High Energy Physics and the Early Universe

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Abstract

This thesis comprises five papers written during the author's tenure as a PhD candidate at McGill University. Building on the author's M.Sc. work, these papers make progress towards understanding our universe as it emerges from string theory, and on implications of established high energy physics for cosmology. The thesis is split into 3 parts: (1) string theory, (2) string cosmology, and (3) cosmology. Part (1) investigates a particular class of backgrounds in string theory, and studies the breaking of supersymmetry in these backgrounds. With this mind, part (2) studies cosmology in these setups, working directly in the four-dimensional description. Finally, part (3) departs from (1) and (2) and instead focuses on quantum field theory effects in curved space. The implications of stochasticity in coupled differential equations, describing UV and IR cosmological perturbations, are studied and corrections to cosmological observables are found.

Abrégé

Cette thèse est composée de cinq articles écrits durant la période de stage des chercheurs à l'Université McGill. Sur la base de M.Sc. Ces documents font des progrès vers la compréhension de notre univers tel qu'il ressort de la théorie des cordes et sur les implications de la physique des hautes énergies pour la cosmologie. La thèse est divisée en 3 parties: (1) théorie des cordes, (2) cosmologie des cordes et (3) cosmologie. La partie (1) étudie une classe particulière d'arrière-plans dans la théorie des cordes et étudie la rupture de la supersymétrie dans ces milieux. Avec cet esprit, une partie (2) étudie la cosmologie dans ces configurations, travaillant directement dans la description en quatre dimensions. Enfin, la partie (3) s'écarte des (1) et (2) et se concentre plutôt sur les effets de la théorie des champs quantiques dans l'espace incurvé. Les implications de la stochasticité dans les équations différentielles couplées, décrivant les perturbations cosmologiques UV et IR, sont étudiées et des corrections aux observables cosmologiques sont trouvées.

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Preface

Preface

This work consists of an introduction and five peer-reviewed and published articles. Chapters 1 and 2 will review cosmological perturbation theory and string theory respectively. The contents of Chapters 3-7 are original scholarship, and are distinct contributions to the body of knowledge. It is customary in high energy physics to list author names in alphabetical order, and not in order of contribution. In this preface I will identify my contribution to each work.

Contributions of the Author

Chapter 3: K. Dasgupta, M. Emelin and E. McDonough, *Non-Kahler Resolved Conifold, Localized Fluxes in M-Theory and Supersymmetry*, JHEP 1502 (2015) 179 [arXiv:1412.3123].

I initiated this project as a study of the M-theory origin of the D-terms used to construct de Sitter solutions in type IIB string theory. The project took a interesting turn when K. Dasgupta pointed out the requisite IIB 'background' solutions were not fully known, though the tools to create them had been proposed. K. Dasgupta, M. Emelin, and myself then proceeded with the calculations to construct these solutions. The writing of the text was shared between myself and K. Dasgupta.

Chapter 4: K. Dasgupta, M. Emelin and E. McDonough, *Fermions on the Anti-Brane: Higher Order Interactions and Spontaneously Broken Supersymmetry*, Phys. Rev. D 95, 026003 , arXiv:1601.03409 [hep-th].

I initiated this work as a follow-up to my previous work with K. Dasgupta and M. Emelin, that would take a step towards cosmology. I performed all calculations in the paper and wrote the majority of the text. K. Dasgupta contributed the insight that an analysis like Chapter 4.4 was possible. The three authors performed the calculations in tandem.

Chapter 5: E. McDonough, H.B. Moghaddam and R.H. Brandenberger, *Preheating and Entropy Perturbations in Axion Monodromy Inflation*, JCAP 1605 (2016) 012 [arXiv:1601.07749]

I initiated and chose the direction for this project. I performed all the calculations, often in tandem with H.B. Moghaddam. The text of the paper was written by myself in collaboration with R. Brandenberger.

Chapter 6: E. McDonough and M. Scalisi, *Inflation from Nilpotent Kahler Corrections*, JCAP 1611, no. 11, 028 (2016). arXiv:1609.00364 [hep-th].

I initiated this project as a follow-up to my earlier work with K. Dasgupta and M. Emelin. I conceived of the idea and performed all calculations. M. Scalisi suggested we also study hyperbolic geometry, and we did the calculations in tandem. The writing of the text was shared between the authors.

Chapter 7: L. Perrault-Levasseur and E. McDonough *Backreaction and Stochastic Effects in Single Field Inflation*, Phys.Rev. D91 (2015) 063513 arXiv:1409.7399[hep-th].

I initiated this project as a minimal working example of the formalism developed in earlier papers by L. Perrault-Levasseur. I performed all analytical calculations while L. Perrault-Levasseur performed the numerical calculations. I wrote the majority of the text of the paper.

Part I

Introduction

Chapter 1

Review of Cosmological Perturbation Theory

1.1 Introduction

This chapter will review the basics of *Cosmological Perturbation Theory*: the 'workhorse' of modern theoretical cosmology. This is not a theory unto itself in the traditional sense, but rather a prescription for the combined application of quantum field theory and general relativity in a cosmological spacetime. This is different from the usual quantum field theory in curved space [1], as cosmological perturbation theory allows the spacetime itself to fluctuate, with the dynamics of these fluctuations governed by general relativity.

1.2 Prelude: Inflation

In this thesis we will focus on *inflationary cosmology*, although much of the discussion carries over to more general cosmologies (e.g. bouncing cosmology). With this in mind, we will motivate a thorough treatment of cosmological perturbation theory by first studying inflation.

Standard big bang cosmology was immensely successful in explaining the elemental abundances and formation of structure in our universe. However this theory suffers from serious initial conditions problems, both at an aesthetic level (e.g. tuning the initial state to be spatially flat), and a conceptual level (e.g. by assuming large scale correlations but providing no mechanism to produce them). The idea of inflationary cosmology is to insert a period of accelerated expansion before the beginning of standard big bang cosmology, which serves to flatten and homogenize the universe, and provides a causal mechanism for the generation of large scale correlations in the spectrum of energy density fluctuations we see today.

The universe in this phase is described by the metric of the FRW type,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\mathrm{d}x^2, \tag{1.1}$$

where $a(t)$ is the scale factor, which during inflation has the form

$$a(t) = e^{Ht} \tag{1.2}$$

where H, the Hubble parameter, is nearly constant. It is important to note that distances measured with respect to the x coordinate appearing in the above metric are *comoving* length scales, which are related to *physical* length scales be a factor of $a(t)$, i.e. $x_p = a(t)x_c$ where p and c denote physical and comoving respectively. Similarly, in momentum space one can define the physical wave number $k_p = 1/x_p$, which is related to the comoving wave number by $k_c = a(t)k_p$. This distinction will become important when considering the evolution of perturbations during inflation. However, in keeping with tradition, we henceforth drop all p and c subscripts, and x and k will always denote comoving length and momentum scales.

The geometry described above can be easily generated by the dynamics of a scalar field minimally coupled to gravity, described by the action[1]

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(R + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)\right). \tag{1.3}$$

This leads to the equations of motion

$$\ddot{\varphi} + 3H\dot{\varphi} + V(\varphi)_{,\varphi} = 0, \tag{1.4}$$

$$H^2 = \frac{1}{3M_{Pl}^2}\left[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right]. \tag{1.5}$$

[1]In the 'mostly plus' sign convention for the metric.

In a certain regime, referred to as 'slow roll',

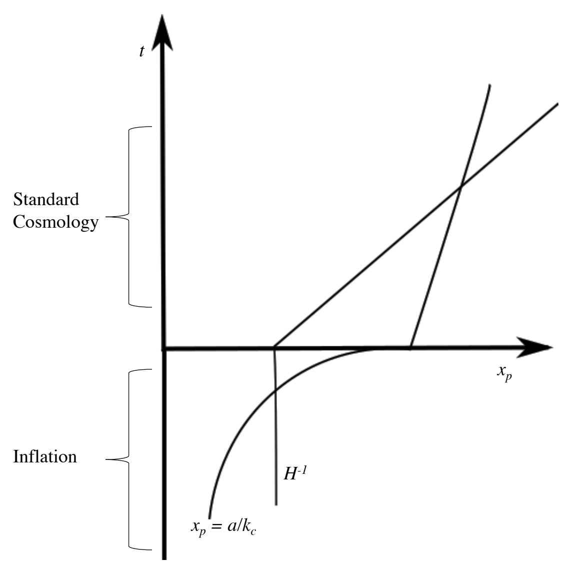
$$3H\dot{\varphi} \gg \ddot{\varphi} \ \ , \ \ V \gg \dot{\varphi}^2, \tag{1.6}$$

the above equations will lead to a period of inflation. A classic example of this given by the potential

$$V(\varphi) = \frac{1}{2}m^2\phi^2. \tag{1.7}$$

This potential is flat, relative to its magnitude, at large values of ϕ, $\partial_\varphi V/V \sim 1/\phi$, allowing for slow-roll inflation, wherein the field ϕ is approximately constant.

Figure 1.1 The evolution of the physical length of a fixed comoving length scale and the Hubble length scale during Inflation.



The physics of the inflationary universe is neatly summarized in Figure 1.1, and we dedicate the rest of the section to explaining this picture. In this figure we compare the Hubble length H^{-1} to the physical length scale of a fixed comoving wave number. This may seem like an odd comparison to make, but we will later see (e.g. equation (1.50)) that the solutions for fluctuations in the above scalar field φ are naturally functions of the quantity,

$$\left(\frac{k}{aH}\right), \tag{1.8}$$

where k here corresponds to the space coordinate that appears in the metric (1.1), i.e. it is a *comoving* quantity. The comparison of $x_p = a/k$ to the Hubble length H^{-1} is then a natural comparison to make, as it encodes the amplitude of fluctuations of a given $k \equiv k_c$. Alternatively, one may think of this as comparing k_p and H.

The physics on small length scales (i.e. $k/aH \gg 1$), as seen in the LHS of the plot in Figure 1.1, is intrinsically *quantum*. In fact, there has been much work on the possibility that the x_p probed by inflation might reveal something about *quantum gravity*. In this regime, the field φ can be described by canonical quantization in quantum field theory,

$$\varphi(x,t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\varphi_k a_k e^{ik\cdot x} + \quad h.c., \tag{1.9}$$

where the $1/\sqrt{2E_k}$ of canonical quantization is now replaced the "mode function" φ_k, and a_k is the standard annhilation operator. Details of this quantization procedure and subtleties of the vacuum state will be discussed later.

As in all quantum theory, the vacuum is not empty. Empty space is in fact *teeming* with fluctuations $\delta\varphi_k$. During inflation, the physical length scale of a mode $\delta\varphi_k$ increases exponentially, and eventually becomes greater than the Hubble length. We refer to the time this occurs as the moment the mode "exits the Horizon", where horizon here is more precisely the Hubble radius. The constancy of H during inflation, and the schematic solution (1.8), ensures that all modes have the same amplitude at the moment they exit the horizon; this is the origin of the "scale-invariant" spectrum of fluctuations in inflationary cosmology. We will come back to this later.

After exiting the horizon, fluctuations are quickly stretched to scales that are relevant for general relativity. In this regime, the fluctuations are assumed to be *classicalized*, with the requisite decoherence occurring at or around horizon exit [2]. The mode expansion of the field is now given by

$$\varphi(x,t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\varphi_k \alpha_k e^{ik\cdot x} + \quad h.c., \tag{1.10}$$

where the α_k are a set of classical random variables with statistics determined by those of the annhilation/creation operators at horizon exit. The vacuum expectation values of the annhilation/creation operators are mapped to ensemble averages of the α_k, or by the ergodic hypothesis, spatial averages. The cosmological consequences of this classicalization were explored in my paper [3], which can be found in Chapter 7.

Once inflation ends, standard big bang cosmology takes over, with the scale factor $a(t)$ growing as a power-law in time. The Hubble length $H^{-1}(t) = a/\dot{a}$ grows linearly with time, while physical length scales still redshift as $a(t)$, which scales with $t^{1/2}$ during radiation domination and then $t^{2/3}$ during matter domination. Modes φ_k no longer exit the horizon, but rather *re-enter* the Hubble radius, where they are imprinted in the spectrum of density fluctuations we see today (e.g. in the cosmic microwave background).

The charaterization of these density fluctuations in terms of their statistical correlations forms a large part of modern cosmology. It it thus worthwhile to quickly go over the key concepts. Free fields can be straight-forwardly quantized as above, in which case the field is *Gaussian*,

$$\langle a_k^\dagger a_{k'} \rangle = (2\pi)^3 \delta^{(3)}(k - k'). \tag{1.11}$$

This is equivalent to the statement that the α_k are drawn from a Gaussian distribution, which implies that the field ϕ is a Gaussian random field on large scales. From this fact we can begin to compute correlation functions. For example, the two point function of a free-field ϕ is given by

$$\begin{aligned}
\langle \phi(x)\phi(y) \rangle = & \int \frac{\mathrm{d}^3 k_1}{(2\pi)^3} \frac{\mathrm{d}^3 k_2}{(2\pi)^3} \, \phi_{k_1}^* \phi_{k_2} \, e^{i(k_1 x - k_2 y)} \langle a_{k_1}^\dagger a_{k_2} \rangle \\
= & \int \frac{\mathrm{d}^3 k_1}{(2\pi)^3} \frac{\mathrm{d}^3 k_2}{(2\pi)^3} \, \phi_{k_1}^* \phi_{k_2} \, e^{i(k_1 x - k_2 y)} \, (2\pi)^3 \delta^{(3)}(k - k') \\
= & \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \, |\phi_k|^2 \, e^{ik(x-y)} \\
= & \int \mathrm{d} \log k \, \frac{k^3}{2\pi^2} |\phi_k|^2 \, e^{ik(x-y)}
\end{aligned} \tag{1.12}$$

This last line defines the *power spectrum*,

$$P_\phi(k) \equiv \frac{k^3}{2\pi^2} |\phi_k|^2. \tag{1.13}$$

A special case of the above is a *scale-invariant spectrum*. In this case,

$$P_\phi(k) \sim constant. \tag{1.14}$$

which owes its name to the implied scaling symmetry of the two-point function,

$$\langle \phi(x)\phi(y)\rangle = \langle \phi(\lambda x)\varphi(\lambda y)\rangle. \tag{1.15}$$

Note however that a Gaussian random field need not be scale-invariant. The Gaussian nature implies a lack of correlation between modes of different k, but does not make any claim as to the variance of the probability distribution at fixed values of k, which is measured by the two-point correlation function or equivalently the power spectrum. For example, a Gaussian random field with $|\phi_k|^2 = constant$ is known as *white noise*.

For cosmological purposes we frequently parametrize the power spectrum as follows,

$$P(k) = \mathcal{A}\left(\frac{k}{k_0}\right)^{n_s-1} \tag{1.16}$$

where $\mathcal{A}$ is the amplitude of power spectrum as measured at a reference scale k_0, referred to as the "pivot scale", and n_s is the spectral index.

The brief discussion thus far is enough to intuit the value of n_s in inflation. During inflation, e.g. the $m^2\phi^2$ model (1.7), the field is slowly decreasing in value. From equation (1.5) we know that the Hubble constant must also be slowly decreasing. Now recall that the amplitude of a mode φ_k is set by the Hubble constant at the time of horizon exit, and modes with smaller values of k exit the horizon earlier, as they need less stretching to do so. This then implies that the amplitude of the power spectrum increases, ever so slightly, as k decreases, i.e.

$$n_s < 1 \;\;,\;\; |n_s - 1| \ll 1. \tag{1.17}$$

This slight "red tilt" of the power spectrum is a key prediction of inflation. The name "red" refers to the fact that the power spectrum, in comparison to a scale-invariant spectrum, has excess power in the infrared, as opposed to a "blue spectrum" which has excess power in the ultraviolet.

1.3 Relativistic Cosmological Perturbation Theory

We will now study the theory of cosmological perturbations, and to do so we will work in the context of general relativity (hence the term 'relativistic' in the section title). It is worth noting that standard cosmology, which comprises the last 13.8 billion years of the universe, can largely be done without leaving Newtonian physics: the evolution of dark matter, baryons, and radiation, and the power spectrum of matter fluctuations, can all be computed fairly accurately in Newtonian perturbation theory in a fluid expanding with velocity H. However, for *early universe* physics, e.g. inflation, both quantum and general relativistic effects are of the utmost importance.

The physics of general relativity are captured by Einstein's Equation,

$$G_{\mu\nu} = \frac{8\pi}{3M_{Pl}^2} T_{\mu\nu}, \tag{1.18}$$

where $G_{\mu\nu}$ is the Einstein tensor, comprised of the metric and its derivatives, that describes the geometry of spacetime, and $T_{\mu\nu}$ is the matter stress tensor, which describes the matter content of the universe. We would like to consider fluctuations around some background solution to the above equation. However, due to the diffeomorphism invariance of general relativity, it is not immediately obvious what are the "physical" degrees of freedom that we should track when considering perturbations. In principle the metric has 10 degrees of freedom, since it is a symmetric rank-2 tensor in 4-dimensions. However we will see that several of these can be removed by a clever choice of coordinate system (or "gauge").

The canonical way to proceed in this analysis is to introduce the so called scalar-vector-tensor (SVT) decomposition, which at linear order in perturbation theory splits the Einstein equation into decoupled systems. This ends up being an incredibly intelligent way to organize the equations, and only once this is done do the equations governing perturbations become amenable to solving. However, to gain some insight into why one might think of doing this in the first place, we can take some insight from effective quantum field theory.

In effective quantum field theory we allow in the action to appear all operators not forbidden by any symmetries of the problem. Quantum field theory has Lorentz-invariance, which means that there can be no uncontracted Lorentz indices appearing in the action.

For massless fields this implies conservation of helicity. For example, operators like

$$\phi^2 A_\mu A^\mu$$

can appear (arising from a $|D\phi|^2$ coupling).

However, if we are only interested in the equation of motion up to *linear order* in pertubations things are a little different. If we take our field content to be a scalar fluctuation $\delta\phi$, vector fliuctuation A_μ, and tensor fluctuation $h_{\mu\nu}$, then operators like

$$\delta\phi\, \partial^\mu A_\mu \partial^\nu A_\nu \quad , \quad \delta\phi\, h_{\mu\nu} h^{\mu\nu} \quad , \quad \delta\phi\, A_\mu A_\nu h^{\mu\nu} \quad ... \tag{1.19}$$

only enter the equations of motion beyond linear order. In fact, it is easy to show that at linear order in perturbation theory, the fields $\delta\phi$, A_μ, and $h_{\mu\nu}$, are totally decoupled. This is referred to as the SVT decomposition, which we will now present in more detail.

We will specialize to perturbations on an FRW background,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}\Omega_3^2. \tag{1.20}$$

This backround breaks the diffeomorphism invariance of GR to the group of spatial rotations. With the above discussion of quantum field theory in mind, but with Lorentz invariance now replaced by invariance under spatial rotations, we then decompose a metric fluctuation $\delta g_{\mu\nu}$ into scalar,vector, and tensor representations. This gives

$$\delta g_{\mu\nu} = \begin{cases} \delta g_{00} & = \text{scalar} = a^2(t)\phi(x,t) \\ \delta g_{0i} & = \text{vector} \to \text{scalar} + \text{vector} = a^2(t)\partial_i B(x,t) + S_i(x,t) \\ \delta g_{ij} & = \text{tensor} \to \text{scalar} + \text{vector} + \text{tensor} \\ & \quad\; = a^2(t)\left[2\psi(x,t)\delta_{ij} + 2E_{,ij}(x,t) + \partial_i F_j(x,t) + \partial_j F_i(x,t) + h_{ij}(x,t)\right] \end{cases}$$

The representation of each piece can be deduced from the number of spatial indices, and the right arrows “ $\to$″ denote a decomposition of a generic vector or tensor into *irreducible* representations.

We can re-arrange the above expression to collect the scalar, vector, and tensor pieces.

The scalar components are,

$$\delta g^S_{\mu\nu} = a^2 \begin{pmatrix} 2\phi & -B_{,i} \\ -B_{,i} & 2(\psi\delta_{ij} - E_{,ij}) \end{pmatrix}, \tag{1.21}$$

where ϕ, ψ, E, and B are scalars, and $_{,i} = \partial/\partial x^i$. Vector perturbations take the form:

$$\delta g^V_{\mu\nu} = a^2 \begin{pmatrix} 0 & -S_i \\ -S_i & F_{i,j} + F_{j,i} \end{pmatrix}, \tag{1.22}$$

where S and F are divergence-less vectors. Finally, tensor perturbations take the form:

$$\delta g^T_{\mu\nu} = a^2 \begin{pmatrix} 0 & 0 \\ 0 & h_{ij} \end{pmatrix}, \tag{1.23}$$

where h_{ij} is traceless and divergence-less. From the above expressions we see that there are 4 scalar degrees of freedom, $(3-1)+(3-1) = 4$ vector degrees of freedom (from two divergence-less 3d vectors), and $9-3-3-1 = 2$ tensor degrees of freedom (for one transverse traceless rank-2 3d tensor). This gives a total of 10 degrees of freedom, which is precisely the number we expect for the metric, a symmetric rank-2 tensor in 4 dimensions.

However, not all of these degrees of freedom are independent, as the theory has an underlying gauge redundancy. They are thus related by gauge transformations. In the scalar sector, these are

$$t \to t' = t + \alpha(x,t), \tag{1.24}$$

and

$$x^i \to x^i + \partial^i \beta(x,t), \tag{1.25}$$

where both α and β are scalars.

As a warning against neglecting this gauge freedom, let's consider what happens to density perturbations under a time reparametrization. Consider a universe with homogeneous energy density in $\{t, x\}$ coordinates:

$$\rho(x,t) = \rho_0(t), \tag{1.26}$$

where ρ_0 is a 'background' energy density, free from perturbations ($\delta\rho = 0$). Now consider the above transformation $t \to t'$, under which the background ρ becomes

$$\rho_0' = \rho_0 + \dot{\rho}_0 \alpha + \mathcal{O}(\alpha^2). \tag{1.27}$$

The second term is a perturbation $\delta\rho = \dot{\rho}_0\alpha$, where previously there was none! This is referred to as a "fictitious perturbation" [4]. To proceed with cosmological perturbation theory we must then make an effort to work consistently in one gauge, or else to work in an entirely gauge-invariant formalism. Both of these approaches are popular in the literature.

To fix the gauge freedom we can make a explicit choice of α and β that sets to 0 two of the four scalar degrees of freedom. One particularly useful gauge is to choose α and β such that $\delta\rho = 0$, meaning that fixed-time ($t = t_*$) slices of spacetime have a uniform density $\rho = \rho_0(t_*)$. We can then define the curvature perturbation ζ on these uniform density hypersurfaces, which on large scales satisfies

$$\dot{\zeta} = 0 \tag{1.28}$$

This property makes this is a very useful gauge. We will show how this relation arises in the next section.

1.4 Cosmological Perturbations in Spatially Flat Gauge

There are two popular gauges for studying cosmological perturbations. The 'uniform field gauge' in which the Arnowitt-Deser-Misner formalism [5] is utilized, provides very efficient computation of correlation functions, while the 'spatially flat gauge' makes manifest the importance of metric perturbations, and is more amenable to discussions of adiabatic and entropy perturbations (which will be the focus of Chapter 5). In this thesis we will present only the spatially flat gauge, and we refer readers to the lecture notes by Daniel Baumann [7] for details on the uniform field gauge. The presentation of spatially flat gauge is based upon [8].

Our starting point is a scalar field minimally coupled to gravity. Again, the action is

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(R + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)\right). \tag{1.29}$$

We will expand both our field and metric into a background piece and fluctuation piece, e.g. $\varphi = \varphi_0(t) + \delta\varphi(x,t)$. (We will drop the 0 subscript from now on). The equations of motion for the background are,

$$\ddot{\varphi} + 3H\dot{\varphi} + V(\varphi)_{,\varphi} = 0, \tag{1.30}$$

$$H^2 = \frac{1}{3M_{Pl}^2}\left[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right]. \tag{1.31}$$

We will study only the scalar fluctuations in the metric. The metric is given by,

$$\mathrm{d}s^2 = -(1+2A)\mathrm{d}t^2 + 2aB_{,i}\mathrm{d}x^i\mathrm{d}t + a^2\left[(1-2\psi)\delta_{ij} + 2E_{,ij}\right]\mathrm{d}x^i\mathrm{d}x^j, \tag{1.32}$$

where we changed notation from $\delta g_{00} = \phi$ to A, and absorbed various factors of a. At this point we have not fixed the gauge at all. The Einstein equations can be expanded to first order in fluctuations, which yields one equation motion

$$\ddot{\delta\varphi} + 3H\dot{\delta\varphi} + \frac{k^2}{a^2}\delta\varphi + V_{,\varphi\varphi} = -2V_{,\varphi}A + \dot{\varphi}\left[\dot{A} + 3\dot{\psi} + \frac{k^2}{a^2}(a^2\dot{E} - aB)\right], \tag{1.33}$$

and two constraint equations,

$$3H\left(\dot{\psi} + HA\right) + \frac{k^2}{a^2}\left[\psi + H(a^2\dot{E} - aB)\right] = -\frac{1}{3M_{Pl}^2}\left[\dot{\varphi}(\dot{\delta\varphi} - \dot{\varphi}A) + V_{,\varphi}\delta\varphi\right], \tag{1.34}$$

$$\dot{\psi} + HA = \frac{1}{3M_{Pl}^2}\dot{\varphi}\delta\varphi. \tag{1.35}$$

There is an additional would-be equation of motion, which in the absence of anisotropic stress acts as an additional constraint equation:

$$(a^2\dot{E} - aB)\dot{} + H(a^2\dot{E} - aB) + \psi - A = 0. \tag{1.36}$$

At first glance we have five degrees of freedom: four from the metric and one from the scalar field. However, two can be removed via a gauge transformation and two are removed by the constraint equations[2]. This leaves a single dynamical degree of freedom. The gauge choice we will make is the *spatially flat gauge*, wherein the spatial metric fluctuations ψ

[2]The three constraint equations (1.34) (1.35) (1.36) in fact only form two linearly independent constraints.

and E are set to 0.

To be extra careful with our calculations, we can construct a gauge invariant perturbation variable. The most common is the 'Mukhanov-Sasaki' variable, which is defined as

$$Q = \delta\varphi + \frac{\dot{\varphi}}{H}\psi. \tag{1.37}$$

In the spatially flat gauge Q is simply $\delta\varphi$. After imposing our gauge and the constraint equations, the equation of motion becomes

$$\ddot{Q} + 3H\dot{Q} + \left[\frac{k^2}{a^2} + V_{,\varphi\varphi} - \frac{1}{3M_{Pl^2}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a^3}{H}\dot{\varphi}^2\right)\right] Q = 0. \tag{1.38}$$

The third term in the square brackets is the result of including metric perturbations into our analysis. Thus one effect of metric perturbations is to introduce couplings of the scalar fluctuations to the background. The origin of this term is made manifest by the prefactor of $1/M_{Pl}$: in the limit $M_{Pl} \to \infty$ the spacetime and matter decouple in the Einstein equation, and the term in question vanishes, as expected.

We can recast this equation in a simpler form by defining the quantity,

$$\zeta = \frac{H}{\dot{\varphi}}Q, \tag{1.39}$$

which is precisely the curvature perturbation on uniform density hypersurfaces: in this context, $\rho = \rho(\varphi)$, and hence constant-ρ slices are slices of constant φ, in which case $\zeta = \psi$. On large scales $k^2 \to 0$, this is related to the Ricci scalar on three-dimensional slices by[3] $R_{(3)} = \frac{1}{a^2}\nabla^2\zeta$. We then re-scale this variable,

$$v = z\zeta \quad , \quad z = a\frac{\dot{\varphi}}{H}, \tag{1.40}$$

work in conformal time $\mathrm{d}\tau = a\mathrm{d}t$, and transform to Fourier space, to arrive at the equation of motion

$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0, \tag{1.41}$$

where $'$ denotes a derivative with respect to conformal time. This is known as the 'Mukhanov-

[3]For the gauge dependence of this statement see e.g. [8]. This issue disappears in the limit $k^2 \to 0$, since in this limit $-\zeta = \psi = \mathcal{R}$.

Sasaki equation.' It contains in it all the information of the Figure 1.1 which we studied in Section 1.2.

The first important property of the above equation is the 'freeze-out' that occurs in the large-scale (i.e. $k/aH \to 0$) limit. In this case, the Mukhanov-Sasaki equation becomes

$$v_k'' - \frac{z''}{z} v_k = 0. \tag{1.42}$$

One solution to this equation is[4],

$$v_k = (constant) \times z. \tag{1.43}$$

Which corresponds to $\zeta = constant$, i.e.

$$\dot{\zeta} = 0. \tag{1.44}$$

Remarkably, this statement is independent of the background evolution of the universe. This, to a large extent, decouples the curvature perturbation from the detailed dynamics of the end of inflation and the transition to standard big bang cosmology, a phase known as 'reheating.' For example, Hubble constant could oscillate about some fixed value, and the curvature perturbation on large scales would be *unchanged*.

This is both a blessing and a curse: it makes theoretical predictions for ζ robust, but also limits our ability to learn about reheating. An important exception to this is the presence of *entropy perturbations*, which provide a source term on the right-hand-side of the above equation, and are generally present when there are multiple fields present. This will be discussed in detail in Chapter 5.

Yet more physics can be learned from the Mukhanov-Sasaki equation without actually solving it. We can use the constancy of ζ to compute the amplitude of the power spectrum of fluctuations produced by inflation. From the definition of ζ, one can show that amplitude of ζ is related to $\delta\varphi$ via[5]

$$\zeta \simeq \frac{1}{M_{Pl}^3} \frac{V}{V'} \delta\varphi \tag{1.45}$$

[4]This is the dominant mode in an expanding universe. There is also a decaying mode, but for an expanding universe the constant mode dominates on large scales.

[5]This follows from the slow-roll relations $\varepsilon = (1/2) M_{Pl}^2 (V'/V)^2$ and $\varepsilon = \dot{\varphi}^2/(2M_{Pl}^2 H^2)$. Also note that ζ in our convention is dimensionless.

As we argued earlier, the amplitude of $\delta\varphi$ at horizon crossing is roughly H, and thus the amplitude of ζ at horizon crossing is

$$\zeta_k(t = t_H) \simeq \frac{1}{M_{pl}^3}\frac{V}{V'}H, \tag{1.46}$$

where t_H is the time at which $k = aH$. Since $\dot{\zeta} = 0$ after horizon crossing by equation (5.52), this is valid at all later times during inflation. Furthermore, from the definition of H we can rewrite this as

$$\zeta_k(t_f) \sim \frac{1}{M_{Pl}^3}\frac{V^{3/2}}{V'} \tag{1.47}$$

where t_f is some later time, e.g. the end of inflation, and the quantities on the RHS are evaluated at the time the mode leaves the horizon. The power spectrum is then given by the square of this quantity. This expression encodes the spectral index of fluctuations, in an identical manner to Figure 1.1 as described around equation (1.17).

We can make this more precise by finding the exact solution to the Mukhanov-Sasaki Equation. This presentation follows the lecture notes by Daniel Baumann [7]. We define the slow-roll parameters ε and η as

$$\varepsilon = -\frac{\dot{H}}{H^2} \quad , \quad \eta = \frac{\dot{\varepsilon}}{H\varepsilon}. \tag{1.48}$$

Slow-roll inflation occurs provided both these parameters are much less than one. The z''/z factor in the Mukhanov-Sasaki equation can then be rewritten as

$$\frac{z''}{z} = \frac{\nu^2 - \frac{1}{4}}{\tau^2} \quad ; \quad \nu \equiv \frac{3}{2} + \varepsilon + \frac{1}{2}\eta. \tag{1.49}$$

where τ is conformal time, given by $\tau = -(1+\varepsilon)/aH$ during inflation. The exact solution to (1.41) is then given in terms of Hankel functions:

$$v_k(\tau) = \sqrt{-\tau}\left[C_1(k)H_\nu^{(1)}(-k\tau) + C_2(k)H_\nu^{(2)}(-k\tau)\right]. \tag{1.50}$$

As promised in equation (1.8), the above solution depends on k only via the combination $-k\tau = k/aH$.

The coefficients C_1 and C_2 need to be fixed by imposing suitable initial conditions. The standard procedure is to impose initial conditions at a time τ such that $|k\tau| \gg 1$, corresponding to a time when the fluctuation was deep inside the Hubble radius. In this regime we expect quantum field theory to be good approximation and we canonically quantize the field. However, to quantize the field we need to deal with subtleties involving a choice of vacuum state. There is a canonical choice, the “Bunch-Davies" vacuum state, which respects the isometries of de Sitter space, however this is not the unique vacuum. For discussion of these details we refer the reader Birrell and Davies [1].

Here we will show how to intuit the correct form of the vacuum state and the values of the coefficients $C_{1,2}$. The equation of motion in the limit $|k\tau| \gg 1$ is,

$$v_k'' + k^2 v_k = 0. \tag{1.51}$$

Recall that in canonical quantization, we expand fields as

$$\phi = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \left(a_k e^{ikx} + a_k^\dagger e^{-ikx} \right), \tag{1.52}$$

where E_k is given by special relativity,

$$E_k^2 = k^2 + m^2. \tag{1.53}$$

For large values of k, this is reduces to

$$\frac{1}{\sqrt{2E_k}} \sim \frac{1}{\sqrt{2k}}. \tag{1.54}$$

Meanwhile, in our mode expansion of v,

$$v = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \left(v_k a_k e^{ikx} + v_k^* a_k^\dagger e^{-ikx} \right), \tag{1.55}$$

we have generalized $\frac{1}{\sqrt{2E_k}} \to v_k$, where v_k will be chosen such that the above equation satisfies the symmetries of the spacetime. In order to have a reasonable sense of quantization of the field v, the two descriptions must agree in the limit $|k\tau| \gg 1$, i.e.

$$\lim_{|k\tau| \to \infty} v_k \propto \frac{1}{\sqrt{2k}}. \tag{1.56}$$

This serves as the initial condition for solving the Mukhanov-Sasaki equation (1.41).

After applying the above initial condition, and keeping track of factors of 2 and π, the coefficients C_1 and C_2 are fixed to be $\sqrt{\pi/2}$ and 0 respectively. Thus the mode function is given by

$$v_k = \sqrt{\frac{\pi}{2}}\sqrt{-\tau}H_\nu^{(1)}(-k\tau). \tag{1.57}$$

We now write down the power spectrum of curvature fluctuations on large scales. Using $\lim_{x\to 0} H_\nu^{(1)}(x) = x^{-\nu}$, we arrive at

$$P_\zeta(k) = \frac{k^3}{2\pi^2}|\frac{v_k}{z}|^2 \propto k^{3-2\nu}. \tag{1.58}$$

From our definition of the power spectrum (1.16), and our definition of ν (1.49), this is a nearly scale-invariant spectrum with spectral tilt,

$$n_s - 1 = -2\varepsilon - \eta. \tag{1.59}$$

As expected, $n_s - 1$ is negative and small. Thus the spectrum of curvature perturbations is slightly red-tilted, as anticipated in previous sections.

Repeating the calculation, but keeping factors of 2 and π, the power spectrum is given by

$$P_\zeta(k) = \frac{1}{8\pi^2 M_{Pl}^2}\frac{H^2}{\epsilon} \tag{1.60}$$

where the RHS is evaluated at horizon crossing.

1.5 Gravitational Waves

No review of cosmological perturbations would be complete without a discussion of gravitational waves. These correspond to tensor fluctuations of metric, see equation (1.23). Tensor fluctuations can be expanded into two polarizations:

$$h_{ij} = h_+ e_{ij}^+ + h_\times e_{ij}^\times \tag{1.61}$$

where $e_{ij}^{+,\times}$ are polarization tensors. The states h_+ and $h_\times$ evolve as massless scalars, and their equation of motion is given by

$$u_k'' + \left(k^2 - \frac{a''}{a}\right) u_k = 0. \tag{1.62}$$

This differs from the Mukhanov-Sasaki equation by the replacement $z \to a$. The presence of z was necessary to bring the scalar metric and scalar field fluctuations together into a unified expression. However, as discussed in section 1.2, tensor perturbations decouple from scalar perturbations. It follows that the only 'background' the tensors feel is governed by a, and not z.

We can solve (1.62) in the same way we solved the Mukhanov-Sasaki equation. In this case, we find the power spectrum,

$$P_h = \frac{H^2}{M_{Pl}^2}. \tag{1.63}$$

We can quantify the amplitude of this by defining the 'tensor-to-scalar ratio' r,

$$r = \frac{P_t}{P_s} \tag{1.64}$$

where t and s are for scalar and tensor respectively, and both power spectra are evaluated at the pivot scale k_0. Using the spectrum of scalar fluctuations (1.60), and noting that P_t is 2 times P_h (due to the two polarization states), we find the inflationary prediction for r:

$$r = 16\epsilon. \tag{1.65}$$

We can go one step further, and relate the value of r to the dynamics of φ during inflation. From the relation $\varepsilon = \dot{\varphi}^2/(2M_{Pl}^2 H^2)$, we have

$$r = \frac{8}{M_{Pl}^2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}N}\right)^2, \tag{1.66}$$

where $\mathrm{d}N = H\mathrm{d}t$. Assuming r to be constant, we can integrate this over the last 60 e-folds of inflation to find the total variation in the field φ during this period,

$$\frac{\Delta\varphi}{M_{Pl}} = \mathcal{O}(1)\sqrt{\frac{r}{0.01}}. \tag{1.67}$$

We thus find Planckian field excursions for $r \geq 0.01$. This relation is known as the *Lyth Bound*.

The above relation implies that models of inflation with $\Delta\varphi \gg M_{Pl}$, so-called 'large-field inflation', will have non-negligible gravitational waves, which might be observable in upcoming CMB experiments. This has led to much excitement regarding large-field inflation, and the popularity of models such as axion monodromy inflation.

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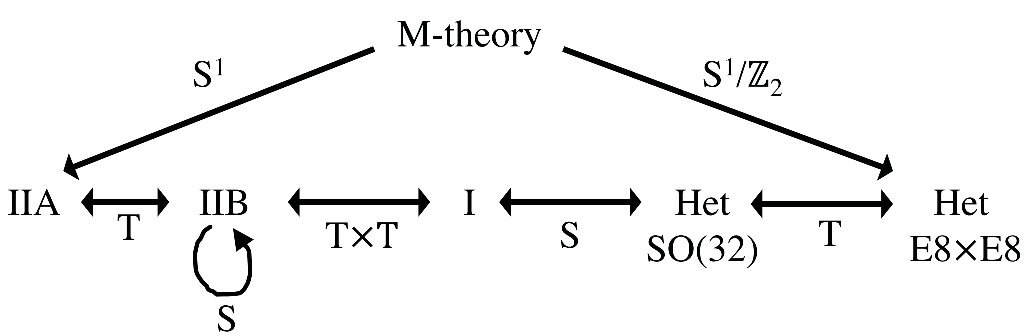
Chapter 2

Review of String Theory

2.1 Introduction

In this section we will perform a lightning review of string theory. The goal of this will be to provide the reader with an understanding and appreciation of Figure 2.1. The presentation will draw heavily on the textbooks [1] [2].

Figure 2.1 The Dualities of String Theory



2.2 The construction of superstring theories

There are five unique consistent superstring theories: Type IIA, type IIB, type I, Heterotic SO(32), and Heterotic $E8 \times E8$. All of these theories live in 10 spacetime dimensions and are supersymmetric, but differ by their spacetime field content. The spacetime field content arises from quantizing the action of a supersymmetric string, which leaves open different choices of boundary conditions and fermion content. In addition, type I and type II theories have *open and closed* strings, while the Heterotic theories have *only closed* strings. As we will see in the next section, these superstring theories are in fact a limit of an 11-dimensional theory with *no strings at all*: M-theory. In this section we will briefly summarize the construction of the superstring theories, and in the following section we will show how the are connected.

Let's start with the action of a classical non-relativistic string:

$$S = \int \mathrm{d}\tau\, \mathrm{d}\sigma\, \frac{1}{2}\left[\mu_0 \left(\frac{\partial y(\tau,\sigma)}{\partial \tau}\right)^2 - T_0 \left(\frac{\partial y(\tau,\sigma)}{\partial \sigma}\right)^2\right], \tag{2.1}$$

where y is a scalar field, and τ and σ characterize the "worldsheet' that the string sweeps out in time, analagous to the worldline of a point particle. The parameters μ_0 and T_0 are the energy density per unit length and tension respectively. The equation of motion that follows from this is the wave equation,

$$\mu_0 \frac{\partial^2 y(\tau,\sigma)}{\partial \tau^2} - T_0 \frac{\partial^2 y(\tau,\sigma)}{\partial \sigma^2} = 0. \tag{2.2}$$

The general solution to the wave equation can be separated in to a 'right-moving' and 'left-moving' piece:

$$y(x,t) = y_+ + y_-, \tag{2.3}$$

with

$$y_+ \equiv y_+(\sigma + v\tau) \ \ , \ \ y_- \equiv y_-(\sigma - v\tau) \ \ , \ \ v^2 = \frac{T_0}{\mu_0}. \tag{2.4}$$

The quantization of this action can then proceed via separately quantizing the excitations of y_+ and y_-, subject to boundary conditions placed on each y_+ and y_-. Different choices of boundary conditions lead to distinct string theories. In superstring theory, we add in fermions. For the fermions we can choose the representation under the spacetime Lorentz group. The different choices of fermion representation and boundary conditions lead to the 5 superstring theories.

We can generalize this is a bit and take y to be comprised of a *set* of fields: a set of scalars $\{X^\mu\}$ and spinors $\{\psi^\mu\}$ (with a Dirac-like action), with $\mu = 0, 1, ..9$. This is the action of the relativisitic superstring:

$$S = \frac{1}{4\pi} \int \mathrm{d}^2 z \left(\frac{2}{\alpha'} \partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial} \psi_\mu + \widetilde{\psi}^\mu \partial \widetilde{\psi}_\mu \right). \tag{2.5}$$

where we have defined

$$z = \tau + i\sigma \qquad , \quad \bar{z} = \tau - i\sigma, \tag{2.6}$$

$$\partial = \frac{1}{2}\left(\partial_\tau - i\partial_\sigma\right) \qquad , \quad \bar{\partial} = \frac{1}{2}\left(\partial_\tau + i\partial_\sigma\right) \tag{2.7}$$

The fields are defined as

$$X^\mu = X^\mu(z, \bar{z}) \ \ , \ \ \psi^\mu = \psi^\mu(z) \ \ , \ \ \widetilde{\psi}^\mu = \widetilde{\psi}^\mu(\bar{z}) \tag{2.8}$$

such that the ψ^μ are 'left-movers' and the $\widetilde{\psi}^\mu$ are 'right movers'. Importantly, the fermions ψ^μ sit in a representation of the 10d Lorentz group $SO(9,1)$, and massless excitations are representations of $SO(8)$. We have not yet chosen a representation; in combination with a choice of boundary conditions, this will completely specify the theory.

To quantize this theory we perform a Laurent expansion in the left and right sectors. For the scalars X^μ, this has the form

$$X^\mu_L(z) \quad = \frac{1}{2} x^\mu + \frac{l_s^2}{2} p^\mu z + \frac{i l_s}{\sqrt{2}} \sum_{k \neq 0} \frac{\alpha^\mu_k}{k}\, e^{-ikz}, \tag{2.9}$$

$$X^\mu_R(\bar{z}) \quad = \frac{1}{2} x^\mu + \frac{l_s^2}{2} \bar{p}^\mu z + \frac{i l_s}{\sqrt{2}} \sum_{k \neq 0} \frac{\bar{\alpha}^\mu_k}{k}\, e^{-ikz}, \tag{2.10}$$

where the x^μ and $p^\mu, \bar{p}^\mu$, are the zero-mode position and momentum respectively. The α^μ_k and $\bar{\alpha}^\mu_k$ are annhilation/creation operators for X_L and X_R respectively, with an independent set for each value of μ. There are also similar expressions for the fermionic variables.

The bosonic variables X^μ give rise to the geometry (i.e. the metric) of spacetime, through the vanishing of the β function for the target space metric. This is reviewed in e.g. the lecture notes by David Tong [3]. The the fermions ψ^μ, on the other hand, give rise to the spacetime field content, both bosonic and fermionic.

Let's focus on closed strings, and normalize the length to 2π. To fully specify the solution we need to impose boundary conditions, of which we have two choices:

$$\text{Ramond:} \quad \psi^\mu(z+2\pi i) = \psi^\mu(z) \tag{2.11}$$

$$\text{Neveu-Schwarz:} \quad \psi^\mu(z+2\pi i) = -\psi^\mu(z) \tag{2.12}$$

and similarly for the 'right-movers' $\widetilde{\psi}^\mu$.

The bosonic variables X^μ must have symmetric (Ramond) boundary conditions in both the right and left sectors, in order to form a Poincare-invariant metric[1]. However, the fermions may have R or NS in the left and/or right. This leaves four sectors of fermion boundary conditions:

Left	Right
NS	NS
NS	R
R	NS
R	R

We can build a full quantum theory by combining sectors.

We also have the freedom to choose the representation of ψ^μ and $\widetilde{\psi}^\mu$ under SO(9,1), or equivalently the representation of massless excitations under SO(8). This effectively specifies the *ground state* of theory. This is made precise by defining the (worldsheet) *fermion number* F, defined via

$$\{(-1)^F, \psi^\mu_n\} = 0, \tag{2.13}$$

where ψ^μ_n is the n^{th} excitation of ψ^μ, and define $\widetilde{F}$ as the analogue acting on $\widetilde{\psi}$. This counts the number of fermionic excitations mod 2.

When acting on the ground state, F counts the dimension of the representation of SO(8) under which ψ^μ falls, and hence a choice of F effectively fixes the representation. The NS and R sectors are quite different in this respect. In the NS sector we have:

$$\text{NS}: \begin{cases} F = +1 & \Rightarrow \mathbf{8}_V \;\text{ of }\; SO(8) \\ F = -1 & \Rightarrow \mathbf{1} \;\text{ of }\; SO(8) \end{cases} \tag{2.14}$$

where $\mathbf{8}_V$ is the vector representation, and $\mathbf{1}$ is the singlet representation. However, one can

[1] Although so-called "twisted sectors" do arise and very important in the description of strings on orbifolds

show that the $F = -1$ choice in fact has a negative mass ground state, i.e. it is tachyonic. Thus we do not use this choice in our theory-bulding. In the R-sector we have

$$\mathrm{R}: \begin{cases} F = +1 & \Rightarrow \mathbf{8} \;\text{ of }\; SO(8) \\ F = -1 & \Rightarrow \mathbf{8}' \;\text{ of }\; SO(8) \end{cases} \tag{2.15}$$

where $\mathbf{8}$ and $\mathbf{8}'$ are the spinor and conjugate spinor representations respectively. Hence the massless states of our superstring theory come from the sectors NS+ , R+, and R–, where the $\pm$ denotes the value of F.

To build the ground state of the theory we make a choice of NS or R and $F = \pm 1$ in both the left and right sectors. Bosonic field content arises from combining NS in the left and NS in the right, or else R in the left and R in the right, since both of these cases lead to bosonic statistics of the massless excitations. In particular, the choices for bosonic field content are:

(Left,Right)	SO(8) spinors	tensors
(NS+,NS+)	$\mathbf{8}_V \times \mathbf{8}_V$	$= [0] + [2] + (2)$
(R+,R+)	$\mathbf{8} \times \mathbf{8}$	$= [0] + [2] + [4]$
(R+,R–)	$\mathbf{8} \times \mathbf{8}'$	$= [1] + [3]$

where on the right-most column we decompose into irreducible representations. The square brackets [...] denote antisymmetric tensors while the round brackets (...) denote symmetric tensors. Note that we could also take $(-) \leftrightarrow (+)$ in all the R sectors, but this does not produce physically distinct theories.

Of particular interest is the (NS+,NS+) sector: this includes a symmetric rank-2 tensor: the graviton! It also includes a scalar (the "dilaton") and an antisymmetric two-form (the "Kalb-Ramond" field). This sector will need to be included in all our string theories.

The (R+,R+) and (R+,R–) sectors both contain antisymmetric tensors of varying rank. Each antisymmetric tensor is governed by p-form electrodynamics, where the vector potential A_μ of electrodynamics is generalized to $A_{\mu_1\mu_2\ldots\mu_p}$. These are referred to as the "fluxes" of string theory.

We now come to building string theories. To do this we combine sectors, (NS/R,$\pm$), and enforce conditions to maintain the consistency of the theory. We will not go into details here, but the conditions are: (1) no tachyon, (2) modular invariance, (3) closure, and (4) mutual locality. From this we are led to minimal models, which contain four sectors. The two physically distinct minimal models are

$$\text{IIB}: \qquad (\text{NS}+,\text{NS}+) \ , \ (\text{R}+,\text{NS}+) \ , \ (\text{NS}+,\text{R}+) \ , \ (\text{R}+,\text{R}+) \tag{2.16}$$

$$\text{IIA}: \qquad (\text{NS}+,\text{NS}+) \ , \ (\text{R}+,\text{NS}+) \ , \ (\text{NS}+,\text{R}-) \ , \ (\text{R}+,\text{R}-) \tag{2.17}$$

The difference between these two theories is captured by the so-called “GSO projection":

$$\text{GSO IIB}: \qquad F = \tilde{F} = 1 \tag{2.18}$$

$$\text{GSO IIA}: \qquad F = +1 \ , \ \{\tilde{F} = +1 \ \text{ if right=NS} \ , \ \tilde{F} = -1 \ \text{ if right=R}\} \tag{2.19}$$

The two choices have important consequences for the fermions: the IIB theory is *chiral*, while IIA theory is not. This can be seen from the (R+,NS+) and (NS+,R+) of IIB, and the (R+,NS+) and (NS+,R–) of IIA. Both theories have two gravitinos, however in IIB they are of the same chirality, while in IIA they are of opposite chirality.

At this point we can construct the type I superstring theory. The type IIB theory has the same parity for both left and right movers, which provides it with a *worldsheet parity* symmetry Ω. We can gauge this symmetry by choosing a sign for Ω. If we fix $\Omega = 1$, we remove from type IIB all states that have (–) parity and arrive at the unoriented type I theory.

This projection removes the [2] from (NS+,NS+) and the (0) and (4) from (R+,R+), leaving the only closed string bosonic field content,

$$[0] + (2) + [2]. \tag{2.20}$$

For reasons that will not explained here, it turns out that this theory is only consistent if open strings are included (see e.g. [1] for a thorough discussion). This introduces an additional gauge field,

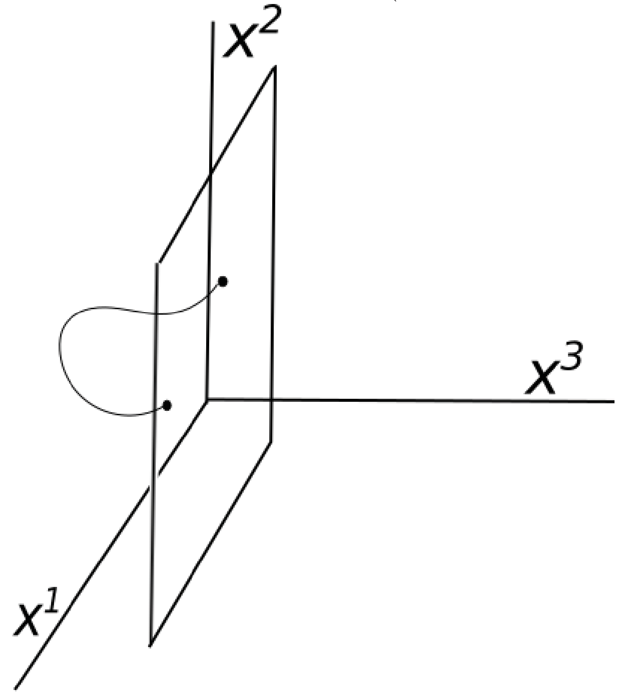
$$[1]. \tag{2.21}$$

The presence of this gauge field can be understood via T-duality with type IIB, as we will discuss shortly. For the moment, we note that the presence of open-strings in type I in turn indicates that open strings must also live in the type II theory. This leads us to a fundamental piece of the type II theories: *D-branes*.

The existence of open strings of type II theory can be inferred from looking at the

(R+,R+) sector of type IIB, which leads to p-form electrodynamics. Analogous to a monopole of electromagnetism, the charged objects of these fluxes are known as Dp-branes, or often simply referred to as D-branes. Whereas a monopole is magnetically charged under A_μ, a Dp-brane is electrically charged under $C_{\mu_1..\mu_{p+1}}$. And whereas a monopole has no spatial volume, a Dp-brane fills p space dimensions. For example, a D3 brane fills a 3+1 dimensional spacetime. As we will show, open strings arise as excitations of D-branes.

Figure 2.2 A D2 brane in 3+1 dimensions (the time-direction is suppressed)



The connection between open strings and D-branes, made by Polchinksi in 1995 [4], was at the heart of the "second superstring revolution." This can be seen (in a simpler form) as follows: in our analysis of the non-relativistic string (2.1), we had two choices for boundary conditions of y. We can fix the velocity of the end points of the string, known as *Neumann* boundary conditions:

$$\text{Neumann:} \quad \partial_\tau y(\tau, \sigma = 0) = \partial_\tau y(\tau, \sigma = L) = 0 \tag{2.22}$$

where L is the string length, or we can fix the position of the endpoints, known as *Dirichlet* boundary conditions:

$$\text{Dirichlet:} \quad \partial_\sigma y(\tau, \sigma = 0) = \partial_\sigma y(\tau, \sigma = L). \tag{2.23}$$

We can make a choice of N (Neumann) or D (Dirichlet) for each scalar X^μ of our superstring. For example, consider three spatial dimensions, $\mu = 0, 1, 2, 3$, and impose Neumann boundary conditions for X^1 and X^2, and Dirichlet boundary conditions for X^3. (The boundary conditions are always Neumann for X^0). This is usually represented as

0	1	2	3
N	N	N	D
×	×	×	

This diagram describes a string whose position is fixed to be along a hyperplane transverse to the x^3 direction. See Figure 2.2. This hyperplane is referred to as a D2-brane. A general D-brane has N boundary conditions along the brane and D boundary conditions transverse to it.

Coming back to the field content of the type IIB and IIA string theories, we can use the respective (R+,R+) and (R+,R–) sectors to read off the brane content of the theories:

$$\text{IIB}: \quad D1 \ , \ D3 \ , \ D5 \ , \ D7 \ , \ D9 \tag{2.24}$$

$$\text{IIA}: \quad D0 \ , \ D2 \ , \ D4 \ , \ D6 \ , \ D8 \tag{2.25}$$

Type IIB has Dp-branes of odd p while IIA has Dp-branes of even p. The electric and magnetic charges of each D-brane source the fluxes allowed by each theory; a Dp-brane electrically sources C_{p+1} and magnetically sources C_{7-p}, where C_p is the p-form RR flux. For example, a D3-brane electrically and magnetically sources [4].

Another key detail of D-branes is their existence as non-perturbative states in the quantized superstring theory. One can show that D-branes have a mass that is inversely proportional to the string coupling,

$$m_{Dp}^2 \sim \frac{1}{g_s}. \tag{2.26}$$

It follows that the branes are infinitely heavy in the weak coupling limit. However, in the strong coupling limit they become massless.

We have now studied type IIA, IIB, and type I. However, we have not yet touched upon the Heterotic string. As the Heterotic string will not appear in this thesis , we will only mention the crucial features.

The idea of the Heterotic string is to quantize the left and right sectors differently. The left sector is quantized as in the *bosonic string*, in 26 spacetime dimensions, while the right sector is quantized as in the 10-dimensional superstring. In this theory there are *no open strings*: this occurs because the quantization of open strings requires a matching between left and right excitations, so that left-moving waves ‘reflect’ off the boundary and become right-moving modes. With the left and right modes quantized differently, no such matching is possible.

The extra degrees of freedom from quantizing the left movers in 26 dimensions are wrapped up into a non-abelian vector field A_μ. The gauge group of A_μ is constrained by anomaly cancellation to be $E8 \times E8$ or SO(32). This discovery was a the heart of the first superstring revolution.

2.3 Dualities and M-theory

In the previous section we constructed the 5 superstring theories. In this section we will demonstrate that these theories can all be realized as limits of another theory, known as *M-theory*. That such a correspondence exists is at first sight miraculous: M-theory is a theory without strings, and yet it has hidden inside it the five consistent string theories. One might wonder if there is a string worldsheet description of M-theory, and the fault lies with physicists who have not been smart enough to find it. However, cancellation of the Weyl anomaly on the superstring worldsheet fixes the number of dimensions to be 10; the theory is not consistent in 11 dimensions. While M-theory does in fact contain some stringy objects, it does not have a description in terms of a quantized string worldsheet.

2.3.1 S and T-duality

Let’s start our discussion of dualities with T-duality. Consider a closed string moving on a cylinder of radius R, see Figure 2.3. The momentum of this string is quantized in units of

$1/R$:

$$p = \frac{n}{R} \ , \quad n \,\epsilon\, \mathbb{R}. \tag{2.27}$$

We can allow this string to wrap around the cylinder multiple times, that is, to have *winding*. See Figure 2.3. This gives an extra contribution to the momentum that scales with R:

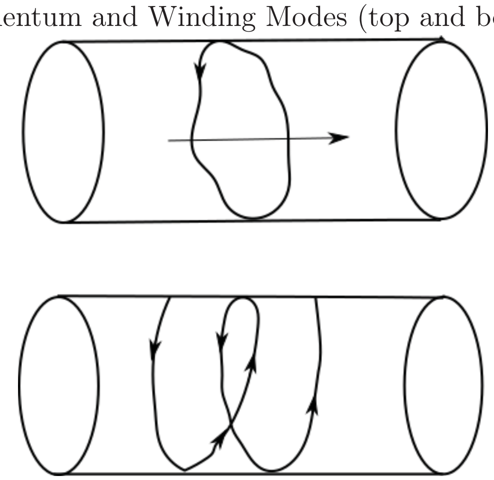
$$p = \frac{n}{R} + mR \ , \quad n, m \,\epsilon\, \mathbb{R}. \tag{2.28}$$

Note that winding can be positive or negative, corresponding to a clockwise or anticlockwise wrapping of the string. As in our previous discussions of the superstring, we can define the momentum left-moving and right-moving excitations:

$$p_L = \frac{n}{R} + mR \ , \ p_R = \frac{n}{R} - mR \ , \quad n, m \,\epsilon\, \mathbb{R} \tag{2.29}$$

A right-mover with positive winding is equivalent to a left-mover with negative winding.

Figure 2.3 Momentum and Winding Modes (top and bottom respectively).



In this context, we can define T-duality as the transformation:

$$\text{T:} \quad R \rightarrow \frac{1}{R}. \tag{2.30}$$

From (2.29), we see that this induces a transformation of the right- and left-moving momenta:

$$p_L \rightarrow p_L \ , \ p_R \rightarrow -p_R. \tag{2.31}$$

This transformation changes the string boundary conditions. Consider a string on a cylinder, parametrized by coordinates τ and σ, described by the scalar

$$X(\sigma,\tau) = X_L(\sigma-\tau) + X_R(\sigma+\tau) \tag{2.32}$$

Let's assume that X_L and X_R have only momenta, and no other complications. T-duality then takes $X_R \to -X_R$. We can define another scalar

$$Y(\sigma,\tau) = X_L(\sigma-\tau) - X_R(\sigma+\tau). \tag{2.33}$$

Under T-duality, these are exchanged:

$$\text{T:}\;\; X \to Y. \tag{2.34}$$

Now consider boundary conditions. From the above the above definition of X and Y, one can show the relations

$$\partial_\tau X = \partial_\sigma Y \quad , \quad \partial_\sigma X = \partial_\tau Y. \tag{2.35}$$

It follows that if we take Dirichlet boundary conditions for X, $\partial_\sigma X = 0$, then Y has *Neumann* boundary conditions $\partial_\tau Y{=}0$. Similarly for Neumann boundary conditions on X. Thus T-duality acts on boundary conditions as

$$\text{T:} \quad \text{Dirichlet} \leftrightarrow \text{Neumann}. \tag{2.36}$$

Let us now consider a D-brane. Consider a D3 brane oriented along the 0123 directions. This is specified by the boundary conditions:

0	1	2	3	4	5	6	7	8	9
N	N	N	N	D	D	D	D	D	D

Now perform a T-duality along x^4. This becomes

0	1	2	3	4	5	6	7	8	9
N	N	N	N	N	D	D	D	D	D

which is a D4-brane! Now recall that we deduced in (2.24) from the field content of IIB and IIA that IIB has odd-p Dp-branes while IIA has even-p Dp-branes. The above transformation then indicates that T-duality exchanges the two type II theories.

We can make this more precise: T-duality along x^μ induces a transformation in the fermions,

$$\psi^\mu \to \psi^\mu \quad , \quad \widetilde{\psi}^\mu \to -\widetilde{\psi}^\mu \tag{2.37}$$

This exchanges the GSO projection of type IIA and type IIB. Thus T-duality relates the two type II theories.

There is another important application of T-duality, which relates type IIB to type I. The parity operation which takes IIB to type I can be realized via two T-dualities applied to a special limit of type IIB. More precisely, this occurs in type IIB with D7-branes and a geometry which projects out the parity (−) states, with T-duality applied along both directions transverse to the 7-branes. The the gauge field of type I arises as the open string excitations of the D7-branes in type IIB.

Let's now move on to S-duality. This is a strong-weak coupling, which exchanges the string coupling with its inverse:

$$\text{S:} \;\; g_s \leftrightarrow \frac{1}{g_s}. \tag{2.38}$$

In terms of the dilaton, the [0] of (NS+,NS+), ϕ, this takes $\phi \leftrightarrow -\phi$ (recall that $g_s = e^\phi$).

The type I and heterotic string are equivalent under S-duality. A leading order in the string coupling and α', this can be seen by comparing the supergravity actions. Let's start with the type I action, given by

$$S_I = \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x \sqrt{-g} \left[R - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{4} e^{-\phi/2} F_2^2 - \frac{1}{12} e^{-\phi} F_3^2 \right]. \tag{2.39}$$

Here F_2 is vector field from open strings and F_3 is the field strength of [2] in (R+,R+). The heterotic action is given by

$$S_{het} = \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x \sqrt{-g} \left[R - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{4} e^{\phi/2} F_2^2 - \frac{1}{12} e^{\phi} H_3^2 \right]. \tag{2.40}$$

where now H_3 is the field strength of the Kalb-Ramond (NSNS) two-form and F_2 is the bulk gauge field that arose from the extra degrees of freedom for left-movers in the heterotic string (where left-movers are quantized as in the 26-dimensional boson string).

We can see from the above actions that type I and heterotic theory are related by the transformation

$$\phi \leftrightarrow -\phi \quad , \quad F_3 \leftrightarrow H_3 \quad , \quad F_2^I \leftrightarrow F_2^{(het)} \tag{2.41}$$

The transformation of the dilaton tells us this is a strong-weak duality, $g_s \to 1/g_s$. The exchange of F_3 and H_3 indicates a switch of NSNS and RR sectors. The final piece of the transformation, the exchange of F_2, indicates something deeper: *open strings* are being exchanged for *closed strings*. More details on this can be found in Chapter 14 of Polchinski [1].

More precise checks of this duality exist, as well as more pairs of theories dual under S-duality. In particular, IIB theory is self-dual under S-duality. That is, the S-dual of IIB theory is IIB theory.

2.3.2 M-theory

We have shown that the five consistent superstring theories are related by S- and T-duality. There is one remaining puzzle piece in Figure 2.1: the relation to M-theory. To begin we follow appendix H of Kiritsis [2], and also include select details from [5] and [6].

In its simplest formulation M-theory is 11-dimensional supergravity. This information alone is enough to deduce the connection to string theory. Consider gravity in eleven dimensions, described by the Einstein-Hilbert action,

$$S_M = \int \mathrm{d}^{11}x\,\sqrt{-g}R. \tag{2.42}$$

Now consider putting the theory on a circle. This is done by taking a metric ansatz,

$$\mathrm{d}s_{11}^2 = \mathrm{d}s_{10}^2 + R_{11}^2\,\mathrm{d}\theta^2, \tag{2.43}$$

where θ is the angular coordinate of the circle, which has an (x^μ-dependent) radius R_{11} (with $\mu = 0, 1..9$ the coordinates on the 10d space). The remaining 10 dimensions are described by $\mathrm{d}s_{10}^2$. For future convenience we redefine R_{11} as:

$$R_{11} = e^\sigma. \tag{2.44}$$

Now let us perform a Kaluza-Klein reduction on this circle. This can be done for small values of R_{11}, which leads to an infinite tower of excitations with a mass spectrum of the form

$$m_n^2 \sim \frac{n}{R} \;\;,\;\; n \,\epsilon\, \mathbb{Z}. \tag{2.45}$$

In the limit $R \to 0$, we can integrate out all massive modes and keep only the massless excitations. The theory is then well described by 10-dimensions. In the opposite limit $R \to \infty$, the Kaluza-Klein modes become massless and we are forced to work with the full 11-dimensional description (2.42).

In the limit $R \to 0$, the metric component g_{1010}, which is simply $e^{2\sigma}$, will give rise to a scalar degree of freedom, while the metric components with one leg along the M-theory circle, $g_{\mu 10}$, give rise to a vector field A_μ. The metric is decomposed as,

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} + e^{2\sigma} A_\mu A_\nu & e^{2\sigma} A_\mu \\ e^{2\sigma} A_\mu & e^{2\sigma} \end{pmatrix}. \tag{2.46}$$

The action (2.42) now takes the form, defining $\phi = 3\sigma/2$,

$$S_M \to \int \mathrm{d}^{10}x \sqrt{-g} e^{-2\phi} \left(R + 4(\nabla\phi)^2 - \frac{1}{12} H_3^2 \right) \tag{2.47}$$

which is *precisely* the action of the NSNS sector of type IIA string theory (at leading order in α' and g_s). As we will see, the RR sector (i.e. fluxes) comes from 11-dimensional fluxes. The string coupling is given in terms of σ by,

$$g_s = e^{\frac{3\sigma}{2}} \equiv R_{11}^{\frac{3}{2}}. \tag{2.48}$$

From the above relation we see that in the strong coupling limit, $g_s \to \infty$, the radius R is large and type IIA string theory effectively behaves as 11-dimensional M-theory. In other words: the strong coupling limit of type IIA string theory is a theory *without strings*.

The correspondence between M-theory and type IIA theory can be made more precise. The field content of 11-dimensional supergravity is constrained by supersymmetry to be only three fields:

$$g_{\mu\nu} \quad , \quad C_{\mu\nu\rho} \quad , \quad \psi^\mu_{3/2} \tag{2.49}$$

The field $C_{\mu\nu\rho}$ is quantized. From this, one can deduce that the allowed values of $C_{\mu\,10\,10}$ match precisely the monopole spectrum of A_μ in IIA. There are further checks one can do, but we will not investigate them here.

Let us now state the action of M-theory The bulk supergravity action for M-theory is given by [8]

$$S_{bulk} = \frac{1}{2\kappa^2} \int \mathrm{d}^{11}x \sqrt{-g} \left[R - \frac{1}{48} G^2 \right] - \frac{1}{12\kappa^2} \int C \wedge G \wedge G + \frac{1}{2\kappa^2} \int C \wedge X_8, \tag{2.50}$$

where $G_4 = \mathrm{d}C_3$, the $C \wedge X_8$ term is a Chern-Simons term required for cancellation of the gravitational anomaly (which occurs via 'anomaly inflow' with 5-branes), and X_8 is a contraction of four Reimannn tensors $X_8 \sim trR^4$.

We saw above that M-theory is related to IIA via dimensional reduction on a circle. This is then related to IIB by T-duality, and hence IIB is realized as M-theory on a torus. In a similar manner, which we will not present here, M-theory is related to the $E8 \times E8$ heterotic theory via dimensional reduction on line segment $S^1/\mathbb{Z}_2$.

The fluxes of the type II theories are contained within the M-theory 3-form potential $C_{\mu\nu\rho}$. For example, in reducing M-theory on two torus directions x and y, C_3 decomposes as

$$C_3 = C_3' + B_2 \wedge L\mathrm{d}x + C_2 \wedge L\mathrm{d}y + A_1 \wedge L\mathrm{d}x \wedge L\mathrm{d}y. \tag{2.51}$$

Upon reducing to IIB, B_2 and C_2 becomes the NSNS and RR two-forms respectively, C_3' becomes the 4-form via $C_4 = C_3' \wedge \mathrm{d}y$, and B_1 (which becomes the IIA 1-form C_1) enters the IIB metric as a leg along the y-direction.

M-theory does not have the branes and planes of the Type II theory, although it does its own similar objects, M2 and M5 branes. In general, the branes and planes of IIB enter M-theory as modifications to the geometry, and contribute to the fluxes as pieces localized at the would-be location of the brane/plane. For example, the IIB D7-branes lead to singularities in the M-theory torus, which come with localized M-theory fluxes of the form,

$$\frac{G_4}{2\pi} = \sum_{i=1}^{k} F_i \wedge \Omega_i, \tag{2.52}$$

where $G_4 = \mathrm{d}C_3$, the Ω_i are a basis harmonic forms that encode the brane geometry and are localized at the singularities in the M-theory torus, and F_i are the gauge fields that live on the worldvolume of the D7-branes. As another example, the D3 branes of IIB lift to M2 branes in M-theory. For further details we refer the reader to [5] and [6].

2.4 Moduli and The Connection to 4d Physics

In this thesis the eventual goal is to study cosmology. To do this, we need to find an effective four-dimensional description of the string compactifications discussed thus far. This is done via Kaluza-Klein reduction.

The simplest case is that of a massless scalar in five dimensions, with the fifth dimension a circle of radius R. This can be reduced to a tower of massive scalars in four dimensions, i.e.

$$S = \int \mathrm{d}^5x\,(\partial\phi)^2 \to 2\pi R \sum_n \int \mathrm{d}^4x \left(\partial_\mu \phi_n \partial^\mu \phi_{-n} + \frac{n^2}{R^2}|\phi_n|^2\right) \quad , \quad n \,\epsilon\, \mathbb{Z} \tag{2.53}$$

where the ϕ_n are the coefficients in the expansion into plane-waves on the circle:

$$\phi(x) = \sum_n \phi_n(x^\mu) e^{iny/R} \tag{2.54}$$

where $\mu = 0, 1, 2, 3$ and $y \equiv x^4$. From the above action we see that the four-dimensional description has a single massless degree of freedom, ϕ_0. At energies far below $\Lambda_{UV} = 1/R$, we can integrate out the massless modes leaving ϕ_0 as the only dynamical degree of freedom,

$$S \approx 2\pi R \int \mathrm{d}^4x\, \partial_\mu \phi_0 \partial^\mu \phi_0. \tag{2.55}$$

This is similar to our analysis of the M-theory/IIA duality, wherein we saw that the reduction of gravity on a circle leads to additional scalar and vector degrees of freedom.

An alternative way to approach this is to take the field content in the higher dimensional theory to be on-shell, and work directly with the equations of motion. For the scalar field example, the equation of motion is

$$\Box_5 \phi = 0, \tag{2.56}$$

where $\Box_5$ is the five-dimensional d'Alembertian operator. For a direct product space, the d'Alembertian splits into a 4d piece and the piece along the fifth dimension, i.e.

$$\Box_5 = \Box_4 + \Box_y, \tag{2.57}$$

where again $y \equiv x^4$. Decomposing $\phi = \sum_n \phi_n(x^\mu)\phi_n^{(5)}(y)$, the above equation of motion can rearranged to solve for ϕ_n,

$$\Box_4 \phi_n = m_n^2 \phi_n, \tag{2.58}$$

with

$$m_n^2 = \frac{1}{\phi_n^{(5)}} \Box_y \phi_n^{(5)}. \tag{2.59}$$

If we take $\phi_n^{(5)} = e^{iny/R}$, this reproduces our earlier result. The massless modes correspond to zero-modes of the d'Alembertian on the fifth dimension, i.e.

$$\Box_y \phi_n^{(5)} = 0. \tag{2.60}$$

We can generalize this to a differential form on a direct product space $\mathcal{M}_4 \times X$. Consider a p-form A_p, with equation of motion

$$\Box A_p = 0. \tag{2.61}$$

Now perform a similar decomposition as before

$$\Box = \Box_4 + \Box_X \quad , \quad A_p = \sum_{r+q=p} \sum_n A_r^n(x^\mu) \wedge \omega_q^n(y^m), \tag{2.62}$$

where subscripts p and q denote the rank of a differential form, and y^m are the coordinates on the internal space X. As before, "n" is not an index but a label for the expansion. We thus see that we have four dimensional fields $A_r(x^\mu)$ of varying rank, from scalars to p-forms.

The equation of motion of a given A_r^n is

$$\left(\Box_4 - m_{n,r}^2\right) A_r^n = 0. \tag{2.63}$$

Again zero modes satisfy,

$$\Box_X \omega_q^n = 0. \tag{2.64}$$

Thus massless r-forms correspond to harmonic $(p-r)$-forms on X. To get a further grasp on the 4d physics, we would like to count how many there are.

First, notice that harmonic forms are neccessarily closed, i.e.

$$d\omega_q^n = 0. \tag{2.65}$$

Next, notice that *independent* solutions to $\Box\omega = 0$ are those which do not trivially solve the above equation, and hence are not exact $\omega \neq d\alpha$ for some α, and also are not related to one another by an exact form $d\alpha$. The set of forms that are closed but not exact form a group; this is the *cohomology*. Thus the number of r-forms that are closed but not exact is precisely the dimension of the rank-$(p-r)$ cohomology group, denoted by the Hodge number h^{p-r}.

When applied to the metric, this leads to the moduli of string theory. There are additional moduli coming from fluxes, but here we will consider only the metric. See e.g. Becker Becker Schwartz [8] for more details.

The vacuum Einstein equations are

$$R_{MN} = R_{\mu\nu} + R_{mn} = 0. \tag{2.66}$$

where in the first equality we have assumed a direct product space. The massless degrees of freedom satisfy $R_{mn} = 0$, or in terms of linearized metric fluctuations

$$\Box_X \delta g_{pq} = 0. \tag{2.67}$$

To classify the solutions, we work in complex coordinates, denoted by $i = 1, 2, 3$, and $\bar{i} = \bar{1}, \bar{2}, \bar{3}$. The metric g_{pq} can be split into "pure" components g_{ij} and "mixed components" $g_{i\bar{j}}$.

We can then define forms that are dual to the metric. We define a complex (2,1)-form via,

$$\Omega = \Omega^{\bar{i}}_{kl}\delta g_{\bar{i}\bar{j}}\, \mathrm{d}x^k \wedge \mathrm{d}x^l \wedge \mathrm{d}x^{\bar{j}}, \tag{2.68}$$

which captures the pure components, and a (1,1) form

$$J = \delta g_{i\bar{j}} \mathrm{d}x^i \wedge \mathrm{d}x^{\bar{j}} \tag{2.69}$$

which captures the mixed components. For a *Calabi-Yau* manifold, both Ω and J are closed. This, and other special cases will be discussed in Chapter 3. For now, let's consider the canonical Calabi-Yau case.

The massless scalars arising from Ω are referred to as *Kahler moduli*, and there are $2(1+h^{(2,1)})$ of them, where $h^{p,q}$ is denotes a Hodge number of the complexified manifold. The massless scalars arising from J are referred to as complex structure moduli, and there are $h^{(1,1)}$ of them. This counting arises from the topological structure of a Calabi-Yau manifold.

As an example, consider an orbifold of a six-torus, $T^6/\mathbb{Z}_3$ [7]. The orbifold operation serves to remove from T^6 the non-trivial one-cycles, making it into a Calabi-Yau manifold. In this case there are 27 complex structure moduli and 6 Kahler moduli. In generic examples, and with fluxes included, there can be hundreds or thousands of moduli fields. Each of these appear in the four-dimensional fields as massless scalars.

The resulting four dimensional theory is a supergravity theory, with a rescaled Newton's constant,

$$G_N^{(4)} = \frac{G^{(10)}}{Vol} \tag{2.70}$$

where $G^{(4)}$ and $G^{(10)}$ are the Newton's constant in four and ten dimensions respectively, and Vol is the volume of the internal space. While the ten-dimensional supergravity description is only applicable for Vol much greater than the string scale, the above relation indicates that Vol cannot be *too* large, as this would effectively 'turn off' gravity in four dimensions.

For compactification of type II string theory on a torus, the resulting theory has $\mathcal{N}=4$ supersymmetry. The amount of supersymmetry can be reduced by orbifolding the internal space, and the most common approach is to study four-dimensional physics with $\mathcal{N}=1$ supersymmetry.

The moduli fields appear in the supergravity theory as the scalar components of chiral superfields Φ^i, whose physics are governed by a superpotential $W(\Phi^i)$ and Kähler potential $K(\Phi,\bar{\Phi}^i)$. The action for these superfields is given by

$$\mathcal{L} = -\frac{1}{2}K_{I\bar{J}}\partial_\mu\Phi^I\partial^\mu\bar{\Phi}^{\bar{J}} - V, \tag{2.71}$$

where $K_{I\bar{J}}$ denotes $\partial_I\partial_{\bar{J}}K$, and $\partial_I \equiv \partial_{\Phi^I}$, $\partial_{\bar{I}} \equiv \partial_{\bar{\Phi}^{\bar{I}}}$. The scalar potential V is given by

$$V = e^K\left(|DW|^2 - 3|W|^2\right), \tag{2.72}$$

where D is the Kahler covariant derivative, $D_I = \partial_I + \partial_I K$, and the contraction in $|DW|^2$

is done via $K_{I\bar{J}}$. To make studies of four-dimensional physics tractable, we can specify choices for K and W which give many or most of the moduli fields a mass. These heavy degrees of freedom can then be integrated out. This is known as *moduli stabilization*.

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Part II

String Theory

Chapter 3

Non-Kähler resolved conifold, localized fluxes in M-theory and supersymmetry

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Addendum for thesis

This chapter studies the internal manifold of string theory in the presence of branes. In particular, it studies so-called ‘conifold’ geometries with wrapped D5-branes.

The everyday cone is a product of the real line and a circle, with the radius of the circle shrinking as one moves along the real line. A conifold replaces the circle with a more general "base manifold", which can have arbitrary dimension or topology. This chapter will focus on conifolds with base manifold $S^2 \times S^3$, and in particular, on the ‘resolved conifold,’ where the size of the S^2 is kept finite at the tip of the conifold.

This chapter will also study the M-theory description of these geometries. For a D5-brane in *flat* space, the M-theory description is remarkably simple (which we will present here but not derive). The D5-brane is first T-dualized to a D6-brane in type IIA, which is accompanied by RR 1-form flux C_1 (magnetically sourced by the brane). The brane disappears under the lift to M-theory and the transverse space is converted to a Taub-Nut space, defined by metric

$$\mathrm{d}s^2 = \left(1 + \frac{1}{2|\vec{x} - \vec{x}_0|}\right) \mathrm{d}\vec{x}^2 + \left(1 + \frac{1}{2|\vec{x} - \vec{x}_0|}\right)^{-1} \left(\mathrm{d}x_{11} + C_i \mathrm{d}x^i\right)^2 , \tag{3.1}$$

where $\vec{x} = (x^7, x^8, x^9)$, $i = 7, 8, 9$, C_i is the RR 1-form in IIA, and x_0 corresponds to the brane position under dimensional reduction back to IIA. This is a circle fibration over $\mathbb{R}^3$, where the circle degenerates (shrinks to zero size) at the position of the brane. Of crucial importance is that the geometry of the Taub-Nut space allows for a unique normalizable two-form, which allows for well defined fluxes despite the singularity in the metric. In more complicated examples, such as those that will be studied in this chapter, the Taub-Nut space is modified and the internal space ceases to be a direct product of a Taub-Nut space and some other space. However, one may still construct the normalizable two-form, allowing for a consistent study of fluxes at the would-be brane position.

Finally, the geometries studied here will be connected to four-dimensional physics in Chapter 4, where we study supersymmetry breaking in these backgrounds.

Abstract

The known supergravity solution for wrapped D5-branes on the two-cycle of a Kähler resolved conifold is in general ISD but not supersymmetric, with the supersymmetry being broken by the presence of (1, 2) fluxes. However if we allow a non-Kähler metric on the resolved conifold, supersymmetry can easily be restored. The vanishing of the (1, 2) fluxes here corresponds to, under certain conformal rescalings of the metric, the torsion class constraints. We construct a class of explicit non-Kähler metrics on the resolved conifold satisfying the constraints. All this can also be studied from M-theory, where the fluxes and branes become non-localized G-fluxes on deformed Taub-NUT spaces. Interestingly, the gauge fluctuations on the wrapped D5-branes appear now as localized G-fluxes in M-theory. These localized fluxes are related to certain harmonic two-forms that are normalizable. We compute these forms explicitly and discuss how new constraints on the geometry of the non-Kähler manifolds may appear from M-theory considerations.

3.1 Introduction

The resolved conifold, originally discussed in the work of [1] supports, for a given complex structure and a given Kähler class, a unique Ricci flat metric with vanishing first Chern class. This is the well known Calabi-Yau metric that has been used in string theory to understand various aspects of dualities and compactifications. However, in most of the studies the effect of the background fluxes on the metric of the Kähler resolved conifold has not been discussed. In certain interesting cases, which will be the subject of this paper, the combined effect of string equations of motion and supersymmetry may lead to metrics on the resolved conifold that are neither Kähler nor Calabi-Yau. These non-Kähler metrics on a resolved conifold have not been used much in the literature, despite their apparent ubiquity, partly because they do not satisfy the nice properties one encounters for the Kähler case, and partly because of their underlining technicalities.

The situation changed once simple examples of non-Kähler manifolds that satisfy torsional equations and supersymmetry [4] were constructed in [11]. This was followed by the classic work of Chiossi and Salamon [26] who essentially laid out the criteria for constructing torsional solutions. The condition for supersymmetry of these solutions were replaced, from the standard $SU(3)$ and G_2 holonomies [6], to the corresponding $SU(3)$ and G_2 structures. In simpler terms, the non-closure of the fundamental two form or the holomorpic three-form were measured by the torsion classes: essentially telling us that there are five torsion classes $\mathcal{W}_i$ that form various representations of an $SU(3)$ structure. The well-know Calabi-Yau case arises when all the torison classes vanish. In M-theory, the equivalent picture with G_2 structure leads to four torsion classes.

The work of Chiossi and Salamon were immediately applied to string theory in a series of beautiful papers [3,8,28,29] that clarified the role of complexity, torsion, supersymmetry and their interconnections in constructing six dimensional manifolds in string theory. One of the early outcomes of this was the realization that the internal six-dimensional manifolds do not have to be Kähler or even complex to satisfy stringy EOM or supersymmetry. If the manifolds are endowed with an almost complex structure that is non-integrable, consistent solutions can be constructed. The earlier work of [11] argued the existence of a six-dimensional compact space that has an integrable complex structure but is non-Kähler. Combining the stories together, we can now construct internal spaces that are neither Kähler nor complex, yet preserve supersymmetry.

One may also go back to familiar territory, for example the resolved conifold, and ask if it is possible to put a non-Kähler metric on it. This is the subject of this paper, and our answer will be in the affirmative: we will be able to construct explicit non-Kähler metrics on a resolved conifold in sec. 3.5. In fact in sec. 3.5.1 we will argue that there is a large class of possible solutions with and without dilaton profiles.

Our key result, presented in section 3.5.1, is two classes of explicit solutions for the supersymmetric non-Kähler resolved conifold, with metric given by equation (3.10),

$$
\begin{aligned}
ds^2 &= \frac{1}{e^{2\phi/3}\sqrt{e^{2\phi/3}+\Delta}}\, ds^2_{0123} + e^{2\phi/3}\sqrt{e^{2\phi/3}+\Delta}\, ds^2_6, \\
ds^2_6 &= F_1\, dr^2 + F_2(d\psi + \cos\,\theta_1 d\phi_1 + \cos\,\theta_2 d\phi_2)^2 + \sum_{i=1}^{2} F_{2+i}(d\theta_i^2 + \sin^2\theta_i d\phi_i^2).
\end{aligned}
\tag{3.2}
$$

We derive a class of solutions with constant dilaton, with warp factors given by:

$$
\begin{aligned}
F_2 &= F_2(r), \qquad \phi = \phi_0, \\
F_1 &= \frac{4F_{2r}^2}{F_2^{5/3}(3+2F_2)^{10/3}}, \quad F_3 = 1 - \frac{2+F_2}{F_2^{1/3}(3+2F_2)^{2/3}}, \\
F_4 &= \left[1 - \frac{2+F_2}{F_2^{1/3}(3+2F_2)^{2/3}}\right]\frac{F_2^{2/3}}{(3+2F_2)^{2/3}},
\end{aligned}
\tag{3.3}
$$

where $F_{2r} = dF_2/dr$, in terms of a general $F_2(r)$, as well as a class of solutions with varying dilaton, given by

$$
F_1 = \frac{1}{2F}, \qquad F_2 = \frac{r^2 F}{2}, \qquad F_3 = \frac{r^2}{4} + a^2 e^{-2\phi}, \qquad F_4 = \frac{r^2}{4}, \qquad \phi = \phi(r), \tag{3.4}
$$

for a general function $F(r)$.

There are many advantages in constructing non-Kähler metrics on a resolved conifold, and in the following we mention a few. The foremost is the study of gauge/gravity duality in the geometic transition setting [13], where the starting point is the gauge theory on wrapped D5-branes on the two-cycle of a resolved conifold. As discussed in [11–13,25], the metric on the resolved conifold has to be non-Kähler to satisfy all the EOMs and supersymmetry. In general wrapping D5-branes on a Calabi-Yau resolved conifold would lead to a background that satisfies EOM but breaks supersymmetry [16, 17, 41, 55]. Using the criteria of [12], this means the background (admitting an integrable complex structure), will have a (1, 2) piece in addition to the supersymmetry preserving (2, 1) piece of the three-form flux (see [16, 17, 41] for more details).

A related construction with non-Kähler resolved conifold appears when we take the mirror of the wrapped D5-branes background. The mirror, or SYZ transformation [20], leads to a non-Kähler deformed conifold with D6-branes wrapping the three-cycle of the manifold. Under geometric transition [13] this will give us another non-Kähler resolved conifold with fluxes in type IIA theory. The difference now is, other than the fact that we are in type IIA instead of type IIB theory, that there are no wrapped branes. The branes have disappeared and are replaced by fluxes [11, 13, 25].

Another place where non-Kähler resolved conifold shows up is in the gravity dual of little string theories (LSTs) recently studied in [32]. The LSTs are constructed in $SO(32)$ and $E_8 \times E_8$ heterotic theories by wrapping heterotic five-branes on the two-cycle of a non-Kähler resolved conifold. These LSTs are not scale invariant, and their degrees of freedom confine at low energies. In our class of solutions, the backgrounds studied in [32] fall in the category with non-trivial dilaton profiles. In fact we study two kinds of solutions with varying dilaton profiles, exemplified by (3.137) and (3.145); and the ones studied in [32] fall in the latter category.

The non-Kähler resolved conifold has also appeared in the study of large N thermal QCD, that confines in the far IR and becomes scale invariant at the highest energies [22, 23]. The background to study QGP properties involve the resolved conifold with three-form fluxes in type IIB theories, but in general there are no wrapped D5-branes. However wrapped anti-D5 branes appear once we demand UV completions with asymptotic AdS spaces [24], and the full construction becomes more involved than the simple cases that we discuss here. Nevertheless, the starting point is still a non-Kähler resolved conifold.

Last, but not the least, the non-Kähler resolved conifold can also be used to generate D-terms in type IIB theory using an embedding of D7-brane in this background [25]. The subtlety here is to generate supersymmetry breaking bulk fluxes that are *not* ISD to allow for non-zero F-terms, as ISD (1,2) fluxes will break supersymmetry without generating a bulk cosmological constant. This criteria is essential, otherwise no D-terms could appear in the theory [26, 27]. These D-terms appear from certain 'localized fluxes' in M-theory on a four-fold. Locally the four-fold will be a Taub-NUT fibered over a four-dimensional base.

On the other hand, we can also study localized fluxes on a *seven*-dimensional manifold in M-theory that is locally a Taub-NUT fibered over a three-dimensional base. We expect the seven-dimensional manifold to have a G_2 structure, and appear from a specific $\mathbf{S}^1$ fibration over the non-Kähler resolved conifold. These two descriptions should match up, with the G_2 structure manifold being constructed by T-dualizing the type IIB solution, and by rewriting it as a warped Taub-NUT fibration over a three-dimensional base. The localized fluxes then are related to certain harmonic two-form on the warped Taub-NUT space.

The paper is organized in the following way. In sec. 3.2 we study the basic construction of fluxes and D5-branes on a non-Kähler resolved conifold using various dualities, and then in sec. 3.2.1 argue for supersymmetry and corresponding constraints on the warp factors. The issue of supersymmetry is dealt with again in sec. 3.3, now using the detailed machinery of the torsion classes both before, in sec. 3.3.1, and after, in sec. 3.3.2, certain solution-generating duality transformations. The system is then lifted to M-theory in sec. 3.4 where Taub-NUT spaces appear prominently. Simple warm-up examples for generating localized fluxes using harmonic two-forms are discussed in sec. 3.4.1 and sec. 3.4.2. We head on to the explicit construction of solutions in sec. 3.5, and in sec. 3.5.1 solutions for the warp factors of non-Kähler resolved conifolds for the two class of examples, with and without dilaton profiles, are derived. In sec. 3.6 we discuss localized fluxes and DBI gauge fields on the brane worldvolume, and conclude in sec. 3.7 with some discussion of directions for future work.

3.2 D5-branes on a Non-Kähler Resolved Conifold

The supersymmetric case of a wrapped D5-brane on a resolved conifold is not hard to construct if we assume that the metric on a resolved conifold is a non-Kähler one. This has been discussed in a different context in [24, 25], and we will first outline the general technique. The starting point is a non-Kähler resolved cone solution in type IIB in the presence of H_{NS} fluxes. This is supersymmetric and the solution is given by the following form:

$$\begin{aligned} ds^2 &= ds^2_{0123} + e^{2\phi} ds^2_6, \\ \mathcal{H} &= e^{2\phi} *_6 d\left(e^{-2\phi} J\right), \end{aligned} \tag{3.5}$$

where ϕ is the usual type IIB dilaton and J is the fundamental two-form of the warped internal six-dimensional manifold whose unwarped metric is given by:

$$ds_6^2 = F_1\ dr^2 + F_2(d\psi + \cos\ \theta_1 d\phi_1 + \cos\ \theta_2 d\phi_2)^2 + \sum_{i=1}^{2} F_{2+i}(d\theta_i^2 + \sin^2\theta_i d\phi_i^2), \tag{3.6}$$

where F_i are warp-factors whose values will be determined later. For simplicity we will consider them to be functions of the radial coordinate r only.

The steps for creating a supersymmetric wrapped D5 brane with three-form fluxes now follow the trick laid out by [24,25]. We S-dualize the background (3.5), followed by three T-dualities along the $x^{1,2,3}$ directions. The resulting type IIA configuration will now become:

$$\begin{aligned} ds^2 &= -e^{-\phi}dt^2 + e^{\phi}ds_{123}^2 + e^{\phi}ds_6^2, \\ G_4 &= d(e^{-2\phi}J) \wedge dt, \end{aligned} \tag{3.7}$$

with a dilaton expressed as $e^{\phi/2}$. We can lift the configuration (3.7) to M-theory and perform a boost along the eleventh direction. Using the boost parameter β, the resulting M-theory configuration is given by:

$$\begin{aligned} ds^2 &= -dt^2(e^{-4\phi/3} - \Delta) + dx_{11}^2(e^{2\phi/3} + \Delta) + e^{2\phi/3}\left(ds_{123}^2 + ds_6^2\right), \\ G_4 &= (G_4)_{0mnp} \cosh\ \beta\ dt \wedge dx^m \wedge dx^n \wedge dx^p - (G_4)_{0mnp} \sinh\ \beta\ dx_{11} \wedge dx^m \wedge dx^n \wedge dx^p, \end{aligned} \tag{3.8}$$

where (m, n, p) in the subscript of G_4 denote the coordinates of the internal non-Kähler manifold. We have also defined Δ as:

$$\Delta = \sinh^2\beta\left(e^{2\phi/3} - e^{-4\phi/3}\right), \tag{3.9}$$

which vanishes when there is no dilaton or no boost as expected.

Once we dimensionally reduce this to type IIA and then make the three T-dualities along directions $x^{1,2,3}$, the type IIB configuration takes the following form:

$$\begin{aligned} ds^2 &= \frac{1}{e^{2\phi/3}\sqrt{e^{2\phi/3}+\Delta}}\, ds^2_{0123} + e^{2\phi/3}\sqrt{e^{2\phi/3}+\Delta}\; ds^2_6, \\ \mathcal{F}_3 &= \cosh\,\beta e^{2\phi} *_6 d\left(e^{-2\phi}J\right), \qquad \mathcal{H}_3 = -\sinh\,\beta\; d\left(e^{-2\phi}J\right), \end{aligned} \tag{3.10}$$

with dilaton $e^{\phi_B} = e^{-\phi}$ and the Hodge-star above is with respect to the non-Kähler metric (3.5). Note that if the underlying metric on the resolved conifold was Kähler, then it *wouldn't* have been possible to have a supersymmetric configuration like (3.7). The above metric describes D5 branes wrapped on the warped resolved conifold, the detailed study of which is the study of our present work.

We will also have a five-form given by:

$$\widetilde{\mathcal{F}}_5 = -\sinh\,\beta\,\cosh\,\beta\,(1 + *_{10})\,\mathcal{C}_5(r)\; d\psi \wedge \prod_{i=1}^{2} \sin\;\theta_i\; d\theta_i \wedge d\phi_i \tag{3.11}$$

where the Hodge-star is now with respect to the metric (3.10) and $\mathcal{C}_5(r)$ will be determined below. Thus combining (3.10) and (3.11) we should get our supersymmetric background. Also note that the 3-form fluxes obey a *modified* ISD condition:

$$\mathcal{F}_3 + e^{2\phi}\tanh\,\beta\;\; *_6\,\mathcal{H}_3 = 0. \tag{3.12}$$

This guarantees the configuration is a solution to the equations of motion, but does not guarantee supersymmetry. For supersymmetry we will need extra conditions on the warp factors. We will discuss this in sec. 3.2.1.

It will also be useful to expand the $\mathcal{H}_3$ and $\mathcal{F}_3$ forms from (3.10) in terms of the coordinate one-forms. Using the metric (3.6), we find $\mathcal{H}_3$ to be given by the following expression:

$$\begin{aligned} \frac{\mathcal{H}_3}{\sinh\,\beta} \quad &= \quad \left(\sqrt{F_1F_2}\sin\theta_1 - F_{3r}\sin\theta_1\right) dr \wedge d\theta_1 \wedge d\phi_1 \\ &\quad + \left(\sqrt{F_1F_2}\sin\theta_2 - F_{4r}\sin\theta_2\right) dr \wedge d\theta_2 \wedge d\phi_2, \end{aligned} \tag{3.13}$$

in terms of $F_i(r)$, $F_{nr} \equiv dF_n/dr$, $\phi_r \equiv d\phi/dr$, and we have assumed $\phi = \phi(r)$ is independent of the angular coordinates. The RR three-form $\mathcal{F}_3$ also simplifies when the dilaton is independent of the angular coordinates, taking the following form:

$$\begin{aligned}\frac{\mathcal{F}_3}{\cosh\,\beta} = \quad & k_1 F_2 \cos\,\theta_2(\sqrt{F_1F_2} - F_{4r})d\theta_1\wedge d\phi_1\wedge d\phi_2 \\ & + k_2\, F_2 \cos\,\theta_1(\sqrt{F_1F_2} - F_{3r})d\theta_2\wedge d\phi_1\wedge d\phi_2 \\ & + k_3\,\sin\,\theta_2(\sqrt{F_1F_2} - F_{4r})d\psi\wedge d\theta_1\wedge d\phi_1 \\ & + k_4\,\sin\,\theta_1(\sqrt{F_1F_2} - F_{3r})d\psi\wedge d\theta_2\wedge d\phi_2,\end{aligned} \tag{3.14}$$

where again $F_{nr} \equiv dF_n/dr$ and the k_i are defined by the following expressions:

$$\begin{aligned} k_1 &= -\frac{F_3e^{2\phi}}{F_4\sqrt{F_1F_2}}\cdot\sin\,\theta_1, \qquad k_2 = \frac{F_4e^{2\phi}}{F_3\sqrt{F_1F_2}}\cdot\sin\,\theta_2, \\ k_3 &= -e^{2\phi}\sqrt{\frac{F_2}{F_1}}\cdot\frac{F_3}{F_4}\cdot\frac{\sin\,\theta_1}{\sin\,\theta_2}, \qquad k_4 = -e^{2\phi}\sqrt{\frac{F_2}{F_1}}\cdot\frac{F_4}{F_3}\cdot\frac{\sin\,\theta_2}{\sin\,\theta_1}.\end{aligned} \tag{3.15}$$

Note that $d\mathcal{H}_3 = 0$, whereas $d\mathcal{F}_3$ does *not* vanish: an indication that there are wrapped five-brane sources. Finally using (3.14) and (3.13), $\mathcal{C}_5(r)$ in (3.11) can be expressed as:

$$\mathcal{C}_5(r) = \int^r \frac{e^{2\phi}F_3F_4\sqrt{F_1F_2}}{F_1}\left[\left(\frac{\sqrt{F_1F_2}-F_{3r}}{F_3}\right)^2 + \left(\frac{\sqrt{F_1F_2}-F_{4r}}{F_4}\right)^2\right]dr. \tag{3.16}$$

To lift this configuration to M-theory, we will first T-dualize along ψ direction to generate a six-brane configuration in IIA. The IIB $\mathcal{F}_3$ gives rise to the following RR two-form flux in the dual type IIA theory:

$$\begin{aligned}\mathcal{F}_2 = \quad & -e^{2\phi}\sqrt{\frac{F_2}{F_1}}\cdot\frac{F_3}{F_4}(\sqrt{F_1F_2} - F_{4r})\cosh\,\beta\,\,\sin\,\theta_1 d\theta_1\wedge d\phi_1 \\ & -e^{2\phi}\sqrt{\frac{F_2}{F_1}}\cdot\frac{F_4}{F_3}(\sqrt{F_1F_2} - F_{3r})\cosh\,\beta\,\,\sin\,\theta_2 d\theta_2\wedge d\phi_2.\end{aligned} \tag{3.17}$$

We will impose non-closure of $\mathcal{F}_2$ to allow for a Taub-NUT background along the angular directions in the lift to M-theory. We also define a warp factor H in the following way from (3.10):

$$H = e^{4\phi/3}\left(e^{2\phi/3} + \Delta\right), \tag{3.18}$$

where Δ is defined in (3.9). Using this, the M-theory metric can be expressed as:

$$\begin{aligned} ds^2_{11} &= \frac{e^{2\phi/3}F_2^{1/3}}{H^{1/3}}\left(ds^2_{0123}+\frac{1}{F_2}d\psi^2\right)+\frac{1}{e^{4\phi/3}F_2^{2/3}H^{1/3}}(dx_{11}+\mathcal{A}_{1\mu}dx^\mu)^2 \\ &+ e^{2\phi/3}F_2^{1/3}H^{2/3}\left[F_1dr^2+F_3(d\theta_1^2+\sin^2\theta_1\ d\phi_1^2)+F_4(d\theta_2^2+\sin^2\theta_2\ d\phi_2^2)\right] \\ &= G_1\left(ds^2_{0123}+\frac{1}{F_2}d\psi^2\right)+G_2\left(d\theta_1^2+\sin^2\theta_1\ d\phi_1^2\right) \\ &+ G_3dr^2+G_4\left(d\theta_2^2+\frac{G_5}{G_4}d\phi_2^2\right)+G_6\left(dx_{11}+\mathcal{A}_{1\mu}dx^\mu\right)^2, \end{aligned} \tag{3.19}$$

where $\mathcal{A}_1$ is the gauge-field coming from (3.17). This is a deformed Taub-NUT geometry that stretches along directions $(r,\theta_2,\phi_2,x_{11})$. We will discuss this geometry in more details in sec. 3.4. Additionally, comparing (3.19) to (3.85) we see that the relevant G_i coefficients are defined as:

$$\begin{aligned} &G_1=e^{2\phi/3}F_2^{1/3}H^{-1/3}, && G_2=e^{2\phi/3}F_2^{1/3}H^{2/3}F_3, \\ &G_3=e^{2\phi/3}H^{2/3}F_1F_2^{1/3}, && G_4=e^{2\phi/3}H^{2/3}F_4F_2^{1/3}, \\ &G_5=e^{2\phi/3}H^{2/3}F_4F_2^{1/3}\sin^2\theta_2, && G_6=e^{-4\phi/3}H^{-1/3}F_2^{-2/3}. \end{aligned} \tag{3.20}$$

There is also background G-flux $\mathcal{G}_4$ to support this configuration. This is given by:

$$\begin{aligned} \frac{\mathcal{G}_4}{\sinh\beta} = &\left(\sqrt{F_1F_2}-F_{3r}\right)\sin\ \theta_1\ dr\wedge d\theta_1\wedge d\phi_1\wedge dx_{11}+\frac{\sin\ \theta_1}{\sinh\ \beta}\ d\psi\wedge d\theta_1\wedge d\phi_1\wedge dx_{11} \\ &+\left(\sqrt{F_1F_2}-F_{4r}\right)\sin\ \theta_2\ dr\wedge d\theta_2\wedge d\phi_2\wedge dx_{11}+\frac{\sin\ \theta_2}{\sinh\ \beta}\ d\psi\wedge d\theta_2\wedge d\phi_2\wedge dx_{11} \\ &+k_1F_2\left(\sqrt{F_1F_2}-F_{4r}\right)\coth\ \beta\ \cos\ \theta_2\ d\theta_1\wedge d\phi_1\wedge d\phi_2\wedge d\psi \\ &+k_2F_2\left(\sqrt{F_1F_2}-F_{3r}\right)\coth\ \beta\ \cos\ \theta_1\ d\theta_2\wedge d\phi_1\wedge d\phi_2\wedge d\psi \\ &+\mathcal{C}_1(r,\theta_1,\theta_2)\ dr\wedge d\theta_1\wedge d\phi_1\wedge d\phi_2+\mathcal{C}_2(r,\theta_1,\theta_2)\ dr\wedge d\theta_2\wedge d\phi_2\wedge d\phi_1, \end{aligned} \tag{3.21}$$

where the last two terms are from the five-form (3.11) with the coefficients $\mathcal{C}_i$ derivable directly from (3.11) and (3.16). We will discuss more on this soon.

3.2.1 Supersymmetry Constraints on the Warp Factors

We can now explicitly demonstrate supersymmetry for the type IIB background with D5-branes wrapping a resolved conifold, equation (3.10). This requires the $\mathcal{G}_3$ flux:

$$\mathcal{G}_3 = \mathcal{F}_3 - ie^{-\phi_B}\mathcal{H}_3, \tag{3.22}$$

to be of a (2, 1) form, and not a (1, 2) form. We will first need the vielbeins for the background (3.10). They are given by:

$$\begin{aligned}
e_1 &= \sqrt{F_1\sqrt{H}}e_r, \qquad e_2 = \sqrt{F_2\sqrt{H}}(d\psi + \cos\ \theta_1 d\phi_1 + \cos\ \theta_2 d\phi_2) = \sqrt{F_2\sqrt{H}}e_\psi,\\
e_3 &= \sqrt{F_3\sqrt{H}}\left(-\sin\ \frac{\psi}{2}\ e_{\phi_1} + \cos\ \frac{\psi}{2}\ e_{\theta_1}\right), \quad e_4 = \sqrt{F_3\sqrt{H}}\left(\cos\ \frac{\psi}{2}\ e_{\phi_1} + \sin\ \frac{\psi}{2}\ e_{\theta_1}\right),\\
e_5 &= \sqrt{F_4\sqrt{H}}\left(-\sin\ \frac{\psi}{2}\ e_{\phi_2} + \cos\ \frac{\psi}{2}\ e_{\theta_2}\right), \quad e_6 = \sqrt{F_4\sqrt{H}}\left(\cos\ \frac{\psi}{2}\ e_{\phi_2} + \sin\ \frac{\psi}{2}\ e_{\theta_2}\right),
\end{aligned} \tag{3.23}$$

where H is the warp factor (3.18). Now using the vielbeins we can define three complex one-forms in the following way:

$$E_1 = e_1 + i\gamma e_2, \qquad E_2 = e_3 + ie_4, \qquad E_3 = e_5 + ie_6, \tag{3.24}$$

where we have inserted a non-trivial complex structure $(i\gamma, i, i)$ respectively. The functional form of γ will be derived soon. Using the complex one-forms (3.24), we can rewrite the $\mathcal{G}_3$ flux (3.22) in the following way:

$$\begin{aligned}
\mathcal{G}_3 \ = \ & -\frac{1}{4}\left[\frac{e^{\phi}(\sqrt{F_1F_2} - F_{3r})\sinh\ \beta}{F_3\sqrt{H}\sqrt{F_1\sqrt{H}}} - \frac{e^{2\phi}(\sqrt{F_1F_2} - F_{4r})\cosh\ \beta}{\gamma F_4\sqrt{H}\sqrt{F_1\sqrt{H}}}\right] E_1\wedge E_2\wedge \bar{E}_2\\
& + \frac{1}{4}\left[\frac{e^{\phi}(\sqrt{F_1F_2} - F_{3r})\sinh\ \beta}{F_3\sqrt{H}\sqrt{F_1\sqrt{H}}} + \frac{e^{2\phi}(\sqrt{F_1F_2} - F_{4r})\cosh\ \beta}{\gamma F_4\sqrt{H}\sqrt{F_1\sqrt{H}}}\right] E_2\wedge \bar{E}_1\wedge \bar{E}_2\\
& - \frac{1}{4}\left[\frac{e^{\phi}(\sqrt{F_1F_2} - F_{4r})\sinh\ \beta}{F_4\sqrt{H}\sqrt{F_1\sqrt{H}}} - \frac{e^{2\phi}(\sqrt{F_1F_2} - F_{3r})\cosh\ \beta}{\gamma F_3\sqrt{H}\sqrt{F_1\sqrt{H}}}\right] E_1\wedge E_3\wedge \bar{E}_3\\
& + \frac{1}{4}\left[\frac{e^{\phi}(\sqrt{F_1F_2} - F_{4r})\sinh\ \beta}{F_4\sqrt{H}\sqrt{F_1\sqrt{H}}} + \frac{e^{2\phi}(\sqrt{F_1F_2} - F_{3r})\cosh\ \beta}{\gamma F_3\sqrt{H}\sqrt{F_1\sqrt{H}}}\right] E_3\wedge \bar{E}_1\wedge \bar{E}_3.
\end{aligned}$$

$$(3.25)$$

We note that the above rewriting of the $\mathcal{G}_3$ flux shows that there are both $(2,1)$ as well as $(1,2)$ pieces, although the total flux is ISD. However we can make the $(1,2)$ piece vanish by choosing an appropriate γ.

Vanishing of the $E_2 \wedge \bar{E}_1 \wedge \bar{E}_2$ term requires γ is required to be:

$$\gamma = -\frac{e^{\phi}\left(\sqrt{F_1 F_2} - F_{4r}\right)}{\sqrt{F_1 F_2} - F_{3r}} \cdot \frac{F_3}{F_4} \coth \beta, \tag{3.26}$$

whereas vanishing of the $E_3 \wedge \bar{E}_1 \wedge \bar{E}_3$ part requires γ to be:

$$\gamma = -\frac{e^{\phi}\left(\sqrt{F_1 F_2} - F_{3r}\right)}{\sqrt{F_1 F_2} - F_{4r}} \cdot \frac{F_4}{F_3} \coth \beta. \tag{3.27}$$

It is easy to see that both (3.26) and (3.27) are satisfied when we choose γ to be the following:

$$\gamma = \pm e^{\phi} \coth \beta, \tag{3.28}$$

which allows us to choose the complex structure as $(\pm i e^{\phi} \coth \beta, i, i)$ for our choice of vielbeins.

Note that this analysis is only valid if the internal manifold is complex. In our case, this requires a constant dilaton, $\phi = \phi_0$, which can be seen by computing the Nijenhuis tensor, and also from the torsion class analysis that we will present in sec. 3.3. Of course, if we relax this condition on the dilaton we can allow for non-complex manifolds, however we will not do just yet, and will instead take γ to be:

$$\gamma = \pm e^{\phi_0} \coth \beta. \tag{3.29}$$

Let us first consider the option with a minus sign in (3.28). This choice leads us to the following constraint on the warp factors F_3 and F_4:

$$\frac{\sqrt{F_1 F_2} - F_{3r}}{\sqrt{F_1 F_2} - F_{4r}} = \frac{F_3}{F_4}. \tag{3.30}$$

A sub-class of solutions satisfying (3.30), for the choice of (3.28), will be the case where the warp factors satisfy:

$$F_3(r) \;=\; F_4(r) \tag{3.31}$$

which corresponds to a *singular* non-Kähler resolved conifold geometry, provided (F_3, F_4) do not vanish at $r = 0$. In general this equality cannot hold for the resolved conifold case as we expect locally $F_3 - F_4 = a^2$ where a^2 is the constant resolution parameter. To allow for a resolved conifold one could in principle demand:

$$F_{3r} \;=\; F_{4r}, \tag{3.32}$$

however such a choice leads to a contradiction, unless we put $F_3 = F_4$. The underlying reason for this is because our case is restrictive, i.e. we have made all the warp factors functions of the radial coordinate r only. If we keep the warp factors functions of the angular coordinates (θ_1, θ_2), we will not have to impose the equality (3.31). However we will not do the most generic case here.

For the moment, let's proceed with the singular conifold background (3.31), and compute the RR gauge field (3.17). Under the constraint (3.31) the form of the gauge field changes from (3.17) to the following:

$$\mathcal{F}_2 = -\sqrt{\frac{F_2}{F_1}} \cdot \left(\sqrt{F_1 F_2} - F_{4r}\right) \cosh\,\beta \left(e_{\theta_1} \wedge e_{\phi_1} + e_{\theta_2} \wedge e_{\phi_2}\right). \tag{3.33}$$

One simple solution for the Taub-NUT to be along the angular directions, i.e. for closed $\mathcal{F}_2$, is that the warp factors F_i satisfy the following differential equation:

$$\frac{dF_4}{dr} = \sqrt{F_1 F_2}\left(1 - \frac{e^{-2\phi_0}}{F_2}\right). \tag{3.34}$$

The above relation, together with (3.31) and (3.32), succinctly summarizes the constraints on the warp factors of the internal manifold (3.6) and the dilaton ϕ to allow for supersymmetric solutions of the form (3.10) with a non-Kähler singular conifold.

The RR gauge field for the singular conifold, using the conditions (3.34), takes the form:

$$\mathcal{A}_1 = \cosh\,\beta \left(\cos\;\theta_1\; d\phi_1 + \cos\;\theta_2\; d\phi_2\right). \tag{3.35}$$

At any given point on the base manifold parametrized by (θ_1, ϕ_1, ψ) and $x^{0,1,2,3}$ the gauge field is:

$$\mathcal{A}_1 = \cosh\beta\ \cos\ \theta_2\ d\phi_2, \tag{3.36}$$

which is the familiar Taub-NUT form as expected. Thus our Taub-NUT space could be thought of as fibered over the two-dimensional sphere (θ_1, ϕ_1).

To study the *resolved conifold*, let's consider a second choice for γ, which is the plus sign solution in (3.28). For this case the constraint equation will change from (3.30) to:

$$\frac{\sqrt{F_1F_2} - F_{3r}}{F_3} + \frac{\sqrt{F_1F_2} - F_{4r}}{F_4} = 0. \tag{3.37}$$

Now of course (3.31) or (3.32) cannot be implemented[1]. However we can still impose a slight variant of (3.34), but now for F_3 as:

$$\frac{dF_3}{dr} = \sqrt{F_1F_2}\left(1 + \frac{e^{-2\phi_0}}{F_2}\right). \tag{3.38}$$

However since $F_3 \neq \pm F_4$, we will assume:

$$F_4 = -g_1F_3 = |g_1|F_3, \tag{3.39}$$

where $g_1(r) = -|g_1|$ is a function of r satisfying the following relation in terms of the warp factors:

$$\frac{dg_1}{dr} = -\frac{\sqrt{F_1F_2}}{F_3}\left[1 + g_1\left(1 + \frac{2e^{-2\phi_0}}{F_2}\right)\right]. \tag{3.40}$$

This is a generalization of the resolved conifold background, where the resolution parameter $(F_3 - F_4)$ is given by the function $(1 - |g_1(r)|)F_3$.

Given the functional forms of (F_1, F_2, ϕ), we can compute F_3 from (3.34). Once F_3 is determined, $g_1(r)$ can be found from (3.40) above, and knowing $g_1(r)$ will give us F_4 from (3.39)[2]. Therefore after the dust settles, the background RR gauge field will change from

[1]The case $F_3 = -F_4$ would lead to an inconsistency in (3.37). This makes sense as the warp factors cannot be negative.

[2]Tighter constraints on the warp factors will be discussed later.

(3.35) to take the following field strength:

$$\mathcal{F}_2 = -\cosh\beta\left[e_{\theta_1}\wedge e_{\phi_1} + g_1(r)e_{\theta_2}\wedge e_{\phi_2}\right] \equiv d\mathcal{A}_1 + \text{sources}. \tag{3.41}$$

As before, at any given point on the two-sphere parametrized by (θ_1, ϕ_1), the gauge field (3.41) will resemble somewhat (3.36) but with the following source equation:

$$d\mathcal{F}_2 = \cosh\beta\frac{\sqrt{F_1F_2}}{F_3}\left[1 + g_1\left(1 + \frac{2e^{-2\phi_0}}{F_2}\right)\right] e_r\wedge e_{\theta_2}\wedge e_{\phi_2}, \tag{3.42}$$

which implies that there are delocalized sources along these directions. For more details on the delocalized sources, one may refer to [29].

3.3 Torsion Classes, Complexity, and Supersymmetry

Having discussed in details a special case of supersymmetry and other constraints in the previous section, let us analyze a more generic case using torsion classes for the background (3.5) and (3.6). We will then specialize our construction to the type IIB background (3.10) and argue for the consistency. To make the picture more succinct, we will divide our analysis for the type IIB background in two parts: before duality and after duality.

3.3.1 Type IIB background before duality

The story before duality transformations begins from the background (3.5) and (3.6). With generic choices of the warp factors F_i and the dilaton e^{ϕ} the manifold (3.6) will be a non-Kähler manifold with an almost-complex structure that may or may not be integrable. In the language of torsion classes $\mathcal{W}_i$ [26] we have two key defining equations:

$$\begin{aligned} dJ &= \frac{3}{4}i\left(\mathcal{W}_1\overline{\Omega} - \overline{\mathcal{W}}_1\Omega\right) + \mathcal{W}_3 + J\wedge\mathcal{W}_4, \\ d\Omega &= \mathcal{W}_1 J\wedge J + J\wedge\mathcal{W}_2 + \Omega\wedge\text{Re}\,\mathcal{W}_5, \end{aligned} \tag{3.43}$$

with the following additional constraints:

$$J\wedge\mathcal{W}_3 = J\wedge J\wedge\mathcal{W}_2 = \Omega\wedge\mathcal{W}_3 = 0. \tag{3.44}$$

Using the above constraints (3.44) and an additional condition $J \wedge \Omega = 0$, that will be consistent with our choice of (J, Ω), it is easy to infer from (3.43) that (equation 2.8 of [3]):

$$\mathcal{W}_1 J \wedge J \wedge J = d\Omega \wedge J = J \wedge d\Omega, \tag{3.45}$$

which will help us to determine $\mathcal{W}_1$ once we know the fundamental form J and the (3, 0) form Ω. Furthermore, from the (2, 2) part of $d\Omega$ we can determine $\mathcal{W}_2$ via [3]:

$$[d\Omega]^{(2,2)} = \mathcal{W}_1 J \wedge J + \mathcal{W}_2 \wedge J. \tag{3.46}$$

Alternatively, all components of the Nijenhuis tensor are completely determined by the torsion classes $\mathcal{W}_1$ and $\mathcal{W}_2$, i.e

$$\mathcal{W}_1 \ \oplus \ \mathcal{W}_2. \tag{3.47}$$

To proceed towards an explicit determination of the torsion components, we will need the complex vielbeins for the background (3.5) and (3.6). Using a slight variant of (3.23), the complex vielbeins now are:

$$\begin{aligned}
\mathcal{E}_1 &= e^{\phi}\left(\sqrt{F_1} e_r + i\sqrt{F_2} e_\psi\right), \\
\mathcal{E}_2 &= e^{\phi + i\psi/2}\sqrt{F_3}\left(e_{\theta_1} + i e_{\phi_1}\right), \\
\mathcal{E}_3 &= e^{\phi + i\psi/2}\sqrt{F_4}\left(e_{\theta_2} + i e_{\phi_2}\right).
\end{aligned} \tag{3.48}$$

Note that these vielbeins differ from the ones discussed in [16, 55], and we will argue that the choice (3.48) give consistent results for the corresponding Calabi-Yau case. Using the convention we have been using, the fundamental form (1, 1) J is defined as:

$$\begin{aligned}
J &= \bar{\mathcal{E}}_1 \wedge \mathcal{E}_1 + \mathcal{E}_2 \wedge \bar{\mathcal{E}}_2 + \mathcal{E}_3 \wedge \bar{\mathcal{E}}_3 \\
&= 2ie^{2\phi}\left(\sqrt{F_1 F_2}\; e_r \wedge e_\psi + F_3 e_{\phi_1} \wedge e_{\theta_1} + F_4 e_{\phi_2} \wedge e_{\theta_2}\right),
\end{aligned} \tag{3.49}$$

[3]Addendum for thesis: where $[A]^{(p,q)}$ denotes the (p, q)-form piece of a differential form A.

such that dJ will become:

$$\begin{aligned} dJ &= 2ie^{2\phi}F_3\left(\frac{\sqrt{F_1F_2}-F_{3r}}{F_3}-2\phi_r\right)e_r\wedge e_{\theta_1}\wedge e_{\phi_1} \\ &+ 2ie^{2\phi}F_4\left(\frac{\sqrt{F_1F_2}-F_{4r}}{F_4}-2\phi_r\right)e_r\wedge e_{\theta_2}\wedge e_{\phi_2}, \end{aligned} \tag{3.50}$$

implying that $dJ = 0$ when the following two conditions are met:

$$\frac{\sqrt{F_1F_2}-F_{3r}}{F_3} = 2\phi_r, \qquad \frac{\sqrt{F_1F_2}-F_{4r}}{F_4} = 2\phi_r. \tag{3.51}$$

One may check that for the Calabi-Yau resolved conifold, where the warp factors F_i take the following values [55]:

$$F_1 = \frac{r^2+6a^2}{r^2+9a^2}, \qquad F_2 = \left(\frac{r^2+9a^2}{r^2+6a^2}\right)\frac{r^2}{9}, \qquad F_3 = \frac{r^2}{6}+a^2, \qquad F_4 = \frac{r^2}{6}, \tag{3.52}$$

with a being the resolution parameter, (3.51) is satisfied with a constant dilaton for both $a = 0$, the singular conifold case, and for $a \neq 0$, the standard resolved conifold case.

Note that due to our choice of the complex structure (i, i, i), dJ has only (2, 1) and (1, 2) pieces. Thus using the same vielbeins we can also to compute the (3, 0) form Ω. For our case this is defined as:

$$\begin{aligned} \Omega &= \mathcal{E}_1\wedge\mathcal{E}_2\wedge\mathcal{E}_3 \\ &= e^{i\psi}\mathcal{A}(r)e_r\wedge(e_{\theta_1}\wedge e_{\theta_2}-e_{\phi_1}\wedge e_{\phi_2}+ie_{\theta_1}\wedge e_{\phi_2}+ie_{\phi_1}\wedge e_{\theta_2}) \\ &+ ie^{i\psi}\mathcal{B}(r)d\psi\wedge(e_{\theta_1}\wedge e_{\theta_2}-e_{\phi_1}\wedge e_{\phi_2}+ie_{\theta_1}\wedge e_{\phi_2}+ie_{\phi_1}\wedge e_{\theta_2}) \\ &+ ie^{i\psi}\mathcal{B}(r)\left(\cot\,\theta_1e_{\phi_1}\wedge e_{\theta_1}\wedge e_{\theta_2}+i\cot\,\theta_1e_{\phi_1}\wedge e_{\theta_1}\wedge e_{\phi_2}\right) \\ &+ ie^{i\psi}\mathcal{B}(r)\left(\cot\,\theta_2e_{\phi_2}\wedge e_{\theta_1}\wedge e_{\theta_2}+i\cot\,\theta_2e_{\phi_2}\wedge e_{\phi_1}\wedge e_{\theta_2}\right), \end{aligned} \tag{3.53}$$

where $\mathcal{A}(r)$ and $\mathcal{B}(r)$ are defined in the following way:

$$\mathcal{A}(r) \equiv \sqrt{F_1F_3F_4}e^{3\phi}, \qquad \mathcal{B}(r) \equiv \sqrt{F_2F_3F_4}e^{3\phi}. \tag{3.54}$$

Using this, one can easily show that $\Omega\wedge dJ = 0$ and hence the first torsion class vanishes. In addition to this, since Ω is a $(3,0)$ form there will be no $(2,2)$ piece to $d\Omega$, and hence

$\mathcal{W}_2$ vanishes as well. Hence we have:

$$\mathcal{W}_1 \;=\; \mathcal{W}_2 \;=\; 0. \tag{3.55}$$

This is as one would expect, since the warp factors are just functions of r [32]. Vanishing of these torsion classes mean that the underlying manifold (3.6) can allow integrable complex structures. Note that this result is independent of $\phi(r)$, and hence the the background (3.5) and (3.6) will be complex for both a constant dilaton and a varying dilaton.

To see how the warp factors in the metric (3.6) are constrained we need to investigate the other three torsion classes. The second equation in (3.43) can now be written as:

$$d\Omega = \Omega \wedge \text{Re}\,\mathcal{W}_5. \tag{3.56}$$

Under a chain of identifications Re $\mathcal{W}_5$ is now related to the dilaton profile in the following way [3, 4, 8, 28, 29][4]:

$$\text{Re}\,\mathcal{W}_5 = \frac{1}{8}\left(\Omega + \overline{\Omega}\right) \lrcorner \left(d\Omega + d\overline{\Omega}\right) = d\text{log}\,|\Omega| = -2d\phi. \tag{3.57}$$

Plugging (3.57) in (3.56) we get:

$$d\left(e^{-2\phi}\Omega\right) \;=\; 0, \tag{3.58}$$

which is a familiar condition for the manifold (3.6) to have a $SU(3)$ structure. It is comforting to see that it appears here naturally.

The only remaining detail is to compute $d\Omega$ explicitly and compare the result with (3.58). This will help us to determine Re $\mathcal{W}_5$. Using (3.53), we can easily compute $d\Omega$. This is given by:

$$\begin{aligned} d\Omega = \quad & ie^{i\psi}\left[\mathcal{A}(r) - \mathcal{B}'(r)\right] d\psi \wedge e_r \wedge \left(e_{\theta_1} \wedge e_{\theta_2} - e_{\phi_1} \wedge e_{\phi_2} + ie_{\theta_1} \wedge e_{\phi_2} + ie_{\phi_1} \wedge e_{\theta_2}\right) \\ & + e^{i\psi}\left[\mathcal{A}(r) - \mathcal{B}'(r)\right] \left(\cot\,\theta_1\; e_{\theta_1} + \cot\,\theta_2\; e_{\theta_2}\right) \wedge e_r \wedge e_{\phi_1} \wedge e_{\phi_2} \end{aligned}$$

[4]Addendum for thesis: The operator $\lrcorner$ is a 'contraction' operator, defined in [3], as:

$$L_k \lrcorner M_n = \frac{1}{n!}\binom{n}{k} L^{a_1 \ldots a_k} M_{a_1 \ldots a_n} e^{a_{k+1}} \wedge \ldots \wedge e^{a_n}$$

where L_k and M_n are arbitrary k and n forms respectively.

$$+ie^{i\psi}\left[\mathcal{A}(r)-\mathcal{B}'(r)\right]\left(\cot\,\theta_1\; e_{\phi_1}+\cot\,\theta_2\; e_{\phi_2}\right)\wedge e_r\wedge e_{\theta_1}\wedge e_{\theta_2}, \tag{3.59}$$

where prime denotes derivative with respect to the radial coordinate r. It is now interesting to note that when the dilaton is a constant, (3.58) $SU(3)$ structure requires $d\Omega$ to vanish. In general, the condition $d\Omega = 0$ requires:

$$\mathcal{A}(r)=\mathcal{B}'(r), \tag{3.60}$$

which in the language of the warp factors F_i and the dilaton ϕ gives the following constraint:

$$\frac{F_{3r}}{F_3}+\frac{F_{4r}}{F_4}+\frac{F_{2r}-2\sqrt{F_1F_2}}{F_2}+6\phi_r \;=\; 0. \tag{3.61}$$

Again it is easy to see that the Calabi-Yau resolved conifold or the singular conifold with warp factors given in (3.52), for $a \neq 0$ and $a = 0$ respectively, satisfy the constraint (3.60). Thus they are Kähler manifolds as expected[5]. Note that (3.6) is definitely not Kähler because the constraints (3.51) are not satisfied.

Let's now consider supersymmetry. The interesting elements of the torsion classes that are responsible for determining supersymmetry are the $\mathcal{W}_4$ and the $\mathcal{W}_5$ torsion classes. The $(\mathcal{W}_4, \mathcal{W}_5)$ torsion classes are:

$$\begin{aligned}\mathcal{W}_4 &= \frac{F_{3r}-\sqrt{F_1F_2}}{4F_3}+\frac{F_{4r}-\sqrt{F_1F_2}}{4F_4}+\phi_r,\\ \text{Re}\;\mathcal{W}_5 &= \frac{F_{3r}}{12F_3}+\frac{F_{4r}}{12F_4}+\frac{F_{2r}-2\sqrt{F_1F_2}}{12F_2}+\frac{\phi_r}{2}.\end{aligned} \tag{3.62}$$

Comparing the $\mathcal{W}_5$ torsion class with the constraint (3.61), and now assuming that the dilaton is constant, $\phi = \phi_0$, then it is no surprise that we have:

$$\text{Re}\;\mathcal{W}_5 = 0, \tag{3.63}$$

and thus the supersymmetry condition for the background (3.5) is simply:

$$2\mathcal{W}_4+\text{Re}\;\mathcal{W}_5 = 2\mathcal{W}_4 = \frac{1}{2}\left(\frac{F_{3r}-\sqrt{F_1F_2}}{F_3}+\frac{F_{4r}-\sqrt{F_1F_2}}{F_4}\right)=0, \tag{3.64}$$

[5] Note that with wrong choice of the vielbeins this will not be obvious.

which is precisely the susy condition that we had in (3.37) for the constant dilaton case! Note that supersymmetry is unbroken as long as (3.64) vanishes up to a total derivative, since such a term can be absorbed as a rescaling of the metric [3], and hence supersymmetry will be unbroken even in the varying dilaton case.

So far we managed to determine all the torsion classes except $\mathcal{W}_3$. The value of $\mathcal{W}_3$ can now be directly read off from the $(2,1)$ piece of dJ, namely:

$$[dJ]^{(2,1)} = [J \wedge \mathcal{W}_4]^{(2,1)} + \mathcal{W}_3 = \mathcal{W}_3, \tag{3.65}$$

where the second equality is due to $\mathcal{W}_4 = 0$, as per equation (3.64). It then follows from (3.50) that $\mathcal{W}_3$ is non-zero, which one would expect since $\mathcal{H}$ in (3.5) is non-vanishing.

Before we move on to the dualized IIB background, lets collect our current results for the torsion classes before any dualities for the special case where the background dilaton has no profile:

$$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_4 = \text{Re } \mathcal{W}_5 = 0, \qquad \mathcal{W}_3 \neq 0, \qquad 2\mathcal{W}_4 + \text{Re } \mathcal{W}_5 = 0. \tag{3.66}$$

Therefore the original background (3.6), with a constant dilaton profile, is a supersymmetric non-Kähler *special-Hermitian manifold*, i.e. a complex manifold with a closed holomorphic (3, 0) form and torsion determined only by the $\mathcal{W}_3$ class.

3.3.2 Type IIB background after duality

So far we have seen how the type IIB background (3.5) and (3.6) with $\mathcal{H}$ torsion can be duality chased to another type IIB background, now with both $\mathcal{H}_3$ and $\mathcal{F}_3$ three-form fluxes. We will now see that the type IIB background after duality can have an integrable complex structure provided the original type IIB background before duality chasing is a special-Hermitian manifold: a manifold with constant dilaton profile and torsion only in $\mathcal{W}_3$ class. In this case, the precise background is:

$$\begin{aligned}
ds^2 &= \frac{1}{\sqrt{H}} ds^2_{0123} + \sqrt{H} ds^2_6 \\
\mathcal{F}_3 &= -\cosh\beta \; e_\psi \wedge (e_{\theta_1} \wedge e_{\phi_1} + g_1 \; e_{\theta_2} \wedge e_{\phi_2}) \\
\mathcal{H}_3 &= -e^{-2\phi} \sinh\beta \sqrt{\frac{F_1}{F_2}} \; e_r \wedge (e_{\theta_1} \wedge e_{\phi_1} + g_1 \; e_{\theta_2} \wedge e_{\phi_2})
\end{aligned} \tag{3.67}$$

where H and Δ are both constants because the dilaton has no profile. For the case where $e^\phi = 1$, the background (3.67) has only delocalized sources with:

$$\Delta = 0, \qquad H = 1. \tag{3.68}$$

One way to study complexity of the dual to above background is to compute the holomorphic (3, 0) form Ω using the vielbeins (3.24). We can define two functions $\mathcal{C}(r)$ and $\mathcal{D}(r)$ similar to the ones defined earlier in (3.54):

$$\mathcal{C}(r) \equiv H^{3/4}\sqrt{F_1 F_3 F_4}, \qquad \mathcal{D}(r) \equiv \gamma H^{3/4}\sqrt{F_2 F_3 F_4}, \tag{3.69}$$

such that the condition $d\Omega = 0$ is precisely (3.60), but with $\mathcal{A}$ and $\mathcal{B}$ are replaced by $\mathcal{C}$ and $\mathcal{D}$ respectively. In terms of the warp factors appearing (3.10), $d\Omega$ turns out to be:

$$d\Omega = \frac{F_{3r}}{F_3} + \frac{F_{4r}}{F_4} + \frac{\gamma F_{2r} - 2\sqrt{F_1 F_2}}{\gamma F_2} + \frac{2\gamma_r}{\gamma} + \frac{3}{2}\cdot\frac{H_r}{H}. \tag{3.70}$$

For the special case of constant dilaton, we have $\mathcal{A} = \mathcal{C}$ and $\mathcal{B} = \mathcal{C}$, and hence $d\Omega = 0$. Similarly, dJ has only (2,1) and (1,2) components, and hence the dualized manifold is complex.

However, comparing (3.70) with (3.62), we note that an equation like (3.58) cannot generically be satisfied, i.e. for $\phi = \phi(r)$, unless $\mathcal{W}_2$ is switched on. Thus the type IIB manifold in (3.10) is in general a non-complex non-Kähler manifold. This should not be a surprise: under duality transformations a complex manifold can become a non-complex one.

The solution (3.67) (3.68) is an interesting example with non-Kähler resolved confold background, and probably the *simplest* non-trivial extension of the well known Calabi-Yau case. However as we show below, this is not the only one. There are numerous choices of non-Kähler metric on a resolved conifold possible if we allow for non-trivial dilaton profile. In fact such examples will have an added advantage: we will be able to argue for localized sources.

With this in mind, let us now assume that the original background (3.6) is a complex non-Kähler manifold but not of the special-Hermitian kind, i.e we allow a dilaton profile in (3.5). The question now is: under the duality transformation that converted (3.5) to

(3.10), is the six-dimensional manifold in (3.10) also a complex non-Kähler manifold?

To be more specific, we will take a concrete example motivated by [55], but with varying dilaton. We will assume without loss of generality a before-duality metric of the form:

$$ds^2 = ds^2_{0123} + \frac{e^{2\phi}}{2}\left[\frac{e_r^2}{F(r)} + r^2 F(r) e_\psi^2 + \frac{1}{2}r^2(e^2_{\theta_2} + e^2_{\phi_2}) + \frac{1}{2}(r^2 + 4a^2 e^{-2\phi})(e^2_{\theta_1} + e^2_{\phi_1})\right], \tag{3.71}$$

where $F(r)$ is a function of r and e^ϕ is the background dilaton. The above metric clearly falls in the class of metrics (3.5) with D5-brane wrapping the resolved two-sphere parametrized by (θ_1, ϕ_1). Note that in the language of (3.6) the constant resolution parameter a^2 is defined as:

$$F_3(r) - F_4(r) = a^2. \tag{3.72}$$

The internal manifold in (3.71) is a complex non-Kähler manifold, and the condition for supersymmetry as before will become[6]:

$$\mathcal{W}_4 = d\phi, \qquad \text{Re}\ \mathcal{W}_5 = -2d\phi, \tag{3.73}$$

with non-trivial dilaton and upto $\mathcal{O}(a^2)$ corrections to RHS of (3.73). Thus the internal manifold is no longer a special-Hermitian manifold, and it is easy to see using (3.62) that the first condition on $\mathcal{W}_4$ is satisfied up to terms proportional to $\mathcal{O}(a^2)$:

$$\mathcal{W}_4 = \left(\frac{r^2 + 2a^2 e^{-2\phi}}{r^2 + 4a^2 e^{-2\phi}}\right)\phi_r = \phi_r - \left(\frac{2\phi_r e^{-2\phi}}{r^2}\right)a^2 + \mathcal{O}(a^4). \tag{3.74}$$

The second condtion in (3.62) is more non-trivial and it leads to the following constraint on dilaton ϕ and the warp factor F:

$$r\frac{d\phi}{dr} + \frac{r}{30}\frac{d}{dr}(\log\ F) + \frac{1}{5}\left(1 - \frac{1}{3F}\right) + \mathcal{O}(a^2) = 0. \tag{3.75}$$

Once we dualize this background to generate the NS and RR three-form fluxes, the back-

[6] Note that our convention differs from [3] as well as [32]. In the latter $(\mathcal{W}_4, \mathcal{W}_5)$ as computed therein are equated to $(d\phi/2, d\phi)$ respectively. See also footnotes 11 and 15 in [32].

ground takes the form (3.10), but now the NS and RR three-form fluxes are simpler:

$$\begin{aligned}\mathcal{H}_3 &= -2a^2 e^{-2\phi}\phi_r \sinh\beta\; e_r\wedge e_{\theta_1}\wedge e_{\phi_1},\\ \mathcal{F}_3 &= -\left(\frac{2a^2r^3F\phi_r}{r^2+4a^2e^{-2\phi}}\right)\cosh\beta\; e_\psi\wedge e_{\theta_2}\wedge e_{\phi_2},\end{aligned} \tag{3.76}$$

where the non-vanishing of $d\mathcal{F}_3$ clearly reflects sources to be along the right directions. Note that the way we supported the D5-brane is via non-Kählerity generated by varying resolution parameter $a^2e^{-2\phi}$. Thus when a vanishes, we have no D5-brane. The metric, of course. still takes the form (3.67), but now both H and Δ are non-trivial functions of the radial coordinate and the resolution parameter. Finally, the type IIA gauge field coming from the T-dual D6-brane along (θ_1, ϕ_1, ψ) is now given by:

$$\mathcal{F}_2 = -2a^2rF\phi_r\cosh\beta\; e_{\theta_2}\wedge e_{\phi_2} + \mathcal{O}(a^4) \equiv \widetilde{g}_1 \cosh\beta\; e_{\theta_2}\wedge e_{\phi_2}, \tag{3.77}$$

from which the corresponding gauge field can be easily determined as before.

It is clear that the dualized manifold, with warp-factor H instead of $e^{2\phi}$, cannot be Kähler, however it remains to be checked if complexity is maintained. To check this, let us assume that the complex vielbeins are of the form (3.24) with an almost complex structure γ. This complex structure cannot be integrable because if it were then (3.25) will have both (2, 1) and (1, 2) components implying breaking of supersymmetry. However the fluxes in (3.76) are explicitly supersymmetric because we have used (3.73) to compute them. The resolution of this is that the manifold after duality does not have an integrable complex structure, at least using the choice of vielbeins that we have taken.

3.4 Branes Lifted to M-theory: Geometry and Harmonic Forms

We would now like to study these solutions from M-theory. However, before we do so, we will consider some simple examples. Let us begin with a very basic scenario of the lift of a D6-brane to M-theory. This is typically given by a Taub-NUT space with the following metric:

$$ds^2 = ds^2_{012.....6} + H_\alpha(dr^2 + r^2 d\Omega^2) + H_\alpha^{-1}(d\psi + \alpha\cdot\cos\theta d\phi)^2, \tag{3.78}$$

where H_α is the standard harmonic function whose value is given by $H_\alpha = 1 + \frac{\alpha}{r}$ with α being a constant. Note that if (3.78) is the localized solution got from T-dualizing a

D5-brane to a D6-brane (the case that we are interested in), then it has the correct warp factor (or, in this language, the correct harmonic function). The constant α is defined as:

$$\alpha = \frac{g_s l_s^2}{2R}, \tag{3.79}$$

where R is the radius of the single-centred Taub-NUT space at $r \to \infty$ i.e at spatial infinity.

The D6-brane world-volume theory is encoded in the M-theory geometry via a normalizable two-form ω [30–33]. In simple examples where the M-theory geometry is a 4d Taub-Nut space in a M-theory fourfold, the task of finding a normalizable two-form is simplified by the fact that the space of 2-forms on a 4d space decomposes into two subspaces, corresponding to self-dual and anti-self dual forms. Hence it suffices to search for such a form, and then test for normalizability. This method has been applied in the past, see for example [33]. We will compute this form explicitly, both in the absence and presence of fluxes.

In the presence of G-fluxes, the background metric (3.78) changes. For a generic choice of G-fluxes, the change in the metric components can be determined by solving the EOMs. This is in general hard as the background EOMs are highly non-trivial (we will discuss this soon). There is, however, a simple *trick* by which one may determine certain aspects of the change in metric, using the type IIB D5-brane. The idea is to take the five-brane solution and *twist* the background solution along the orthogonal direction of the five brane. This twist is effectively performed for the case where the D5-brane solution is delocalized along the orthogonal direction. We then T-dualize the twisted solution along the twist-direction and lift the solution to M-theory. In M-theory we get the required deformed Taub-NUT metric in the presence of certain components of the G-flux.

To define the twist properly we need to analyze the asymptotic behavior of the G-flux. We use $G = dC_3$ to define the twist as:

$$C_{z_1 z_2 \psi}(r \to \infty) - C_{z_1 z_2 \psi}(r \to 0) \equiv \tan\,\alpha, \tag{3.80}$$

where the directions (z_1, z_2) are related to the directions (x^5, x^6) in (3.78) as:

$$\begin{pmatrix} x^5 \\ x^6 \end{pmatrix} = \begin{pmatrix} \sec\,\alpha & R\,\sin\,\alpha \\ 0 & R\,\cos\,\alpha \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \tag{3.81}$$

The G-flux associated with this twist can be expressed in the following way:

$$
\begin{aligned}
G = \quad & \frac{2R'\sin\,\alpha}{(2r+R'\cos\,\alpha)^2}\; dr\wedge dz_1\wedge dz_2\wedge\left(d\psi+\frac{R'}{2}\cos\,\theta\; d\phi\right)\\
& -\frac{rR'\tan\,\alpha\sin\,\theta}{2r+R'\cos\,\alpha}dz_1\wedge dz_2\wedge d\theta\wedge d\phi,
\end{aligned}
\tag{3.82}
$$

with $R' = \frac{g_s l_s^2}{R}$ as the new scale, and we see clearly that the flux vanishes in the limit $\alpha \to 0$. Note that this G-flux when reduced to type IIA will give rise to the necessary B_{NS} field. The M-theory metric that solves EOM with the flux choice (3.82) can be expressed as:

$$
\begin{aligned}
ds^2 = \quad & \left(\frac{2r\cos\,\alpha+R'\cos^2\alpha}{2r\cos\,\alpha+R'}\right)^{1/3} ds^2_{01234}+\left(\frac{2r\cos\,\alpha+R'}{2r\cos\,\alpha+R'\cos^2\alpha}\right)^{2/3} ds^2_{z_1z_2}\\
& +\frac{\left(2r\cos\,\alpha+R'\right)^{2/3}\left(2r+R'\cos\,\alpha\right)^{1/3}}{2r\;\cos^{2/3}\alpha}\left(dr^2+r^2d\Omega_2^2\right)\\
& +\frac{2r\;\cos^{1/3}\alpha}{\left(2r\cos\,\alpha+R'\right)^{1/3}\left(2r+R'\cos\,\alpha\right)^{2/3}}\left(d\psi+\frac{R'}{2}\cos\,\theta\; d\phi\right)^2,
\end{aligned}
\tag{3.83}
$$

where R' is inversely proportional to R, the asymptotic radius of the Taub-NUT space, i.e

$$
R' \equiv \frac{g_s l_s^2}{R}. \tag{3.84}
$$

The M-theory metric that we are dealing with now has the following form:

$$
\begin{aligned}
ds^2 = \quad & G_1(r)ds^2_{01234}+G_2(r)ds^2_{z_1z_2}+G_3(r,\theta)dr^2+G_4(r,\theta)d\theta^2+G_5(r,\theta)d\phi^2\\
& +G_6(r,\theta)\left(d\psi+\frac{1}{2}R'\cos\,\theta\; d\phi\right)^2,
\end{aligned}
\tag{3.85}
$$

where $G_i(r)$ are the warp factors that could, for example, be read from a variant of the metric (3.83). The above metric (3.85) could allow for wrapped D7-branes also.

Following [33] we can construct the normalizable two-form ω by first defining a one-form ζ in the following way:

$$
\zeta \;\equiv g_1(r,\theta)\left(d\psi+\frac{1}{2}R'\cos\,\theta\; d\phi\right)+g_2(r,\theta)d\phi, \tag{3.86}
$$

and we define $\omega \equiv d\zeta$. If we now demand ω is self-dual or anti-self-dual on the Taub-

Nut space, i.e $\omega = \pm *_4 \omega$, and also normalizable, then $g_1(r,\theta)$ and $g_2(r)$ must satisfy the following relations:

$$\frac{\partial g_1}{\partial r}\sqrt{\frac{G_4G_5}{G_3G_6}} = \left(-\frac{1}{2}R' g_1 \sin\theta + \frac{\partial g_2}{\partial \theta}\right)$$
$$\frac{\partial g_1}{\partial \theta}\sqrt{\frac{G_3G_5}{G_4G_6}} = -\frac{\partial g_2}{\partial r}, \tag{3.87}$$

provided ω is self-dual (SD), i.e $\omega = *_4\omega$. For ω anti-self-dual, it is not possible to find a normalizable harmonic two-form [7].

A solution for the set of equations (3.87) can be constructed in the following way. First let us assume that $\frac{\partial g_1}{\partial \theta}$ is non-zero. For this case, we can have:

$$g_2(r,\theta) = g_2(\infty,\theta) + \int_r^\infty dr\ \frac{\partial g_1}{\partial \theta}\sqrt{\frac{G_3G_5}{G_4G_6}}. \tag{3.88}$$

Interestingly if (3.88) is *independent* of θ, then g_1 takes the following form in terms of the warp factors G_i:

$$g_1(r,\theta) = g_0\ \exp\left[-\frac{R'}{2}\int_r^\infty dr\ \sin\theta\sqrt{\frac{G_3G_6}{G_4G_5}}\right], \tag{3.89}$$

which would happen if

$$\frac{\partial g_2(\infty,\theta)}{\partial \theta} = -\int_r^\infty dr \frac{\partial}{\partial \theta}\left(\frac{\partial g_1}{\partial \theta}\sqrt{\frac{G_3G_5}{G_4G_6}}\right). \tag{3.90}$$

The above equation (3.90) looks highly constrained because the LHS is independent of the radial coordinate r whereas the RHS could in principle depend on r for more generic choices of the warp factors $G_i(r,\theta)$. Thus one would have to tread more carefully here. In the following let us take few examples to clarify the scenario.

[7]This statement is dependent on the choice of veilbeins. For a different choice of veilbeins, it may be the anti-self-dual solution which is normalizable.

3.4.1 A warm-up example: Regular D6-brane

The simplest case of a regular D6-brane, i.e. a D6 brane without background flux, is easy. We will take the twist parameter $\alpha = 0$ in both (3.82) and (3.83) giving us vanishing G-flux. For this case we have:

$$\begin{aligned} G_3(r,\theta) &= 1 + \frac{R'}{2r}, \qquad G_6(r,\theta) = \left(1 + \frac{R'}{2r}\right)^{-1}, \\ G_4(r,\theta) &= r^2 G_3(r), \qquad G_5(r,\theta) = r^2 \sin^2\theta \; G_3(r), \end{aligned} \tag{3.91}$$

as the background warp-factors. Note that this converts the metric (3.83) completely to a standard Taub-NUT space, and the differential equations (3.87) to the following:

$$\begin{aligned} r^2 \sin\theta \left(1 + \frac{R'}{2r}\right) \frac{\partial g_1}{\partial r} &= -\frac{1}{2} R' g_1 \sin\theta + \frac{\partial g_2}{\partial \theta}, \\ \frac{\partial g_1}{\partial \theta} \left(1 + \frac{R'}{2r}\right) \sin\theta &= -\frac{\partial g_2}{\partial r}. \end{aligned} \tag{3.92}$$

The above two first-order equations give rise to the following second-order equation for $g_1(r,\theta)$ in terms of the variables appearing in them:

$$r^2 \frac{\partial^2 g_1}{\partial r^2} + 2r \frac{\partial g_1}{\partial r} = -\frac{\partial^2 g_1}{\partial \theta^2} - \frac{\partial g_1}{\partial \theta} \cot\theta. \tag{3.93}$$

It is interesting that the above differential equation is independent of R', but this constant will appear when we fix the boundary conditions. The above differential equation may be solved using separation of variables in r and θ coordinates. This amounts to the assumption that $g_1(r,\theta) = g_a(r) g_b(\theta)$, leading us to the following equations for g_a and g_b:

$$\begin{aligned} r^2 \frac{d^2 g_a}{dr^2} + 2r \frac{d g_a}{dr} &= \lambda g_a, \\ \frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \, \frac{d g_b}{d\theta}\right) &= -\lambda g_b, \end{aligned} \tag{3.94}$$

where λ is the eigenvalue.

Let's first consider the zero-mode, i.e. $\lambda = 0$, and come back to general λ soon. For

$\lambda = 0$ the second equation in (3.94) gives us:

$$g_b(\theta) = c_0 + c_1 \log\left(\tan\frac{\theta}{2}\right), \tag{3.95}$$

where (c_0, c_1) are constants. The above solution blows up at $\theta = 0$ and so to avoid pathologies we can put $c_1 = 0$. Thus g_b is just a constant and we can normalize this to $g_b = 1$. This way $g_1(r,\theta)$ is completely independent of the angular coordinate, and is given by:

$$g_1(r,\theta) \equiv g_a(r) = c_3\left(1 + \frac{R'}{2r}\right), \tag{3.96}$$

with c_3 the normalization constant. This can be seen from the fact that once g_1 is independent of θ, the second equation in (3.92) implies that g_2 is independent of the radial coordinate r. Now plugging the first equation (3.94) in the first equation of (3.92) implies that:

$$\frac{1}{\sin\,\theta}\cdot\frac{\partial}{\partial r}\left(\frac{\partial g_2}{\partial\theta}\right) = \left(1 + \frac{R'}{2r}\right)\lambda g_1(r,\theta), \tag{3.97}$$

and hence $\lambda = 0$ implies g_2 is independent of θ also! This means the g_2 part in (3.86) is a total derivative and is therefore trivial in cohomology. Finally, the normalizable one-form ζ is given by:

$$\zeta = c_3\left(1 + \frac{R'}{2r}\right)\left(d\psi + \frac{1}{2}R'\cos\,\theta\; d\phi\right). \tag{3.98}$$

So far our discussion was related to the zero mode only. What about other values of λ? To study this, note that the second equation in (3.94) is a Legendre equation in θ variable, implying that:

$$g_b(\theta) = P_n(\cos\,\theta), \qquad \lambda = n(n+1). \tag{3.99}$$

Using the above value for λ, we can formulate the first equation in (3.94):

$$\frac{d}{dr}\left(r^2\frac{dg_a}{dr}\right) = n(n+1)g_a, \tag{3.100}$$

as the large x limit of the Legendre equation for $P_n(x)$. This means that the large r limit for $g_a(r)$ will be given by:

$$g_a(r) \approx 2^n \binom{\frac{2n-1}{2}}{n} r^n, \tag{3.101}$$

for $\lambda = n(n+1)$. These functions are clearly non-normalizable, and therefore the corresponding two-form ω will not be a normalizable harmonic form. Thus the *only* choice is for $n = 0$ or $\lambda = 0$, which is (3.98), in accordance with the previous results that a single-centered Taub-NUT space has a unique normalizable harmonic two-form [30–33].

3.4.2 Another warm-up example: D6-brane with background fluxes

Let us now discuss the case of the D6-brane in the presence of fluxes. In M-theory this will be the background studied in (3.83) and (3.82). The warp factors are given by:

$$\begin{aligned}
G_4 &= r^2 G_3, \qquad\qquad G_5 = r^2 \sin^2\theta\ G_3 \\
G_3 &= \frac{(2r\cos\,\alpha + R')^{2/3}\,(2r + R'\cos\,\alpha)^{1/3}}{2r\cos^{2/3}\alpha} \\
G_6 &= \frac{2r\cos^{1/3}\alpha}{(2r\cos\,\alpha + R')^{1/3}\,(2r + R'\cos\,\alpha)^{2/3}},
\end{aligned} \tag{3.102}$$

where α is the required twist parameter. The equation for $g_1(r,\theta)$ follows from the same procedure outlined in (3.87). Thus as before, decomposing $g_1(r,\theta) = g_a(r)g_b(\theta)$, the equation for g_a is given by:

$$\begin{aligned}
\lambda g_a = \quad & r^2 \frac{d^2 g_a}{dr^2} + 2r\,\frac{dg_a}{dr} \\
& - \frac{R'}{4}\frac{dg_a}{dr}\frac{1}{\sqrt{1 + \frac{R'}{2r}(\cos\,\alpha + \sec\,\alpha) + \left(\frac{R'}{2r}\right)^2}}\left[\frac{\cos\,\alpha + \sec\,\alpha + \frac{R'}{r}}{\sqrt{1 + \frac{R'}{2r}(\cos\,\alpha + \sec\,\alpha) + \left(\frac{R'}{2r}\right)^2}} - 2\right],
\end{aligned} \tag{3.103}$$

and the equation for $g_b(\theta)$ is similar to the second equation in (3.94). This means we can again take the $\lambda = 0$ solution, again implying the θ independence of g_1. Additionally, using arguments mentioned earlier, $g_2(r,\theta)$ is just a constant as before. Therefore the solution for $g_a(r)$ or $g_1(r,\theta)$ is then given by:

$$g_1(r,\theta) \equiv g_a(r) = \frac{c_3}{4}\left[\cos\ \alpha + \sec\ \alpha + \frac{R'}{r} + 2\sqrt{1 + \frac{R'}{2r}\left(\cos\ \alpha + \sec\ \alpha\right) + \left(\frac{R'}{2r}\right)^2}\right], \tag{3.104}$$

where c_3 is the same constant that appeared before. Its comforting to see that $\alpha = 0$ limit reproduces the result for the vanilla (i.e. flux less) Taub-NUT. Finally, in the limit of large r, or more concretely, for

$$r \ >> \ \frac{R'}{8}\left(\cos\ \alpha + \sec\ \alpha - 2\right), \tag{3.105}$$

normalizable solution only exists for $\lambda = 0$. Thus, expectedly, there exists a unique normalizable harmonic form for the Taub-NUT background with G-flux.

3.5 M-theory Lift of the Five-brane on a Warped Resolved Conifold

Our next example is a more complicated one: a D5-brane wrapped on a warped resolved conifold, or D6-brane embedded in a related background. We will continue using the M-theory description as the analysis will be easier to perform. The wrapped D5-branes are converted to the D6-branes which are then lifted to M-theory. As we discussed earlier, the D5-brane should be delocalized along the orthogonal T-duality direction.

The conifold and its cousins, the resolved and deformed conifolds, can be Calabi-Yau spaces if one allows Ricci flat metrics on them. Generically, however, they will allow non-Kähler metrics. One may wrap branes on appropriate cycles of the conifolds and get the corresponding world-volume dynamics and effective theory on the non-compact directions of the branes.

As an example of lifting a conifold to M-theory, let us consider the case of D6-branes wrapped on the three-cycle of a deformed conifold. We reach this configuration by taking the SYZ [20] mirror of the wrapped D5-brane on a resolved conifold. How do the dynamics look from M-theory? This is not a new question and has been addressed in recent papers like [25], where the M-theory lift for a special case, with appropriate G-fluxes, is expressed as:

$$
\begin{aligned}
ds^2_{11} = \quad & e^{-\frac{2\phi}{3}}\Bigg\{F_0 ds^2_{0123} + F_1 dr^2 + \frac{\alpha F_2}{\Delta_1\Delta_2}\Big[d\psi - b_{\psi r}dr - b_{\psi\theta_2}d\theta_2 \\
& +\Delta_1 \cos\,\theta_1\Big(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr\Big) + \Delta_2 \cos\,\theta_2 \cos\,\psi_0\Big(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr\Big)\Big]^2 \\
& +\alpha\mathcal{E}_1\Big[d\theta_1^2 + \Big(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr\Big)^2\Big] \\
& +\alpha\mathcal{E}_2\Big[d\theta_2^2 + \Big(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr\Big)^2\Big] \\
& +2\alpha\mathcal{E}_3\,\cos\,\psi_0\Big[d\theta_1 d\theta_2 - \Big(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr\Big)\Big(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr\Big)\Big] \\
& +2\alpha\mathcal{E}_3\,\sin\,\psi_0\Big[\Big(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr\Big)d\theta_2 + \Big(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr\Big)d\theta_1\Big]\Bigg\} \\
& +e^{\frac{4\phi}{3}}\Big[dx_{11} + A_{\phi_1}d\phi_1 + A_{\phi_2}d\phi_2 + A_{\theta_1}d\theta_1 + A_{\theta_2}d\theta_2 + A_r dr\Big]^2, \qquad (3.106)
\end{aligned}
$$

where F_i are the warp factors such that $F_0 = F_0(r, \theta_1, \theta_2)$ and $F_k = F_k(r)$ for $k \neq 0$, and (θ_i, ϕ_i) with ψ are the usual deformed conifold coordinates [1]. Infact the $F_i(r)$ are exactly the same F_i described in (3.6) and the metric components used in (3.106) are

$$
\begin{aligned}
& \mathcal{E}_1 = F_2\cos^2\theta_2 + F_4\sin^2\theta_2, \quad \mathcal{E}_2 = F_2\cos^2\theta_1 + F_3\sin^2\theta_1, \\
& \alpha^{-1} = F_3F_4\sin^2\theta_1\sin^2\theta_2 + F_2F_4\cos^2\theta_1\sin^2\theta_2 + F_2F_3\sin^2\theta_1\cos^2\theta_2, \\
& \mathcal{E}_3 = F_2\cos\,\theta_1\,\cos\,\theta_2, \quad \Delta_1 = \alpha F_2F_4\sin^2\theta_2, \quad \Delta_2 = \alpha F_2F_3\sin^2\theta_1. \qquad (3.107)
\end{aligned}
$$

The $(b_{mn}, \mathcal{E}_i, \phi)$ are respectively the components of the B_{NS} field, the metric and the dilaton in the dual type IIB side and A_M are the type IIA $U(1)$ gauge field components. We have also defined $\psi_0 = \langle\psi\rangle$ and Δ_i to be the warp factors that depend on all the above parameters. For details the readers may refer to [25].

The wrapped D6-branes in type IIA are oriented along (θ_2, ϕ_1) and ψ in the internal space, and spread along spacetime directions $x^{0,1,2,3}$. In M-theory, the deformed Taub-NUT space will take the following form:

$$
ds^2 = e^{-2\phi/3}\left[F_1 dr^2 + A\big|d\theta_1 + \tau d\phi_2\big|^2\right] + e^{4\phi/3}\left(dx_{11} + A_{\phi_2}d\phi_2 + A_{\theta_1}d\theta_1\right)^2, \qquad (3.108)
$$

where we have restricted ourselves to the case with $b_{rm} = 0$; and switched on a complex structure τ defined as:

$$\begin{aligned} \left|\tau\right|^2 &= \alpha A^{-1}\left[\mathcal{E}_2 + F_2\Delta_2\Delta_1^{-1}\cos^2\theta_2\ \cos^2\ \psi_0\right] \\ \text{Re}\ \tau &= \alpha A^{-1}\left[\mathcal{E}_3(\sin\ \psi_0 - b\ \cos\ \psi_0) + b\ F_2\cos\ \theta_1\cos\ \theta_2\ \cos\ \psi_0\right], \end{aligned} \tag{3.109}$$

where $b = b_{\theta_1\phi_1}$ and the coefficient A appearing in (3.108) and (3.109) is defined as:

$$A \equiv \alpha\left[F_2\Delta_1\Delta_2^{-1}b^2\cos^2\theta_1 + \mathcal{E}_1(1+b^2)\right]. \tag{3.110}$$

Hence lift of the D6 brane wrapped on a deformed conifold is not of a simple Taub-NUT form, and additionally there are cross-terms that would make the analysis of the harmonic form highly non-trivial.

Let us now return to the problem at hand: D5 branes wrapped on the resolved conifold. We have already developed the details of the solution, including its M-theory lift, so our next step should be obvious. However due to the existence of the non-trivial functions $g_1(r)$ in the RR gauge field (3.41) and $g_2(r)$ in (3.77), new subtleties appear that we elaborate below.

To start, let us consider the special-Hermitian case studied in (3.67) with background values (3.68). With some minor alterations we can extend our technique to encompass (3.71), as we will discuss later. We also have a non-trivial five-form (3.11) but a trivial dilaton $\phi_B = -\phi_0$. Note that $\mathcal{H}_3$ is closed but $\mathcal{F}_3$ is not, as expected. In fact:

$$d\mathcal{F}_3 = -\cosh\ \beta\ g_1'(r)\ e_r\wedge e_\psi\wedge e_{\theta_2}\wedge e_{\phi_2} + \cosh\ \beta\ e_{\theta_1}\wedge e_{\phi_1}\wedge e_{\theta_2}\wedge e_{\phi_2}, \tag{3.111}$$

where the first term is the *delocalized* source term for the wrapped five-brane along the (θ_1, ϕ_1) sphere and stretched along the Minkowski directions $x^{0,1,2,3}$. The second term is a form that vanishes at every point on the two-sphere (θ_1, ϕ_1). This is more obvious in the T-dual type IIA description where the source equation (3.42) is precisely the first term of (3.111).

We also need to determine the type IIA gauge field from the field strength (3.41). Since $\mathcal{F}_2$ is not closed, as per equation (3.42), and depending on how we distribute charges in (3.41), we can rewrite $\mathcal{F}_2$ in at least two possible ways, i.e as:

$$\mathcal{F}_2 = d\mathcal{A}_1 - \cosh\ \beta\ \cot\ \theta_2\ g_1'(r)\ e_r\wedge e_{\phi_2} \tag{3.112}$$

with components along (r, ϕ_2), or as:

$$\mathcal{F}_2 = d\mathcal{A}_1 + (1 - g_1)\cosh\beta \; e_{\theta_2} \wedge e_{\phi_2} \tag{3.113}$$

with components along (θ_2, ϕ_2). For the distribution (3.112), the one-form gauge-field $\mathcal{A}_1$ is given by:

$$\mathcal{A}_1 = \cosh\beta\left(\cos\ \theta_1 d\phi_1 + g_1(r)\cos\ \theta_2 d\phi_2\right), \tag{3.114}$$

whereas for the distribution (3.113) the one-form gauge-field $\mathcal{A}_1$ is given by:

$$\mathcal{A}_2 = \cosh\beta\left(\cos\ \theta_1 d\phi_1 + \cos\ \theta_2 d\phi_2\right). \tag{3.115}$$

Observe that the function g_1 does not appear in the definition of the gauge-field in (3.115). Finally note that there is a possible variant of (3.114) in which the g_1 function is shifted in the following way:

$$\begin{aligned} \mathcal{A}_3 &= \cosh\beta\left(g_1 \cos\ \theta_1 d\phi_1 + \cos\ \theta_2 d\phi_2\right) \\ &= \mathcal{A}_1 + \cosh\beta(1 - g_1)\left(\cos\ \theta_2\ d\phi_2 - \cos\ \theta_1\ d\phi_1\right), \end{aligned} \tag{3.116}$$

where the angular piece in the second equality, appearing as a difference between two one-forms, will be the representative of the second cohomology under d-action.

The output of our discussion above reveals that at every point (θ_1, ϕ_1) there is a deformed Taub-NUT space parametrized by the warp factors (G_3, G_4, G_5, G_6) in (3.19) with the following ansatze for the one-form:

$$\zeta = g_2(r, \theta_2)\Big[d\Psi + g_1(r)\cos\ \theta_2 d\phi_2\Big], \tag{3.117}$$

where $d\Psi = dx_{11}/\cosh\beta$, and we have used $g_1(r)$ to represent all the choices (3.114), (3.115) and (3.116), with the understanding that for the latter two choices $g_1 \to 1$ in (3.117). Self-duality and anti self-duality of the two-form $\omega \equiv d\zeta$ then imply the following two conditions on the coefficient g_2, using $G_5 = G_4 \sin^2\theta_2$:

$$\frac{1}{g_2}\frac{\partial g_2}{\partial r} = \pm g_1 \frac{\sqrt{G_3 G_6}}{G_4}, \qquad \frac{1}{g_2\ \cot\ \theta_2}\frac{\partial g_2}{\partial \theta_2} = \pm\frac{\partial g_1}{\partial r}\sqrt{\frac{G_6}{G_3}}. \tag{3.118}$$

To solve the set of equations (3.118), we will use the usual separation of variables trick, defining:

$$g_2(r, \theta_2) \equiv \Lambda_1(r)\Lambda_2(\theta_2). \tag{3.119}$$

It is easy to solve for $\Lambda_1(r)$ once we plug-in (3.119) in (3.118). Using the warp factors G_i in (3.19), $\Lambda_1(r)$ is given by:

$$\Lambda_1(r) = \Lambda_0 \exp\left[-\int^r dr\ |g_1(r)|\frac{\sqrt{G_3 G_6}}{G_4}\right], \tag{3.120}$$

with Λ_0 a constant and we have chosen the *anti*-self-duality condition on ω, to allow for normalizability.

So far the analysis has followed more or less the path laid out in the previous section. However, a subtlety appears once we study the Λ_2 equation. This is given by:

$$\frac{1}{\Lambda_2}\frac{d\Lambda_2}{d\theta_2} = -\frac{\partial |g_1|}{\partial r}\sqrt{\frac{G_6}{G_3}}\ \cot\ \theta_2, \tag{3.121}$$

where we see that the separation of variables trick has failed because of the r dependent terms. Clearly this problem disappears when g_1 is a constant for the choices (3.115) and (3.116).

The simplest way out of this conundrum would be to evaluate $\partial^2 g_2/\partial r \partial\theta_2$ for both the equations in (3.118) and compare. This immediately leads us to the following constraint on $g_1(r)$ of the form:

$$\frac{\partial |g_1|}{\partial r} = \sqrt{\frac{G_3}{G_6}} = e^{\phi}\sqrt{HF_1F_2} \tag{3.122}$$

The above constraint is in *addition* to the earlier constraints (3.37), (3.38) and (3.40) imposed by supersymmetry.

We are now ready to determine the one-form ζ, given in (3.117), from the information we have gathered thus far. Using (3.120), (3.121) and the constraint (3.122), the one-form is:

$$\zeta = g_0 \ \sin \ \theta_2 \ \exp\left[-\int^r dr \ |g_1(r)| e^{-5\phi/3} \sqrt{\frac{F_1}{\left(e^{2\phi/3} + \Delta\right) F_2 F_4^2}} \right] (d\Psi + g_1 \ \cos \ \theta_2 \ d\phi_2) \tag{3.123}$$

where g_0 is a normalization constant, and Δ is defined in (3.9). Note that the normalizable form $\omega = d\zeta$ would work for any non-Kähler resolved conifold background whose warp factors appearing in the metric ansatze (3.6) are (F_1, F_2, F_3, F_4) respectively with the constraints (3.37), (3.38), (3.39), (3.40) and (3.122). As mentioned, its equally easy to work out the generic case with the warp factors functions of the angular coordinates (θ_1, θ_2) in addition to r, but we will not do so here. In fact this will be left as an exercise for the reader.

3.5.1 Towards explicit solutions for the background

Let's now consider explicit examples which obey the constraints described in sec. 3.2 (warp factor constraints), sec. 3.3 (torsion constraints), and sec. 3.5 (constraint to solve the one-form ODE via separation of variables). We can organize these constraints in a relatively simple way, to provide a systematic method to generate solutions for warp factors and dilaton.

We start with the special-Hermitian manifolds given as (3.67), with a certain appropriate functional form for $F_2(r)$ that is well defined in the regime $0 \le r < \infty$. With this input form for F_2 we can determine the functional form for $g_1(r)$ satisfying (3.40) and (3.122) using the following differential equation:

$$\frac{dg_1}{dr} = \frac{(1+g_1)\left(\phi_r + \frac{H_r}{2H}\right) + \frac{2g_1 e^{-2\phi}}{F_2}\left(3\phi_r + \frac{H_r}{2H} + \frac{F_{2r}}{F_2}\right)}{\left(2 + \frac{3e^{-2\phi}}{F_2}\right)} \equiv \frac{2g_1}{F_2(3+2F_2)} \frac{dF_2}{dr}. \tag{3.124}$$

In the second equality we have inserted the background choice (3.68) which is appropriate for the special-Hermitian manifolds, where both the dilaton and H are constants.

Once we determine g_1 from above, we can determine the warp factor F_1 using the functional forms for (g_1, F_2) via the following equation:

$$F_1 = \left(\frac{dg_1}{dr}\right)^2 \cdot \left(\frac{e^{-2\phi}}{F_2 H}\right) \equiv \frac{1}{F_2}\left(\frac{dg_1}{dr}\right)^2, \tag{3.125}$$

where the boundary values for the warp factors may be specified by demanding asymptotic regularity along the radial direction. Thus we have a solvable system, requiring only the input functional forms for F_2. Note that there are no additional constraints coming from quantization of $\mathcal{H}_3$ and $\mathcal{F}_3$ fluxes because the underlying manifold (3.6) is non-compact.

We can now solve all the constraint equations with vanishing dilaton i.e $e^{\phi_0} = 1$ and implementing the ODE constraint. A closed form expression can be found for all the warp factors in terms of the input function $F_2(r)$. We will first need $g_1(r) = -|g_1(r)|$. This is given by:

$$|g_1(r)| = \left[\frac{F_2(r)}{3 + 2F_2(r)}\right]^{2/3}. \tag{3.126}$$

Clearly by construction $|g_1| < 1$ implying that the resolution parameter $(1 - |g_1|)F_3$ is always positive definite. Let us assume that $F_2(r)$ is a monotonically increasing function of r such that $\min(F_2) \ll 3/2$. In this case $|g_1|$ is bounded as:

$$\left(\frac{\min(F_2)}{3}\right)^{2/3} \le |g_1| \le \left(\frac{1}{2}\right)^{2/3}. \tag{3.127}$$

Interestingly, if $\min(F_2) \gg 3/2$, we can ignore the 3 in the denominator of (3.126) for all values of F_2, and $|g_1|$ then takes the following approximate value for all r:

$$|g_1| \approx 2^{-2/3}, \tag{3.128}$$

in appropriate units[8]. This means that the resolution parameter is approximately 0.37 times the warp factor F_3.

Once we have the functional form for $g_1(r)$ in terms of the input function $F_2(r)$, we can determine the rest of the warp factors satisfying the constraint equations (3.124), (3.37),

[8]We haven't kept track of the units, but they can be inserted in by the careful and diligent readers.

(3.38) and (3.40). They are now expressed as:

$$
\begin{gathered}
F_1 = \frac{4F_{2r}^2}{F_2^{5/3}(3+2F_2)^{10/3}}, \quad F_3 = 1 - \frac{2+F_2}{F_2^{1/3}(3+2F_2)^{2/3}}, \\
F_4 = \left[1 - \frac{2+F_2}{F_2^{1/3}(3+2F_2)^{2/3}}\right]\frac{F_2^{2/3}}{(3+2F_2)^{2/3}},
\end{gathered}
\tag{3.129}
$$

where again $F_{2r} = dF_2/dr$. Note that we also require the warp factors to be positive definite, which can act as an additional constraint on the warp factors. Since F_1 is positive definite, the requirement that F_3 be positive definite is:

$$
\left(2+\frac{3}{F_2}\right)^{2/3} > \left(1+\frac{2}{F_2}\right). \tag{3.130}
$$

For a monotonically increasing function F_2 with large $\min(F_2)$, this inequality is automatically satisfied. For small values of F_2, i.e. at small r, this inequality becomes hard to satisfy unless:

$$
\min(F_2) \ \geq \ 2. \tag{3.131}
$$

Unfortunately this doesn't quite match-up with the lower bound of $|g_1|$ discussed in (3.127), although it is closer to (3.128). Taking (3.131) into account, (3.127) changes to:

$$
\left(\frac{2}{7}\right)^{2/3} \ \leq \ |g_1| \ \leq \ \left(\frac{1}{2}\right)^{2/3}. \tag{3.132}
$$

Therefore with (3.132) and (3.131) in mind, a generic monotonically increasing functional form for F_2 can be constructed in the following way:

$$
F_2(r) \ \equiv \ p(r) + 2q(r), \tag{3.133}
$$

with $p(r)$ and $q(r)$ as two monotonically increasing functions such that $p(0) = 0$ and $q(0) = 1$.

Therefore with the choice (3.133) for $F_2(r)$, we can determine all the warp factors (3.129). This class of solutions corresponds to a class of supersymmetric resolved conifold solutions (3.6), on non-Kähler special-Hermitian manifolds. After lifting to M-theory, performing a boost, and dimensionally reducing back to IIB, they give a class of IIB solutions describing delocalized five-brane sources on resolved conifolds, which are again complex, non-Kähler, and supersymmetric. On the other hand if we take more generic F_2, not necessarily monotonically increasing, we can still find solutions with postive definite warp factors. Additionally, if we relax the ODE constraint (3.124), e.g. by resorting to gauge-field choices (3.115) and (3.116), more solutions could be found satisfying the constraints (3.37), (3.38) and (3.40).

With the solution (3.129) we are ready to determine the one-form. The one-form ζ can then be worked out from equation (3.123), and is given by,

$$\zeta = g_0 \ \sin \ \theta_2 \ A_\zeta \left(d\Psi - |g_1| \ \cos \ \theta_2 \ d\phi_2\right), \tag{3.134}$$

where g_0 is the normalization constant and the two-form ω is given by $d\zeta$. The functional form for A_ζ is given as:

$$A_\zeta = \exp\left[-\int^r \left(\frac{2F_{2r}}{F_2(3+2F_2)}\right)\cdot \frac{dr}{(3+2F_2)^{2/3}F_2^{1/3} - 2 - F_2}\right]. \tag{3.135}$$

The asymptotic behavior for A_ζ can be easily determined using the monotonically increasing function $F_2(r)$. For large values of F_2, i.e. at large r, A_ζ approaches the following limit:

$$A_\zeta \ = \ \exp\left[\left(\frac{2}{2^{2/3}-1}\right)\cdot\frac{1}{F_2^2}\right] \ \rightarrow \ 1. \tag{3.136}$$

Thus both $\mathcal{A}_\zeta$ in (3.135) and g_1 in (3.126) approach a constant at $r \to \infty$, and hence $\omega \to 0$ asymptotically, as is required for ω to be normalizable. Interestingly, at the origin $r \to 0$ again A_ζ and g_1 approach constant values and therefore ω vanishes. This is different from the blow-up behavior that we saw for the earlier cases and is perfectly consistent with the fact that there are only delocalized sources for this background: thus no singularities from localized branes.

Our next example is a more interesting one because of non-trivial dilaton profile, and is given by the metric (3.71) and three-form fluxes (3.76). In the language of (3.5) the warp factors F_i for (3.71) can be expressed as:

$$F_1 = \frac{1}{2F}, \qquad F_2 = \frac{r^2 F}{2}, \qquad F_3 = \frac{r^2}{4} + a^2 e^{-2\phi}, \qquad F_4 = \frac{r^2}{4}. \tag{3.137}$$

We have developed the analysis in the previous section, so we will be brief here. To start we will need the gauge field from (3.77). This is given by:

$$\mathcal{A}_1 = \cosh\beta\ \widetilde{g}_1(r)\ \cos\ \theta_2\ d\phi_2, \tag{3.138}$$

where $\widetilde{g}_1$ can be read off from (3.77) as:

$$\widetilde{g}_1(r) = -2a^2 r F(r)\ \frac{d\phi}{dr}, \tag{3.139}$$

which vanishes when either the resolution parameter vanishes or the dilaton is constant. In the limit of vanishing resolution parameter but non-vanishing dilaton profile, the internal manifold in (3.71) is still non-Kähler because (3.51) is not satisfied and hence dJ in (3.50) is non-zero. However both $\mathcal{H}_3$ and $\mathcal{F}_3$ (3.76) tend to vanish for this background, and hence the non-Kählerity is supported purely by the dilaton profile. Furthermore, putting branes in this background would break supersymmetry exactly like in Pando Zayas Tseytlin [55].

However, for non-zero resolution parameter, we can support branes because the three-form fluxes in (3.76) do not vanish. In this case, which is the focus of the present work, we allow the following ansatze for the one-form ζ:

$$\zeta = \widetilde{g}_2(r,\theta_2)\left(d\Psi + \widetilde{g}_1\ \cos\theta_2 d\phi_2\right). \tag{3.140}$$

The coefficient $\widetilde{g}_2(r,\theta_2)$ must satisfy constraint equations similar to (3.118), and therefore we allow for separation of variables as in (3.119). We would face a similar subtlety as in (3.121), unless we allow:

$$\frac{\partial \widetilde{g}_1}{\partial r} = -2a^2\sqrt{\frac{G_3}{G_6}} \tag{3.141}$$

which differs from (3.122) by the coefficient a^2. This coefficient is essential because to zeroth order in a^2, $\widetilde{g}_1$ in (3.139) vanishes, whereas (G_3, G_6) do not. This is consistent with the fact that to zeroth order in a^2 we do not expect a Taub-NUT space in M-theory for the supersymmetric case.

On the other hand, the radial part in the decomposition (3.119) must satisfy an equation similar to (3.120). The condition (3.141) leads to the following constraint on the warp factors:

$$\frac{d}{dr}\left(rF\frac{d\phi}{dr}\right) = \frac{1}{2}re^{\phi}\sqrt{e^{2\phi}\cosh\,\beta - \sinh\,\beta}, \tag{3.142}$$

which is in addition to the constraint equation (3.75) on the warp factors from the supersymmetry conditions. Thus after the dust settles, the one-form ζ will now be given by:

$$\zeta = \widetilde{g}_0\ a^2\ \sin\ \theta_2\ \exp\left[-\int^r \frac{4a^2}{r}\cdot\frac{\phi_r}{\left(rF\phi_r\right)_r}dr\right]\left(d\Psi + \widetilde{g}_1\cos\ \theta_2\ d\phi_2\right), \tag{3.143}$$

where $\widetilde{g}_0$ is a constant independent of the resolution parameter a^2, and the subscript r denote derivative with respect to the radial coordinate r.

Note that if we chose the gauge field $\mathcal{A}_1$ such that it satisfies:

$$\mathcal{F}_2 = d\mathcal{A}_1 + (1+\widetilde{g}_1)\cosh\,\beta\ e_{\theta_2}\wedge e_{\phi_2}, \tag{3.144}$$

instead of (3.138), then the only constraint on the warp factor would be (3.75), i.e. (3.142) will not apply, similar to what we saw earlier. In the absence of (3.142) we will require input functions for the dilaton e^{ϕ} or the warp-factor $F(r)$.

Finally, we note that in the case of a vanishing resolution parameter (i.e. a singular conifold), a wrapped D5-brane solution *could* be constructed with ISD fluxes, but the resulting construction may not be supersymmetric. This scenario can be easily rectified by altering slightly the warp factor choices in (3.137) in the following way:

$$F_1 = \frac{e^{-\phi}}{2F}, \qquad F_2 = \frac{e^{-\phi}r^2F}{2}, \qquad F_3 = \frac{e^{-\phi}r^2}{4} + \mathcal{O}(a^2), \qquad F_4 = \frac{e^{-\phi}r^2}{4}. \tag{3.145}$$

The disadvantage of this approach is that, in the zeroth order in a^2, we will have components of three-form fluxes along both (θ_1,ϕ_1) and (θ_2,ϕ_2) directions. The results (3.138) and (3.139) will appear in the next order in a^2. An analysis of this case can be easily performed along the lines of the previous section, but we will not do it here.

3.6 Discussion of Localized Fluxes

We have considered two classes of examples. In the first class, we have a D6-brane with and without without background fluxes. In the second class, we have a D5-brane wrapped on the two-cycle of a non-Kähler resolved conifold in the presence of fluxes, with and without a dilaton profile. In the latter category, i.e the one without a dilaton profile, the sources only appear as delocalized. In the former category, i.e the one with a dilaton profile, many non-trivial localized supersymmetric solutions can be constructed.

In each of the two classes, the brane world-volume theory is encoded in the M-theory geometry via the normalizable two-form ω. However, we have not considered the effect of non-zero DBI gauge fields on the brane, which would arise in M-theory via the term

$$\mathcal{G}_4 = \mathcal{F} \wedge \omega, \tag{3.146}$$

where $\mathcal{F} = d\mathcal{A}$ is the U(1) field strength of the gauge theory localized on the brane. Since ω quickly decays to zero away from the brane, we refer to fluxes of the above form as *localized fluxes*. These have been mentioned in many previous works, see for example [30–33], but no detailed discussion has yet been presented. These fluxes are intimately related to de Sitter solutions in string theory, as they are the key ingredient in D-term uplifting [34], which makes use of the D-term potential later derived by Louis and Jockers in [35], and Haack et al. in [36].

Up to this point all of our examples have implicitly assumed zero localized flux, i.e. $\mathcal{F} = 0$. In particular, we have only studied our solutions at lowest order in α', such that the bulk IIA RR field $\mathcal{F}_2 = d\mathcal{A}_1$ does not induce any non-zero $\mathcal{F} = d\mathcal{A}$ on the brane. We would like to study the effect of including non-zero localized fluxes, as one would expect $\mathcal{F}$ to be non-zero in a generic flux compactification ignoring, for the moment, the backreaction of the localized flux. That is, we will keep the background solution (specified by the metric and non-localized flux) fixed, and consider the effect of including a small localized flux $\mathcal{F} \wedge \omega$. In an upcoming work, we will consider the full solution, including the corrections to the metric and complex structure.

Our strategy to discuss the localized flux in M-theory relied on the existence of a one-form ζ, from which the harmonic two-form $\omega = d\zeta$. For the first class of examples, we see that ζ takes the following form, up to possible scalings:

$$\zeta = g(r)\,(d\Psi + \cos\ \theta_2\ d\phi_2) \tag{3.147}$$

where $g(r)$ takes the form (3.98) for the vanilla (i.e. flux less) D6-brane and (3.104) for the case of D6-brane with fluxes.

For the second class of examples, the one-form ζ is more non-trivial and takes the following form:

$$\zeta = g_2(r,\theta_2)\,[d\Psi + g_1(r)\ \cos\ \theta_2\ d\phi_2] \tag{3.148}$$

where $g_1(r)$ is a non-trivial function of the radial coordinate and is given by (3.126) for the constant dilaton case and by (3.139) for the case with a dilaton profile. The other function $g_2(r,\theta_2)$ takes the generic form (3.123), and is given by (3.134) for the constant dilaton case, and by (3.143) for the case with a dilaton profile.

At this stage, we can study the M-theory picture either in terms of a four-fold $\mathcal{M}_8$ with a $SU(4)$ structure or in terms of a seven-dimensional manifold $\mathcal{M}_7$ with a G_2 structure. In the former case the four-fold will be parametrized by coordinates: (θ_1,ϕ_1), (θ_2,ϕ_2), (r,ψ), (x_{11},x_3), whereas in the latter case the seven-dimensional manifold will be parametrized by coordinates (θ_1,ϕ_1), (θ_2,ϕ_2), (r,ψ,x_{11}). The seven-dimensional manifold with G_2 structure is locally a Taub-NUT space oriented along $(\theta_2,\phi_2,r,x_{11})$ fibered over a three-dimensional base parametrized by (θ_1,ϕ_1,ψ). In the language of the $\mathcal{M}_7$, the G-flux (3.21) can be re-written as:

$$\begin{aligned}\frac{\mathcal{G}_4}{\sinh\,\beta} &= \left(\sqrt{F_1F_2} - F_{3r}\right) e_r\wedge e_{\theta_1}\wedge e_{\phi_1}\wedge e_{11} + \widetilde{e}_\psi\wedge e_{\theta_1}\wedge e_{\phi_1}\wedge \widetilde{e}_{11}\\ &+ \left(\sqrt{F_1F_2} - F_{4r}\right) e_r\wedge e_{\theta_2}\wedge e_{\phi_2}\wedge e_{11} + \widetilde{e}_\psi\wedge e_{\theta_2}\wedge e_{\phi_2}\wedge \widetilde{e}_{11}\end{aligned} \tag{3.149}$$

where the new vielbeins are defined in the following way:

$$\widetilde{e}_\psi = d\psi, \qquad e_{11} = dx_{11} + \mathcal{A}_1, \qquad \widetilde{e}_{11} = dx_{11} + \mathcal{A}_3 \tag{3.150}$$

with $\mathcal{A}_1$ and $\mathcal{A}_3$ are given in (3.114) and (3.116) respectively. Note that the way we constructed the fluxes, they are naturally defined on $\mathcal{M}_7$ instead of $\mathcal{M}_8$. The flux (3.149) does not have a x_3 component, so if we use a four-fold, we cannot define the self-duality naturally, although (3.149) is supersymmetric by construction.

The localized flux for the vanilla D6-brane, on the other hand, can easily be made self-dual when defined on a four-fold, as is required to satisfy the equations of motion for a compactification to (warped) Minkowski space, see for example [11, 12]. This occurs because the gauge field $\mathcal{F}$ is necessarily transverse to the Taub-NUT space, while $\omega = d\zeta$ lives strictly *on* the Taub-NUT space. Hence the dual of $\mathcal{G}_4$ on the fourfold is given by

$$*_8\, \mathcal{G}_4 = *_{\rm NT}\, \mathcal{F} \wedge *_{\rm TN}\, \omega, \tag{3.151}$$

where we have denoted the hodge star on the Taub-NUT directions with a subscript TN, and the remaining four orthogonal directions as NT. The above equation is satisfied for self-dual $\mathcal{F}$, and thus the equations of motion can easily be satisfied without resorting to additional effects such as the generation of a cosmological constant. One may also easily check that this flux will not break supersymmetry, namely that it is primitive with respect to the simplest choice of complex structure on the manifold with metric given by equation (3.91).

The above argument extends easily to the D6-brane in a flux background, although the Taub-NUT space becomes slightly deformed due to the non-zero twist parameter α encoding the presence of fluxes. The M-theory fourfold is split into a Taub-NUT and four orthogonal directions, and the normalizable two-form that spans the Taub-NUT space can be computed by demanding self-duality and normalizability. As before, the introduction of localized flux will not break supersymmetry and will not generate a cosmological constant, provided suitable conditions are placed on $\mathcal{F}$.

The second case that we study here, however, cannot be lifted in M-theory on a four-fold because any duality to M-theory will convert the wrapped D5-brane into an M-theory five-brane and not into geometry. Thus the seven-dimensional manifold with G_2 structure, which in-turn is a warped Taub-NUT space fibered over a three-dimensional base, is probably the best way to analyze this picture. However, if we allow an additional D7-brane in the type IIB side, a four-fold description will become useful. In fact this is also where we can study D-term uplifting [34].

3.7 Conclusion

In this work we have constructed explicit supersymmetric solutions for D5 branes wrapping a resolved conifold. We accomplished this by duality chasing a supersymmetric conifold solution with general warp factors, and only $\mathcal{H}_3$ flux, to a solution with $\mathcal{H}_3$, $\mathcal{F}_3$, and $\mathcal{F}_5$ flux, as well as five-brane sources. In this way, supersymmetry is built in to the final solution, provided we satisfy certain constraints on the warp factors, which we have verified explicitly.

Interestingly, both the 'before duality' and 'after duality' solutions are non-Kähler, but the detailed properties depend intimately on the dilaton: a constant dilaton corresponds to a special-Hermitian manifold which dualizes to a complex manifold with delocalized sources, while a non-constant dilaton $\phi = \phi(r)$ corresponds to a complex manifold which dualizes to a non-complex manifold with localized sources (i.e. D5 branes). The solutions we found have the following form for the metric, equation (3.10),

$$ds^2 = \frac{1}{e^{2\phi/3}\sqrt{e^{2\phi/3}+\Delta}}\, ds^2_{0123} + e^{2\phi/3}\sqrt{e^{2\phi/3}+\Delta}\, ds^2_6, \tag{3.152}$$

$$ds^2_6 = F_1\, dr^2 + F_2(d\psi + \cos\,\theta_1 d\phi_1 + \cos\,\theta_2 d\phi_2)^2 + \sum_{i=1}^{2} F_{2+i}(d\theta_i^2 + \sin^2\theta_i d\phi_i^2).$$

We derived a class of solutions with constant dilaton, with warp factors given by:

$$\begin{aligned}
&F_2 = F_2(r), \qquad \phi = \phi_0,\\
&F_1 = \frac{4F_{2r}^2}{F_2^{5/3}(3+2F_2)^{10/3}}, \qquad F_3 = 1 - \frac{2+F_2}{F_2^{1/3}(3+2F_2)^{2/3}},\\
&F_4 = \left[1 - \frac{2+F_2}{F_2^{1/3}(3+2F_2)^{2/3}}\right]\frac{F_2^{2/3}}{(3+2F_2)^{2/3}},
\end{aligned} \tag{3.153}$$

where $F_{2r} = dF_2/dr$, in terms of a general $F_2(r)$, as well as a class of solutions with varying dilaton, given by

$$F_1 = \frac{1}{2F}, \qquad F_2 = \frac{r^2 F}{2}, \qquad F_3 = \frac{r^2}{4} + a^2 e^{-2\phi}, \qquad F_4 = \frac{r^2}{4}, \qquad \phi = \phi(r), \tag{3.154}$$

for a general function $F(r)$.

We then studied the M-theory lift of these solutions, and related examples. The world-volume theory of the brane is encoded into a normalizable two-form $\omega = d\zeta$ which describes the M-theory geometry around the position of the brane. In simple cases, such as a D6 brane, the M-theory geometry is a simple Taub-Nut space embedded in a fourfold, and the calculation of ω is straightforward. For the solutions described above, the lift to a M-theory is much less trivial; while we were able to compute the normalizable two-form, a meaningful discussion of localized fluxes (which encode the DBI gauge field on the brane) requires studying M-theory on a seven-dimensional manifold with G_2 structure. We leave this discussion, as well as an extension to include D7 branes, to future work.

Despite this apparent set-back, we have achieved quite a bit using our M-theory construction. The extra constraints on the warp factors of non-Kähler resolved conifold (3.6), that appear from analyzing M-theory harmonic forms are useful for predicting certain interesting geometric behaviors of these manifolds. There are a large class of these manifolds admitting supersymmetric fluxes that are useful for studying new aspects of string compactifications. We have simply scratched the surface.

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Chapter 4

Fermions on the Anti-Brane: Higher Order Interactions and Spontaneously Broken Supersymmetry

K. Dasgupta, M. Emelin and E. McDonough, *Fermions on the Anti-Brane: Higher Order Interactions and Spontaneously Broken Supersymmetry*, Phys. Rev. D 95, 026003 , arXiv:1601.03409 [hep-th].

Addendum for Thesis

In the previous chapter we constructed new conifold geometries with wrapped branes. In this chapter we put these geometries to use, and take steps towards constructing our universe in string theory. We do not arrive at any solution resembling our expanding universe, but study the breaking of supersymmetry; a necessary phenomenon in order to realize the inflationary universe.

In particular, we consider the breaking of supersymmetry due to "anti-branes"; the negatively charged companions to branes. This mechanism for supersymmetry breaking has been widely appealed to in the cosmology literature but very little has been done by way of ten dimensional calculations (beyond educated guesses for the four dimensional equations). With this in mind we present different examples, and study in detail the form of and perturbative corrections to the antibrane action.

The spirit of this chapter is carried over in to Chapter 6, which we motivate as emerging from the ten dimensional framework developed in this chapter.

Abstract

It has been recently argued that inserting a probe $\overline{\text{D3}}$-brane in a flux background breaks supersymmetry spontaneously instead of explicitly, as previously thought. In this paper we argue that such spontaneous breaking of supersymmetry persists even when the probe $\overline{\text{D3}}$-brane is kept in a curved background with an internal space that doesn't have to be a Calabi-Yau manifold. To show this we take a specific curved background generated by fractional three-branes and fluxes on a non-Kähler resolved conifold where supersymmetry breaking appears directly from certain world-volume fermions becoming massive. In fact this turns out to be a generic property even if we change the dimensionality of the anti-brane, or allow higher order fermionic interactions on the anti-brane. We argue for the former by taking a probe $\overline{\text{D7}}$-brane in a flux background and demonstrate the spontaneous breaking of supersymmetry using world-volume fermions. We argue for the latter by constructing an all order fermionic action for the $\overline{\text{D3}}$-brane from which the spontaneous nature of supersymmetry breaking can be demonstrated by bringing it to a κ-symmetric form.

4.1 Introduction

It has recently been shown [1, 2] that a probe $\overline{\text{D3}}$-brane in a flux background breaks supersymmetry spontaneously, and furthermore, if the $\overline{\text{D3}}$ is placed on an orientifold plane, the only low-energy field content is a single massless fermion[1]. The implications of this are two-fold: (1) that SUSY breaking is spontaneous, as opposed to explicit, indicates that there is no perturbative instability in the D3-$\overline{\text{D3}}$ system famously used to construct the KKLT de Sitter solution [6], and (2) as the only four-dimensional field content is a single massless fermion, which can be expressed in the $d = 4$ $\mathcal{N} = 1$ supergravity theory as the spinor component of a nilpotent multiplet, this provides a natural starting point for a string theory embedding of the inflation models proposed in [8–10] and other works.

[1]See also [3, 5], and especially the key papers [4], that motivated the research on spontaneous susy breaking in the presence of a $\overline{\text{D3}}$-brane.

This result, and the connection to string cosmology, provides impetus to further investigate $\overline{\mathrm{Dp}}$-brane systems; in order to populate the landscape of stable non-supersymmetric compactifications with $\overline{\mathrm{Dp}}$-branes, to better understand supersymmetry breaking in these models, and to perhaps stumble upon new string theory settings where de Sitter space and inflation naturally arise. It is with these goals in mind that we present three interconnected analyses, which generalize and build upon the work of [1–3].

4.1.1 Spontaneous vs. explicit supersymmetry breaking with anti-branes

Before we proceed with our analysis, let us start with a discussion of spontaneous supersymmetry breaking.

Spontaneous supersymmetry breaking is a crucial element of string theory model building. This is because a consistent study of four dimensional physics requires that all or almost all moduli be stabilized, and all known mechanisms of moduli stabilization[2] are understood in terms of a supersymmetric four dimensional theory, e.g. the complex structure moduli are fixed via the flux induxed superpotential as in [12]. Without an underlying supersymmetric theory, i.e. in the case that supersymmetry is explicitly broken, it is not clear to what extent the known methods of moduli stabilization are applicable.

Spontaneous symmetry breaking occurs when the ground state of a theory does not respect the symmetries of the action. This is an essential part of model building in particle physics, supergravity, and string theory, as it gives theoretical control over corrections to the action. The situation in string theory is slightly more complicated than in particle physics, since proposed de Sitter solutions in string theory (for example KKLT [54]) rarely exist as the ground state of the theory, but rather as metastable minima. Given this, we will drop the phrase 'ground state' from our definition, and instead refer to non-supersymmetric states in a supersymmetric theory as spontaneously breaking the supersymmetry.

In simple cases, for example [2], there is a smoking gun of spontaneous supersymmetry breaking by antibranes: a worldvolume fermion remains massless, which one can identify with the goldstino of SUSY breaking. However, as discussed in [3], it will not in general be true that a worldvolume fermion remains massless. Instead, the goldstino of SUSY breaking will be some combination of open and closed string modes. Thus a more general diagnostic of spontaneous breaking is needed, which we will now develop. We will see that

[2] with the exception of 'string gas' moduli stabilization, see e.g. [74]

even in the absence of a massless fermion on the brane, supersymmetry breaking can still be shown to be spontaneous.

Our diagnostic for spontaneous supersymmetry breaking by a probe $\overline{\mathrm{Dp}}$ brane is the following: a solution breaks supersymmetry spontaneously if it is a solution of the theory with action:

$$S = S_{\mathrm{IIB}} + S_{\overline{\mathrm{Dp}}}, \tag{4.1}$$

where S_{IIB} is action of type IIB supergravity. The above action is explicitly supersymmetric, since an anti-brane is 1/2 BPS, and thus negates the requirement to 'find' the goldstino in order to deduce that supersymmetry breaking is spontaneous. A probe $\overline{\mathrm{D3}}$ in a non-compact GKP background without sources can be studied in this way. This reasoning applies directly to our second example: an $\overline{\mathrm{D7}}$ in a warped bosonic background without sources, which we will study in Section 3.

However, this diagnostic is limited in its applicability, as many interesting backgrounds have explicit brane or orientifold content in addition to the probe $\overline{\mathrm{Dp}}$. Fortunately, the condition (4.1) can in fact be extended to apply to a subset of these cases, by making use of string dualities to relate a flux background with branes to a background without branes. Again, this makes no recourse to the goldstino being a pure open-string mode, i.e. a worldvolume fermion.

Our first example in this paper, a $\overline{\mathrm{D3}}$ in a resolved conifold background with wrapped five-branes, which we study in Section 2, is an example where dualities must be used to make sense of (4.1). One way to arrive at the resolved conifold with wrapped five-branes background is as a solution to $S = S_{\mathrm{IIB}} + S_{\mathrm{D5}}$, in which case the addition of a $\overline{\mathrm{D3}}$ would break supersymmetry explicitly, since the D5 and $\overline{\mathrm{D3}}$ are invariant under different κ-symmetries. However, the resolved conifold background can alternatively be found as the dual to the deformed conifold with fluxes and no branes[3], see for example [14, 19]. In this dual frame the underlying action is source-free, and the addition of an $\overline{\mathrm{D3}}$ (again in the dual deformed conifold) will break SUSY spontaneously. The deformed conifold with $\overline{\mathrm{D3}}$ can then dualized back to a resolved conifold with wrapped D5 along with a $\overline{\mathrm{D3}}$, but the spontaneous (as opposed to explicit) nature of SUSY breaking is only manifest in the dual frame.

As we will see, backreaction of the $\overline{\mathrm{D3}}$ on the resolved conifold induces masses for

[3]The dual is succinctly described in supergravity when the number of wrapped D5-branes is very large [13, 14].

all the fermions, so there is no obvious candidate for the goldstino; this further indicates that the resolved conifold with wrapped D5 and a $\overline{\rm D3}$ system exhibits explicit breaking of supersymmetry. This is consistent with our discussion above: the spontaneous nature of SUSY breaking is only manifest in the dual deformed conifold description. In terms of moduli stabilization, a dual description in terms of spontaneous breaking allows one to consistently define a superpotential for both the Kähler and complex structure moduli, which is precisely the feature of 'spontaneous breaking' that is useful for studying $4d$ physics from string theory.

4.1.2 Outline of the paper

Our first analysis, studied in section 2, considers a probe $\overline{\rm D3}$-brane, not in a Calabi-Yau background [11,12] as studied in [2], but in a non-Kähler resolved conifold background with integer and fractional three-branes. We will construct a supersymmetric deformation to the Calabi-Yau resolved conifold that converts it to a non-Kähler resolved conifold, provides a non-zero curvature to the internal space, and which induces a non-zero amount of ISD fluxes. Once a probe $\overline{\rm D3}$ is introduced, supersymmetry is spontaneously broken by the coupling of ISD fluxes to the worldvolume fermions, giving masses to the world-volume fermions. This breaking is in fact 'soft' as the fluxes and fermion masses are set by the non-Kahlerity of the internal space, which is in turn a tune-able parameter. The picture is somewhat similar to the case with Calabi-Yau internal space as studied in [2] but the analysis differs in terms of fluxes and backreaction. In particular, the analysis in the probe approximation now yields *two* massless fermions, as opposed to one in [2]. This result is modified upon considering backreaction of the $\overline{\rm D3}$ on the bulk fluxes, which generates both (2, 1) and (1, 2) three-form fluxes, inducing masses for *all* the worldvolume fermions, i.e. there are *zero* massless fermions remaining in the spectrum. We also study certain aspects of de Sitter vacua from our analysis. It interesting to note that a curved internal space appears to be a requirement for de Sitter solutions in string theory, at least in many contexts, especially negatively curved internal spaces (see for example [15] and references therein). With this in mind, we consider moduli stabilization in this background, and the connection to de Sitter space in this model.

The physics discussed above remains largely unchanged even if we change the dimensionality of the anti-brane. In section 3, we consider a second application of anti-brane fermionic actions and take a probe[4] $\overline{\text{D7}}$-brane, this time working with a Calabi-Yau background. Supersymmetry is again broken spontaneously via flux-induced fermion masses, and the masses are proportional to the piece of the three-form flux which is ISD in the space transverse to the brane. In the $\overline{\text{D3}}$ case, where the transverse space is the entire internal space, this flux is precisely the flux of the GKP background[5]. However, in the $\overline{\text{D7}}$ case, the fermion masses are now sourced by the subset of these fluxes which are ISD in the two-directions transverse to the brane. In other words, the fermion masses are now determined solely by fluxes that have two legs on the brane, and one leg off. We show that for a special class of flux background there can be many massless fermions in the low energy spectrum, while in a general flux background there may be none. This provides yet another instance of a string theory realization of nilpotent goldstinos[6], and a possible starting point for inflation and de Sitter solutions.

Our final application is actually closer to a derivation; we study the fermionic $\overline{\text{D3}}$ action at *all* orders in the fermionic expansion. To do this, we promote the bosonic fields to superfields, and discuss the physics at the self-dual point. At the self-dual point we can use U-dualities to relate various pieces of the multiplet and consequently determine the fermionic completions of the different fields. Once we move away from the self-dual point, we can determine the fermionic completions of all the bosonic fields in a compact form. As an added bonus, we find that the all-order fermionic action can be written in a manifestly κ-symmetric form, even without precise details of the form of the terms in the action. The orientifolding action can then be easily incorporated in the action. This indicates that the spontaneous nature of supersymmetry breaking by anti-branes, both in the presence and in the absence of an orientifold plane, is not a leading order effect, but in fact continues to be true to all orders. This puts the conclusions of [1, 2], and its implications for KKLT, on solid footing.

We conclude with a short discussion of the implications of our work and directions for future research.

[4]By assuming such a heavy object as probe simply means that the logarithmic backreactions of the $\overline{\text{D7}}$-brane on geometry and fluxes are suppressed by powers of g_s.

[5]Henceforth by GKP background we will always mean the background proposed in [11, 12].

[6]See [16–18] for even more examples.

4.2 $\overline{\text{D3}}$-brane in a Resolved Conifold Background: Soft (and Spontaneous) Breaking of Supersymmetry

The breaking of supersymmetry by a probe $\overline{\text{D3}}$-brane in a warped bosonic background was studied recently in [2]. They studied a $\overline{\text{D3}}$-brane in a GKP background, and found that supersymmetry was spontaneously broken by the coupling of ISD fluxes to the worldvolume fermions. In this section we perform a similar analysis, focusing instead on a probe $\overline{\text{D3}}$-brane in a resolved conifold background. We will consider a deformation to the Calabi-Yau resolved conifold which maintains supersymmetry but provides a non-zero curvature to the internal space, and which induces ISD three-form fluxes from a set of integer and fractional D3-branes. Once a probe $\overline{\text{D3}}$ is introduced, supersymmetry is again spontaneously (and softly) broken by the coupling of ISD fluxes to the worldvolume fermions, and the fermion masses can be straightforwardly computed. As we will see, the 'soft' nature of supersymmetry breaking is due to the tune-able nature of the non-Kählerity of the internal manifold.

The key details of the fermionic action for a $\overline{\text{D3}}$-brane in a warped bosonic background are given in [2]. These will be the starting point of our analysis, so here we merely quote them. The worldvolume action is given, in a convenient κ-symmetry gauge, by

$$\mathcal{L}_f^{\overline{\text{D3}}} = T_3 e^{4A_0}\,\bar{\theta}^1\left[2e^{-\phi}\Gamma^\mu\nabla_\mu - \frac{i}{12}\left(\mathcal{G}^{\text{ISD}}_{mnp} - \bar{\mathcal{G}}^{\text{ISD}}_{mnp}\right)\Gamma^{mnp}\right]\theta^1\,. \tag{4.2}$$

where θ^1 is a 16-component[7] $10d$ Majorana-Weyl spinor[8], and we have defined the three from flux $\mathcal{G}_3$ as $\mathcal{G}_{(3)} = F_{(3)} - \tau H_{(3)}$. The 16-component spinor θ^1 can be decomposed into four $4d$ Dirac spinors λ^0, λ^i with $i = 1,2,3$. On a Calabi-Yau manifold, the λ^0 is a singlet under the $SU(3)$ holonomy group of the internal Calabi-Yau manifold while the λ^i transform as a triplet.

We can now rewrite the $\overline{\text{D3}}$ brane action (4.2) using the $4d$ decomposition of the θ^1 spinor in the following way:

$$\begin{aligned}\mathcal{L}_f^{\overline{\text{D3}}} \;=\; & 2T_3 e^{4A_0-\phi}\Big[\bar{\lambda}^{\bar{0}}_-\gamma^\mu\nabla_\mu\lambda^0_+ + \bar{\lambda}^{\bar{j}}_-\gamma^\mu\nabla_\mu\lambda^i_+\delta_{i\bar{j}} \\ & + \tfrac{1}{2}m_0\bar{\lambda}^0_+\lambda^0_+ + \tfrac{1}{2}\overline{m}_0\bar{\lambda}^{\bar{0}}_-\lambda^{\bar{0}}_- + m_i\bar{\lambda}^0_+\lambda^i_+ + \overline{m}_{\bar{\imath}}\bar{\lambda}^{\bar{0}}_-\lambda^{\bar{\imath}}_- + \tfrac{1}{2}m_{ij}\bar{\lambda}^i_+\lambda^j_+ + \tfrac{1}{2}\overline{m}_{\bar{\imath}\bar{\jmath}}\bar{\lambda}^{\bar{\imath}}_-\lambda^{\bar{\jmath}}_-\Big]\,,\end{aligned} \tag{4.3}$$

[7] 16 complex components, or 32 real components.
[8] We have already fixed κ-symmetry.

where we use $\pm$ subscripts to denote $4d$ Dirac spinors that satisfy $\lambda_\pm = \frac{1}{2}(1 \pm i\widetilde{\Gamma}_{0123})\lambda$, and the masses are defined as

$$m_0 = \frac{\sqrt{2}}{12}\mathrm{i}e^{\phi}\bar{\Omega}^{uvw}\bar{\mathcal{G}}^{\mathrm{ISD}}_{uvw}\,, \qquad \text{from } (0,3) \text{ flux}, \tag{4.4}$$

$$m_i = -\frac{\sqrt{2}}{4}e^{\phi}e_i^u\,\bar{\mathcal{G}}^{\mathrm{ISD}}_{uv\bar{w}}J^{v\bar{w}}\,, \qquad \text{from non-primitive } (1,2) \text{ flux}, \tag{4.5}$$

$$m_{ij} = \frac{\sqrt{2}}{8}\mathrm{i}e^{\phi}\left(e_i^w e_j^t + e_j^w e_i^t\right)\Omega_{uvw}g^{u\bar{u}}g^{v\bar{v}}\bar{\mathcal{G}}^{\mathrm{ISD}}_{t\bar{u}\bar{v}}\,, \qquad \text{from primitive } (2,1) \text{ flux}, \tag{4.6}$$

where J and Ω are the Kähler form and holomorphic 3-form respectively.

We are interested in a more general background, where the $SU(3)$ holonomy will be broken by a perturbation to the geometry. Compactifications on manifolds with $SU(3)$ structure but not $SU(3)$ holonomy have been studied in, for example, [20] and [21]. These are non-Kähler manifolds, which in general may or may not have an integrable complex structure, and are classified by five torsion classes $\mathcal{W}_i$ [26–28]. The simplest case, where all five torsion classes vanish, is a Calabi-Yau manifold that supports no fluxes. We are looking for the case with fluxes, so that we can make use of equations (4.4), (4.6), and (4.5), and therefore some of the torsion classes must be non-zero.

Moreover, the non-Kähler manifold that we need has to be a complex manifold, otherwise the flux decomposition in terms of (2, 1), (1, 2) or (0, 3) forms would not make any sense. In addition, the manifold should to be non-compact, so as to avoid any tension with Gauss' law. The simplest internal manifold that satisfies our requirements is the resolved conifold with a non-Kähler metric which allows an integrable complex structure (and by definition doesn't have a conifold singularity).

The goal of this section will be to study the action (4.2) or (4.3) in a resolved conifold with an arbitrary amount of D3 branes and delocalized five branes (see [23] and [22] for more details on delocalized sources). More precisely, we will put a $\overline{\text{D3}}$-brane in a supersymmetric background with metric given by:

$$ds^2 = \frac{1}{e^{2\phi/3}\sqrt{e^{2\phi/3}+\Delta}}\,ds^2_{0123} + e^{2\phi/3}\sqrt{e^{2\phi/3}+\Delta}\,ds^2_6, \tag{4.7}$$

where e^{ϕ} is related to type IIB dilaton e^{ϕ}_B as $\phi_B = -\phi$ and the factor Δ encodes the backreaction of the 3-branes. It is defined using a parameter β as:

$$\Delta = \sinh^2\beta \left(e^{2\phi/3} - e^{-4\phi/3}\right). \tag{4.8}$$

The other piece appearing in (4.7) is ds_6^2, which is the metric of the internal six-dimensional non-Kähler resolved conifold. This is expressed in terms of the coordinates $(r, \psi, \theta_i, \phi_i)$ in the following way:

$$ds_6^2 = F_1\ dr^2 + F_2(d\psi + \cos\,\theta_1 d\phi_1 + \cos\,\theta_2 d\phi_2)^2 + \sum_{i=1}^{2} F_{2+i}(d\theta_i^2 + \sin^2\theta_i d\phi_i^2), \tag{4.9}$$

where the resolution parameter is proportional to $F_3 - F_4$.

We will start by making an ansatze for the warp-factors $F_i(r)$ appearing in (4.9) which will allow us to see how to go from a Ricci-flat Calabi-Yau metric to a non-Kähler metric on a resolved conifold. A more generic class of solutions for the warp-factors exists and has been discussed in [22], but we will only consider a subset given by:

$$F_1 = \frac{1}{F} + \delta F, \qquad F_2 = \frac{r^2 F}{9}, \qquad F_3 = \frac{r^2}{6} + a_1^2(r), \qquad F_4 = \frac{r^2}{6} + a_2^2(r), \quad \phi = \phi(r), \tag{4.10}$$

where F, $\delta F(r)$, $a_1(r)$, and $a_2(r)$, are functions of the radial coordinate only. From the above ansatze, it is easy to see where the Calabi-Yau case fits in. It is given by:

$$F(r) \equiv F_{CY} = \left(\frac{r^2 + 9a^2}{r^2 + 6a^2}\right), \qquad \delta F(r) = 0, \quad a_1(r) = a, \quad a_2(r) = 0, \quad \phi = 0. \tag{4.11}$$

The Calabi-Yau case is fluxless (with the vanishing of the flux enforced by supersymmetry), and has a constant dilaton. Once we switch on fluxes, we can no longer assume that the other pieces of the warp-factors appearing in (4.10) vanish.

As a cautionary tale, let us first consider whether we can perturb away from Calabi-Yau resolved conifold simply by allowing for a small perturbation in $F(r)$ and $\phi(r)$. We will see that this in fact does not lead to useful results, and thus we will need to be more careful in constructing our geometry. Nonetheless, it is useful for establishing an algorithm for constructing solutions.

Consider a small perturbation to (4.11) of the form:

$$F(r) = F_{CY} + \sigma f(r), \qquad \delta F(r) = 0, \qquad a_1(r) = ae^{-\phi}, \qquad a_2(r) = 0, \tag{4.12}$$

where σ is a dimensionless expansion parameter, that satisfies the the EOMs and takes the solution from the Calabi-Yau resolved conifold to the non-Kähler resolved conifold. We can narrow down our perturbation scheme by allowing the dilaton field to behave in the following way:

$$\phi(r) = \log\left(\frac{1}{r^\sigma}\right), \tag{4.13}$$

which would guarantee the existence of a small parameter σ that, while preserving supersymmetry, would be responsible in taking us away from the Calabi-Yau case. In the limit $\sigma \to 0$, we go back to the fluxless Calabi-Yau case. This geometry is of course singular in the $r \to \infty$ limit, but we will assume for this discussion that the geometry is capped off at some sufficiently large r. In any case, this issue will not be important, as this perturbation fails for other reasons.

A way to construct such a background has already been discussed in [22], and therefore we will simply quote some of the steps. The best and probably the easiest way to analyze such a background is by using the torsion classes. For us the relevant torsion classes are $\mathcal{W}_4$ and $\mathcal{W}_5$. They can be expressed in terms of the warp-factors $F_i(r)$ and the dilaton $\phi(r)$ in the following way:

$$\begin{aligned}
\mathcal{W}_4 &= \frac{F_{3r} - \sqrt{F_1F_2}}{4F_3} + \frac{F_{4r} - \sqrt{F_1F_2}}{4F_4} + \phi_r,\\
\mathrm{Re}\,\mathcal{W}_5 &= \frac{F_{3r}}{12F_3} + \frac{F_{4r}}{12F_4} + \frac{F_{2r} - 2\sqrt{F_1F_2}}{12F_2} + \frac{\phi_r}{2}.
\end{aligned} \tag{4.14}$$

The other torsion classes take specific values, with $\mathcal{W}_3$ determining the torsion. This solution is generated by following the duality chain described in [22], which generates both the RR and the NS three-forms $\mathcal{F}_3$ and $\mathcal{H}_3$ respectively.

Our aim then is to use these torsion classes to determine the functional form for the warp-factors F_i using the specific variation of the ansatze (4.10) i.e (4.12) and (4.13). The key relation, that allows us to find the connection between $F(r)$ and the dilaton $\phi(r)$, is the supersymmetry condition:

$$2\mathcal{W}_4 + \mathrm{Re}\,\mathcal{W}_5 = 0. \tag{4.15}$$

Plugging in the ansatze (4.12) and (4.13) in (4.15) will allow us to determine $f(r)$ completely in terms of the radial coordinate r and the resolution parameter a^2. The functional form for $f(r)$ turns out to be a non-trivial function of r:

$$\begin{aligned} f(r) &= \frac{2}{(6a^2+r^2)}\left\{27a^2(6a^2+r^2)\left[\sum_{i=1}^{3}\Phi_i(r;a^2)+r^2\log r\right]\right. \\ &- \left.(9a^2+r^2)(6a^2+r^2)\left[3\log\left(\frac{r^2}{6a^2}+1\right)+2-\frac{r^2\log r}{6a^2+r^2}\right]\right\}, \end{aligned} \tag{4.16}$$

which is defined for $a^2 > 0$. For vanishing a^2 the functional form for $f(r)$ simplifies and has been studied earlier in [32]. The other variables appearing in (4.16) are defined in the following way:

$$\begin{aligned} \Phi_1(r;a^2) &= {}_2F_1^{(0,0,1,0)}\left(-1,2,3,-\frac{r^2}{6a^2}\right), \\ \Phi_2(r;a^2) &= {}_2F_1^{(0,1,0,0)}\left(-1,2,3,-\frac{r^2}{6a^2}\right), \\ \Phi_3(r;a^2) &= {}_2F_1^{(1,0,0,0)}\left(-1,2,3,-\frac{r^2}{6a^2}\right), \end{aligned} \tag{4.17}$$

where the notation ${}_2F_1^{(0,1,0,0)}$ refers to $\partial_y\,{}_2F_1[x;y;z;w]$, and similarly for ${}_2F_1^{(1,0,0,0)}$ and ${}_2F_1^{(0,0,1,0)}$. This perturbation to $F(r)$ corresponds to introducing a small Ricci scalar on the internal space. This could computed using the torsion classes ([56]), or computed directly using standard GR techniques. Using GR techniques, we find a simple expression emerges for small resolution parameter a^2 and small value for the parameter σ:

$$\delta R_6 = -\frac{72\sigma}{r^2}\left[3-2\log\left(\frac{6a^2}{r^2}\right)\right], \tag{4.18}$$

which is negative for $r \geq 1.2a$. Furthermore one can check that for *general* a, i.e. not small a, while the expression for δR_6 is no longer simple, it is negative definite. It is interesting to note that negatively curved internal spaces have been widely studied as a mechanism for finding de Sitter solutions in string theory, see the discussion and references in [15].

The above analysis, although interesting because of the control we can have on the non-Kählerity of the internal manifold, is ultimately *not* useful for finding the masses of the

$\overline{\text{D3}}$ world-volume fermions, as it in fact renders the internal manifold with a non-integrable complex structure. Thus, there exists an almost complex structure but the manifold itself may not be complex[9]. This means we cannot decompose our $\mathcal{G}_3$ flux in terms of (1, 2), (2, 1) or (0, 3) forms in a global sense, making the fermionic mass decompositions given in (4.6), (4.5) and (4.4), not very practical in analyzing the fermions on the probe $\overline{\text{D3}}$. This of course doesn't mean that we cannot study the spontaneous susy breaking; we can, but the analysis will not be so straightforward as was with the complex decomposition of the three-form fluxes.

The question then is: can we have a *complex* non-Kähler resolved conifold satisfying a more generic ansatze like (4.10) where we can use equations (4.4), (4.6), and (4.5), to study spontaneous susy breaking with a probe $\overline{\text{D3}}$? The answer turns out to be in the affirmative, and in the following section we elaborate the story[10].

4.2.1 A SUSY perturbation of the resolved conifold

Let us start with a simple example of a D3-brane located at a point in an internal manifold specified by the metric ds_6^2 where ds_6^2 is given by:

$$ds_6^2 = dr^2 + g_{mn} dy^m dy^n, \tag{4.19}$$

where (r, y^m) are the coordinates of the internal six-dimensional space. To avoid contradiction with Gauss' law, the internal manifold has to be non-compact, although a compact example could be constructed by either inserting orientifold planes, or anti-branes. Details of this will be discussed later. The backreaction of the D3-brane converts the vacuum manifold:

$$ds_{\text{vac}}^2 = ds_{0123}^2 + ds_6^2, \tag{4.20}$$

[9] There might exist a non-trivial *integrable* complex structure, but we haven't been able to find one.

[10] Note that there is some subtlety with the mapping to [55] at this stage, for example the possibility of a non-Kähler special Hermitian solution with a constant dilaton that we get here demanding supersymmetry as opposed to a Calabi-Yau resolved conifold with a constant dilaton studied in [55]. This has been discussed in details in [22] so we will not dwell on this any further.

with ds^2_{0123} being the Minkowski metric along the space-time directions, to the following:

$$ds^2_{10} = \frac{1}{\sqrt{h}} ds^2_{0123} + \sqrt{h} ds^2_6, \tag{4.21}$$

where h is the warp-factor. The five-form flux in the background (4.21) is now given as:

$$\mathcal{F}_5 = \frac{1}{g_s} (1 + *_{10})\, dh^{-1} \wedge dx^4. \tag{4.22}$$

The above analysis is generic, but it is highly non-trivial to actually compute the warp-factor h. For a complicated internal space, the equation for h typically becomes an involved second-order PDE. Furthermore, in the presence of other type IIB fluxes, for example the three-form fluxes $\mathcal{H}_3$ and $\mathcal{F}_3$, the metric is more complicated than (4.21). Additionally, the string coupling constant generically will not be constant.

There is, however, a way out of the above conundrum if we analyze the picture from a more general setting. We can use the powerful machinery of torsional analysis [27–29] to write the background of a D5-brane wrapped on some two-cycle, parametrized by (θ_1, ϕ_1), of a generic six-dimensional internal space. Assuming that the size of the wrapped cycle is smaller than some chosen scale, any fluctuations along the (θ_1, ϕ_1) will take very high energy to excite. This means at low energies the theory will be of an effective D3-brane[11] and the source charge of the wrapped D5-brane C_6 will decompose as:

$$C_6(\overrightarrow{\mathbf{x}}, \theta_1, \phi_1) = C_4(\overrightarrow{\mathbf{x}}) \wedge \left(\frac{e_{\theta_1} \wedge e_{\phi_1}}{\sqrt{V}} \right), \tag{4.23}$$

where V is the volume of the two-cycle on which we have the wrapped D5-brane. Therefore using the criteria (4.23), the supergravity background for the configuration of the effective D3-brane is given by:

$$\begin{aligned} ds^2 &= e^{-\phi} ds^2_{0123} + e^{\phi} ds^2_6, \\ \mathcal{F}_3 &= e^{2\phi} *_6 d\left(e^{-2\phi} J\right), \end{aligned} \tag{4.24}$$

[11] Also known as a fractional D3-brane. There is yet another way to generate a fractional D3-brane which we don't explore here. For example if we take wrapped D5-$\overline{\text{D5}}$-branes with (n_1, n_2) amount of gauge fluxes on each of them, then we can have bound D3-branes with charges n_1 and n_2 respectively. If n_i are fractional, these give fractional three-branes with vanishing global five-brane charges. See [30, 31] for more details.

where ϕ is the dilaton and the Hodge star and the fundamental form J are wrt to the dilaton deformed metric $e^{2\phi}ds_2^6$. The five-brane charge in (4.24) decomposes as (4.23) once we express it as a seven-form $\mathcal{F}_7 = *_{10}\mathcal{F}_3$. The metric ds_6^2 is in general a noncompact non-Kähler metric that may not even have an integrable complex structure.

If we allow for background three-forms $\mathcal{F}_3$ and $\mathcal{H}_3$, the above background (4.24) changes. One way to see the change would be to work out the precise EOMs. However there exists another way, using a series of duality transformations, to study the background in the presence of the three-form fluxes. The steps have been elaborated in [22, 24, 25]. The solutions we will study here are specific realizations of the general solutions found and analyzed in [22], where supersymmetry of the final 'dualized' solution was explicitly confirmed[12]. The idea is to:

• Compactify the spatial coordinates $x^{1,2,3}$ and T-dualize three times along these directions. The resulting picture will now be in type IIA theory.

• Lift the type IIA configuration to M-theory and make a boost along the eleventh direction using a boost parameter β. This boosting will create the necessary gauge charges.

• Reduce this down to type IIA and T-dualize three times along the spatial coordinates to go to type IIB theory. The IIB background now automatically has the three-form fluxes, as well as a five-form flux.

The result of this duality procedure is that the type IIB background (4.24) now converts to exactly what we expect in (4.21), namely[13]:

$$ds^2 = \frac{1}{\sqrt{h}}ds_{0123}^2 + \sqrt{h}\, ds_6^2 = \frac{1}{e^{2\phi/3}\sqrt{e^{2\phi/3}+\Delta}}\, ds_{0123}^2 + e^{2\phi/3}\sqrt{e^{2\phi/3}+\Delta}\, ds_6^2, \qquad (4.25)$$

confirming the low-energy effective D3-brane behavior, and the following background for

[12] In addition, the fact that the T-duality transformations lead to solutions that solve explicitly the supergravity EOMs has been shown earlier in [44, 45, 47]. In [21] and [22], this was confirmed using torsion classes. The subtlety that such transformations *do not* lead to non-trivial Jacobians follows from the fact that the supergravity fields have no dependence on the T-duality directions. If the supergravity fields start depends on the T-duality directions, there will arise non-trivial Jacobians as discussed in some details in [75]. We thank the referee for raising this question.

[13] There is some subtlety in interpreting the final background with fluxes or with sources. This has been discussed in [25] which the readers may refer to for details.

the three- and the five-form fluxes:

$$\mathcal{F}_3 = \cosh\,\beta\; e^{2\phi} *_6 d\left(e^{-2\phi}J\right), \quad \mathcal{H}_3 = -\sinh\,\beta\; d\left(e^{-2\phi}J\right),$$
$$d\widetilde{\mathcal{F}}_5 = -\sinh\,\beta\,\cosh\,\beta\; e^{2\phi}\; d\left(e^{-2\phi}J\right) \wedge *_6 d\left(e^{-2\phi}J\right), \tag{4.26}$$

with the type IIB dilaton $e^{\phi_B} = e^{-\phi}$. One may verify that (4.25) and (4.26) together solve the type IIB EOMs.

We will concentrate on a specific background given by a (generically non-Kähler) singular, resolved or deformed conifold. The typical internal metric ds_6^2 in this class is given by a variant of (4.9) as:

$$\begin{aligned} ds_6^2 \;&=\; F_1\; dr^2 + F_2\left(d\psi + \cos\,\theta_1\; d\phi_1 + \cos\,\theta_2\; d\phi_2\right)^2 + \sum_{i=1}^{2} F_{2+i}\left(d\theta_i^2 + \sin^2\theta_i\; d\phi^2\right) \\ &+\; F_5\;\sin\,\psi\left(d\phi_1\; d\theta_2\;\sin\,\theta_1 + d\phi_2\; d\theta_1\;\sin\,\theta_2\right) + F_6\;\cos\,\psi\left(d\theta_1\; d\theta_2 - d\phi_1\; d\phi_2\;\sin\,\theta_1\;\sin\,\theta_2\right), \end{aligned} \tag{4.27}$$

where $F_i(r)$ are warp factors that are functions of the radial coordinate r only[14] and in the following, unless mentioned otherwise, we will only consider the resolved conifold, i.e we take $F_5 = F_6 = 0$ henceforth. The above background (4.27) can be easily converted to a background with both $\mathcal{H}_3$ and $\mathcal{F}_3$ fluxes by the series of duality specified above. Using (4.25), our background becomes:

$$ds^2 = \frac{1}{e^{2\phi/3}\sqrt{e^{2\phi/3}+\Delta}}\; ds^2_{0123} + e^{2\phi/3}\sqrt{e^{2\phi/3}+\Delta}\; ds_6^2, \tag{4.28}$$
$$\mathcal{F}_3 = -e^{2\phi}\cosh\,\beta\sqrt{\frac{F_2}{F_1}}\left(g_1\; e_\psi \wedge e_{\theta_1} \wedge e_{\phi_1} + g_2\; e_\psi \wedge e_{\theta_2} \wedge e_{\phi_2}\right),$$
$$\widetilde{\mathcal{F}}_5 = -\sinh\,\beta\,\cosh\,\beta\left(1 + *_{10}\right)\mathcal{C}_5(r)\; d\psi \wedge \prod_{i=1}^{2}\,\sin\,\theta_i\; d\theta_i \wedge d\phi_i,$$
$$\mathcal{H}_3 = \sinh\,\beta\Big[\left(\sqrt{F_1F_2} - F_{3r}\right) e_r \wedge e_{\theta_1} \wedge e_{\phi_1} + \left(\sqrt{F_1F_2} - F_{4r}\right) e_r \wedge e_{\theta_2} \wedge e_{\phi_2}\Big],$$

with a dilaton $e^{\phi_B} = e^{-\phi}$ and with Δ defined as in (4.8),

$$\Delta = \sinh^2\beta\left(e^{2\phi/3} - e^{-4\phi/3}\right), \tag{4.29}$$

[14]One may generalize this to make the warp factors F_i functions of all coordinates except (θ_1, ϕ_1), i.e the directions of the wrapped brane. We will not discuss the generalization here.

and β is the boost parameter discussed above while the others, namely $(g_1, g_2, \mathcal{C}_5)$ are given by:

$$g_1(r) = F_3\left(\frac{\sqrt{F_1F_2} - F_{4r}}{F_4}\right), \quad g_2(r) = F_4\left(\frac{\sqrt{F_1F_2} - F_{3r}}{F_3}\right), \tag{4.30}$$
$$\mathcal{C}_5(r) = \int^r \frac{e^{2\phi}F_3F_4\sqrt{F_1F_2}}{F_1}\left[\left(\frac{\sqrt{F_1F_2} - F_{3r}}{F_3}\right)^2 + \left(\frac{\sqrt{F_1F_2} - F_{4r}}{F_4}\right)^2\right] dr.$$

The above background for the D3-brane is consistent as long as the energy is less than the inverse size of the sphere parametrized by (θ_1, ϕ_1). For vanishing size of the sphere, which would happen for a singular conifold, our analysis continues to hold to arbitrary energies.

Equation (4.28) contains all the information that we need, so now the relevant question is to find appropriate warp-factors that allow us to have a non-Kähler resolved conifold with an integrable complex structure. A simple analysis of the fluxes along the lines of [22] will tell us that an integrable complex structure is possible when the dilaton has no profile in the internal direction. This means we can take, without any loss of generality, a vanishing dilaton inducing the following complex structure on the internal space:

$$\tau_k \equiv (i \coth \beta, i, i). \tag{4.31}$$

The metric on the internal space now is not too hard to find if one takes care of all the subtleties pointed out in [22]. The subtleties are generically related to flux quantization and integrability conditions. Once the dust settles the metric becomes:

$$\begin{aligned} ds^2 &= 4F_{2r}^2\left(\frac{1-G}{2+F_2}\right)dr^2 + F_2(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 \\ &+ G(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + G(1-G)\left(\frac{F_2}{2+F_2}\right)(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2), \end{aligned} \tag{4.32}$$

where $F_2(r)$ is taken to be dimensionless. This means all terms of the metric are dimensionless, and thus if r has a dimension of length, the warp-factor should have inverse length dimension. This works out fine because the coefficient of dr^2 is indeed the derivative of F_2. We could also rewrite the metric with dimensionful warp-factors but this would not change any of the physics. Note also that $G(r)$ appearing in (4.32) is not an independent function, but depends on F_2 in the following way:

$$(1-G)^3 = \frac{(2+F_2)^3}{F_2(3+2F_2)^2}, \tag{4.33}$$

and therefore an appropriate choice of F_2 will fix the functional form for G. Furthermore, the resolution parameter for the resolved conifold is no longer a constant, but a function of the radial coordinate r that takes the following form:

$$a^2(r) \equiv \frac{(2+GF_2)G}{2+F_2}, \tag{4.34}$$

which is by construction a positive definite function provided G remains positive definite everywhere. It is definitely a well-behaved function at any point in r since $F_2 > 0$ and if F_2 is chosen to be a well-behaved function of r. Positivity of G implies that at any point in r, F_2 should satisfy:

$$F_2^3 + 2F_2^2 - F_2 > \frac{8}{3}, \tag{4.35}$$

which is not hard to satisfy. This also imples $G < 1$ at any point in r. A simple choice of $F_2(r)$ would be to consider the following functional form that should make all the warp-factors positive definite:

$$F_2(r) = 1.1022 + \widetilde{F}_2^2(r), \tag{4.36}$$

assuming $\widetilde{F}_2(r)$ never hits zero at any point in r. We can also bring our metric (4.32) to the form (4.10) by appropriately defining $\delta F, a_1(r)$ and $a_2(r)$.

It is now time to determine the fluxes that preserve the background supersymmetry. As is well known, the fluxes should be ISD and primitive, so the appropriate choice is to take (2, 1) forms. This can be easily worked out from (4.28), and once we fix the complex structure to be (4.31), and with the above warp factors and dilaton, the three-form flux takes particularly simple form [15]:

$$\begin{aligned}
\mathcal{G}_3 &= \frac{\sinh\beta}{4\sqrt{H}\sqrt{F_1\sqrt{H}}}\left[\left(\frac{\sqrt{F_1F_2}-F_{3r}}{F_3}\right)-\left(\frac{\sqrt{F_1F_2}-F_{4r}}{F_4}\right)\right]\left(E_1\wedge E_3\wedge\overline{E}_3-E_1\wedge E_2\wedge\overline{E}_2\right),\\
&= \frac{\sqrt{F}\,(2-F\delta F)}{8\ \mathrm{cosech}\ \beta}\left[\left(\frac{rF\delta F-12a_1a_{1r}}{r^2+6a_1^2}\right)-\left(\frac{rF\delta F-12a_2a_{2r}}{r^2+6a_2^2}\right)\right]\\
&\quad\cdot\left(E_1\wedge E_3\wedge\overline{E}_3-E_1\wedge E_2\wedge\overline{E}_2\right),
\end{aligned}\tag{4.37}$$

which is ISD, primitive, and a $(2,1)$ form. In the second line we have used the ansatze (4.10) with vanishing dilaton. Note also that the three functions δF, a_1, and a_2 are constrained by supersymmetry, via (4.15), which is a first order ODE. The SUSY condition also forces the (1, 2) components of $\mathcal{G}_3$ to vanish identically.

One can see that the boost parameter β, which counts the units of $\mathcal{F}_3$ flux, or equivalently the number of delocalized ([23]) five-branes, in the resolved conifold background, controls the amount of ISD flux. Naively, if we take $\beta\to 0$, the flux vanishes. However the complex structure (4.31) also blows up in this limit, so vanishing β case has to be studied differently. This is indeed the case because, in the language of [22], taking $\beta\to 0$ takes us to the "before duality" picture where only RR three-form fluxes are present. Therefore the way we derived our background, we can take β arbitrarily small but not zero.

This completes our analysis of the supersymmetric fluxes on a non-Kähler resolved conifold bacground that allows an integrable complex structure. In the following section we will insert a $\overline{\mathrm{D3}}$-brane in this background and study the fluxes and the corresponding supersymmetry breaking scenario using the world-volume action. We start with the bosonic action for a $\overline{\mathrm{D3}}$-brane in this background.

---

[15] where the E_i are defined as:

$$E_1 = e_1 + i\coth\beta\, e_2, \qquad E_2 = e_3 + ie_4, \qquad E_3 = e_5 + ie_6,$$

with

$$\begin{aligned}
&e_1=\sqrt{F_1\sqrt{H}}e_r, \qquad e_2=\sqrt{F_2\sqrt{H}}(d\psi+\cos\ \theta_1 d\phi_1+\cos\ \theta_2 d\phi_2)=\sqrt{F_2\sqrt{H}}e_\psi,\\
&e_3=\sqrt{F_3\sqrt{H}}\left(-\sin\ \frac{\psi}{2}\ e_{\phi_1}+\cos\ \frac{\psi}{2}\ e_{\theta_1}\right),\quad e_4=\sqrt{F_3\sqrt{H}}\left(\cos\ \frac{\psi}{2}\ e_{\phi_1}+\sin\ \frac{\psi}{2}\ e_{\theta_1}\right),\\
&e_5=\sqrt{F_4\sqrt{H}}\left(-\sin\ \frac{\psi}{2}\ e_{\phi_2}+\cos\ \frac{\psi}{2}\ e_{\theta_2}\right),\quad e_6=\sqrt{F_4\sqrt{H}}\left(\cos\ \frac{\psi}{2}\ e_{\phi_2}+\sin\ \frac{\psi}{2}\ e_{\theta_2}\right),
\end{aligned}$$

4.2.2 Bosonic action for a $\overline{\text{D3}}$-brane

Before considering a $\overline{\text{D3}}$, let us consider a D3. In the previous section we saw how to incorporate the backreaction of a single (or generically N) *effective* D3-branes in flux background. We can compute the bosonic action of the D3-brane in this background, not as a probe, but as an actual backreacted object. This is *different* from what has been done earlier in [3, 36–40, 44] where the D3-brane has been considered as a probe in a GKP background [11, 12] of the form:

$$
\begin{aligned}
&ds^2 = e^{2A} g_{\mu\nu} dx^\mu dx^\nu + e^{-2A} g_{mn} dy^m dy^n, \\
&\mathcal{G}_3 = \mathcal{F}_3 + \tau\, \mathcal{H}_3, \quad \mathcal{F}_5 = (1 + *_{10}) d\alpha \wedge \mathrm{dvol}_{R^{3,1}},
\end{aligned}
\tag{4.38}
$$

where $\tau = C_0 + ie^{-\phi_B}$ and $\alpha = e^{4A}$. For our case, with the backreaction of the D3-branes taken into account, we can define the following quantities:

$$
\begin{aligned}
e^{2A} &= \sqrt{\alpha} = \frac{1}{e^{2\phi/3}\sqrt{e^{2\phi/3} + \Delta}}, \quad g_{\mu\nu} = \eta_{\mu\nu}, \\
\Phi_+ &= \frac{2}{e^{2\phi}\,\cosh^2\beta - \sinh^2\beta}, \qquad \Phi_- = 0.
\end{aligned}
\tag{4.39}
$$

The above equation implies that the scalar fields on a D3-brane are completely massless (as the masses of the scalar fields are determined by Φ_- [3]). Other details regarding the action can be worked out from [3, 36–40, 44].

Let us now consider a $\overline{\text{D3}}$ in this background. We will take this as a probe so that the backreaction of the anti-brane will not be felt strongly in (4.28). Details of this will be discussed in the next section. For the time being we shall assume that a small profile for the dilaton is now switched on, along with small changes in the three-form fluxes. Furthermore, the tachyonic instability of the anti-brane will not be visible in the probe limit. The world-volume multiplet on the anti-brane will have the usual vector field A_μ and six scalars φ^m associated with the six internal directions of the resolved conifold (4.27). The bosonic action in the Einstein frame is then given by:

$$
S_{\overline{\text{D3}}} = -\tau_{D3} l_s^4 \int d^4x \left(\frac{\pi}{2g_s^2} f_{\mu\nu} f^{\mu\nu} + \frac{\pi}{g_s} g_{mn} \mathcal{D}_\mu \varphi^m \mathcal{D}^\mu \varphi^n + \frac{\pi}{g_s} \partial_m \partial_n \Phi_+ \varphi^m \varphi^n + \mathcal{L}_{int} \right), \tag{4.40}
$$

where the interaction lagrangian $\mathcal{L}_{int}$ is given by the following expression:

$$\mathcal{L}_{int} = \frac{2\pi}{l_s^2 g_s}\partial_m \Phi_+ \varphi^m + \frac{i\pi}{12}\Phi_+ \left(\text{Re } G_+\right)_{mnp} \varphi^m \varphi^n \varphi^p + \frac{\pi}{l_s^4 g_s}\Phi_+, \tag{4.41}$$

with g_{mn} to be the metric of the internal non-Kähler resolved conifold (4.27) and $G_+ = (*_6 + i)G_3$ where $*_6$ is the Hodge star with respect to the warped metric (4.38).

For a conifold background, there are five compact scalars, namely: $(\varphi^{\theta_1}, \varphi^{\phi_1}, \varphi^{\theta_2}, \varphi^{\phi_2}, \varphi^{\psi})$, and one non-compact scalar φ^r. The compact scalars are all massless, and the mass of the non-compact scalar is given by:

$$\begin{aligned} m^2_{\varphi^r} &= \frac{\pi}{g_s}\left(\frac{\partial^2 \Phi_+}{\partial r^2}\right) \\ &= \frac{8\pi e^{2\phi}\cosh^2\beta}{g_s\left(e^{2\phi}\cosh^2\beta - \sinh^2\beta\right)^2}\left[\left(\frac{e^{2\phi}\cosh^2\beta + \sinh^2\beta}{e^{2\phi}\cosh^2\beta - \sinh^2\beta}\right)\left(\frac{\partial\phi}{\partial r}\right)^2 - \frac{1}{2}\frac{\partial^2\phi}{\partial r^2}\right], \end{aligned} \tag{4.42}$$

where due to the presence of the linear interaction in (4.40), the non-compact scalar is shifted from its original value φ^r to the following:

$$\widetilde{\varphi}^r \equiv \varphi^r + \frac{1}{l_s^2}\left[\frac{\partial}{\partial r}\log\left(\frac{\partial \Phi_+}{\partial r}\right)\right]^{-1}. \tag{4.43}$$

In a generic setting, where the warp-factors and the dilaton ϕ are functions of all the internal coordinates, all the six-scalars would be massive and the anti-brane will be fixed at a point in the internal space where the mass matrix is extremised.

However, the background we have constructed has a *constant* dilaton, and thus Φ_+ is constant and φ^r is massless. If one allows for a small dilaton profile, for example by perturbing beyond the probe limit, a mass is generated for φ^r. In the limit where β is small, this happens at the point where the dilaton satisfies the following differential equation:

$$\frac{\partial^3\phi}{\partial r^3} - 6\left[\frac{\partial^2\phi}{\partial r^2} - \frac{2}{3}\left(\frac{\partial\phi}{\partial r}\right)^2\right]\frac{\partial\phi}{\partial r} + \mathcal{O}(\beta) = 0. \tag{4.44}$$

For the solution discussed above, and allowing for some $\overline{\text{D3}}$ backreaction in the form of a small profile for the dilaton, Φ_+ takes the following simple form (for arbitrary values of β):

$$\Phi_+ = 2 - 4\phi(r)\,\cosh^2\!\beta + \mathcal{O}(\phi^2). \tag{4.45}$$

This form of Φ_+ will fix φ^r to be 0. The remaining scalars can be stabilized along the lines of [58]; the angular moduli recieve masses upon 'glueing' the non-compact throat geometry on to a compact Calabi-Yau. Alternatively, one can place the $\overline{\mathrm{D3}}$ directly on an orientifold plane, as in [16], which fixes all the scalars and gauge fields[16].

4.2.3 SUSY breaking and the fermionic action for a $\overline{\mathrm{D3}}$

Now let us return to the fermionic action, which we gave in equation (4.3). The masses of the fermions are dictated by ISD three-form flux $\mathcal{G}_3$ given in (4.37), which is valid strictly in the probe approximation. The backreaction of the $\overline{\mathrm{D3}}$ induces corrections to the flux, which we will come back to shortly.

Staying within the probe approximation, the flux is given by equation (4.37),

$$\mathcal{G}_3 = \frac{\sqrt{F}\,(2 - F\delta F)}{8\ \mathrm{cosech}\ \beta}\left[\left(\frac{rF\delta F - 12a_1a_{1r}}{r^2+6a_1^2}\right) - \left(\frac{rF\delta F - 12a_2a_{2r}}{r^2+6a_2^2}\right)\right]\left(E_1\wedge E_3\wedge\overline{E}_3 - E_1\wedge E_2\wedge\overline{E}_2\right).$$

Clearly the masses m_0 and m_i will be zero (since $\mathcal{G}_3$ is ISD and primitive). The breaking of supersymmetry is done purely through the mass matrix m_{ij}, defined in equation (4.6). Evaluating these masses explicitly, we find

$$m_{23} = m_{32} = \frac{\sqrt{2}}{8}i|\mathcal{G}_3|\ \ ,\ \ m_{12} = m_{21} = m_{13} = m_{31} = 0, \tag{4.46}$$

where $|\mathcal{G}_3|$ is

$$|\mathcal{G}_3| = \frac{\sinh\beta}{8}\sqrt{F}\,(2 - F\delta F)\left[\left(\frac{rF\delta F - 12a_1a_{1r}}{r^2+6a_1^2}\right) - \left(\frac{rF\delta F - 12a_2a_{2r}}{r^2+6a_2^2}\right)\right]. \tag{4.47}$$

From this we see that the λ^2 and λ^3 fermions will have a mass induced by $\mathcal{G}_3$, which spontaneously breaks the $\mathcal{N}=1$ supersymmetry of the resolved conifold. This leaves *two* massless fermions, λ^0 and λ^1, as the low energy field content. This is in contrast to an

[16]For more details on orientifolding conifolds see [68, 69], and for the consistency of placing anti-branes on orientifolds of conifolds see [16].

$\overline{\text{D3}}$ in a GKP background, studied in [2], where there was only a single massless fermion. Interestingly, the scale of SUSY breaking is controlled by $\delta F(r)$, a_{1r}, and a_{2r}, and thus we can easily allow for soft breaking of supersymmetry.

4.2.4 Perturbing away from the probe limit

Let us now consider perturbing away from the probe limit, which corresponds to taking the $\overline{\text{D3}}$ to be a large yet finite distance away from the D5-brane (fractional D3). We will neglect subtleties regarding boundary conditions, which can lead to divergences in the fluxes when a stack of $\overline{\text{D3}}$'s is considered, see e.g. [72] and more recently [73], and also continue to study only a single $\overline{\text{D3}}$. As we will see, even with this issue neglected, backreaction changes the story considerably. In the presence of a probe $\overline{\text{D3}}$, the background changes from what we have thus far studied. The question then is to compute the changes in the background metric and fluxes to account for the fermionic masses on the anti-brane world-volume. We will *not* attempt to find an exact backreacted solution with an $\overline{\text{D3}}$, but rather take on a simpler task; we can compute the leading corrections to the fluxes and thus fermion masses by perturbing away from the probe limit.

The situation is not as hard as it sounds. Due to the (perturbatively) probe nature of the $\overline{\text{D3}}$, and as we hinted before, the tachyonic degree of freedom will not be visible at the supergravity level. Furthermore the backreaction of the $\overline{\text{D3}}$-brane will appear from its energy-momentum tensor that comes solely from the Born-Infeld part (the Chern-Simons piece, that can distinguish between a brane and an anti-brane, does not contribute to the energy-momentum tensor). This is good because then at the supergravity level we are effectively inserting a three-brane in a wrapped D5-brane background. To compensate for this new source of energy-momentum tensor the warp-factors change slightly as:

$$F_i \;\rightarrow\; F_i + \delta F_i, \tag{4.48}$$

where this change is over and above the δF change in (4.10) that was there in the absence of $\overline{\text{D3}}$-brane[17]. The dilaton ϕ also changes from zero to $\delta\phi$, but, as a first trial, we keep the complex structure of the non-Kähler resolved conifold fixed to (4.31) (as we shall see, this will have to be changed). Note that for a supersymmetric perturbation, the complex

[17]Note that due to the probe nature, $\delta F_5 = \delta F_6 = 0$ along with vanishing (F_5, F_6), so that the *form* of the metric remains (4.27) and the topology doesn't change.

structure would have also changed exactly in a way so as to remove any $(1,2)$ fluxes. Taking this into account, the ISD primitive $(2,1)$ flux (4.37) now changes to the following additional piece:

$$\begin{aligned}\delta\mathcal{G}_3^{(1)} \;&=\; \frac{\sinh\beta}{4\sqrt{H}\sqrt{F_1\sqrt{H}}}\left(1+\frac{\delta F_1}{2F_1}+\frac{3}{4}\frac{\delta H}{H}\right)\left[\frac{\sqrt{F_1F_2}}{2}\left(\frac{\delta F_1}{F_1}+\frac{\delta F_2}{F_2}\right)\left(\frac{1}{F_3}-\frac{1}{F_4}\right)+\left(\frac{\delta F_{4r}}{F_4}-\frac{\delta F_{3r}}{F_3}\right)\right.\\ &+\left.\left(\frac{\sqrt{F_1F_2}-F_{4r}}{F_4}\right)\left(\frac{\delta F_4}{F_4}-\delta\phi\right)-\left(\frac{\sqrt{F_1F_2}-F_{3r}}{F_3}\right)\left(\frac{\delta F_3}{F_3}-\delta\phi\right)\right]\\ &\quad\cdot\left(E_1\wedge E_3\wedge\overline{E}_3-E_1\wedge E_2\wedge\overline{E}_2\right), \end{aligned}\tag{4.49}$$

which is again a primitive $(2,1)$ form. When combined with the primitive $(2,1)$ piece that we had in (4.37), this would enter the mass formula given in (4.6) to give masses to the corresponding fermions. Note that, the E_i's appearing above are the *original* vielbein used earlier to write the (2, 1) flux (4.37), but could be replaced by the modified vielbein under (4.48), i.e:

$$E_i \;\to\; E_i+\delta E_i, \tag{4.50}$$

without changing any physics. This will be also be the case for all other (2, 1) and (1, 2) perturbations that we shall discuss below: we will express them in terms of old vielbeins although we could also use (4.50). Using the old vielbeins E_i, we do however develop an *additional* contribution to the (2, 1) flux, other than (4.37) and (4.49), that typically takes the following form:

$$\delta\mathcal{G}_3^{(2)} = \left(\alpha_1\delta F+\alpha_2 a_{1r}+\alpha_3 a_{3r}\right)\left(E_i\wedge\delta E_j\wedge\overline{E}_k\pm\sigma\; E_i\wedge E_j\wedge\delta\overline{E}_k\right), \tag{4.51}$$

where $\alpha_i(r)$ and $\sigma(r)$ are certain well defined functions of r that could be derived from our flux formulae discussed above. We cannot simply ignore this term as it is of the same order as the second line in (4.49) above, but we can absorb this in (4.37) by resorting to the modified veilbein (4.50). The conclusion then remains unchanged: all $\delta\mathcal{G}_3^{(k)}$ will be expressed in terms of E_i, but the original (2, 1) flux (4.37) will now be expressed in terms of (4.50) under perturbative backreaction of $\overline{\text{D3}}$-brane.

Coming back to our analysis, the primitive $(2,1)$ pieces are responsible in determining the masses, but we do also get another (2, 1) piece that is *neither* primitive nor ISD. This appears because we haven't changed our complex structure, and it is given by the following form:

$$\delta\mathcal{G}_3^{(3)} = \mathcal{G}_0^{(\delta\phi)}\left[\left(\frac{\sqrt{F_1F_2}-F_{4r}}{F_4}\right)E_1\wedge E_2\wedge\overline{E}_2+\left(\frac{\sqrt{F_1F_2}-F_{3r}}{F_3}\right)E_1\wedge E_3\wedge\overline{E}_3\right], \quad (4.52)$$

which becomes an ISD primitive form when the sum of the coefficents of the two terms vanish. This is no surprise because it is exactly the supersymmetry condition that we had in [22]. We have also defined the coefficient $\mathcal{G}(r)$ in terms of the warp-factors H and F_1 in the following way:

$$\mathcal{G}_0^{(\delta\phi)} \equiv -\frac{\delta\phi\,\sinh\,\beta}{4\sqrt{H}\sqrt{F_1\sqrt{H}}}\left(1+\frac{\delta F_1}{2F_1}+\frac{3}{4}\frac{\delta H}{H}\right). \quad (4.53)$$

Additionally, under supersymmetry $\delta\phi$ vanishes, so this term never shows up. For the present case, clearly we cannot impose the supersymmetry conditions. However if we change the complex structure (4.31) a bit in the following way:

$$\delta\tau_k \;=\; (i\delta\phi\,\coth\,\beta, 0, 0), \quad (4.54)$$

instead of keeping it completely rigid as we discussed above, we can make this term vanish. Note that some care is required to interpret this result. As mentioned earlier, we can change the complex structure to absorb any appearance of (1, 2) forms so that supersymmetry is restored. This case should then be interpreted differently. As we shall see below, we do get (1, 2) forms and they will be non-zero for the shifted complex structure (4.54) as well as for the original complex structure (4.31).

The (1, 2) piece is given by the following form:

$$\begin{aligned}\delta\mathcal{G}_3^{(4)} \;=\;& \frac{\sinh\,\beta}{4\sqrt{H}\sqrt{F_1\sqrt{H}}}\left(1+\frac{\delta F_1}{2F_1}+\frac{3}{4}\frac{\delta H}{H}\right)\left[\frac{\sqrt{F_1F_2}}{2}\left(\frac{\delta F_1}{F_1}+\frac{\delta F_2}{F_2}\right)\left(\frac{1}{F_3}+\frac{1}{F_4}\right)-\left(\frac{\delta F_{4r}}{F_4}+\frac{\delta F_{3r}}{F_3}\right)\right.\\ &\left.-\left(\frac{\sqrt{F_1F_2}-F_{4r}}{F_4}\right)\frac{\delta F_4}{F_4}-\left(\frac{\sqrt{F_1F_2}-F_{3r}}{F_3}\right)\frac{\delta F_3}{F_3}\right]\left(E_2\wedge\overline{E}_1\wedge\overline{E}_2+E_3\wedge\overline{E}_1\wedge\overline{E}_3\right), \quad (4.55)\end{aligned}$$

which is an ISD but non-primitive form, and therefore breaks supersymmetry. As before, we have ignored terms of the form $\delta F_i\delta F_j$ and $\delta F_i\delta\phi$, as we are assuming the perturbations to be small. When the perturbations are not small we need to use more exact expressions which can be derived with some effort, but we will not do this here. The above (1, 2) form (4.55) enters the mass formula (4.5), inducing a non-zero $m_{\bar{1}}$. This acts as an interaction between $\lambda^{\bar{1}}$ and $\lambda^{\bar{0}}$. Similarly, $\overline{\delta\mathcal{G}}_3^{(3)}$ induces an interaction $m_1\lambda^0\lambda^1$. This is given by

$$m_1 = \frac{1}{\sqrt{2}}e^{\delta\phi}|\overline{\delta\mathcal{G}}_3^{(3)}|, \tag{4.56}$$

where $|\overline{\delta\mathcal{G}}_3^{(3)}|$ is the coefficient of $\left(E_2 \wedge \overline{E}_1 \wedge \overline{E}_2 + E_3 \wedge \overline{E}_1 \wedge \overline{E}_3\right)$ in equation (4.55).

Note that in deriving the perturbations to our background we did not find any (0, 3) or IASD forms. This is expected from the probe nature of our analysis. On the other hand the (1, 2) form that we got above in (4.55) cannot be absorbed by the change in the complex structure (4.54). However one might ask if a more generic analysis could be performed. In other words, is it possible to find the most generic (2, 1) and (1, 2) perturbations in the non-Kähler resolved conifold background?

The way to answer this question would be to first find the complete basis for the (2, 1) and (1, 2) forms in the resolved conifold background. This has been studied in [41], and we reproduce it here for completeness. The basis for the (2, 1) forms are:

$$\begin{aligned}
&u_1 \equiv E_1 \wedge E_2 \wedge \overline{E}_2 - E_1 \wedge E_3 \wedge \overline{E}_3, \qquad && u_2 \equiv E_1 \wedge E_2 \wedge \overline{E}_3 - E_1 \wedge E_3 \wedge \overline{E}_2\\
&u_3 \equiv E_1 \wedge E_2 \wedge \overline{E}_1 + E_2 \wedge E_3 \wedge \overline{E}_3, \qquad && u_4 \equiv E_1 \wedge E_3 \wedge \overline{E}_1 - E_2 \wedge E_3 \wedge \overline{E}_2\\
&u_5 \equiv E_2 \wedge E_3 \wedge \overline{E}_1,
\end{aligned} \tag{4.57}$$

where all of them are ISD and primitive. The first basis, u_1, was used earlier to write both the original and the perturbed (2, 1) forms. The bases $(u_2, ..., u_5)$ are useful when the $\overline{\text{D3}}$ backreaction is not as simple as (4.48). Thus a generic (2, 1) perturbation can be of the form:

$$\delta\mathcal{G}_3^{(2,1)} = \sum_{n=1}^{5} a_n u_n, \tag{4.58}$$

where a_n could be functions of all the coordinates of the internal non-Kähler resolved conifold. We can then use (4.58) in (4.6) to expresses the masses of the relevant fermions on the $\overline{\text{D3}}$-brane. Most importantly, it will in general no longer be the case that λ^1 is massless, since more general $(2,1)$ fluxes induces non-zero masses, i.e. we will now have

$$m_{12} \neq 0, \quad m_{13} \neq 0. \tag{4.59}$$

One may similarly construct the complete basis for the (1, 2) forms for the resolved conifold background. We will again require our basis forms to be ISD to solve the background EOMs. For a (1, 2) form this is possible only if it is proportional to the fundamental form J, thus restricting the number of such forms to be just three. They are given by [41]:

$$\begin{aligned} &w_1 \equiv E_1 \wedge \overline{E}_1 \wedge \overline{E}_3 + E_2 \wedge \overline{E}_2 \wedge \overline{E}_3, \qquad w_2 \equiv E_1 \wedge \overline{E}_1 \wedge \overline{E}_2 - E_3 \wedge \overline{E}_2 \wedge \overline{E}_3, \\ &w_3 \equiv E_2 \wedge \overline{E}_1 \wedge \overline{E}_2 + E_3 \wedge \overline{E}_1 \wedge \overline{E}_3, \end{aligned} \tag{4.60}$$

where one may check that they are ISD but not primitive. We had used w_3 earlier to express the (1, 2) perturbation in (4.55). Thus a more generic non-supersymmetric perturbation in the presence of a $\overline{\text{D3}}$-brane can be expressed by the following (1, 2) form:

$$\delta\mathcal{G}_3^{(1,2)} = \sum_{n=1}^{3} b_n w_n, \tag{4.61}$$

where b_n, as for a_n above, could be generic functions of all the coordinates of the internal non-Kähler resolved conifold. This could now be inserted into (4.5) to determine the mixing between the $\lambda^0_\pm$ and $\lambda^i_\pm$ fermions, i.e.

$$m_1 \neq 0. \tag{4.62}$$

The consequence of this is that the backreaction-induced fluxes give a mass to λ^0 and λ^1, and hence there are *no massless fermions left in the spectrum.* This is a striking difference to the probe approximation, where there were two massless fermions.

Let us take a moment to consider why this is the case. From the supergravity perspective, a $\overline{\text{D3}}$ is equivalent to a D3. The background we are considering has a wrapped D5-brane, and since a D3-D5 system is non-supersymmetric, the induced fluxes will include supersymmetry breaking fluxes. It is these fluxes which give a mass to the would-be massless fermions on the $\overline{\text{D3}}$ worldvolume. In the GKP analysis of [2], there was no D5-brane, and thus this issue will not arise when considering backreaction.

This completes our discussion of spontaneous supersymmetry breaking via massive fermions on the $\overline{\text{D3}}$-brane world-volume. In the following section we will briefly dwell on certain aspects of moduli stabilization and de Sitter space.

4.2.5 Moduli stabilization and de Sitter vacua

In order to construct a concrete phenomelogical model, the resolved conifold geometry we have studied should be *glued* on to a compact, non-Kähler space. As discussed in [58], and also [59], this glueing induces corrections to the $\overline{\text{D3}}$ scalar moduli masses.

In addition to this, a compact space requires charge cancellation. Since charge cancellation is a global requirement, the necessary fluxes can be placed far from the resolved conifold which contains the $\overline{\text{D3}}$, so as not to disrupt the local dynamics we have studied. In other words, for the case that we study here, the internal six-dimensional manifold (4.27) should be thought of as extending to a fixed radius $r = r_0$, and beyond which a compact manifold is attached. The boundary condition implies that at $r = r_0$, the compact manifold should have a topology of $\mathbf{S}^2 \times \mathbf{S}^3$. The compact manifold is equipped with the right amount of fluxes etc that is necessary for global charge cancellation.

Finally, we note that moduli stabilization should be included in to this picture. We need to consider two sets of moduli: the Kähler and the complex structure moduli of our non-Kähler space. The moduli of compactifications on non-Kähler manifolds was discussed in [60], and reviewed in [61]. An interesting feature of these models is that the radial modulus and the complex structure moduli can be stabilized at tree-level whereas the other Kähler moduli, including the axio-dilaton need additional non-perturbative effects for stabilization. There are also other moduli, namely the moduli of the $\overline{\text{D3}}$-brane, fractional three-branes and possible seven-branes (that we didn't discuss here, but are nonetheless important).

From the point of view of Einstein equations, the existence of de Sitter vacua is rather non-trivial to see. Switching on (4.58) and (4.61) gives masses to worldvolume fermions and simultaneously fixes the complex structure moduli (including the radial modulus) of our non-Kähler space. However the potential generated by the susy breaking flux (4.61):

$$V = \frac{1}{2\kappa_{10}^2} \int \frac{\delta\mathcal{G}_3^{(1,2)} \wedge *\overline{\delta\mathcal{G}_3^{(1,2)}}}{\text{Im }\tau}, \tag{4.63}$$

where τ is the axio-dilaton, vanishes identically. This means the presence of a $\overline{\rm D3}$-brane takes a supersymmetric AdS space to a non-supersymmetric one, and therefore doesn't contribute any positive vacuum energy to the system. This conclusion is not new and is another manifestation of the no-go condition of Gibbons-Maldacena-Nunez [50], recently updated in [52]. This means to allow for a positive cosmological solution in the four space-time direction, the no-go condition should be averted[18].

This then brings us to the recent study done in [52] from an uplift in M-theory. Quantum corrections play an important role, and positive cosmological constant is only achieved in four space-time directions if the following condition is satisfied:

$$\langle \mathcal{T}^\mu_\mu \rangle_q \;>\; \langle \mathcal{T}^m_m \rangle_q, \tag{4.64}$$

which is a generalization of the classical condition studied in [50]. Here $\mathcal{T}_{mn}$ is the energy-momentum tensor and the subscript q denote the quantum part of it. For more details, and the derivation of this, the readers may want to refer to [52].

This indicates that a concrete realization of de Sitter vacua in this context, and a precise connection to KKLT [54], would thus require including at least a subset of the above corrections (similar to 'Kähler Uplifting' [70]). Note that our setup would not involve to the KPV process [71], whereby a stack of $\overline{\rm D3}$'s polarize into an NS5, as we are only considering a single anti-brane.

4.3 Probe $\overline{\rm D7}$ in a GKP Background

In the previous section we generalized the work of [1, 2] to a more general background, and found several interesting features. We now consider a different generalization: we turn our attention to an $\overline{\rm D7}$ brane in a GKP background. Similar to the $\overline{\rm D3}$ case, the $\overline{\rm D7}$ brane differs from the D7 brane only in the sign of κ-symmetry projector, and the charge under the RR fields. The embedding of D7 branes into flux compactifications has been the focus of many works; for example [62], [63], [41], and [64]. In particular, many details of the D7

[18] All the energy-momentum tensors are computed using both the bosonic and the fermionic terms on the branes and the planes. Note that the no-go conditions in [50, 52] were derived exclusively using the bosonic terms on the branes and the planes. However if we use (4.144) (see section 4) to define the pullbacks of the type IIB fields on the branes and the planes, we can easily see that the conclusions of [50, 52] remain unchanged in the presence of the fermionic terms.

and $\overline{\mathrm{D7}}$ fermionic action were worked out in [65] and [66].

Placing a $\overline{\mathrm{D7}}$ in a warped $\mathcal{N}=1$ background will spontaneously break supersymmetry. The breaking of supersymmetry manifests itself in the fermionic action via a mass for the fermions (see [65] for details), and the spontaneous nature of SUSY breaking can be deduced via the condition discussed in Section 4.1.1. Furthermore, for general background fluxes, all the $\overline{\mathrm{D7}}$ worldvolume fermions are massive. Only under special circumstances will there remain a massless fermion in the low energy spectrum; demonstrating this will be the focus of this section. We will find that, under suitable conditions, we have not only one massless fermion, but many. This is similar to the the $\overline{\mathrm{D3}}$ in a resolved conifold case studied in Section 2, where (in the probe approximation) we found not one but two massless fermions.

4.3.1 The fermionic action for a $\overline{\mathrm{D7}}$ in a flux background

The quadratic fermionic action for a single Dp-brane (in string frame) is detailed in [40], we will follow their conventions in what follows. The only difference for an anti-brane is in the κ-symmetry projector, which flips sign relative to the brane case. For the case of $p=7$ this reads:

$$S_f^{\overline{\mathrm{D7}}} = -\frac{1}{2}T_7(2\pi\alpha')^2 \int \mathrm{d}^8\xi\, e^{\phi}\, \sqrt{-\det(G+\mathcal{F})}\, \bar{\theta}\left[1-\Gamma_{\overline{\mathrm{D7}}}(\mathcal{F})\right]\left(\not{\mathcal{D}}-\Delta\right)\,\theta, \tag{4.65}$$

where we scaled our action by an overall factor of $(2\pi\alpha)'^2$ (to match with the convention of writing the gauge field as $2\pi\alpha' F_{\mu\nu}$). As before, the spinor θ is a 10-dimensional 64(32) real(complex) component Majorana spinor, which is a doublet of 10-dimensional (left-handed) 32(16) real(complex) component Majorana-Weyl spinors.

The factor $[1-\Gamma_{\overline{\mathrm{D7}}}(\mathcal{F})]$ is the κ-symmetry projector, which depends on the worldvolume flux $\mathcal{F}$, and we have defined:

$$\Gamma_{\overline{\mathrm{D7}}} = -i\sigma_2\frac{1}{\sqrt{-g}}\Gamma_{01234567} + \mathcal{O}(\mathcal{F}), \tag{4.66}$$

and we take the brane to be along the $x^0,...,x^7$ coordinate directions. The covariant derivative $\widetilde{\mathcal{D}}$ on the brane is defined as:

$$\not{\mathcal{D}} = (M^{-1})^{\alpha\beta}\Gamma_\beta\widetilde{D}_\alpha, \tag{4.67}$$

where M_{ab} is defined using $\mathcal{F}_{ab}$ and the pull-back of the metric g_{ab} as:

$$M_{ab} = g_{ab} + \mathcal{F}_{ab}, \tag{4.68}$$

with $\mathcal{F} = P[\mathcal{B}_{(2)}] + 2\pi\alpha' F_2$. We have also defined $\widetilde{D}_\alpha$ as a shifted covariant derivative,

$$\widetilde{D}_m = D_m \mathbb{I}_2 + \sigma_1 W_m, \tag{4.69}$$

which we shall define in more detail momentarily. It is important to note that the contraction $\not{D} = \Gamma^m D_m$ sums only over the indices on the brane-worldvolume, and as mentioned above, we will take the brane to be oriented along the $(x^0, x^1, ...x^7)$ directions. In contrast to this, the contractions appearing in Δ will sum over *all* indices[19], for example Δ contains the term $\Gamma^{MNP}\mathcal{H}_{MNP}$ where $M, N, P = 0..9$. We can further decompose $\mathcal{H}_{MNP}$ into pieces with 0, 1, and 2, indices along the transverse two-dimensional space parametrized by (x^8, x^9) coordinates.

In a general GKP background the worldvolume flux $\mathcal{F}$ will be non-zero, and this cannot be gauged away. To make our analysis simple, we will focus on a class of backgrounds with the property that $\mathcal{B}_2$ is constant along the brane worldvolume, i.e. $\mathcal{B}_2 = \mathcal{B}_2(x^8, x^9)$, and there is an equal and opposite DBI gauge F_2, such that $\mathcal{F} = 0$. This allows us to take the M_{ab} appearing in equation (4.68) as simply g_{ab}, and $\Gamma_{\overline{\text{D7}}}$ to be $-i\sigma_2 \frac{1}{\sqrt{-g}}\Gamma_{01234567}$. Recall that a GKP background also comes equipped with a self-dual five-form flux $\widetilde{\mathcal{F}}_5$, given by

$$\widetilde{\mathcal{F}}_5 = (1 + *)\left(\mathrm{d}\alpha \wedge \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\right), \tag{4.70}$$

where the function α depends on the coordinates of the internal space, and is responsible for setting the profile of the warp factor, i.e. $\alpha = e^{4A}$. We will see that $\widetilde{\mathcal{F}}_5$ generically contributes to the fermion masses, unless $\alpha = \alpha(x^8, x^9)$, i.e. α is independent of the brane coordinates.

Let us consider an explicit choice of background flux which realizes this. We again define in the standard way $\mathcal{G}_3 = \mathcal{F}_3 - \tau\mathcal{H}_3$. A choice of $\mathcal{G}_3$ which meets the above criteria is:

$$\mathcal{G}_3 = N\ E_1 \wedge E_2 \wedge \overline{E}_3, \tag{4.71}$$

[19]We take our three-form fluxes to be only in the internal space.

where N is a constant and we take complex structure $J = (i, i, i)$, i.e. $z^1 = x^4 + ix^5$ and so on. One can easily check that this is ISD and primitive[20]. The corresponding $\mathcal{B}_2$ and $\mathcal{C}_2$ which generate this $\mathcal{G}_3$ are:

$$\begin{aligned}\mathcal{C}_2 &= N\left(x^4\, dx^6 \wedge dx^8 - x^4\, dx^7 \wedge dx^9 - x^5\, dx^6 \wedge dx^9 - x^5\, dx^7 \wedge dx^8\right)\\ \mathcal{B}_2 &= Ne^{\phi_0}\left(x^9\, dx^4 \wedge dx^6 + x^8\, dx^4 \wedge dx^7 + x^8\, dx^5 \wedge dx^6 + x^9\, dx^5 \wedge dx^7\right), \end{aligned} \tag{4.72}$$

where we take the dilaton to be constant $\phi = \phi_0$. With the above example in mind, we will proceed in our analysis with a general $\mathcal{G}_3$, but with the assumption that $F_2 = -P[B_2]$ and hence $\mathcal{F} = 0$.

As mentioned above, the IIB spinor θ is actually a doublet of 16-component left-handed (i.e. same chirality) Majorana-Weyl spinors; this 'doublet' is a 32 component Majorana spinor, note that it is *not* Weyl. The gamma matrices in this 64 component representations are related to the 16 component representations by:

$$\Gamma_m^{\text{doublet}} = \Gamma_m \otimes \mathbb{I}_2, \tag{4.73}$$

as in, e.g. below equation 85 in [40].

We gauge fix κ-symmetry by enforcing the κ-symmetry projection to satisfy the following condition, namely:

$$\bar{\theta}\left(1 + \Gamma_{\overline{\text{D7}}}\right) = 0. \tag{4.74}$$

This enforces a relation between $\theta_{1,2}$ componenst of the doublet θ, given by:

$$\theta_2 = \Gamma_{012\ldots7}\theta_1. \tag{4.75}$$

This choice of gauge fixing was used in recent papers by Kallosh et al., for example [1, 2], as it is consistent with an orientifold projection. Alternatively, one could use a condition $\theta_2 = 0$, as was used in papers by Martucci et al., e.g. [40] and [38, 39]. Here, we will only use the condition above, namely, $\theta_2 = \Gamma_{012\ldots7}\theta_1$.

Lastly, we note that the operators W_m and Δ appearing in equation (4.65) are given by

[20] To avoid clutter we are using the same symbol E_i to denote the vielbeins as before although now the definitions of the vielbeins are very different. Furthermore since the background is no longer a non-Kähler resolved conifold we are not restricted to the basis (4.60) to express the three-form $\mathcal{G}_3$.

(see for example [2]):

$$\Delta = -\frac{1}{2}\Gamma^M\partial_M\phi - \frac{1}{24}\left(\mathcal{H}_{MNP}\sigma_3 - e^{\phi}\mathcal{F}_{MNP}\sigma_1\right)\Gamma^{MNP} \tag{4.76}$$
$$W_m = -\frac{1}{4}e^{\phi}(i\sigma_2)\mathcal{F}_m + \frac{1}{8}\left(\mathcal{H}_{mNP}\sigma_3 - e^{\phi}\mathcal{F}_{mNP}\sigma_1\right)\Gamma^{NP} - \frac{1}{8\cdot 4!}(i\sigma_2)e^{\phi}\mathcal{F}_{NPQRS}\Gamma^{NPQRS}\Gamma_m$$

where $m = 4,5,6,7$, and $M,N = 0,1,...,9$. Additionally any quantity not appearing with a σ_i is implicitly a tensor product with the 2×2 identity matrix.

We can now expand our action (4.65), using the operators (4.76) and the κ-symmetry fixing condition (4.75). We use the fact that the fluxes are only in the internal space, and that the only non-vanishing bilinears for $10d$ Majorana-Weyl spinors have 3 or 7 gamma matrices. The action can be written in terms of θ_1 as:

$$S_f^{\overline{\mathrm{D7}}} = -\frac{1}{2}T_7(2\pi\alpha')^2\int \mathrm{d}^8\xi\; e^{\phi}\,\sqrt{-\mathrm{det}G}\;\mathcal{L}_{\theta}, \tag{4.77}$$

where G is the warped metric, and $\mathcal{L}_\theta$ is given purely in terms of θ_1 as:

$$\mathcal{L}_\theta = 2\bar{\theta}_1\left[\Gamma^m D_m - \frac{3}{16}e^{\phi}\Gamma^{mna}\left(\mathcal{F}_{mna}\Gamma_{012...7} + e^{-\phi}\mathcal{H}_{mna}\right) - \frac{5}{16}e^{\phi}\Gamma^{mnp}\mathcal{F}_{0123}{}^{q}\epsilon_{mnpq}\right]\theta_1, \tag{4.78}$$

with the indices running as $m,n,p,q = 4,5,6,7$ and $a = 8,9$. Note the interesting feature that the only 3-form fluxes which contribute to the action are those with two-legs along the brane, and one leg transverse to the brane. The other contributions, (1) 3 legs along the brane, 0 transverse and (2) 1 leg along the brane, 2 transverse, cancel out of the action. As we see, there is a possible contribution from the 5-form flux when all legs of the flux lie along the brane. This can be made to vanish if we impose that α depend only on the transverse directions to brane. This is different from the $\overline{\mathrm{D3}}$ case, where the $\widetilde{\mathcal{F}}_5$ term simply did not contribute, regardless of the choice of α. We will return to this point in Section 4.3.4; for the moment we will take $\alpha = \alpha(x^8, x^9)$ and hence $\widetilde{\mathcal{F}}_5$ will not contribute to the masses. There can generally also be a contribution from the 1-form flux, but a GKP background doesn't have these, due to the lack of 1-cycles on a CY manifold[21].

[21] Note that we are putting a $\overline{\mathrm{D7}}$ in a GKP background with a constant dilaton and zero axion. The backreacted axionic source of the $\overline{\mathrm{D7}}$ is suppressed by g_s and to this order we are not taking this to backreact on the $\overline{\mathrm{D7}}$ world-volume (the axion will only be along (x^8, x^9) directions). This differs slightly in spirit of the previous section where due to the non-supersymmetric nature of the D3-D5 system, it was essential to take the perturbative backreactions into account, otherwise certain aspects of the physics would not have

The action (4.77) can be simplified further by using $\Gamma_{0\ldots9}\theta_1 = \theta_1$, which implies that $\mathcal{F}_{mna}\Gamma_{012\ldots7}\theta_1 = (*_2\mathcal{F}_3)_{mna}\theta_1$, where $*_2$ is hodge duality in the (x^8, x^9) directions. We can also write this in terms of the familiar $\mathcal{G}_3 = \mathcal{F}_3 - ie^{-\phi}\mathcal{H}_3$ along with the following nomenclatures: ISD2 as the "imaginary self-dual" along the transverse two-cycle and IASD2 as the "imaginary anti-self-dual" again along the transverse two cycle pieces of $\mathcal{G}_3$ as

$$\mathcal{G}_3 = \mathcal{G}_3^{\rm IASD2} + \mathcal{G}_3^{\rm ISD2}, \quad \mathcal{G}_3^{\rm ISD2} = \frac{1}{2}\left(\mathcal{G}_3 - i *_2 \mathcal{G}_3\right), \quad \mathcal{G}_3^{\rm IASD2} = \frac{1}{2}\left(\mathcal{G}_3 + i *_2 \mathcal{G}_3\right), \tag{4.79}$$

which is equivalent to the decomposition

$$\mathcal{H}_3 = \frac{i}{2}e^{\phi}\left(\mathcal{G}_3 - \bar{\mathcal{G}}_3\right) \ , \ \mathcal{F}_3 = \frac{1}{2}\left(\mathcal{G}_3 + \bar{\mathcal{G}}_3\right). \tag{4.80}$$

With these definitions the action becomes

$$\mathcal{L}_\theta = 2\bar{\theta}_1\left[\Gamma^m D_m - \frac{3i}{32}e^{\phi}\Gamma^{mna}\left(\mathcal{G}_3^{\rm ISD2} - \bar{\mathcal{G}}_3^{\rm ISD2}\right)_{mna}\right]\theta_1; \quad (m,n) = 4,5,6,7; \quad a = 8,9. \tag{4.81}$$

Thus the worldvolume fermions on the $\overline{\rm D7}$ brane will have masses determined by ISD2 $\mathcal{G}_3$ flux, where the 'dual' in ISD2 refers to space *transverse to the brane* (and not the full internal space). For our example $\mathcal{G}_3$ given in equation (4.71), the flux is purely ISD2 and thus will contribute to the masses. These masses spontaneously break the background $\mathcal{N} = 1$ supersymmetry.

We could also include flux which is ISD – and thus solves the equations of motion for a GKP background – but which is *not* ISD2, and hence will not contribute to the fermion masses. An example of such a flux is

$$\mathcal{G}_3 = M\left(E_1 \wedge \overline{E}_1 - E_2 \wedge \overline{E}_2\right) \wedge E_3 \tag{4.82}$$

which is purely IASD2, and thus will not enter equation (4.81). Such a flux would come from a $\mathcal{B}_2$ of the form

$$\mathcal{B}_2 = -Me^{\phi_0}x^9 \cdot \left({\rm d}x^4 \wedge {\rm d}x^5 - {\rm d}x^6 \wedge {\rm d}x^7\right), \tag{4.83}$$

and a similar form for $\mathcal{C}_2$.

been visible.

4.3.2 Fermions in $4d$ and spontaneous SUSY breaking in a GKP background

We can already see that supersymmetry will be spontaneously broken by the $\overline{\text{D7}}$ in the presence of three-form fluxes. What remains to be checked is if there remains a massless fermion in the four dimensional effective theory.

In the absence of $\mathcal{G}_3$ flux, the massless fermions in the $4d$ theory are those who's dependence on the coordinates of the internal 4-cycle wrapped by the brane is harmonic. The exact spectrum of effective $4d$ fermions is therefore given by the cohomology classes of the wrapped cycle. On the other hand the coupling of the $\mathcal{G}_3$ flux to the fermions is governed by the structure of the spinors, so we do not need to know the full details of the topology of the wrapped cycle to know whether some of these fermions remain massless. Indeed, most of our calculation proceeds in the same fashion and certainly in the same spirit as the $\overline{\text{D3}}$ case[22].

The 16 component spinor θ_1 can decomposed into two 8 component spinors θ_{1+} and θ_{1-} where the $\pm$ denotes the chirality in the transverse space, i.e. under $SO(2)$. In terms of Γ matrices, $\Gamma^3\theta_{1+} = \theta_{1-}$ and $\Gamma^{\bar{3}}\theta_{1-} = \theta_{1+}$. The four dimensional fermions can be obtained via dimensional reduction of θ_{1+} and θ_{1-}, according to the cohomology classes of the cycle wrapped by the brane, as depicted below:

$$\begin{aligned}
\theta_{1+} &= \sum_a \psi^a_{\pm\pm+} \otimes \chi^a_{\pm\pm+} \\
\theta_{1-} &= \sum_a \psi^a_{\pm\pm-} \otimes \chi^a_{\pm\pm-},
\end{aligned} \tag{4.84}$$

where the ψ^a are $4d$ spinors while the χ^a are internal spinors; the index a simply counts the number of $4d$ spinors. The unspecified $\pm\pm$ indices correspond their chirality under $SU(2)$, i.e. corresponding to their behaviour under the action of Γ^1 and Γ^2. This allows us to group all the fields precisely as done in [1, 2]. We define

$$\begin{aligned}
\lambda^0 &= \sum \psi^a_{---} \quad , \quad \lambda^{\bar{0}} = \sum \psi^a_{+++} \\
\lambda^1 &= \sum \psi^a_{+--} \quad , \quad \lambda^{\bar{1}} = \sum \psi^a_{-++} \\
\lambda^2 &= \sum \psi^a_{-+-} \quad , \quad \lambda^{\bar{2}} = \sum \psi^a_{+-+} \\
\lambda^3 &= \sum \psi^a_{--+} \quad , \quad \lambda^{\bar{3}} = \sum \psi^a_{++-}.
\end{aligned} \tag{4.85}$$

[22] Without the (1, 2) perturbations of course.

We can now perform the fermion decomposition exactly as in [1,2], except now the fermions λ actually refer to the *set* of fermions which transform according the corresponding chirality. We have

$$\begin{aligned}\frac{\sqrt{2}}{12}\bar{\theta}^1\Gamma^{MNP}\hat{\mathcal{G}}_{MNP}\theta^1 \;&=\; \bar{\lambda}^0_+\lambda^0_+\hat{\mathcal{G}}_{123}+\bar{\lambda}^{\bar{0}}_-\lambda^{\bar{0}}_-\hat{\mathcal{G}}_{\bar{1}\bar{2}\bar{3}}+\left(\bar{\lambda}^0_+\lambda^i_+\hat{\mathcal{G}}_{ij\bar{\jmath}}-\bar{\lambda}^{\bar{0}}_-\lambda^{\bar{\imath}}_-\hat{\mathcal{G}}_{\bar{\imath}j\bar{\jmath}}\right)\delta^{j\bar{\jmath}}\\ &\quad+\tfrac{1}{2}\left(\bar{\lambda}^i_+\lambda^j_+\varepsilon_{jk\ell}\hat{\mathcal{G}}_{i\bar{k}\bar{\ell}}+\bar{\lambda}^{\bar{\imath}}_-\lambda^{\bar{\jmath}}_-\varepsilon_{\bar{\jmath}\bar{k}\bar{\ell}}\hat{\mathcal{G}}_{\bar{\imath}k\ell}\right)\delta^{k\bar{k}}\delta^{\ell\bar{\ell}}\,,\end{aligned}\tag{4.86}$$

where in our case $\hat{\mathcal{G}}_{MNP}\equiv\left(\mathcal{G}_3^{\mathrm{ISD2}}-\bar{\mathcal{G}}_3^{\mathrm{ISD2}}\right)_{MNP}$, and in an abuse of notation, we now use M,N,P to refer to internal space, $M=4,5,...,9$.

The $\mathcal{G}_3$ flux must be (2, 1) and primitive, since we only want supersymmetry to be broken by the presence of the brane. This on its own immediately implies that λ^0 remains massless and that the mass cross-terms with λ^i vanish as well, as in the $\overline{\mathrm{D3}}$ case. The additional feature that the flux which couples to the fermions is 'ISD2' further reduces the allowed components to only those that have a $\bar{3}$ index, and hence the only non-vanishing mass terms are:

$$m_3=m_{\bar{3}}\propto\left(\mathcal{G}_3^{\mathrm{ISD2}}\right)_{12\bar{3}}\,,\tag{4.87}$$

where λ^3 gets its mass from $\mathcal{G}_3^{\mathrm{ISD2}}$ while $\lambda^{\bar{3}}$ gets its mass from $\bar{\mathcal{G}}_3^{\mathrm{ISD2}}$. The other fermions remain massless, i.e.

$$m_0=m_i=m_{0i}=m_{ij}=0\;\;;\;\;i,j=1,2,\tag{4.88}$$

and similarly for barred indices.

Thus the resulting four-dimensional massless fermionic field content consists of λ^0, λ^1 and λ^2. We emphasize that the λ's refer to *sets* of 4d fermions, the precise details of which can be found via dimensional reduction. Thus there are *many* massless fermions in this case, in contrast to the $\overline{\mathrm{D3}}$ in a GKP background, which has only one [2]. However, both examples illustrate how supersymmetry is broken spontaneously by a probe anti-brane. Finally, we note that the bosonic field content on the brane can be taken care of as in the $\overline{\mathrm{D3}}$ case, by placing the $\overline{\mathrm{D7}}$ on an O7 plane.

4.3.3 Inclusion of $\mathcal{F}$

There is good reason to study non-zero $\mathcal{F}$: worldvolume fluxes on D7 branes generate D-terms and F-terms in the 4d theory [42], and may even allow for de Sitter solutions along the lines of [43]. With this in mind, let us see what happens on the anti-brane side of this story, i.e. what happens when we allow worldvolume fluxes on a $\overline{\text{D7}}$. Non-zero $\mathcal{F}$ modifies our previous analysis in two ways. One, it modifies the kinetic term via the matrix M_{ab} defined earlier in (4.68) and two, it also modifies the κ-symmetry projector, which in turn induces new mass terms.

The equations of motion require $\mathcal{F}$ to be anti self-dual on the cycle wrapped by the anti-brane, which we take to be in the (x^4, x^5, x^6, x^7) directions, with $\epsilon_{4567} = -1$ to be consistent with our conventions in the previoius section. A judicious choice of vielbeins along the cycle can put the flux into the simple form,

$$\mathcal{F} = f(e_4 \wedge e_5 + e_6 \wedge e_7). \tag{4.89}$$

Note that in this approach we first choose a worldvolume flux, which then guides our choice of vielbeins and complex structure. This of course also affects the spacetime Γ-matrices and the definitions of the fermions in the $SU(3)$ triplet. At the end of the day, this amounts to an $SU(3)$ transformation and does not affect the number of massless fermions, which is what we are ultimately interested in, nor does it affect the masses of the massive ones.

The modified kinetic term can be recast as a canonical kinetic term plus a generalized electromagnetic coupling by a (generally non-isotropic) rescaling of the vielbeins, as described in [40]. For our above choice of $\mathcal{F}$, the rescaling of the vielbeins to obtain a canonical kinetic term is simple. The matrix $M = g + \mathcal{F}$ now has off-diagonal terms, and in the vielbein basis its inverse is given by,

$$M^{-1} = \frac{1}{1+f^2}\begin{pmatrix} 1 & -f & 0 & 0 \\ f & 1 & 0 & 0 \\ 0 & 0 & 1 & -f \\ 0 & 0 & f & 1 \end{pmatrix}. \tag{4.90}$$

By defining rescaled vielbeins,

$$\hat{e}^m = \frac{1}{\sqrt{1+f^2}} e^a \qquad m = 4, 5, 6, 7, \tag{4.91}$$

the kinetic term becomes

$$\bar{\theta}\Gamma_m \mathcal{D}_n M^{mn}\theta = \bar{\theta}\left(\hat{g}^{mn} + \hat{\mathcal{F}}^{mn}\right)\Gamma_m \mathcal{D}_n \theta, \tag{4.92}$$

where the 'hatted' quantities are expressed in terms of the rescaled vielbeins, e.g. $\hat{g}^{mn} = \eta^{jk}\hat{e}^m_j \hat{e}^n_k$. We see that the kinetic term splits into a canonical kinetic term and a derivative coupling of the fermions to the worldvolume flux.

This derivative coupling complicates the dimensional reduction of θ_1. The underlying SU(3) structure guarantees that there is are solutions to $g^{mn}\Gamma_m D_n \chi_6 = 0$, i.e. there exist zero-modes of the Dirac operator on the internal space, however it will generically *not* be true that there are solutions to $(g^{mn} + \mathcal{F}^{mn})\Gamma_m D_n \chi_6 = 0$, particularly for non-small $\mathcal{F}$. If no zero-modes exist for this 'modified Dirac operator' then there will be no massless degrees of freedom. Thus the effect of the modified kinetic terms is to give mass to some, if not all, of the fermions.

We still have yet to consider the modification of the couplings to $\mathcal{G}_3$. Before doing so, we must incorporate the rescaling of the vielbeins that we performed. This is simply done by putting a factor of $\sqrt{1+f^2}$ for every lower index along the brane directions in all the quantities. To avoid notation clutter, we will assume for the remainder of this section that the spacetime fluxes are implicitely 'hatted' and contractions are made using the rescaled metric. This rescaling ultimately does not affect the tensor structure of the fluxes, and therefore will not affect which fermions acquire masses.

The inclusion of $\mathcal{F}$ also modifies the κ-symmetry projector, in the following way:

$$\begin{aligned}\Gamma_{\overline{\mathrm{D7}}} &= \frac{1}{\sqrt{|g+\mathcal{F}|}}\left[-i\sigma_2\Gamma_{01234567} + \sigma_3 i\sigma_2(\Gamma_{012345}\mathcal{F}_{67} - \Gamma_{012367}\mathcal{F}_{45}) - i\sigma_2\Gamma_{0123}\mathcal{F}_{45}\mathcal{F}_{67}\right] \\ &= \frac{1}{\sqrt{|g+\mathcal{F}|}}\left[-i\sigma_2\Gamma_{01234567} + \hat{f}\sigma_3 i\sigma_2(\Gamma_{012345} - \Gamma_{012367}) - i\sigma_2\Gamma_{0123}\hat{f}^2\right],\end{aligned} \tag{4.93}$$

which in turn modifes the relation between $\theta_{1,2}$ imposed by the gauge fixing condition $\bar{\theta}(1+\Gamma_{\overline{\mathrm{D7}}}) = 0$, in the following way:

$$\theta_2 = \left[\Gamma_{01234567} + \hat{f}(\Gamma_{012345} - \Gamma_{012367}) + \hat{f}^2\Gamma_{0123}\right]\theta_1. \tag{4.94}$$

The outcome of all these changes is that now new coupling arise as:

$$\bar{\theta}_1 e^{-\phi}\Big[\Gamma^{mna}\left(\mathcal{F}_{mna}\Gamma_{0\ldots7} + e^{-\phi}\mathcal{H}_{mna}\right) + \hat{f}\Gamma^{mab}\left(\mathcal{F}_{mab}(\Gamma_{012345} - \Gamma_{012367}) + e^{-\phi}\mathcal{H}_{mab}\right) \\ + \hat{f}^2\Gamma^{mnl}\left(\mathcal{F}_{mnl}\Gamma_{0123} + e^{-\phi}\mathcal{H}_{mnl}\right)\Big]\theta_1, \tag{4.95}$$

where the indices (m, n, l) now take values 4,5,6,7 and (a, b) as before take values (8,9).

These new terms include fluxes that have one leg or all three legs along the brane, which were not presence for $\mathcal{F} = 0$. In fact, the last term is the coupling we get for an $\overline{\mathrm{D3}}$ brane. This is to be expected, since worldvolume fluxes induce lower-dimensional brane charge. The term linear in $\hat{f}$ is the coupling due to the induced five-brane charge and is similar to what we would obtain if we studied an $\overline{\mathrm{D5}}$ in a GKP background. It produces couplings to fluxes which obey a self-duality condition in the directions transverse to the cycles threaded by the flux. As in the pure $\overline{\mathrm{D7}}$ case, this simply restricts which subset of fermions get masses and produces no new unexpected couplings. The presence of the $\overline{\mathrm{D3}}$-like coupling means that the $SU(3)$ triplet fermions will generically all acquire a mass (in addition to any mass they receive from the modified kinetic term), though some may remain massless due to the specific form of the flux as we saw in the previous section. The singlet fermions, however, receive no new $\mathcal{G}_3$ induced mass, for the same reason as before: its mass term does not arise from primitive (2, 1) fluxes, which we require by construction. However, as mentioned already, the singlet *does* in general receive a mass from the modified kinetic term, and hence there will generically remain no massless degrees of freedom.

4.3.4 Effect of more general $\mathcal{F}_5$

Before we close this section we wish to comment on how the scenario changes once we allow for more general $\mathcal{F}_5$. The combination

$$\widetilde{\mathcal{F}}_5 = \mathcal{F}_5 + \mathcal{B}_2 \wedge \mathcal{F}_3 + \mathcal{C}_2 \wedge \mathcal{H}_3, \tag{4.96}$$

needs to be self-dual in the full $10d$ space. If we demand that the 3-form fluxes have only *one* leg transverse to the brane, which is necessary for them to give fermion masses, then the 5-form flux must have a leg off the brane as well and therefore will not generate a mass for the fermions! Conversely, if $\mathcal{F}_5$ is entirely along the brane directions, the corresponding 3-forms will not be of the appropriate form to generate masses. It is therefore possible to consider embeddings of the $\overline{\text{D7}}$ such that only one or the other type of mass contributions are present or combine both.

Let's consider a non-zero $\mathcal{F}_{0123m}$ component, where m is along the brane worldvolume. The fermion decomposition analysis is very similar to before. The contribution to the action is of the form

$$\begin{aligned}\frac{\sqrt{2}}{12}\bar{\theta}^1\Gamma^{MNP}\varepsilon_{MNPQ}\mathcal{F}_{0123}{}^{Q}\theta^1 \quad = \quad & \left(\bar{\lambda}^0_+\lambda^i_+\varepsilon_{ij\bar{j}\bar{k}}\mathcal{F}_{0123}{}^{\bar{k}} - \bar{\lambda}^{\bar{0}}_-\lambda^{\bar{\imath}}_-\varepsilon_{\bar{\imath}j\bar{j}k}\mathcal{F}_{0123}{}^{k}\right)\delta^{j\bar{j}} \\ & +\tfrac{1}{2}\left(\bar{\lambda}^i_+\lambda^a_+\varepsilon_{ak\ell}\varepsilon_{i\bar{k}\bar{\ell}m}\mathcal{F}_{0123}{}^{m} + \bar{\lambda}^{\bar{\imath}}_-\lambda^{\bar{a}}_-\varepsilon_{\bar{a}\bar{k}\bar{\ell}}\varepsilon_{\bar{\imath}k\ell\bar{m}}\mathcal{F}_{0123}{}^{\bar{m}}\right)\delta^{k\bar{k}}\delta^{\ell\bar{\ell}},\end{aligned} \tag{4.97}$$

where the $i, j, k...$ indices are restricted to lie along the brane (but a has no such restriction). This results in non-zero m_{13}, m_{23} and even more notably m_{01}, m_{02}. Note that m_{11}, m_{12}, m_{22} remain vanishing, so even when both the 3-form and the 5-form fluxes contribute mass terms, there is still a massless degree of freedom remaining.

Finally, the modification of the κ-projector in the scenario with worldvolume fluxes does not introduce new contributions from the 5-form flux. Indeed, the second term in $\Gamma_{\overline{\text{D7}}}$, which gives the coupling to the induced five-brane charge, can only conspire to give 3 or 7 gamma matrices inside the resulting fermion bilinear if $\mathcal{F}_5$ has two legs in the internal space, but it must have four legs along the spacetime directions. Similarly, the third term necessarily results in a single gamma matrix, yielding a vanishing bilinear, exactly as in the $\overline{\text{D3}}$ case. Note however, that in combining both worldvolume fluxes and an $\mathcal{F}_5$ without transverse legs results in all the fermions acquiring a mass.

Let us also note that if we had taken the internal space to be a non-Kähler resolved conifold with fractional branes, and then inserted a $\overline{\text{D7}}$-brane wrapping a four-cycle inside the non-Kähler space, the background fluxes and also the physics would have been quite different. We will however not explore this further here, but instead go to another interesting aspects of our analysis: the all-order fermionic action on a $\overline{\text{D3}}$-brane.

4.4 Towards the κ-symmetric All-Order Fermionic Action for a $\overline{\text{D3}}$-brane

The previous two sections detailed the spontaneous breaking of supersymmetry by probe anti-branes in otherwise supersymmetric compactifications. The starting point of both of these analyses has been the fermionic brane action at lowest order in θ, which takes a manifestly κ-symmetric form.

We would now like to see if this result continues to hold at higher orders in θ. As we will see, the answer to this question is in affirmative, and to show this we need only minimal knowledge of brane actions[23]. In particular, we can use string dualities to deduce the structure of the all order fermionic action, without needing precise information as to the form of the higher order operators. To do so, we will define a (completely general) fermionic completion of the $\overline{\text{D3}}$-brane action, as was done at lowest order in θ in [38, 39], and use certain duality tricks to generate the higher order fermionic counterparts of the bosonic fields. Note that under RG the higher order terms are generically irrelevant, but they are nevertheless needed to realize the full κ-symmetry.

The bosonic components of the NS and RR sectors are connected by the type IIB equations of motion, and therefore once a certain set of field components are known, others can be generated from the corresponding EOMs. On the other hand, for the fermionic components no additional work is needed: knowing the fermionic fields $(\theta, \bar{\theta})$ and the bosonic fields, one should be able to predict the fermionic completions of the bosonic fields to all orders in θ and $\bar{\theta}$. This means the fermionic completions of higher p-form fields should *at least* be related to the lower p-form field (including the graviton, anti-symmetric tensor and dilaton) by certain U-duality transformations at the self-dual points $g_s = 1$ and $R_i = 1$ for $i = 1, .., 2k$ with R_i being the radii of the compact directions. To see why this is the case, let us study two corners of type IIB moduli space.

• We can go to $g_s = e^\phi = 1$ point where we should be able to exchange $\mathbf{B}^{(1)}_{mn}$ with $\mathbf{B}^{(2)}_{mn}$, as shown as point C in **figure 4.1**.

• We can go to self-dual radii of the compact target space $R_i = 1$ where we should be able to exchange the p-form fields with $(p + 2k)$-form fields, as shown as point B in **figure 4.1**.

This is only possible if at least a subset of the fermionic counterparts of the $(p + 2k)$-form fields are the ones got via U-duality transformations. This trick could then be used to generate all the fermionic counterparts of the higher form fields at least at the self-dual corner $g_s = R_i = 1$ of type IIB moduli space, i.e around the point A in **figure 4.1**. Once we move away from the self-dual point, we can study the fermionic counterparts of the bososnic fields at generic point in the type IIB moduli space.

[23]See [33], [34], and [35], for more recent related works on the Volkov-Akulov actions.

On the other hand the scenario is subtle in the presence of branes. It is known that the D3 or the $\overline{\text{D3}}$-branes are S-duality neutral although the world-volume degrees of freedom differ. However they are not T-duality neutral. The other D-branes (or NS-branes) are neither S nor T-duality neutrals. So, to effectively use the duality trick, no branes should be present. This is good because now we can determine the fermionic completion of the background without worrying about the backreactions from the branes, and then insert D-branes to study the world-volume theory.

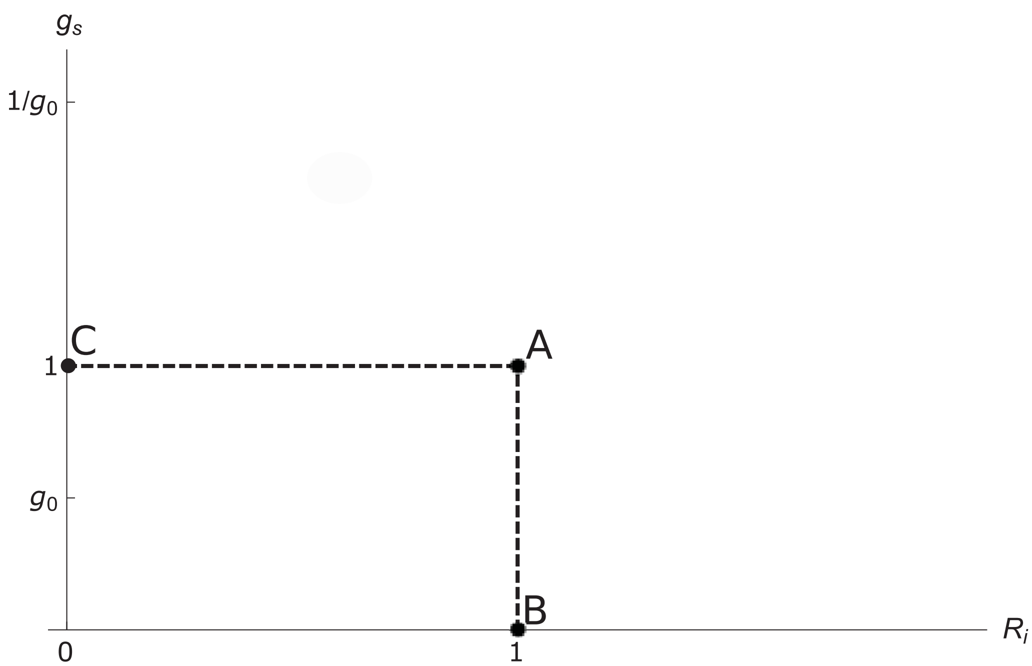

Figure 4.1 The Type IIB moduli space with the self-dual point denoted by A. The point B is for all $R_i = 1$ and the point C is for $g_s = 1$. Our duality mappings are defined for the point A. Going away from the point A in any direction in the moduli space will imply switching on non-trivial values for the axio-dilaton.

4.4.1 Towards all-order θ expansion from dualities

Let us now proceed with our analysis. We start by redefining the all order fermionic completion of type IIB scalar fields in the following way:

$$\begin{aligned}\mathbf{\Phi}^{(i)} &= \varphi^{(i)} + \bar{\theta}\Delta^{(i)}\theta \\ &\equiv \varphi^{(i)} + \sum_j \prod_{k=1}^{j} \bar{\theta}\Delta^{(i)jk}\theta \\ &= \varphi^{(i)} + \bar{\theta}\Delta^{(11i)}\theta + \bar{\theta}\Delta^{(21i)}_{m..p}\theta\bar{\theta}\Delta^{(22i)}_{q..n}\theta g^{pq}..g^{mn} + \bar{\theta}\Delta^{(31i)}_{m..p}\theta\bar{\theta}\Delta^{(32i)}_{q..l}\theta\bar{\theta}\Delta^{(33i)}_{s..n}\theta g^{pq}..g^{ls}..g^{mn} + ...\end{aligned} \tag{4.98}$$

where $\mathbf{\Phi}^{(1)} = \phi_B$ and $\mathbf{\Phi}^{(2)} = C^{(0)}$ are the dilaton and the axion respectively; and the dotted terms are of $\mathcal{O}(\theta^8)$. The fermion products in (4.98) are defined in terms of components in the following way:

$$\bar{\theta}\Delta^{(21i)}_{m..p}\theta\bar{\theta}\Delta^{(22i)}_{q..n}\theta \equiv \bar{\theta}^{\alpha}\Delta^{(21i)}_{m..p\alpha\beta}\theta^{\beta}\bar{\theta}^{\delta}\Delta^{(22i)}_{q..n\delta\gamma}\theta^{\gamma}, \tag{4.99}$$

where the Greek indices span the 32 (complex) component[24] fermions θ. The IIB spinor θ is a doublet of 16 (complex) component Majorana spinors of the same chirality, i.e this doublet is a 32 component Majorana spinor but is *not* Weyl. We decompose θ into the two 16 (complex) component fermions θ_1 and θ_2 as:

$$\theta = \begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix}, \tag{4.100}$$

with θ_2 generically non-vanishing. The $\Delta^{(abi)}$ are all operators that can be represented in the matrix form in the following way:

$$\Delta^{(i)} \equiv \begin{pmatrix} \Delta^{(11i)} & \Delta^{(12i)} & \Delta^{(13i)} & .... \\ \Delta^{(21i)} & \Delta^{(22i)} & \Delta^{(23i)} & .... \\ \Delta^{(31i)} & \Delta^{(32i)} & \Delta^{(33i)} & .... \\ .... & & & \end{pmatrix}, \tag{4.101}$$

[24]Or 64 real component. Note that the series in (4.98) and in the following, terminate at some finite number of terms because of finite number of fermionic components as well as because of the Grassmannian nature of the fermions.

where every element of the matrix should be viewed as an operator with its own matrix representation in some appropriate Hilbert space. The complete form of the matrix (4.101) is not known, but a few elements have been worked out in the literature [38–40, 44]. For example it is known that:

$$\Delta^{(111)}\theta = -\frac{i}{2}\bar{\delta}\lambda\ \theta, \qquad \Delta^{(112)}\theta = \frac{1}{2}e^{-\phi}\sigma_2\bar{\delta}\lambda\ \theta, \tag{4.102}$$

where $\bar{\delta}\lambda$ is the supersymmetric variation of the type IIB spinor λ in the presence of an $\overline{\rm D3}$ and σ_2 is the second Pauli matrix that act on the $\theta_{1,2}$ components of (4.100). It should also be clear, from the way we constructed the matrix, that:

$$\Delta^{(abi)} = \Delta^{(bai)}. \tag{4.103}$$

Additionally, in the ensuing analysis we will resort to the following simplification: instead of considering the $\Delta^{(abi)}$ operators to have an arbitrary rank q as $\Delta^{(abi)}_{m_1m_2....m_q}$ for $a \geq 2$, we will only take them to have a maximal rank 2. As will be clear from the context, this simplification will not change any of the physics, and one may easily switch to arbitrary rank $\Delta^{(abi)}$ operators without loss of generalities. On the other hand, this simplification will avoid unnecessary cluttering of indices. Henceforth unless mentioned otherwise, we will take only this simplified version.

With this in mind, let us now consider the type IIB metric g_{mn}. We can expand the all order fermionic completion in a way analogous to the scalar field:

$$\begin{aligned} \mathbf{G}_{mn} &= g_{mn} + \bar{\theta}M_{(mn)}\theta \\ &= g_{mn} + \bar{\theta}M^{(11)}_{(mn)}\theta + g^{pq}\bar{\theta}M^{(21)}_{(m|p}\theta\bar{\theta}M^{(22)}_{q|n)}\theta + g^{pq}g^{ls}\bar{\theta}M^{(31)}_{(m|p}\theta\bar{\theta}M^{(32)}_{ql}\theta\bar{\theta}M^{(33)}_{s|n)}\theta + \mathcal{O}(\theta^8), \end{aligned} \tag{4.104}$$

which is again a sum over products of contractions of the fermions with matrix elements of the operator M_{mn}. The four-component M operator can be written using two bosonic and two fermionic components as:

$$M_{(mn)\alpha\beta} = M^{(11)}_{(mn)\alpha\beta} + M^{(21)p}_{(m|\alpha\gamma}\theta^\gamma\bar{\theta}^\delta M^{(22)}_{p|n)\delta\beta} + M^{(31)p}_{(m|\alpha\gamma}\theta^\gamma\bar{\theta}^\delta M^{(32)}_{ps\delta\sigma}\theta^\sigma\bar{\theta}^\rho M^{(33)s}_{n)\rho\beta} + ... \tag{4.105}$$

where the first term in the above expansion is well-known in terms of the supersymmetric variation of Rarita-Schwinger fermion ψ_m [38–40, 44]:

$$M^{(11)}_{\alpha\beta(mn)} = -i\Gamma_{\alpha\beta(m}D_{n)} \equiv -i\Gamma_{\alpha\beta(m}\bar{\delta}\psi_{n)}. \tag{4.106}$$

The anti-symmetric rank two tensor can also be expanded in terms of the fermionic components like the symmetric tensor (4.104). We can define $\mathbf{B}^{(i)}_{mn}$ as the generalized anti-symmetric tensors, where $B^{(1)}_{mn} = B_{mn}$ and $B^{(2)}_{mn} = C^{(2)}_{mn}$ as the NS and RR two-forms respectively, using certain anti-symmetric tensor $N^{(i)}_{[mn]}$ in the following way:

$$\begin{aligned}\mathbf{B}^{(i)}_{mn} &= B^{(i)}_{mn} + \bar{\theta}N^{(i)}_{[mn]}\theta \\ &= B^{(i)}_{mn} + \bar{\theta}N^{(11i)}_{[mn]}\theta + g^{pq}\bar{\theta}N^{(21i)}_{[m|p}\theta\bar{\theta}N^{(22i)}_{q|n]}\theta + g^{pq}g^{ls}\bar{\theta}N^{(31i)}_{[m|p}\theta\bar{\theta}N^{(32i)}_{ql}\theta\bar{\theta}N^{(33i)}_{s|n]}\theta + \mathcal{O}(\theta^8).\end{aligned} \tag{4.107}$$

To see the connection between $M_{(mn)}$ and $N^{(i)}_{[mn]}$ operators let us revisit the T-duality rules of [45, 46]. The powerful thing about the fermionic completion is that the T-duality rules follow *exactly* the formula laid out for the bosonic fields, except now all the fields are replaced by their fermionic completions. This can be illustrated as[25]:

$$\begin{aligned}
&\widetilde{\Phi}^{(1)} = \Phi^{(1)} - \frac{1}{2}\ln \mathbf{G}_{xx} && \widetilde{\mathbf{G}}_{xx} = \frac{1}{\mathbf{G}_{xx}} \\
&\widetilde{\mathbf{G}}_{mn} = \mathbf{G}_{mn} - \frac{\mathbf{G}_{mx}\mathbf{G}_{nx} - \mathbf{B}^{(1)}_{mx}\mathbf{B}^{(1)}_{nx}}{\mathbf{G}_{xx}} && \widetilde{\mathbf{G}}_{mx} = \frac{\mathbf{B}^{(1)}_{mx}}{\mathbf{G}_{xx}} \\
&\widetilde{\mathbf{B}}^{(1)}_{mn} = \mathbf{B}^{(1)}_{mn} - \frac{\mathbf{B}^{(1)}_{mx}\mathbf{G}_{nx} - \mathbf{G}_{mx}\mathbf{B}^{(1)}_{nx}}{\mathbf{G}_{xx}} && \widetilde{\mathbf{B}}^{(1)}_{mx} = \frac{\mathbf{G}_{mx}}{\mathbf{G}_{xx}}
\end{aligned} \tag{4.108}$$

where x is the T-duality direction. From the T-duality rule we see that, in the presence of cross-terms of $\mathbf{G}$ in type IIA, $\mathbf{B}^{(1)}$ could be generated in type IIB using (4.108). Since

[25] There seems to be two ways of analyzing the T-duality transformations in the literature. One, is to assume that the Buscher's rules are *exact* to all orders in α' and only the supergravity fields receive α' corrections. This way, the Busher's rule could be used to study supergravity field transformations order by order in α'. Two, both the T-duality transformations *and* the supergravity fields receive α' corrections. There is some confusion of which one should be considered, but in our opinion the more conservative picture is the latter one where *both*, the T-duality rules as well as the supergravity fields, receive α' corrections. Since T-duality transformations preserve supersymmetry, the α' corrections to the T-duality transformations would imply α' corrections to the supersymmetry transformations: a result consistent with the known facts. See for example [47, 48] for the lowest order corrections, where somewhat similar arguments have appeared; and [49] for more recent discussions. However as we will see soon, our results will not be very sensitive to this.

both IIA or IIB metric uses $M_{(mn)}$, this is possible if:

$$\bar{\theta} N^{(1)}_{[mx]} \theta \ \equiv \bar{\theta} \ c \sigma_3^p M_{[mx]} \theta, \tag{4.109}$$

where the operator $M_{[mx]}$ is now expressed with respect to the T-dual fields, i.e the IIB bosonic fields. We have also inserted the third Pauli matrix σ_3 in (4.109), with $p = 1$ or 2, to take care of certain subtleties that will be explained later[26], and c is a constant matrix. The only constant matrices for our case, that do not change the chirality, are the identity and the chirality matrix Γ^{10}, so we will choose $c = \Gamma^{10}$. Since we can make T-duality along any direction, x appearing in (4.109) could span all directions. This means we can generalize (4.109) to the following:

$$\bar{\theta} N^{(1)}_{[mn]} \theta \ \equiv \bar{\theta} \ \sigma_3^p \otimes \Gamma^{10} M_{[mn]} \theta, \tag{4.110}$$

implying that the symmetric matrix $M_{(mn)}$ determines the generalized metric $\mathbf{G}_{mn}$, whereas the anti-symmetric matrix $M_{[mn]}$ determines the generalized B-field $\mathbf{B}^{(1)}_{mn}$. In terms of components, we expect:

$$N^{(111)}_{\alpha\beta[mn]} \quad = \quad -i\sigma_3 \otimes \Gamma^{10} \Gamma_{\alpha\beta[m} \delta\psi_{n]}, \tag{4.111}$$

which is consistent with the results in [38–40, 44]. However the relation (4.110) predicts the form of *all* the operators appearing in (4.107) once all the corresponding operators appearing in (4.104) are known, not just the component given above.

To find the form of $\mathbf{B}^{(2)}_{mn}$, or the operator $N^{(2)}_{[nm]}$, we will use the T-duality trick discussed above, assuming that the T-duality rules go for the RR fields with fermionic completions exactly as their bosonic counterparts [38, 39]. To proceed we will need $\mathbf{\Phi}^{(2)}$ and $\mathbf{B}^{(1)}_{mn}$ from (4.98) and (4.107), rewritten as:

$$\mathbf{\Phi}^{(2)} = C^{(0)} + \bar{\theta}\sigma_2 \widetilde{\Delta}^{(2)} \theta, \qquad \mathbf{B}^{(1)}_{mn} = B_{mn} + \bar{\theta}\sigma_3 \otimes \Gamma^{10} M_{[mn]} \theta, \tag{4.112}$$

where we have extracted a Pauli matrix σ_2 in defining $\Delta^{(2)} = \sigma_2 \widetilde{\Delta}^{(2)}$. The other components appearing in (4.112) are the corresponding bosonic backgrounds. The T-duality rules for

[26] See discussions after (4.126).

the RR fields are given as[27]:

$$
\begin{aligned}
\widetilde{\mathbf{C}}^{(n)}_{xm_2\cdots m_n} &= \mathbf{C}^{(n-1)}_{m_2\cdots m_n} - (n-1)\widetilde{\mathbf{B}}^{(1)}_{x[m_2}\mathbf{C}^{(n-1)}_{|x|m_3\cdots m_n]},\\
\widetilde{\mathbf{C}}^{(n)}_{m_1\cdots m_n} &= \mathbf{C}^{(n+1)}_{xm_1\cdots m_n} - n\mathbf{B}^{(1)}_{x[m_1}\widetilde{\mathbf{C}}^{(n)}_{|x|m_2\cdots m_n]}.
\end{aligned} \tag{4.113}
$$

There are now two possible ways to get the fermionic part of $\mathbf{B}^{(2)}_{mn}$: we can T-dualize twice the scalar $\mathbf{\Phi}^{(2)}$ using the T-duality rule (4.113), and we can S-dualize $\mathbf{B}^{(1)}_{mn}$. Let us start by discussing the first possibility, namely the T-duality way of getting part of $\mathbf{B}^{(2)}_{mn}$. T-dualizing once we get a vector field in type IIA as:

$$
\widetilde{\mathbf{C}}^{(1)}_{x} = \mathbf{\Phi}^{(2)}, \tag{4.114}
$$

and then another T-duality will give us the required RR two-form field in the following way:

$$
\hat{\mathbf{C}}^{(2)}_{yx} = \widetilde{\mathbf{C}}^{(1)}_{x} - \hat{\mathbf{B}}^{(1)}_{yx}\widetilde{\mathbf{C}}^{(1)}_{y} = \mathbf{\Phi}^{(2)} - \hat{\mathbf{B}}^{(1)}_{yx}\widetilde{\mathbf{C}}^{(1)}_{y} = \mathbf{\Phi}^{(2)} \tag{4.115}
$$

because $\widetilde{\mathbf{C}}^{(1)}_{y} = 0$ according to (4.114), and therefore the field $\mathbf{\Phi}^{(2)}$ should determine the required two-form. However before proceeding we should determine how the 32 component Majorana fermion (4.100) change under the two T-dualities. It is easy to show that:

$$
\theta \ \to \ \Sigma_1\,\theta, \qquad \bar{\theta} \ \to \ \bar{\theta}\,\Sigma_2, \tag{4.116}
$$

where Σ_i are two 32×32 component matrices, i.e. the act on the doublet basis, given in terms of the sixteen component Gamma matrices[28] Γ_x and Γ_y by:

$$
\Sigma_1 \ = \ \begin{pmatrix} \mathbf{I}_{16} & 0 \\ 0 & \Gamma_x\Gamma_y \end{pmatrix}, \qquad \Sigma_2 \ = \ \begin{pmatrix} \mathbf{I}_{16} & 0 \\ 0 & \Gamma_y\Gamma_x \end{pmatrix}, \tag{4.117}
$$

and leading to the following set of algebras that will be useful soon:

$$
\Sigma_2(\sigma_2 \otimes \mathbf{I}_{16})\Sigma_1 \ = \ \sigma_3\sigma_2 \otimes \Gamma_x\Gamma_y, \qquad \Sigma_2 \cdot \Sigma_1 \ = \ \mathbf{I}_{32}
$$

[27] As before, we expect the T-duality rules for the RR fields to also receive α' corrections. We will discuss the consequence of this on our analysis soon.

[28] We are using the flat-space Γ matrices.

$$\Sigma_2 \begin{pmatrix} 0 & \mp i\mathbf{C} \\ \pm i\mathbf{C} & 0 \end{pmatrix} \Sigma_1 = \begin{pmatrix} \pm\mathbf{C} & 0 \\ 0 & \pm\mathbf{C} \end{pmatrix} (\sigma_3\sigma_2 \otimes \Gamma_x\Gamma_y)$$
$$\Sigma_2 \begin{pmatrix} \pm\mathbf{C} & 0 \\ 0 & \pm\mathbf{C} \end{pmatrix} \Sigma_1 = \begin{pmatrix} \pm\mathbf{C} & 0 \\ 0 & \pm\mathbf{C} \end{pmatrix}, \qquad (\sigma_3\sigma_2)^2 = -\mathbf{I}_2$$
$$\begin{pmatrix} \mathbf{1} & 0 \\ 0 & \Gamma_b\Gamma_a \end{pmatrix} (\sigma_3\sigma_2 \otimes \Gamma_x\Gamma_y) \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \Gamma_a\Gamma_b \end{pmatrix} = \sigma_2 \otimes \Gamma_x\Gamma_y\Gamma_a\Gamma_b. \tag{4.118}$$

Therefore using (4.115) and (4.116) with the algebras (4.118), we can get one part of the two-form $\mathbf{B}^{(2)}_{mn}$ in the following way:

$$\hat{\mathbf{C}}^{(2)}_{mn} = \bar{\theta} c \sigma_3\sigma_2 \otimes \Gamma_{mn}\widetilde{\Delta}^{(2)}\theta, \tag{4.119}$$

where, as before, we can take $c = \Gamma^{10}$ i.e the chirality matrix, and $\widetilde{\Delta}^{(2)}$ can either be expressed in terms of the T-dual fields or the original fields.

In deriving (4.119) we haven't actually looked at the form of $\widetilde{\Delta}^{(2)}$. Depending on the representation of Gamma matrices in the definition of $\widetilde{\Delta}^{(2)}$, our simple expression (4.119) could in principle change to a more involved one. The scenario is subtle so let us tread carefully here. We start by rewriting the RR scalar (4.98) field as:

$$\begin{aligned}
\mathbf{\Phi}^{(2)} &= C^{(0)} + \left(\bar{\theta}\sigma_2\right)^\alpha \widetilde{\Delta}^{(2)}_{\alpha\beta}\theta^\beta \\
&= C^{(0)} + (\bar{\theta}\sigma_2)^\alpha \widetilde{\Delta}^{(112)}_{\alpha\beta}\theta^\beta + (\bar{\theta}\sigma_2)^\alpha \widetilde{\Delta}^{(212)p}_{\alpha\chi m}\theta^\chi\bar{\theta}^\sigma\widetilde{\Delta}^{(222)m}_{\sigma\beta p}\theta^\beta \\
&\quad + (\bar{\theta}\sigma_2)^\alpha \widetilde{\Delta}^{(312)p}_{\alpha\gamma m}\theta^\gamma\bar{\theta}^\sigma\widetilde{\Delta}^{(322)l}_{\sigma\chi p}\theta^\chi\bar{\theta}^\delta\widetilde{\Delta}^{(332)m}_{\delta\beta l}\theta^\beta + \mathcal{O}(\theta^8),
\end{aligned} \tag{4.120}$$

where we have assumed that the generic operator $\widetilde{\Delta}^{(ab2)m}_{\alpha\beta n}$ is constructed from the products of 16 dimensional Gamma matrices, the type IIB bosonic fields and covariant derivatives $\mathbf{A}_{16\times 16}$ as:

$$\left(\bar{\theta}\sigma_2^p\right)_\alpha \widetilde{\Delta}^{(ab2)}_{mn\alpha\beta}\theta_\beta \equiv \bar{\theta}\sigma_2^p \begin{pmatrix} \mathbf{A}^{(ab)}_{16\times 16} & 0 \\ 0 & \mathbf{A}^{(ab)}_{16\times 16} \end{pmatrix}_{mn} \theta, \tag{4.121}$$

where p can be 0 or 1 depending on what fermion combination we are looking at in (4.120). Using our T-duality ideas, and using the Gamma matrix algebras (4.118), it is easy to see that the two-form (4.119) appears naturally with an overall $\Gamma_m\Gamma_n$ matrix provided we impose:

$$[\mathbf{A},\ \Gamma_x\Gamma_y] = 0, \tag{4.122}$$

without loss of generalities as transformations with even number of Gamma matrices will not change any results. The puzzle however is if (4.121) takes the following form:

$$\begin{aligned}\left(\bar\theta\sigma_2^p\right)^\alpha \widetilde{\Delta}^{(ab2)}_{mn\alpha\beta}\theta^\beta \quad & \equiv \bar\theta(\sigma_2^p\otimes\mathbf{I}_{16})\left(\mathbf{I}_2\otimes\mathbf{A}^{(ab)}+\sigma_1\otimes\mathbf{C}^{(ab)}\right)_{mn}\theta \\ & = \bar\theta(\sigma_2^p\otimes\mathbf{I}_{16})\begin{pmatrix}\mathbf{A}^{(ab)}_{16\times16} & \mathbf{C}^{(ab)}_{16\times16}\\ \mathbf{C}^{(ab)}_{16\times16} & \mathbf{A}^{(ab)}_{16\times16}\end{pmatrix}_{mn}\theta,\end{aligned} \tag{4.123}$$

where $(\sigma_1, \mathbf{I}_2)$ are the first Pauli matrix and 2 dimensional identity matrix respectively; and $\mathbf{C}_{16\times16}$ is another 16 dimensional matrix constructed out of Gamma matrices, IIB fields and covariant derivatives.

To understand the consequence of the above mentioned representations of the operators, let us discuss a few additional Gamma matrix algebras under our T-duality transformations:

$$\begin{aligned}\Sigma_2(\sigma_2\sigma_1\otimes\mathbf{C})\Sigma_1 \ & = -i\sigma_3\otimes\mathbf{C}\\ \Sigma_2(\sigma_1\otimes\mathbf{C})\Sigma_1 \ & = i(\sigma_2\otimes\mathbf{C}\Gamma_x\Gamma_y)\\ \Sigma_2(\sigma_2\otimes\mathbf{I}_{16})(\mathbf{I}_2\otimes\mathbf{A})\Sigma_1 \ & = \ \sigma_3\sigma_2\otimes\mathbf{A}\Gamma_x\Gamma_y.\end{aligned} \tag{4.124}$$

Using these algebras, it is now easy to see that under T-dualities the operators (4.121) and (4.123) transform in the following way:

$$\begin{aligned}\bar\theta\left(\mathbf{I}_2\otimes\mathbf{A}+\sigma_1\otimes\mathbf{C}\right)\theta \ & \to \ \bar\theta\left(\mathbf{I}_2\otimes\mathbf{A}+i\sigma_2\otimes\mathbf{C}\Gamma_x\Gamma_y\right)\theta\\ \bar\theta\sigma_2\left(\mathbf{I}_2\otimes\mathbf{A}+\sigma_1\otimes\mathbf{C}\right)\theta \ & \to \ \bar\theta\left(\sigma_3\sigma_2\otimes\mathbf{A}\Gamma_x\Gamma_y-i\sigma_3\otimes\mathbf{C}\right)\theta,\end{aligned} \tag{4.125}$$

from where we see that the first terms in (4.125) are clearly consistent with the duality rules that lead us to the result (4.119). However it is the second term in the two expressions above in (4.125) which would *not* fit with the generic result (4.119). Clearly when $\mathbf{C} = 0$ this problem does not arise.

A way out of this conundrum is in fact clear from the transformations themselves. Existence of $\mathbf{C}_{16\times16}$ in (4.123) would imply that this piece is T-duality neutral, and *doesn't* transform as a rank 2 tensor under T-duality. Thus this piece cannot be part of a RR axionic scalar whose T-duality transformations are well known. In fact its neutrality to the T-duality transformation hints that $\mathbf{C}_{16\times 16}$ could be a part of the NS scalar i.e the dilaton, unless of course we can use $\bar{\psi} \equiv \bar{\theta}\sigma_1$ to transform

$$\bar{\psi}\mathbf{C} \otimes \sigma_1\theta \ \rightarrow \ \bar{\theta}\mathbf{C}\theta, \tag{4.126}$$

under two T-dualities. This way the issues raised in (4.125) will not arise and the generic result (4.119) will continue to hold to arbitrary orders in θ expansion.

Let us now come to the second possibility of getting the fermionic part of $\mathbf{B}^{(2)}_{mn}$ namely, S-dualizing $\mathbf{B}^{(1)}_{mn}$ i.e the NS part of the two-form (with its fermionic completion). In light of our earlier discussion, this would be like moving up the type IIB coupling, at fixed self-dual radii of the compact spaces, so as to reach $g_s \to 1^-$ point. In other words, we are moving from region B to region A in **figure 4.1**.

We will however start by first fulfilling the promise that we made earlier, namely discuss the appearance of σ_3, the third Pauli matrix, in (4.109) for the NS B-field $\mathbf{B}^{(1)}_{mn}$. Recall that our argument was to motivate the result from T-dualizing the metric component with cross-terms from type IIA to type IIB theory. Under T-duality the 32 component type IIA chiral fermion θ_A transforms as:

$$\begin{gathered}\theta_A = \begin{pmatrix}\theta_+ \\ \theta_-\end{pmatrix} \ \rightarrow \ \begin{pmatrix}1 & 0 \\ 0 & -\Gamma^{10}\Gamma_x\end{pmatrix}\begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix} \equiv \widetilde{\Sigma}_1\theta \\ \bar{\theta}_A = \begin{pmatrix}\bar{\theta}_+ & \bar{\theta}_-\end{pmatrix} \ \rightarrow \ \begin{pmatrix}\bar{\theta}_1 & \bar{\theta}_2\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & \Gamma^{10}\Gamma_x\end{pmatrix} \equiv \bar{\theta}\widetilde{\Sigma}_2,\end{gathered} \tag{4.127}$$

where the T-duality is performed along direction x to go from IIA to IIB. The above transformations immediately implies the following algebra, similar to the algebras that we discussed earlier in (4.118):

$$\widetilde{\Sigma}_2 \otimes \begin{pmatrix}\mathbf{C}_{16\times 16} & 0 \\ 0 & \mathbf{C}_{16\times 16}\end{pmatrix} \otimes \widetilde{\Sigma}_1 \ = \ \begin{pmatrix}\mathbf{C}_{16\times 16} & 0 \\ 0 & -\Gamma^{10}\Gamma_x\mathbf{C}_{16\times 16}\Gamma^{10}\Gamma_x\end{pmatrix} = \sigma_3^p \otimes \mathbf{C}_{16\times 16}, \tag{4.128}$$

where σ_3 is the third Pauli matrix with $p = 1$ or 2 depending on the specific representation of the 16-dimensional $\mathbf{C}$ matrix. To fix the value of p, we can go to our self-dual point such that the transformation (4.127) becomes an intermediate transformation at $R_x = R_\perp = 1$, where $R_\perp$ is the radius of an orthogonal circle. We can choose $\mathbf{C}$ matrix to be of the form: $\mathbf{C}_{xm} \equiv \Gamma_x \mathcal{O}_m$, with $\mathcal{O}_m$ being a combination of type IIB fields and covariant derivatives with *even* or *odd* number of Gamma matrices. In that case $p = 1$ in (4.128). Even when the intermediate matrix, in the θ expansion, is of the form $\mathbf{C} + \sigma_1 \otimes \widetilde{\mathbf{C}}$, result of the form (4.128) will continue to hold because we can absorb σ_1 in the transformation matrices as in (4.126). Therefore, combining the results together, and assuming $p = 1$, we can express the fermionic part of the NS B-field $\mathbf{B}^{(1f)}_{mn}$ as:

$$\mathbf{B}^{(1f)}_{mn} = \bar{\theta}\sigma_3 \otimes \Gamma^{10} M_{[mn]}\theta. \tag{4.129}$$

As discussed earlier, we can now go to a corner of type IIB moduli space where the string coupling is strong i.e $g_s \to 1$. Here we expect the RR B-field $\mathbf{B}^{(2)}_{mn}$ to be given at least by the S-dual of $\mathbf{B}^{(1)}_{mn}$. The S-duality matrix that concerns us here is:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{4.130}$$

which squares to $-\mathbf{I}_2$. This is the perturbative piece of the duality that keeps the string coupling unchanged, but changes the signs of the two-form fields. To incorporate S-duality in our fermionic part of the NS B-field $\mathbf{B}^{(1)}_{mn}$ one needs only to insert $-i\sigma_2$ in (4.129) to give us the following fermionic piece[29]:

$$\hat{\mathbf{D}}^{(2)}_{mn} = -i\bar{\theta}\sigma_3\sigma_2 \otimes \Gamma^{10} M_{[mn]}\theta, \tag{4.131}$$

such that S-dualizing twice will yield $(-i\sigma_2)^2 = -\mathbf{I}_2$. This way we will get back the same result as (4.130) after two S-dualities that allow for a $\mathbf{Z}_2$ phase factor. Combining (4.119) and (4.131) together we get our final expression for the RR two-form field along with its fermionic completion as:

$$\mathbf{B}^{(2)}_{mn} = C^{(2)}_{mn} - i\bar{\theta}\sigma_3\sigma_2 \otimes \Gamma^{10}\left(M_{[mn]} + i\Gamma_{mn}\widetilde{\Delta}^{(2)}\right)\theta. \tag{4.132}$$

[29] The sign is chosen for later convenience.

From the above expression we expect the fermionic terms to be suppressed by powers of string coupling *away* from the self-dual points, so that at the self-dual point (the region A in **figure 4.1**) we can exchange $\mathbf{B}^{(2)}$ and $\mathbf{B}^{(1)}$ and simultaneously perform two T-dualities. To see whether this is indeed true, we need to expand (4.132) to higher orders in θ. This can be easily worked out using earlier expressions for $M_{[mn]}$ and $\widetilde{\Delta}$ in (4.105) and (4.120) respectively, and the result is given by:

$$\begin{aligned}\mathbf{B}^{(2)}_{mn} &= C^{(2)}_{mn} - i\bar{\theta}e^{-\phi}\sigma_3\sigma_2 \otimes \Gamma^{10}\left(M^{(11)}_{[mn]} + i\Gamma_{mn}\widetilde{\Delta}^{(112)}\right)\theta \\ &- i\bar{\theta}e^{-\phi}\sigma_3\sigma_2 \otimes \Gamma^{10}\left(M^{(21)}_{[m|p}\theta\bar{\theta}M^{(22)}_{q|n]}g^{pq} + i\Gamma_{mn}\widetilde{\Delta}^{(212)}_{rp}\theta\bar{\theta}\widetilde{\Delta}^{(222)}_{qs}g^{pq}g^{rs}\right)\theta + \mathcal{O}(\theta^8),\end{aligned} \tag{4.133}$$

where to $\mathcal{O}(\theta^2)$ the coefficients can be read off from (4.106) and (4.111) as (see also [38,39] for more details):

$$M^{(11)}_{[mn]} = -\Gamma_{[m}\bar{\delta}\psi_{n]}, \qquad \widetilde{\Delta}^{(112)} = \frac{1}{2}\bar{\delta}\lambda. \tag{4.134}$$

We can see that the string coupling appears correctly in (4.133) as to allow for the right behavior of the form-fields in the full IIB moduli space. The fermion variations $(\bar{\delta}\psi_m, \bar{\delta}\lambda)$ are with respect to either the original type IIB variables or the T-dual type IIB variables in our transformation scheme. Note that once we know the functional form of $\hat{\Delta}^{(ab2)}_{mn}$ for generic values of (a, b), we will know the θ expansion of (4.133) to arbitrary orders. This is of course a challenging exercise which we will not perform here. Instead we will use our results for $\mathbf{B}^{(1)}_{mn}$ and $\mathbf{B}^{(2)}_{mn}$ etc to determine the fermionic structure of the four-form $\mathbf{C}_{mnpq}$ around the self-dual point.

The fermionic structure of the four-form can be determined using similar trick as before by scanning the IIB moduli space. There are two differents points in the moduli space that would give us the four-form. First, at weak string coupling, we can go to the small compactification radii (or more appropriately the self-dual radii) where the four-form can get contributions from the T-dual of $\mathbf{B}^{(2)}_{mn}$. Secondly, at strong string coupling i.e $g_s \to 1$, we can again go to self-dual radii where the four-form can now get contributions from the U-dual of $\mathbf{B}^{(1)}_{mn}$. For the first case, we can T-dualize twice the RR field $\mathbf{B}^{(2)}_{mn}$ along directions (a, b); and for the second case, we can S-dualize the $\mathbf{B}^{(1)}_{mn}$ field and then T-dualize twice along directions (a, b). The Gamma matrix algebra useful for us are now the following:

$$\begin{pmatrix} 1 & 0 \\ 0 & \Gamma_b\Gamma_a \end{pmatrix} (\sigma_3\sigma_2 \otimes \Gamma^{10}) \begin{pmatrix} 1 & 0 \\ 0 & \Gamma_a\Gamma_b \end{pmatrix} = \sigma_2 \otimes \Gamma^{10}\Gamma_a\Gamma_b \tag{4.135}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & \Gamma_b\Gamma_a \end{pmatrix} (\sigma_3\sigma_2 \otimes \Gamma^{10}\Gamma_x\Gamma_y) \begin{pmatrix} 1 & 0 \\ 0 & \Gamma_a\Gamma_b \end{pmatrix} = \sigma_2 \otimes \Gamma^{10}\Gamma_a\Gamma_b\Gamma_x\Gamma_y.$$

Using these algebras, which are basically the T-duality rules, for both strong and weak string couplings will immediately provide us the contributions to the four-form from the two sources mentioned above around $g_s = R_a = R_b = 1$. The result is:

$$\begin{aligned} \mathbf{C}_{mnpq} &= C^{(4)}_{mnpq} - i\bar{\theta}\sigma_2 \otimes \Gamma^{10} \left(2\Gamma_{[mn}M_{pq]} + i\Gamma_{mnpq}\widetilde{\Delta}^{(2)}\right)\theta \\ &= C^{(4)}_{mnpq} - i\bar{\theta}\sigma_2 \otimes \Gamma^{10} \left(2\Gamma_{[mn}M^{(11)}_{pq]} + i\Gamma_{mnpq}\widetilde{\Delta}^{(112)}\right)\theta + \mathcal{O}(\theta^4) \\ &= C^{(4)}_{mnpq} - \frac{1}{2}\bar{\theta}\sigma_2 \otimes \Gamma^{10} \left(4\Gamma_{[mnp}\bar{\delta}\psi_{q]} - \Gamma_{mnpq}\bar{\delta}\lambda\right)\theta + \mathcal{O}(\theta^4), \end{aligned} \tag{4.136}$$

where the factor of 2 signifies the contributions from the U-dual of the two B-fields, and we have determined the results upto $\mathcal{O}(\theta^2)$. One may verify with [3, 38, 39, 44] that the result quoted above matches well with the literature at the self-dual point. It is interesting that to this order the match is exact, therefore other possible corners of the type IIB moduli space do not contribute anything else to the fermionic parts of the bososnic RR and NS fields. At higher orders in θ there could be contributions that we cannot determine using out U-duality trick. Nevertheless, the U-duality transformations are powerful enough to extract out the fermionic contributions from various corners of the moduli space.

So far however we have not discussed the connection between $\Delta^{(1)}$ appearing in the dilaton and $\widetilde{\Delta}^{(2)}$ appearing in the axion, as in (4.144). The fact that they are related can be seen from M-theory on a torus $\mathbf{T}^2$ in the limit when the torus size is shrunk to zero. Of course the scenario that we have envisioned here at the self-dual point cannot be uplifted to M-theory because we are not allowed to shrink the M-theory torus to zero size (as $g_s = 1$). However away from the self-dual point we *can* lift our configuration to M-theory, so let us discuss this point briefly. In M-theory we expect the metric to take a form similar to (4.104) or (4.144), i.e:

$$\hat{\mathbf{G}}_{mn} = G^{(11)}_{mn} + \bar{\theta}\hat{M}_{mn}\theta, \tag{4.137}$$

where the superscript denotes the bosonic part of the metric, and θ is the corresponding fermionic variable. If we parametrize the torus direction by (x^3, x^a) where x^a denotes the eleventh-direction, then it is easy to see that in the limit of vanishing size of the torus, the type IIB axio-dilaton, with their fermionic completions, are related via:

$$\exp\left[-2\mathbf{\Phi}^{(1)}\right] + \left[\mathbf{\Phi}^{(2)}\right]^2 = \frac{\hat{\mathbf{G}}_{33}}{\hat{\mathbf{G}}_{aa}}, \tag{4.138}$$

implying the connection between $\Delta^{(1)}$ and $\widetilde{\Delta}^{(2)}$ *away from the self-dual point*. Using this one should be able to derive the $\mathcal{O}(\theta^2)$ result similar to (4.145) but away from the self-dual point, as also given in [38, 39].

What happens at the self-dual point? The self-dual point is defined for $C^{(0)} = \phi = 0$, and therefore we should at least assume that this continues to be the case for the fermionic completions of the dilaton and axion too. In other words we should expect:

$$\tau \equiv \mathbf{\Phi}^{(2)} + ie^{-\mathbf{\Phi}^{(1)}} = i \quad \text{(at the self-dual point)}, \tag{4.139}$$

to all orders in $(\theta, \bar{\theta})$. Interestingly the condition $|\tau|^2 = 1$ is similar to the M-theory condition (4.138) in the limit $\hat{\mathbf{G}}_{33} = \hat{\mathbf{G}}_{aa}$. To lowest order in $\theta, \bar{\theta}$ it is easy to see that (4.139) reduces to the following condition:

$$\bar{\theta}^\alpha \Delta^{(111)}_{\alpha\beta} \theta^\beta = -i\bar{\theta}^\alpha \left(\sigma_2\right)^\gamma_\alpha \widetilde{\Delta}^{(112)}_{\gamma\beta} \theta^\beta \quad \text{(at the self-dual point)}. \tag{4.140}$$

In general, to all orders in $(\theta, \bar{\theta})$, the relation between $\Delta^{(1)}$ and $\widetilde{\Delta}^{(2)}$ at the self-dual point can be directly seen from (4.139) as:

$$\bar{\theta}\Delta^{(1)}\theta = -\log\left(1 + i\bar{\theta}\sigma_2\widetilde{\Delta}^{(2)}\theta\right) \quad \text{(at the self-dual point)}. \tag{4.141}$$

We expect (4.140) and (4.141) to reproduce the condition (4.102) or (4.145) discussed in [38–40, 44] at the self-dual point also. To this effect we will start by defining:

$$\Delta^{(1)} = -i\widetilde{\Delta}^{(2)} + \hat{\Delta}, \tag{4.142}$$

generically, both *at* and *away* from the self-dual point. Plugging (4.142) in (4.141), and taking into account the lowest order results in [38–40, 44], we expect $\hat{\Delta}$ to vanish to lowest order in $(\bar{\theta}, \theta)$ and the following constraint on the fermionic coordinate:

$$\bar{\theta}\,(1 - \sigma_2) = 0 \quad \text{(at the self-dual point)}, \tag{4.143}$$

which would naturally explain the invariance under U-dualities at region A in **figure 4.1**. Of course away from the self-dual point we do not expect (4.141) and (4.143) to hold, although (4.138) should continue to hold.

We now conclude this section by collecting together all of our results. The fermionic completions of the type IIB fields, away from the self-dual point, can be expressed in the following compact notations:

$$\begin{aligned}
&\mathbf{\Phi}^{(1)} = \phi + \bar{\theta}\Delta^{(1)}\theta, \qquad \mathbf{\Phi}^{(2)} = C^{(0)} + \bar{\theta}e^{-\phi}\sigma_2\widetilde{\Delta}^{(2)}\theta \\
&\mathbf{B}^{(1)}_{mn} = B_{mn} + \bar{\theta}\sigma_3 \otimes \Gamma^{10} M_{[mn]}\theta, \quad \mathbf{G}_{mn} = g_{mn} + \bar{\theta}M_{(mn)}\theta \\
&\mathbf{B}^{(2)}_{mn} = C^{(2)}_{mn} - i\bar{\theta}e^{-\phi}\sigma_3\sigma_2 \otimes \Gamma^{10}\left(M_{[mn]} + i\Gamma_{mn}\widetilde{\Delta}^{(2)}\right)\theta \\
&\mathbf{C}_{mnpq} = C^{(4)}_{mnpq} - i\bar{\theta}e^{-\phi}\sigma_2 \otimes \Gamma^{10}\left(2\Gamma_{[mn}M_{pq]} + i\Gamma_{mnpq}\widetilde{\Delta}^{(2)}\right)\theta,
\end{aligned} \tag{4.144}$$

where the θ expansion for $\Delta^{(1)}$ is given by (4.98), for $\widetilde{\Delta}^{(2)}$ is given by (4.120) and for $M_{(mn)}$ and $M_{[mn]}$ are given by (4.104). We will take $(C^{(0)}, \phi) \to 0$, such that $g_s = e^{\phi} \to 1$ at the self-dual point. Knowing these series expansions we can in principle determine the type IIB fields to arbitrary orders in θ (provided of course there are no additional terms other than the ones got via U-duality transformations). In the presence of an $\overline{\text{D3}}$, the functional forms for $\Delta^{(1)}, \widetilde{\Delta}^{(2)}$ and M_{mn} become fixed. Henceforth this is the choice that we will consider, unless mentioned otherwise[30]. For example, to $\mathcal{O}(\theta^2)$, $\Delta^{(1)}, \widetilde{\Delta}^{(2)}$ and M_{mn} are known to be:

$$\Delta^{(1)} = -\frac{i}{2}\bar{\delta}\psi, \qquad \widetilde{\Delta}^{(2)} = \frac{1}{2}\bar{\delta}\psi, \qquad M_{mn} = -i\Gamma_m\bar{\delta}\psi_n, \tag{4.145}$$

and therefore plugging them in (4.144) will determine the type IIB fields to $\mathcal{O}(\theta^2)$ in the presence of an $\overline{\text{D3}}$-brane. The above values should be understood as operators acting on θ,

[30]For simplicity we will only concentrate on the integer $\overline{\text{D3}}$ brane (including D3-brane), and not discuss the fractional branes as we did for the resolved conifold case. Although with our formalism it is easy to extend to any D-brane, integer or fractional, one needs to be careful when fractional branes are present along with integer D3 or $\overline{\text{D3}}$-branes. However in the presence of only fractional branes, but no integer branes, the story proceeds in exactly the same way as discussed here as long as we are below the energy scale proportional to the inverse size of the two sphere on which we have our wrapped branes.

and therefore to higher orders in θ one would need to express in terms of components:

$$\left(\Delta^{(111)}_{\alpha\beta}, \Delta^{(ab1)}_{mn\alpha\beta}\right), \qquad \left(\widetilde{\Delta}^{(112)}_{\alpha\beta}, \widetilde{\Delta}^{(ab2)}_{mn\alpha\beta}\right), \qquad M^{(ab)}_{mn\alpha\beta}, \tag{4.146}$$

as elucidated in (4.98), (4.120) and (4.104) to properly write the higher order terms. Also, in (4.146) (m, n) are Lorentz indices, and (α, β) are spinor indices. One may easily check that these results match with the ones known in the literature [36–40, 44] for $e^{\phi} = 1$. The interesting thing about (4.146) is that, knowing these coefficients, one might be able to go to higher orders in θ as discussed above.

4.4.2 κ-symmetry at all orders in θ

In the previous section we managed to get the full fermionic action for the $\overline{\rm D3}$ branes using certain U-duality transformations at the self-dual point in the type IIB moduli space. The result is extendable to the D3-brane also, modulo certain subtleties that we want to elaborate here. Our answer is given in (4.156) which is derived for the special case of $\mathcal{F}_{mn} = 0$. The most generic case, given as (4.147), could also be worked out using the representations (4.144) for the type IIB fields, but we will not do so here.

Another issue that we briefly talked about earlier is the behavior of these higher order terms under Renormalization Group flow. Under RG we expect these terms to be irrelevant. However as we will discuss momentarily, to argue for the full κ-symmetry, all the higher order terms are essential. Therefore for our purpose it may be useful to work with the *exact* renormalization group equations [67] to keep track of the irrelevant operators. In the following however we will not discuss the quantum behavior and concentrate only on the classical action (4.156) with all the higher order terms.

The question that we want to answer here is the following: under what condition will the action (4.156) take the κ-symmetric form, i.e a form like $\mathcal{L} \sim \bar{\theta}(1 - \mathbf{\Gamma}^{\pm}_{\rm D3})[\,...\,]\theta$, where $\mathbf{\Gamma}^{\pm}_{\rm D3}$ is the κ-symmetry operator[31]? The condition, as we shall see, turns out to be rather subtle so we will have to tread carefully. Therefore as a start we will take the world-volume action, for a single D3 or $\overline{\rm D3}$, in the presence of the fermionic terms, to be given by:

$$S = -T_3 \int d^4\zeta e^{-\mathbf{\Phi}^{(1)}} \sqrt{-\det\left(\mathbf{G}_{ab} + \mathbf{B}^{(1)}_{ab} + \alpha'\mathbf{F}_{ab}\right)} \pm T_3 \int \mathbf{C} \wedge e^{\mathbf{B} + \alpha'\mathbf{F}}, \tag{4.147}$$

[31] See (4.152) for the definition of $\mathbf{\Gamma}^{\pm}_{\rm D3}$.

where the first term is the Born-Infeld (BI) piece and the second one is the Chern-Simons (CS) piece. The only difference now is that they both include the fermionic completions that we developed earlier which are in general different for D3 and $\overline{\rm D3}$ branes[32]. We can choose the gauge field $\mathbf{F}_{ab}$ in such a way as to cancel the fermionic contributions of the NS B-field $\mathbf{B}^{(1)}_{ab}$. This way we can write a bosonic combination $\mathcal{F}_{ab} \equiv \mathbf{B}^{(1)}_{ab} + \alpha'\mathbf{F}_{ab}$ to represent the gauge field. We can also define a matrix A in the following way:

$$A_{mn} \equiv \left[(g+\mathcal{F})^{-1}\right]^p_m \bar{\theta}^\alpha M_{pn\alpha\beta}\theta^\beta, \tag{4.148}$$

with M_{mn} matrix defined earlier in (4.144) to study the fermionic parts of the metric and the NS B-field. With this definition, the BI part of the antibrane action takes the following form:

$$S_{\rm BI} = -T_3 \int d^4\zeta e^{-\phi}\sqrt{-{\rm det}(g+\mathcal{F})}\ {\rm exp}\left[\frac{1}{2}{\rm tr}\ {\rm log}\left(\mathbf{I}+A\right) - \bar{\theta}\Delta^{(1)}\theta\right], \tag{4.149}$$

where $\mathbf{I}$ is the identity matrix in four-dimension, and A is the same matrix defined earlier in (4.148). As usual, at the self-dual point we put $\phi = 0$ to be consistent with our U-dualities. Moving away from the self-dual points, as exemplified in (4.133), (4.134) and (4.144), the action has the necessary dilaton piece.

We now come to the Chern-Simons part of the brane action for both the D3 and $\overline{\rm D3}$ using the fermionic completions developed above. The action can be written as:

$$S_{\rm CS} = T_3 \int d^4\zeta \epsilon^{mnpq}\left(\mathbf{C}^{\pm}_{mnpq} + \mathbf{B}^{(2\pm)}_{mn}\mathcal{F}_{pq} + \frac{1}{2}\mathbf{\Phi}^{(2\pm)}\mathcal{F}_{mn}\mathcal{F}_{pq}\right), \tag{4.150}$$

where the superscript represent D3 and $\overline{\rm D3}$ respectively, and $\mathbf{C}^{-}_{mnpq} \equiv \mathbf{C}_{mnpq}$, $\mathbf{B}^{(2-)}_{mn} \equiv \mathbf{B}^{(2)}_{mn}$ and $\mathbf{\Phi}^{(2-)} \equiv \mathbf{\Phi}^{(2)}$ for an $\overline{\rm D3}$ as we developed here. We have assumed that the background is flat along spacetime directions so that the curvature terms do not appear above. In general, for curved background, the curvature terms with their fermionic completions (from the metric) should also appear. For our case this should only change the last term in the

[32]We have used three kind of matrices, namely $M_{mn}, \Delta^{(1)}$ and $\widetilde{\Delta}^{(2)}$ to express the fermionic pieces in the presence of an $\overline{\rm D3}$-brane. One may choose similar matrices to express the fermionic pieces in the presence of a D3-brane. For example we will use $M^+_{mn}, \Delta^{(1+)}$ and $\widetilde{\Delta}^{(2+)}$ as the corresponding matrices for a D3-brane to represent the fermionic parts, whereas $M^-_{mn} = M_{mn}, \Delta^{(1-)} = \Delta^{(1)}$ and $\widetilde{\Delta}^{(2-)} = \widetilde{\Delta}^{(2)}$ will be reserved for the $\overline{\rm D3}$-brane to avoid clutter.

above action (4.150).

We can simplify the action (4.150) further by assuming $\mathcal{F}_{mn} = 0$. This would also imply that A_{mn} in (4.148) simplifies. This is the case we will consider here. A more generic scenario with $\mathcal{F}_{mn}$, or even with the fermionic pieces of $\mathcal{F}_{mn}$ (that we cancelled here) can be studied. This will make the system more involved but won't change the physics. Therefore, for this special case we have:

$$S_{\rm CS} = T_3 \int d^4\zeta \epsilon^{mnpq} C^{(4)}_{mnpq} + T_3 \int d^4\zeta e^{-\phi}\sqrt{-\det\, g}\; \bar\theta\; \mathbf{\Gamma}^{\pm}_{\rm D3}\left(\frac{1}{2}\Gamma^{ba} M^{\pm}_{ab} + i\widetilde\Delta^{(2\pm)}\right)\theta, \tag{4.151}$$

where, as before, $M^{-}_{ab} \equiv M_{ab}$ and $\widetilde\Delta^{(2-)} \equiv \widetilde\Delta^{(2)}$ represent the corresponding matrices for an $\overline{\rm D3}$; and $\mathbf{\Gamma}^{\pm}_{\rm D3}$ is defined as:

$$\mathbf{\Gamma}^{\pm}_{\rm D3} = \pm \frac{i\sigma_2 \otimes \Gamma^{10}\Gamma_{mnpq}\epsilon^{mnpq}}{4!\sqrt{-\det\, g}}. \tag{4.152}$$

Let us now come back to the BI piece of the action (4.149). To analyze this we will use the well known expansion for log as:

$$\mathrm{tr}\, \log\left(\mathbf{I} + A\right) = \mathrm{tr}\; A - \frac{1}{2}\mathrm{tr}\; A^2 + \frac{1}{3}\mathrm{tr}\; A^3 + ... = \sum_{k=1}^{k_{max}} \frac{(-1)^{k+1}\mathrm{tr}\; A^k}{k}, \tag{4.153}$$

where k_{max} is determined by the rank of the matrix. Plugging this in the BI action (4.149) and rearranging the action appropriately, we get for an $\overline{\rm D3}$:

$$\begin{aligned} S_{\rm BI} &= -T_3 \int d^4\zeta e^{-\phi}\sqrt{-\det\, g}\left[1 + \sum_{k=1}^{k_{max}}\left(\frac{1}{2}\mathrm{tr}\; A - \bar\theta\Delta^{(1)}\theta - \frac{1}{2}\sum_{l=1}^{l_{max}}\frac{\mathrm{tr}\; (-A)^{l+1}}{l}\right)^k \cdot \frac{1}{k!}\right] \\ &= -T_3 \int d^4\zeta e^{-\phi}\sqrt{-\det\, g}\;\left[1 + \sum_{k=1}^{k_{max}} \frac{\left(\frac{1}{2}\mathrm{tr}\; A + i\bar\theta\widetilde\Delta^{(2)}\theta + \mathcal{S}(A, \hat\Delta)\right)^k}{k!}\right] \end{aligned} \tag{4.154}$$

where the first term is the standard BI term for the bosonic piece and the second term is the fermionic extension. We have also used (4.142) to replace $\Delta^{(1)}$ by $\widetilde\Delta^{(2)}$ and defined the other variable appearing above in the following way:

$$\mathcal{S}(A, \hat{\Delta}) = -\frac{1}{2}\sum_{l=1}^{l_{max}} \frac{\text{tr } (-A)^{l+1}}{l} - \bar{\theta}\hat{\Delta}\theta. \tag{4.155}$$

Combining the Chern-Simons and the Born-Infeld parts, i.e (4.151) and (4.154) respectively, we can extract the fermionic completions of the brane and anti-brane actions. The result is given by:

$$\begin{aligned} S^f_\pm &= -T_3 \int d^4\zeta e^{-\phi}\sqrt{-\text{det } g}\ \mathcal{L}_\pm \qquad (4.156)\\ \mathcal{L}_\pm &\equiv \left[\sum_{k=1}^{k_{max}} \frac{\left(\frac{1}{2}\text{tr } A^\pm + i\bar{\theta}\widetilde{\Delta}^{(2\pm)}\theta + \mathcal{S}^\pm(A, \hat{\Delta})\right)^k}{k!} - \bar{\theta}\ \mathbf{\Gamma}^\pm_{\text{D3}}\left(\frac{1}{2}\Gamma^{ba}M^\pm_{ab} + i\widetilde{\Delta}^{(2\pm)}\right)\theta\right], \end{aligned}$$

where $\pm$ subscript denote D3 brane and $\overline{\text{D3}}$ respectively and $A^-_{mn} \equiv A_{mn}$ as in (4.148). The bosonic parts of the action for the brane and the anti-brane remain the same as the standard ones, as one can easily verify. It is also easy to see that:

$$\frac{1}{2}\text{tr } A^\pm = \frac{1}{2}\bar{\theta}\Gamma^{ba}M^\pm_{ab}\theta \equiv \bar{\theta}\left(\mathbf{N}_\pm - i\widetilde{\Delta}^{(2\pm)}\right)\theta, \tag{4.157}$$

where $\mathbf{N}_\pm$ is defined in such a way that the fermionic action (4.156) takes the following form:

$$S^f_\pm = -T_3 \int d^4\zeta e^{-\phi}\sqrt{-\text{det } g}\left(e^{\bar{\theta}\mathbf{N}_\pm\theta + \mathcal{O}(\mathbf{N}^2_\pm)} - 1 - \bar{\theta}\mathbf{\Gamma}^\pm_{\text{D3}}\mathbf{N}_\pm\theta\right). \tag{4.158}$$

In the absence of any other information about the series $\mathbf{N}_\pm$, the above action for the fermionic terms for the D3 or the $\overline{\text{D3}}$ is probably the best we can say at this stage. Simplification can occur when $\mathbf{N}_\pm$ remains small to all orders in $(\theta, \bar{\theta})$, which in-turn would guarantee the smallness of the $\mathcal{O}(\mathbf{N}^2_\pm)$ terms in the exponential, as well as the exponential itself. If this is the case then:

$$\begin{aligned} S^f_\pm &= -T_3 \int d^4\zeta e^{-\phi}\sqrt{-\text{det } g}\ \bar{\theta}\left(1 + \mathbf{\Gamma}^\pm_{\text{D3}}\right)\mathbf{N}_\pm\theta \\ &= -T_3 \int d^4\zeta e^{-\phi}\sqrt{-\text{det } g}\ \bar{\theta}\left(1 + \mathbf{\Gamma}^\pm_{\text{D3}}\right)\left(\frac{1}{2}\Gamma^{ba}M^\pm_{ab} + i\widetilde{\Delta}^{(2\pm)}\right)\theta, \end{aligned} \tag{4.159}$$

which would provide a strong confirmation of the recent work of [1], which was originally done to $\mathcal{O}(\theta^2)$. For our case we can use the θ-expansions for $M^-_{ab} = M_{ab}$ and $\widetilde{\Delta}^{(2-)} = \widetilde{\Delta}^{(2)}$ for an $\overline{\text{D3}}$ to express:

$$
\begin{aligned}
\bar{\theta}\left(\frac{1}{2}\Gamma^{ba}M_{ab} + i\widetilde{\Delta}^{(2)}\right)\theta &= \bar{\theta}^{\alpha}\left(\frac{1}{2}\Gamma^{ba\gamma}_{\alpha}M^{(11)}_{ab\gamma\beta} + i\widetilde{\Delta}^{(112)}_{\alpha\beta}\right)\theta^{\beta} \\
&+ \bar{\theta}^{\alpha}\left(\frac{1}{2}\Gamma^{ba\gamma}_{\alpha}M^{(21)}_{ac\gamma\delta}\theta^{\delta}\bar{\theta}^{\sigma}M^{(22)c}_{b\sigma\beta} + i\widetilde{\Delta}^{(212)}_{\alpha\delta m}\theta^{\delta}\bar{\theta}^{\sigma}\widetilde{\Delta}^{(222)m}_{\sigma\beta}\right)\theta^{\beta} + \mathcal{O}(\theta^6) \\
&= -\frac{1}{2}i\bar{\theta}\left(\Gamma^{a}\bar{\delta}\psi_a - \bar{\delta}\lambda\right)\theta + \mathcal{O}(\theta^4),
\end{aligned}
\tag{4.160}
$$

which is consistent with what we know to $\mathcal{O}(\theta^2)$ from the literature [3, 38, 39, 44]. Now if we define $\mathbf{\Gamma}^-_{\text{D3}} = \mathbf{\Gamma}_{\text{D3}}$ and $\mathbf{\Gamma}^+_{\text{D3}} = -\mathbf{\Gamma}_{\text{D3}}$ from (4.152) and $\delta^+ = \delta$ and $\delta^- = \bar{\delta}$ from [1]; and using the fermionic actions (4.156) or (4.159) for the D3 and the $\overline{\text{D3}}$ branes, then to $\mathcal{O}(\theta^2)$ we can easily reproduce the expected result in κ-symmetric form:

$$
S_{\pm} = \frac{1}{2}T_3\int d^4\zeta e^{-\phi}\sqrt{-\det g}\; i\bar{\theta}\left(1 \mp \mathbf{\Gamma}_{\text{D3}}\right)\left(\Gamma^{a}\delta^{\pm}\psi_a - \delta^{\pm}\lambda\right)\theta + \mathcal{O}(\theta^4). \tag{4.161}
$$

At the orientifold point, if we assume that the action is given by (4.159), then to all orders in θ the fermionic coordinate satisfy $\bar{\theta}\left(1 - \mathbf{\Gamma}_{\text{D3}}\right) = 0$. This way S_+ vanishes identically and S_- remains non-zero. This result seems to be valid only if the fermionic action takes the form (4.159), but is not obvious from the fermionic action (4.156) that this will continue to be the case. In fact the action (4.156) has many terms, coming from the log and from the exponential pieces, that do not in any obvious way give us $S_+ = 0$ at the orientifold point. In the following we will try to see how we can adjust the background, for example (4.144), to get the required form of the action.

Clearly adjusting the background should effect the definition of the type IIB fields (4.144). From the way we derived (4.144), we cannot arbitarily change the field definitions since they are related by certain U-duality transformations at a self-dual point. Thus for example, knowing $\mathbf{B}^{(1)}_{mn}, \mathbf{\Phi}^{(1)}$ and $\mathbf{\Phi}^{(2)}$, we pretty much derived the rest of the RR fields using U-dualitites. All the fields and their corresponding fermionic completions depend on three set of functional forms: $M_{mn}, \Delta^{(1)}$ and $\widetilde{\Delta}^{(2)}$. In fact the anti-symmetric part of the operator M_{mn}, namely $M_{[mn]}$, is essential to describe the fermionic completions of the p-form fields in type IIB. The symmetric part, $M_{(mn)}$, on the other hand is reserved for the fermionic completion of the metric. At the self-dual radii, $M_{(mn)}$ and $M_{[mn]}$, could be related by T-dualities along one parallel and one orthogonal spatial directions. The temporal directions however are not connected via simple T-dualities. This distinction may help us to construct the κ-symmetric form of the action from (4.156). To this end, we start by redefining the temporal components of the metric $\mathbf{G}_{0\mu}$ in the following way:

$$\mathbf{G}_{00} \equiv \left(g_{00} + \bar{\theta} M_{00} \theta\right) \exp\left(2\bar{\theta}\Omega\theta\right), \quad \mathbf{G}_{0i} \equiv \left(g_{0i} + \bar{\theta} M_{0i} \theta\right) \exp\left(\frac{1}{5}\bar{\theta}\Omega\theta\right), \tag{4.162}$$

keeping $\mathbf{G}_{ij}$ and all other type IIB fields exactly as in (4.144). The $\Omega(\theta, \bar{\theta})$ appearing above is again a series defined by powers of $(\theta, \bar{\theta})$ as:

$$\bar{\theta}\Omega\theta = \bar{\theta}^\alpha \Omega^{(11)}_{\alpha\beta} \theta^\beta + \bar{\theta}^\alpha \Omega^{(21)}_{m...q\alpha\gamma} \theta^\gamma \bar{\theta}^\delta \Omega^{(22)}_{p...n\delta\beta} \theta^\beta g^{qp} ... g^{mn} + \mathcal{O}(\theta^6) \tag{4.163}$$

where the coefficients can be defined in a similar way as the variables appearing in (4.144). As before, we could resort to rank two tensor representations for $\Omega^{(21)}$ and $\Omega^{(22)}$ etc., without losing much of the physics here.

Let us now revisit the Born-Infeld part of the action (4.147). Taking (4.162) and (4.144) into account, it is easy to see that the BI action now takes the following form:

$$\begin{aligned} S_{BI} &= -T_3 \int d^4\zeta e^{-\mathbf{\Phi}^{(1)}} \sqrt{-\det\left(\mathbf{G}_{ab} + \mathbf{B}^{(1)}_{ab} + \alpha' \mathbf{F}_{ab}\right)}\Bigg|_{\mathbf{B}^{(1)}_{ab} + \alpha'\mathbf{F}_{ab} \equiv 0} \\ &= -T_3 \int d^4\zeta e^{-\phi} \sqrt{-\det g}\, \exp\left[\frac{1}{2}\text{tr}\log\left(\mathbf{I} + A\right) + i\bar{\theta}\widetilde{\Delta}^{(2)}\theta - \bar{\theta}\hat{\Delta}\theta + \bar{\theta}\Omega\theta\right] \\ &= -T_3 \int d^4\zeta e^{-\phi} \sqrt{-\det g}\, \exp\left[\frac{1}{2}\text{tr}\ A + i\bar{\theta}\widetilde{\Delta}^{(2)}\theta - \left(\frac{1}{2}\sum_{k=2}^{k_{max}} \frac{(-1)^k\ \text{tr}\ A^k}{k} + \bar{\theta}\hat{\Delta}\theta\right) + \bar{\theta}\Omega\theta\right] \\ &\equiv -T_3 \int d^4\zeta e^{-\phi} \sqrt{-\det g}\, \exp\left[\sum_{k=1}^{k_{max}} \frac{(-1)^{k+1}}{k} \left(\frac{1}{2}\text{tr}\ A + i\bar{\theta}\widetilde{\Delta}^{(2)}\theta\right)^k + \bar{\theta}\left(\Theta + \Omega\right)\theta\right] \end{aligned} \tag{4.164}$$

where going from the second-last to the last line of (4.164), we have used the following mathematical identity:

$$\frac{1}{2}\sum_{k=2}^{k_{max}} \frac{(-1)^k \text{tr}\ A^k}{k} + \bar{\theta}\hat{\Delta}\theta \equiv \sum_{k=2}^{k_{max}} \frac{(-1)^k}{k} \left(\frac{1}{2}\text{tr}\ A + i\bar{\theta}\widetilde{\Delta}^{(2)}\theta\right)^k + \bar{\theta}\Theta\theta, \tag{4.165}$$

implying that the functional forms of Θ and $\hat{\Delta}$ can be used to express all tr A^k in terms of $(\text{tr } A)^k$ to allow for (4.165). Additionally, since Ω in (4.162) is arbitrary, we can as well as absorb Θ in the definition of Ω to give us:

$$\bar{\theta}\,(\Theta + \Omega)\,\theta \;= 0. \tag{4.166}$$

The above two conditions (4.165) and (4.166) are essential for expressing the $\overline{\text{D3}}$-brane action in the κ-symmetric form. Putting (4.165) and (4.166) in (4.164), we get:

$$\begin{aligned} S_{BI} &= -T_3 \int d^4\zeta e^{-\mathbf{\Phi}^{(1)}} \sqrt{-\det \mathbf{G}_{ab}} \qquad (4.167)\\ &= -T_3 \int d^4\zeta e^{-\phi} \sqrt{-\det g} - T_3 \int d^4\zeta e^{-\phi} \sqrt{-\det g}\,\bar{\theta} \left(\frac{1}{2}\Gamma^{ba} M^{\pm}_{ab} + i\widetilde{\Delta}^{(2)}\right)\theta, \end{aligned}$$

which is precisely the condition that is required for the BI action to take the κ-symmetric form when combined with the Chern-Simons part of the action (4.151). Thus putting (4.167) and (4.151) together, we get our final expression for the $\overline{\text{D3}}$-brane action:

$$\begin{aligned} S_{\overline{\text{D3}}} = \quad & -T_3 \int d^4\zeta e^{-\phi} \sqrt{-\det g} - T_3 \int d^4\zeta\; \epsilon^{mnpq} C^{(4)}_{mnpq} \\ & -T_3 \int d^4\zeta e^{-\phi} \sqrt{-\det g}\; \bar{\theta}\left(1 - \mathbf{\Gamma}^{-}_{\text{D3}}\right) \left(\frac{1}{2}\Gamma^{ba} M^{-}_{ab} + i\widetilde{\Delta}^{(2)}\right)\theta, \end{aligned} \tag{4.168}$$

in a manifestly κ-symmetric form. Equivalently, the above action indicates that the $\overline{\text{D3}}$ κ-symmetry projector

$$\left(1 - \mathbf{\Gamma}^{-}_{\text{D3}}\right), \tag{4.169}$$

continues to be the κ-symmetry projector at *all* orders in θ. Recall that the κ-symmetry variation of $\bar{\theta}$ is given by

$$\delta_\kappa \bar{\theta} = \bar{\kappa}(1 + \mathbf{\Gamma}^{-}_{\text{D3}}). \tag{4.170}$$

It follows from this that $\overline{\text{D3}}$ action is manifestly κ-symmetric at *all orders* in θ.

In deriving our result we have relied on the fact that at the self dual point we do not have extra fermionic operators other than the ones given by our U-duality transformations. This seems to be the case in any given background, otherwise we will end up with extra fermionic condensates which would appear to violate equations of motion. On the other hand, the U-duality rules that we used here also have α' corrections [47–49] so one might worry that this could change our result. A careful thought will tell us that this is not the case, as in deriving our results we have only used generic properties of T-duality. To see this in more details, let us investigate the two key relations where some aspects of the T-duality rules have been used, namely (4.109) and (4.115). The first relation i.e (4.109) relates $N^{(1)}_{[mx]}$ with $M_{[mx]}$ under one T-duality along direction x. This is one of the Buscher's rule derived for the limit $\alpha' \to 0$, so one would ask what happens under α' corrections. Before we go about discussing α' corrections to this, let us ask what does it mean to have a relation like (4.109). Since the piece M_{mn} comes from the metric and the piece N_{mn} comes from the NS B-field, the relation, or at least the bosonic part of it, implies the connection between the momentum and the winding modes under one T-duality. Thus, this is in the spirit of charge conservation: momentum charges being exchanged with winding charges or vice-versa and we can take this to be the defining property of T-duality. Since (4.109) implies the fermionic version of this, we will assume that (4.109) do not have any additional α' pieces.

Similar argument unfortunately cannot be given for (4.115), where the RR two-form appears from the axion under two T-dualities, as unlike the previous argument − where momentum and winding modes appear automatically − we do not have the advantage of invoking charge conservation *a priori*. We do however notice that there is a possible *alternative* way of expressing the fermionic parts of the background fields, namely that the background fields are functions of $(\theta, \bar{\theta})$ with the tensorial parts being specified by certain functions of the spacetime coordinates. In this language the T-duality rules are simply given by the way $(\theta, \bar{\theta})$ change, i.e the transformation rules given in (4.116). This way we don't have to worry about the explicit α' dependences appearing from the T-duality transformations, and the all-order result (4.144) should be exact with the α' dependences now appearing from the order-by-order expansions of the $(\theta, \bar{\theta})$ terms for every components of the type IIB fields in (4.144).

4.5 Conclusion and Discussion

In this work we have studied the interplay of $\mathcal{N} = 1$ supersymmetric backgrounds and anti-branes. We found two new examples where supersymmetry is spontaneously broken by a probe anti-brane: a $\overline{\text{D3}}$ in a resolved conifold, and a $\overline{\text{D7}}$ in a GKP background. In the first case, the low-energy spectrum in the probe approximation has two massless fermions.

However, once backreaction of the $\overline{\mathrm{D3}}$ on bulk fluxes is taken into account (perturbatively), the would-be massless fermions in fact become massive; this is a consequence of having a wrapped five-brane in the background (an issue which does not arise when studying GKP-type backgrounds). In the second case, we found there can in fact be *many* massless fermions, and the precise number depends on the Hodge numbers of the 4-cycle wrapped by the $\overline{\mathrm{D7}}$, although we did not extend the analysis to include backreaction. We also studied the effect of worldvolume fluxes, which provide extra mass terms. It is possible that for the most general worldvolume fluxes background there are no $\overline{\mathrm{D7}}$ fermions which remain massless.

As a step towards a more complete understanding of anti-branes and supersymmetry breaking, we studied the brane fermionic action at all orders in the fermionic expansion. In other words, we studied the all-order α' expansion of the fermionic action, while working at *leading order* in the bosonic α' expansion. This allowed us to neglect curvature corrections to the action, as well as purely bosonic α' corrections to the string duality transformations. Our result is that the all-order fermionic action can be written in a manifestly κ-symmetric form, which implies that our previous two analyses (and the results of [1,2]) are not simply a leading-order effect. In this analysis we neglected the effect of worldvolume flux, and while we don't expect this to qualitatively change the result (see, for example, [40]), it would be interesting to see the precise details of how this changes the all-order fermionic calculation.

There are many directions for future work. It would be interesting to see what types of inflationary scenarios can be built from the two examples we have studied, and if the interaction of the fermions with worldvolume fluxes can lead to a modification of the inflationary dynamics. In a totally different direction, we would like to see how the all-order fermionic action can be expressed in a Volkov-Akulov form, which should in principle be possible given the recent results of [34]. We plan to study all these effects in future works.

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Part III

String (Inspired) Cosmology

Chapter 5

Preheating and Entropy Perturbations in Axion Monodromy Inflation

E. McDonough, H.B. Moghaddam and R.H. Brandenberger, *Preheating and Entropy Perturbations in Axion Monodromy Inflation*, JCAP 1605 (2016) 012 [arXiv:1601.07749]

Addendum for Thesis

This chapter deviates substantially from the previous two and works directly with four-dimensional physics, namely a model of inflation in string theory known as "axion monodromy inflation."

The name "axion" refers to the underlying shift-symmetry of the field, which in this case arises from a higher dimensional gauge invariance: the axion is the scalar-component of a higher dimensional (tensor) gauge field, and since gauge fields can only appear in the action derivatively (e.g. via $F_{\mu\nu}F^{\mu\nu}$), the axion inherits a shift symmetry. This shift symmetry is very useful for constructing self-consistent models of inflation, as this protects the inflaton potential from other corrections that would violate the 'slow-roll' conditions (and thus forbidding inflation).

The axion in question has couplings to other fields which are intrinsic to the underlying string compactification. For example, in the axion monodromy setup, the inflationary potential is generated by interactions of the axion with a D5-brane. The D5-brane comes equipped with a gauge field (corresponding to open string excitations on the brane), which then are also coupled to the inflaton. The current chapter will investigate the dynamics of this gauge field during and after inflation.

Abstract

We study the preheating of gauge fields in a simple axion monodromy model and compute the induced entropy perturbations and their effect on the curvature fluctuations. We find that the correction to the spectrum of curvature perturbations has a blue spectrum with index $n_s = 5/2$. Hence, these induced modes are harmless for the observed structure of the universe. Since the spectrum is blue, there is the danger of overproduction of primordial black holes. However, we show that the observational constraints are easily satisfied.

5.1 Introduction

Axion monodromy inflation [1] (see also [2]) may be the most promising way to obtain large field inflation in the context of superstring theory [1]. Large field inflation models have the advantage over most small field models in that the inflationary slow-roll trajectory is a local attractor in initial condition space [5] (see e.g. [6] for a recent review of this issue).

Axion monodromy models contain, in addition to the axion field (which plays the role of the inflaton), gauge fields to which the axion couples via a Pontryagin term in the effective action. As a consequence, during the post-inflationary phase when the inflaton starts to oscillate, there is a preheating type instability in the gauge field equation of motion, and there can be explosive gauge field particle production. This, at second order in the amplitude of the gauge field perturbations, induces a growing curvature fluctuation mode.

The amplitude of the induced curvature fluctuations is constrained by observations. On one hand, on cosmological scales the amplitude of the induced curvature fluctuations must be smaller than the observed perturbations [2]. This is easy to satisfy if the spectrum of the induced fluctuations is blue, as in our case. If the spectrum is blue then, on the other hand, we must worry about the possible over-production of primordial black holes.

In this paper we will show that both sets of constraints are satisfied. We will first study the preheating of the gauge field fluctuations. Then, we compute the resulting entropy

[1] Note, however, that there may be constraints on the scenario coming from string back-reaction effects [3] and from the “Weak Gravity Conjecture" [4].

[2] They must be strictly smaller since the observed fluctuations are well described by a Gaussian process, whereas the induced fluctuations due to the gauge field perturbations have non-Gaussian statistics.

fluctuations and determine the induced curvature fluctuations. We find that the power spectrum of these perturbations is deeply blue, with a spectral index of $n_s = 4$, which gives a leading correction to the curvature power spectrum with index $n_s = 5/2$. Hence, these perturbations have a completely negligible effect on cosmological scales. The amplitude of the perturbations on small scales is influenced both by the tachyonic growth of the modes during inflation, and by the instability during the preheating phase. However, for parameter values used in axion monodromy models, we find that the constraints from possible over-abundance of primordial black holes are easily satisfied.

The Pontryagin term which couples the axion to a gauge field and which is playing the key role in our study has been studied in detail in recent years. The tachyonic instability which the gauge field experiences in the presence of a rolling axion field during inflation has been investigated in [7]. The amplified gauge fields, in turn, lead to axion fluctuations which induce non-Gaussianities in the adiabatic curvature perturbations [9]. It was realized that the spectrum of these fluctuations is blue, and there hence are potential constraints on the theory due to possible over-production of primordial black holes. This has been studied e.g. in [10, 11] (see also [12]). The Pontryagin term in the joint action of the axion and gauge field can lead to overdamped motion of the axion in which the axion field value is set by the coupling to the gauge field, and both the acceleration term and the velocity term in the axion equation of motion are negligible [7]. This can lead to inflation on steep potentials for sub-Planckian field values [7, 13]. The same Pontryagin term also leads to an inverse cascade [14] for cosmological magnetic fields, and it can be used to provide a scaling quintessence model for dark energy [15].

5.2 Review of Axion Monodromy Inflation

Axions are ubiquitous in string theory [16, 17]. They arise in string compactifications by integrating gauge potentials over non-trivial cycles of the compactification. In the absense of branes, the potential for the axions is classically flat, and obtains periodic terms from non-perturbative effects. However, in the presence of branes the periodicity of the axion potential is broken. The axion acquires an infinite field range with a potential which is slowly rising as the absolute value of the field increases. In the original example studied in [1], the potential is linear at large field values.

A ‘realistic’ construction of Axion Monodromy Inflation requires three distinct sectors:

(1) the monodromy brane, (2) moduli stabilization, and (3) a realization of the Standard Model. Typically these each are achieved via a brane construction: the DBI action of a D5 brane induces monodromy for the axion associated with the NS two-form (the axion we will focus our attention on), a stack of D7 branes induces gaugino condensation which fixes the radial modulus of the internal space, and a set of intersecting branes realizes the (extension of the) Standard Model. Each of these sectors comes with its own gauge theory: the monodromy brane has a U(1) Super-Yang-Mills (SYM) theory, the stack of D7 branes has a SU(N) SYM, while the intersecting branes have either a GUT group (e.g. SU(5)) or the Standard Model group SU(3)×SU(2)×U(1). The axion of axion monodromy inflation is a bulk field, and thus couples to and can lose energy to each sector. There may be phenomenological issues which arise when the energy loss of the axion into other sectors is considered, but we will not address this issue here.

In this work we will focus on a minimal setup of axion monodromy inflation in which we only consider the monodromy brane and its associated $U(1)$ gauge field. For the case of the B_2 axion ϕ, this gives the following 4d action [3] (in (-,+,+,+) signature)

$$\mathcal{L} \;=\; -(1/2)(\partial\phi)^2 - V(\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{\Lambda}\,\phi\;F_{\mu\nu}\widetilde{F}^{\mu\nu}, \tag{5.2}$$

where Λ is a UV scale, and is different from the axion decay constant. The potential $V(\phi)$ is the monodromy potential:

$$V(\phi) \;=\; \mu^3\sqrt{\phi_c^2+\phi^2}\,, \tag{5.3}$$

where μ is an energy scale whose value can be determined from the observed magnitude of the cosmic microwave background anisotropies, and where $\phi_c < m_{pl}$ is a constant, m_{pl} denoting the Planck mass. The field strength

$$F_{\mu\nu} \;=\; \partial_\mu A_\nu - \partial_\nu A_\mu \tag{5.4}$$

[3] There is additionally a coupling ϕ to $F_{\mu\nu}F^{\mu\nu}$, of the form

$$\frac{\beta}{\Sigma}V(\phi)\;F_{\mu\nu}F^{\mu\nu}, \tag{5.1}$$

which comes from the $(\alpha')^2$ correction to the DBI action. The coupling constant of this term is smaller than the coupling to $F\tilde{F}$ by a factor of C_0, where $C_0 \sim 10^2$ in cosmological models based on compactifications which stabilize moduli via gaugino condensation on D7 branes (see e.g. [57]). We will ignore the effect of this term in the current work.

is that of the abelian gauge field which lives on the brane world-volume, and we have neglected fermions, as preheating into fermions is inefficient. String models of axion monodromy and the resulting values of Λ are discussed in Appendix A (see also [18]). From the point of view of effective field theory, we would expect Λ to be given either by the string scale or the Planck scale. More stringent, though indirect, constraints come from models of early universe cosmology based upon this coupling. The gaussianity of the CMB constrains the parameter ξ, which we will define shortly, to be $\xi_* \lesssim 2.2$ at the moment when the pivot scale k_* exits the horizon [8], see also [9, 20], which can be translated to a bound $\Lambda^{-1} \leq 12\, m_{pl}$. Recent results on the validity of perturbation theory during inflation [19] constrain ξ to be $\xi \leq 3.5$, which correspond to an even tighter constraint on ξ_* (if the whole inflationary trajectory is to be treated perturbatively). Given these considerations we will take a conservative approach, and work with an upper bound $\Lambda^{-1} \leq \mathcal{O}(1){m_{pl}}^{-1}$.

5.3 Background Evolution

We assume that the axion starts out in the large field region $\phi \gg m_{pl}$ where the slow-roll approximation

$$3H\dot{\phi} \;=\; -V'(\phi) \;\simeq\; -\mu^3 \tag{5.5}$$

of the equation of motion is self-consistent. The end of inflation occurs at the field value when the slow-roll approximation breaks down, at which point $(1/2)\dot{\phi}^2 = V$. This takes place when

$$|\phi| \;\equiv\; \phi_e \;=\; \frac{1}{\sqrt{6}} m_{pl}\,, \tag{5.6}$$

and the kinetic energy at this point is

$$\frac{1}{2}\dot{\phi}^2_{\phi=\phi_e} \;=\; \frac{1}{\sqrt{6}}\mu^3 m_{pl}\,. \tag{5.7}$$

The value of the Hubble constant at the end of inflation is $H = H_e$ with

$$H_e \;=\; 2^{-1/4} 3^{-3/4} m_{pl}^{-1/2} \mu^{3/2}\,. \tag{5.8}$$

After inflation ends ϕ begins anharmonic motion about the ground state $\phi = 0$. As long as we can neglect the expansion of space and the loss of energy by particle production, the motion is periodic but anharmonic.

The value of μ is set by the observed amplitude of the cosmic microwave background (CMB) anisotropies. A simple application of the usual theory of cosmological perturbations (see e.g. [21] for a detailed review, and [22] for an overview) shows that the power spectrum $\mathcal{P}_\zeta$ of the primordial[4] curvature fluctuation ζ has the amplitude

$$\mathcal{P}_\zeta \sim \big(\frac{\mu}{m_{pl}}\big)^3, \tag{5.9}$$

from which it follows that

$$\mu \sim 6 \times 10^{-4} m_{pl} \,. \tag{5.10}$$

5.4 Preheating of Gauge Field Fluctuations

As first pointed out in [23] and [24], a periodic axion background can lead to explosive particle production for all fields coupled to the axion. This effect is called "preheating" [25–27] (see also [28, 29] for reviews). Here we will consider the resonance of the gauge field fluctuations [5]

The equation of motion for the linear fluctuations of A_μ is (see e.g. [7, 9, 31])

$$\frac{d^2 A_{k\pm}}{d\tau^2} + \left(k^2 \pm 2k\frac{\xi}{\tau}\right) A_{k\pm} = 0 \,, \tag{5.11}$$

where $\pm$ denote the two polarizations of the gauge field, τ is conformal time, k indicates a comoving mode, and ξ is given by[6]

$$\xi = \frac{2\dot{\phi}}{\Lambda H} \,, \tag{5.12}$$

where H is the Hubble expansion rate and ϕ is the background field, and an overdot denotes

[4]We add the word "primordial" to make a distinction between the original fluctuations and the induced ones which will be the focus of this paper.

[5]There is the also a possibility that there is an efficient self-resonance of the inflaton, leading to oscillons [30]. Oscillon formation occurs once the amplitude of ϕ oscillations falls below ϕ_c, as defined in equation (5.3). Provided that ϕ_c is small compared to the initial amplitude of oscillations, which is indeed the case in realistic string embeddings, oscillon formation will not occur until preheating in to gauge fields has ceased to be efficient, and will not occur at all if preheating into gauge fields is efficient enough to halt the oscillatory motion of ϕ. Given this, we will not consider oscillon formation in this work, although this does deserve further attention.

[6]Our definition of ξ is equivalent to the definition used in [7, 9, 31] with the identification $\alpha/f = 4/\Lambda$.

the derivative with respect to physical time. As long as the slow-roll approximation is valid, ξ can be taken to be constant. This is the equation relevant during the inflationary period.

As Eq. (5.11) shows, for one of the polarization states there is a tachyonic instability (see e.g. [32] for an initial discussion of tachyonic instabilities in reheating) already during inflation for long wavelength modes, i.e. modes which obey

$$k - \frac{2\xi}{|\tau|} = k - \frac{4|\dot{\phi}_I|}{\Lambda H|\tau|} < 0\,, \tag{5.13}$$

where the subscript I indicates that the time derivative is evaluated during slow-roll inflation. The critical wavelength beyond which there is a tachyonic instability has a fixed value in physical coordinates if we take H and $\dot{\phi}$ to be constant in time. The critical wavelength can be called a "gauge horizon" and it plays a similar role as the Hubble radius (Hubble horizon) for cosmological perturbations. The gauge horizon is proportional to the Hubble radius, its physical wavenumber k_p being given by

$$k_p = 2\xi H\,. \tag{5.14}$$

For modes which start in their vacuum state deep inside the horizon, the tachyonic resonance [32] leads to squeezing of the mode function. The Floquet exponent is proportional to k, and hence, among all the modes which become super-horizon (meaning super-gauge horizon) by the end of inflation, the ones which undergo the most squeezing are the ones which exit shortly before the end of inflation, i.e. whose comoving wavenumbers is given by

$$k = k_* \equiv 2\xi H\,, \tag{5.15}$$

if we normalize the cosmological scale factor to be $a(t) = 1$ at the end of inflation. The value of k_* is determined by the Hubble rate and the axion field velocity at the end of the period of inflation.

It can be shown [7] that the mode function prepared by inflation is

$$\begin{aligned} A^{(0)}_{k+} &= \frac{2^{-1/4}}{\sqrt{2k}}\left(\frac{k}{\xi aH}\right)^{1/4} e^{\pi\xi - 4\xi\sqrt{k/2\xi aH}} \\ A^{(0)}_{k-} &= 0, \end{aligned} \tag{5.16}$$

where $+/-$ denote the positive/negative chirality mode (the $-$ mode is not amplified during inflation). This corresponds to a highly blue spectrum of gauge field fluctuations with an ultraviolet cutoff which is set by the gauge horizon; the cutoff comes from the second term in the exponential on the right hand side of (5.16). The major amplification factor F_I of the amplitude is

$$F_I \,=\, e^{\pi\xi}\,. \tag{5.17}$$

For the specific potential (5.3) of axion monodromy inflation the values of k_* and ξ are (making use of (5.7) and (5.8))

$$k_* \;=\; 4\big(\frac{2}{3}\big)^{1/4} m_{pl}^{1/2}\mu^{3/2}\Lambda^{-1} \tag{5.18}$$

$$\xi \;=\; 2\sqrt{6}\frac{m_{pl}}{\Lambda}\,. \tag{5.19}$$

This shows that if $\Lambda \ll m_{pl}$ there is a large enhancement of the amplitude of A_k during inflation. On the other hand, if $\Lambda \gg m_{pl}$, then the growth is negligible. For small values of Λ (i.e. large values of ξ), the "gauge horizon" is smaller than the Hubble horizon, whereas for large values of Λ the opposite is true.

As mentioned above, the power spectrum $\mathcal{P}_A$ of gauge field fluctuations is blue. On length scales larger that the gauge horizon we have

$$\mathcal{P}_A(k) \,\equiv\, k^3|A_k|^2 \,\sim\, k^{5/2}\,. \tag{5.20}$$

During reheating the expansion of space can be neglected [23] and the equation (5.11) becomes

$$\ddot{A}_{k\pm} + \left(k^2 \pm 4\frac{k}{\Lambda}\dot{\phi}\right) A_{k\pm} \,=\, 0. \tag{5.21}$$

We immediately see that the tachyonic resonance which was present during the period of inflation persists during the preheating period when ϕ undergoes damped anharmonic oscillations about $\phi = 0$. While $\dot{\phi}$ is negative, then the same polarization mode gets amplified as during inflation. During the second half cycle, when $\dot{\phi} > 0$, it is the other mode which is amplified while the original mode oscillates.

To obtain an order of magnitude estimate of the amplification of A_k during preheating, we focus on the first oscillation period (when the Floquet exponent of the instability is largest). We focus on the first quarter of the oscillation period T when ϕ is decreasing from

$\phi = \phi_e$ to $\phi = 0$. The velocity during most of this time interval is approximately $\dot{\phi}_e$ (see (5.7)). The amplitude of A_k grows exponentially at a rate (for $k/k_* < 1$),

$$\mu_k = 2\left(\frac{k}{\Lambda}\right)^{1/2}\sqrt{\dot{\phi}_e} = 2\left(\frac{2}{3}\right)^{1/8}\left(\frac{k}{\Lambda}\right)^{1/2} m_{pl}^{1/4}\mu^{3/4}. \tag{5.22}$$

The factor F_k by which the amplitude of A_k is amplified is

$$F_k = e^{X_k}, \tag{5.23}$$

with

$$X_k = \frac{1}{4}T\mu_k, \tag{5.24}$$

where T is the period. The quarter period is given by

$$\frac{1}{4}T = \frac{\phi_e}{\dot{\phi}_e}. \tag{5.25}$$

Combining these equations yields

$$X_k = X_{k_*}\left(\frac{k}{k_*}\right)^{1/2}, \tag{5.26}$$

with

$$X_{k_*} = 2\left(\frac{2}{3}\right)^{1/2}\frac{m_{pl}}{\Lambda}. \tag{5.27}$$

Comparing the amplification factors F_I and F_k (see (5.17) and (5.27) one sees that at the value $k = k_*$ they have similar magnitudes.

The mode function after one period of oscillation of ϕ is thus given by

$$A_{k+} = \frac{2^{-1/4}}{\sqrt{2k}}e^{X_k}\left(\frac{k}{\xi aH}\right)^{1/4} e^{\pi\xi - 4\xi\sqrt{k/2\xi aH}}. \tag{5.28}$$

As long as the expansion of the universe can be neglected, and before back-reaction shuts off the resonance, the gauge field fluctuations grow by the same factor in each period. Hence, after N periods we obtain

$$A_{k+} = \frac{2^{-1/4}}{\sqrt{2k}} e^{NX_k} \left(\frac{k}{\xi a H}\right)^{1/4} e^{\pi\xi - 4\xi\sqrt{k/2\xi a H}} . \tag{5.29}$$

Comparing the expressions for the period T and the Hubble expansion rate H at the end of inflation we see that right at the end of inflation $T \sim H^{-1}$ and hence the expansion of space cannot be neglected. However, once reheating starts, ϕ decreases and hence T decreases and the expansion of space becomes negligible. The Floquet exponent can be taken to be approximately constant during half of each period, and vanishing for the other half. Hence, over a period $(0, t)$ of reheating, the increase in the amplitude is

$$F_k \sim e^{\frac{1}{2}\mu_k t} , \tag{5.30}$$

and the gauge field amplitude becomes

$$A_{k+} = \frac{2^{-1/4}}{\sqrt{2k}} e^{\frac{1}{2}\mu_k t} \left(\frac{k}{\xi a H}\right)^{1/4} e^{\pi\xi - 2\sqrt{2}\xi\sqrt{k/\xi a H}} . \tag{5.31}$$

There is also an amplification for the $(-)$ polarization, A_{k-}, but this mode is suppressed during inflation, and enters preheating with a different mode function.

5.5 Gauge Field Energy Density Fluctuations

We have thus far computed the gauge fields produced during preheating. This sources an energy density perturbation, $\delta\rho_A$, which we will now focus on. The gauge field energy density is defined as (in $(-,+,+,+)$ signature)

$$\rho_A(x,t) = -T^0_0 , \tag{5.32}$$

where $T_{\mu\nu}$ is given by (again in $(-,+,+,+)$ signature, and assuming a Lagrangian $\mathcal{L} = (1/4)F^2$),

$$T_{\mu\nu} = -\frac{1}{4} g_{\mu\nu} F^2 + F_{\mu\lambda} F^{\lambda}_{\nu} . \tag{5.33}$$

In terms of the gauge field A_μ, and without any gauge fixing, this reduces to

$$\rho_A(x,t) \;=\; -\frac{1}{2}(\partial^0 A^i - \partial^i A^0)(\partial_0 A_i - \partial_i A_0) + \frac{1}{4}(\partial^i A^j - \partial^j A^i)(\partial_i A_j - \partial_j A_i)\,. \tag{5.34}$$

We can fix the gauge by setting $A_0 = 0$. The leading term on cosmological scales is given by

$$\rho_A(x,t) \;\simeq\; -\frac{1}{2}\partial^0 A^i \partial_0 A_i\,. \tag{5.35}$$

To find the Fourier modes of $\rho_A(x,t)$, we first expand A_μ in terms of *classical* oscillators

$$A_\mu(x,t) \;=\; \sum_{\lambda=+,-} \int \frac{\mathrm{d}^3k}{(2\pi)^3}\left[\epsilon_\mu^\lambda A_\lambda(k,t)\alpha_k e^{ikx} + \epsilon_\mu^{\lambda *} A_\lambda(k,t)\alpha_k^\dagger e^{-ikx}\right]\,, \tag{5.36}$$

where α_k are classical oscillators drawn from a nearly Gaussian distribution, satisfying

$$\langle \alpha_k \alpha_{k'}\rangle \;=\; (2\pi)^3\delta^3(k+k')\,, \tag{5.37}$$

where the angular brackets stand for ensemble averaging. We can expand ρ in a similar fashion

$$\rho_A(x,t) \;=\; \int \frac{\mathrm{d}^3k}{(2\pi)^3}\rho_{Ak}\beta_k e^{ikx} + c.c.\,, \tag{5.38}$$

where β_k are a different set of classical oscillators, whose distribution function can be determined in terms of the α_k. The Fourier modes of $\rho(x,t)$ are simply a convolution of Fourier modes of the gauge field A_μ

$$\rho_{Ak}\beta_k \;=\; +\frac{1}{2}a^{-2}\int \frac{\mathrm{d}^3k'}{(2\pi^3)}\dot{A}_{k'+}\dot{A}_{(k-k')+}\alpha_{k'}\alpha_{k-k'}\,, \tag{5.39}$$

where the mode function A_k is given by equation (5.31). There is a gradient term $k^2A_k^2$ which is comparable in magnitude to the time-derivative term, and thus changes ρ_{Ak} by a factor of two.

We can use the above expression to straightforwardly calculate the background energy density in the gauge field and the spectrum of the gauge fluctuations. The homogenous background energy density is simply $\langle\rho_A(x,t)\rangle$, and we define the fluctuations $\delta\rho_A(x,t)$ about this background as $\delta\rho_A = \rho_A - \langle\rho_A\rangle$, such that $\langle\delta\rho_A\rangle = 0$, and the variance of fluctuations is simply $\langle\delta\rho_A^2\rangle = \langle\rho_A^2\rangle - \langle\rho_A\rangle^2$. A simple calculation shows that the background is given by

$$\langle \rho_A(x,t) \rangle = \frac{1}{2} a^{-2} \int \mathrm{d}^3 k |\dot{A}_{k+}|^2 \,. \tag{5.40}$$

The dominant contribution to the integral comes from the maximally amplified mode $k = k_*$, and we can hence approximate it as

$$\langle \rho_A(x,t) \rangle \sim \sqrt{2}\, a^{-2}\, e^{2\mu_* t} (\mu_* k_*)^2 e^{-2\sqrt{2}} \cdot e^{2\pi\xi} \,, \tag{5.41}$$

where $\mu_* \equiv \mu_{k_*}$ From this we see that the amplitude of $\langle \rho \rangle$ depends inversely on the UV scale Λ, since a smaller Λ means an increased k_*.

The mode function of fluctuations can be straightforwardly computed using the definition $\delta\rho_A = \rho_A - \langle \rho_A \rangle$ in conjunction with equation (5.39) and the approximation that the β_k are drawn from a nearly Gaussian distribution, i.e. $\langle \beta_k \beta_{k'} \rangle = (2\pi)^3 \delta^3(k + k')$. The exact β_k are *not* drawn from a Gaussian distribution, but as we have the modest goal of computing power spectra (i.e. two-point statistics), this is not an important distinction. The dominant term in $\delta\rho_{Ak}$ is

$$|\delta\rho_{Ak}|^2 \simeq \frac{1}{4}\, a^{-4} \int \mathrm{d}^3 q\, |\dot{A}_q|^2 |\dot{A}_{k-q}|^2 \,. \tag{5.42}$$

For modes in the IR, i.e. $k \ll k_*$, this integral is highly peaked at $q = k_*$ and we can find

$$|\delta\rho_{Ak}|^2 \simeq \frac{\langle \rho_A \rangle^2}{k_*^3} \,. \tag{5.43}$$

Note, in particular, that the resulting power spectrum of gauge field fluctuations is highly blue. The spectral index is $n_s = 4$.

5.6 Back-Reaction Considerations

The exponential increase in the gauge field value cannot continue forever. Eventually, the tachyonic resonance will be shut off by back-reaction effects. Back-reaction in a two field toy model of parametric resonance was considered in [33], where it was concluded that back-reaction does not prevent the exponential production of entropy fluctuations before these perturbations become important. In this subsection we estimate how long the tachyonic resonance in our model will last until back-reaction becomes important.

We will consider the two most important back-reaction effects involving gauge field production. The first is the effect of gauge field production on the axion field dynamics, the dynamics driving the instability. Recall that the axion equation of motion is given by

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = \frac{1}{\Lambda}\langle F\tilde{F}\rangle, \tag{5.44}$$

where $\langle F\tilde{F}\rangle$ refers to enseble or spatial averaging as was done to determine $\langle \rho_A \rangle$ in the previous section. To obtain an order of magnitude estimate of when back-reaction becomes important, we can compare the term on the right hand side of (5.44) with the force driving the oscillations. The first condition of 'small backreaction' comes from demanding that the force term dominates. This translates to

$$\langle V_{,\phi}\rangle_{rms} \gg \langle \frac{F\tilde{F}}{\Lambda}\rangle_{RMS} . \tag{5.45}$$

We can estimate the order of magnitude of the right-hand side of the above equation by $\Lambda^{-1}\rho_A$, and hence the condition (5.45) becomes

$$V' \gg \frac{1}{\Lambda}\rho_A . \tag{5.46}$$

The second back-reaction condition comes from demanding that the energy density is dominated by the scalar field, i.e.

$$V \gg \rho_A . \tag{5.47}$$

In this equation, the value of ϕ appears. We will use the value at the end of inflation.

For the axion monodromy potential we are using, the two conditions differ by a factor Λ/m_{pl}. Inserting the expression (5.41) into the first back-reaction criterium (5.46) yields

$$2\mu_* t = -2\pi\xi + 3\ln\left(\frac{\Lambda}{\mu}\right) + 2\ln\left(\frac{\Lambda}{m_{pl}}\right) \tag{5.48}$$

for the time interval t before back-reaction becomes important, whereas the second condition (5.47) yields

$$2\mu_* t = -2\pi\xi + 3\ln\left(\frac{\Lambda}{\mu}\right) + 3\ln\left(\frac{\Lambda}{m_{pl}}\right) \tag{5.49}$$

which is a stronger condition if $\Lambda < m_{pl}$ and weaker otherwise.

The amplitude of the gauge field energy density fluctuations when back-reaction becomes important then is bounded from above by

$$\begin{aligned} \delta\rho_{Ak} &\sim \frac{V}{k_*^{3/2}} \quad \text{for } \Lambda > m_{pl} \\ \delta\rho_{Ak} &\sim \frac{V}{k_*^{3/2}}\frac{\Lambda}{m_{pl}} \quad \text{for } \Lambda < m_{pl} \,. \end{aligned} \tag{5.50}$$

Note that there can be back-reaction effects from the production of other fields which may turn off the resonance much earlier. Since we are interested in obtaining upper bounds on the effects generated by gauge field production, we will work with the above upper bounds.

5.7 Induced Curvature Perturbations

During reheating purely adiabatic fluctuations on super-Hubble scales cannot be amplified since it can be shown that the curvature fluctuation variable ζ is conserved. This can be shown in linear cosmological perturbation theory [34–37], but the result holds more generally (see e.g. [38, 39]). On the other hand, entropy fluctuations can be parametrically amplified during reheating [40, 41] (see also [42]). Entropy fluctuations inevitably seed a growing curvature perturbation. Thus, in the presence of entropy modes it is possible to obtain an exponentially growing curvature fluctuation on super-Hubble scales (see e.g. [43] for some studies of this question in earlier string-motivated models of inflation).

Consider ζ, the curvature perturbation on uniform density hypersurfaces. This is the variable which determines the amplitude of the CMB anisotropies at late times (see [21] for a detailed overview of the theory of cosmological perturbations). In the absence of entropy fluctuations, this variable is conserved on super-Hubble scales [34–36, 39]. However, in the presence of entropy perturbations, a growing mode of ζ is induced on super-Hubble scales, as already discussed in the classic review articles on cosmological perturbations [21, 44] and as applied to axion inflation in [45]. For more modern discussions the reader is referred to [46, 47]. The equation of motion for ζ_k (k denotes the comoving wavenumber) is given by Equation 3.29 of [47]

$$\dot{\zeta}_k = -\frac{H}{p+\rho}\delta P_{nad,k} + \frac{1}{3H}\frac{k^2}{a^2}\left(\Psi_k - \zeta_k\right) + \frac{k^4}{9H\dot{H}}\Psi_k \,, \tag{5.51}$$

where Ψ_k is the gauge invariant curvature perturbation in longitudinal gauge, p and ρ are the total pressure and energy densities, respectively and $\delta P_{nab,k}$ is the non-adiabatic pressure perturbation On large length scales the dependence on Ψ disappears and the evolution equation is simply

$$\dot{\zeta} = -\frac{H}{p+\rho}\delta P_{nad} \,. \tag{5.52}$$

Note that ζ is dimensionless.

The non-adiabatic pressure perturbation δP_{nad} is the sum of an intrinsic and a relative perturbation

$$\delta P_{nad} = \delta P_{int} + \delta P_{rel} \,. \tag{5.53}$$

The intrinsic non-adiabatic pressure perturbation is the sum

$$\delta P_{int} = \sum_{\alpha} \delta P_{int,\alpha} = \sum_{\alpha} \left(\delta p_\alpha - c_\alpha^2 \delta\rho_\alpha\right) \,, \tag{5.54}$$

while the relative non-adiabatic pressure perturbation is given by

$$\delta P_{rel} = -\frac{1}{6H\dot{\rho}} \sum_{\alpha\beta} \dot{\rho}_\alpha \dot{\rho}_\beta (c_\alpha^2 - c_\beta^2) S_{\alpha\beta} \,, \tag{5.55}$$

where $S_{\alpha\beta}$ is the relative entropy perturbation

$$S_{\alpha\beta} = -3H\left(\frac{\delta\rho_\alpha}{\dot{\rho}_\alpha} - \frac{\delta\rho_\beta}{\dot{\rho}_\beta}\right) \,. \tag{5.56}$$

In the above equations, the sum runs over the different components of matter, and c_α^2 is the square of the speed of sound of the α component of matter.

The above set of equations can be rewritten in a more compact form (see e.g. [48])

$$\delta P_{nad} = \dot{p}\left(\frac{\delta p}{\dot{p}} - \frac{\delta\rho}{\dot{\rho}}\right) \,, \tag{5.57}$$

where in our case the total pressure is the sum of the contributions from the ϕ field and from the gauge field, i.e. $p = p_\phi + p_A$, and similarly for ρ, and we have set the intrinsic entropy perturbations to zero. For a background that is dominated by ϕ, and with $\delta\rho_A > \delta\rho_\phi$, the above non-adiabatic pressure perturbation is simply

$$\delta P_{nad} = \dot{p_\phi} \left(\frac{\delta p_A}{\dot{p_\phi}} - \frac{\delta \rho_A}{\dot{\rho}_\phi} \right) , \tag{5.58}$$

and the evolution equation of ζ is given by

$$\dot{\zeta} = - \frac{H}{\rho_\phi + p_\phi} \left(\frac{1}{3} - c_{s\phi}^2 \right) \delta \rho_A . \tag{5.59}$$

In our case, the gauge field energy density fluctuations $\delta\rho_A$ is increasing exponentially with a Floquet exponent $2\mu_*$ during the preheating phase, as shown in earlier sections. Hence, integrating over time, we get

$$\Delta \zeta_k = -\mu_*^{-1} \frac{H}{\rho_\phi + p_\phi} \left(\frac{1}{3} - c_{s\phi}^2 \right) \delta \rho_{Ak} , \tag{5.60}$$

where the wavenumber k and the density fluctuation $\delta\rho_A$ are Fourier space quantities. However, since it follows from Section V that $\delta\rho_A$ is independent of k, we find that the power spectrum of the induced fluctuations of ζ is

$$\mathcal{P}_{\Delta\zeta}(k) \sim k^3 , \tag{5.61}$$

which corresponds to a highly blue tilted spectrum with index $n_s = 4$. Since the spectrum has such a large blue tilt, there are no constraints on our model coming from demanding that the induced curvature fluctuations do not exceed the observational upper bounds.

5.8 Primordial Black Hole Constraints

Since the power spectrum of induced curvature fluctuations is highly blue, we have to worry about the possible constraints on the model coming from over-production of primordial black holes. Primodial black holes are constrained by a set of cosmological observations, beginning with the original constraints coming from the observational bounds on cosmic rays produced by radiating black holes [49]. Primordial black hole production during reheating has been considered in simple two field inflation models in [50], and in models with spectra with a distinguished scale in [51].

In the context of an inflationary cosmology, primordial black holes of mass M can form when the length scale associated with this mass (i.e. the length l for which the mass inside a sphere of radius l equals M) enters the Hubble radius. The number density of black holes of this mass will depend on the amplitude of the primordial power spectrum [7].

Since in our case the power spectrum is highly blue, the tightest constraints will come from the smallest mass for which cosmological constraints exist. These correspond to black holes with a mass such that they evaporate during nucleosynthesis. The extra radiation from these black holes would act as an extra species of radiation, and would destroy the agreement between the theory of nucleosynthesis and observations (see [53] for reviews). The smallest length scale (i.e. largest wavenumber k) for which constraints exist is [54]

$$k_{max} \sim 10^{19}\mathrm{Mpc}^{-1}, \tag{5.62}$$

and the approximate bound on the power spectrum is

$$\mathcal{P}_\zeta(k_{max}) < 10^{-1.5}\,. \tag{5.63}$$

In fact, the bound for smaller values of k has comparable amplitude.

The power spectrum including the induced curvature perturbations is given by

$$P_\zeta(k) = \frac{k^3}{(2\pi)^2}|\mathcal{A}_0 k^{-3/2} + \Delta\zeta_k|^2, \tag{5.64}$$

where $\mathcal{A}_0 \sim 10^{-10}$ is the amplitude of the power spectrum at the pivot scale $k = k_0 = 0.05\mathrm{Mpc}^{-1}$, and we have approximated the spectrum of curvature perturbations from inflation to be scale invariant. We already computed the value of the induced curvature fluctuations $\Delta\zeta$ in the previous section in Eq. (5.60). Inserting the values from (5.50), (5.18) and (5.19) we obtain the following expressions for the leading order correction to the power spectrum of curvature fluctuations

$$\Delta\mathcal{P}_\zeta(k) \quad = \quad \mathcal{O}(10^{-3})\sqrt{\mathcal{A}_0}\; k^{3/2}\,\frac{\Lambda^{5/2}}{m_{pl}^{7/4}\,\mu^{9/4}} \quad \text{for } \Lambda > m_{pl}$$

[7]The are numerous subtleties in computing the precise number density, which tend to suppress the number of primordial black holes formed, see e.g. [52] and references therein. These details will not be important for our analysis.

$$\Delta\mathcal{P}_\zeta(k) \quad = \quad \mathcal{O}(10^{-3})\,\sqrt{\mathcal{A}_0}\;k^{3/2}\,\frac{\Lambda^{7/2}}{m_{pl}^{11/4}\mu^{9/4}} \quad \text{for } \Lambda < m_{pl}\,, \tag{5.65}$$

corresponding to a spectrum with index $n_s = 5/2$. These expressions hold if the exponential growth of the curvature fluctuations is only limited by the back-reaction effects studied in Section VI. Other effects may terminate the growth earlier. Hence, the above equations provide upper bounds on the amplitude of the induced curvature perturbations.

For the largest value of k for which the primordial black hole constraints apply we have

$$\frac{k}{m_{pl}} \sim 10^{-39}\,. \tag{5.66}$$

Inserting this value into (5.65) we find that the primordial black hole constraint (5.63) is trivially satisfied for the realistic range of values of Λ.

5.9 Conclusions

In this paper we have considered a minimal axion monodromy model and have calculated the spectrum of the curvature perturbations induced by the entropy mode associated with the gauge field to which the axion couples. We find that the leading correction to the curvature spectrum is blue with spectral index $n_s = 5/2$. Hence, there are no constraints from large scale cosmological observations. On the other hand, since the spectrum is blue, there is a danger of overproduction of primordial black holes. We find, however, that the amplitude of the spectrum is too low even on the smallest scales for which cosmological constraints exist.

Realistic axion monodromy models, on the other hand, typically contain many other scalar fields which can source entropy modes, and these modes could, in principle, pose cosmological problems. It is reassuring, however, that the prototypical minimal axion monodromy model is safe from the constraints studied in this paper.

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Appendix A: String Theory Model Building Constraints on the Coupling of ϕ to $F\tilde{F}$

A realistic universe built from axion monodromy neccessarily has three sectors: (1) the inflation sector, (2) the moduli stabilization sector, and (3) the standard model. However, without considering all three, there is already interesting couplings purely in the inflation sector.

In (the standard story of) axion monodromy inflation [18], a potential an axion field is induced by the DBI action of a D5 action. The world volume action of the D5-brane receives corrections at each order in the string coupling constant α'. In particular, the Chern-Simons part of the action has an α'^2 correction:

$$\delta S_{CS} = -\mu_5(2\pi\alpha')^2 \int C_0 B_2 \wedge F \wedge F \,, \tag{5.67}$$

where C_0 is the RR 0-form potential, B_2 is the NSNS two-form, F is the world volume gauge field (see, for example, Equation 2.6 of [55]). There are also corrections to the DBI action, but the coupling of these corrections to the two-form B_2 is not known (see for example equation (2.4) of [55]).

From the above action we can derive the 4d interaction,

$$\mathcal{L}_{int} = \frac{1}{\Lambda}\phi F\tilde{F} \,, \tag{5.68}$$

where Λ is given by

$$\frac{1}{\Lambda} \equiv 2\mu^3 l^3 (2\pi\alpha')^2 C_0 \,, \tag{5.69}$$

where μ^3 is the coupling constant appearing in the axion potential, and l^2 is the size of Σ_2 (the two-cycle wrapped by the brane) in string units.The constants in Λ are constrained by the consistency of the model. We can write α' in terms of the string mass scale, $M_s = 1/\sqrt{2\pi\alpha'}$, such that the coupling takes the form

$$\frac{1}{\Lambda} = 2l^3 C_0 \left(\frac{\mu}{M_s}\right)^3 \frac{1}{M_s} \,. \tag{5.70}$$

Let us consider the size of the parameters. Firstly, moduli stabilization requires a stack of D7 branes. These branes source C_0, and the value of C_0 (which is dimenionless) is roughly equal to the number of D7 branes. For more details on the supergravity background in the presence of a stack of D7 branes, see e.g. [56]. In realistic models of 4d physics (see e.g. [57])

$$C_0 \sim 10^2 \,. \tag{5.71}$$

Secondly, the value of μ is chosen such that the amplitude of cosmological perturbations arising from axion monodromy inflation matches observations. The potential is

$$V_{int} = \mu^3 \phi \,, \tag{5.72}$$

where μ is given by

$$\mu = \frac{\mu_5 \alpha'}{f g_s} \,, \tag{5.73}$$

where f is the axion decay constant. Note that the axion decay constant enters the potential and the interaction term with the same power, and thus it is impossible to change the relative strength of the interaction by fine-tuning the axion decay constant. Consistency with observations requires

$$\mu^3 = \left(6 \times 10^{-4} m_{pl}\right)^3 \,. \tag{5.74}$$

Thirdly, the internal space must be of an 'intermediate' size: if the internal space is too small then the supergravity approximation (and the DBI action) ceases to be the correct description of physics, while if the internal space is too large then the 4d Newton's constant is too small. A reasonable value of the volume of the internal space is $Vol(X_6) \sim 10^6$ in units of α', which corresponds to a length scale $l_{X_6} \sim 10$.

These constraints determine to a great degree the allowed value of the parameter Λ. A consistent value is:

$$\frac{1}{\Lambda} \sim 10^{-5} \left(\frac{m_{pl}}{M_s}\right)^4 \frac{1}{m_{pl}} \,, \tag{5.75}$$

where we took $C_0 = 50$, $l = 10$, and $\mu = 6 \times 10^{-4} m_{pl}$.

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Chapter 6

Inflation from Nilpotent Kähler Corrections

E. McDonough and M. Scalisi, *Inflation from Nilpotent Kahler Corrections*, JCAP 1611, no. 11, 028 (2016). arXiv:1609.00364 [hep-th].

Addendum for Thesis

This chapter will continue in a similar spirit to Chapter 5, and study only four-dimensional physics. However, in an attempt to connect with ten-dimensional physics, we will work in a supergravity framework conjectured to arise from the anti-brane scenario presented in Chapter 4.

We consider whether corrections to the supergravity description of anti-branes can lead to inflation, taking motivation from the corrections to the anti-brane action considered in Chapter 4.4 and the knowledge that Kähler potentials in supergravity receive quantum corrections at each loop order.

Abstract

We develop a new class of supergravity cosmological models where inflation is induced by terms in the Kähler potential which mix a nilpotent superfield S with a chiral sector Φ. As the new terms are non-(anti)holomorphic, and hence cannot be removed by a Kähler transformation, these models are intrinsically Kähler potential driven. Such terms could

arise for example due to a backreaction of an anti-D3 brane on the string theory bulk geometry. We show that this mechanism is very general and allows for a unified description of inflation and dark energy, with controllable SUSY breaking at the vacuum. When the internal geometry of the bulk field is hyperbolic, we prove that small perturbative Kähler corrections naturally lead to α-attractor behaviour, with inflationary predictions in excellent agreement with the latest Planck data.

6.1 Introduction

One of the key goals of modern theoretical physics is to find a UV complete description of our Universe, unifying particle physics with cosmology at both early and late times. There has recently been significant advancements towards this goal: It has indeed been realized that obtaining a pure *acceleration* phase, in the context of supergravity and/or string theory, often involves the appearance of a *nilpotent superfield* S. The latter is constrained by the condition [1]

$$S^2 = 0\,, \tag{6.1}$$

which implies the absence of scalar degrees of freedom[1]. This fact has turned out to be beneficial for cosmological applications: one can indeed show that coupling a nilpotent field to an inflationary sector generally simplifies the overall dynamics and allows for a unified description of inflation and dark energy [4, 5].

The initial investigations of the nilpotent chiral multiplet in the context of global supersymmetry [1] (see also [6]), have been by now extended to the regime when this symmetry becomes local [7, 8]. The recent discovery of 'de Sitter supergravity' [7], nearly forty years after the advent of AdS supergravity [9], has marked a serious development in the field. Coupling a nilpotent multiplet to supergravity indeed gives rise to a pure de Sitter phase, with no scalar fields involved.

Some of the most interesting aspects have however emerged in the context of string theory. It was realized in [10] that the four dimensional description of anti-D3 branes in an $\mathcal{N} = 1$ flux background, famously used by 'KKLT' to construct de Sitter vacua in string theory [11], is indeed a supergravity theory of a nilpotent superfield, wherein su-

[1] After writing S in terms of the superspace coordinates, one can easily check that the scalar part is replaced by a bilinear fermion. This simply reflects that, in this case, supersymmetry is non-linearly realized (see also the recent investigations [2]). One can then recover the original Volkov-Akulov action [3].

persymmetry is non-linearly realized. This demonstrated that supersymmetry breaking by $\overline{\text{D3}}$'s is spontaneous rather than explicit, providing strong evidence for the compatibility of 'uplifting' via $\overline{\text{D3}}$'s and moduli stabilization via various contributions to the superpotential.

The string theory origins of constrained superfields, and connections to D-brane physics, were then further worked out in [12–16]. The fermions arising when one or more anti-branes, placed in certain geometries, break supersymmetry spontaneously (see e.g. [13]) can often be packaged into constrained superfields. An example is provided by the recent works [15], where both a nilpotent superfield and an 'orthogonal nilpotent superfield', as used to construct inflationary models in [17], emerge when an $\overline{\text{D3}}$ is placed on intersecting O7-planes. It thus appears that constrained superfields are ubiquitous in string theory, and not just the simple example of a single nilpotent superfield originally studied in [10].

In terms of physical applications, the nilpotent superfield S has proven to be an optimal tool in the construction of cosmological scenarios [4, 5] (see also the recent work [18]). On the one hand, it allows to easily uplift models of inflation in supergravity, analogous to what happens in string scenarios with the addition of one or many $\overline{\text{D3}}$'s (see e.g. [19]). On the other hand, it generically ameliorates the stability properties of the model and yields better control over the phenomenology. Then, by means of an inflaton sector Φ and a nilpotent one S, it is possible to obtain a comprehensive physical framework which describes the primordial expansion of the Universe together with controllable level of dark energy and SUSY breaking.

However, in general there is no reason to expect the nilpotent field to be totally decoupled from the inflationary physics. In the context of string theory, this becomes a question of backreaction of the $\overline{\text{D3}}$ on the bulk geometry, which would manifest itself in the $d = 4$ supergravity theory as couplings between the nilpotent superfield and the bulk moduli. Typically, it is the task of model builders to argue that such corrections do not affect the dynamics of the model under consideration, and in particular, that the corrections to the Kähler potential are suppressed and do not lead to an η-problem [6, 20]. The effect on inflation due to non-negligible corrections to the Kähler potential has been considered in e.g. [22]. Computing these corrections explicitly in a concrete string compactification setting is a notoriously difficult task and has been done in only a small number of cases, see e.g. [23].

In this letter, we take the opposite approach. We will show that inflation can be driven by corrections to the Kähler potential of the form

$$\delta K = \delta K(\Phi, \bar{\Phi}, S, \bar{S}), \tag{6.2}$$

which mix the nilpotent superfield S with a bulk modulus Φ, even in the *absence of a superpotential* for Φ. This is similar in spirit to 'Kähler uplifting' [24], where the term responsible for the uplift to dS is an α' correction to the Kähler potential.

We will prove that this procedure provides a unified description of early and late time cosmology, where, at least qualitatively, the inflationary dynamics is due to the backreaction of the $\overline{\mathrm{D3}}$ on the bulk manifold and/or fluxes.

We will focus our investigation on the case of both *flat* and *hyperbolic* Kähler geometry, the latter typically arising from string theory compactifications. Therefore, in Sec. 6.2, we will thoroughly study the effects of the possible Kähler corrections to a flat Kähler geometry, providing all the relevant formulas and results. In Sec. 6.3, we will discuss the hyperbolic case highlighting the differences and similarities with the previous flat case. Interestingly, when the internal manifold is hyperbolic, we will prove that small perturbative Kähler corrections automatically lead to *α-attractor* [25] behaviour with cosmological predictions in great agreement with the latest observational data [26]. We will conclude in Sec. 6.4 with a summary of the main findings and perspectives for future directions. Throughout the paper, we will work in reduced Planck mass units ($M_{Pl} = 1$).

6.2 Inflation From Kähler Corrections to Flat Geometry

In this section, we would like to show how inflation can arise simply from corrections to a Kähler potential with *zero curvature*, while keeping the superpotential independent of the inflaton superfield.

Our starting point is the simplest realization of de Sitter phase in supergravity. This can be encoded in the following set of Kähler and superpotential [4, 7]:

$$K = S\bar{S}\,, \qquad W = W_0 + MS\,, \tag{6.3}$$

where W_0 is the flux induced superpotential [27] and M parametrizes the contribution of the $\overline{\mathrm{D3}}$. Then, the resulting scalar potential is a cosmological constant of the form

$$V = M^2 - 3W_0^2 \,. \tag{6.4}$$

Note that one can obtain the latter Eq. (6.4) by employing the usual formula for the scalar potential in supergravity and declaring that $S = 0$ (since the bilinear fermion, replacing the scalar part of S, cannot get any vev). The constant phase described by Eq. (6.4) is then the result of the delicate balance between the scale of spontaneous supersymmetry breaking of S

$$D_S W = M \,, \tag{6.5}$$

and the gravitino mass

$$m_{3/2} = W_0 \,. \tag{6.6}$$

Now, let us just extend the internal Kähler geometry with a chiral multiplet Φ which eventually will play the role of the inflaton. In a string theory interpretation, this framework would describe a $\overline{\mathrm{D3}}$ brane (encoded in S) and a bulk geometry and/or fluxes (encoded in Φ). Specifically, for the sake of simplicity, we choose a flat shift-symmetric Kähler function for Φ and then have

$$K = -\tfrac{1}{2}\left(\Phi - \bar{\Phi}\right)^2 + S\bar{S} \,, \qquad W = W_0 + MS \,. \tag{6.7}$$

The latter setting still provides the same constant value (6.4), along the real axis $\mathrm{Im}\Phi = 0$. This is obviously a flat direction as both K and W do not depend on $\mathrm{Re}\Phi$. On the other hand, the orthogonal field $\mathrm{Im}\Phi$ has a positive mass when $|M| > |W_0|$. However, this direction turns out to be not suitable for inflation, as it is too steep (due to the typical exponential dependence e^K in the scalar potential).

In order to produce inflationary dynamics, one must break the shift symmetry of K. Traditionally, this has been done by introducing a Φ-dependence in the superpotential (see for example the pioneering work [28] and the subsequent developments [29]). In the context where S is nilpotent, this approach has been put forward by [4]. The basic idea is to promote W_0 and M in Eq. (6.7) to functions of the field Φ. This breaks the shift symmetry along the real axis of Φ and perturbs the original flat direction creating an inflationary slope.

In this letter, we intend to explore an alternative possibility to induce inflation: while keeping a Φ-independent superpotential, we can add terms to the Kähler potential which mix the two sectors, break the original shift-symmetry and encode the interaction between the antibrane and the bulk modulus.

In full generality, the only possible allowed corrections are either *bilinear* or *linear* in S and $\bar{S}$, such as

$$\delta K = f(\Phi, \bar{\Phi}) S\bar{S} + g(\Phi, \bar{\Phi}) S + \bar{g}(\Phi, \bar{\Phi}) \bar{S}\,, \tag{6.8}$$

where f and g are arbitrary functions of their arguments, whose non-zero values can break the shift symmetry for[2] ReΦ. Note that higher order terms in S are forbidden since this field is nilpotent and Eq. (6.1) holds. In addition, the above couplings are in general non-(anti)holomorphic, and so cannot be gauged away by a Kähler transformation (whereas this is possible when S satisfies also an orthogonal nilpotency constraint [17]).

It is interesting to notice that the corrections (6.8) will affect the form of the Kähler metric such as

$$K_{I\bar{J}} = \begin{pmatrix} 1 & \partial_{\bar{\Phi}}\bar{g} \\ \partial_{\Phi} g & 1+f \end{pmatrix}\,, \tag{6.9}$$

thus inducing non-zero off-diagonal terms and modifying the originally canonical $K_{S\bar{S}}$.

However, this turns out not to be an issue for the cosmological dynamics of the model as the field S is nilpotent (fermion interactions are subdominant during inflation) and the only scalars involved are the real and imaginary components of Φ. The off-diagonal terms may have some relevant consequences for the post-inflationary evolution, as we comment in the concluding section of this paper.

In the following, we analyse the effects of the bilinear and linear nilpotent corrections separately.

6.2.1 Bilinear nilpotent corrections

Let us focus on the effects of the sole bilinear corrections while keeping $g = 0$. The model is then characterized by the same Φ-independent superpotential given in Eq. (6.7) and a

[2] In the context where S is an unconstrained chiral multiplet, the works [29] already considered bilinear couplings. However, these were taken to be independent on ReΦ, thus not affecting the form of the inflationary potential.

Kähler potential such as

$$K = -\tfrac{1}{2}\left(\Phi - \bar{\Phi}\right)^2 + \left[1 + f(\Phi, \bar{\Phi})\right] S\bar{S}\,. \tag{6.10}$$

This class of couplings is well motivated from string theory as the Kähler potential for D-brane matter fields generically appears as a bilinear combination of the fields and their complex conjugate.

This model still allows for an extremum along $\Phi = \bar{\Phi}$ (i.e. $\mathrm{Im}\Phi = 0$) if the function f satisfies

$$\partial_\Phi f(\Phi, \bar{\Phi})|_{\Phi=\bar{\Phi}} = \partial_{\bar{\Phi}} f(\Phi, \bar{\Phi})|_{\Phi=\bar{\Phi}}\,. \tag{6.11}$$

A sufficient condition for Eq. (6.11) to be valid is that f is symmetric under[3] $\Phi \leftrightarrow \bar{\Phi}$. Then, typical corrections are the ones depending on $(\Phi + \bar{\Phi})$ or $\Phi\bar{\Phi}$. In addition, in order for $\mathrm{Im}\Phi = 0$ to be a consistent truncation, one must ensure positive mass of the orthogonal direction (we discuss this later).

Supersymmetry is spontaneously broken just in the S-direction as the F-terms are equal to

$$D_\Phi W = 0\,, \qquad D_S W = M\,, \tag{6.12}$$

for any value of Φ and then for the entire cosmological evolution. Note the difference with respect to the previously developed nilpotent cosmological models [4, 5], which yield a positive potential thanks to the supersymmetry breaking along both directions.

One can simplify the following discussion by defining the function

$$F(\Phi) \equiv \frac{1}{1 + f(\Phi, \Phi)}\,, \tag{6.13}$$

along the extremum $\Phi = \bar{\Phi}$.

The combined effects of the SUSY breaking in S and of the non-zero Kähler correction generates a scalar potential for Φ, along the real axis, given by

$$V(\Phi) = -3W_0^2 + M^2 F(\Phi)\,, \tag{6.14}$$

which clearly allows for arbitrary inflation and a residual cosmological constant (CC). At

[3]This is analogous to the reality property imposed on the holomorphic function $f(\Phi)$ in the superpotential of the models [29] in order to assure consistent truncation along the real direction.

the minimum of the potential (which is placed at $\Phi = 0$, provided $F'(0) = 0$), we have indeed

$$\Lambda = -3W_0^2 + M^2 F(0)\,. \tag{6.15}$$

The cosmological constant and amplitude of the inflationary potential are thus determined in terms of the same underlying parameters. The cosmological constant is constrained to be very small by late-time cosmology, while the size of the inflationary potential is fixed (albeit in a model-dependent way) by the amplitude of the curvature perturbations in the cosmic microwave background (CMB).

Note that, within this framework, a large value of M does not necessarily correspond to a very high gravitino mass, which is still equal to W_0, as in Eq. (6.6). At the vacuum, the SUSY breaking scale is indeed given by

$$K^{S\bar{S}}|D_S W|^2 = M^2 F(0)\,, \tag{6.16}$$

where the Kähler metric term $K^{S\bar{S}}$ is non-canonical, unlike the dS model defined by Eq. (6.3) and the models of [4,5] (in these cases the almost vanishing CC forces M and $m_{3/2}$ to be of the same order). A small fine-tuned value of $F(0)$ can still allow for a desirable low gravitino mass (e.g. order TeV) and a negligible cosmological constant (in the spirit of the string theory landscape). Nevertheless, the latter case (small $F(0)$ and $m_{3/2}$) implies a large Kähler correction f at the minimum and thus a considerable deviation of K from its canonical form Eq. (6.7).

The regime of small Kähler corrections $|f| \ll 1$ corresponds instead to an F of order unity. In this case, Eq. (6.14) implies that the parameter M must be of the same of order of the Hubble scale of inflation H or even higher, such as

$$M \geq H\,. \tag{6.17}$$

This holds during the whole cosmological evolution until the minimum of the potential, since M is a constant. In this limit, the scalar potential can be indeed expanded as

$$V = (M^2 - 3W_0^2) - M^2 f + \mathcal{O}(f^2)\,, \tag{6.18}$$

which makes once more explicit what we just said about the magnitude of M. Note that, in this regime, the CC is given again as compensation between M and the gravitino mass, as in Eq.(6.4). Therefore, the latter $m_{3/2} = W_0$ is necessarily large.

To summarize, bilinear nilpotent corrections to a flat Kähler potential, such as the ones of Eq. (6.29), can account for both inflation and dark energy. Both acceleration phases are solely due to spontaneous SUSY breaking of the nilpotent field S. The non-trivial structure of the Kähler correction still allows to have great control over the phenomenology of the cosmological model, with tunable level of the CC and the scale of SUSY breaking.

Stability

It is important that we check the stabilization of ImΦ. The mass of ImΦ is given by,

$$m^2_{\text{Im}\Phi} = -4W_0^2 + 4M^2 F(\Phi)\,. \tag{6.19}$$

The mass at late times, at the minimum of the potential, is given by

$$m^2_{\text{Im}\Phi} = 8W_0^2 + 4\Lambda\,, \tag{6.20}$$

where we have used Eq.(6.15) to relate M and W_0.

The mass during inflation, expressed as a ratio to the Hubble constant $H^2 \sim \frac{1}{3}V$, is given by

$$\frac{m^2_{\text{Im}\Phi}}{H^2} = 12 + \frac{24W_0^2}{M^2 F(\Phi) - 3W_0^2}. \tag{6.21}$$

Both the above terms are positive, and the first term is dominant. The above ratio is large and ImΦ is effectively stabilized during inflation, regardless of the precise details of F, W_0 or M.

Example: quadratic inflation

As a concrete example, let us consider the classic model of quadratic inflation. In the following, we explicitly construct this model in our framework. We do so in two ways, which have low and high scale supersymmetry breaking respectively.

First consider the following Kähler potential,

$$K = -\tfrac{1}{2}\left(\Phi - \bar{\Phi}\right)^2 + \frac{M^2}{M^2 + m^2\Phi\overline{\Phi}}S\bar{S}, \tag{6.22}$$

which corresponds to the choice

$$F(\Phi) = \frac{m^2}{M^2}\Phi^2 + 1. \tag{6.23}$$

This gives the scalar potential

$$V = \left(M^2 - 3W_0^2\right) + \frac{1}{2}m^2\varphi^2\,, \tag{6.24}$$

with $\varphi = \sqrt{2}\mathrm{Re}\Phi$. The normalization of the inflationary potential depends only on m, and hence the only constraint on M and W_0 comes from the condition that Λ be small. Thus this model allows for low-scale supersymmetry breaking and a small gravitino mass. In this case, the magnitude of $f = (1/F) - 1$ is necessarily very large during inflation and hence the model is a large deviation from a canonical Kähler potential.

We can also construct this model as a small perturbative correction away from a flat Kähler potential. Consider the following Kähler potential:

$$K = -\tfrac{1}{2}\left(\Phi - \bar{\Phi}\right)^2 + S\bar{S} - \frac{m^2}{2M^2}\Phi\bar{\Phi}\cdot S\bar{S}, \tag{6.25}$$

which corresponds to the choice of $f(\Phi, \bar{\Phi})$,

$$f(\Phi, \bar{\Phi}) = -\frac{m^2}{2M^2}\Phi\bar{\Phi}. \tag{6.26}$$

If we impose the condition that $|f| \ll 1$, so as to be a small correction to a flat Kähler potential, this again gives the same quadratic potential (6.24).

The normalization $V \sim 10^{-10}$ when the CMB pivot scale exits the horizon during inflation (see e.g. [30]), along with the the condition $|f| \ll 1$, then imposes a condition on M:

$$M^2 \gg 10^{-10}. \tag{6.27}$$

Since $|f| \ll 1$ implies $F \sim 1$, this corresponds to high-scale supersymmetry breaking and (due to Eq. (6.24) and the smallness of the CC) also to a very large gravitino mass $m_{3/2}$.

6.2.2 Linear nilpotent corrections

Let us now consider terms in the Kähler potential which are linear in S and $\bar{S}$ as given by Eq. (6.8), while neglecting the effects of the bilinear correction ($f = 0$). The most general form of this correction is

$$\delta K = g(\Phi, \bar{\Phi})S + \bar{g}(\Phi, \bar{\Phi})\bar{S}\,. \tag{6.28}$$

If we make the simplifying assumption that g is purely real ($g = \bar{g}$), then this is a coupling of Φ to $\mathrm{Re}S$, while if g is purely imaginary ($g = -\bar{g}$) then the coupling is to $\mathrm{Im}S$.

In the former case, the model is characterized by a Kähler potential such as

$$K = -\tfrac{1}{2}\left(\Phi - \bar{\Phi}\right)^2 + S\bar{S} + g(\Phi, \bar{\Phi})(S + \bar{S})\,. \tag{6.29}$$

and the same Φ-independent W as in Eq. (6.7). Similar to the previous case of Sec. 6.2.1, we have a consistent truncation along $\Phi = \bar{\Phi}$ provided the function g is symmetric under the exchange $\Phi \leftrightarrow \bar{\Phi}$. In the following, we will then assume g to be a real and symmetric function of its arguments.

Supersymmetry is still broken purely along the S-direction, with the F-terms equal to

$$D_\Phi W = 0 \qquad D_S W = M + g(\Phi, \bar{\Phi})W_0\,. \tag{6.30}$$

Note that the term in S now receives a Φ-dependent correction.

The scalar potential of this model now involves derivatives of g, and is given by

$$V(\Phi) = -3W_0^2 + \frac{\left[M + W_0\; g(\Phi, \Phi)\right]^2}{1 - g'(\Phi, \Phi)^2}, \tag{6.31}$$

along the inflationary trajectory $\Phi = \bar{\Phi}$.

Contrary to the models with a correction coupling to $S\overline{S}$, the task of finding the form of g which yields the desired inflationary potential V now requires solving a non-linear differential equation. This makes constructing models with low-scale supersymmetry breaking, such as the example (6.22), an intractable problem.

However, we can make some progress. In particular, in the regime

$$|g| \ll 1 \;\;,\;\; |g'| \ll 1, \tag{6.32}$$

the potential can be expanded perturbatively in g and g', as follows

$$V = (M^2 - 3W_0^2) + 2MW_0\, g + \mathcal{O}(g^2, g'^2)\,, \tag{6.33}$$

which is similar to the expansion (6.18). Therefore, the same quadratic model Eq. (6.24) can be constructed here via the choice

$$g(\Phi,\bar{\Phi}) = \frac{m^2}{2MW_0}\Phi\bar{\Phi}. \tag{6.34}$$

As in (6.25), the normalization of the inflationary potential in conjunction with the requirement that $|g| \ll 1$ forces M and W_0 to unobservably large values, corresponding to high-scale supersymmetry breaking.

6.3 α-Attractors from Kähler corrections to hyperbolic geometry

In the previous section we have considered the case of Kähler corrections which mix a nilpotent superfield S and a chiral one Φ, where the latter spans a flat internal manifold (i.e. zero Kähler curvature). However, typical Kähler potentials arising from string theory compactifications have often a logarithmic dependence on the moduli and describe a *hyperbolic geometry* (see [31] for an analysis of its properties in relation with the physical implications). When the latter is expressed in terms of half-plane variables, the presence of an $\overline{\mathrm{D3}}$ brane and a bulk field may be described by the following Kähler potential:

$$K = -3\alpha \log\left(\Phi + \bar{\Phi}\right) + S\bar{S}\,, \tag{6.35}$$

where the parameter α controls the value of curvature of the internal field-space, given by $R_K = -2/3\alpha$.

One can make the inversion and rescaling symmetries of this Kähler potential explicit by performing a Kähler transformation (which leaves the physics invariant) and obtain [32]

$$K = -3\alpha \log\left(\frac{\Phi + \bar{\Phi}}{2|\Phi|}\right) + S\bar{S}\,. \tag{6.36}$$

The latter can be regarded as the *curved* analogue of the flat and shift-symmetric Kähler potential (6.7). It indeed vanishes at $\Phi = \bar{\Phi}$ and $S = 0$ and it is again explicitly symmetric with respect to a shift of the canonically normalized field

$$\varphi = \pm\sqrt{\frac{3\alpha}{2}}\log\Phi\,. \tag{6.37}$$

The Kähler potential (6.36) then implies non-trivial kinetic terms of the field Φ, such as

$$K_{\Phi\bar{\Phi}}\,\partial\Phi\partial\bar{\Phi} = \frac{3\alpha}{\left(\Phi+\bar{\Phi}\right)^2}\,\partial\Phi\partial\bar{\Phi}\,, \tag{6.38}$$

thus inducing a boundary in moduli space, placed at both $\Phi \to 0$ and $\Phi \to \infty$ (note the symmetry under $\Phi \leftrightarrow 1/\Phi$).

When the field Φ moves away from this boundary, in the direction $\Phi = \bar{\Phi}$, the inflationary implications are very peculiar as they generically lead to a scalar potential which is an exponential deviation from a dS phase such as

$$V = V_0 + V_1\exp\left(-\sqrt{2/3\alpha}\;\varphi\right) + \ldots\,, \tag{6.39}$$

when expanded at large values of the canonical field φ. This yields universal cosmological predictions in excellent agreement with the latest observational data [26].

Some working examples of this phenomenon were already found in [33]. However, the general framework with a varying Kähler curvature in terms of the parameter α was developed by [25] and the corresponding family of models has been dubbed 'α-attractors'. Further studies have clarified that the attractor nature is simply a peculiar feature of the Kähler geometry of the sole inflaton sector, independently of the SUSY breaking directions and with a certain special resistance to the other fields involved [34]. It can indeed be realized by means of a single-superfield setup [35] (see also [36], in the case of flat geometry). The case where the bulk field Φ is coupled to a nilpotent sector S, via a Kähler potential equal to (6.35) or (6.36), has been investigated by [5].

In all the works cited above, the inflationary attractor dynamics arises due to a Φ-dependent superpotential $W = W(\Phi, S)$. In the Kähler frame defined by Eq. (6.36), it becomes manifest that such a W simply breaks the original scale-symmetry in Φ of the system (corresponding to a shift-symmetry in φ) thus generating non-trivial cosmological dynamics. Conversely, a Φ-independent W produces again a pure de Sitter phase such as the one given by Eq. (6.4).

One can then proceed in analogy to the previous Sec. 6.2 by including mixing terms of the form

$$\delta K = f(\Phi, \bar{\Phi}) S\bar{S} + g(\Phi, \bar{\Phi})(S + \bar{S}) \,, \tag{6.40}$$

in the Kähler potential (6.36), while keeping a superpotential just dependent on S, such as $W = W_0 + MS$. The resulting situation strikingly resembles the flat one and we find simply the same formulas in terms of the geometric variable Φ. Therefore, SUSY is broken just in the S direction as given by Eq. (6.12), in the case of bilinear nilpotent corrections, and as given by Eq. (6.30), in the case of terms linear in S. In these two cases, the scalar potential takes the form (6.14) and (6.31), respectively. The stability conditions results to be identical to the flat case as well.

There are, however, some important differences with respect to the flat case, which are worth highlighting:

- Inflation happens around the boundary of moduli space at $\Phi \to 0$ (or $\Phi \to \infty$). This implies that any polynomial Kähler correction in Φ (or in $1/\Phi$, if we expand around infinity), e.g. such as

  $$f = \textstyle\sum_{n=1}^{\infty} f_n |\Phi|^n \,, \quad g = \textstyle\sum_{n=1}^{\infty} g_n |\Phi|^n \,, \tag{6.41}$$

  with f_n and g_n some coefficients, is naturally small during inflation (i.e. $|f| \ll 1$ and $|g| \ll 1$, for bilinear and linear corrections respectively). In the inflationary regime, one can then consider the expansions (6.18) and (6.33).

- Unlike the flat case, the geometric variable Φ has non-trivial kinetic terms, being related to the canonical field φ by means of Eq. (6.37). This implies that any perturbative Kähler correction in Φ will naturally turn into exponential terms in the scalar potential $V(\varphi)$, thus easily allowing for plateau inflation as given by Eq. (6.39).

- In the case of small Kähler corrections, one realizes an exponential fall-off such as Eq. (6.39) with

  $$V_0 = M^2 - 3W_0^2 \,, \tag{6.42}$$

  and $V_1 = M^2$, for bilinear nilpotent terms (see Eq. (6.18)), and $V_1 = 2MW_0$, for linear nilpotent terms (see Eq. (6.33)).

- The pure nilpotent acceleration phase, equal to (6.42), thus serves as the Hubble inflationary energy rather than the CC (see e.g. Eq. (6.24)), whereas the perturbative Kähler mixing terms induce the inflationary slope. Qualitatively, an $\overline{\mathrm{D3}}$ brane provides the primordial acceleration which then gets a dependence on Φ, due to the interaction with the bulk geometry.

On the other hand, similar to the case of flat Kähler geometry, small corrections correspond to very high SUSY breaking scale, which is order Hubble or higher (the compensation between M and the gravitino mass $m_{3/2} = W_0$ determines indeed the inflationary plateau as given by Eq. (6.42)). Nevertheless, one can still obtain a desirable low value of the gravitino mass as this results again to be decoupled from the M. The contribution of the Kähler corrections to the SUSY breaking scale might indeed become important at the minimum of the potential (see Eq.(6.15) for bilinear corrections).

Finally, also in the hyperbolic case, the model allows for a residual cosmological constant. This is given by the finite contributions of the Kähler correction terms at the minimum of the potential, which can be placed at $\Phi = 1$ (provided we impose some conditions on the first derivatives of the functions f and g).

Examples

We conclude this section with some concrete examples by focusing just on bilinear nilpotent corrections ($g = 0$).

One can easily obtain a simple α-attractor model by considering

$$F(\Phi) = 1 + \sum_{n=1}^{\infty} \frac{c_n}{M^2} \Phi^n \,, \tag{6.43}$$

which is still related to f by means of Eq. (6.13). Therefore, during inflation (at $\Phi \simeq 0$), $F \simeq 1$ which corresponds to very small Kähler corrections f. Note that, for some choices of the coefficients c_n, Eq. (6.43) has provided quadratic inflation in the flat case (see Eq. (6.23)). However, once we assume hyperbolic Kähler geometry for the bulk field Φ, the corresponding scalar potential reads

$$V = M^2 - 3W_0^2 + \sum_{n=1}^{\infty} c_n e^{-\sqrt{\frac{2n^2}{3\alpha}}\varphi} \,, \tag{6.44}$$

in terms of the canonical inflaton φ and obtained by means of Eq. (6.14). The minimum of the potential can be set at $\varphi = 0$ (i.e. $\Phi = 1$), provided $\partial_\Phi F|_{\Phi=1} = 0$, that is

$$\sum_{n=1}^{\infty} n\; c_n = 0\,. \tag{6.45}$$

One can then control the residual cosmological constant of this model, at the vacuum of the potential, by tuning the several contributions, which add to

$$\Lambda = V(\varphi = 0) = M^2 - 3W_0^2 + \sum_{n=1}^{\infty} c_n\,. \tag{6.46}$$

Although the magnitude of M is order Hubble (or higher), the gravitino mass $m_{3/2} = W_0$ can be still tuned to phenomenologically desiderable values (e.g. order TeV).

This framework allows for remarkable phenomenological flexibility and one can reproduce several other known models of inflation. Another example is given by the so-called 'E-model' [37], defined by the potential

$$V = V_0 \left(1 - e^{-\sqrt{\frac{2}{3\alpha}}\varphi}\right)^{2n}\,, \tag{6.47}$$

which for $n = 1$ and $\alpha = 1$ returns the original Starobinsky model of inflation [38]. This is realized via the choice

$$F(\Phi) = \frac{V_0}{M^2}\,(1 - \Phi)^{2n} + F_0, \tag{6.48}$$

where the constant F_0 can be tuned in order to change the residual CC ($F_0 = 3W_0^2/M^2$ in the case of Minkowski vacuum).

6.4 Discussion

In this work we have developed models of inflation in supergravity where inflation is driven by terms in the Kähler potential which mix the inflaton field with a nilpotent superfield, even in the absence of a superpotential for the inflationary sector. The physical situation one would have in mind is given by an anti-D3 brane interacting with a bulk geometry. We have studied the effects of these additional terms when the internal geometry of the bulk field is either flat or hyperbolic, and found that this generically allows for inflation that exits to de Sitter space. The outcome is a scenario which allows for flexible phenomenology in terms of inflation, dark energy and supersymmetry breaking.

A general feature of these models is that SUSY is broken purely in the direction of the nilpotent superfield[4] S for the entire cosmological evolution, thus providing alone the necessary acceleration for inflation and the residual CC. Interestingly, the non-trivial Kähler corrections (which cannot be gauged away by a Kähler transformation) become the fundamental ingredient in order to have controllable level of supersymmetry breaking and dark energy at the vacuum of the potential (see the example defined by Eq. (6.43)).

The regime of small Kähler corrections is definitively important as one would expect these terms arising as subleading dynamical effects. The case of hyperbolic geometry is particularly interesting as perturbative Kähler corrections in the inflaton Φ are naturally small (as inflation happens at $\Phi \simeq 0$) and the consequent cosmological dynamics is an exponential deviation from a dS plateau at the Hubble scale. The physical picture is that of an anti-D3 brane, responsible for the inflationary acceleration, whose interaction with the bulk geometry induces the typical behaviour of α-attractors.

While we have explicitly studied the case of a Φ-independent superpotential, where the inflationary dynamics is purely Kähler driven, one may wonder what happens if the Kähler corrections considered here are incorporated into a model of inflation that is driven by the superpotential. In this case, W acquires a dependence on the inflaton, such as

$$W(\Phi, S) = A(\Phi) + B(\Phi)S\,. \tag{6.49}$$

One can prove that, in the case of hyperbolic Kähler geometry defined by Eq. (6.36), any Taylor expansion of the functions A and B in the geometric field Φ will contribute to the scalar potential $V(\varphi)$ with exponential terms, thus preserving the typical attractor behaviour (6.39). The situation is different in the case of flat Kähler geometry, as one generically needs a higher amount of fine-tuning in order to preserve the original superpotential-driven model of inflation. As a clear example of this circumstance, the famous model [28] of quadratic inflation, defined by $W = m\Phi S$, will be immediately spoiled by any generic polynomial Kähler correction in Φ.

Another interesting avenue of research would be to investigate the consequences of the

[4]This simplifies the description of the fermionic sector, whose non-linear terms disappear from the supegravity action, as already pointed out in the last reference of [4].

modified Kähler potentials studied here for the fermions, both at early and late times. Such Kähler potentials will lead to derivative interactions of the inflaton with the fermion bilinear, e.g.

$$\partial\Phi\,\partial\,(\psi\psi) \tag{6.50}$$

which may have important consequences for (p)reheating, or leave an imprint in the spectrum of primordial curvature perturbations.

Finally, we note that there are many possible generalizations of the models presented here, similar to the series of developments of superpotential-driven models of inflation considered in [4]. It would be interesting to understand the extent to which the same is possible for Kähler potential driven models of inflation.

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Part IV

Cosmology

Chapter 7

Backreaction and Stochastic Effects in Single Field Inflation

L. Perrault-Levasseur and E. McDonough *Backreaction and Stochastic Effects in Single Field Inflation*, Phys.Rev. D91 (2015) 063513 arXiv:1409.7399[hep-th].

Addendum for Thesis

This final chapter deviates significantly from the previous four, and is unrelated to string theory. Instead, this chapter concerns itself with the semi-classical quantum field theory description of cosmological perturbations.

In standard Wilsonian effective field theory, the detailed physics of UV degrees of freedom is decoupled from the physics of IR degrees of freedom. The UV degrees of freedom (e.g. heavy particles) can be integrated out to arrive at an effective description for the IR degrees of freedom, at which point the UV degrees of freedom can be completely neglected.

The situation is quite different in the inflationary universe. The energy of a field excitation redshifts, and hence UV modes will later become IR modes. The UV and IR degrees of freedom are thus intrinsically coupled. However, one would still hope to construct an effective field theory for IR degrees of freedom, and the present chapter makes progress in doing this.

Abstract

The formalism of stochastic inflation is a powerful tool for analyzing the backreaction of cosmological perturbations, and making precise predictions for inflationary observables. We demonstrate this with the simple $m^2\phi^2$ model of inflation, wherein we obtain an effective field theory for IR modes of the inflaton, which remains coupled to UV modes through a classical noise. We compute slow-roll corrections to the evolution of UV modes (i.e. quantum fluctuations), and track this effect from the UV theory to the IR theory, where it manifests as a correction to the classical noise. We compute the stochastic correction to the spectral index of primordial perturbations, finding a small effect, and discuss models in which this effect can become large. We extend our analysis to tensor modes, and demonstrate that the stochastic approach allows us to recover the standard tensor tilt n_T, plus corrections.

7.1 Introduction

Inflation has been tremendously successful in explaining the physics of the very early Universe. It was the first compelling cosmological model to provide a causal mechanism for generating fluctuations on cosmological scales, and it predicted that their spectrum should be almost scale invariant, with small deviations from scale invariance that can be traced back to the precise microphysics of inflation [1–3]. These predictions provide a way of connecting theoretical physics to observational cosmology; this has been a very fruitful venture, as has lead to particle-physics based models of inflation [4], inflation in supergravity [5], and string inflation [6], to name a few. There is still much to be learned from the CMB, and if the large tensor-to-scalar ratio of [7] is a hint of good things to come, then the CMB may yet give us an unprecedented opportunity to test models of inflation and quantum gravity.

With the ever-increasing precision of experiments probing the CMB, for example [8], it becomes imperative to develop self-consistent methods of calculation for inflationary predictions. The formalism of stochastic inflation is a promising avenue in this direction. It allows for the constant renormalization of background dynamics and in this way circumvents one of the main difficulties of traditional methods: backreaction [9–14]. This is achieved by separating the dynamics of long, classical wavelengths from short, quantum fluctuation-dominated wavelengths, and studying the interplay of the two sectors. The stochastic formalism then allows for the resummation of corrections to the background dynamics as modes of fluctuations are stretched from the quantum regime into the coarse-grained effective theory.

The resulting theory describes the effective classical dynamics of a large-scale gravitational system, in the presence of a 'bath' where all the quantum fluctuations are collected in a classical noise term, through a set of Langevin equations. As required by the fluctuation-dissipation theorem, this noise term comes hand in hand with a dissipation term, which in turn allows for irreversibility and approach to equilibrium. The effective theory therefore belongs to a new class of non-Hamiltonian theories [15], which have not been studied much so far in the context of cosmology. (However, see [16] and references therein in the context of warm inflation.)

Stochastic inflation has a long history. Originally proposed by Starobinsky [17, 18], stochastic inflation as studied in the early work of [19–32] was a simple way to include quantum effects into inflation. The idea was this: quantum fluctuations are generated deep inside the horizon and, at zeroth order in slow-roll, evolve as quantum fields on a fixed de Sitter background. The quantum modes grow and exit the horizon. Doing so, due to their random phase, they provide a kick of a random amplitude to the long-wavelength physics. It follows that the quantum modes act as a source for the classical background, and the physics of this source is probabilistic in nature. More precisely, stochastic inflation provided an 'educated guess' that this source should be white noise. The physics of slow-roll inflation can then be studied as per the usual treatment, with the noise included as a source in the equation of motion for the classical (long-wavelength) field.

Stochastic inflation was put on a more solid footing by [33] and [34], where the equations of motion for stochastic inflation were derived from a path integral [35–41]. Given these equations of motion, the vast majority of modern applications of stochastic inflation take the same approach as Starobinsky: calculate the variance of quantum modes in a pure de Sitter background, include this as white noise in the Klein-Gordon equation for long-wavelength modes, and study slow-roll inflation in the presence of this white noise (see, e.g., [42–51]). However, this method misses a key element of the physics: as pointed out in [52] the short-wavelength and long-wavelength physics are coupled. Namely, the quantum modes do not evolve on a pure de Sitter background, but rather on a background that is both slow-roll and stochastically corrected. In terms of the path integral, the coupling of the two sectors (long-wavelength, or 'coarse-grained' fields, and short-wavelength, or 'bath' fields) manifests itself as loop diagrams calculated in the Schwinger-Keldysh 'in-in' formalism of quantum field theory, which has become widely applied in cosmology since [53, 54], after the early work of [55, 56] (however, see [57, 58] and references within for an introduction in the context of out-of-equilibrium QFT and open systems). This approach was developed in [59], where it was dubbed the 'recursive formalism of stochastic inflation'.

Cosmological perturbations have also been studied in the context of stochastic inflation, see for example [46] and [60, 61]. We will use a method inspired from the approach used in [46], with some modifications that will be discussed in section VI. An alternative, and relatively recent, proposal [60,61] is to apply the δN formalism to stochastic inflation. This makes intuitive sense: the δN formalism can be qualitatively understood as a 'separate universe approach', and one would not expect a local noise to invalidate this approach. This approached will also be touched upon in section VI.

In [62], the recursive formalism was applied to hybrid inflation [63]. In this scenario, the spectral index is strongly dependent on the duration of the 'waterfall phase' of inflation [64], where the field dominating the energy density of the Universe during inflation becomes tachyonic and 'waterfalls' down the side of the potential. This generates a red tilt, provided that the waterfall phase lasts for a suitable number of e-folds. It was found in [62] that the recursive corrections caused the tilt of the inflaton perturbations to become bluer in the valley, while also causing the waterfall phase to end earlier than otherwise expected, making a red tilt much more difficult, if not impossible, to achieve.

In the present paper we have more modest goals, that is, to study recursive stochastic effects in single field inflation, both analytically and numerically, in particular the simple $m^2\phi^2$ model [31, 65, 66], away from the regime of eternal chaotic inflation [67–69].[1] We find that the recursive approach gives corrections to quantum modes that could not be deduced from naively including slow-roll effects alone. We then study the effect of this on long-wavelength perturbations, and calculate the power spectrum of primordial perturbations. We extend this approach to include tensor perturbations, and discuss the effect of couplings to heavy fields.

The outline is as follows: in section II we outline the usual approach to stochastic inflation and review the recursive formalism. In section III we calculate the classical noise induced by quantum fluctuations on a classical background which is zeroth order in slow-

[1] See also [70–73] and references therein for existing studies of stochastic eternal inflation.

roll. In section IV we study the effect of this noise by computing the stochastic (and slow-roll) corrected classical background, and continue in section V to compute the backreaction on the quantum modes. We then use this in section VI to compute the backreaction on IR modes and the spectrum of curvature perturbations. We extend this to a class of simple multifield models in section VII, and to tensor modes in section VIII. We conclude and discuss our results in section IX.

7.2 Stochastic Inflation: Basic Setup and Recursive Strategy

Let us first consider the action of a single scalar field in a fixed background. The matter part of the action is given by:

$$S_M = \int d^4x\sqrt{-g}\left(-\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - V(\Phi)\right) , \tag{7.1}$$

which leads to the equation of motion

$$-\Box\Phi + V_{,\Phi} = 0 \,, \tag{7.2}$$

$$\Box = -\partial_{tt} - 3H\partial_t + \frac{\nabla^2}{a^2} \,. \tag{7.3}$$

In the present paper, we will more specifically be interested in the chaotic potential

$$V(\Phi) = \frac{1}{2}m^2\Phi^2 \,. \tag{7.4}$$

Moreover, to ensure that we remain away from the eternal inflation regime throughout our analysis, we impose the condition $m\Phi_0^2/M_{\rm Pl}^3 \ll 8\pi$ throughout this paper [69], where Φ_0 is the initial value of the inflaton at the beginning of inflation.

The starting point of stochastic inflation is to split the field Φ into long-wavelength modes ϕ_c (c for classical), and short-wavelength modes ϕ_q (q for quantum). Note that both ϕ_q and ϕ_c are quantum fields in nature; ϕ_c technically corresponds to a quantum averaged field, coarse grained on a radius of constant physical size. We choose this coarse-graining scale to be the scale at which quantum fields undergo squeezing, i.e. the Hubble scale., at which point the commutators of the fields and their derivatives scale as k/a and are therefore exponentially suppressed (see [15, 30, 36–38, 41, 51, 55, 74–79] and references therein concerning the topics of quantum average versus classical fields, decoherence, and the conditions required for classicalization).

The splitting into ϕ_c and ϕ_q is defined by

$$\Phi = \phi_c + \phi_q \, , \tag{7.5}$$

$$\phi_q = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} W(k,t) \hat{\Phi}_k e^{-ik\cdot x} \, , \tag{7.6}$$

where the $\hat{\Phi}_k = \phi_k \hat{a}_k + \phi^*_{-k} \hat{a}^\dagger_{-k}$ is the mode expansion of the quantum fields in terms of creation and annihilation operators, and $W(k,t)$ is a time-dependent window function. The window function acts to project onto ϕ_q only the modes with a comoving wavenumber somewhat larger than the physical Hubble scale. To be precise, we take the filtering scale to be large enough to make sure that all modes that are part of ϕ_c have undergone squeezing and classicalization. To see this splitting at the level of the equation of motion, we can Taylor expand the equation of motion[2] about $\Phi = \phi_c$,

$$-\Box\phi_c + V_{,\Phi}(\phi_c) + [-\Box\phi_q + V_{,\Phi\Phi}(\phi_c)\phi_q] = -\frac{1}{2} V_{,\Phi\Phi\Phi}\phi_q\phi_q + \dots \, , \tag{7.7}$$

where our perturbation variable has been chosen to be the number of quantum fields (which can be seen from the path integral formulation to be equivalent to counting powers of $\hbar$ in a Schwinger-Keldysh loop expansion).

Given that the coarse-graining radius is chosen to correspond to the classicalization radius, the quantum-averaged field ϕ_c corresponds to an effective classical field which we call $\tilde{\phi}_c$, endowed with a probability density function (PDF). This effective classical field $\tilde{\phi}_c$ allows us to treat collectively all realizations of the universe with consistent histories. Its PDF gives different probabilistic weights to classical realizations coming from different sets of random phases of the mode functions as they successively cross the Hubble radius and freeze. The PDF allows for a notion of ensemble average, which is equal to the quantum expectation value provided the ergodic hypothesis is satisfied. This point is further clarified in [80] and discussed in more details in [59].

It follows that alongside the system $\{\phi_c \, , \, \phi_q\}$, {coarse-grained quantum field , small

[2] In a dynamical spacetime H depends on the *full* quantum field Φ, and hence the extension of this heuristic argument to realistic inflationary setups is slightly more involved, although conceptually the derivation is identical.

scale quantum fluctuations}, we can write a corresponding classical, probabilistic system consisting of $\tilde{\phi}_c$ and a set of classical Gaussian noises $\xi_{1,2}$ modeling the effects of the incoming modes of ϕ_q joining the coarse-grained theory. Using the definition (7.6) to rewrite the ϕ_q in square brackets in terms of their linear mode expansion, as well as the fact that the linearized mode functions ϕ_k satisfy their linearized equation of motion, equation (7.7) can be rewritten as

$$-\Box\tilde{\phi}_c + V_{,\Phi}(\tilde{\phi}_c) = 3H\xi_1 + \dot{\xi}_1 - \xi_2 \quad + \left(-\frac{1}{2}V_{,\Phi\Phi\Phi}\phi_q\phi_q + ...\right) , \tag{7.8}$$

where the only surviving terms in the square brackets (i.e. the ones containing at least one time derivative acting on the time-dependent window function $W(k,t)$) have been defined as the classical noise. A simple calculation, again using only equation (7.6), reveals that the noise terms are drawn from a random Gaussian probability distribution given by

$$\mathcal{P}\left[\xi_1, \xi_2\right] = \exp\left\{-\frac{1}{2}\int d^4x d^4x' [\xi_1(x)\,\xi_2(x)]\mathbf{A}^{-1}(x,x')\left[\begin{array}{c}\xi_1(x')\\ \xi_2(x')\end{array}\right]\right\} , \tag{7.9}$$

and we have defined, letting $r = |\mathbf{x} - \mathbf{x}'|$, the matrix A to have components given by

$$\mathbf{A}^{i,j}(x,x') = \int \frac{dk}{2\pi^2}\frac{\sin(kr)}{kr}\partial_t W\,(k,t)\,\partial_{t'}W\,(k,t')\,\mathrm{Re}\left[\mathbf{M}^{i,j}(k,t,t')\right] , \tag{7.10}$$

with

$$\mathbf{M}^{i,j}(\mathbf{k},t,t') = \begin{pmatrix} \phi_{\mathbf{k}}(t)\phi^*_{\mathbf{k}}(t') & \phi_{\mathbf{k}}(t)\dot{\phi}^*_{\mathbf{k}}(t') \\ \dot{\phi}_{\mathbf{k}}(t)\phi^*_{\mathbf{k}}(t') & \dot{\phi}_{\mathbf{k}}(t)\dot{\phi}^*_{\mathbf{k}}(t') \end{pmatrix} , \tag{7.11}$$

where $\mathbf{k}$ in the above equation is larger than the coarse-graining scale. The terms written in parenthesis on the right-hand side (r.h.s.) of (7.8), which are the only ones still containing quantum fields, can also be rewritten in terms of classical noise terms,[3] as was done in [59]. However, for the case of a quadratic potential, which we shall consider here, these higher-order terms vanish and hence will not contribute to equation (7.8). To emphasize the split between quantum and classical: the modes ϕ_q are quantum, while the noises $\xi_{1,2}$ are *classical*, as the noise terms appearing in equation (7.8) are evaluated at the moment the modes ϕ_q exit the horizon and give a 'kick' to the IR (classical) theory.

[3] In fact, equation (7.8) is inconsistent *unless* this is done, since as written they are a quantum contribution to a classical equation of motion.

The variance of ξ_1 and ξ_2 can be read directly from this definition, by equating ensemble averages and quantum expectation values under the ergodicity assumption. To solve for the stochastic background, it is necessary to solve simultaneously for the linear mode function of the bath field, which satisfies the equation:

$$\left(\partial_t^2 + 3H\partial_t - \frac{k^2}{a^2} + m^2\right)\phi_{\mathbf{k}}(t) = 0\,, \tag{7.12}$$

where the wavenumber $\mathbf{k}$ is larger than the coarse-graining scale, i.e. for wavelengths smaller than the coarse-graining radius. In what follows, we will be interested in solving the classical system $\{\tilde{\phi}_c, \xi_q\}$ perturbatively and will not look any further at the quantum averaged field ϕ_c. We therefore drop the tilde for the sake of simplicity, and from now on by ϕ_c we mean the classical, stochastic coarse-grained field.

This is not an easy system to solve: the coarse-grained field ϕ_c, which obeys (7.8), depends on the amplitude and statistics of the noise terms $\xi_{1,2}$, which are given in terms of the mode functions of the quantum field, $\phi_{\mathbf{k}}$. These mode functions in turn depend on a specific realization of the background in which they evolve, through their self-energy, which acts as a ϕ_c-dependent mass term.

A word on the precise structure of the perturbative expansion: we will solve the system $\{\phi_c, \xi\}$ perturbatively in the number of quantum fields, N_q, and in the slow-roll parameter ϵ. For example, the solution for ϕ_c at order $(\epsilon^0, N_q = 0)$ corresponds to a constant background field with no stochastic corrections. A calculation of the quantum mode function can be done in this background, which is now at order $(\epsilon^0, N_q = 1)$. An equivalent counting is in powers of $\sqrt{\hbar}$, which counts loops in the Schwinger-Keldysh formalism, i.e. $N_q = 1$ corresponds to $\mathcal{O}(\sqrt{\hbar})$. This was shown in [59]. Given a quantum mode valid at order $\sqrt{\hbar}$, the variance of the noise can consistently be computed at order $\hbar$, and hence the PDF of IR modes can also be computed at order $\hbar$. In particular, the variance of IR modes (which encodes the spectral tilt), is valid at order $\hbar$. We will use this notation extensively in this paper.

7.3 First Step of the Recursion: Stochastic Noise

To make progress with the system (7.8) and (7.12) while maintaining the consistency of the solution, and in order to capture the fact that the quantum modes sit in a stochastic background, we use the recursive method from [59]. At step 0 of the method, we start by approximating the background to zeroth order in slow-roll (i.e. pure de Sitter space). Step one of the method is to find the amplitude of the noise in this zeroth-order background.

More precisely, we want to calculate the amplitude of the classical noise arising from quantum fluctuations evolving in such a background and joining the coarse-grained theory. That is, we want to calculate $\xi_{1,2} = \xi_{1,2}^{(1)}$ to order $\{\epsilon^0, \sqrt{\hbar}\}$ by solving equation (7.12) and then use equation (7.10) to find the statistical properties of the noise terms. Fortunately, it is possible to show (after simple algebra) that ξ_2 is always suppressed by some power of the slow-roll parameters, since it is proportional to at least one time derivative of the quantum field mode function ϕ_k, and hence, at this order in perturbation theory, it is sufficient to calculate ξ_1 alone.

We start by making an explicit choice of the window function

$$W(k,t) = \theta(k - \gamma a H)\,, \tag{7.13}$$

where $\gamma \ll 1$ parametrizes how long after their Hubble crossing modes can be considered classicalized, and so acts as an 'ignorance parameter' (however, see [81–83] for discussions of subtleties concerning this choice and explorations of different possibilities).

Using this, and changing time variables to the number of e-folds (for reasons that will be discussed in sections 7.4 and 7.6), the variance can be computed to the required order in slow-roll at this stage of the recursive method

$$\langle \xi_1^{(1)}(\mathbf{x}, N)\xi_1^{(1)}(\mathbf{x}', N')\rangle = \frac{\gamma^3 H^5}{2\pi^2}\frac{\sin(\gamma a H r)}{\gamma a H r} a^3 \left|\phi_{\mathbf{k}}(N)\right|^2_{k=\gamma a H} \delta(N - N'), \tag{7.14}$$

where we have used the PDF (7.9), leaving the mode function unspecified in (7.11), to explicitly calculate the variance of the noise ξ_1, keeping only terms of order ϵ^0 (where we define the first slow-roll by $\epsilon = -\dot{H}/H^2$). Here the modes $\phi_{\mathbf{k}}$ need to be evaluated at the time when they join the coarse-grained scales, which can be evaluated using the usual expression for mode functions in the $\mathcal{O}(\epsilon^0)$ de Sitter background,

$$\left|\phi_{\mathbf{k}}(N)\right|^2_{k=\gamma a H} = \frac{H^2}{2(\gamma a H)^3}, \tag{7.15}$$

and it follows that the mean and variance of the noise are given by

$$\langle \xi_1^{(1)}(\mathbf{x}, N) \rangle = 0 \quad , \quad \langle \xi_1^{(1)}(\mathbf{x}, N) \xi_1^{(1)}(\mathbf{x}', N') \rangle = \frac{H^4}{4\pi^2} \frac{\sin(\gamma a H r)}{\gamma a H r} \delta(N - N'). \tag{7.16}$$

The variance is constant and proportional to $\delta(N-N')$, and hence ξ_1 acts as white Gaussian noise[4] with zero mean. The noise variance is local in time, and although it might appear to be nonlocal in space, the $\frac{\sin(\gamma aHr)}{\gamma aHr}$ factor in fact acts as a theta function at the coarse-graining radius, being one within the coarse-graining length, and zero outside. This ensures that the noise is only (100%) correlated within each coarse-graining region, but is not correlated between different regions. Equivalently, this can be stated by saying the nonlocalities are only within the coarse-graining scale, and so the coarse-grained theory remains local.

7.4 Step Two: Stochastically-Corrected Coarse-Grained Theory

7.4.1 Analytic solution

In the previous subsection we assumed a classical nondynamical background and used it to calculate the noise $\xi_1 = \xi_1^{(1)}$ to order $(\epsilon^0, \hbar^{1/2})$. Using this, we can calculate the corrected classical background $\phi_c^{(1)}$ at order $(\epsilon^1, \hbar)$. To do this, we solve equation (7.8), which is a Langevin equation for $\phi_c^{(1)}$ at this order [5]:

$$\dot{\phi}_c^{(1)}(\mathbf{x}, t) + \frac{V_{,\Phi}\left(\phi_c^{(1)}(\mathbf{x}, t)\right)}{3H_0} = \xi_q^{(1)}(\mathbf{x}, t) \ . \tag{7.17}$$

In the above, all quantities are valid to zeroth order in slow-roll. In particular, as should be explicit from the previous section, the variance of the noise term ξ_1 is valid to leading order in $\hbar$ but to zeroth order in slow-roll. Consistent perturbation theory then requires that the Hubble parameter appearing in (7.17) also be evaluated at zeroth order in slow-roll, and hence is simply constant.

Solving (7.17) gives a solution valid at $\mathcal{O}(\epsilon, \hbar)$. At this stage in the recursive method, we are considering quasi-de Sitter space rather than a nondynamical de Sitter spacetime. Gauge fixing therefore becomes necessary and we choose to work with gauge-invariant

[4] This will not remain true at higher orders in the recursive method: the noise will become colored due to interactions of the bath and the system, as was discussed in [32].

[5] Recall, as mentioned above equation (7.13), that the noise terms $\dot{\xi}_1$ and ξ_2 are higher order in slow-roll.

variables. In the stochastic formalism, this can be achieved by using the number of e-folds elapsed since the beginning of inflation, N, as the time variable. Recalling that we are working with the potential

$$V(\Phi) = \frac{1}{2}m^2\Phi^2, \tag{7.18}$$

we get

$$\frac{\mathrm{d}\phi_c^{(1)}(\mathbf{x},N)}{\mathrm{d}N} = -\frac{m^2\phi_c^{(1)}(\mathbf{x},N)}{3H_0^2} + \frac{\xi_1^{(1)}(\mathbf{x},N)}{H_0}\,. \tag{7.19}$$

The advantage of using N as the time variable is that linear order perturbations of the resulting stochastic process $\phi_c^{(1)}$ then coincide with the Mukhanov gauge-independent variable, as shown in [47]. This is because in terms of the number of e-folds, Taylor expanding to linear order the full fields equations of motion yields the gauge-fixed perturbation equations. This will be discussed further in section 7.6.

The solution to equation (7.19) can easily be written in terms of an integral equation:

$$\phi_c^{(1)}(\mathbf{x},N) = \phi_c^{(1)}(\mathbf{x},0)\exp\left[-\frac{m^2}{3H_0^2}N\right] + \frac{H_0}{2\pi}\exp\left[-\frac{m^2}{3H_0^2}N\right]\int e^{\left[\frac{m^2}{3H_0^2}N\right]}\tilde{\xi}(\mathbf{x},N)dN\,, \tag{7.20}$$

where we have made the rescaling $\xi_1^{(1)} = \frac{H_0^2}{2\pi}\tilde{\xi}$, and $\tilde{\xi}$ is therefore a regular Brownian motion with unit variance. From this, along with equation (7.16) for the statistics of the noise, we see that the incoming quantum modes leave the mean (i.e. the first-order moment) of the effective classical background unaffected and only modify higher moments of the background PDF.

Other derived stochastic quantities can also be calculated from the coarse-grained field $\phi_c^{(1)}$ at this order. For example, the slow-roll parameter ϵ will now have a stochastic piece (the same is true of the time-dependent Hubble parameter $H(t)$), which can be expressed as:

$$\epsilon \equiv -\frac{\dot{H}(t)}{H(t)^2} = \frac{\left(\dot{\phi}_c^{(1)}\right)^2}{2H^2(t)} - \frac{\xi_1\dot{\phi}_c^{(1)}}{2H^2(t)} = \left(\frac{\mathrm{d}\phi_c^{(1)}}{\mathrm{d}N}\right)^2\frac{H_0^2}{2H^2(t)} - \xi_1\frac{\mathrm{d}\phi_c^{(1)}}{\mathrm{d}N}\frac{H_0}{2H^2(t)}\,. \tag{7.21}$$

This can be rewritten as,

$$\epsilon = \epsilon^C + \epsilon^\xi = \frac{m^2}{3H_0^2} - \frac{H_0}{2\pi}\frac{\tilde{\xi}}{\phi_c^{(1)}}\,, \tag{7.22}$$

where we have separated the first slow-roll parameter into a classical piece ϵ^C and a stochastic piece ϵ^ξ .

It is important to note that (7.20) is not the most general expression to characterize solutions to (7.19), since each individual solution is only one realization the stochastic process $\phi_c^{(1)}$, as is made explicit by the presence of the Wiener process $\tilde{\xi}$ in the expression. Alternatively, we can solve for the probability density function $\rho(\phi_c^{(1)}(\mathbf{x}, N))$, which gives the probability of a field configuration over the whole length of inflation, using a Fokker-Planck equation. Another, perhaps simpler, option is to solve (7.19) numerically, by solving many different realizations and from there inferring the shape of the underlying PDF using Bayes' theorem. This can be done by maximizing the likelihood on the μ-σ space (provided we assume the PDF is Gaussian) or, in the absence of Gaussianity, by finding the 68% confidence levels. In the following, this is the strategy we will adopt.

7.4.2 Numerical solution to the coarse-grained theory

In terms of the rescaled noise $\tilde{\xi}$, the stochastic differential equation (SDE) to solve is:

$$\frac{d\phi_c^{(1)}}{dN} = -\frac{m^2\phi_c^{(1)}}{3H_0^2} + \frac{H}{2\pi}\tilde{\xi}(\mathbf{x}, N)\,, \tag{7.23}$$

$$\langle\tilde{\xi}(N)\tilde{\xi}(N')\rangle = \delta(N - N')\,. \tag{7.24}$$

In order to discretize the SDE, we need to discretize the time delta function in (7.24):

$$\delta(N - N') = \begin{cases} 1/\delta N, & \text{if } N \text{ and } N' \text{ are in the same time step } \delta N\,, \\ 0, & \text{otherwise.} \end{cases} \tag{7.25}$$

Here, δN is the integration time step used in the numerical solver. This simple SDE can be solved using the Euler method to integrate:

$$\phi_{n+1}^{(1)} = \phi_n^{(1)} - \left[\frac{m^2\phi_n^{(1)}}{3H_0^2}\right]\delta N + \frac{H_0}{2\pi}\tilde{\xi}_n\,, \tag{7.26}$$

where the $\tilde{\xi}_n$ are independent random numbers drawn from a random normal distribution with standard deviation $\sqrt{\delta N}$.

After simulating a large number of realizations of this coarse-grained background, the underlying PDF of the random variable $\phi_c^{(1)}$ can be reconstructed. This can be done

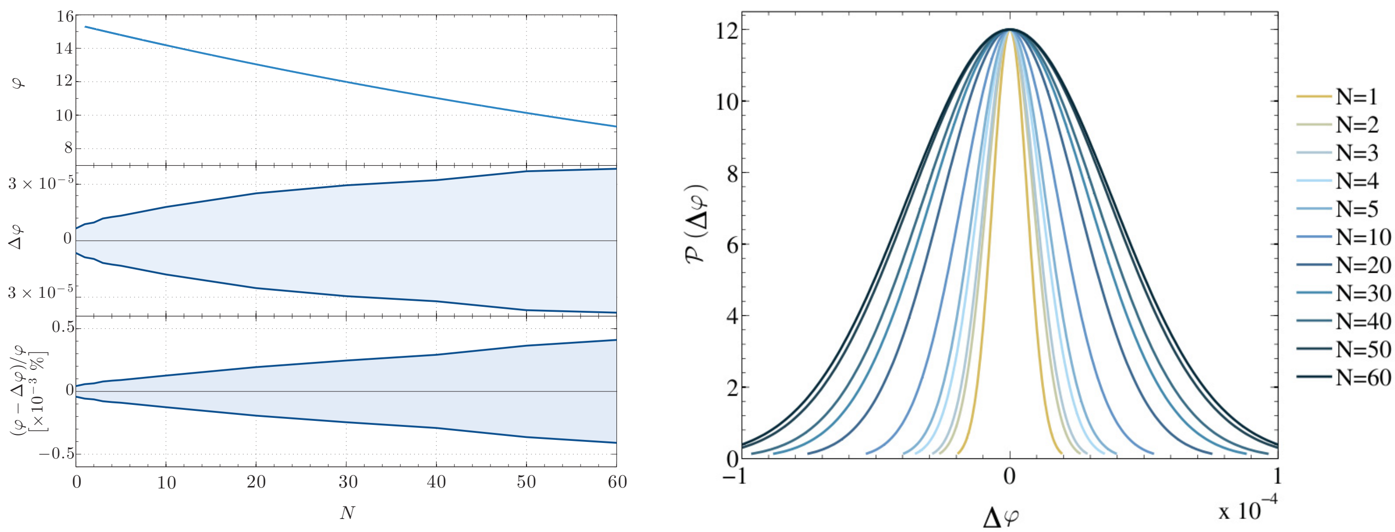

Figure 7.1 *Left panel*: Top: mean trajectory of the coarse grained inflaton φ. Note that the large discrepancy of $\varphi(N = 60)$ compared to the standard slow-roll value is simply due to the fact that at this level in the recursion, we solved the EoM for φ keeping $H = H_0$, a constant. This means that by the end of inflation, we can expect corrections to our coarse-grained solution as large as $\epsilon_1 N$. Middle: 1σ deviation from the mean trajectory, i.e. standard deviation of maximum likelihood. Bottom: percentage error at 1σ. *Right panel*: The reconstructed (un-normalized) PDF of the random variable $\varphi(N) - \langle\varphi(N)\rangle$ for a few fixed e -folds during inflation, and assuming inflation lasted 60 e-folds. These values were inferred from 250 realizations, with $m = 6 \times 10^{-6} M_{\text{Pl}}$, $N_{tot} = 60$, and $\phi_c(0) = \sqrt{2(2N_{tot} - 1)}$. All fields shown are in units of M_{Pl}.

by assuming an underlying Gaussian PDF and sampling the likelihood of the μ-σ space parametrizing the possible Gaussians to find the maximum likelihood. This means, at each time step, and for every plausible value of μ and σ, we apply Bayes theorem to find the probability that values from all realizations of $\phi_c^{(1)}$ are drawn from the Gaussian defined by a given choice of μ and σ.

The resulting PDF for a few fixed e-folds over the course of inflation are shown for the $m^2\phi_c^2$ potential in Figure 7.1. One can see that the variance of the long-wavelength field is initially zero, as one would expect from the fact that the PDF was initialized as a delta function in probability space. Its variance then grows as more modes exit the horizon and join the coarse-grained theory, which can be seen in the middle panel on the left-hand side (l.h.s.), as well as the plot on the r.h.s. The fractional variance of the field also grows during this time period, as shown in the bottom panel of the l.h.s.

The absolute variance does not grow indefinitely. As can be seen from both the l.h.s. and r.h.s. panels, the variance saturates during the last 10 e-folds of inflation, approaching a maximal value that can easily be estimated from (7.19):

$$\sigma^2_{\phi_c^{(1)}} \equiv \left\langle (\phi_c^{(1)})^2 \right\rangle - \left\langle (\phi_c^{(1)}) \right\rangle^2 \;\rightarrow\; \frac{3H_0^4}{8\pi^2 m^2} \,. \tag{7.27}$$

Since the field is massive, this is as one should expect: quantum fluctuations becoming classical push the field fluctuations to roll up their potential, but the shape of the potential tends to make the field roll back down to its minimum. These two competing effects eventually reach an equilibrium point, which can be calculated in standard perturbative analysis to coincide with (7.27).

As a final comment, note that, (7.27) being a constant, the power spectrum of curvature perturbations that we obtain at this stage in the recursive method is exactly flat. That is to say, the maximal equilibrium value that the field fluctuations reach is constant with time, which is consistent with a constant push from the incoming quantum modes. If the spectrum were tilted, this would correspond to kicks with time-dependent amplitude, and this would in turn modify the quasiequilibrium position for $\sigma^2_{\phi_c^{(1)}}$, making it time dependent. This is what we will observe in the next level of recursion.

We can apply the same method to infer the underlying PDF of the slow-roll parameter ϵ, displayed in Figure 7.2, which exhibits a qualitative behavior similar to ϕ_c. These graphs depict an interesting perspective: the super-Hubble classical theory we obtain is a 'fuzzy' one, in the sense that the classical parameters have an inherent uncertainty stemming from the constant incoming quantum modes. Therefore, on the large scales of the coarse-grained theory (by which we mean on scales of many Hubble volumes), the value of ϵ varies from point to point with a standard deviation shown in Figure 7.2.

Furthermore, even at a *single point* the value of the classical parameters, such as ϵ, are constantly fluctuating. In particular, this means that, when one averages over macroscopic timescales, there is a minimum possible value for the slow-roll parameter ϵ. Indeed, even in the limit where H_0 (or equivalently ϕ_0) starts out very large, in such a way that ϵ^C as defined in (7.22) tends to 0, the contribution from ϵ^ξ will always remain finite. This is true in general for any model of inflation: the root-mean-square of ϵ^ξ always provides a

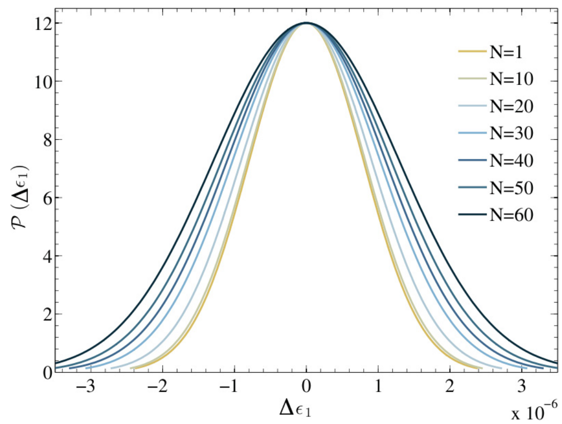

Figure 7.2 The reconstructed (un-normalized) probability density function of the slow-roll parameter ϵ_1 (the subscript 1 denotes first slow-roll parameter) for a few fixed e-folds during the last 60 e-folds of inflation. These values were inferred from 250 realizations, with $m = 6 \times 10^{-6} M_{\rm Pl}$, $N_{tot} = 60$.

minimal value of the first slow-roll parameter, regardless of how small it is engineered to be classically. It is worth stressing how the picture that we obtain differs from the standard one: the super-Hubble theory is now fundamentally probabilistic, and each realization of the quantum modes in the bath sees one of this ensemble of fluctuating field trajectories as a background.

7.5 Step Three: Quantum Fields Evolving on a Stochastic Background

7.5.1 Analytic solution

Now that a solution to (7.8) valid to $\mathcal{O}(\epsilon, \hbar)$ has been found, we can go back to the bath (i.e. short-wavelength) sector of the theory and solve the linearized mode function of the quantum field to $\mathcal{O}(\epsilon, \hbar)$. This will allow us to find the noise variance to leading order in slow-roll.

To do this, we treat the twice-corrected quantum modes, which we denote $\phi_q^{(2)}$, as perturbations about a fixed background field $\phi_c^{(1)}$. It is important to note that this procedure requires a careful treatment of metric perturbations, the necessity of which was realized in [84], where the authors considered the backreaction of cosmological perturbations (i.e. the effect of second-order perturbations on the background) in the presence of stochastic effects. However they did *not* consider the backreaction of the shifted background on the noise itself, which is *precisely* what we are interested in here.

Before we progress further, we must first fix a gauge. The most natural choice is the same gauge as classical perturbations of ϕ_c, that is, the spatially flat gauge. Following the treatment of [85], we fix the gauge to the spatially flat gauge, and find the equation of motion for the field perturbations:

$$\ddot{\phi}_q^{(2)} + 3H\dot{\phi}_q^{(2)} + \left[-\frac{\nabla^2}{a^2} + m^2 - \frac{1}{M_{\rm Pl}^2 a^3}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a^3\left(\dot{\phi}_c^{(1)}\right)^2}{H} \right) \right]\phi_q^{(2)} = 0\,, \tag{7.28}$$

It is important to note that $\phi_q^{(2)}$ is *a corrected version* of $\phi_q^{(1)}$, as opposed to an additional contribution to the mode function ϕ_q. Hence, we impose the Bunch-Davies initial condition on $\phi_q^{(2)}$.

Solving the gauge constraints, transforming to Fourier space, and using the canonically normalized variable $v_k = a\phi_{\mathbf{k}}$, we find the equation of motion:

$$v_k'' - \left\{ k^2 - \frac{2}{\tau^2} - \frac{9\epsilon}{\tau^2} + \frac{V_{\Phi\Phi}(1+\epsilon)^2}{H^2(t)\tau^2} + 4\frac{\dot{H}(t)}{H(t)}\frac{\ddot{\phi}_c^{(1)}}{\dot{\phi}_c^{(1)}}\frac{(1+\epsilon)^2}{H^2(t)\tau^2} - 14\frac{\epsilon^2}{\tau^2} \right\} v_k = 0\,, \tag{7.29}$$

where prime denotes a derivative with respect to τ, the conformal time defined by $a(t)\mathrm{d}\tau = \mathrm{d}t$. The solution to this, at first order in slow-roll, and after matching to Bunch-Davies initial conditions, is given by:

$$v_k = \frac{\sqrt{\pi}}{2}\sqrt{-\tau}H_\nu^{(1)}(-k\tau)\,, \tag{7.30}$$

$$\nu^2 = \frac{9}{4} + 9\epsilon - \left(\frac{m}{H}\right)^2(1+2\epsilon)\,, \tag{7.31}$$

where we have ignored the time dependence of ν coming from $\dot{\epsilon} \sim \dot{\xi}_1$ since it is suppressed relative to ϵ.

Now that we have an expression for the mode function evolution, we can obtain the expression for the noise variance, $\langle \xi_1, \xi_1 \rangle = \langle \xi_1^{(2)} \xi_1^{(2)} \rangle$ valid to order $(\epsilon^1, \hbar^2)$. After some computation, see Appendix 7.9, and doing a combined expansion in $(m/H)^2$ and ϵ, the variance simplifies to (at first order in both ϵ and $(m/H)^2$)

$$\langle \xi_1^{(2)}(N)\xi_1^{(2)}(N')\rangle = \frac{H^4(t)}{4\pi^2}\left[1+\Delta\right]\delta(N-N')\,, \tag{7.32}$$

where Δ is defined as

$$\Delta = \frac{2}{3}\left(\frac{m}{H}\right)^2(-2+\gamma_E+\log 2\gamma) - 3\epsilon(-3+2\gamma_E+2\log 2\gamma)\,. \tag{7.33}$$

Here, γ_E denotes the Euler-Mascheroni constant. Note that the Hubble parameter H appearing in the above two equations stands for the full H, i.e. stochastically and slow-roll corrected. Stochastic corrections to the variance of the noise are therefore included in ϵ, as per equation (7.21), as well as in H.

Also, note that in order to impose that the variance of the noise is independent of the choice of coarse-graining radius, that is, independent of γ, we must impose the hierarchy $\exp\left[-\frac{H^2}{m^2}\right] \ll \gamma \ll 1$ [32]. This is consistent with the expressions found in [62], and ensures that the effective theory we obtain through the coarse-graining process is a sensible and physical one.

7.5.2 Numerical solution to the mode function equation

To proceed with the numerical solution, we first recast equation (7.28) in a more useful form. This equation can easily be rewritten in terms of the number of e-folds N:

$$\frac{\mathrm{d}^2 v_k}{\mathrm{d}N^2}+\frac{\mathrm{d}v_k}{\mathrm{d}N}+\left\{\frac{k^2e^{-2N}}{H^2(t)}(1+\epsilon)^2-2-9\epsilon+\frac{V_{\Phi\Phi}}{H^2(t)}(1+\epsilon)^2+4\frac{\dot{H}(t)}{H(t)}\frac{\ddot{\phi}_c^{(1)}}{\dot{\phi}_c^{(1)}}\frac{(1+\epsilon)^2}{H^2(t)}-14\epsilon^2\right\}v_k=0\,, \tag{7.34}$$

where a dot refers to a derivative with respect to cosmic time t. Retaining only terms up to leading order in ϵ^C, the classical piece of ϵ, and up to $\mathcal{O}(\tilde{\xi}^2)$, we obtain the equation

$$\begin{aligned}\frac{\mathrm{d}^2 v_k}{\mathrm{d}N^2} &+ \frac{\mathrm{d}v_k}{\mathrm{d}N}+\left\{\frac{k^2e^{-2N}}{H^2(t)}(1+2\epsilon^C)-2-9\epsilon_1^C+\frac{m^2}{H^2(t)}(1+2\epsilon_1^C)\right\}v_k \qquad (7.35)\\ &+ \frac{H_0}{\pi}\left\{9-2\frac{m^2}{H^2(t)}-2\frac{k^2e^{-2N}}{H^2(t)}\right\}\frac{\tilde{\xi}}{\phi_c^{(1)}}v_k+\frac{H_0^2}{\pi^2}\left\{\frac{k^2e^{-2N}}{H^2(t)}-14+\frac{m^2}{H^2(t)}\right\}\left(\frac{\tilde{\xi}}{\phi_c^{(1)}}\right)^2 v_k=0\,.\end{aligned}$$

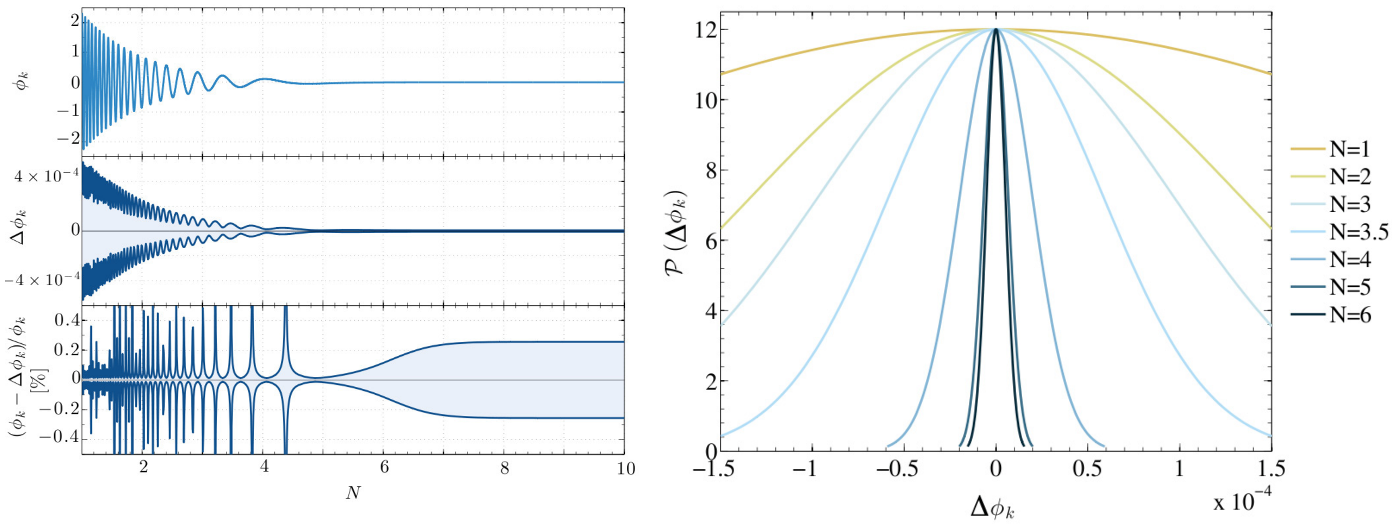

Figure 7.3 *Left panel*: Top: mean trajectory of the real part of the linearized mode function ϕ_k with $k = 1.26 \times 10^{-2} M_{\rm Pl}$, freezing at $N \approx 5.5$. Middle: 1-σ deviation form the mean trajectory, i.e. standard deviation of maximum likelihood. Bottom: percentage error at 1σ. *Right*: The reconstructed (unnormalized) PDF of the random variable $\phi_k(N) - \langle \phi_k(N) \rangle$ for a few fixed e-folds during the last 60 e-folds of inflation. These values were inferred from 250 realizations, with $m = 6 \times 10^{-6} M_{\rm Pl}$, $N_{tot} = 60$. Fields are shown in units of $M_{\rm Pl}$.

To solve for the PDF of the stochastic linearized quantum mode function corresponding to (7.34) is very difficult, since (7.34) is now proportional to the square of the noise[6]. We therefore proceed numerically, using a modified version of the Runge-Kutta method for solving SDEs (which reduces to the improved Euler method in the absence of a stochastic term), as explained in Appendix B.

In order to solve for *each* realization of the mode function in a given realization of the background (solved for in step two of the recursive method, see section 7.4.1), for every realization, the background and the mode function equation must be solved simultaneously (such that each realization of the mode function 'sees' the background generated by the right Wiener process $\tilde{\xi}$). The result for a fixed mode is displayed in Figure 7.3.

Figure 3 highlights the generic behavior of a mode of the gauge-invariant Mukhanov

[6] In this case, an analytical solution for the PDF of the mode functions through a Fokker-Planck equation is not possible anymore.

variable ϕ_k inside and outside the Hubble horizon. Before horizon crossing, the mode v_k has a constant norm. Therefore, ϕ_k plotted here has an amplitude that decays as a^{-1}. The reason why it appears to be oscillating widely in the top panel of Figure 7.3 is that only the real part to the mode function is shown, and the real and imaginary parts oscillate at identical speed with a $\pi/2$ phase shift. Around $N = 5$, the mode plotted crosses the Hubble radius and freezes, and its amplitude remains constant from then on (real and imaginary parts independently).

The absolute variance of the mode also decays as a^{-1} while the mode is inside the horizon, as shown in the middle panel of the l.h.s., as well as the plot on the r.h.s. The fractional variance, displayed in the bottom panel of the l.h.s., diverges each time the real part of the mode crosses zero (which should not be interpreted as a physical effect), However, after the mode has exited the horizon, both the field and the variance approach a constant (this can be seen in the r.h.s. and the middle panel of the l.h.s.), and the fractional variance converges to roughly 0.25%. Note that, for our purpose in the present paper, we only apply the description of ϕ_k as a UV mode up until this mode joins the coarse-grained theory via the noise ξ, which occurs a few e-folds after horizon exit to ensure classicalization, around $N = 7$ in Figure 7.3.

Repeating this procedure for every k mode exiting the coarse-graining radius during the last 60 e-folds of inflation in a given realization of the background, we obtain the corrected power spectrum of the stochastic noise. Figure 7.4 shows the resulting average noise correlator (thick blue, top panel), when averaging over 100 realizations, and the 1-σ error on this correlator on the middle (absolute) and bottom (fractional) panels. The red dot-dashed line in the top panel, representing the analytical calculation from equation (7.33), shows very good agreement between our numerical and analytical treatments. The top panel of the figure also shows, for comparison, the result at zeroth order in slow-roll which was obtained in section 7.3 and used in section 7.4, as well as the naive slow-roll correction obtained by taking $H \to H(t)$ in the zeroth-order result, as one would obtain by following the procedure of, e.g., [47] (yellow line).

7.6 Step Four: Corrected Coarse-Grained Theory

7.6.1 Overview and numerical approach

We have thus far completed two levels of recursion: 1) for our first 'guess', we began with a nondynamical de Sitter background, then calculated the amplitude of the noise generated by quantum modes evolving on such a background in section 7.3; 2) using this noise (valid to leading order in $\hbar$ and zeroth order in slow-roll) as a source, we went back to the large scales and solved for the statistics of the coarse-grained classical inflaton, $\phi_c^{(1)}$, in section 7.4. Using this as a background (valid to first order in slow-roll and $\hbar$) for the short-scale physics, we then evaluated the corrected quantum modes in section 7.5. This then allowed us to find the variance of the noise arising from this bath, $\langle \xi_1 \ \xi_1 \rangle = \langle \xi_1^{(2)} \xi_1^{(2)} \rangle$, valid to $\mathcal{O}(\epsilon^1, \hbar^2)$.

Next, we shall use this noise to, once more, come back to the large-scale physics and source the coarse-grained theory. This will allow us to obtain a coarse-grained field $\phi_c^{(2)}$ valid to $\mathcal{O}(\epsilon^2,\ \hbar^2)$. That is, we must now solve:

$$\frac{\mathrm{d}\phi_c^{(2)}}{\mathrm{d}N} = -\frac{V_\Phi\left(\phi_c^{(2)}\right)}{3H^2} + \frac{\xi_1^{(2)}}{H}\,, \tag{7.36}$$

where now $H = m\phi_c^{(2)}/\sqrt{6}M_{\mathrm{Pl}}$ and $\xi_1^{(2)}$ is a random Gaussian variable sampled from a distribution with mean 0 and variance given by equation (7.32).

Recall that, although it should be thought of as a background when discussing the short-scale dynamics of the quantum mode functions inside the bath, the resulting ϕ_c is *not* homogeneous, i.e. the stochastic contribution to ϕ_c is inherently inhomogeneous. Rather, the PDF for ϕ_c contains all the information about the classicalized field, including perturbations. This is an elegant way to encode a large amount of information; however, we are left with the problem of calculating the standard phenomenological parameters of inflationary cosmology, such as the spectral tilt.

Numerically, however, solving this equation is quite easy. Using a method analogous to what was done in section 7.4, we solve for each realization of the coarse-grained theory using a realization of the noise output that was used toward the construction of Figure 7.4. After constructing 100 realizations of ϕ_c, we then use Bayes theorem to infer the two first moments of the underlying PDF (assuming Gaussianity), i.e. its mean and variance as functions of time.

As one should expect with a choice of parameters excluding eternal inflation, this additional step in the recursive method does not give significant corrections to the mean trajectory of the inflaton. Its variance, however, is the quantity capturing the integrated power of the classicalized field fluctuations, and is of great interest to us. This is the quantity presented in Figure 7.5, where the numerical result is the black solid line.

7.6.2 Inflaton fluctuations beyond leading order

Perturbation equations for the random variable ϕ_c

As alluded to in section 7.4, the analysis of fluctuations in stochastic inflation can be done quite simply by using the number of e-folds N as the time variable. This fact can also be seen by considering the δN formalism. As an example, let us consider single field inflation, following [86]. The background equation of motion is given by (where ρ is the energy density):

$$3H^2 M_{\rm Pl}^2 = \rho, \tag{7.37}$$

$$H\partial_N(H\partial_N\phi) + 3H^2\partial_N\phi + \partial_\phi V(\phi) = 0. \tag{7.38}$$

The power of the δN formalism comes from realizing that, in the absence of entropy perturbations, the above equation applies *nonperturbatively*. This leads to a statement of the 'separate universe approach' to perturbations,

$$\phi(N) = \phi_0(N, \phi_{init}, (\partial_N\phi)_{init}). \tag{7.39}$$

This equation states that the nonperturbative dynamics in one region of spacetime are captured by solving for the background ϕ_0, but given a set of perturbed initial conditions $\{\phi_{init}, (\partial_N\phi)_{init}\}$. In fact, the full δN formalism is much more powerful than this, as it is easily generalizable to a gradient expansion.

This same formalism can (and has, see [60, 61]) be applied to stochastic inflation, since the noise evolves independently in different Hubble patches, and hence does not spoil the 'separate universe approach'.[7] In fact, this locality was shown to be a necessary condition

[7] To be more precise: fluctuation and dissipation terms that arise at 3rd order are *non-local*, meaning this approach would need to be modified. However, these non-localities are at most at the coarse-graining radius, and therefore do not spill into neighboring Hubble patches. Furthermore, here we are considering

for stochastic inflation in [87]. The applicability of the δN formalism could in principle be shown more rigorously by expanding the action for the stochastic inflation, which was derived in [59].

This approach can be applied here as follows. The equation of motion for the classical coarse-grained field $\phi_c^{(2)}$ is given by equation (7.36). We can split the field $\phi_c^{(2)}$ into a homogeneous mode, which is just the expectation value $\langle\phi_c^{(2)}\rangle$, and an inhomogeneous piece $\delta\phi_c$ containing all the *classical* fluctuations.[8] We therefore expand (7.36) around $\langle\phi_c^{(2)}\rangle$ using:

$$\phi_c^{(2)} = \langle\phi_c^{(2)}\rangle + \delta\phi_c\,, \tag{7.40}$$

to find the equation for the first-order fluctuations [47]:

$$\frac{\mathrm{d}\delta\phi_c}{\mathrm{d}N} + 2M_{\mathrm{Pl}}\left(\frac{H_{,\langle\phi_c^{(2)}\rangle}}{H}\right)_{,\langle\phi_c^{(2)}\rangle}\delta\phi_c = \frac{\xi_1^{(2)}}{H}\,, \tag{7.41}$$

where, as in equation (7.36), H is defined as including *both* the slow-roll and stochastic corrections. The above equation can be solved to give the PDF for $\delta\phi_c$, from which we would like to extract information about the power spectrum.

Alternatively, the variance of $\delta\phi_c$ can be calculated from (7.41) by multiplying both sides by $\delta\phi_c$ and averaging, without having to solve for the full PDF of the classical field,

$$\frac{\mathrm{d}\langle\delta\phi_c^2\rangle}{\mathrm{d}N}+4M_{\mathrm{Pl}}\left(\frac{H_{,\langle\phi_c^{(2)}\rangle}}{H}\right)_{,\langle\phi_c^{(2)}\rangle}\langle\delta\phi_c^2\rangle = \frac{S}{4\pi^2 H}, \qquad \text{with} \qquad S = \left(H|_{\phi_c^{(1)}}\right)^3(1+\Delta)\,, \tag{7.42}$$

where, apart for the occurrences of H in S, all other powers of H are evaluated at $\langle\delta\phi_c^{(2)}\rangle$. To arrive at this equation, we have used the relation $\langle\delta\phi_c\xi_1\rangle = (H|_{\phi_c^{(1)}})^3(1+\Delta)/(8\pi^2)$, which can be deduced by expanding ξ_1 and $\delta\phi_c$ in terms of their Fourier modes, and enforcing continuity of the (amplitude of the) full field Φ across the horizon $k=\gamma aH$. We emphasize that the cumbersome notation is necessary: the slow-roll correction to the variance of the noise was calculated with respect to the background $\phi_c^{(1)}$, where as the occurrences of H in equation (7.36) are defined with respect to $\phi_c^{(2)}$.

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a free scalar field, and are neglecting the coupling between tensor and scalar perturbations.

[8] We will not write a subscript $^{(2)}$ on the inhomogeneous piece of $\phi_c^{(2)}$ for the sake of simplifying the notation.

This equation is easily solved in terms of the homogeneous solution for $\langle\phi_c\rangle$,

$$\langle\delta\phi_c^2\rangle = -\frac{H^2_{,\langle\phi_c^{(2)}\rangle}}{8\pi^2\left(H|_{\langle\phi_c^{(2)}\rangle}\right)^2 M_{\rm Pl}^2}\int\frac{H|_{\langle\phi_c^{(2)}\rangle}}{H^2_{,\langle\phi_c^{(2)}\rangle}}S\mathrm{d}N \approx -\frac{H^2_{,\langle\phi_c^{(2)}\rangle}}{8\pi^2\left(H|_{\langle\phi_c^{(2)}\rangle}\right)^2 M_{\rm Pl}^2}\int\frac{\left(H|_{\langle\phi_c\rangle}\right)^5}{H^3_{,\langle\phi_c\rangle}}(1+\Delta)\mathrm{d}\langle\phi_c\rangle\,. \tag{7.43}$$

In the last step, we have assumed that the time evolution of $H|_{\langle\phi_c^{(2)}\rangle}$ and $H|_{\phi_c^{(1)}}$ are the same. To be precise, they differ by terms that are higher in slow-roll than the precision to which $H|_{\phi_c^{(1)}}$ is defined.

Substituting the form of $H|_{\langle\phi_c^{(2)}\rangle}$ in the above equation yields the following solution for the total power in the fluctuations of the coarse-grained inflaton as a function of time:[9]

$$\langle\delta\phi_c^2\rangle = \frac{H_0^6 - H^6}{8\pi^2 m^2 H^2}(1+\Delta) \to \frac{H_0^6}{8\pi^2 m^2 H^2}(1+\Delta)\,, \tag{7.44}$$

where the right arrow denotes the asymptotic value approached towards the end of inflation (as the field approaches its minimum).

The first equality in the above equation (before the limit is taken) is the analytic result of the recursive formalism, and is shown in Figure 4 (dashed dark-blue line). The pale-blue dashed line shows the first equality in equation (7.43), before any approximation on $H|_{\langle\phi_c^{(1)}\rangle}$ is done. The good agreement between this, the final analytical result, and the variance of the classical field obtained numerically (full black line) supports the validity of our approach. For comparison, we have also plotted (dashed yellow line) the obtained total integrated power in classical fluctuations obtained in [47, 48], by using the slow-roll corrected $H(t)$ to account for the fact that the bath evolves in a slow-rolling background (by enforcing it by hand). In contrast, the recursive method we apply here self-consistently accounts for this correction, in a natural way.

[9] If one had instead solved the integral exactly, i.e. *kept* the occurrences of $H|_{\phi_c^{(1)}}$ in (7.43), the integral to solve would have had the form $\sim \int_0^N \langle\phi_c^{(2)}(N')\rangle\phi_0^3 \exp\left[-\frac{m^2}{H^2}N'\right] dN'$. Solving this integral, (7.43) would become $-\frac{i\phi_0^6\left(\Gamma\left[\frac{3}{2},-\frac{3}{2}\right]-\Gamma\left[\frac{3}{2},-\frac{3}{2}+\frac{6N}{\phi_0^2}\right]\right)}{3\sqrt{6}\mathrm{e}^{\frac{3}{2}}}\frac{m^4\phi_0^3}{6^2}\frac{1}{4\pi^2}\frac{1}{H|^2_{\langle\phi_c^{(2)}\rangle}}$. The theoretical prediction from this result for $\langle\delta\phi_c\rangle$ is shown in Figure 7.5 (dark blue dashed line), and is negligibly close to the result from the expression in (7.44).

Recovering the power spectrum of scalar fluctuations

Clearly, when calculating the power spectrum of scalar perturbations, the often-used procedure of quantizing the fluctuations deep inside the horizon and then evaluating at horizon crossing cannot be applied here, as the quantum fluctuations have been replaced by a classical noise which, at every given time, is only nonzero at the coarse-graining scale. The power spectrum of perturbations can instead be calculated by noting that the variance of fluctuations is the integral of the power spectrum from an IR cutoff to the Hubble horizon, or more precisely, the classicalization radius, which corresponds to the coarse-graining scale. That is,

$$\langle \delta\phi_c^2 \rangle = \int_l^{\gamma a H} \mathcal{P}_{\delta\phi_c}(k)\, \mathrm{d}\log k\,, \tag{7.45}$$

which is a standard textbook result (see, for example, [88]). The power spectrum in the above expression can be written in terms of mode functions as

$$\mathcal{P}_{\delta\phi_c}(k) = \frac{k^3}{2\pi^2} |\delta\tilde{\phi}_k|^2\,, \tag{7.46}$$

where the tilde is to denote that $\delta\tilde{\phi}_k$ is not a Fourier mode of $\delta\phi_c$, but rather the mode function that one would find in the standard procedure of quantizing perturbations and computing the power spectrum.

Another approach consists in using the following trick: we can parametrize a generic power spectrum in terms of a general spectral index n_s, and explicitly compute the integral on the r.h.s. of equation (7.45). Then, by solving equation (7.42) for $\langle \delta\phi_c^2 \rangle$, we can deduce the value of n_s. More explicitly, in standard perturbation theory, a generic power spectrum of field fluctuations far outside the Hubble radius can be written as (see for example [89,90], where we keep only the terms that depend on k):

$$\mathcal{P}_{\delta\phi_c}(k) = \mathcal{A}_s(t) k^3 \left[1 + \frac{(n_s - 1)}{2} \log\left(\frac{k}{k^*}\right)\right]^2\,, \tag{7.47}$$

where $\mathcal{A}_s(t)$ is a time-dependent amplitude and n_s is the spectral index. Therefore, integrating over all super-Hubble modes as in (7.45), we obtain

$$\langle \delta\phi_c^2 \rangle = \frac{1}{4\pi^2}\mathcal{A}_s(t)\frac{2}{3(n_s-1)}\left[1+\frac{(n_s-1)}{2}\log\left(\frac{k}{k^*}\right)\right]^3\Bigg|_l^{\gamma aH}, \tag{7.48}$$

where a standard calculation would take $\gamma = 1$. Performing the renormalisation of this quantity through adiabatic subtraction as done in [90], we can obtain a value for $\langle \delta\varphi^2 \rangle_{\mathrm{REN}}$ that is independent of the IR cutoff. From there, evaluating this result after a sufficiently long period of inflation (in order for the one-point correlator $\langle \delta\varphi^2 \rangle$ to saturate to its maximal asymptotic value and for the memory of initial conditions to disappear), the terms in square brackets in equation (7.48) (plus counter terms) simplify to $\left\{ \left[1+(n_s-1)\log(\frac{k}{k^*})\right]\Big|_l^{\gamma aH} + c.t.\right\} \to -1$. Therefore, given $\langle \delta\phi_c^2 \rangle_{\mathrm{REN}}$, one can solve for the spectral index:

$$n_s - 1 = -\frac{1}{6\pi^2}\frac{\mathcal{A}_s(t)}{\langle \delta\phi_c^2 \rangle_{\mathrm{REN}}}\,. \tag{7.49}$$

Computation of the time-dependent amplitude

In the specific case of $m^2\Phi^2$ inflation, the standard theory of cosmological perturbations gives the following result for the time dependence of mode functions far outside of the Hubble horizon:

$$\delta\tilde{\phi}_k = \frac{1}{a^{\frac{3}{2}}}\left(\frac{\pi(1+\epsilon)}{4H(t)}\right)^{\frac{1}{2}}\left(\frac{H(t_k)}{H(t)}\right)^2 \mathcal{H}^{(1)}_{3/2}\left((1+\epsilon)\frac{k}{aH}\right), \tag{7.50}$$

$$\text{with } H(t_k) = H_0\sqrt{1+2\frac{\dot{H}_0}{H_0^2}\log\left(\frac{(1+\epsilon_0)k}{H_0\nu_0}\right)}, \tag{7.51}$$

where $\mathcal{H}^{(1)}$ is the Hankel function of the first kind, and $\epsilon_0 = -\dot{H}/H_0^2$. At small $-k\tau$ or $(1+\epsilon)\frac{k}{aH}$, i.e. outside the Hubble radius, the mode functions can be asymptotically approximated by:

$$\delta\tilde{\phi}_k \approx \frac{1}{(aH)^{3/2}}\frac{1}{2^{1/2}}\frac{1}{(1+\epsilon)}\frac{H_0^2}{H(t)}\left(1-2\epsilon_0\log\left(\frac{(1+\epsilon_0)k}{H_0\nu_0}\right)\right)\left(\frac{k}{aH}\right)^{-3/2}. \tag{7.52}$$

This allows us to calculate the r.h.s. of equation (7.45), which gives the total amount of power in the inflaton fluctuations:

$$\begin{aligned}\int_l^{\gamma a H} \mathcal{P}_{\delta\tilde{\phi}_c}(k)\,\mathrm{d}\log k &= \frac{1}{4\pi^2}\frac{1}{(1+\epsilon)^2}\frac{H_0^4}{H^2(t)}\frac{(-1)}{6\epsilon_0}\left[\left(1-2\epsilon_0\log\left(a\frac{(1+\epsilon_0)H\nu}{(1+\epsilon)H_0\nu_0}\right)\right)^3\right.\\ &\quad \left.-\left(1-2\epsilon_0\log\left(\frac{(1+\epsilon_0)l}{H_0\nu_0}\right)\right)^3\right],\\ &\to -\frac{1}{4\pi^2}\frac{1}{(1+\epsilon)^2}\left(\frac{H_0^4}{H^2(t)}\right)\frac{1}{3(-2\epsilon_0)}\,. \end{aligned} \tag{7.53}$$

The arrow in the last line denotes the value at which the correlator saturates towards the end of inflation. From this, we can deduce an explicit form for the time-dependent amplitude of the power spectrum:

$$\mathcal{A}_s(t) = \frac{1}{(1+\epsilon)^2}\left(\frac{H_0^4}{H^2(t)}\right). \tag{7.54}$$

The spectral index

The final result for the stochastically-corrected spectral tilt n_s can now be obtained from equations (7.44) and (7.49), in combination with the above result (7.54) for the amplitude, to arrive at

$$n_s - 1 \approx -4\epsilon_0(1-2\epsilon)(1-\Delta) + \mathcal{O}\left((\epsilon_1^2)_{\text{std pert}}\right). \tag{7.55}$$

Here, $\mathcal{O}\left((\epsilon_1^2)_{\text{std pert}}\right)$ represents additional second-order terms which appear in the standard slow-roll calculation, but do not bear any stochastic contributions. From this expression, one can see that (n_s-1) is manifestly negative, corresponding to a red tilt of scalar perturbations, as one would expect for this simple inflationary model. Also, the specific value of the tilt to leading order in slow-roll matches that obtained from standard methods of calculations for $m^2\phi^2$ inflation.

The corrections to the standard slow-roll result appear at second order in ϵ. In particular, the slow-roll correction to the noise is captured by Δ, which is easily checked to be positive definite. Hence we find a positive second-order correction to n_s. This may come in addition to other second-order slow-roll corrections, as denoted by the term $\mathcal{O}\left((\epsilon_1^2)_{\text{std pert}}\right)$, which come from the fact that we have neglected higher-order terms in (7.53) when calculating $\mathcal{A}_s(t)$. Retaining those, it is in principle possible to recover standard second-order results (see, for example, [91, 92]), plus stochastic corrections.

The additional effect of stochastic corrections to n_s could have been anticipated from Figure 7.5, where the recursive calculations done in this paper are compared to the $\langle\delta\phi_c^2\rangle$ one would get from simply taking $H \to H(t)$ in the variance of ξ. The recursive calculation is positively shifted with respect to the naive slow-roll correction, indicating that stochastic effects push the spectral index towards scale invariance, i.e., closer to zero. This is a general result for stochastic corrections in single field models: recursive corrections become more important in the late stages of inflation, and generically increase the variance of the field fluctuations. This is a second-order 'blue' (i.e. positive) contribution to the spectral tilt.

Equation (7.55) is the main result of this section. While the above correction to standard slow-roll inflation occurs only at second order in slow-roll, this serves as a proof-of-principle that stochastic corrections can indeed affect inflationary observables. As we will discuss in section 7.7, these corrections can be large in multifield models.

7.7 Inflation with Extra Heavy Fields

The corrections due to stochastic effects can be more dramatic in multifield inflation. In particular, a direct coupling of the inflaton to extra heavy fields will give a stochastic correction to the mass of the inflaton, which enters the spectral index at first order in slow-roll. However, one should be careful not to choose the masses to be so large that their dynamics can simply be integrated out, i.e. we want to look at the case $m_\phi \ll m_\chi \lesssim H$. This is the case in hybrid inflation, as has been studied in [62], and can easily be generalized to N fields.

Consider $m^2\Phi^2$ inflation in the presence of a heavy field Ψ, which couples directly to the inflaton through a potential $V(\Phi, \Psi)$. For simplicity, let's take the potential

$$V(\Phi, \Psi) = \frac{1}{2} g^2 \Phi^2 \Psi^2 \, , \tag{7.56}$$

where g is a dimensionless coupling constant. The classical dynamics is determined by the equations

$$3H^2 \frac{\mathrm{d}\Phi}{\mathrm{d}N} \quad = \quad -m_\Phi^2 \phi - g^2 \Phi \Psi^2 \, ; \tag{7.57}$$

$$3H^2\frac{\mathrm{d}\Psi}{\mathrm{d}N} \;=\; -m_\Psi^2\psi - g^2\Phi^2\Psi\,. \tag{7.58}$$

Let us consider a hierarchy of VEVs, that is $\langle\Phi\rangle \gg \langle\Psi\rangle$, which physically corresponds to a slowly rolling field Φ and the field Ψ oscillating about its minimum. As we assume Ψ is a relatively heavy field, the oscillations of Ψ are suppressed by the large mass m_Ψ, and this in turn will induce corrections to spectrum of Φ that are suppressed by powers of m_Ψ. Similar scenarios have been discussed in many works, for example [93–96].

However, the picture in stochastic inflation is quite different: for sufficiently large (though less than the Hubble energy scale) m_Ψ, the dynamics of the coarse-grained part of Ψ, χ, can easily be dominated by stochastic effects. In this case, and since the field Ψ has vanishing VEV, the value of the field is well characterized by its standard deviation:

$$\chi(N) \sim \sqrt{\langle\chi^2\rangle} = \sigma_\chi. \tag{7.59}$$

The equation of motion for ϕ_c can now be rewritten

$$3H^2\frac{\mathrm{d}\phi_c}{\mathrm{d}N} = -m_\Phi^2\phi_c\left(1+\frac{g^2\sigma_\chi^2}{m_\Phi^2}\right) + 3H\xi_1 = -\tilde{m}_\Phi^2\phi_c + 3H\xi_1\,, \tag{7.60}$$

where we have defined the stochastic-corrected mass $\tilde{m}_\Phi^2 = m_\Phi^2 + g^2\sigma_\chi^2$. The recursive formalism then allows for a consistent and precise calculation of the statistics of ϕ_c, as per the single field case.

This can be easily generalized to the case with a set of N heavy fields, with variances denoted by σ_i,

$$\tilde{m}_\Phi^2 = m_\Phi^2 + \sum_i g_i^2\sigma_i^2, \tag{7.61}$$

and the spectral index of inflaton perturbations can be read off from the single field case,

$$n_s - 1 = -\frac{4\tilde{m}_\Phi^2}{3H^2}(1+\delta) \approx -\frac{4m_\Phi^2}{3H^2} - \sum_i \frac{4g_i^2\sigma_i^2}{3H^2}, \tag{7.62}$$

where δ is a recursive correction, analogous to the single field case, that enters at second order in slow-roll, and the $\approx$ denotes that we have truncated to first order in slow-roll. We thus find a non-negligible correction to the inflaton spectrum in the presence of direct couplings to heavy fields, and interestingly, the stochastic correction in this case is red.

Note that, here, we were careful to choose the number of fields and their masses in such a way that the inflaton itself always dominates the dynamics of the Hubble constant, even after the massive fields have developed a stochastic VEV. That is, writing the effective potential for the inflaton as $V = V_0 + \frac{1}{2}\tilde{m}_\Phi^2\Phi^2$, V_0 remains negligible throughout the analysis.

If, on the contrary, we had chosen the initial potential to have a large VEV, $V_0 \gg m_\Phi^2\Phi^2$ throughout inflation, the VEV that the massive fields develop due to stochastic effects would still renormalize the mass of the inflaton in a fashion identical to (7.61). However, as in the standard analysis, the tilt we would obtain would be blue, and the stochastic corrections would make it bluer at linear order in slow-roll.

More subtle scenarios can also be examined, in particular two-field inflation with a turning trajectory [97], and more general multifield inflation models [85]. We leave the analysis of such setups to future work.

7.8 Tensor Fluctuations

The analysis of scalar perturbations in the stochastic formalism can be straightforwardly extended to include tensor perturbations, as is most easily seen from the functional derivation [98]. However, the phenomenology can be easily worked out without such detailed knowledge, by noting that tensor modes evolve as massless scalar fields. The analysis is nearly identical to the scalar case already studied. Tensor perturbations are defined as

$$ds^2 = a^2(\tau)[d^2\tau^2 - (\delta_{ij} + \mathbf{h}_{ij})dx^i dx^j] \,. \tag{7.63}$$

The second order action for dimensionless tensor perturbations is given by

$$S = \frac{M_{\text{Pl}}}{8}\int d\tau d^3x a^2 \left[\left(\mathbf{h}'_{ij}\right)^2 - \left(\nabla \mathbf{h}_{ij}\right)^2\right] \,. \tag{7.64}$$

Assuming no anisotropic stress, the gauge-invariant Einstein equation for the tensor mode is given by

$$\ddot{\mathbf{h}}_{ij} + 3H\dot{\mathbf{h}}_{ij} - \frac{\nabla^2}{a^2}\mathbf{h}_{ij} = 0. \tag{7.65}$$

Decomposing the tensor perturbations into eigenmodes of the Laplacian with $\mathbf{h}_{ij} = \mathbf{h}^\lambda(t)e^\lambda_{ij}(x)$, where $\lambda \,\varepsilon\, \{\times, +\}$ are the two polarization states of the fluctuations, and $\nabla^2 e^\lambda_{ij} = -k^2 e^\lambda_{ij}$, it becomes easy to split the full field $\mathbf{h}_{ij}$ into a coarse-grained part, h_{ij}, and a bath part,

w_{ij},

$$\begin{aligned} \mathbf{h}_{ij} &= h_{ij} + w_{ij}\,, \\ w_{ij} &= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\theta\left(\frac{k}{\gamma a H} - 1\right)\sum_{\lambda=1,2}\hat{w}^{\lambda}_{\mathbf{k}}(\tau)e^{\lambda}_{ij}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}\,. \end{aligned} \tag{7.66}$$

Far outside the Hubble radius, the IR modes (which are part of the coarse-grained theory) are frozen out, and their equation of motion can therefore be approximated by

$$\frac{\mathrm{d}h_{ij}}{\mathrm{d}N} = \frac{\xi_{h_{ij}}}{H}, \tag{7.67}$$

where $\xi_{h_{ij}}$ is the tensor noise. Following the scalar analysis, the variance of the noise at first order in slow-roll is given in terms of the tensor mode functions $w_{\mathbf{k},\lambda}$ by

$$\langle \xi_{h_{ij}}(N)\xi_{h_{ij}}(N')\rangle = \frac{(\gamma a)^3 H^5}{2\pi^2}\sum_{\lambda=1,2}\left|w^{\lambda}_{\mathbf{k}}\right|^2_{k=\gamma a H}(1-\epsilon)\delta(N-N')\,, \tag{7.68}$$

where we have used the polarization tensors identities $e^{\lambda}_{ij}(-\mathbf{k}) = \left(e^{\lambda}_{ij}(\mathbf{k})\right)^*$ and $e^{\lambda}_{ij}(\mathbf{k},\lambda)\left(e^{\mu}_{ij}(\mathbf{k})\right)^* = \delta^{\lambda\mu}$.

From this, we can readily calculate the amplitude of the slow-roll corrected noise. To do so, we need to compute the mode functions $w^{\lambda}_{\mathbf{k}}$ evolving on the slow-roll and stochastically-corrected background calculated in section 7.4, $\phi_c^{(1)}$. Defining the canonically quantized variable $u^{\lambda}_{\mathbf{k}} = \frac{a}{2}M_{\rm Pl}w^{\lambda}_{\mathbf{k}}$, we can write

$$\left(u^{\lambda}_{k}\right)'' + \left(k^2 - \frac{a''}{a}\right)u^{\lambda}_{\mathbf{k}} = 0. \tag{7.69}$$

Using $\frac{a''}{a} = (aH)^2(2-\epsilon) = \frac{2+3\epsilon}{\tau^2}$ and following the scalar case, this leads to the variance at first order in slow-roll,

$$\langle \xi_{h_{ij}}(N)\xi_{h_{ij}}(N')\rangle = \frac{2\left.H\right|^4_{\phi_c^{(1)}}}{M_{\rm Pl}^2\pi^2}[1+\epsilon(1-2\gamma_E-2\log 2\gamma)]\delta(N-N'), \tag{7.70}$$

where the additional factor of 2 comes from summing over possible polarization states.

Since the homogeneous mode of tensor perturbations is zero, $\langle h_{ij}\rangle = 0$, we can straightforwardly obtain an equation for the total amount of power in classical tensor fluctuations by a procedure analogous to what allowed us to go from (7.41) to (7.42),

$$\frac{d\langle h_{ij}^2\rangle}{dN} = \frac{2}{\pi^2 M_{\rm Pl}^2}\frac{H|^3_{\phi_c^{(1)}}}{H|_{\phi_c^{(2)}}}(1+\Delta)\,. \tag{7.71}$$

The solution to this equation is

$$\langle h_{ij}^2\rangle = \frac{\sqrt{6}}{\pi^2}\frac{2H_0^3}{mM_{\rm Pl}^2}\int_0^N \frac{\exp\left[-\frac{m^2}{H_0^2}N'\right]}{\phi^{(2)}(N')}dN' = \frac{1}{M_{\rm Pl}^2}\frac{\sqrt{6}H_0^4}{\pi^2 m^2}\frac{\sqrt{\pi}\left[erfi\left(\sqrt{\frac{3}{2}}\right)-erfi\left(\frac{\phi_c^{(2)}(N)}{\phi_c^{(2)}(0)}\sqrt{\frac{3}{2}}\right)\right]}{e^{3/2}}\,, \tag{7.72}$$

which can be expanded, to leading order in slow-roll,

$$\frac{\sqrt{6\pi}}{e^{3/2}}\left[erfi\left(\sqrt{\frac{3}{2}}\right)-erfi\left(\sqrt{\frac{3}{2}}\frac{\phi_c^{(2)}(N)}{\phi_c^{(2)}(0)}\right)\right]\approx 2\left(1-\left(\frac{\phi_c^{(2)}}{\phi_0}\right)^3\right)\,. \tag{7.73}$$

Taking the limit towards the end of inflation, we find

$$\langle h_{ij}^2\rangle \to 2\frac{H_0^4}{\pi^2 m^2 M_{\rm Pl}^2}\,. \tag{7.74}$$

On the other hand, a general power spectrum of tensor fluctuations can be parametrized by

$$k^3\mathcal{P}_h(k) = \mathcal{A}_T(t)k^3\left[1+\frac{n_T}{2}\log\left(\frac{k}{\gamma aH}\right)\right]^2\,, \tag{7.75}$$

with $\mathcal{A}_T(t) = 8\mathcal{A}_s(t)$ (as can be shown from a straightforward calculation, including the two possible polarizations of the tensor modes [10]). Therefore, we find that the tensor spectral index is:

$$n_T = -2\epsilon + \mathcal{O}(\epsilon^2)\,, \tag{7.76}$$

which is precisely the standard slow-roll result, see for example [91]. The stochastic corrections to this can be calculated in a similar manner to the scalar case, and we leave this to further work. Note that, as in the scalar case, the stochastic correction will enter as a blue

[10] Note that $\mathcal{A}_s$ refers to the amplitude of field fluctuations, which is related to the amplitude of curvature perturbations by $\zeta^2 = \frac{1}{2\epsilon}\delta\phi^2$.

contribution to n_T, and hence pushes the spectral index towards scale invariance.[11] As the precision of CMB B-mode experiments increases, these corrections (and, in general, precise phenomenology) will become an increasingly important consideration.

7.9 Conclusion

In this work we have put forward a detailed example of how the recursive formalism of stochastic inflation [59, 62] can be applied to models of single field inflation. A key difference between our analysis here and the recursive prescription of the original papers is our treatment at step three of the recursive method. Indeed, when computing the dynamics of the bath's quantum modes on a stochastically corrected background, we did not impose that the modes evolve on a representative realization of the background (constructed by replacing all occurrences of the coarse-grained fields by their average in the bath propagator). Rather, we kept the background seen by these quantum modes purely stochastic. In [62], this approach was sufficient to capture the leading contribution of stochastic effects to observables, since the dominant effect arose from a spectator field, while the inflaton was well approximated as deterministic. However, this statement is not true in general, and in the present analysis we went beyond this approximation in a fiducial example of a single field model.

The picture we put forward here is that, during inflation, modes inside the Hubble horizon evolve on a background that is 'fuzzy' in field space on scales $\lambda < H_0/(2\pi)$. This is because, within each realization, the background seen by those modes is getting kicked at every moment in time, in a random direction with a rms of that size. This approach allows one to capture nontrivial stochastic corrections, even when the average of the coarse-grained field remains identical to that of the classical standard approach, or when no even powers of the coarse-grained fields[12] appear in the time-dependent mass of the quantum modes.

[11] Note that our method can also capture the backreaction of tensor modes on scalar modes (which occurs at second order in standard perturbation theory) through the appearance of higher-order noise terms in (7.67), in a similar fashion to the additional noise terms which were shown to appear at the 1-loop level in [59]. However, as shown in, e.g., [99], the induced second order perturbations have the form $\epsilon\left(\mathcal{R}^{(1)}\right)^2$, and so such an effect only appears at 3rd order in slow-roll. To the level of accuracy relevant for the present calculation, this effect is negligible.

[12] Recall that Gaussianity of the field implies that the expectation value of an odd power of fields is zero.

Throughout the present paper, we have studied, both analytically and numerically, the slow-roll correction to the growth of quantum fluctuations in $m^2\phi^2$ inflation away from the regime of eternal inflation, and found a nontrivial stochastic correction to their variance which could not have been deduced by simply correcting the Hubble constant to first order in slow-roll. We then used to this to compute the corrected long-wavelength modes, which includes both the slow-roll and stochastic effects.

We separated the long-wavelength physics into a homogeneous component (i.e. the background) and an inhomogeneous component (i.e. primordial classical perturbations), from which we could compute the spectral tilt of classical scalar perturbations. We found a stochastic correction to the spectral tilt in $m^2\phi^2$ inflation, which enters n_s at second order in slow-roll, as a blue (i.e. positive) contribution. While this is a small effect in the case of a single field inflation, the same formalism can be straightforwardly applied to other models of inflation, and the machinery used to compute the corresponding corrections. One example is the case of inflation with direct couplings to heavy fields, wherein stochastic effects induce a shift in the effective mass of the inflaton, leading to a red contribution to the spectral tilt.

Extending the current analysis to the regime of eternal inflation would be an interesting followup to this work, however, it is not immediately straightforward because of a number of issues. The first and main reason for this is the need for a 'reading rule' for the SDE, which can be explained as follows: as the eternal inflation regime is entered, the local value of H and of the classical coarse-grained field ϕ_c become dominated by their stochastic contribution (that is to say, the time evolution of the coarse-graining radius itself becomes a stochastic process). To obtain the incoming noise at each time N, which is proportional to the local Hubble radius, one needs to know the amplitude of all the modes that previously crossed the Hubble radius[13]. This can be summarized by saying that, in the eternal inflation regime, the noise is *multiplicative* (and cannot be considered additive), i.e., the importance of the noise term in the SDE depends on the stochastic process ϕ_c itself.

This affects the previous analysis in the sense that it renders the SDE (7.8) or (7.19) undefined without a so-called reading rule. That is, when defining a SDE as a limit of

[13]Note that this does not make the process non-Markovian, since the precise history of how $H(t)$ acquired that value is irrelevant, but to calculate $H(t)$ at a fixed time the prior history of a realization becomes essential.

discretized equations as

$$\begin{aligned}\phi_c(N+\Delta N)-\phi_c(N) &= -\left.\frac{V_{,\Phi}}{3H^2}\right|_{\phi_c(t)_\alpha}\Delta N+\left.\frac{H}{2\pi}\right|_{\phi_c(t)_\alpha}\int_N^{N+\Delta N}dN'\hat{\xi}(N')\,,\\ \langle\hat{\xi}(N)\hat{\xi}(N')\rangle &= \delta(N-N')\,,\end{aligned} \tag{7.77}$$

where the weighted average $\phi_c(N)_\alpha$ is defined by:

$$\phi_c(N)_\alpha=(1-\alpha)\phi_c(N)+\alpha\phi_c(N+\Delta N)\,, \tag{7.78}$$

one must specify a value for α between 0 and 1. Failing to do so when the noise coefficient depends on ϕ_c explicitly makes the SDE undefined, since in general different reading rules (i.e. different choices of α) give rise to different stochastic processes ϕ_c [100]. Typically, $\alpha=0$ (referred to as an Ito process) corresponds to discrete systems, whereas $\alpha=1/2$ (referred to as a Stratonovich processes) corresponds to continuous physical systems; however, cases with exotic α values have also been found [101]. Determining the correct value of α for eternal inflation falls beyond the scope of this work, and we plan to return to this issue in a follow-up paper.

In the last section of this paper, we finished by extending our analysis to include tensor perturbations, and again found a correction only at second order in slow-roll. Our analysis was simple, although a rigorous path integral derivation of this approach is still in progress. We leave this, and the analysis of general multifield models, to future work. We conclude on one final remark: The formalism of stochastic inflation is an effective field theory, where the UV modes are integrated out, and an IR theory is obtained with extra operators (i.e. noise) capturing the effect of UV physics. However, in this case, the UV and IR sectors of the theory remain coupled, and the 'cutoff' (the horizon) that separates these two sectors is time dependent. It follows that one must proceed with caution when solving this system.

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Variance of the slow-roll corrected noise

The variance of the noise is given in terms of the mode function by

$$\langle \xi(N)\xi(N')\rangle = \frac{\gamma^3 H^5}{2\pi^2} a^3 |\phi_k|^2_{k=\gamma a H}(1-\epsilon)\delta(N-N'), \tag{A 1}$$

where H is the full stochastic and slow-roll correct Hubble constant, hence the corrected variance is itself stochastic. Now we get $|\phi_k|^2$

$$|\phi_k|^2 = |\frac{v_k}{a}|^2, \tag{A 2}$$

where v_k is the solution to the mode function equation with Bunch-Davies initial conditions. The mode function is given by

$$v_k = \frac{\sqrt{\pi}}{2}\sqrt{-\tau}H_\nu^{(1)}(-k\tau). \tag{A 3}$$

Note that $aH = -(1+\epsilon)/\tau$, and hence

$$\tau = -\frac{(1+\epsilon)}{aH}, \tag{A 4}$$

which is an exact relation, as it follows from the definition of ϵ. Also note the asymptotic form of the Hankel function,

$$\lim_{k\tau\to 0} H_\nu^{(1)}(-k\tau) = \frac{i}{\pi}\Gamma(\nu)\left(\frac{2}{-k\tau}\right)^\nu. \tag{A 5}$$

We plug all this into $|\phi_k|^2$, and evaluate at $k = \gamma a H$,

$$|\phi_k|^2 \quad = \quad \frac{\Gamma(\nu)^2}{4\pi}\frac{2^{2\nu}}{a^3 H}(1+\epsilon)^{1-2\nu}\frac{1}{\gamma^{2\nu}}. \tag{A 6}$$

Now we use the expression for ν^2 to expand ν as:

$$\nu^2 = \frac{9}{4} + \delta \rightarrow \nu \sim \frac{3}{2} + \frac{\delta}{3}, \tag{A 7}$$

where $\delta = 9\epsilon - (1+2\epsilon)(m/H_0)^2$. We do a combined expansion in $(m/H)^2$ and ϵ, and the variance, given in equation (A 1), simplifies to (to first order in both ϵ and $(m/H)^2$)

$$\langle \xi(N)\xi(N')\rangle = \frac{H^4}{4\pi^2}\left[1+\Delta\right]\delta(N-N'), \tag{A 8}$$

where Δ is defined as

$$\Delta = \frac{2}{3}\left(\frac{m}{H}\right)^2(-2+\gamma_E+\log 2\gamma) - 3\epsilon(-3+2\gamma_E+2\log 2\gamma). \tag{A 9}$$

Note that the stochastic correction is hidden inside ϵ, as per equation (7.21). Also note that the H appearing in the two above equations is the full H, i.e. stochastic and slow-roll corrected.

Numerical Integration Method for Stochastic Differential Equations

In order to solve equation (7.34) numerically, we need to modify the standard Runge-Kutta method for SDEs, in order to allow for equations with more than linear powers of the noise. The method we propose still reduces to the improved Euler method in the absence of a stochastic term.

It works as follows. For a system of SDEs given by

$$d\vec{X} = \vec{a}(t,\vec{X})dt + \vec{b}(t,\vec{X})dW + \vec{c}(t,\vec{X})dW^2\,. \tag{B 1}$$

with $\vec{X}$ a system of stochastic processes and W the Wiener process (i.e. just regular

Brownian motion). We can define the following Runge-Kutta scheme:

$$\vec{K}_1 = \vec{a}(t_n, \vec{X}_n)\delta t + \vec{b}(t_n, \vec{X}_n)\Delta W + \vec{c}(t_n, \vec{X}_n)\Delta W^2/\delta t\,; \tag{B 2}$$

$$\begin{aligned}\vec{K}_2 &= \vec{a}(t_{n+1}, \vec{X}_n + \vec{K}_1)\delta t + \vec{b}(t_{n+1}, \vec{X}_n + \vec{K}_1)\Delta W \\ &\quad + \vec{c}(t_{n+1}, \vec{X}_n + \vec{K}_1)\Delta W^2/\delta t\,;\end{aligned} \tag{B 3}$$

$$\vec{X}_{n+1} = \vec{X}_n + \frac{1}{2}\left(\vec{K}_1 + \vec{K}_2\right), \tag{B 4}$$

where ΔW is now a random variable sampled at each time step from a normal distribution of mean 0 and standard variation $\sqrt{\delta t}$.

In our case, the second-order SDE that needs to be solved can be cast as a system of first-order SDEs and solved using the above scheme. The components of the vector coefficients $\vec{a}$, $\vec{b}$, and $\vec{c}$ can be read from (7.34).

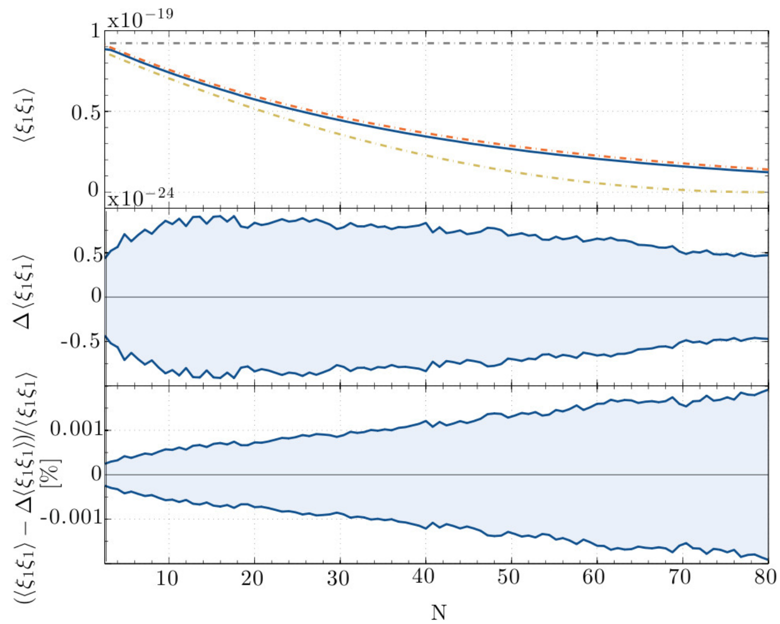

Figure 7.4 Power spectrum of the noise correlator at step 3 of the recursive method, $\langle\xi_1\xi_1\rangle$. *Top panel*: Average power spectrum, $\langle\xi_1\xi_1\rangle$, in units of $M_{\rm Pl}^4$. The full (blue) line is the corrected power spectrum obtained numerically at step 3, while the red line is the analytic computation. The grey dashed line is the variance obtained at the previous step of the recursion, $H_0^4/(4\pi^2)$, and the yellow dashed line is $H^4(t)/(4\pi^2)$, i.e. the naive correction to the noise to account for the fact that inflation happens is quasi-de Sitter space by simply making the Hubble radius a time-dependent quantity. *Middle panel*: 1σ deviation from the mean trajectory, i.e. standard deviation of maximum likelihood, again in units of $M_{\rm Pl}^4$. *Bottom panel*: Percentage error at 1-σ. These graphs were obtained by simulating 100 realizations, and, assuming a Gaussian underlying PDF, the mean and variances were inferred by maximizing the likelihood of the $\mu - \sigma$ space. Simulations were performed using $m = 6 \times 10^{-6} M_{\rm Pl}$, $N_{tot} = 80$.

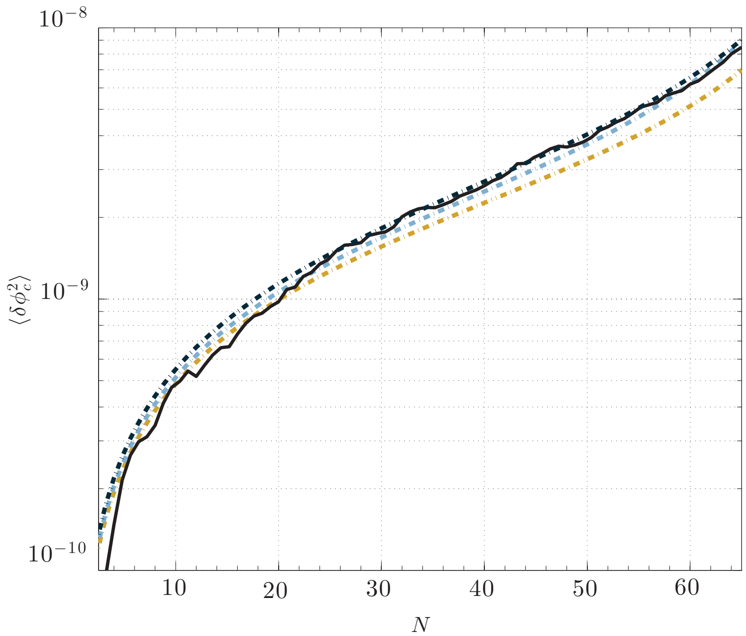

Figure 7.5 Total integrated power in, or variance of, the classical long-wavelength fluctuations, $\langle\delta\phi_c^2\rangle$, in units of M_{Pl}^2. The full (black) line is the numerical result, while the dashed light- and dark-blue dashed lines correspond to different analytic approximations to the integral in equation (7.43), see footnote 8 for details. The yellow dashed line corresponds to a noise variance of $H^4(t)/(4\pi^2)$, i.e. the naive approximation obtained by slow-roll corrected $H(t)$ to account for the fact that the bath evolves in a slow-rolling background [47]. This graph was obtained by simulating 100 realizations, and, assuming a Gaussian underlying PDF, the mean and variances were inferred by maximizing the likelihood of the $\mu - \sigma$ space. Simulations were performed using $m = 6 \times 10^{-6} M_{\mathrm{Pl}}$, $N_{tot} = 80$.

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Afterword

Afterword

This thesis has spanned many topics at the interface of high energy physics and cosmology; a reflection of the evolution in the author's interests over the course of his PhD. With such breadth, there are many directions for new research, and many projects to be done. Foremost among these is developing the connection between higher order anti-brane actions studied in Chapter 4 and the supergravity inflation models studied in Chapter 6, and the application of a preheating analysis like in Chapter 5 to the inflation models presented in Chapter 6. Another new direction for research, spurred on by the primordial black hole analysis of Chapter 5, is to study and develop new observational signatures of preheating, such as gravitational waves. All of these, and more, are currently in progress.

Finally, we thank the reader for their patience in reaching to the end of this thesis.