SET POINT FEEDBACK STABILIZATION OF DRIFT FREE SYSTEMS

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July 1997

A Thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfilment of the requirements for the degree of
Doctor of Philosophy

0-612-37036-4

To

the memory of my father,
my dear wife and son.

Abstract

This dissertation presents two new systematic feedback approaches for ***set point feedback stabilization of drift free systems*** which employ Lie algebraic techniques. Due to some special features, drift free systems are known to be difficult to stabilize. For such systems, linearization, and state feedback linearization techniques, fail to be useful. Moreover, such systems cannot be stabilized by continuous static state feedback as they fail to satisfy Brockett's necessary condition for smooth stabilization. In the absence of continuous static state feedback laws, most of the existing methods utilize piece-wise constant feedback or time-varying feedback controls which usually necessitate transformation of the system models into chained, power or nilpotentized forms. In this dissertation two new feedback approaches: the guiding functions approach and the trajectory interception approach are introduced. The importance of these approaches is due to their simplicity and the fact that they do not require any special transformation techniques.

The guiding functions approach delivers piece-wise constant control sequences and relies on the construction of special guiding functions which are not Lyapunov functions. However, a comparison of their values allows to determine a desired direction of system motion and permits to construct a sequence of controls such that the sum of these guiding functions decreases in an average sense. The individual guiding functions are hence not restricted to decrease monotonically but their oscillations are limited and coordinated in a way to guarantee convergence.

The guiding functions strategy is first analysed with reference to systems which appear in a special *rectified form* and requires the construction of as many guiding functions as there are control variables. Later, this approach is extended to apply to general drift free systems and usually results in the construction of only a pair of guiding functions. The strategy is general and can be employed to control a variety of mechanical systems with velocity constraints. The most important feature of this strategy is that it often leads to dead beat control. For higher order systems, a combined strategy which employs sinusoidal steering in conjunction with the guiding functions approach is also examined.

The trajectory interception approach provides a universal method for the construction of time varying stabilizing feedback control for drift free systems in the sense that it is independent of the vector fields determining the motion of the system, or of the choice of a Lyapunov function. The resulting feedback law is a composition of a standard stabilising feedback control for a Lie bracket extension of the original system and a periodic continuation of a specific solution to an open loop, finite horizon control problem stated in terms of a formal equation on a Lie group - an equation which (via an evaluation homomorphism) describes the evolution of the flows of the original as well as the extended system. The open loop problem is solved as a trajectory interception problem in logarithmic coordinates of flows.

The construction proposed in this approach demonstrates that synthesis of time varying feedback stabilizers for drift free systems can be viewed as a procedure of combining static feedback laws for a Lie bracket extension of the system with a solution of an open loop control problem on the associated Lie group. This approach is employed first to a subclass of drift free systems which are characterized by nilpotent controllability Lie algebras. Later, the approach is extended to apply to other drift free systems which possess non-nilpotent controllability Lie algebras. This is accomplished by introducing approximate system models which possess nilpotent controllability Lie algebras.

The applicability of both approaches is demonstrated on a variety of drift free systems with different algebraic structures: a unicycle, Brockett's system, a front wheel drive car, a rigid spacecraft, a hopping robot in flight phase, an underwater vehicle, a fire truck, a mobile robot with trailer, and a class of wheeled mobile robots. The examples confirm the effectiveness of both approaches beyond any doubt.

RÉSUMÉ

Dans cette thèse on présente deux méthodes directes pour la stabilisation par rétro-action des systèmes sans dérives. Ces méthodes sont basées sur les méthodes algébriques de Lie. Due à certaines particularités ces systémes sont difficiles à stabiliser. Ainsi la linéarisation et les techniques de linéarisation par retour d' etat ne sont pas utiles. De plus, de tels systèms ne peuvent pas être stabilisés par un retour d'etat statique et continu puisqu'ils ne satisfont pas à la condition nécessaire et suffisante de Brockett pour la stabilisation lisse. En absence de commande par retour d'etat statique et continu, la majeure partie des méthodes existantes utilisent des commandes à retro-action constantes par morceaux ou variantes dans le temps qui d'habitude nécessitent une transformation des modéles du système en chaines, et formes de puissances ou nilpotantes. Dans cette dissertation on introduit deux approches rétro-actives nouvelles: l'approche par fonctions de guidances et l'approche d'interception de trajectoire. L'importance de ces approches est due à leur simplicité, et le fait qu'elles n'ont pas besoin de technique de transformation spéciale.

L'approche des fontions de guidance génerent des suites de contrôle constantes par morceaux, et s'appuie sur la construction de fonctions de guidance spéciales qui sont des fonctions de Liapounov. Cependant, une comparaison de leurs valeurs permet de détérminer une direction désirée du mouvement du système et de construire une suite de commandes telle que la somme de ces fontions de guidance décro$\hat{i}$t en moyenne. Les fontions de guidance individuelles ne sont pas donc restraintes à décro$\hat{\iota o}ta$tre de facon monotone mais leurs oscillations sont limitées et coordonnées dans un sense qui garantie la convergence.

La stratégie des fontions de guidance est d'abord analyser avec référence aux systémes apparaissant dans une forme rectifiée particulière et exigant la construction d'autant de fonctions de guidance que de variables de contrôle. Plus tard, cette approche est prolongée de facon à s'appliquer en général aux systémes sans dérive, il en résulte comme d'habitude la construction de seulement une paire des fonctions de guidance. La stratégie est générale et peut être employée à contrôler une variété de systémes mécaniques avec des constraintes de vélocité. La caractéristique la plus importante de cette stratégie est qu'elle méne souvent à la commande pile. Pour les systémes d'ordre élevé, une stratégie combinée qui emploie un entrainement sinusoidal en conjonction avec l'approche des fontions de guidance est examinée.

L'approche d'interception de trajectoire fournit une méthode universelle de constructi on de commandes par rétro-action variante dans le temps pour la stabilisation des systèmes sans dérive, dans le sense qu'elle est indépendante du champs vectoriel qui détermine le mouvement du système, ou du choix d'une fonction de Liapounov. La loi de rétro-action résultante est une composition d'une stabilisation par rétro-action standard spéci fique au crochet de Lie du systéme original, et un prolongement périodique d'une solution spécifique à un problème de commande en boucle ouverte d'horizon fini, posé en termes d'une equation formelle sur un groupe de Lie - une equation qui décrit (via un homomorphisme d'evaluation) l'évolution du flux du système original et prolongé. Le problème en boucle ouverte est résolu comme un problème d'interception de trajectoire dans les coordonnées logarithmiques du flux.

La construction proposée dans cette approche démontre que la synthèse de stabilisa teurs par rétro-action variant dans le temps pour des systémes sans dérive peut être considérée comme une procédure combinant les lois de rétro- action statiques pour une extension du crochet de Lie du système avec solution d'un problèm e de contrôle en boucle ouverte sur le groupe de Lie associé. Cette approche est employée en premier lieu à une sous-classe de systèmes sans dérive qui sont caractérisés par des algébres de contrôllabilité de Lie nilpotantes. Plus tard, cette approche est prolongée de facon à s'appliquer à d'autres systèmes sans dérive qui possédent des algébres de contrõllabilité de Lie nilpotantes. Ceci est accomplie en introduisant des modéles approximatifs qui possédent des algébres de contôllabilité de Lie nilpotante.

L'applicabilité de ces deux approches est démontrée sur une variété de systè mes sans dérive avec des structures algébriques différentes: un unicycle, le système de Brockett, une auto à traction avant, un engin spatial rig id, un robo sautillant dans une phase de saut, un vehicule sous-marin, une pompe à incendi e, un robot mobile avec remorque, et une classe de robots mobiles avec des roues. Les exemples confirment l'efficacité des deux approches hors de doute.

Acknowledgements

I would like to express my deepest gratitude to my supervisor Professor Hannah Michalska for her continuous guidance, assistance and inspiration without which this thesis could not have been completed. Her depth of knowledge, clear explanation and elegant presentation have always helped me to understand, analyse and solve problems in a unified way. During our discussions, she provided a friendly environment and was very generous with her time. I would like to thank her for her rigorous reading of the manuscripts. Needless to say, this dissertation has been significantly improved by her comments that she made after reading the manuscripts.

I am grateful to the Government of Pakistan, Ministry of Education, Islamabad, for awarding me a Merit Scholarship for higher education, and also NSERC for providing my further financial support.

I would like to express my special thanks to Carlos Martinez-Mascarua for his continuous technical and moral support during my studies at McGill. Whenever I had problem, he always encouraged and helped me.

I am really thankful to Saddik Djouadi who translated and typed the abstract of the thesis into French, and Michael Glaum for his careful proofreading of the thesis. I am also thankful to Kouther Benameur and Tom Mackling for their useful discussion during my studies. I am grateful to my friends Mohsen Ahmadi, Amir Zia, Kashif Zia and Imran for their encouragement and moral support.

I am deeply indebted to my mother, brothers and sisters who faced many crises due to my father's sudden death but they did not allow me to quit my studies. They always supported and encouraged me. I will always remember their sacrifices. I cannot forget my father who provided me higher education and always supported me. May God give his soul peace and rest.

Finally, I would like to express my appreciation to my wife and son for their continuous six years sacrifice and patience. Without their support and patience this dissertation would not have been possible.

Claims of Originality

The following contributions are made in this dissertation:

- The two feedback control strategies introduced are *new* and do not require conversion of the system models either to power or chained forms. In principle, no transformation techniques are needed. Both strategies can thus be applied to systems which fail to satisfy the conditions for the existence of special transformations, and to systems which are not flat. Both control strategies possess strong robustness properties with respect to model inaccuracies.

- *The concept of guiding functions as a tool for feedback control design has not appeared in previous literature.* The guiding functions approach is particularly simple and often leads to very effective feedback control laws such as 'dead beat control'.

- The trajectory interception approach involves an *original* idea of employing the Lie algebraic techniques of [51] and [50] in a systematic synthesis of time-varying feedback control for drift free systems.

- The trajectory interception approach provides for exponential rates of convergence to a desired set point. The results contained in this thesis open a new area of research with the goal of rendering this synthesis approach computationally simpler, more effective, and extending its applicability to systems with drift.

This research work has been partially reported by H. Michalska and F. U. Rehman ([63], [64], [65], [66],[67], [68],[69], [70], [71], [72], and [84]) in journals and conference proceedings.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1

Introduction

It is hardly possible to avoid contact with nonholonomic systems. Nonholonomic control problems arise in everyday life: while driving a car to work, pushing a baby stroller, or riding a bicycle to school. Some of these control problems are simple for a human being to solve, after some training. There are however control problems which are more intricate, for example: parking a tractor with multiple trailers or reorientation of a body of a cat in mid-air while respecting the law of angular momentum conservation. It is indeed amazing that a cat, dropped from an upside down configuration, is usually able to land on her feet by using an interesting combination of maneuvers, which is one example of a nonholonomic control problem solved successfully.

These examples seem to be superficially unrelated. However, from a mechanical or mathematical stand point, they are examples of nonholonomic systems. Such systems arise due to the presence of either nonholonomic constraints or non-integrable conservation laws in their motion. A nonholonomic constraint, such as a rolling contact constraint in the instance of parallel-parking, is a constraint on the velocity of the system which cannot be integrated into position constraints, (if it could be integrated, it would then be referred to as a holonomic constraint). Similarly, a non-integrable conservation law, such as the angular momentum conservation law in the case of a falling cat, is a physical law that constrains the velocities of a system.

As pointed out, nonholonomic constraints appear very frequently in our daily lives and in fact are much more common than holonomic constraints. Unlike a holonomic constraint which constrains motion of a system away from a certain region of its configuration space, a nonholonomic constraint limits only the freedom of motion. The case of parallel parking subject to rolling contact constraints serves as a perfect illustrative example: with such constraint a car can move backwards and forwards but not sideways.

Much research interest was taken recently in the control of nonholonomic systems as such control problems are of practical importance and are theoretically challenging. The literature on the control of nonholonomic systems has grown enormously, see the survey paper [**43**].

In this work, we will restrict our attention to systems which represent mechanical systems with *linear* velocity constraints. Such constraints can arise in a number of different ways; a few typical examples are given below:

(1) *Mobile robots navigating in a cluttered environment:*

The kinematics of the drive mechanisms of robot carts results in constraints on the instantaneous velocities that can be achieved. For instance, a cart with two forward drive wheels and two back wheels is often required to move without slipping sideways.

(2) *Multi-fingered hands manipulating a grasped object:*

If an object is twirled through a cyclic motion that returns the object to its initial position and orientation, the fingers are constrained to roll without slipping on the surface of the object.

(3) *Space robotics:*

Unanchored robots in space are difficult to control with either thrusters or internal motors since they conserve total angular momentum. The latter is a nonholonomic constraint. The motion of astronauts on space walks is of this ilk, so that planning a strategy to reorient an astronaut is a nonholonomic control problem. Other examples of this effect include gymnasts and springboard divers in flight phase.

1. Linear velocity constraints and their integrability

Most of the velocity constraints mentioned above have the form of linear constraints expressed by the following system of equations:

$$\omega_j(q)\,\dot{q} = 0, \quad q \in \mathbb{R}^n, \quad j = 1, 2, ...k \tag{1.1}$$

where the vector $q \in \mathbb{R}^n$ describes the configuration of the system to be controlled, and $\omega_j(q)$, $j = 1, 2, ...k$, are row vectors in $\mathbb{R}^n$. The following example illustrates, how linear velocity constraints can be written in the form of (1.1).

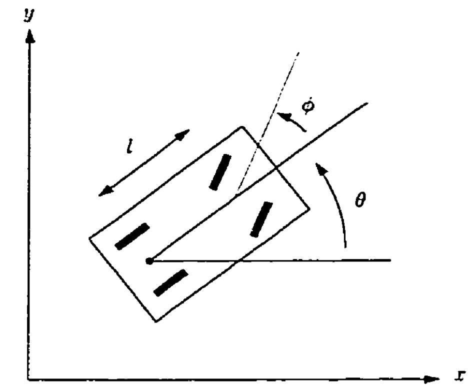

FIGURE 1.1. Model of an automobile with front and rear wheels.

Example 1.1

Consider a simple model of an automobile with front and rear wheels, as presented in [**79**]. The rear wheels are aligned with the car, while the front tires are allowed to spin about the vertical axes. The constraints on the system arise by allowing the wheels to roll and spin, but not slip. Let $q = (\phi, x, y, \theta)$ denote the configuration of the car, parameterized by: ϕ - the steering angle with respect to the car body, (x, y) - the xy - location of the rear wheels, and θ - the angle of the car body with respect to the horizontal. Let l be the distance between the front and the rear wheels, see Figure 1.1. The constraints for the front and rear wheels are formed by setting the 'sideways velocity' of the wheels to zero. In particular, the velocity of the back wheels perpendicular to their direction is $sin\theta\ \dot{x} - cos\theta\ \dot{y}$, and the velocity of the front wheels perpendicular to the direction they are pointing is $sin(\theta+\phi)\ \dot{x} - cos(\theta+\phi)\ \dot{y} - l\ cos\phi\ \dot{\theta}$. These constraints can thus be written as:

$$sin\theta\ \dot{x} - cos\theta\ \dot{y} = 0 \tag{1.2}$$

$$sin(\theta+\phi)\ \dot{x} - cos(\theta+\phi)\ \dot{y} - l\ cos\phi\ \dot{\theta} = 0 \tag{1.3}$$

$$\text{or} \qquad \omega_1(q)\ \dot{q} = 0, \qquad \omega_2(q)\ \dot{q} = 0 \tag{1.4}$$

$$\text{where,} \quad \omega_1(q) = [0,\ sin\theta,\ -cos\theta,\ 0] \tag{1.5}$$

$$\omega_2(q) = [0,\ sin(\theta+\phi),\ -cos(\theta+\phi),\ -l\ cos\phi] \tag{1.6}$$

The constraints are said to be *integrable* if for each q there exist scalar functions $h_j : N(q) \to I\!R$, $j = 1, 2, ...k$, (defined on some neighbourhood $N(q)$ of q), such that (1.1) can be written as:

$$\frac{d}{dt} h_j(q) = \nabla h_j(q)\dot{q} = 0, \quad j = 1, 2, ...k, \quad \text{for} \quad q \in N(q) \tag{1.7}$$

where ∇ denotes the gradient of h_j. Integrating (1.7) yields:

$$h_j(q) = 0, \quad j = 1, 2, ...k, \quad \text{for} \quad q \in N(q) \tag{1.8}$$

It follows that integrable constraints can be substituted by algebraic constraints which do not involve velocities. The constraints are said to be *non-integrable* if they cannot be written as algebraic constraints involving only configuration variables q. Integrable constraints are known as *holonomic* constraints and, non-integrable constraints are called *nonholonomic* constraints.

Constraints can be classified either as *holonomic* or *nonholonomic* by using the Frobenius Theorem which gives a necessary and sufficient condition for the existence of at least locally defined scalar functions h_j in (1.7).

Before we state the Frobenius Theorem, the reader is advised to see Appendix A for the definitions of a distribution, a regular distribution, a codistribution, involutiveness and integrability of distributions, etc.

Frobenius Theorem [79]

Suppose a distribution $\Delta(q) = span\{g_1(q), g_2(q), ..., g_m(q)\}$ is regular, so that the dimension of $\Delta(q)$, $dim(\Delta(q))$, is a constant. Then such distribution is integrable if and only if it is involutive.

To check *integrability* of constraints (1.1) by employing the Frobenius Theorem it is first necessary to find a distribution $\Delta(q) = span\{g_1(q), g_2(q), ..., g_{n-k}(q)\}$ such that whenever

$$\dot{q} \in span\{g_1(q), g_2(q), ..., g_{n-k}(q)\}$$

then (1.1) is satisfied at q. The latter is clearly equivalent to the problem of finding an annihilator $\Omega^{\perp}(q)$ to the codistribution $\Omega(q)$ defined by the covector fields ω_i of (1.1):

$$\Omega(q) = span\{\omega_1(q), ..., \omega_k(q)\}$$

The existence of $\Omega^{\perp}(q)$ is guaranteed by the Proposition below:

Proposition 1.1 [79]

Assume that ω_i, $i = 1, ..., k$ are smooth and linearly independent covector fields on $I\!R^n$ which form a codistribution $\Omega(q) = span\{\omega_1(q), ..., \omega_k(q)\}$ of constant dimension. There exist smooth, linearly *independent vectors fields g_j, $j = 1, ..., n-k$, such that the distribution*

$$\Delta(q) = span\{g_1(q), g_2(q), ..., g_{n-k}(q)\}, \quad q \in I\!R^n$$

is the annihilator of $\Omega(q)$ at q, i.e., $\Delta(q) = \Omega^{\perp}(q)$, which implies that $\omega_i(q)g_j(q) = 0$, for $i = 1, ..., k$, $j = 1, ..., n-k$.

The following is an easy consequence of the Frobenius Theorem and Proposition 1.1.

Corollary 1.1 [79]

A set of smooth constraints of the type (1.1) is integrable if and only if the distribution $\Delta(q) = \Omega^{\perp}(q)$ of Proposition 1.1 is involutive.

We continue to consider Example 1.1 and show that the constraints (1.2)-(1.3) are nonholonomic.

Example 1.1 (continued)

It is easily seen that the codistribution $\Omega(q) = span\{\omega_1(q), \omega_2(q)\}$, involving the covector fields ω_i, $i = 1, 2$, of constraints (1.2)-(1.3) is annihilated by the distribution

$$\Delta(q) = span\{g_1(q),\ g_2(q)\} \tag{1.9}$$

$$\text{where,} \qquad g_1(q) = [1,\ 0,\ 0,\ 0]^T, \qquad g_2(q) = [0,\ cos\theta,\ sin\theta,\ \frac{1}{l}\ tan\phi]^T \tag{1.10}$$

It is easily verified that distribution (1.9) is not involutive as $[g_1, g_2](q) \notin \Delta(q)$. Therefore these constraints are *non-integrable*, and thus nonholonomic.

In the next section, we describe, how nonholonomic systems with linear velocity constraints give rise to control systems known as "drift free systems". The study of such systems is the main interest of this thesis.

2. Drift free systems as nonholonomic systems

Consider the problem of constructing a path $q(t) \in I\!R^n$ between given points q_0 and q_1, subject to constraints (1.1). Without the loss of generality, it can be assumed that the ω_i, $i = 1, ..., k$, are linearly independent and smooth covector fields. Intuitively, constructing such a path requires converting the constraint specification from describing the directions in which the system cannot move to those in which it can. To do this, we first construct the codistribution $\Omega(q) = span\{\omega_1(q), ..., \omega_k(q)\}$.

By Proposition 1.1, there exists $n-k$ smooth, linearly independent vector fields $g_1, ..., g_{n-k}$ such that the $(n-k)$-dimensional distribution $\Delta(q) = span\{g_1(q), ..., g_{n-k}(q)\}$, spans the annihilator $\Omega^\perp$ of Ω, so that:

$$\Delta = \Omega^\perp \text{ i.e. } \omega(q)g(q) = 0, \quad \text{for all } \omega \in \Omega, \quad g \in \Delta, \quad \forall\, q.$$

It is now clear that the nonholonomic constraints $\omega_i(q)\dot{q} = 0$, $i = 1, ..., k$, are equivalent to the statement that $\dot{q} \in \Delta$, which requires that $\dot{q}$ is a linear combination of the vector fields of Δ:

$$\dot{q} = g_1(q)u_1 + ... + g_{n-k}(q)u_{n-k} \tag{1.11}$$

with some coefficients $u_1, ..., u_{n-k}$ which generally depend on time, (as q and $\dot{q}$ vary with time). The above equation represents a control system in which q is the controlled state and $u_1, ..., u_{n-k}$ are the controls. Assuming that the velocity $\dot{q}$ can be actuated directly, the path planning problem becomes as finding the control, $u(t) \stackrel{def}{=} [u_1, ..., u_{n-k}]\,(t) \in I\!R^{n-k}$, which steers q_0 to q_1 along the trajectory of (1.11).

The control system (1.11) is said to be a *drift free* system; a system which is at rest if all its controls are zero, (so that $\dot{q} = 0$ if $u(t) = 0$).

The construction of a drift free system from kinematic constraints is illustrated on the previous Example 1.1.

Example 1.1 (continued)

Since

$$\Delta(q) = span\{g_1(q),\ g_2(q)\}$$

with g_1 and g_2 given by (1.10), then (1.11) can be written as:

$$\dot{q} = \begin{bmatrix} \dot{\phi} \\ \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ cos\ \theta \\ sin\ \theta \\ \frac{1}{l}\ tan\ \phi \end{bmatrix} u_2 \tag{1.12}$$

$$\stackrel{def}{=} g_1(q)u_1 + g_2(q)u_2$$

and represents a kinematic model of the four wheel car. The controls u_1 and u_2 influence: the angular steering, and forward velocities of the car, respectively, both of which were assumed to be actuated directly.

3. Important features of drift free systems and difficulties arising in their control

Generally, drift free systems can be defined as systems described by equation of the form:

$$\dot{x} = \sum_{i=1}^{m} g_i(x)\ u_i, \quad x \in I\!R^n \tag{1.13}$$

where, $g_i(x)$, $i \in \underline{m} \stackrel{def}{=} \{1, ..., m\}$, $m = n - k$, $m < n$, can be assumed to be linearly independent, smooth vector fields in $I\!R^n$, and u_i can be assumed to belong to the class of Lebesgue integrable functions on the interval $[0, \infty)$.

The most important features of drift free systems are described below:

(a) The number of control variables u_i, $i = 1, ..., m$ is smaller than the number of state variables x_i, $i = 1, ..., n$.

(b) Equation (1.13) has no equilibrium points in the usual sense; setting $u_1 = ... = u_m = 0$ gives $\dot{x} = 0$, indicating that every point is an "equilibrium point".

(c) The linearization of (1.13) around any operating point is uncontrollable. To see this, let x_0 be any operating point, and $u_1 = ... = u_m = 0$ be the nominal controls. Then the linearization of equation (1.13) gives:

$$\dot{\delta x} = \sum_{i=1}^{m} g_i(x_0)\ \delta u_i \tag{1.14}$$

which is a linear system of the form $\dot{\delta x} = A\delta x + B\delta u$ in which the matrix A is identically zero and B is a matrix of dimension n by m, where $n \neq m$. Therefore (1.14) does not satisfy Kalman's controllability rank condition.

(d) It is easily seen that drift free systems have the special feature that every trajectory of (1.13), run backwards in time, is also a trajectory of this system. To see this, suppose that the controls $t \mapsto u_1(t), ..., t \mapsto u_m(t)$ steer x_0 to x_f in time T. Then the controls reversed in time $t \mapsto u_1(T - t), ..., t \mapsto u_m(T - t)$, steer x_f to x_0. Additionally, it is possible to re-scale $u_1(t), u_2(t), ..., u_m(t)$ so that x_f is attained from x_0 in any time $\tilde{T} \neq T$. This is explained as follows. Suppose $u = (u_1, ..., u_m)$ steers (1.13) from x_0 to x_f in time T. Then $\tilde{u}$ defined by rescaling the original u in time:

$$\tilde{u}(s) = \frac{T}{\tilde{T}}\, u(s\frac{T}{\tilde{T}}), \quad \text{for } 0 \leq s < \infty$$

steers (1.13) x_0 to x_f in time $\tilde{T}$. (This is because the solution of (1.13) with controls $\tilde{u}$, $\tilde{x}$, can be expressed by:

$$\tilde{x}(\tau) = x_0 + \int_0^{\tau} \sum_{i=0}^{m} g_i(\tilde{x}(s))\; u_i(s\frac{T}{\tilde{T}})\; \frac{T}{\tilde{T}}\; ds$$

Then substituting $\xi \overset{def}{=} s\; T/\tilde{T}$, yields

$$\tilde{x}(\tilde{T}) = x_0 + \int_0^{T} \sum_{i=0}^{m} g_i(x(\xi))\; u_i(\xi)\; d\xi = x(T) = x_f$$

where x is the solution of (1.13) with the original controls $u_1, ..., u_m$.)

(e) Despite the fact that linearization of (1.13) is uncontrollable, the controllability of (1.13) can be easily established by the famous Chow's Theorem [79]. Before we can state it, some rigorous definitions of nonlinear controllability are in place.

Given an open set $V \subseteq I\!R^n$, define $\mathcal{R}^V(x_0, T)$ to be the set of states x such that there exist admissible controls $u_1, ..., u_m$, defined on $[0, T]$, that steer a given system from $x(0) = x_0$ to $x(T) = x_f$ and such that $x(t) \in V$ for $0 \leq t \leq T$. Also define

$$\mathcal{R}^V(x_0, \leq T) = \bigcup_{0 \leq \tau \leq T} \mathcal{R}^V(x_0, \tau)$$

to be the set of states reachable up to time T.

Definition 1.1 (small-time local controllability)

A system is *small-time locally controllable at* x_0 if $\mathcal{R}^V(x_0, \leq T)$ contains a neighbourhood of x_0 for all neighbourhoods V of x_0 and $T > 0$.

In simple terms, small time local controllability at $x \in I\!R^n$ implies that for any given bound on the time T, and any given neighbourhood V of x, there exists another neighbourhood of x, $N \subseteq V$, such that any $\tilde{x} \in N$ can be reached from x, in time not exceeding T, while the corresponding trajectory remains in V.

Definition 1.2 (local controllability)

A system is said to be *locally controllable* at x if there exists a neighbourhood N of x such that for any x_0, $x_f \in N$ there exists a $T > 0$, and admissible controls $u_1, ..., u_m$ defined on $[0, T]$, which steer x_0 to x_f in time T, i.e. $x(0) = x_0$ and $x(T) = x_f$.

Let $\mathcal{L}(g_1, ..., g_m)$ denote the Lie algebra of vector fields generated by $g_1, ..., g_m$, (see the Appendix A for a definition of a Lie algebra of vector fields). Also, let $\mathcal{L}(g_1, ..., g_m)(x)$ denote the Lie algebra $\mathcal{L}(g_1, ..., g_m)$ "evaluated" at x; i.e.

$$\mathcal{L}(g_1, ..., g_m)(x) \stackrel{def}{=} \{g(x) \ \in I\!R^n; \ g \in \mathcal{L}(g_1, ..., g_m)\}$$

Chow Theorem (local version) [79]

Suppose the vector fields $g_1, ..., g_m$ in (1.13) are real analytic, linearly independent and complete (in that the solutions of (1.13) are defined for all initial conditions and all times). The system (1.13) is locally small-time controllable at $x \in I\!R^n$ if

$$\mathcal{L}(g_1, ..., g_m)(x) = I\!R^n$$

Clearly, small time local controllability at x implies *local controllability* of (1.13) at x. This is because small time local controllability guarantees the existence of a neighbourhood N of x and time $T > 0$, such that any $x_0 \in N$ can be attained from x in time T. Then by (d), the control u reversed in time, u_R, steers x_0 to x. Since $x_f \in N$, there exists another control $\tilde{u}$ which steers x to x_f in time less or equal to T. Thus the concatenated control: $u_R \circ \tilde{u}$ steers x_0 to x_f in time no greater than $2T$. Therefore, for drift free systems, small time local controllability implies local controllability in the usual sense.

Definition 1.3 (controllability on open sets)

A system (1.13) is said to be *controllable* on an open set $U \subset I\!R^n$, if for any x_0, $x_f \in I\!R^n$ there exists a $T > 0$ and admissible controls $u_1, ..., u_m$, defined on $[0, T]$, which steer x_0 to x_f in time T, i.e. $x(0) = x_0$ and $x(T) = x_f$.

Chow Theorem (global version) [79]

Suppose the vector fields $g_1, ..., g_m$ in (1.13) are real analytic, linearly independent and complete. The system (1.13) is controllable on an open neighbourhood of the origin, $U \subset I\!R^n$, if

$$\mathcal{L}(g_1, ..., g_m)(x) = I\!R^n \ \text{ for all } \ x \in U.$$

(f) Systems of the form (1.13) do not satisfy Brockett's necessary condition [10] for the existence of smooth (or even continuous) time invariant feedback laws:

Brockett's necessary condition for smooth static stabilization [10]:

Consider the control system

$$\dot{x} = f(x,u), \quad x(0) = x_0 \in I\!R^n, \quad f(0,0) = 0 \tag{1.15}$$

where $f : I\!R^n \times I\!R^m \to I\!R^n$ is continuously differentiable, which is denoted by $f \in C^1$. If (1.15) is C^1 stabilizable (in the sense that there exists a time-invariant C^1 feedback that renders the origin to be both Lyapunov stable, and an attractor), then the image of the map f contains some neighbourhood of the origin.

To illustrate that the above condition fails to hold for systems of type (1.13), consider the famous example known as Brockett's system:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \\ -x_2 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ x_1 \end{bmatrix} u_2 \\ &\stackrel{def}{=} g_1(x)u_1 + g_2(x)u_2 = f(x,u) \end{aligned} \tag{1.16}$$

Let $x_0 = 0$ be the equilibrium point for this system. For any $\epsilon \neq 0$, a point of the form $[0\ 0\ \epsilon]^T$, belongs to the neighbourhood of x_0 but is not a member of the image of f. This is because $f(x,u) = [0\ 0\ \epsilon]^T$ implies that $u_1 = u_2 = 0$, but then $f(x,u) \equiv 0$. Consequently, system (1.16) cannot be asymptotically stabilized to $x_0 = 0$ by a C^1 static state feedback.

(g) It is easy to see that drift free systems (1.13) are invariant with respect to diffeomorphic state transformations. It follows that the standard feedback linearization techniques cannot be applied to construct stabilizing feedback controls for (1.13). (For suppose that there is a diffeomorphic state-space transformation and nonlinear feedback which brings (1.13) to a linear system form. It would then follow that, after coordinate change, the smooth feedback constructed for the transformed linear system is also a smooth stabilizing control law for (1.13). This however is not possible since there exist no smooth static feedback control laws for (1.13).)

Due to the above difficulties, the control of drift free systems is a challenging problem which has attracted the attention of many researchers. Control strategies for such systems can naturally be classified into two groups:

- open loop strategies (referred to under the name of motion planning)

- closed loop strategies (typically used for stabilization purposes).

Since open loop strategies often lead to closed loop strategies, the next section summarizes previously obtained results in both areas.

4. Literature pertinent to the control of drift free systems

The relative difficulty of the control problem depends not only on the nonholonomic nature of the system but also on the control objective. For some control objectives, classical nonlinear control approaches (e.g., feedback linearization and dynamic inversion, as developed in [**38**]) are effective. Examples of such control objectives include stabilization to a suitably defined manifold [**8, 14**], stabilization to certain trajectories [**115**], dynamic path following [**92**], and output tracking [**31, 87**]. Consequently, there are classes of control problems for nonholonomic systems for which standard nonlinear control methods can be applied.

However, many of the most common control objectives, e.g., motion planning and stabilization to a point, cannot be solved using the standard nonlinear control methods, and new approaches have been developed. Substantial research has been devoted to motion planning, i.e., the study of (open loop) controls that transfer the system from a specified initial state to a specified final state. A variety of construction procedures for determining such controls have been proposed. Feedback control of nonholonomic systems has also been studied where the goal has been to accomplish specified closed loop performance objectives, including the classical control objectives of stabilization, asymptotic tracking, disturbance rejection, robustness improvement, etc.

In the next section, we describe recent developments in motion planning as these are relevant to feedback synthesis results presented later.

4.1. Open loop control strategies (motion planning)

Motion planning problems are concerned with obtaining *open loop* controls which steer a nonholonomic control system from an initial state to a final state over a given finite time interval. To understand why nonholonomic motion planning may be difficult, it is convenient to compare it with motion planning for holonomic mechanical systems. For a holonomic system, a set of independent generalized coordinates can be found, and thus an arbitrary motion in the space of independent generalized coordinates is feasible.

In contrast, for a nonholonomic system, a set of independent generalized coordinates does not exist. Consequently, not every motion is feasible, but only those motions which satisfy the instantaneous nonholonomic constraints. Nevertheless, the controllability condition of Chow's Theorem guarantees that feasible motions do exist which steer an arbitrary initial state to an arbitrary final state. Efficient techniques for such steering have been developed.

A variety of motion planning techniques are described in the book [**58**], which is a collection of research articles on nonholonomic motion planning. Besides [**58**], an excellent introduction to motion planning for nonholonomic robots is contained in the book by Murray, Li, and Sastry [**79**]. The book by Latombe [**49**] also contains a chapter on nonholonomic motion planning. The motion planning methodologies can be categorized into the following three groups according to which mathematical methods are used :

- strategies derived by employing differential-geometric and differential-algebraic techniques;
- strategies based on special control parametrization;
- strategies employing methods of optimal control.

Although, at first glance, the above appear to be very different, however, there are many connections between them, and they all lead to similar developments.

4.1.1. Strategies derived by employing differential-geometric and differential-algebraic techniques

Many of the available open loop strategies are based on Lie-algebraic techniques in which motion in the directions of iterated Lie brackets is generated by using piecewise constant inputs. This is explained below.

It is well known, see [**79**], that if g_i, $1 \leq i \leq m$ are smooth vector fields associated with a drift free system of the type (1.13), then the motion of the system in any Lie bracket direction $[g_i, g_j]$, $1 \leq i, j \leq m$ can be achieved by applying the following control sequence for a time Δt:

$$\begin{aligned}
(a) \qquad (u_i, u_j) &= (1, 0) \\
(b) \qquad (u_i, u_j) &= (0, 1) \\
(c) \qquad (u_i, u_j) &= (-1, 0) \\
(d) \qquad (u_i, u_j) &= (0, -1)
\end{aligned}$$

To make the idea more concrete, we consider a simplification of the model used in Example 1.1.

Example 1.2: (two wheel car or unicycle)

Neglecting the equation for the θ angle in (1.12) of Example 1.1, and substituting θ for ϕ (which is equivalent to assuming that the car model has only the rear wheels), leads to the following equation:

$$\dot{x} \stackrel{def}{=} \begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} v_1 + \begin{bmatrix} 0 \\ cos\theta \\ sin\theta \end{bmatrix} v_2 \stackrel{def}{=} g_1(\bar{q})v_1 + g_2(\bar{q})v_2 \tag{1.17}$$

where θ represents the orientation of the car with respect to the x-axis, (x, y) are the Cartesian coordinates of the centre of the mass of the car, v_1 and v_2 denote the angular velocity, and the translational velocity of the car, respectively, and $\bar{q} \stackrel{def}{=} [\theta\ x\ y]^T$.

The motion along g_1 corresponds to forward translation of the car, and the motion along g_2 corresponds to counterclockwise rotation of the car about its mass centre. It is clear that controls (a) and (b) result respectively in forward translation and counterclockwise rotation of the car (both controls are applied for a time Δt). Similarly, controls (c) and (d) result in backward translation and clockwise rotation of the car (also applied for a time Δt). It is then easy to verify that for a small Δt the net motion of the car is essentially a sideways translation with respect to its original configuration and in fact, the Lie bracket, $[g_1, g_2](\bar{q}) = [0\ \ -sin\theta\ \ cos\theta]^T$, predicts precisely this motion. The example thus illustrates that, although instantaneous sideways motion is impossible because of the imposed no-slip condition, sideways motion can be generated by switching between the motions which satisfy the instantaneous nonholonomic constraint.

By using more complex switchings it is possible to generate net motions in the directions provided by the iterated Lie brackets of g_i and g_j. The idea of employing piecewise constant inputs to generate motions in the directions of iterated Lie brackets has been exploited by Lafferriere [**51**] and Lafferriere and Sussmann [**52**]. Their algorithm is based on expressing the flow resulting from piecewise constant inputs as a formal exponential product expansion involving iterated Lie brackets. If the initial and final states are sufficiently close, the algorithm of Lafferriere and Sussmann moves the original system closer to the goal by at least a half of the initial distance. By repeated application of the algorithm, it is possible to move the system into an arbitrary neighbourhood of the desired state. For nilpotent systems (systems for which all iterated Lie brackets of sufficiently high order are zero) the algorithm provides exact steering. Examples of nilpotent systems include systems in chained and in power forms which are *special cases* of (1.13).

Systems in *chained form* are systems whose equations are given by:

$$\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_2\, u_1 \\
\dot{x}_4 &= x_3\, u_1 \\
&\vdots \\
\dot{x}_n &= x_{n-1}\, u_1
\end{aligned} \tag{1.18}$$

Systems in *power form* are systems whose equations are given by:

$$\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_1\, u_2 \\
\dot{x}_4 &= \frac{1}{2} x_1^2\, u_2 \\
&\vdots \\
\dot{x}_n &= \frac{1}{(n-2)!} x_1^{n-2}\, u_2
\end{aligned}$$

Example 1.2 (continued)

It is easily seen that the following state and control transformation:

$$\begin{aligned}
x_1 &= \theta \\
x_2 &= x\ cos\ \theta + y\ sin\ \theta \\
x_3 &= x\ sin\ \theta - y\ cos\ \theta \\
u_1 &= v_1 \\
u_2 &= v_2 - v_1 x_3
\end{aligned}$$

brings the car system (1.17) into chained form:

$$\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_2\, u_1
\end{aligned} \tag{1.19}$$

Similarly the following transformation:

$$\begin{aligned} x_1 &= x \cos\theta + y \sin\theta \\ x_2 &= \theta \\ x_3 &= x \sin\theta - y \cos\theta \\ u_1 &= v_1 - v_2 x_3 \\ u_2 &= v_1 \end{aligned}$$

brings the car system (1.17) into power form:

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1\, u_2 \end{aligned} \tag{1.20}$$

It is worth pointing out that the chained form and the power form are equivalent via a state transformation [**78, 114**]. Sufficient conditions on the vector fields in (1.13) that guarantee that (1.13) can be transformed into the chained form via state and control transformations have been developed by Murray and Sastray in [**77, 80**] for $m = 2$, and by Bushnell et al. in [**13**] for $m > 2$.

It should also be pointed out that the algorithm of Lafferriere and Sussmann [**51, 52**], can be based on other types of switching inputs, not necessarily piecewise constant inputs, see [**52**] for details. In a related paper, Jacob [**39**] proposed an algorithm for exact steering of nilpotent systems using piecewise constant or polynomial inputs. His algorithm is similar to Lafferriere and Sussmann's but with some modification in the construction procedure which results in simpler paths.

Another set of tools for motion planning, based on averaging theory, has been developed by Gurvits and Li [**34**], Leonard and Krishnaprasad [**54**], Liu [**57**], and Sussmann and Liu [**106**]. The basic idea there is to use high-frequency, high-amplitude periodic control inputs to generate motions in the directions of the iterated Lie brackets. Employing this technique, the averaged system, obtained in the limit as the frequency of the inputs increases, is steered exactly to a given desired point.

Tilbury et al. [**109**] examine a variety of implementation issues pertinent to the asymptotic sinusoidal steering algorithm of Sussman and Liu [**106**], in the context of steering kinematic car-like systems with trailers. Specifically, it is shown that preliminary state and control transformations may facilitate convergence to the averaged trajectory. Although high-frequency control inputs may be undesirable from an implementation point of view, the high frequency can be avoided by selecting the time interval to be large, over which the system is steered approximately.

The concept of a flat nonlinear system [**29, 30, 112**] is useful in solving certain nonholonomic motion planning problems. To explain the notion of a flat system, consider the nonholonomic system (1.13). If there exists an output function $y(x(t), u(t), \dot{u}(t))$, with same dimension as control input u, such that the state x and the control input u can be expressed as functions of the output y and its derivative $\dot{y}$, then (1.13) is called *differentially flat* and the output is called the *flat output*. For a differentially flat system motion planning reduces to prescribing a smooth output function satisfying the boundary conditions imposed by the initial and final state specification. The desired control input and the trajectory can be obtained by differentiating the prescribed output function and no integration is required. Rouchon et al. [**85, 86**] showed that systems such as, an automobile with multiple trailers, are flat. The flat output is provided by the Cartesian coordinates of the last trailer. For system (1.20), the flat output is given by $(y_1, y_2) \stackrel{def}{=} (x_2, x_3)$, and thus:

$$
\begin{aligned}
x_1 &= \dot{y_2}/\dot{y_1}, \qquad x_2 = y_1, \qquad x_3 = y_2 \\
u_1 &= (\ddot{y_2}\dot{y_1} - \dot{y_2}\ddot{y_1})/(\dot{y_1})^2, \qquad u_2 = \dot{y_1}
\end{aligned}
$$

The motion planning problem hence reduces to prescribing output functions $y_1(t), y_2(t)$, satisfying the boundary conditions imposed by the initial and final state specification and $\dot{y_1}(t) \neq 0$. In [**60**] it is shown that any kinematic nonholonomic system of the form (1.13) with $n = 5$ and $m = 2$ is flat. An example of a system which is not flat is provided by a ball rolling on a plane without slipping [**12, 59**].

4.1.2. Strategies based on special control parametrization

A more elementary method for motion planning is also available. This method is based on parametrization of the inputs within a given finite dimensional family of functions such as sinusoidal functions. Consider the kinematic model of a nonholonomic control system of the form (1.13). The objective is to steer the system from a given initial state $x_0 \in I\!R^n$ to a pre-specified final state $x_f \in I\!R^n$, over a time interval $[0, T]$. Let $\{U(\alpha; .) : \alpha \in I\!R^q\}$ be a parameter-dependent family of control inputs $U(\alpha; .) : [0, 1] \to I\!R^m$, where $\alpha \in I\!R^q$ is a parameter. Let $\tilde{x}(\alpha; t)$, $0 \leq t \leq 1$, denote the solution to (1.13) with $\tilde{x}(\alpha; 0) = 0$ and $u(t) = U(\alpha; t)$, $0 \leq t \leq 1$. Let $G : I\!R^q \to I\!R^n$ be defined by $G(\alpha) = \tilde{x}(\alpha; 1)$. If the control family $\{U(\alpha; .) : \alpha \in I\!R^q\}$ is sufficiently rich, G is onto $I\!R^n$. In this case the control input $\tilde{u}(\alpha; t)$, $0 \leq t \leq 1$, which steers the system from the origin to $x \in I\!R^n$, can be defined by setting $\tilde{u}(\alpha; t) = U(\alpha; t)$, $0 \leq t \leq 1$, where α is a solution to $G(\alpha) = x$. Since (1.13) is drift-free, it is then possible to re-scale $\tilde{u}$ to obtain a control:

$$u(t) = \begin{cases} -\frac{2}{T}\tilde{u}(x_0; 1 - 2t/T), & 0 \leq t \leq .5T, \\ \frac{2}{T}\tilde{u}(x_f; 2t/T - 1), & .5T \leq t \leq T \end{cases}$$

which steers system (1.13) from x_0 to x_f over the time-interval $[0, T]$ (see (d) of section 3). For example, consider steering (1.20) using a family of control inputs $U(\alpha; t) = (U_1(\alpha; t), U_2(\alpha; t))$:

$$\begin{aligned} U_1(\alpha; t) &= \begin{cases} \alpha_1, & 0 \leq t \leq .5, \\ \alpha_3 \; sin 4\pi t, & .5 \leq t \leq 1 \end{cases} \\ U_2(\alpha; t) &= \begin{cases} \alpha_2, & 0 \leq t \leq .5, \\ |\alpha_3| \; cos 4\pi t, & .5 \leq t \leq 1 \end{cases} \end{aligned}$$

Integrating (1.20) with the above controls and with $x_1(0) = x_2(0) = x_3(0) = 0$ over the interval $[0, 1]$, yields

$$G(\alpha) = \begin{bmatrix} \frac{\alpha_1}{2} \\ \frac{\alpha_2}{2} \\ \frac{\alpha_1 \alpha_2}{2} - \alpha_3 |\alpha_3| / (8\pi) \end{bmatrix}$$

Clearly, G is onto $I\!R^3$ and the system can be steered to any configuration as described above.

The above idea appears in the work of many researchers: Bushnell et al. [**13**], Murray [**78**], Murray and Sastry [**77**]. In this approach a system in power or chained form can be steered to any given desired point by using a family of sinusoids at integrally related frequencies. Lewis et al. [**56**] showed that sinusoids at integrally related frequencies can be used to steer a snakeboard. The use of other control functions, e.g., piecewise constant functions or polynomials, has also been investigated by Jacob [**39**], Tilbury [**110**], and Tilbury et al. [**111**]

The multirate digital control approaches developed by Chelouah et al. [**20**], Monaco and Norman-Cyrot [**74**], Sordalen and Egeland [**101**], Tilbury and Chelouah [**108**], can be also viewed as a way of steering a system via parameterization of the input within a family of piecewise constant inputs. The basic idea of the multirate digital control approach is to sample the input by a zero order hold and steer the resulting discrete time system, typically, different sampling rates are used for different input channels. Depending on the nature of control strategies and their interpretation, the multirate digital control approaches may be viewed as feedback strategies [**101**].

4.1.3. Strategies employing methods of optimal control

Although the methodologies of the previous section provide a solution to the motion planning problem, there often exist many solutions. A specific solution can be selected using optimization. Brockett [11], and Brockett and Dai [12] demonstrated the optimality of sinusoidal and elliptic control functions for certain minimum norm nonholonomic optimal control problems. The optimality of elliptic functions has been also addressed by Krishnaprasad and Yang in [48]. Reeds and Shepp [83] obtained a complete characterization of the shortest paths connecting any two given configurations for the car model (1.17). They showed that the shortest path is one of 48 extremal paths that can be explicitly computed. Each of the extremal paths has no more than five segments and requires no more than two direction reversals.

Conditions for optimality in various nonholonomic optimal control problems are discussed by Bloch and Crouch [9], Sastry and Montgomery [93] and Montgomery [75]. In particular, Sastry and Montgomery [93] and Montgomery [75] study the optimal control problem of minimizing the L_2-norm of control subject to given initial and final states and subject to equation (1.13). Using the maximum principle, they show that the optimal control is such that the quantity $\sum_{i=1}^{m} |u_i(t)|^2$ remains constant. For the same problem, Montgomery [75] considers in detail the case of abnormal (singular) extremals. He demonstrates that abnormal extremals may provide an optimal solution and thus cannot be neglected in analysis. He also considers a time-optimal control problem for nonholonomic control system (1.13). Walsh et al. [116] studied the minimal norm control problems for kinematic systems evolving on Lie groups. Numerical techniques for constructing optimal trajectories for a variety of nonholonomic control problems are proposed by Agrawal and Xu [1], Fernandes et al. [27, 28] and Hussein and Kane [36].

4.2. Closed loop control strategies

A majority of the closed loop control strategies developed for nonholonomic control systems serve the purpose of stabilization of such systems to a point. In the absence of smooth static or even continuous stabilizing feedback, (see Brockett's necessary condition [10]), the closed-loop synthesis methods concentrate on either:

- synthesis of discontinuous state feedback,
- synthesis of time-varying state feedback.

The literature pertinent to the above methods is summarized below.

4.2.1. Synthesis of discontinuous state feedback

Discontinuous state feedback control for stabilization of drift free systems to the origin can be further classified into three types:

(a) piecewise continuous feedback

(b) sliding mode control

(c) hybrid feedback control.

(a) Piecewise continuous feedback

In [**102**], Sussmann proved the existence of stabilizing piecewise continuous static state feedback control for a class of nonlinear controllable systems. The class mentioned includes nonholonomic control systems which satisfy the real analyticity assumption (namely that all the vector fields in the drift free system equation are real and analytic). Lafferierre and Sontag [**53**] presented a formula for a piecewise continuous feedback law, obtained from a piecewise smooth control Lyapunov function. The resulting feedback is globally stabilizing and is discontinuous on a surface of a lower dimension than the state space. However, there are no general methods for constructing control Lyapunov functions satisfying the assumptions of [**53**].

Piecewise continuous feedback has been constructed for specific examples as reported by Lafferierre and Sontag [**53**], Canudas de Wit and Sordalen [**16**] and Khennouf and Canudas de Wit [**46, 17**]. Exponential convergence of the states to the equilibrium point has been demonstrated in all these examples. Sordalen et al. [**98**] also proposed a piecewise continuous feedback law for local stabilization of the attitude of an under-actuated rigid spacecraft with only two angular velocity controls. The feedback law results in exponential convergence rates of the states to the equilibrium.

A different approach for the construction of piecewise continuous controllers has been developed by Aicardi et al. [**2**], Astolfi [**4, 5**], and Badreddin and Mansour [**6**]. A non-smooth state transformation is employed there and a smooth time-invariant feedback is constructed to stabilize the transformed system. In the original coordinates, the resulting feedback law is discontinuous. In [**2, 4**] this approach has been used for stabilization of kinematic and dynamic models of simple mobile robots. For these examples, the non-smooth state transformation is provided by simply changing the Cartesian coordinates to polar coordinates. The potential of this approach for application to more complicated nonholonomic control problems remains to be investigated.

(b) Sliding mode control

Discontinuous time-invariant feedback laws can also be developed using sliding mode control approaches as proposed by Bloch and Drakunov [7] and by Guldner and Utkin [**32**]. The resulting discontinuous feedback laws force the trajectory to slide along a certain manifold towards the equilibrium. Consider, for example, the problem of stabilizing system (1.20) to the origin. Define the feedback law according to [**7**]:

$$\begin{aligned} u_1 &= -x_1 + 2x_2 sign(x_3 - \frac{x_1 x_2}{2}), \\ u_2 &= -x_2 - \frac{1}{2} x_1 sign(x_3 - \frac{x_1 x_2}{2}) \end{aligned} \tag{1.21}$$

where $sign(\cdot)$ denotes the signum function. Let $V(x_1, x_2) = \frac{1}{2}(\frac{x_1^2}{4} + x_2^2)$. Then the derivative of V along the closed-loop trajectories of (1.20) satisfies $\dot{V} = \frac{1}{4}x_1 u_1 + x_2 u_2 = -2V$. Thus $V(t) = V(0)e^{-2t} \to 0$ as $t \to \infty$ and $x_1 \to 0$, $x_2 \to 0$ as $t \to \infty$. Let $\theta = x_3 - \frac{x_1 x_2}{2}$. Then, $\dot{\theta} = -2V sign(\theta)$. Clearly, $|\theta(t)|$ is non-increasing and, in fact, can reach zero in finite time provided that

$$V(x_1(0), x_2(0)) > |\theta(0)| \tag{1.22}$$

Once $\theta(t)$ reaches the origin, it must stay at the origin and, hence, the trajectory will slide along the surface $x_3 = \frac{x_1 x_2}{2}$ toward the origin. If the initial conditions do not satisfy inequality (1.22), a preliminary control can be used to force the trajectory into the region where inequality (1.22) holds and then the feedback law (1.21) can be switched on.

The sliding mode control approach can only be applied to certain classes of higher dimensional nonholonomic control systems [**7, 32**]. Generally, however, it is not suitable for stabilization of nonholonomic control systems and remains a subject for future research.

(c) Hybrid feedback control

Typically, hybrid controllers combine continuous-time features with either discrete-event features or discrete-time features. The operation of hybrid controllers is based on switching at discrete-time instants between various low-level continuous-time controllers. The time-instants at which switches occur may either be specified a priori or also be determined in the process of controller operation.

Controllers which combine continuous time features with discrete event features have been proposed by Bloch et al. [8], and by Kolmanovsky et al. [**41, 42**], for a sub-class of (1.13). The controllers developed there consist of a discrete event supervisor and low-level time-invariant feedback controllers. The supervisor configures the low-level feedback controllers and accomplishes switchings between them in a way that provides stabilization of the system. Each of the low-level feedback controllers forces the base variables to trace a specific straight line segment of the base space path, which is selected by the supervisor to produce the desired geometric phase change. The feedback law provides finite time (dead-beat) responses. This approach is also used by Krishnan et al. [**47**] for attitude stabilization of a rigid under-actuated spacecraft model with only two control torques.

Sordalen et al. [**99, 100**] developed a hybrid controller for stabilization of kinematic nonholonomic systems in *chained form* and showed that such controllers result in exponential convergence rates of the states to the origin. Also hybrid controllers of a different type, which apply to systems specifically in *chained form* have been proposed by Canudas de Wit et al. [**18**]. The control of [**18**] provides only for practical stabilization (stabilization to a small neighbourhood of the origin). The hybrid approach proposed by Sontag in [**95, 96**] is more general as it applicable to a large class of nonholonomic control systems, but is less explicit. Sontag makes use of a family of periodic inputs that are universal nonsingular control [**95**] and result in periodic trajectories. Linearization about each of these trajectories is controllable. Consequently, a perturbation of a periodic input can be constructed to bring the state closer to the origin at the end of each cycle. A good introduction to some of hybrid stabilization techniques in mobile robot context is contained in the article [**19**].

4.2.2. Synthesis of time-varying state feedback

The use of time-varying feedback controls in application to nonholonomic systems was probably first proposed by Samson [**88, 89, 90, 91**] in his research work concerning mobile robots. Coron [**24**] was to the first to show rigorously that kinematic nonholonomic control systems can be asymptotically stabilized to an equilibrium point by smooth time-periodic static state feedback. The existence proof of [**24**], however, does not provide for the construction of feedback laws.

Some explicit feedback construction procedures are developed later. Murray et al. [**77**], Teel et al. [**107**], and Walsh and Bushnell [**114**] used the method of averaging and saturation type functions to construct smooth time-periodic feedback laws for systems in *power* and *chained forms*. The feedback laws of [**77**], [**107**], and [**114**] achieve global asymptotic stabilization. In [**114**] numerical simulations illustrate the resulting feedback laws for a fire truck example, a nonholonomic system with three inputs and five states.

Samson and Ait-Abder-rahim [**90**], and Walsh et al. [**115**] provided a different asymptotic stabilization scheme based on construction of a "nominal trajectory" which asymptotically approaches the equilibrium. In [**90**] and [**115**] linear controllers are constructed which stabilize a variational system about the nominal trajectory. This approach is easy to use but requires an a priori selection of a nominal trajectory.

Another constructive approach has been proposed by Pomet [82], and Coron et al. [25]. This approach, widely known as Pomet's method, is based on Lyapunov's direct method and is to some extent, similar to the well-known technique of Jurdjevic and Quinn [40]. Pomet's method generates smooth time-periodic feedback laws by constructing suitable Lyapunov functions. As an illustration, consider the following smooth feedback law for the chained system (1.20) provided by Pomet's method:

$$\begin{aligned} u_1(x,t) &= -x_1 + x_3(sint - cost) \\ u_2(x,t) &= -x_2 - x_1x_3 - 2(x_1 + x_3cost) \end{aligned}$$

This feedback law is obtained from the closed-loop Lyapunov function given by

$$V(x,t) = \frac{1}{2}(x_1 + x_3cost)^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2$$

which gives

$$\dot{V}(x,t) = -\{(x_1 + x_3cost) + (x_2 + x_1x_3)\}^2 \le 0, \quad \forall\, x \in I\!R^3$$

along any trajectory of (1.20). By applying the Krasovskii-LaSalle invariance principle for periodic systems it can be verified that the origin is the globally asymptotically stable equilibrium of the closed loop.

Unfortunately, the rates of convergence provided by smooth time-periodic feedback laws are necessarily non-exponential [78]. For system (1.20) smooth time-periodic controllers can provide time rates of convergence of at most $1/\sqrt{t}$, see [88]. Furthermore, in experiment work [21], M'Closkey and Murray have demonstrated that smooth time-periodic feedback laws do not steer mobile robots to a small neighbourhood of the desired configuration in a reasonable amount of time. Thus feedback laws which provide faster convergence rates are desirable. These feedback laws must be necessarily be nonsmooth (non-differentiable). Further information on connections between the rates of convergence and smoothness of feedback laws can be found in references [34, 22].

A construction procedure which provides nonsmooth feedback laws with exponential convergence rates has been proposed by M'Closkey and Murray in [23]. The resulting feedback laws are continuous and smooth everywhere except at the origin. The construction procedure can be viewed as an extension of Pomet's algorithm to the case of nonsmooth feedback laws. For systems in power form, explicit expressions for the feedback laws can be obtained. For example, for system (1.20) a nonsmooth time-periodic feedback law which results in exponential convergence rates is of the form:

$$\begin{aligned} u_1(x,t) &= -x_1 + \frac{x_3}{\rho(x)} cost, \quad x \neq 0 \\ u_2(x,t) &= -x_2 + \frac{x_3^2}{\rho^3(x)} sint, \quad x \neq 0 \\ u_1(0,t) &= u_2(0,t) = 0, \quad \text{where} \quad \rho(x) = (x_1^4 + x_2^4 + x_3^2)^{(1/4)} \end{aligned}$$

The closed-loop system is globally exponentially stable with respect to a homogeneous norm $\rho(x)$, in that, there exist constants $\lambda_1 > 0$ and $\lambda_2 > 0$ such that

$$\rho(x(t)) \leq \lambda_1 \rho(x(0)) exp(-\lambda_2 t)$$

This notion of exponential stability "with respect to a homogeneous norm" is only slightly different from the standard notion of exponential stability.

Time-periodic feedback laws for stabilization of dynamic models of nonholonomic control systems can be derived from kinematic controllers using the integrator backstepping or "error tracking" approaches, see [**45, 114**] for details.

Besides mobile robots [**21, 88, 89, 90, 91, 114**], time-varying stabilization has been used for knife-edge models with augmented actuator dynamics [**45**], under-actuated rigid spacecraft controlled by only two rotors [**76, 115**], and free-floating multi-body spacecraft [**44**].

4.3. Difficulties arising in the previously existing methods

(a) Most of the existing methods for both motion planning and feedback stabilization necessitate the construction of diffeomorphic state transformations which convert the systems into either chained or power form, see [**13, 77, 78, 80, 114**]. Although, there are necessary and sufficient conditions available for the existence of such diffeomorphic state transformations, see [**77, 80, 13**], the transformations are usually defined only locally.

(b) The disadvantage of the sliding mode controllers is that they may cause chattering. Guldner et al. [**33**] have proposed to use smoothing to prevent chattering, see [**94**] for the definition of smoothing. Piecewise continuous controllers usually avoid chattering as the trajectory does not "stick" to the discontinuities. It should also be pointed out that controlling kinematic nonholonomic systems with discontinuous (velocity) controls may be difficult to implement.

Formulations involving dynamic nonholonomic control systems seem preferable if discontinuous controllers are used.

(c) As mentioned earlier, Coron [**24**] showed that the nonholonomic control systems can be asymptotically stabilized to an equilibrium point by smooth time-periodic static state feedback but the existence proof of [**24**] does not provide for the construction of feedback laws. Other time-varying feedback approaches [**24, 77, 107, 114, 82, 25**] rely on the existence of suitable time varying Lyapunov functions which are not easy to find.

(d) Time varying feedback laws do not provide for exponential convergence rates in asymptotic stabilization.

(e) There are hardly any results available concerning robustness of the control methods developed (with respect to model uncertainties, as arise from parameter variations or from neglected dynamics). Only a few preliminary results are available in [**17**]. The difficulties are primarily technical and general methods for the study of robustness for this class of nonlinear systems are not available. Consequently, methods for design of robust controllers for nonholonomic systems are unknown. Open loop approaches are less likely to produce solutions which are robust with respect to modelling uncertainties and censoring error, as compared with feedback approaches.

5. Research objective

In the light of the difficulties arising in the control of nonholonomic systems listed above, and the intrinsic features of nonholonomic systems outlined in section 3, the objective of this thesis was to explore novel and more effective feedback synthesis approaches for stabilization of such systems to a point. In the absence of continuous static feedback laws, attention was focused on developing simple and systematic approaches for the construction of:

1- *piece-wise constant* feedback control laws

2- *time varying* feedback control laws

3- investigating the possibility of employing a mixture of the above approaches and sinusoidal steering [**77, 109**].

Since the existence of transformations to either power or chained form is not generally obvious and even if such transformations exist they are usually defined locally, the ultimate aim was to propose synthesis methods of 1-3 which abstract from any specific form of a drift free system. Such transformation free approaches would naturally result in simpler and more direct feedback synthesis.

6. Contribution of the thesis

With respect to objectives 1-3 listed above, two novel feedback synthesis approaches were introduced, analysed, and their utility for applications was explored:

(a) a *guiding functions approach* [**63, 64, 65, 67, 69, 71, 72**]

(b) a *trajectory interception approach* [**66, 68, 70, 84**].

The possibility of employing sinusoidal steering [**77, 109**], in conjunction with either of the above approaches was also explored, see [**64, 67, 69, 70, 71, 72**]. The above approaches resulted in the construction of new and effective feedback control strategies for drift free systems.

A brief description of the new approaches (a) and (b) follows next.

6.1. The guiding functions approach

In the guiding functions approach a number of semi-positive definite functions called "guiding functions" are introduced to determine a desired direction of system motion. These guiding functions permit to construct a sequence of controls such that the sum of the guiding functions decreases in an average sense. This approach delivers bounded, piece-wise constant control laws. The approach is first applied and analysed for systems which appear in a special *rectified form*, and is later extended to apply to general drift free systems. The guiding functions approach often leads to dead beat control and provides discontinuous stabilizing feedback laws.

6.2. The trajectory interception approach

The trajectory interception approach is based on considering of what is known as the Lie bracket extension of the original system. The resulting feedback law can be viewed as a composition of a standard stabilizing feedback control for the extended system and a periodic continuation of a parametrized solution to an open loop, finite horizon control problem stated in logarithmic coordinates of flows. In this approach an arbitrary Lyapunov function is used to construct a time-varying stabilizing feedback law.

7. Thesis outline

The thesis is organized as follows:

- Chapter 2. **Guiding functions as a tool for stabilization of systems in rectified form.**

  A novel concept of guiding functions is introduced which can be used as a tool for construction of new and effective feedback control strategies for drift free systems, [**63**]. A stabilizing control strategy based on this concept is developed and analysed for systems of control deficiency order one, in rectified form. The strategy is shown to be applicable also to systems of higher order of control deficiency. Under reasonable assumptions, the feedback control strategy yields global asymptotic stabilization to a set point.

  The guiding functions control strategy is tested on two examples on which it proves to be very effective. The geometric insight into the steering problem, gained by employing the guiding functions, is demonstrated to lead to a yet simpler and more effective feedback control laws [**63, 65**].

- Chapter 3. **The guiding functions stabilizing strategy for general drift free systems.**

  The guiding functions control strategy is next extended to apply to general drift free systems which need not be transformable to a rectified form. A systematic method for the construction of a pair of guiding functions is introduced [**63**], and conditions are stated which guarantee that the resulting feedback control strategy yields global asymptotic convergence to a desired set point.

  A few applications of the strategy are discussed and tested on models of drift free systems which are characterized by different algebraic structures: an underwater vehicle, a general system with five state variables and three controls, and a spacecraft model [**64, 69**].

  The possibility of employing the guiding functions approach to systems whose controllability Lie algebra involves higher order Lie brackets is also investigated. The idea of combining sinusoidal steering [**77, 109**] with the guiding functions approach is explored in [**64, 69, 72**]. In [**64, 69**] a feedback controller is constructed for an underwater vehicle in actuator failure mode, and in [**72**] a feedback controller is constructed for a mobile robot with trailer.

The guiding functions approach for general drift free systems is further extended to allow for the construction of several rather than two guiding functions and its applicability is demonstrated on a fire truck model [**67**] and on a class of wheeled mobile robots [**71**].

- Chapter 4. **The trajectory interception approach**

The trajectory interception approach is first introduced for a class of drift free systems for which the associated controllability Lie algebra is nilpotent. The concept of a Lie bracket extension of the system, see [**51**], is employed and an arbitrary Lyapunov function is used to construct a closed loop stabilizing controller for the extended system. This classical static feedback is then combined with a periodic continuation of a parametrized solution to an open loop steering problem for the comparison of flows of the original and extended systems. This approach is applied to stabilize several examples of drift free systems possessing different algebraic structures.

It is shown that the application of the trajectory interception approach is not limited to systems whose controllability Lie algebra is *nilpotent*. The approach can successfully be applied to systems with *non nilpotent* controllability Lie algebras by introducing approximate models which generate nilpotent controllability Lie algebras. This approximation idea is employed to stabilize a number of drift free systems possessing different algebraic structures: a rigid spacecraft in actuator failure mode [**68**], a hopping robot in flight phase [**84**], an underwater vehicle [**66**], and a class of wheeled mobile robots [**70**].

It is shown that introducing approximate models often permits significant simplification of the differential equations describing the evolution of the logarithmic coordinates in the open-loop problem formulation (which are usually difficult to solve analytically).

Since the computation of the solutions to the open loop trajectory interception problem may be elaborate if the extended system contains high order Lie brackets, the possibility of introducing decomposition into control synthesis is explored. This idea involves decomposing a complex system model into subsystems of which one can be controlled by the trajectory interception approach and the other by simple sinusoidally varying inputs. The feasibility of this approach is demonstrated using a few examples.

As in the case of the guiding functions approach, the feedback synthesis method based on the trajectory interception idea does not necessitate conversion of the models into chained or power forms.

- Chapter 5. **Conclusions and Future Research**

  In conclusion, a brief review of the results of the preceding chapters is presented and some general observations are commented. The two feedback design approaches are compared and their utility for different applications is explained. Some suggestions are also given for future work.

8. Originality of research contribution and its potential advantages for applications

- The two feedback control strategies introduced are *new* and do not require conversion of the system models either to power or chained forms. In principle, no transformation techniques are needed. Both strategies can thus be applied to systems which fail to satisfy the conditions for the existence of special transformations, and to systems which are not flat. Both control strategies possess strong robustness properties with respect to model inaccuracies.

- *The concept of guiding functions as a tool for feedback control design has not appeared in previous literature.* The guiding functions approach is particularly simple and often leads to very effective feedback control laws such as 'dead beat control'.

- The trajectory interception approach involves an *original* idea of employing the Lie algebraic techniques of [**51**] and [**50**] in a systematic synthesis of time-varying feedback control for drift free systems.

- The trajectory interception approach provides for exponential rates of convergence to a desired set point. The results contained in this thesis open a new area of research with the goal of rendering this synthesis approach computationally simpler, more effective, and extending its applicability to systems with drift.

CHAPTER 2

Guiding functions as a tool for stabilization of systems in rectified form

A novel and systematic approach to the construction of feedback control for stabilization to a set point of drift free systems is introduced, [**63, 64**]. The approach is based on a new concept of guiding functions whose sum vanishes only at the reference set point. The guiding functions are not Lyapunov functions, however, a comparison of their values allows to determine a desired direction of system motion and permits to construct a sequence of controls such that the sum of the guiding functions decreases in an average sense. The individual guiding functions are hence not restricted to decrease monotonically but their oscillations are limited and coordinated in a way to guarantee convergence. The guiding functions control strategy is tested on two examples; a unicycle and a front wheel drive, on which it proves to be very effective. In both cases, the choice of the guiding functions is straightforward and gives additional, geometric insight into the steering problem. The guiding functions approach presented is general and can be employed to control a variety of mechanical systems with velocity constraints.

1. Introduction

The feedback control method presented in this Chapter applies to drift free systems of the form (1.13), which appear in a special rectified form (see section 3). The method employs a new concept of guiding functions in place of a single Lyapunov function. Before to explain this concept, consider the system (1.13) with slightly different notaions; $\xi = x$, $v_i = u_i$, and $f_i(\xi) = g_i(x)$:

$$\dot{\xi} = \sum_{i=1}^{m} f_i(\xi) v_i, \tag{2.1}$$

and also for any integer $m > 1$, let $\underline{m} \stackrel{def}{=} \{1, ..., m\}$. The principal idea of this approach is explained below for the simple case when the number of controls is by one less than the number of state variables ($m = n - 1$).

Design of nonlinear stabilizing feedback typically involves a search for a suitable 'control' Lyapunov function $V(\xi)$ and a control law $v(\xi) \stackrel{def}{=} [v_1(\xi), ..., v_m(\xi)]$ which renders $\frac{d}{dt}V(\xi) < 0$ along the trajectories of the controlled system, see [**50, 97**]. For systems of type (2.1) this approach is not possible as there does not exist any function V for which the set $\mathcal{T} \stackrel{def}{=} \{\xi \in I\!R^n : \nabla V(\xi) f_i(\xi) = 0, i \in \underline{m}\} = \{0\}$. Hence we take a different route. We attempt to find $n-1$ functions $V_i(\xi)$, $i \in \underline{n-1}$, henceforth called 'guiding functions', whose behaviour along the trajectories of the controlled system is not limited to $\frac{d}{dt}V_i(\xi) < 0$. While allowing some guiding functions to increase, we design controls $v_i(\xi)$ $i \in \underline{m}$ such that their "synchronized action" causes the sum $V(\xi) \stackrel{def}{=} \sum_{i=1}^{n-1} V_i(\xi)$ to decrease *on average*. The latter is indeed possible if the functions V_i are chosen to satisfy the following conditions:

Condition (a): Each V_i, $i \in \underline{n-1}$, is semi-positive definite on $I\!R^n$ while the sum $V = \sum_{i=1}^{m} V_i$ is positive definite, decrescent and proper on $I\!R^n$.

Condition (b): The value of each V_i, $i \in \underline{n-2}$, can be manipulated independently of the value of V_{n-1} in that if $V_i(p) \neq 0$ for some $i \in \underline{n-2}$, at some point p, then there exist controls $v_i, i \in \underline{m}$, which steer V_i, $i \in \underline{n-2}$, to zero in finite time while V_{n-1} maintains its value at p.

Condition (c): The value of V_{n-1} can be decreased over a finite interval of time if the remaining V_i, $i \in \underline{n-2}$, are allowed to vary freely.

The above assumptions suggest a feedback strategy which focuses on the decrease on V_{n-1} alone. To begin with, the strategy attempts to employ controls which provide for the satisfaction of the usual condition that $\frac{d}{dt}V(\xi) = \sum_{i=1}^{m} \frac{d}{dt}V_i(\xi) < 0$. If the last becomes impossible, due to the fact that $\frac{d}{dt}V_i(p) = 0$, for $i \in \underline{n-1}$, regardless to the values of the controls $u_i, i \in \underline{m}$, then a sequence of controls is employed which results in a decrease of V_{n-1} while the remaining V_i are permitted to increase by an amount proportional to the current value of V (see assumption (c)). Next, another sequence of controls is employed which maintains the current value of V_{n-1} and steers V_i, $i \in \underline{n-2}$, to zero (see assumption (b)). Repeating the above procedure results in asymptotic convergence of V to zero.

It is shown here that the above strategy is indeed feasible, and that the guiding functions, satisfying the desired properties (a)-(c), are especially easy to define in the case when the vector fields $f_1, ..., f_m$

are simultaneously "rectifiable" (see hypothesis H1 in section 3). The strategy is first developed and analysed for systems in which the difference between the number of control and state variables is one (i.e. $m = n - 1$). Such systems are referred to under the name of systems of "control deficiency order one". A well known example of such systems is the model of a unicycle. For this example, when the set point is chosen to be the origin, the strategy is shown to achieve what could be regarded as the 'intuitively best' type of control. In the absence of disturbances, the control is dead-beat and is accomplished in three steps. At the end of the first step the car assumes a position sideways to its goal - the origin. This position of the car 'requires' the car to displace sideways, (in the direction of the Lie bracket of the vector fields corresponding to the rotation and rolling movements of the car). In its second step the strategy makes the car rotate 90 degrees and then drives it straight to the origin.

It is shown next that the guiding functions approach can be generalized to apply to systems of control deficiency order two. In this case considering a Lie bracket extension of the original system is necessary. A more complex example of a car is used this time (a front wheel drive for which $m = n - 2$). Simulations confirm that the strategy is very effective also in this case. Finally, a yet better control construction for this particular example is shown to follow from the guiding functions approach.

The guiding functions control strategy has several advantages which make it potentially useful for applications (examples include the control of mobile robots in which it is important to control the entire state vector) :

(i) Without the loss of generality, the strategy employs bounded, piecewise constant controls. The bound on the control can be adjusted as necessary if control constraints have to be satisfied.

(ii) The strategy is based on simple principles; the values of the guiding functions provide an on-line convergence verification test.

(iii) Control efficiency in terms of the convergence speed can be improved in special cases as it is dictated by particular realizations of assumptions (b) and (c) which are model dependent.

The novel contribution of this Chapter can be summarized as follows:

- A novel concept of guiding functions is introduced which can be used as a tool for the construction of new and effective feedback control strategies for drift free systems.

- A stabilizing control strategy based on this concept is developed and analysed for systems of control deficiency order one in rectified form. The strategy is shown to be applicable also to systems of higher order of control deficiency.
- It is shown that, under reasonable assumptions, the feedback control strategy yields global asymptotic stabilization to a set point.
- The guiding function controller is tested on two examples on which it proves to be very effective. The geometric insight into the steering problem, gained by employing the guiding functions, is demonstrated to lead to a yet simpler and more effective feedback control laws.

2. The control problem and basic assumptions

We aim to solve the following:

Set point control problem.

SPC: Find a feedback control strategy in terms of piecewise constant controls $v_i(x)$, $i \in \underline{m}$, such that for any two points $\xi_0, \xi_s \in I\!R^n$, $\xi(t;\xi_0,0) \to \xi_s$ as $t \to \infty$, where $\xi(t;\xi_0,0)$ denotes the trajectory of the controlled system (2.1) emanating from ξ_0 at time $t = 0$.

Without the loss of generality, we assume that $\xi_s = 0$, or else the original coordinate system can be translated as required. Two sub-classes of drift free systems will be considered: systems of control deficiency order one, for which $m = n - 1$, and systems of control deficiency order two, for which $m = n - 2$.

We need the following, basic assumptions:

A1. **Complete controllability :**

For system of control deficiency order one, $(m = n - 1)$:

$$span\{f_i(\xi), [f_i, f_j](\xi), i, j \in \underline{n-1}\} = I\!R^n \quad \forall\, \xi \in I\!R^n. \tag{2.2}$$

For system of control deficiency order two, $(m = n - 2)$:

$$span\{f_i(\xi), [f_i, f_j](\xi), [f_i, [f_j, f_k]](\xi), i, j, k \in \underline{n-2}\} = I\!R^n \quad \forall\, \xi \in I\!R^n. \tag{2.3}$$

A2. **Absence of singular points :**

The vector fields in (2.2)-(2.3) do not have asymptotic singular points in that there exists a constant $c > 0$ such that

$$inf\{||f_i(\xi)||, ||[f_i, f_j](\xi)||, ||[f_i, [f_j, f_k]](\xi)||, i, j, k \in \underline{n-2}, \xi \in I\!R^n\} > c \tag{2.4}$$

3. Construction of feedback using guiding functions

3.1. Construction of guiding functions

In the case when the number of state variables exceeds the number of controls by one and, additionally, the system equations appear in a 'rectified form' (see below), the set of guiding functions satisfying the conditions (a)-(c) of section 1 can be introduced in a particularly easy way. For systems in a general form we need to impose the following hypothesis :

H1. **Rectifiability:**

There exist diffeomorphic state feedback transformations $\xi = T_{n-1}(x)$, $v = U_{n-1}(\xi, u)$, such that, in the new coordinates x, and in terms of the new control u, the system with $m = n - 1 \geq 2$ assumes the form :

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_2 + \ldots + \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} u_{n-2} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ h_1(x) \\ h_2(x) \end{bmatrix} u_{n-1} \\ &\stackrel{def}{=} g_1(x)u_1 + g_2(x)u_2 + \ldots + g_{n-2}(x)u_{n-2} + g_{n-1}(x)u_{n-1} \end{aligned} \tag{2.5}$$

where h_1, h_2, are some smooth functions of the new state variable x.

REMARK 2.1. Sufficient conditions for local existence of similar transformations were discussed in [**37**]. Hypothesis H1 is not very restrictive; many systems which are important for applications, appear a priori in this form, or else the rectifying transformations can be found very easily. At times the rectifying transformations are not needed at all as a set of guiding functions $V_i, i \in \underline{n-1}$ which satisfy conditions (a)-(c) of the Introduction can be found for the original system, see Remark 2.2. This is also confirmed by examples in the next sections.

Assuming the satisfaction of H1, we will concentrate on the construction of the guiding functions and the associated feedback control strategy for system (2.5) in place of the original system (2.1). Clearly, if (2.1) satisfies the controllability assumption A1, so does the transformed system (2.5). We will hence assume that assumption A1 is made with respect to system (2.5).

For a system in a rectified form (2.5), we introduce a set of the following semi-positive definite guiding functions:

$$V_i(x) \stackrel{def}{=} \frac{1}{2}x_i^2, \quad i \in \underline{n-2}, \quad x \stackrel{def}{=} (x_1, ..., x_n)^T \tag{2.6}$$

$$V_{n-1}(x) \stackrel{def}{=} \frac{1}{2}[x_{n-1}^2 + x_n^2] \tag{2.7}$$

The above functions indeed satisfy the conditions (a)-(c), as explained below:

Condition (a):

The sum of all guiding functions, defined as:

$$V(x) \stackrel{def}{=} \sum_{i=1}^{n-1} V_i(x) = \frac{1}{2}x^T x \tag{2.8}$$

is clearly positive definite, proper and decrescent in $I\!R^n$.

Condition (b):

Calculating

$$\frac{d}{dt}V_i(x) = \nabla V_i(x) g_i(x) u_i = \begin{cases} x_i u_i & i \in \underline{n-2} \\ x^T g_{n-1}(x) u_{n-1} & i = n-1 \end{cases}, \quad i \in \underline{n-1} \tag{2.9}$$

shows that the value of each guiding function V_i, for $i \in \underline{n-1}$, can be changed only by the corresponding control u_i but not by any other control u_j, $j \neq i$. Now, suppose that $V_k(p) \neq 0$, for some $p \in I\!R^n$ and $k \in \underline{n-2}$. The standard feedback control

$$u_k(x) \stackrel{def}{=} -sign(x_k), \quad u_i(x) = 0, \quad i \neq k, i \in \underline{n-1}, \quad x \in I\!R^n, \tag{2.10}$$

with the *sign* function defined in the usual way, so that $sign(0) = 0$, yields

$$\frac{d}{dt}V_i(x) = \begin{cases} -|x_k| & i = k \\ 0 & i \neq k \end{cases}, \quad i \in \underline{n-1} \tag{2.11}$$

and

$$\frac{d}{dt}x_i(t) = \begin{cases} -sign(x_k) & i = k \\ 0 & i \neq k \end{cases}, \quad i \in \underline{n} \tag{2.12}$$

for all $t \geq 0$, where $x(0) \stackrel{def}{=} p$. It follows that there exists a finite time $t^* > 0$ such that $x_k(t^*) = 0$, and hence that $V_k(x(t^*)) = \frac{1}{2}x_k^2(t^*) = 0$, while $V_j(x(t^*)) = V_j(p)$, for all $j \neq k$, $j \in \underline{n-1}$. This shows that condition (b) is satisfied.

Condition (c):

If $p \in \mathbb{R}^n$ is such that $V_{n-1}(p) \neq 0$ and $p^T g_{n-1}(p) \neq 0$, then the value of V_{n-1} can be decreased by the standard controls

$$u_{n-1}(x) \stackrel{def}{=} -sign(x^T g_{n-1}(x)), \quad u_i(x) = 0, \quad i \in \underline{n-2}, \quad x \in \mathbb{R}^n \tag{2.13}$$

When $p^T g_{n-1}(p) = 0$, but $V_{n-1}(p) \neq 0$, then, there may exist no controls which can further decrease V_{n-1}, while simultaneously preserving the values of the other guiding functions V_i, $i \neq n-1$. (Such an impasse situation occurs when, additionally, $V_i(p) = 0$, for $i \in \underline{n-2}$.) A decrease in V_{n-1} is thus guaranteed only when the values of V_i, $i \neq n-1$, are permitted to increase temporarily. A method for achieving such a decrease can be obtained by inspecting the second derivatives of the guiding functions $V_i, i \in \underline{n-1}$, *while assuming that the controls can only take constant values.*

Noticing that, for all $x \in \mathbb{R}^n$,

$$||g_i(x)||^2 = 1, \quad \nabla g_i(x) = 0 \quad \text{for } i \in \underline{n-2}, \text{ and all } x \in \mathbb{R}^n \tag{2.14}$$

$$g_i(x)^T g_j(x) = 0 \quad \text{for } i \neq j, \; i,j \in \underline{n-1} \text{ and } x \in \mathbb{R}^n \tag{2.15}$$

$$[g_j, g_{n-1}](x) = \nabla g_{n-1}(x) g_j(x), \text{ for } j \in \underline{n-2}, \quad \text{and } x \in \mathbb{R}^n \tag{2.16}$$

gives

$$\tfrac{d^2}{dt^2} V_i(x) = u_i^2, \quad i \in \underline{n-2} \tag{2.17}$$

$$\begin{aligned} \tfrac{d^2}{dt^2} V_{n-1}(x) &= \textstyle\sum_{j=1}^{n-1} g_{n-1}^T(x) g_j(x) u_j u_{n-1} + \sum_{j=1}^{n-1} x^T \nabla g_{n-1}(x) g_j(x) u_j u_{n-1} \\ &= \{||g_{n-1}(x)||^2 + x^T \nabla g_{n-1}(x) g_{n-1}(x)\} u_{n-1}^2 + \textstyle\sum_{j=1}^{n-2} x^T [g_j, g_{n-1}](x) u_j u_{n-1} \end{aligned} \tag{2.18}$$

From (2.9), and (2.14)-(2.18) it further follows that

$$\tfrac{d}{dt} x^T g_i(x) = u_i, \quad i \in \underline{n-2} \tag{2.19}$$

$$\tfrac{d}{dt} x^T g_{n-1}(x) = \{||g_{n-1}(x)||^2 + x^T \nabla g_{n-1}(x) g_{n-1}(x)\} u_{n-1} + \textstyle\sum_{j=1}^{n-2} x^T [g_j, g_{n-1}](x) u_j \tag{2.20}$$

If $p^T g_{n-1}(p) = 0$ but $V_{n-1}(p) \neq 0$, then $\nabla V_{n-1}(p) \perp g_{n-1}(p)$ *and* $\nabla V_{n-1}(p) \neq 0$, and since $\nabla V_{n-1}(p) \perp g_i(p)$, for $i \in \underline{n-2}$, then $\nabla V_{n-1}(p) \in span\{[g_i, g_{n-1}], i \in \underline{n-2}\}$, by virtue of the controllability assumption A1. It is then possible to choose an index $i \in \underline{n-2}$ such that

$$|x^T [g_i, g_{n-1}](x)| = max\{|x^T [g_j, g_{n-1}](x)|, j \in \underline{n-2}\} > 0 \tag{2.21}$$

It is then clear that the controls

$$u_i(x) = 1, \quad \text{and} \quad u_j(x) = 0 \quad \text{for} \quad j \neq i, \; j \in \underline{n-1} \tag{2.22}$$

change the value of V_i along the controlled trajectory, (increasing it, if $V_i(p) = 0$), while V_j for $j \neq i$ stays constant. Most importantly, in the process of the above, the 'coefficient' $x^T g_{n-1}(x)$ in $\frac{d}{dt}V_{n-1}$ changes value from zero to nonzero since, by virtue of (2.19)-(2.20),

$$\frac{d}{dt} x^T g_{n-1}(x) = x^T [g_i, g_{n-1}](x) \neq 0 \tag{2.23}$$

along the controlled trajectory, with controls as in (2.22).

At this point, i.e. when $x^T g_{n-1}(x) \neq 0$, the controls of (2.13) can be re-employed, resulting in a further decrease of V_{n-1}, while the values of the other guiding functions stay uneffected. After $x^T g_{n-1}(x)$ reaches zero again, V_i can be restored to its previous value by 'reverse controls' :

$$u_i(x) = -1, \quad \text{and} \quad u_j(x) = 0 \quad \text{for} \quad j \neq i, \; j \in \underline{n-1} \tag{2.24}$$

This demonstrates that the choice of the guiding functions satisfies condition (c).

3.2. Feedback control strategy for systems of control deficiency order one

A control strategy which employs the above guiding functions can now easily be constructed. Its principles are summarized below.

In the initial stage of the strategy, the standard controls (compare with (2.10))

$$\begin{aligned} u_i(x) &\stackrel{def}{=} -sign(x^T g_i(x)) = -sign(x_i), \qquad i \in \underline{n-2} \\ u_{n-1}(x) &\stackrel{def}{=} -sign(x^T g_{n-1}(x)), \qquad \text{for all } x \in I\!R^n, \end{aligned} \tag{2.25}$$

where the *sign* function is defined in the usual way, so that $sign(0) = 0$, are employed, to steer the system to the set $\mathcal{T}$, defined by

$$\begin{aligned} \mathcal{T} &\stackrel{def}{=} \{x \in I\!R^n : x^T g_i(x) = 0, i \in \underline{n-1}\} \\ &= \{x \in I\!R^n : x_i = 0, i \in \underline{n-2}, \; x^T g_{n-1}(x) = 0\} \end{aligned} \tag{2.26}$$

The conditions which guarantee this, will be given later in Propositions 2.1 and 2.2. It is important to note that system (2.5) with controls in (2.25) is essentially a variable structure system and hence it is necessary to define precisely its solutions. No difficulty arises if the *sign* function can be realized faithfully (instantaneously), in the absence of any model system error, or disturbances. This is because, it follows from the definition of the guiding functions and equation (2.13) that, for each $i \in \underline{n-2}$, there exists a finite time t_i^* at which $u_i = -sign(x_i) = 0$ and thus $\frac{d}{dt}x_i(t) = 0$, for all $t \geq t_i^*$, regardless of the action of the remaining controls. (If, in addition, the control u_{n-1} takes a zero value, then also $\frac{d}{dt}x_i(t) = 0$, for $i = n-1$, and for $i = n$). It follows that, in such case of " disturbance free, faithful realization", any control u_i can switch value only once (to zero). For simplicity of exposition, and to avoid the discussion of the chattering effect, we assume henceforth that, in the presence of errors and disturbances, a control u_i takes zero value for all times after the instant at which its argument changes sign for the first time. The latter creates no additional problems, as the system need not be steered to the set $\mathcal{T}$ exactly, see Remark 2.4.

Clearly,

$$\frac{d}{dt}V(x) = -\sum_{i=1}^{n-1} |x^T g_i(x)| = -\sum_{i=1}^{n-2} |x_i| - |x^T g_{n-1}(x)| \leq 0, \quad \text{for all } x \notin \mathcal{T} \tag{2.27}$$

along the trajectory of (2.5) with controls (2.25), which implies that V decreases in the complement of $\mathcal{T}$. Additionally,

$$V_i(p) = 0, \quad i \in \underline{n-2}, \quad \text{for all } p \in \mathcal{T} \tag{2.28}$$

$$V(p) = V_{n-1}(p) \quad \text{for all } p \in \mathcal{T} \tag{2.29}$$

and $0 \in \mathcal{T}$. $\mathcal{T}$ is hence a set of impasse points at which none of the guiding functions V_i, $i \in \underline{n-1}$, can be further decreased instantaneously, as, for $x(t) = p \in \mathcal{T}$, $\frac{d}{dt}V_i(p) = 0$, $i \in \underline{n-1}$, regardless of the controls. At this point, the control strategy enters its second stage in which an index $i \in \underline{n-2}$ is selected, as in (2.21), and followed by application of the controls (2.22) until $x^T g_{n-1}(x)$ reaches its maximal value, or else until the value of $V_i(x)$ becomes comparable with the value of $V(p)$ at a point p at which $\mathcal{T}$ was last traversed, and so until the controlled trajectory reaches a point x at which either of the following conditions is satisfied

$$x^T[g_i, g_{n-1}](x) = 0 \quad \text{or} \quad V(x) = \alpha V(p) \tag{2.30}$$

where $\alpha >> 1$ is a given constant. The controls in (2.13) are employed next to decrease V_{n-1} while the values of the other guiding functions stay unchanged. After $x^T g_{n-1}(x)$ reaches zero, V_i is restored to its previous value (zero) by

$$u_i(x) = -sign(x^T g_i(x)) = -sign(x_i), \quad \text{and} \quad u_j(x) = 0 \text{ for } j \neq i, \ j \in \underline{n-1} \tag{2.31}$$

which is applied until $x^T g_i(x) = x_i = 0$, and hence until the state of the controlled system returns to the set $\mathcal{T}$. The next control cycle is then initiated by choosing a possibly new index value i which satisfies (2.21).

REMARK 2.2. The guiding functions approach is not limited to systems which are appearing in a "rectified" form. Finding a suitable set of guiding functions can often be easy even if the system is not in the form stated in hypothesis H1. To see this, consider the well known example

$$\begin{aligned}
\dot{x} &= u \\
\dot{y} &= v \\
\dot{z} &= xv - yu
\end{aligned} \tag{2.32}$$

The two guiding functions needed in this case can, for example, be introduced as follows:

$$\begin{aligned}
V_1 &\stackrel{def}{=} \frac{1}{2}y^2 \\
V_2 &\stackrel{def}{=} \frac{1}{2}x^2 + \frac{1}{4}(z - xy)^2
\end{aligned} \tag{2.33}$$

An easy calculation shows that $\mathcal{T} = \{p \stackrel{def}{=} (x, y, z) \in \mathbb{R}^3 : x = y = 0\}$ as

$$\frac{d}{dt}V(x,y,z) = [x - zy + xy^2]\ u + yv \tag{2.34}$$

and the controls $u = -sign(x - zy + xy^2)$ and $v = -sign(y)$ can be applied until the system trajectory traverses $\mathcal{T}$.

For constant controls u, v:

$$\begin{aligned}
\frac{d^2}{dt^2}V_1(x,y,z) &= v^2 \\
\frac{d^2}{dt^2}V_2(x,y,z) &= (1 + 2y^2)u^2 - (z - xy)uv
\end{aligned} \tag{2.35}$$

It follows that, whenever $p \in \mathcal{T}$, then setting $u = 0$ and $v = 1$ produces a change in V_1 while V_2 stays unchanged. Also, the time derivative of $x - zy + xy^2$, the 'coefficient' in $\frac{d}{dt}V$, associated with u, is given by

$$\frac{d}{dt}(x - zy + xy^2) = (1 + 2y^2)u - (z - xy)v = -z \neq 0 \tag{2.36}$$

for all $p = (x, y, z) \in \mathcal{T}$ such that $p \neq 0$, and provided that $v = 1$ and $u = 0$ are selected as control values. Hence $|x - zy|$ grows away from zero, as required. The guiding functions, as introduced by (2.33), satisfy the assumptions (a)-(c) of the Introduction and our control strategy can be applied.

Let $t \mapsto x(t)$ denote the trajectory of (2.5) with controls (2.25) and let x denote the current (measured) state of this system. The above discussion can be formalised into the following algorithmic feedback strategy.

Stabilizing feedback strategy

- Data: $\alpha >> 1$.
  - 1 If $x \in I\!R^n \setminus \mathcal{T}$, apply the controls

$$u_i(x) = \begin{cases} -sign(x_i), & i \neq n-1 \\ -sign(x^T g_{n-1}(x)), & i = n-1 \end{cases} \qquad i \in \underline{n-1}.$$

  - 2 If at some time instant t, $p \overset{def}{=} x(t) \in \mathcal{T}$, then stop if $p = 0$; else proceed if $p \neq 0$
  - 2a Select an index $i \in \underline{n-2}$ which satisfies

$$|p^T[g_i, g_{n-1}](p)| = max\{|p^T[g_j, g_{n-1}](p)|, j \in \underline{n-2}\}$$

  - 2b Until $x^T[g_i, g_{n-1}](x) = 0$ or else until $V(x) = \alpha V(p)$, employ the controls

$$u_i(x) = 1, \quad \text{and} \quad u_j(x) = 0 \quad \text{for all } j \neq i,\ j \in \underline{n-1}$$

  - 2c Until $x^T g_{n-1}(x) = 0$, employ the controls

$$u_{n-1}(x) = -sign(x^T g_{n-1}(x)) \ \text{ and } \ u_j(x) = 0 \quad \text{for } j \in \underline{n-2}$$

  - 2d Until $x^T g_i(x) = 0$, employ the controls

$$u_i(x) = -sign(x_i), \quad \text{and} \quad u_j(x) = 0 \quad \text{for all } j \neq i,\ j \in \underline{n-1}$$

  and repeat Step 2.

REMARK 2.3. Clearly, if p_k, $k \in I\!N$, denotes the value of the state at the entrance to Step 2b, in iteration k of the algorithm, then $V(p_k) = V_{n-1}(p_k)$, $k \in I\!N$. In Step 2b of the above strategy, the state of the controlled system is driven away from the set $\mathcal{T}$ while $x^T g_{n-1}(x)$ changes from zero to non-zero. Simultaneously, V increases since V_i increases from zero to non-zero. The value of V_{n-1} stays unchanged during the execution of Step 2b. In Step 2c, the increase of V_i is halted while V_{n-1} is decreased beyond its value at p_k. When further decrease in V_{n-1} becomes impossible due to $x^T g_{n-1}(x) = 0$, Step 2c is entered in which the value of V_i is restored to zero. Clearly, at the exit of Step 2c, the state of the controlled system returns to the set $\mathcal{T}$, so that the condition of Step 2 can again be verified. By the controllability assumption A1, whenever $p_k \neq 0$ then the value of the maximum in Step 2a is positive, so that Step 2b is non-trivial. The strategy hence produces a sequence of points $\{p_k\}$ which is finite (if for some finite value of the index $k = k_*$, $p_{k_*} = 0$) , or else an infinite sequence $\{p_k\}_{k\in\mathbb{N}}$, for which the corresponding sequence of values $\{V(p_k)\}_{k\in\mathbb{N}}$ is monotonically decreasing.

It is worth noticing that the "oscillations" in the x_i , $i \neq n-1$, components of the state (as caused by controls (2.22) and (2.31)) can be big. The evolution of V_{n-1} consists of intervals in which V_{n-1} stays constant, alternated by intervals in which V_{n-1} is strictly decreasing. In the meantime, the remaining guiding functions are oscillating freely.

Since the magnitudes of the non-zero controls are constant, the control switches increase in frequency as V_{n-1} decreases. This can easily be prevented by scaling each u_i by a factor of the corresponding value of V_i, which may be practical but is a trade-off with the convergence rate of the strategy.

REMARK 2.4. It is not essential that the system is steered to the set $\mathcal{T}$ exactly. In this respect, several relaxed, alternative control strategies can be constructed which result in convergence to a pre-specified neighbourhood of the origin, rather than the origin itself (practical stabilization). We omit the details as these would further complicate the analysis, but notice that "disturbances" such as numerical errors in computer simulations do not prejudice convergence.

The parameter $\alpha \in (1, \infty)$ can be selected arbitrarily, however, its value is correlated with the rate of convergence of the strategy. Large values of α, permitting large oscillations in guiding functions values, are preferable when the possibility of achieving convergence in finite time needs to be explored. This is explained in Example 1 of this section.

3.3. Convergence analysis

The properties of the controls of Step 1, which guarantee the feasibility of this Step, are stated in the next two Propositions.

PROPOSITION 2.1. *Any trajectory of system (2.5) with controls given in (2.25) converges the set $\mathcal{T}$, (where convergence is defined in the sense of the Euclidean distance between a point on the system trajectory and the set $\mathcal{T}$).*

Proof. Let $x_0 \in \mathbb{R}^n$ be an initial state of (2.5) at $t = 0$ and $t \mapsto x(t)$ denote the corresponding trajectory when controls (2.10) are employed. Further, let Ω denote a level set of V which contains x_0, i.e. $\Omega \stackrel{def}{=} \{x \in \mathbb{R} \,:\, V(x) \leq V(x_0)\}$.

Suppose, contrary to what needs to be shown, that $x(t)$ does not approach $\mathcal{T}$. Hence, there exists an $\epsilon > 0$ and a sequence of time instants $\{t_k\}_{k\in\mathbb{N}}$ such that

$$dist(x(t_k);\ \mathcal{T}) > \epsilon \quad \text{for all } k \in \mathbb{N} \tag{2.37}$$

where the function $x \mapsto dist(x;\ \mathcal{T})$ is a measure of the distance of a point x from the hypersurface $\mathcal{T}$, and is defined by

$$dist(x;\ \mathcal{T}) \stackrel{def}{=} \sum_{i=1}^{n-1} |x^T g_i(x)| \quad \text{for all } x \in \mathbb{R}^n \tag{2.38}$$

The above definition is meaningful since $dist(x;\ \mathcal{T}) > 0$ for all $x \in \mathbb{R}^n \setminus \mathcal{T}$ and $dist(x;\ \mathcal{T}) = 0$ for all $x \in \mathcal{T}$.

Clearly, $x(t) \in \Omega$ for all times t, and Ω is compact. Hence, by smoothness of the vector fields g_i, and boundedness of the controls in (2.25), there exists a constant $c_1 > 0$ such that

$$\|\ \dot{x}(t)\ \| \le c_1 \quad \text{for all } t > 0 \tag{2.39}$$

It follows that there exists a constant $\delta > 0$ such that for all $k \in \mathbb{N}$:

$$dist(x(\tau);\ \mathcal{T}) \ge 0.5\ \epsilon \quad \text{for all } \tau \in [t_k, t_k + 2\delta]. \tag{2.40}$$

Therefore,

$$\begin{aligned} V(x(t_k+2\delta)) &= V(x(t_k)) + \int_{t_k}^{t_k+2\delta} \frac{d}{dt} V(x(\tau))\ d\tau \\ &= V(x(t_k)) - \int_{t_k}^{t_k+2\delta} \sum_{i=1}^{n-1} |x(\tau)^T g_i(x(\tau))|\ d\tau \\ &\le V(x(t_k)) - 0.5 \int_{t_k}^{t_k+2\delta} \epsilon\ d\tau \\ &\le V(x(t_k)) - \delta\epsilon \quad \text{for all } k \in \mathbb{N} \end{aligned} \tag{2.41}$$

Since V is non-increasing along $x(t)$ then the latter implies the existence of a finite time $t^* < \infty$ such that $V(x(t^*)) = 0$. Thus $x(t^*) = 0 \in \mathcal{T}$, which is a contradiction with the assumption that $x(t)$ never approaches $\mathcal{T}$. ∎

In fact we can show a stronger result under an additional assumption which is somewhat stronger than the one requesting that the motion of system (2.5) with controls (2.25) is not confined to any non-void level surface $\mathcal{T}_V \stackrel{def}{=} \{x \in \mathbb{R}^n\ :\ V(x) = r\}$, $r > 0$; equivalently, requesting that $\mathcal{T}_V$ does not contain any invariant sets of $\dot{x} = g_{n-1}(x)$. This is shown in the following proposition.

PROPOSITION 2.2. *Suppose that*

$$A(x) \stackrel{def}{=} \nabla(x^T g_{n-1}(x)) g_{n-1}(x) \neq 0$$
$$\text{for all } x \in \mathcal{A} \stackrel{def}{=} \{x \in \mathbb{R}^n \mid x_i = 0,\ i \in \underline{n-2}\} \tag{2.42}$$

Under this condition, any trajectory of system (2.5), with controls (2.25), reaches $\mathcal{T}$ in finite time.

Proof. Let $t \mapsto x(t)$ be a trajectory of (2.5) with controls (2.25) emanating from some initial condition x_0 at $t = 0$. As before, let Ω be a level set of V which contains x_0. Clearly, by virtue of the fact that $\frac{d}{dt}V(x(t)) \leq 0$, for all $t \geq 0$, $x(t) \in \Omega$, for all times $t > 0$.

By virtue of (2.25) and the definition of g_i, $i \in \underline{n-2}$,

$$\dot{x}_i(t) = -sign(x(t)^T g_i(x(t))) = -sign(x_i(t)) \quad \text{for all } i \in \underline{n-2} \tag{2.43}$$

which implies the existence of a finite time $t^* < \infty$ such that $x_i(t^*) = 0$ for all $i \in \underline{n-2}$, and consequently, that $x_i(t) = 0$ for all $t > t^*$, $i \in \underline{n-2}$.

Suppose, contrary to what needs to be shown, that the trajectory $x(t)$ never reaches $\mathcal{T}$. Since Ω is compact and $x \mapsto A(x)$ is continuous, then

$$\delta \stackrel{def}{=} min\{\ |A(x)| \mid x \in \Omega \cap \mathcal{A}\} > 0 \tag{2.44}$$

Also,

$$\frac{d}{dt}V(x(t)) = x(t)^T g_{n-1}(x(t)) u_{n-1} < 0 \quad \text{for all } t \geq t^* \tag{2.45}$$

in which $u_{n-1} = -1$ or else $u_{n-1} = 1$. Clearly, since for all $t > t^*$, $u_i = 0$, $i \in \underline{n-2}$, and $x(t) \in \mathcal{A}$, then

$$\frac{d^2}{dt^2}V(x(t)) = \nabla\{x(t)^T g_{n-1}(x(t))\} u_{n-1}\ \dot{x}(t) = A(x(t)) u_{n-1}^2 = A(x(t)), \quad t > t^* \tag{2.46}$$

By assumption, for all t , $A(x(t))$ has a constant sign. Suppose first that $A(x(t)) > 0$, for all t. Then

$$\frac{d}{dt}V(x(t)) = \frac{d}{dt}V(x(t^*)) + \int_{t^*}^{t} \frac{d^2}{d\tau^2}V(x(\tau))\, d\tau \geq \frac{d}{dt}V(x(t^*)) + (t - t^*)\ \delta \tag{2.47}$$

for all $t \geq t^*$. Equations (2.45) and (2.47) imply that there exists a finite time $t' > t^*$ such that $\frac{d}{dt}V(x(t')) = 0$ which contradicts (2.45).

Next, suppose that $A(x(t)) < 0$ for all t. Equation (2.44) then implies that

$$max\{\ A(x) \mid x \in \Omega \cap \mathcal{A}\} = -\delta < 0 \tag{2.48}$$

and thus that

$$\frac{d}{dt}V(x(t)) = \frac{d}{dt}V(x(t^*)) + \int_{t^*}^{t} \frac{d^2}{d\tau^2}V(x(\tau))\, d\tau \leq \frac{d}{dt}V(x(t^*)) - (t - t^*)\, \delta \tag{2.49}$$

for all $t \geq t^*$. It follows that there exists a constant $c_1 > 0$ and a time $\tilde{t} \geq t^*$ such that

$$\frac{d}{dt}V(x(t)) < -c_1 \quad \text{for all } t > \tilde{t} \tag{2.50}$$

The latter implies the existence of a finite time $t' > \tilde{t}$ such that $V(x(t')) = 0$, so that $x(t') = 0 \in \mathcal{T}$ which contradicts the assumption that $x(t)$ never reaches $\mathcal{T}$. ■

A quantitative analysis of the decrements in V_{n-1} in Step 2c leads to the following result.

THEOREM 2.1. *Let the assumption of Proposition 2.2 and assumptions A1 and A2 be satisfied with respect to the system (2.5), and assume the absence of any model-system error and disturbances. Under these conditions,*

(a): *the stabilization feedback strategy is well defined,*

(b): *every trajectory of system (2.5) employing the stabilizing feedback strategy converges to the origin (the origin is globally attractive).*

For the proof of the Theorem we will need the following auxiliary result.

PROPOSITION 2.3. *If the vector fields g_j, $j \in \underline{n-1}$, are smooth and satisfy assumptions A1 and A2, then for any compact set $\mathcal{B} \in I\!R^n$ which does not include the origin (i.e. $0 \notin \mathcal{B}$), there exists a constant $\gamma > 0$ such that*

$$|p^T[g_i, g_{n-1}](p)| > \gamma\ ||p||\ ||[g_i, g_{n-1}](p)|| \tag{2.51}$$

for any index $i \in \underline{n-2}$ and any point $p \in \mathcal{B} \cap \mathcal{T}$ which satisfies

$$|p^T[g_i, g_{n-1}](p)| = max\{|p^T[g_j, g_{n-1}](p)|, j \in \underline{n-2}\}. \tag{2.52}$$

Proof. Suppose that the assertion of the proposition is not true. Then there exists a sequence $\{\gamma_l\}_{l \in I\!N}$ such that $\gamma_l \to 0$ as $l \to \infty$ and corresponding sequences of indices $\{i_l\}_{l \in I\!N}$, and points $p_l \in \mathcal{B} \cap \mathcal{T}$, $\{p_l\}_{l \in I\!N}$, satisfying (2.52) , and such that (2.51) is violated, so that

$$|p_l^T[g_{i_l}, g_{n-1}](p_l)| \leq \gamma_l\ ||p_l||\ ||[g_{i_l}, g_{n-1}](p_l)||, \quad l \in I\!N \tag{2.53}$$

together with

$$p_l^T g_j(p_l) = 0 \quad \text{for } j \in \underline{n-1},\ l \in I\!N \tag{2.54}$$

$$|p_l^T[g_{i_l}, g_{n-1}](p_l)| \geq |p_l^T[g_j, g_{n-1}](p_l)|, \quad j \in \underline{n-2},\ l \in \mathbb{N}. \tag{2.55}$$

Since $\mathcal{B}$ is compact, there exists a convergent subsequence of $\{p_l\}_{l\in\mathbb{N}}$ for which we will use the same symbol to simplify notation. Let $p_* \in \mathcal{B}$ denote its limit, so that $p_l \to p_*$ as $l \to \infty$. Additionally, suppose that the latter subsequence is chosen in such a way that each of its elements corresponds to the same value of the index i of (2.52). Denote this value by i_*. Letting l tend to infinity in (2.53)-(2.55) yields

$$p_*^T[g_{i_*}, g_{n-1}](p_*) = 0 \tag{2.56}$$

and

$$p_*^T g_j(p_*) = 0 \quad \text{for } j \in \underline{n-1} \tag{2.57}$$

$$0 = |p_*^T[g_{i_*}, g_{n-1}](p_*)| \geq |p_*^T[g_j, g_{n-1}](p_*)|, \quad j \in \underline{n-2}, \tag{2.58}$$

which implies that p_* is orthogonal to the set $span\{g_i(p_*), [g_i, g_j](p_*), i, j \in \underline{n-1}\}$. By virtue of the controllability assumption, $p_* = 0$, which contradicts the fact that $p_* \in \mathcal{B}$ and that $\mathcal{B}$ does not contain the origin. This shows the validity of (2.51). ■

Proof of Theorem 2.1.

Part (a) :

This part is clearly true by virtue of Propositions 2.2 and the discussion in Remark 2.3.

Part (b) :

Let x_0 be an arbitrary initial condition for the controlled system (2.5). In the k-th iteration of the stabilizing strategy, let p_k and p_k' denote the values of the state of the system at the entrance and the exit of Step 2b, respectively. Further, let τ_k and τ_k' be the time instants at which Step 2b is entered and exited, respectively. Similarly, let t_k and t_k' be the time instants at which Step 2c is entered and exited. Finally, let $\bar{t}_k$ be the time at the exit of Step 2d. Clearly, $t_k = \tau_k'$.

If the sequence $\{p_k\}$ produced by the strategy is finite, the Theorem is trivially true , so that only the case when $\{p_k\}_{k\in\mathbb{N}}$ is infinite requires analysis.

First, suppose that $V(p_k) = V_{n-1}(p_k) \to 0$ as $k \to \infty$. By virtue of the condition in Step 2b,

$$V(x(t)) \leq \alpha V(p_k) \quad \text{for all } t \in [\tau_k, \tau_k'], \quad k \in \mathbb{N} \tag{2.59}$$

Since V decreases in all the steps other than Step 2b then, from the above analysis, it also follows that

$$V(x(t)) \leq \alpha V(p_k) \quad \text{for all } t \in [\tau_k, \bar{t}_k], \quad k \in I\!N \tag{2.60}$$

and hence that $V(x(t)) \to 0$ as $t \to \infty$.

Next, suppose, contrary to what needs to be shown, that $V(x(t))$ does not converge to zero as $t \to \infty$. From the previous discussion it follows that there exists a constant $a > 0$ such that $V(p_k) = V_{n-1}(p_k) > a$, for all $k \in I\!N$. Since the sequence $\{V_{n-1}(p_k)\}_{k \in I\!N}$ is bounded from below and monotonically decreasing, it is convergent.

Since $V(x(t))$ increases only in Step 2b, then by virtue of the condition of this Step, the trajectory $x(t)$ remains in the set $\Omega_\alpha \stackrel{def}{=} \{x \in I\!R^n \ : \ V(x) \leq \alpha V(x_0) \ \}$. Since V_{n-1} is decreasing monotonically, the latter implies that the trajectory $x(t)$ remains for all times in the compact annulus $\mathcal{B} \stackrel{def}{=} \{x \in I\!R \ : \ a \leq V(x) \leq \alpha V(x_0)\}$.

In order to estimate the decrease in V_{n-1} which takes place in Step 2c, we will show the validity of the following :

(i): There exists a constant $K_1 > 0$ such that

$$|(p_k')^T g_{n-1}(p_k')| \geq K_1 \tag{2.61}$$

for any $p_k \in \mathcal{B}$, and $k \in I\!N$.

(ii): There exists a constant $K_2 > 0$ such that

$$\int_{t_k}^{t_k'} |x(s)^T g_{n-1}(x(s))| \, ds \geq K_2 \ |(p_k')^T g_{n-1}(p_k')|^2, \quad k \in I\!N \tag{2.62}$$

whenever $x(\tau) \in \mathcal{B}$, for all $\tau \in [t_k, t_k']$.

Part (i) :

At the entrance of Step 2b, $x(\tau_k) = p_k$ and $p_k^T g_{n-1}(p_k) = 0$. By virtue of (2.20) and the fact that in Step 2b the only nonzero control is $u_i = 1$, we have that

$$\begin{aligned} (p_k')^T g_{n-1}(p_k') &= \int_{\tau_k}^{\tau_k'} \frac{d}{ds}[x(s)^T g_{n-1}(x(s))] \, ds \\ &= \int_{\tau_k}^{\tau_k'} x(s)^T [g_i, g_{n-1}](x(s)) \, ds \end{aligned} \tag{2.63}$$

The rate of change of the integrand in (2.63) is limited if the trajectory $x(t)$, for $t \in [\tau_k, \tau_k']$, remains in the annulus $\mathcal{B}$. Since in Step 2b the controlled system equation is given by $\dot{x} = g_i(x)$, then

$$
\begin{aligned}
&|\frac{d}{dt}\{x(t)^T[g_i, g_{n-1}](x(t))\}| = |[g_i, g_{n-1}](x(t))^T\dot{x}(t) + x(t)^T\nabla[g_i, g_{n-1}](x(t))\dot{x}(t)| \\
&\leq \{||[g_i, g_{n-1}](x(t))|| + \sqrt{\alpha V(x_0)}\ ||\nabla[g_i, g_{n-1}](x(t))||\}\ ||g_i(x(t))|| \\
&\leq M_1^2\{1 + \sqrt{\alpha V(x_0)}\} \stackrel{def}{=} v_2 \qquad (2.64)
\end{aligned}
$$

in which $M_1 > 0$ is a common bound for the values of $||[g_i, g_{n-1}](x)||$, $||\nabla[g_i, g_{n-1}](x)||$, and $||g_i(x)||$ for all $x \in \mathcal{B}$ and all $i \in \underline{n-2}$.

Assuming that Step 2b is exited due to the satisfaction of the condition $x(\tau_k')^T[g_i, g_{n-1}](x(\tau_k')) = 0$, the time $\tau_k' - \tau_k$ can be estimated as follows :

$$
\tau_k' - \tau_k \geq \tau_{min}^1 \stackrel{def}{=} \frac{1}{v_2}\ |p_k^T[g_i, g_{n-1}](p_k)| \qquad (2.65)
$$

If Step 2b is exited due to the satisfaction of $V(x(\tau_k')) = \alpha V(p_k)$, then, at the exit of Step 2b, it holds that $0.5\ x_i(\tau_k')^2 + V(p_k) = \alpha V(p_k)$. Since $x_i(\tau_k) = 0$, then the time $\tau_k' - \tau_k$ can be estimated as follows :

$$
\tau_k' - \tau_k \geq \frac{1}{v_2}|x_i(\tau_k')| = \frac{1}{v_2}\sqrt{2(\alpha-1)V(p_k)} \geq \frac{1}{v_2}\sqrt{2(\alpha-1)a} \stackrel{def}{=} \tau_{min}^2 \qquad (2.66)
$$

Denoting $\tau_{min} \stackrel{def}{=} min\{\tau_{min}^1, \tau_{min}^2\}$, we obtain the following bound for $|(p_k')^T g_{n-1}(p_k')|$:

$$
\begin{aligned}
|(p_k')^T g_{n-1}(p_k')| &= |\int_{\tau_k}^{\tau_k'} x(s)^T[g_i, g_{n-1}](x(s))\ ds| \\
&\geq \int_{\tau_k}^{\tau_k+\tau_{min}} \{|p_k^T[g_i, g_{n-1}](p_k)| - v_2(s-\tau_k)\}\ ds \\
&\geq \frac{1}{2}\tau_{min}\ |p_k^T[g_i, g_{n-1}](p_k)| \\
&\geq \frac{1}{2v_2}|p_k^T[g_i, g_{n-1}](p_k)|^2 \qquad (2.67)
\end{aligned}
$$

The latter is valid since $x(s)^T[g_i, g_{n-1}](x(s))$ does not change sign for $s \in [\tau_k, \tau_k']$. By virtue of Proposition 2.3 and assumption A2, it follows that

$$
\begin{aligned}
|(p_k')^T g_{n-1}(p_k')| &\geq \frac{1}{2v_2}\gamma^2||p_k||^2||[g_i, g_{n-1}](p_k)||^2 \\
&\geq \frac{(\gamma r c)^2}{2v_2} \stackrel{def}{=} K_1 \qquad (2.68)
\end{aligned}
$$

where c is the constant of assumption A2 and r is the radius of the largest ball $B(0; r)$ contained in the level set $\{x \in I\!R^n\ :\ V(x) \leq a\}$.

Part (ii) :

We first notice that $x(t_k) = p'_k$ and that $x(t'_k)^T g_{n-1}(x(t'_k)) = 0$. The rate of change in $|x(t)^T g_{n-1}(x(t))|$ under the action of u_{n-1}, which is the only nonzero control in Step 2c, is limited as the trajectory $x(t)$ evolves in $\mathcal{B}$. Since $|u_{n-1}| = 1$ then, by virtue of (2.20),

$$
\begin{aligned}
|\frac{d}{dt}x^T g_{n-1}(x)| &\leq ||g_{n-1}(x)||^2 + ||x||\ ||\nabla g_{n-1}(x)||\ ||g_{n-1}(x)|| \\
&\leq M_2^2\ \{1 + \sqrt{\alpha V(x_0)}\} \stackrel{def}{=} v_1
\end{aligned}
\tag{2.69}
$$

where M_2 is a common bound for $||\nabla g_{n-1}(x)||$ and $||g_{n-1}(x)||$ in the annulus $\mathcal{B}$. Hence

$$
\int_{t_k}^{t'_k} |x(s)^T g_{n-1}(x(s))|\ ds \geq \int_{t_k}^{t''_k} \{|(p'_k)^T g_{n-1}(p'_k)| - v_1(s - t_k)\}\ ds \tag{2.70}
$$

where t''_k is the time for which the integrand in (2.70) becomes equal to zero, so that

$$
t''_k - t_k = t_{min} \stackrel{def}{=} \frac{1}{v_1}\ |(p'_k)^T g_{n-1}(p'_k)| \tag{2.71}
$$

Therefore

$$
\int_{t_k}^{t'_k} |x(s)^T g_{n-1}(x(s))|\ ds \geq \frac{1}{2} t_{min}\ |(p'_k)^T g_{n-1}(p'_k)| = \frac{1}{2v_1}\ |(p'_k)^T g_{n-1}(p'_k)|^2 \tag{2.72}
$$

for any $k \in I\!N$, which proves part (ii).

We are now ready to estimate the decrease in V_{n-1} in Step 2c. We first note that

$$
V_{n-1}(x(t'_k)) = V_{n-1}(x(t_k)) + \int_{t_k}^{t'_k} |x(s)^T g_{n-1}(x(s))|\ ds, \quad k \in I\!N \tag{2.73}
$$

Hence, and from parts (i) and (ii) of this proof, it follows that

$$
\begin{aligned}
V_{n-1}(x(t'_k)) &\geq V_{n-1}(x(t_k)) + \frac{1}{2v_1}\ |(p'_k)^T g_{n-1}(p'_k)|^2 \\
&\geq V_{n-1}(x(t_k)) + \frac{1}{2v_1}\ K_1^2,\ k \in I\!N
\end{aligned}
\tag{2.74}
$$

Finally, the latter implies

$$
\begin{aligned}
|V_{n-1}(p_{k+1}) - V_{n-1}(p_k)| &= |V_{n-1}(x(t'_k)) - V_{n-1}(x(t_k))| \\
&\geq \frac{1}{2v_1}\ K_1^2 > 0,\ k \in I\!N,
\end{aligned}
\tag{2.75}
$$

which contradicts the convergence of $\{V(p_k)\}_{k \in I\!N}$.

Therefore, $V(p_k) \to 0$ as $k \to \infty$, and consequently $V(x(t)) \to 0$ as $t \to \infty$, as claimed. ∎

4. Stabilizing feedback control of a unicycle model

It is interesting to apply the above feedback strategy to a unicycle model. As shown in Example 1.2 of Chapter 1, the kinematic model of a unicycle can be written as:

$$\dot{x}(t) = g_1(x(t))u_1 + g_2(x(t))u_2 \tag{2.76}$$

$$\text{where,} \quad x(t) \stackrel{def}{=} [x_1(t), x_2(t), x_3(t)]^T \in I\!R^3$$

$$g_1(x) = [1, 0, 0]^T, \qquad g_2(x) = [0, cosx_1, sinx_1]^T \tag{2.77}$$

The model of this system appears originally in a rectified form, so no transformation is needed. The feedback control strategy with constant $\alpha = 10$ is employed to steer the system to the origin with the initial condition $[x_1, x_2, x_3](0) = [1., 3., 3.]$. (In this case, the constant α is chosen to be sufficiently large as to permit Step 2b of the strategy to be exited due to the satisfaction of $x^T[g_1, g_2](x) = 0$, rather than due to the condition that $V(x) = \alpha V(p)$. The latter guarantees finite time convergence of the strategy, as explained below.)

The assumptions of Proposition 2.2 are satisfied here since $\nabla(x^T g_2(x))g_2(x) = ||g_2(x)||^2 = 1$ for all $x \in I\!R^3$. The surface $\mathcal{T} = \{x \in I\!R^3 : x_1 = 0\} \cap \{x \in I\!R^3 : x_2 cosx_1 + x_3 sinx_1 = 0\} = \{x \in I\!R^3 : x_1 = x_2 = 0\}$ is hence reached in finial time.

Figure 2.1 shows the trajectories of the controlled system versus time. It is visible that the set $\mathcal{T}$ is reached approximately at time $t = 3.2$. The strategy then enters its second phase . The desired (but inaccessible) direction of motion is $[g_1, g_2]$ at any point $(0, 0, x_3)$, $x_3 > 0$. (or else $-[g_1, g_2]$ when $x_3 < 0$).

Figure 2.2 shows the actual trajectory of the car's center of mass. At the end of the first phase of the control strategy the car is positioned sideways to its goal - the origin. Any further decrease of the global guiding function V is impossible at this point since a car with no slipping cannot perform instantaneous sideways motion. In Step 2b of the strategy the car is rotated in place by an angle of $(\pi/2)$ which is the point at which $x^T[g_1, g_2](x) = 0$, and at which $x^T g_2(x)$ achieves its maximum. The application of Step 2c then results in a straight line motion of the car to the origin. In this case, the controller achieves its goal in a finite number of steps, which demonstrates its effectiveness.

Figure 2.3 shows the plot of the guiding functions $V_1 \stackrel{def}{=} \frac{1}{2}x_1^2$ and $V_2 \stackrel{def}{=} \frac{1}{2}(x_2^2 + x_3^2)$ and Figure 2.4 shows the plot of their sum V.

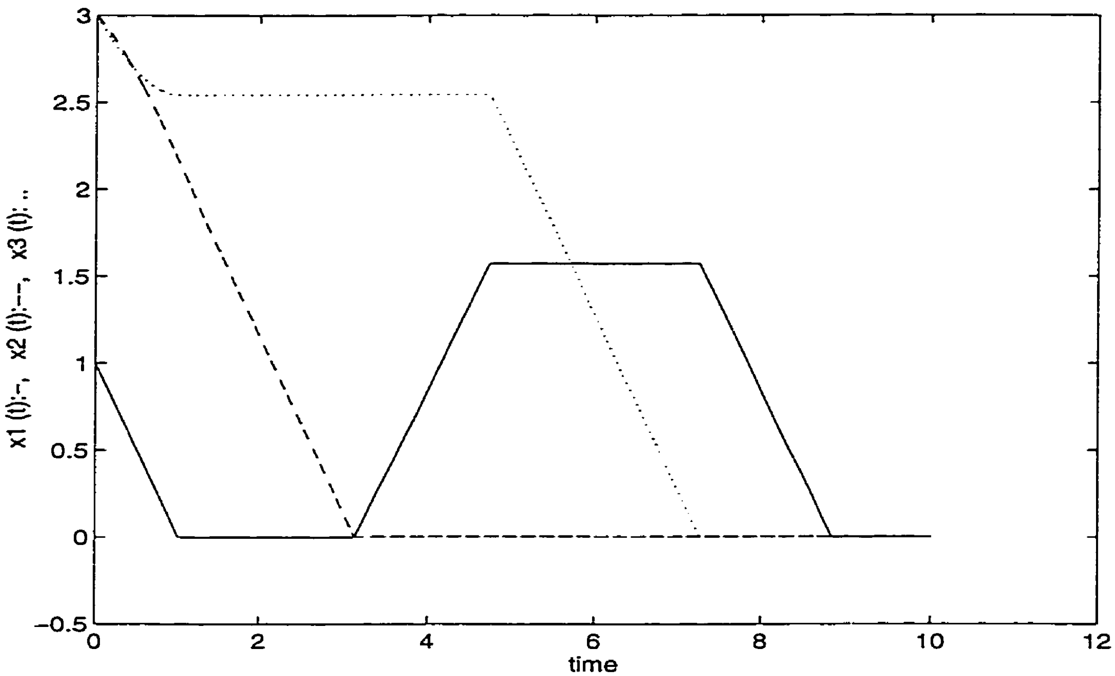

FIGURE 2.1. *Unicycle model*: Trajectories $(x_1(t), x_2(t), x_3(t))$ of the unicycle versus time.

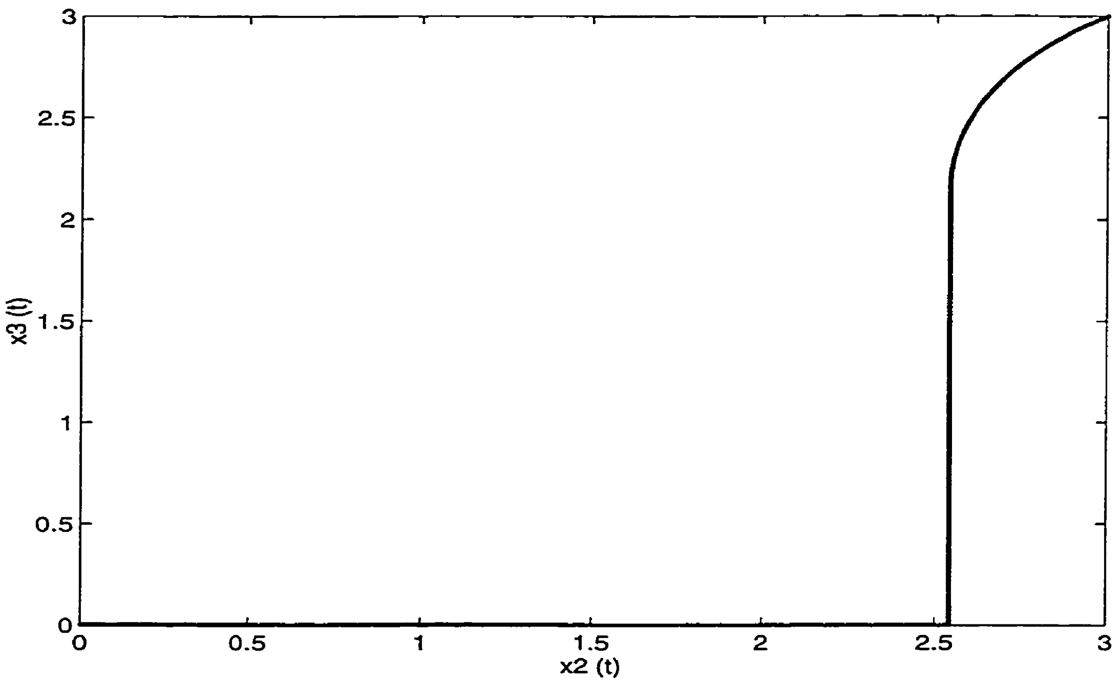

FIGURE 2.2. *Unicycle model*: Plot of the position of the unicycle $(x_2(t), x_3(t))$.

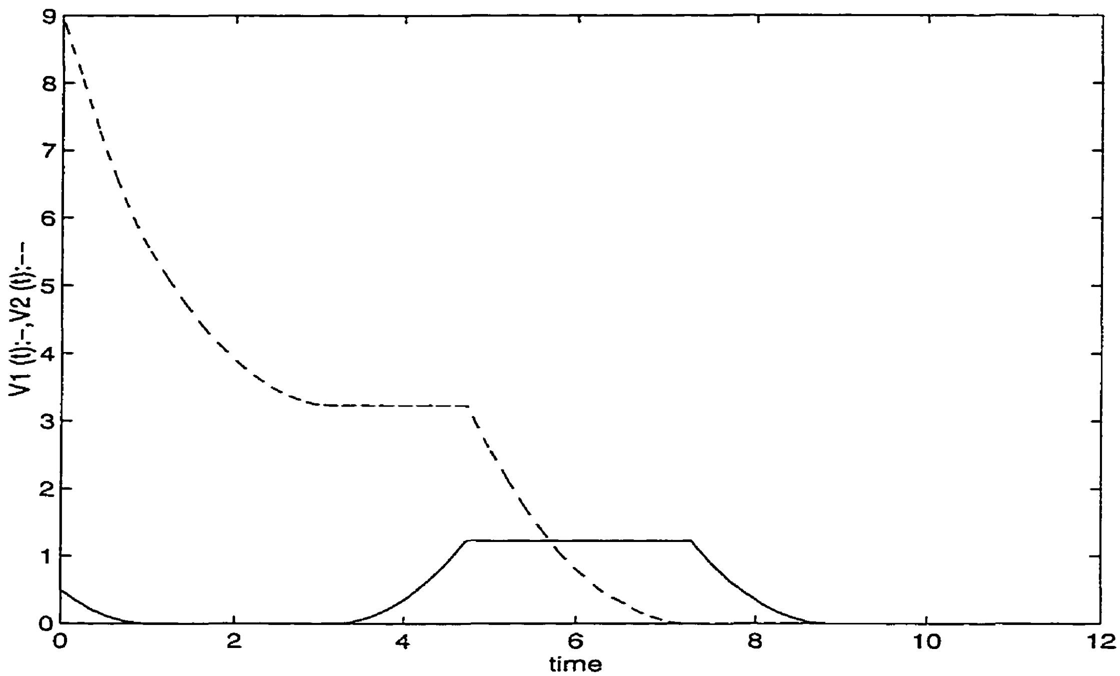

FIGURE 2.3. *Unicycle model*: Plots of the guiding functions $V_1 = \frac{1}{2}{x_1}^2$ and $V_2 = \frac{1}{2}({x_2}^2 + {x_3}^2)$ versus time.

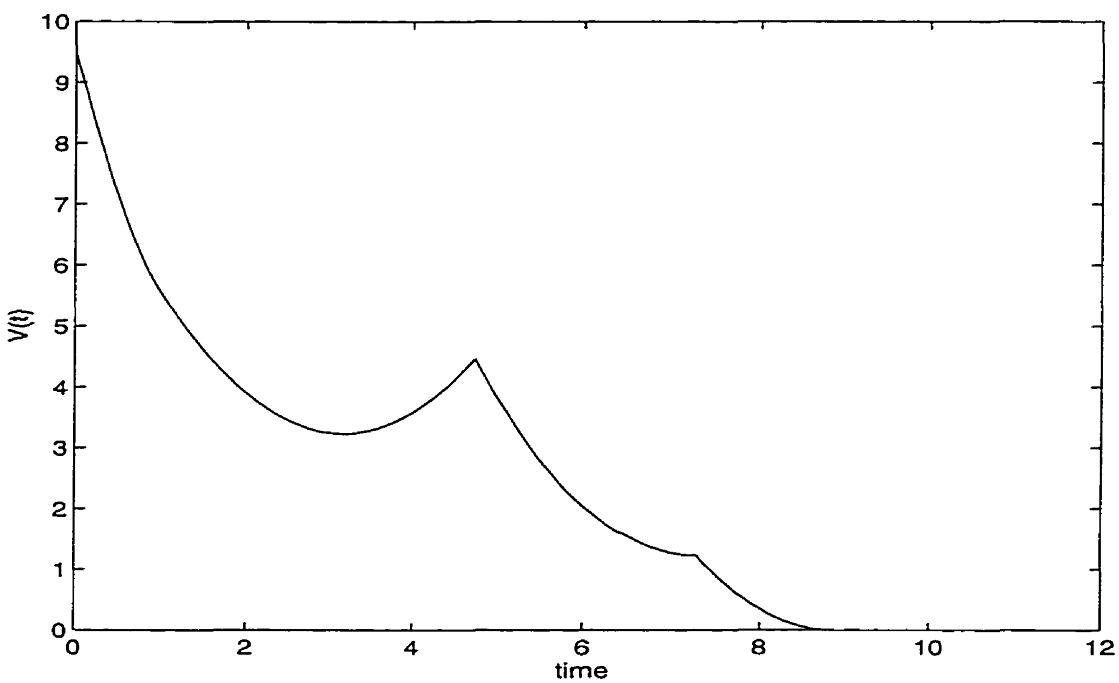

FIGURE 2.4. *Unicycle model*: Plot of the sum of the guiding functions $V = V_1 + V_2$ versus time.

5. Extension of the strategy to systems of higher order control deficiency

In this section we explain how the guiding functions approach can be extended to apply to systems of control deficiency order two (and, by analogy, to systems of higher order control deficiency).

For systems for which $m = n-2$, we introduce the following family of Lie bracket extension systems

$$\dot{\xi} = \sum_{i=1}^{n-2} f_i(\xi)v_i + [f_j, f_k](\xi)v_{n-1}, \quad \text{for any } j, k \in \underline{n-2} \tag{2.78}$$

Any member of this family can be regarded as a system of control deficiency order one to which the previous strategy is applicable.

However, we now require that the rectifiability Hypothesis H1 holds for all members of the family (2.78) :

Rectifiability of the family of extended systems :

H1'. For any first order extension of the original system (2.78) there exist diffeomorphic state feedback transformations $\xi = T(x)$, $u = U(\xi, v)$, such that in the new coordinates x and in terms of the new control u, system (2.1) takes the rectified form (2.5).

The strategy of the previous section can now be applied to individual members of the family of extended systems. The detailed "adapted version" of the control strategy is omitted here; instead, its interpretation for a particular example is presented in the next section.

Clearly, for the stabilizing feedback strategy constructed for the extended system to work with the original system, it is necessary to provide a way in which the original system can "move" in the directions corresponding to the first order Lie brackets $[f_i, f_j]$. Motion involving such directions is needed whenever the system traverses the set $\mathcal{T}$ given by (2.26) and which, in terms of the original system vector fields, is given by

$$\mathcal{T} = \{x \in I\!R^n : x^T f_i(x) = 0, x^T [f_j, f_k](x) = 0, \ i, j, k \in \underline{n-2}\} \tag{2.79}$$

If, for example, $g_i = [f_j, f_k]$, then Step 2b of the strategy requires the system motion to take place purely in the direction $[f_j, f_k]$. For the original system, such direction of motion is not directly accessible and has to be achieved only approximately. This can be done in a number of ways, of which a standard one relies on repetitive use of the following four control pairs (each over a fixed interval of time δ) : $(u_j, u_k) = (1,0), (0,1), (-1,0), (0,-1)$. It is well known, see for example [**113**], that the latter control sequence results in the following estimated value for the state x of the system at time $t = 4\delta$:

$$x(4\delta) = x(0) + \delta^2\ [f_j, f_k](x(0)) + 0(\delta^3) \tag{2.80}$$

where the precision in maintaining a motion in the direction of the Lie bracket can be increased as desired by letting $\delta \to 0$. The latter can be adjusted on line or converted into a proper feedback strategy in the case of particular models as shown in the next section. It is not important to follow the Lie bracket direction exactly but to keep track of the relative increments and decrements of the associated guiding functions, as required by the stabilizing strategy in order to observe periodic decrease in V_{n-1}.

Finally, the original controls are derived by using inverse transformations to T and U which convert the extended system into its rectified form. This may be simplified in particular cases, as demonstrated below.

6. Stabilizing feedback control of a front wheel drive

The kinematic model of a front wheel drive (car) as given in Example 1.1 of Chapter 1, can be written as:

$$\begin{bmatrix} \dot{\phi} \\ \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} v_1 + \begin{bmatrix} 0 \\ cos\theta \\ sin\theta \\ tan\phi \end{bmatrix} v_2 \stackrel{def}{=} f_1 v_1 + f_2 v_2 \tag{2.81}$$

$$\text{where,}\quad f_1 = \frac{\partial}{\partial\phi}, \qquad f_2 = cos\theta\frac{\partial}{\partial x} + sin\theta\frac{\partial}{\partial y} + tan\phi\frac{\partial}{\partial\theta}$$

Calculating the Lie brackets yields

$$f_3 \stackrel{def}{=} [f_1, f_2] = \frac{1}{cos^2\phi}\frac{\partial}{\partial\theta}$$

$$f_4 \stackrel{def}{=} [f_2, [f_1, f_2]] = \frac{sin\theta}{cos^2\phi}\frac{\partial}{\partial x} - \frac{cos\theta}{cos^2\phi}\frac{\partial}{\partial y}$$

$$f_5 \stackrel{def}{=} [f_1, [f_1, f_2]] = \frac{2tan\phi}{cos^2\phi}\frac{\partial}{\partial\theta}$$

and shows that if the motion of the system is restricted to the manifold

$$\mathcal{M} = \{\xi \stackrel{def}{=} (\phi, x, y, \theta) \in \mathbb{R}^4 : |\phi| < \pi/2\} \tag{2.82}$$

then $\{f_1, f_2, f_3, f_4\}$ are linearly independent and hence the system represented by (2.81) satisfies condition A1 for complete controllability on the manifold $\mathcal{M}$. As we will see, this is sufficient for control purposes.

By virtue of the dimension of the system ($n = 4$ and $n - m = 2$), there exists only one extended system to (2.81) which is given in terms of the equation

$$\begin{aligned}
\begin{bmatrix} \dot{\phi} \\ \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} v_1 + \begin{bmatrix} 0 \\ cos\theta \\ sin\theta \\ tan\phi \end{bmatrix} v_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{cos^2\phi} \end{bmatrix} v_3 \\
&\stackrel{def}{=} f_1(\xi)v_1 + f_2(\xi)v_2 + f_3(\xi)v_3
\end{aligned} \tag{2.83}$$

It is easy to see that the transformation $\bar{x} \stackrel{def}{=} (x_1, x_2, x_3, x_4) = (\phi, \theta, x, y)$ and $(u_1, u_2, u_3) = (v_1, \frac{1}{cos^2 x_1} v_3 + tan x_1 v_2, v_2)$ brings system (2.83) into the following "rectified" form

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ 0 \\ cos x_2 \\ sin x_2 \end{bmatrix} u_3 \\
&\stackrel{def}{=} g_1(\bar{x})u_1 + g_2(\bar{x})u_2 + g_3(\bar{x})u_3
\end{aligned} \tag{2.84}$$

The model of the front wheel drive hence satisfies assumption H1 on the manifold $\mathcal{M}$.

6.1. The guiding functions and the original strategy

The extended system (2.84) induces the following guiding functions:

$$\begin{aligned}
V_1(\bar{x}) &\stackrel{def}{=} \frac{1}{2}x_1^2 = \frac{1}{2}\phi^2 \\
V_2(\bar{x}) &\stackrel{def}{=} \frac{1}{2}x_2^2 = \frac{1}{2}\theta^2 \\
V_3(\bar{x}) &\stackrel{def}{=} \frac{1}{2}(x_3^2 + x_4^2) = \frac{1}{2}(x^2 + y^2)
\end{aligned} \tag{2.85}$$

It follows from (2.79) that the set of "impass points", $\mathcal{T}$, is given by:

$$\mathcal{T} = \{\bar{x} \in I\!R^4 : x_1 = 0, x_2 = 0,\ x_3 cos x_2 + x_4 sin x_2 = 0\}$$
$$= \{\xi \stackrel{def}{=} (\phi, x, y, \theta) \in I\!R^4 : \phi = \theta = x = 0\} \tag{2.86}$$

From the discussion of section 3 and the model equations, it follows that the guiding function V_1 can be manipulated independently of the remaining ones. The guiding function V_2 must be manipulated through motion in the Lie bracket direction $[f_1, f_2]$, while V_3 can be manipulated by $u_3 = v_2$, unless $x \in \mathcal{T}$. At any point $\bar{x} \in \mathcal{T}$:

$$V_1(\bar{x}) = V_2(\bar{x}) = 0,\quad V_3(\bar{x}) = \frac{1}{2}y^2 \tag{2.87}$$

Hence, the stabilizing strategy of section 3 can be applied, without change, to the extended system (2.84). In Steps 1 and 2 of the strategy the system is supposed to be steered to the set $\mathcal{T}$. This can be easily achieved for the extended system but requires the use of the control $u_3 = -sign(\bar{x}^T g_3(\bar{x}))$ $= -sign(x_3 cos x_2 + x_4 sin x_2) = -sign(x cos\theta + y sin\theta)$ which cannot be implemented directly in the original system. The action of this control must thus be translated into appropriate controls in terms of v_1 and v_2 which are the only inputs in the original system. For the car model (2.81) it is easy to suggest a possible control strategy which can accomplish such a task , as can be verified by direct inspection :

Subalgorithm 1 (feedback control for steering the car to the set $\mathcal{T}$)

1a $v_i = -sign(\xi^T f_i(\xi))$, $i = 1, 2$, until $\xi^T f_i(\xi) = 0$, for $i = 1, 2$.
1b $v_1 = 1, v_2 = 0$, until $\phi = \pi/4$.
1c $v_1 = 0, v_2 = -sign(\theta tan\phi)$, until $\theta = 0$.
1d $v_1 = -sign(\phi), v_2 = 0$, until $\phi = 0$.
1e $v_1 = 0, v_2 = -sign(x cos\theta + y sin\theta)$ until $\xi \in \mathcal{T}$.

The index i of Step 2a is constant and equal 2 since, in the case of the extended model (2.83),

$$\bar{x}^T[g_2, g_3](\bar{x}) = \xi^T[f_2, [f_1, f_2]](\xi) \neq 0 \quad \text{for } \xi \in \mathcal{T} \text{ and } \xi \neq 0, \tag{2.88}$$

and, additionally, $\xi^T[f_1, f_2](\xi) = 0$ for $\xi \in \mathcal{T}$, so that $[f_1, [f_1, f_2]](\xi) = 0$, as it is linearly dependent with $[f_1, f_2](\xi)$.

The control of Step 2b then requires the use of $u_2 = 1$ which corresponds to the motion of the system in the direction of increasing θ, while keeping the remaining state variables constant. The latter corresponds to pure motion in the Lie bracket direction $[f_1, f_2]$. In view of our previous discussion, such motion can be achieved only approximately, by cyclic switching between the vector fields f_1 and f_2. Here, we employ the following simple 'feedback' control scheme, by repeating N times the following sequence of controls:

Subalgorithm 2 (for achieving motion in the θ direction)

(i) Increase ϕ until $\phi = min\{\pi/4, |y|\}$ by using $v_1 = 1, v_2 = 0$.

(ii) Increase θ by $\frac{1}{N}min\{\pi/2, (\alpha-1)^{1/2}|y|\}$, by using $v_1 = 0, v_2 = 1$.

(iii) Restore ϕ to zero by using $v_1 = -sign(\phi), v_2 = 0$.

(ixv) Steer (x, y), as closely as possible, to their values prior to (i) by using the controls $v_1 = 0$, and $v_2 = -sign(xcos\theta + ysin\theta)$.

A decrease in θ , which is required in Step 2d, can be obtained by reversing the sign of v_2 in (ii). Scaling is introduced in (i) and (ii) to prevent excessive deviations in the variables ϕ and θ. Clearly, at the exit of the N-th cycle of Subalgorithm 2, $\theta = \pi/2$, or else $\theta = (\alpha-1)^{1/2}|y|$. The latter correspond precisely to the exit conditions of Step 2b because, $\bar{x}^T[g_2, g_3](\bar{x}) = 0$ for any $\bar{x} = (0, \frac{\pi}{2}, 0, y)$, and $V(\bar{x}) \approx \frac{1}{2}\theta^2 + \frac{1}{2}y^2 = \alpha\frac{1}{2}y^2 = \alpha V(p)$ whenever $p \in \mathcal{T}$, and $\bar{x}$ is the value of the state of the system at the exit of Subalgorithm 2.

Step 2c is easy to implement on the original system since, at the exit of Step 2b, $\phi = 0$ and hence $tan\phi = 0$. The control $v_2 = -sign(x_3cosx_2 + x_4sinx_2) = -sign(xcos\theta + ysin\theta)$ is thus exactly equal to $u_3 = -sign(\bar{x}g_3(\bar{x}))$.

The stabilizing feedback control for the car model (2.81) in the original variables, hence takes the following form :

Stabilizing feedback for the front wheel drive

- •1 Steer the system to the set $\mathcal{T}$ by employing Subalgorithm 1.
- •2b Increase θ until $\theta = \pi/2$ or until $\theta = (\alpha)^{1/2}|y|$ by employing Subalgorithm 2.
- •2c Employ the controls $v_1 = 0$, and $v_2 = -sign(xcos\theta + ysin\theta)$ until $xcos\theta + ysin\theta = 0$.
- •2d Restore θ to zero by employing Subalgorithm 1 and repeat Steps 2b-2d.

The simulation results are depicted in Figures 2.5-2.8. Figure 2.5 shows the trajectories of (ϕ, x, y, θ) of the controlled system while the guiding functions V_1, V_2 and V_3, and their sum V, are depicted in Figures 2.7 and 2.8, respectively. Figure 2.6 shows the position of the car (x, y) . It should be added that during the simulation a practical modification was introduced to the original stabilizing strategy : the controls in all steps were scaled by a factor of the current value of the function V.

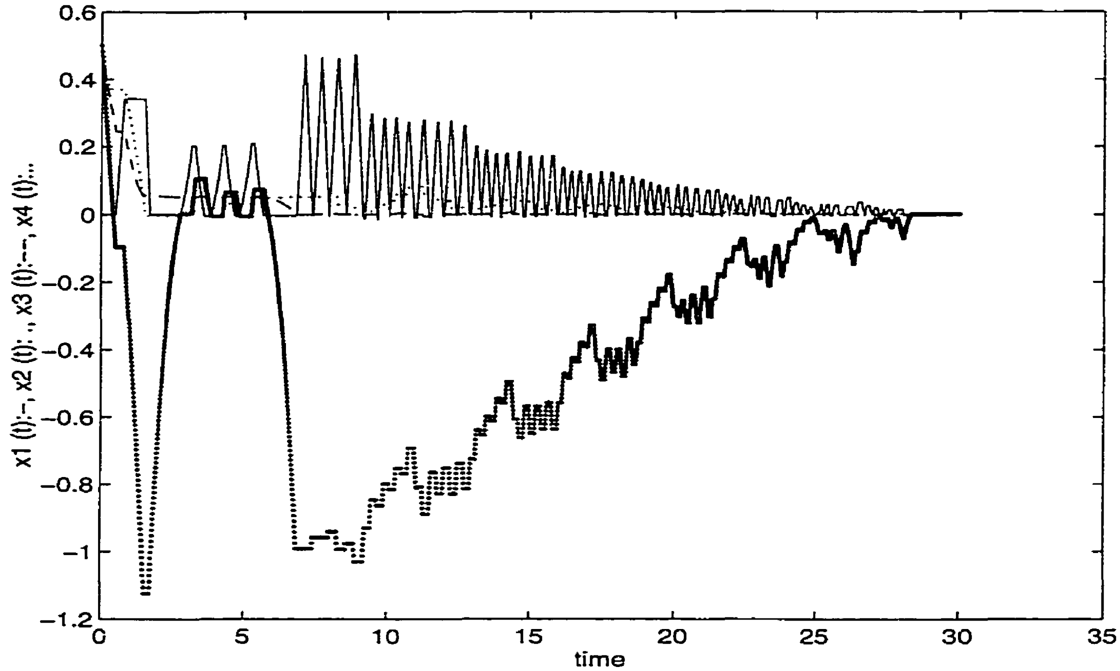

FIGURE 2.5. *Front wheel drive model:* Trajectories $(x_1(t), x_2(t), x_3(t), x_4(t)) \stackrel{def}{=} (\phi(t), x(t), y(t), \theta(t))$ versus time of the car while using the original stabilising control.

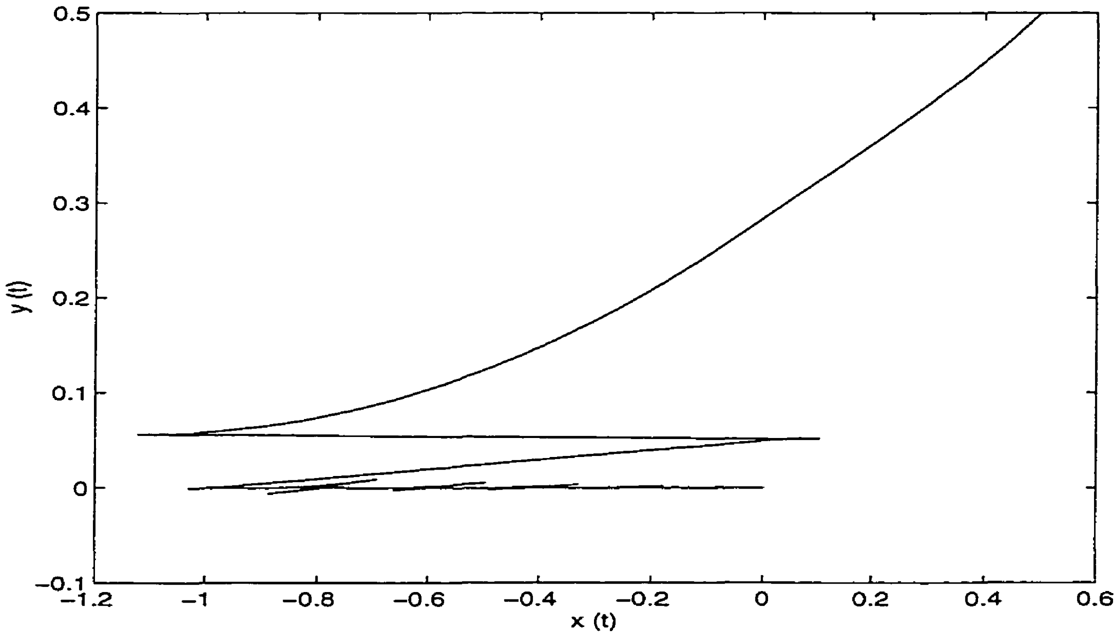

FIGURE 2.6. *Front wheel drive model:* Plot of the position of the car $(x(t), y(t))$, using the original stabilising control.

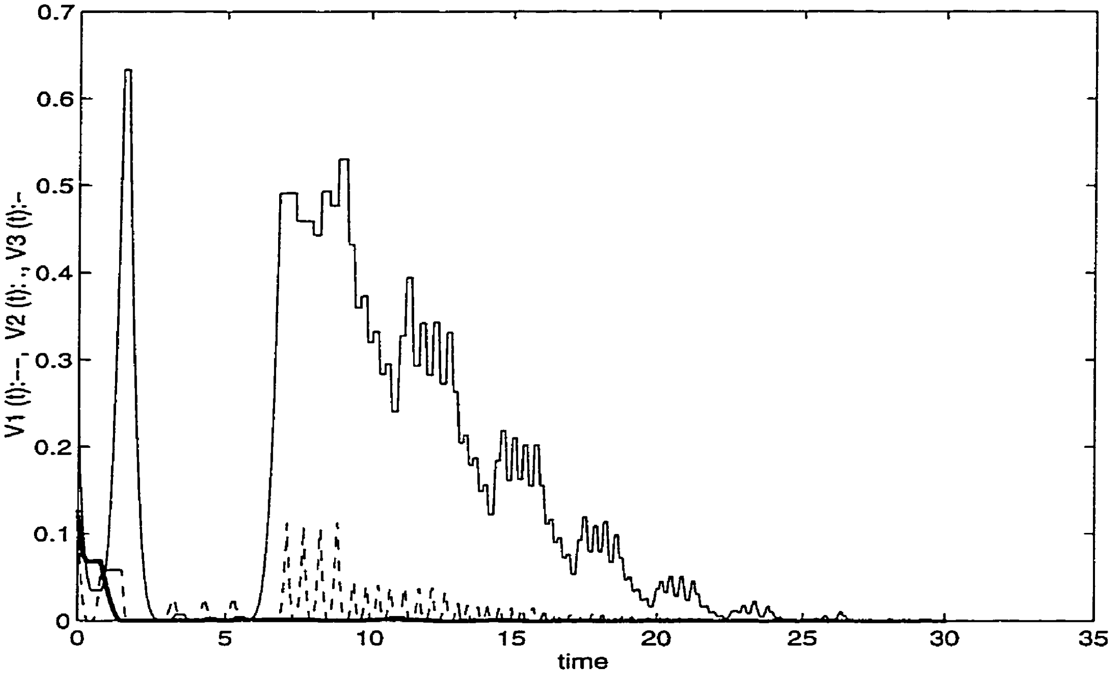

FIGURE 2.7. *Front wheel drive model*: Plots of the guiding functions V_1, V_2 and V_3 versus time when original strategy is applied.

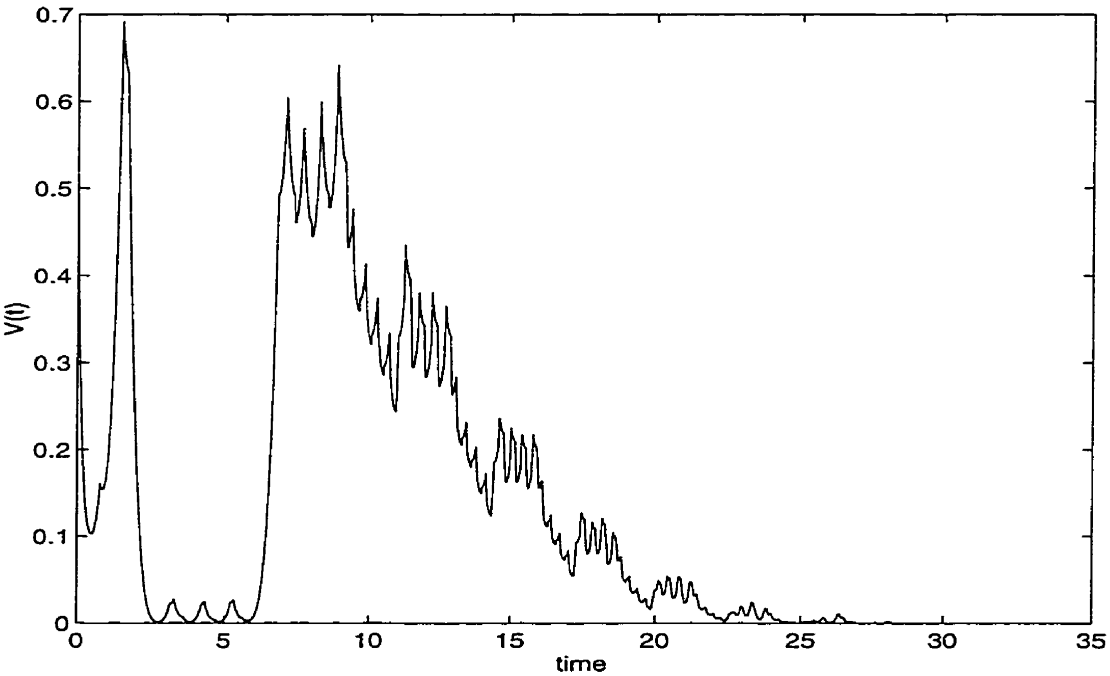

FIGURE 2.8. *Front wheel drive model*: Plot of the sum of the guiding functions $V = V_1 + V_2 + V_3$ versus time when original strategy is applied.

Other simulation experiments show that also in this case, in the absence of disturbances, the control is essentially dead-beat in that the origin can be achieved with an accuracy reflected by $V \leq 10^{-5}$ in only one cycle of the strategy (Steps 1-2d).

6.2. Further simplifications resulting from the guiding functions approach

The guiding functions approach reveals that the greatest difficulty in steering the car to the origin arises when the trajectory of the controlled system traverses the set $\mathcal{T}$. The corresponding, "desired" direction of system motion is then the y axes in the configuration space. Motion in y can be achieved only indirectly, by increasing θ to a nonzero value. The configuration variables y and θ are clearly the ones which are the most difficult to manipulate (Steps 2b-2d of the control strategy). On the other hand, once y and θ are both zero, steering ϕ and x to zero is easy. In attempt to simplify Steps 2b-2d we consider a "reduced system" which consists only of the two last model equations in y and θ :

$$\dot{y} = sin\theta \; v_2$$

$$\dot{\theta} = \bar{v}_1 v_2 \tag{2.89}$$

in which $tan\phi$ is replaced by "an additional" constant control $\bar{v}_1$. Assuming that $\bar{v}_1$ and v_2 are constant and that $\bar{v}_1 \neq 0$, integration of (2.89) yields

$$\theta(t) = \theta_0 + \bar{v}_1 v_2 t \tag{2.90}$$

$$y(t) = y_0 + \frac{1}{\bar{v}_1}[cos\theta_0 - cos(\theta_0 + \bar{v}_1 v_2 t)] \tag{2.91}$$

where θ_0 and y_0 are the initial values of $\theta(t)$ and $y(t)$. Clearly, if $y_0 \neq 0$ and $\theta_0 \neq \frac{\pi}{2} \pm k\pi$, $k \in I\!N$, then

$$\bar{v}_1 = \frac{1}{y_0}[1 - cos\theta_0] \neq 0 \tag{2.92}$$

$$v_2 = -sign[\theta(t)\bar{v}_1] \tag{2.93}$$

steer θ and y exactly to zero in finite time. The latter suggests the following, surprisingly simple, control law which stabilizes the car configuration:

Simplified feedback strategy

- 1 Steer the system to the set $\mathcal{T}$ using Subalgorithm 1 as above.

- 2 By employing the controls $v_1 = 1$, $v_2 = 1$ steer the system to a point (ϕ, x, y, θ) at which

$$\phi_{des} \stackrel{def}{=} atan[(1 - cos\theta)/y] \tag{2.94}$$

  satisfies $\phi_{des} \neq 0$ and $\phi_{des} \leq \alpha V(p)$, where p denotes a point at which $\mathcal{T}$ is last traversed, and $\alpha >> 1$ is a constant.

- 3 By employing the controls $v_1 = -sign(\phi - \phi_{des})$, $v_2 = 0$, steer the system to a point at which $\phi = \phi_{des}$.

- 4 Emply $v_1 = 0$ and $v_2 = -sign(\theta\bar{v}_1)$, with $\bar{v}_1 = tan(\phi_{des})$, until $\theta = 0$.

- 5 Employ $v_1 = -sign(\phi)$, and $v_2 = 0$ until $\phi = 0$.

- 6 Employ $v_1 = 0$ and $v_2 = -sign(x)$ until $x = 0$. Repeat from Step 2 in case when $(\phi, x, y, \theta) \neq 0$.

It is easy to see that the above strategy is feasible. Once the system reaches $\mathcal{T}$, then $\phi = x = \theta = 0$ but $y \neq 0$ so Step 2 is well defined. In Step 3 , ϕ is steered to a value required by equation (2.92), (a value such that $tan\phi \neq 0$). Hence, in the absence of any disturbances or model system error, the controls of Step 4 steer the system to a point at which both θ and y are zero. Due to disturbances or model system error, only the condition that $\theta = 0$ is met in finite time. In Steps 5 and 6 the system is steered back to the set $\mathcal{T}$, permitting repetition of Step 2 , if the system's state is not equal zero exactly. In this way, the strategy acquires the properties of a feedback control. It follows that in the absence of disturbances and model system error, the system is steered to the origin in finite time (at the exit of Step 6), regardless of its initial configuration.

Simulation results confirm the above analysis. Figure 2.9 shows the trajectories $(\phi(t), x(t), y(t), \theta(t))$ versus time, during the parallel parking maneuver. The trajectories showing the Cartesian position (x, y) of the car's center of mass, and the variations in the corresponding values of the sum of all guiding functions V, while performing a parallel parking maneuver, are shown in Figure 2.10.

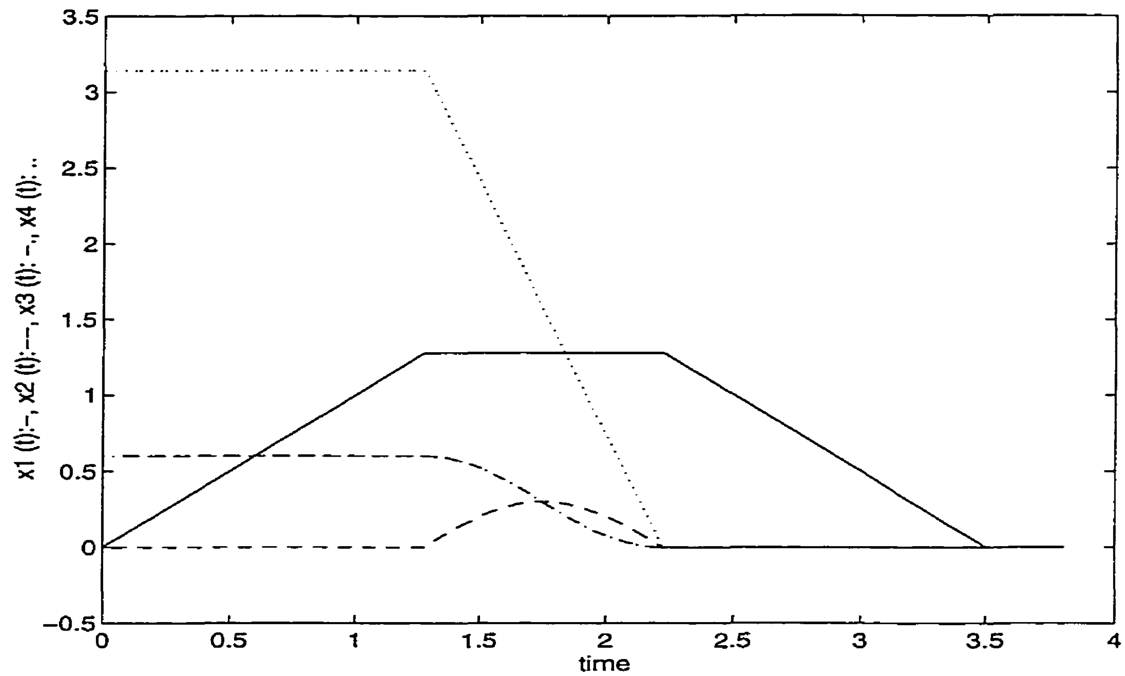

FIGURE 2.9. *Simplified control* (*car*) *example*: Trajectories $(x_1(t),\ x_2(t),\ x_3(t),\ x_4(t)) \stackrel{def}{=} (\phi(t),\ x(t),\ y(t),\ \theta(t))$ versus time in the parallel parking maneuver.

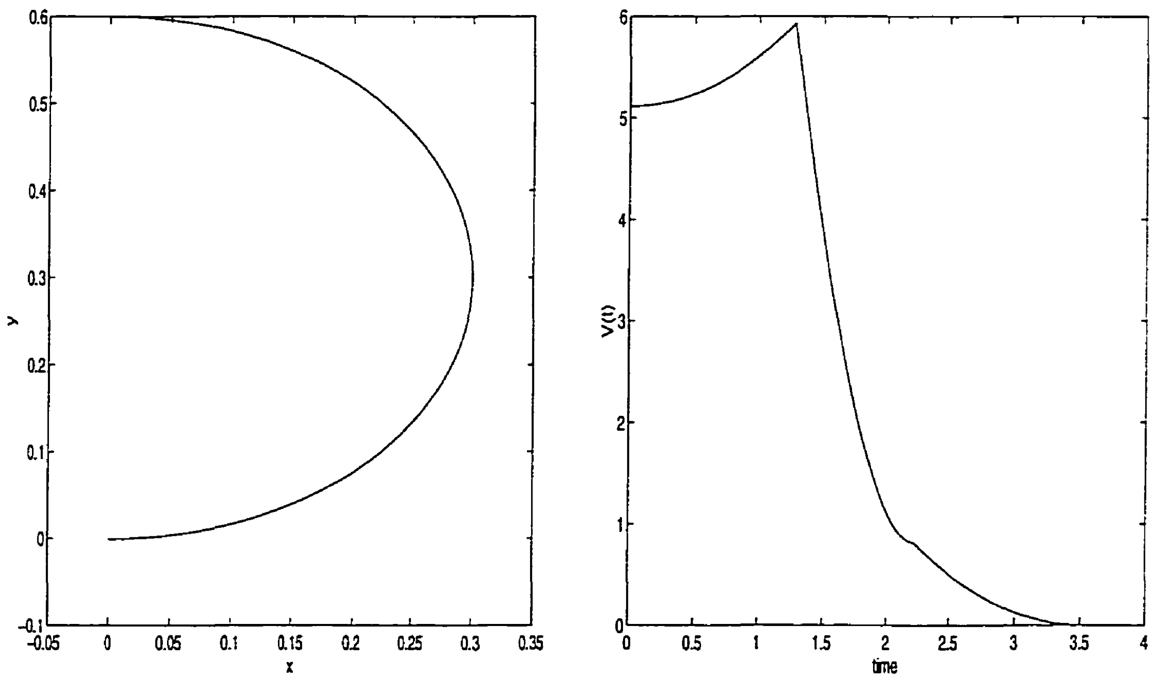

FIGURE 2.10. *Simplified control* (*car*) *example*: Plots of the position of the car $(x(t), y(t))$ and the sum of the guiding functions V versus time, in a parallel parking maneuver, when the simplified strategy is employed.

CHAPTER 3

The guiding functions stabilizing strategy for general drift free systems

In this chapter, the guiding functions control strategy of Chapter 2 is extended to apply to general drift free systems which need not be transformable to a rectified form [**64, 69**]. A systematic method for the construction of a pair of guiding functions is introduced and conditions are stated which guarantee that the resulting feedback control strategy yields global asymptotic convergence to a desired set point. Applications of the strategy are discussed and tested on different models of drift free systems such as: an underwater vehicle model, a general drift free system with five state variables and three controls, and a model of a rigid spacecraft in actuator failure mode [**64, 69**].

The possibility of employing the guiding functions approach to systems whose controllability Lie algebra involves higher order Lie brackets is also investigated. The idea of combining sinusoidal steering with the guiding functions approach is explored using models of an underwater vehicle in actuator failure mode, a fire truck model, and a mobile robot with trailer [**64, 72**].

The guiding functions approach for general drift free systems is further extended [**67, 71**], to allow for the construction of several rather than two guiding functions and its applicability is demonstrated on several examples of drift free systems: two different general drift free systems, a model of a hopping robot in flight phase, a fire truck model, and a class of wheeled mobile robots.

The approach presented in this Chapter is general and can be employed to control a variety of mechanical systems with velocity constraints.

1. Introduction

The theory developed in the previous Chapter requires the introduction of as many guiding functions as there are inputs in the rectified system. It is shown here that, in general, this is not always necessary. A class of drift free systems in general form is specified for which it is sufficient to introduce only two guiding functions $V_i(x)$, $i \in \{1,2\}$. The latter, serve a similar purpose as the guiding functions of the previous chapter and so their behaviour along the trajectories of the controlled system is not limited to $\frac{d}{dt}V_i(x) < 0$, $i \in \{1,2\}$. While allowing one of the guiding functions to increase, feedback controls $v_i(x)$, $i \in \underline{m}$, are constructed in such a way that, as before, the sum $V(x) \stackrel{def}{=} V_1(x) + V_2(x)$ decreases *on average*. The functions V_i, $i \in \{1,2\}$, must be chosen to satisfy similar conditions as those in Chapter 2, namely:

Condition (a): Each V_i, $i \in \{1,2\}$, is semi-positive definite, while the sum $V = V_1 + V_2$ is strictly positive definite in $I\!R^n$. The level sets $V^r \stackrel{def}{=} \{x \in I\!R^n : V(x) \leq r\}$, are bounded for all $r \geq 0$, and, additionally, $dV_i(x) = 0$ (where $dV_i(x)$ denotes the gradient of $V_i(x)$) implies that $V_i(x) = 0$, $i \in \{1,2\}$.

Condition (b): The value of each V_i, $i \in \{1,2\}$, can be manipulated by a fixed subset of the controls which have no effect on the other function V_k, $k \in \{1,2\}$, $k \neq i$. Additionally, for any constant $r > 0$, there exists a feedback control strategy which steers the system, in finite time, to the level set $V_1^r \stackrel{def}{=} \{x \in I\!R^n : V_1(x) \leq r\}$, while the value of V_2 remains unchanged.

Condition (c): The value of the second function, V_2, can be decreased over a finite interval of time if the first function, V_1, is allowed to vary freely.

The above conditions suggest a feedback synthesis which focuses on the decrease on V_2 alone. To begin with, such a strategy employs controls which provide for $\frac{d}{dt}V(x) = \sum_{i=1}^{2} \frac{d}{dt}V_i(x) < 0$. If this becomes impossible, due to the fact that $\frac{d}{dt}V_i(x) = 0$, for $i \in \{1,2\}$, regardless to the values of the controls $v_i, i \in \underline{m}$, then a sequence of controls is employed which results in a decrease of V_2 while the first function, V_1, is permitted to increase (see condition (c)). Next, another sequence of controls is employed whose task is to maintain the current value of V_2 while restoring V_1 to zero (see condition (b)). Repeating the above procedure results in asymptotic convergence of V to zero.

The guiding functions control strategy introduced in this Chapter has similar advantages (i)-(iii) as mentioned in the introduction of Chapter 2.

The novel contribution of this Chapter can be summarized as follows:

- The guiding functions control strategy, introduced in Chapter 2, is extended to a general class of drift free systems, which need not be transformable to any special form, and in which the difference between the number of state variables and controls can exceed one.
- A much improved strategy is presented here, in which exact steering of the system to the set $\mathcal{T} \stackrel{def}{=} \{x \in I\!R^n : L_{g_i}V(x) = 0,\ i \in \underline{m}\}$ is no longer necessary.
- A systematic method for the construction of guiding functions is introduced, and conditions are stated which guarantee that the resulting feedback control strategy yields global asymptotic convergence to a desired set point.
- The idea of combining sinusoidal steering with the guiding functions approach is also explored.
- Applications of the strategy are discussed involving set point stabilization of different types of models of drift free system possessing different algebraic structures. In all these examples, the strategy proves very efficient in that it effectively leads to dead-beat control.

2. Problem statement and assumptions

In this Chapter, the set point control problem is stated as a practical stabilization problem:

(SPC) : Given a desired set point $x_{des} \in I\!R^n$, and any constant $\epsilon > 0$, construct a (possibly discontinuous) feedback strategy in terms of the controls $v_i : I\!R^n \to [0,1]$, $i \in \underline{m}$, such that every trajectory $t \mapsto x(t; t_0, x_0)$ of the controlled system

$$\dot{x} = \sum_{i=1}^{m} g_i(x)v_i(x), \tag{3.1}$$

reaches $B(x_{des}; \epsilon)$ in finite time, where $(t_0, x_0) \in I\!R^+ \times I\!R^n$ denotes an arbitrary initial condition for (3.1) and $B(x_{des}; \epsilon)$ is a ball of radius ϵ, centred at x_{des}.

Without the loss of generality, it is also assumed that $x_{des} = 0$, which can be achieved by a suitable translation of the coordinate system. The guiding functions idea, permitting an effective synthesis of such stabilizing feedback, is first explained with reference to a subclass of systems of the type (3.1), whose Lie bracket extension contains only brackets of depth one; see the definition below, (of which the models of an underwater vehicle, a general system with five state variables and three controls, and a rigid spacecraft in actuator failure mode are typical examples). The application of the basic idea is then extended to systems whose Lie bracket extensions also contain brackets of depth ≥ 2, by using the combination of the guiding functions idea and sinusoidal steering, (the models of an underwater vehicle, a hopping robot in the flight phase, a fire truck, and a mobile robot with trailer are such examples).

The following controllability assumptions are assumed to hold for members of the class of systems considered:

A1. (a) : The vector fields g_i, $i \in \underline{m}$, are real analytic, complete, and linearly independent at all $x \in I\!R^n$. The Lie algebraic rank controllability condition (LARC) for these systems is assumed to take the form :

$$span\{g_i(x), [g_j, g_k](x), i, j, k \in \underline{m}\} = I\!R^n \qquad \text{for all } x \in I\!R^n. \tag{3.2}$$

A1. (b) : There exists a subset of indices, $\mathcal{J} \subset \underline{m} \times \underline{m}$, of cardinality $n - m$, such that

$$\mathcal{S}(x) \overset{def}{=} \{g_i(x), [g_j, g_k](x), i \in \underline{m}, (j,k) \in \mathcal{J}\} \tag{3.3}$$

$$span\{\mathcal{S}(x)\} = I\!R^n \qquad \text{for all } x \in I\!R^n. \tag{3.4}$$

Due to the skew symmetry property for Lie brackets we will not distinguish between $(j,k) \in \mathcal{J}$ and $(k,j) \in \mathcal{J}$. Next, suppose that the set of indices $\mathcal{J}$ has the following property :

A2. : The vector fields of the set $\mathcal{S}$ can be arranged in two groups, $\mathcal{G}_i \overset{def}{=} \{g_{i,k},\ k \in \underline{p_i}\}$, $i \in \{1,2\}$, while adhering to the following rules:

(R1): For any $(j,k) \in \mathcal{J}$, $[g_j, g_k] \in \mathcal{G}_2$, and either $g_j \in \mathcal{G}_2$, or $g_k \in \mathcal{G}_2$, but never both.

(R2): Each vector field of $\mathcal{S}$ belongs only to one group and $\mathcal{G}_1 \cup \mathcal{G}_2 = \mathcal{S}$, so that $p_1 + p_2 = n$.

For the construction of guiding functions we additionally need the following involutiveness condition:

A3. : The distributions $x \mapsto \Delta_i(x)$:

$$\Delta_i(x) \overset{def}{=} span\{\mathcal{G}_i(x)\}, \qquad i \in \{1,2\},\ \ x \in I\!R^n \tag{3.5}$$

are involutive, and thus are completely integrable.

It follows from the Frobenius Theorem that there exists a neighbourhood of the origin $\Omega \subset I\!R^n$, and two sets of scalar functions

$$\begin{aligned} \mathcal{P}_1(x) &\overset{def}{=} \{\lambda_{2,k}(x), k \in \underline{r_2}\}, \qquad \underline{r_2} \overset{def}{=} \underline{n - p_2}, \qquad x \in \Omega \\ \mathcal{P}_2(x) &\overset{def}{=} \{\lambda_{1,k}(x), k \in \underline{r_1}\}, \qquad \underline{r_1} \overset{def}{=} \underline{n - p_1}, \qquad x \in \Omega \end{aligned} \tag{3.6}$$

such that the codistributions $\Delta_1^{\perp}(x)$, and $\Delta_2^{\perp}(x)$, are spanned by exact differentials of (3.6), so that

$$\Delta_i^{\perp}(x) = span\{d\mathcal{P}_i(x)\}, \qquad \text{for all } i \in \{1,2\} \quad \text{and all } x \in \Omega \tag{3.7}$$

where $d\mathcal{P}_i(x) = \{d\lambda_{i,k}(x), k \in \underline{r_i}\}$. By virtue of rule (R2), $r_1 + r_2 = n$. For simplicity of exposition, we additionally assume the following.

A4. : The scalar functions $\lambda_{i,k}(x), k \in \underline{r_i}, i \in \{1,2\}$, are defined globally in $\mathbb{R}^n$, so that

$$\Delta_i^{\perp}(x) = span\{d\mathcal{P}_i(x)\}, \qquad \text{for all } i \in \{1,2\} \quad \text{and all } x \in \mathbb{R}^n \tag{3.8}$$

3. The guiding functions and their properties

At this stage, it is convenient to introduce the following notation. For any column vectors (or matrices) v_1, v_2 the symbol $\lceil v_1, v_2 \rfloor$ denotes a matrix with columns v_1 and v_2 (or a matrix whose columns are those of v_1 and v_2). In particular, if v_1 and v_2 are one-dimensional, then $\lceil v_1, v_2 \rfloor$ is a row vector. For any vectors v_1 and v_2 the symbol $col\lceil v_1, v_2 \rfloor$ denotes a column vector formed by listing the elements of v_1 and v_2 in a single sequence. For each index $i \in \{1,2\}$ and any $x \in \mathbb{R}^n$, let $\Lambda_i(x) \in \mathbb{R}^{r_i}$, and $L_i(x) \in \mathbb{R}^{p_i}$ be vectors defined by

$$\Lambda_i(x) \stackrel{def}{=} \lceil \lambda_{i,1}, ..., \lambda_{i,r_i} \rfloor (x)^T, \qquad i \in \{1,2\} \tag{3.9}$$

$$L_i(x) \stackrel{def}{=} \lceil L_{g_{i,1}} V, ..., L_{g_{i,p_i}} V \rfloor (x)^T, \quad i \in \{1,2\} \tag{3.10}$$

where $g_{i,j}$ are members of the groups $\mathcal{G}_i$, for $i \in \{1,2\}$ and V is a real, analytic, positive definite function. Let $d\Lambda_i(x)^T$ be the Jacobians of $x \mapsto \Lambda_i(x)$, $i \in \{1,2\}$, so that

$$d\Lambda_i(x) \stackrel{def}{=} \lceil d\lambda_{i,1}(x)^T, ..., d\lambda_{i,r_i}(x)^T \rfloor, \qquad x \in \mathbb{R}^n, \;\; i \in \{1,2\} \tag{3.11}$$

and $G_i(x)$ be matrices whose columns are vector fields from $\mathcal{G}_i(x)$, $i \in \{1,2\}$, respectively :

$$G_i(x) \stackrel{def}{=} \lceil g_{i,1}(x), ..., g_{i,p_i}(x) \rfloor, \quad x \in \mathbb{R}^n, \quad i \in \{1,2\} \tag{3.12}$$

Further, for any $x \in \mathbb{R}^n$, let

$$\Lambda(x) \stackrel{def}{=} col\lceil \Lambda_1, \Lambda_2 \rfloor (x), \qquad L(x) = col\lceil L_1, L_2 \rfloor (x) \tag{3.13}$$

$$d\Lambda(x) \stackrel{def}{=} \lceil d\Lambda_1, d\Lambda_2 \rfloor (x) \qquad G(x) = \lceil G_1, G_2 \rfloor (x) \tag{3.14}$$

be aggregated vectors and matrices constructed from components Λ_i, L_i, $d\Lambda_i$, and G_i, respectively.

Using the above notation, the semi-positive definite guiding functions are introduced as follows:

$$\begin{aligned} V_i(x) &\stackrel{def}{=} \frac{1}{2} \sum_{k \in \underline{r_i}} (\lambda_{i,k}(x) - \lambda_{i,k}(0))^2 \\ &= \frac{1}{2} \lceil \Lambda_i(x) - \Lambda_i(0) \rfloor^T \lceil \Lambda_i(x) - \Lambda_i(0) \rfloor, \quad i \in \{1,2\}, \quad x \in \mathbb{R}^n \end{aligned} \tag{3.15}$$

$$V(x) \stackrel{def}{=} V_1(x) + V_2(x) = \frac{1}{2}\lceil\Lambda(x) - \Lambda(0)\rfloor^T\lceil\Lambda(x) - \Lambda(0)\rfloor, \quad x \in \mathbb{R}^n \tag{3.16}$$

REMARK 3.1. Assumptions A1 (b) and A4 which insure that the guiding functions in (3.15) can be defined in the entire $\mathbb{R}^n$, were made primarily for the simplicity of exposition, and are essential only if *global* convergence to the set point is required. It can be shown that if these assumptions are omitted then all the results of this Chapter are still valid locally. This is due to the fact that, in general, the set of indices $\mathcal{J}$, and the associated set $\mathcal{S}(x)$ of assumption A1 are guaranteed to exist only locally, (such local existence is guaranteed by the satisfaction of the LARC condition for controllability). Similarly, as pointed out in equations (3.6) and (3.7), the scalar functions $\lambda_{i,k}(x), k \in \underline{r_i}, i \in \{1,2\}$, and the associated codistributions $\Delta_i^{\perp}(x)$, $i \in \{1,2\}$, are also, generally, defined locally. The satisfaction of assumption A4 is related to all of the vector fields in the controllability distribution (3.2) being complete.

Assumptions A1-A4 are sufficient to insure that the guiding functions V_i, $i \in \{1,2\}$, possess the desired properties (a)-(c) of the previous section, which we state in the form of Propositions 1-3, below. Auxiliary results are included in lemmas.

LEMMA 3.1. *Under assumptions A1-A4, the mapping $x \mapsto \Lambda(x)$, is a local diffeomorphism.*

Proof. First, we will show that, by construction,

$$\Delta_1^{\perp}(x) \oplus \Delta_2^{\perp}(x) = \mathbb{R}^n \qquad \text{for all } x \in \mathbb{R}^n \tag{3.17}$$

where the symbol $\oplus$ denotes a direct sum of subspaces. To see this, we note that, for all $x \in \mathbb{R}^n$, $\Delta_i^{\perp}(x)$, $i = 1,2$, are closed linear subspaces. Recalling the definitions of the distributions $\Delta_i(x)$, $i = 1,2$, it is then easy to verify that, for all $x \in \mathbb{R}^n$:

$$\begin{aligned}(\Delta_1^{\perp}(x) \oplus \Delta_2^{\perp}(x))^{\perp} &= \Delta_1^{\perp\perp}(x) \cap \Delta_2^{\perp\perp}(x) \\ &= \Delta_1(x) \cap \Delta_2(x) = span\{\mathcal{G}_1(x)\} \cap span\{\mathcal{G}_2(x)\} = \{0\}\end{aligned} \tag{3.18}$$

where the last equality holds by virtue of the construction of the sets of vector fields $\mathcal{G}_i(x)$, $i = 1,2$, (because the vector fields in the set $\mathcal{S}(x)$ are all linearly independent, span $\mathbb{R}^n$, and $\mathcal{G}_1(x) \cap \mathcal{G}_2(x) = \emptyset$). Equation (3.18) shows (3.17). By definition,

$$\begin{aligned}\Delta_1^{\perp} &= span\{d\lambda_{2,k}(x), k \in \underline{r_2}\}, \\ \Delta_2^{\perp} &= span\{d\lambda_{1,k}(x), k \in \underline{r_1}\}, \qquad x \in \mathbb{R}^n\end{aligned} \tag{3.19}$$

so that (3.17) implies that the matrix $d\Lambda(x) = \lfloor d\Lambda_1, d\Lambda_2\rceil(x)$ is non-singular for all $x \in \mathbb{R}^n$. The result of the Lemma follows readily, since $d\Lambda(x)^T$ is the jacobian of the mapping $x \mapsto \Lambda(x)$. ∎

LEMMA 3.2. *For any compact set $C \in I\!R^n$ there exist constants $\gamma_1(C) > 0$, and $\gamma_2(C) > 0$ such that*

$$\gamma_1 V_i(x) \leq \tilde{V}_i(x) \leq \gamma_2 V_i(x), \qquad \text{for all } \; x \in C, \quad i \in \{1,2\} \tag{3.20}$$

$$\text{where,} \quad \tilde{V}_i(x) \stackrel{def}{=} L_i(x)^T L_i(x), \qquad \text{for all } \; x \in I\!R^n, \quad i \in \{1,2\} \tag{3.21}$$

Proof. By direct calculation it is easy to verify that for all $x \in I\!R^n$, and for $i \in \{1,2\}$:

$$\tilde{V}_i(x) = \frac{1}{2}\lceil \Lambda_i(x) - \Lambda_i(0)\rfloor^T \, d\Lambda_i(x)^T G_i(x) G_i(x)^T d\Lambda_i(x) \, \lceil \Lambda_i(x) - \Lambda_i(0)\rfloor \tag{3.22}$$

For any $x \in I\!R^n$, the matrix $M(x) \stackrel{def}{=} G(x)^T d\Lambda(x) = \lceil G_1, G_2 \rfloor(x)^T \lceil d\Lambda_1, d\Lambda_2 \rfloor(x)$, is nonsingular by virtue of assumption A1(b), and the fact that the jacobian $d\Lambda(x)$ is non-singular (see Lemma 3.1). Hence the matrix $M(x) \stackrel{def}{=} G(x)^T d\Lambda(x)$ is non-singular, and $M(x)^T M(x)$ is positive definite and symmetric. Moreover, by construction, $G_1(x)^T d\Lambda_2(x)$ and $G_2(x)^T d\Lambda_1(x)$ are matrices containing only zero elements, so that $M(x)$ is block diagonal and is given by $M(x) = diag\{G_1(x)^T d\Lambda_1(x), G_2(x)^T d\Lambda_2(x)\}$. It follows that the blocks $M_i(x) \stackrel{def}{=} G_i(x)^T d\Lambda_i(x)$, $i \in \{1,2\}$, are also non-singular and that $M_i(x)^T M_i(x)$, $i \in \{1,2\}$, are both positive definite and symmetric. Let $\sigma^{(i)}_{min}(x)$ and $\sigma^{(i)}_{max}(x)$ denote the smallest and largest eigenvalues of $M_i(x)^T M_i(x)$, and choose the constants γ_1 and γ_2 as follows :

$$\gamma_1 \stackrel{def}{=} min\{\sigma^{(i)}_{min}(x) \mid x \in C, \; i \in \{1,2\}\} \tag{3.23}$$

$$\gamma_2 \stackrel{def}{=} max\{\sigma^{(i)}_{max}(x) \mid x \in C, \; i \in \{1,2\}\} \tag{3.24}$$

which are well defined, as these eigenvalues are continuous functions of x. Hence, for any $x \in C$, and $y \in I\!R^n$, we have that $\frac{1}{2}\gamma_1 ||y||^2 \leq (1/2) y^T M_i(x)^T M_i(x) y \leq \frac{1}{2}\gamma_2 ||y||^2$, $i \in \{1,2\}$, and (3.20) follows from (3.22). ■

PROPOSITION 3.1. *Suppose that assumptions A1-A4 are valid and, additionally, the mapping $x \mapsto \Lambda(x)$, is a global diffeomorphism. Under these conditions :*

(a): *The sum of the guiding functions $V(x) = V_1(x) + V_2(x)$ is strictly positive definite in $I\!R^n$.*

(b): *The level sets $V^r \stackrel{def}{=} \{x \in I\!R^n : V(x) \leq r\}$, are bounded for all $r \geq 0$.*

(c): *For any $i \in \{1,2\}$, and any $x \in I\!R^n$, the condition $dV_i(x) = 0$ implies that $V_i(x) = 0$, and $dV(x) = 0$ implies that $V(x) = 0$.*

Proof. (a): By contradiction, suppose that there exists a point $x_0 \neq 0$ for which $V(x_0) = 0$, so that $\lambda_{i,k}(x_0) = \lambda_{i,k}(0)$, $i \in \{1,2\}$, $k \in \underline{r_i}$. This implies that the mapping $x \mapsto \Lambda(x)$ is not injective which contradicts the assumption that it is a global diffeomorphism. This demonstrates positive definiteness of V in $I\!R^n$.

(b): By contradiction, suppose that there exists a constant $r > 0$ such that the corresponding level set V^r is not compact. Since V^r is closed, it is then possible to extract a sequence of points $\{x_i\}_{i \in I\!N} \subset V^r$ such that

$$x_i \to \infty \quad \text{as} \quad i \to \infty \tag{3.25}$$

By definition of the level set, the corresponding values satisfy :

$$V(x_i) = \frac{1}{2} \sum_{i=1,2} \sum_{k \in \underline{r_i}} (\lambda_{i,k}(x_i) - \lambda_{i,k}(0))^2 \leq r \tag{3.26}$$

Hence $\Lambda(x_i) \in cl\{B(\Lambda(0); (2r)^{1/2})\}$, where $cl\{B(\Lambda(0); (2r)^{1/2})\}$ is a compact ball of radius $(2r)^{1/2}$, centred at $\Lambda(0)$. Let the inverse mapping to $x \mapsto \Lambda(x)$ be denoted by Λ^{-1}. Since Λ^{-1} is continuous, then the image $\Lambda^{-1}(cl\{B(\Lambda(0); (2r)^{1/2})\})$ is a compact set. Clearly, $x_i \in \Lambda^{-1}(cl\{B(\Lambda(0); (2r)^{1/2})\})$, for all $i \in I\!N$, which contradicts (3.25).

(c): Since, for each $i \in \{1, 2\}$, the matrix $d\Lambda_i(x) \stackrel{def}{=} \lceil d\lambda_{i,1}(x), ..., d\lambda_{i,r_i}(x) \rfloor$ contains only vectors which are linearly independent at every $x \in I\!R^n$, then the equality

$$dV_i(x) = \sum_{k \in \underline{r_i}} (\lambda_{i,k}(x) - \lambda_{i,k}(0))\ d\lambda_{i,k}(x) = 0 \tag{3.27}$$

implies that $\lambda_{i,k}(x) = \lambda_{i,k}(0)$, $k \in \underline{r_i}$, and thus, $V_i(x) = 0$, by definition of the guiding function V_i. By a similar argument, the fact that the columns of the matrix $d\Lambda(x)$ are linearly independent also implies that the condition $dV(x) = 0$ entails $V(x) = 0$. ∎

To show that the guiding functions construction satisfies Conditions (b) and (c), further definitions will be of help.

For any constant $\rho > 0$, and for any subset of indices $J \subset \underline{m}$ we define

$$\mathcal{T}_J(\rho) \stackrel{def}{=} \{x \in I\!R^n : |L_{g_k} V(x)| < \rho,\ k \in J\} \tag{3.28}$$

Clearly, if $J = \underline{m}$ then $\mathcal{T} = \mathcal{T}_J(0)$, where $\mathcal{T}$ is the set of "impasse points" defined in the Introduction to this Chapter. Let the sets J_i, $i \in \{1, 2\}$, contain the indices of the vector fields $g_k, k \in \underline{m}$, which correspond to members of the groups $\mathcal{G}_i$, $i \in \{1, 2\}$, respectively, so that

$$\mathcal{T}_{J_i}(\rho) \stackrel{def}{=} \{x \in I\!R^n : |L_{g_k} V(x)| < \rho,\ k \in \underline{m}, g_k \in \mathcal{G}_i\}, \quad i \in \{1, 2\} \tag{3.29}$$

also, let

$$\mathcal{T}_{lb}(\rho) \stackrel{def}{=} \{x \in I\!R^n : |L_{[g_k, g_j]} V(x)| < \rho,\ (k, j) \in \mathcal{J}\} \tag{3.30}$$

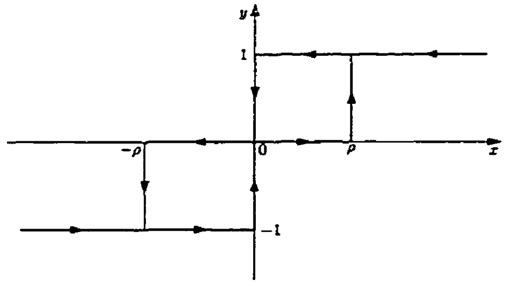

FIGURE 3.1. A hysteresis loop of width 2ρ

where $\mathcal{J}$ is defined in assumption A1. For any subset of indices $J \subset \underline{m}$ and any constant $\rho > 0$, let a control $v^J(x,\rho)$ be defined component-wise by :

$$v_j^J(x,\rho) = \begin{cases} -sign_\rho(L_{g_j}V(x)), & \text{if } j \in J, \\ 0, & \text{if } j \notin J, \end{cases} \qquad j \in \underline{m} \qquad x \in I\!R^n \tag{3.31}$$

where $sign_\rho$ denotes the usual signum function with a hysteresis loop of width 2ρ; preferably of the shape shown in Figure 3.1. With reference to Figure 3.1 it is assumed that at an initial point $x_0 \neq 0$, either $sign_\rho(x_0) = -1$ or else that $sign_\rho(x_0) = 1$, and that $sign_\rho(x_0) = 0$ if $x_0 = 0$. This control definition is practical in that existence and uniqueness of classical solutions to a closed loop system using this type of control is not prejudiced (a "sharp" switching control of the type $sign(x)$ would normally require special definitions of solutions to the closed-loop system equation and would necessitate considerations related to chattering). The presence of the hysteresis loop induces the possibility of oscillations, but, for a given value of ρ, the latter will have finite frequency, or else will not occur at all. The selection of the shape of the hysteresis loop in Figure 3.1 is motivated by Lemma 3.3 which provides conditions which guarantee that the associated control does not exhibit any oscillations.

PROPOSITION 3.2. *Suppose assumptions A1-A4 hold and the mapping $x \mapsto \Lambda(x)$ is a global diffeomorphism. Under these conditions:*

(a) *The value of each V_i, $i \in \{1,2\}$, can be changed by a fixed subset of the controls which have no effect on the other function V_k, $k \in \{1,2\}$, $k \neq i$.*

(b) *For any subset of indices $J \subset \underline{m}$, and for any constant $\rho > 0$, the control $v^J(x,\rho)$ steers the system to the set $\mathcal{T}_J(\rho)$ in finite time. Consequently, for any $r > 0$, there exists a control which steers the system to the level set $V_1^r = \{x \in I\!R^n : V_1(x) \leq r\}$, (in finite time), while the value of V_2 stays unchanged.*

Proof. (a): By virtue of assumption A4,

$$d\lambda_{1,k}(x) \perp \mathcal{G}_2(x), \quad k \in \underline{r_1}, \quad d\lambda_{2,k}(x) \perp \mathcal{G}_1(x), \quad k \in \underline{r_2}, \quad x \in \mathbb{R}^n \tag{3.32}$$

which implies that for all $x \in \mathbb{R}^n$

$$L_{g_j} V_1(x) = 0, \quad \text{if } g_j \in \mathcal{G}_2, \quad \text{and} \quad L_{g_j} V_2(x) = 0, \quad \text{if } g_j \in \mathcal{G}_1 \tag{3.33}$$

Therefore, a control $v \stackrel{def}{=} \lceil v_1, ..., v_m \rfloor$ in which $v_j = 0$ for all j such that $g_j \in \mathcal{G}_1$, has no effect on V_1, while a control v in which $v_j = 0$, for all j such that $g_j \in \mathcal{G}_2$, cannot change the value of V_2.

(b): Suppose, contrary to what needs to be shown, that the trajectory $t \mapsto x(t)$ of the closed loop system controlled by v^J never reaches $\mathcal{T}_J(\rho)$. It follows that at any time $t > 0$ there exists an index $k \in J$ such that $|L_{g_k} V(x(t))| \geq \rho$, and hence that

$$\frac{d}{dt} V(x(t)) \leq -|L_{g_k} V(x(t))| \leq -\rho < 0, \tag{3.34}$$

Consequently, $\frac{d}{dt} V(x(t)) \leq -\rho$, for all times t, and therefore $V(x(t)) \to -\infty$ as $t \to \infty$, which contradicts positive definiteness of V.

Now, suppose that the index set J coincides with the set of indices of vector fields which belong to $\mathcal{G}_1$. From the definition of the control v^J it follows that $v_j^J(x, \rho) = 0$ whenever j corresponds to a vector field which belongs to $\mathcal{G}_2$, in which ρ can be selected freely. From part (a) of this proposition we conjecture that the guiding function V_2 cannot change under the action of such v^J. Clearly, along any trajectory $t \mapsto x(t)$ of the system so controlled, $\frac{d}{dt} V_1(x(t)) \leq 0$, so that $x(t) \in \mathcal{C} \stackrel{def}{=} \{x \in \mathbb{R}^n : V_1(x) \leq V_1(x(0))\}$, for all t. Select $\rho = (2r\gamma_1/m)^{\frac{1}{2}}$, where γ_1 is the constant of Lemma 3.2 which corresponds to the set $\mathcal{C}$. It further follows that v^J steers the system to the set $\mathcal{T}_J(\rho) = \mathcal{T}_{J_1}(\rho)$ in finite time. Since $L_{g_k} V(x) = L_{g_k} V_1(x)$, if $g_k \in \mathcal{G}_1$, and $L_{g_{2,j}} V_1(x) = 0$, for all $g_{2,j} \in \mathcal{G}_2$, $j \in \underline{p_2}$, then v^J steers the system to the set $\{x \in \mathbb{R}^n : |L_{g_k} V_1(x)| \leq \rho, k \in \underline{m}, \ g_k \in \mathcal{G}_1\}$ in finite time. It follows from Lemma 3.2 that the control v^J steers the system to the set $\{x \in \mathbb{R}^n : \tilde{V}_1(x) \leq (1/2)\rho^2 m\}$ $\subset \{x \in \mathbb{R}^n : V_1(x) \leq (1/2\gamma_1)\rho^2 m = r\} = V^r$ in finite time. ■

LEMMA 3.3. *Suppose that the assumptions of Proposition 3.2 are valid where the index set J is written as $J = \{i_1, ..., i_l\}$, $l \leq m$. Let the following conditions hold :*

$$L^2_{g_{i_k}} V(x) \neq 0 \quad \textit{for all } x \in \mathbb{R}^n, \quad k \in \{1, ..., l\} \tag{3.35}$$

$$L_{g_{i_k}} L_{g_{i_j}} V(x) = 0, \quad \textit{for all } x \in \mathcal{T}_{I_{k-1}}(0), \quad 1 \leq j < k, \quad k \in \{2, ..., l\} \tag{3.36}$$

where $I_{k-1} = \{i_1, ..., i_{k-1}\}$. Further suppose that the component controls in v^J, as defined in (3.31), are activated sequentially in that for any $1 < k \leq l$, $v^J_{i_k}(x, \rho)$ becomes non-zero only if $L_{g_{i_j}} V(x) = 0$, *for all $j \leq k-1$, while $L_{g_{i_k}}V(x)$ changes sign. Under these conditions, and under the absence of disturbances and model-system error, such sequentially activated control v^J does not generate oscillations and steers the system to the set $\mathcal{T}_J(0)$ in finite time.*

Proof. Let x_0 be the value of the state of the controlled system at the initial time $t = 0$. If $L_{g_{i_1}}V(x_0) \neq 0$, then $v^J_{i_1}$ is the only active component of v^J on some interval of time $[0, \epsilon)$, so that

$$\frac{d}{dt}V(x(t)) = -|L_{g_{i_1}}V(x(t))| < 0 \tag{3.37}$$

along the controlled system trajectory $t \mapsto x(t)$, for $t \in [0, \epsilon)$. It follows that the trajectory remains in the level set $V^0 \stackrel{def}{=} \{x \in I\!R : V(x) \leq V(x_0)\}$ and converges to $\mathcal{T}_{I_1}(0)$, $I_1 = \{i_1\}$, (see Proposition 3.2 (b)). In fact, due to the additional assumption (3.35), the trajectory reaches $\mathcal{T}_{I_1}(0)$ in finite time. For if this is not true, then $L_{g_{i_1}}V(x(t)) \to 0$ as $t \to \infty$ but $L_{g_{i_1}}V(x(t)) \neq 0$ for all t. The latter is impossible since, both $L_{g_{i_1}}V(x(t))$ and $L^2_{g_{i_1}}V(x(t))$ are continuous, cannot change their sign, and, when $v^J_{i_1}$ is the only non-zero component of v^J, then

$$|\frac{d}{dt}L_{g_{i_1}}V(x(t))| = |L^2_{g_{i_1}}V(x(t))| \geq \delta > 0 \tag{3.38}$$

for all $t \geq 0$, where $\delta \stackrel{def}{=} min\{|L^2_{g_{i_1}}V(x)| \mid x \in V^0\}$. Hence $x(t)$ reaches $\mathcal{T}_{I_1}(0)$ at some finite time t_1, at which $v^J_{i_1}(x(t_1), \rho) = 0$, and the next control component $v^J_{i_2}$ becomes active. Now, condition (3.36) entails that, at any $x \in \mathcal{T}_{I_1}(0)$, the vector field g_2 is tangential to the hypersurface $\mathcal{T}_{I_1}(0) = \{x \in I\!R^n : L_{g_{i_1}}V(x) = 0\}$, which further implies that if at any time t, $x(t) \in \mathcal{T}_{I_1}(0)$ then $x(t)$ can never leave $\mathcal{T}_{I_1}(0)$ (provided that the only non-zero control component is $v^J_{i_2}$). The latter is indeed the case at $t = t_1$ because $x(t_1) \in \mathcal{T}_{I_1}(0)$. Hence, $v^J_{i_2}$ is the only non-zero control component for $t \geq t_1$, and

$$L_{g_{i_1}}V(x(t)) = 0 \quad \text{so that} \quad v^J_{i_1}(x(t), \rho) = 0 \qquad \text{for } t \geq t_1 \tag{3.39}$$

Therefore, (3.37) and (3.38) are valid for the index value i_1 substituted by i_2, and there exists a finite time $t_2 \geq t_1$ at which $L_{g_{i_1}}V(x(t_2)) = L_{g_{i_2}}V(x(t_2)) = 0$, i.e. $x(t_2) \in \mathcal{T}_{I_2}(0)$.

Now, suppose that due to such sequentially activated control, for some index $i_k < i_l$, there exists a time instant t_k such that $x(t_k) \in \mathcal{T}_{I_k}(0)$. Clearly, $v^J(x(t_k), \rho) = 0$ for all $j \leq k$, and again, by virtue of (3.36), $x(t) \in \mathcal{T}_{I_k}(0)$ for all $t \geq t_k$, if the only non-zero control component of v^J is $v^J_{i_{k+1}}$. Since this is indeed the case, we have as before

$$L_{g_{i_j}}V(x(t)) = 0 \quad \text{and} \quad v^J_{i_j}(x(t), \rho) = 0 \qquad \text{for } t \geq t_k, \;\; 1 \leq j < k \tag{3.40}$$

and equations (3.37) and (3.38) are valid for the index value i_k. By an argument identical to the one used for i_1 and i_2, there exists a finite time $t_{k+1} \geq t_k$ at which $x(t_{k+1}) \in \mathcal{T}_{I_{k+1}}(0)$. The result follows by induction. ■

Contrary to what might seem, the assumptions of Lemma 3.3 are not restrictive and are satisfied for most examples considered in this Chapter.

For any open set $\mathcal{T}$, let $cl\mathcal{T}$ denote its closure and let the scalar switching function $x \mapsto sign^+(x)$ be defined by : $sign^+(x) = 1$ if $x \geq 0$, and $sign^+(x) = -1$ if $x < 0$.

LEMMA 3.4. *For an arbitrary $\rho > 0$ and any pair of indices $(k, i) \in \mathcal{J}$ such that $g_k \in \mathcal{G}_1$, the control $x \mapsto u^{(i)}(x, \rho)$, defined component-wise by :*

$$u_j^{(i)}(x,\rho) \stackrel{def}{=} \begin{cases} sign_\rho\{L_{[g_k,g_i]}V_2(x)\}\ sign_+\{L_{g_i}V_2(x)\}, & \text{if } j = k, \\ 0, & \text{if } j \neq k, \end{cases} \quad j \in \underline{m},\ x \in \mathbb{R}^n \tag{3.41}$$

is regular in the sense that the closed loop system equation with this control has unique, classical solutions. Furthermore, $u^{(i)}$ steers the closed loop system from the set $\mathcal{A}_{k,i}(\rho) \stackrel{def}{=} \{x \in \mathbb{R}^n : |L_{g_i}V_2(x)| \leq \rho,\ |L_{[g_k,g_i]}V_2(x)| \geq \rho\}$ to the set $\mathcal{B}_{k,i}(\rho) \stackrel{def}{=} \{x \in \mathbb{R}^n : |L_{g_i}V_2(x)| \geq 2\rho\} \cup \{x \in \mathbb{R}^n : |L_{[g_k,g_i]}V_2(x)| \leq \rho/2\}$ in finite time (any trajectory emanating from $\mathcal{A}_{k,i}(\rho)$ terminates in $\mathcal{B}_{k,i}(\rho)$). For any such controlled trajectory $t \mapsto x(t)$, emanating from $\mathcal{A}_{k,i}(\rho)$, the value of $|L_{g_i}V_2(x(t))|$ increases with t, while the value of the guiding function V_2 stays constant.

Proof. By virtue of the definition of the switching function $sign_\rho$, and smoothness of the vector fields g_i, $i \in \underline{m}$, the existence and uniqueness of classical solutions to the closed loop system equation employing control $u^{(i)}$ can only be endangered by the presence of the other switching function $sign^+$. However, as will be shown soon, the Lie derivative $L_{g_i}V_2$ cannot change its sign if the system is controlled by $u^{(i)}$. The right hand side of the controlled system equation is hence well defined and the discontinuities in the control are encountered in isolated moments of time which does not prejudice existence and uniqueness of its solutions.

Now, suppose that $x_0 \in \mathcal{A}_{k,i}(\rho)$ and that the closed loop system trajectory $t \mapsto x(t)$, emanating from x_0 never reaches the set $\mathcal{B}_{k,i}(\rho)$ and, in particular, that it never reaches the set:

$$\mathcal{C}_{k,i}(\rho) \stackrel{def}{=} \{x \in \mathbb{R}^n : |L_{[g_k,g_i]}V_2(x)| \leq \rho/2\}.$$

To see how the Lie derivative $L_{g_i}V_2$ changes along $x(t)$ we calculate

$$\begin{aligned}\frac{d}{dt}L_{g_i}V_2(x(t)) &= L_{g_k}L_{g_i}V_2(x(t))u_k^{(i)}(x(t)) \\ &= L_{[g_k,g_i]}V_2(x(t))u_k^{(i)}(x(t)) + L_{g_i}L_{g_k}V_2(x(t))u_k^{(i)}(x(t)) \\ &= L_{[g_k,g_i]}V_2(x(t))u_k^{(i)}(x(t)) \\ &= |L_{[g_k,g_i]}V_2(x(t))|sign_+(L_{g_i}V_2(x(t)))\end{aligned} \tag{3.42}$$

The latter equality is due to the application of the Jacobi identity, and the fact that $L_{g_k}V_2 \equiv 0$ for all k such that $g_k \in \mathcal{G}_1$. Since $x(t)$ never reaches $\mathcal{C}_{k,i}(\rho)$, it follows that there exists an interval $[0,\delta]$ such that, if $L_{g_i}V_2(x_0) \geq 0$, then

$$\frac{d}{dt}L_{g_i}V_2(x(t)) > \rho/2, \qquad \text{for } t \in [0,\delta] \tag{3.43}$$

and if $L_{g_i}V_2(x_0) < 0$, then

$$\frac{d}{dt}L_{g_i}V_2(x(t)) < -\rho/2, \qquad \text{for } t \in [0,\delta] \tag{3.44}$$

Clearly, the value of $|L_{g_i}V_2(x(t))|$ increases over the interval $[0,\delta]$ and thus (3.43), or else (3.44), remains valid for all times $t > 0$ as $sign^+(L_{g_i}V_2(x(t))$ is of constant sign. The latter implies that there exists a finite time $t^* > 0$ such that $|L_{g_i}V_2(x(t^*))| = 2\rho$, which contradicts the assumption that $x(t)$ never reaches $\mathcal{B}_{k,i}(\rho)$. So $x(t)$ reaches $\mathcal{B}_{k,i}(\rho)$ and V_2 is uneffected by the control $u^{(i)}$, as $u_j^{(i)} \equiv 0$ for all j such that $g_j \in \mathcal{G}_2$. ■

PROPOSITION 3.3. *Suppose assumptions A1-A4 hold and the mapping $x \mapsto \Lambda(x)$ is a global diffeomorphism. Then the value of the second function, V_2, can be decreased (over any finite interval of time) if the first function, V_1, is allowed to vary freely.*

Proof. It follows from Proposition 3.2 that if $x \notin cl\mathcal{T}_{J_2}(0)$ then V_2 can be decreased by applying a control $v^J(\cdot,\rho)$ in which J includes all indices $k \in \underline{m}$ such that $g_k \in \mathcal{G}_2$. Hence, difficulty in generating controls which decrease V_2 arises only at points $x_0 \in cl\mathcal{T}_{J_2}(0)$. To show how this can be resolved suppose that $x_0 \in cl\mathcal{T}_{J_2}(0)$ but that $V_2(x_0) \neq 0$. It follows from the complete controllability condition that there exists indices $(k,i) \in \mathcal{J}$, and a constant $\rho > 0$, such that

$$|L_{[g_k,g_i]}V_2(x_0)| \geq \rho \tag{3.45}$$

(as, otherwise, $L_gV_2(x_0) = 0$ for all $g \in \mathcal{S}(x_0)$ which implies that $dV_2(x_0) = 0$, and contradicts the assumption that $V_2(x_0) \neq 0$, see Proposition 3.1 (c)).

At this point the control $u^{(i)}$, defined in Lemma 3.4, can be employed to change the value of $|L_{g_i}V_2|$

from zero to nonzero (in finite time) without having any effect on the value of V_2. Suppose that $t^* > 0$ is a time instant such that $|L_{g_i} V_2(x(t^*))| = \epsilon > 0$, where $t \mapsto x(t)$ is the controlled system trajectory emanating from x_0 and ϵ is some positive constant. The control $v^J(\cdot, \epsilon/2)$ in which J includes all indices $k \in \underline{m}$ such that $g_k \in \mathcal{G}_2$ can again be used for $t > t^*$ to achieve a decrease in V_2. ∎

REMARK 3.2. It should be clear that the constant ρ in the definition of the the control $u^{(i)}$ of Lemma 3.4 can be taken to be zero without prejudicing the existence and uniqueness of solutions to the closed loop system equation. The latter follows readily from the fact that the only non-zero component of $u^{(i)}$ is $u_k^{(i)}$, so, in the absence of disturbances and model error, once $L_{[g_k, g_i]} V_2$ becomes zero, the control $u_k^{(i)}$ is set to zero, and remains zero for all later times.

3.1. Some simple examples

Assumption A2 to A4 are seemingly complicated, however, they are not restrictive, as demonstrated by the examples below.

The unicycle

The model of the unicycle is perhaps the most widely known nonholonomic system. Its model, as given by equation (1.17) of Chapter 1, can be written as:

$$\begin{aligned} \dot{x} &= g_1(x)u_1 + g_2(x)u_2, \qquad x \stackrel{def}{=} [x_1, x_2, x_3]^T \in \mathbb{R}^3 \\ &\text{where,} \quad g_1(x) \stackrel{def}{=} \frac{\partial}{\partial x_1}, \qquad g_2(x) \stackrel{def}{=} cos(x_1)\frac{\partial}{\partial x_2} + sin(x_1)\frac{\partial}{\partial x_3} \end{aligned} \tag{3.46}$$

The first Lie bracket of g_1 and g_2 is given by

$$\begin{aligned} g_3(x) \stackrel{def}{=} [g_1, g_2](x) = -sin(x_1)\frac{\partial}{\partial x_2} + cos(x_1)\frac{\partial}{\partial x_3}, \quad \text{and} \\ span\{g_1(x), g_2(x), g_3(x)\} = \mathbb{R}^3, \qquad \text{for all} \quad x \in \mathbb{R}^3 \end{aligned} \tag{3.47}$$

Adhering to the rules R1-R2 of assumption A2, the groups $\mathcal{G}_1$ and $\mathcal{G}_2$ can be defined as follows :

$$\mathcal{G}_1(x) \stackrel{def}{=} \{g_1(x)\}, \qquad \mathcal{G}_2(x) \stackrel{def}{=} \{g_2(x), g_3(x)\} \tag{3.48}$$

It is easy to see that the distribution $\Delta_2(x) \stackrel{def}{=} span\{g_2(x), g_3(x)\}$ is involutive, and the distribution $\Delta_1(x)$, as spanned only by a single vector field g_1, is also involutive. Hence, the codistributions $\Delta_1^{\perp}$

and $\Delta_2^\perp$ are (at least locally) spanned by exact differentials

$$\begin{aligned}\Delta_1^\perp(x) &= span\{d\lambda_{2,1}(x), d\lambda_{2,2}\}\\ \Delta_2^\perp(x) &= span\{d\lambda_{1,1}(x)\}\end{aligned}$$

The choice of the scalar functions $\lambda_{i,k}$ is immediate :

$$\lambda_{1,1}(x) = x_1, \quad \lambda_{2,1}(x) \stackrel{def}{=} x_2, \quad \lambda_{2,2}(x) \stackrel{def}{=} x_3 \qquad \text{for all } x \in I\!R^3 \tag{3.49}$$

The guiding functions are hence defined globally :

$$V_1(x) \stackrel{def}{=} \frac{1}{2}x_1^2, \quad V_2(x) \stackrel{def}{=} \frac{1}{2}(x_2^2 + x_3^2), \quad x \in I\!R^3, \tag{3.50}$$

and

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2), \quad x \in I\!R^3 \tag{3.51}$$

is clearly positive definite in $I\!R^3$.

Brockett's system

Consider the famous system:

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2, \qquad x \stackrel{def}{=} [x_1, x_2, x_3]^T \in I\!R^3 \tag{3.52}$$

$$\begin{aligned}g_1(x) &\stackrel{def}{=} \frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_3}\\ g_2(x) &\stackrel{def}{=} \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3}\end{aligned}$$

with

$$g_3(x) \stackrel{def}{=} [g_1, g_2](x) = 2\frac{\partial}{\partial x_3}$$

Hence

$$span\{g_1(x), g_2(x), g_3(x)\} = I\!R^3 \quad \text{for all } x \in I\!R^3 \tag{3.53}$$

In this case, two different group divisions can be considered :

$$\mathcal{G}_1'(x) \stackrel{def}{=} \{g_1(x)\}, \quad \mathcal{G}_2'(x) \stackrel{def}{=} \{g_2(x), g_3(x)\} \tag{3.54}$$

$$\mathcal{G}_1(x) \stackrel{def}{=} \{g_2(x)\}, \quad \mathcal{G}_2(x) \stackrel{def}{=} \{g_1(x), g_3(x)\} \tag{3.55}$$

and an easy caculation shows that either of the distributions $\Delta_2'(x) \stackrel{def}{=} \{g_2(x), g_3(x)\}$ or $\Delta_2(x) \stackrel{def}{=} \{g_1(x), g_3(x)\}$, is involutive. Thus, there are two possible choices of the guiding functions generated by the following selections of scalar functions :

$$\begin{cases} \lambda_{1,1}'(x) = x_1 \\ \lambda_{2,1}'(x) = x_1x_2 + x_3 \\ \lambda_{2,2}'(x) = x_2 \end{cases} \qquad \begin{cases} \lambda_{1,1}(x) = x_2 \\ \lambda_{2,1}(x) = x_1x_2 - x_3 \\ \lambda_{2,2}(x) = x_1 \end{cases} \qquad x \in \mathbb{R}^3 \tag{3.56}$$

Note that the following are globally valid :

$$\begin{array}{ll} d\lambda_{1,1}' \perp span\{g_2, g_3\} & d\lambda_{1,1} \perp span\{g_1, g_3\} \\ span\{d\lambda_{2,1}', d\lambda_{2,2}'\} \perp g_1 & span\{d\lambda_{2,1}, d\lambda_{2,2}\} \perp g_2 \end{array} \qquad x \in \mathbb{R}^3 \tag{3.57}$$

The two sets of guiding functions are then also globally defined :

$$\begin{cases} V_1'(x) \stackrel{def}{=} \frac{1}{2}x_1^2 \\ V_2'(x) \stackrel{def}{=} \frac{1}{2}[x_2^2 + (x_1x_2 + x_3)^2] \end{cases} \qquad \begin{cases} V_1(x) \stackrel{def}{=} \frac{1}{2}x_2^2 \\ V_2(x) \stackrel{def}{=} \frac{1}{2}[x_1^2 + (x_1x_2 - x_3)^2] \end{cases} \tag{3.58}$$

for all $x \in \mathbb{R}^3$, and prove to be equally effective for stabilization purposes.

4. The stabilizing control strategy and its convergence analysis

As demonstrated by Lemmas 3.1-3.4, and Propositions 3.1-3.3, the constructed guiding functions possess the desired properties, which easily suggests an algorithmic feedback strategy for the solution of the SPC.

Before we can state it formally, we first recall the definitions of the sets : $\mathcal{T}_{J_1}$, $\mathcal{T}_{J_2}$, and $\mathcal{T}_{lb}$, defined in (3.29) and (3.30), and notice that for any given constant $\epsilon > 0$, and level set V^r, there exists a constant $\rho > 0$ such that if $x \in \mathcal{T}_{J_1}(\rho) \cap \mathcal{T}_{J_2}(\rho) \cap \mathcal{T}_{lb}(\rho) \cap V^r$ then $x \in B(0;\epsilon)$. To see this, note that since V^r is compact, and V is strictly positive definite and analytic (see A1(a)), there exists a constant $\gamma_3 > 0$ and an integer $q \geq 1$, (possibly dependent on V^r) such that $V(x) \geq \gamma_3||x||^{2q}$, for all $x \in V^{3r}$. Let γ_1 be the constant of Lemma 3.2 corresponding to the compact level set V^{3r} and suppose that the constant $\rho > 0$ is selected to satisfy

$$\rho \leq min\{(\frac{2}{n}\gamma_1\gamma_3)^{\frac{1}{2}}\epsilon^q, (\frac{2r\gamma_1}{m})^{\frac{1}{2}}\} \stackrel{def}{=} \rho_{min} \tag{3.59}$$

It then follows from Lemma 3.2 that, whenever $x \in V^{3r}$, then

$$\begin{aligned} ||x||^2 &\leq (\frac{V(x)}{\gamma_3})^{\frac{1}{q}} \leq (\frac{\bar{V}(x)}{\gamma_1\gamma_3})^{\frac{1}{q}} \\ &\leq (\frac{n\rho^2}{2\gamma_3\gamma_1})^{\frac{1}{q}} \leq \epsilon^2 \end{aligned} \tag{3.60}$$

because, by assumption, $\bar{V}(x) \leq (n/2)\rho^2$, when $x \in \mathcal{T}_{J_1}(\rho) \cap \mathcal{T}_{J_2}(\rho) \cap \mathcal{T}_{lb}(\rho)$.

Assumming that ρ, r and ϵ are related by (3.59), the controls $v^J(\cdot,\rho)$ and $u^{(i)}(\cdot,\rho)$, for $i \in \underline{m}$, are defined as in (3.31) and (3.41), respectively, the set $\mathcal{B}_{k,i}(\rho)$ is defined as in Lemma 3.4, $t \mapsto x(t)$, for $t \geq 0$, denotes the trajectory of the controlled system, and for any subset $J \subset \underline{m}$, the set $\mathcal{T}_J$ is defined by (3.28), the stabilizing strategy is stated as follows.

Stabilizing feedback control strategy:

- Data: $r > 0$, $x(0) \in V^r$, and $\rho \leq \rho_{min}$.
  Until $x(t) \in \mathcal{T}_{J_1}(\rho) \cap \mathcal{T}_{J_2}(\rho) \cap \mathcal{T}_{lb}(\rho)$ repeat the following :
- 1 Until $x(t) \in \mathcal{T}_{J_1}(\rho) \cap \mathcal{T}_{J_2}(\rho)$, employ the control $v^J(x,\rho)$, in which $J = \{1,...,m\}$.
- 2 Find a pair of indices $(k,i) \in \mathcal{J}$ such that $g_i \in \mathcal{G}_2$, and $|L_{[g_k,g_i]}V_2(x(t))| > \rho$, and perform steps 2a-2b :
  - 2a Employ the control $u^{(i)}(x,\rho)$ until $x(t) \in \mathcal{B}_{k,i}(\rho)\}$, or else until $x(t)$ reaches the boundary of the level set V^{3r}. Set $\delta \overset{def}{=} |L_{g_i}V_2(x(t_a))|$ in which t_a is the time at the exit of this step.
  - 2b For $J = \{i\}$, and until $x(t) \in \mathcal{T}_J(\delta/2)$, employ the control $v^J(x,\delta/2)$.

Before proceeding with the convergence analysis it is helpful to explain how a skillful application of this strategy can result in constructing a dead-beat control for the unicycle example .

Stabilizing control for the unicycle

For this example, the assumptions of Lemma 3.3 are satisfied permitting exact steering to the set $\mathcal{T}_J(0)$, $J = \{1,2\}$. For an arbitrary initial configuration of the unicycle $x(0)$ at time $t = 0$, and the parameter ρ taken to be zero (with the set $\mathcal{B}_{k,i}(0) \overset{def}{=} \{x \in I\!R^n : |L_{[g_k,g_i]}V_2(x)| = 0\}$), the strategy results in the following control actions.

In step 1, the controls $v_1^J(x,0) = -sign(x_1)$ and $v_2^J(x,0) = -sign[x_2cos(x_1) + x_3sin(x_1)]$ are used to steer the system to the set $\mathcal{T}_{J_1}(0) \cap \mathcal{T}_{J_2}(0) = \{x \in I\!R^3 : x_1 = 0,\ x_2cos(x_1) + x_3sin(x_1) = 0\}$ $= \{x \in I\!R^3 : x_1 = x_2 = 0\}$, in finite time. Since there is only one Lie bracket in the group $\mathcal{G}_2$, and the control u_1 cannot change the values of neither x_2 nor x_3, then the controls of step 2a are : $u_1^2(x,0) = sign(-x_2 sin(x_1) + x_3 cos(x_1))\ sign^+(x_2 cos(x_1) + x_3 sin(x_1)) = sign(x_3 cos(x_1))$ $sign^+(x_3 sin(x_1))$, and $u_2^2(x,0) = 0$, which, provided that the constant r is chosen large enough, can be employed until $L_{[g_1,g_2]}V_2(x) = -x_2 sin(x_1) + x_3 cos(x_1) = 0$. At this point $x \in \mathcal{T}_{lb}(0)$ and this occurs when the system reaches a point at which $x_1 = \pm\frac{\pi}{2}$ (recall that, at the entrance to step 2a, $x_1 = x_2 = 0$ and that u_1^2 can only influence the value of x_1). Step 2b is hence entered with $x_1 = \frac{\pi}{2}$, or with $x_1 = -\frac{\pi}{2}$, and $x_2 = 0$, so that $L_{g_2}V_2(x) = x_3 \neq 0$. The controls $v_1^{\{2\}}(x,0) = 0$ and $v_2^{\{2\}}(x,0) = -sign(x_2 cos(x_1) + x_3 sin(x_1))$ (the components of the control $v^J(x,0)$ with $J = \{2\}$) thus decrease V_2 while maintaining $x_1 = \frac{\pi}{2}$ or $x_1 = -\frac{\pi}{2}$. Since

$$\begin{aligned} &L_{g_2}L_{[g_1,g_2]}V_2(x) = L_{g_2}(-x_2 sin(x_1) + x_3 cos(x_1)) \\ &= \lceil -x_2 cos(x_1) - x_3 sin(x_1), -sin(x_1), cos(x_1)\rfloor^T \lceil 0, cos(x_1), sin(x_1)\rfloor = 0 \end{aligned} \tag{3.61}$$

the value of $L_{[g_1,g_2]}V_2(x)$ stays unchanged (and equal to zero) over the duration of step 2b. At the end of this step the system reaches a point at which $L_{g_2}V_2(x) = x_2 cos(x_1) + x_3 sin(x_1) = 0$ again. Since $L_{g_1}V_2(x) \equiv 0$, by construction, the latter implies that $V_2(x) = 0$ at the end of step 2b. Finally, in step 1, the controls $v_1^J(x,0) = -sign(x_1)$ and $v_2^J(x,0) = 0$ restore V_1 to zero, while maintaining V_2 at zero. At this point , $V(x) = 0$, demonstrating that any "parking maneuver" of the unicycle can be realized by the above strategy in 5 steps.

A quantitative analysis of the decrements in V_2 yields the final stabilization result.

THEOREM 3.1. *Under assumptions A1-A4, for any constant $\epsilon > 0$ and any desired set of attraction V^r, the stabilization feedback control strategy is well defined in that each of its steps is feasible, and is exited in finite time. Any trajectory of the controlled system, emanating from a point in V^r reaches the ball $B(0;\epsilon)$ in finite time.*

Proof. Steps 1, and 2b, of the strategy are feasible and are exited in finite time as guaranteed by Proposition 3.2. Step 2a of the strategy is of finite duration, by virtue of Lemma 3.4. Hence the overall strategy is well defined.

If follows from Proposition 3.2 (b) and Lemma 3.4 that V increases only in Step 2a. Hence the controlled trajectory $t \mapsto x(t)$ remains in the level set V^{3r} for all times $t \geq 0$, and, by virtue of (3.60), a trajectory emanating from V^r reaches $B(0;\epsilon)$ if it reaches the set $\mathcal{T}_{J_1}(\rho) \cap \mathcal{T}_{J_2}(\rho) \cap \mathcal{T}_{lb}(\rho)$. It is thus enough to show that the strategy is exited in a finite number of steps.

Suppose, contrary to what needs to be shown, that $x(t)$ never reaches $\mathcal{T}_{J_1}(\rho) \cap \mathcal{T}_{J_2}(\rho) \cap \mathcal{T}_{lb}(\rho)$. By virtue of the result of Proposition 3.2 , at the end of step 1, $x(t) \in \mathcal{T}_{J_1}(\rho) \cap \mathcal{T}_{J_2}(\rho)$ and thus, at the entrance to step 2a, $x(t)$ is never a member of $\mathcal{T}_{lb}(\rho)$. As a result, there always exists an index pair $(k, i) \in \mathcal{J}$ such that $|L_{[g_k, g_i]} V_2(x(t))| > \rho$, at the entrance of this step. From Lemma 3.2 it then follows that

$$V_2(x(t)) \geq \frac{1}{\gamma_2} \bar{V}_2(x(t)) \geq \frac{\rho^2}{\gamma_2} \tag{3.62}$$

for all times $t \geq 0$, where the constant γ_2 corresponds to the level set V^{3r}. However, the guiding function V_2 decreases in step 2b and stays constant under the control actions of the remaining steps of the strategy. The decrease of V_2 in step 2b can be estimated as follows. Along the controlled trajectory with the control of step 2b

$$\frac{d}{dt} V_2(x(t)) = -|L_{g_i} V_2(x(t))|, \quad t \geq t_a \tag{3.63}$$

where, initially, $|L_{g_i} V_2(x(t_a))| = \delta > 0$. Let

$$c_1 \stackrel{def}{=} max\{|L^2_{g_i} V_2(x)| \mid x \in V^{3r}, i \in \underline{m}\} \tag{3.64}$$

which provides an upper bound for the speed of change of the Lie derivative $L_{g_i} V_2$ under the action of the control $v^J(x, \delta/2)$ with $J = \{i\}$. It thus follows that

$$|L_{g_i} V_2(x(t))| \geq \delta - c_1(t - t_a), \qquad \text{for all } t \in [t_a, t_*] \tag{3.65}$$

where $t_* - t_a$ is the minimal time in which the controlled system trajectory reaches the set $\mathcal{T}_{\{i\}}(\delta/2)$, so that $|L_{g_i} V_2(x(t^*))| = \delta/2$. From (3.65)

$$t_* - t_a = \frac{\delta}{2c_1} \tag{3.66}$$

Equations (3.63) and (3.65) imply that

$$\frac{d}{dt} V_2(x(t)) \leq -\delta + c_1(t - t_a) \tag{3.67}$$

and, consequently, that

$$V_2(x(t_*)) - V_2(x(t_a)) \leq -\delta(t_* - t_a) + \frac{c_1}{2}(t_* - t_a)^2 \leq -\frac{3}{8c_1}\,\delta^2 \tag{3.68}$$

Since t_* is the smallest time at which the step 2b can be exited, equation (3.68) gives an estimate for the decrease in V_2 in this step in terms of δ - the value of $|L_{g_i} V_2|$ at its entrance.

Next, we will show that δ is bounded from below (if $x(t)$ never reaches $\mathcal{T}_{lb}(\rho)$). By definition, δ is always greater than the increment in $|L_{g_i}V_2|$ in step 2a. Step 2a can be exited if either of the three situations occur :

(a) $x(t_a) \in \partial V^{3r}$

(b) $x(t_a) \in \partial \mathcal{T}_{Ib}(\rho/2)$

(c) $x(t_a) \in \partial \mathcal{T}_{J_2}(2\rho)$

in which ∂S denotes the boundary of any given set S. Case (a) is similar to case (b) in that the increment in the magnitude of the Lie derivative $|L_{g_i}V_2|$ can be estimated from the time needed to execute step 2a. Let this time be denoted by T_a, and T_b, for cases (a) and (b), respectively. Let t_1 and t_2 denote the times at the entrance and at the exit of step 2a, respectively. Then, by construction, $x(t_1) \in \mathcal{T}_{J_1}(\rho) \cap \mathcal{T}_{J_2}(\rho)$, and, as V_2 never increases, then

$$\begin{aligned} V(x(t_1)) &\leq V_2(x(t_1)) + \frac{1}{\gamma_1}\bar{V}_1(x(t_1)) \\ &\leq r + \frac{\rho^2 m}{2\gamma_1} \end{aligned} \tag{3.69}$$

In case (a), $V(x(t_2)) = 3r$, and since $\rho \leq \rho_{min}$ and, in particular, $\rho \leq (2\gamma_1 r/m)^{\frac{1}{2}}$, the total increase in V in step 2a is estimated as

$$V(x(t_2)) - V(x(t_1)) \geq 3r - r - \frac{\rho^2 m}{2\gamma_1} \geq r \tag{3.70}$$

It follows that the time T_a can be estimated from below as the minimal time needed for the system to reach the boundary of V^{3r} from the boundary of V^{2r}. By definition of the control $u^{(i)}(\cdot, \rho)$, the value function V_2 stays constant during this transition, and $|u_k^{(i)}(x, \rho)| = 1$, so the speed of the change in V, or equivalently in V_1 is limited by the value of the Lie derivative $|L_{g_k}V_1(x(t))|$, because

$$\frac{d}{dt}V_1(x(t)) = L_{g_k}V_1(x(t))u_k^{(i)}(x(t), \rho) \tag{3.71}$$

along any trajectory of the system using control $u^{(i)}(\cdot, \rho)$. Hence, if

$$c_2 \stackrel{def}{=} max\{|L_{g_k}V_1(x)| \mid x \in V^{3r},\ k \in \underline{m}\} \tag{3.72}$$

then $T_a \geq r/c_2$.

In case (b), the time T_b can be estimated as the shortest time needed for the Lie derivative $|L_{[g_k,g_i]}V_2|$ to decrease by the value of $\rho/2$. As the system trajectory remains for all times in the level set V^{3r}, and $u_k^{(i)}$ is the only nonzero component of $u^{(i)}$, then speed of change in this Lie derivative is again limited by the largest value of $|L_{g_k}L_{[g_k,g_i]}V_2(x(t))|$ along $x(t) \in V^{3r}$. Since $g_k \in \mathcal{G}_1$, then by virtue of the Jacobi equality, this speed is limited by

$$c_3 \stackrel{def}{=} max\{|L_{[g_k,[g_k,g_i]]}V_2(x)| \mid x \in V^{3r},\ k, i \in \underline{m}\} \tag{3.73}$$

Therefore, $T_b \geq \rho/(2c_3)$. A lower bound for the time of execution of step 2a in cases (a) and (b) is hence given by:

$$T_{ab} \stackrel{def}{=} min\{\frac{r}{c_2}, \frac{\rho}{2c_3}\} \tag{3.74}$$

A lower bound δ_{ab} for the increase in $|L_{g_i}V_2|$ in step 2b, in the case when step 2a is exited in situations (a) and (b), can now be obtained from equations (3.42) through (3.43)-(3.44) of the proof of Lemma 3.4, by which

$$\delta \geq \delta_{ab} = \frac{\rho T_{ab}}{2} \tag{3.75}$$

Case (c) is straightforward because, by definition of the set $\mathcal{T}_{J_2}$, and the fact that at the exit of step 1, $x(t) \in \mathcal{T}_{J_2}(\rho)$, the increment in $|L_{g_i}V_2|$ is at least ρ. Therefore, in all the cases, the increase in the Lie derivative $|L_{g_i}V_2|$ is bounded from below by δ_* :

$$\delta \geq \delta_* \stackrel{def}{=} min\{\delta_{ab}, \rho\} \tag{3.76}$$

Recalling equation (3.68), the minimal decrement in V_2 in Step 2b is therefore bounded from below by $3\delta_*^2/(8c_1) > 0$. It follows inevitably that, after a finite number of repetitions of step 2b, $V_2 < \rho^2/\gamma_2$ which contradicts (3.62) and completes the proof. ∎

REMARK 3.3. As was pointed out using the example of the unicycle, in certain cases, and provided that there are no disturbances nor model error, the stabilizing feedback strategy can be adjusted to produce trajectories which pass through the origin in a finite time, thus acting as a dead-beat stabilizing controller.

5. Applications of the feedback strategy

In this section several examples are provided iluminating different features of the guiding functions strategy and explaining its applicability in non-standard situations such as when the controllability Lie algebra contains Lie brackets of order higher than one. Each example is preceded by a brief motivation.

The strategy presented in section 4 is formulated with refrence to general systems and takes no account of any specific properties of these systems nor of their particular algebraic structures. In most individual cases, however, it is possible to introduce straightforward modifications to Step 1 of the strategy which consequently lead to dead-beat control. For example, such "intelligent" application of the strategy is possible when :

(i) the assumptions of Lemma 3.3 are satisfied

(ii) there exists a subset $\{x_{i_j}\}$, $j \in \underline{m}$, of state variables (of cardinality equal to the number of control variables) such that each x_{i_j} can be changed only by a single control u_j while the remaining controls u_i, $i \neq j$, have no effect on $\{x_{i_j}\}$.

In case of (i), Lemma 3.3 indicates that sequential activation of the components of the control v^J of Step 1 of the strategy is possible and allows for the selection of $\rho = 0$.

Similarly, in case (ii) introduction of the hysteresis loop in the definition of v^J is also redundant as the individual control u_j, $j \in \underline{m}$, can be switched off sequentially as the corresponding state variables x_{i_j} achieve zero values.

Indeed, all the examples discussed below fall into either of the categories (i) or (ii) permitting the selection of $\rho = 0$. Thus in all the considered cases the controllers exhibit the dead beat property.

5.1. Stabilizing feedback control for a model of an underwater vehicle (all controls available) [64, 69]

The model of an underwater vehicle is standard in that it satisfies all the assumptions of section 2 of this Chapter. This example also illustrates, how to use the strategy in a typical situation, in particular how to construct the guiding functions when the system under consideration is defined on a manifold.

A kinematic model of an underwater vehicle, as described in [**81**], involves six configuration variables and four inputs (velocities), of which three are the angular velocity components, and the fourth represents the forward velocity of the vehicle. If the velocity vector of the vehicle is constrained so that only its forward component can be nonzero, the vehicle exhibits nonholonomic behaviour, for details see [**26**]. Feedback control of the autonomous underwater vehicle with this type of nonholonomic constraint was previously studied in ([**81**], [**26**], [**55**]). In [**118**], Yoerger and Slotine applied sliding modes to trajectory control of such a vehicle.

In the derivation of the model of the underwater vehicle, two frames of reference are considered, as shown in Figure 3.2. The $O - XYZ$ is the inertial frame, while the local frame, $C - xyz$, is attached to the vehicle at its centre of mass C, with the x axis pointing along the OZ direction,

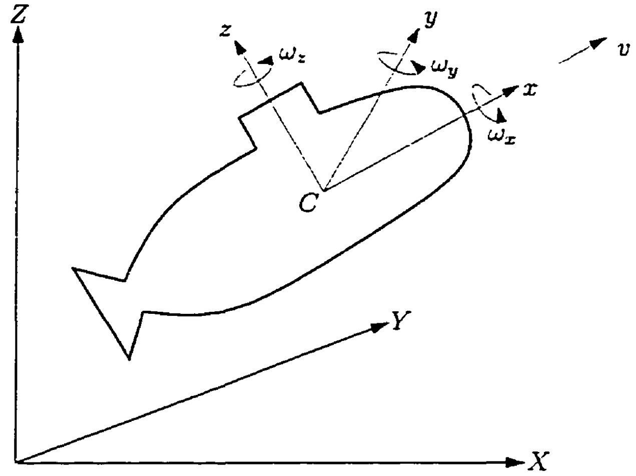

FIGURE 3.2. Model of an underwater vehicle

when the vehicle is in the upright position. Six coordinates are used to describe the motion; three to specify the position of the centre of mass, described by coordinates (x, y, z) and three to describe the orientation. The $Z - Y - X$ Euler angles are denoted by (ϕ, θ, ψ). When the angles are small, ϕ corresponds to what is commonly called the roll motion, while θ and ψ correspond to the pich and yaw motions, respectively.

As in [**81**], it is assumed that the vehicle is moving with velocity v, whose direction is the $C - x$ axis in the local frame, so the components of this velocity along the x, y, and z axes are given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} v \cos\psi \cos\theta \\ v \sin\psi \cos\theta \\ -v \sin\theta \end{pmatrix} \tag{3.77}$$

The relation between the time rate of the Euler angles and the angular velocity in the local frame, $\omega = (\omega_x, \omega_y, \omega_z)^T$, is given by, [**81**],

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi \tan\theta & \cos\phi \tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi \sec\theta & \cos\phi \sec\theta \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \tag{3.78}$$

Combining equations (3.77) and (3.78), and introducing a new set of state and control variables : $(x_1, x_2, x_3, x_4, x_5, x_6) = (x, y, z, \phi, \theta, \psi)$ and $(u_1, u_2, u_3, u_4) = (v, \omega_x, \omega_y, \omega_z)$, yields the kinematic model for the underwater vehicle.

Model 1:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} cos\ x_6\ cos\ x_5 \\ sin\ x_6\ cos\ x_5 \\ -sin\ x_5 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ sin\ x_4\ tan\ x_5 \\ cos\ x_4 \\ sin\ x_4\ sec\ x_5 \end{bmatrix} u_3 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ cos\ x_4\ tan\ x_5 \\ -sin\ x_4 \\ cos\ x_4\ sec\ x_5 \end{bmatrix} u_4$$

$$\stackrel{def}{=} g_1(x)\ u_1 + g_2(x)\ u_2 + g_3(x)\ u_3 + g_4(x)\ u_4 \tag{3.79}$$

$$\begin{aligned} \text{where,}\quad g_1(x) &= cos\ x_6\ cos\ x_5\ \frac{\partial}{\partial x_1} + sin\ x_6\ cos\ x_5\ \frac{\partial}{\partial x_2} - sin\ x_5\ \frac{\partial}{\partial x_3},\quad g_2(x) = \frac{\partial}{\partial x_4} \\ g_3(x) &= sin\ x_4\ tan\ x_5\ \frac{\partial}{\partial x_4} + cos\ x_4\ \frac{\partial}{\partial x_5} + sin\ x_4\ sec\ x_5\ \frac{\partial}{\partial x_6} \\ g_4(x) &= cos\ x_4\ tan\ x_5\ \frac{\partial}{\partial x_4} - sin\ x_4\ \frac{\partial}{\partial x_5} + cos\ x_4\ sec\ x_5\ \frac{\partial}{\partial x_6} \end{aligned}$$

We refer to the kinematic model of the underwater vehicle given by (3.79) as Model 1.

It is easy to see that g_1, g_2, g_3, g_4 are smooth as vector fields defined on the manifold $\mathcal{M}$:

$$\mathcal{M} = \{x \stackrel{def}{=} (x_1, x_2, x_3, x_4, x_5, x_6) \in I\!R^6\ :\ |x_5| < \pi/2\} \tag{3.80}$$

The solution to (3.79) exist for all times as long as the system trajectories remain in $\mathcal{M}$. The system defined by (3.79) is also completely controllable on the manifold $\mathcal{M}$ as it satisfies the LARC (Lie algebraic controllability rank condition) on $\mathcal{M}$. To see this, it is necessary to verify that the controllability Lie algebra, $L(g_1, g_2, g_3, g_4)$ for system (3.79), span $I\!R^6$ at each point $x \in \mathcal{M}$. An easy calculation shows that

$$span\{g_1(x), g_2(x), g_3(x), g_4(x), g_5(x), g_6(x)\} = I\!R^6 \quad \text{for } x \in \mathcal{M} \tag{3.81}$$

in which the vector fields g_5 and g_6 are given by :

$$\begin{aligned} g_5(x) \stackrel{def}{=} [g_1, g_3](x) &= (sin\ x_5\ cos\ x_6\ cos\ x_4 + sin\ x_6\ sin\ x_4)\frac{\partial}{\partial x_1} \\ &\quad + (sin\ x_5\ sin\ x_6\ cos\ x_4 - cos\ x_6\ sin\ x_4)\ \frac{\partial}{\partial x_2} + cos\ x_5\ cos\ x_4\ \frac{\partial}{\partial x_3} \\ g_6(x) \stackrel{def}{=} [g_1, g_4](x) &= (-sin\ x_5\ cos\ x_6\ sin\ x_4 + sin\ x_6\ cos\ x_4)\ \frac{\partial}{\partial x_1} \\ &\quad - (sin\ x_5\ sin\ x_6\ sin\ x_4 + cos\ x_6\ cos\ x_4)\ \frac{\partial}{\partial x_2} - cos\ x_5\ sin\ x_4\ \frac{\partial}{\partial x_3} \end{aligned}$$

The Lie brackets multiplication table for $L(g_1, g_2, g_3, g_4)$ is:

$$
\begin{array}{lll}
[g_1, g_2] = 0 & [g_2, g_3] = g_4 & [g_2, g_4] = -g_3 \\
[g_3, g_4] = g_2 & [g_1, g_5] = 0 & [g_1, g_6] = 0 \\
[g_2, g_5] = g_6 & [g_2, g_6] = -g_5 & [g_3, g_5] = g_1 \\
[g_3, g_6] = 0 & [g_4, g_5] = [g_4, g_6] = [g_5, g_6] = 0 &
\end{array}
\tag{3.82}
$$

which shows that the controllability Lie algebra $L(g_1, g_2, g_3, g_4)$ is finite dimensional but not nilpotent. The set $\mathcal{S}(x)$ is clearly defined by :

$$
\mathcal{S}(x) = \{g_1(x), g_2(x), g_3(x), g_4(x), [g_1, g_3](x), [g_1, g_4](x)\}, \quad x \in \mathbb{R}^6 \tag{3.83}
$$

and the groups $\mathcal{G}_1$ and $\mathcal{G}_2$ are easily formed while obeying the rules (R1) - (R2) :

$$
\begin{aligned}
\mathcal{G}_1(x) &= \{g_2, g_3, g_4\}(x), \quad x \in \mathbb{R}^6 \\
\mathcal{G}_2(x) &= \{g_1, [g_1, g_3], [g_1, g_4]\}(x), \quad x \in \mathbb{R}^6
\end{aligned}
$$

From Lie brackets multiplication table (3.82), it is clear that the distributions

$$
\begin{aligned}
\Delta_1(x) &\stackrel{def}{=} span\{g_2, g_3, g_4\}(x) \\
\Delta_2(x) &\stackrel{def}{=} span\{g_1, [g_1, g_3], [g_1, g_4]\}(x)
\end{aligned}
\tag{3.84}
$$

are involutive, and the corresponding codistributions have the following expressions as linear spans of exact differentials :

$$
\begin{aligned}
\Delta_1^{\perp}(x) &\stackrel{def}{=} span\{d\lambda_{2,1}(x), d\lambda_{2,2}(x), d\lambda_{2,3}(x)\}, \quad x \in \mathbb{R}^6 \\
\Delta_2^{\perp}(x) &\stackrel{def}{=} span\{d\lambda_{1,1}(x), d\lambda_{1,2}(x), d\lambda_{1,3}(x)\}, \quad x \in \mathbb{R}^6
\end{aligned}
\tag{3.85}
$$

where the choices for the scalar functions $\lambda_{i,j}$, are immediate, and are valid in the entire $\mathbb{R}^6$:

$$
\begin{array}{lll}
\lambda_{2,1}(x) \stackrel{def}{=} x_1, & \lambda_{2,2}(x) \stackrel{def}{=} x_2, & \lambda_{2,3}(x) \stackrel{def}{=} x_3 \\
\lambda_{1,1}(x) \stackrel{def}{=} x_4, & \lambda_{1,2}(x) \stackrel{def}{=} x_5, & \lambda_{1,3}(x) \stackrel{def}{=} x_6
\end{array}
$$

so that the mapping $x \mapsto [\lambda_1, ..., \lambda_6]$ is obviously a global diffeomorphism. The resulting guiding functions V_1, V_2, and their sum V, are hence defined for all $x \in \mathbb{R}^6$:

$$
\begin{aligned}
&V_1(x) \stackrel{def}{=} \frac{1}{2}(x_4^2 + x_5^2 + x_6^2), \quad V_2(x) \stackrel{def}{=} \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) \\
&V(x) = V_1(x) + V_2(x)
\end{aligned}
\tag{3.86}
$$

With the guiding functions constructed above, the feedback control strategy of this Chapter is employed to Model 1, for which it proves to be very effective. Simulation results are shown in Figures 3.3 - 3.6. Figure 3.3 shows the state trajectories $x_i(t)$, $i = 1, ..., 6$, of the controlled system corresponding to Model 1, and Figures 3.5 and 3.6 show the associated trajectories of the guiding functions V_1 and V_2 and their sum V. The results also confirm that the origin is achieved in finite time, and thus the constructed controller is dead beat.

It is easy to see that for Model 1, steering to the set $\mathcal{T}_{J_1}(0) \cap \mathcal{T}_{J_2}(0)$, where

$$\mathcal{T}_{J_1}(0) \cap \mathcal{T}_{J_2}(0) = \{x \in I\!R^6 : x_1 = x_4 = x_5 = x_6 = 0\} \tag{3.87}$$

can be realized in finite time by sequential application of the following controls:

- $u_2(x) \stackrel{def}{=} -sign(x_4)$, and $u_1(x) = u_3(x) = u_4(x) \equiv 0$ until $x_4 = 0$.
- $u_3(x) \stackrel{def}{=} -sign(x_5)$, and $u_1(x) = u_2(x) = u_4(x) \equiv 0$ until $x_5 = 0$.
- $u_4(x) \stackrel{def}{=} -sign(x_6)$, and $u_1(x) = u_2(x) = u_3(x) \equiv 0$ until $x_6 = 0$.
- $u_1(x) \stackrel{def}{=} -sign(x_1)$, and $u_2(x) = u_3(x) = u_4(x) \equiv 0$ until $x_1 = 0$.

Therefore, the insight gained by construction of the guiding functions construction leads to yet a simpler stabilizing strategy for Model 1. Employing this strategy in the absence of disturbances, the origin can be achieved exactly in 9 steps, regardless of the initial condition of the system. These steps are stated below.

Simplified strategy for Model 1:

- 1 Until $x_4 = 0$ employ the controls $u_2 = -sign(x_4)$ and $u_1 = u_3 = u_4 \equiv 0$
- 2 Until $x_5 = 0$ employ $u_3 = -sign(x_5)$ and $u_1 = u_2 = u_4 \equiv 0$
- 3 If $x_6 \leq \pi/6$, employ $u_4 = 1$ and $u_1 = u_2 = u_3 \equiv 0$ until $x_6 = \pi/6$.
- 4 Until $x_2 = 0$ employ $u_1 = -sign(L_{[g_1,g_4]}V(x)) = -sign(x_2)$ and $u_2 = u_3 = u_4 \equiv 0$
- 5 Until $x_6 = 0$ employ $u_4 = -sign(x_6)$ and $u_1 = u_2 = u_3 \equiv 0$
- 6 Until $x_5 = \pi/6$, employ $u_3 = 1$ and $u_1 = u_2 = u_4 \equiv 0$
- 7 Until $x_3 = 0$ employ $u_1 = -sign(L_{[g_1,g_3]}V(x)) = -sign(x_3)$ and $u_2 = u_3 = u_4 \equiv 0$
- 8 Until $x_5 = 0$ employ $u_3 = -sign(x_5)$ and $u_1 = u_3 = u_3 \equiv 0$
- 9 Until $x_1 = 0$ employ $u_1 = -sign(x_1)$ and $u_2 = u_3 = u_4 \equiv 0$

The above stabilization strategy is tested on Model 1 and the controlled trajectories are shown in Figure 3.7, while the plots of the variations of the corresponding guiding functions are depicted in Figures 3.9 and 3.10.

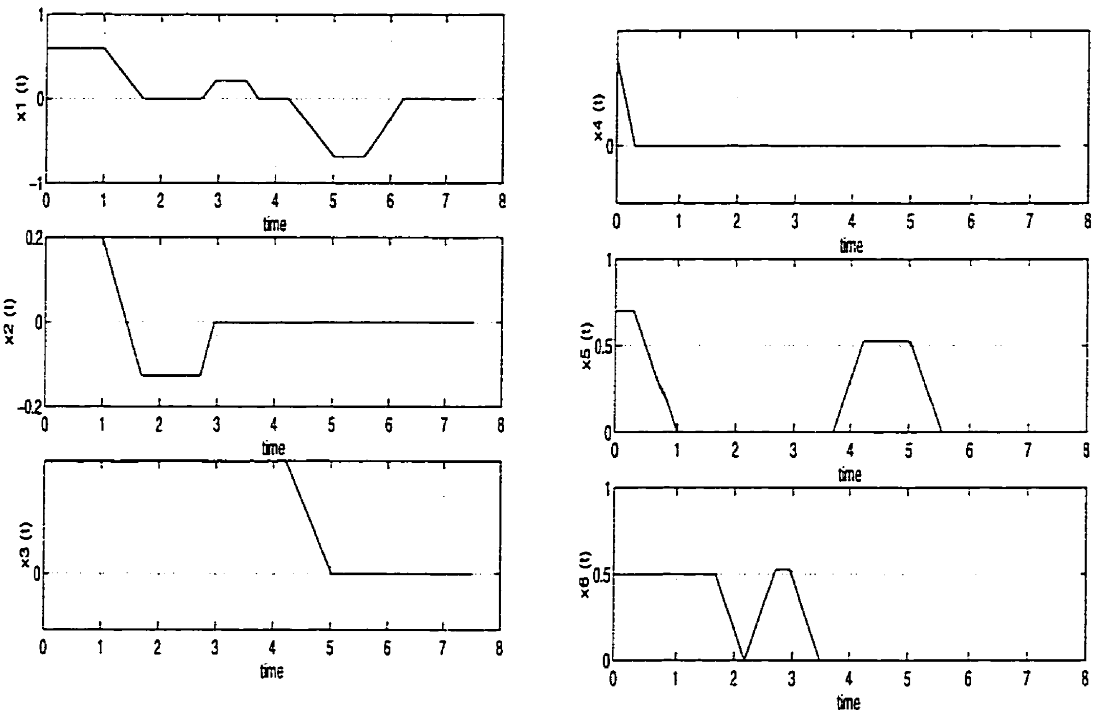

FIGURE 3.3. *Underwater vehicle Model 1:* Plots of the controlled state trajectories $t \mapsto (x_1(t), ..., x_6(t))$ versus time.

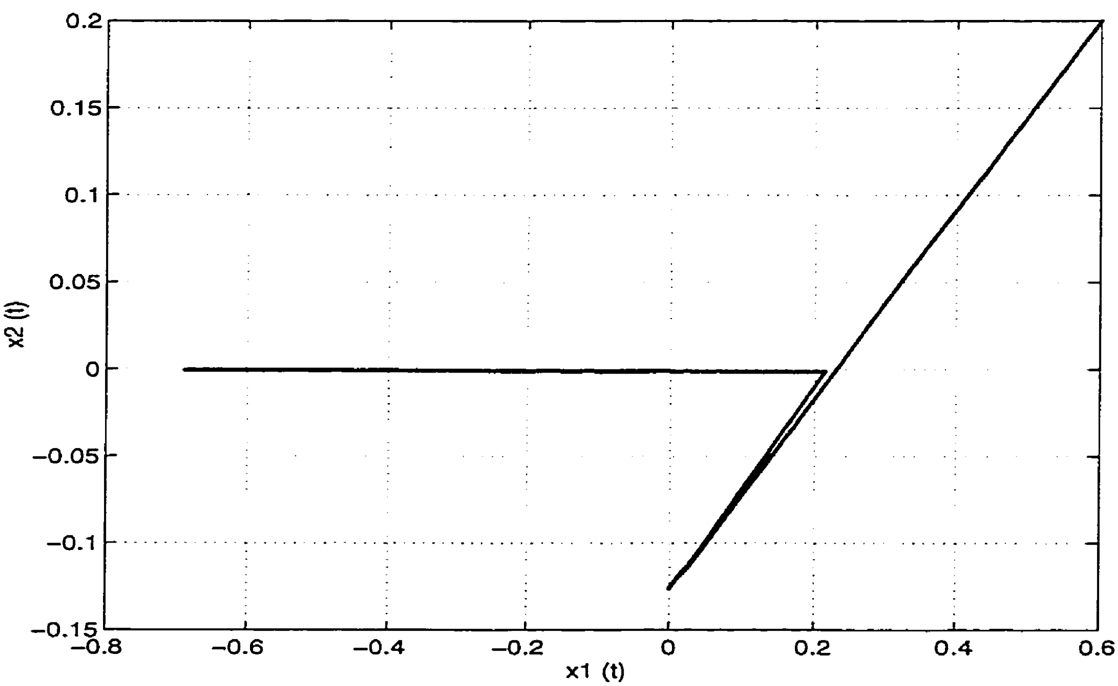

FIGURE 3.4. *Underwater vehicle Model 1:* Plot of the controlled state trajectory $x_1(t)$ versus $x_2(t)$.

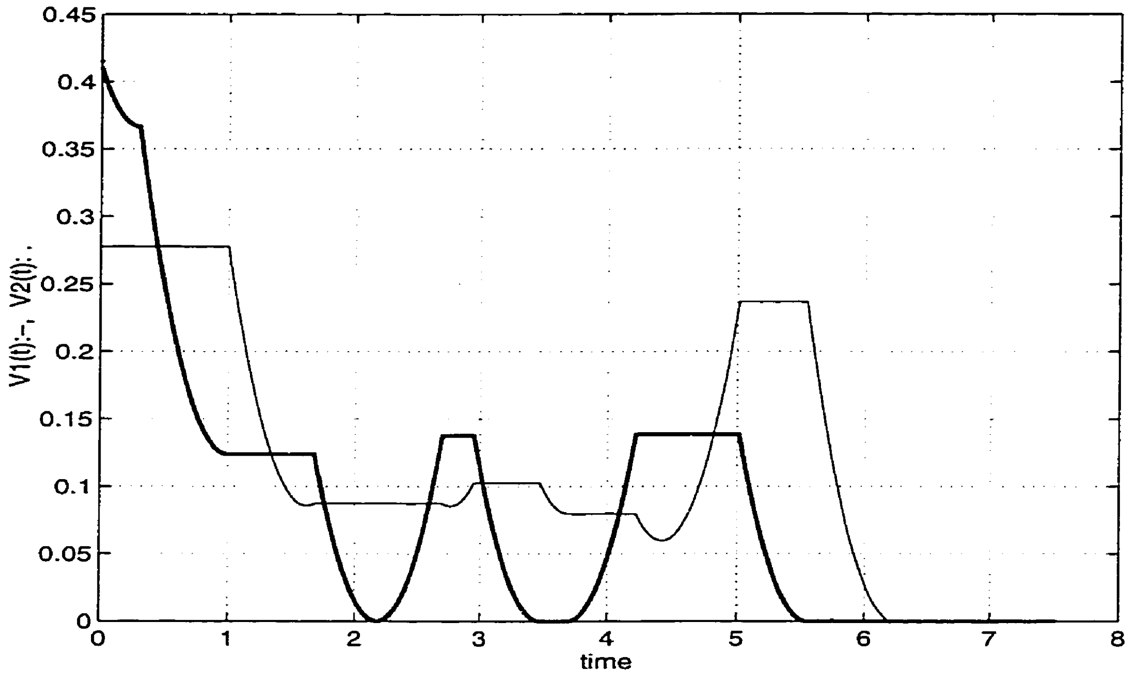

FIGURE 3.5. *Underwater vehicle Model 1:* Plots of the guiding functions $V_1(t)$ and $V_2(t)$ versus time.

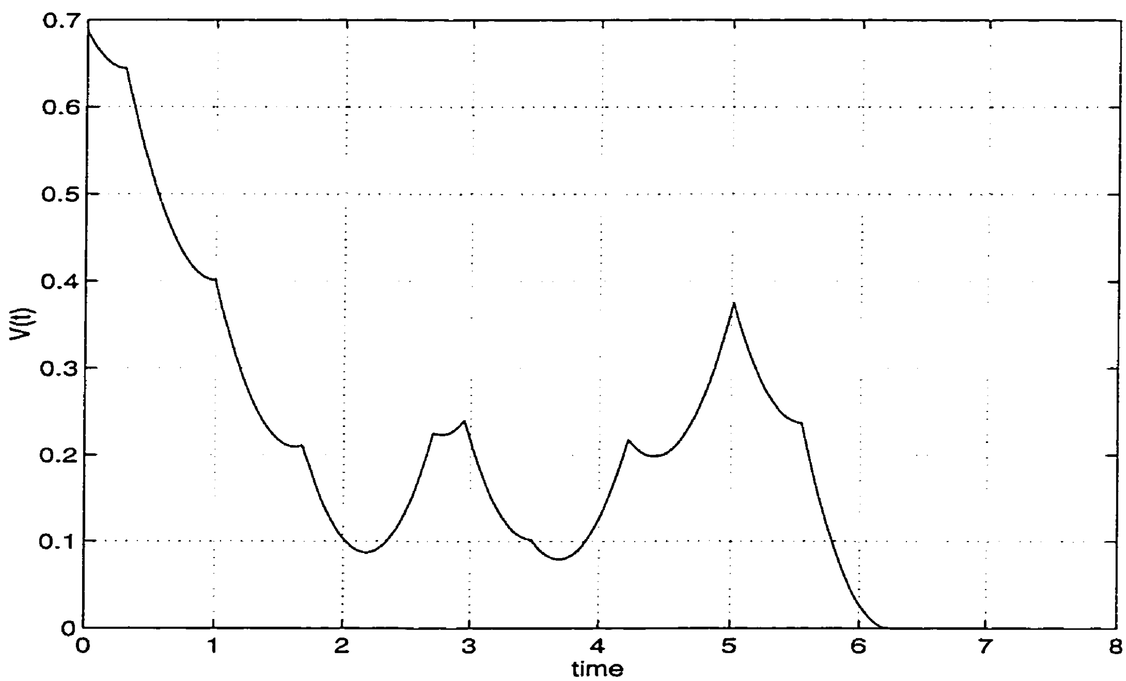

FIGURE 3.6. *Underwater vehicle Model 1:* Plot of the guiding function $V(t) = V_1(t) + V_2(t)$ versus time.

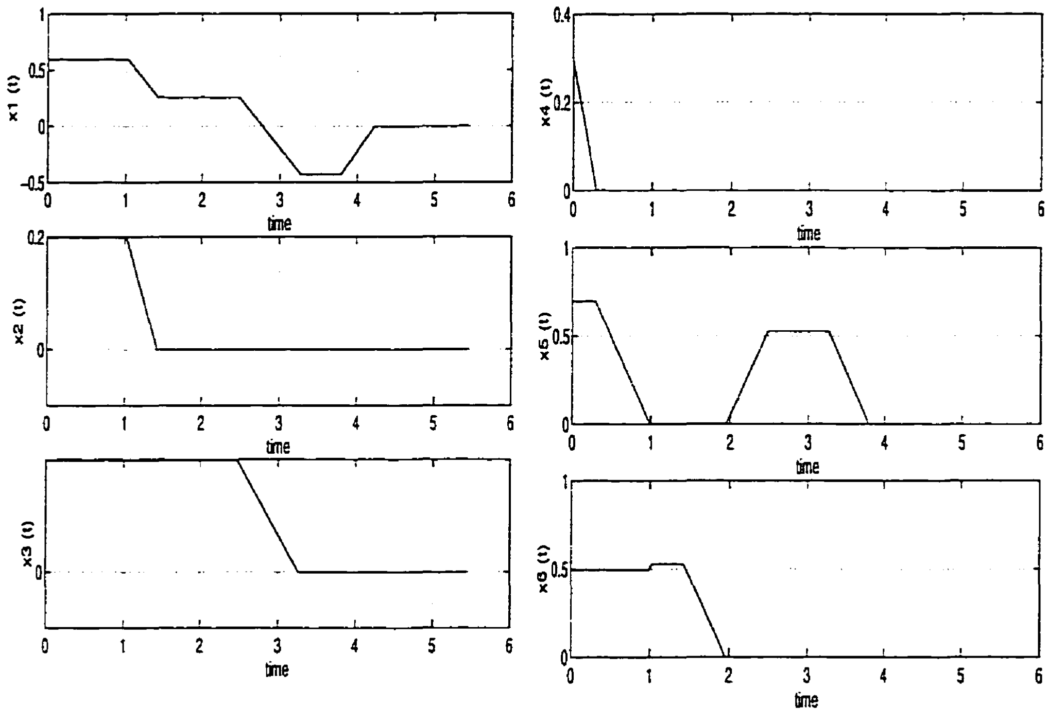

FIGURE 3.7. *Underwater vehicle Model 1:* Plots of the controlled state trajectories $t \mapsto (x_1(t), ..., x_6(t))$ versus time corresponding to simplified strategy.

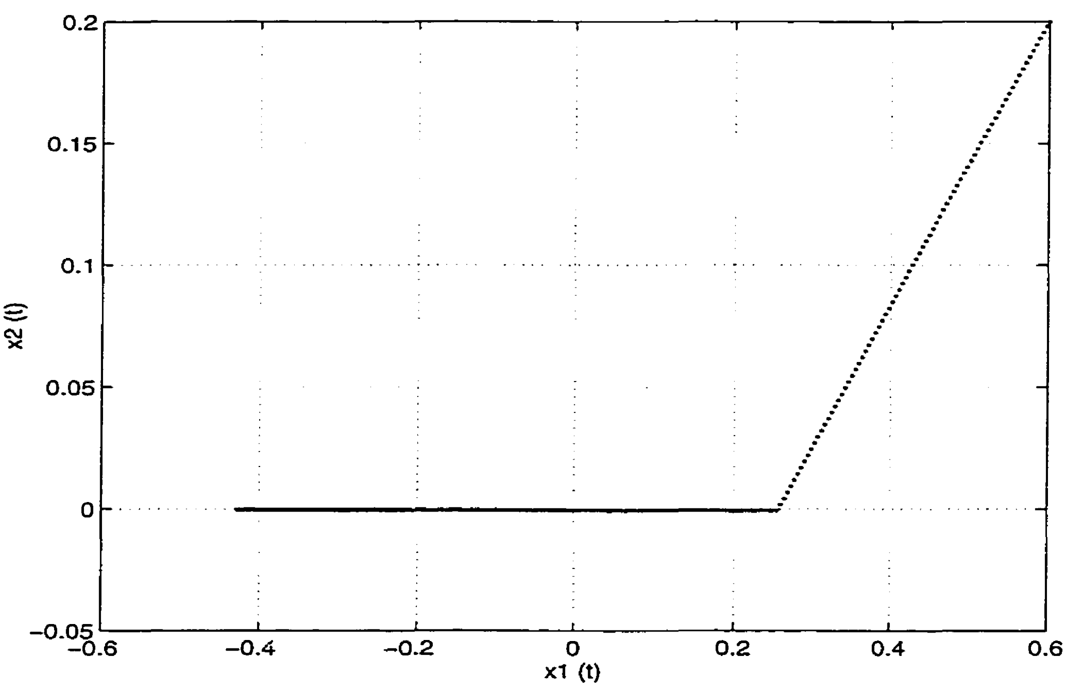

FIGURE 3.8. *Underwater vehicle Model 1:* Plot of the controlled state trajectory $x_1(t)$ versus $x_2(t)$ corresponding to simplified strategy.

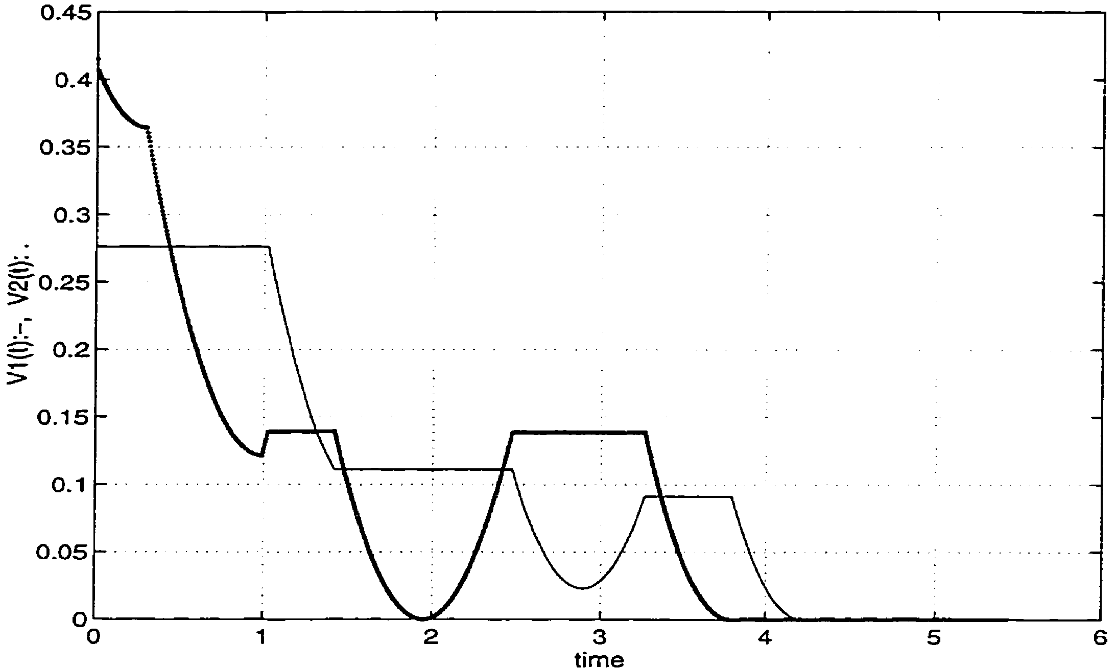

FIGURE 3.9. *Underwater vehicle Model 1:* Plots of the guiding functions $V_1(t)$ and $V_2(t)$ versus time corresponding to simplified strategy.

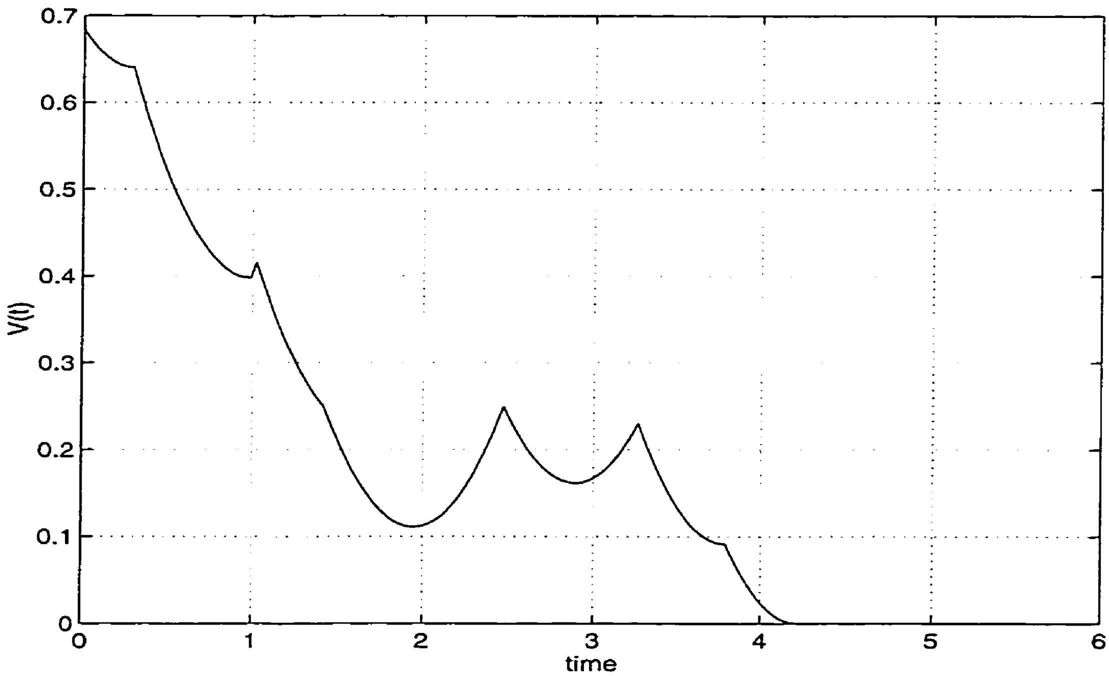

FIGURE 3.10. *Underwater vehicle Model 1:* Plot of the guiding function $V(t) = V_1(t) + V_2(t)$ versus time corresponding to simplified strategy.

5.2. Stabilizing feedback control for a model of drift free system with five state variables and three controls

The following example of a drift free system demonstrates that the guiding functions need not be simple quadratics of the coordinate variables. The equations of this system are:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -x_2 \\ 0 \\ x_4 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ x_1 \\ 0 \\ 0 \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u_3$$

$$\stackrel{def}{=} g_1(x)\ u_1 + g_2(x)\ u_2 + g_3(x)\ u_3 \tag{3.88}$$

$$\begin{aligned} \text{where,} \quad g_1(x) &= \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_5} \\ g_2(x) &= \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} \\ g_3(x) &= \frac{\partial}{\partial x_4} \end{aligned}$$

To satisfy the LARC condition, we need to calculate the following Lie brackets:

$$\begin{aligned} g_4(x) &\stackrel{def}{=} [g_1, g_2](x) = 2 \frac{\partial}{\partial x_3} \\ g_5(x) &\stackrel{def}{=} [g_1, g_3](x) = -\frac{\partial}{\partial x_5} \end{aligned}$$

which yields

$$span\{g_1(x), g_2(x), g_3(x), g_4(x), g_5(x)\} = I\!R^5, \qquad \text{for all} \quad x \in I\!R^5 \tag{3.89}$$

The set $\mathcal{S}$ can then be defined:

$$\mathcal{S} = \{g_1(x), g_2(x), g_3(x), g_4(x), g_5(x)\}, \qquad x \in I\!R^5 \tag{3.90}$$

The Lie brackets multiplication table for $L(g_1, g_2, g_3)$ is:

$$\begin{aligned} &[g_1, g_2] = g_4 \qquad && [g_1, g_3] = g_5 \\ &[g_2, g_3] = 0 && [g_4, g_5] = 0 \\ &[g_i, g_4] = [g_i, g_5] = 0, && i = 1, 2, 3 \end{aligned} \tag{3.91}$$

which shows that the controllability Lie algebra $L(g_1, g_2, g_3)$ is nilpotent and hence finite dimensional. By adhering to the rules R1-R2, we can find two groups $\mathcal{G}_1$ and $\mathcal{G}_2$ as follows:

$$\mathcal{G}_1(x) \stackrel{def}{=} \{g_1(x), g_4(x), g_5(x)\}$$

$$\mathcal{G}_2(x) \stackrel{def}{=} \{g_2(x), g_3(x)\}$$

The multiplication table (3.91) shows that the distributions:

$$\Delta_1(x) \stackrel{def}{=} span\{g_1(x), g_4(x), g_5(x)\}$$

$$\Delta_2(x) \stackrel{def}{=} span\{g_2(x), g_3(x)\}$$

are involutive, and hence, the corresponding codistributions $\Delta_1^{\perp}$ and $\Delta_2^{\perp}$ are:

$$\Delta_1^{\perp}(x) = span\{d\lambda_{2,1}(x), d\lambda_{2,2}(x)\}$$

$$\Delta_2^{\perp}(x) = span\{d\lambda_{1,1}(x), \lambda_{1,2}(x), d\lambda_{1,3}(x)\}$$

By using the Frobenius theorem, the scalar functions $\lambda_{i,k}$ are easily found:

$$\begin{aligned}
&\lambda_{1,1}(x) \stackrel{def}{=} x_1, \qquad \lambda_{1,2}(x) \stackrel{def}{=} x_5 \\
&\lambda_{1,3}(x) \stackrel{def}{=} (x_3 - x_1 x_2), \\
&\lambda_{2,1}(x) \stackrel{def}{=} x_2, \qquad \lambda_{2,2}(x) \stackrel{def}{=} x_4, \quad \text{for all} \quad x \in I\!R^5
\end{aligned}$$

Therefore, the guiding functions for this system are defined globally:

$$\begin{aligned}
V_1(x) &\stackrel{def}{=} \frac{1}{2}\{x_1^2 + x_5^2 + (x_3 - x_1 x_2)^2\}, \\
V_2(x) &\stackrel{def}{=} \frac{1}{2}\{x_2^2 + x_4^2\} \\
V(x) &= \frac{1}{2}\{x_1^2 + x_2^2 + (x_3 - x_1 x_2)^2 + x_4^2 + x_5^2\}, \quad x \in I\!R^5 \qquad (3.92)
\end{aligned}$$

The feedback strategy is then applied with conjunction of the guiding functions (3.92). The simulation results are shown in Figures 3.11-3.13 which conform the effectiveness of this strategy. Figure 3.11 shows that all state trajectories $x_i(t)$, $i = 1, ..., 5$, of the controlled system, reach the origin in finite time. Figures 3.12 and 3.13 show the associated trajectories of the guiding functions V_1 and V_2 and their sum V. The control is again essentially dead-beat.

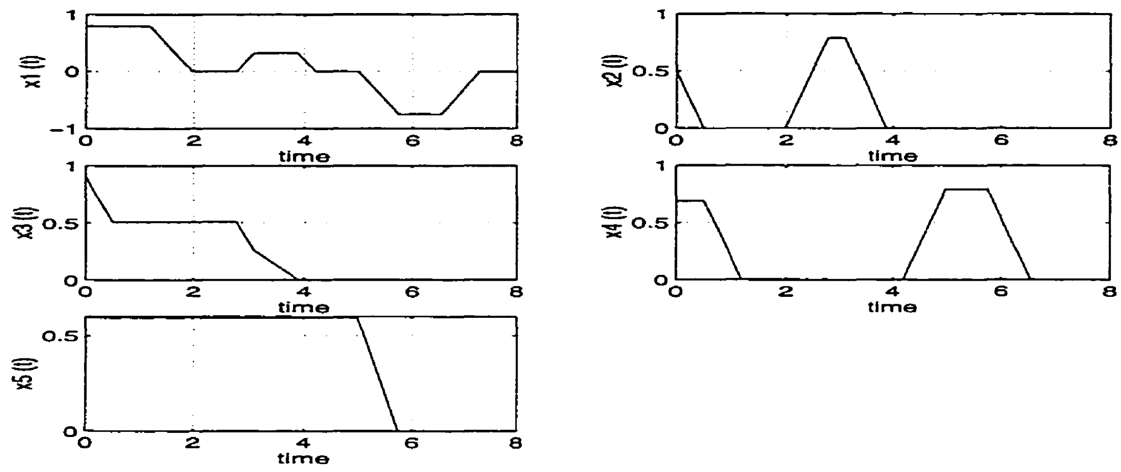

FIGURE 3.11. *General drift free system with $n-m=2$:* Plots of the controlled state trajectories $t \mapsto (x_1(t), ..., x_5(t))$ versus time.

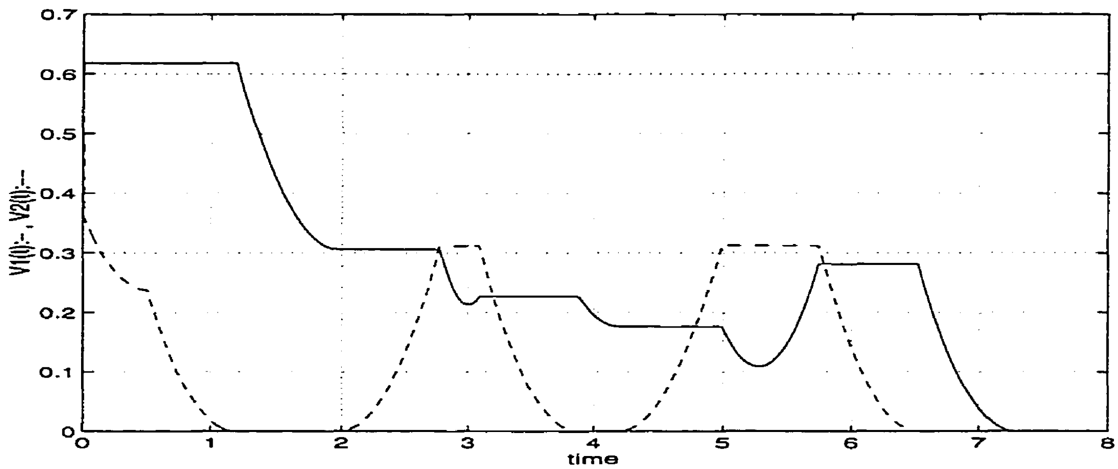

FIGURE 3.12. *General drift free system with $n-m=2$:* Plots of the guiding functions $V_1(t)$ and $V_2(t)$ v.s. time.

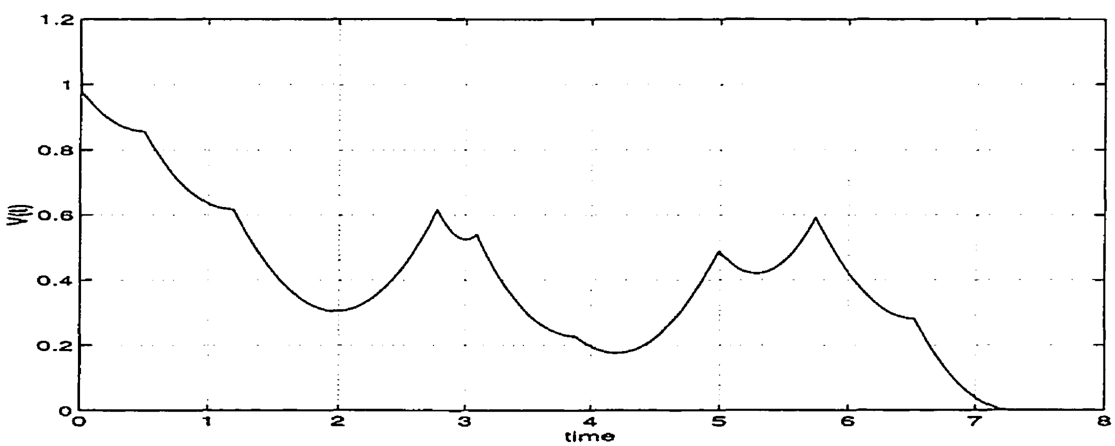

FIGURE 3.13. *General drift free system with $n-m=2$:* Plot of the guiding function $V(t) = V_1(t) + V_2(t)$ v.s. time.

5.3. Stabilizing feedback control for a model of a rigid spacecraft in actuator failure mode

This example illustrates that the guiding functions strategy developed in this Chapter, appears to be robust with respect to model error. There are some drift free systems which do not satisfy the assumption A2, so that, direct construction of guiding functions for such systems is not possible. Instead, an approximation technique can be utilized first, in which the original system is approximated (by using truncated Taylor series expansion at zero) by a model which preserves controllability and such that additionally, $g_i(0) = \tilde{g}_i(0)$, for any $g_i \in L(g_1, ..., g_m)$ and $\tilde{g}_i \in L(\tilde{g}_1, ..., \tilde{g}_m)$, where $L(g_1, ..., g_m)$ and $L(\tilde{g}_1, ..., \tilde{g}_m)$ are the controllability Lie algebras for the original and approximate systems, respectively. If the approximate system satisfies assumption A2, the guiding functions can be constructed for this approximate system and applied in the feedback control to the original system. In all the cases considered, simulations confirm that such feedback control is stabilizing for the original system, thus demonstrating a robustness property of the guiding functions strategy. A quantitative assessment of such robustness margin with respect to model error is beyond the scope of this thesis. Due to the approximation, the convergence of the controlled system trajectories to the origin is generally expected to be local. However, in most cases analysed, the region of attraction appears to be practically unlimited. This is going to be confirmed on the example of the hopping robot in section 7.3 of this Chapter.

The approach described above is applied to a model of a rigid spacecraft as given by, see [47]:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ \sin\theta \tan\phi & 1 & -\cos\theta \tan\phi \\ -\sin\theta \sec\phi & 0 & \cos\theta \sec\phi \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

where $[\phi, \theta, \psi]^T$ are Euler angles describing the orientation and $[\omega_1, \omega_2, \omega_3]^T$ is the angular velocity vector. Assuming that one of the rotation velocities, say ω_3, is constrained to be equal to zero, and by introducing a new set of state and control variables : $(x_1, x_2, x_3) = (\phi, \theta, \psi)$ and $(u_1, u_2) = (\omega_1, \omega_2)$, yields the following kinematic model for the spacecraft:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} \cos x_2 \\ \sin x_2 \tan x_1 \\ -\sin x_2 \sec x_1 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_2 \\ &\stackrel{def}{=} g_1(x)\, u_1 + g_2(x)\, u_2 \end{aligned} \tag{3.93}$$

where

$$g_1(x) = cos\ x_2\ \frac{\partial}{\partial x_1} + sin\ x_2\ tan\ x_1\ \frac{\partial}{\partial x_2} - sin\ x_2\ sec\ x_1\ \frac{\partial}{\partial x_3}, \qquad g_2(x) = \frac{\partial}{\partial x_2}$$

The Lie bracket of g_1 and g_2 is given by:

$$g_3(x) \overset{def}{=} [g_1, g_2](x) = sin\ x_2\ \frac{\partial}{\partial x_1} - cos\ x_2\ tan\ x_1 \frac{\partial}{\partial x_2} + cos\ x_2\ sec\ x_1\ \frac{\partial}{\partial x_3}$$

The kinematic model (3.93) satisfies the LARC condition if the motion of the system is constrained to the manifold:

$$\mathcal{M} = \{x \overset{def}{=} (x_1, x_2, x_3) \in I\!R^3\ :\ |x_1| < \pi/2\}$$

that is

$$span\{g_1(x), g_2(x), g_3(x)\} = I\!R^3,\ x \in \mathcal{M} \tag{3.94}$$

The Lie brackets multiplication table for $L(g_1,\ g_2)$ is given by:

$$[g_1, g_2] = g_3, \qquad [g_1, g_3] = -g_2, \qquad [g_2, g_3] = g_1 \tag{3.95}$$

which shows that the controllability Lie algebra $L(g_1,\ g_2)$ is finite dimensional, but not nilpotent. By consulting table (3.95), it can easily be verified that it is not possible to construct groups $\mathcal{G}_1$ and $\mathcal{G}_2$, which give rise to involutive distributions. It is thus impossible to construct guiding functions directly for system (3.93).

For this reason an approximation of (3.93) is considered in which the nonlinear terms in the vector field g_1 are substituted by their truncated (of order one) Taylor series expansions at zero. In doing so, $sinx \approx x$ and $cosx \approx 1$, which results in the following system:

$$\begin{aligned} \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \\ -x_2 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_2 \\ &\overset{def}{=} \tilde{g}_1(x)\ u_1 + \tilde{g}_2(x)\ u_2 \end{aligned} \tag{3.96}$$

where, $$\tilde{g}_1(x) = \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_3}, \qquad \tilde{g}_2(x) = \frac{\partial}{\partial x_2}$$

The Lie bracket of $\tilde{g}_1$ and $\tilde{g}_2$ is now given by:

$$\tilde{g}_3(x) \overset{def}{=} [\tilde{g}_1, \tilde{g}_2](x) = \frac{\partial}{\partial x_3}$$

The approximate model (3.96) also satisfies the LARC condition since :

$$span\{\bar{g}_1(x), \bar{g}_2(x), \bar{g}_3(x)\} = \mathbb{R}^3, \quad x \in \mathbb{R}^3 \tag{3.97}$$

and the Lie brackets multiplication table for $L(\bar{g}_1, \bar{g}_2)$:

$$[\bar{g}_1, \bar{g}_2] = \bar{g}_3, \quad [\bar{g}_1, \bar{g}_3] = 0, \quad [\bar{g}_2, \bar{g}_3] = 0 \tag{3.98}$$

shows that the Lie algebra $L(\bar{g}_1, \bar{g}_2)$ is nilpotent. By adhering to the rules R1-R2, the groups $\mathcal{G}_1$ and $\mathcal{G}_2$ can easily be defined for the approximate system (3.96) as follows:

$$\mathcal{G}_1(x) \stackrel{def}{=} \{\bar{g}_1(x), \bar{g}_3(x)\}, \quad \mathcal{G}_2(x) \stackrel{def}{=} \{\bar{g}_2(x)\} \tag{3.99}$$

From table (3.98), it is clear that the distributions:

$$\begin{aligned} \Delta_1(x) &\stackrel{def}{=} span\{\bar{g}_1(x), \bar{g}_3(x)\} \\ \Delta_2(x) &\stackrel{def}{=} span\{\bar{g}_2(x)\} \end{aligned} \tag{3.100}$$

are involutive and hence, the corresponding codistributions are:

$$\begin{aligned} \Delta_1^{\perp}(x) &= span\{d\lambda_{2,1}(x)\} \\ \Delta_2^{\perp}(x) &= span\{d\lambda_{1,1}(x), d\lambda_{1,2}(x)\} \end{aligned} \tag{3.101}$$

One choice of the scalar functions $\lambda_{i,k}$ is:

$$\lambda_{1,1}(x) \stackrel{def}{=} x_1, \quad \lambda_{1,2}(x) \stackrel{def}{=} x_3, \quad \lambda_{2,2}(x) \stackrel{def}{=} x_2, \qquad \text{for all } x \in \mathbb{R}^3 \tag{3.102}$$

which yield globally defined guiding functions:

$$\begin{aligned} V_1(x) &\stackrel{def}{=} \frac{1}{2}(x_1^2 + x_3^2), \quad x \in \mathbb{R}^3 \\ V_2(x) &\stackrel{def}{=} \frac{1}{2}(x_2^2), \quad x \in \mathbb{R}^3 \\ V(x) &= \frac{1}{2}(x_1^2 + x_2^2 + x_3^2), \quad x \in \mathbb{R}^3 \end{aligned} \tag{3.103}$$

The above guiding functions can next be incorporated into the stabilizing strategy which is applied to the original system (3.93). Simulation results conform that the strategy is robust with respect to model error in the sense that although the guiding functions are constructed with reference to an approximate model (3.96), yet they are generating stabilizing controls for the original system (3.93). In this case simulations also show that the set point is reached in a finite number of steps (in finite time). The simulated trajectories are depicted in Figures 3.14 - 3.16.

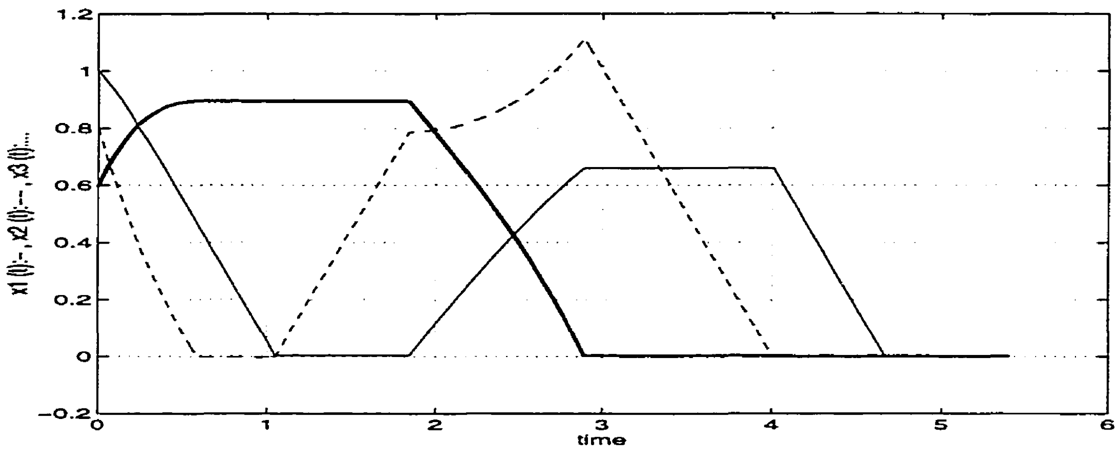

FIGURE 3.14. *Spacecraft model:* Plots of the controlled state trajectories $t \mapsto (x_1(t), x_2(t), x_3(t))$ versus time.

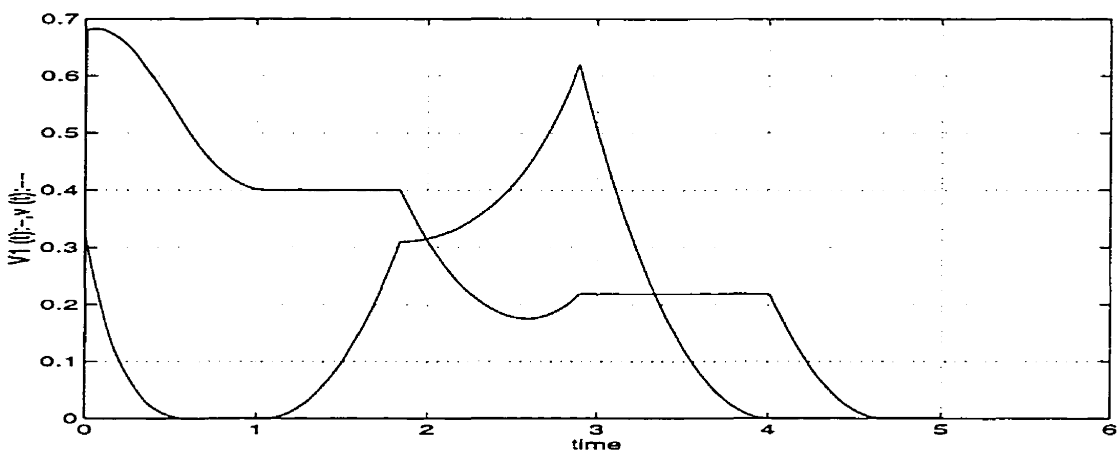

FIGURE 3.15. *Spacecraft model:* Plots of the guiding functions $V_1(t)$ and $V_2(t)$ versus time.

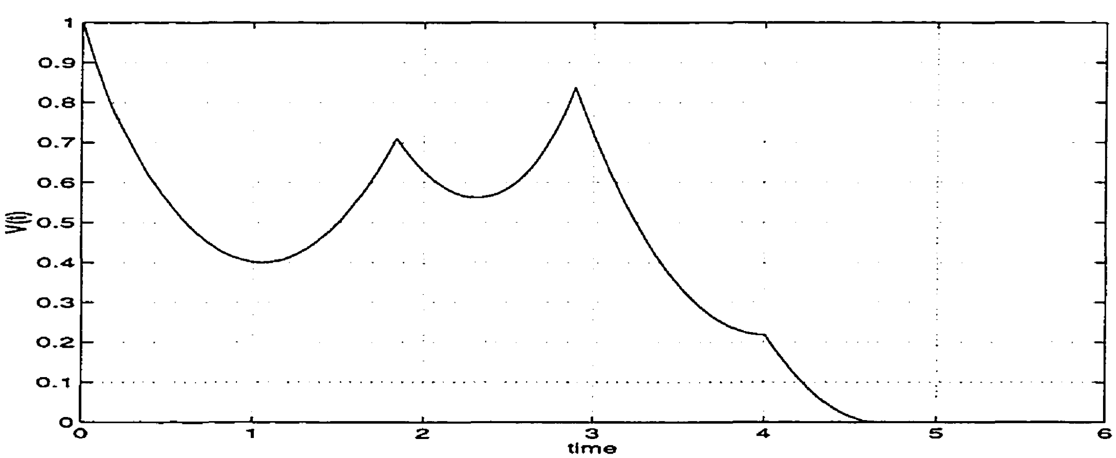

FIGURE 3.16. *Spacecraft model:* Plot of the guiding function $V(t) = V_1(t) + V_2(t)$ versus time.

6. Sinusoidal steering and guiding functions

In this section, the possibility of combining the guiding functions approach with sinusoidal steering of Murray and Sastry [77], and Tilbury et al. [109], as applied to systems whose controllability Lie algebra involves higher order Lie brackets, is demonstrated. It is well known, see Tilbury et al. [109] that the motion along the Lie bracket $[g_1, [g_1, [g_1, ..., [g_1, g_2], ...]]] = ad^k_{g_1} g_2$, of depth k can be generated by using the following sinusoidal controls:

$$\begin{aligned} u_1(t) &= k_1 \; sin(2\pi \frac{t}{T}) \\ u_2(t) &= k_2 \; cos(k \; 2\pi \frac{t}{T}) \end{aligned} \tag{3.104}$$

where k_1 and k_2 are some constants. By combining this idea with the guiding functions approach, stabilizing controllers are constructed for different types of drift free systems possessing controllability Lie algebras with higher order Lie brackets.

6.1. Stabilizing feedback control for a model of an underwater vehicle (in actuator failure mode) [64, 69]

The example below explains, how the idea of guiding functions can be combined with sinusoidal steering. It is hence shown that the guiding functions approach is not limited to the class of systems which satisfy assumptions A1-A4.

A model of an underwater vehicle (3.79) is considered in which the actuator corresponding to control u_4 fails to be operational. The model of the underwater vehicle with such reduced number of controls, is referred to as Model 2:

Model 2:

$$\begin{aligned} \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \\ \dot{x_5} \\ \dot{x_6} \end{bmatrix} &= \begin{bmatrix} cos\; x_6 \; cos\; x_5 \\ sin\; x_6 \; cos\; x_5 \\ -sin\; x_5 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ sin\; x_4 \; tan\; x_5 \\ cos\; x_4 \\ sin\; x_4 \; sec\; x_5 \end{bmatrix} u_3 \\ &\stackrel{def}{=} g_1(x)\; u_1 + g_2(x)\; u_2 + g_3(x)\; u_3 \end{aligned} \tag{3.105}$$

To verify complete controllability of (3.105), we need the following Lie brackets:

$$
\begin{aligned}
g_4(x) \overset{def}{=} [g_2, g_3](x) &= (cos\ x_4\ tan\ x_5)\ \frac{\partial}{\partial x_4} - (sin\ x_4)\ \frac{\partial}{\partial x_5} + (cos\ x_4\ sec\ x_5)\ \frac{\partial}{\partial x_6} \\
g_5(x) \overset{def}{=} [g_1, g_3](x) &= (sin\ x_5\ cos\ x_6\ cos\ x_4 + sin\ x_6\ sin\ x_4)\ \frac{\partial}{\partial x_1} \\
&+ (sin\ x_5\ sin\ x_6\ cos\ x_4 - cos\ x_6\ sin\ x_4)\ \frac{\partial}{\partial x_2} \\
&\quad +cos\ x_5\ cos\ x_4\ \frac{\partial}{\partial x_3} \\
g_6(x) \overset{def}{=} [g_1, [g_2, g_3]](x) &= (-sin\ x_5\ cos\ x_6\ sin\ x_4 + sin\ x_6\ cos\ x_4)\ \frac{\partial}{\partial x_1} \\
&- (sin\ x_5\ sin\ x_6\ sin\ x_4 + cos\ x_6\ cos\ x_4)\ \frac{\partial}{\partial x_2} \\
&\quad -cos\ x_5\ sin\ x_4\ \frac{\partial}{\partial x_3}
\end{aligned}
$$

which satisfy the LARC condition:

$$span\{g_1(x), g_2(x), g_3(x), g_4(x), g_5(x), g_6(x)\} = I\!R^6 \quad \text{for } x \in \mathcal{M} \tag{3.106}$$

The Lie brackets multiplication table for $L(g_1, g_2, g_3)$ is the same as given in (3.82). The set corresponding to $\mathcal{S}(x)$, can thus be defined by

$$\mathcal{S}(x) = \{g_1(x), g_2(x), g_3(x), [g_1, g_3](x), [g_2, g_3](x), [g_1, [g_2, g_3]](x)\}, \qquad x \in I\!R^6 \tag{3.107}$$

and contains Lie brackets of depth one as well as a Lie bracket of depth two. The stabilization strategy and the associated guiding functions construction must hence be modified to take account of this complication. An immediate remedy for this situation comes to mind and relies on substituting the original system by its extension of the form:

$$\dot{x} = g_1(x)\ v_1 + g_2(x)\ v_2 + g_3(x)\ v_3 + [g_2, g_3](x)\ v_4 \tag{3.108}$$

The control v_4 is clearly not accessible but, assuming that the motion of the real system along the Lie bracket direction $[g_2, g_3]$ can be realized, at least approximately, through controls v_2, v_3, in an indirect way, and over a finite interval of time, allows the introduction of the vector field groups $\mathcal{G}_1$ and $\mathcal{G}_2$ for the extended system (3.108):

$$\mathcal{G}_1(x) = \{g_2, g_3, [g_2, g_3]\}(x), \qquad x \in I\!R^6$$

$$\mathcal{G}_2(x) = \{g_1, [g_1, g_3], [g_1, [g_2, g_3]]\}(x), \quad x \in I\!R^6$$

as if the motion of the real system in the direction $[g_2, g_3]$ was instantaneously feasible. By using the Lie brackets multiplication table (3.82), it can be easily seen that the following distributions

$$\Delta_1(x) \stackrel{def}{=} span\{g_2, g_3, [g_2, g_3]\}(x)$$
$$\Delta_2(x) \stackrel{def}{=} span\{g_1, [g_1, g_3], [g_1, [g_2, g_3]]\}(x)$$

are involutive, and the corresponding codistributions have the following expressions as linear spans of exact differentials, valid in the entire $\mathbb{R}^6$:

$$\Delta_1^{\perp}(x) \stackrel{def}{=} span\{d\lambda_{2,1}(x), d\lambda_{2,2}(x), d\lambda_{2,3}(x)\}, \quad x \in \mathbb{R}^6$$
$$\Delta_2^{\perp}(x) \stackrel{def}{=} span\{d\lambda_{1,1}(x), d\lambda_{1,2}(x), d\lambda_{1,3}(x)\}, \quad x \in \mathbb{R}^6$$

where the choices for the scalar functions $\lambda_{i,j}$, are identical as for Model 1 :

$$\lambda_{2,1}(x) \stackrel{def}{=} x_1, \quad \lambda_{2,2}(x) \stackrel{def}{=} x_2, \quad \lambda_{2,3}(x) \stackrel{def}{=} x_3,$$
$$\lambda_{1,1}(x) \stackrel{def}{=} x_4, \quad \lambda_{1,2}(x) \stackrel{def}{=} x_5, \quad \lambda_{1,3}(x) \stackrel{def}{=} x_6.$$

The corresponding guiding functions are defined for all $x \in \mathbb{R}^6$ and given by (3.86). The motion along the Lie bracket $[g_2, g_3]$ is realized indirectly by using the following standard controls:

$$\begin{aligned} u_2(t) &= sin(2\pi\frac{t}{T}) \\ u_3(t) &= cos(2\pi\frac{t}{T}), \quad \text{while } u_1 = 0 \end{aligned} \tag{3.109}$$

where T is a positive constant (the value $T = 1$ was used in simulations). The trajectories $t \mapsto x_i(t)$, $i = 1, ..., 6$, of the controlled system incorporating Model 2 are shown in Figure 3.17, while the corresponding plots of the guiding functions V_1, V_2, and V are depicted in Figures 3.20 and 3.21. Also in this case, the control is essentially dead-beat.

6.2. Stabilizing feedback control for a model of a fire truck

The example below demonstrates that the combination of the guiding functions strategy with sinusoidal steering is also robust with respect to model error. Such robustness property is important in cases when the extended system (see section 6.1) fails to satisfy assumption A2, hence disallowing direct construction of the guiding functions. Proceeding similarly as in section 5.3, an approximation of the extended system is sought. If such approximate extended system satisfies assumption A2, the guiding functions can be constructed with reference to this approximation. The latter can later be used in a combined strategy and applied to the original system. Several simulations, see sections

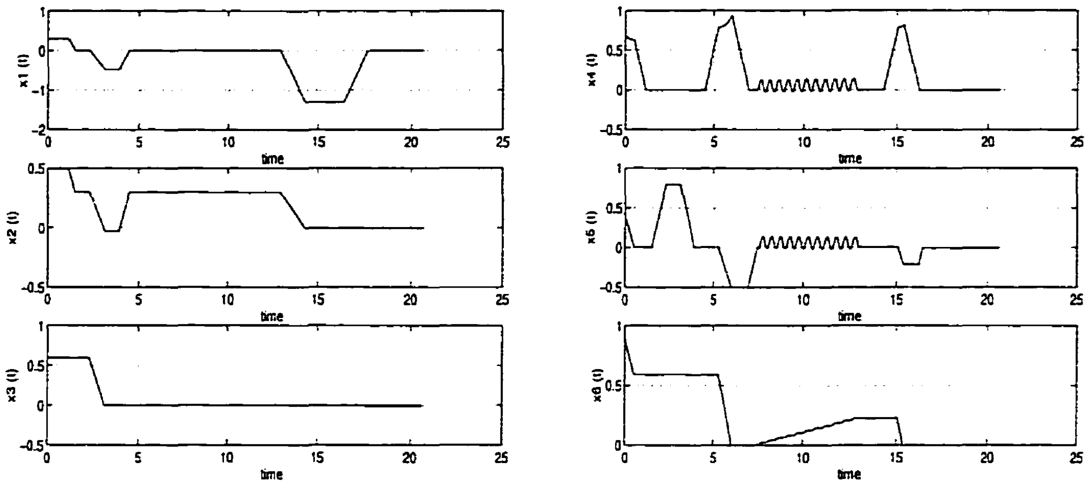

FIGURE 3.17. *Underwater vehicle Model 2:* Plots of the controlled state trajectories $t \mapsto (x_1(t), ..., x_6(t))$ versus time.

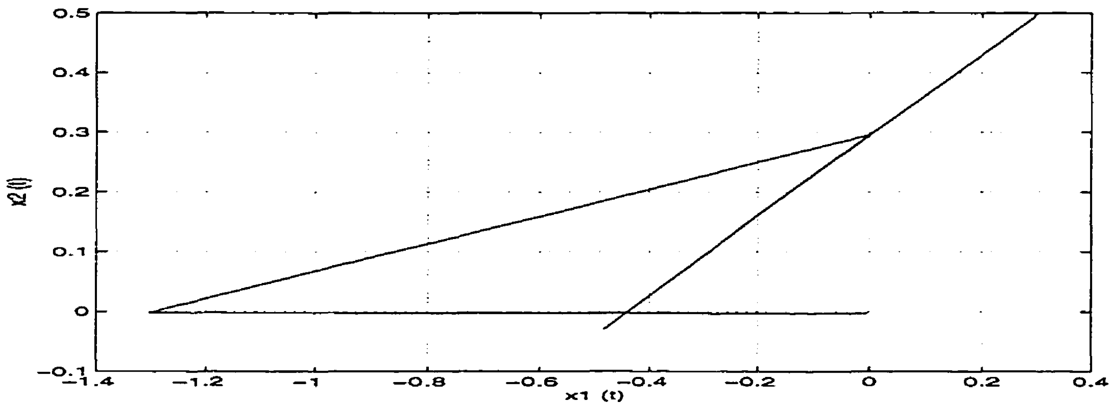

FIGURE 3.18. *Underwater vehicle Model 2:* Plot of the controlled state trajectory $x_1(t)$ versus $x_2(t)$.

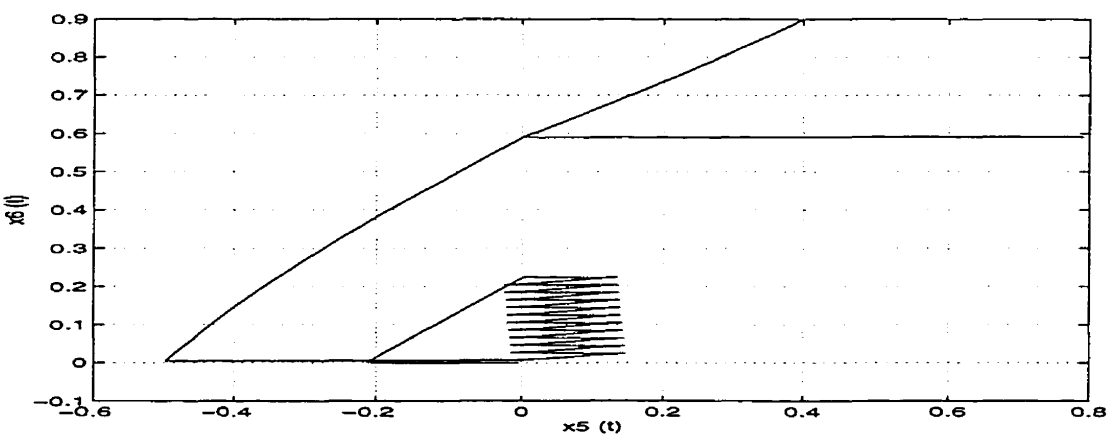

FIGURE 3.19. *Underwater vehicle Model 2:* Plot of the controlled state trajectory $x_5(t)$ versus $x_6(t)$.

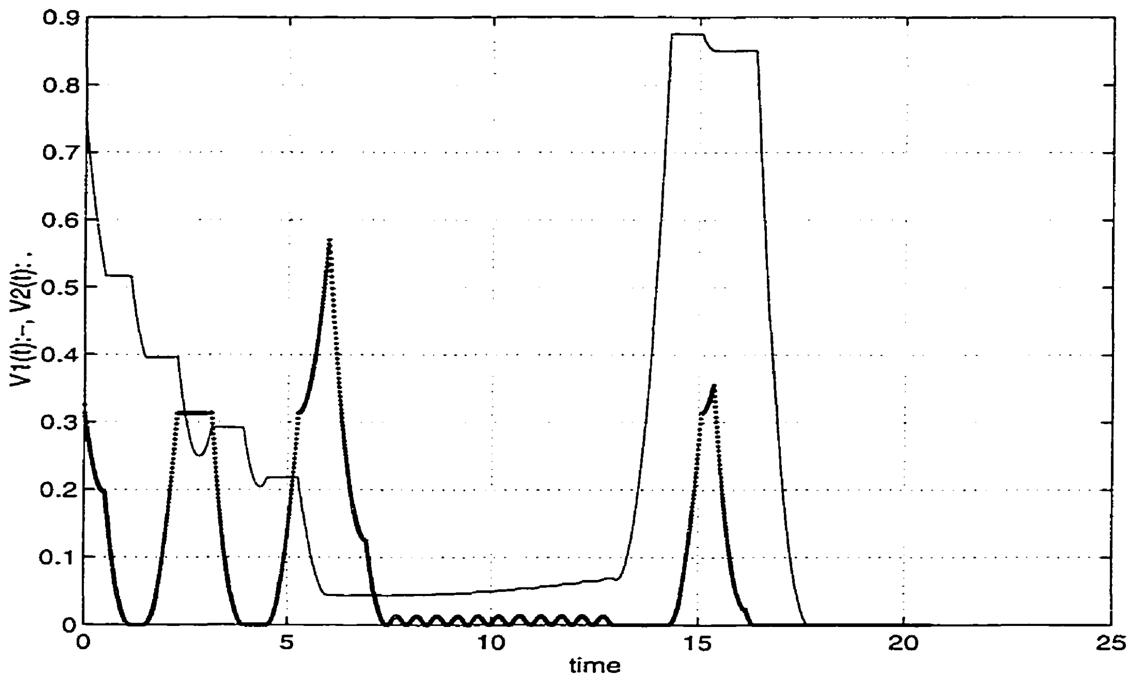

FIGURE 3.20. *Underwater vehicle Model 2:* Plots of the guiding functions $V_1(t)$ and $V_2(t)$ versus time.

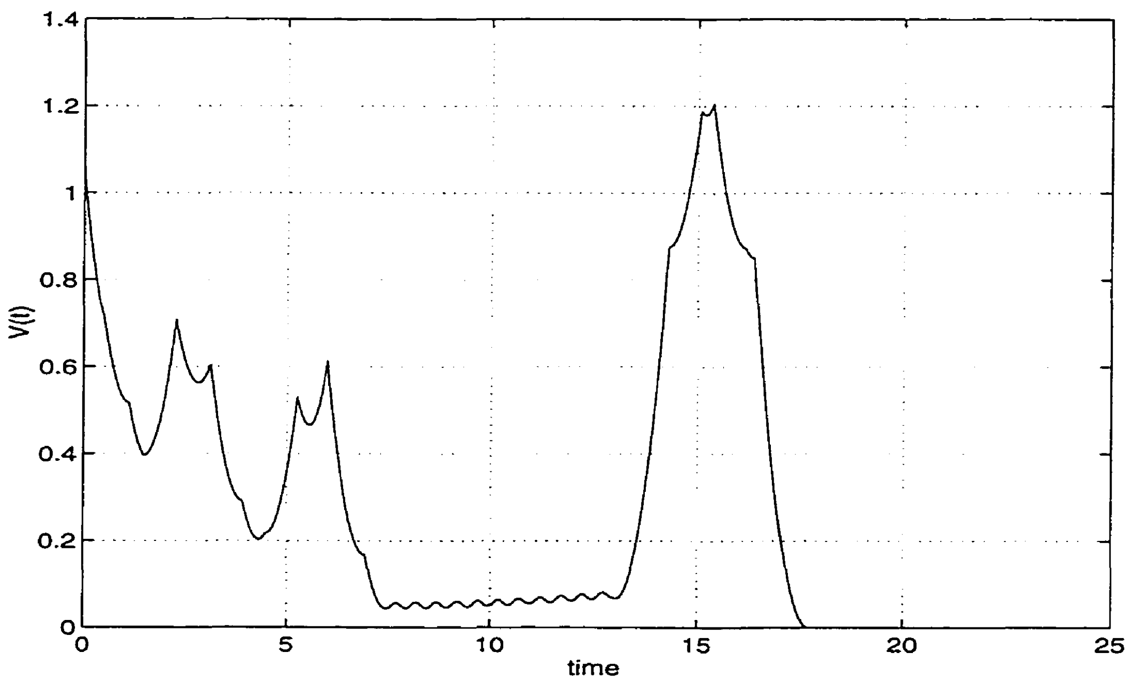

FIGURE 3.21. *Underwater vehicle Model 2:* Plot of the guiding function $V(t) = V_1(t) + V_2(t)$ versus time.

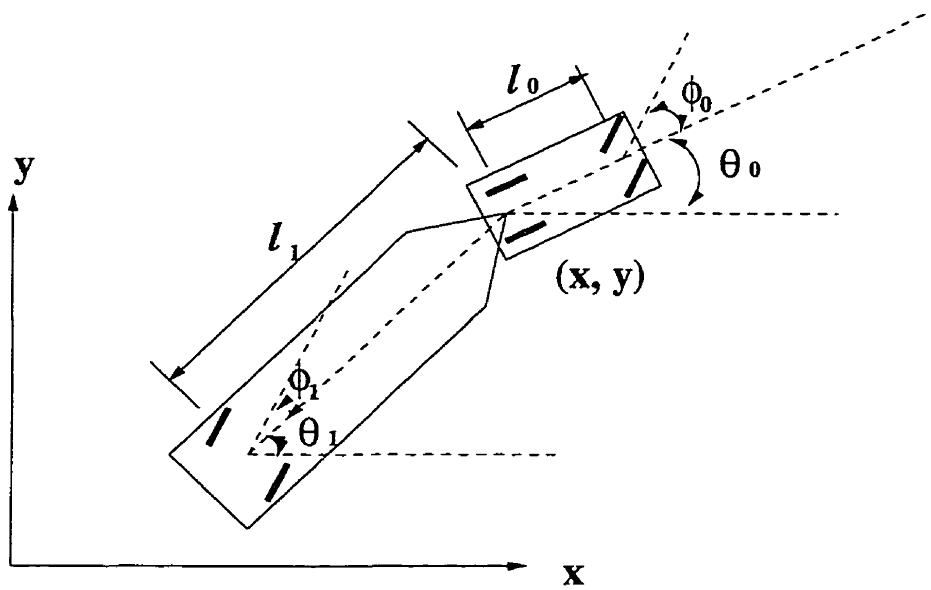

FIGURE 3.22. Model of a fire truck

6.1, 6.2, and 7.4 of this Chapter, show that the stabilizing property of the guiding functions strategy, when combined with sinusoidal steering, is preserved under a skillful approximation. The approach outlined above is fully explained by the example below.

The fire truck is an example of a nonholonomic system with three inputs and six configuration variables, for which the Lie bracket extended system also involves second order Lie brackets. A model given in [**13**], consists of two planar rigid bodies supported by three axles, see Figure 3.22. The support of the rear body, or trailer, is over the center of the rear axle of the front body, or cab (axle-to-axle hitching). The first and third axles are allowed to pivot, while the middle axle is rigidly fixed to the cab body. The wheels are assumed to roll but not slip, thus giving velocity constraints. The selected configuration variables (states variables) in this system, $(x, y, \phi_0, \theta_0, \phi_1, \theta_1) \in I\!R^6$, have the following description:

(x, y) – the Cartesian location of the center of the rear axle of the cab,

ϕ_0 – the steering angle of the front wheels with respect to the cab body,

θ_0 – the orientation of the cab body with respect to the horizontal axis of the inertial frame,

ϕ_1 – the angle of the rear wheels with respect to the trailer body,

θ_1 – the orientation of the trailer body with respect to the horizontal axis.

Denoting by l_0 and l_1 the distance between the front and rear axles of the cab, and distance between the centers of the rear axles of the cab and the trailer, respectively, the model of the fire truck can be written as (see also [**114**]):

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi}_0 \\ \dot{\theta}_0 \\ \dot{\phi}_1 \\ \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} 1 \\ \tan\,\theta_0 \\ 0 \\ (1/l_0)\ \tan\,\phi_0\ \sec\,\theta_0 \\ 0 \\ (-1/l_1)\ \sin(\phi_1 - \theta_0 + \theta_1)\ \sec\,\phi_1\ \sec\,\theta_0 \end{bmatrix} w_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} w_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} w_3 \tag{3.110}$$

where the inputs (w_1, w_2, w_3) correspond to: the forward driving velocity of the truck, the steering velocity of the front wheels of the cab, and the steering velocity of the rear wheels of the trailer, respectively.

It is convenient to redefine the state and control variables by putting $x \stackrel{def}{=} (x_1, x_2, x_3, x_4, x_5, x_6) \stackrel{def}{=} (\phi_1, \phi_0, x, y, \theta_0, \theta_1)$, and $v \stackrel{def}{=} (v_1, v_2, v_3) \stackrel{def}{=} (w_3, w_2, w_1)$. With respect to this new set of variables, additionally assuming that $l_0 = l_1 = 1$, the system (3.110) takes the following, simpler form:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} v_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} v_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ \tan\,x_5 \\ \tan\,x_2\ \sec\,x_5 \\ -\sin(x_1 - x_5 + x_6)\ \sec\,x_1\ \sec\,x_5 \end{bmatrix} v_3 \\ &\stackrel{def}{=} g_1(x)v_1 + g_2(x)v_2 + g_3(x)v_3 \end{aligned} \tag{3.111}$$

where, $g_1(x) = \dfrac{\partial}{\partial x_1}$, $\qquad g_2(x) = \dfrac{\partial}{\partial x_2}$

$$g_3(x) = \frac{\partial}{\partial x_3} + \tan\,x_5\ \frac{\partial}{\partial x_4} + \tan\,x_2\ \sec\,x_5\ \frac{\partial}{\partial x_5} - \sin(x_1 - x_5 + x_6)\ \sec\,x_1\ \sec\,x_5\ \frac{\partial}{\partial x_6}$$

Calculating the Lie brackets which are linearly independent at the origin yields:

$$g_4(x) \stackrel{def}{=} [g_1, g_3](x) = [-\cos(x_1 - x_5 + x_6)\sec x_1 \sec x_5 + \sin(x_1 - x_5 + x_6)\sec x_1 \tan x_1 \sec\,x_5]\frac{\partial}{\partial x_6}$$

$$g_5(x) \stackrel{def}{=} [g_2, g_3](x) = (\sec\,x_2)^2\ \sec\,x_5\ \frac{\partial}{\partial x_5}$$

$$g_6(x) \stackrel{def}{=} [[g_2, g_3], g_3](x) = (\sec\,x_2)^2\ (\sec\,x_5)^3\ \frac{\partial}{\partial x_4}$$

$$+[(\sec\,x_2)^2\ \sec\,x_5\ (\cos(x_1 - x_5 + x_6)\ \sec\,x_1\ \sec\,x_5 - \sin(x_1 - x_5 + x_6)\ \sec\,x_1\ \sec\,x_5\ \tan\,x_5]\ \frac{\partial}{\partial x_6}$$

It is hence clear that, if the motion of the system is restricted to the manifold

$$\mathcal{M} = \{x \in I\!R^6 \ : |x_i| < \frac{\pi}{2}, \ i = 1, 2, 5\}$$

then the LARC condition:

$$span\{g_1, g_2, g_3, [g_1, g_3], [g_2, g_3], [[g_2, g_3], g_3]\}(x) = I\!R^6 \quad \text{for } x \in \mathcal{M} \tag{3.112}$$

is satisfied. For system (3.111), the set $\mathcal{S}(x)$ is given by:

$$\mathcal{S}(x) = \{g_1(x), g_2(x), g_3, [g_1, g_3], [g_2, g_3], [[g_2, g_3], g_3]\}(x) \quad x \in \mathcal{M} \tag{3.113}$$

The Lie brackets multiplication table for $L(g_1, g_2, g_3)$ is:

$$\begin{aligned} &[g_1, g_3] = g_4 \qquad [g_2, g_3] = g_5 \qquad [g_5, g_3] = g_6 \\ &[g_i, g_j] \neq 0, \quad i = 1, 2, 3, \ j = 4, 5, 6 \end{aligned} \tag{3.114}$$

which shows that the Lie algebra $L(g_1, g_2, g_3)$ is neither nilpotent nor finite dimensional. It is also clear from the table (3.114) that the extended system for the model (3.111)

$$\dot{x} = g_1(x)\ v_1 + g_2(x)\ v_2 + g_3(x)\ v_3 + [g_2, g_3](x)\ v_4 \tag{3.115}$$

does not lead to vector field groups $\mathcal{G}_1$ and $\mathcal{G}_2$ which satisfy assumption A2. Similarly, as in section 5.3, we hence consider an approximate system (which preserves controllability) as follows:

$$\dot{x} \stackrel{def}{=} \bar{g}_1(x)\ u_1 + \bar{g}_2(x)\ u_2 + \bar{g}_3(x)\ u_3 \tag{3.116}$$

$$\begin{aligned} &\bar{g}_1(x) = \frac{\partial}{\partial x_1}, \qquad \bar{g}_2(x) = \frac{\partial}{\partial x_2} \\ &\bar{g}_3(x) = \frac{\partial}{\partial x_3} + x_5\ \frac{\partial}{\partial x_4} + x_2\ \frac{\partial}{\partial x_5} - (x_1 - x_5 + x_6)\ \frac{\partial}{\partial x_6} \end{aligned}$$

By defining

$$\begin{aligned} \bar{g}_4(x) &\stackrel{def}{=} [\bar{g}_1, \bar{g}_3](x) = -\frac{\partial}{\partial x_6} \\ \bar{g}_5(x) &\stackrel{def}{=} [\bar{g}_2, \bar{g}_3](x) = \frac{\partial}{\partial x_5} \\ \bar{g}_6(x) &\stackrel{def}{=} [\bar{g}_3, [\bar{g}_2, \bar{g}_3]](x) = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_6} \end{aligned}$$

the LARC condition:

$$span\{\bar{g}_1, \bar{g}_2, \bar{g}_3, [\bar{g}_1, \bar{g}_3], [\bar{g}_2, \bar{g}_3], [\bar{g}_3, [\bar{g}_2, \bar{g}_3]]\}(x) = I\!R^6, \quad \text{for } x \in I\!R^6$$

is satisfied. The Lie brackets multiplication table for $L(\bar{g}_1, \bar{g}_2, \bar{g}_3)$ is:

$$
\begin{aligned}
&[\bar{g}_1, \bar{g}_2] = 0, \qquad [\bar{g}_1, \bar{g}_3] = \bar{g}_4, \qquad [\bar{g}_2, \bar{g}_3] = \bar{g}_5, \qquad [\bar{g}_3, \bar{g}_5] = g_6 \\
&[\bar{g}_3, \bar{g}_4] = [\bar{g}_3, \bar{g}_6] = \bar{g}_4, \qquad [\bar{g}_1, \bar{g}_i] = [\bar{g}_2, \bar{g}_i] = 0, \quad i = 4, 5, 6
\end{aligned}
\tag{3.117}
$$

and shows that the Lie algebra $L(\bar{g}_1, \bar{g}_2, \bar{g}_3)$ is finite dimensional but not nilpotent. By using this table, the extension to the approximate model (3.116) is of the form:

$$
\dot{x} = \bar{g}_1(x)\ v_1 + \bar{g}_2(x)\ v_2 + \bar{g}_3(x)\ v_3 + [\bar{g}_2, \bar{g}_3](x)\ v_4 \tag{3.118}
$$

and provides for two vector field groups $\mathcal{G}_1$ and $\mathcal{G}_2$:

$$
\mathcal{G}_1 = \{\bar{g}_1, \bar{g}_2, [\bar{g}_2, \bar{g}_3]\}, \qquad \mathcal{G}_2 = \{\bar{g}_3, [\bar{g}_1, \bar{g}_3], [\bar{g}_3, [\bar{g}_2, \bar{g}_3]]\}
$$

It can be checked that the following distributions

$$
\Delta_1(x) \stackrel{def}{=} span\{\bar{g}_1, \bar{g}_2, [\bar{g}_2, \bar{g}_3]\}(x), \quad \Delta_2(x) \stackrel{def}{=} span\{\bar{g}_3, [\bar{g}_1, \bar{g}_3], [\bar{g}_3, [\bar{g}_2, \bar{g}_3]]\}(x)
$$

are involutive, and the corresponding codistributions have the following expressions:

$$
\Delta_1^{\perp}(x) \stackrel{def}{=} span\{d\lambda_{2,1}(x), d\lambda_{2,2}(x), d\lambda_{2,3}(x)\}, \quad \Delta_2^{\perp}(x) \stackrel{def}{=} span\{d\lambda_{1,1}(x), d\lambda_{1,2}(x), d\lambda_{1,3}(x)\}
$$

By Frobenius theorem the scalar functions $\lambda_{i,k}$ can be computed:

$$
\begin{aligned}
&\lambda_{2,1}(x) \stackrel{def}{=} x_3, \quad \lambda_{2,2}(x) \stackrel{def}{=} x_4, \quad \lambda_{2,3}(x) \stackrel{def}{=} x_6 \\
&\lambda_{1,1}(x) \stackrel{def}{=} x_1, \quad \lambda_{1,2}(x) \stackrel{def}{=} x_2, \quad \lambda_{1,3}(x) \stackrel{def}{=} (x_5 - x_2 x_3) \quad \text{for all } x \in I\!R^6
\end{aligned}
$$

The guiding functions for the approximate extended system can thus be selected:

$$
\begin{aligned}
&V_1(x) \stackrel{def}{=} \frac{1}{2}\{x_1^2 + x_2^2 + (x_5 - x_2 x_3)^2\}, \qquad V_2(x) \stackrel{def}{=} \frac{1}{2}(x_3^2 + x_4^2 + x_6^2), \quad x \in I\!R^6 \\
&V(x) = \frac{1}{2}\{x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_6^2 + (x_5 - x_2 x_3)^2\}, \quad x \in I\!R^6
\end{aligned}
$$

These guiding functions are next incorporated into the stabilizing strategy which is applied to the original system (3.111). The motion along the Lie bracket $[g_2, g_3]$ is realized indirectly by using the following controls:

$$
u_2(t) = sin(2\pi \frac{t}{T}), \quad u_3(t) = cos(2\pi \frac{t}{T}), \quad \text{while } u_1 = 0 \tag{3.119}
$$

Simulation results are shown in Figures 3.23- 3.26, which confirm that the constructed controller is essentially dead beat. In simulations, the value $T = 1$ was used in the controls (3.119).

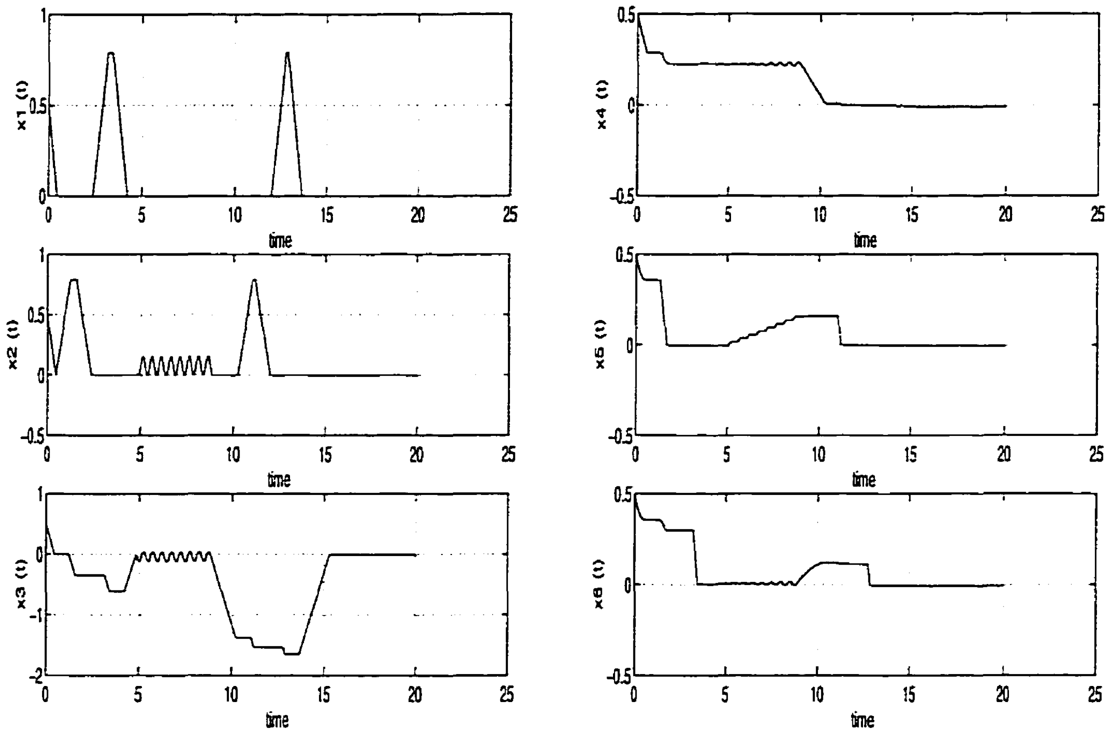

FIGURE 3.23. *Fire truck model:* Plots of the controlled state trajectories $t \mapsto (x_1(t), \ldots, x_6(t))$ versus time.

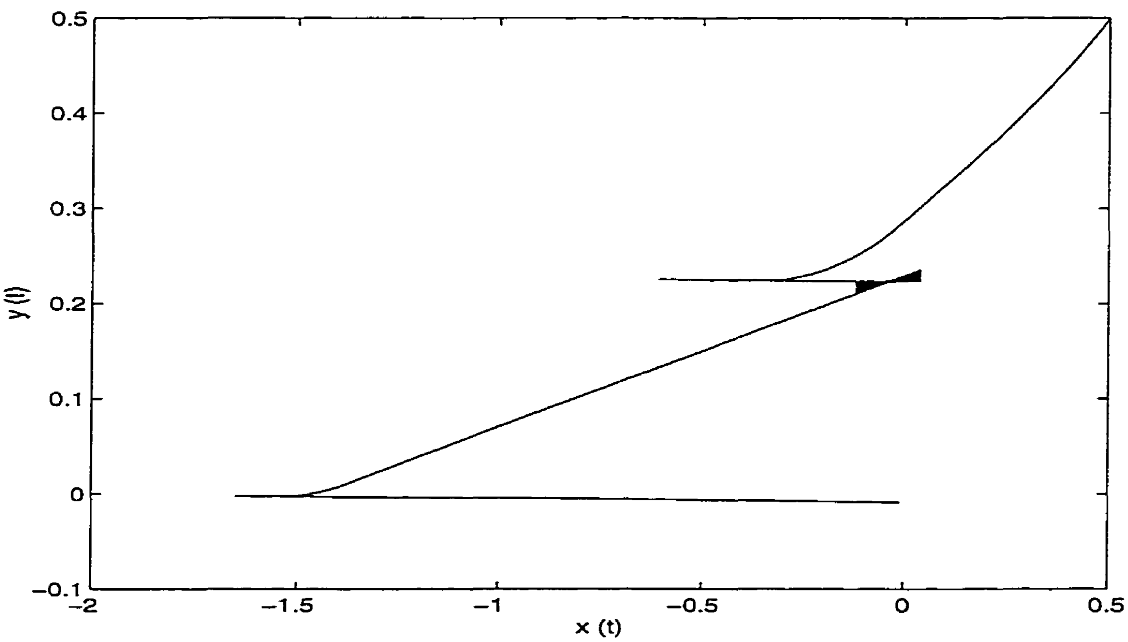

FIGURE 3.24. *Fire truck model:* Plot of the controlled state trajectory $x_3(t) = x(t)$ versus $x_4(t) = y(t)$.

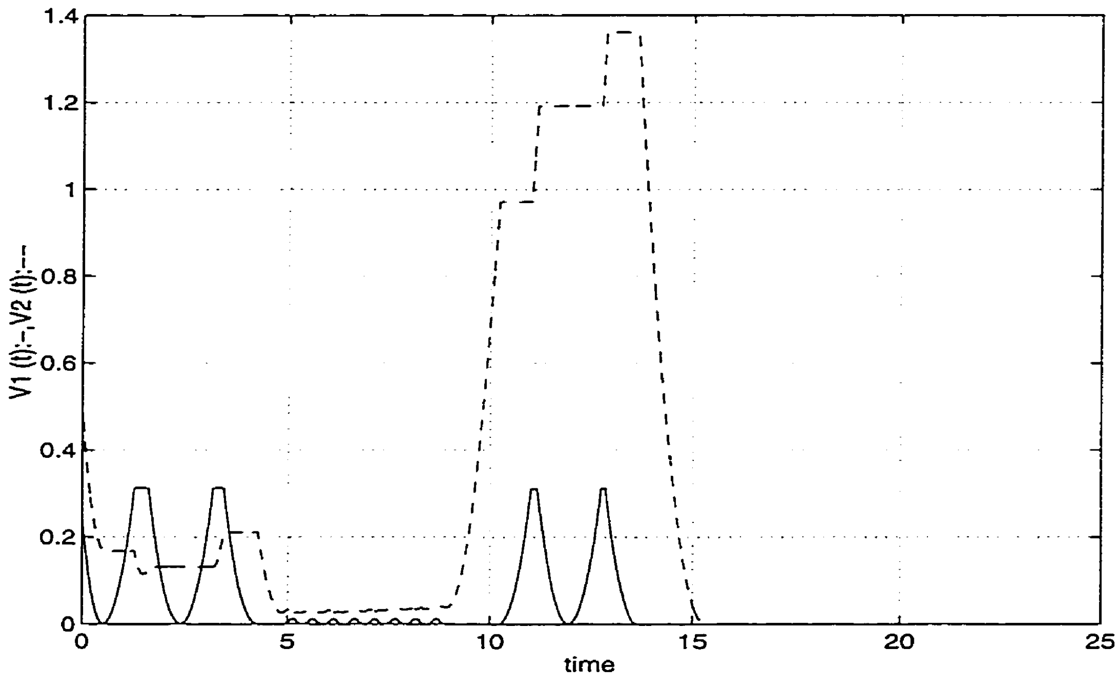

FIGURE 3.25. *Fire truck model:* Plots of the guiding functions $V_1(t)$ and $V_2(t)$ versus time.

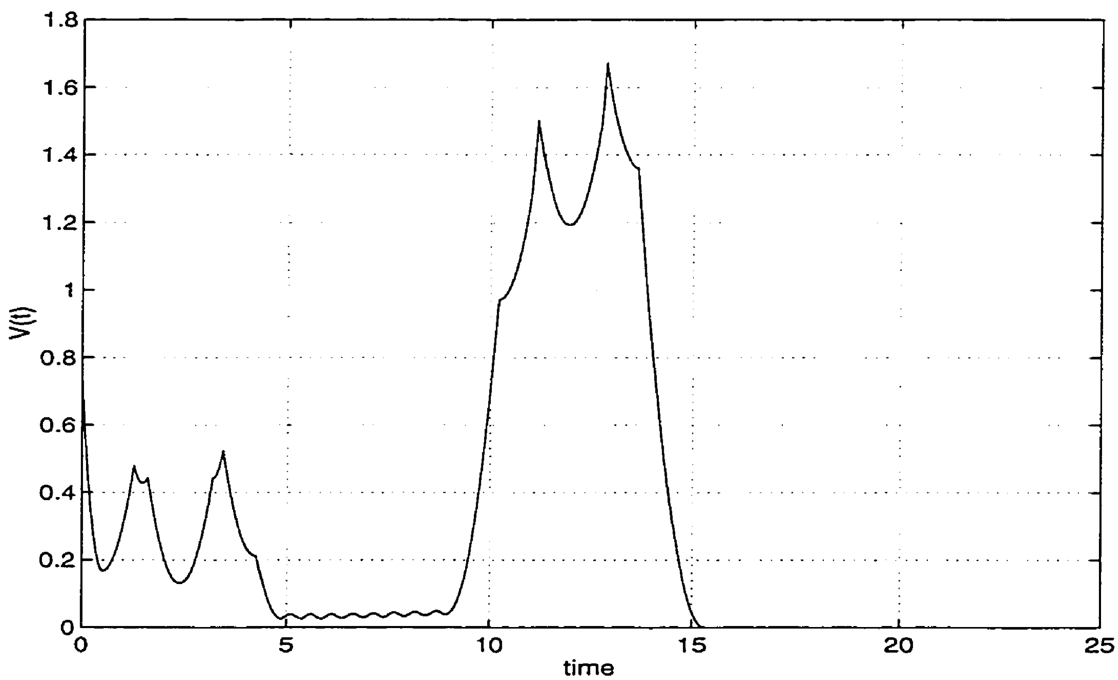

FIGURE 3.26. *Fire truck model:* Plot of the guiding function $V(t) = V_1(t) + V_2(t)$ versus time.

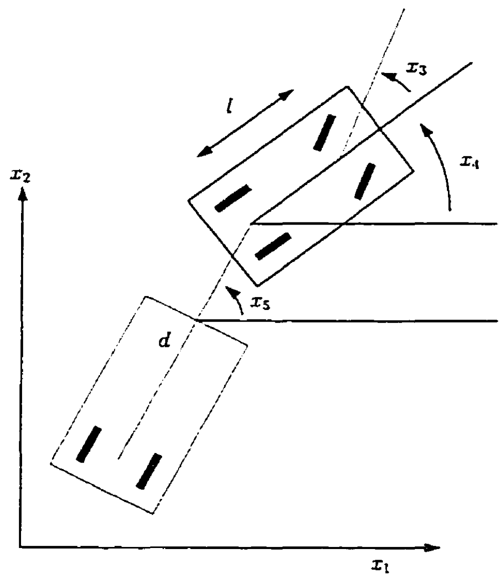

FIGURE 3.27. Model of a mobile robot with trailer

6.3. Stabilizing feedback control for a mobile robot with trailer [72]

In this section yet another example is provided which demonstrates the robustness property of the combined strategy based on guiding functions and sinusoidal steering. This example is more complex than these of the previous of sections since the corresponding controllability Lie algebra involves also a bracket of depth three.

The kinematic model of car-like robot with trailer, see [52], is given below.

$$
\begin{aligned}
\dot{x}_1 &= cos\ x_3\ cos\ x_4\ u_1 \\
\dot{x}_2 &= cos\ x_3\ sin\ x_4\ u_1 \\
\dot{x}_3 &= u_2 \\
\dot{x}_4 &= \frac{1}{l}\ sin\ x_3\ u_1 \\
\dot{x}_5 &= \frac{1}{d}\ sin\ (x_4 - x_5)\ cos\ x_3\ u_1
\end{aligned}
\tag{3.120}
$$

where x_1, x_2 are the Cartesian coordinates of the centre of mass of the car, x_3 is the steering angle, x_4 and x_5 are the angles which the main axes of the car and trailer make with the x_1 axis, respectively, see Figure 3.27. The above can be rewritten in a compact form as

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2, \quad x \in I\!R^5 \tag{3.121}$$

$$g_1(x) = cos\ x_3\ cos\ x_4\ \frac{\partial}{\partial x_1} + cos\ x_3\ sin\ x_4\ \frac{\partial}{\partial x_2} + sin\ x_3\ \frac{\partial}{\partial x_4} + \ cos\ x_3\ sin\ (x_4 - x_5)\ \frac{\partial}{\partial x_4}$$

$$g_2(x) = \frac{\partial}{\partial x_3}$$

where, for simplicity, it is assumed that $l = d = 1$. The following Lie brackets are needed:

$$g_3(x) \stackrel{def}{=} [g_1, g_2](x) = sin\ x_3\ cos\ x_4\ \frac{\partial}{\partial x_1} + sin\ x_3\ sin\ x_4\ \frac{\partial}{\partial x_2} - cos\ x_3\ \frac{\partial}{\partial x_4} + sin\ x_3\ sin\ (x_4 - x_5)\ \frac{\partial}{\partial x_5}$$

$$g_4(x) \stackrel{def}{=} [g_1, g_3](x) = sin\ x_4\ \frac{\partial}{\partial x_1} + cos\ x_4\ \frac{\partial}{\partial x_2} + cos\ (x_4 - x_5)\ \frac{\partial}{\partial x_5}$$

$$g_5(x) \stackrel{def}{=} [g_1, g_4](x) = -sin\ x_3\ cos\ x_4\ \frac{\partial}{\partial x_1} - sin\ x_3\ sin\ x_4\ \frac{\partial}{\partial x_2} -(sin\ x_3\ sin\ (x_4 - x_5) - cos\ x_3)\ \frac{\partial}{\partial x_5}$$

to satisfy the LARC condition:

$$span\{g_i(x), i = 1, ..., 5\} = I\!R^5, \quad \text{for all } x \in I\!R^5$$

The set $\mathcal{S}(x)$ can be defined as:

$$\mathcal{S}(x) = \{g_i(x), i = 1, ..., 5\} = I\!R^5, \quad \text{for all } x \in I\!R^5$$

The Lie brackets multiplication table for $L(g_1, g_2)$ is:

$$[g_1, g_2] = g_3 \qquad [g_1, g_3] = g_4 \qquad [g_1, g_4] = g_5$$
$$[g_i, g_j] \neq 0, \quad i = 1, 2,\ j = 3, 4, 5 \tag{3.122}$$

which indicates that the Lie algebra $L(g_1, g_2)$ is neither nilpotent nor finite dimensional. The table (3.122) also shows that the extended system for (3.121) :

$$\dot{x} = g_1(x)\ v_1 + g_2(x)\ v_2 + [g_1, g_2](x)\ v_3 + [g_1, g_3](x)\ v_4 \tag{3.123}$$

does not allow for the introduction of two vector field groups $\mathcal{G}_1$ and $\mathcal{G}_2$ which give rise to involutive distributions. We thus consider the following approximate system:

$$\dot{x} = \bar{g}_1(x)\ u_1 + \bar{g}_2(x)\ u_2 \tag{3.124}$$

where, $$\bar{g}_1(x) = \frac{\partial}{\partial x_1} + x_4\ \frac{\partial}{\partial x_2} + x_3\ \frac{\partial}{\partial x_4} + (x_4 - x_5)\ \frac{\partial}{\partial x_5}, \quad \bar{g}_2(x) = \frac{\partial}{\partial x_3}.$$

Clearly

$$\bar{g}_3 \stackrel{def}{=} [\bar{g}_1, \bar{g}_2] = -\frac{\partial}{\partial x_4}, \qquad \bar{g}_4(x) \stackrel{def}{=} [\bar{g}_1, \bar{g}_3](x) = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_5}$$
$$\bar{g}_5(x) \stackrel{def}{=} [\bar{g}_1, \bar{g}_4](x) = \frac{\partial}{\partial x_5}$$

which also satisfy the LARC condition:

$$span\{\bar{g}_i(x), i = 1, ..., 5\} = I\!R^5, \quad \text{for all} \quad x \in I\!R^5 \tag{3.125}$$

and form the corresponding extended system to the approximation (3.124):

$$\begin{aligned} \dot{x} &= \bar{g}_1(x)\ v_1 + \bar{g}_2(x)\ v_2 + [\bar{g}_1, \bar{g}_2](x)\ v_3 + [\bar{g}_1, [\bar{g}_1, \bar{g}_2]](x)\ v_4 \\ &= \bar{g}_1(x)\ v_1 + \bar{g}_2(x)\ v_2 + \bar{g}_3(x)\ v_3 + \bar{g}_4(x)\ v_4 \end{aligned} \tag{3.126}$$

The controls v_3 and v_4 are clearly not directly accessible but motion of the system along the corresponding Lie bracket directions $[\bar{g}_1, \bar{g}_2]$, and $[\bar{g}_1, [\bar{g}_1, \bar{g}_2]]$, can be realized through controls u_1 and u_2, in an indirect way, and over a finite interval of time. The Lie brackets multiplication table for $L(\bar{g}_1, \bar{g}_2)$ is:

$$\begin{aligned} &[\bar{g}_1, \bar{g}_2] = g_3, \qquad [\bar{g}_1, \bar{g}_3] = \bar{g}_4 \\ &[\bar{g}_1, \bar{g}_4] = \bar{g}_5, \qquad [\bar{g}_1, \bar{g}_5] = \bar{g}_5 \\ &[\bar{g}_j, \bar{g}_i] = 0, \quad j = 2, ..., 5, \quad i = 3, 4, 5 \end{aligned} \tag{3.127}$$

which shows that Lie algebra $L(\bar{g}_1, \bar{g}_2)$ is nilpotent. This table also allows for the introduction of the vector field groups $\mathcal{G}_1$ and $\mathcal{G}_2$ for the extended system (3.126), as follows:

$$\begin{aligned} \mathcal{G}_1 &= \{\bar{g}_2, [\bar{g}_1, \bar{g}_2], [\bar{g}_1, [\bar{g}_1, \bar{g}_2]]\ \} = \{\bar{g}_2, \bar{g}_3, \bar{g}_4\ \} \\ \mathcal{G}_2 &= \{\bar{g}_1, [\bar{g}_1, [\bar{g}_1, [\bar{g}_1, \bar{g}_2]]]\ \} = \{\bar{g}_1, \bar{g}_5\ \} \end{aligned}$$

which give rise to the involutive distributions:

$$\Delta_1(x) \stackrel{def}{=} span\{\bar{g}_2, \bar{g}_3, \bar{g}_4\}(x)$$
$$\Delta_2(x) \stackrel{def}{=} span\{\bar{g}_1, \bar{g}_5\}(x)$$

The corresponding codistributions thus have the following expressions:

$$\Delta_1^{\perp}(x) = span\{d\lambda_{1,1}(x), d\lambda_{1,2}(x)\}$$
$$\Delta_2^{\perp}(x) = span\{d\lambda_{2,1}(x), d\lambda_{2,2}(x), d\lambda_{2,3}(x)\}$$

and the scalar functions $\lambda_{i,j}$, can be obtained by using the Frobenius Theorem:

$$\lambda_{1,1}(x) = x_1, \quad \lambda_{1,2}(x) = x_2 - x_5$$
$$\lambda_{2,1}(x) = x_3, \quad \lambda_{2,2}(x) = (x_4 - x_1 x_3)$$
$$\lambda_{2,3}(x) = (x_2 - x_1 x_3 + \frac{1}{2} x_1^2 x_3)$$

The resulting guiding functions are given by:

$$\begin{aligned} V_1(x) &\stackrel{def}{=} \frac{1}{2}\{x_1^2 + (x_2 - x_5)^2\} \\ V_2(x) &\stackrel{def}{=} \frac{1}{2}\{x_3^2 + (x_4 - x_1 x_3)^2 + (x_2 - x_1 x_3 + \frac{1}{2} x_1^2 x_3)^2\} \\ V(x) &\stackrel{def}{=} V_1(x) + V_2(x) \end{aligned}$$

and can be used in a combined feedback strategy as applied to the original system (3.121). The system motion along the Lie bracket direction $[g_1, g_2]$ can be achieved by employing sinusoidal controls :

$$\begin{aligned} u_1(t) &= sin(2\pi \frac{t}{T}) \\ u_2(t) &= cos(2\pi \frac{t}{T}) \end{aligned} \tag{3.128}$$

while motion along the Lie bracket direction $[g_1, [g_1, g_2]]$ can be achieved by employing:

$$\begin{aligned} u_1(t) &= k_1 \; sin(2\pi \frac{t}{T}) \\ u_2(t) &= k_2 \; cos(4\pi \frac{t}{T}) \end{aligned} \tag{3.129}$$

where k_1, k_2 and T are some non-zero constants. Three sets of simulation results are shown in Figures 3.28 - 3.30, 3.31 - 3.33, and 3.34 - 3.36, respectively.

Figures 3.28 - 3.30 correspond to the situation when the robot and trailer are steered to the origin from an arbitrary initial condition in the configuration space (specifically, the trajectories shown are obtained when $x_0 = [0.6, 0.8, 0.4, 0.7, 0.5]^T$ and $k_1 = 2$, $k_2 = 3$, and $T = 0.9$).

Figures 3.31 - 3.33 and 3.34 - 3.36 show the controlled system trajectories during two parallel parking maneuvers, corresponding to the initial conditions $x_0 = [0, 1, 0, 0, 0]^T$ and $x_0 = [0, -1, 0, 0, 0]^T$, respectively (here, $k_1 = 2$, $k_2 = 3$ and $T = 1.5$ were used).

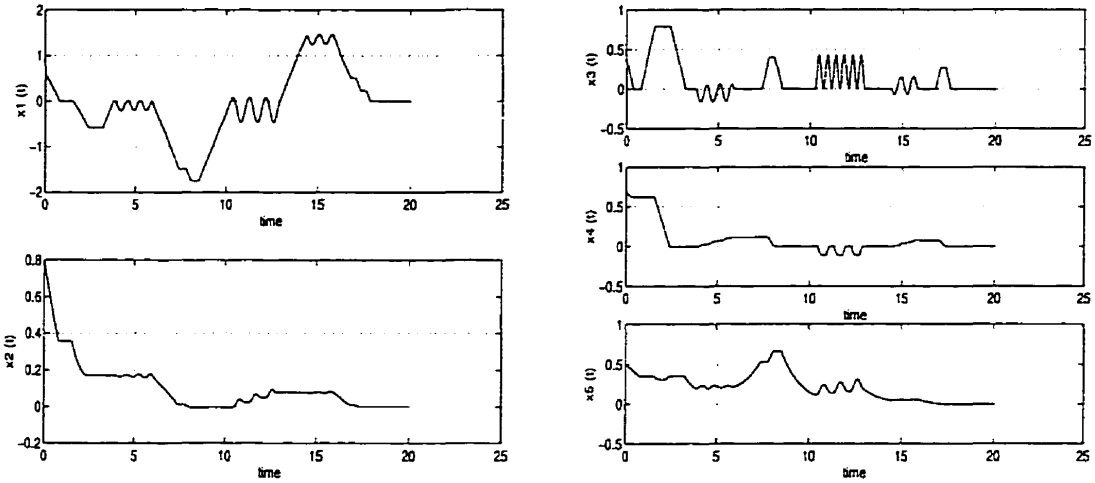

FIGURE 3.28. Steering from an arbitrary initial configuration. Plots of the controlled state trajectories $t \mapsto (x_1(t), ..., x_5(t))$ versus time.

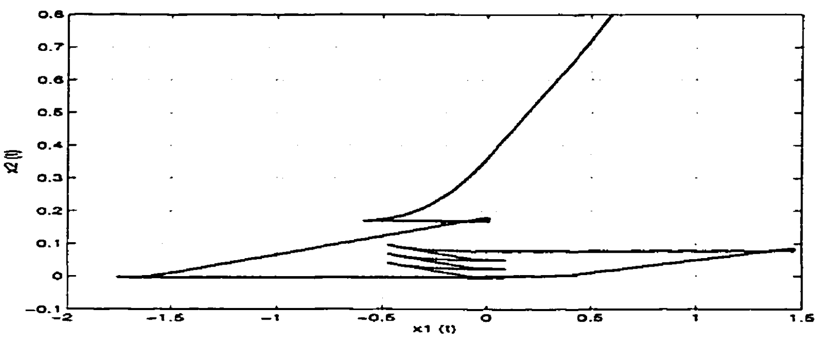

FIGURE 3.29. Plot of the controlled state trajectory $x_1(t)$ versus $x_2(t)$ corresponding to Figure 3.28.

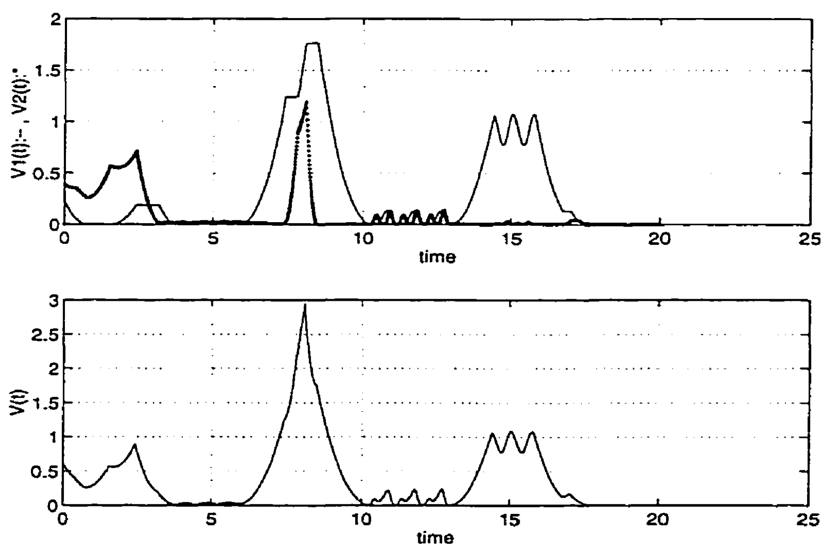

FIGURE 3.30. Plots of the guiding functions $V_1(t)$, $V_2(t)$ and $V(t) = V_1(t) + V_2(t)$ versus time, corresponding to Figure 3.28.

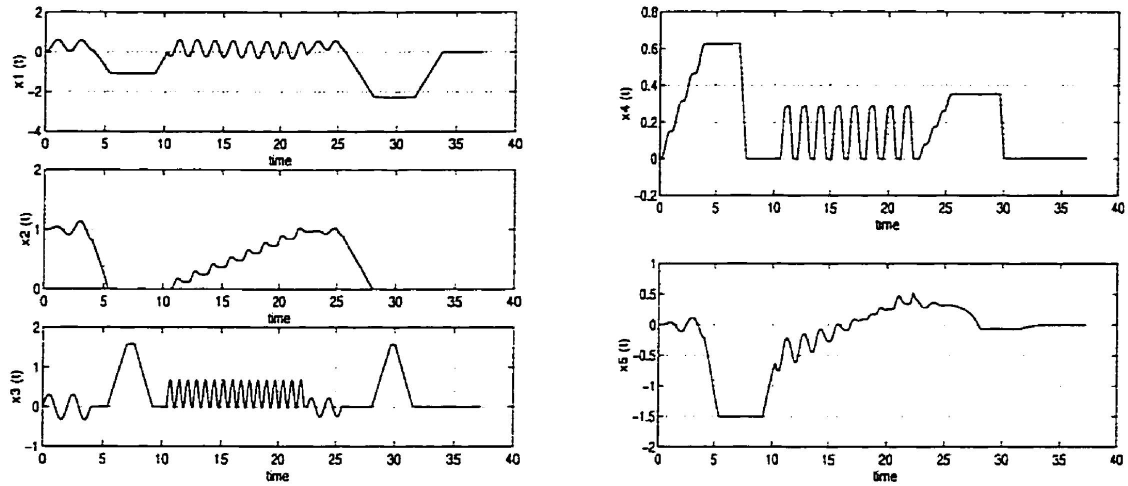

FIGURE 3.31. Parallel parking maneuver I: Plots of the controlled state trajectories $t \mapsto (x_1(t), ..., x_5(t))$ versus time.

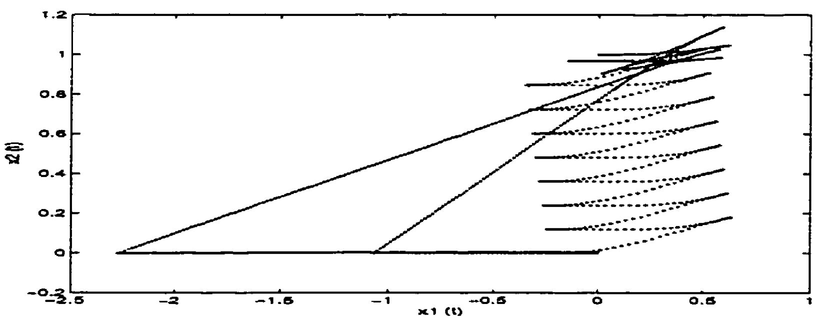

FIGURE 3.32. Plot of the controlled state trajectory $x_1(t)$ versus $x_2(t)$ corresponding to Figure 3.31.

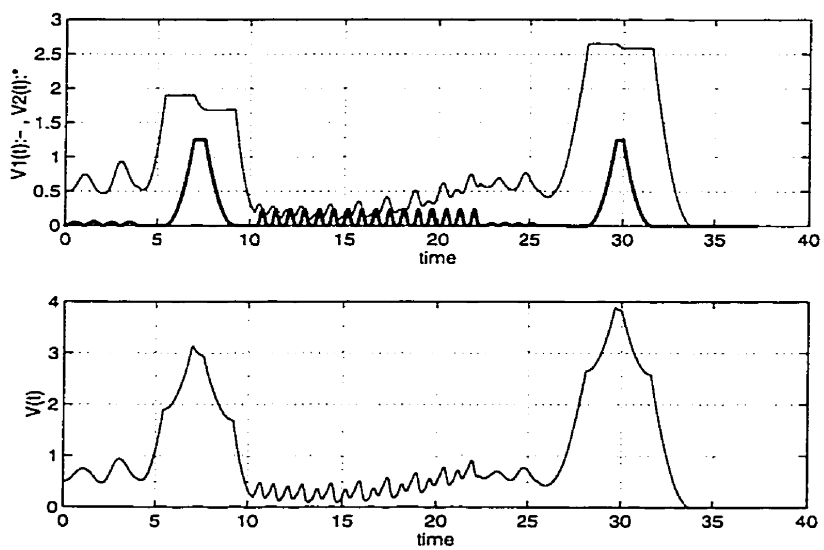

FIGURE 3.33. Plots of the guiding functions $V_1(t)$, $V_2(t)$ and $V(t) = V_1(t) + V_2(t)$ versus time, corresponding to Figure 3.31.

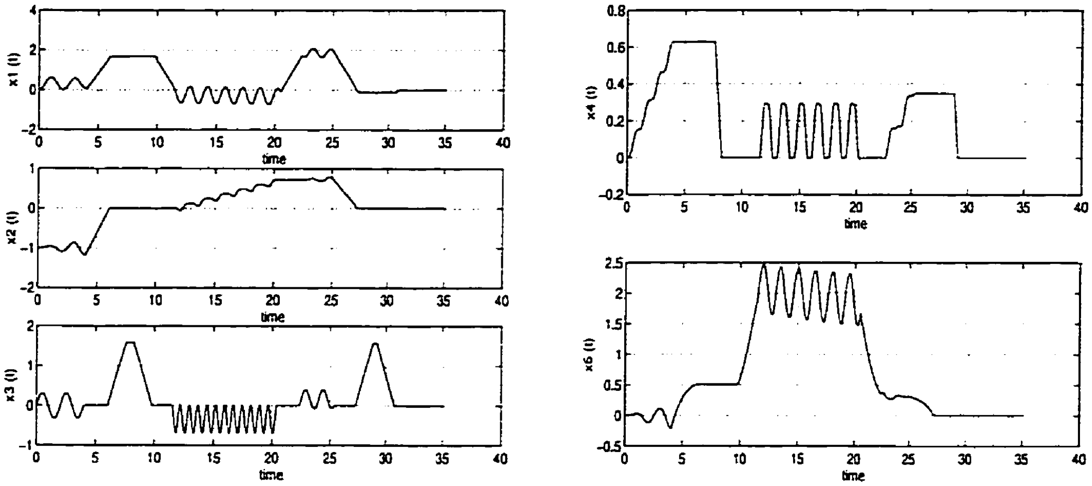

FIGURE 3.34. Parallel parking maneuver II. Plots of the controlled state trajectories $t \mapsto (x_1(t), \ldots, x_5(t))$ versus time.

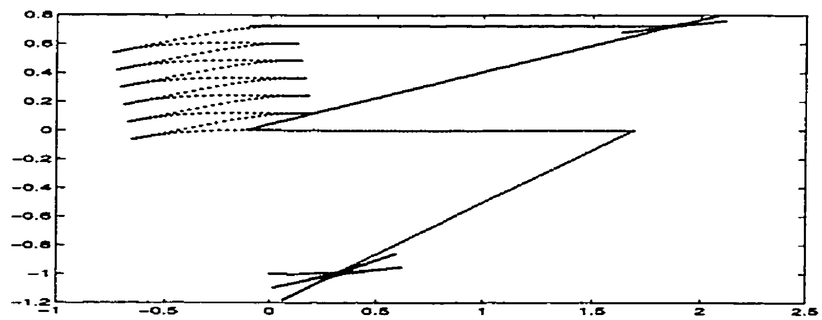

FIGURE 3.35. Plot of the controlled state trajectory $x_1(t)$ versus $x_2(t)$ corresponding to Figure 3.34.

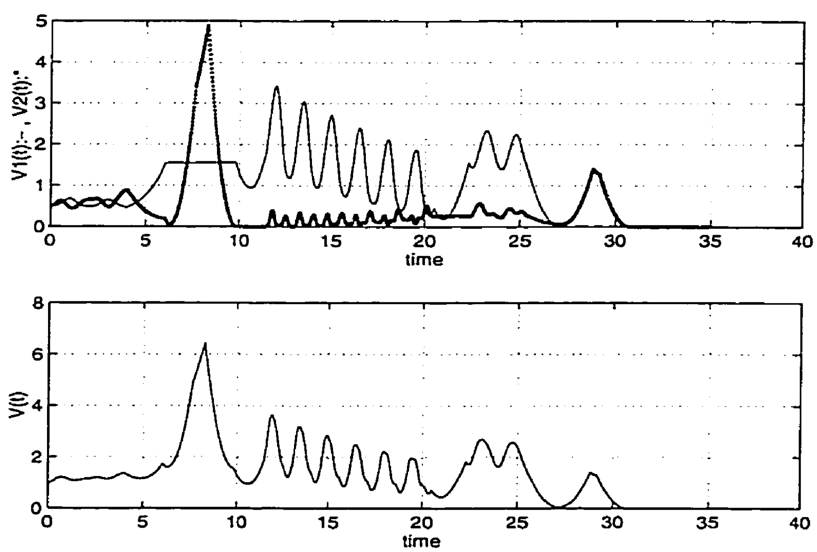

FIGURE 3.36. Plots of the guiding functions $V_1(t)$, $V_2(t)$ and $V(t) = V_1(t) + V_2(t)$ versus time, corresponding to Figure 3.34.

7. A further extension of the guiding functions strategy

The guiding functions approach for general drift free systems can be further extended to allow for the construction of m rather than two guiding functions. Its applicability is demonstrated on two general drift free systems (one with six state variables and three controls, and another with ten state variables and four control variables), a hopping robot in flight phase, a fire truck model [**67**], and a class of wheeled mobile robots [**71**].

For the construction of m guiding functions, the assumption A2 is replaced by assumption A5 below:

A5. The distributions

$$\begin{aligned}
\Delta_1(x) &\overset{def}{=} span\{g_i(x), i \neq 1, i \in \underline{m}, [g_k, g_m](x), k \in \underline{m}\} \\
\Delta_2(x) &\overset{def}{=} span\{g_i(x), i \neq 2, i \in \underline{m}, [g_k, g_m](x), k \in \underline{m}\} \\
\Delta_3(x) &\overset{def}{=} span\{g_i(x), i \neq 3, i \in \underline{m}, [g_k, g_m](x), k \in \underline{m}\} \\
&\vdots \\
\Delta_{m-1}(x) &\overset{def}{=} span\{g_i(x), i \neq m-1, i \in \underline{m}, [g_k, g_m](x), k \in \underline{m}\} \\
\Delta_m(x) &\overset{def}{=} span\{g_i(x), i \in \underline{m-1}\}
\end{aligned} \tag{3.130}$$

are involutive, and therefore completely integrable.

Let $\lambda_1(x), \lambda_2(x), ..., \lambda_{m-1}(x)$ be scalar functions such that the differentials $d\lambda_1(x), ..., d\lambda_{m-1}(x)$ span the codistributions $\Delta_1^\perp(x), \Delta_2^\perp(x), ..., \Delta_{m-1}^\perp(x)$, respectively, and let $\lambda_m(x), ..., \lambda_n(x)$ be such that $d\lambda_m(x), ..., d\lambda_n(x)$ span the codistribution $\Delta_m^\perp(x)$, so that

$$d\lambda_i \perp \Delta_i, \quad i \in \underline{m-1} \tag{3.131}$$

$$d\lambda_k \perp \Delta_m, \quad k = m, ..., n \tag{3.132}$$

The following semi-positive definite guiding functions can then be introduced:

$$V_i(x) \overset{def}{=} \frac{1}{2}[\lambda_i(x) - \lambda_i(0)]^2, \quad i \in \underline{m-1} \tag{3.133}$$

$$V_m(x) \overset{def}{=} \frac{1}{2}\sum_{k=m}^{n}[\lambda_k(x) - \lambda_k(0)]^2 \tag{3.134}$$

The guiding functions strategy of this Chapter can now be modified to account for several rather that two guiding functions, as follows (recall the definition of the set of impasse points: $\mathcal{T} \overset{def}{=} \{x \in I\!R^n : L_{g_i}V(x) = 0,\ i \in \underline{m}\}$).

Extended guiding functions strategy:

- Data: $\alpha \geq 1$.

- 1 If $x \in \mathbb{R}^n \setminus \mathcal{T}$, then for each $i \in \underline{m}$ employ the control

$$v_k(x) = \begin{cases} -sign_\rho[L_{g_i} V_i(x)], & \text{for } k = i \\ 0, & \text{for } k \neq i \end{cases}$$

until $L_{g_i} V_i(x) = 0$.

- 2 Define $p \stackrel{def}{=} x(t)$ in which t is the time at the exit of Step 1 (when the set $\mathcal{T}$ is traversed). If $p = 0$ then stop, else if $p \neq 0$, then

- 2a Select a set of indices $J \in \underline{m-1}$, such that

$$i \in J \quad \text{if} \quad L_{[g_i, g_m]} V_m(p) \neq 0,$$

- 2b Employ the controls

$$v_k(x) = \begin{cases} 1, & \text{for } k \in J \\ 0, & \text{for } k \notin J \end{cases}$$

until, for each $i \in J : L_{[g_i, g_m]} V_m(x) = 0$, or else until $V_i(x) \geq \alpha V(p)$.

- 2c Until $L_{g_m} V_m(x) = 0$, employ the controls

$$v_k(x) = \begin{cases} -sign[L_{g_m} V_m(x)], & \text{for } k = m \\ 0, & \text{for } k \neq m \end{cases}$$

- 2d For each of the indices $i \in J$, employ the controls

$$v_k(x) = \begin{cases} -sign[L_{g_i} V_i(x)], & \text{for } k = i \\ 0, & \text{for } k \neq i \end{cases}$$

until $L_{g_i} V_i(x) = 0$, for each i. Repeat Step 2.

The convergence analysis for the above strategy is similar to the one found in section 4 and is omitted here as the proofs are direct analogs of the ones already presented. The efficiency of the strategy is demonstrated using a few representative examples.

7.1. Stabilizing feedback control for general drift free system with six state variables and three controls

The following example motivates the necessity for the extension of the guiding functions strategy. For this example, it is not possible to find two vector field groups $\mathcal{G}_1$ and $\mathcal{G}_2$ which give rise to involutive distributions.

The system equations are given by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ x_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ x_1 \\ x_2 \end{bmatrix} u_3$$

$$\stackrel{def}{=} g_1(x)\, u_1 + g_2(x)\, u_2 + g_3(x)\, u_3 \tag{3.135}$$

$$\text{where,} \quad g_1(x) = \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_3}, \qquad g_2(x) = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3}$$

$$g_3(x) = \frac{\partial}{\partial x_4} + x_1 \frac{\partial}{\partial x_5} + x_2 \frac{\partial}{\partial x_6}$$

With:

$$g_4(x) \stackrel{def}{=} [g_1, g_2](x) = 2\frac{\partial}{\partial x_3}, \quad g_5(x) \stackrel{def}{=} [g_1, g_3](x) = \frac{\partial}{\partial x_5}, \quad g_6(x) \stackrel{def}{=} [g_2, g_3](x) = \frac{\partial}{\partial x_6}$$

yield

$$span\{g_i(x),\ i = 1, ..., 6\} = I\!R^6, \qquad \text{for all} \quad x \in I\!R^6. \tag{3.136}$$

The Lie brackets multiplication table for $L(g_1, g_2, g_3)$ is :

$$[g_1, g_2] = g_4 \qquad [g_1, g_3] = g_5 \qquad [g_2, g_3] = g_6$$

$$[g_i, g_j] = 0, \quad i = 1, ..., 6 \qquad j = 4, ..., 6 \tag{3.137}$$

which shows that the controllability Lie algebra $L(g_1, g_2, g_3)$ is nilpotent. The set $\mathcal{S}(x)$ is defined by

$$\mathcal{S}(x) = \{g_1(x), g_2(x), g_3(x), [g_1, g_2](x), [g_1, g_3](x), [g_2, g_3](x)\}, \quad x \in I\!R^6 \tag{3.138}$$

By using table (3.137) it is clear that vector fields groups $\mathcal{G}_1$ and $\mathcal{G}_2$ cannot be formed while obeying

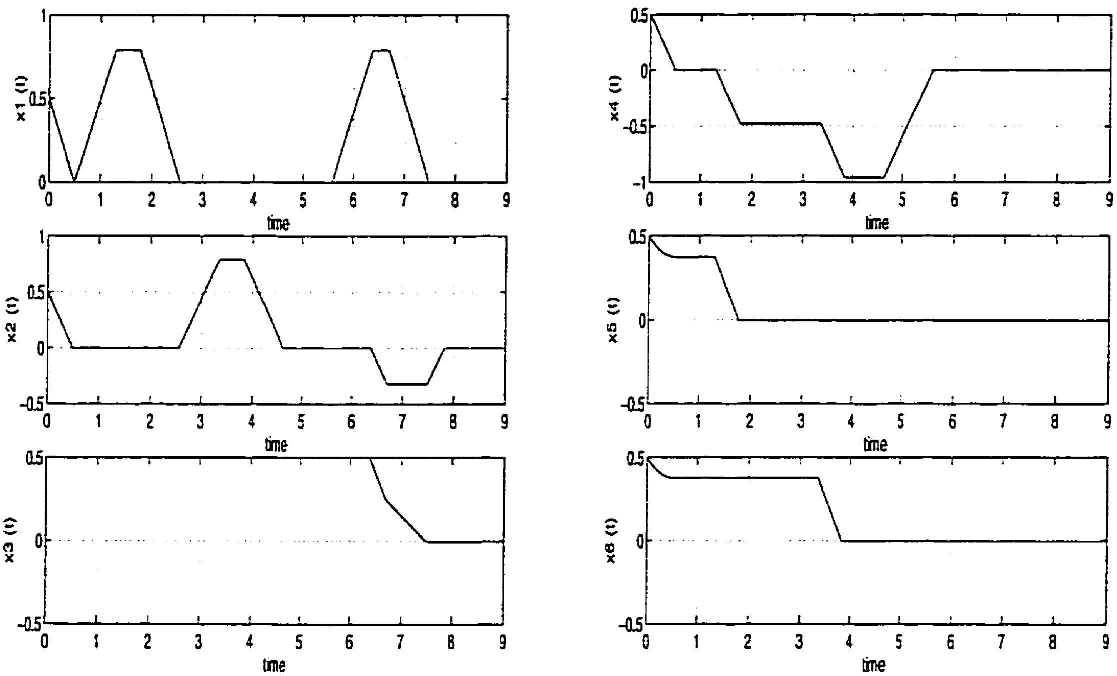

FIGURE 3.37. *General drift free system with $n - m = 3$:* Plots of the controlled state trajectories $t \mapsto (x_1(t), ..., x_6(t))$ versus time.

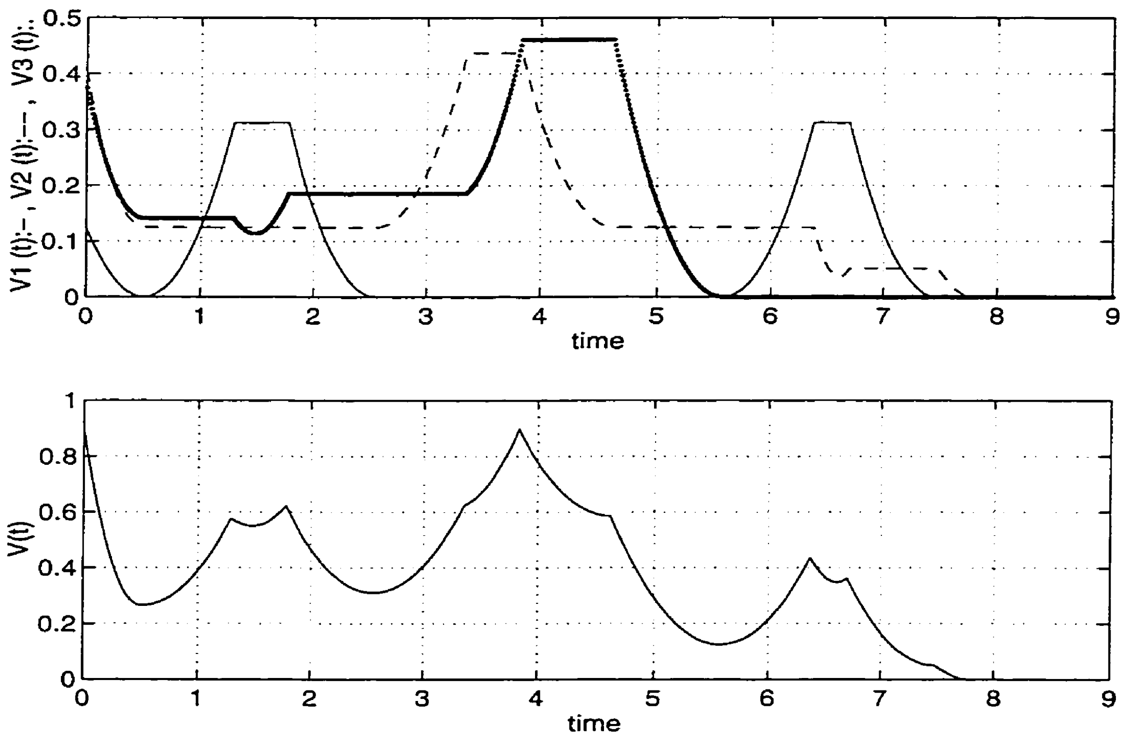

FIGURE 3.38. *General drift free system with $n - m = 3$:* Plots of the guiding functions $V_1(t)$, $V_2(t)$ and $V(t) = \sum_{i=1}^{3} V_i(t)$ versus time.

the rules (R1) - (R2). This is the reason for which the introduction of several guiding functions is attempted as guaranteed by assumption A5. Indeed, the following distributions:

$$\Delta_1(x) \stackrel{def}{=} span\{g_2, g_3, [g_1, g_2], [g_1, g_3], [g_2, g_3]\}(x)$$
$$\Delta_2(x) \stackrel{def}{=} span\{g_1, g_3, [g_2, g_3], [g_1, g_3]\}(x)$$
$$\Delta_3(x) \stackrel{def}{=} span\{g_1, g_2, [g_1, g_2]\}(x)$$

are involutive, and the corresponding codistributions are:

$$\Delta_1^{\perp}(x) = \{d\lambda_{1,1}(x)\}$$
$$\Delta_2^{\perp}(x) = \{d\lambda_{2,1}(x), d\lambda_{2,2}(x)\}$$
$$\Delta_3^{\perp}(x) = \{d\lambda_{3,1}(x), d\lambda_{3,2}(x), d\lambda_{3,3}(x)\}$$

By using the Frobenius Theorem, one choice of the scalar functions $\lambda_{i,k}$ is:

$$\lambda_{1,1}(x) = x_1, \quad \lambda_{2,1}(x) = x_2, \quad \lambda_{2,2}(x) = x_3 + x_1 x_2$$
$$\lambda_{3,1}(x) = x_4, \quad \lambda_{3,2}(x) = x_5, \quad \lambda_{3,3}(x) = x_6$$

which leads to the following guiding functions:

$$\begin{aligned} V_1(x) &= \frac{1}{2}(x_1^2) \\ V_2(x) &= \frac{1}{2}x_2^2 + \frac{1}{2}(x_3 + x_1 x_2)^2 \\ V_3(x) &= \frac{1}{2}(x_4^2 + x_5^2 + x_6^2) \\ V(x) &= V_1(x) + V_2(x) + V_3(x) \end{aligned}$$

These guiding functions are then used into the extended strategy which is applied to this system. Simulation results are shown in Figures 3.37 - 3.38 which confirm that the controller constructed by this extended strategy is also dead beat.

7.2. Stabilizing feedback control for general drift free system with ten state variables and four controls

The example considered below demonstrates that the extended guiding functions strategy introduced in this section is successfully applicable to systems with higher order control deficiency (in this case $n - m = 6$).

The equations of this example are:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{x}_7 \\ \dot{x}_8 \\ \dot{x}_9 \\ \dot{x}_{10} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ x_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ x_1 \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ x_2 \\ x_4 \\ 0 \end{bmatrix} u_3 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ x_1 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_4$$

$$\stackrel{def}{=} g_1(x)\ u_1 + g_2(x)\ u_2 + g_3(x)\ u_3 + g_4(x)\ u_4 \tag{3.139}$$

$$\text{where,}\quad \begin{aligned} g_1(x) &= \frac{\partial}{\partial x_1} + x_3\ \frac{\partial}{\partial x_5} \\ g_2(x) &= \frac{\partial}{\partial x_2} + x_1\ \frac{\partial}{\partial x_{10}} \\ g_3(x) &= \frac{\partial}{\partial x_3} + x_2\ \frac{\partial}{\partial x_8} + x_4\ \frac{\partial}{\partial x_9} \\ g_4(x) &= \frac{\partial}{\partial x_4} + x_1\ \frac{\partial}{\partial x_6} + x_2\ \frac{\partial}{\partial x_7} \end{aligned} \tag{3.140}$$

The following Lie brackets:

$$\begin{aligned} g_5(x) &\stackrel{def}{=} [g_1, g_2](x) = \frac{\partial}{\partial x_{10}}, & g_6(x) &\stackrel{def}{=} [g_1, g_3](x) = -\frac{\partial}{\partial x_5} \\ g_7(x) &\stackrel{def}{=} [g_1, g_4](x) = \frac{\partial}{\partial x_6}, & g_8(x) &\stackrel{def}{=} [g_2, g_3](x) = \frac{\partial}{\partial x_8} \\ g_9(x) &\stackrel{def}{=} [g_2, g_4](x) = \frac{\partial}{\partial x_7}, & g_{10}(x) &\stackrel{def}{=} [g_3, g_4](x) = -\frac{\partial}{\partial x_9} \end{aligned}$$

yield

$$span\{g_i(x),\ i = 1, ..., 10\} = I\!R^{10} \qquad \text{for all} \qquad x \in I\!R^{10}.$$

The Lie brackets multiplication table for $L(g_1, g_2, g_3, g_4)$ is :

$$\begin{aligned} &[g_1, g_2] = g_5 \qquad && [g_1, g_3] = g_6 \qquad && [g_1, g_4] = g_7 \\ &[g_2, g_3] = g_8 && [g_2, g_4] = g_9 && [g_3, g_4] = g_{10} \\ &[g_i, g_j] = 0, \quad i = 1, ..., 10, && j = 5, ..., 10 && \end{aligned} \tag{3.141}$$

which shows that the Lie algebra $L(g_1, g_2, g_3, g_4)$ is nilpotent. From table (3.141) it is also clear that the following distributions are involutive:

$$\Delta_1(x) \stackrel{def}{=} span\{g_2, g_3, g_4, [g_1, g_2], [g_1, g_4], [g_2, g_3], [g_2, g_4], [g_3, g_4]\}(x)$$
$$\Delta_2(x) \stackrel{def}{=} span\{g_1, g_3, g_4, [g_1, g_3], [g_1, g_4], [g_2, g_3], [g_2, g_4], [g_3, g_4]\}(x)$$
$$\Delta_3(x) \stackrel{def}{=} span\{g_1, g_2, g_4, [g_1, g_2], [g_1, g_3], [g_1, g_4], [g_2, g_4]\}(x)$$
$$\Delta_4(x) \stackrel{def}{=} span\{g_1, g_2, g_3, [g_1, g_2], [g_1, g_3], [g_2, g_3], [g_3, g_4]\}(x)$$

The corresponding codistributions are:

$$\Delta_1^{\perp}(x) = \{d\lambda_{1,1}(x), d\lambda_{1,2}(x)\}$$
$$\Delta_2^{\perp}(x) = \{d\lambda_{2,1}(x), d\lambda_{2,2}(x)\}$$
$$\Delta_3^{\perp}(x) = \{d\lambda_{3,1}(x), d\lambda_{3,2}(x), d\lambda_{3,3}(x)\}$$
$$\Delta_4^{\perp}(x) = \{d\lambda_{4,1}(x), d\lambda_{4,2}(x), d\lambda_{4,3}(x)\}$$

One choice of the scalar functions $\lambda_{i,k}$ is:

$$\lambda_{1,1}(x) = x_1, \qquad \lambda_{1,2}(x) = x_5$$
$$\lambda_{2,1}(x) = x_2, \qquad \lambda_{2,2}(x) = x_{10}$$
$$\lambda_{3,1}(x) = x_3, \qquad \lambda_{3,2}(x) = x_8, \qquad \lambda_{3,3}(x) = x_9$$
$$\lambda_{4,1}(x) = x_4, \qquad \lambda_{4,2}(x) = x_6, \qquad \lambda_{4,3}(x) = x_7$$

which result in the following guiding functions:

$$V_1(x) = \frac{1}{2}(x_1^2 + x_5^2)$$
$$V_2(x) = \frac{1}{2}(x_2^2 + x_{10}^2)$$
$$V_3(x) = \frac{1}{2}(x_3^2 + x_8^2 + x_9^2)$$
$$V_4(x) = \frac{1}{2}(x_4^2 + x_6^2 + x_7^2)$$
$$V(x) = V_1(x) + V_2(x) + V_3(x) + V_4(x).$$

By using these guiding functions, the extended strategy is applied to the system (3.139). Simulation results are shown in Figures 3.39 - 3.40 and demonstrate the effectiveness of the extended guiding functions strategy.

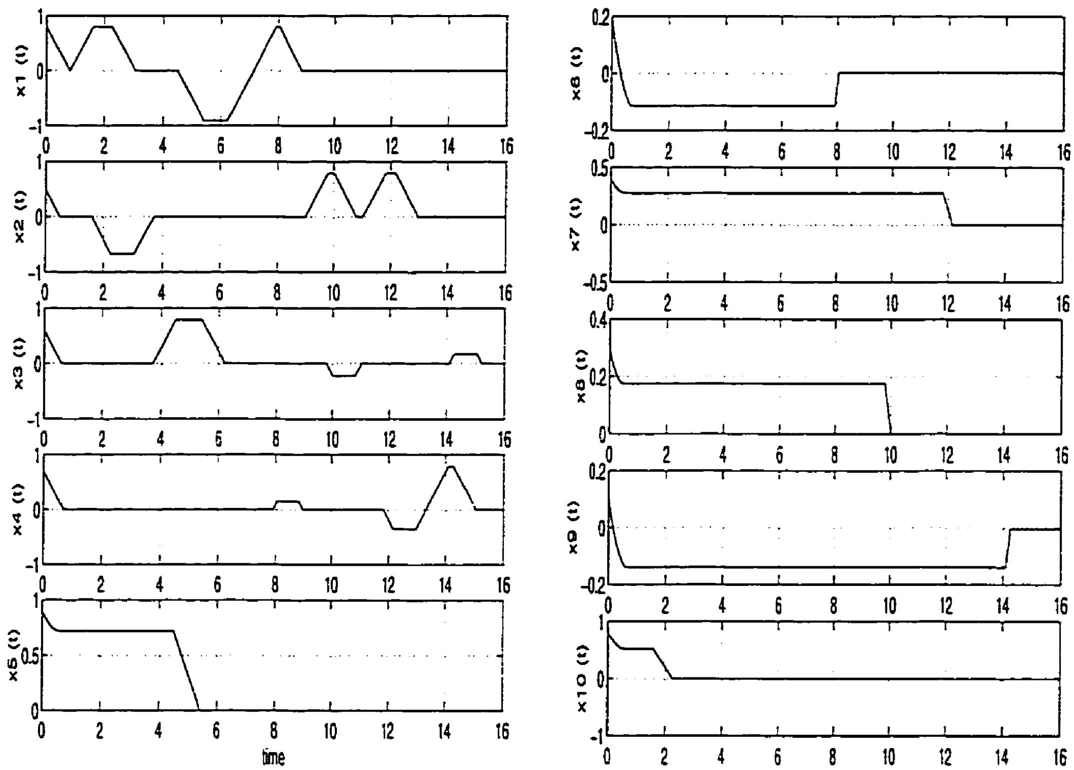

FIGURE 3.39. *General drift free system with $n - m = 6$:* Plots of the controlled state trajectories $t \mapsto (x_1(t), ..., x_{10}(t))$ versus time.

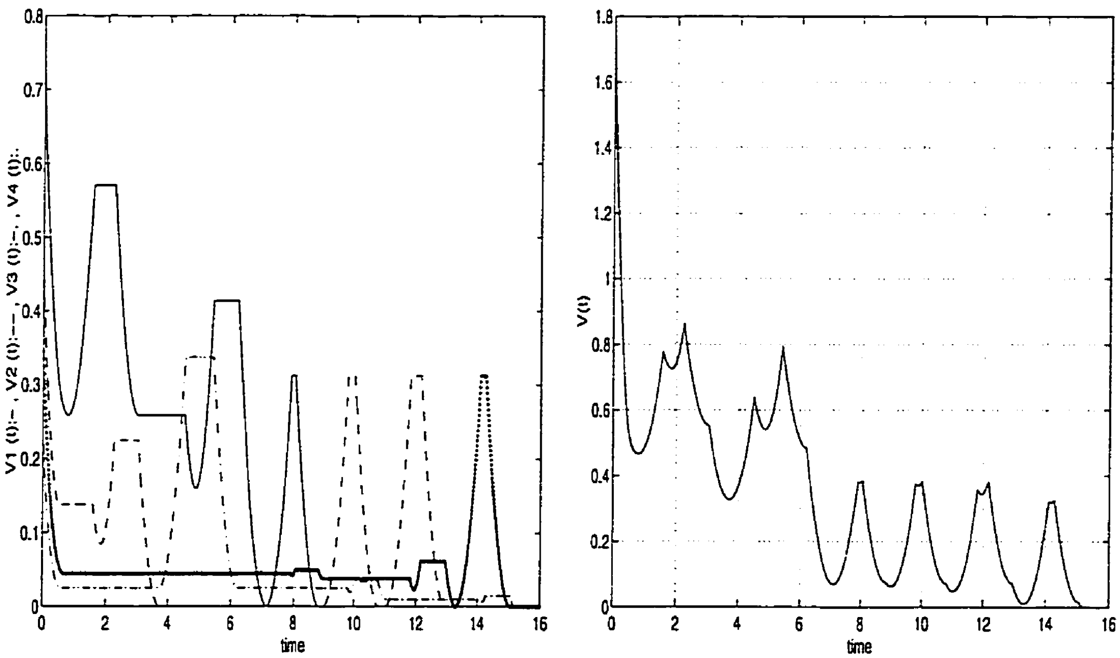

FIGURE 3.40. *General drift free system with $n - m = 6$:* Plots of the guiding functions $V_1(t)$,..., $V_4(t)$ and $V(t) = \sum_{i=1}^{4} V_i(t)$ versus time.

7.3. Stabilizing feedback control for a hopping robot in flight phase

This example demonstrates that the extended guiding functions strategy can be applied if the controllability Lie algebra of the system contains one Lie bracket of order two and in which $n-m=1$, without the necessity of introducing sinusoidal steering (compare with the examples of section 6).

A simplified kinematic model of a hopping robot in the flight phase can be given in the form of the following state space equations (see [79]):

$$\begin{aligned} \dot{\psi} &= u_1 \\ \dot{l} &= u_2 \\ \dot{\theta} &= -\frac{m(l+d)^2}{I+m(l+d)^2}u_1 \end{aligned} \tag{3.142}$$

The configuration variables ψ, l and θ have the following description: ψ is the angle of the hip of the hopping robot in the flight phase, l the length of the leg extension, and θ is the angle of the body of the robot, as shown in Figure 3.41. The remaining symbols represent constants: I is the moment of inertia of the body, m is the mass of the leg concentrating at the foot, and d is the upper leg length.

Assuming for simplicity that $m = I = d = 1$, and introducing a new set of state variables $x = (x_1, x_2, x_3) = (\psi, l+1, \theta)$, the kinematic model can be written as:

$$\begin{aligned} \dot{x_1} &= u_1 \\ \dot{x_2} &= u_2 \\ \dot{x_3} &= -\frac{x_2{}^2}{1+x_2{}^2}u_1 \\ \text{or} \quad \dot{x} &= g_1(x)u_1 + g_2(x)u_2, \qquad x \in I\!R^3 \end{aligned} \tag{3.143}$$

$$\text{where,} \quad g_1(x) = \frac{\partial}{\partial x_1} - \frac{x_2{}^2}{1+x_2{}^2}\frac{\partial}{\partial x_3}, \qquad g_2(x) = \frac{\partial}{\partial x_2}$$

To verify the LARC condition, we need the following Lie brackets of g_1 and g_2:

$$\begin{aligned} g_3(x) \overset{def}{=} [g_1, g_2](x) &= \frac{2x_2}{(1+x_2{}^2)^2}\frac{\partial}{\partial x_3} \\ g_4(x) \overset{def}{=} [g_2, [g_1, g_2]](x) &= \frac{2-6x_2{}^2}{(1+x_2{}^2)^3}\frac{\partial}{\partial x_3} \end{aligned}$$

which satisfy

$$span\{g_1(x), g_2(x), g_4(x)\} = I\!R^3 \qquad \text{for all} \qquad x \in I\!R^3 \tag{3.144}$$

The Lie brackets multiplication table for $L(g_1, g_2)$ is:

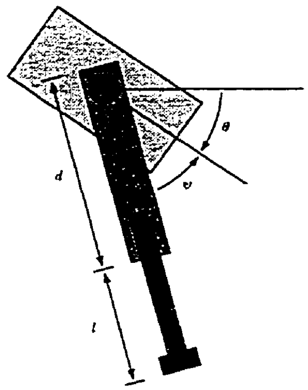

FIGURE 3.41. A simple hopping robot

$$[g_1, g_2] = g_3 \qquad [g_1, g_3] = 0 \qquad [g_2, g_3] = g_4$$
$$[g_1, g_4] = 0 \qquad [g_2, g_4] \neq 0 \tag{3.145}$$

which shows that the Lie algebra $L(g_1, g_2)$ is neither nilpotent nor finite dimensional. The table (3.145) also shows that the distributions

$$\Delta_1(x) \stackrel{def}{=} span\{g_1(x), g_4(x)\}, \qquad \Delta_2(x) \stackrel{def}{=} span\{g_2(x)\} \tag{3.146}$$

are involutive, and the corresponding codistributions can be represented as:

$$\Delta_1^{\perp}(x) = span\{d\lambda_{2,1}(x)\}, \qquad \Delta_2^{\perp}(x) = span\{d\lambda_{1,1}(x), d\lambda_{1,2}(x)\} \tag{3.147}$$

where the scalar functions $\lambda_{i,k}$ can be selected as follows:

$$\lambda_{1,1}(x) = x_1, \quad \lambda_{1,2}(x) = x_3, \quad \lambda_{2,1}(x) = x_2 \qquad \text{for all} \quad x \in I\!R^3 \tag{3.148}$$

The guiding functions for this system are hence defined globally :

$$V_1(x) \stackrel{def}{=} \frac{1}{2}(x_1^2 + x_3^2), \qquad V_2(x) \stackrel{def}{=} \frac{1}{2}(x_2^2), \quad x \in I\!R^3$$
$$V(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2), \quad x \in I\!R^3 \tag{3.149}$$

These guiding functions can be directly incorporated in a slight modification of the extended guiding functions strategy (which is basically applicable to those systems whose Lie algebras contain only first order Lie brackets) as explained below:

•a By employing Step 1 of the extended strategy in which:

$$u_i = -sign_\rho[L_{g_i} V_i(x)], \quad i = 1, 2, \quad \text{until} \quad L_{g_i} V_i(x) = 0$$

the system trajectories reach the ρ-neighbourhood of the set:

$$\mathcal{T} = \{x \in I\!R^3 : L_{g_i} V_i(x) = 0, \quad i = 1, 2\} = \{x \in I\!R^3 : x_1 = x_2 = 0, \quad x_3 \neq 0\}$$

in finite time.

•b As follows from the convergence analysis of section 4 of this Chapter the aim of the control of Step 2 of the feedback strategy is to steer the system away from the set of impasse points $\mathcal{T}$. This is achieved if $L_{g_i} V_i(x) \neq 0$ for some index i. The control which achieves this, is proposed by considering the first time derivative of $L_{g_i} V_i(x)$. In the case of the model of the hopping robot, the first time derivative of $L_{g_1} V_1(x)$ is computed as:

$$\frac{d}{dt} L_{g_1} V_1(x) = L^2_{g_1} V_1(x) u_1 + L_{g_2} L_{g_1} V_1(x) u_2 = L^2_{g_1} V_1(x) u_1 + L_{[g_1,g_2]} V_1(x) u_2$$

and $L_{[g_1,g_2]} V_1(x) = 2x_2 x_3/(1 + {x_2}^2)^2$ which is equal to zero at $x \in \mathcal{T}$. Therefore, this first derivative is of little help in determining the control which is able to increase the value of $|L_{g_1} V_1(x)|$. However, considering the second time derivative of $L_{g_1} V_1(x)$ provides more information about the way to choose the controls u_1 and u_2. Clearly,

$$\frac{d^2}{dt^2} L_{g_1} V_1(x) = L^3_{g_1} V_1(x) u_1^2 + L_{g_2} L_{g_1} V_1(x) u_1 u_2 + L_{g_1} L_{[g_1,g_2]} V_1(x) u_1 u_2 + L_{g_2} L_{[g_1,g_2]} V_1(x) u_2^2$$

and by choosing $u_1 = 0$ and $u_2 = 1$, while recalling that $L_{g_j} V_i(x) = 0$, if $i \neq j$, we obtain

$$\begin{aligned} \frac{d^2}{dt^2} L_{g_1} V_1(x) &= L_{g_2} L_{[g_1,g_2]} V_1(x) = L_{[g_2,[g_1,g_2]]} V_1(x) + L_{[g_1,g_2]} L_{g_2} V_1(x) \\ &= L_{[g_2,[g_1,g_2]]} V_1(x) \neq 0 \end{aligned}$$

as is guaranteed by the LARC condition (3.144) if only $x \neq 0$. Hence, Step 2 of the extended strategy can be stated as :

Employ the controls $u_1 = 0$ and $u_2 = 1$ until $L_{[g_2,[g_1,g_2]]} V_1(x) = 0$, or else until $V_2(x) \geq \alpha V(p)$.

At this stage, Steps 2c-2d of the extended guiding functions strategy can be applied without change.

Three sets of simulation results are depicted in Figures 3.42 - 3.43, 3.44 - 3.45, and 3.46 - 3.47 corresponding to initial conditions $[.5, .5, .5]^T$, $[100, 100, 100]^T$ and $[-100, -100, -100]^T$ respectively. Figures 3.44 - 3.45, and 3.46 - 3.47 demonstrate the global convergence property of the extended guiding functions strategy. In all these simulations the set point is reached in four steps of strategy.

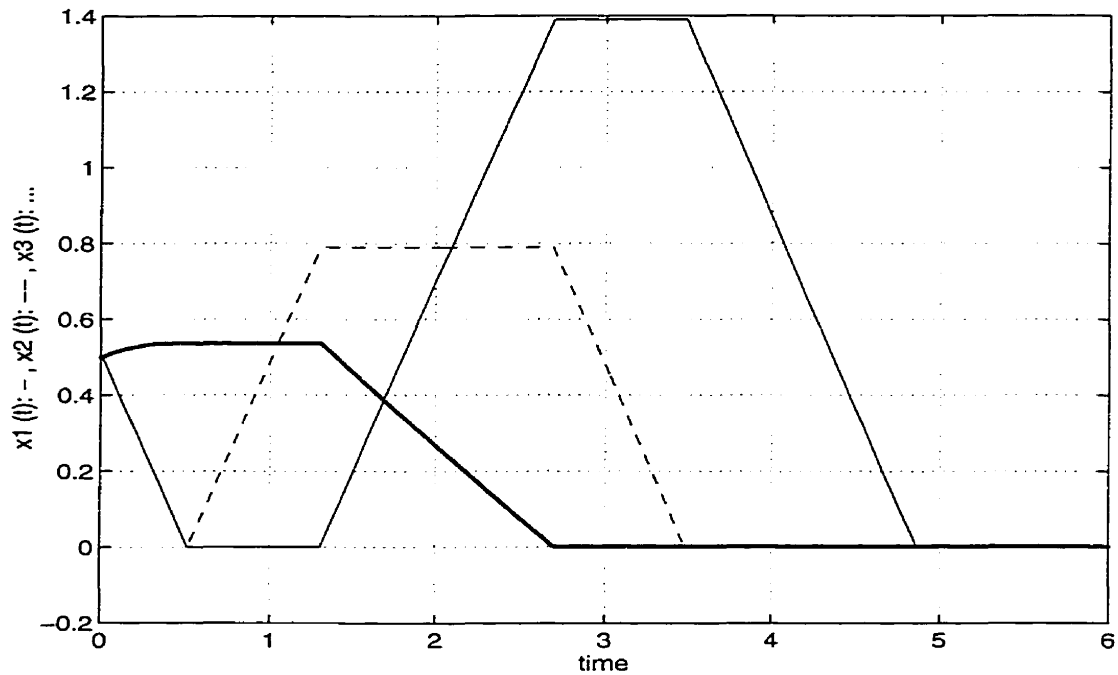

FIGURE 3.42. *Hopping robot:* Plots of the controlled state trajectories $t \mapsto (x_1(t), x_2(t), x_3(t))$ versus time corresponding to initial condition $(0.5, 0.5, 0.5)$.

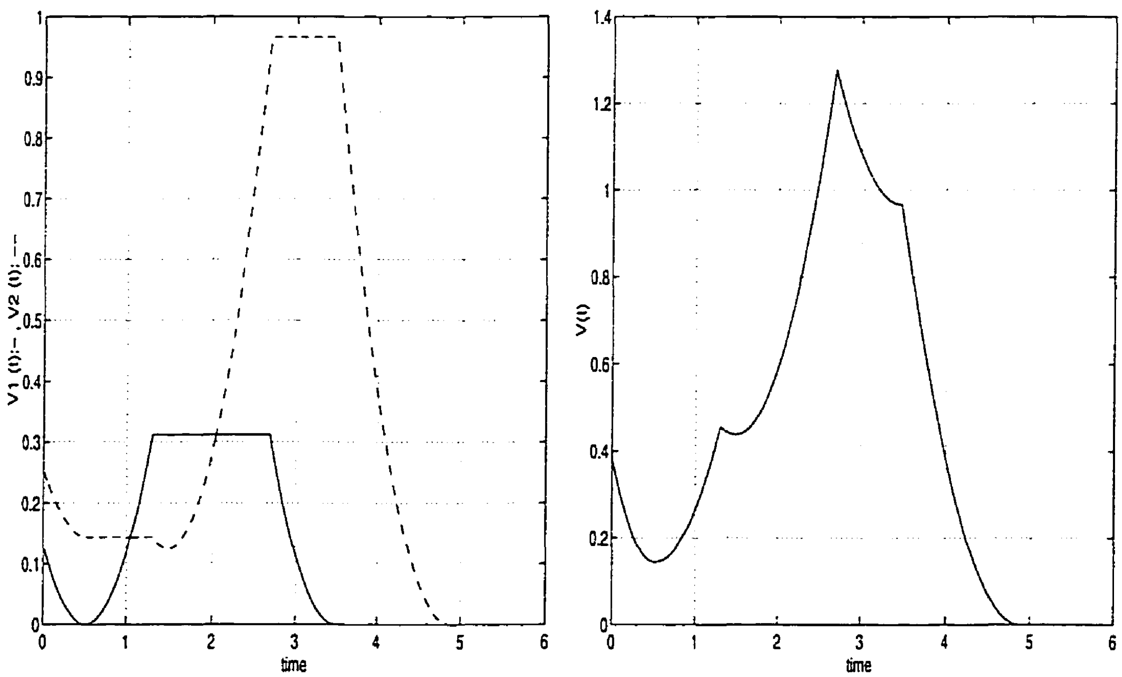

FIGURE 3.43. *Hopping robot:* Plots of the guiding functions $V_1(t)$, $V_2(t)$ and $V(t) = V_1(t) + V_2(t)$ versus time corresponding to Figure 3.42.

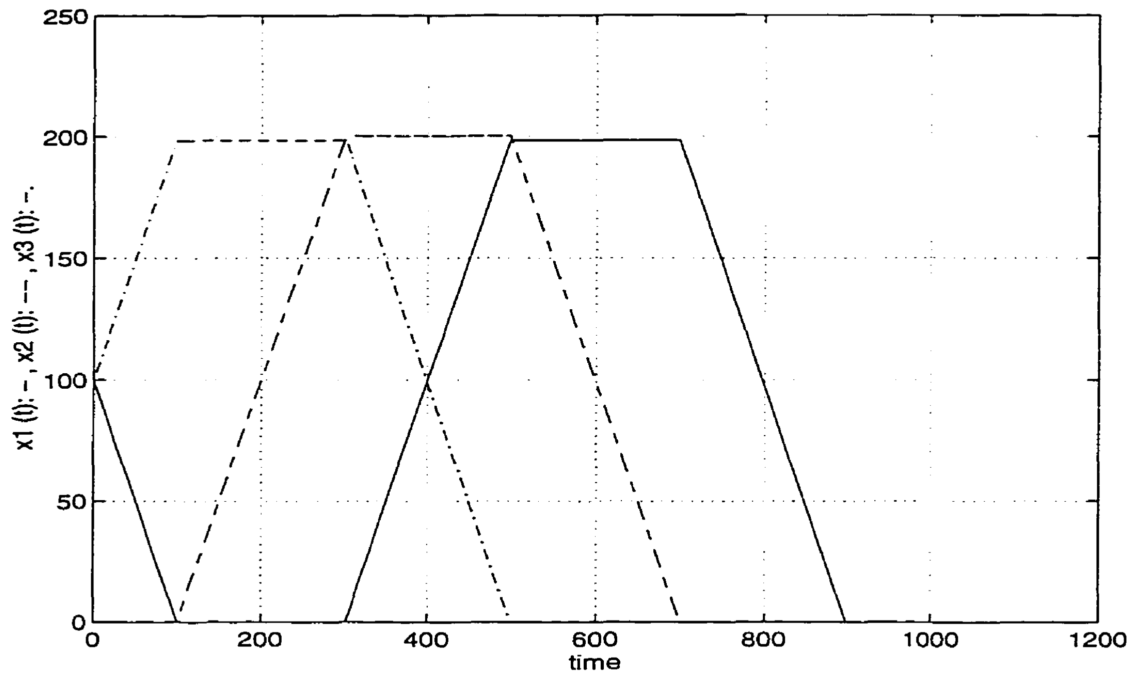

FIGURE 3.44. *Hopping robot:* Plots of the controlled state trajectories $t \mapsto (x_1(t), x_2(t), x_3(t))$ versus time corresponding to initial condition $(100, 100, 100)$.

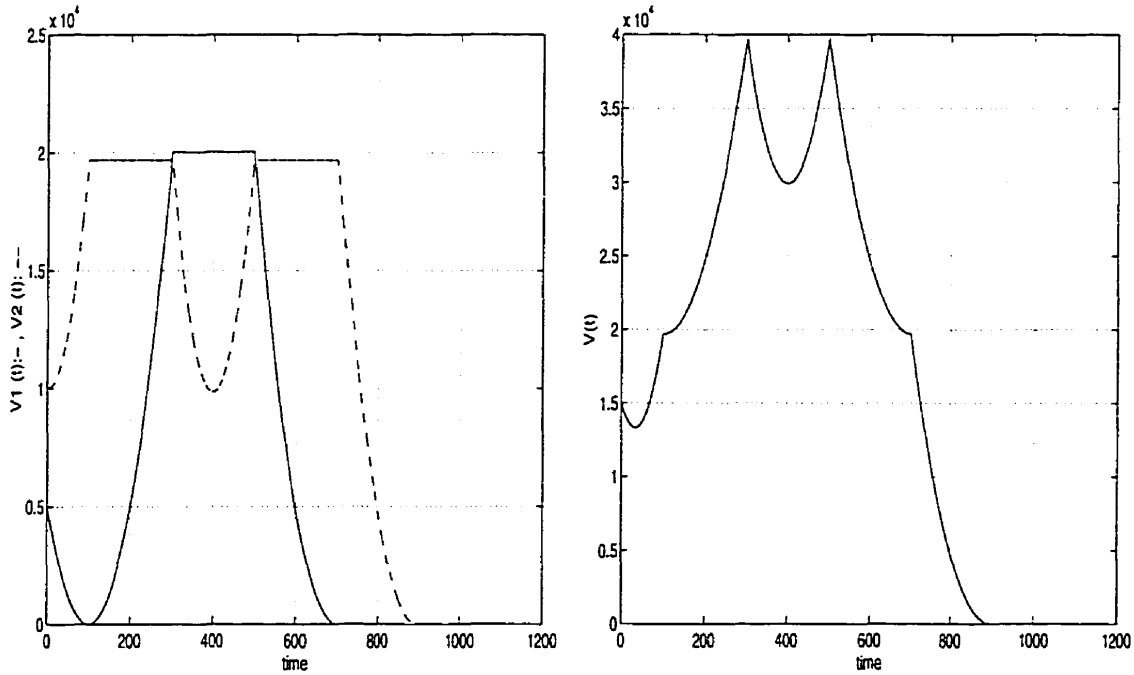

FIGURE 3.45. *Hopping robot:* Plots of the guiding functions $V_1(t)$, $V_2(t)$ and $V(t) = V_1(t) + V_2(t)$ versus time corresponding to Figure 3.44.

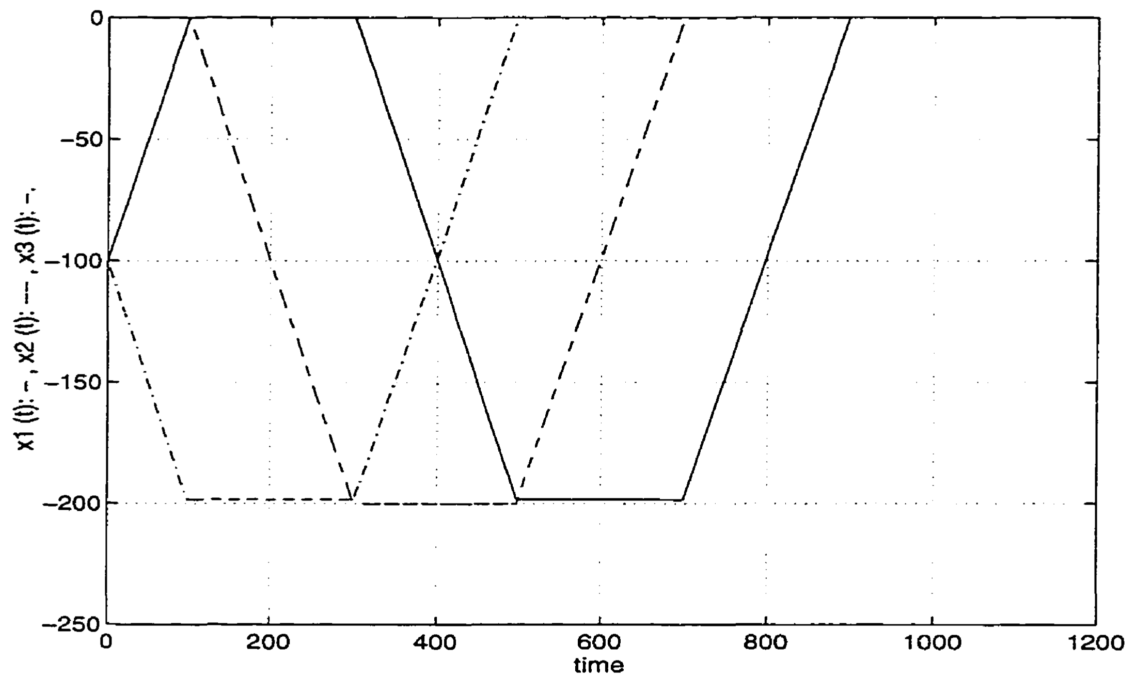

FIGURE 3.46. *Hopping robot:* Plots of the controlled state trajectories $t \mapsto (x_1(t), x_2(t), x_3(t))$ versus time corresponding to initial condition $(-100, -100, -100)$.

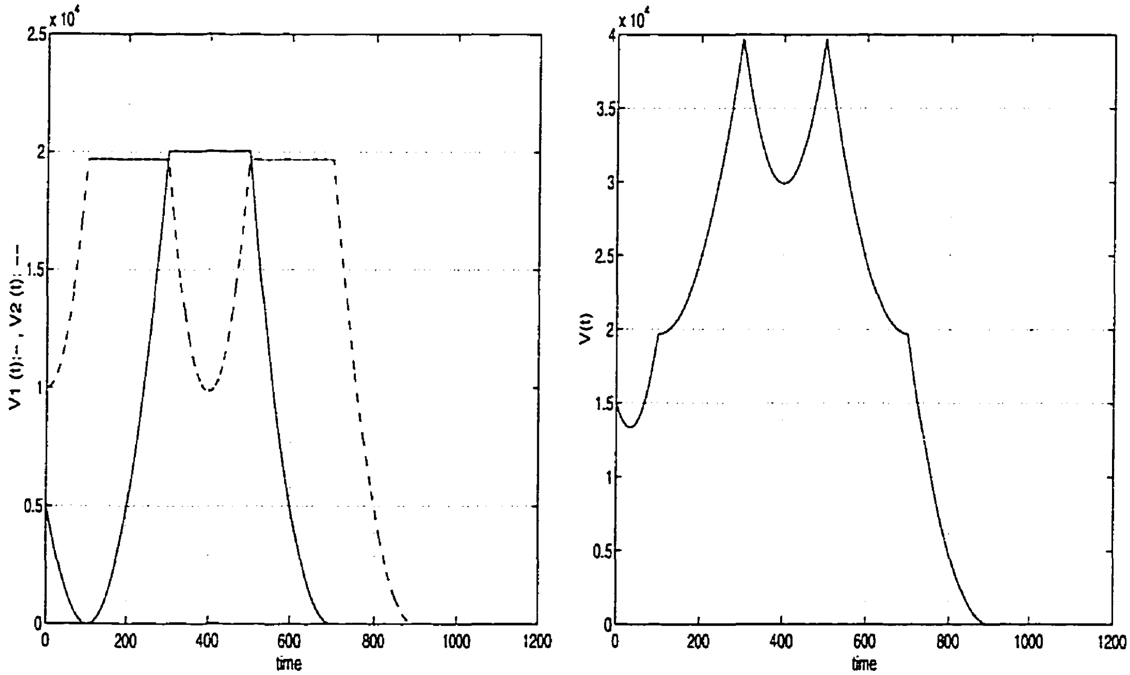

FIGURE 3.47. *Hopping robot:* Plots of the guiding functions $V_1(t)$, $V_2(t)$ and $V(t) = V_1(t) + V_2(t)$ versus time corresponding to Figure 3.46.

7.4. Stabilizing feedback control for a model of a fire truck by using the extended strategy [67]

The following example illustrates, how the extended strategy can be applied when the controllability Lie algebra of the system involves higher order Lie brackets. This example also demonstrates the robustness property of the extended strategy.

Consider the fire truck model as given in (3.111) and its extended system (3.123). The Lie brackets multiplication table (3.114) shows that, it is not possible to find three involutive distributions for the extended system (3.123), which satisfy assumption A5. For this reason we consider the approximate model (3.116). The extended system for this approximate model is:

$$\dot{x} = \tilde{g}_1 v_1 + \tilde{g}_2 v_2 + \tilde{g}_3 v_3 + [\tilde{g}_2, \tilde{g}_3] v_4 \tag{3.150}$$

where it is assumed that v_4 is realized indirectly by using the following standard sequence of controls:

$$(v_2, v_3) = ((1,0), (0,1), (-1,0), (0,-1))$$

The above extended system now provides for the following involutive distributions:

$$\begin{aligned}
\Delta_1(x) &\stackrel{def}{=} span\{\tilde{g}_2, \tilde{g}_3, [\tilde{g}_1, \tilde{g}_3], [\tilde{g}_2, \tilde{g}_3], [[\tilde{g}_2, \tilde{g}_3], \tilde{g}_3]\}(x) \\
\Delta_2(x) &\stackrel{def}{=} span\{\tilde{g}_1, \tilde{g}_3, [\tilde{g}_1, \tilde{g}_3], [\tilde{g}_2, \tilde{g}_3], [[\tilde{g}_2, \tilde{g}_3], \tilde{g}_3]\}(x) \\
\Delta_3(x) &\stackrel{def}{=} span\{\tilde{g}_1, \tilde{g}_3\}(x)
\end{aligned}$$

and the corresponding codistributions have the following expressions as linear spans of exact differentials :

$$\begin{aligned}
&\Delta_1^{\perp}(x) \stackrel{def}{=} span\{d\lambda_{1,1}(x)\}, \qquad \Delta_2^{\perp}(x) \stackrel{def}{=} span\{d\lambda_{2,1}(x)\} \\
&\Delta_3^{\perp}(x) \stackrel{def}{=} span\{d\lambda_{3,1}(x), d\lambda_{3,2}(x), d\lambda_{3,3}(x), d\lambda_{3,4}(x)\}
\end{aligned}$$

The choice of the scalar functions $\lambda_{i,k}$, is immediate:

$$\begin{aligned}
&\lambda_{1,1}(x) \stackrel{def}{=} x_1, \quad \lambda_{2,1}(x) \stackrel{def}{=} x_2, \\
&\lambda_{3,1}(x) \stackrel{def}{=} x_3, \quad \lambda_{3,2}(x) \stackrel{def}{=} x_4, \quad \lambda_{3,3}(x) \stackrel{def}{=} x_5, \quad \lambda_{3,4}(x) \stackrel{def}{=} x_6.
\end{aligned}$$

Hence the guiding functions for the approximate system (3.116) are:

$$\begin{aligned}
V_1(x) &\stackrel{def}{=} \frac{1}{2}x_1^2, \quad V_2(x) \stackrel{def}{=} \frac{1}{2}x_2^2 \\
V_3(x) &\stackrel{def}{=} \frac{1}{2}(x_3^2 + x_4^2 + x_5^2 + x_6^2)
\end{aligned}$$

These guiding functions are next used in extended strategy and applied to the original model. The motion along the Lie bracket $[g_2, g_3]$ is realized indirectly by using the standard sequence of controls:

$$(u_2, u_3) = ((1,0), (0,1), (-1,0), (0,-1))$$

Simulation results are shown in Figures 3.48 - 3.51 and clearly demonstrate the robustness of this strategy. By comparing the simulation results 3.48 - 3.51, 3.48 - 3.51 obtained by the extended strategy and the combined strategy with sinusoidal steering, it is evident that the desired set point is reached faster by using the extended strategy.

7.5. Stabilizing feedback control for a class of wheeled mobile robots [71]

In this section, the extended guiding functions strategy is employed to three categories of mobile robots. There is a growing interest in feedback control design for such mobile robots, as it raises practical and theoretically challenging issues.

In the sequel, the abbreviation "WMR of type (δ_m, δ_s)" is used to denote wheeled mobile robots of degree of mobility δ_m and degree of steeribility δ_s; see [3] for the definition of degree of mobility and steeribilty. The application of our strategy to three types of WMR is discussed below.

7.5.1. Stabilizing feedback control for a WMR of type (2, 1)

This type of wheeled mobile robots is easy to control as its controllability Lie algebra contains only Lie brackets of depth one.

The kinematic model of WMR of type (2, 1) is given by, (see [3]):

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} &= \begin{pmatrix} -sin(\theta+\beta) & 0 \\ cos(\theta+\beta) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \\ \dot{\beta} &= \eta_3 \end{aligned} \tag{3.151}$$

The notation $(x, y, \theta, \beta) = (x_1, x_2, x_3, x_4)$ and $(\eta_1, \eta_2, \eta_3) = (u_3, u_2, u_1)$, is used for simplicity, so that (3.151) becomes:

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2 + g_3(x)u_3, \quad x \in I\!R^4 \tag{3.152}$$

$$g_1(x) = \frac{\partial}{\partial x_4}, \qquad g_2(x) = \frac{\partial}{\partial x_3}$$

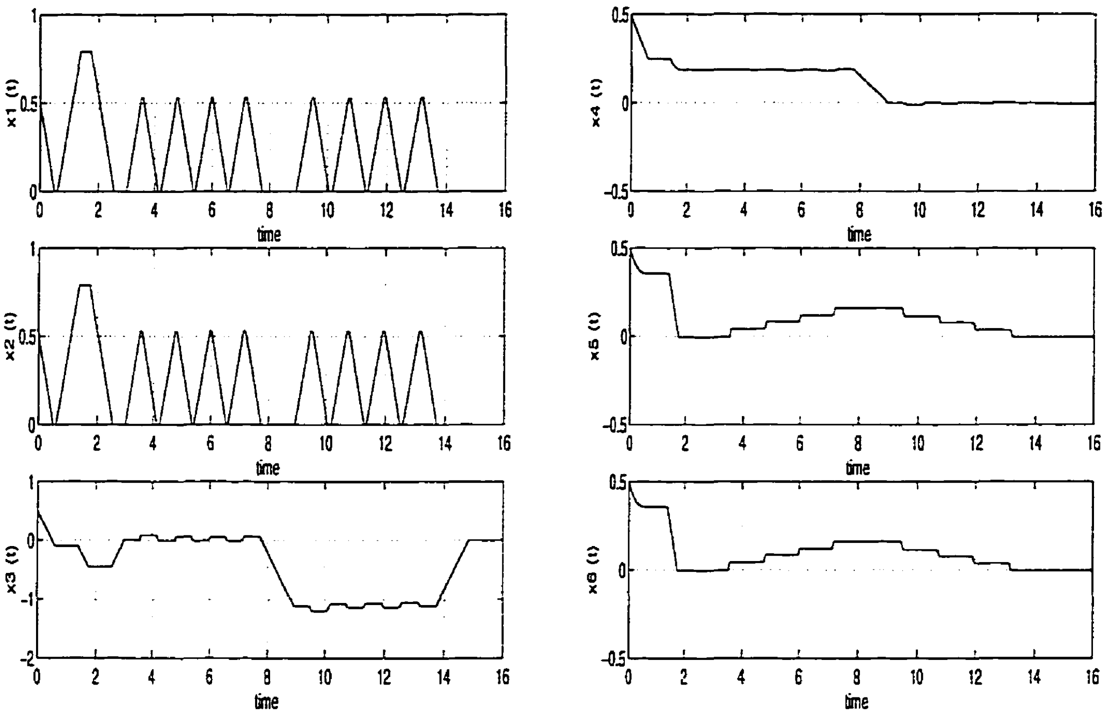

FIGURE 3.48. *Fire truck model :* Plots of the controlled state trajectories $t \mapsto (x_1(t), ..., x_6(t))$ versus time.

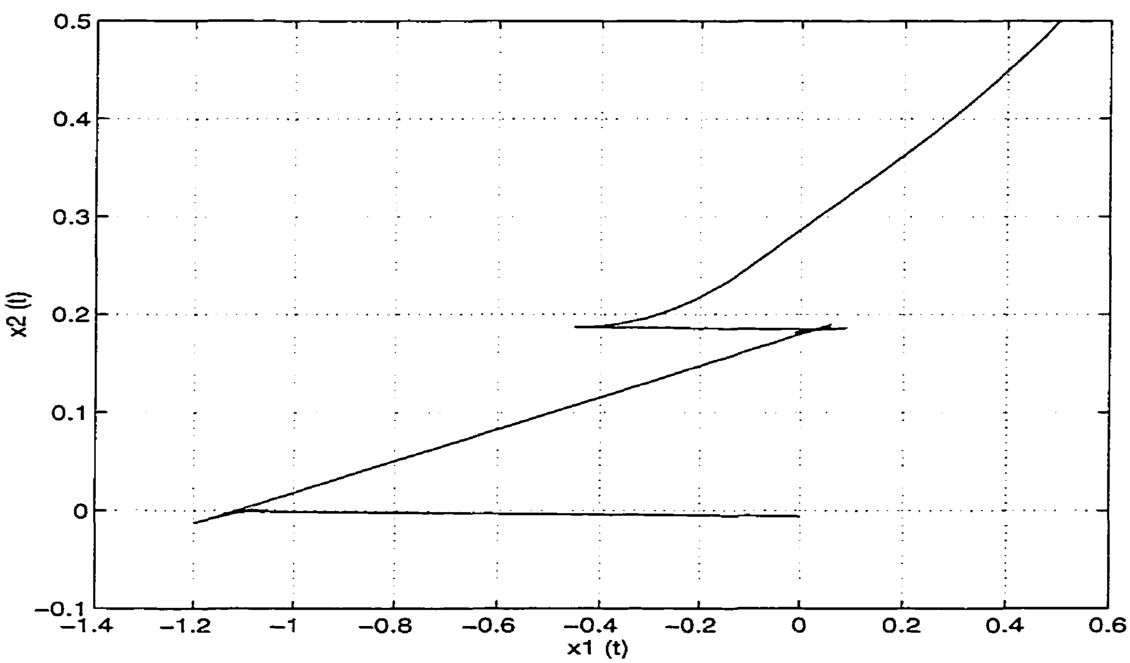

FIGURE 3.49. *Fire truck model :* Plot of the controlled state trajectory $x_3(t) = x(t)$ versus $x_4(t) = y(t)$.

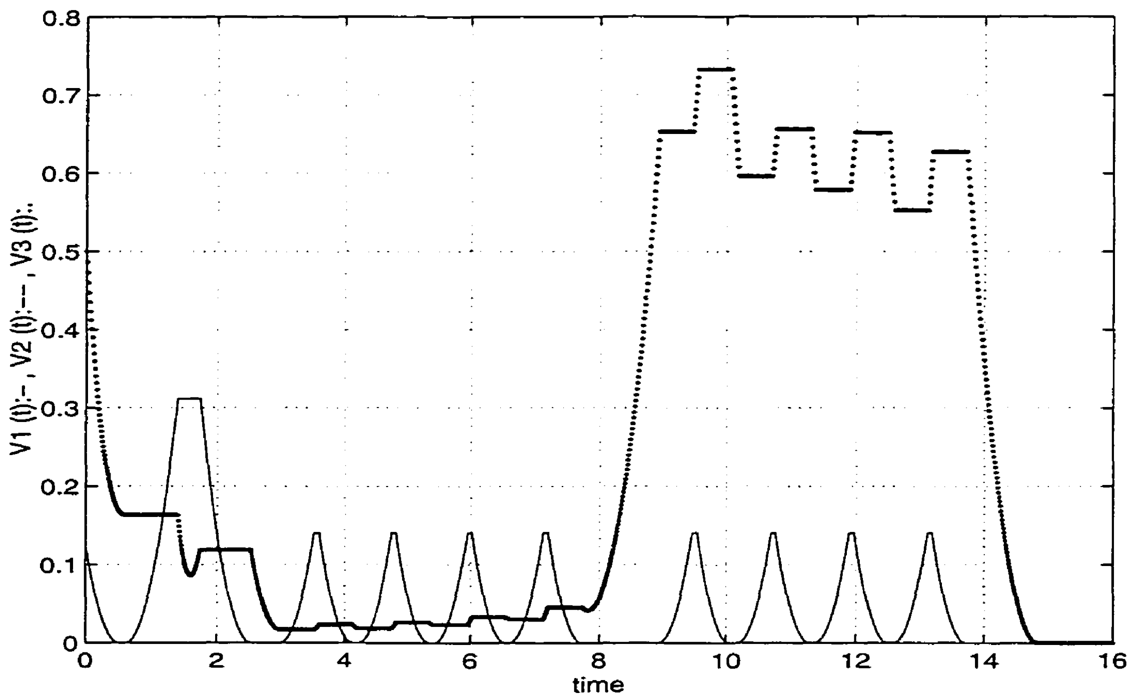

FIGURE 3.50. *Fire truck model :* Plots of the guiding functions $V_1(t)$ and $V_2(t)$ versus time.

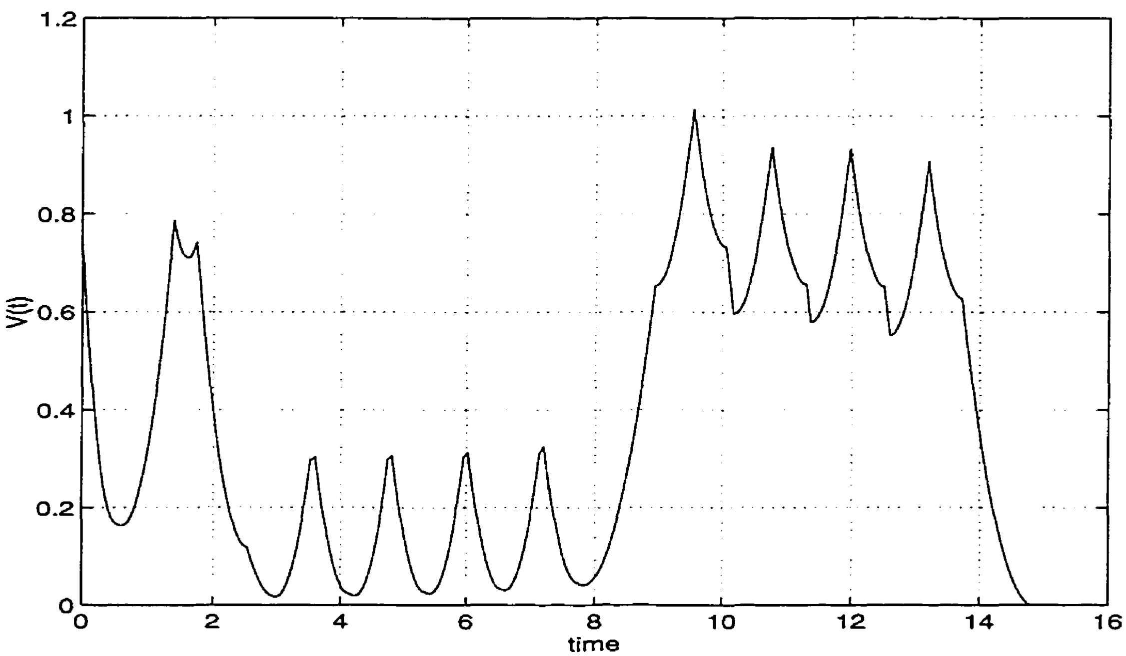

FIGURE 3.51. *Fire truck model :* Plot of the guiding function $V(t) = V_1(t) + V_2(t) + V_3(t)$ versus time.

$$g_3(x) = -sin(x_3 + x_4)\,\frac{\partial}{\partial x_1} + cos(x_3 + x_4)\,\frac{\partial}{\partial x_2}$$

The LARC condition is satisfied since

$$span\{g_1, g_2, g_3, g_4\}(x) = I\!R^4, \quad \text{for all} \;\; x \in I\!R^4 \tag{3.153}$$

$$\text{where,} \quad g_4(x) = -cos(x_3 + x_4)\,\frac{\partial}{\partial x_1} - sin(x_3 + x_4)\,\frac{\partial}{\partial x_2}$$

The Lie brackets multiplication table for $L(g_1, g_2, g_3)$ is given by

$$\begin{aligned} &[g_1, g_3] = g_4, \quad [g_2, g_3] = g_4, \quad [g_1, g_2] = 0 \\ &[g_3, g_4] = 0, \quad [g_1, g_4] = -g_3, \quad [g_2, g_4] = -g_3 \end{aligned} \tag{3.154}$$

which shows that the Lie algebra $L(g_1, g_2, g_3)$ is finite dimensional but not nilpotent. It also follows directly from the multiplication table (3.154), that the following distributions are involutive :

$$\begin{aligned} \Delta_1(x) &\overset{def}{=} span\{g_2, g_3, g_4\}(x) \\ \Delta_2(x) &\overset{def}{=} span\{g_1, g_3, g_4]\}(x) \\ \Delta_3(x) &\overset{def}{=} span\{g_1, g_2\}(x) \end{aligned}$$

From the Frobenius Theorem, the corresponding codistributions have the following expressions as linear spans of exact differentials :

$$\begin{aligned} \Delta_1^{\perp}(x) &= span\{d\lambda_{1,1}(x)\} \\ \Delta_2^{\perp}(x) &= span\{d\lambda_{2,1}(x)\} \\ \Delta_3^{\perp}(x) &= span\{d\lambda_{3,1}(x), d\lambda_{3,2}(x)\} \end{aligned}$$

where the scalar functions $\lambda_{i,k}$, can be chosen to be :

$$\lambda_{1,1}(x) = x_4, \;\; \lambda_{2,1}(x) = x_3, \;\; \lambda_{3,1}(x) = x_2, \;\; \lambda_{3,2}(x) = x_1$$

The guiding functions are hence given by :

$$V_1(x) \overset{def}{=} \frac{1}{2}x_4^2, \quad V_2(x) \overset{def}{=} \frac{1}{2}x_3^2, \quad V_3(x) \overset{def}{=} \frac{1}{2}(x_1^2 + x_2^2) \tag{3.155}$$

and are used in the application of the stabilizing strategy to model (3.151).

The simulation results are shown in Figures 3.52 - 3.54 from which it can be seen that the stabilization control task is performed in finite time.

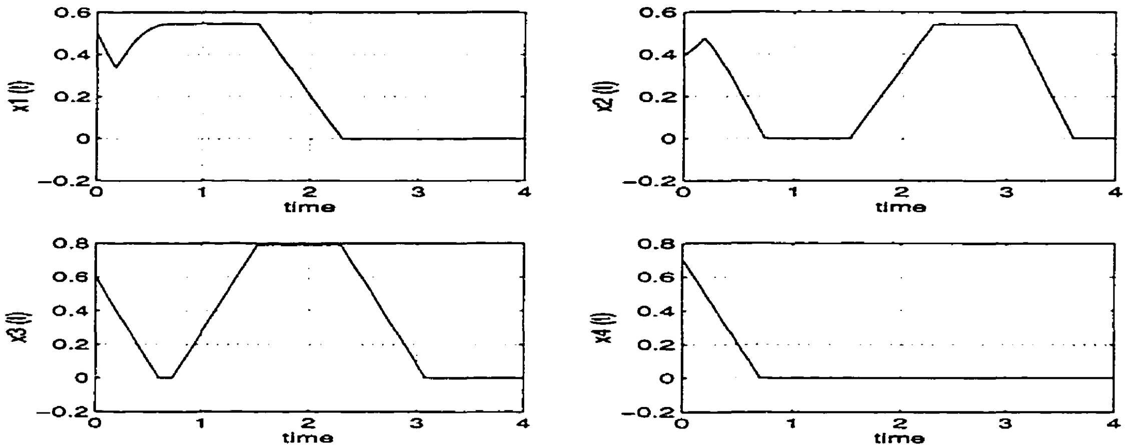

FIGURE 3.52. *WMR of type (2,1):* Plots of the controlled state trajectories $t \mapsto (x_1(t), \ldots, x_4(t))$ versus time.

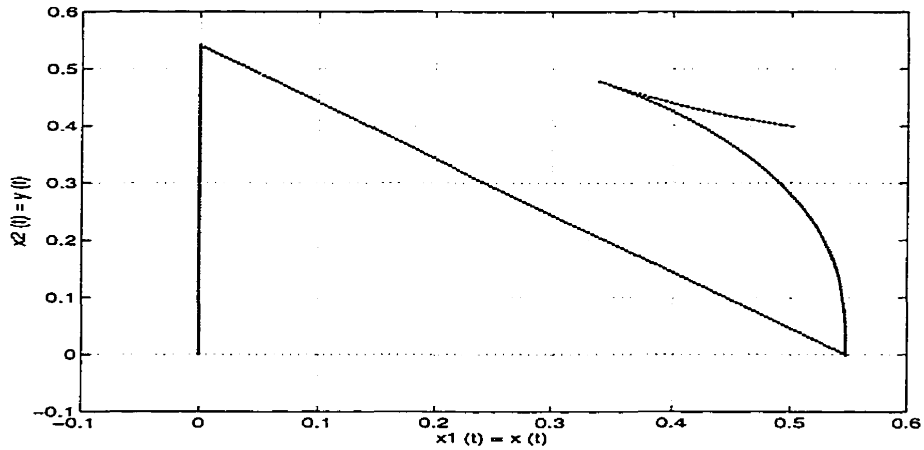

FIGURE 3.53. *WMR of type (2,1):* Plot of the controlled state trajectory $x_1(t)$ versus $x_2(t)$.

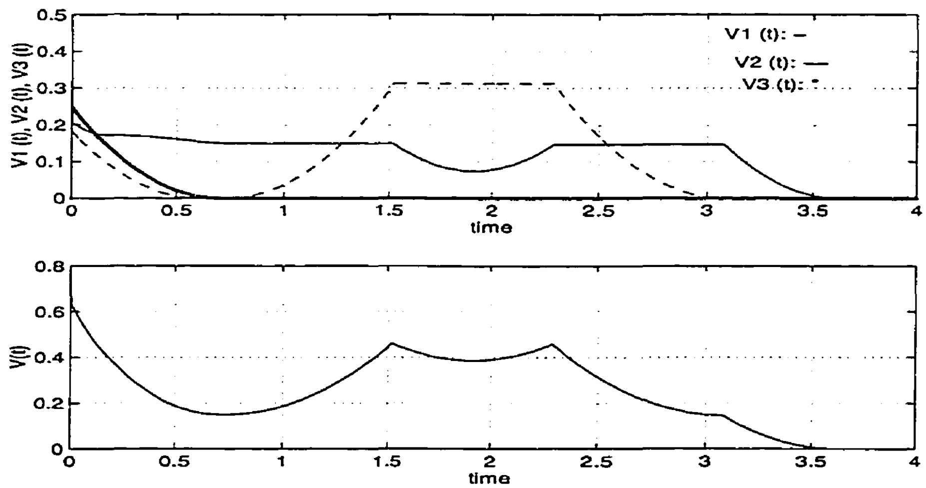

FIGURE 3.54. *WMR of type (2,1):* Plots of the guiding functions $V_1(t)$, $V_2(t)$, $V_3(t)$ and their sum $V(t)$ versus time.

7.5.2. Control of WMR of type $(1,2)$

Although the controllability Lie algebra of this class of wheeled mobile robots contains Lie brackets of depth one, it is not possible to find m distributions which satisfy assumption A5. Proceeding similarly as before an approximate model is sought which satisfies A5.

In this example, the desired set point is chosen to be different from the origin as at the origin, the kinematic model of this class of robots (see [3]) does not satisfy the LARC condition.

The kinematic model of the WMR of type $(1,2)$ is given by, see [3] :

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -2\ L\ cos\ \theta\ sin\ \beta_1\ sin\ \beta_2\ - L\ sin\theta\ sin\ (\beta_1+\beta_2) \\ -2\ L\ sin\ \theta\ sin\ \beta_1\ sin\ \beta_2 + L\ cos\ \theta\ sin\ (\beta_1+\beta_2) \\ sin\ (\beta_2-\beta_1) \end{bmatrix} \eta_1$$

$$\dot{\beta}_1 = \xi_1$$

$$\dot{\beta}_2 = \xi_2 \tag{3.156}$$

Here (x, y) are the Cartesian coordinates of a point P of the wheeled robot platform, θ is the orientation of the platform with respect to the horizontal axis, β_i, $i = 1,2$ are the orientation angles of the independent steering wheels and L is the distance between P and the centre of the master wheel.

Without the loss of generality, let the desired rest point be given by:

$$(x, y, \theta, \beta_1, \beta_2) \quad = \quad (0, 0, 0, \pi/2, \pi/2)$$

The set point stabilization problem for a WMR of type $(1,2)$ can now be stated :

Find a feedback control which stabilizes the system described by (3.156) on the manifold $\mathcal{M}$:

$$\mathcal{M} \stackrel{def}{=} \{(x, y, \theta, \beta_1, \beta_2)^T \in I\!R^5 : \beta_1, \beta_2 \neq 0[\pi]\} \tag{3.157}$$

to the set point $x_0 = (0,0,0,\pi/2,\pi/2) \in \mathcal{M}$.

It should be noted that restricting the motion of the robot to manifold $\mathcal{M}$ is necessary for controllability purposes; at points $(0,0,0,0[\pi],0[\pi])$ the system fails to satisfy the LARC condition.

For simplicity, we assume that $L = 1$, and define:

$$\begin{aligned} (x, y, \theta, \beta_1, \beta_2) \quad &= \quad (x_1, x_2, x_3, x_4 + \pi/2, x_5 + \pi/2) \\ (u_3, u_2, u_1) \quad &= \quad (\eta_1, \xi_1, \xi_2) \end{aligned}$$

With this notation, (3.156) takes the following vector form :

$$\dot{x} \;=\; g_1(x)u_1 + g_2(x)u_2 + g_3(x)u_3, \quad x \in \mathcal{M} \tag{3.158}$$

$$\text{where,} \quad g_1(x) = \frac{\partial}{\partial x_5}, \qquad g_2(x) = \frac{\partial}{\partial x_4}$$

$$g_3(x) = \{-2cosx_3cosx_4cosx_5 + sinx_3sin(x_4+x_5)\}\frac{\partial}{\partial x_1}$$

$$-\{2sinx_3cosx_4cosx_5 + cosx_3sin(x_4+x_5)\}\frac{\partial}{\partial x_2} + sin(x_5 - x_4)\frac{\partial}{\partial x_3}$$

To satisfy the LARC condition, we need the following Lie brackets:

$$\begin{aligned} g_4(x) \stackrel{def}{=} [g_3, g_2](x) &= \{-2cosx_3sinx_4cosx_5 - sinx_3cos(x_4+x_5)\}\ \frac{\partial}{\partial x_1} \\ &- \{2sinx_3sinx_4cosx_5 - cosx_3cos(x_4+x_5)\}\ \frac{\partial}{\partial x_2} + cos(x_5 - x_4)\ \frac{\partial}{\partial x_3} \\ g_5(x) \stackrel{def}{=} [g_3, g_1](x) &= \{-2cosx_3cosx_4sinx_5 - sinx_3cos(x_4+x_5)\}\ \frac{\partial}{\partial x_1} \\ &- \{2sinx_3cosx_4sinx_5 - cosx_3cos(x_4+x_5)\}\ \frac{\partial}{\partial x_2} - cos(x_5 - x_4)\ \frac{\partial}{\partial x_3} \end{aligned}$$

so that

$$span\{g_i(x),\ i = 1,...,5\} = I\!R^5 \qquad \text{for all} \quad x \in \mathcal{M}. \tag{3.159}$$

The Lie brackets multiplication table for $L(g_1, g_2, g_3)$ is computed

$$[g_3, g_2] = g_4, \quad [g_3, g_1] = g_5, \quad [g_1, g_2] = 0, \quad [g_i, g_j] \neq 0, \quad i = 1,2,3, \quad j = 4,5 \tag{3.160}$$

and shows that the Lie algebra $L(g_1, g_2, g_3)$ is neither nilpotent nor finite dimensional. It is also clear from the table (3.160) that for system (3.158), it is not possible to find three involutive distributions satisfying A5.

Consider the following approximation to model (3.158):

$$\dot{x} = \bar{g}_1\bar{u}_1 + \bar{g}_2\bar{u}_2 + \bar{g}_3\bar{u}_3 \tag{3.161}$$

$$\bar{g}_3(x) = -2\frac{\partial}{\partial x_1} - (x_4 + x_5)\frac{\partial}{\partial x_2} + (x_5 - x_4)\frac{\partial}{\partial x_3}$$

$$\bar{g}_2(x) = \frac{\partial}{\partial x_4}, \qquad \bar{g}_1(x) = \frac{\partial}{\partial x_5}$$

with

$$\bar{g}_4(x) \stackrel{def}{=} [\bar{g}_1, \bar{g}_3](x) = -\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}, \qquad \bar{g}_5(x) \stackrel{def}{=} [\bar{g}_2, \bar{g}_3](x) = -\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}$$

yields

$$span\{\tilde{g}_i(x),\ i = 1, ..., 5\} = I\!R^5, \qquad \text{for all} \quad x \in I\!R^5 \tag{3.162}$$

and a new Lie brackets multiplication table for the Lie algebra $L(\tilde{g}_1, \tilde{g}_2, \tilde{g}_3)$:

$$[\tilde{g}_1, \tilde{g}_3] = \tilde{g}_4, \quad [\tilde{g}_2, \tilde{g}_3] = \tilde{g}_5, \quad [\tilde{g}_1, \tilde{g}_2] = 0, \quad [\tilde{g}_i, \tilde{g}_j] = 0, \quad j = 4, 5, \quad i = 1, 2, 3. \tag{3.163}$$

It follows that the distributions :

$$\Delta_1(x) \stackrel{def}{=} span\{\tilde{g}_2, \tilde{g}_3, \tilde{g}_4, \tilde{g}_5\}(x)$$
$$\Delta_2(x) \stackrel{def}{=} span\{\tilde{g}_1, \tilde{g}_3, \tilde{g}_4, \tilde{g}_5\}(x), \qquad \Delta_3(x) \stackrel{def}{=} span\{\tilde{g}_1, \tilde{g}_2\}$$

are all involutive and the corresponding codistributions thus have the following expressions:

$$\Delta_1^{\perp}(x) = span\{d\lambda_{1,1}(x)\}, \qquad \Delta_2^{\perp}(x) = span\{d\lambda_{2,1}(x)\}$$
$$\Delta_3^{\perp}(x) = span\{d\lambda_{3,1}(x), d\lambda_{3,2}(x), d\lambda_{3,3}(x)\}, \quad \text{in which}$$
$$\lambda_{1,1}(x) = x_5, \quad \lambda_{2,1}(x) = x_4, \quad \lambda_{3,1}(x) = x_3, \quad \lambda_{3,2}(x) = x_2, \quad \lambda_{3,3}(x) = x_1$$

can be selected. The resulting guiding functions are hence defined by:

$$V_1(x) \stackrel{def}{=} \frac{1}{2}x_5^2, \quad V_2(x) \stackrel{def}{=} \frac{1}{2}x_4^2, \quad V_3(x) \stackrel{def}{=} \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) \tag{3.164}$$

and incorporated into the stabilizing strategy which is applied to the original system (3.156). Simulation results are shown in Figures 3.55 - 3.57 and again confirm the robustness property of the strategy and its fast convergence properties.

7.5.3. Control of WMR of type (1, 1)

The controllability Lie algebra of this type of mobile robots contains Lie brackets of depth one as well as depth two. It is hence not possible to employ the extended strategy directly. The approximation technique is thus applied in inconjunction with sinusoidal steering.

The kinematic state space model of WMR of type (1, 1) is given by, see [**3**]

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} &= \begin{pmatrix} -L\ sin\ \theta\ sin\ \beta \\ L\ cos\ \theta\ sin\ \beta \\ cos\ \beta \end{pmatrix} \eta_1 \\ \dot{\beta} &= \xi_1 \end{aligned} \tag{3.165}$$

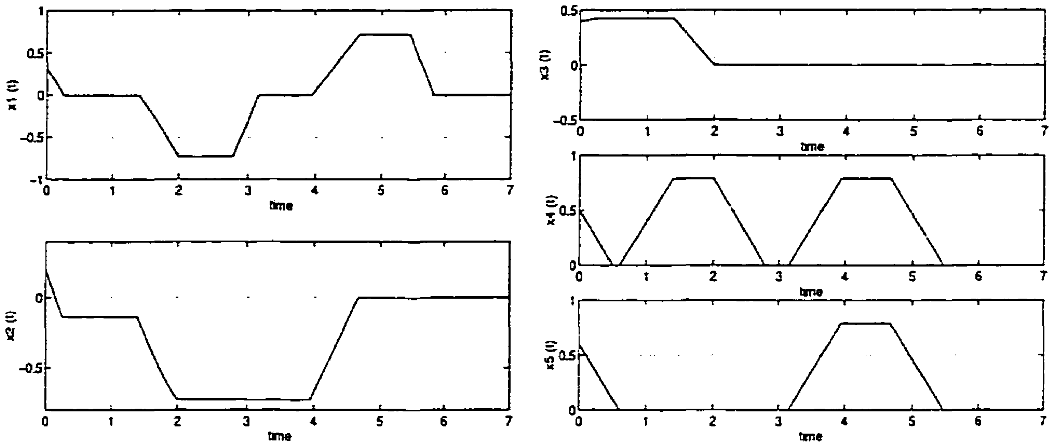

FIGURE 3.55. *WMR of type (1,2):* Plots of the controlled state trajectories $t \mapsto (x_1(t), ..., x_5(t))$ versus time.

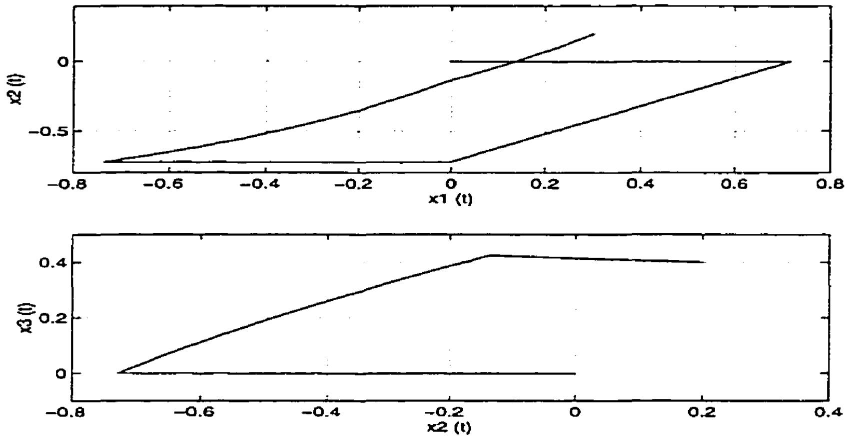

FIGURE 3.56. *WMR of type (1,2):* Plots of the controlled state variables $x_1(t)$ versus $x_2(t)$ and $x_2(t)$ versus $x_3(t)$.

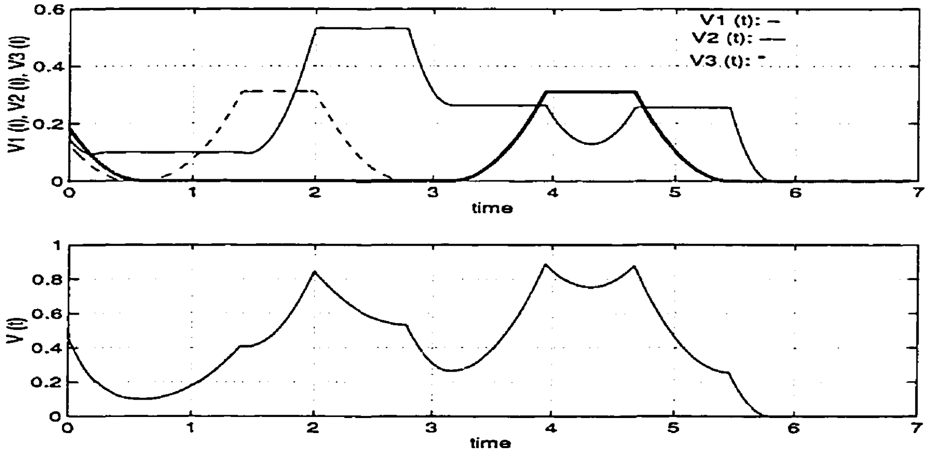

FIGURE 3.57. *WMR of type (1,2):* Plots of the guiding functions $V_1(t)$, $V_2(t)$, $V_3(t)$ and their sum $V(t)$ versus time.

where (x, y) are the Cartesian coordinates of a point P of the wheeled mobile robots platform, θ is the orientation of the platform with respect to the horizontal axis, β is the orientation angle of the independent steering wheel, and L is the distance between P and the centre of wheel. For simplicity, we take $L = 1$ and denote :

$$(x, y, \theta, \beta) = (x_1, x_2, x_3, x_4), \qquad (u_1, u_2) = (\eta_1, \xi_1)$$

so that (3.165) is written as :

$$\begin{aligned} \dot{x} &= g_1(x)\, u_1 + g_2(x)\, u_2, \qquad x \in I\!R^4 \\ g_1(x) &= -sin\ x_3 sin\ x_4 \frac{\partial}{\partial x_1} + cos\ x_3\ sin\ x_4 \frac{\partial}{\partial x_2} + cos\ x_4 \frac{\partial}{\partial x_3}, \quad g_2(x) = \frac{\partial}{\partial x_4} \end{aligned} \tag{3.166}$$

Calculation of the Lie brackets of g_1 and g_2:

$$g_3(x) \stackrel{def}{=} [g_1, g_2](x) = sin\ x_3\ cos\ x_4 \frac{\partial}{\partial x_1} - cos\ x_3\ cos\ x_4 \frac{\partial}{\partial x_2} + sin\ x_4 \frac{\partial}{\partial x_3}$$

$$g_4(x) \stackrel{def}{=} [g_1, [g_1, g_2]](x) = cos\ x_3 \frac{\partial}{\partial x_1} + sin\ x_3 \frac{\partial}{\partial x_2}$$

shows that the LARC condition is satisfied:

$$span\{g_1(x), g_2(x), g_3(x), g_4(x)\} = I\!R^4, \quad \text{for all } x \in I\!R^4$$

and the Lie brackets multiplication table for $L(g_1, g_2)$

$$[g_1, g_2] = g_3, \quad [g_1, g_3] = g_4, \quad [g_2, g_4] = 0, \quad [g_2, g_3] = g_1, \quad [g_1, g_4] \neq 0, \tag{3.167}$$

From table (3.167) it follows that finding two involutive distributions for system (3.166) is not possible. However, a controllable approximation to (3.166), such as, for example,

$$\dot{x} = \bar{g}_1(x)\, v_1 + \bar{g}_3(x)\, v_3 \tag{3.168}$$

$$\text{where,} \quad g_1(x) \approx \bar{g}_3(x), \; g_2(x) = \bar{g}_1(x), \; v_1 = u_2, \; v_3 = u_1,$$

$$\bar{g}_3(x) = -x_3 x_4 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}, \quad \bar{g}_1(x) = \frac{\partial}{\partial x_4}$$

can be constructed and satisfies the LARC condition:

$$span\{\bar{g}_1(x), \bar{g}_2(x), \bar{g}_3(x), \bar{g}_4(x)\} = I\!R^4$$

where the new vector fields $\bar{g}_2$ and $\bar{g}_4$ are given by

$$\bar{g}_2(x) = x_3 \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}, \quad \bar{g}_4(x) = \frac{\partial}{\partial x_1}$$

The Lie brackets multiplication table for $L(\bar{g}_1, \bar{g}_3)$ shows that :

$$[\bar{g}_3, \bar{g}_1] = \bar{g}_2, \quad [\bar{g}_3, \bar{g}_2] = \bar{g}_4, \quad [\bar{g}_1, \bar{g}_2] = 0, \quad [\bar{g}_i, \bar{g}_4] = 0, \quad i = 1, 2, 3 \tag{3.169}$$

the Lie algebra $L(\bar{g}_1, \bar{g}_3)$ is nilpotent. Since the vector field $\bar{g}_4$ is a bracket of depth two: $\bar{g}_4 = [\bar{g}_3, [\bar{g}_3, \bar{g}_1]]$, direct application of the extended guiding functions strategy is still not possible. The extended system to (3.168) is :

$$\dot{x} = \bar{g}_1(x)\ v_1 + \bar{g}_2(x)\ v_2 + \bar{g}_3(x)\ v_3 \tag{3.170}$$

in which the control v_2 (or equivalently motion along the Lie bracket direction $\bar{g}_2 = [\bar{g}_3, \bar{g}_1]$) can be realized approximately and indirectly, by employing sinusoidal inputs of the type (3.104).

Clearly, the guiding functions approach can now be applied without change, to the extension (3.170), as the multiplication table (3.169) indicates that the distributions

$$\begin{aligned} &\Delta_1(x) = span\{\bar{g}_2, \bar{g}_3, \bar{g}_4\}(x) \\ &\Delta_2(x) = span\{\bar{g}_1, \bar{g}_2, \bar{g}_4\}(x), \qquad \Delta_3(x) = span\{\bar{g}_1, \bar{g}_2\}(x) \end{aligned} \tag{3.171}$$

are involutive. The corresponding codistributions are:

$$\begin{aligned} \Delta_1^{\perp}(x) &\stackrel{def}{=} span\{d\lambda_{1,1}(x)\} \\ \Delta_2^{\perp}(x) &\stackrel{def}{=} span\{d\lambda_{2,1}(x)\}, \qquad \Delta_3^{\perp}(x) \stackrel{def}{=} span\{d\lambda_{3,1}(x), d\lambda_{3,2}(x)\} \end{aligned}$$

with

$$\lambda_{1,1}(x) \stackrel{def}{=} x_4 \quad \lambda_{2,1}(x) \stackrel{def}{=} x_2 \quad \lambda_{3,1}(x) \stackrel{def}{=} x_3 \quad \lambda_{3,2}(x) \stackrel{def}{=} (x_1 + x_2 x_3)$$

The guiding functions employed are hence defined by :

$$V_1(x) \stackrel{def}{=} \frac{1}{2} x_4^2, \quad V_2(x) \stackrel{def}{=} \frac{1}{2} x_2^2, \quad V_3(x) \stackrel{def}{=} \frac{1}{2}((x_1 + x_2 x_3)^2 + x_3^2)$$

These guiding functions are next used in the extended strategy as applied to the original model. The system motion along the Lie bracket $[g_1, g_2]$ is realized indirectly by using

$$u_1 = sin(\frac{2\pi}{T} t), \qquad u_2 = cos(\frac{2\pi}{T} t) \tag{3.172}$$

with $T = 1$. Simulation results are shown in Figures 3.58 - 3.60 and clearly demonstrate the effectiveness of the approach.

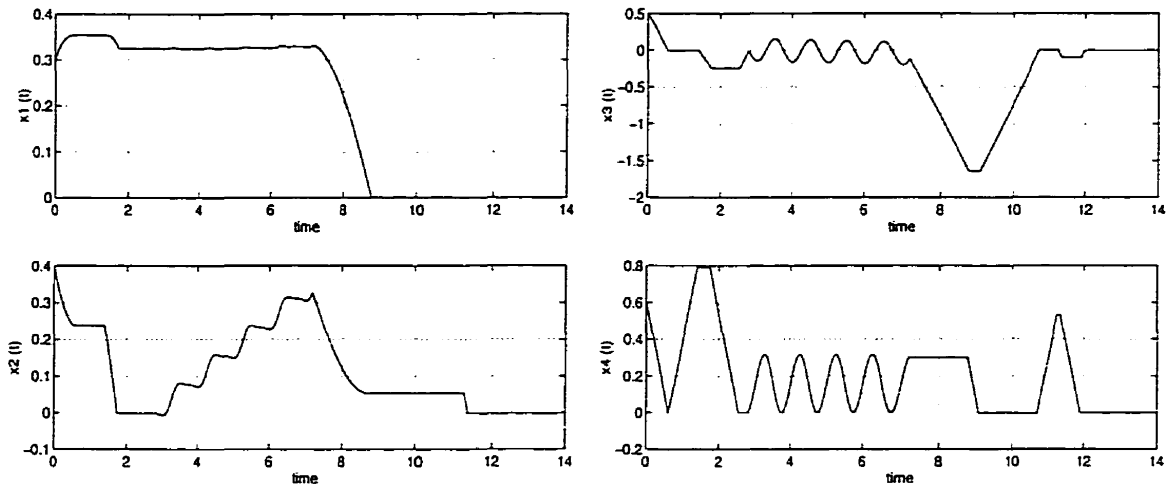

FIGURE 3.58. *WMR of type (1,1):* Plots of the controlled state trajectories $t \mapsto (x_1(t), ..., x_4(t))$ versus time.

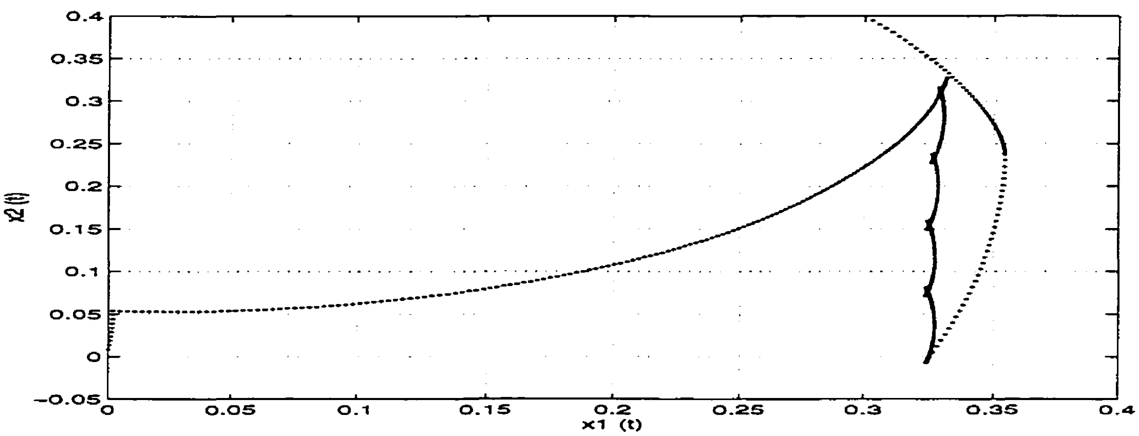

FIGURE 3.59. *WMR of type (1,1):* Plots of the controlled state variables $x_1(t)$ versus $x_2(t)$.

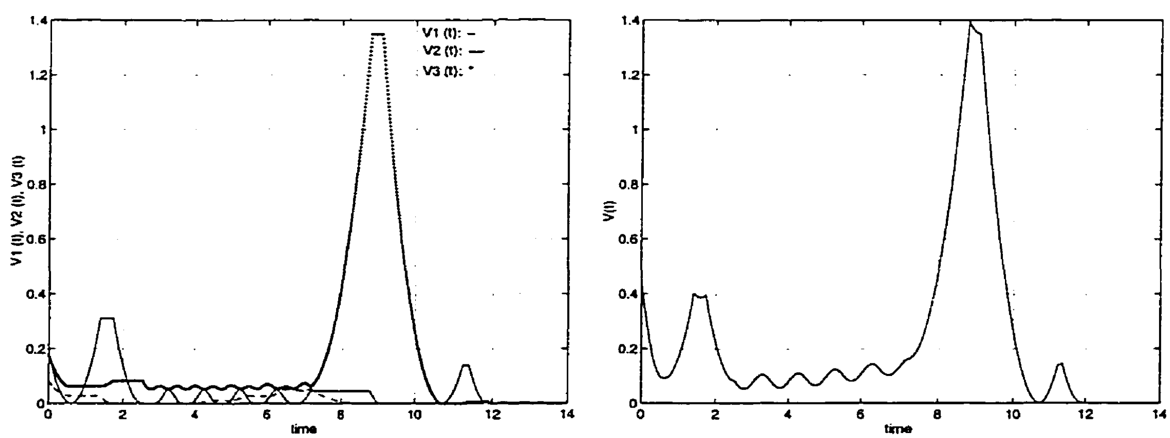

FIGURE 3.60. *WMR of type (1,1):* Plots of the guiding functions $V_1(t)$, $V_2(t)$, and their sum $V(t)$ versus time.

CHAPTER 4

The trajectory interception approach

This chapter provides a simple and systematic method for the construction of time varying stabilizing feedback control for drift free systems in the spirit of the idea first presented in [**61**]. The method is universal in the sense that it is independent of the vector fields determining the motion of the system, or of the choice of a Lyapunov function. The resulting feedback law is a composition of a standard stabilizing feedback control for a Lie bracket extension of the original system and a periodic continuation of a specific solution to an open loop, finite horizon control problem stated in terms of a formal equation on a Lie group - an equation which (via an evaluation homomorphism) describes the evolution of the flows of the original as well as the extended system. The open loop problem is solved as a trajectory interception problem in logarithmic coordinates of flows.

The trajectory interception approach is first explained in application to a sub-class of drift free systems with solvable or nilpotent controllability Lie algebras. The approach is further extended, see [**73**] and also [**66, 68, 70, 84**], to other drift free systems whose controllability Lie algebras need not be nilpotent nor solvable. This is done by introducing approximate models whose controllability Lie algebras possess the desired properties. The time varying feedback constructed for such an approximation can then be successfully applied to the original system in that the model error does not prejudice local stabilization provided that an adequately large stability robustness margin for the extended controlled system is insured.

The trajectory interception approach does not require transformation of the system model to chained or power forms.

1. Introduction

The method proposed here elaborates on the ideas contained in [**61**] and [**62**], and is inspired by the results and techniques contained in [**52**] - in which a piece-wise constant open loop control is constructed to achieve point to point steering. A similar attempt of feedback through system extension can be found in [**34**]. The feedback law constructed there allows to track trajectories with arbitrary precision but is assumed to be applied "on a finite time interval" only. The precision of steering (and hence the error within which the final point is attained) is adjusted by decreasing (to zero) a parameter ϵ in the time varying part of the control. The construction is based on the result contained in [**106**] which states that on finite intervals of time, trajectories of extended systems can be uniformly approximated by ordinary trajectories with oscillatory controls. The feedback of [**34**] is not suitable for stabilization purposes.

The trajectory interception approach primarily applies to systems which are nilpotent or at least solvable. It is well known that many systems can be made nilpotent by application of a smooth feedback, see for example [**35**].

The method proposed is based on considering of what is known as the Lie bracket extension of the original system, see [**52**]. An arbitrary Lyapunov function is first employed to furnish a closed loop stabilizing control for the extended system. The stabilizing time-invariant feedback control for the extended system is then combined with a periodic continuation of a specific solution of a formal, open loop, finite horizon control problem. This open loop control problem is posed in terms of the logarithmic coordinates for flows, see [**117**], and its purpose is to generate open loop controls such that the trajectories of the controlled extended system and the original system intersect after a finite time T, independent of their common initial condition. While the time-invariant feedback for the extended system dictates the speed of convergence of the system trajectory to the desired terminal point, the open loop solution serves the *averaging* purpose in that it ensures that the "average motion" of the original system is that of the controlled extended system.

The construction proposed demonstrates that synthesis of time varying feedback stabilizers for drift free systems can be viewed as a procedure of combining static feedback laws for a Lie bracket extension of the system with a solution of an open loop control problem on the associated Lie group.

The contribution of this chapter can be summarized as follows:

- A systematic method for the synthesis of stabilizing, time-varying feedback for a large class of drift-free systems is presented. The feedback provides for exponential rate of convergence of the system trajectories to a desired set point.

- The method shows how the averaging effect can be achieved by a (periodically repeated) open loop solution to a control problem in logarithmic coordinates.
- It is shown that the application of the trajectory interception approach is not limited to systems whose controllability Lie algebra is nilpotent. The approach can successfully be applied to systems with non-nilpotent controllability Lie algebras by introducing approximate models which generate nilpotent controllability Lie algebras. The error in the solution of the open-loop problem, resulting from such an approximation, can be compensated (without prejudicing stabilization) by adjusting the stability robustness margin of the feedback control for the extended system.
- It is shown that introducing approximate models often permits significant simplification of the differential equations describing the evolution of the logarithmic coordinates in the open-loop problem formulation (which are usually difficult to solve analytically).
- The approach is first applied to stabilize drift free systems which are characterized by nilpotent (solvable) controllability Lie algebras such as: a general drift free system with 5 states and 3 controls, a unicycle model in chained form, and Brockett's system.

  The approach is also applied to stabilize drift free systems whose controllability Lie algebras fail to satisfy the solvability assumption. Example systems such as: an underwater vehicle [**66**], a unicycle, a rigid spacecraft in actuator failure mode [**70**], a class of wheeled mobile robots [**70**], and a hopping robot in flight phase [**84**], are considered.

  Since computation of the solutions to the open loop trajectory interception problem may be elaborate if the extended system contains high order Lie brackets, the possibility of introducing decomposition into control synthesis is explored. This idea involves decomposing a complex system model into sub-systems of which one can be controlled by the trajectory interception approach and the other is controlled by sinusoidally varying inputs. The feasibility of this approach is demonstrated using: a model of a wheeled mobile robot [**70**], a fire truck, an underwater vehicle in actuator failure mode, and a mobile robot with trailer.

2. Notation and hypotheses

The symbol $C(I)$ denotes the space of continuous functions on a closed sub-interval $I \subset [0, \infty)$, and $\mathcal{PC}(I)$ denotes the class of piece-wise continuous functions on a closed interval I, (with a finite number of discontinuities in I). The symbol $B(x;\epsilon)$ denotes a ball of radius ϵ, centred at x.

For a set of (real) analytic vector fields $g_1,...,g_m$, the symbol $L(g_1,...,g_m)$ denotes the Lie algebra of vector fields generated by $g_1,...,g_m$. If $L(g_1,...,g_m)$ is nilpotent of order k, we write $L_k(g_1,...,g_m)$.

Consider the following system on $\mathbb{R}^n$:

$$\dot{x}(t) = \sum_{i=1}^{m} g_i(x(t))u_i(t), \qquad x \in \mathbb{R}^n \tag{4.1}$$

where

H0. $g_1,...,g_m$ are complete, (real) analytic, and linearly independent vector fields on $\mathbb{R}^n$, and $u_1,...u_m$ are locally bounded and Lebesgue integrable functions on $\mathbb{R}$.

The objective is to construct controls $u_i(x,t) : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$, $i = 1,...,m$, such that system (4.1) is Lyapunov asymptotically stable (the continuity properties of u are to be specified later).

For our construction to be valid, we need to impose the following basic controllability hypotheses :

H1. System (4.1) satisfies the LARC (Lie algebra rank condition) for accessibility, namely that $L(g_1,...,g_m)(x)$ spans $\mathbb{R}^n$ at each point $x \in \mathbb{R}^n$.

H2. The controllability Lie algebra $L(g_1,...,g_m)$ is nilpotent of order k, so that $L(g_1,...,g_m) = L_k(g_1,...,g_m)$.

Since drift free systems are "symmetric" in the sense that every trajectory run backwards in time is also a system trajectory, it is well known that the accessibility hypothesis H1 implies complete controllability of (4.1) in $\mathbb{R}^n$. Hypothesis H2 is quite restrictive but will be needed only in section 3. In section 5 it will be shown that H2 can be removed at the cost of increasing the frequency of oscillation in the time varying part of the constructed feedback control.

3. Solution of the stabilization problem for systems with nilpotent controllability Lie algebras

The solution of the stabilization problem, as first suggested in [**61**], involves two steps:

1). The construction of a time invariant feedback law for stabilization of a Lie bracket extension of the original system based on the choice of an arbitrary Lyapunov function.

2). The solution of a formal open loop control problem in logarithmic coordinates which effectively provides for 'pointwise equivalence' of the flows of the original and extended systems.

A periodic continuation of this solution is composed with the feedback for the extended system to yield the final time-varying feedback control. The oscillatory behaviour of the closed loop system results in an "average" decrease of the Lyapunov function selected in step 1).

3.1. Stabilization of the Lie bracket extension of the original system

To solve the stabilization problem for the "extended system", we first select a scalar Lyapunov function $V : I\!R^n \to I\!R^+$. Without the loss of generality, let V be quadratic i.e. $V(x) \stackrel{def}{=} \frac{1}{2}x^TQx$, $x \in I\!R^n$, for some positive definite and symmetric matrix Q.

As in [**61**], a Lie bracket extension of (4.1) is considered next, and is given by

$$\dot{x}(t) = \sum_{i=1}^{r} g_i(x(t))v_i(t) \qquad \text{for all } x \in \Omega \tag{4.2}$$

where $\Omega \subset I\!R^n$ is a sufficiently large, compact neighbourhood of the origin, and the vector fields g_i, $i = m+1, ..., r$, are Lie brackets of $g_1, ..., g_m$ which are necessary to complete the span of $I\!R^n$ for all $x \in \Omega$. Generally, $r \geq n$, since for a given Ω there may not exist precisely $n-m$ brackets such that $span\{g_1, ..., g_n\}(x) = I\!R^n$ for all $x \in \Omega$ (however, the compactness of Ω guarantees that r is finite).

Let $G(x) \stackrel{def}{=} [g_1, ..., g_r](x)$ denote the state dependent matrix whose columns are the $g_1, ...g_r$ of (4.2). Since (4.2) is instantaneously locally controllable in any direction in $I\!R^n$, a variety of static stabilizing feedback controls for (4.2) can be constructed easily. Consider, for example, the following controls:

$$(C1): \quad v(x) \stackrel{def}{=} -G(x)^{\dagger}x, \quad v(x) \stackrel{def}{=} [v_1, ..., v_r](x)^T \tag{4.3}$$

in which $G(x)^{\dagger}$ is the pseudo-inverse of $G(x)$, or else:

$$(C2): \quad v_i(x) \stackrel{def}{=} -L_{g_i}V(x), \quad i \in \{1, ..., r\} \tag{4.4}$$

The result stated below can be found in [**62**] but is cited here for the reason of completeness.

PROPOSITION 4.1. *The control laws (C1) and (C2) are exponentially stabilizing for system (4.2), (globally exponentially stabilizing if $\Omega = I\!R^n$).*

Proof. First, note that both of the above controls satisfy $v(0) = 0$, and the control (C2) can be written as $v(x) = -G(x)^TQx$. By construction, $rank\{G(x)\} = n$, for all $x \in \Omega$. Thus $G(x)G(x)^T$ is invertible for $x \in \Omega$, as in fact $G(x)G(x)^T = [r_ir_j^T]_{i\leq n, j\leq n}(x)$ is the Grammian matrix for the n linearly independent rows $r_i, i = 1, ..., n$, of $G(x)$. Therefore, $G^{\dagger}(x)$ is the right inverse of $G(x)$ since then $G(x)^\dagger = G(x)^T(G(x)G(x)^T)^{-1}$. Since the Grammian matrix for a linearly independent set of vectors is positive definite, thus the time derivative of V along the trajectories of (4.2) with controls (C1) and (C2), respectively, can be bounded as follows:

$$\begin{aligned}
\frac{d}{dt}V(x(t)) &= -\frac{1}{2}[G(x(t))G^\dagger(x(t))x(t)]^T Qx(t) - \frac{1}{2}x(t)^T QG(x(t))G^\dagger(x(t))x(t) \\
&= -x(t)^T Qx(t) \\
&= -2V(x(t)) \qquad\qquad (4.5) \\
\frac{d}{dt}V(x(t)) &= -\frac{1}{2}[G(x(t))G(x(t))^T Qx(t)]Qx(t) - \frac{1}{2}x(t)^T QG(x(t))G(x(t))^T Qx(t) \\
&\leq -\gamma x(t)^T x(t) \\
&\leq -\frac{2\gamma}{\lambda_{max}(Q)}V(x(t)) \qquad\qquad (4.6)
\end{aligned}$$

for all t such that $x(t) \in \Omega$, where $\gamma > 0$ is a lower bound for the eigenvalues of the positive definite and symmetric matrices $QG(x)G(x)^TQ$ on the set Ω, and $\lambda_{max} > 0$ is the largest eigenvalue of Q. By a standard Lyapunov argument, both controls (C1) and (C2) are exponentially stabilizing for the extended system (and globally exponentially stabilizing if $\Omega = I\!R^n$). ■

While the "controlled" vector fields g_iv_i are still analytic, they may not be complete and the new Lie algebra of vector fields $L(g_1v_1, ..., g_rv_r)$ may not be nilpotent. As this complicates our principal construction it is convenient to assume that the feedback controls are "updated" discretely in time which also simplifies the calculation of the time varying part of the stabilizing control for the original system. Instead of (4.2) we thus consider an extended system with "discretised" controls:

$$\dot{x}(t) = \sum_{i=1}^{r} g_i(x(t))\bar{v}_i(T, x(t)) \qquad (4.7)$$

where the functions $\bar{v}_i, i = 1, ..., r$, are obtained from $v_i, i = 1, ..., r$, by the formula

$$\bar{v}_i(T, x(t)) \stackrel{def}{=} v_i(x(nT)) \qquad t \in [nT, (n+1)T), \quad n = 0, 1, 2, ... \quad i = 1, ..., r \qquad (4.8)$$

and thus are constant over each interval $[nT, (n+1)T)$. (In practical terms, the control (4.8) can be viewed as a cascade of the smooth controller $v(x)$ and a zero order extrapolator.)

Not to complicate further analysis, it will henceforth be assumed that $\Omega = I\!R^n$. In cases when this assumption cannot be made, all the subsequent results hold only locally.

It can be shown that such "discretization" of the control v does not prejudice exponential stability of the controlled extended system.

PROPOSITION 4.2. [62] *Suppose $\Omega = \mathbb{R}^n$, the hypotheses H0-H2 are valid, and the extended system (4.2) employs any of the controls (C1) or (C2). Under these conditions, for any compact region C containing the origin there exists a maximal discretization step $T_{max} > 0$ such that the corresponding discretized controlled extended system (4.7) is exponentially stable with region of attraction C, for any $T \leq T_{max}$.*

Proof. First we note that discontinuities in the control of equation (4.7) do not prejudice existence or uniqueness of its solutions (on any interval of time) since, by construction, such discontinuities occur at isolated moments of time, and solutions of (4.7) with constant controls exist and are unique (on any interval of time), by virtue of hypotheses H0-H2.

Without the loss of generality, let C be a level set of the Lyapunov function $V = \frac{1}{2}x^T Qx$, i.e. $C = \{x \in \mathbb{R}^n : V(x) \leq \alpha\}$, for some positive α. Let $t \mapsto x\ exp\{t\ \sum_{i=1}^r g_i u_i\}$ denote the integral curve of (4.7), passing through x at time $t = 0$, and due to *constant* controls u_i, $i = 1, ..., r$. From standard results in differential equations concerning the sensitivity of solutions to perturbations in initial conditions and parameters, it follows, by virtue of hypotheses H0 - H2, that the mapping $\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^r \ni (t, x, u_1, ..., u_r) \mapsto x\ exp\{t\ \sum_{i=1}^r g_i u_i\} \in \mathbb{R}^n$ is well defined and at least continuous. If $\mathbb{R}^n \ni x \mapsto (u_1(x), ..., u_r(x)) \in \mathbb{R}^r$ is any continuous mapping, then the mapping $\mathbb{R}^+ \times \mathbb{R}^n \ni (t, x) \mapsto x\ exp\{t\ \sum_{i=1}^r g_i u_i(x)\} \in \mathbb{R}^n$ is continuous as a composition of continuous mappings. It follows that the mapping $\mathbb{R}^+ \times \mathbb{R}^n \ni (t, x) \mapsto x\ exp\{t\ \sum_{i=1}^r g_i \bar{v}_i(T_0, x)\} \in \mathbb{R}^n$ is continuous for an arbitrary discretization step T_0 (note that here the $\bar{v}_i(T_0, x)$ signify controls which depend only on x but not on t). By continuity, it maps compact sets onto compact sets. Thus the image of the set $[0, T_0] \times C$ under this map is some compact set $\mathcal{R}$. In fact $\mathcal{R}$ is the set of states reachable (in times not exceeding T_0) by trajectories of (4.7) which emanate from C. This implies that any trajectory of the extended system with discretized control, emanating from C, remains in $\mathcal{R}$ for time T_0. Let $\bar{B}(0; \rho)$, $\rho > 0$, be a closed ball containing all the trajectories of the extended system with a smooth feedback control $v(x)$ which emanate from C, and a superset of $\mathcal{R}$ (such a ball exists as the extended system with smooth controls is stable). Let M_g denote a uniform bound for g_i, $i = 1, ..., r$, on $\bar{B}(0; \rho)$, and let L_v be a common Lipschitz constant for v_i, $i = 1, ..., r$, also on $\bar{B}(0; \rho)$. Let the symbols $x(\cdot; t_0, x_0)$ and $\bar{x}(\cdot; t_0, x_0)$ (or simply $x(\cdot)$ and $\bar{x}(\cdot)$) denote the trajectories passing through $x_0 \in C$ at time t_0 of the extended systems (4.2) and (4.7) with continuous and discretized controls, respectively. For an arbitrary $(t_0, x_0) \in \mathbb{R}^+ \times C$,

$$||\bar{x}(t_0 + \tau; t_0, x_0) - x_0|| \leq \sum_{i=1}^{r} \int_{t_0}^{t_0+\tau} ||g_i(\bar{x}) v_i(x_0)||\ dt \leq \tau\ r M_g L_v ||x_0|| \tag{4.9}$$

for all $\tau \in [0, T_0]$, as $v(0) = 0$. Define $c \stackrel{def}{=} rM_gL_v$ and let $T_1 \leq min\{T_0, \frac{1}{2c}\}$. Then for all $(t_0, x_0) \in \mathbb{R}^+ \times \mathcal{C}$, and all $\tau \in [0, T_1]$,

$$\begin{aligned} ||\bar{x}(t_0+\tau; t_0, x_0) - x_0)|| &\leq \tau c||x_0|| \leq T_1 c||\bar{x}(t_0+\tau; t_0, x_0) - x_0)|| + \tau c||\bar{x}(t_0+\tau; t_0, x_0)|| \\ &\leq \frac{1}{2}||\bar{x}(t_0+\tau; t_0, x_0) - x_0)|| + \tau c||\bar{x}(t_0+\tau; t_0, x_0)|| \end{aligned} \tag{4.10}$$

$$\text{so that} \qquad ||\bar{x}(t_0+\tau; t_0, x_0) - x_0)|| \leq \tau\ 2c||\bar{x}(t_0+\tau; t_0, x_0)||, \quad \tau \in [0, T_1] \tag{4.11}$$

From the proof of Proposition 4.1 it follows that the extended system with smooth feedback controls v_i, as given by (C1) or (C2), is exponentially stable, and there exists a constant $\beta > 0$ such that

$$\frac{d}{dt}V(x) = \nabla V(x)^T \sum_{i=1}^{r} g_i(x)v_i(x) \leq -\beta V(x) \tag{4.12}$$

for all $x \in \Omega = \mathbb{R}^n$, where ∇V denotes the gradient of V. This yields the following bound for the time derivative of V, this time along the trajectory $\bar{x}(\cdot; t_0, x_0)$ of the extended system with discretized control using a discretization step $T_2 \leq T_1$:

$$\begin{aligned} &\frac{d}{dt}V(\bar{x}) = \nabla V(\bar{x})^T \sum_{i=1}^{r} g_i(\bar{x})v_i(x_0) \\ &\leq \nabla V(\bar{x})^T \sum_{i=1}^{r} g_i(\bar{x})v_i(\bar{x}) + ||\nabla V(\bar{x})^T|| || \sum_{i=1}^{r} g_i(\bar{x})v_i(x_0) - \sum_{i=1}^{r} g_i(\bar{x})v_i(\bar{x})|| \\ &\leq -\beta V(\bar{x}) + M_g||\nabla V(\bar{x})^T|| \sum_{i=1}^{r} ||v_i(x_0) - v_i(\bar{x})|| \\ &\leq -\beta V(\bar{x}) + rM_gL_v||Q|| ||\bar{x}|| ||\bar{x} - x_0|| \\ &\leq -\beta V(\bar{x}) + T_2 2c^2||Q|| ||\bar{x}||^2 \leq -\beta V(\bar{x}) + T_2 \frac{4c^2||Q||}{\lambda_{min}(Q)} V(\bar{x}) \end{aligned} \tag{4.13}$$

for all $\tau \in [0, T_2]$, by virtue of (4.12) and (4.11). Clearly, it is possible to select a $T_2 \leq T_1$ such that

$$\frac{d}{dt}V(\bar{x}(t_0+\tau; t_0, x_0)) \leq -\frac{1}{2}\beta V(\bar{x}(t_0+\tau; t_0, x_0)) \tag{4.14}$$

for all $\tau \in [0, T_2]$, along a trajectory of the controlled extended system using discretization step T_2. It follows that $x_1 \stackrel{def}{=} \bar{x}(t_1; t_0, x_0) \in \mathcal{C}$, with $t_1 \stackrel{def}{=} t_0 + T_2$. As equations (4.9) - (4.14) were obtained for an arbitrary initial condition (t_0, x_0), inequality (4.14) is thus again valid for $\bar{x}(t_1 + \tau; t_1, x_1)$, $\tau \in [0, T_2]$, and $x_2 \stackrel{def}{=} \bar{x}(t_2; t_1, x_1) \in \mathcal{C}$, with $t_2 \stackrel{def}{=} t_1 + T_2$. By a simple inductive argument, $x_n = \bar{x}(t_0 + nT_2; t_0, x_0) \in \mathcal{C}$, for all $n = 1, 2, ..$, and, consequently, (4.14) is valid for all times $\tau \geq t_0$. As x_0 was arbitrary, this implies exponential stability of the extended controlled system using discretization step $T_{max} = T_2$.

Finally, for any discretization step $T < T_{max}$, and because $T_{max} \leq T_0$, the trajectories $x(t; t_0, x_0)$, and $\bar{x}(t; t_0, x_0)$, $t \geq t_0$, (where the latter corresponds to a discretisation step T), still remain in $\bar{B}(0; \rho)$, provided that $(t_0, x_0) \in I\!R^+ \times C$. Hence the previous argument applies also for T, as equations (4.9) - (4.14) can be re-written without change. The extended system using discretized control is thus exponentially stable for any $T \leq T_{max}$. ∎

The following definition, cited from **[62]**, specifies a somewhat different type of exponential stability which will be found useful in the analysis of the stabilizing properties of the feedback law constructed later.

DEFINITION 1. *A time varying system given by*

$$\dot{x}(t) = f(x(t), t), \qquad x(t_0) = x_0 \qquad (t_0, x_0) \in I\!R^+ \times I\!R^n \tag{4.15}$$

whose solutions $x(\cdot; t_0, x_0)$ through any (t_0, x_0) exist and are unique for all times $t \geq t_0$, and in which $f(0, t) \equiv 0$, for all t, is said to be ρ-exponentially stable with $\rho \in (0, 1]$ and a region of attraction Ω iff there exist positive constants C and γ such that for all $t \geq t_0$ and all $(t_0, x_0) \in I\!R^+ \times \Omega$:

$$||x(t; t_0, x_0)|| \leq C||x_0||^{\mu(x_0)} \, e^{-\gamma(t-t_0)} \tag{4.16}$$

where $\mu(x_0) = \rho$ for $x_0 \in B(0; 1)$, and $\mu(x_0) = 1$ for $x_0 \notin \ B(0; 1)$.
Clearly, if $\rho = 1$ then (4.15) is exponentially stable in the usual sense and, in any case, (4.16) differs from the usual definition of exponential stability only for $x_0 \in B(0; 1)$.

The following elementary lemma, taken from **[62]**, will also be found useful, and basically re-states the well known fact that asymptotic stability of a system is guaranteed by the existence of a Lyapunov function which decreases "on average" (rather than monotonically) along the trajectories of this system.

LEMMA 4.1. *Consider the system of Definition 1. Suppose there exists a Lipschitz continuous function $V : I\!R^n \to I\!R^+$ which, for some $p > 0$, satisfies:*

$$\gamma_1 ||x||^p \leq V(x) \leq \gamma_2 ||x||^p \qquad \textit{for all } x \in I\!R^n \tag{4.17}$$

with some positive constants γ_1 and γ_2. Let $\rho \in (0, 1]$, and let C be a level set of V, so that $C = \{x \in I\!R^n : V(x) \leq \alpha\}$, for some positive α. Suppose that there exist constants $M > 1$, $T > 0$, and $\beta \in (0, 1)$, and a function $\mu : I\!R^+ \times C \to \{1, \rho\}$, such that for all $(t_0, x_0) \in I\!R^+ \times C$

$$(a) \;\; V(x(t_0 + T; t_0, x_0)) \;\; \leq \;\; \beta V(x_0) \tag{4.18}$$

$$(b) \;\; V(x(t_0 + \tau; t_0, x_0)) \;\; \leq \;\; M V(x_0)^{\mu(x_0)} \qquad \textit{for all } \tau \in [0, T] \tag{4.19}$$

Under these conditions, system (4.15) is ρ-exponentially stable with region of attraction $\mathcal{C}$.

Proof. Take any $(t_0, x_0) \in \mathbb{R}^+ \times \mathcal{C}$ and define

$$x_k \stackrel{def}{=} x(t_0 + Tk; t_0, x_0), \quad t_k \stackrel{def}{=} t_0 + Tk \qquad k = 0, 1, 2, ... \tag{4.20}$$

By uniqueness of trajectories of (4.15), if $x_k \in \mathcal{C}$ then

$$V(x_{k+1}) = V(x(t_0 + T(k+1); t_0, x_0)) = V(x(t_k + T; t_k, x_k)) \leq \beta V(x_k) \tag{4.21}$$

which implies that $x_{k+1} \in \mathcal{C}$, as $\beta \in (0,1)$. Since x_k was arbitrary, it follows that

$$\begin{aligned} & V(x_{k+1}) = V(x(t_0 + T(k+1); t_0, x_0)) = V(x(t_k + T; t_k, x_k)) \\ & \leq \beta V(x_k) \leq ... \leq \beta^{k+1} V(x_0), \quad k = 0, 1, 2, ... \end{aligned} \tag{4.22}$$

for any $(t_0, x_0) \in \mathbb{R}^+ \times \mathcal{C}$, as $x_1, ..., x_{k+1}$ are all members of $\mathcal{C}$. Further, by condition (b),

$$\begin{aligned} & V(x(t_0 + Tk + \tau; t_0, x_0)) = V(x(t_k + \tau; t_k, x_k)) \\ & \leq M V(x_k)^\mu \leq M \beta^{k\mu} V(x_0)^\mu, \qquad \text{for all } \tau \in [0, T] \end{aligned} \tag{4.23}$$

and for any $k = 0, 1, 2, ...$, with either: $\mu = 1$, or $\mu = \rho$, (depending on k,t_0, and x_0). Choosing $\gamma > 0$ such that $\beta = e^{-\gamma T \frac{p}{\rho}}$ and defining $C \stackrel{def}{=} (M\gamma_2/\gamma_1)^{\frac{1}{p}} e^{\gamma T}$ yields

$$\begin{aligned} V(x(t_0 + Tk + \tau; t_0, x_0)) & \leq Me^{-\gamma T k p \frac{\mu}{\rho}} \, V(x_0)^\mu \leq Me^{-\gamma T k p} \, V(x_0)^\mu \\ & \leq C^p \frac{\gamma_1}{\gamma_2} e^{-\gamma p T(k+1)} \, V(x_0)^\mu \\ & \leq C^p \frac{\gamma_1}{\gamma_2} e^{-\gamma p (Tk+\tau)} V(x_0)^\mu \end{aligned} \tag{4.24}$$

so using (4.17) gives

$$||x(t_0 + Tk + \tau; t_0, x_0)|| \leq C e^{-\gamma(Tk+\tau)} ||x_0||^\mu \tag{4.25}$$

which holds for any $k = 0, 1, 2, ..$, any $\tau \in [0, T]$, and any $(t_0, x_0) \in \mathbb{R}^+ \times \mathcal{C}$, with some value of $\mu \in \{\rho, 1\}$ (dependent on k, t_0 and x_0). However, if $||x_0|| \geq 1$ then $||x_0|| \geq ||x_0||^\mu$ for any $\mu \leq 1$, and if $||x_0|| < 1$ then $||x_0|| < ||x_0||^\mu$ for any $\mu \leq 1$, so inequality (4.25) can be further re-written as

$$||x(t_0 + Tk + \tau; t_0, x_0)|| \leq \begin{cases} C||x_0|| \, e^{-\gamma(Tk+\tau)} & \text{for } x_0 \notin \; B(0;1) \\ C||x_0||^\rho \, e^{-\gamma(Tk+\tau)} & \text{for } x_0 \in B(0;1) \end{cases} \tag{4.26}$$

for all $k = 0, 1, 2, ..$, all $\tau \in [0, T]$, and all $(t_0, x_0) \in \mathbb{R}^+ \times \mathcal{C}$. Putting $t \stackrel{def}{=} Tk + \tau$, yields

$$||x(t_0 + t; t_0, x_0)|| \leq C||x_0||^{\mu(x_0)} \, e^{-\gamma t} \tag{4.27}$$

for all $t \geq t_0$ and all $(t_0, x_0) \in I\!R^+ \times \mathcal{C}$, with $\mu(x_0) = \rho$ if $x_0 \in B(0;1)$ and $\mu(x_0) = 1$ if $x_0 \notin B(0;1)$. System (4.15) is hence ρ-exponentially stable with region of attraction $\mathcal{C}$. ∎

3.2. An open loop control problem on a Lie group

The objective here is to use the stabilizing feedback control for the extended system as a base upon which to construct a time varying stabilizing feedback but, this time, for the original system. Consider the discretized system (4.7) on the interval $[nT, (n+1)T)$. Since the controls $\bar{v}_i(T, x(t))$, $i = 1, ..., r$ are constant over each interval $[nT, (n+1)T)$, $n = 0, 1, 2, ...$, these can be regarded as parameters $a_i \in I\!R$, $i = 1, ..., r$, yielding a parametrized extended system

$$\dot{x} = \sum_{i=1}^{r} g_i(x(t))a_i, \quad a_i \stackrel{def}{=} \bar{v}_i(T, x(t)), \quad t \in [nT, (n+1)T) \tag{4.28}$$

The task of the open loop control problem is to generate the time varying part of the feedback controls $u_i(x,t)$, $i = 1, ..., m$, so that the trajectories of the original system (4.1) with controls u_i, and the parameterized controlled extended system (4.28) intersect periodically with a freequency $1/T$, for some fixed $T \leq T_{max}$, where T_{max} is the constant whose existence was established in Proposition 4.2. An open loop "trajectory interception" problem (TIP) can now be posed as follows:

TIP: For a fixed value of the time horizon $T \leq T_{max}$, find control functions $w_i(a,t)$, $i = 1, ..., m$, in the class of functions which are Holder continuous in $a \stackrel{def}{=} [a_1, ..., a_r]$ at zero, so that there exist constants $K > 0$ and $\gamma \in (0,1)$ such that for any fixed t

$$|w_i(a,t)| \leq K||a||^{\gamma} \quad i = 1, ..., m \tag{4.29}$$

for all a, and piece-wise continuous, and locally bounded in t, such that, for *any initial condition* $x(0) = x$ at time $t_s = 0$, the trajectory $x^a(t; x, 0)$ of the extended, parametrized system (4.28) intersects the trajectory $x^w(t; x, 0)$ of the original system with controls w_i, $i = 1, ..., m$

$$\dot{x} = \sum_{i=1}^{m} g_i w_i(a,t) \tag{4.30}$$

precisely at time $t_f = T$, so that $x^a(T; x, 0) = x^w(T; x, 0)$.

REMARK 4.1. It is important that the solution of the TIP is independent of the "parameter" n as well as the actual values of the trajectory interception points $x(t_s)$ and $x(t_f)$.

Employing the formalism described in [62] and [52] is essential in the solution of the TIP. This formalism, as it applies to the TIP, is briefly summarized below.

Given a set of controls $w_i : [0,T] \to I\!R$, $i = 1, ..., m$, one can find the corresponding trajectory $t \mapsto x(t)$ of (4.30), through an initial condition $x(0)$ by first considering a formal initial value problem primarily stated on the algebra $\hat{A}(X_1, ..., X_m)$ of formal power series in a set of objects X_i, $i = 1, ..., m$, called indeterminates (see [**52**]):

$$\dot{S}(t) = S(t) \ (\sum_{i=1}^{m} w_i(t) X_i)$$
$$S(0) = 1 \in \hat{A}(X_1, ..., X_m) \tag{4.31}$$

Since the systems considered are assumed to be nilpotent, it is actually sufficient to consider (4.31) as an evolution equation on a nilpotent version of $\hat{A}(X_1, ..., X_m)$, namely $A_k(X_1, ..., X_m)$ - the free nilpotent associative algebra of order k whose elements are finite series in the indeterminates $X_1, ..., X_m$ (in which all the monomials in more than k indeterminates are assumed to be zero). It is then possible to define the free nilpotent Lie algebra $L_k(X_1, ..., X_m)$ of order k, which is a subalgebra of $A_k(X_1, ..., X_m)$ consisting of those elements of $A_k(X_1, ..., X_m)$ which are actually Lie polynomials, and also the set

$$G_k(X_1, ..., X_m) \stackrel{def}{=} \{P \in A_k(X_1, ..., X_m) : log P \in L_k(X_1, ..., X_m)\} \tag{4.32}$$

where the mapping log is the inverse of the exponential mapping from $L_k(X_1, ..., X_m)$ to $G_k(X_1, ..., X_m)$ defined in terms of the usual power series. With this definition, $G_k(X_1, ..., X_m)$ is actually the connected simply connected Lie group with Lie algebra $L_k(X_1, ..., X_m)$. It is well known (see [**52**]) that solutions to (4.31) exist and are unique for all times, and that the "trajectories" $t \mapsto S(t)$ remain in $G_k(X_1, ..., X_m)$ for all times, as $1 \in G_k(X_1, ..., X_m)$. Now it is important to note that the controllability Lie algebra, $L_k(g_1, ..., g_m)$, is nilpotent of order k. It is thus possible to define an evaluation homomorphism (a Lie algebra homomorphism) as a mapping $\nu : L_k(X_1, ..., X_m) \to L_k(g_1, ..., g_m)$ which maps each element of $L_k(X_1, ..., X_m)$ into an element of $L_k(g_1, ..., g_m)$ by substituting any X_j for the corresponding g_j, $j = 1, ..., m$. For example, $\nu([X_{i_1}, [X_{i_2}, X_{i_3}]]) = [g_{i_1}, [g_{i_2}, g_{i_3}]]$. This mapping extends to a Lie group homomorphism (also denoted by ν), from $G_k(X_1, ..., X_m)$ to G - the connected simply connected Lie group with Lie algebra $L_k(g_1, ..., g_m)$. It is well known, see [**52**], that the function $t \mapsto x(t)$, defined by $x(t) \stackrel{def}{=} x(0)\nu(S(t))$ (where $\nu(S(t)) \in G$ for all $t \geq 0$) is in fact the unique trajectory of (4.30) through $x(0)$. Furthermore, it is clear that a similar conclusion holds for the discretized extended system (4.28) if the formal initial value problem (4.31) is substituted by an "extended" initial value problem

$$\dot{S}(t) = S(t) \ (\sum_{i=1}^{r} a_i X_i), \qquad S(0) = 1 \tag{4.33}$$

whose solution also evolves in $G_k(X_1, ..., X_m)$ and in which the elements X_i, for $i \in \{m+1, ..., r\}$ are all members of $L_k(X_1, ..., X_m)$ and correspond to the vector fields g_j, $j = m+1, ..., r$, in the extended system.

The above leads to a conclusion that the "trajectory interception" problem TIP translates into a formal interception problem (FIP) on the Lie group $G_k(X_1, ..., X_m)$, stated below.

FIP: Consider the two formal initial value problems on $G_k(X_1, ..., X_m)$:

$$S1: \quad \dot{S}^a(t) = S^a(t)\ (\sum_{i=1}^{r} a_i X_i), \qquad S^a(0) = 1$$

$$S2: \quad \dot{S}^w(t) = S^w(t)\ (\sum_{i=1}^{m} w_i(a,t) X_i), \qquad S^w(0) = 1 \tag{4.34}$$

where the constants a_i, $i = 1, ..., r$, of S1 are known from (4.28), and in which the indeterminates $X_{m+1}, ..., X_r$ are understood to be Lie brackets of the first $X_1, ..., X_m$. For a fixed time horizon $T \leq T_{max}$, find control functions w_i , $i = 1, ..., m$, which are Holder continuous in a (at zero) and piece-wise continuous in t, such that $S^a(T) = S^w(T)$.

REMARK 4.2. For any value of the time horizon T, a solution to FIP can always be found but is generally non-unique. The existence of solutions follows directly from the general theory of accessibility (see a version of Chow's theorem for systems on manifolds in [**105**]).

A solution to FIP can be calculated in many ways. One such way, as presented in [**117**], is summarized below.

It is a well known fact, see [**117**], that for sufficiently small t, the solution $S(t)$ of (4.31) or (4.33) can be represented as a product of exponentials, so that

$$S(t) = e^{t_1 B_1} e^{t_2 B_2} .... e^{t_p B_p} \tag{4.35}$$

for all $t \in B(0;\epsilon)$, where $\epsilon > 0$ is a small constant, where the elements $B_1, ..., B_p$ constitute a finite basis for $L_k(X_1, ..., X_m)$, and $(t_1, ..., t_p)$ is a real vector (dependent on t). Moreover, it can be shown, see [**117**], that because the algebra $L_k(X_1, ..., X_m)$ is solvable (which is the fact since $L_k(X_1, ..., X_m)$ is nilpotent), it is possible to select an *ordered* basis of $L_k(X_1, ..., X_m)$, such that representation (4.35) is global (holds for any t). Without the loss of generality, it can be assumed that the indeterminants $X_1, ..., X_r$ form a such basis for $L_k(X_1, ..., X_m)$, and that $\nu(g_i) = X_i$, for $i = 1, ..., r$. It follows that each $S(t)$ has the unique representation:

$$S(t) = e^{\gamma_1(t) X_1} e^{\gamma_2(t) X_2} .... e^{\gamma_s(t) X_r} \tag{4.36}$$

where the real functions $t \mapsto \gamma_i(t)$, $i = 1, ..., r$, are known as "the logarithmic coordinates" of $S(t)$.

For a given set of controls a_i, $i = 1, ..., r$, the evolution of the corresponding logarithmic coordinates of $S^a(t)$, $t \geq 0$, can be determined easily. Since (4.36) is required to satisfy the formal extended equation (S1), then the left hand side of (S1) is (omitting explicit time dependence):

$$\begin{aligned}
&\frac{d}{dt}(e^{\gamma_1 X_1} e^{\gamma_2 X_2} \cdots e^{\gamma_r X_r}) \\
&= \dot{\gamma}_1 X_1 e^{\gamma_1 X_1} e^{\gamma_2 X_2} \cdots e^{\gamma_r X_r} + e^{\gamma_1 X_1} \dot{\gamma}_2 X_2 e^{\gamma_2 X_2} \cdots e^{\gamma_s X_r} \\
&+ \cdots + e^{\gamma_1 X_1} e^{\gamma_2 X_2} \cdots \dot{\gamma}_r X_r e^{\gamma_r X_r} \qquad (4.37)
\end{aligned}$$

It would then be desirable to collect the common factor $e^{\gamma_1 X_1} \cdots e^{\gamma_r X_r}$ on the left so as to be able to equate coefficients of the basis elements X_i, $i = 1, ..., r$ on both sides of (S1). This is however complicated by the fact that the X_i generally do not commute ($[X_i, X_j] \neq 0$ for $i \neq j$). A variation of the Campbell-Hausdorff formula turns out helpful to overcome this difficulty :

$$e^{tX_i} X_j = \sum_{k=0}^{\infty} \frac{t^k}{k!}\, ad_{X_i}^k X_j\; e^{tX_i} \qquad (4.38)$$

for any t, where the symbols ad on the right hand side are defined recursively by:

$$ad_A^0 B = B, \quad ad_A^{k+1} B = [A, ad_A^k B], \quad \text{for } k = 1, 2, ... \qquad (4.39)$$

Since $L_k(X_1, ..., X_r)$ is nilpotent, (4.38) can actually be re-written as a finite sum:

$$e^{tX_i} X_j = \sum_{l=1}^{r} c_l^{ij} X_l\; e^{tX_i} \qquad (4.40)$$

where c_l^{ij} are computed from (4.38). This formula allows to move the X_j past the $e^{\gamma_i X_i}$ in (4.37) and collect the common product of exponentials on the left. Equating coefficients of $X_1, ..., X_r$ in the formal equation (S1) so transformed, yields a set of ordinary differential equations for the logarithmic coordinates $\gamma_1, ..., \gamma_r$, which have the form:

$$\begin{aligned}
\dot{\gamma}_1(t) &= F_1(\gamma_t, a) \\
&\vdots \\
\dot{\gamma}_r(t) &= F_r(\gamma_t, a) \qquad (4.41)
\end{aligned}$$

where the F_i, $i = 1, ..., r$, are analytic functions of $\gamma_t \overset{def}{=} (\gamma_1(t), ..., \gamma_r(t))$ and $a \overset{def}{=} (a_1, ..., a_r)$. The initial condition for (4.41) is clearly $\gamma_1(0) = ... = \gamma_r(0) = 0$, because $S^a(0) = 1$. It is also clear that a similar set of equations can be obtained for the formal system S2 with unknown controls w_i, $i = 1, ..., m$, and differs from (4.41) only in that the controls $a = (a_1, ..., a_r)$ need to be substituted by $w_t \stackrel{def}{=} (w_1(a,t), ..., w_m(a,t), 0, 0, .., 0)$:

$$\begin{aligned}
\dot{\gamma}_1'(t) &= F_1(\gamma_t', w_t) \\
&\vdots \\
\dot{\gamma}_r'(t) &= F_r(\gamma_t', w_t), \quad \gamma_1'(0) = ... = \gamma_r'(0) = 0
\end{aligned} \tag{4.42}$$

The solution of the FIP, as re-stated in the logarithmic coordinates, is then an ordinary trajectory interception problem for systems (4.41) and (4.42) satisfying zero initial conditions, and can be solved by different methods.

The method adopted here is the following. The solution to the FIP can often be obtained by imposing that the w_i are some linear combinations of a set of known time functions $\hat{w}_k(T,t)$ $k = 1, ..., l$, so that $w_i(t) = \sum_{k=1}^{l} b_{k,i}(a)\hat{w}_k(T,t)$ for $i = 1, ..., m$. The unknown coefficients $b_{k,i}(a)$ can be found in terms of the known parameters a_i by solving the equations (4.41) and (4.42) symbolically and comparing the solutions at time T and should be Holder continuous in a at zero. An example illustrating such procedure is presented at the end of this section.

Remark 4.3. It is perhaps desirable to seek solutions of FIP in the subspace of $\mathcal{C}[0,T]$, $\tilde{\mathcal{C}}[0,T] \stackrel{def}{=} \{f \in \mathcal{C} : f(0) = f(T)\}$, consisting of functions with equal end-points. As will soon become clear, such restriction of the class of admissible w_i's leads to feedback controls which are continuous in t. The result contained in [**103**] seems helpful for this purpose as it establishes that "motion" in Lie bracket directions can be realised with an arbitrary precision using controls which are linear combinations of sinusoidal functions of adequately chosen frequencies, which are members of $\tilde{\mathcal{C}}[0,T]$. The examples considered later demonstrate this possibility.

3.3. The time varying stabilizing feedback

For any given $T \leq T_{max}$, where T_{max} is the maximal discretization step defined in Proposition 4.2, let a solution to the TIP be denoted by $w_k(T,a,t), t \in [0,T], k = 1, ..., m$, and be of the form:

$$w_i(T,a,t) = \sum_{k=1}^{l} b_{k,i}(a)\ \hat{w}_k(T,t), \qquad i = 1, ..., m \tag{4.43}$$

Its substitution into (4.30) yields a system :

$$\begin{aligned}
\dot{x} &= \sum_{i=1}^{m} g_i(x) w_i(T,a,t-t_0) \\
&= \sum_{i=1}^{m} g_i(x) \sum_{k=1}^{l} b_{k,i}(a(x))\hat{w}_k(T,t),\ t \in [t_0, t_0+T],\ x(t_0) = x_0
\end{aligned} \tag{4.44}$$

This suggests the following definition of a time-varying feedback law for the original system (4.1):

$$u_i(t,x) \stackrel{def}{=} \sum_{k=1}^{l} b_{k,i}(a(\bar{v}(T,x))\hat{w}_k^P(T,t), \quad i=1,...,m, \quad t \in [t_0,\infty) \tag{4.45}$$

where $\hat{w}_i^P$ denote periodic continuations of $\hat{w}_k$, given by

$$\hat{w}_k^P(T,t-t_0) \stackrel{def}{=} \hat{w}_k(T,t-t_0-nT), \qquad t \in [\ t_0+nT, t_0+(n+1)T\ ),$$
$$\text{for all } n=0,1,2,..., \quad k=1,...,l \tag{4.46}$$

The following stabilization result follows readily.

THEOREM 4.1. *Let T be such that the extended system using discretized controls is exponentially stable with a desired region of attraction C. Under hypotheses H0-H2, a solution to the TIP problem exists and the original system (4.1) with feedback control defined by (4.45) is ρ-exponentially stable with the same region of attraction C.*

Proof. The existence of solutions to the TIP follows from Remark 4.2.

Without the loss of generality, let C be a level set of V - the Lyapunov function for the extended system. Let $(t_0,x_0) \in I\!R^+ \times C$ be arbitrary and denote by $x(\cdot;t_0,x_0)$, and $\bar{x}(\cdot;t_0,x_0)$, the trajectories of the original system (4.1) using feedback control (4.45), and the extended system with discretized control (4.7), through (t_0,x_0), respectively. Due to assumption H0, and because the functions a_i, $i=1,...,r$, are piece-wise constant along the trajectories of the extended system, the trajectories x and $\bar{x}$ exist and are unique for all times.

From the discussion of the previous section it follows that :

$$x(t_0+t;t_0,x_0) = x_0\nu(S^w(t)) \quad \text{and} \quad \bar{x}(t_0+t;t_0,x_0) = x_0\nu(S^a(t)), \quad t \in [0,T] \tag{4.47}$$

and since $S^a(T) = S^w(T)$, then

$$x(t_0+T;t_0,x_0) = \bar{x}(t_0+T;t_0,x_0) \in C \tag{4.48}$$

Since the extended system with discretized control is exponentially stable with Lyapunov function V, there exists a constant $\gamma > 0$ such that

$$V(x(t_0+T;t_0,x_0)) = V(\bar{x}(t_0+T;t_0,x_0)) \leq e^{-\gamma T}V(x_0) \tag{4.49}$$

By construction, for any fixed initial point $(t_0,x_0) \in I\!R^+ \times C$ the corresponding feedback controls u_i, are linear combinations of the time functions $\hat{w}_i(T,\cdot)$ with constant coefficients $b_{k,i}(a) =$

$b_{k,i}(\bar{v}(T,x_0))$ for all $t \in [t_0, t_0+T]$. Hence $u_i(t,x) \equiv u_i(t,x_0)$ for $t \in [t_0,t_0+T]$. Let $t \mapsto x\ exp(t\sum_{i=1}^{m} g_i u_i(\tau,x_0))$ denote the integral curve of the original system (4.1) through the point (t_0,x), with control $u_i(\cdot,x_0)$. Consider the mapping : $(t,x) \mapsto x\ exp(t\sum_{i=1}^{m} g_i u_i(\tau,x_0))$ on the domain $[t_0,\infty)\times \mathbb{R}^n$. By virtue of the definition of the control $u_i(\cdot,x_0)$, and completeness of the vector fields $g_1,...,g_m$, this mapping is at least continuous , as it is a composition of smooth mappings and in fact can be equivalently expressed by

$$\begin{aligned}
(t,x) \mapsto x\ \nu(S^u(t)) &= x\ \nu[\exp(\gamma_1(t,x_0)X_1)...\exp(\gamma_r(t,x_0)X_r)] \\
&= x\ \exp(\gamma_1(t,x_0)\nu(X_1))...\exp(\gamma_r(t,x_0)\nu(X_r)) \\
&= x\ \exp(\gamma_1(t,x_0)g_1)...\exp(\gamma_r(t,x_0)g_r) \qquad (4.50)
\end{aligned}$$

where S^u is the solution of a corresponding formal initial value problem on a group $G_k(X_1,...,X_r)$ with $\nu(X_i) = g_i$, $i = 1,...,r$, and controls $u_i(T,x_0)$, $i = 1,...,r$ (the corresponding logarithmic coordinates γ_i are smooth functions of the time t and depend on x_0 via $u_i(T,x_0)$).

Hence, the image of the compact set $[t_0,t_0+T]\times C$ under this mapping is a subset of some compact ball $\bar{B}(0;\rho) \subset \mathbb{R}^n$. It follows that

$$x(t_0+t;t_0,x_0) \in \bar{B}(0;\rho), \qquad \text{for all } t \in [t_0,t_0+T], \qquad (4.51)$$

and for any $(t_0,x_0) \in \mathbb{R}^+ \times C$. Let M_g and $L_v > 1$ be: a common bound for g_i, $i = 1,...,m$, and a common Lipschitz constant for all $v_k(T,x)$, $k = 1,...,r$, on $\bar{B}(0;\rho)$, respectively. Further, let $K > 1$ and $p_{k,i} > 0$ be the Holder constants for the $b_{k,i}$, $i = 1,...,m$, $k = 1,...,l$. Finally, let $\hat{w}_k$, $k = 1,...,l$, be bounded by M_c on $[0,T]$. Then

$$|b_{k,i}(\bar{v}(x))| \le KL_v^{p_{k,i}}||x||^{p_{k,i}} \le \begin{cases} KL_v||x||^{\rho} & x \in B(0;1) \\ KL_v||x|| & x \notin\ B(0;1) \end{cases} \qquad (4.52)$$

for $i \in \{1,...,m\}$, $k \in \{1,...,l\}$, and all $x \in \bar{B}(0,\rho)$, where

$$\rho \stackrel{def}{=} min\{p_{k,i}, i \in \{1,...,m\}, k \in \{1,...,l\}\} \qquad (4.53)$$

Hence the integral curve $x(t;t_0,x_0)$, for $t \in [t_0,t_0+T]$, can be bounded as follows :

$$\begin{aligned}
||x(t_0+t;t_0,x_0)|| &\le ||x_0|| \\
&+ \sum_{i=1}^{m}\int_{t_0}^{t_0+t} ||g_i(x)||\sum_{k=1}^{l}|b_{k,i}(a(x))||\hat{w}_k(T,\tau)|d\tau \\
&\le ||x_0|| + mlM_gKL_vM_cT||x_0||^{\mu(x_0)} \qquad (4.54)
\end{aligned}$$

in which $\mu(x_0) = \rho$ if $x_0 \in B(0;1)$, and $\mu(x_0) = 1$ if $x_0 \notin B(0;1)$. Since for all $x \in I\!R^n$, $2\lambda_{min}||x||^2 \leq V(x) \leq 2\lambda_{max}||x||^2$, then there exists a constant $M > 1$ such that

$$V(x(t_0 + t; t_0, x_0)) \leq M\ V(x_0)^{\mu(x_0)} \tag{4.55}$$

for all $t \in [t_0, t_0 + T]$, and all $(t_0, x_0) \in I\!R^+ \times \mathcal{C}$. Equations (4.49) and (4.55), together with Lemma 4.1, prove that the original system (4.1) with controls (4.45) is ρ-exponentially stable with region of attraction $\mathcal{C}$. ■

REMARK 4.4. An obvious question comes to mind and is concerned with determining the effect of reversing the discretization of the feedback function $v(x)$, a posteriori to the design, which involves substituting the piece-wise constant functions $(\bar{v}_i(x))$ by their continuous counterparts $v_i(x)$ (in which $v(x)$ is the smooth feedback for the extended system). If, additionally, the solution to the FIP is sought in the class of continuous functions with equal end-points, then the resulting feedback law is continuous in t and smooth in x. Since the extended controlled system has strong stabilizing properties (the rate of exponential stability can be adjusted arbitrarily), it is natural to expect that the robustness margin of the extended system can compensate for such difference in the values of the functions v_i. The latter is indeed confirmed by numerous simulation experiments, which show that asymptotic stabilizing properties of the feedback (4.45) using continuous instead if discretized controls, are preserved (with the proviso that the discretization step T is sufficiently small and the magnitudes of the solutions to the FIP increase with T at a rate not exceeding that of $T^{-\rho}$, for $\rho \in (0,1)$). A quantitative assessment of the influence of such reversal to continuous feedback on the rate of stabilization is technically involved and will not be presented here.

4. Examples

In this section, the application of the trajectory interception approach is demonstrated on some examples of drift free systems characterized by nilpotent controllability Lie algebras.

4.1. Time varying stabilizing feedback control for a system with five state variables and three controls

The example considered below represents a drift free system of control deficiency order two whose controllability Lie algebra is nilpotent. This example is general enough to illustrate the procedure for deriving the equations describing the evolution of the logarithmic coordinates of flows.

The equations of the system are given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} -x_4 \\ -x_5 \\ 1 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_3$$

$$\begin{aligned} &\stackrel{def}{=} g_1 u_1 + g_2 u_2 + g_3 u_3 \qquad (4.56) \\ \text{where,} \quad g_1(x) &= -x_4 \frac{\partial}{\partial x_1} - x_5 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \\ g_2(x) &= \frac{\partial}{\partial x_4}, \qquad g_3(x) = \frac{\partial}{\partial x_5} \end{aligned}$$

To satisfy the LARC condition, we need to calculate the following Lie brackets:

$$\begin{aligned} g_4(x) &\stackrel{def}{=} [g_1, g_2](x) = \frac{\partial}{\partial x_1} \\ g_5(x) &\stackrel{def}{=} [g_1, g_3](x) = \frac{\partial}{\partial x_2} \end{aligned}$$

which yields

$$span\{g_1(x), g_2(x), g_3(x), g_4(x), g_5(x)\} = I\!R^5, \qquad \text{for all} \quad x \in I\!R^5 \qquad (4.57)$$

The Lie brackets multiplication table for $L(g_1, g_2, g_3)$ is:

$$\begin{aligned} &[g_1, g_2] = g_4 \qquad [g_1, g_3] = g_5 \qquad [g_2, g_3] = 0 \\ &[g_4, g_5] = 0 \qquad [g_i, g_4] = [g_i, g_5] = 0, \quad i = 1, 2, 3 \qquad (4.58) \end{aligned}$$

which shows that the controllability Lie algebra $L(g_1, g_2, g_3)$ is nilpotent. The extended system can be defined as :

$$\dot{x} = g_1(x)v_1 + g_2(x)v_2 + g_3(x)v_3 + g_4(x)v_4 + g_5(x)v_5, \quad x \in I\!R^5 \qquad (4.59)$$

and the controls are taken to be

$$v_i(x) \stackrel{def}{=} -L_{g_i} V(x), \quad i = 1, ..., 5, \qquad \text{with} \qquad V(x) \stackrel{def}{=} \frac{1}{2} x^T x, \quad x \in I\!R^5 \qquad (4.60)$$

which give the following extended system with discretized controls:

$$\begin{aligned} \dot{x} &= g_1(x)\bar{v}_1(T,x) + g_2(x)\bar{v}_2(T,x) + g_3(x)\bar{v}_3(T,x) + g_4(x)\bar{v}_4(T,x) + g_5(x)\bar{v}_5(T,x), \; x \in I\!R^5 \\ &= g_1(x)a_1 + g_2(x)a_2 + g_3(x)a_3 + g_4(x)a_4 + g_5(x)a_5, \; x \in I\!R^5 \qquad (4.61) \end{aligned}$$

with $a_i = \bar{v}_i(T, x)$, $i = 1, ...5$. The formal equation for this system becomes:

$$\dot{S}(t) = S(t) \; (\sum_{i=1}^{5} X_i a_i) \tag{4.62}$$

$$S(0) = I \tag{4.63}$$

where each X_i corresponds to g_i, $i = 1, ..., 5$, via an evaluation homomorphism ν. The solution of (4.62) is assumed in the form:

$$S(t) = \prod_{i=1}^{5} exp(\gamma_i(t) X_i) \tag{4.64}$$

and the logarithmic coordinates, γ_i, $i = 1, ..., 5$, are computed as follows. Expression (4.64) is first substituted into (4.62) which yields:

$$\begin{aligned} X_1 a_1 + ... + X_5 a_5 &= \dot{\gamma}_1 X_1 + \dot{\gamma}_2 (e^{\gamma_1 AdX_1}) X_2 + \dot{\gamma}_3 (e^{\gamma_1 AdX_1} e^{\gamma_2 AdX_2}) X_3 \\ &+ \; ... + \dot{\gamma}_5 (e^{\gamma_1 AdX_1} e^{\gamma_2 AdX_2} \; ... \; e^{\gamma_4 AdX_4}) X_5 \end{aligned} \tag{4.65}$$

where

$$(e^{AdX}) Y \stackrel{def}{=} e^X Y e^{-X} \quad \text{and} \quad (AdX) Y \stackrel{def}{=} [X, Y]$$

Employing the Baker-Hausdorff formula:

$$(e^{AdX}) Y \;\; = \;\; e^X Y e^{-X} = Y + [X, Y] + [X, [X, Y]]/2! + ...$$

gives

$$\begin{aligned} (e^{\gamma_1 AdX_1}) X_2 &= e^{\gamma_1 X_1} X_2 e^{-\gamma_1 X_1} \\ &= X_2 + (\gamma_1/1!)[X_1, X_2] + (\gamma_1^2/2!)[X_1, [X_1, X_2]] + ... \\ &= X_2 + \gamma_1 X_4 + (\gamma_0^2/2!) \; 0 + 0 = X_2 + \gamma_1 X_4 \end{aligned} \tag{4.66}$$

Similarly

$$(e^{\gamma_1 AdX_1} e^{\gamma_1 AdX_2}) X_3 \;\; = \;\; X_3 + \gamma_1 X_5$$

$$(e^{\gamma_1 AdX_1} e^{\gamma_2 AdX_2} e^{\gamma_3 AdX_3}) X_4 \;\; = \;\; X_4$$

$$(e^{\gamma_1 AdX_1} e^{\gamma_2 AdX_2} e^{\gamma_3 AdX_3} e^{\gamma_4 AdX_4}) X_5 \;\; = \;\; X_5 \tag{4.67}$$

Substituting (4.66) and (4.67) into equation (4.65) and comparing the coefficients of X_i, $i = 1, .., 5$, yields the following equations for the evolution of the logarithmic coordinates γ_i, $i = 1, ..., 5$:

$$\dot{\gamma}_1 = a_1$$
$$\dot{\gamma}_2 = a_2$$
$$\dot{\gamma}_3 = a_3$$
$$\dot{\gamma}_4 + \gamma_1 \dot{\gamma}_2 = a_4$$
$$\dot{\gamma}_5 + \gamma_1 \dot{\gamma}_3 = a_5$$

with initial conditions $\gamma_i(0) = 0$, $i = 1, .., 5$, corresponding to the identity in (4.63). The latter is easily solved with respect to $\dot{\gamma}_1$, ..., $\dot{\gamma}_5$, yielding:

$$\begin{aligned} \dot{\gamma}_1 &= a_1 \\ \dot{\gamma}_2 &= a_2 \\ \dot{\gamma}_3 &= a_3 \\ \dot{\gamma}_4 &= -\gamma_1 a_2 + a_4 \\ \dot{\gamma}_5 &= -\gamma_1 a_3 + a_5, \quad \gamma_i(0) = 0, \ i = 1, .., 5 \end{aligned} \tag{4.68}$$

The TIP in logarithmic coordinates now takes the form of a trajectory interception problem for the following two "control systems":

$$CS1: \begin{cases} \dot{\gamma}_1 = a_1 \\ \dot{\gamma}_2 = a_2 \\ \dot{\gamma}_3 = a_3 \\ \dot{\gamma}_4 = -\gamma_1 a_2 + a_4 \\ \dot{\gamma}_5 = -\gamma_1 a_3 + a_5 \end{cases} \qquad CS2: \begin{cases} \dot{\gamma}_1 = w_1 \\ \dot{\gamma}_2 = w_2 \\ \dot{\gamma}_3 = w_3 \\ \dot{\gamma}_4 = -\gamma_1 w_2 \\ \dot{\gamma}_5 = -\gamma_1 w_3 \end{cases}$$

with common initial conditions $\gamma_i(0) = 0$, $i = 1, .., 5$.

The controls $w_i(a, t)$, $i = 1, 2, 3$, can be sought in the form

$$\begin{aligned} w_1 &= b_1 + (b_4 + b_5)\ sin(\frac{2\pi}{T}t) \\ w_2 &= b_2 + b_4\ cos(\frac{2\pi}{T}t) \\ w_3 &= b_3 + b_5\ cos(\frac{2\pi}{T}t) \end{aligned} \tag{4.69}$$

where b_i, $i = 1,..,5$ are some unknown coefficients. The above are substituted into CS2, and the systems CS1 and CS2 are integrated symbolically, to yield respective solutions $\gamma^a(T)$ and $\gamma^w(T)$ in terms of parameters a and b. The equation $\gamma^a(T) = \gamma^w(T)$ is then also solved symbolically to deliver the values for the unknown coefficients $b_i(a)$, $i = 1, ..., 5$ as functions of the control parameters $a = [a_1, a_2, a_3, a_4, a_5]$ and T:

$$b_1 = a_1, \quad b_2 = a_2, \quad b_3 = a_3$$
$$b_4 = \pm 3.54491\sqrt{(a_4)}/\sqrt{(T)}. \quad b_5 = \pm 3.54491\sqrt{(a_5)}/\sqrt{(T)}$$

which reflects that two solutions were found.

In this and all further examples, at the implementation stage of the final feedback control, all the terms involving square roots of the extended discretized controls a_i, such as $\sqrt{a_i}$, must naturally be substituted by $sign(a_i)\sqrt{|a_i|}$.

The time varying stabilizing controls are then finally given by

$$\begin{aligned} u_1(T,x) &= \bar{v}_1(T,x) + \{b_4(\bar{v}(T,x)) + b_5(\bar{v}(T,x))\}\ sin(\frac{2\pi}{T}t) \\ u_2(T,x) &= \bar{v}_2(T,x) + b_4(\bar{v}(T,x))\ cos(\frac{2\pi}{T}t) \\ u_3(T,x) &= \bar{v}_3(T,x) + b_5(\bar{v}(T,x))\ cos(\frac{2\pi}{T}t) \end{aligned} \tag{4.70}$$

The above controls, as applied to the system model (4.56), result in controlled trajectories $t \mapsto x_i(t)$, $i = 1, ..., 5$, depicted in Figure 4.1. Plot of the "Lyapunov function" V along the controlled trajectories and plot of $x_3(t)$ versus $x_4(t)$ is shown in Figure 4.2.

4.2. Time varying stabilizing feedback control for a unicycle in chained form

This example is a particular case of a three dimensional system in chained form. Systems in chained form have the important property of being characterized by nilpotent controllability Lie algebras.

The model of a unicycle in chained form is given by equation (1.19) of Chapter 1 and represents a system with control deficiency order one. The controllability Lie algebra $L(g_1, g_2)$ for this system is nilpotent, as can be verified by inspection of the following Lie brackets multiplication table for $L(g_1, g_2)$:

$$[g_1, g_2] = g_3 \qquad [g_1, g_3] = 0 \qquad [g_2, g_3] = 0 \tag{4.71}$$

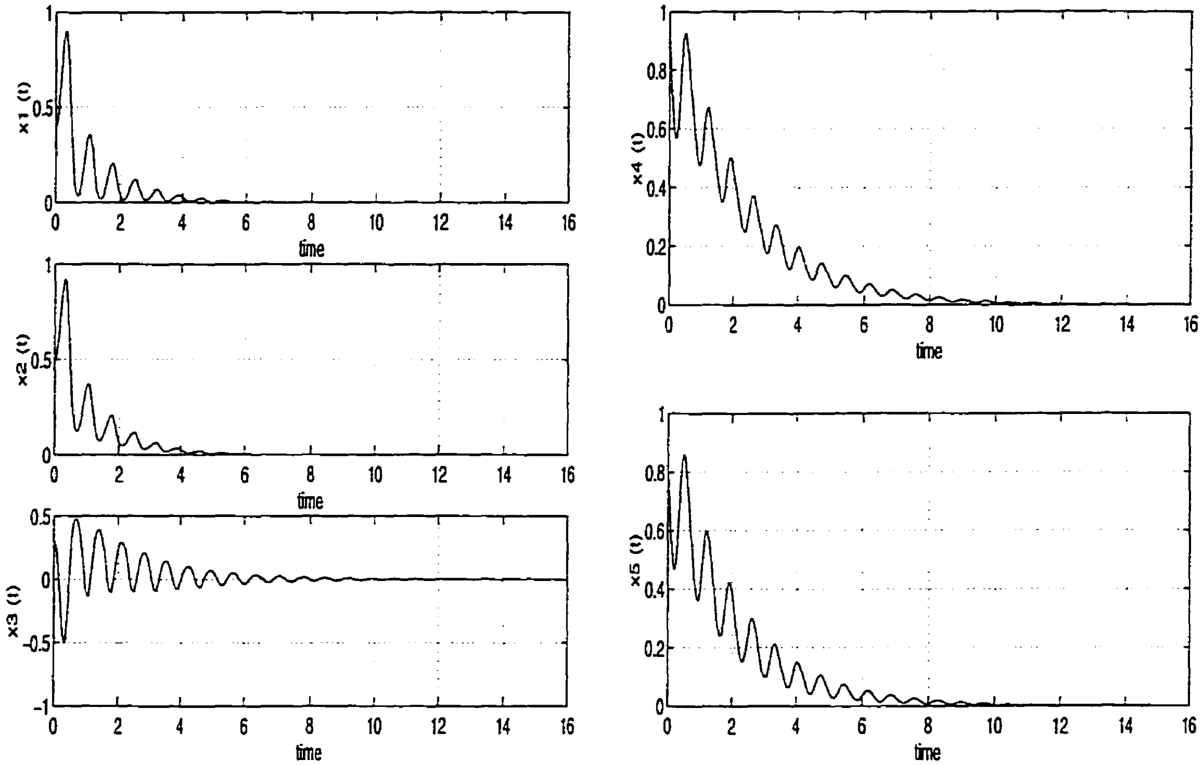

FIGURE 4.1. *Five dimensional drift free system:* Plots of the controlled state trajectories $t \mapsto (x_1(t), \dots, x_5(t))$ versus time.

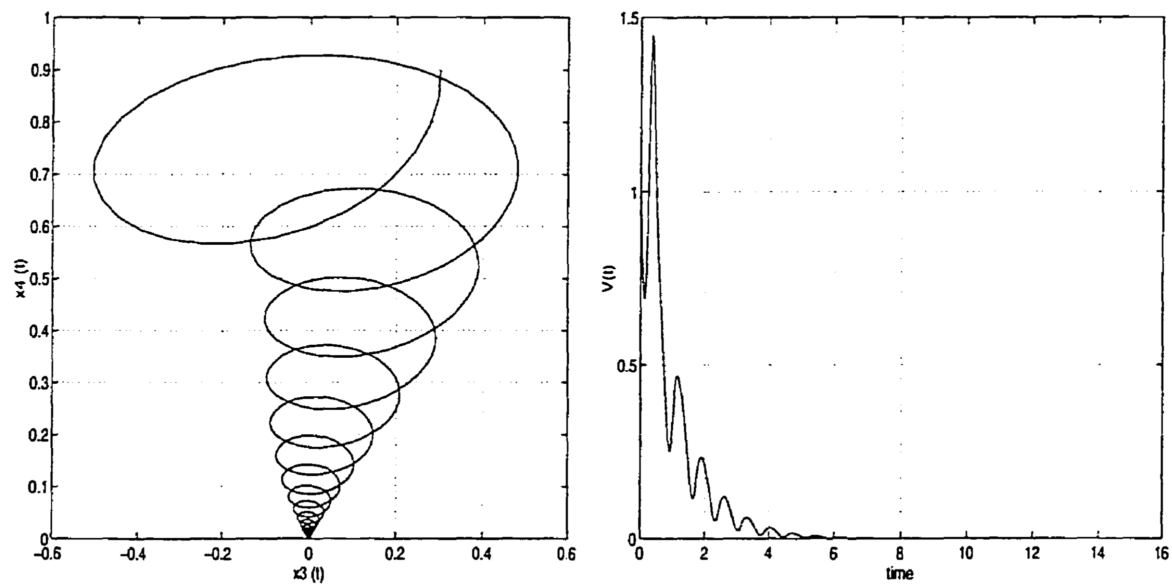

FIGURE 4.2. *Five dimensional drift free system:* Plots of the controlled state trajectories $x_3(t)$ versus $x_4(t)$ and the Lyapunov function $V(x(t)) = \frac{1}{2}\sum_{i=1}^{5} x_i^2(t)$ along the controlled state trajectories.

The extended system can thus be defined as :

$$\dot{x} = g_1(x)v_1 + g_2(x)v_2 + g_3(x)v_3, \; x \in I\!R^3 \tag{4.72}$$

and the extended controls can be defined by:

$$v_i(x) \overset{def}{=} -L_{g_i}V(x), \quad i = 1, ..., 3, \quad \text{with} \quad V(x) \overset{def}{=} \frac{1}{2}x^T x, \quad x \in I\!R^3 \tag{4.73}$$

and result in the following discretized extended system:

$$\begin{aligned} \dot{x} &= g_1(x)\bar{v}_1(T,x) + g_2(x)\bar{v}_2(T,x) + g_3(x)\bar{v}_3(T,x), \; x \in I\!R^3 \\ &= g_1(x)a_1 + g_2(x)a_2 + g_3(x)a_3, \; x \in I\!R^3 \end{aligned} \tag{4.74}$$

where $a_i = \bar{v}_i(T,x)$, $i = 1, 2, 3$.

Proceeding similarly as in section 4.1, the equations describing the evolution of the logarithmic coordinates of the flow of this system can be found :

$$\begin{aligned} \dot{\gamma}_1 &= a_1 \\ \dot{\gamma}_2 &= a_2 \\ \dot{\gamma}_3 &= -\gamma_1 a_2 + a_3, \; \gamma_i(0) = 0, \; i = 1, 2, 3 \end{aligned} \tag{4.75}$$

The TIP in logarithmic coordinates thus takes the form of a trajectory interception problem for the following two control systems:

$$CS1: \begin{cases} \dot{\gamma}_1 = a_1 \\ \dot{\gamma}_2 = a_2 \\ \dot{\gamma}_3 = -\gamma_1 a_2 + a_3 \end{cases} \qquad CS2: \begin{cases} \dot{\gamma}_1 = w_1 \\ \dot{\gamma}_2 = w_2 \\ \dot{\gamma}_3 = -\gamma_1 w_2 \end{cases} \tag{4.76}$$

with common initial conditions $\gamma_i(0) = 0$, $i = 1, 2, 3$.

The controls $w_i(a,t)$, $i = 1, 2$ can be sought in the form

$$w_1 = b_1 + b_3 \; sin(\frac{2\pi}{T}t), \qquad w_2 = b_2 + b_3 \; cos(\frac{2\pi}{T}t) \tag{4.77}$$

and indeed $b_1 = a_1$, $b_2 = a_2$, $b_3 = \pm 3.54491\sqrt{(}a_3)/\sqrt{(}T)$ are found.

The time varying stabilizing controls for the unicycle in chained form are finally given by

$$\begin{aligned} u_1(T,x) &= \bar{v}_1(T,x) + b_3(\bar{v}(T,x)) \; sin(\frac{2\pi}{T}t) \\ u_2(T,x) &= \bar{v}_2(T,x) + b_3(\bar{v}(T,x)) \; cos(\frac{2\pi}{T}t) \end{aligned} \tag{4.78}$$

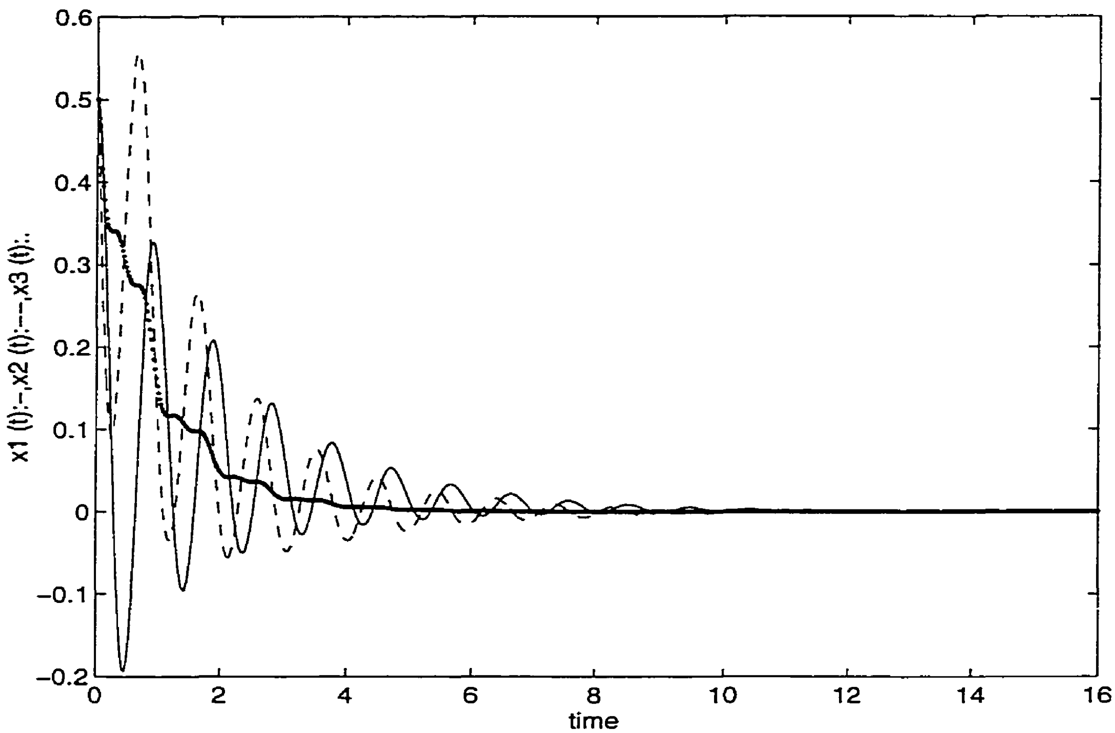

FIGURE 4.3. *Unicycle model in chained form:* Plots of the controlled state trajectories $t \mapsto (x_1(t), x_2(t), x_3(t))$ versus time.

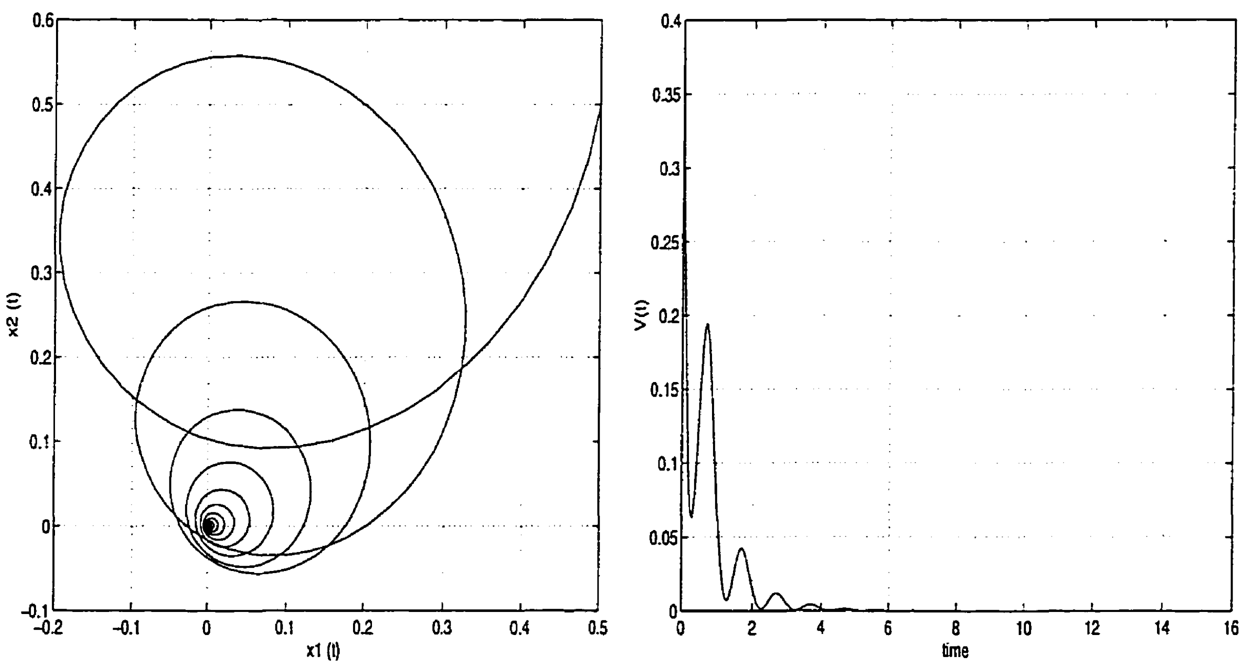

FIGURE 4.4. *Unicycle model in chained form:* Plots of the controlled state trajectories $x_1(t)$ versus $x_2(t)$ and the Lyapunov function $V(x(t)) = \frac{1}{2}\sum_{i=1}^{3} x_i^2(t)$ along the controlled state trajectories.

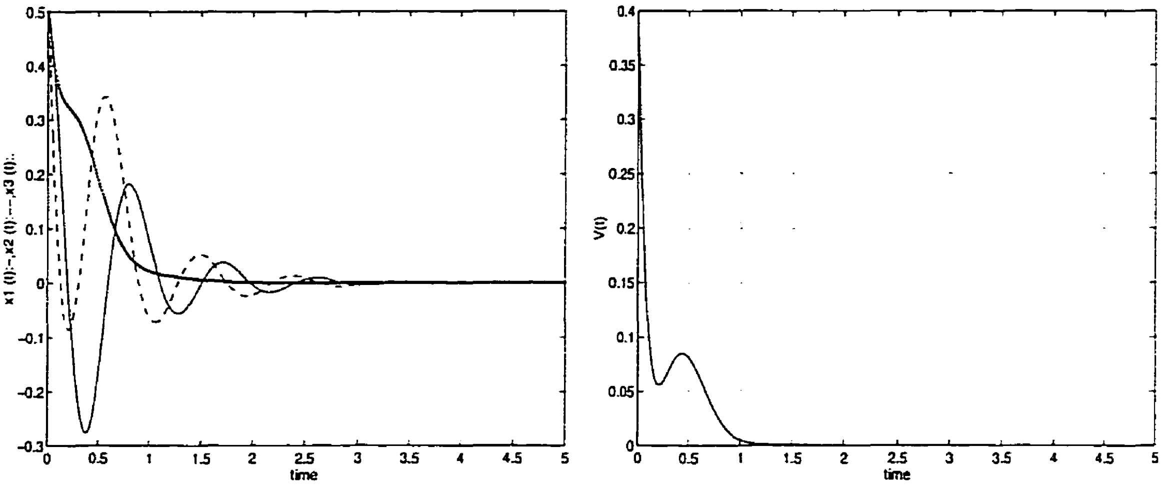

FIGURE 4.5. *Brockett's system:* Plots of the controlled state trajectories $t \mapsto (x_1(t), x_2(t), x_3(t))$ versus time and the Lyapunov function $V(x(t)) = \frac{1}{2}\sum_{i=1}^{3} x_i^2(t)$ along the controlled state trajectories.

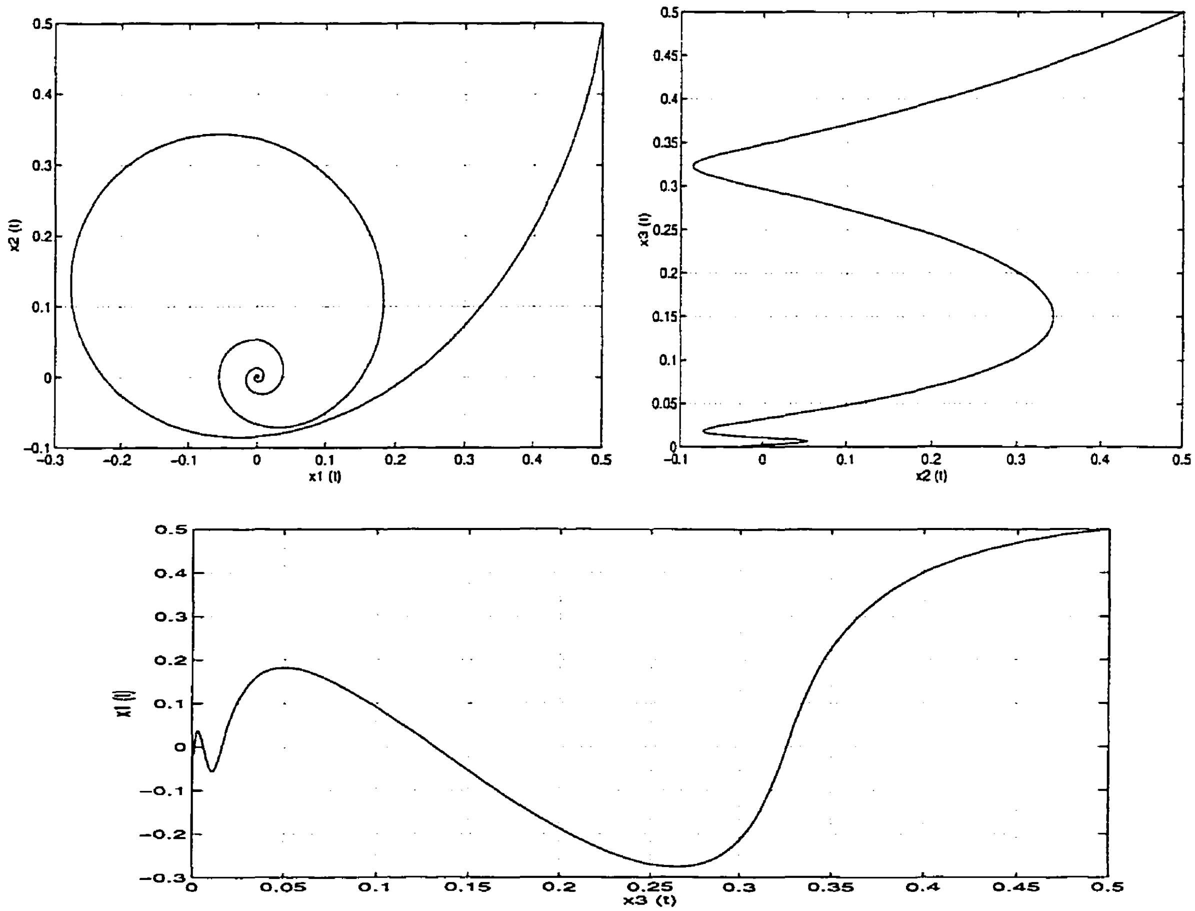

FIGURE 4.6. *Brockett's system:* Plots of the controlled state trajectories $x_1(t)$ versus $x_2(t)$, $x_2(t)$ versus $x_3(t)$, and $x_3(t)$ versus $x_1(t)$.

The controls given in (4.78) are ready to apply to the system model (1.20), and produce results depicted in Figures 4.3 - 4.4.

4.3. Time varying stabilizing feedback control for Brockett's system

Brockett's system whose equations of motion are given in (1.16) of Chapter 1 is another famous example of a three dimensional drift free system which does not, however, appear in a chained form. The controllability Lie algebra $L(g_1, g_2)$ for Brockett's system is given by (1.16) of Chapter 1, and is nilpotent. The equations for the evolution of the logarithmic coordinates of the flow of this system are the same as the ones given in (4.75). This is because the controllability Lie algebras of the unicycle and Brockett's system are isomorphic. The following controls with $k = 2$ and $T = 0.8$ are used in simulations:

$$\begin{aligned} u_1(T,x) &= k\ \bar{v}_1(T,x) + b_3(\bar{v}(T,x))\ sin(\frac{2\pi}{T}t) \\ u_2(T,x) &= k\ \bar{v}_2(T,x) + b_3(\bar{v}(T,x))\ cos(\frac{2\pi}{T}t) \end{aligned} \tag{4.79}$$

where, $b_3 = \pm 3.54491\sqrt(a_3)/\sqrt(T)$. Simulation results are shown in Figures 4.5 - 4.6.

5. Time varying feedback for general systems

In the case when the algebra of vector fields $L(g_1, ..., g_m)$ is not nilpotent, the TIP or, equivalently the FIP, cannot be solved exactly, since the basis for $L(g_1, ..., g_m)$ is then not necessarily finite. It is possible often, however, to solve the FIP "approximately" as will be explained shortly. The success of the approximate approach relies on the ability to construct a type of nilpotent approximation to the original system which is defined below.

DEFINITION 2. *(***Approximately nilpotent system***) A system $\dot{x} = \sum_{i=1}^{m} g_i(x)u_i$, with a non-nilpotent controllability Lie algebra $L(g_1, ..., g_m)$ is said to be approximately nilpotent if the vector fields can be approximated by their truncated Taylor series expansions $\tilde{g}_1, ..., \tilde{g}_m$ at zero in such a way that the controllability Lie algebra $L_k(\tilde{g}_1, ..., \tilde{g}_m)$ for the approximate system $\dot{x} = \sum_{i=1}^{m} \tilde{g}_i u_i$ is nilpotent of some order k, and, additionally, the Lie brackets multiplication tables for both Lie algebras are identical for all brackets of depth up to k.*

Once an approximately nilpotent substitute for the original system is selected, the FIP problem can be solved for the approximate system, and produces a kind of approximate solution to the original TIP. The computation of such an approximate solution to FIP clearly does not differ from the one discussed above but the trajectories of the extended and the original system with time varying feedback are no longer guaranteed to intercept with frequency $1/T$. It is therefore not entirely clear whether the stabilizing properties of the constructed time varying feedback are preserved. To answer this question we make use of a result from [**52**] which delivers an error estimate for open loop steering while neglecting higher order Lie brackets. This result is relevant also to our analysis because, loosely speaking, the controllability Lie algebras $L(g_1, ..., g_m)$ and $L_k(\bar{g}_1, ..., \bar{g}_m)$ differ only in the brackets of depths higher than k. The aforementioned result, adequately interpreted in the context of the TIP, can be restated as follows.

THEOREM 4.2. [**52**] *Suppose that the algebra of vector fields $L(g_1, ..., g_m)$ is not nilpotent but that the FIP is still solved on a Lie group $G_k(X_1, ..., X_r)$ of order $k < \infty$, using the structure of the nilpotent approximation $L_k(\bar{g}_1, ..., \bar{g}_m)$ of the original $L(g_1, ..., g_m)$. Let $\bar{w}$ denote such an approximate solution to the FIP, and let $t \mapsto \bar{x}(t; t_0, x_0)$ and $t \mapsto x(t; t_0, x_0)$ denote integral curves, through (t_0, x_0), of the extended system with discretised control, and the original system (4.1) with the time varying feedback incorporating $\bar{w}$, respectively. Further, let $\bar{x}_T \overset{def}{=} \bar{x}(t_0 + T; t_0, x_0)$, and $x_T \overset{def}{=} x(t_0 + T; t_0, x_0)$. Finally, let $\mathcal{R}$ be a bounded region in $I\!R^n$. Under these conditions, there exists a function $F : [0, \infty) \to [0, \infty]$, which is finite and bounded near zero, such that if $x_0, x_T \in \mathcal{R}$, $t_0 \geq 0$, then*

$$||x_T - \bar{x}_T|| \leq F(||\bar{x}_T - x_0||) \; ||\bar{x}_T - x_0||^{1+\frac{1}{k}} \tag{4.80}$$

The stabilizing properties of the time varying feedback control incorporating an approximate solution to the FIP can now be specified.

COROLLARY 1. *Suppose that the hypotheses H0-H1 are valid but the controllability Lie algebra $L(g_1, ..., g_m)$ is not necessarily nilpotent. Let $\bar{w}$ denote an approximate solution to the FIP, as obtained in $G_k(X_1, ..., X_m)$, for some finite k. Then for any given level set $\mathcal{C}$ of the Lyapunov function V for the extended system, there exists a $T^* > 0$ such that the time varying feedback using a discretisation step $T \leq T^*$, and incorporating $\bar{w}$, is ρ-exponentially stabilizing for the original system (4.1), with region of attraction $\mathcal{C}$.*

Proof. Proceeding similarly as in the proof of Theorem 4.1, let $t \mapsto \bar{x}(t; t_0, x_0)$ and $t \mapsto x(t; t_0, x_0)$ denote the integral curves of the extended system (4.7), with discretized controls, and the original system (4.1) with the time varying feedback (4.45) employing an approximate solution $\bar{w}$ to FIP, respectively. (The dependence on the initial conditions of x and $\bar{x}$ will be omitted, when it is clear from the context.)

As the extended system is Lyapunov stable with Lyapunov function V, there exists a closed ball $\bar{B}(0; \rho)$, $\rho > 0$, which contains all the trajectories $\bar{x}(t; t_0, x_0)$, $t \geq t_0$, as (t_0, x_0) ranges over $I\!R^+ \times \mathcal{C}$.

Additionally, there exists a constant $\gamma > 0$ such that for any $T > 0$, and any $(t_0, x_0) \in I\!R^+ \times \mathcal{C}$

$$V(\bar{x}(t_0 + T; t_0, x_0)) \leq e^{-\gamma T} V(x_0) \tag{4.81}$$

Let M_g and L_v be: a common bound for $g_i, i = 1, ..., m$, and, a common Lipschitz constant for $v_k, k = 1, ..., r$, on $\bar{B}(0; \rho)$, respectively. The following estimates are immediate :

$$||\bar{x}(t_0 + T; t_0, x_0) - x_0|| \leq \sum_{k=1}^{r} \int_{t_0}^{t_0+T} ||g_k(\bar{x})|| ||\bar{v}_k(x_0)|| d\tau \leq r M_g L_v T ||x_0|| \stackrel{def}{=} cT||x_0|| \tag{4.82}$$

and

$$||\bar{x}(t_0 + T; t_0, x_0)|| \leq ||\bar{x}(t_0 + T; t_0, x_0) - x_0|| + ||x_0|| \leq (1 + cT)||x_0|| \tag{4.83}$$

which hold for any $(t_0, x_0) \in I\!R^+ \times \mathcal{C}$, and any $T > 0$. For brevity, let $x_T \stackrel{def}{=} x(t_0 + T; t_0, x_0)$, $\bar{x}_T \stackrel{def}{=} \bar{x}(t_0 + T; t_0, x_0)$, and also $\Delta x_T \stackrel{def}{=} x_T - \bar{x}_T$. From (4.81) it follows that :

$$\begin{aligned} & V(x(t_0 + T; t_0, x_0)) = V(\bar{x}_T) + (V(x_T) - V(\bar{x}_T)) \\ & \leq e^{-\gamma T} V(x_0) + \frac{1}{2}(\bar{x}_T + \Delta x_T)^T Q (\bar{x}_T + \Delta x_T) - \frac{1}{2}(\bar{x}_T)^T Q \bar{x}_T \\ & \leq e^{-\gamma T} V(x_0) + (\Delta x_T)^T Q \bar{x}_T + \frac{1}{2}(\Delta x_T)^T Q \Delta x_T \\ & \leq e^{-\gamma T} V(x_0) + ||Q||(||\bar{x}_T|| ||\Delta x_T|| + \frac{1}{2}||\Delta x_T||^2) \end{aligned} \tag{4.84}$$

Let $\delta \in (0, 1)$ be such that the function $\xi \mapsto F(\xi)$ of Theorem 4.2 is bounded by some constant M_F for $\xi \in [0, \delta]$. Let T_1 be such that

$$max\{cT_1, T_1^{\frac{1}{2}}\}||x_0|| \leq \delta \qquad \text{for all } x_0 \in \mathcal{C} \tag{4.85}$$

(such a T_1 exists because $\mathcal{C}$ is bounded). Then, by virtue of the definition of Δx_T, equation (4.82) and the result of Theorem 4.2

$$\begin{aligned} & ||\Delta x_T|| \leq M_F ||\bar{x}(t_0 + T; t_0, x_0) - x_0||^{1+\frac{1}{k}} \\ & \leq M_F c^{1+\frac{1}{k}}\ T^{1+\frac{1}{2k}} ||x_0||\ (T^{\frac{1}{2}}||x_0||)^{\frac{1}{k}} \leq M_F c^{1+\frac{1}{k}}\ T^{1+\frac{1}{2k}} ||x_0|| \end{aligned} \tag{4.86}$$

for all $T \leq T_1$, and all $x_0 \in \mathcal{C}$, as $\delta < 1$. Using (4.82) and (4.83), the "error" on right hand side of (4.84) can thus be bounded as follows:

$$\begin{aligned} & ||Q||(||\bar{x}_T|| ||\Delta x_T|| + \frac{1}{2}||\Delta x_T||^2) \\ & \quad \leq M_F c^{1+\frac{1}{k}}(1 + cT_1)\ T^{1+\frac{1}{2k}} ||x_0||^2 + \frac{1}{2} M_F^2 c^{2+\frac{2}{k}}\ T^{2+\frac{1}{k}} ||x_0||^2 \end{aligned} \tag{4.87}$$

for all $T \leq T_1$, and $(t_0, x_0) \in \mathbb{R}^+ \times \mathcal{C}$. Since $V(x) \geq 2\lambda_{min}(Q)||x||^2$, then (4.84) and (4.87) imply that there exist constants $N_1, N_2 > 0$ such that

$$V(x(t_0+T;t_0,x_0)) \leq f(T)V(x_0)$$
$$\text{where } f(T) \stackrel{def}{=} e^{-\gamma T} + N_1 T^{1+\frac{1}{2k}} + N_2 T^{2+\frac{1}{k}} \tag{4.88}$$

Clearly, $f(0) = 1$, and $f'(0) = -\gamma < 0$, so f is decreasing in some neighbourhood of zero. It follows that there exists an interval $(0, T^*]$, $T^* < T_1$, such that $f(T) < 1$ for all $T \in (0, T^*]$. Hence for any fixed $T \in (0, T^*]$ there exists a $\beta < 1$ such that

$$V(x(t_0+T;t_0,x_0)) \leq \beta V(x_0) \tag{4.89}$$

for all $(t_0, x_0) \in \mathbb{R}^+ \times \mathcal{C}$. Inequality (4.89) implies that the original system, using the time varying control with $\bar{w}_i$ and discretization step $T < T^*$, satisfies condition (a) of Lemma 4.1 in the region $\mathcal{C}$.

To show that condition (b) of this Lemma is also satisfied, we proceed similarly as in the proof of Theorem 4.1. Let $t \mapsto x\ exp(t\sum_{i=1}^m g_i u_i(\tau, x_0))$ denote the integral curve of the original system (4.1) through the point (t_0, x), with feedback control given by (4.45), in which $\hat{w}_i$ are replaced by $\bar{w}_i$ and $\bar{v}_i(T,x)$, are constant and equal to $\bar{v}_i(T, x_0)$, $i = 1, ..., r$. By the definition of the time varying feedback control u_i, and completeness of the vector fields $g_1, ..., g_m$, the mapping : $(t,x) \mapsto x\ exp(t\sum_{i=1}^m g_i u_i(\tau, x_0))$, with the domain $[t_0, \infty) \times \mathbb{R}^n$, is continuous, as a composition of continuous mappings (see (4.50). Consider a fixed value of $T \in (0, T^*]$, and let the image of the compact set $[t_0, t_0+T] \times \mathcal{C}$ under this mapping be contained in $\bar{B}(0;\rho)$ (the radius ρ can always be chosen large enough to ensure this). It follows that

$$x(t_0+t;t_0,x_0) \in \bar{B}(0;\rho), \qquad \text{for all } t \in [t_0, t_0+T], \tag{4.90}$$

and any $(t_0, x_0) \in \mathbb{R}^+ \times \mathcal{C}$. Let M_g and $L_v > 1$ be: a common bound for g_i, $i = 1, ..., m$, and a Lipschitz constant for all the feedback control functions v_k, $k = 1, ..., m$, on $\bar{B}(0;\rho)$, respectively. Further, let $\bar{w}_i$, $i = 1, ..., l$ be bounded by M_c on $[0, T]$ (regardless of the value of T). As before, (see (4.52)) the functions $b_{k,i}$ can be bounded by :

$$|b_{k,i}(\bar{v}(x))| \leq K L_v^{p_{k,i}} ||x||^{p_{k,i}} \leq \begin{cases} K L_v ||x||^{\rho} & x \in B(0;1) \\ K L_v ||x|| & x \notin B(0;1) \end{cases} \tag{4.91}$$

for all $i \in \{1, ..., m\}$, $k \in \{1, ..., l\}$, and all $x \in \mathcal{C}$, where

$$\rho \stackrel{def}{=} min\{p_{k,i}, i \in \{1, ..., m\}, k \in \{1, ..., l\}\} \tag{4.92}$$

and K and $p_{k,i}$ are the Holder constants for $b_{k,i}$. Hence the integral curve $x(t;t_0,x_0)$, for $t \in [t_0, t_0+T]$ can be bounded as follows :

$$\begin{aligned}
||x(t_0+t;t_0,x_0)|| &\leq ||x_0|| \\
&+ \sum_{i=1}^{m} \int_{t_0}^{t_0+t} ||g_i(x)|| \sum_{k=1}^{l} |b_{k,i}(v(x_0))||\hat{w}_k(T,\tau)|d\tau \\
&\leq ||x_0|| + mlM_g K L_v M_c T ||x_0||^{\mu(x_0)} \qquad (4.93)
\end{aligned}$$

in which $\mu(x_0) = \rho$ if $x_0 \in B(0;1)$, and $\mu(x_0) = 1$ if $x_0 \in B(0;1)$. It follows from the definition of V that there exists a constant $M > 1$ such that

$$V(x(t_0+t;t_0,x_0)) \leq M\ V(x_0)^{\mu(x_0)} \qquad (4.94)$$

for all $t \in [t_0, t_0+T]$, and all $(t_0, x_0) \in I\!R^+ \times \mathcal{C}$. Equation (4.94) shows that condition (b) of Lemma 4.1 is satisfied so the original system controlled by (4.45), with $\hat{w}_i$ substituted by $\bar{w}_i$, is ρ-exponentially stable with region of attraction $\mathcal{C}$. ■

6. Examples of time varying control of non-nilpotent systems

The procedure described above, and involving nilpotent approximations in the sense of definition 2, is applied to stabilize several drift free systems whose controllability Lie algebras fail to be nilpotent : a unicycle, an underwater vehicle [**66**], a rigid spacecraft in actuator failure mode [**70**], a class of wheeled mobile robots [**70**], and a hopping robot in flight phase [**84**].

Simulations confirm that the error in the solution of the open-loop problem, resulting from a nilpotent approximation, can be compensated (without prejudicing stabilization) by adjusting the stability robustness margin of the feedback control for the extended system.

6.1. Time varying stabilizing feedback control for a unicycle

Using the idea of nilpotent approximation, the trajectory interception approach can be successfully applied to control a unicycle without converting the model into a chained form.

The kinematic model of a unicycle is given by equation (1.17) of Chapter 1, and satisfies the LARC condition:

$$span\{g_1(x), g_2(x), g_3(x)\} = I\!R^3, \quad \text{for all} \ \ x \in I\!R^3$$

where

$$
\begin{aligned}
g_1(x) &= \frac{\partial}{\partial x_1}, \qquad g_2(x) = \cos x_1 \frac{\partial}{\partial x_2} + \sin x_1 \frac{\partial}{\partial x_3} \\
g_3(x) &\stackrel{def}{=} [g_1, g_2](x) = -\sin x_1 \frac{\partial}{\partial x_2} + \cos x_1 \frac{\partial}{\partial x_3}
\end{aligned}
$$

The first three terms of the Lie brackets multiplication table for $L(g_1, g_2)$ are :

$$
[g_1, g_2] = g_3 \qquad [g_2, g_3] = 0 \qquad [g_1, g_3] = -g_2 \tag{4.95}
$$

and confirm that $L(g_1, g_2)$ is not nilpotent. The following approximation to the original model (1.17) is therefore considered:

$$
\dot{x} = \bar{g}_1(x)u_1 + \bar{g}_2(x)u_2, \quad x \in I\!R^3 \tag{4.96}
$$

where

$$
\begin{aligned}
\bar{g}_1(x) &= \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2}, \qquad \bar{g}_2(x) = \frac{\partial}{\partial x_3} \\
\bar{g}_3(x) &\stackrel{def}{=} [\bar{g}_1, \bar{g}_2](x) = -\frac{\partial}{\partial x_2}
\end{aligned}
$$

The Lie brackets multiplication table for $L(\bar{g}_1, \bar{g}_2)$ is now:

$$
[\bar{g}_1, \bar{g}_2] = \bar{g}_3 \qquad [\bar{g}_1, \bar{g}_3] = 0 \qquad [\bar{g}_2, \bar{g}_3] = 0 \tag{4.97}
$$

which shows that $L(\bar{g}_1, \bar{g}_2)$ is nilpotent of order 2 and its structure coincides with that of $L(g_1, g_2)$ for brackets up to order 2. The extended system for the original model is :

$$
\dot{x} = g_1(x)v_1 + g_2(x)v_2 + g_3(x)v_3, \; x \in I\!R^3 \tag{4.98}
$$

and the extended controls:

$$
v_i(x) \stackrel{def}{=} -L_{g_i}V(x), \quad i = 1, ..., 3, \quad \text{with} \quad V(x) \stackrel{def}{=} \frac{1}{2}x^T x, \quad x \in I\!R^3 \tag{4.99}
$$

give the following discretized extended system:

$$
\dot{x} = g_1(x)a_1 + g_2(x)a_2 + g_3(x)a_3, \; x \in I\!R^3 \tag{4.100}
$$

where $a_i = \bar{v}_i(T, x)$, $i = 1, 2, 3$.

The differential equations describing the evolution of the logarithmic coordinates of the flow of the nilpotent approximation (4.96) are obtained easily and are obviously identical to those of sections

4.2 and 4.3, i.e.

$$\begin{aligned}
\dot{\gamma}_1 &= a_1 \\
\dot{\gamma}_2 &= a_2 \\
\dot{\gamma}_3 &= -\gamma_1 a_2 + a_3, \quad \gamma_i(0) = 0, \; i = 1,2,3
\end{aligned} \tag{4.101}$$

The TIP in logarithmic coordinates is hence a trajectory interception problem for the following two control systems:

$$CS1: \quad \begin{cases} \dot{\gamma}_1 = a_1 \\ \dot{\gamma}_2 = a_2 \\ \dot{\gamma}_3 = -\gamma_1 a_2 + a_3 \end{cases} \qquad CS2: \quad \begin{cases} \dot{\gamma}_1 = w_1 \\ \dot{\gamma}_2 = w_2 \\ \dot{\gamma}_3 = -\gamma_1 w_2 \end{cases} \tag{4.102}$$

with common initial conditions $\gamma_i(0) = 0, \; i = 1,2,3$.

The controls $w_i(a,t)$, $i = 1,2$ can be found by assuming :

$$w_1 = b_1 + b_3 \; sin(\frac{2\pi}{T}t), \quad w_2 = b_2 + b_3 \; cos(\frac{2\pi}{T}t) \tag{4.103}$$

and the unknown coefficients $b_i(a)$, $i = 1,2,3$ are computed analytically as:

$$b_1 = a_1 \quad b_2 = a_2, \quad b_3 = \pm 3.54491\sqrt{(}a_3)/\sqrt{(}T)$$

and are identical to those in (4.77). The time varying stabilizing controls for the unicycle are :

$$\begin{aligned}
u_1(T,x) &= \bar{v}_1(T,x) + b_3(\bar{v}(T,x)) \; sin(\frac{2\pi}{T}t) \\
u_2(T,x) &= \bar{v}_2(T,x) + b_3(\bar{v}(T,x)) \; cos(\frac{2\pi}{T}t)
\end{aligned} \tag{4.104}$$

Simulation results are shown in Figures 4.7 - 4.8 and clearly confirm the result obtained in Corollary 1. The trajectories of the controlled unicycle are similar to those of (Figures 4.3 - 4.4) which were obtained by converting the system model into a chained form a priori to control design.

6.2. Time varying feedback stabilizing control for a rigid spacecraft in actuator failure mode [68]

This example demonstrates the applicability of the trajectory interception approach when the system under consideration is defined on a manifold rather than in a linear space, and is characterized by a non-nilpotent Lie algebra.

The kinematic model of a rigid spacecraft in actuator failure mode as given by (3.93) of Chapter 3, is

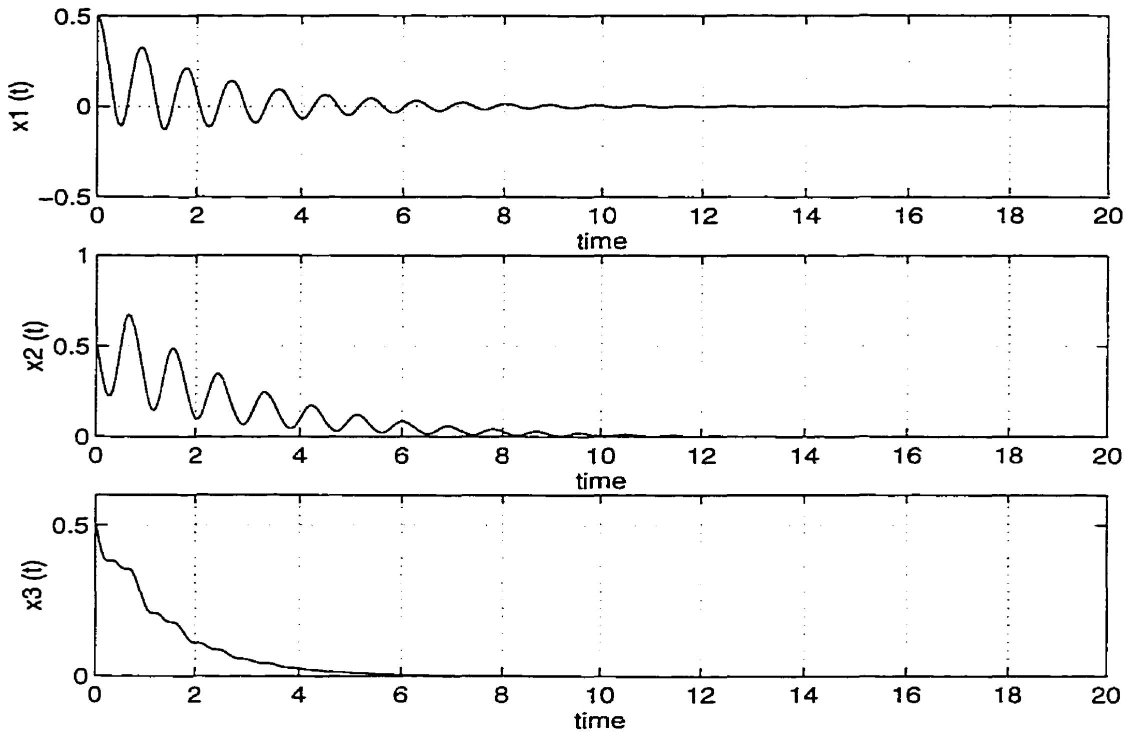

FIGURE 4.7. *Unicycle model:* Plots of the controlled state trajectories $t \mapsto (x_1(t), x_2(t), x_3(t))$ versus time.

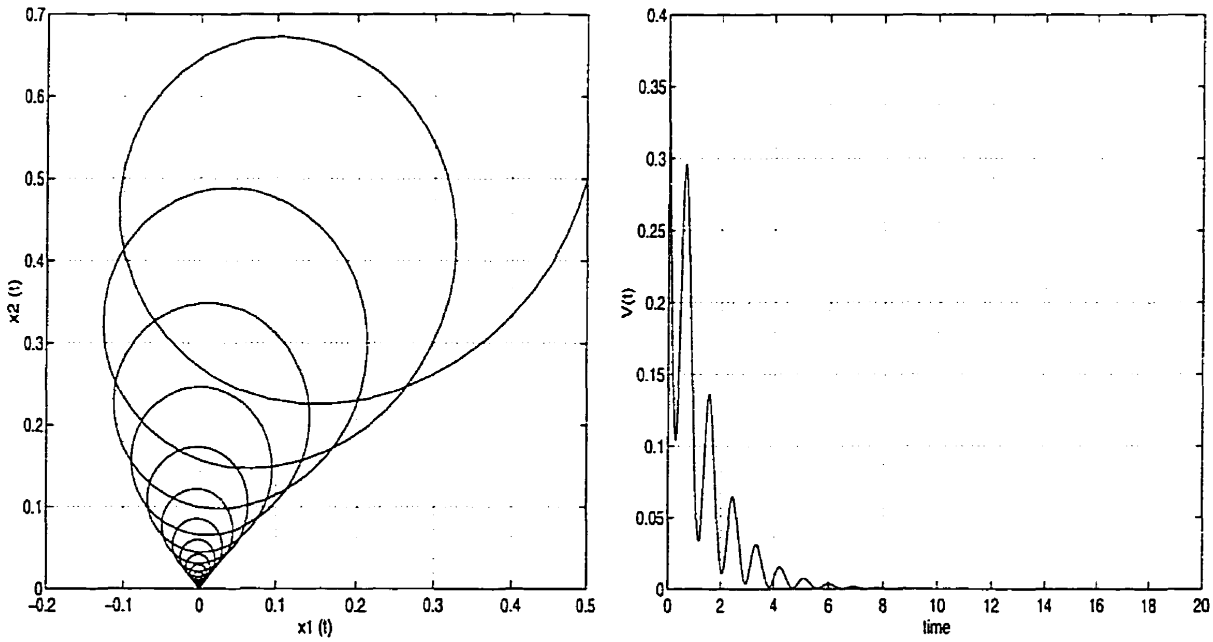

FIGURE 4.8. *Unicycle model:* Plots of the controlled state trajectories $x_1(t)$ versus $x_3(t)$ and the Lyapunov function $V(x(t)) = \frac{1}{2}\sum_{i=1}^{3} x_i^2(t)$ along the controlled state trajectories.

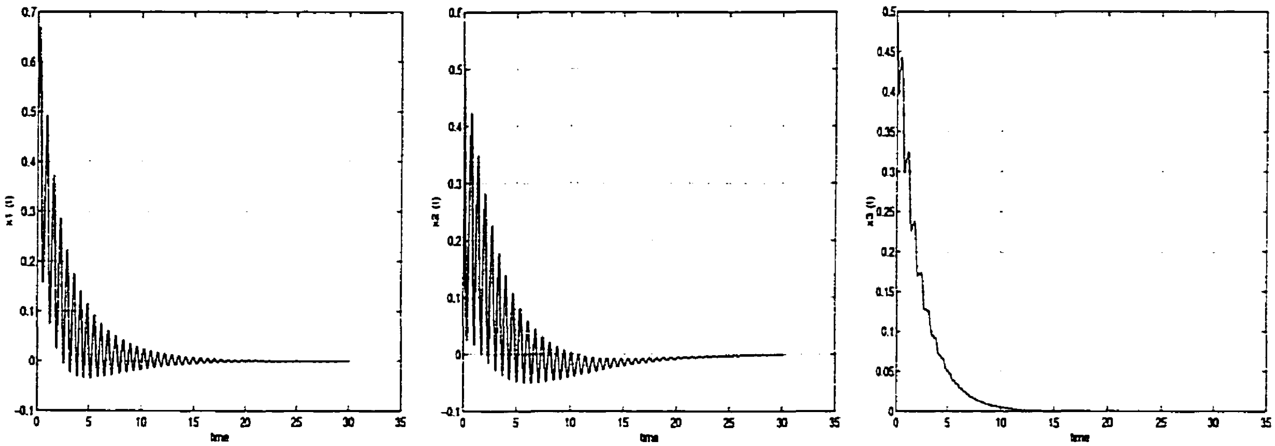

FIGURE 4.9. *Spacecraft model:* Plots of the controlled state trajectories $t \mapsto (x_1(t), x_2(t), x_3(t))$ versus time.

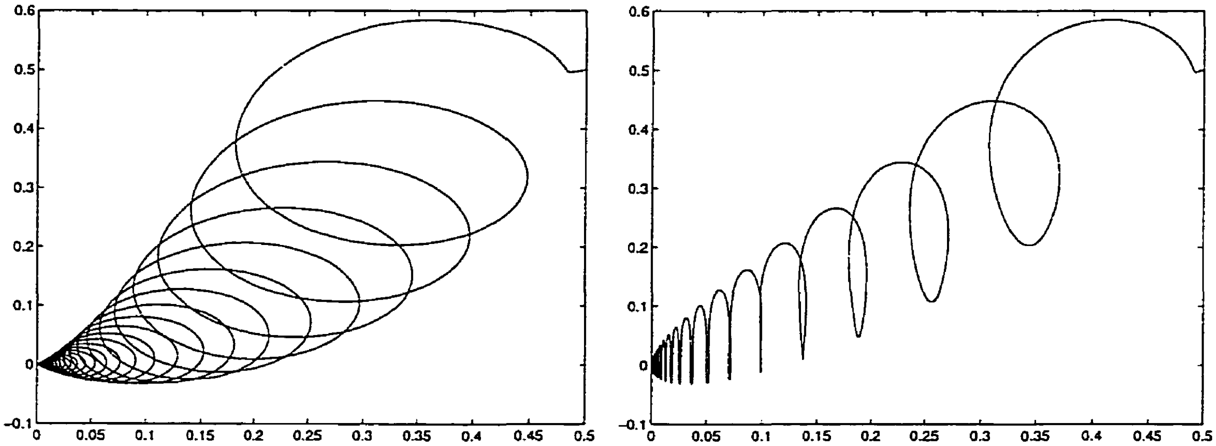

FIGURE 4.10. *Spacecraft model:* Plots of the controlled state trajectories $x_1(t)$ versus $x_2(t)$, and $x_2(t)$ versus $x_3(t)$.

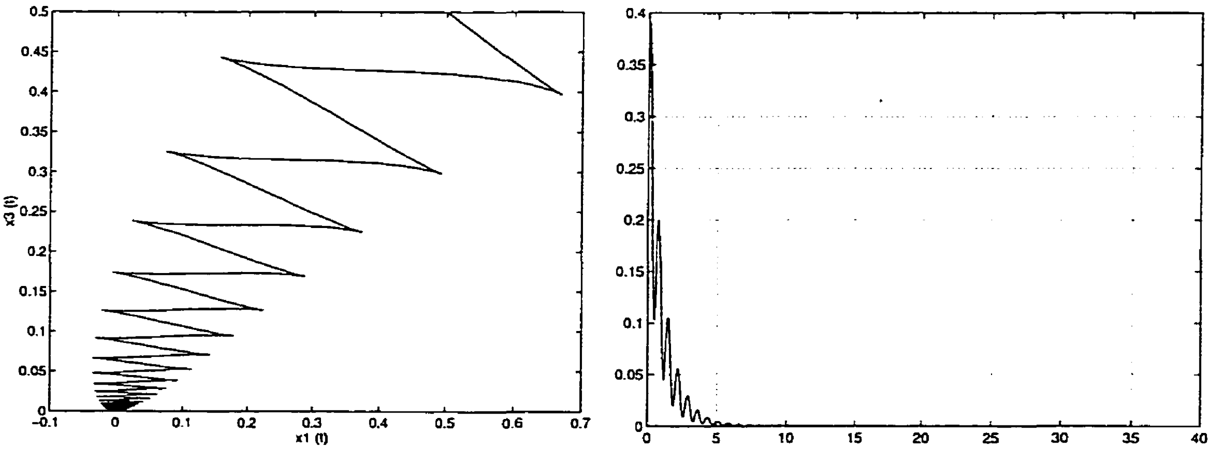

FIGURE 4.11. *Spacecraft model:* Plots of the controlled state trajectories $x_1(t)$ versus $x_3(t)$, and the Lyapunov function $V(x(t)) = \frac{1}{2}\sum_{i=1}^{3} x_i^2(t)$ along the controlled state trajectories.

controllable on a manifold $\mathcal{M}$ defined by (3.94). The Lie brackets multiplication table (3.95) shows that the Lie algebra $L(g_1, g_2)$ is not nilpotent. The approximate model can be taken to be that of (3.96), for which the Lie brackets multiplication table (3.98) establishes nilpotency. As the Lie algebraic structure of the nilpotent approximation is the same as that of a unicycle, the equations for the evolution of the logarithmic coordinates are again given by (4.101) and the controls are of the same form as those of (4.104).

The simulations results are shown in Figures 4.9 - 4.11.

6.3. Time varying stabilizing feedback control for an underwater vehicle [66]

This example demonstrates that the trajectory interception approach is also successful when applied to non-nilpotent systems with higher order of control deficiency. The Lie brackets multiplication table for the controllability Lie algebra $L(g_1, g_2, g_3, g_4)$ for the underwater vehicle is computed previously in (3.82) of in Chapter 3, and clearly shows that the controllability Lie algebra fails to be nilpotent. The following approximate model is adopted by using truncated Taylor series of order one for the vector field g_1 and of order zero for the vector fields g_3 and g_4, (each evaluated at zero):

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} &= \begin{bmatrix} 1 \\ x_6 \\ -x_5 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u_3 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_4 \\
&\overset{def}{=} \bar{g}_1(x)u_1 + \bar{g}_2(x)u_2 + \bar{g}_3(x)u_3 + \bar{g}_4(x)u_4 \qquad (4.105) \\
\text{where,} \quad \bar{g}_1(x) &= \frac{\partial}{\partial x_1} + x_6 \frac{\partial}{\partial x_2} - x_5 \frac{\partial}{\partial x_3}, \qquad \bar{g}_2(x) = \frac{\partial}{\partial x_4}, \\
\bar{g}_3(x) &= \frac{\partial}{\partial x_5}, \qquad \bar{g}_4(x) = \frac{\partial}{\partial x_6}
\end{aligned}$$

The approximate system is controllable as it satisfied the LARC condition:

$$span\{\bar{g}_1(x), \bar{g}_2(x), \bar{g}_3(x), \bar{g}_4(x), \bar{g}_5(x), \bar{g}_6(x)\} = I\!R^6, \quad \text{for all} \quad x \in I\!R^6$$

where the vector fields $\bar{g}_5(x)$ and $\bar{g}_6(x)$ are given by :

$$\bar{g}_5(x) \overset{def}{=} [\bar{g}_1, \bar{g}_3](x) = \frac{\partial}{\partial x_3}, \qquad \bar{g}_6(x) \overset{def}{=} [\bar{g}_1, \bar{g}_4](x) = -\frac{\partial}{\partial x_2}$$

The Lie brackets multiplication table for $L(\bar{g}_1, \bar{g}_2, \bar{g}_3, \bar{g}_4)$ is:

$$
\begin{aligned}
&[\bar{g}_1, \bar{g}_3] = \bar{g}_5 \qquad [\bar{g}_1, \bar{g}_4] = \bar{g}_6, \qquad [\bar{g}_j, \bar{g}_2] = 0,\ j = 1, ..., 4 \\
&[\bar{g}_i, \bar{g}_5] = [\bar{g}_i, \bar{g}_6] = 0,\ i = 1, ..., 6
\end{aligned}
\tag{4.106}
$$

so that $L(\bar{g}_1, \bar{g}_2, \bar{g}_3, \bar{g}_4)$ is nilpotent.

The extended system for the original model (3.79) can be defined by:

$$\dot{x} = g_1(x)v_1 + g_2(x)v_2 + g_3(x)v_3 + g_4(x)v_4 + g_5(x)v_5 + g_6(x)v_6, \quad x \in I\!R^6 \tag{4.107}$$

and the extended controls:

$$v_i(x) \stackrel{def}{=} -L_{g_i}V(x), \quad i = 1, ..., 6, \quad \text{with} \quad V(x) \stackrel{def}{=} \frac{1}{2}x^T x, \quad x \in I\!R^6 \tag{4.108}$$

give the following discretized extended system:

$$\dot{x} \ = \ g_1(x)a_1 + g_2(x)a_2 + g_3(x)a_3 + g_4(x)a_4 + g_5(x)a_5 + g_6(x)a_6, \ x \in I\!R^6 \tag{4.109}$$

where $a_i = \bar{v}_i(T, x)$, $i = 1, ...6$. It can be verified that the equations for the logarithmic coordinates are given by:

$$
\begin{aligned}
\dot{\gamma}_1 &= a_1 \\
\dot{\gamma}_2 &= a_2 \\
\dot{\gamma}_3 &= a_3 \\
\dot{\gamma}_4 &= a_4 \\
\dot{\gamma}_5 &= -\gamma_1 a_3 + a_5 \\
\dot{\gamma}_6 &= -\gamma_1 a_4 + a_6, \quad \gamma_i(0) = 0,\ i = 1, .., 6
\end{aligned}
\tag{4.110}
$$

so the TIP in logarithmic coordinates is a trajectory interception problem for the following two control systems:

$$
CS1: \begin{cases}
\dot{\gamma}_1 = a_1 \\
\dot{\gamma}_2 = a_2 \\
\dot{\gamma}_3 = a_3 \\
\dot{\gamma}_4 = a_4 \\
\dot{\gamma}_5 = -\gamma_1 a_3 + a_5 \\
\dot{\gamma}_6 = -\gamma_1 a_4 + a_6
\end{cases}
\qquad
CS2: \begin{cases}
\dot{\gamma}_1 = w_1 \\
\dot{\gamma}_2 = w_2 \\
\dot{\gamma}_3 = w_3 \\
\dot{\gamma}_4 = w_4 \\
\dot{\gamma}_5 = -\gamma_1 w_3 \\
\dot{\gamma}_6 = -\gamma_1 w_4
\end{cases}
$$

with common initial conditions $\gamma_i(0) = 0$, $i = 1, .., 6$.

It is reasonable to seek the controls $w_i(a,t)$, $i = 1, ..., 4$, in the following form :

$$\begin{aligned} w_1 &= b_1 + b_5 \, sin(\frac{2\pi}{T}t) + b_6 \, cos(\frac{2\pi}{T}t), \quad w_2 = b_2 \\ w_3 &= b_3 + b_5 \, cos(\frac{2\pi}{T}t), \quad w_4 = b_4 + b_6 \, sin(\frac{2\pi}{T}t) \end{aligned} \tag{4.111}$$

The constants b_i, $i = 1, .., 6$ can be computed as:

$$b_1 = a_1, \quad b_2 = a_2, \quad b_3 = a_3, \quad b_4 = a_4$$

$$b_5 = \pm 3.54491\sqrt{(a_5)}/\sqrt{(T)}$$

$$b_6 = (0.5(2a_1T^2 \pm \sqrt{(-50.2655a_6T^3 + 4a_1^2T^4)}))/(T^2)$$

The time varying stabilizing controls for underwater vehicle model are hence given by

$$\begin{aligned} u_1(T,x) &= \bar{v}_1(T,x) + b_5(\bar{v}(T,x)) \, sin(\frac{2\pi}{T}t) + b_6(\bar{v}(T,x)) \, cos(\frac{2\pi}{T}t) \\ u_2(T,x) &= \bar{v}_2(T,x) \\ u_3(T,x) &= \bar{v}_3(T,x) + b_5(\bar{v}(T,x)) \, cos(\frac{2\pi}{T}t) \\ u_4(T,x) &= \bar{v}_4(T,x) + b_6(\bar{v}(T,x)) \, sin(\frac{2\pi}{T}t) \end{aligned} \tag{4.112}$$

Simulation results are depicted in Figures 4.12 - 4.15 and confirm the effectiveness of the approach.

6.4. Time varying stabilizing feedback control for a WMR of type (2, 1)

This and the next section discuss the application of the trajectory interception approach to two classes of wheeled mobile robots which are important in industry.

The kinematic model of a WMR of type (2, 1) is given by (3.152) of Chapter 3 and is characterized by a controllability Lie algebra which fails to be nilpotent (see the multiplication table (3.154)). The following approximate model:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u_2 + \begin{bmatrix} -x_4 \\ 1 \\ 0 \\ 0 \end{bmatrix} u_3 \\ &\stackrel{def}{=} \bar{g}_1 u_1 + \bar{g}_2 u_2 + \bar{g}_3 u_3 \end{aligned} \tag{4.113}$$

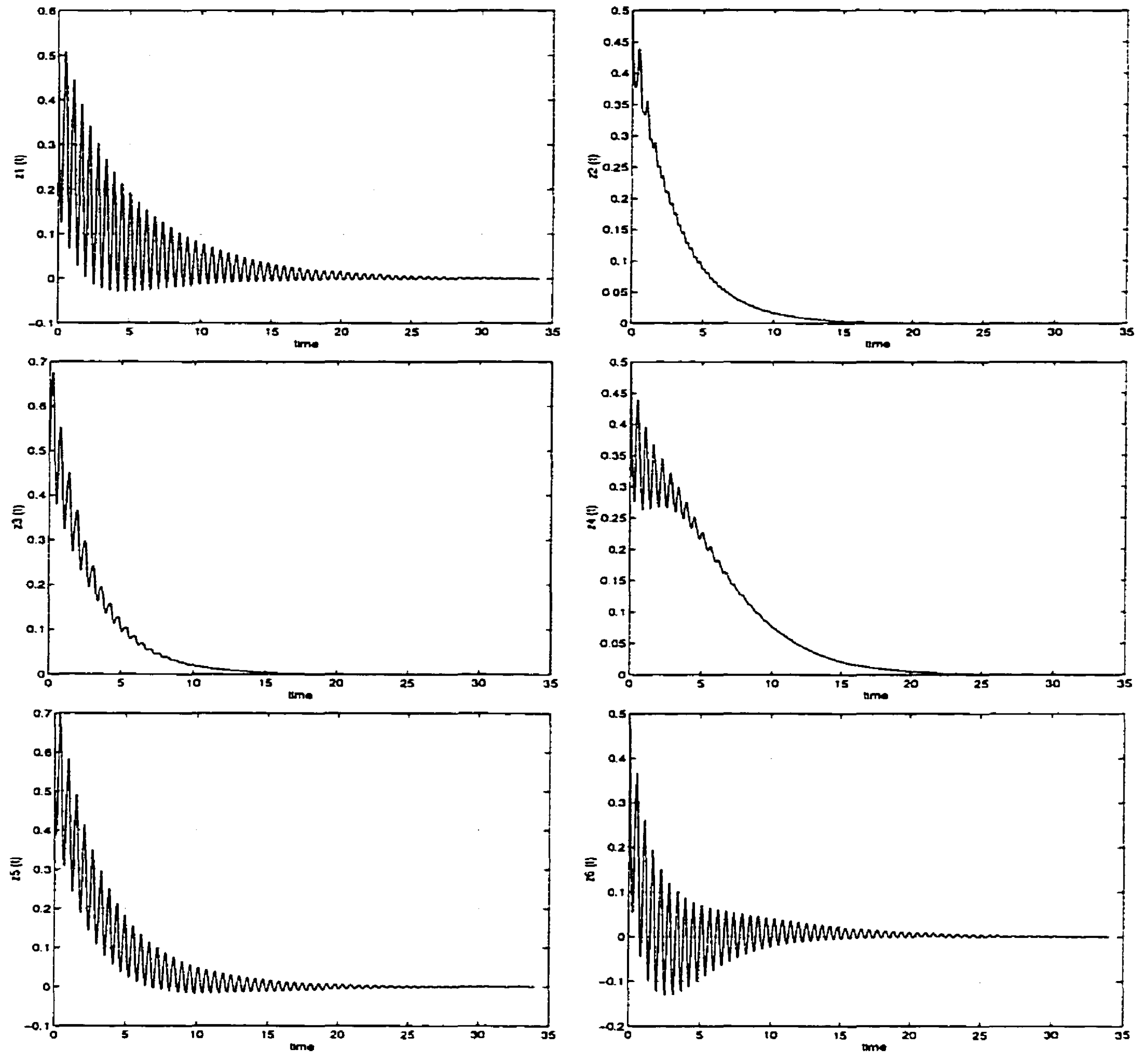

FIGURE 4.12. *Underwater vehicle Model 1*: Plots of the controlled state trajectories $t \mapsto (x_1(t), \ldots, x_6(t)) \overset{def}{=} (z_1(t), \ldots, z_6(t))$ versus time.

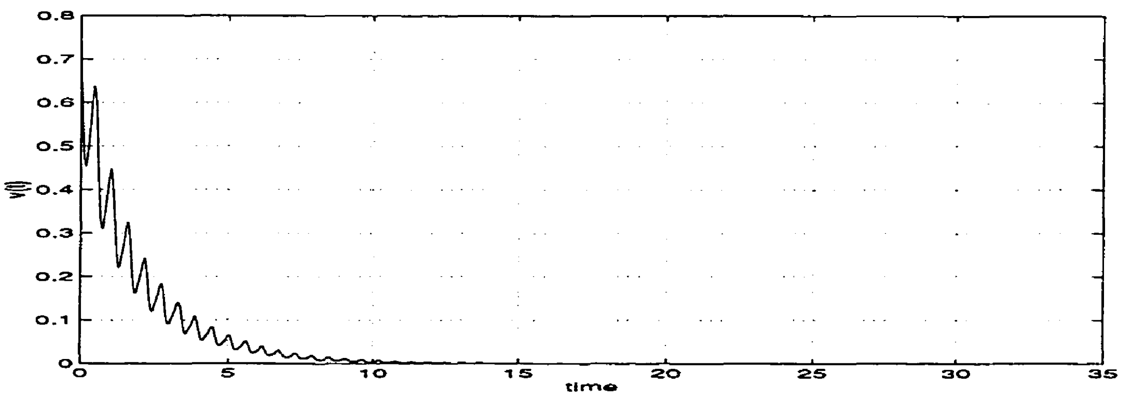

FIGURE 4.13. *Underwater vehicle Model 1*: Plot of the Lyapunov function $V(z(t)) = \frac{1}{2}\sum_{i=1}^{6} z_i^2(t)$ along the controlled state trajectories.

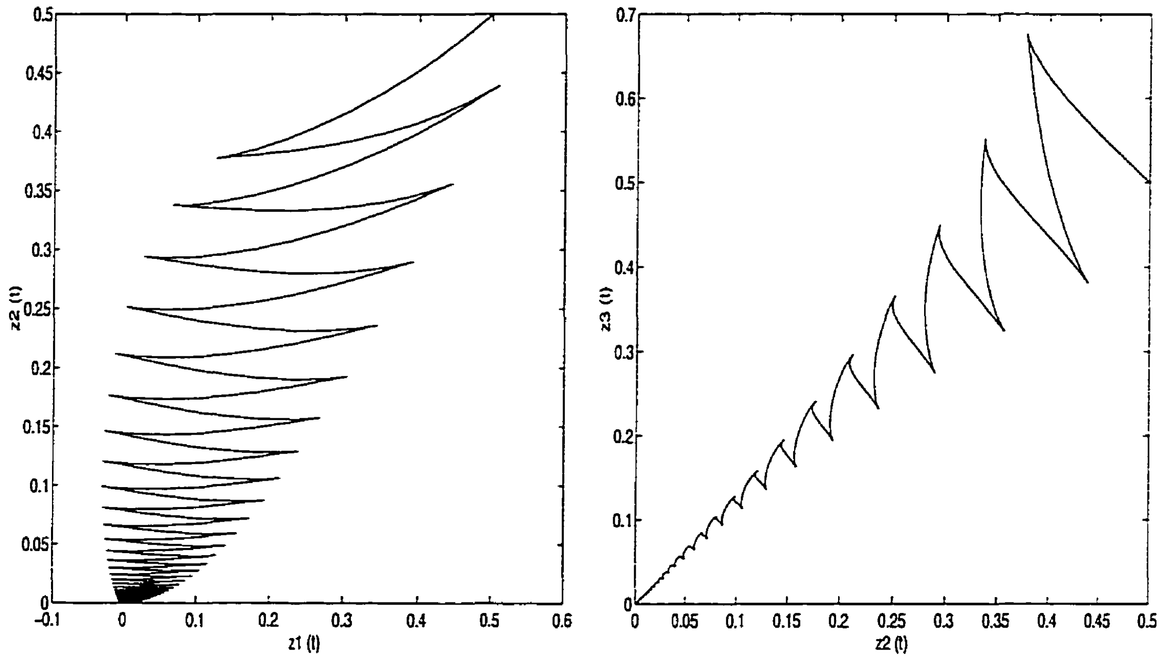

FIGURE 4.14. *Underwater vehicle Model 1*: Plots of the controlled state trajectories $z_1(t)$ versus $z_2(t)$, and $z_2(t)$ versus $z_3(t)$.

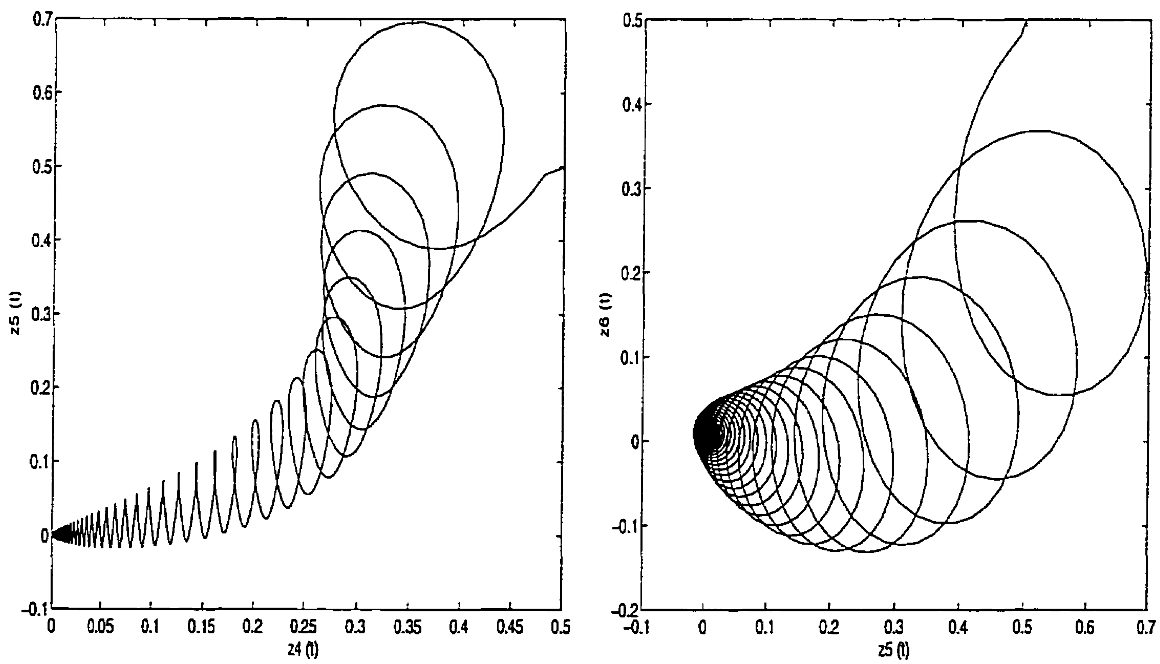

FIGURE 4.15. *Underwater vehicle Model 1*: Plots of the controlled state trajectories $z_4(t)$ versus $z_5(t)$, and $z_5(t)$ versus $z_6(t)$.

is found to be adequate as it satisfies the LARC condition:

$$span\{\bar{g}_1, \bar{g}_2, \bar{g}_3, \bar{g}_4\} = \mathbb{R}^4, \quad \text{for all} \quad x \in \mathbb{R}^4 \tag{4.114}$$

$$\text{where,} \quad \bar{g}_4 \stackrel{def}{=} [\bar{g}_1, \bar{g}_3](x) = -\frac{\partial}{\partial x_1}$$

and the Lie brackets multiplication table for $L(\bar{g}_1, \bar{g}_2, \bar{g}_3)$:

$$\begin{aligned} &[\bar{g}_1, \bar{g}_3] = \bar{g}_4, \quad [\bar{g}_2, \bar{g}_3] = 0, \quad [\bar{g}_1, \bar{g}_2] = 0 \\ &[\bar{g}_i, \bar{g}_4] = 0, \quad i = 1, ..., 4 \end{aligned} \tag{4.115}$$

clearly shows that the Lie algebra $L(\bar{g}_1, \bar{g}_2, \bar{g}_3)$ is nilpotent.

The extended system is clearly given by :

$$\dot{x} = g_1(x)v_1 + g_2(x)v_2 + g_3(x)v_3 + g_4(x)v_4, \quad x \in \mathbb{R}^4 \tag{4.116}$$

and standard extended controls can again be adopted :

$$v_i(x) \stackrel{def}{=} -L_{g_i}V(x), \quad i = 1, ..., 4, \quad \text{with} \quad V(x) \stackrel{def}{=} \frac{1}{2}x^T x, \quad x \in \mathbb{R}^4 \tag{4.117}$$

These result in the following discretized extended system:

$$\dot{x} = g_1(x)a_1 + g_2(x)a_2 + g_3(x)a_3 + g_4(x)a_4, \quad x \in \mathbb{R}^4 \tag{4.118}$$

where $a_i = \bar{v}_i(T, x)$, $i = 1, ...4$. It can be verified that the logarithmic coordinates for the approximate system satisfy the following differential equations:

$$\begin{aligned} \dot{\gamma}_1 &= a_1 \\ \dot{\gamma}_2 &= a_2 \\ \dot{\gamma}_3 &= a_3 \\ \dot{\gamma}_4 &= -\gamma_1 a_3 + a_4, \qquad \gamma_i(0) = 0, \ i = 1, .., 4 \end{aligned} \tag{4.119}$$

The corresponding control systems are:

$$CS1: \begin{cases} \dot{\gamma}_1 = a_1 \\ \dot{\gamma}_2 = a_2 \\ \dot{\gamma}_3 = a_3 \\ \dot{\gamma}_4 = -\gamma_1 a_3 + a_4 \end{cases} \qquad CS2: \begin{cases} \dot{\gamma}_1 = w_1 \\ \dot{\gamma}_2 = w_2 \\ \dot{\gamma}_3 = w_3 \\ \dot{\gamma}_4 = -\gamma_1 w_3 \end{cases} \tag{4.120}$$

with common initial conditions $\gamma_i(0) = 0$, $i = 1, .., 4$.

The controls $w_i(a,t)$, $i = 1, ..., 4$ are sought in the following form

$$\begin{aligned} w_1 &= b_1 + b_4 \, sin(\frac{2\pi}{T}t), \qquad w_2 = b_2 \\ w_3 &= b_3 + b_4 \, cos(\frac{2\pi}{T}t) \\ \text{and} \quad b_1 &= a_1, \quad b_2 = a_2, \quad b_3 = a_3, \quad b_4 = \pm 3.54491\sqrt{(a_4)}/\sqrt{(T)} \quad \text{are found.} \end{aligned} \tag{4.121}$$

The time varying stabilizing controls are hence given by :

$$\begin{aligned} u_1(T,x) &= \bar{v}_1(T,x) + b_4(\bar{v}(T,x)) \, sin(\frac{2\pi}{T}t), \quad u_2(T,x) = \bar{v}_2(T,x) \\ u_3(T,x) &= \bar{v}_3(T,x) + b_4(\bar{v}(T,x)) \, cos(\frac{2\pi}{T}t) \end{aligned} \tag{4.122}$$

Simulation results are depicted in Figures 4.16 - 4.18.

6.5. Time varying stabilizing feedback control for a WMR of type $(1,2)$ [70]

The example of WMR of type $(1,2)$ represents a typical five dimensional system with control deficiency order two and with a non-nilpotent controllability Lie algebra.

The kinematic model of a WMR of type $(1,2)$ is given in (3.158) of Chapter 3 and the Lie brackets multiplication table for its controllability Lie algebra is given in (3.160). An approximate model (3.161) with multiplication table (3.163) proves sufficient in that its controllability Lie algebra satisfies the conditions of Definition 2.

The extended system is clearly :

$$\dot{x} = g_1(x)v_1 + g_2(x)v_2 + g_3(x)v_3 + g_4(x)v_4 + g_5(x)v_5, \quad x \in I\!R^5 \tag{4.123}$$

and the usual extended controls:

$$v_i(x) \stackrel{def}{=} -L_{g_i}V(x), \quad i = 1, ..., 5, \quad \text{with} \quad V(x) \stackrel{def}{=} \frac{1}{2}x^T x, \quad x \in I\!R^5 \tag{4.124}$$

give the following discretized extended system:

$$\dot{x} = g_1(x)a_1 + g_2(x)a_2 + g_3(x)a_3 + g_4(x)a_4 + g_5(x)a_5, \quad x \in I\!R^5 \tag{4.125}$$

where $a_i = \bar{v}_i(T,x)$, $i = 1, ...5$. The equations describing the evolution of the logarithmic coordinates for the approximate system are the same as those of (4.68), as the system has the same Lie algebraic structure as that of section 4.1. The controls $w_i(a,t)$, $i = 1, ..., 3$ are therefore again given by (4.70).

Simulation results are shown in Figures 4.19 - 4.20.

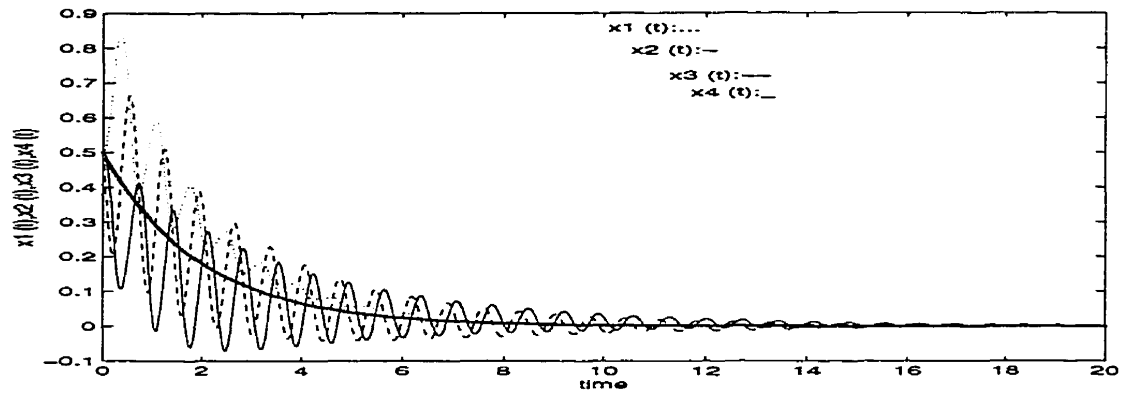

FIGURE 4.16. *WMR of type* (2, 1): Plots of the controlled state trajectories $t \mapsto (x_1(t), \ldots, x_4(t))$ versus time.

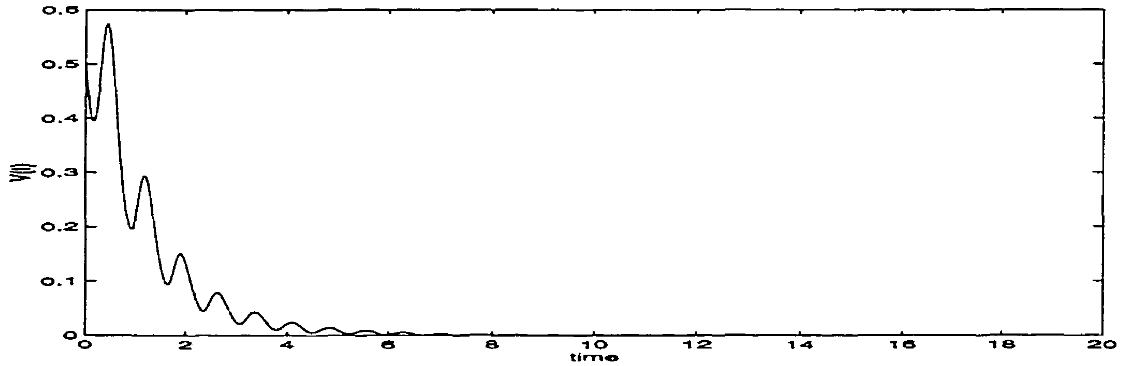

FIGURE 4.17. *WMR of type* (2, 1): Plot of the Lyapunov function $V(x(t)) = \frac{1}{2}\sum_{i=1}^{4} x_i^2(t)$ along the controlled state trajectories.

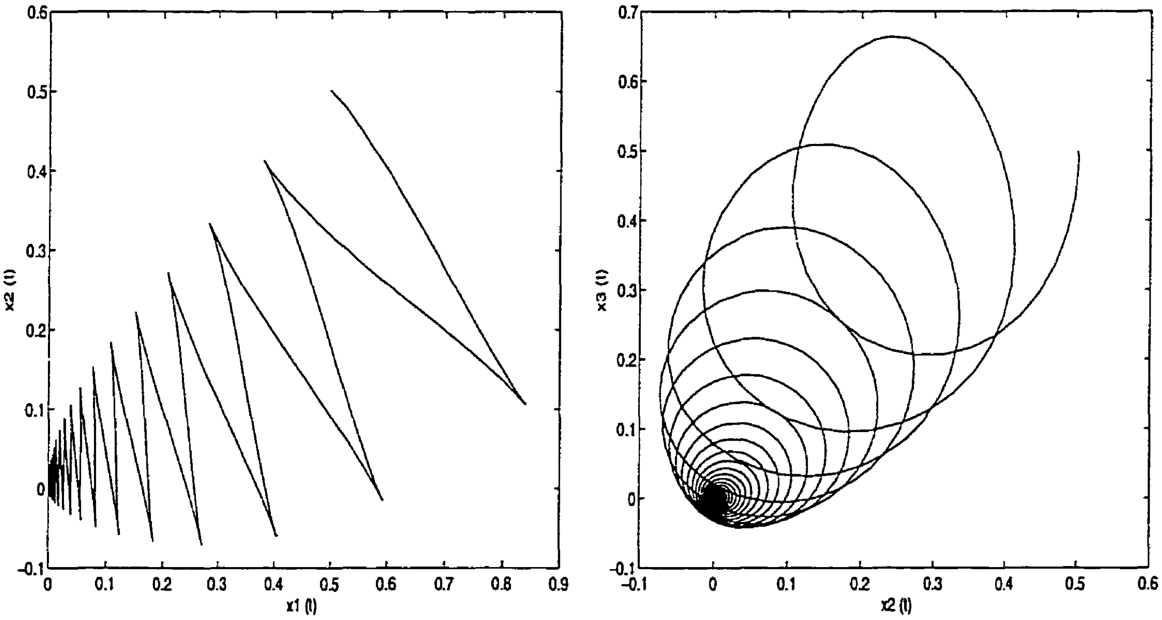

FIGURE 4.18. *WMR of type* (2, 1): Plots of the controlled state trajectories $x_1(t)$ versus $x_2(t)$, and $x_2(t)$ versus $x_3(t)$.

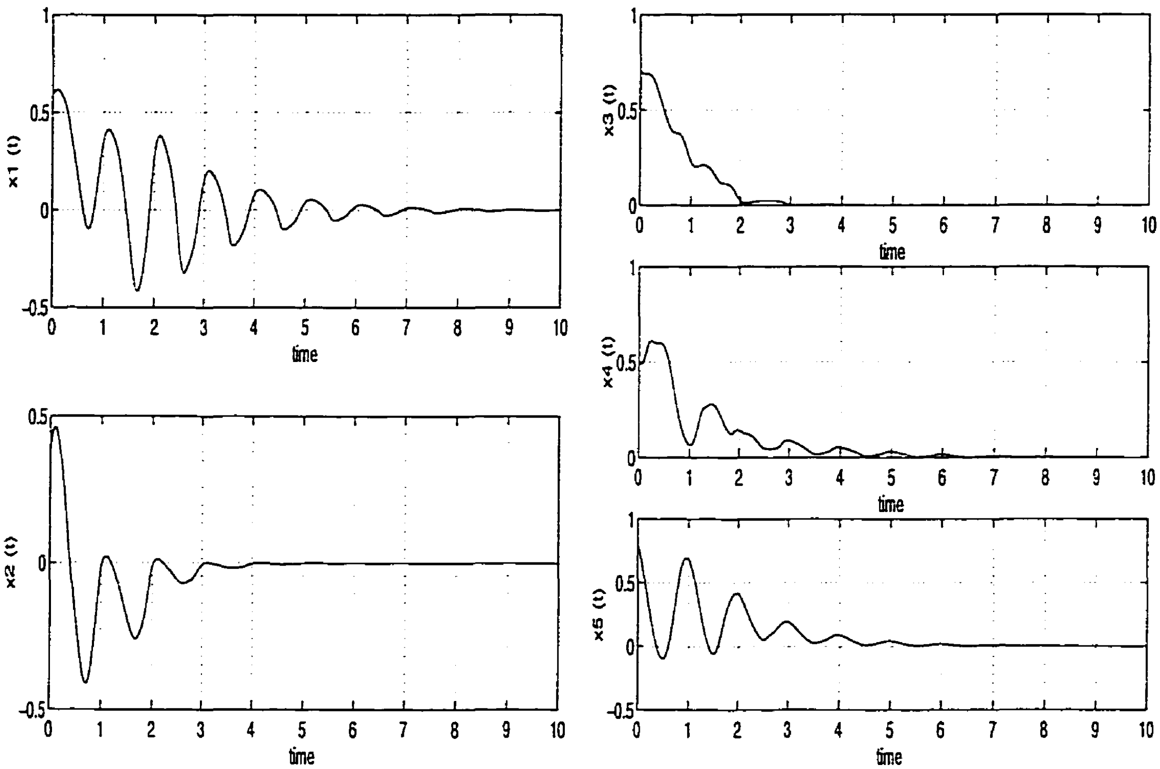

FIGURE 4.19. *WMR of type* (1, 2): Plots of the controlled state trajectories $t \mapsto (x_1(t), \ldots, x_5(t))$ versus time.

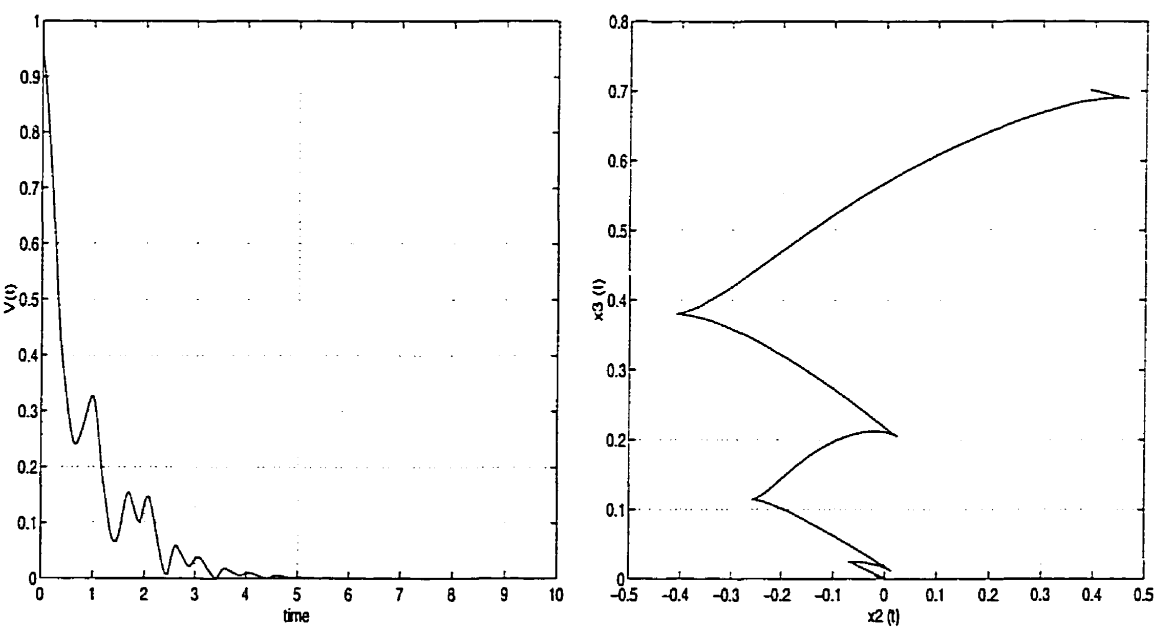

FIGURE 4.20. *WMR of type* (1, 2): Plots of the controlled state trajectories $x_2(t)$ versus $x_3(t)$, and Lyapunov function $V(x(t)) = \frac{1}{2}\sum_{i=1}^{5} x_i^2(t)$ along the controlled state trajectories.

6.6. Time varying feedback stabilizing control for a hopping robot in flight phase [84]

This example illustrates the applicability of the trajectory interception approach to systems whose controllability Lie algebra contains higher order Lie brackets and fails to be nilpotent. It is also shown that it is important that the approximate models satisfy all the conditions of Definition 2, specifically, the Lie brackets multiplication tables for the original and approximate controllability Lie algebras ought to agree for brackets of depth up to k - the order of nilpotency of the approximate system.

The kinematic model of a hopping robot is stated in (3.143) of Chapter 3. The Lie brackets multiplication table (3.145) for the controllability Lie algebra of this system shows that the nilpotency condition fails to hold.

An approximate model of the form

$$\begin{aligned} \dot{x} &= \tilde{g}_1(x)u_1 + \tilde{g}_2(x)u_2, \qquad x \in I\!R^3 \\ \tilde{g}_1(x) &= \frac{\partial}{\partial x_1} - {x_2}^2\frac{\partial}{\partial x_3}, \qquad \tilde{g}_2(x) = \frac{\partial}{\partial x_2} \end{aligned} \tag{4.126}$$

is therefore considered. We refer to this model as to "approximate model 1". The latter proves to be controllable as:

$$span\{\tilde{g}_1(x), \tilde{g}_2(x), \tilde{g}_4(x)\} = I\!R^3, \quad \text{for all} \quad x \in I\!R^3$$

$$\text{where,} \qquad \tilde{g}_3 = [\tilde{g}_1, \tilde{g}_2] = 2x_2\frac{\partial}{\partial x_3}, \qquad \tilde{g}_4 = [\tilde{g}_2, [\tilde{g}_1, \tilde{g}_2]] = 2\frac{\partial}{\partial x_3}$$

The Lie brackets multiplication table for $L(\tilde{g}_1, \tilde{g}_2)$ is :

$$\begin{aligned} &[\tilde{g}_1, \tilde{g}_2] = \tilde{g}_3 \qquad [\tilde{g}_1, \tilde{g}_3] = 0 \qquad [\tilde{g}_2, \tilde{g}_3] = \tilde{g}_4 \\ &[\tilde{g}_1, \tilde{g}_4] = 0 \qquad [\tilde{g}_2, \tilde{g}_4] = 0 \end{aligned} \tag{4.127}$$

and shows that the Lie algebra $L(\tilde{g}_1, \tilde{g}_2)$ is nilpotent.

The Lie algebraic extension of the original system, (3.143) is taken to be :

$$\dot{x} = g_1(x)v_1 + g_2(x)v_2 + g_4(x)v_4 \tag{4.128}$$

With the usual extended controls:

$$v_i(x) \stackrel{def}{=} -L_{g_i}V(x), \quad i = 1,2,4, \qquad \text{with} \qquad V(x) \stackrel{def}{=} \frac{1}{2}x^Tx, \quad x \in I\!R^3 \tag{4.129}$$

the discretized extended system becomes :

$$\dot{x} \;=\; g_1(x)a_1 + g_2(x)a_2 + g_4(x)a_4,\; x \in I\!R^3, \quad \text{where,} \quad a_i = \bar{v}_i(T,x), \quad i = 1,2,4 \quad (4.130)$$

It can be verified that the logarithmic coordinates for the approximate system satisfy the following differential equations:

$$\begin{aligned}
\dot{\gamma}_1 &= a_1 \\
\dot{\gamma}_2 &= a_2 \\
\dot{\gamma}_3 &= -\gamma_1 a_2 + a_3, \quad \text{with} \quad a_3 = 0 \\
\dot{\gamma}_4 &= \gamma_1\gamma_2 a_2 + a_4, \quad \gamma_i(0) = 0, \; i = 1,..,4
\end{aligned} \qquad (4.131)$$

The third equation of (4.131), $\dot{\gamma}_3 = -\gamma_1 a_2 + a_3$, is discarded as it corresponds to the component of the flow along the vector field $\bar{g}_3 = [\bar{g}_1, \bar{g}_2]$, which is linearly dependent with the remaining vector fields $\bar{g}_1, \bar{g}_2$ and $\bar{g}_4$. Then the corresponding control systems for the TIP are hence :

$$CS1: \begin{cases} \dot{\gamma}_1 = a_1 \\ \dot{\gamma}_2 = a_2 \\ \dot{\gamma}_4 = \gamma_1\gamma_2 a_2 + a_4 \end{cases} \qquad CS2: \begin{cases} \dot{\gamma}_1 = w_1 \\ \dot{\gamma}_2 = w_2 \\ \dot{\gamma}_4 = \gamma_1\gamma_2 w_2 \end{cases} \qquad (4.132)$$

with common initial conditions $\gamma_i(0) = 0$, $i = 1, 2, 4$.

Since flows of $\dot{x} = [g_2, [g_1, g_2]]$ can be "approximated" by the flows of $\dot{y} = g_1 sin(2\pi\frac{t}{T}) + g_2 cos(2\pi\frac{t}{T})$, the controls $w_i(a,t)$, $i = 1,2$ are assumed to take the form :

$$w_1 = b_1 + b_4 \; sin(\frac{2\pi}{T}t), \qquad w_2 = b_2 + b_4 \; cos(\frac{2\pi}{T}t) \qquad (4.133)$$

where b_i, $i = 1, 2, 4$ are found to be :

$$\begin{aligned}
&b_1 = a_1, \qquad b_2 = a_2, \qquad b_4 = (4a_1a_2T^2 \pm d)/2(a_1 + 2a_2\pi)T^2, \\
&\text{where,} \quad d = \{-64a_4\pi^2(a_1 + 2a_2\pi)T^2 + 16{a_1}^2{a_2}^2T^4\}^{1/2}
\end{aligned}$$

The time varying stabilizing controls for the original system, (3.143), are thus given by

$$\begin{aligned}
u_1(x) &= b_1(v^T(x)) + b_4(v^T(x)) \; sin(\frac{2\pi}{T}t) \\
u_2(x) &= b_2(v^T(x)) + b_4(v^T(x)) \; cos(\frac{2\pi}{T}t)
\end{aligned} \qquad (4.134)$$

Simulation results obtained when such controls are applied to the system model (3.143) are presented in Figures 4.21 - 4.22. Continuous extended controls are used in place of v^T , and the time horizon chosen is $T = 4$. Simulation experiments confirm that the constructed time varying feedback possesses a wide robustness margin with respect to inaccuracies in the solution to the TIP.

A comparison result:

For the sake of comparison, we consider a different approximation to the system model (3.143) which is used in [**77**], to construct open-loop controls to steer the hopping robot to a set point. The approximate system obtained by this approximation is in chained form:

$$\dot{x} = \tilde{f}_1(x)u_1 + \tilde{f}_2(x)u_2, \qquad x \in \mathbb{R}^3 \tag{4.135}$$

where

$$\tilde{f}_1(x) = \frac{\partial}{\partial x_1} - \frac{1}{2}\, x_2 \,\frac{\partial}{\partial x_3}, \qquad \tilde{f}_2(x) = \frac{\partial}{\partial x_2}$$

$$\text{and} \qquad \tilde{f}_3 \overset{def}{=} [\tilde{f}_1, \tilde{f}_2] = 1/2\, \frac{\partial}{\partial x_3}$$

It is easy to see that the above model, to which we refer as to an approximate model 2, has a different algebraic structure. The latter is reflected by the fact that the Lie algebraic controllability rank condition for system (4.135) involves only a Lie bracket of depth one, so that :

$$span\{\tilde{f}_1(x), \tilde{f}_2(x), [\tilde{f}_1, \tilde{f}_2](x)\} = \mathbb{R}^3, \quad \text{for all} \quad x \in \mathbb{R}^3 \tag{4.136}$$

This leads to simpler equations for the evolution of the logarithmic coordinates of flows, which for (4.135) take the following form :

$$\begin{aligned} \dot{\gamma}_1 &= a_1 \\ \dot{\gamma}_2 &= a_2 \\ \dot{\gamma}_3 &= -\gamma_1 a_2 + a_3, \quad \gamma_i(0) = 0, \; i = 1, 2, 3 \end{aligned} \tag{4.137}$$

The calculation of a solution to the corresponding TIP also simplifies; the controls w_1 and w_2 can be obtained in following form:

$$w_1 = b_1 + b_3 \, sin(\frac{2\pi}{T}t)$$
$$w_2 = b_2 + b_3 \, cos(\frac{2\pi}{T}t)$$

where the unknown coefficients b_1, b_2, b_3 can easily be expressed in terms of a_1, a_2, a_3. The resulting time varying stabilizing controls for the original system obtained by using approximate logarithmic

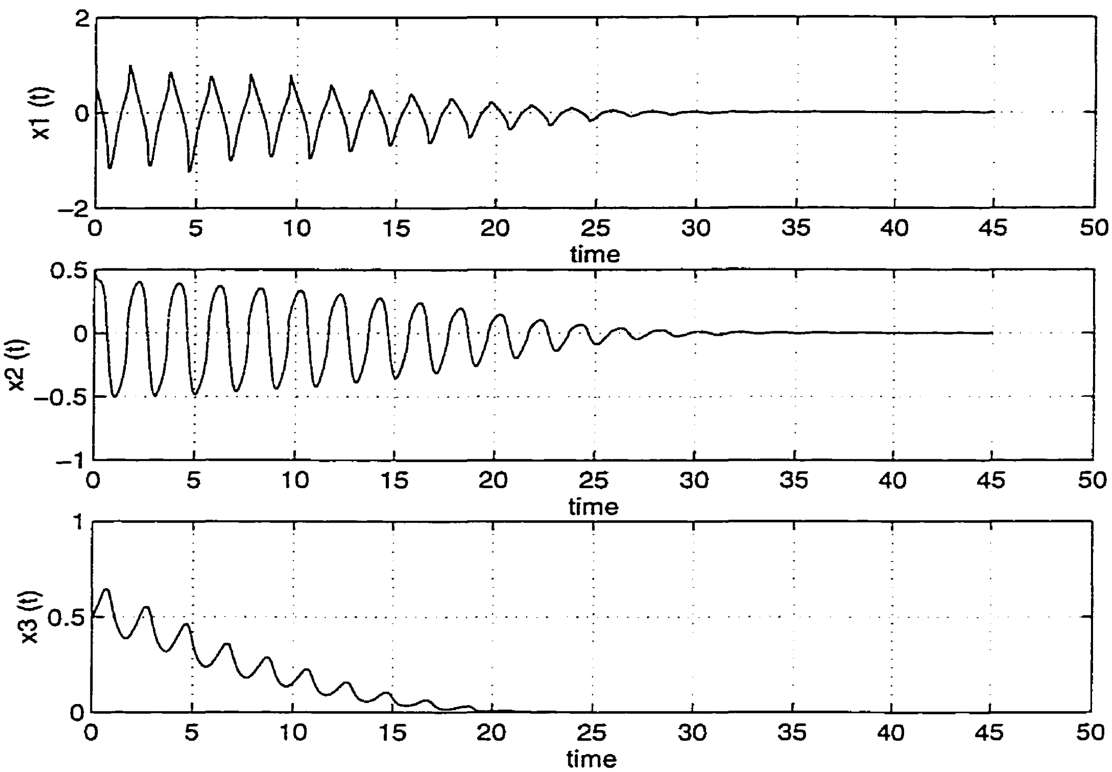

FIGURE 4.21. *Hopping robot model*: Plots of the controlled state trajectories $t \mapsto (x_1(t), ..., x_3(t))$ versus time.

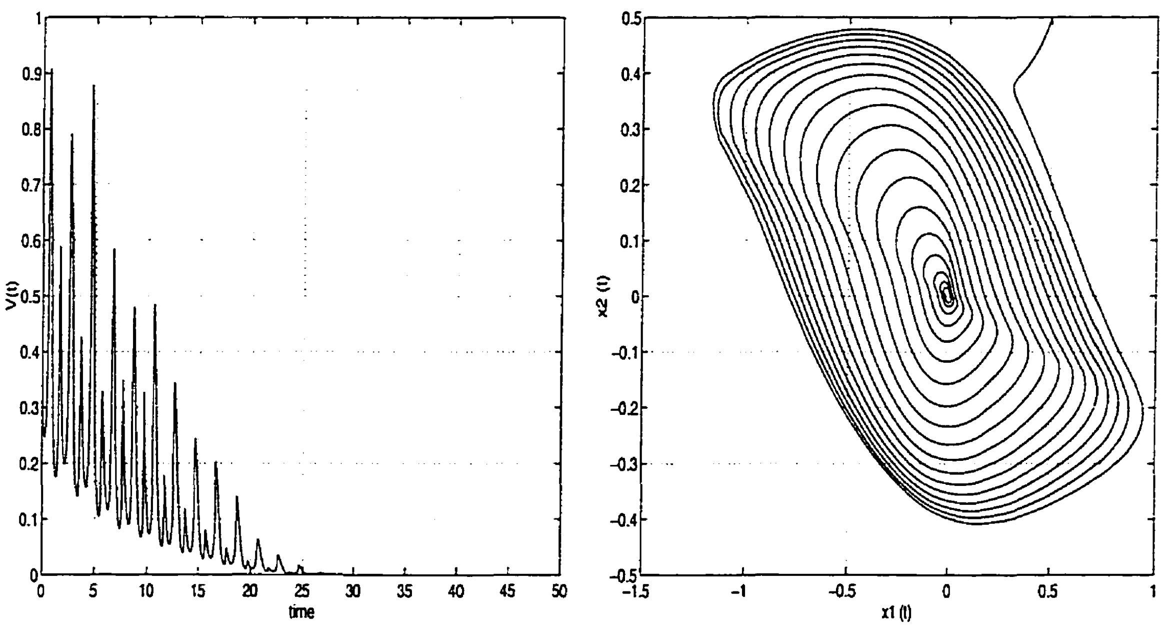

FIGURE 4.22. *Hopping robot model*: Plots of the controlled state trajectories $x_1(t)$ versus $x_2(t)$, and Lyapunov function $V(x(t)) = \frac{1}{2}\sum_{i=1}^{3} x_i^2(t)$ along the controlled state trajectories.

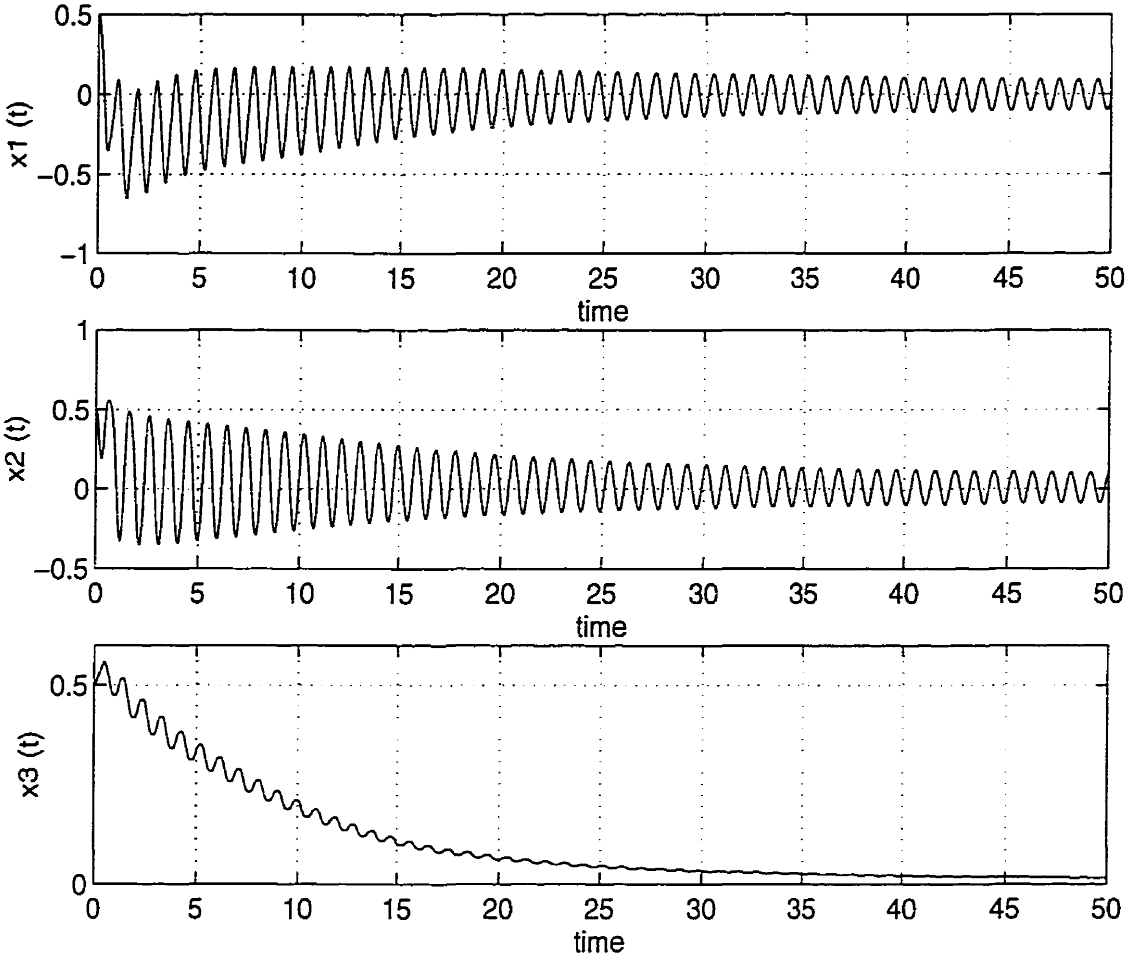

FIGURE 4.23. *Hopping robot model*: Plots of the controlled state trajectories $t \mapsto (x_1(t), ..., x_3(t))$ versus time.

coordinates (4.137) are hence given by:

$$\tilde{u}_1(x) = b_1(v_1^T(x)) + b_3(v_3^T(x))\ sin(\frac{2\pi}{T}t)$$
$$\tilde{u}_2(x) = b_2(v_2^T(x) + b_3(v_3^T(x))\ cos(\frac{2\pi}{T}t) \tag{4.138}$$

The controls in (4.138) are applied to the original system model (3.143) for the sake of comparison of the stabilizing properties of the controls derived using model 1 and model 2. The simulation results are shown in Figure 4.23 and clearly indicate that model 2 is too crude an approximation: the controls based on model 2 do not have as good stabilizing properties as those derived by using model 1.

7. Sinusoidal steering and the trajectory interception approach

Using a few examples we demonstrate here that the trajectory interception approach can sometimes be successfully combined with sinusoidal steering.

Such combined control relies on decomposing a complex system model into subsystems of which one can be controlled by the trajectory interception approach and the other by sinusoidally varying inputs. The decomposition idea proves especially useful when the Lie algebraic structure of the higher dimensional sub-system is sufficiently simple as to permit an easy application of the trajectory interception approach.

The purpose of this section is not to present a rigorous strategy based on such decomposition but rather to demonstrate that combined strategies using TIP may also be successful. Theoretical investigation involving possible decomposition methods can be a topic for future research.

7.1. Stabilizing feedback control for a WMR of type (1,1) [70]

A wheeled mobile robot model of type $(1,1)$ represents a four dimensional systems with control deficiency order two and with a non-nilpotent controllability Lie algebra which contains Lie brackets of depth one as well as depth two. Interestingly, the model of a WMR of type $(1,1)$ has a similar algebraic structure as the model of a front wheel drive; the stabilizing controller constructed for a WMR of type $(1,1)$ can therefore also be used to stabilize a front wheel drive.

The kinematic model of a WMR of type (1,1) is given by (3.165) of Chapter 3, in which $(z_1, z_2, z_3, z_4) = (\beta, y, \theta, x)$ and $(u_1, u_2) = (\eta_1, \xi_1)$ are defined as new sets of state and control variables :

$$\dot{z} = g_1(z)u_1 + g_2(z)u_2, \qquad z \overset{def}{=} (z_1, z_2, z_3, z_4) \in I\!R^4 \tag{4.139}$$

$$g_1(z) = \frac{\partial}{\partial z_1}, \qquad g_2(z) = cos z_3 \; sin z_1 \frac{\partial}{\partial z_2} + cos z_1 \frac{\partial}{\partial z_3} - sin z_3 \; sin z_1 \frac{\partial}{\partial z_4}$$

Calculating the following Lie brackets:

$$g_3(z) \overset{def}{=} [g_1, g_2](z) = cos\ z_1\ cos\ z3 \frac{\partial}{\partial z_2} - sin\ z_1 \frac{\partial}{\partial z_2} - cos\ z_1\ sin\ z3 \frac{\partial}{\partial z_3}$$

$$g_4(z) \overset{def}{=} [g_2, [g_1, g_2]](z) = -sin\ z_3 \frac{\partial}{\partial z_2} - cos\ z_3 \frac{\partial}{\partial z_4}$$

shows that the LARC condition is satisfied for this system :

$$span\{g_1, g_2, g_3, g_4\}(z) = I\!R^4, \quad \text{for all} \quad z \in I\!R^4$$

A few moments of reflection leads to the conclusion that (4.139) can be decomposed into the following two subsystems:

$$S1: \quad \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ cos\ z_3\ sin\ z_1 \\ cos\ z_1 \end{bmatrix} u_2 \tag{4.140}$$

$$S2: \quad \dot{z}_4 = -sin\ z_3\ sin\ z_1 u_2 \stackrel{def}{=} f(z)\ u_2 \tag{4.141}$$

Next we observe that defining $x \stackrel{def}{=} (z_1, z_2, z_3)$ allows to re-write subsystem S1 as:

$$\begin{aligned} \dot{x} &= f_1(x)u_1 + f_2(x)u_2, \qquad x \in I\!R^3 \\ f_1(x) &= \frac{\partial}{\partial z_1}, \quad f_2(x) = cos\ z_3\ sin\ z_1 \frac{\partial}{\partial z_2} + cos\ z_1 \frac{\partial}{\partial z_3} \end{aligned} \tag{4.142}$$

Subsystem (4.142) is controllable as it clearly satisfies:

$$span\{f_1, f_2, f_3\}(x) = I\!R^3, \quad \text{for all} \quad x \in I\!R^3$$

where

$$f_3(x) \stackrel{def}{=} [f_1, f_2](x) = cos\ z_1\ cos\ z_3 \frac{\partial}{\partial z_2} - sin\ z_1\ \frac{\partial}{\partial z_3}.$$

It can be easily checked that the Lie algebra $L(f_1, f_2)$ is not nilpotent, but the following approximation to S1 :

$$\begin{aligned} \tilde{S}1: \quad \dot{x} &= \tilde{f}_1(x)u_1 + \tilde{f}_2(x)u_2, \qquad x \in I\!R^3 \\ \tilde{f}_1(x) &= \frac{\partial}{\partial z_1} \\ \tilde{f}_2(x) &= z_1 \frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3} \\ \tilde{f}_3(x) &\stackrel{def}{=} [\tilde{f}_1, \tilde{f}_2](x) = \frac{\partial}{\partial z_2} \end{aligned} \tag{4.143}$$

gives

$$span\{\tilde{f}_1, \tilde{f}_2, \tilde{f}_3\}(x) = I\!R^3, \quad \text{for all} \quad x \in I\!R^3$$

and is nilpotent as shown by the Lie brackets multiplication table for $L(\tilde{f}_1, \tilde{f}_2)$:

$$[\tilde{f}_1, \tilde{f}_2] = \tilde{f}_3 \qquad [\tilde{f}_1, \tilde{f}_3] = 0 \qquad [\tilde{f}_2, \tilde{f}_3] = 0$$

The trajectory interception approach can thus be employed to steer $\tilde{S}1$.

The extended system for $S1$ is given by

$$\dot{x} = f_1(x)v_1 + f_2(x)v_2 + f_3(x)v_3 \tag{4.144}$$

in which $v_i(x) = L_{f_i}W(x)$, $i = 1,2,3$ and $W(x) = \frac{1}{2}\sum_{i=1}^{3} z_i^2$.

The logarithmic coordinates for $\bar{S}1$ satisfy the following differential equations:

$$\begin{aligned}
\dot{\gamma}_1 &= a_1 \\
\dot{\gamma}_2 &= a_2 \\
\dot{\gamma}_3 &= -\gamma_1 a_2 + a_3, \quad \gamma_i(0) = 0, \ i = 1,2,3
\end{aligned}$$

and the following controls stabilize subsystem $S1$

$$\begin{aligned}
u_1(x) &= (v_1^T(x) + b_3(v_3^T(x)) \ sin(\frac{2\pi}{T}t)) \\
u_2(x) &= (v_2^T(x) + b_3(v_3^T(x)) \ cos(\frac{2\pi}{T}t))
\end{aligned} \tag{4.145}$$

where b_3 is $\pm 3.54491\sqrt{(}v_3^T(x))/\sqrt{(}T)$ are easily computed.

It can be seen that the control (4.145) obtained using the trajectory interception approach, steers the original system (4.139), to any $\epsilon - neighbourhood$ of the manifold $\mathcal{M}' \stackrel{def}{=} \{z \in I\!R^4 : z_i = 0, i = 1,2,3\}$, and further decrease in the cost function V can be obtained only through system motion in the direction of the Lie bracket $g_4 \stackrel{def}{=} [g_1,[g_1,g_2]]$. Such motion can be achieved only indirectly, for example by using an open-loop control of the type :

$$\begin{aligned}
u_1 &= k_1 sin(\frac{2\pi}{T}t) \\
u_2 &= k_2 cos(\frac{4\pi}{T}t)
\end{aligned} \tag{4.146}$$

where k_1, k_2 are constants. Introducing the following definitions :

$$\begin{aligned}
\mathcal{S}_1 &\stackrel{def}{=} \{z \in I\!R^4 : z_1 = z_2 = z_3 = 0, z_4 \neq 0\} \\
\mathcal{S}_2 &\stackrel{def}{=} \{z \in I\!R^4 : z_4 = 0 \ \& \ f(z) = 0\} \\
&= \{z \in I\!R^4 : z_4 = 0 \ \& \ sin \ z_3 \ sin \ z_1 = 0\} \\
&= \{z \in I\!R^4 : z_4 = z_3 = z_1 = 0\}
\end{aligned}$$

and for any set $S \subset I\!R^n$ and any constant $\epsilon > 0$, let the symbol $\mathcal{N}(\mathcal{S};\epsilon)$ denotes the usual $\epsilon - neighbourhood$ of S. This leads to the following feedback stabilization strategy for the original system.

Stabilization strategy for a WMR of type (1,1)

Repeat the steps below until sufficient accuracy is achieved in reaching the origin:

Data : $\epsilon > 0$

Step a: Steer the original system (4.139) to $\mathcal{N}(\mathcal{S}_1;\epsilon)$ by applying the control of (4.145).

Step b: Employ the control (4.146) until the system trajectories converge to $\mathcal{N}(\mathcal{S}_2;\epsilon)$.

Step c: Set $\epsilon := \frac{\epsilon}{2}$.

Three sets of simulation results are shown in Figures 4.24 - 4.25, 4.26 - 4.27, and 4.28 - 4.29, respectively.

Figures 4.24 - 4.25 correspond to the situation when the mobile robot is steered to the origin from an arbitrary initial condition in the configuration space (specifically, the trajectories shown were achieved when $x_0 = [0.4, 0.7, 0.6, 0.5]^T$ and $k_1 = 2$, $k_2 = -3.5$ and $T = 1$, were used).

Figures 4.26 - 4.27 and 4.28 - 4.29 show the controlled system trajectories during two parallel parking maneuvers, corresponding to the initial conditions $x_0 = [0, 1, 0, 0]^T$ ($k_1 = -2.5$, $k_2 = 4$ and $T = 1.5$ were used), and $x_0 = [0, -1, 0, 0]^T$ ($k_1 = 2$, $k_2 = 3$ and $T = 1.2$ were used), respectively.

7.2. Stabilizing feedback control for a fire truck model

The fire truck model represents a six dimensional systems with control deficiency order three and with a non-nilpotent controllability Lie algebra which contains Lie brackets of depth one as well as depth two.

The kinematic model of a fire truck is given by (3.110) of Chapter 3, in which $z = (z_1, z_2, z_3, z_4, z_5, z_6) \overset{def}{=} (x, \phi_0, \phi_1, \theta_0, \theta_1, y)$ and which can thus be re-written as :

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \\ \dot{z}_5 \\ \dot{z}_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ tan\ z_2\ sec\ z_4 \\ -sin(z_3 - z_4 + z_5)\ sec\ z_3\ sec\ z_4 \\ tan\ z_4 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_3$$

$$\overset{def}{=} g_1(z)u_1 + g_2(z)u_2 + g_3(z)u_3 \tag{4.147}$$

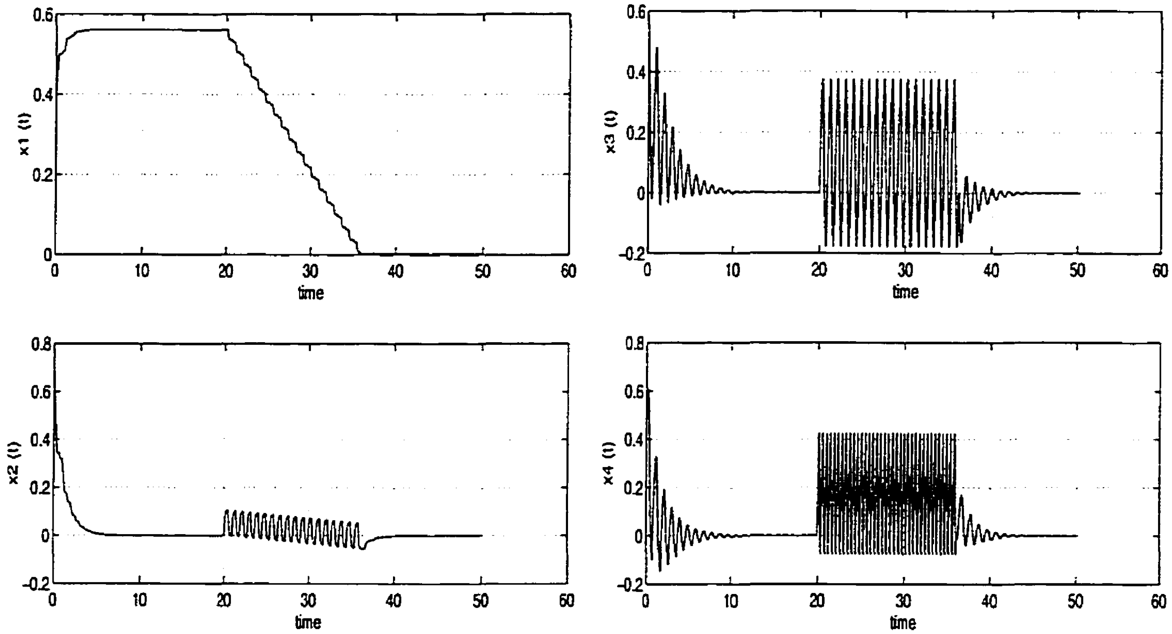

FIGURE 4.24. *WMR of type* $(1,1)$: Plots of the controlled state trajectories $t \mapsto ((z_1(t), ..., z_4(t)) = (x_1(t), ..., x_4(t))$ versus time.

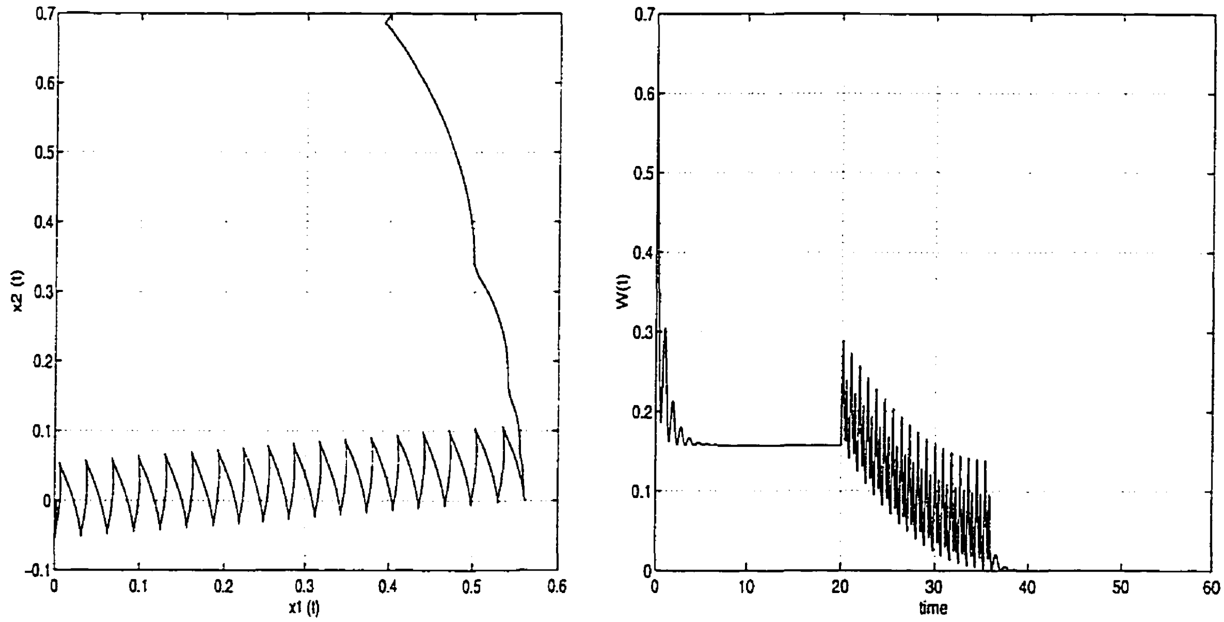

FIGURE 4.25. *WMR of type* $(1,1)$: Plots of the controlled state trajectories $x_1(t) = z_1(t)$ versus $x_2(t) = z_2(t)$, and the Lyapunov function $V(x(t)) = \frac{1}{2}\sum_{i=1}^{4} x_i^2(t)$ along the controlled state trajectories.

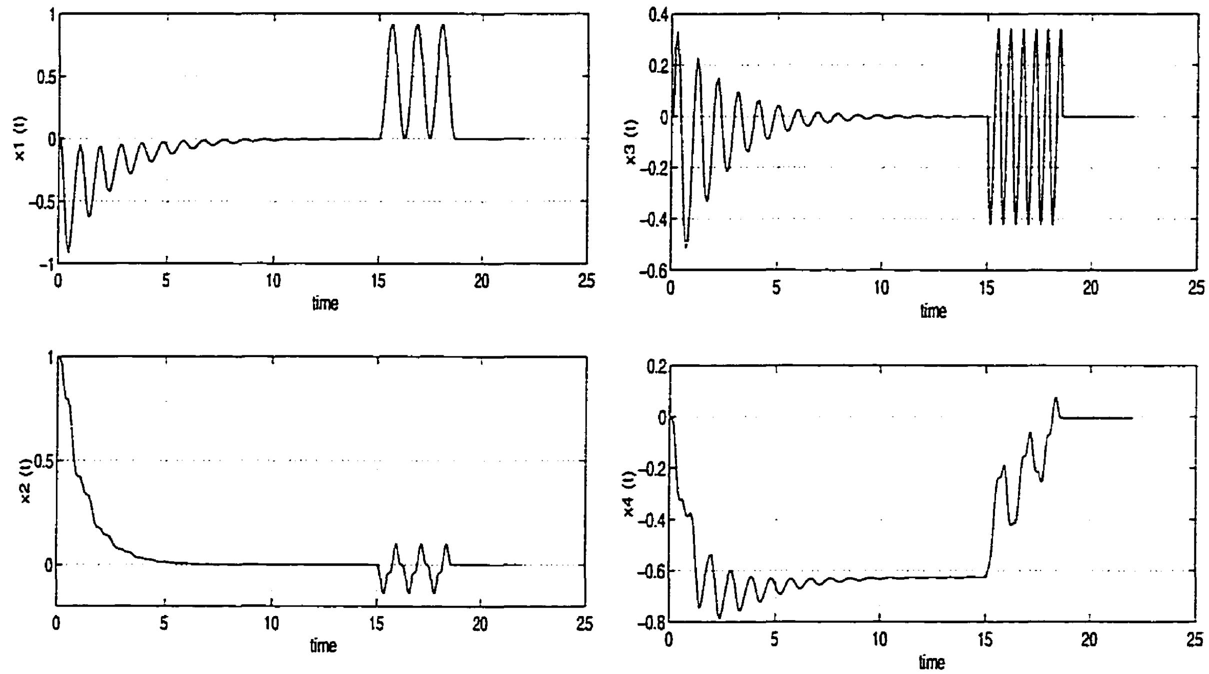

FIGURE 4.26. *WMR of type* $(1,1)$: Plots of the controlled state trajectories $t \mapsto ((z_1(t), \ldots, z_4(t)) = (x_1(t), \ldots, x_4(t))$ versus time in parallel parking maneuver.

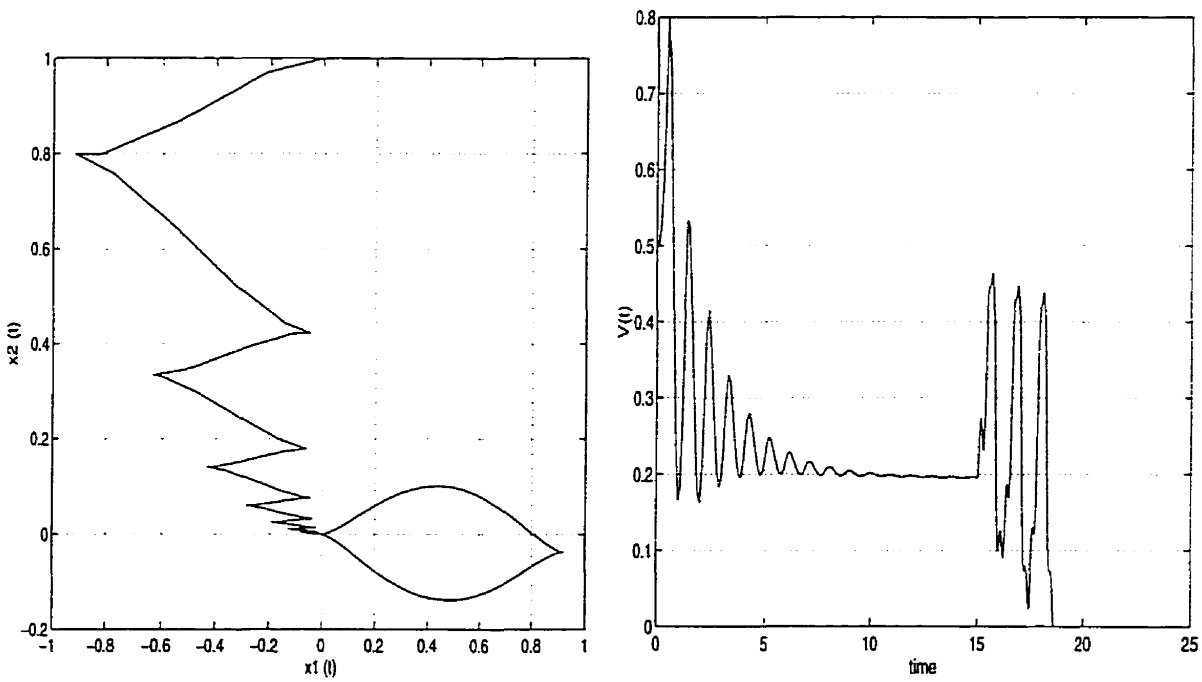

FIGURE 4.27. *WMR of type* $(1,1)$: Plots of the controlled state trajectories $x_1(t) = z_1(t)$ versus $x_2(t) = z_2(t)$, and the Lyapunov function $V(x(t)) = \frac{1}{2}\sum_{i=1}^{4} x_i^2(t)$ along the controlled state trajectories in parallel parking maneuver.

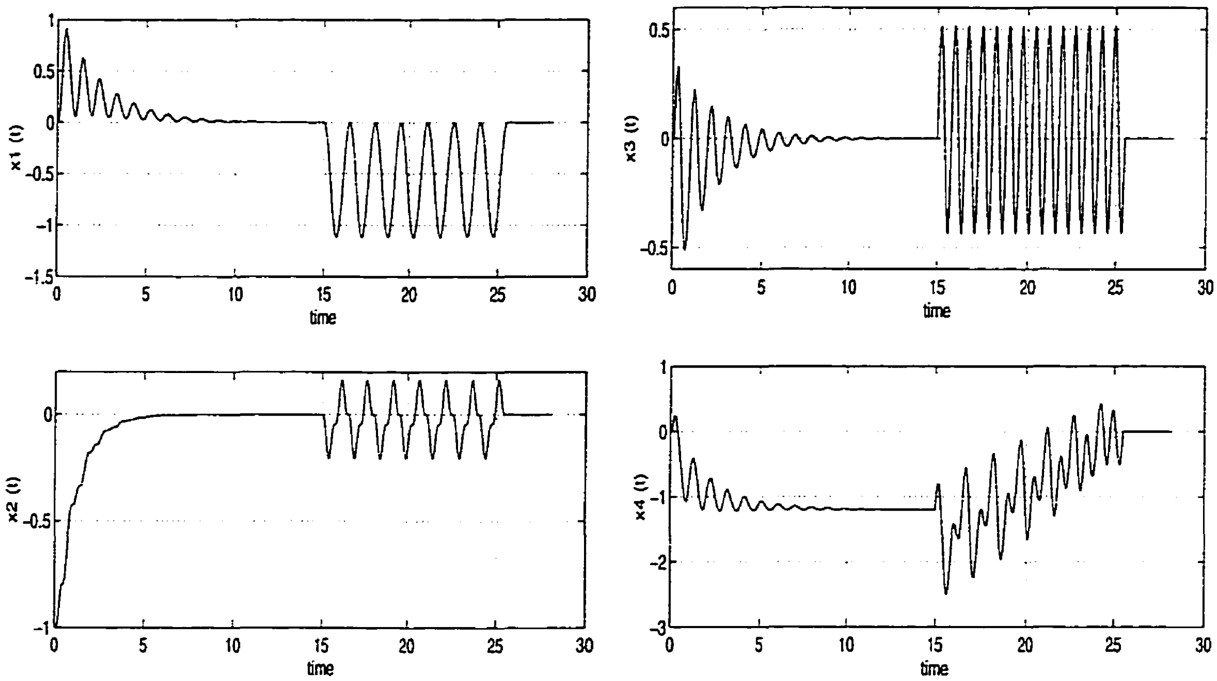

FIGURE 4.28. *WMR of type* (1, 1): Plots of the controlled state trajectories $t \mapsto ((z_1(t), ..., z_4(t)) = (x_1(t), ..., x_4(t))$ versus time in parallel parking maneuver.

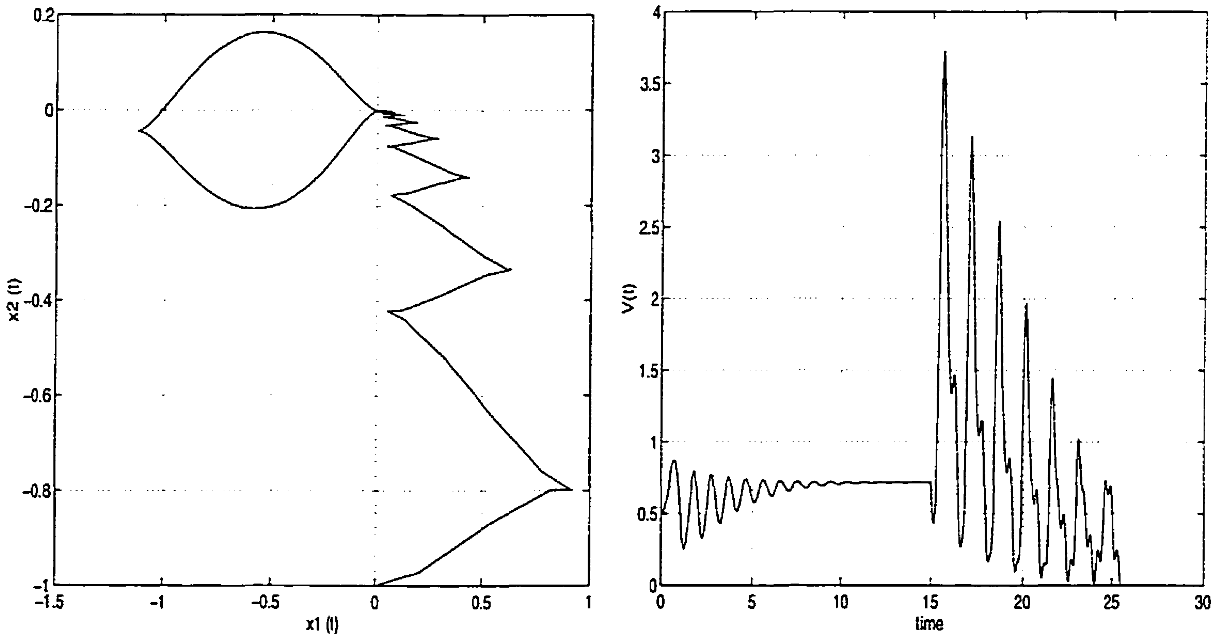

FIGURE 4.29. *WMR of type* (1, 1): Plots of the controlled state trajectories $x_1(t) = z_1(t)$ versus $x_2(t) = z_2(t)$, and the Lyapunov function $V(x(t)) = \frac{1}{2}\sum_{i=1}^{4} x_i^2(t)$ along the controlled state trajectories in parallel parking maneuver.

$$
\begin{aligned}
g_1(z) &= \frac{\partial}{\partial z_1} + tan\ z_2\ sec\ z_4\ \frac{\partial}{\partial z_4} - sin(z_3 - z_4 + z_5)\ sec\ z_3\ sec\ z_4\ \frac{\partial}{\partial z_5} + tan\ z_4\ \frac{\partial}{\partial z_6} \\
g_2(z) &= \frac{\partial}{\partial z_2} \qquad g_3(z) = \frac{\partial}{\partial z_3}
\end{aligned}
$$

Computing the following Lie brackets:

$$
\begin{aligned}
&g_4(z) \stackrel{def}{=} [g_1, g_2](z) = -(sec\ z_2)^2\ sec\ z_4 \frac{\partial}{\partial z_4} \\
&g_5(z) \stackrel{def}{=} [g_1, g_3](z) = [\ sec\ z_3\ sec\ z_4(cos(z_3 - z_4 + z_5) + sin(z_3 - z_4 + z_5)\ tan\ z_3)]\ \frac{\partial}{\partial z_5} \\
&g_6(z) \stackrel{def}{=} [g_1, [g_1, g_2]](z) = (sec\ z_2)^2\ (sec\ z_5)^3\ \frac{\partial}{\partial z_6} \\
&+[(sec\ z_2)^2\ (sec\ z_4)^2\ sec\ z_3\ (cos(z_3 - z_4 + z_5) - sin(z_3 - z_4 + z_5)\ tan\ z_4)]\ \frac{\partial}{\partial z_5}
\end{aligned}
$$

demonstrates that, if the motion of the system is restricted to the manifold:

$$
\mathcal{M} = \{z \in I\!\!R^6\ : |z_i| < \frac{\pi}{2},\ i = 2, 3, 4\},
$$

then the LARC condition is satisfied :

$$
span\{g_1, g_2, g_3, [g_1, g_2], [g_1, g_3], [g_1, [g_1, g_2]]\}(z) = I\!\!R^6 \quad \text{for } z \in \mathcal{M} \tag{4.148}
$$

Consider the following decomposition:

$$
S1: \quad \begin{bmatrix} \dot z_1 \\ \dot z_2 \\ \dot z_3 \\ \dot z_4 \\ \dot z_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ tan\ z_2\ sec\ z_4 \\ -sin(z_3 - z_4 + z_5)\ sec\ z_3\ sec\ z_4 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u_3 \tag{4.149}
$$

$$
S2: \qquad \dot z_6 = tan\ z_4\ u_3 \stackrel{def}{=} f(z)\ u_3 \tag{4.150}
$$

By defining $x \stackrel{def}{=} (z_1, z_2, z_3, z_4, z_5)$, the subsystem S1 can be written as:

$$
\begin{aligned}
\dot x &= f_1(x)u_1 + f_2(x)u_2 + f_3(x)u_3, \qquad x \in I\!\!R^5 \qquad (4.151) \\
f_1(x) &= \frac{\partial}{\partial z_1} + tan\ z_2\ sec\ z_4\ \frac{\partial}{\partial z_4} - sin\ (z_3 - z_4 + z_5)\ sec\ z_3\ sec\ z_4\ \frac{\partial}{\partial z_5} \\
f_2(x) &= \frac{\partial}{\partial z_2}, \qquad f_3(x) = \frac{\partial}{\partial z_3}
\end{aligned}
$$

Subsystem $S1$ is controllable as it satisfies the LARC condition:

$$
span\{f_1(x), f_2(x), f_3(x), f_4(x), f_5(x)\} = I\!\!R^5, \qquad \text{for all } x \in \tilde{\mathcal{M}}
$$

where, $\quad \bar{\mathcal{M}} = \{x \stackrel{def}{=} (z_1, ..., z_5) \in I\!R^5 \ : |z_i| < \frac{\pi}{2},\ i = 2,3,4\}$

$$f_4(x) \stackrel{def}{=} [f_1, f_2](x) = -(sec\ z_2)^2\ sec\ z_4 \frac{\partial}{\partial z_4}$$

$$f_5(x) \stackrel{def}{=} [f_1, f_3](x) = [\ sec\ z_3\ sec\ z_4(cos\ (z_3 - z_4 + z_5) + sin\ (z_3 - z_4 + z_5)\ tan\ z_3)]\ \frac{\partial}{\partial z_5}$$

It can be easily checked that the Lie algebra $L(f_1, f_2, f_3)$ is not nilpotent. The following approximation to subsystem S1:

$$\dot{x} \ = \ \tilde{f}_1(x)u_1 + \tilde{f}_2(x)u_2 + \tilde{f}_3(x)u_3, \qquad x \in I\!R^5 \tag{4.152}$$

$$\begin{aligned} \tilde{f}_1(x) &= \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_4} - (z_3 - z_4 + z_5) \frac{\partial}{\partial z_5} \\ \tilde{f}_2(x) &= \frac{\partial}{\partial z_2}, \qquad \tilde{f}_3(x) = \frac{\partial}{\partial z_3} \\ \tilde{f}_4(x) &\stackrel{def}{=} [\tilde{f}_1, \tilde{f}_2](x) = -\frac{\partial}{\partial z_4} \qquad \tilde{f}_5(x) \stackrel{def}{=} [\tilde{f}_1, \tilde{f}_3](x) = \frac{\partial}{\partial z_5} \end{aligned}$$

gives

$$span\{\tilde{f}_1(x), \tilde{f}_2(x), \tilde{f}_3(x), \tilde{f}_4(x), \tilde{f}_5(x)\} \ = \ I\!R^5, \qquad \text{for all} \ \ x \in I\!R^5$$

and is nilpotent as shown by Lie brackets multiplication table $L(\tilde{f}_1, \tilde{f}_2, \tilde{f}_3)$:

$$[\tilde{f}_1, \tilde{f}_2] = \tilde{f}_4 \qquad [\tilde{f}_1, \tilde{f}_3] = \tilde{f}_5 \qquad [\tilde{f}_2, \tilde{f}_3] = 0$$
$$[\tilde{f}_i, \tilde{f}_4] = [\tilde{f}_i, \tilde{f}_5] = 0, i = 1, 2, 3$$

Therefore the trajectory interception approach can be applied to steer $S1$. The extended system for $S1$ is given by:

$$\dot{x} \ = \ f_1(x)v_1 + f_2(x)v_2 + f_3(x)v_3 + f_4(x)v_4 + f_5(x)v_5, \qquad x \in I\!R^5 \tag{4.153}$$

in which $v_i(x) = L_{f_i} W(x),\ i = 1, ..., 5$, and $W(x) = \frac{1}{2}\sum_{i=1}^{5} z_i^2$. The approximate logarithmic coordinates for $\tilde{S}1$ satisfy the following differential equations :

$$\begin{aligned} \dot{\gamma}_1 &= a_1 \\ \dot{\gamma}_2 &= a_2 \\ \dot{\gamma}_3 &= a_3 \\ \dot{\gamma}_4 &= -\gamma_1 a_2 + a_4 \\ \dot{\gamma}_5 &= -\gamma_1 a_3 + a_5,\ \gamma_i(0) = 0,\ i = 1, ..., 5 \end{aligned}$$

and the following controls stabilize subsystem S1:

$$\begin{aligned}
u_1(x) &= (v_1^T(x) + b_4(v_4^T(x))\ sin(\frac{2\pi}{T}t) + b_5(v_5^T(x))\ sin(\frac{2\pi}{T}t)) \\
u_2(x) &= (v_2^T(x) + b_4(v_4^T(x))\ cos(\frac{2\pi}{T}t)) \\
u_3(x) &= (v_3^T(x) + b_5(v_5^T(x))\ cos(\frac{2\pi}{T}t))
\end{aligned} \tag{4.154}$$

where $b_4 = \pm 3.54491\sqrt{(}v_4^T)/\sqrt{(}T)$ and $b_5 = \pm 3.54491\sqrt{(}v_5^T)/\sqrt{(}T)$.

For faster convergence, v_i^T were replaced by $k\ v_i^T$, $i = 1, 2, 3.$ and in simulation, $k = 3$ was used.

Stabilizing algorithm for a fire truck model:

Repeat the following steps until sufficient accuracy is achieved in reaching the origin:

Data : $\epsilon > 0$

Step a: Apply the controls (4.154) to original system (4.147) until its trajectories converge to $\mathcal{N}(\mathcal{S}_1; \epsilon)$, where :

$$\mathcal{S}_1 \stackrel{def}{=} \{z \in I\!R^6 : z_1 = z_2 = z_3 = z_4 = z_5 = 0, \quad z_6 \neq 0\}$$

Step b: To generate motion along $g_6 = [g_1, [g_1, g_2]]$, apply the sinusoidal controls

$$u_1 = sin(\frac{2\pi}{T}t), \qquad u_2 = cos(\frac{4\pi}{T}t), \qquad u_3 = 0$$

until the system trajectories converge to $\mathcal{N}(\mathcal{S}_2; \epsilon)$, where :

$$\begin{aligned}
\mathcal{S}_2 &\stackrel{def}{=} \{z \in I\!R^6 : z_6\ \&\ f(z) = 0\} \\
&= \{z \in I\!R^6 : z_6\ \&\ tan\ z_4 = 0\} \\
&= \{z \in I\!R^6 : z_6 = z_4 = 0\}
\end{aligned}$$

Step c: Set $\epsilon := \frac{\epsilon}{2}$.

Two sets of simulation results are shown in Figures 4.30 - 4.22, and 4.32 - 4.33, respectively.

Figures 4.30 - 4.31 correspond to the situation when the fire truck is steered to the origin from an arbitrary initial condition in the configuration space.

Figures 4.32 - 4.33 show the controlled system trajectories during a parallel parking maneuver, corresponding to the initial conditions $z_0 = [0, .4, 0, 0, 0, 0]^T$.

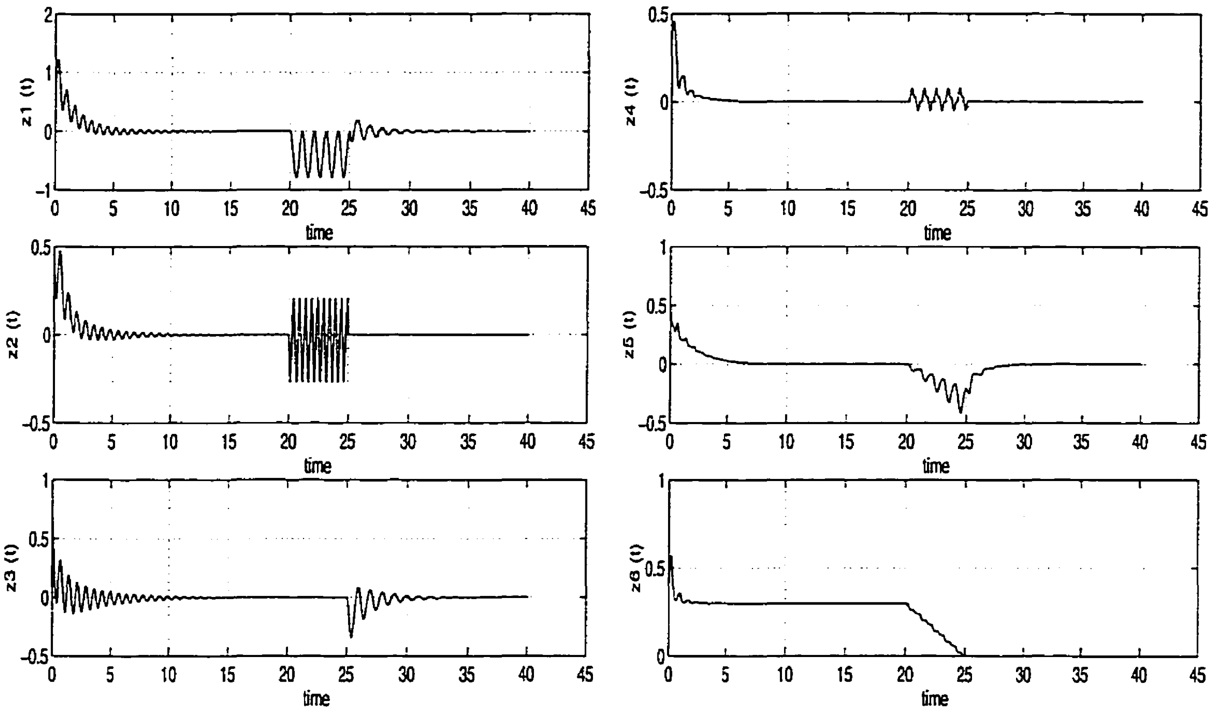

FIGURE 4.30. *Fire truck* : Plots of the controlled state trajectories $t \mapsto ((z_1(t), \ldots, z_6(t))$ versus time.

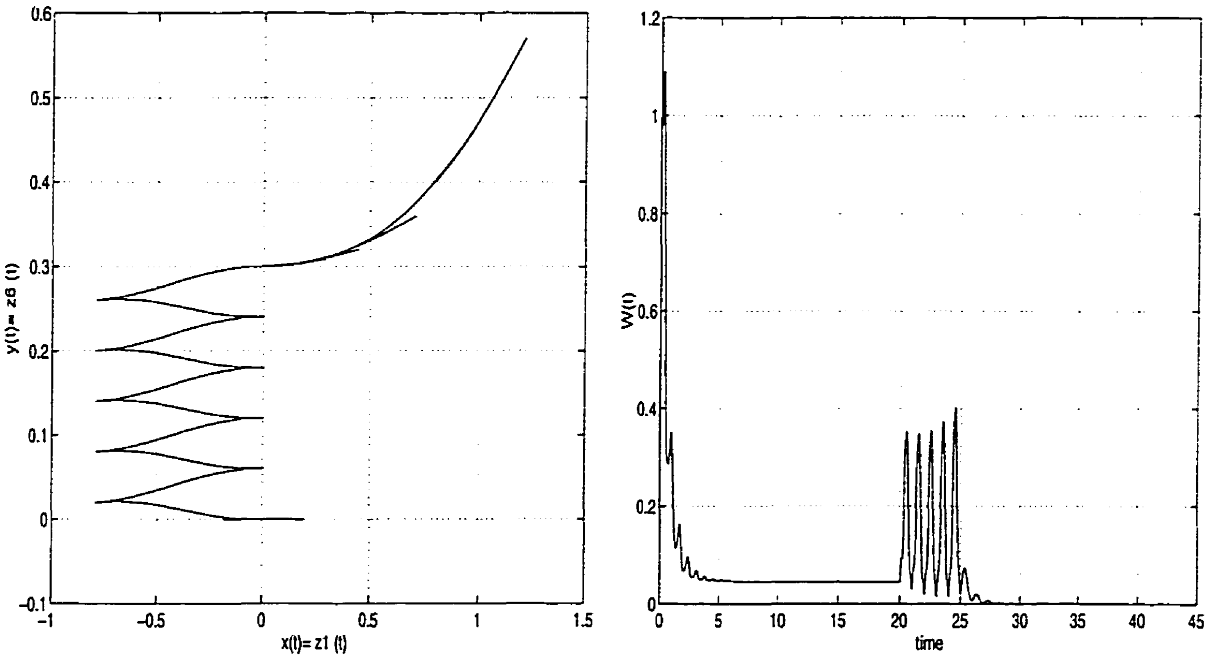

FIGURE 4.31. *Fire truck* : Plots of the controlled state trajectories $z_1(t)$ versus $z_3(t)$, and Lyapunov function $V(z(t)) = \frac{1}{2}\sum_{i=1}^{6} z_i^2(t)$ along the controlled state trajectories.

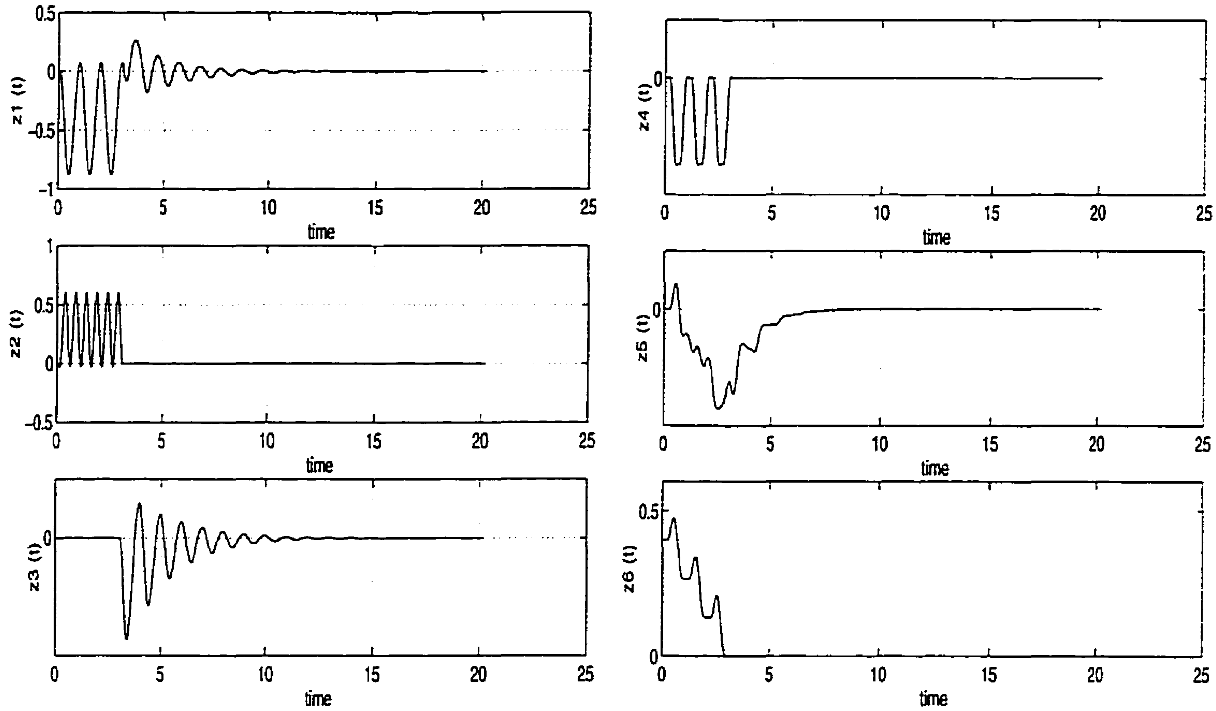

FIGURE 4.32. *Fire truck* : Plots of the controlled state trajectories $t \mapsto ((z_1(t), ..., z_6(t))$ versus time in parallel parking maneuver.

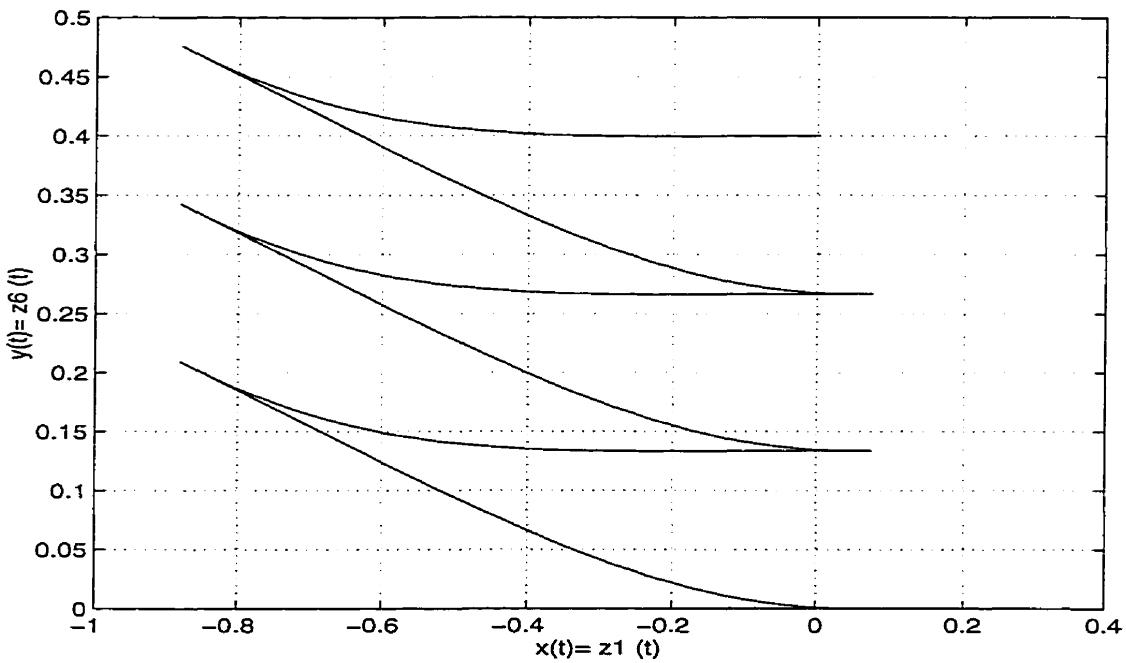

FIGURE 4.33. *Fire truck* : Plots of the controlled state trajectories $z_1(t) = x(t)$ versus $z_6(t) = y(t)$ in parallel parking maneuver.

7.3. Stabilizing algorithm for an underwater vehicle in actuator failure mode

By defining

$$(z_1, z_2, z_3, z_4, z_5, z_6) \stackrel{def}{=} (x_5, x_4, x_1, x_6, x_3, x_2) \;\;\&\;\; (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) \stackrel{def}{=} (u_3, u_2, u_1)$$

the kinematic model of underwater vehicle with one actuator failure mode, as given by (3.105) of Chapter 3, can be decomposed as:

$$S1: \quad \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \\ \dot{z}_5 \end{bmatrix} = \begin{bmatrix} \cos z_2 \\ \sin z_2 \tan z_1 \\ 0 \\ \sin z_2 \sec z_1 \\ 0 \end{bmatrix} \tilde{u}_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tilde{u}_2 + \begin{bmatrix} 0 \\ 0 \\ \cos z_4 \cos z_1 \\ 0 \\ -\sin z_1 \end{bmatrix} \tilde{u}_3 \qquad (4.155)$$

$$S2: \quad \dot{z}_6 = \sin z_4 \cos z_1 \, \tilde{u}_3 \stackrel{def}{=} f(z)\, \tilde{u}_3 \qquad (4.156)$$

Stabilizing feedback control for an underwater vehicle in actuator failure mode:

Repeat the following steps until sufficient accuracy is achieved in reaching the origin:

Data : $\epsilon > 0$

Step a: Apply the controls (4.154) to original system (4.147) until its trajectories converge to $\mathcal{N}(\mathcal{S}_1; \epsilon)$, where :

$$\mathcal{S}_1 \stackrel{def}{=} \{z \in I\!R^6 : z_1 = z_2 = z_3 = z_4 = z_5 = 0, \;\; z_6 \neq 0\}$$

Step b: To generate motion along $g_6 = [g_2, [g_1, g_3]]$, apply the sinusoidal controls

$$u_1 = k_1 sin(\frac{2\pi}{T}t), \qquad u_2 = k_2 sin(\frac{2\pi}{T}t), \qquad u_3 = k_3 cos(\frac{4\pi}{T}t) \qquad (4.157)$$

until the system trajectories converge to $\mathcal{N}(\mathcal{S}_2; \epsilon)$, where :

$$\begin{aligned} \mathcal{S}_2 &\stackrel{def}{=} \{z \in I\!R^6 : z_6 = 0 \;\&\; f(z) = 0\} \\ &= \{z \in I\!R^6 : z_6 = 0 \;\&\; \sin z_4 \cos z_1 = 0\} = \{z \in I\!R^6 : z_6 = z_4 = 0\} \end{aligned}$$

Step c: Set $\epsilon := \frac{\epsilon}{2}$.

Simulation results are shown in Figures 4.34 - 4.35. In simulation results, the values $k_1 = 1$, $k_2 = -3$, $k_3 = 4$ and $T = 1.6$ were used.

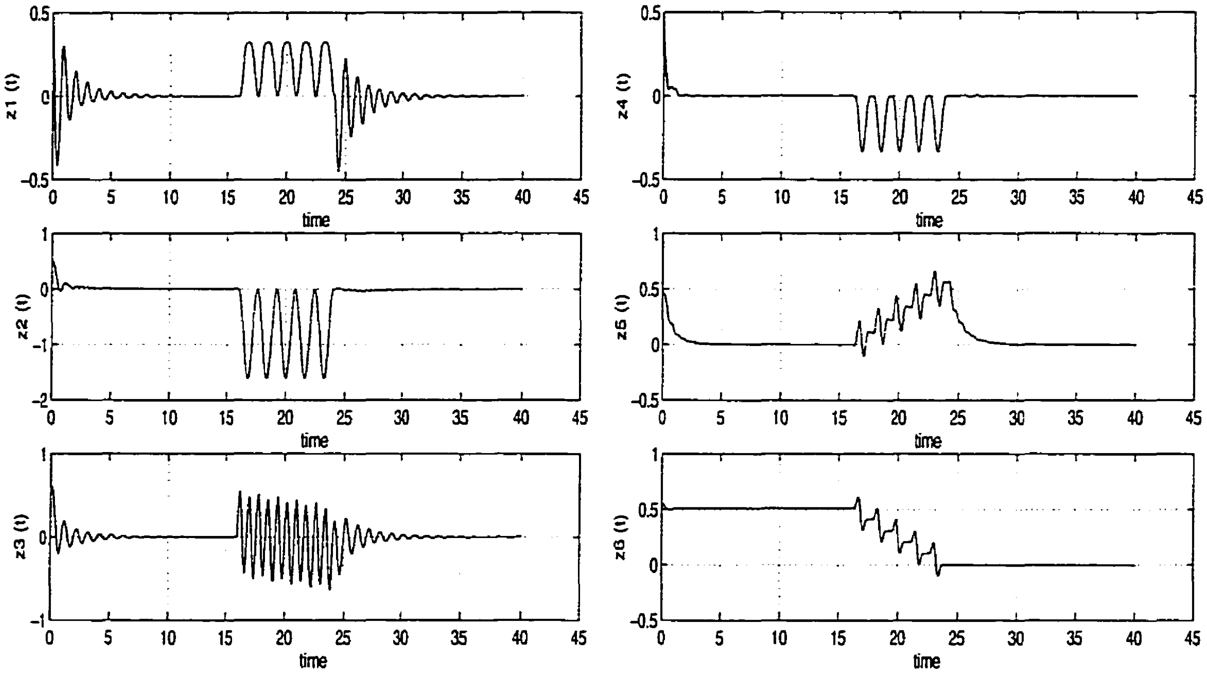

FIGURE 4.34. *Underwater vehicle Model 2* : Plots of the controlled state trajectories $t \mapsto ((z_1(t), ..., z_6(t))$ versus time.

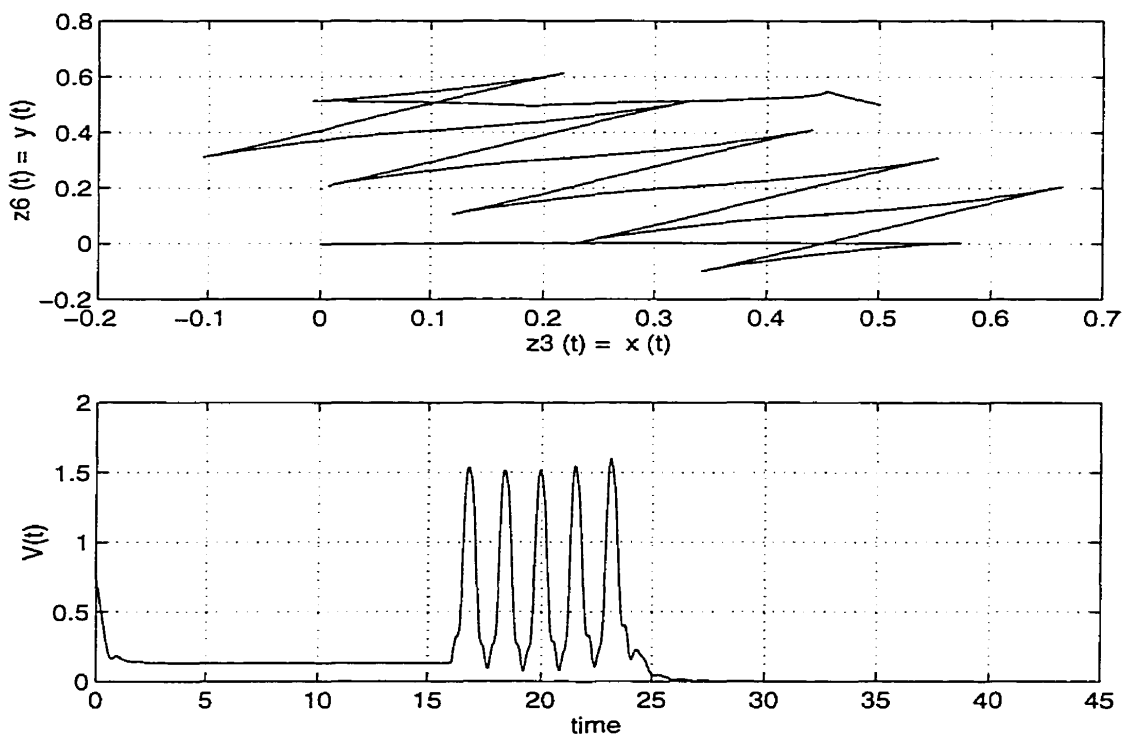

FIGURE 4.35. *Underwater vehicle Model 2* : Plots of the controlled state trajectories $z_3(t)$ versus $z_6(t)$, and Lyapunov function $V(z(t)) = 0.5 \sum_{i=1}^{6} z_i^2(t)$ along the controlled state trajectories.

7.4. Stabilizing feedback control for a mobile robot with trailer

The example considered below represents a five dimensional systems with control deficiency order three, possessing a non-nilpotent controllability Lie algebra which contains Lie brackets of depth one, two, and three. Although, the algebraic structure of mobile robot with trailer is more complicated, the decomposition idea can still be employed successfully.

The kinematic model of a mobile robot with trailer as given by (3.121) of Chapter 3 and can be suitably re-written by defining $(x_1, x_2, x_3, x_4, x_5) = (z_1, z_4, z_3, z_2, z_5)$:

$$\dot{z} = g_1(z)u_1 + g_2(z)u_2, \quad z \in I\!R^5 \tag{4.158}$$

$$\begin{aligned} g_1(z) &= \cos z_3 \cos z_2 \frac{\partial}{\partial z_1} + \sin z_3 \frac{\partial}{\partial z_2} + \cos z_3 \sin z_2 \frac{\partial}{\partial z_4} + \cos z_3 \sin(z_2 - z_5) \frac{\partial}{\partial z_5} \\ g_2(z) &= \frac{\partial}{\partial z_3} \end{aligned}$$

The following Lie brackets:

$$\begin{aligned} g_3(z) &\stackrel{def}{=} [g_1, g_2](z) = \sin z_3 \cos z_2 \frac{\partial}{\partial z_1} - \cos z_3 \frac{\partial}{\partial x_2} + \sin z_3 \sin z_2 \frac{\partial}{\partial z_4} + \sin z_3 \sin(z_2 - z_5) \frac{\partial}{\partial z_5} \\ g_4(z) &\stackrel{def}{=} [g_1, g_3](z) = -\sin z_2 \frac{\partial}{\partial z_1} + \cos z_2 \frac{\partial}{\partial z_4} + \cos(z_2 - z_5) \frac{\partial}{\partial z_5} \\ g_5(z) &\stackrel{def}{=} [g_1, g_4](z) = -\sin z_3 \cos z_2 \frac{\partial}{\partial z_1} - \sin z_3 \sin z_2 \frac{\partial}{\partial z_4} - (\sin z_3 \sin(z_2 - z_5) - \cos z_3) \frac{\partial}{\partial z_5} \end{aligned}$$

show that the LARC condition is satisfied :

$$span\{g_i(z), i = 1, ..., 5\} = I\!R^5, \quad \text{for all } z \in I\!R^5$$

For this model the following decomposition is considered:

$$S1: \quad \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} \cos z_2 \cos z_3 \\ \sin z_3 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2 \tag{4.159}$$

$$S2: \quad \begin{bmatrix} \dot{z}_4 \\ \dot{z}_5 \end{bmatrix} = \begin{bmatrix} \sin z_2 \cos z_3 \\ \cos z_3 \sin(z_2 - z_5) \end{bmatrix} u_1 \stackrel{def}{=} \begin{bmatrix} f_1(z) \\ f_2(z) \end{bmatrix} u_1 \tag{4.160}$$

By defining $x \stackrel{def}{=} (z_1, z_2, z_3)$, subsystem S1 can be written as:

$$\dot{x} = f_1(x)u_1 + f_2(x)u_2, \quad x \in I\!R^3 \tag{4.161}$$

$$\begin{aligned} f_1(x) &= cos\ z_2\ cos\ z_3\ \frac{\partial}{\partial z_1} + sin\ z_3\ \frac{\partial}{\partial z_2} \\ f_2(x) &= \frac{\partial}{\partial z_3} \end{aligned}$$

Subsystem $S1$ is controllable as it satisfies:

$$span\{f_1(x), f_2(x), f_3(x)\} \quad = \quad I\!R^3, \text{ for all } \ x \in I\!R^3$$

where

$$f_3(x) \stackrel{def}{=} [f_1, f_2](x) = sin\ z_3\ cos\ z_2\ \frac{\partial}{\partial z_1} - cos\ z_3\ \frac{\partial}{\partial z_2}$$

It can be easily verified that the Lie algebra $L(f_1, f_2)$ is not nilpotent.

The following approximation to $S1$:

$$\dot{x} = \tilde{f}_1(x)u_1 + \tilde{f}_2(x)u_2, \quad x \in I\!R^3 \tag{4.162}$$

$$\tilde{f}_1(x) \quad = \quad \frac{\partial}{\partial z_1} + z_3\ \frac{\partial}{\partial z_2}, \qquad \tilde{f}_2(x) = \frac{\partial}{\partial z_3}$$

satisfies the LARC condition:

$$span\{\tilde{f}_1(x), \tilde{f}_2(x), \tilde{f}_3(x)\} \quad = \quad I\!R^3, \qquad \text{for all} \qquad x \in I\!R^3$$

$$\text{where,} \qquad \tilde{f}_3(x) \stackrel{def}{=} [\tilde{f}_1, \tilde{f}_2](x) = -\frac{\partial}{\partial z_2}$$

and the Lie brackets multiplication table for $L(\tilde{f}_1, \tilde{f}_2)$:

$$[\tilde{f}_1, \tilde{f}_2] = \tilde{f}_3 \qquad\qquad [\tilde{f}_1, \tilde{f}_3] = [\tilde{f}_2, \tilde{f}_3] = 0$$

shows that the controllability Lie algebra $L(\tilde{f}_1, \tilde{f}_2)$ is nilpotent. The extended system for $S1$ is given by

$$\dot{x} \quad = \quad f_1(x)v_1 + f_2(x)v_2 + f_3(x)v_3 \tag{4.163}$$

$$\text{with} \quad v_i(x) = L_{f_i}W(x), \quad i = 1, 2, 3, \quad \text{where} \quad W(x) = \tfrac{1}{2}\textstyle\sum_{i=1}^{3} z_i^2,$$

and the logarithmic coordinates for $\tilde{S}1$ satisfy the following equations:

$$\begin{aligned} \dot{\gamma}_1 &= a_1 \\ \dot{\gamma}_2 &= a_2 \\ \dot{\gamma}_3 &= -\gamma_1 a_2 + a_3 \end{aligned}$$

The following controls stabilize subsystem $S1$

$$\begin{aligned} u_1(x) &= (v_1^T(x) + b_3(v_3^T(x))\ sin(\frac{2\pi}{T}t)) \\ u_2(x) &= (v_2^T(x) + b_3(v_3^T(x))\ cos(\frac{2\pi}{T}t)) \end{aligned} \tag{4.164}$$

where, $b_3 = \pm 3.54491\sqrt{(}v_3^T(x))/\sqrt{(}T)$.

Stabilizing algorithm for a mobile robot with trailer:

Repeat the following steps until sufficient accuracy is achieved in reaching the origin:

Data : $\epsilon > 0$

Step a: Apply the controls (4.164) to original system (4.158) until its trajectories converge to $\mathcal{N}(\mathcal{S}_1;\epsilon)$, where :

$$\mathcal{S}_1 \stackrel{def}{=} \{z \in I\!R^5 : z_1 = z_2 = z_3 = 0, \quad z_4 \neq 0,\ z_5 \neq 0\}$$

(b): To generate motion along $g_4 = [g_1,[g_1,g_2]]$, apply the following controls

$$\begin{aligned} u_1 &= k_1\ sin(\frac{2\pi}{T}t) \\ u_2 &= k_2\ cos(\frac{4\pi}{T}t) \end{aligned} \tag{4.165}$$

until the system trajectories converge to $\mathcal{N}(\mathcal{S}_2;\epsilon)$, where :

$$\begin{aligned} \mathcal{S}_2 &\stackrel{def}{=} \{z \in I\!R^5 : z_4 = 0\ \&\ f_1(z) = 0\} \\ &= \{z \in I\!R^5 : z_4 = 0\ \&\ sin\ z_2\ cos\ z_3 = 0\} \\ &= \{z \in I\!R^5 : z_4 = z_2 = 0\} \end{aligned}$$

(c): Again apply the control (4.164) until the system trajectories converge to $\mathcal{N}(\mathcal{S}_3;\epsilon)$:

$$\mathcal{S}_3 \stackrel{def}{=} \{z \in I\!R^5 : z_1 = z_2 = z_3 = z_4 = 0, \quad z_5 \neq 0\}$$

(d): To generate motion along $g_5 = [g_1,[g_1,[g_1,g_2]]]$, apply the following controls

$$\begin{aligned} u_1 &= k_3\ sin(\frac{2\pi}{T}t) \\ u_2 &= k_4\ cos(\frac{6\pi}{T}t) \end{aligned} \tag{4.166}$$

until its trajectories converge to $\mathcal{N}(\mathcal{S}_4;\epsilon)$:

$$\begin{aligned}\mathcal{S}_4 &\stackrel{def}{=} \{z \in \mathbb{R}^5 : z_5 = 0 \ \& \ f_2(z) = 0\} \\ &= \{z \in \mathbb{R}^5 : z_5 = 0 \ \& \ sin\ (z_2 - z_5)\ cos\ z_3 = 0\} = \{z \in \mathbb{R}^5 : z_5 = z_2 = 0\}\end{aligned}$$

(e): Set $\epsilon := \frac{\epsilon}{2}$.

Simulation results are depicted in Figures 4.36 - 4.37 which confirm the applicability of combining strategy. In simulation, the values $k_1 = -2$, $k_1 = -3$, $k_3 = -2.8$, $k_4 = 5$, and $T = 1.2$ were used.

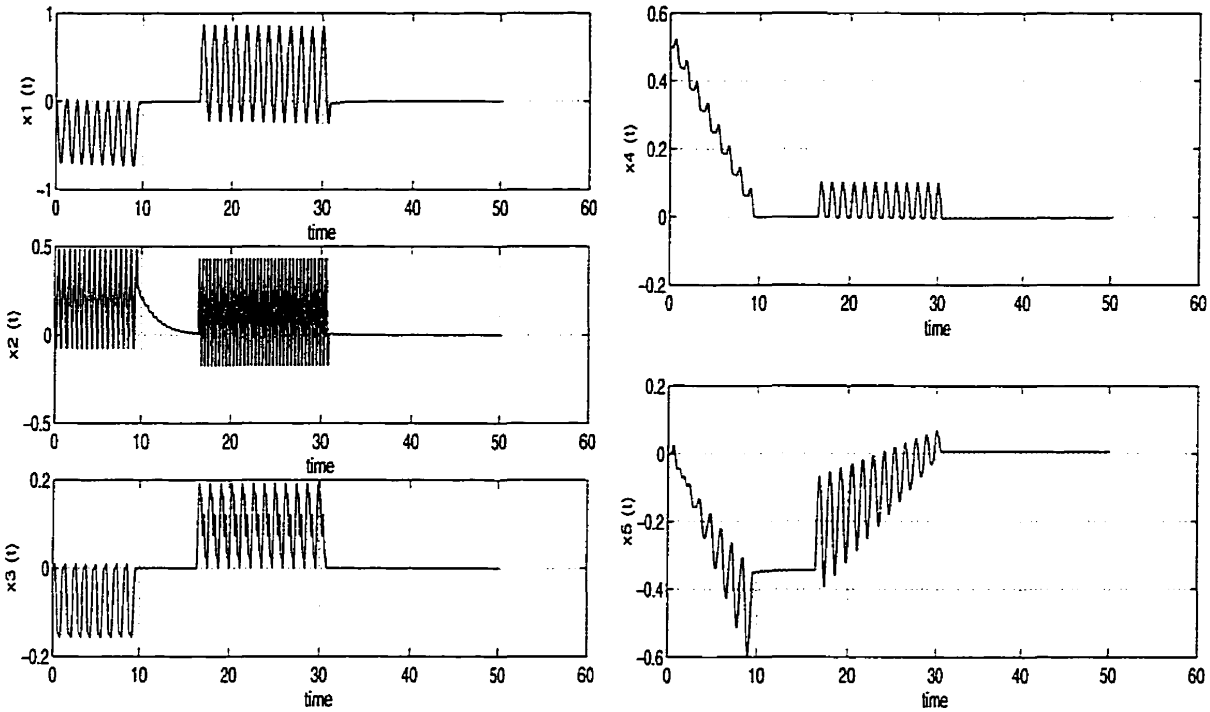

FIGURE 4.36. *Mobile robot with trailer* : Plots of the controlled state trajectories $t \mapsto ((z_1(t), \ldots, z_6(t))$ versus time.

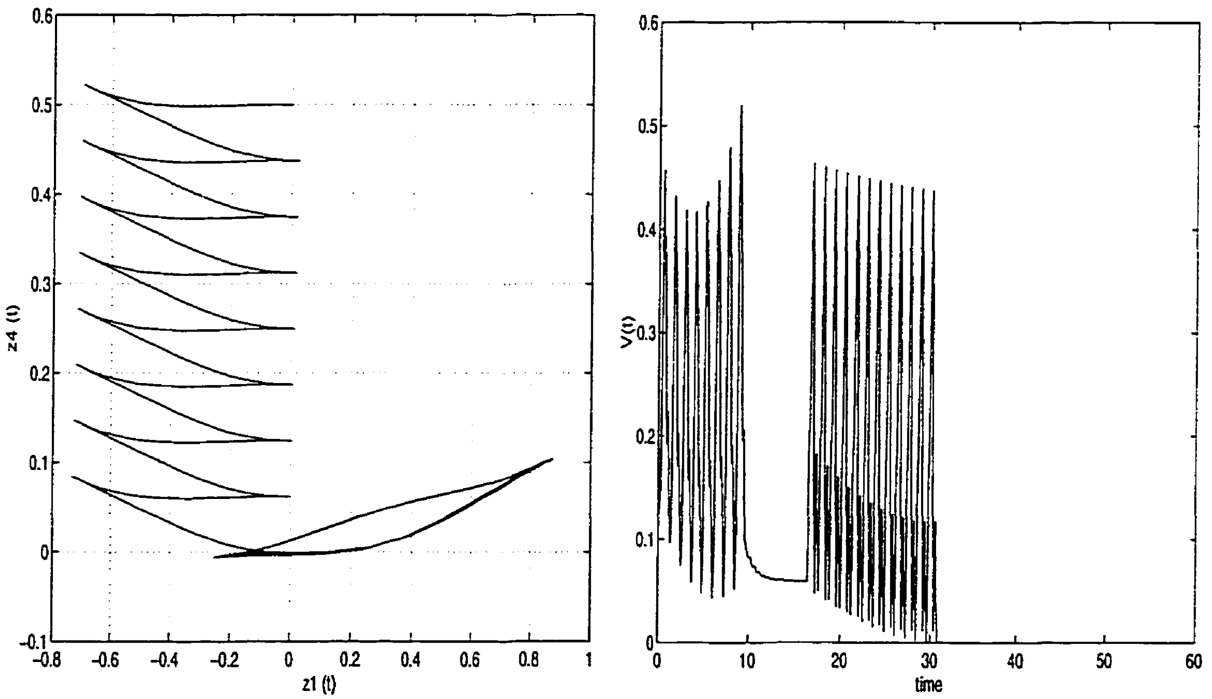

FIGURE 4.37. *Mobile robot with trailer* : Plots of the controlled state trajectories $z_1(t)$ versus $z_4(t)$, and Lyapunov function $V(z(t)) = \frac{1}{2}\sum_{i=1}^{5} z_i^2(t)$ along the controlled state trajectories.

CHAPTER 5

Conclusions and Future Research

In this dissertation, the problem of set point feedback stabilization of drift free systems was discussed in the context of introducing two new feedback synthesis approaches :

(i) the guiding functions approach, and

(ii) the trajectory interception approach

Lie algebraic techniques were used and the importance of the introduced methods lies in the fact that they do not necessitate converting system models into chained or power forms. The applicability of both approaches was demonstrated on a variety of examples.

In this conclusion we briefly review the results of the preceding chapters, give some general observations, and make some suggestions for future work.

1. Review of the results

In **Chapter 2:**

- A novel concept of guiding functions is introduced which can be used as a tool for construction of new and effective feedback control strategies for drift free systems.
- A stabilizing control strategy based on this concept is first developed and analysed for systems of control deficiency order one in rectified form, but is also shown to apply to systems of higher order control deficiency. The strategy is based on simple principles and employs bounded, piecewise constant controls. The values of the guiding functions provide an on-line convergence verification test.

- It is shown that, under reasonable assumptions, the feedback control strategy yields global asymptotic stabilization to a set point.

In **Chapter 3:**

- The guiding functions control strategy, introduced in Chapter 2, is first extended to a general class of drift free systems, which need not be transformable to any special form, and requires constructing two guiding functions. The strategy is extended further to a general class of drift free systems by constructing m guiding functions.
- A systematic method for the construction of guiding functions is introduced, and conditions are stated which guarantee that the resulting feedback control strategy yields global asymptotic convergence to a desired set point.
- The idea of combining sinusoidal steering with the guiding functions approach is also explored.
- Applications of the strategy are discussed involving set point stabilization of several types of models of drift free systems possessing different algebraic structures. In all examples, the strategy proves very efficient in that it effectively leads to dead-beat control.

In **Chapter 4:**

- A systematic method for the synthesis of stabilising, time-varying feedback for a large class of drift-free systems is presented.
- The method shows how the averaging effect can be achieved by a (periodically repeated) open loop solution to a control problem in logarithmic coordinates.
- It is shown that the application of the trajectory interception approach is not limited to systems whose controllability Lie algebra is nilpotent. The approach can successfully be applied to systems with non-nilpotent controllability Lie algebras by introducing approximate models which generate nilpotent controllability Lie algebras. This is confirmed by several examples.
- For higher orders systems, the idea of decomposing the system model into subsystems (of which one can be controlled by the trajectory interception approach and the other by simple sinusoidally varying inputs) is explored.

2. Observations

Generally, the concept of guiding functions, as introduced for the purpose of the construction of stabilizing feedback, gives rise to piecewise constant controls. The first version of the method presented in Chapter 2, however, has two disadvantages :

(i) To be effective, it requires the system to be written in a certain *rectified* form (guiding functions are then chosen to be *quadratic* functions). To apply the technique of this Chapter to a broader class of drift free systems again necessitates the construction of a diffeomorphic state feedback transformation which bring the system to a rectified form.

(ii) Although the control strategy of Chapter 2 is proved to asymptotically stabilize the system, no bounds on the number of switchings are provided (which can be infinite) leading possibly to non-practicable chattering controls.

The extended guiding functions strategy of Chapter 3 brings improvements exactly about the two points listed above :

(a) Under the hypotheses of *analytic* vector fields and involutive distributions, the candidate guiding functions $\{V_1, V_2\}$ are constructed directly by the application of the Frobenius theorem (and thus need not be quadratic) and, more importantly, it is not require that the system is written in a rectified form.

(b) Estimates of the minimum decrease of $|L_{g_i}V_1|$ (while V_2 stays constant) and the maximum increase of $|L_{g_i}V_1|$ (while V_2 is decreasing) are provided. Thus for a given $\epsilon > 0$, bounds on the number of switchings and the time to reach the set $B(0,\epsilon)$ can be computed off-line, depending only of the size of the level set which contains the initial condition. Furthermore, it is shown that introducing hysteresis on the switching controls leads to practical controllers.

(c) It is important to notice, however, that the guiding functions strategy is not a "feedback control" in the classical sense which is understood to be given in terms of a single function $x \mapsto u(x)$ (see Step 2 of the strategy). For this reason, the terms "feedback control strategy" or "control strategy" are used instead of terms "feedback control" or "feedback control law". However, the word "feedback" is justified as the control action of the strategy clearly depends on $x(t) \in \mathcal{T}$ - the point at which $\mathcal{T}$ is traversed, and the value of the state needs to be accessible for measurement in order to implement the control.

The trajectory interception approach introduced in Chapter 4, provides time-varying feedback control laws. The following can be listed as its main properties :

(1) The trajectory interception approach appears to be very effective for systems whose controllability Lie algebras contain only brackets of depth one. For systems whose controllability Lie algebra contains brackets of higher order, the equations describing the evolution of the logarithmic coordinates are more complicated and usually difficult to solve analytically. In such cases numerical solutions should be sought or else the idea of decomposing the system into simpler subsystems should be further explored.

(2) The trajectory interception approach provides for exponential rates of convergence to a desired set point.

(3) The introduction of approximate models often permits significant simplification of the differential equations describing the evolution of the logarithmic coordinates in the open-loop problem formulation.

3. Comparison of the strategies

The control approaches developed can be compared as follows :

- The guiding functions approach provides feedback controls which are discontinuous in the state while the trajectory interception approach leads to feedback control which are continuous in the state. Further, the trajectory interception approach can result in controls which are also continuous in the time if the solution to the OCP is chosen in the sub-class of continuous functions with equal end point values.

- Generally, the guiding functions strategy provides controls which are stabilizing only in the sense of practical stabilization while the trajectory interception approach provides for stabilization in the Lyapunov sense.

- The feedback controls obtained by the guiding functions approach yield global asymptotic convergence to a desired set point and are often dead-beat. On the other hand, the trajectory interception approach provides feedback controls which 'generally' result in only local asymptotic convergence to a set point.

- For systems whose controllability Lie algebra contains Lie brackets of higher order, the guiding functions approach is generally easier to implement than the trajectory interception approach.

- The guiding functions approach appears to be more robust with respect to model error as compared with the trajectory interception approach.
- In the trajectory interception approach arbitrary Lyapunov functions can be used while in the guiding functions approach the construction of a Lyapunov function is a part of the feedback control synthesis.
- The trajectory interception approach is more general in the sense that it applies directly (at least theoretically) to systems whose controllability Lie algebra contains brackets of higher order.

Both approaches lend themselves well to various improvements and generalizations.

4. Future research topics

A list of a few topics for future research is :

(i) Analysis of the robustness properties (with respect to both model error and external disturbances) of the strategies developed.

(ii) Generalization of the control strategies to systems with drift.

(iii) Accommodation of other control objectives such as trajectory tracking and steering to set points under control and state constraints.

(iv) Observer based control when states are not accessible for measurement.

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APPENDIX A

1. Basic review of differential geometry

Diffeomorphism:

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets. A mapping $f : U \to V$ is a *smooth map* if all partial derivatives of f, of any order, exist and are continuous. If $m = n$ and f is bijective and both f and f^{-1} are smooth, then f is called a *diffeomorphism* and U and V are said to be *diffeomorphic.*

Manifolds:

A *manifold of dimension* n is a set M which is locally homeomorphic to $\mathbb{R}^n$.

Local coordinate chart:

A *local coordinate chart* is a pair (ϕ, U), where ϕ is a function which maps points in the set $U \subset M$ to an open subset of $\mathbb{R}^n$.

C^∞ related and smooth atlas:

Two overlapping charts (ϕ, U) and (ψ, V) are C^∞ *related* if $(\psi)^{-1} \circ \phi$ is a diffeomorphism where it is defined. A collection of such charts with the additional property that the U's cover M is called a *smooth atlas.*

Smooth manifolds:

A manifold M is a *smooth manifold* if it admits a smooth atlas.

Smooth map between smooth manifolds:

Let $F : M \to N$ be a mapping between two smooth manifolds and let (ϕ, U) and (ψ, V) be coordinate charts for M and N respectively. The mapping $F : M \to N$ is smooth if $\tilde{F} = \psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$ is smooth for all choices of coordinate charts on M and N. Similarly, F is a diffeomorphism if $\bar{F}$ is a diffeomorphism for all coordinate charts.

Derivation:

Let M be a smooth manifold of dimension n and let p be a point in M. We write $C^\infty(p)$ for the set of smooth, real-valued functions on M whose domain of definition includes some open neighbourhood of p. A map $X_p : C^\infty(p) \to I\!R$ is called a *derivation* if, for all $\alpha, \beta \in I\!R$ and $f, g \in C^\infty(p)$, it satisfies

$$(i) \quad X_p(\alpha f + \beta g) = \alpha(X_p f) + \beta(X_p g) \quad \text{(linearity)}$$

$$(ii) \quad X_p(fg) = (X_p f)g(p) + f(p)(X_p g) \quad \text{(Leibniz rule)}$$

The set of all *derivations* $X_p : C^\infty(p) \to I\!R$ defines a vector space over the reals with the operations

$$\begin{aligned} (X_p + Y_p)f &= X_p f + Y_p f \\ (\alpha X_p)f &= \alpha(X_p f) \end{aligned}$$

Tangent space:

The *tangent space* of M at a point p, denoted T_pM, is the set of all derivations $X_p : C^\infty(p) \to I\!R$. Elements of the tangent space are called *tangent vectors.* Let (ϕ, U) be a coordinate chart on M with local coordinates $(x_1, ..., x_n)$. Then, the set of derivations $\{\frac{\partial}{\partial x}\}$ forms a basis for T_pM and hence we can write

$$X_p = X_1 \frac{\partial}{\partial x_1} + ... + X_n \frac{\partial}{\partial x_n}$$

The vector $(X_1, ..., X_n) \in I\!R^n$ is a local coordinate representation of $X_p \in T_pM$.

Cotangent space:

Given the tangent space T_pM to a manifold M at a point p, we define the *cotangent space* of M at p, denoted T_p^*M, as the set of all linear functions $\omega_p : T_pM \to I\!R$. T_p^*M is a vector space having the same dimension as T_pM and the elements of T_p^*M are called *cotangent vectors.* We write $< \omega_p, X_p >$ for the action of a cotangent vector $\omega_p \in T_p^*M$ on a tangent vector $X_p \in T_pM$. If $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\}$ is a basis for T_pM corresponding to local coordinates $(x_1, ..., x_n)$, the dual basis for T_p^*M is given by $\{dx_1, ..., dx_n\}$, where

$$< dx_i, \frac{\partial}{\partial x_j} >= \delta_{ij}$$

Given a function $f : M \to I\!R$, we define a cotangent vector $df(p) \in T_p^*M$ by

$$< df(p), X_p >= X_p(f), \qquad X_p \in T_pM$$

$df(p)$ is called the *differential* of f. Relative to a chart (ϕ, U) with local coordinates $(x_1, ..., x_n)$, $df(p)$ is written as

$$df(x) = \frac{\partial f}{\partial x_1}(x)dx_1 + ... + \frac{\partial f}{\partial x_n}(x)dx_n \quad \text{where} \quad x = \phi(p)$$

Vector Field:

A *vector field* on $\mathbb{R}^n$ is a smooth map which assign to each point $x \in \mathbb{R}^n$ a tangent vector $f(x) \in T_x\mathbb{R}^n$, where $T_x\mathbb{R}^n$ is the tangent space to $\mathbb{R}^n$ at a point $x \in \mathbb{R}^n$. In local coordinates, we represent f as a column vector whose elements depend on x,

$$f(x) \quad = \quad \begin{bmatrix} f_1(x) \\ . \\ . \\ . \\ f_n(x) \end{bmatrix}$$

A vector field is *smooth* if each $f_i(x)$ is *smooth.* Vector fields are to be thought of as right-hand sides of differential equations:

$$\dot{x} = f(x) \tag{A.1}$$

The *rate of change of a smooth* function $V : \mathbb{R}^n \to \mathbb{R}$ along the flow of f is given by

$$\dot{V} = \frac{\partial V}{\partial x} f(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x)$$

Covector fields :

The dual space of the tangent space $T_x\mathbb{R}^n$ is the set of linear functions on $T_x\mathbb{R}^n$ and it is denoted by $T_x^*\mathbb{R}^n$. The elements of $T_x^*\mathbb{R}^n$ are called *cotangent vectors.* A *covector field* or *one-form* on $\mathbb{R}^n$ is a smooth map which assign to each point $x \in \mathbb{R}^n$ a cotangent vector $\omega(x) \in T_x^*\mathbb{R}^n$. In local coordinates, we represent a smooth *one-form* ω as a row vector

$$\omega(x) = [\omega_1(x) \; \omega_2(x) \; ... \; \omega_n(x)]$$

where $\omega_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, ...n$ are smooth functions. Differentials of smooth functions are examples of *one-forms.* For example, if $\beta : \mathbb{R}^n \to \mathbb{R}$, then the *one-form* $d\beta$ is given by

$$d\beta = \left[\frac{\partial \beta}{\partial x_1} \; \frac{\partial \beta}{\partial x_2} \; \cdots \; \frac{\partial \beta}{\partial x_n}\right]$$

Note, however, the all *one-forms* are not necessarily the differentials of smooth functions (a *one-form* which does happen to be the derivative of a function is said to be exact).

Flow of a vector field:

If f is a vector field, we denote the parametrized maximal integral curve of the differential equation $\dot{x} = f(x)$, passing through $x \in I\!R^n$ at time zero, by $\phi_t^f(x)$, and call the mapping $(t, x) \mapsto \phi_t^f(x)$ the *flow* generated by f. Thus $\phi_t^f(x) : I\!R^n \to I\!R^m$ satisfies

$$\frac{d\phi_t^f(x)}{dt} = f(\phi_t^f(x)) \qquad x \in I\!R^n$$

Lie derivative :

The time derivative of V along the flow of f is called the *Lie derivative* of V along f and is denoted $L_f V$:

$$L_f V \stackrel{def}{=} \frac{\partial V}{\partial x} f(x)$$

Complete vector field:

A vector field is said to be *complete* if its *flow* is defined for all t.

Remark:

By the existence and uniqueness theorem of ordinary differential equations, for each fixed t, ϕ_t^f is a local diffeomorphism of $I\!R^{n+1}$ onto itself. Further, it satisfies the following group property:

$$\phi_t^f \circ \phi_s^f = \phi_{t+s}^f$$

for all t and s, where $\circ$ means the composition of the two flows, namely $\phi_t^f(\phi_s^f(x))$.
If there are two vector fields g_1 and g_2, the map $\phi_t^{g_1} \circ \phi_s^{g_2}$ stands for the composition of the flow of g_2 for s seconds with the flow of g_1 for t seconds. In general,

$$\phi_s^{g_2} \circ \phi_t^{g_1} \neq \phi_t^{g_1} \circ \phi_s^{g_2}$$

Motivation for the definition of a Lie bracket:

Consider the flow depicted in Figure A.1 starting from x_0. It consists of a flow along g_1 for ϵ seconds followed by a flow along g_1 for ϵ seconds, $-g_1$ for ϵ seconds, and $-g_2$ for ϵ seconds. For small ϵ, we can evaluate the Taylors series in ϵ for the value of the state of the differential equation A.1 as:

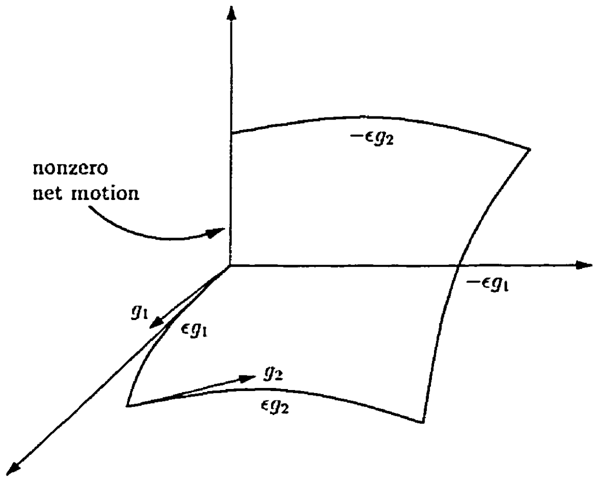

FIGURE A.1. A Lie bracket motion.

$$\begin{aligned}
x(\epsilon) &= \phi_\epsilon^{g_1}(x(0)) \\
&= x(0) + \epsilon\dot{x}(0) + \frac{1}{2}\epsilon^2\ddot{x}(0) + o(\epsilon^3) \\
&= x_0 + \epsilon g_1(x_0) + \frac{1}{2}\epsilon^2\frac{\partial g_1}{\partial x}g_1(x_0) + o(\epsilon^3)
\end{aligned}$$

where the partial derivative of g_1 is evaluated at x_0 and the notation $o(\epsilon^k)$ represents terms of order ϵ^k. Similarly

$$\begin{aligned}
x(2\epsilon) &= \phi_\epsilon^{g_2}\circ\phi_\epsilon^{g_1}(x(0)) \\
&= \phi_\epsilon^{g_2}[x_0 + \epsilon g_1(x_0) + \frac{1}{2}\epsilon^2\frac{\partial g_1}{\partial x}g_1(x_0) + o(\epsilon^3)] \\
&= x_0 + \epsilon g_1(x_0) + \frac{1}{2}\epsilon^2\frac{\partial g_1}{\partial x}g_1(x_0) + \epsilon g_2(x_0 + \epsilon g_1(x_0)) + \frac{\epsilon^2}{2}\frac{\partial g_2}{\partial x}g_2(x_0) + o(\epsilon^3) \\
&= x_0 + \epsilon[g_1(x_0) + g_2(x_0)] + \frac{1}{2}\epsilon^2[\frac{\partial g_1}{\partial x}g_1(x_0) + \frac{\partial g_2}{\partial x}g_2(x_0) + 2\frac{\partial g_2}{\partial x}g_1(x_0)] + o(\epsilon^3)
\end{aligned}$$

where we have used the Taylor series expansion for

$$g_2(x_0 + \epsilon g_1(x_0)) = g_2(x_0) + \epsilon\frac{\partial g_2}{\partial x}g_1(x_0) + o(\epsilon^2)$$

Further,

$$\begin{aligned}
x(3\epsilon) &= \phi_\epsilon^{-g_1}\circ\phi_\epsilon^{g_2}\circ\phi_\epsilon^{g_1}(x(0)) \\
&= x_0 + \epsilon g_2(x_0) + \frac{\epsilon^2}{2}[\frac{\partial g_2}{\partial x}g_2(x_0) + 2\frac{\partial g_2}{\partial x}g_1(x_0) - 2\frac{\partial g_1}{\partial x}g_2(x_0)] + o(\epsilon^3)
\end{aligned}$$

Finally, we get

$$\begin{aligned} x(4\epsilon) &= \phi_\epsilon^{-g_2} \circ \phi_\epsilon^{-g_1} \circ \phi_\epsilon^{g_2} \circ \phi_\epsilon^{g_1}(x(0)) \\ &= x_0 + \epsilon^2[\frac{\partial g_2}{\partial x} g_1(x_0) - \frac{\partial g_1}{\partial x} g_2(x_0)] + o(\epsilon^3) \end{aligned} \tag{A.2}$$

Motivated by the above calculation is the next definition.

Lie bracket:

A *Lie bracket* of two vector fields f and g is defined (in local coordinates) as:

$$[f,g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x)$$

If $[f,g] = 0$ then the right hand side of equation A.2 is identically equal to x_0 and f and g are said to *commute*.

Properties of Lie brackets:

Given vector fields f, g, h on $I\!R^n$ and smooth functions $\alpha, \beta : I\!R^n \to I\!R$, the Lie bracket satisfies the following properties:

(1) Skew-symmetry: $\quad [f,g] = -[g,f]$

(2) Jacobi identity: $\quad [f,[g,h]] + [h,[f,g]] + [g,[h,f]] = 0$

(3) Chain rule: $\quad [\alpha f, \beta g] = \alpha\beta[f,g] + \alpha(L_f\beta)g - \beta(L_g\alpha)f$

where $(L_f\beta)$ and $(L_g\alpha)$ stand for the Lie derivative of β and α along the vector fields f and g respectively.

(4) Jacobi identity: $\quad L_{[f,g]}\alpha = L_f(L_g\alpha) - L_g(L_f\alpha)$

Lie algebra:

A vector space V (over $I\!R$) is a *Lie algebra* if there exists a bilinear operator $V \times V \to V$, denoted $[.,.]$, satisfying

1. Skew-symmetry:

$$[v,w] = -[w,v] \qquad \text{for all } v, w \in V$$

2. Jacobi identity:

$$[[v,w],z] + [[z,v],w] + [[w,z],v] = 0 \qquad \text{for all } v, w, z \in V$$

The vector space of all smooth vector fields on a manifold M is an infinite-dimensional Lie algebra under Lie bracket operation on vector fields.

Lie sub-algebra:

A subspace $W \subset V$ of a Lie algebra V is called a *Lie sub-algebra* if $[v, w] \in W$ for all $v, w \in W$.

Lie group:

A *Lie group* is a group G which is also a smooth manifold and for which the group operations $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are smooth.

Examples of Lie groups:

1 The Euclidean space under addition.

2 The general linear group, $GL(n, \mathbb{R})$; set of all $n \times n$ nonsingular real matrices, which can be regarded as an open subset of $\mathbb{R}^{n^2}$.

3 The special orthogonal group, $SO(n)$;

$$SO(n) = \{R \in GL(n, \mathbb{R}) : RR^T = I, \quad detR = 1\}$$

The dimension of $SO(n)$ is $n(n-1)/2$. For $n = 3$, the group $SO(3)$ is also referred to as the *rotation group* on $\mathbb{R}^3$.

4 The special Euclidean group, $SE(3)$; the set of mappings $g : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $g(x) = Rx + p$, where $R \in SO(3)$ and $p \in \mathbb{R}^3$. $SE(3)$ can be identified with the space of 4×4 matrices of the form

$$g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

$SE(3)$ is a Lie group of dimension 6.

Distribution:

Let $\{g_1, ..., g_m\}$ be a set of vectors fields. Then for any fixed $x \in \mathbb{R}^n$, the vectors $g_1(x), ..., g_m(x)$ span a vector space called a *distribution*. The distribution at a point x is denoted by:

$$\Delta(x) = span\{g_1(x), ..., g_m(x)\}$$

If the spanning vector fields g_i's are *smooth* then the distribution is called a *smooth distribution*.

Dimension of the distribution:

The *dimension* of a distribution at a point $x \in \mathbb{R}^n$ is the dimension of the vector space $\Delta(x)$.

Regular distribution:

The distribution is said to be *regular* if the dimension of the vector space $\Delta(x)$ does not vary with x i.e. $dim(\Delta(x)) =$ constant, for all $x \in \mathbb{R}^n$.

Involutive distribution:

A distribution $\Delta(x) = span\{f_1(x), ..., f_m(x)\}$ is *involutive* if it is closed under the Lie bracket operation, i.e.,

$$\Delta \quad \text{involutive} \quad \Longleftrightarrow \forall\, f, g \in \Delta, \;\; [f, g] \in \Delta$$

Codistribution:

Let $\{\omega_1, ..., \omega_m\}$ be a set of covectors fields. Then for any fixed $x \in \mathbb{R}^n$, the *codistribution* is defined as: $\Omega(x) = span\{\omega_1(x), ..., \omega_k(x)\}$. If ω_i are smooth then the codistribution is called a *smooth codistribution.*

Annihilator:

The *annihilator of* $\Delta(x)$ is the set of all covectors which annihilates all vectors in $\Delta(x)$

$$\Delta^{\perp}(x) = \{\omega \in (\mathbb{R}^n)^* \; : < \omega, v >= 0 \quad \forall \quad v \in \Delta(x)\}$$

Similarly the *annihilator of* $\Omega(x)$ is defined as:

$$\Omega^{\perp}(x) = \{v \in \mathbb{R}^n \; : < \omega, v >= 0 \quad \forall \quad \omega \in \Delta(x)\}$$

Integrable distribution:

A distribution Δ of constant dimension k is said to be *integrable* if for every point $x \in \mathbb{R}^n$, there exists a set of smooth functions $h_i : \mathbb{R}^n \to \mathbb{R}, \;\; i = 1, ..., n-k$ such that the row vectors $\frac{\partial h_i}{\partial x}$ are linearly independent at x, and for every $f \in \Delta(x)$

$$L_f h_i(x) \stackrel{def}{=} \frac{\partial h_i}{\partial x} f(x) = 0, \quad i = 1, ..., n-k$$

Document Log:

Manuscript Version 1 — March 12, 1994

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-LaTeX — 30 September 1997

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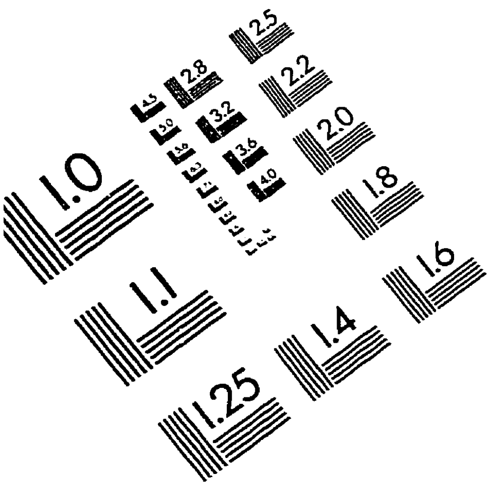

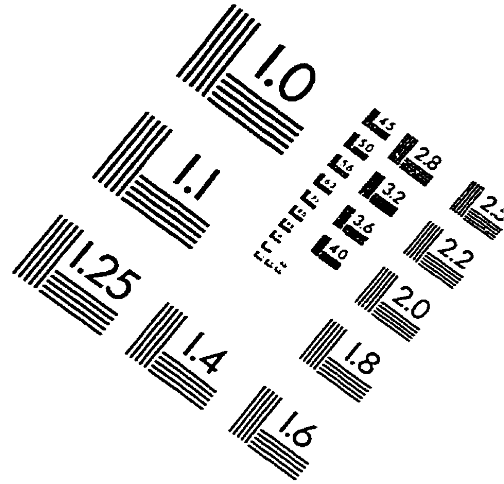

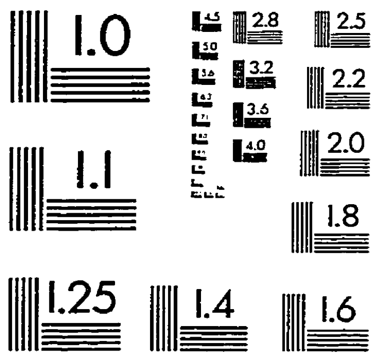

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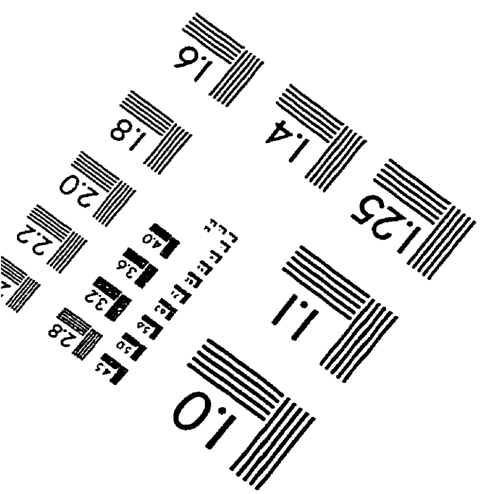

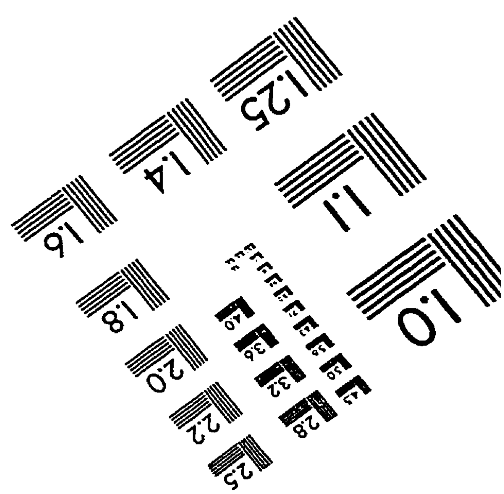

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