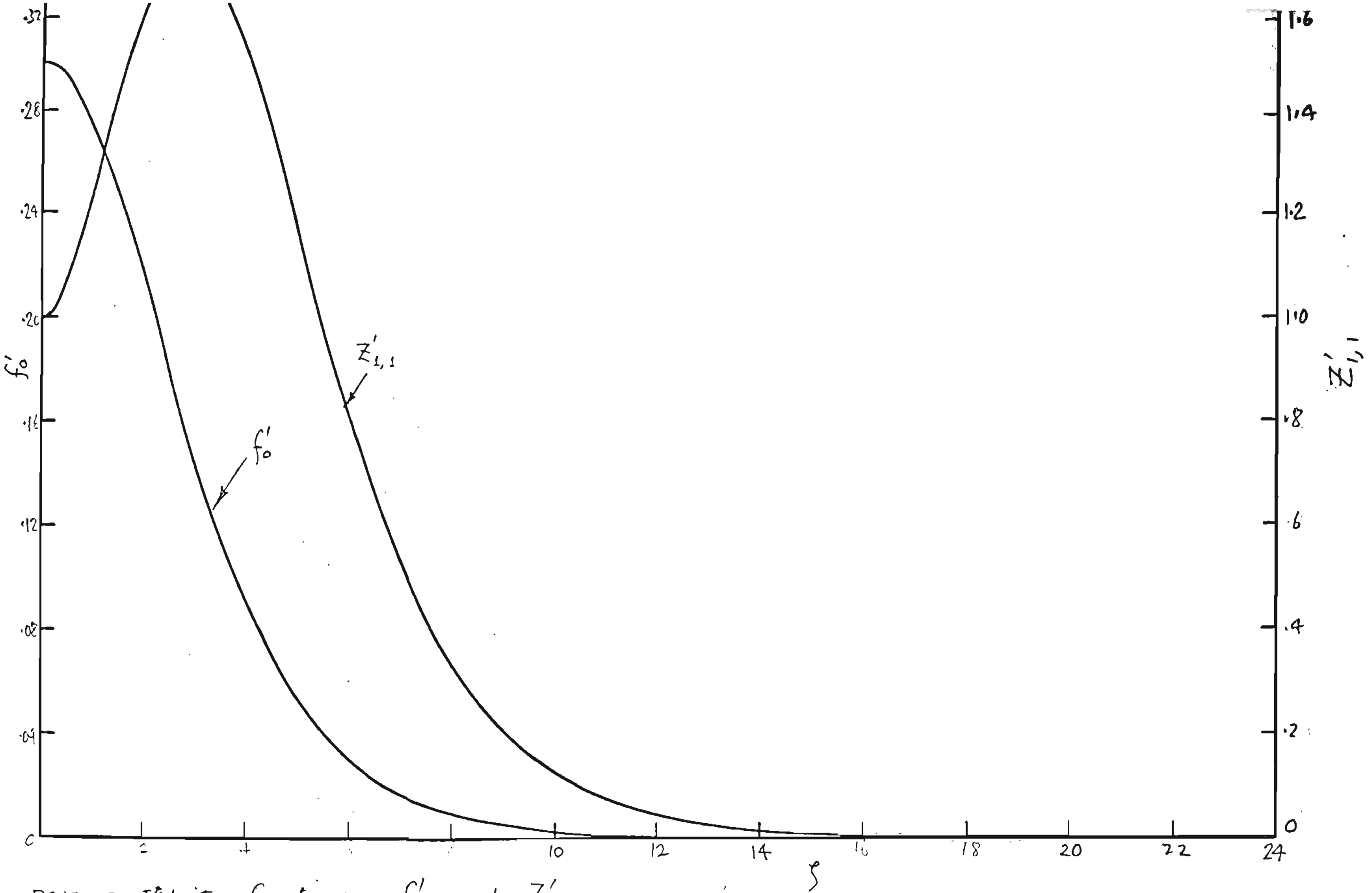

FIGURE 5. Velocity functions, f_0' and $Z_{1,1}'$

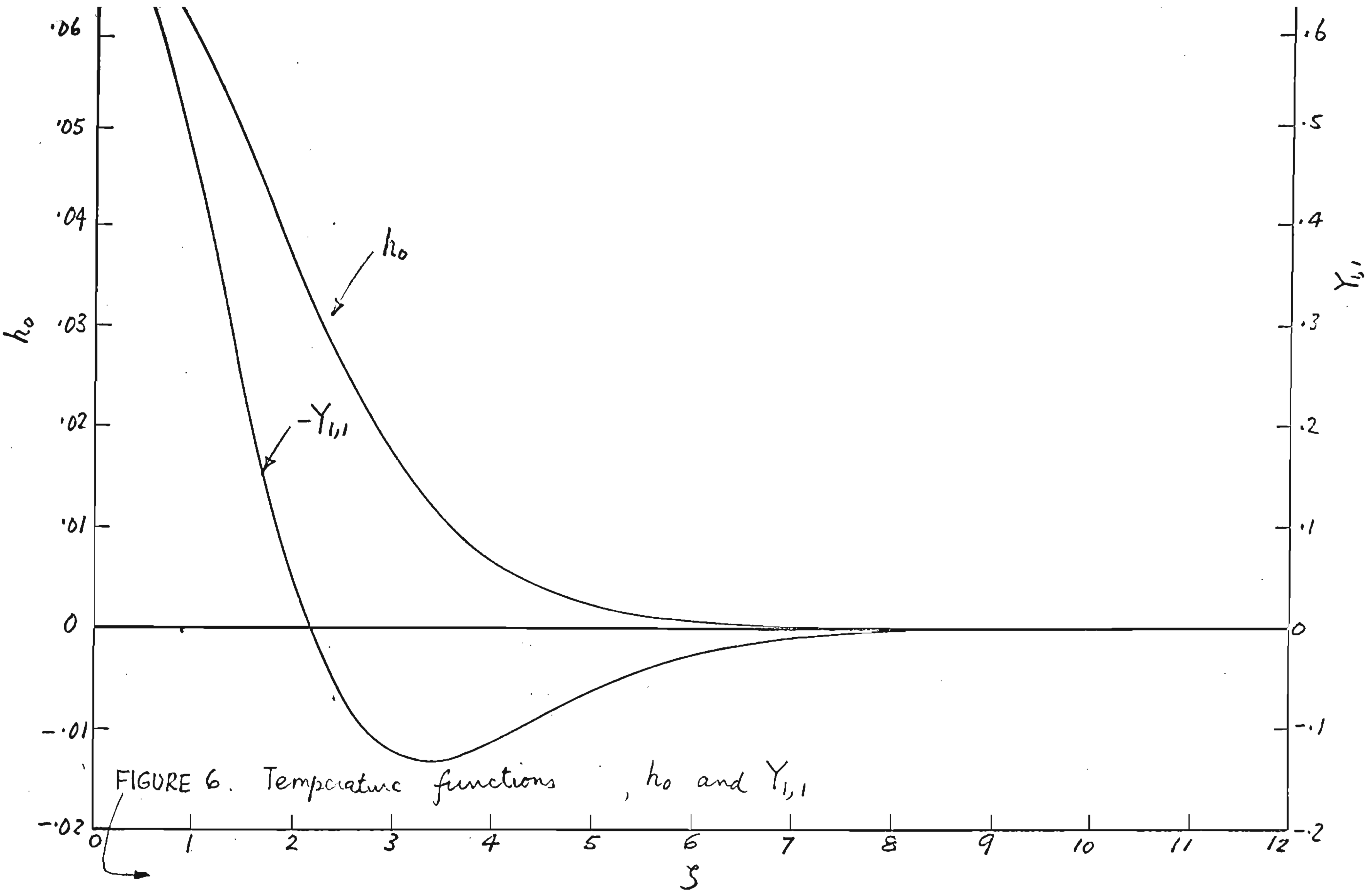

FIGURE 6. Temperature functions , h_0 and $Y_{1,1}$

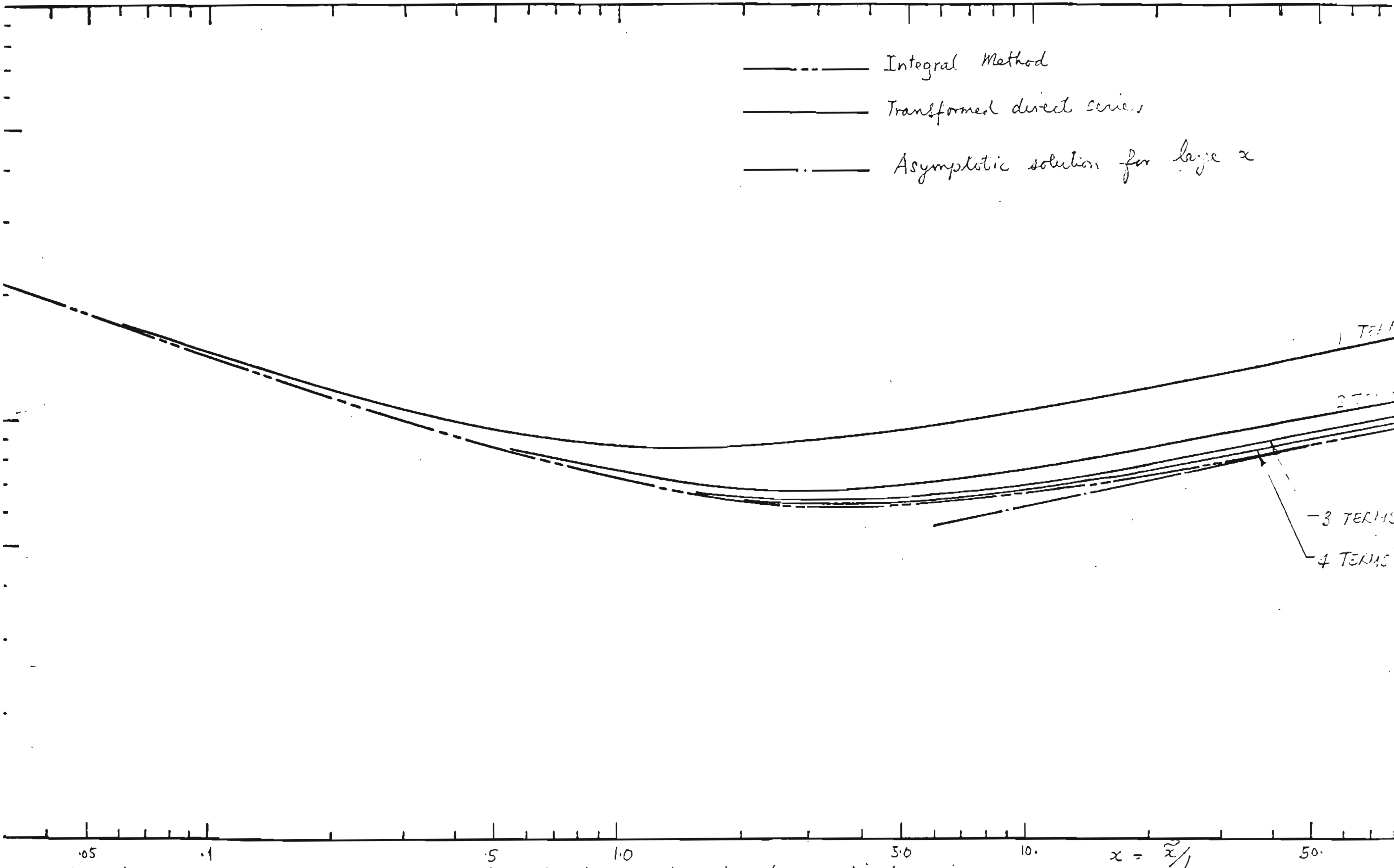

FIGURE 7. Variation of velocity along vertical axis of symmetry, transformed direct series.

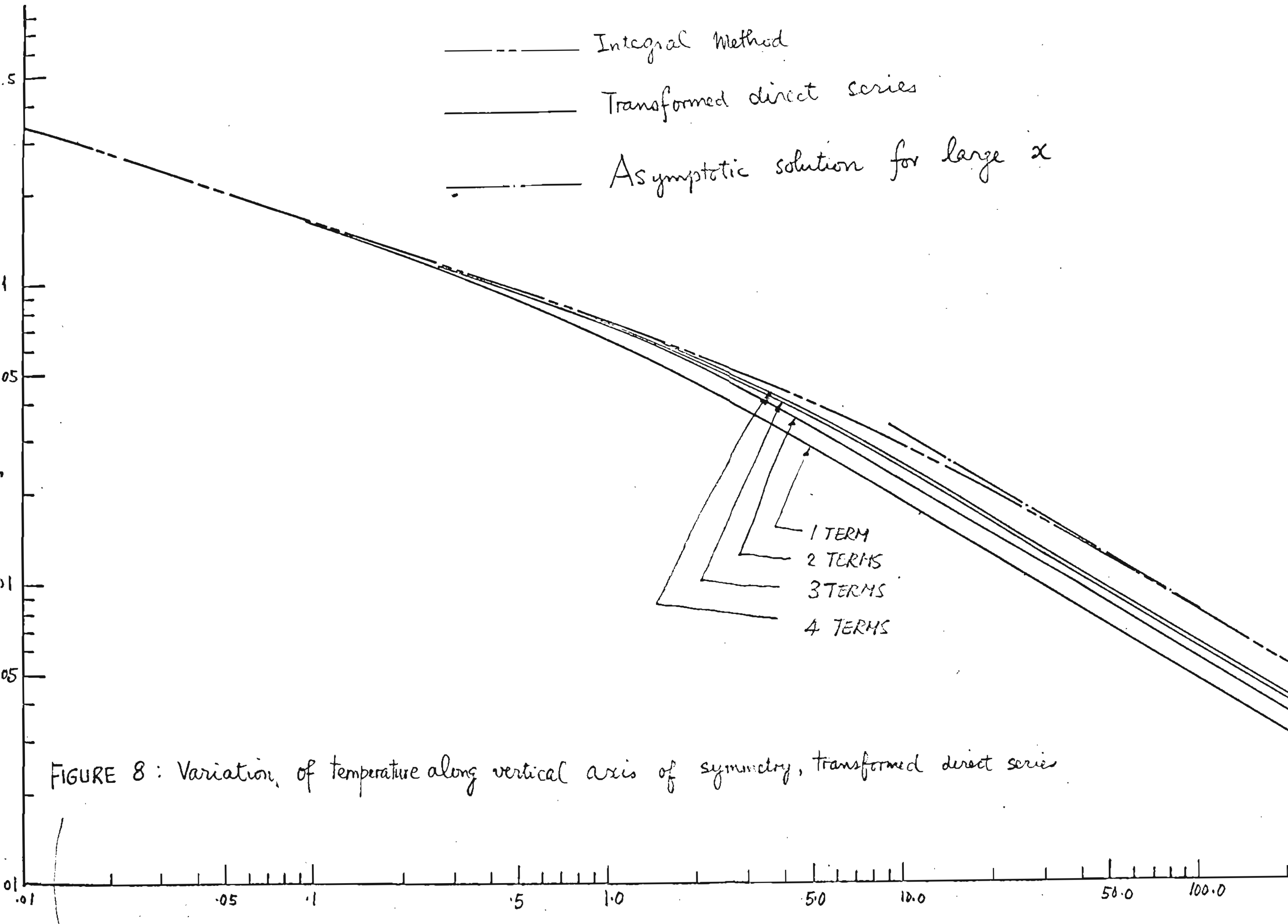

FIGURE 8 : Variation of temperature along vertical axis of symmetry, transformed direct series

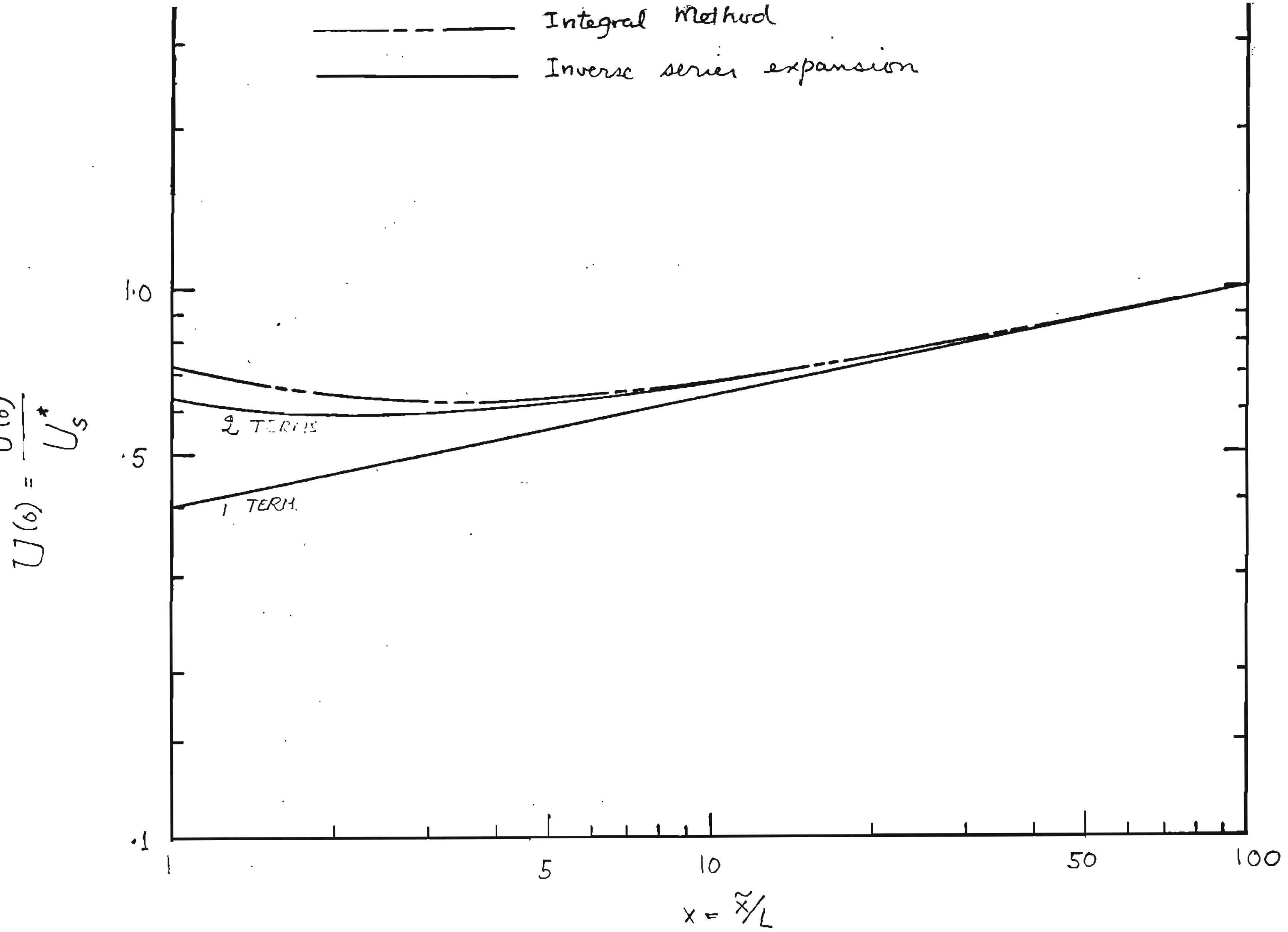

FIGURE 9. Variation of velocity along vertical axis of symmetry at large x.

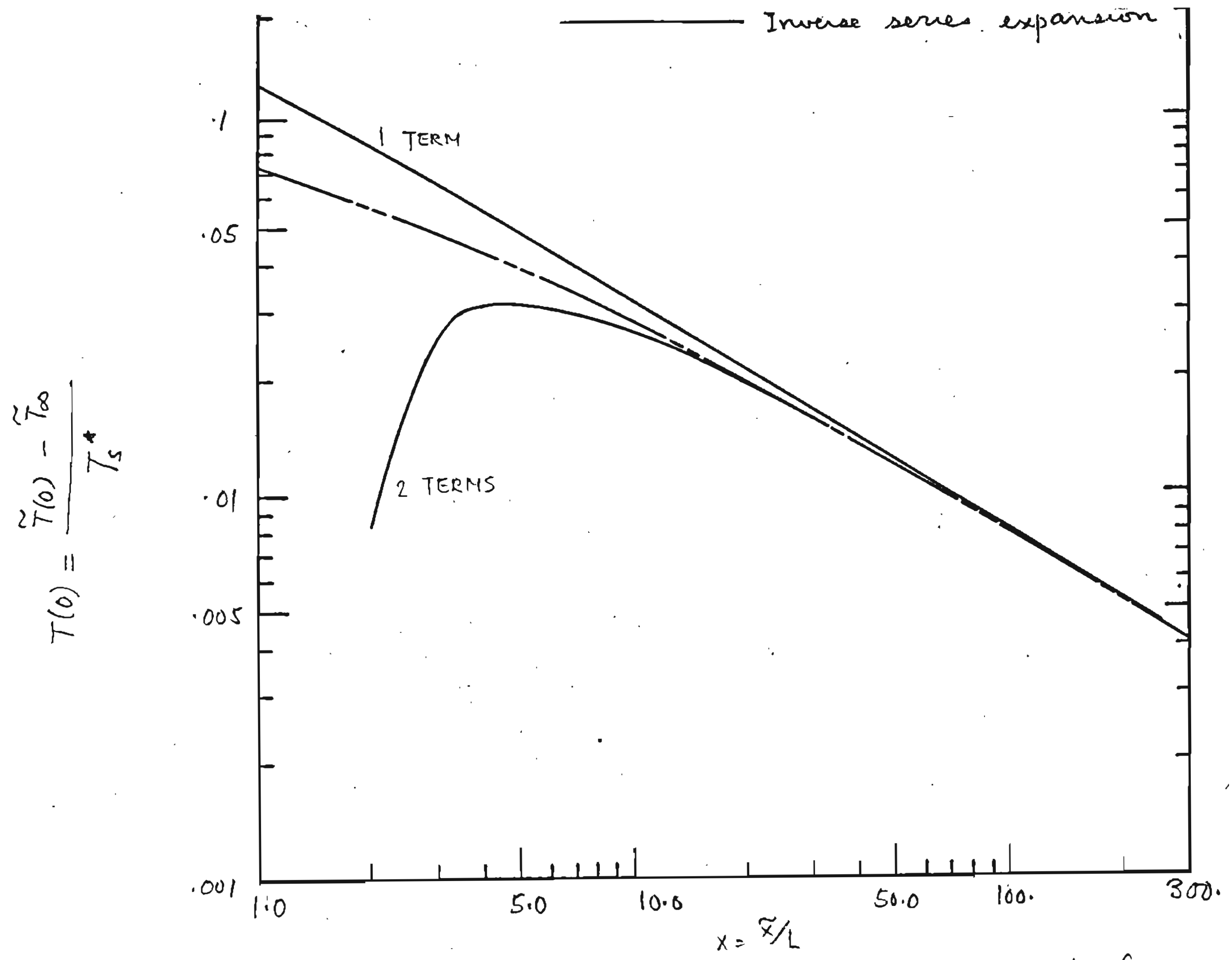

FIGURE 10. Variation of temperature along vertical axis of symmetry at large x.