Mean Field Games and Optimal Execution Problems: Hybrid and Partially Observed Major Minor Systems

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Abstract

In this thesis major minor LQG mean field game (MM LQG MFG) theory is extended in three main directions which are motivated by algorithmic trading (more specifically optimal execution) problems in finance. In financial applications in this thesis, following standard financial models, the market is studied as a large population non-cooperative game where each trader has stochastic linear dynamics with quadratic costs. We consider the case where there exists one institutional investor (interpreted as an MFG major agent) with a large number of high frequency traders (interpreted as MFG minor agents) constituting two subpopulations of liquidators and acquirers. In general, the traders are coupled in their dynamics and cost functions by the market's average trading rate (a component of the system's mean field). In each case, the existence of an ϵ-Nash equilibrium. together with the individual agents' trading strategies which yield the equilibria, are established.

In the first part of the thesis, partially observed (PO) MM LQG MFG problems with general information patterns are investigated where (i) the major agent has partial observations of its own state, and (ii) each minor agent has partial observations of its own state and the major agent's state. The assumption of partial observations by all agents leads to a new situation involving the recursive estimation by each minor agent of the major agent's estimate of its own state. For the general case of indefinite LQG MFG systems, the existence of ϵ-Nash equilibria together with the individual agents' control actions yielding the equilibria are established via the Separation Principle. Numerical experiments are presented. The PO MM LQG MFG theory is then applied to an optimal execution problem where the major trader has partial observations of its own state (which includes its inventory), and each one of minor traders has partial observations of its own state and the major trader's state (which include the corresponding inventories). A simulation example is provided.

The second part of the thesis presents a novel framework that combines LQG MFG theory and hybrid optimal control theory to obtain a unique ϵ-Nash equilibrium for a non-cooperative game with stopping and switching times. We consider the case where there exists one major agent together with a large number of minor agents constituting two subpopulations. Each agent has stochastic linear dynamics with quadratic costs, and the agents are coupled in their dynamics by the average state of minor agents. The hybrid feature enters via the indexing by discrete states: (i) the switching of the major agent between alternative dynamics, or (ii) the termination of the agents' trajectories in one or both of the subpopulations of minor agents. Optimal switchings and

stopping time strategies together with best response control actions for, respectively, the major agent and all minor agents are established with respect to their individual cost criteria by an application of LQG Hybrid MFG theory. Then LQG Hybrid MFG theory is applied to optimal execution problems where minor agents are provided with the option to quit the market if it is optimal for them to do so. Hence, the hybrid feature enters via the indexing of the cessation of trading by one or both subpopulations of minor traders by discrete states.

In the third part of the thesis, first, a convex analysis method is used to rederive the solutions to LQG optimal control problems. Then the methodology is applied to MM LQG MFG systems to retrieve the best response strategies for the major agent and each individual minor agent which collectively yield an ϵ-Nash equilibrium for the entire system. Subsequently a class of (non-cooperative) stochastic games with major and minor agents is investigated where agents interact with a completely observed common process. However, the common process is modulated by a latent Markov chain and a latent Wiener process (common noise) which are not observable to agents. Consequently the Wonham filter is used to generate the posteriori estimates of the latent processes based on the realized trajectories of the common process. Then, the convex analysis is further developed to (i) solve the MFG limit of the problem, (ii) demonstrate that the best response strategies generate an ϵ-Nash equilibrium for the finite player game, and (iii) obtain explicit characterisations of the best response strategies.

Résumé

Dans cette thèse, la théorie du jeux majeur mineur à champ moyen linéaire-quadratique-Gaussienne (MM LQG MFG) est étendue dans trois directions principales qui sont motivées par des problèmes de commerce algorithmique (plus précisément exécution optimale) en finance. Dans les applications financières de cette thèse, qui suivent les modèles financiers classiques, les marchés financiers sont étudiés comme un jeu non coopératif de grande population dans lequel chaque commerçant a une dynamique linéaire stochastique avec des coûts quadratiques. Nous considérons le cas où il existe un seul investisseur institutionnel (interprété comme un agent majeur de MFG) avec un grand nombre de commerçants à haute fréquence (interprété comme des agents mineurs de MFG) constituant deux sous-populations de liquidateurs et d'acquéreurs. En général, les commerçants sont couplés dans leur dynamique et leurs fonctions de coût au taux moyen de commerce du marché (une composante du champ moyen du système). Dans chaque cas, l'existence d'un équilibre de ϵ-Nash ainsi que les stratégies commerciales des agents individuels qui donnent les équilibres sont établies.

Dans la première partie de la thèse, les problèmes partiellement observés (PO) MM LQG MFG avec informations générales sont examinés lorsque (i) l'agent majeur a des observations partielles de son propre état, et (ii) chaque agent mineur a des observations partielles de propre état et l'état de l'agent majeur. L'hypothèse d'observations partielles par tous les agents crée une nouvelle situation impliquant l'estimation récursive par chaque agent mineur de l'estimation par l'agent majeur de son propre état. Dans le cas général des systèmes LQG indéterminée MFG, l'existence d'équilibre de ϵ-Nash et des actions de contrôle des agents individuels générant l'équilibre sont établis via le principe de séparation. Des expériences numériques sont présentées. La théorie de PO MM LQG MFG est ensuite appliquée à un problème d'exécution optimale où le commerçant majeur a des observations partielles de son propre état (ce qui inclut son inventaire), et chaque commerçant mineur dispose d'observations partielles de son propre état et de l'état du commerçant majeur (qui incluent les inventaires correspondants). Un exemple de simulation est fourni.

La deuxième partie de la thèse présente un nouveau cadre combinant la théorie LQG MFG et la théorie du contrôle optimal hybride pour obtenir un équilibre unique de ϵ-Nash pour un jeu non coopératif avec des temps d'arrêt et de commutation. Nous considérons le cas où il existe un agent majeur avec un grand nombre d'agents mineurs constituant deux sous-populations. Chaque agent a une dynamique linéaire stochastique avec des coûts quadratiques, et la dynamique des agents

est couplée à l'état moyen des agents mineurs. La caractéristique hybride entre via l'indexation par états discrets: (i) le commutation de l'agent majeur entre des dynamiques alternatives, ou (ii) la fin des trajectoires des agents dans l'une ou les deux sous-populations d'agents mineurs. Des stratégies optimales de temps de commutation et de temps d'arrêt ainsi que les actions de contrôle de meilleures réponse pour, respectivement, l'agent majeur et tous les agents mineurs sont établies en fonction de leurs critères de coût individuels par l'application de la théorie LQG Hybrid MFG. Ensuite, la théorie LQG Hybrid MFG est appliquée aux problèmes d'exécution optimale lorsque des agents mineurs ont l'option de quitter le marché si cela leur convient le mieux. Par conséquent, la caractéristique hybride entre via l'indexation de la cessation de commerce par une ou les deux sous-populations de commerçants mineurs par des états discrets.

Dans la troisième partie de la thèse, d'abord, une méthode d'analyse convexe est utilisée pour redériver les solutions aux problèmes de contrôle optimal LQG. La méthodologie est ensuite appliquée aux systèmes MM LQG MFG pour extraire les stratégies de meilleure réponse pour l'agent majeur et chaque agent mineur individuel qui produisent collectivement un équilibre de ϵ-Nash pour l'ensemble du système. Ensuite, une classe de jeux stochastiques (non coopératifs) avec des agents majeurs et mineurs est examinée lorsque les agents interagissent avec un processus commun complètement observé. Cependant, le processus commun est modulé par une chaîne de Markov latente et un processus de Wiener latent (bruit commun) qui ne sont pas observables par les agents. Par conséquent, le filtre de Wonham est utilisé pour générer les estimations à posteriori des processus latents basées sur les trajectoires réalisées du processus commun. Ensuite, l'analyse convexe est développée plus avant pour (i) résoudre la limite MFG du problème, (ii) démontrer que les stratégies de meilleure réponse génèrent un équilibre de ϵ-Nash pour le jeu à joueur fini, et (iii) obtenir des caractérisations explicites des stratégies de meilleures réponse.

Claims of Originality and Published Work

Claims of Originality

The following original contributions are presented in this thesis:

Part I

- Partially observed major minor LQG mean field game (PO MM LQG MFG) problems with the following general information patterns are studied where (i) the major agent has partial observations of its own state, and (ii) each minor agent has partial observations of its own state and the major agent's state.
- In the theory the new and general case where (i) the major agent recursively estimates its own state, and (ii) each minor agent recursively estimates its own state, and the major agent's estimate of its own state (in order to estimate the major agent's feedback control input), is presented. In addition, both the major agent and minor agents generate estimates of the system's mean field.
- MFG theory is extended to cover the general case of indefinite LQG MFG systems which alleviates the positive definiteness condition of weight matrices in linear quadratic cost functionals .
- The existence of ϵ-Nash equilibria together with the individual agents' control laws yielding the equilibria is established; this is achieved in the PO MM LQG case by an application of the Separation Principle which also yields computationally tractable solutions which in nonlinear case is far more complex.
- Completely observed major minor LQG mean field game (CO MM LQG MFG) framework is utilized to formulate optimal execution problems in financial markets with the standard linear financial models where there exist one institutional investor, interpreted as major agent, and a large population of high frequency traders (HFTs), interpreted as minor agents, who attempt to maximize their own wealth. Nash equilibrium and ϵ-Nash equilibrium best response trading strategies for all participating traders in the market are obtained.
- PO MM LQG MFG theory is applied to optimal execution problems where an institutional investor aims to liquidate a specific amount of shares and it has only partial observations

of its own state (which includes its inventory). Furthermore, there exists a large population of HFTs who wish to liquidate or acquire shares, and each of them has partial observations of its own state and the major agent's state (which include the corresponding inventories). The existence of an ϵ-Nash equilibrium together with the best response trading strategies are established.

Part II

- A hybrid systems MFG (Hybrid MFG) framework is developed for a general class of LQG mean field game systems with a major agent permitted to switch between different dynamics and subpopulations of minor agents provided with the option to stop at some optimal time. Optimal switching time and stopping time strategies together with best response control actions for, respectively, the major agent and all minor agents are established with respect to their individual cost criteria.
- Conditions under which the stopping and switching times for LQG systems are trajectory independent are derived.
- Hybrid MFG theory is employed in a non-cooperative game formulation of the financial market where HFTs (minor agents) may leave the market before the final time. The best response trading policies for the agents are further shown to yield an ϵ-Nash equilibrium for the the market.

Part III

- A convex analysis method is used to rederive the solutions to LQG optimal control problems. Then the methodology is applied to major minor LQG mean field game (MM LQG MFG) systems to retrieve the best response strategies for the major agent and each individual minor agent
- MM LQG MFG theorem is extended to incorporate the impact of a common process which modulated by a latent Markov chain process and a latent Wiener process (common noise).
- The Wonham filter is used to generate the posteriori estimates of latent processes using the complete observations on the common process.

- Common process (an extended form of common noise) is modeled as a passive major agent in the MM MFG framework.
- Convex analysis method is further developed to obtain the best response strategies which yield an ϵ-Nash equilibrium for the MM LQG MFG systems including common noise and Markovian latent processes.

Publications

Published and Submitted Journal Papers:

[J1] D. Firoozi, P. E. Caines, "ϵ-Nash Equilibria for Partially Observed LQG Mean Field Games with Major Player", *under review for publication in the IEEE Transactions on Automatic Control*, 2017, [available on arXiv]: https://arxiv.org/abs/1810.04369.

[J2] D. Firoozi, P. E. Caines, "The Execution Problem in Finance with Major and Minor Traders: A Mean Field Game Formulation", *Annals of International Symposium on Dynamic Games and Applications: Advances in Dynamic and Mean Field Games*, Volume 15, 2017, pp. 107-130.

Conference Papers:

[C1] D. Firoozi, P. E. Caines, "Mean Field Game Systems with Common and Latent Processes", *(To appear) Proceedings of the 57th IEEE Conference on Decision and Control (CDC 2018)*, Miami, USA, December 2018.

[C2] D. Firoozi, P. E. Caines, "An Optimal Execution Problem in Finance Targeting the Market Trading Speed: an MFG Formulation", *Proceedings of the 56th IEEE Conference on Decision and Control (CDC 2017)*, Melbourne, Australia, December 2017, pp. 7–14.

[C3] D. Firoozi, A. Pakniyat, P. E. Caines, "A Mean Field Game - Hybrid Systems Approach to Optimal Execution Problems in Finance with Stopping Times", *Proceedings of the 56th IEEE Conference on Decision and Control (CDC 2017)*, Melbourne, Australia, December 2017, pp. 3144–3151.

[C4] D. Firoozi, P. E. Caines, "An Optimal Execution Problem in Finance with Acquisition and Liquidation Objectives: an MFG Formulation", *Proceedings of 20th World Congress of the*

International Federation of Automatic Control (IFAC 2017), , Toulouse, France, July 2017, pp 4960-4967.

[C5] D. Firoozi, P. E. Caines, "Mean Field Game ϵ-Nash Equilibria for Partially Observed Optimal Execution Problems in Finance", *Proceedings of 55th IEEE Conference on Decision and Control (CDC 2016)*,Las Vegas, USA, December 2016, pp. 268–275.

[C6] D. Firoozi, P. E. Caines, "ϵ-Nash Equilibria for Partially Observed LQG Mean Field Games with Major Player: Partial Observation by All Agents", *the Proceedings of 54th IEEE Conference on Decision and Control (CDC 2015)*, Osaka, Japan, December 2015, pp. 4430–4437.

Preprints:

[P1] D. Firoozi, A. Pakniyat, P. E. Caines, "A Hybrid Optimal Control Approach to LQG Mean Field Games with Switching and Stopping Strategies", *To be submitted to the SIAM Journal on Control and Optimization*, [available on arXiv]: https://arxiv.org/abs/1810.02920.

[P2] D. Firoozi, P. E. Caines, Sebastian Jaimungal, "Convex Analysis for LQG Systems with Applications to Major Minor LQG Mean Field Game Systems", *To be submitted to the SIAM Journal on Optimization*, [available on arXiv]: https://arxiv.org/abs/1810.07551.

[P3] D. Firoozi, Sebastian Jaimungal, P. E. Caines, "Mean Field Game Systems including Common Noise and Markovian Latent Processes", *to be submitted to the IEEE Transactions on Automatic Control*, [available on arXiv]: https://arxiv.org/abs/1809.07865.

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- Professor Caines amounted to 30% in Chapters 2, 3, 15% in Chapters 4, 5, 7, and 10% in Chapter 6.
- Professor Jaimungal amounted to 10% in Chapter 6 and 15% in Chapter 7.
- Dr. Pakniyat amounted to 15% in Chapters 4 and 5.

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Dena Firoozi, Fall 2018

[1]Groupe d'Études et de Recherche en Analyse des Décisions (Group for Study and Research in Decision Analysis)

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List of Acronyms

MFG	Mean Field Game
MM	Major Minor
PO	Partially Observed
LQG	Linear Quadratic Gaussian
HOC	Hybrid Optimal Control
CO	Completely Observed
PO MM LQG MFG	Partially Observed Major Minor LQG Mean Field Game
CO MM LQG MFG	Completely Observed Major Minor LQG Mean Field Game
OEP	Optimal Execution Problems
HFT	High Frequency Trader

Chapter 1

Introduction

Large population dynamical multi-agent noncooperative and cooperative phenomena occur in a wide range of designed and natural settings such as communications, environmental, epidemiological, transportation and energy systems, and they underlie much economic and financial behaviour. Analysis of such systems with even a moderate number of agents is regarded as being extremely difficult using the finite population game theoretic methods which were developed over several decades for multi-agent control systems (see e.g. [1–4]) [5].

Subsequently, what is now called MFG theory originated in the equations for dynamical games with (i) large finite populations of asymptotically negligible agents together with (ii) their infinite limits, in the work of Caines, Huang and Malhamé ([6–9]), where the framework was called the Nash Certainty Equivalence Principle, and independently in that of Lasry and Lions ([10–12]), where the now standard terminology of Mean Field Games (MFG) was introduced. The closely related notion of Oblivious Equilibria for large population dynamic games was also independently introduced by Weintraub, Benkard, and Van Roy ([13, 14]) within the framework of discrete time Markov Decision Processes (MDP) [5].

Mean Field Game (MFG) theory studies the existence of Nash equilibria, together with the individual strategies which generate them, in games involving a large number of asymptotically negligible agents modelled by controlled stochastic dynamical systems. This is achieved by exploiting the relationship between the finite and corresponding infinite limit population problems. The solution to the infinite population problem is given by (i) the Hamilton-Jacobi-Bellman (HJB) equation of optimal control for a generic agent and (ii) the Fokker-Planck-Kolmogorov (FPK) equation for that agent, where these equations are linked by the distribution of the state of the generic agent, otherwise known as the system's mean field. Moreover, (i) and (ii) have an equivalent expression in terms of the Stochastic Maximum Principle together with a McKean-Vlasov stochastic differential equation, and yet a third characterisation is in terms of the so-called Master Equation. An important feature of MFG solutions is that they have fixed point properties regarding the individual responses to and the formation of the mean field which conceptually correspond to equilibrium solutions of the associated games (see e.g. [5, 8, 15]).

The theory and methodology of MFG systems has rapidly developed since its inception and is still advancing. In [16, 17] the authors analyse and solve the linear quadratic systems case where there is a major agent (i.e. non-asymptotically vanishing as the population size goes to infinity) together with a population of minor agents (i.e. individually asymptotically negligible). The new feature in this case is that the mean field becomes stochastic but by minor agent state extension the existence of ϵ-Nash equilibria is established together with the individual agents' control laws that yield the equilibria [17]. In the purely minor agent case the mean field is deterministic and this obviates the need for observations on other agents' states. This is a separate issue from that of an agent estimating its own state (self state for short) from partial observations on that state, see [18]. However, when a systems has a major agent whose state is partially observed the standard MFG procedure for generating a Nash equilibrium needs to be extended to include estimates of the major agent's state generated by each minor agent.

In [19–21], partially observed LQG mean field games with major and minor agents (PO MM LQG MFG) have been investigated and in [22–24], a nonlinear generalization of this problem is considered. The main results in those papers are obtained with the assumptions that (i) the major agent's state is partially observed by the minor agents and (ii) the major agent has complete observations of its own state.

Single-agent optimal execution problems have been addressed in the literature (see e.g. [25–28]) where an agent must liquidate or acquire a certain amount of shares over a pre-specified time horizon at a trading speed to balance the price impact (from trading quickly) and the price uncertainty (from trading slowly), while it maximizes its final wealth. Further, in [29] the partially observed setting where the market liquidity variable is not observed was studied. This problem with the linear models in [25] was formulated as for the nonlinear major minor (MM) MFG model in [30].

The primary goal of this thesis is to develop and extend the theory of MM LQG MFG systems in three main directions which are motivated by algorithmic trading (more specifically optimal execution) problems in finance. A brief description of each chapter follows.

Chapter 2 formulates PO MM LQG MFG problems with general information patterns where (i) the major agent has partial observations of its own state, and (ii) each minor agent has partial observations of its own state and the major agent's state. The assumption of partial observations by all agents leads to a new situation involving the recursive estimation by each minor agent of the major agent's estimate of its own state. For the general case of indefinite LQG MFG systems, the existence of ϵ-Nash equilibria together with the individual agents' control laws yielding the equilibria are established via the Separation Principle. Numerical experiments are presented.

Chapter 3 applies the theory of partially observed mean field games to an optimal execution problem in finance. Following standard financial models, controlled linear system dynamics are postulated where an institutional investor (interpreted as a major agent) in the market aims to liquidate a specific amount of shares and has partial observations of its own state (which includes its inventory). Furthermore, the market is assumed to have two populations of high frequency traders (interpreted as minor agents) who wish to liquidate or acquire a certain number of shares within a specific time, and each one of them has partial observations of its own state and the major agent's state (which include the corresponding inventories). The objective for each agent is to maximize its own wealth and to avoid the occurrence of large execution prices, large rates of trading and large trading accelerations which are appropriately weighted in the agent's performance function. The existence of ϵ-Nash equilibria together with the individual agents' trading strategies yielding the equilibria, are established. A simulation example is provided.

Chapter 4 presents a novel framework that combines MFG theory and hybrid optimal control theory to obtain a unique ϵ-Nash equilibrium for a non-cooperative game with stopping times. We consider the case where there exists one major agent with a significant influence on the system together with a large number of minor agents constituting two subpopulations, each with individually asymptotically negligible effect on the whole system. Each agent has stochastic linear dynamics with quadratic costs, and the agents are coupled in their dynamics by the average state of minor agents (i.e. the empirical mean field). The hybrid feature enters via the indexing by discrete states: (i) the switching of the major agent between alternative dynamics or (ii) the termination of the agents' trajectories in one or both of the subpopulations of minor agents. Optimal switchings and stopping time strategies together with best response control actions for, respectively, the major agent and all minor agents are established with respect to their individual cost criteria.

Chapter 5 employs LQG Hybrid MFG theory to obtain a unique ϵ-Nash equilibrium for optimal execution problems within the stock market. Following standard financial models, the stock market is studied in this paper as a large population non-cooperative game where each trader has stochastic linear dynamics with quadratic costs. We consider the case where there exists one major trader with a large number of minor traders (in two subpopulations). The traders are coupled in their dynamics and cost functions by the market's average trading rate (a component of the system mean field) and the hybrid feature enters via the indexing of the cessation of trading by one or both subpopulations of minor traders by discrete states. Optimal stopping time strategies together with best response trading policies for all traders are established with respect to their individual cost criteria.

Chapter 6 develops a convex analysis method to rederive the solutions to LQG optimal control problems. Then the methodology is applied to MM LQG MFG systems addressed in [16] to retrieve the best response strategies for the major agent and each individual minor agent which collectively yield an ϵ-Nash equilibrium for the entire system.

Chapter 7 investigates a class of non-cooperative stochastic games with major and minor agents where agents interact with a completely observed common process. However, the common process is modulated by a latent Markov chain and a latent Wiener process (common noise) which are not observable to agents. Consequently the Wonham filter is used to generate the posteriori estimates of the latent processes based on the realized trajectories of the common process. Then, the convex analysis is further developed to (i) solve the MFG limit of the problem, (ii) demonstrate that the best response strategies generate an ϵ-Nash equilibrium for the finite player game, and (iii) obtain explicit characterisations of the best response strategies.

Chapter 8 presents future research directions.

Part I

Major Minor LQG Mean Field Game Systems with Partial Observations

Chapter 2

Partially Observed Major Minor LQG Mean Field Game Systems

2.1 Introduction

In [19–21], partially observed LQG mean field games with major and minor agents (PO MM LQG MFG) have been investigated and in [22–24], a nonlinear generalization of this problem is considered. The main results in those papers are obtained with the assumptions that (i) the major agent's state is partially observed by the minor agents and (ii) the major agent has complete observations of its own state. In this chapter, PO MM LQG MFG problems with general information patterns are studied where (i) the major agent has partial observations of its own state, and (ii) each minor agent has partial observations of its own state and the major agent's state. In the theory we present for this new, general case where (i) the major agent recursively estimates its own state, and (ii) each minor agent recursively estimates its own state, and the major agent's estimate of its own state (in order to estimate the major agent's feedback control input). In addition, both the major agent and minor agents generate estimates of the system's mean field. We remark that an infinite regress does not happen here due to the asymmetric major minor (MM) feature of the MFG problem. Moreover, MFG theory is extended to cover the general case of indefinite LQG MFG systems which alleviates positive definiteness condition of weight matrices in linear quadratic cost functionals. The existence of ϵ-Nash equilibria together with the individual agents' control laws yielding the equilibria is then established; this is achieved in the PO MM LQG case by an application of the Separation Principle which also yields computationally tractable solutions which in nonlinear case is far more complex (see [22–24]). The initial results of this work has been published in [31].

This extension of the situation in [21], where only assumption (ii) holds, is in particular motivated by optimal execution problems in financial markets where there exist one institutional trader (interpreted as major agent) and a large population of high frequency traders (interpreted as minor agents) who attempt to maximize their own wealth. To obtain the Nash equilibrium best response trading strategy, each minor trader estimates the major agent's inventory and trading rate based on its partial observations of market state which entails the estimation of the major trader's self estimates. The reader is referred to Chapter 3 and the works [32–35] for more details on financial applications.

The rest of the chapter is organized as follows. Section 2.2 introduces Partially Observed Major-Minor LQG MFG systems. The estimation and control problems for PO MM LQG MFG systems are addressed in Section 2.3. The simulation results and the concluding remarks are presented in Section 2.4 and Section 7.4, respectively.

2.2 Partially Observed Major-Minor LQG MFG Systems

A class of major-minor LQG MFG (MM LQG MFG) systems including a large population of N stochastic dynamic minor agents with a stochastic dynamic major agent is considered where the agents are coupled through their cost functionals.

2.2.1 Dynamics

The dynamics of the major and minor agents in the class of systems under consideration are, respectively, given by

$$dx_0 = [A_0 x_0 + B_0 u_0]dt + D_0 dw_0, \tag{2.1}$$

$$dx_i = [A(\theta_i)x_i + B(\theta_i)u_i + Gx_0]dt + Ddw_i, \tag{2.2}$$

where $t \geq 0$, $1 \leq i \leq N < \infty$, $\theta_i \in \Theta$, where Θ is a parameter set. Here $x_i \in \mathbb{R}^n,\ 0 \leq i \leq N$, are the states, $u_i \in \mathbb{R}^m,\ 0 \leq i \leq N$, are control inputs, $\{w_i,\ 0 \leq i \leq N\}$ denote $(N+1)$ independent standard Wiener processes in $\mathbb{R}^r$ on an underlying probability space $(\Omega, \mathcal{F}, P)$ which is sufficiently large that w is progressively measurable with respect to the filtration $\mathcal{F}^w \triangleq (\mathcal{F}^w_t;\ t \geq 0)$ on $\mathcal{F}$, and $\mathbb{E}w_i w_i^T = \Sigma$.

Assumption 2.1. *The initial states* $\{x_i(0),\ 0 \le i \le N\}$ *defined on* $(\Omega, \mathcal{F}, P)$ *are identically distributed, mutually independent and also independent of* $\mathcal{F}^w_\infty$*, with* $\mathbb{E}x_i(0) = 0$*. Moreover,* $sup_i \mathbb{E}\|x_i(0)\|^2 \le c < \infty$*,* $0 \le i \le N < \infty$*, with c independent of* N*.*

The matrices $A_0, B_0, D_0, G,$ and D are constant matrices of appropriate dimensions. From (2.2), $A(.)$ and $B(.)$ depend on the parameter θ which specifies the minor agent's type. Minor agents are given in K distinct types with $1 \le K < \infty$. The notation $\mathcal{I}_k$ is defined as

$$\mathcal{I}_k = \{i : \theta_i = k,\ 1 \le i \le N\}, \quad 1 \le k \le K,$$

where the cardinality of $\mathcal{I}_k$ is denoted by $N_k = |\mathcal{I}_k|$. Then, $\pi^N = (\pi^N_1, ..., \pi^N_K),\ \pi^N_k = \frac{N_k}{N},\ 1 \le k \le K$, denotes the empirical distribution of the parameters $(\theta_1, ..., \theta_N)$ sampled independently of the initial conditions and Wiener processes of the agents $A_i, 1 \le i \le N$. The first assumption is as follows.

Assumption 2.2. *There exists* π *such that* $lim_{N\to\infty}\pi^N = \pi$ *a.s.*

We note that except for clarity the time argument for the stochastic and deterministic processes throughout the paper may be dropped for the purpose of notation abbreviation as in (2.1)-(2.2).

2.2.2 Cost Functionals

The individual (finite) large population infinite horizon cost functional for the major agent $\mathcal{A}_0$ is specified by

$$\begin{aligned} J^N_0(u_0, u_{-0}) &= \mathbb{E}\int_0^\infty e^{-\rho t}\Big\{\|x_0 - \Phi(x^{(N)})\|^2_{Q_0} + \|u_0\|^2_{R_0}\Big\}dt, \\ \Phi(.) &:= H_0 x^{(N)} + \eta_0, \end{aligned} \tag{2.3}$$

where $R_0 > 0$, and the individual (finite) large population infinite horizon cost functional for a minor agent $\mathcal{A}_i, 1 \le i \le N$, is given by

$$\begin{aligned} J^N_i(u_i, u_{-i}) =& \mathbb{E}\int_0^\infty e^{-\rho t}\Big\{\|x_i - \Psi(x^{(N)})\|^2_{Q} + \|u_i\|^2_{R}\Big\}dt, \\ \Psi(.) &:= H_1 x_0 + H_2 x^{(N)} + \eta, \end{aligned} \tag{2.4}$$

where $R > 0$. We note that the major agent $\mathcal{A}_0$ and minor agents $\mathcal{A}_i$, $1 \leq i \leq N$, are coupled with each other through the average term $x^{(N)} = \frac{1}{N}\sum_{i=1}^{N} x_i$ in their cost functionals given by (6.48)-(2.4).

2.2.3 Observation Processes

The major agent's partial observations y_0 is given by

$$dy_0 = L_0[x_0^T,\, (x^{(N)})^T]^T dt + R_{v_0}^{\frac{1}{2}} dv_0, \tag{2.5}$$

where v_0 is a standard Wiener process in $\mathbb{R}^\ell$ with $\mathbb{E}[v_0 v_0^T] = R_{v_0}$ and matrix L_0 is given by

$$L_0 = \begin{bmatrix} l_0^1 & 0_{\ell\times n} \end{bmatrix}, \tag{2.6}$$

with $l_0^1 \in \mathbb{R}^{\ell\times n}$. The partial observations for a minor agent $\mathcal{A}_i$, $1 \leq i \leq N$, of type k, $1 \leq k \leq K$, is given by

$$dy_i = L_k[x_i^T,\, x_0^T,\, (x^{(N)})^T]^T dt + R_v^{\frac{1}{2}} dv_i, \tag{2.7}$$

where $\{v_i,\, 1 \leq i \leq N\}$ denotes the set of N independent standard Wiener processes in $\mathbb{R}^\ell$ with $\mathbb{E}[v_i v_i^T] = R_v$, and matrix L_k is given by

$$L_k = \begin{bmatrix} l_k^1 & l_k^2 & 0_{\ell\times n} \end{bmatrix}, \tag{2.8}$$

where $l_k^1,\, l_k^2 \in \mathbb{R}^{\ell\times n}$.

Control σ-Fields

The family of partial observation information sets $\mathcal{F}_0^y$ is defined to be the increasing family of σ-fields of partial observations $\{\mathcal{F}_{0,t}^y, t \geq 0\}$ generated by the major agent $\mathcal{A}_0$'s partial observations $(y_0(\tau), 0 \leq \tau \leq t)$ on its own state as given in (3.39). The set of control inputs $\mathcal{U}_y^{N,L}$ is defined to be the collection of linear feedback control laws adapted to $\mathcal{F}_t^{N,y} = \{\bigvee_{i=0}^{N} \mathcal{F}_i^y\}$.

Assumption 2.3. *Major Agent σ-Fields and Linear Controls: For the major agent $\mathcal{A}_0$ the set of control inputs $\mathcal{U}_{0,y}^L$ is defined to be the collection of linear feedback control laws adapted to the increasing σ-fields of partial observations $\{\mathcal{F}_{0,t}^y, t \geq 0\}$.*

The family of partial observation information sets $\mathcal{F}_i^y, 1 \leq i \leq N$, is defined to be the increasing σ-fields $\{\mathcal{F}_{i,t}^y, t \geq 0\}$ generated by the minor agent $\mathcal{A}_i$'s partial observations $(y_i(\tau), 0 \leq \tau \leq t)$, on its own state and the major agent's state, as given in (3.41).

Assumption 2.4. *Minor Agent σ-Fields and Linear Controls: For each minor agent $\mathcal{A}_i, 1 \leq i \leq N$, the set of control inputs $\mathcal{U}_{i,y}^L$ is defined to be the collection of linear feedback control laws adapted to the increasing σ-fields of partial observations $\{\mathcal{F}_{i,t}^y, t \geq 0\}$.*

2.3 Estimation and Control Solutions for PO MM LQG MFG Systems

In this section we present the solution to partially observed (PO) MM LQG MFG problems where it is assumed that the major agent partially observes its own state, and each generic minor agent partially observes its own state and the major agent's state. The problem is first solved in the infinite population case which is far simpler to solve than the finite large population problem. Because the agents in the infinite population case are decoupled and therefore the problem reduces to the type of indefinite LQG tracking problem whose solution is given in *Theorem 2.1*. Subsequently, the ϵ-Nash equilibrium property is established in *Theorem 2.2* for the system when the infinite population control laws are applied to the finite large population PO MM LQG MFG system.

The following theorem is a restriction to the constant matrix parameter case of the general result in [36].

Theorem 2.1 (Stochastic Indefinite LQ Problem [36])**.** *Let $\breve{T} > 0$ be given. For any $(\breve{s}, \breve{y}) \in [0, \breve{T}) \times \mathbb{R}^n$, consider the following linear system*

$$d\breve{x} = \left[\breve{A}\breve{x} + \breve{B}\breve{u} + \breve{b}\right]dt + \left[\breve{C}\breve{x} + \breve{D}\breve{u} + \breve{\sigma}\right]d\breve{w}, \tag{2.9}$$

where $t \in [\breve{s}, \breve{T}]$, $\breve{x}(\breve{s}) = \breve{y}$ and $\breve{A}$, $\breve{B}$, $\breve{C}$, $\breve{D}$, $\breve{b}$, $\breve{\sigma}$ are matrix valued functions of suitable sizes, $\breve{w}(.) \in \mathbb{R}^r$ is a standard Wiener process. Moreover, $\mathcal{F}_t = \sigma\{\breve{w}(\tau), 0 \leq \tau \leq t\}$, and $\breve{u}(.) \in \mathcal{U}$, where $\mathcal{U}$ is the set of all $\mathcal{F}_t$-adapted $\mathbb{R}^m$-valued processes such that $\mathbb{E}\int_0^T \|u(t)\|^2 dt < \infty$.

A quadratic cost functional is given by

$$J(\breve{s},\breve{y},\breve{u}(.))=\mathbb{E}\Big\{\frac{1}{2}\int_0^{\breve{T}}\big[\langle\breve{P}\breve{x}(t),\breve{x}(t)\rangle+\langle\breve{N}\breve{x}(t),\breve{u}(t)\rangle \\ +\langle\breve{R}\breve{u}(t),\breve{u}(t)\rangle\big]dt+\frac{1}{2}\langle\breve{\bar{P}}\breve{x}(\breve{T}),\breve{x}(\breve{T})\rangle\Big\}, \quad (2.10)$$

with $\breve{P}$, $\breve{N}$ and $\breve{R}$ being $\mathcal{S}^n$, $\mathbb{R}^{m\times n}$ and $\mathcal{S}^m$-valued functions, respectively, and $\breve{G}\in\mathcal{S}^{n\times n}$, where $\mathcal{S}^n$ denotes symmetric matrix space of size n.

We also denote the set of all $\mathbb{R}^n$-valued continuous functions defined on $[s,T]$ by $\mathbf{C}([s,T];\mathbb{R}^n)$. Then, let $\breve{\Pi}(.)\in\mathbf{C}([\breve{s},\breve{T}];\mathcal{S}^n)$ be the solution of the Riccati equation

$$\dot{\breve{\Pi}}+\breve{\Pi}\breve{A}+\breve{A}^T\breve{\Pi}+\breve{C}^T\breve{P}\breve{C}+\breve{P}-(\breve{B}^T\breve{\Pi}+\breve{N}+\breve{D}^T\breve{\Pi}\breve{C})^T(\breve{R}+\breve{D}^T\breve{\Pi}\breve{D})^{-1} \\ \times(\breve{B}^T\breve{\Pi}+\breve{N}+\breve{D}^T\breve{\Pi}\breve{C})=0, \quad a.e. t\in[\breve{s},t],\ \breve{\Pi}(\breve{T})=\breve{\bar{P}}, \quad (2.11)$$

where $\breve{R}+\breve{D}^T\breve{\Pi}\breve{D}>0,\ a.e.\,t\in[\breve{s},\breve{T}]$, and $\breve{s}(.)\in C([\breve{s},\breve{T}];\mathbb{R}^n)$ be the solution of the offset equation given by

$$\dot{\breve{s}}+[\breve{A}-\breve{B}(\breve{R}+\breve{D}^T\breve{\Pi}\breve{D})^{-1}(\breve{B}^T\breve{P}+\breve{s}+\breve{D}^T\breve{P}\breve{C})]^T\breve{s}+[\breve{C}-\breve{D}(\breve{R}+\breve{D}^T\breve{\Pi}\breve{D})^{-1} \\ (\breve{B}^T\breve{\Pi}+\breve{N}+\breve{D}^T\breve{\Pi}\breve{C})]^T\breve{\Pi}\breve{\sigma}+\breve{\Pi}\breve{b}=0, \quad a.e.\,t\in[\breve{s},\breve{T}],\ \breve{s}(\breve{T})=0.$$

Let us define $\breve{\Psi}\triangleq(\breve{R}+\breve{D}^T\breve{\Pi}\breve{D})^{-1}[\breve{B}^T\breve{\Pi}+\breve{N}+\breve{D}^T\breve{\Pi}\breve{C}]$, and $\breve{\psi}\triangleq(\breve{R}+\breve{D}^T\breve{\Pi}\breve{D})^{-1}[\breve{B}^T\breve{s}+\breve{D}^T\breve{\Pi}\breve{\sigma}]$. Then the stochastic LQ problem (2.9)-(2.10) *is solvable at $\breve{s}$ with the optimal control $\breve{u}^\circ(.)$ being in the state feedback form as in*

$$\breve{u}^\circ(t)=-\breve{\Psi}(t)\breve{x}(t)-\breve{\psi}(t), \quad t\in[\breve{s},\breve{T}].$$

□

Henceforth we discuss the stochastic optimal control problem for the major agent, and a generic minor agent.

2.3.1 Mean Field Evolution

We introduce the empirical state average as

$$x^{(N_k)} = \frac{1}{N_k}\sum_{j=1}^{N_k} x_j^k, \qquad 1 \le k \le K,$$

and write $x^{(N)} = [x^{(N_1)}, x^{(N_2)}, ..., x^{(N_K)}]$, where the point-wise in time L^2 limit of $x^{(N)}$, if it exists, is called the mean field of the system and is denoted by $\bar{x} = [\bar{x}^1, ..., \bar{x}^K]$. We consider for each minor agent $\mathcal{A}_i$ of type k, $1 \le k \le K$, a uniform (with respect to i in any subpopulation k, $1 \le k \le K$) feedback control $u_i^k \in \mathcal{U}_{i,y}^L$, which is a function of

(i) minor agent's estimate of its own state, i.e. $\hat{x}_{i|\mathcal{F}_i^y} \triangleq \mathbb{E}_{|\mathcal{F}_i^y} x_i = \mathbb{E}\{x_i|\mathcal{F}_i^y\}$,

(ii) minor agent's estimate of the major agent's state, i.e. $\hat{x}_{0|\mathcal{F}_i^y} \triangleq \mathbb{E}_{|\mathcal{F}_i^y} x_0 = \mathbb{E}\{x_0|\mathcal{F}_i^y\}$,

(iii) minor agent's estimate of x_j, $1 \le j \le N$, $j \ne i$, i.e. $\hat{x}_{j|\mathcal{F}_i^y} \triangleq \mathbb{E}_{|\mathcal{F}_i^y} x_j = \mathbb{E}\{x_j|\mathcal{F}_i^y\}$,

(iv) minor agent's estimate of the major agent's estimate of its own state, i.e. $(\hat{x}_{0|\mathcal{F}_0^y})_{|\mathcal{F}_i^y} \triangleq \mathbb{E}_{|\mathcal{F}_i^y} \hat{x}_{0|\mathcal{F}_0^y} = \mathbb{E}\{\hat{x}_{0|\mathcal{F}_0^y}|\mathcal{F}_i^y\}$,

(v) minor agent's estimate of the major agent's estimate of x_j, $1 \le j \le N$, $j \ne i$, i.e. $(\hat{x}_{j|\mathcal{F}_0^y})_{|\mathcal{F}_i^y} \triangleq \mathbb{E}_{|\mathcal{F}_i^y} \hat{x}_{j|\mathcal{F}_0^y} = \mathbb{E}\{\hat{x}_{j|\mathcal{F}_0^y}|\mathcal{F}_i^y\}$,

(vi) bounded continuous functions of time $m_k(.) \in \mathbf{C}_b([0,\infty);\mathbb{R}^m)$.

Hence u_i^k is given by

$$u_i^k = L_1^k \hat{x}_{i|\mathcal{F}_i^y}^k + L_2^k \hat{x}_{0|\mathcal{F}_i^y} + \sum_{l=1}^{K}\sum_{j=1}^{N_l} L_3^{k,l} \hat{x}_{j|\mathcal{F}_i^y}^l + L_4^k (\hat{x}_{0|\mathcal{F}_0^y})_{|\mathcal{F}_i^y} + \sum_{l=1}^{K}\sum_{j=1}^{N_l} L_5^{k,l} (\hat{x}_{j|\mathcal{F}_0^y}^l)_{|\mathcal{F}_i^y} + m_k, \quad (2.12)$$

for matrices L_1^k, L_2^k, $L_3^{k,l}$, and L_4^k of appropriate dimension, which are time invariant due to the time shift invariance of the infinite horizon performance function (2.4) and the dynamics (2.2), and where $L_3^{k,l}$, $L_5^{k,l}$ are assumed to depend upon N_l, and satisfy $N_l L_3^{k,l} \to \bar{L}_3^{k,l}$, $N_l L_5^{k,l} \to \bar{L}_5^{k,l}$ as

$N_l \to \infty$ for all k, $1 \le k \le K$. Substituting (2.12) in (2.2) yields

$$dx_i = [A_k x_i + B_k L_1^k \hat{x}^k_{i|\mathcal{F}_i^y} + B_k L_2^k \hat{x}_{0|\mathcal{F}_i^y} + B_k \sum_{l=1}^{K} N_l L_3^{k,l} \hat{x}^{(N_l)}_{|\mathcal{F}_i^y} + B_k L_4^k (\hat{x}_{0|\mathcal{F}_0^y})_{|\mathcal{F}_i^y} + B_k \sum_{l=1}^{K} N_l L_5^{k,l} (\hat{x}^{(N_l)}_{|\mathcal{F}_0^y})_{|\mathcal{F}_i^y} + B_k m_k + G x_0] dt + D dw_i, \quad (2.13)$$

Then we take the average over the subpopulation k to obtain

$$dx^{(N_k)} = \Big[A_k x^{(N_k)} + B_k L_1^k \frac{1}{N_k} \sum_{i=1}^{N_k} \hat{x}^k_{i|\mathcal{F}_i^y} + B_k L_2^k \frac{1}{N_k} \sum_{i=1}^{N_k} \hat{x}_{0|\mathcal{F}_i^y} + B_k \sum_{l=1}^{K} N_l L_3^{k,l} \frac{1}{N_k} \sum_{i=1}^{N_k} \hat{x}^{(N_l)}_{|\mathcal{F}_i^y} + B_k L_4^k \frac{1}{N_k} \sum_{i=1}^{N_k} (\hat{x}_{0|\mathcal{F}_0^y})_{|\mathcal{F}_i^y} + B_k \sum_{l=1}^{K} N_l L_5^{k,l} \frac{1}{N_k} \sum_{i=1}^{N_k} (\hat{x}^{(N_l)}_{|\mathcal{F}_0^y})_{|\mathcal{F}_i^y} + B_k m_k + G x_0\Big] dt + D \frac{1}{N_k} \sum_{i=1}^{N_k} dw_i. \quad (2.14)$$

To compute the average of the estimation terms in (2.14), we use the state decomposition

$$\begin{bmatrix} \hat{x}_{i|\mathcal{F}_i^y} \\ \hat{x}_{0|\mathcal{F}_i^y} \\ \hat{x}^{(N_l)}_{|\mathcal{F}_i^y} \\ (\hat{x}_{0|\mathcal{F}_0^y})_{|\mathcal{F}_i^y} \\ (\hat{x}^{(N_l)}_{|\mathcal{F}_0^y})_{|\mathcal{F}_i^y} \end{bmatrix} = \begin{bmatrix} \hat{x}_{i|\mathcal{F}_i^y} - x_i \\ \hat{x}_{0|\mathcal{F}_i^y} - x_0 \\ \hat{x}^{(N_l)}_{|\mathcal{F}_i^y} - x^{(N_l)} \\ (\hat{x}_{0|\mathcal{F}_0^y})_{|\mathcal{F}_i^y} - \hat{x}_{0|\mathcal{F}_0^y} \\ (\hat{x}^{(N_l)}_{|\mathcal{F}_0^y})_{|\mathcal{F}_i^y} - \hat{x}^{(N_l)}_{|\mathcal{F}_0^y} \end{bmatrix} + \begin{bmatrix} x_i \\ x_0 \\ x^{(N_l)} \\ \hat{x}_{0|\mathcal{F}_0^y} \\ \hat{x}^{(N_l)}_{|\mathcal{F}_0^y} \end{bmatrix}, \quad (2.15)$$

which we denote equivalently in the compact form

$$\hat{x}^{k,ex}_{i|\mathcal{F}_i^y} = -\tilde{x}_i^{k,ex} + x_i^{k,ex}, \quad (2.16)$$

for $1 \leq i \leq N$, and $1 \leq k \leq K$. Accordingly we rewrite (2.14) as

$$\begin{aligned} dx^{(N_k)} = \Big[& A_k x^{(N_k)} + B_k L_1^k \frac{1}{N_k} \sum_{i=1}^{N_k} x_i^k + B_k L_2^k x_0 + B_k \sum_{l=1}^{K} N_l L_3^{k,l} x^{(N_l)} \\ & + B_k L_4^k \hat{x}_{0|\mathcal{F}_0^y} + B_k \sum_{l=1}^{K} N_l L_5^{k,l} \hat{x}^{(N_l)}_{|\mathcal{F}_0^y} + B_k m_k + G x_0 \Big] dt \\ & - \Big[B_k L_1^k \frac{1}{N_k} \sum_{i=1}^{N_k} (x_i - \hat{x}_{i|\mathcal{F}_i^y}) + B_k L_2^k \frac{1}{N_k} \sum_{i=1}^{N_k} (x_0 - \hat{x}_{0|\mathcal{F}_i^y}) \\ & + B_k \sum_{l=1}^{K} N_l L_3^{k,l} \frac{1}{N_k} \sum_{i=1}^{N_k} \big(x^{(N_l)} - \hat{x}^{(N_l)}_{|\mathcal{F}_i^y} \big) + B_k L_4^k \frac{1}{N_k} \sum_{i=1}^{N_k} \big(\hat{x}_{0|\mathcal{F}_0^y} - (\hat{x}_{0|\mathcal{F}_0^y})_{|\mathcal{F}_i^y} \big) \\ & + B_k \sum_{l=1}^{K} N_l L_5^{k,l} \frac{1}{N_k} \sum_{i=1}^{N_k} \big(\hat{x}^{(N_l)}_{|\mathcal{F}_0^y} - (\hat{x}^{(N_l)}_{|\mathcal{F}_0^y})_{|\mathcal{F}_i^y} \big) \Big] dt + D \frac{1}{N_k} \sum_{i=1}^{N_k} dw_i. \quad (2.17) \end{aligned}$$

From (2.17) as $N \to \infty$ we obtain the convergence in quadratic mean to the solution to

$$\begin{aligned} d\bar{x}^k = \Big[& (A_k + B_k L_1^k)\bar{x}^k + (G + B_k L_2^k) x_0 + B_k \sum_{l=1}^{K} \bar{L}_3^{k,l} \bar{x}^l + B_k L_4^k \hat{x}_{0|\mathcal{F}_0^y} + B_k \sum_{l=1}^{K} \bar{L}_5^{k,l} \hat{\bar{x}}^l_{|\mathcal{F}_0^y} \\ & + B_k m_k \Big] dt - \Big[B_k L_1^k \overline{(x_i - \hat{x}_{i|\mathcal{F}_i^y})}^k + B_k L_2^k \overline{(x_0 - \hat{x}_{0|\mathcal{F}_i^y})}^k + B_k \sum_{l=1}^{K} \bar{L}_3^{k,l} \overline{\big(\bar{x}^l - \hat{\bar{x}}^l_{|\mathcal{F}_i^y} \big)}^k \\ & + B_k L_4^k \overline{\big(\hat{x}_{0|\mathcal{F}_0^y} - (\hat{x}_{0|\mathcal{F}_0^y})_{|\mathcal{F}_i^y} \big)}^k + B_k \sum_{l=1}^{K} \bar{L}_5^{k,l} \overline{\big(\hat{\bar{x}}^l_{|\mathcal{F}_0^y} - (\hat{\bar{x}}^l_{|\mathcal{F}_0^y})_{|\mathcal{F}_i^y} \big)}^k \Big] dt, \quad (2.18) \end{aligned}$$

where the overline symbol with superscript k, i.e. $\overline{(.)}^k$ denotes the infinite-population limit of the the average over subpopulation k of the corresponding terms, which are the components of $\tilde{x}_i^{ex}$ in (2.16) (see *Proposition 3.1* in [21] for the convergence analysis in quadratic mean).

Subsequently, a compact representation of (2.18) shall be used as in

$$\begin{aligned} d\bar{x}^k = \Big(& (A_k + B_k L_1^k)\bar{x}^k + (G + B_k L_2^k) x_0 + B_k \sum_{l=1}^{K} \bar{L}_3^{k,l} \bar{x}^l + B_k L_4^k \hat{x}_{0|\mathcal{F}_0^y} \\ & + B_k \sum_{l=1}^{K} \bar{L}_5^{k,l} \hat{\bar{x}}^l_{|\mathcal{F}_0^y} + B_k m_k \Big) dt + \bar{J}_k \bar{\tilde{x}}^{k,ex} dt, \quad (2.19) \end{aligned}$$

where we denote by $\bar{\tilde{x}}^{k,ex}$ the average of the estimation errors of the minor agents of subpopulation k as $N_k \to \infty$, and which satisfies the dynamical equation (2.58) in Section 2.3.4. Hence, the second bracket in (2.18) is given by $\bar{J}_k \bar{\tilde{x}}^{k,ex}$. (Here the term $\bar{J}_k \bar{\tilde{x}}^{k,ex}$ corrects its omission in [21].)

Therefore the mean field state vector $\bar{x}$ satisfies

$$d\bar{x} = \bar{A}\bar{x}dt + \bar{G}x_0 dt + \bar{H}\hat{x}_{0|\mathcal{F}_0^y} dt + \bar{L}\hat{\bar{x}}_{|\mathcal{F}_0^y} dt + \bar{J}\bar{\tilde{x}}^{ex} dt + \bar{m}dt, \tag{2.20}$$

where $(\bar{\tilde{x}}^{ex})^T = [(\bar{\tilde{x}}^{1,ex})^T, \ldots, (\bar{\tilde{x}}^{K,ex})^T]$, and the matrices $\bar{A}$, $\bar{G}$, $\bar{H}$, $\bar{L}$, $\bar{J}$, and $\bar{m}$ collect the corresponding terms in (2.19) and have the block matrix form

$$\bar{A} = \begin{bmatrix} \bar{A}_1 \\ \vdots \\ \bar{A}_K \end{bmatrix}, \quad \bar{G} = \begin{bmatrix} \bar{G}_1 \\ \vdots \\ \bar{G}_K \end{bmatrix}, \quad \bar{H} = \begin{bmatrix} \bar{H}_1 \\ \vdots \\ \bar{H}_K \end{bmatrix},$$
$$\bar{L} = \begin{bmatrix} \bar{L}_1 \\ \vdots \\ \bar{L}_K \end{bmatrix}, \quad \bar{m} = \begin{bmatrix} \bar{m}_1 \\ \vdots \\ \bar{m}_K \end{bmatrix}, \quad \bar{J} = \begin{bmatrix} \bar{J}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{J}_K \end{bmatrix}. \tag{2.21}$$

We note that $\bar{A}_k,\ \bar{L}_k \in \mathbb{R}^{n \times nK},\ \bar{G}_k,\ \bar{H}_k \in \mathbb{R}^{n \times n},\ \bar{m}_k \in \mathbb{R}^n,\ \bar{J}_k \in \mathbb{R}^{n \times (3n+2nK)},\ 1 \le k \le K$, are to be solved for using the consistency equations in Section 2.3.4. By abuse of language, the mean value of the system's Gaussian mean field given by the state process $\bar{x} = [\bar{x}^1, \ldots, \bar{x}^K]$ shall also be termed the system's mean field.

2.3.2 Major Agent: Infinite Population

The major agent's infinite population dynamics, as the number of agents goes to infinity ($N \to \infty$), remain the same as in (2.1), while its infinite population individual cost functional is given by

$$J_0^\infty(u_0, u_{-0}) = \mathbb{E}\int_0^\infty e^{-\rho t}\Big\{\|x_0 - \phi(\bar{x})\|^2_{Q_0} + \|u_0\|^2_{R_0}\Big\}dt, \tag{2.22}$$

$$\phi(.) := H_0^\pi \bar{x} + \eta_0, \tag{2.23}$$

$$H_0^\pi = \pi \otimes H_0 \triangleq [\pi_1 H_0, \pi_2 H_0, \ldots, \pi_K H_0], \tag{2.24}$$

where $x^{(N)}$ in (6.48) was replaced by its L^2 limit, i.e. the mean field $\bar{x}$.

To solve the infinite population tracking problem for the major agent, its state is extended with the mean field process $\bar{x}$, where this is assumed to exist, i.e. $x_0^{ex} = [x_0,\, \bar{x}]$.

Then the Kalman filter which generates the estimates of the major agent's state $\hat{x}_{0|\mathcal{F}_0^y}$ and the mean field $\hat{\bar{x}}_{|\mathcal{F}_0^y}$ based on its own observations are, respectively, given by

$$d\hat{x}_{0|\mathcal{F}_0^y} = A_0\hat{x}_{0|\mathcal{F}_0^y}dt + B_0\hat{u}_0dt + K_0^1 d\nu_0, \tag{2.25}$$

$$d\hat{\bar{x}}_{|\mathcal{F}_0^y} = (\bar{G} + \bar{H})\hat{x}_{0|\mathcal{F}_0^y}dt + (\bar{A} + \bar{L})\hat{\bar{x}}_{|\mathcal{F}_0^y}dt + \bar{m}dt + K_0^2 d\nu_0, \tag{2.26}$$

where $\hat{\bar{\bar{x}}}_{|\mathcal{F}_0^y} = 0$ is used (see *Observation 2.4*). Moreover, $\bar{m}$ is a deterministic process according to (2.19), K_0^1 and K_0^2 are the Kalman filter gains, and ν_0 is the innovation process. Therefore the Kalman filter which generates the estimates of the major agent's extended state is given by

$$\begin{aligned}\begin{bmatrix} d\hat{x}_{0|\mathcal{F}_0^y} \\ d\hat{\bar{x}}_{|\mathcal{F}_0^y} \end{bmatrix} &= \begin{bmatrix} A_0 & 0_{n\times nK} \\ \bar{G}+\bar{H} & \bar{A}+\bar{L} \end{bmatrix}\begin{bmatrix} \hat{x}_{0|\mathcal{F}_0^y} \\ \hat{\bar{x}}_{|\mathcal{F}_0^y} \end{bmatrix}dt \\ &+ \begin{bmatrix} B_0 \\ 0_{nK\times m} \end{bmatrix}\hat{u}_0dt + \begin{bmatrix} 0_{n\times 1} \\ \bar{m} \end{bmatrix}dt + K_0 d\nu_0, \end{aligned} \tag{2.27}$$

with the corresponding Kalman filter gain $K_0 = [(K_0^1)^T, (K_0^2)^T]^T$, and the innovation process ν_0, respectively, given by

$$K_0 = V_0\mathbb{L}_0^T R_{v_0}^{-1}, \tag{2.28}$$

$$d\nu_0 = dy_0 - \mathbb{L}_0\left[\hat{x}_{0|\mathcal{F}_0^y}^T, \hat{\bar{x}}_{|\mathcal{F}_0^y}^T\right]^T dt, \tag{2.29}$$

where $\mathbb{L}_0 = \begin{bmatrix} l_0^1 & 0_{\ell\times nK}\end{bmatrix}$, and $V_0(t)$ is the solution to the corresponding Riccati equation (3.56).

From (2.1), (6.54), and (2.27) we denote

$$\begin{aligned}\mathbb{A}_0 &= \begin{bmatrix} A_0 & 0_{n\times nK} \\ \bar{G}+\bar{H} & \bar{A}+\bar{L} \end{bmatrix}, \quad \mathbb{B}_0 = \begin{bmatrix} B_0 \\ 0_{nK\times m} \end{bmatrix}, \quad \mathbb{M}_0 = \begin{bmatrix} 0_{n\times 1} \\ \bar{m} \end{bmatrix}, \\ \mathbb{D}_0 &= \begin{bmatrix} D_0 & 0_{n\times rK} \\ 0_{nK\times r} & 0_{nK\times rK} \end{bmatrix}, \quad \mathbb{J}_0 = \begin{bmatrix} 0_{n\times(3nK+2nK^2)} \\ \bar{J} \end{bmatrix}. \end{aligned} \tag{2.30}$$

Then to guarantee the convergence of the solution to the Riccati equation to a positive definite asymptotically stabilizing solution, we assume:

Assumption 2.5. $[\mathbb{A}_0, \mathbb{D}_0]$ *is stabilizable and* $[\mathbb{L}_0, \mathbb{A}_0]$ *is detectable.*

The corresponding Riccati equation is then given by

$$\dot{V}_0 = \mathbb{A}_0 V_0 + V_0 \mathbb{A}_0^T - K_0 R_{v_0} K_0^T + \mathbb{J}_0 \bar{V} \mathbb{J}_0^T + Q_{w_0}, \tag{2.31}$$

where $Q_{w_0} = \mathbb{D}_0 \mathbb{D}_0^T$, $\bar{V}(t) = \mathbb{E}\big[\bar{\tilde{x}}^{ex}(t)\big(\bar{\tilde{x}}^{ex}(t)\big)^T\big]$ satisfies (2.65), and $V(0) = \mathbb{E}\big[\big(x_0^{ex}(0) - (\widehat{x_0^{ex}(0)})_{|\mathcal{F}_0^y}\big)\big(x_0^{ex}(0) - (\widehat{x_0^{ex}(0)})_{|\mathcal{F}_0^y}\big)^T\big]$.

Then, utilizing the infinite horizon discounted analogy to *Theorem 2.1*, it can be shown (see *Theorem 2.2* in Section 2.3.4) that the optimal control action for the major agent's tracking problem (and hence best response MFG control input) is

$$\hat{u}_0^\circ = -R_0^{-1} \mathbb{B}_0^T [\Pi_0 (\hat{x}_{0|\mathcal{F}_0^y}^T, \hat{\bar{x}}_{|\mathcal{F}_0^y}^T)^T + s_0], \tag{2.32}$$

where Π_0 and s_0 are the solutions to the Riccati and offset equations given by

$$\rho \Pi_0 = \Pi_0 \mathbb{A}_0 + \mathbb{A}_0^T \Pi_0 - \Pi_0 \mathbb{B}_0 R_0^{-1} \mathbb{B}_0^T \Pi_0 + Q_0^\pi, \tag{2.33}$$

$$\rho s_0 = \frac{ds_0}{dt} + (\mathbb{A}_0 - \mathbb{B}_0 R_0^{-1} \mathbb{B}_0^T \Pi_0)^T s_0 + \Pi_0 \mathbb{M}_0 - \bar{\eta}_0, \tag{2.34}$$

with $\bar{\eta}_0 = [I_{n\times n}, -H_0^\pi]^T Q_0 \eta_0$ and $Q_0^\pi = [I_{n\times n}, -H_0^\pi]^T Q_0 [I_{n\times n}, -H_0^\pi]$. We note $\frac{ds_0}{dt} = 0$ in (2.34), since $\mathbb{M}_0$, $\bar{\eta}_0$ are constant.

Finally, the joint dynamics of the major agent's closed-loop system and its Kalman filter system are given by

$$\begin{bmatrix} dx_0 \\ d\bar{x} \\ d\hat{x}_{0|\mathcal{F}_0^y} \\ d\hat{\bar{x}}_{|\mathcal{F}_0^y} \end{bmatrix} = \mathbf{A}_0 \begin{bmatrix} x_0 \\ \bar{x} \\ \hat{x}_{0|\mathcal{F}_0^y} \\ \hat{\bar{x}}_{|\mathcal{F}_0^y} \end{bmatrix} dt + \mathbf{J}_0 \bar{\tilde{x}}^{ex} dt + \mathbf{M}_0 dt + \mathbf{D}_0 \begin{bmatrix} \begin{bmatrix} dw_0 \\ 0_{nK\times 1} \end{bmatrix} \\ dv_0 \end{bmatrix}, \tag{2.35}$$

where

$$\mathbf{A}_0 = \begin{bmatrix} \begin{bmatrix} A_0 & 0_{n\times nK} \\ \bar{G} & \bar{A} \end{bmatrix} & \begin{bmatrix} -B_0 R_0^{-1}\mathbb{B}_0^T \Pi_0 \\ \begin{bmatrix} \bar{H} & \bar{L} \end{bmatrix} \end{bmatrix}, \\ K_0\mathbb{L}_0 & \mathbb{A}_0 - K_0\mathbb{L}_0 - \mathbb{B}_0 R_0^{-1}\mathbb{B}_0^T\Pi_0 \end{bmatrix}, \quad \mathbf{J}_0 = \begin{bmatrix} \mathbb{J}_0 \\ 0_{(n+nK)\times(3nK+2nK^2)} \end{bmatrix},$$

$$\mathbf{M}_0 = \begin{bmatrix} \mathbb{M}_0 - \mathbb{B}_0 R_0^{-1}\mathbb{B}_0^T s_0 \\ \mathbb{M}_0 - \mathbb{B}_0 R_0^{-1}\mathbb{B}_0^T s_0 \end{bmatrix}, \quad \mathbf{D}_0 = \begin{bmatrix} \mathbb{D}_0 & 0_{(n+nK)\times\ell} \\ 0_{(n+nK)\times(r+nK)} & K_0 R_{v_0}^{\frac{1}{2}} \end{bmatrix}.$$

2.3.3 Minor Agent: Infinite Population

A generic minor agent's infinite population dynamics, as the number of agent goes to infinity ($N \to \infty$), remain the same as in (2.2), while its infinite population individual cost functional is given as

$$J_i^\infty(u_i, u_{-i}) = \mathbb{E}\int_0^\infty e^{-\rho t}\Big\{\|x_i - \psi(\bar{x})\|_Q^2 + \|u_i\|_R^2\Big\}dt, \tag{2.36}$$

$$\psi(.) = H_1 x_0 + H_2^\pi \bar{x} + \eta, \tag{2.37}$$

$$H_2^\pi = \pi \otimes H_2 \triangleq [\pi_1 H_0, \pi_2 H_0, ..., \pi_K H_0]. \tag{2.38}$$

In the case where all agents have partial observations on the major agent's state, the joint dynamics of the major agent's closed-loop system and its Kalman filtering recursions are employed in order to solve the minor agent's tracking problem. Therefore, the minor agent's state is next extended to form $x_i^{ex} \triangleq [x_i,\ x_0,\ \bar{x},\ \hat{x}_{0|\mathcal{F}_0^y},\ \hat{\bar{x}}_{|\mathcal{F}_0^y}]$. Specifically this yields

$$dx_i^{ex} = \mathbb{A}_k x_i^{ex}dt + \mathbb{B}_k u_i dt + \mathbb{J}\bar{\tilde{x}}^{ex} + \mathbb{M}dt + \mathbb{D}[dw_i^T, dw_0^T, 0_{1\times nK}, dv_0^T]^T, \tag{2.39}$$

where

$$\mathbb{A}_k = \begin{bmatrix} A_k & \begin{bmatrix} G & 0_{n\times(n+2nK)} \end{bmatrix} \\ 0_{2(n+nK)\times n} & \mathbf{A}_0 \end{bmatrix}, \quad \mathbb{B}_k = \begin{bmatrix} B_k \\ 0_{2(n+nK)\times m} \end{bmatrix},$$

$$\mathbb{J} = \begin{bmatrix} 0_{n\times(3nK+2nK^2)} \\ \mathbf{J}_0 \end{bmatrix}, \quad \mathbb{M} = \begin{bmatrix} 0_{n\times 1} \\ \mathbf{M}_0 \end{bmatrix}, \quad \mathbb{D} = \begin{bmatrix} D & 0_{n\times(r+nK+\ell)} \\ 0_{2(n+nK)\times r} & \mathbf{D}_0 \end{bmatrix}. \tag{2.40}$$

To derive the Kalman filter equations for (2.39), we first define $\mathbb{L}_k = \begin{bmatrix} l_k^1 & l_k^2 & 0_{\ell\times(n+2nK)} \end{bmatrix}$. To guarantee the convergence of the solution to the Riccati equation to a positive definite asymptotically stabilizing solution, we assume:

Assumption 2.6. *The system parameter set* $\Theta = \{1, ..., K\}$ *is such that* $[\mathbb{A}_k, \mathbb{D}]$ *is stabilizable and* $[\mathbb{L}_k, \mathbb{A}_k]$ *is detectable for all* k, $1 \leq k \leq K$.

The Riccati equation associated with the filtering equations for (2.39) is then given by

$$\dot{V}_k = \mathbb{A}_k V_k + V_k \mathbb{A}_k^T - K_k R_v K_k^T + \mathbb{J}\bar{V}\mathbb{J}^T + Q_w, \tag{2.41}$$

where $Q_w = \mathbb{D}\mathbb{D}^T$, $\bar{V}(t) = \mathbb{E}\big[\bar{\tilde{x}}^{ex}(t)\big(\bar{\tilde{x}}^{ex}(t)\big)^T\big]$ satisfies (2.65), and $V_k(0) = \mathbb{E}\big[\big(x_i^{ex}(0) - (\widehat{x_i^{ex}(0)})_{|\mathcal{F}_i^y}\big)\big(x_i^{ex}(0) - (\widehat{x_i^{ex}(0)})_{|\mathcal{F}_i^y}\big)^T\big]$. The Kalman filter gain K_k is in turn given by

$$K_k = V_k \mathbb{L}_k^T R_v^{-1}, \tag{2.42}$$

and the innovation process $\nu_i(t)$ is defined as in

$$d\nu_i = dy_i - \mathbb{L}_k\Big[\hat{x}_{i|\mathcal{F}_i^y}^T, \hat{x}_{0|\mathcal{F}_i^y}^T, \hat{\bar{x}}_{|\mathcal{F}_i^y}^T, (\hat{x}_{0|\mathcal{F}_0^y})_{|\mathcal{F}_i^y}^T, (\hat{\bar{x}}_{|\mathcal{F}_0^y})_{|\mathcal{F}_i^y}^T\Big]^T dt, \tag{2.43}$$

where $(\hat{x}_{0|\mathcal{F}_0^y})_{|\mathcal{F}_i^y}$ and $(\hat{\bar{x}}_{|\mathcal{F}_0^y})_{|\mathcal{F}_i^y}$, respectively, denote the minor agent $\mathcal{A}_i$'s estimates of the major agent's estimates of its own state and the mean field. Then the Kalman filter equations for a generic minor agent $\mathcal{A}_i$, $1 \leq i \leq N$, are given as in

$$d\hat{x}_{i|\mathcal{F}_i^y}^{ex} = \mathbb{A}_k \hat{x}_{i|\mathcal{F}_i^y}^{ex} dt + \mathbb{B}_k \hat{u}_i dt + \mathbb{M} dt + K_k d\nu_i, \tag{2.44}$$

where $\hat{\bar{\bar{x}}}_{|\mathcal{F}_i^y}^{ex} = 0$ (see *Observation 2.4*) is used. Clearly, (2.44) generates the iterated estimates $(\hat{x}_{0|\mathcal{F}_0^y})_{|\mathcal{F}_i^y}$ and $(\hat{\bar{x}}_{|\mathcal{F}_0^y})_{|\mathcal{F}_i^y}$ which are required to calculate $\hat{x}_{0|\mathcal{F}_i^y}$ and $\hat{\bar{x}}_{|\mathcal{F}_i^y}$ (see *Proposition 1* in [37] for a simplified case of Estimates of Estimates Filter).

Remark 2.1. By virtue of the asymmetric information available to the major agent and a generic minor agent, an infinite regress does not occur in the process of estimating other agents' states. In fact to calculate the best response action, the major agent only estimates its own state and hence does not estimate minor agents' states, while each minor agent estimates its own state and the major agent's state.

We note that by *Assumption 2.3* the minor agent $\mathcal{A}_i$ is able to estimate $\hat{u}_0^\circ$ whenever the functional dependence of the major agent's control on it's state is available to the minor agent through forming the conditional expectation of the major agent's control action which by (2.32) is given by the following expression

$$(\hat{u}_0^\circ)_{|\mathcal{F}_i^y} = \mathbb{E}\{\hat{u}_0^\circ|\mathcal{F}_i^y\} = -R^{-1}\mathbb{B}_0^T\Big[\Pi_0\Big((\hat{x}_{0|\mathcal{F}_0^y})^T_{|\mathcal{F}_i^y}, (\hat{\bar{x}}_{|\mathcal{F}_0^y})^T_{|\mathcal{F}_i^y}\Big) + s_0\Big], \tag{2.45}$$

and which is embedded in (2.44). Then, utilizing the infinite horizon discounted analogy to *Theorem 2.1*, it can be shown (see *Theorem 2.2*) that the optimal control action for the minor agent $\mathcal{A}_i$'s tracking problem (and hence best response MFG control input) is given by

$$\hat{u}_i^\circ = -R^{-1}\mathbb{B}_k^T\left[\Pi_k\Big(\hat{x}^T_{i|\mathcal{F}_i^y}, \hat{x}^T_{0|\mathcal{F}_i^y}, \hat{\bar{x}}^T_{|\mathcal{F}_i^y}, (\hat{x}_{0|\mathcal{F}_0^y})^T_{|\mathcal{F}_i^y}, (\hat{\bar{x}}_{|\mathcal{F}_0^y})^T_{|\mathcal{F}_i^y}\Big)^T + s_k\right], \tag{2.46}$$

where the iterated estimation terms $(\hat{x}_{0|\mathcal{F}_0^y})_{|\mathcal{F}_i^y}$, and $(\hat{\bar{x}}_{|\mathcal{F}_0^y})_{|\mathcal{F}_i^y}$ explicitly appear, and the corresponding Riccati and offset equations are given by

$$\rho\Pi_k = \Pi_k\mathbb{A}_k + \mathbb{A}_k^T\Pi_k - \Pi_k\mathbb{B}_kR^{-1}\mathbb{B}_k^T\Pi_k + Q^\pi,\ \forall k, \tag{2.47}$$

$$\rho s_k = \frac{ds_k}{dt} + (\mathbb{A}_k - \mathbb{B}_kR^{-1}\mathbb{B}_k^T\Pi_k)^Ts_k + \Pi_k\mathbb{M} - \bar{\eta},\ \forall k, \tag{2.48}$$

with

$$\bar{\eta} = [I_{n\times n},\ -H_1,\ -H_2^\pi,\ 0_{n\times(n+nK)}]^TQ\eta,$$

$$Q^\pi = [I_{n\times n},\ -H_1,\ -H_2^\pi,\ 0_{n\times(n+nK)}]^TQ[I_{n\times n},\ -H_1,\ -H_2^\pi,\ 0_{n\times(n+nK)}].$$

We note $\frac{ds_k}{dt} = 0$ in (2.48), since $\mathbb{M}$, $\bar{\eta}$ are constant.

2.3.4 Mean Field Consistency Equations

Let us denote the components of Π_k in (2.47) as

$$\Pi_k = \begin{bmatrix} \Pi_{k,11} & \Pi_{k,12} & \Pi_{k,13} & \Pi_{k,14} & \Pi_{k,15} \\ \Pi_{k,21} & \Pi_{k,22} & \Pi_{k,23} & \Pi_{k,24} & \Pi_{k,25} \end{bmatrix}, \tag{2.49}$$

$1 \le k \le K$, and where $\Pi_{k,11}, \Pi_{k,12}, \Pi_{k,14} \in \mathbb{R}^{n\times n}$, $\Pi_{k,13}, \Pi_{k,15} \in \mathbb{R}^{n\times nK}$, $\Pi_{k,21}, \Pi_{k,22}, \Pi_{k,24} \in \mathbb{R}^{2(n+nK)\times n}$, and $\Pi_{k,23}, \Pi_{k,25} \in \mathbb{R}^{2(n+nK)\times nK}$. Let us also define the block matrix $\mathbf{e}_{k,v} = [0_{v\times v}, ..., 0_{v\times v}, I_v, 0_{v\times v}, ..., 0_{v\times v}]$ with K blocks, where the $v\times v$ identity matrix I_v is located at the kth block. Finally we define the block matrix $\mathbf{1}_v = [I_v, ..., I_v, ..., I_v]$ with K blocks of identity matrix. Then we denote by

$$\bar{\mathbf{e}}_k = \mathbf{e}_{k,n}, \tag{2.50}$$

$$\tilde{\mathbf{e}}_k = \mathbf{e}_{k,(3n+2nK)}, \tag{2.51}$$

$$\tilde{\mathbf{1}} = \mathbf{1}_{(3n+2nK)} \tag{2.52}$$

To obtain the mean field consistency equations, we substitute (2.46) in (2.2) to get

$$dx_i = A_k x_i dt + Gx_0 dt - B_k R^{-1}\mathbb{B}_k^T \big[\Pi_k \hat{x}^{ex}_{i|\mathcal{F}^y_i} + s_k\big]dt + Ddw_i. \tag{2.53}$$

Then $\hat{x}^{ex}_{i|\mathcal{F}^y_i}$ can be written as

$$\begin{aligned}\hat{x}^{ex}_{i|\mathcal{F}^y_i} &= -(x^{ex}_i - \hat{x}^{ex}_{i|\mathcal{F}^y_i}) + x^{ex}_i,\\ &= -\tilde{x}^{ex}_i + x^{ex}_i,\end{aligned} \tag{2.54}$$

where $\tilde{x}^{ex}_i$ denotes the estimation error, and the governing dynamics for $1 \le i \le N$, $1 \le k \le K$, are given by

$$d\tilde{x}^{k,ex}_i = (\mathbb{A}_k - K_k\mathbb{L}_k)\tilde{x}^{k,ex}_i + \mathbb{J}\bar{\tilde{x}}^{ex}dt - K_k R_v^{\frac{1}{2}} dv_i + \mathbb{D}[dw_i^T, dw_0^T, 0_{1\times nK}, dv_0^T]^T, \tag{2.55}$$

where $(\bar{\tilde{x}}^{ex})^T = [(\bar{\tilde{x}}^{1,ex})^T, \dots, (\bar{\tilde{x}}^{K,ex})^T]$ satisfies (2.60).

Next the empirical average of (2.53), where (2.54) has been substituted, over the population of the minor agents of type k is given by

$$\begin{aligned} d(\frac{1}{N_k}\sum_{i=1}^{N_k} x_i^k) &= A_k(\frac{1}{N_k}\sum_{i=1}^{N_k} x_i^k)dt + Gx_0 dt\\ &- B_k R^{-1}\mathbb{B}_k^T\Big[\Pi_k\big(\frac{1}{N_k}\sum_{i=1}^{N_k}\tilde{x}_i^{k,ex} + \frac{1}{N_k}\sum_{i=1}^{N_k} x_i^{k,ex}\big) + s_k\Big]dt + D\frac{1}{N_k}\sum_{i=1}^{N_k} dw_i. \end{aligned} \tag{2.56}$$

As $N_k \to \infty$, the solution to (2.56) converges, in quadratic mean, to the solution of

$$d\bar{x}^k = A_k\bar{x}^k dt + Gx_0 dt - B_k R^{-1}\mathbb{B}_k^T\Big[\Pi_k\big(\bar{\tilde{x}}^{k,ex} + \bar{x}^{k,ex}\big) + s_k\Big]dt, \tag{2.57}$$

where $\bar{x}^{k,ex} = \big[(\bar{x}^k)^T, x_0^T, \bar{x}^T, \hat{x}^T_{0|\mathcal{F}^y_0}, \hat{\bar{x}}^T_{|\mathcal{F}^y_0}\big]^T$, and from (2.55) the average of the estimation error

$\tilde{x}_i^{ex}$ over subpopulation $k, 1 \leq k \leq K$, as $N_k \to \infty$, i.e. $\bar{\tilde{x}}^{k,ex}$, is given by

$$d\bar{\tilde{x}}^{k,ex} = (\mathbb{A}_k - K_k \mathbb{L}_k)\bar{\tilde{x}}^{k,ex} + \mathbb{J}\bar{\tilde{x}}^{ex} - \mathbb{D}[0_{1\times r}, dw_0^T, 0_{1\times rK}, dv_0^T]^T. \tag{2.58}$$

Note that in the derivation of (2.58), we use the property that $\frac{1}{N_k}\sum_{i=1}^{N_k} w_0 = w_0$ and $\frac{1}{N_k}\sum_{i=1}^{N_k} \nu_0 = \nu_0$, since w_0 and ν_0 are the common processes shared between all agents of type k. Moreover, the law of large numbers is used to obtain as $N_k \to \infty$

$$\frac{1}{N_k}\sum_{i=1}^{N_k} K_k d\nu_i \xrightarrow{\text{q.m.}} 0, \quad \frac{1}{N_k}\sum_{i=1}^{N_k} dw_i \xrightarrow{\text{q.m.}} 0.$$

Subsequently, from (2.58), $(\bar{\tilde{x}}^{ex})^T = [(\bar{\tilde{x}}^{1,ex})^T, \dots, (\bar{\tilde{x}}^{K,ex})^T]$ satisfies

$$d\bar{\tilde{x}}^{ex} = \begin{bmatrix} (\mathbb{A}_1 - K_1\mathbb{L}_1)\tilde{\mathbf{e}}_1 + \mathbb{J} \\ \vdots \\ (\mathbb{A}_K - K_K\mathbb{L}_K)\tilde{\mathbf{e}}_K + \mathbb{J} \end{bmatrix} \bar{\tilde{x}}^{ex} dt + \begin{bmatrix} -\mathbb{D} \\ \vdots \\ -\mathbb{D} \end{bmatrix} \begin{bmatrix} 0_{r\times 1} \\ dw_0 \\ 0_{rK\times 1} \\ dv_0, \end{bmatrix}, \tag{2.59}$$

or equivalently in the compact form

$$d\bar{\tilde{x}}^{ex} = \tilde{\mathbb{A}}\bar{\tilde{x}}^{ex} dt + \tilde{\mathbb{D}}[0_{1\times r}, dw_0^T, 0_{1\times rK}, dv_0^T]^T. \tag{2.60}$$

Using (2.49) the mean field equation (2.57) can be presented as

$$\begin{aligned} d\bar{x}^k = &\Big([A_k - B_k R^{-1} B_k^T \Pi_{k,11}]\bar{\mathbf{e}}_k - B_k R^{-1} B_k^T \Pi_{k,13}\Big)\bar{x} dt + \Big(G - B_k R^{-1} B_k^T \Pi_{k,12}\Big) x_0 dt \\ &- B_k R^{-1} B_k^T \Pi_{k,14} \hat{x}_{0|\mathcal{F}_0^y} dt - B_k R^{-1} B_k^T \Pi_{k,15} \hat{\bar{x}}_{|\mathcal{F}_0^y} dt - B_k R^{-1} \mathbb{B}_k^T \Pi_k \bar{\tilde{x}}^{k,ex} dt \\ &- B_k R^{-1} \mathbb{B}_k^T s_k dt. \end{aligned} \tag{2.61}$$

Since (2.57) and (6.54) must be identical, we obtain the Consistency Equations, determining the components of $\bar{A}$, $\bar{G}$, $\bar{H}$, $\bar{L}$, $\bar{J}$, and $\bar{m}$ in (6.54), given by the following compact set of equations

$$\begin{aligned} \bar{A}_k &= [A_k - B_k R^{-1} B_k^T \Pi_{k,11}]\bar{\mathbf{e}}_k - B_k R^{-1} B_k^T \Pi_{k,13}, \quad \forall k, \\ \bar{G}_k &= G - B_k R^{-1} B_k^T \Pi_{k,12}, \quad \forall k, \end{aligned}$$

$$\begin{aligned}
\bar{H}_k &= -B_k R^{-1} B_k^T \Pi_{k,14}, \quad \forall k, \\
\bar{L}_k &= -B_k R^{-1} B_k^T \Pi_{k,15}, \quad \forall k, \\
\bar{J}_k &= -B_k R^{-1} \mathbb{B}_k^T \Pi_k, \quad \forall k, \\
\bar{m}_k &= -B_k R^{-1} \mathbb{B}_k^T s_k, \quad \forall k,
\end{aligned} \tag{2.62}$$

where Π_k and s_k satisfy (2.47) and (2.48), respectively. The set of equations (5.51) together with (2.33)-(2.34) and (2.47)-(2.48) form a fixed point problem which must be solved by each individual agent $\mathcal{A}_i, 0 \leq i \leq N$, in order to compute the matrices in the mean field dynamics (6.54).

Finally from (2.39) and (2.57)-(2.60) the Markovian dynamics of $\bar{x}^k$ (i.e. the mean field of subpopulation $k, 1 \leq k \leq K$) are given by

$$\begin{aligned}
\begin{bmatrix} d\bar{x}^{k,ex} \\ d\bar{\tilde{x}}^{ex} \end{bmatrix} &= \begin{bmatrix} \mathbb{A}_k - \mathbb{B}_k R^{-1} \mathbb{B}_k^T \Pi_k & -\mathbb{B}_k R^{-1} \mathbb{B}_k^T \Pi_k \tilde{\mathbf{e}}_k \\ 0 & \tilde{\mathbb{A}}_k \end{bmatrix} \begin{bmatrix} \bar{x}^{k,ex} \\ \bar{\tilde{x}}^{ex} \end{bmatrix} dt \\
&\begin{bmatrix} \mathbb{M} - \mathbb{B}_k R^{-1} \mathbb{B}_k^T s_k \\ 0 \end{bmatrix} dt + \begin{bmatrix} \mathbb{D} & 0 \\ 0 & \tilde{\mathbb{D}} \end{bmatrix} \begin{bmatrix} 0_{r\times 1} \\ dw_0 \\ 0_{rK\times 1} \\ dv_0 \end{bmatrix}.
\end{aligned} \tag{2.63}$$

Remark 2.2. From (2.58) in the infinite population limit the average of the estimation errors of the minor agents of type $k, 1 \leq k \leq K$, is driven by the major agent's Wiener process w_0 and the measurement noise v_0 (or equivalently innovation process ν_0). In other words, it is driven by the non-zero quadratic variation processes in the dynamics of the common processes $x_0^{ex}, \hat{x}_{0|\mathcal{F}_0^y}^{ex}$, with which the minor agents $\mathcal{A}_i, 1 \leq i \leq N$, are coupled.

Subsequently, $\bar{V}(t) = \mathbb{E}\big[\bar{\tilde{x}}^{ex}(t)\big(\bar{\tilde{x}}^{ex}(t)\big)^T\big]$ satisfies

$$\dot{\bar{V}} = \tilde{\mathbb{A}}\bar{V} + \bar{V}\tilde{\mathbb{A}}^T + \tilde{\mathbb{D}} \begin{bmatrix} 0_{r\times r} & & & \\ & I_{r\times r} & & \\ & & 0_{rK\times rK} & \\ & & & I_{r\times r} \end{bmatrix} \tilde{\mathbb{D}}^T, \tag{2.64}$$

and if we put $\tilde{\mathbb{D}} = -\tilde{\mathbf{1}}^T \mathbb{D}$, we obtain

$$\dot{\bar{V}} = \tilde{\mathbb{A}}\bar{V} + \bar{V}\tilde{\mathbb{A}}^T + \tilde{\mathbb{Q}}\tilde{\mathbb{Q}}^T, \tag{2.65}$$

where

$$\tilde{\mathbb{Q}}\tilde{\mathbb{Q}}^T = \tilde{\mathbf{1}}^T \begin{bmatrix} 0_{n\times n} & & \\ & Q_{w_0} & \\ & & K_0 R_{v_0} K_0^T \end{bmatrix} \tilde{\mathbf{1}}. \tag{2.66}$$

To guarantee the convergence of the solution to the corresponding Lyapunov equation to a unique, symmetric and positive definite solution, we assume:

Assumption 2.7. *The pair* $[\tilde{\mathbb{A}}, \tilde{\mathbb{Q}}]$ *is controllable.*

□

Remark 2.3. For the case where the major agent has complete observation on its own state, and each minor agent has complete observations on their own state and the major agent's state we have

$$\bar{\tilde{x}}^{k,ex}(t) = 0, \quad t \geq 0, \tag{2.67}$$

$$\mathbb{E}\{x_0 | \mathcal{F}_0^y\} = x_0, \tag{2.68}$$

$$\mathbb{E}\{\bar{x} | \mathcal{F}_0^y\} = \bar{x}, \tag{2.69}$$

where (2.69) holds since the major agent can compute the real value of $\bar{x}$ by observing its own state. Hence the mean field equation (6.54) reduces to that of completely observed major minor LQG MFG systems (see [16]). □

Remark 2.4 (Estimate of Infinite-Population Average Estimation Error). The solution to (2.60) is given by

$$\bar{\tilde{x}}^{ex}(t) = \Phi(t,0)\bar{\tilde{x}}^{ex}(0) + \int_0^t \Phi(t,\tau)\tilde{\mathbb{D}}[0_{1\times r}, dw_0^T, 0_{1\times rK}, dv_0^T]^T d\tau, \tag{2.70}$$

where $\Phi(t,\tau) = \exp\big(\tilde{\mathbb{A}}(t-\tau)\big)$. The initial estimation error of the minor agent $\mathcal{A}_i$ is given by

$$\tilde{x}_i^{k,ex}(0) = \begin{bmatrix} \hat{x}_{i|\mathcal{F}_i^y}(0) - x_i(0) \\ \hat{x}_{0|\mathcal{F}_i^y}(0) - x_0(0) \\ \hat{\bar{x}}_{|\mathcal{F}_0^y}(0) - \bar{x}(0) \\ (\hat{x}_{0|\mathcal{F}_0^y})_{|\mathcal{F}_i^y}(0) - \hat{x}_{0|\mathcal{F}_0^y}(0) \\ (\hat{\bar{x}}_{|\mathcal{F}_0^y})_{|\mathcal{F}_i^y}(0) - \hat{\bar{x}}_{|\mathcal{F}_0^y}(0) \end{bmatrix} = \begin{bmatrix} -x_i(0) \\ -x_0(0) \\ 0_{nK\times 1} \\ 0_{n\times 1} \\ 0_{nK\times 1} \end{bmatrix}, \tag{2.71}$$

since the partial observation information sets $\mathcal{F}_i^y, 0 \le i \le N$, at time $t_0 = 0$ are null sets, the conditional expectations turn into total expectations which according to *Assumption 2.1* their value is zero. Hence, the infinite-population limit of the average initial estimation error of the minor agents of subpopulation k is given by

$$\bar{\tilde{x}}^{k,ex}(0) = [0_{1\times n}, x_0^T(0), 0_{1\times nK}, 0_{1\times n}, 0_{1\times nK}]^T, \tag{2.72}$$

where *Assumption 2.1* is again used, and hence $\mathbb{E}[\bar{\tilde{x}}^{k,ex}(0)|\mathcal{F}_i^y] = 0$. Then the conditional expectation of $\bar{\tilde{x}}^{ex}(t)$ with respect to $\mathcal{F}_i^y, 0 \le i \le N$, i.e. $\hat{\bar{\tilde{x}}}^{ex}_{|\mathcal{F}_i^y}(t)$, is given by

$$\begin{aligned} \hat{\bar{\tilde{x}}}^{ex}_{|\mathcal{F}_i^y}(t) &\triangleq \mathbb{E}[\bar{\tilde{x}}^{ex}(t)|\mathcal{F}_i^y] \\ &= \Phi(t,0)\mathbb{E}[\bar{\tilde{x}}^{ex}(0)|\mathcal{F}_i^y] + \mathbb{E}\Big[\int_0^t \Phi(t,\tau)\tilde{\mathbb{D}} \begin{bmatrix} 0_{r\times 1} \\ dw_0 \\ 0_{nr\times 1} \\ dv_0 \end{bmatrix} d\tau \Big| \mathcal{F}_i^y\Big] \qquad (2.73) \\ &= 0, \qquad (2.74) \end{aligned}$$

where the second term in (2.73) is zero due to the independence of $\{w_i, 0 \le i \le N\}$ and $\{v_i, 0 \le i \le N\}$. $\square$

Next we define

$$M_1 = \begin{bmatrix} A_1 - B_1 R^{-1} B_1^T \Pi_{1,11} & & \\ & \ddots & \\ & & A_K - B_K R^{-1} B_K^T \Pi_{K,11} \end{bmatrix},$$

$$M_2 = \begin{bmatrix} B_1 R^{-1} B_1^T \Pi_{1,13} \\ \vdots \\ B_K R^{-1} B_K^T \Pi_{K,13} \end{bmatrix}, \quad M_3 = \begin{bmatrix} A_0 & 0 & 0 \\ \bar{G} & \bar{A} & 0 \\ \bar{G} & -M_2 & M_1 \end{bmatrix},$$

$$L_{0,H} = Q_0^{1/2}[I, 0, -H_0^{\pi}]. \tag{2.75}$$

The final set of assumptions is as follows:

Assumption 2.8. *The pair* $(L_{0,H}, M_3)$ *is observable.*

Assumption 2.9. *The pair* $(L_a, \mathbb{A}_0 - (\rho/2)I)$ *is detectable, and for each* $k, 1 \leq k \leq K$*, the pair* $(L_b, \mathbb{A}_k - (\rho/2)I)$ *is detectable, where* $L_a = Q_0^{1/2}[I, -H_0^{\pi}]$ *and* $L_b = Q^{1/2}[I, -H_1, -H_2^{\pi}, 0_{n\times(n+nK)}]$*. The pair* $(\mathbb{A}_0 - (\rho/2)I, \mathbb{B}_0)$ *is stabilizable and* $(\mathbb{A}_k - (\rho/2)I, \mathbb{B}_k)$ *is stabilizable for each* $k, 1 \leq k \leq K$*.*

Assumption 2.10. *There exists a stabilizing solution* Π_0*,* s_0*,* Π_k*,* s_k*,* $\bar{A}_k$*,* $\bar{G}_k$*,* $\bar{H}_k$*,* $\bar{L}_k$*,* $\bar{J}_k$*,* $\bar{m}_k$ *to the major-minor mean field equations (5.51) in the sense that the matrices*

$$\mathbb{A}_0 - \mathbb{B}_0 R_0^{-1} \mathbb{B}_0^T \Pi_0 - \frac{\rho}{2} I,$$
$$\mathbb{A}_k - \mathbb{B}_k R^{-1} \mathbb{B}_k^T \Pi_k - \frac{\rho}{2} I, \quad 1 \leq k \leq K,$$

are asymptotically stable, and

$$\sup_{t\geq 0, 1\leq k\leq K} e^{-\frac{\rho}{2}t}(|s_0(t)| + |s_k(t)| + |\bar{m}_k(t)|) < \infty.$$

Theorem 2.2 (ϵ-Nash Equilibria for PO LQG MM-MFG Systems)**.** *Subject to Assumptions 2.1-2.10, the KF-MFG state estimation scheme* (2.27)-(3.56) *and* (3.65)-(2.44) *together with the MM-MFG equation scheme* (5.51) *generate an infinite family of stochastic control laws* $\hat{\mathcal{U}}_{MF}^{\infty}$*, with finite sub-families* $\hat{\mathcal{U}}_{MF}^{N} \triangleq \{u_i^{\circ}; 0 \leq i < N\}$*,* $1 \leq N < \infty$*, given by* (2.32) *and* (2.46)*, such that*

(i) $\hat{\mathcal{U}}_{MF}^{\infty}$ *yields a unique Nash equilibrium within the set of linear controls* $\mathcal{U}_{i,y}^{\infty,L}$ *and* $\mathcal{U}_{0,y}^{L}$ *such*

that

$$J_i^{\infty}(u_i^{\circ}, u_{-i}^{\circ}) = \inf_{u_i \in \mathcal{U}_{i,y}^{\infty,L}} J_i^{\infty}(u_i, u_{-i}^{\circ});$$

(ii) *All agent systems $0 \le i \le N$, are $e^{-\frac{\rho}{2}t}$ discounted second order stable in the sense that*

$$\sup_{t\ge 0,\, 0\le i\le N} e^{-\frac{\rho}{2}t}\mathbb{E}\Big(\|\hat{x}_{i|\mathcal{F}_i^y}\|^2 + \|\hat{\bar{x}}_{|\mathcal{F}_i^y}\|^2 + \|(\hat{x}_{0|\mathcal{F}_0^y})_{|\mathcal{F}_i^y}\|^2 + \|(\hat{\bar{x}}_{|\mathcal{F}_0^y})_{|\mathcal{F}_i^y}\|^2\Big) < C,$$

with C independent of N;

(iii) $\{\hat{\mathcal{U}}_{MF}^N; 1 \le N < \infty\}$ *yields a unique ϵ-Nash equilibrium within the class of linear control laws $\mathcal{U}_{i,y}^{N,L}$ and $\mathcal{U}_{0,y}^{L}$ for all ϵ, i.e. for all $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $N \ge N(\epsilon)$;*

$$J_i^{s,N}(\hat{u}_i^{\circ}, \hat{u}_{-i}^{\circ}) - \epsilon \le \inf_{u_i \in \mathcal{U}_y^{N,L}} J_i^{s,N}(u_i, \hat{u}_{-i}^{\circ}) \le J_i^{s,N}(\hat{u}_i^{\circ}, \hat{u}_{-i}^{\circ}),$$

where the major agent's and the generic minor agent's performance function $J_i^{s,N}(u_i^{\circ}, u_{-i}^{\circ})$, $u_i \in \mathcal{U}_{i,y}^{N,L}$, $0 \le i \le N$, is given by

$$J_i^N(u_i, u_{-i}) + \hat{E}_N,$$

where $J_i^N(u_i, u_{-i})$ is as in the completely observed case, $\hat{E}_N > 0$, and when $u_i = \hat{u}_i^{\circ}$ the following limits hold:

- $\lim_{N\to\infty} J_i^N(\hat{u}_i^{\circ}, \hat{u}_{-i}^{\circ}) = J_i^{\infty}(\hat{u}_i^{\circ}, \hat{u}_{-i}^{\circ})$,
- $\lim_{N\to\infty} \hat{E}_N = \int_0^{\infty} e^{-\rho t} tr[Q^{\pi} V] dt$,
  where $V(t)$ is the solution to (3.56) for the major agent and the solution to (3.65) for a generic minor agent.

Proof. Generalizing the standard methodology in [38] and [39], we first decompose the state processes into their estimates and their estimation errors orthogonal to the corresponding estimates. Substituting the decomposed states into the performance functions and applying the smoothing property of conditional expectations with respect to the increasing filtration families $\mathcal{F}_i^y$ and $\mathcal{F}_0^y$ to the major and minor cost functionals respectively, we obtain the separated performance functions. This technique is applied to both finite and infinite population cases which yields the best response controls $\{\hat{u}_i^{\circ},\ 0 \le i \le N\}$ as optimal tracking controls for the major and minor agents in the infinite population case (see [21] for the case where only the minor agent has partial observations on the major agent's state). Specifically we form the following decompositions where the superscript 's' on the resulting performance functions indicates the separation into control dependent and control independent summands.

1. Major Agent's State Decomposition
   Finite Population:

$$\begin{bmatrix} x_0 \\ x^{(N)} \end{bmatrix} = \begin{bmatrix} \hat{x}_{0|\mathcal{F}_0^y} \\ \hat{x}^{(N)}_{|\mathcal{F}_0^y} \end{bmatrix} + \begin{bmatrix} x_0 - \hat{x}_{0|\mathcal{F}_0^y} \\ x^{(N)} - \hat{x}^{(N)}_{|\mathcal{F}_0^y} \end{bmatrix}.$$

   Infinite Population:

$$\begin{bmatrix} x_0 \\ \bar{x} \end{bmatrix} = \begin{bmatrix} \hat{x}_{0|\mathcal{F}_0^y} \\ \hat{\bar{x}}_{|\mathcal{F}_0^y} \end{bmatrix} + \begin{bmatrix} x_0 - \hat{x}_{0|\mathcal{F}_0^y} \\ \bar{x} - \hat{\bar{x}}_{|\mathcal{F}_0^y} \end{bmatrix}.$$

2. Major Agent's Cost Functional Separation
   Finite Population:

$$\begin{aligned} J_0^{s,N}(u_0, u_{-0}) = \mathbb{E}\bigg[\int_0^\infty e^{-\rho t}\Big\{\big\|\hat{x}_{0|\mathcal{F}_0^y} - H_0\hat{x}^{(N)}_{|\mathcal{F}_0^y} - \eta_0\big\|^2_{Q_0} + \|u_0\|^2_{R_0}\Big\}dt\bigg] \\ + \mathbb{E}\bigg[\int_0^\infty e^{-\rho t}\big\|(x_0 - \hat{x}_{0|\mathcal{F}_0^y}) - H_0(x^{(N)} - \hat{x}^{(N)}_{|\mathcal{F}_0^y})\big\|^2_{Q_0}dt\bigg]. \end{aligned} \quad (2.76)$$

   Infinite Population:

$$\begin{aligned} J_0^{s,\infty} = \mathbb{E}\bigg[\int_0^\infty e^{-\rho t}\Big\{\big\|\hat{x}_{0|\mathcal{F}_0^y} - H_0^\pi\hat{\bar{x}}_{|\mathcal{F}_0^y} - \eta_0\big\|^2_{Q_0} + \|u_0\|^2_{R_0}\Big\}dt\bigg] \\ + \mathbb{E}\bigg[\int_0^\infty e^{-\rho t}\big\|(x_0 - \hat{x}_{0|\mathcal{F}_0^y}) - H_0^\pi(\bar{x} - \hat{\bar{x}}_{|\mathcal{F}_0^y})\big\|^2_{Q_0}dt\bigg]. \end{aligned} \quad (2.77)$$

3. Minor Agent's State Decomposition
   Finite Population:

$$\begin{bmatrix} x_i \\ x_0 \\ x^{(N)} \end{bmatrix} = \begin{bmatrix} \hat{x}_{i|\mathcal{F}_i^y} \\ \hat{x}_{0|\mathcal{F}_i^y} \\ \hat{x}^{(N)}_{|\mathcal{F}_i^y} \end{bmatrix} + \begin{bmatrix} x_i - \hat{x}_{i|\mathcal{F}_i^y} \\ x_0 - \hat{x}_{0|\mathcal{F}_i^y} \\ x^{(N)} - \hat{x}^{(N)}_{|\mathcal{F}_i^y} \end{bmatrix}.$$

Infinite Population:

$$\begin{bmatrix} x_i \\ x_0 \\ \bar{x} \end{bmatrix} = \begin{bmatrix} \hat{x}_{i|\mathcal{F}_i^y} \\ \hat{x}_{0|\mathcal{F}_i^y} \\ \hat{\bar{x}}_{|\mathcal{F}_i^y} \end{bmatrix} + \begin{bmatrix} x_i - \hat{x}_{i|\mathcal{F}_i^y} \\ x_0 - \hat{x}_{0|\mathcal{F}_i^y} \\ \bar{x} - \hat{\bar{x}}_{|\mathcal{F}_i^y} \end{bmatrix}.$$

4. Minor Agent's Cost Functional Separation
   Finite Population:

$$\begin{aligned} J_i^{s,N}(u_i, u_{-i}) &= \mathbb{E}\Big[\int_0^\infty e^{-\rho t}\Big\{\big\|\hat{x}_{i|\mathcal{F}_i^y} - H_1\hat{x}_{0|\mathcal{F}_i^y} - H_2\hat{x}^{(N)}_{|\mathcal{F}_i^y} - \eta\big\|_Q^2 + \|u_i\|_R^2\Big\}dt\Big] \\ &+ \mathbb{E}\Big[\int_0^\infty e^{-\rho t}\big\|(x_i - \hat{x}_{i|\mathcal{F}_i^y}) - H_1(x_0 - \hat{x}_{0|\mathcal{F}_i^y}) - H_2(x^{(N)} - \hat{x}^{(N)}_{|\mathcal{F}_i^y})\big\|_Q^2 dt\Big]. \quad (2.78) \end{aligned}$$

Infinite Population:

$$\begin{aligned} J_i^{s,\infty} &= \mathbb{E}\Big[\int_0^\infty e^{-\rho t}\Big\{\big\|\hat{x}_{i|\mathcal{F}_i^y} - H_1\hat{x}_{0|\mathcal{F}_i^y} - H_2^\pi\hat{\bar{x}}_{|\mathcal{F}_i^y} - \eta\big\|_Q^2 + \|u_i\|_R^2\Big\}dt\Big] \\ &+ \mathbb{E}\Big[\int_0^\infty e^{-\rho t}\big\|(x_i - \hat{x}_{i|\mathcal{F}_i^y}) - H_1(x_0 - \hat{x}_{0|\mathcal{F}_i^y}) - H_2^\pi(\bar{x} - \hat{\bar{x}}_{|\mathcal{F}_i^y})\big\|_Q^2 dt\Big]. \quad (2.79) \end{aligned}$$

As can be seen, the first integral expressions in (2.76), (2.77), (2.78) and (2.79) depend on the estimated states generated by the estimation schemes (2.27) and (2.44) for the major agent and minor agents respectively, and the second integral expressions depend only upon the respective estimation errors and on the solutions to the associated Riccati equations. The latter expressions are independent of the control actions and generate the additional cost $\hat{E}_N$ in the finite population case incurred by the errors in the estimation process.

Next, the resulting infinite population tracking problems are solved for the major and minor agents in their separated forms. The control dependent summands in (2.77) have exactly the same structure in terms of the functional dependence on the estimated states as the infinite population cost functionals in the complete observation case have on the states. Moreover, the control dependent summands in (2.79) have exactly the same structure in terms of the functional dependence on the estimated states as the infinite population cost functional for the system (2.39) with complete observations on its own state, the major agent's state, and the major agent's estimates of its own state and the mean field. Hence, by the Separation Principle the infinite population Nash Certainly Equivalence equilibrium controls are given by $\{\hat{u}_i^\circ,\ 0 \leq i \leq N\}$ in the theorem statement. Finally the infinite population control actions are applied to the finite population systems and the fact that these yield (i) $e^{-\frac{\rho}{2}t}$ second order system stability, and (ii) ϵ-Nash equilibrium property, is established by the standard approximation analysis parallel to that of completely observed major-minor LQG MFG systems (see [9], [16]). □

Remark 2.5. We note that $(\hat{x}_{0|\mathcal{F}_0^y})_{|\mathcal{F}_i^y}$ and $(\hat{\hat{x}}_{|\mathcal{F}_0^y})_{|\mathcal{F}_i^y}$ do not appear in the minor agent's state decomposition and in its separated performance function but that they are used in the extended estimated state recursion (2.44) and hence appear in the control action for a minor agent in (2.46). □

Remark 2.6. The non-uniqueness of Nash equilibria which may occur in classical LQG stochastic dynamic games with specified information sets [40, 41] does not occur in this analysis. This holds since, for the specified maximal individual information sets, and subject to the hypotheses of *Theorem 2.2* giving unique solutions to the MFG Consistency equations (as functions of the system parameters), a unique linear best response function is obtained for each agent with respect to its stochastic control problem arising from its performance function in the infinite population limit. We note that any set of controls generating a Nash equilibrium will yield the same consistency equations whose solution depends only on the system parameters. □

2.4 Simulations

Consider a system of 100 minor agents and a single major agent. The system matrices $\{A_k,\ B_k,\ 1 \leq k \leq 100\}$ for the minor agents are uniformly defined as

$$A \triangleq \begin{bmatrix} -0.05 & -2 \\ 1 & 0 \end{bmatrix},\ B \triangleq \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and for the major agent we have

$$A_0 \triangleq \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix},\quad B_0 \triangleq \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The parameters used in the simulation are: $t_{final} = 25\,\text{sec}$, $\Delta t = 0.01\,\text{sec}$, $\sigma_{w_0} = \sigma_{w_i} = 0.009$, $\sigma_{v_0} = \sigma_{v_i} = 0.0003$, $\rho = 0.9$, $\eta_0 = \eta = [0.25, 0.25]^T$, $Q_0 = Q = I_{2\times 2}$, $R_0 = R =$

$1,\ H_0 = H_1 = H_2 = 0.6 \times I_{2\times 2},\ G = 0_{2\times 2}$. The true and estimated state trajectories, and the estimation errors for a single realization can be displayed for the entire population of 101 agents together, but in figures 2.1-2.6 only 10 minor agents are shown for the sake of clarity.

2.5 Conclusions

In this chapter, PO MM LQG MFG problems with general information patterns are studied where (i) the major agent has partial observations on its own state, and (ii) each minor agent has partial observations on its own state and the major agent's state. For the general case of indefinite LQG MFG systems, the existence of ϵ-Nash equilibria together with the individual agents' control laws generating them are established via the Separation Principle. The assumption of partial observations for all agents leads to a new situation involving the recursive estimation by each minor agent of the major agent's estimate of its own state. To the best of our knowledge, the dynamic game theoretic equilibrium which is established in this chapter constitutes a rare case wherein agents explicitly generate estimates of another agent's beliefs. Moreover, this does not give rise to an infinite regress due to the information asymmetry of the major and minor agents.

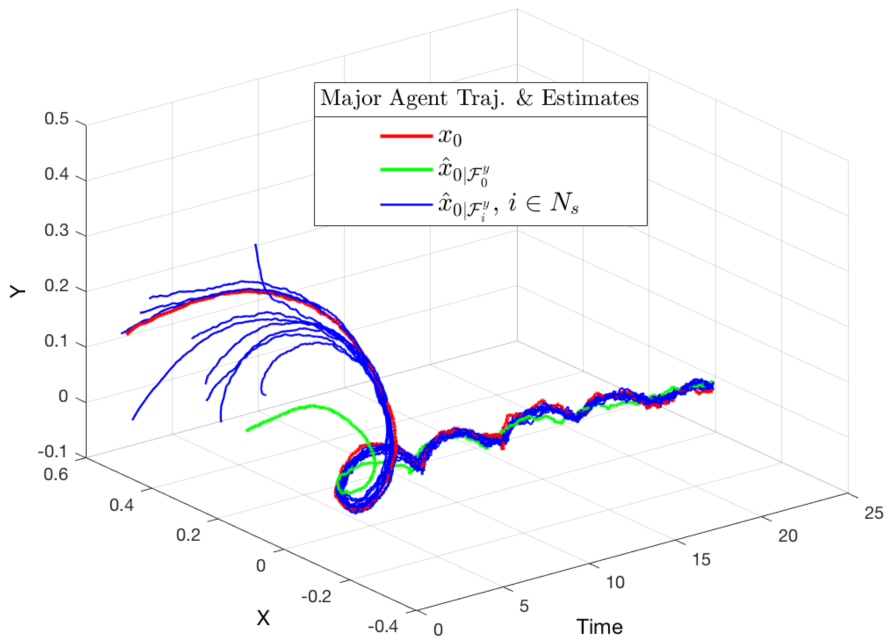

Figure 2.1: The Major agent's true and estimated trajectories.

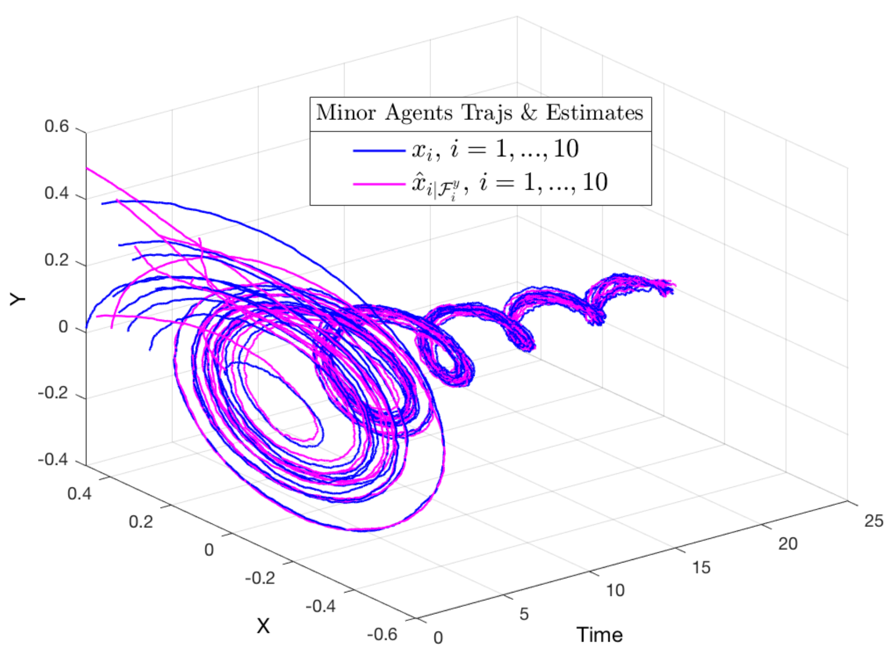

Figure 2.2: 10 Minor agents' true and estimated trajectories.

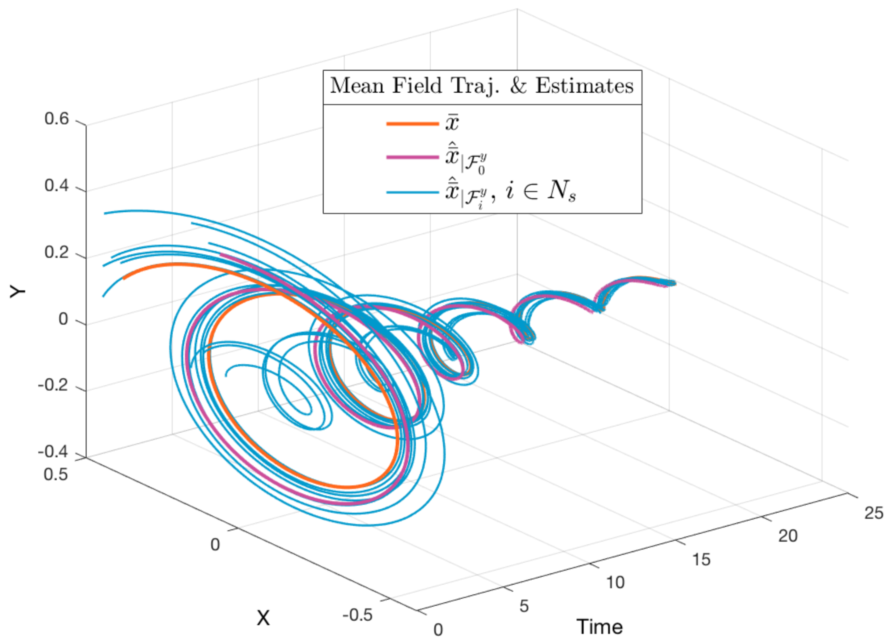

Figure 2.3: The mean field true and estimated trajectories.

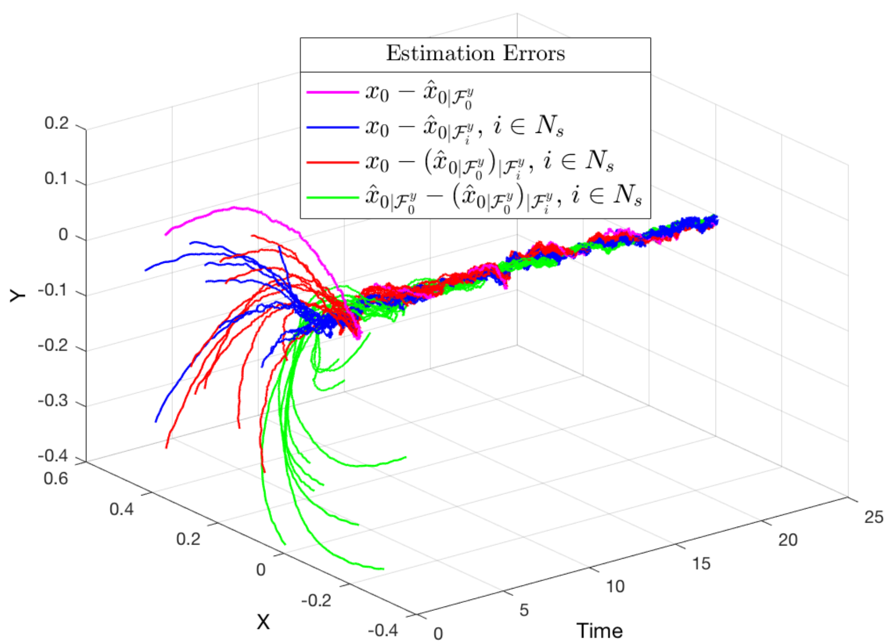

Figure 2.4: The estimation errors of the major agent's trajectory.

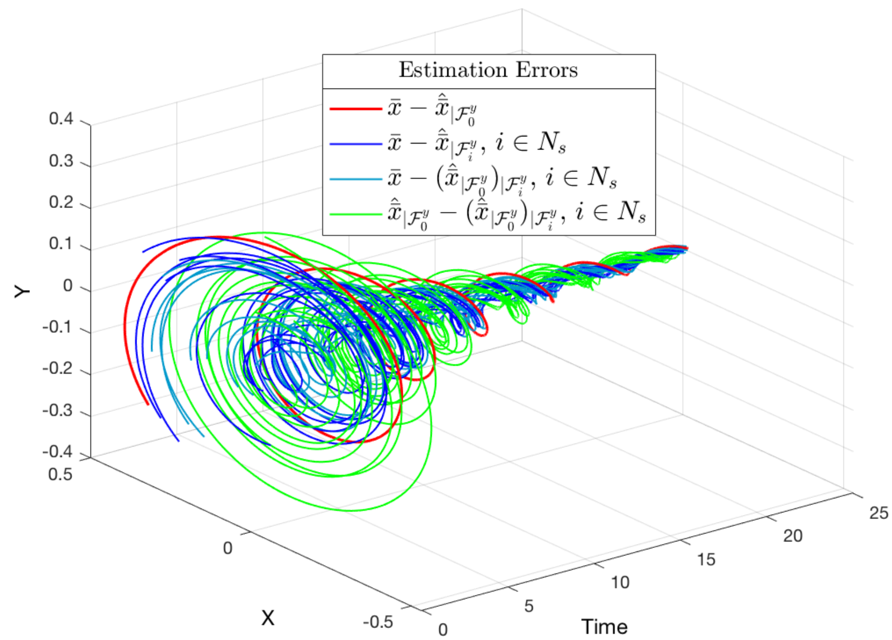

Figure 2.5: The estimation errors of the mean field trajectory.

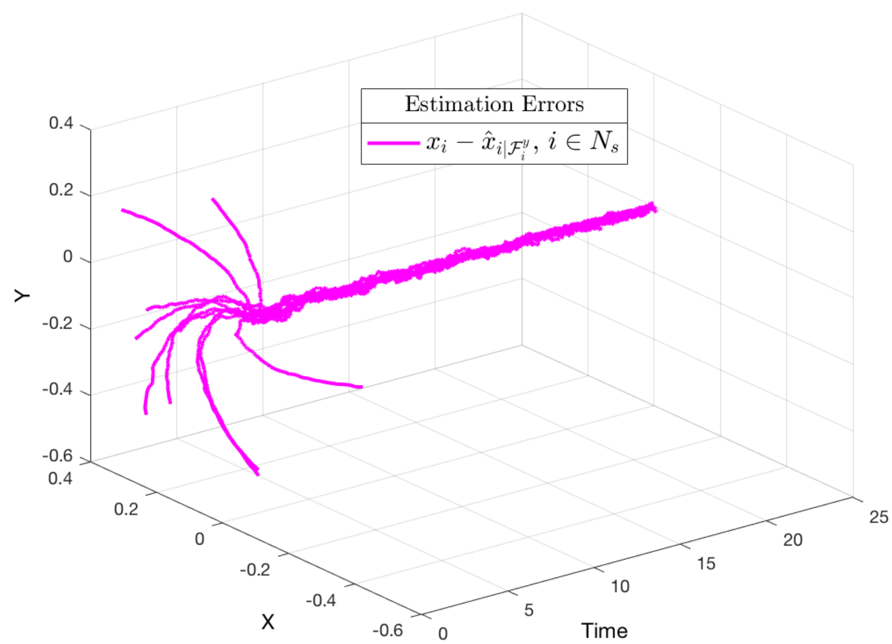

Figure 2.6: The estimation errors of 10 minor agents' trajectories.

Chapter 3

Optimal Execution Problems in Finance with Partial Observations

3.1 Introduction

The PO MM LQG MFG theory was first applied to an optimal execution problem with the linear models of [25] in [32] where an institutional investor, interpreted as a major agent, aims to liquidate a specific amount of shares and it has only partial observations of its own state (which includes its inventory). Furthermore, there is a large population of high frequency traders (HFTs), interpreted as minor agents, who wish to liquidate their shares, and each of them has partial observations of its own state and the major agent's state (which include the corresponding inventories). In the current chapter, this work is refined in the formulation of the market dynamics in the MFG framework, and also is extended to consider two populations of HFTs with liquidation or acquisition objectives who wish to, respectively, liquidate or acquire a certain number of shares within a specific duration of time. MM (indefinite) LQG MFG theory is then utilized to establish the existence of ϵ-Nash equilibria together with the best response trading strategies such that each agent attempts to maximize its own wealth and avoid the occurrence of large execution prices, and large trading accelerations which are appropriately weighted in the agent's performance function. The results of this chapter have been presented in [33, 34].

We note that the terms major trader (respectively, minor trader), and institutional trader (respectively, HFT) are used interchangeably in this chapter.

The chapter is organized as follows. Section 3.2 is devoted to the description of trading dynamics in the market and the execution problem. In Section 3.3 the optimal execution problem is formulated in the mean field game framework. Completely observed and partially observed optimal execution problems are then addressed in Sections 3.4 and 3.5, respectively. Section 3.6 presents the simulation results.

3.2 Trading Dynamics of Agents in Market

As stated in the Introduction, the institutional investor is considered as a major agent in the mean field model of the market which liquidates its shares and the HFTs are considered as minor agents, where two types of them are considered: liquidators and acquirers. Employing the trading model in [25], the trading dynamics of the major agent and any generic minor agent in the market are described by the linear time evolution of the inventories, trading rates and prices while the bilinear cash process appears in the quadratic performance function for each agent.

3.2.1 Inventory Dynamics

It is assumed that the institutional investor liquidates its inventory of shares, $Q_0(t)$, by trading at a rate $\nu_0(t)$ during the trading period $[0, T]$. Hence the major agent's inventory dynamics is given by

$$dQ_0(t) = \nu_0(t)dt + \sigma_0^Q dw_0^Q(t), \quad 0 \leq t \leq T,$$

where w_0^Q is a Wiener process modeling the noise in the inventory information that the institutional trader collects from its branches in different locations; σ_0^Q is a positive scalar and we assume that $Q_0(0) \gg 1$. The same dynamical model is adopted for the trading dynamics of a generic HFT

$$dQ_i(t) = \nu_i(t)dt + \sigma_i^Q dw_i^Q(t), \quad 1 \leq i \leq N_a + N_l,\ 0 \leq t \leq T$$

where N_a and N_l are respectively liquidator and acquirer populations of N minor traders, i.e. $N = N_a + N_l$, w_i^Q is a Wiener process that models the HFT's information noise, σ_i^Q is a positive scalar, $\nu_i(t)$ is the agent's rate of trading which can be positive or negative depending on whether the agent is acquirer or liquidator, respectively; $Q_i(t)$ is the minor liquidator agent's remaining shares at time t, or the shares the minor acquirer agent has bought until time t. However, the initial inventories of the HFTs, $\{Q_i(0), 1 \leq i \leq N_a + N_l\}$, are not considered to be large. We assume that the trading rate of the major agent is controlled via $u_0(t)$ as

$$d\nu_0(t) = u_0(t)dt, \quad 0 \leq t \leq T,$$

where the trading strategy $u_0(t)$ can be seen to be the trading acceleration of the major trader. Correspondingly, $u_i(t)$ controls the trading rate of minor agent, $\mathcal{A}_i$, by

$$d\nu_i(t) = u_i(t)dt, \quad 1 \leq i \leq N_a + N_l,\ 0 \leq t \leq T.$$

3.2.2 Price Dynamics

The trading rate of the major agent and the average trading rate of the minor agents give rise to the asset midprice which models the permanent effect of agents' trading rates on the market price. Further, each agent has a temporary effect on the asset price which only persists during the action of the trade and which determines the execution price, that is to say the price at which each agent can trade.

Asset Midprice

We model the dynamics of the asset midprice, as seen from the major agent's viewpoint, by

$$dF_0(t) = \big(\lambda_0 \nu_0(t) + \frac{\lambda}{N}\sum_{i=1}^{N} \nu_i(t)\big)dt + \sigma dw_0^F(t), \quad 0 \leq t \leq T,$$

where the Wiener process $w_0^F(t)$ models the aggregate effect of all traders in the market which - unlike the major and minor agents $\mathcal{A}_0$, $\mathcal{A}_i$, - have no partial observations on any of the state variables appearing in the dynamical market model (these are termed uninformed traders). Further, σ denotes the intensity of the market volatility and $\lambda_0, \lambda \geq 0$ denote the strength of the linear permanent impact of the major and minor agents' tradings on the asset midprice, respectively. Similarly, we model the asset midprice dynamics, as seen by a minor agent $\mathcal{A}_i$, by

$$dF_i(t) = \big(\lambda_0 \nu_0(t) + \frac{\lambda}{N}\sum_{i=1}^{N} \nu_i(t)\big)dt + \sigma dw_i^F(t), \quad 1 \leq i \leq N_a + N_l, \quad 0 \leq t \leq T,$$

where the Wiener process, $w_i^F(t)$, represents the mass effect of all uninformed traders in the market. The time differences between agents in getting data from fast changing limit order book make the Wiener processes, $w_i^F,\ 0 \le i \le N_a + N_l$ independent.

Execution Price

The major agent's execution price $S_0(t)$ evolution is assumed to be given by

$$dS_0(t) = dF_0(t) + a_0 d\nu_0(t), \quad 0 \le t \le T, \tag{3.1}$$

where $a_0 \ge 0$ is the temporary impact strength of the major agent on the asset midprice. Likewise, a minor agent's execution price, $S_i(t)$, is assumed to evolve as

$$dS_i(t) = dF_i(t) + a d\nu_i(t), \quad 1 \le i \le N_a + N_l,\ 0 \le t \le T, \tag{3.2}$$

where a models the temporary impact of a minor agent's trading on its execution price.

3.2.3 Cash Process

The cash processes for the major agent and a generic minor agent, $Z_0(t)$, $Z_i(t)$, are given by

$$dZ_0(t) = -S_0(t)dQ_0(t), \quad 0 \le t \le T, \tag{3.3}$$

$$dZ_i(t) = -S_i(t)dQ_i(t), \quad 1 \le i \le N_a + N_l,\ 0 \le t \le T, \tag{3.4}$$

where $Z_0(t),\ Z_i(t),\ 1 \le i \le N_l$, are the cash obtained through liquidation of shares, and $Z_i(t), 0 \le i \le N_a$, is the cash paid for acquisition of shares up to time t. We note that the value of $dQ_0(t)$ in a stock sale is negative and hence for positive $S_0(t)$, $Z_0(t)$ increases.

3.2.4 Cost Function

Major Liquidator Trader

The objective for the major trader is to liquidate $\mathcal{N}_0$ shares and maximize the cash it holds at the end of the trading horizon, i.e. maximize $Z_0(T)$, and if the remaining inventory at the final time T is $Q_0(T)$, it can liquidate it at a lower price than the market asset price, reflected in the cost function by $Q_0(T)(F_0(T) - \alpha_0 Q_0(T))$. Further, the major trader's utility in minimizing the inventory over the period $[0,T]$ is modeled by including the penalty $\int_0^T Q_0^2(s)ds$ in its objective function, and the utility of avoiding very high execution prices, large trading intensities and large trading accelerations by including the terms $S_0^2(T)$, $\int_0^T S_0^2(s)ds$, $\nu_0^2(T)$, $\int_0^T \nu_0^2(s)ds$ and $\int_0^T R_0 u_0^2(s)ds$ in the objective function. Therefore, its cost function to be minimized is given by

$$J_0(u_0,u_{-0}) = \mathbb{E}\Big[-\psi_0 Z_0(T) - \mu_0 Q_0(T)\big(F_0(T)-\alpha_0 Q_0(T)\big) + \xi_0 S_0^2(T) + \gamma_0 \nu_0^2(T) + \int_0^T \big(\phi_0 Q_0^2(s) + \delta_0 S_0^2(s) + \theta_0 \nu_0^2(s) + R_0 u_0^2(s)\big)ds\Big], \quad (3.5)$$

where ψ_0, μ_0, α_0, ξ_0, γ_0, ϕ_0, δ_0, θ_0 and R_0 are positive scalars, and $u_{-0} := (u_1, ..., u_{N_a+N_l})$ are trading strategies of the minor traders. Note that for larger values of ϕ_0 the trader attempts to liquidate its inventory more quickly.

Minor Liquidator Trader

In a similar way, the objective function to be minimized for a liquidator HFT who wants to liquidate $\mathcal{N}_l$ shares during the time interval $[0,T]$ is given by

$$J_i(u_i,u_{-i}) = \mathbb{E}\Big[-\psi_l Z_i(T) - \mu_l Q_i(T)\big(F_i(T)-\alpha_l Q_i(T)\big) + \xi_l S_i^2(T) + \gamma_l \nu_i^2(T) + \int_0^T \big(\phi_l Q_i^2(s) + \delta_l S_i^2(s) + \theta_l \nu_i^2(s) + R_l u_i^2(s)\big)ds\Big],\ 1 \le i \le N_l, \quad (3.6)$$

where ψ_l, μ_l, α_l, ξ_l, γ_l, ϕ_l, δ_l, θ_l and R_l are positive scalars, and $u_{-i} := (u_0, u_1, ..., u_{i-1}, u_{i+1}, ..., u_{N_a+N_l})$. Note that $\mathcal{N}_l \ll \mathcal{N}_0$.

Minor Acquirer Trader

The objective for a minor acquirer trader is to buy $\mathcal{N}_a$ shares over the trading horizon $[0,T]$, while it minimizes the execution cost including the cash $Z_i(T)$ paid up to time T, and the cash must be paid at time T to buy the remaining shares at once at a higher price than the market's asset price, i.e. $(\mathcal{N}_a - Q_i(T))\big(F_i(T) + \alpha_a(\mathcal{N} - Q_i(T))\big)$. It also intends to avoid high execution prices, large trading intensities and large trading accelerations modeled by including

$\xi_a S_i^2(T) + \gamma_a \nu_i^2(T) + \int_0^T \big(\delta_a S_i^2(s) + \theta_a \nu_i^2(s) + R_a u_i^2(s)\big) ds$ in its objective function

$$\begin{aligned} J_i(u_i, u_{-i}) = \mathbb{E}\Big[& \psi_a Z_i(T) + \mu_a(\mathcal{N}_a - Q_i(T))\big(F_i(T) + \alpha_a(\mathcal{N}_a - Q_i(T))\big) + \xi_a S_i^2(T) + \\ & \gamma_a \nu_i^2(T) + \int_0^T \big(\phi_a(\mathcal{N}_a - Q_i(s))^2 + \delta_a S_i^2(s) + \theta_a \nu_i^2(s) + R_a u_i^2(s)\big) ds\Big], \ 1 \leq i \leq N_a, \end{aligned} \tag{3.7}$$

where $\int_0^T \phi_a(\mathcal{N}_a - Q_i(s))^2 ds$ is to penalize the agent for the remaining shares to be bought up to T and to expedite the acquisition. The parameters ψ_a, μ_a, α_a, ξ_a, γ_a, ϕ_a, δ_a, θ_a and R_a are positive scalars, and $u_{-i} := (u_0, u_1, ..., u_{i-1}, u_{i+1}, ..., u_{N_a+N_l})$.

3.3 Mean Field Game Formulation of Optimal Execution Problems

In this section we formulate the optimal execution problem in the major minor LQG MFG framework.

3.3.1 Finite populations

Major Agent

The stochastic optimal control problem for the major trader is modeled as

$$d\nu_0(t) = u_0(t)dt, \tag{3.8}$$

$$dQ_0(t) = \nu_0(t)dt + \sigma_0^Q dw_0^Q(t), \tag{3.9}$$

$$dS_0(t) = \big(\lambda_0 \nu_0(t) + \frac{\lambda}{N}\sum_{i=1}^{N} \nu_i(t)\big)dt + a_0 u_0(t)dt + \sigma dw_0^F(t), \tag{3.10}$$

with the cost function

$$\begin{aligned} J_0(u_0, u_{-0}) = \mathbb{E}\Big[& -\mu_0 Q_0(T)\big(S_0(T) - a_0\nu_0(T) - \alpha_0 Q_0(T)\big) + \xi_0 S_0(T)^2 + \gamma_0 \nu_0^2(T) + \\ & \int_0^T \Big(\phi_0 Q_0^2(s) + \psi_0 S_0(s)\nu_0(s) + \delta_0 S_0^2(s) + \theta_0 \nu_0^2(s) + R_0 u_0^2(s)\Big) ds\Big], \end{aligned}$$

wherein the final cash process in (3.5) was replaced by $\mathbb{E}[Z_0(T)] = -\mathbb{E}[\int_0^T S_0(s)\nu_0(s)ds]$, and the asset midprice $F_0(T)$ were replaced using (3.1).

As can be seen, the major agent is coupled with the minor agents by the average term $\frac{\lambda}{N}\sum_{i=1}^{N}\nu_i$ in the execution price dynamics (3.10).
Now let the major agent's state be denoted by

$$x_0 = \begin{bmatrix} \nu_0 \\ Q_0 \\ S_0 \end{bmatrix}.$$

Then the major agent's cost function will be written in the standard quadratic form

$$J_0(u_0) = \mathbb{E}\Big[\|x_0(T)\|^2_{\bar{P}_0} + \int_0^T \big(\|x_0(s)\|^2_{P_0} + \|u_0(s)\|^2_{R_0}\big)ds\Big], \tag{3.11}$$

with

$$\bar{P}_0 = \begin{bmatrix} \gamma_0 & \frac{1}{2}\mu_0 a_0 & 0 \\ \frac{1}{2}\mu_0 a_0 & \mu_0\alpha_0 & -\frac{1}{2}\mu_0 \\ 0 & -\frac{1}{2}\mu_0 & \xi_0 \end{bmatrix}, \quad P_0 = \begin{bmatrix} \theta_0 & 0 & \frac{1}{2}\psi_0 \\ 0 & \phi_0 & 0 \\ \frac{1}{2}\psi_0 & 0 & \delta_0 \end{bmatrix}, \quad R_0 > 0. \tag{3.12}$$

Minor Liquidator Agent

Similarly, the stochastic optimal control problem for a minor liquidator trader $\mathcal{A}_i,\ 1 \le i \le N_l$, is given by the set of dynamical equations

$$d\nu_i(t) = u_i(t)dt, \tag{3.13}$$

$$dQ_i(t) = \nu_i(t)dt + \sigma^Q dw_i^Q(t), \tag{3.14}$$

$$dS_i(t) = \big(\lambda_0\nu_0(t) + \frac{\lambda}{N}\sum_{i=1}^{N}\nu_i(t)\big)dt + au_i(t)dt + \sigma dw_i^F(t), \tag{3.15}$$

The equations above show that a minor agent is coupled with the major agent and other minor agents through the execution price dynamics (3.15).
Similar to the major trader, we define a generic minor trader's state vector as

$$x_i = \begin{bmatrix} \nu_i \\ Q_i \\ S_i \end{bmatrix},$$

and its quadratic cost function where the final cash process in (3.6) has been replaced by $\mathbb{E}[Z_i(T)] = -\mathbb{E}[\int_0^T S_i(s)\nu_i(s)ds]$ using (3.4), and the asset midprice $F_i(T)$ were replaced using (3.2) is given by

$$J_i(u_i, u_{-i}) = \mathbb{E}\Big[\|x_i(T)\|^2_{\bar{P}_l} + \int_0^T \big(\|x_i(s)\|^2_{P_l} + \|u_i(s)\|^2_{R_l}\big)ds\Big], \tag{3.16}$$

where

$$\bar{P}_l = \begin{bmatrix} \gamma_l & \frac{1}{2}\mu_l a & 0 \\ \frac{1}{2}\mu_l a & \mu_l \alpha_l & -\frac{1}{2}\mu_l \\ 0 & -\frac{1}{2}\mu_l & \delta_l \end{bmatrix}, \quad P_l = \begin{bmatrix} \theta_l & 0 & \frac{1}{2}\psi_l \\ 0 & \phi_l & 0 \\ \frac{1}{2}\psi_l & 0 & \delta_l \end{bmatrix}, \quad R_l > 0.$$

Minor Acquirer Agent

The stochastic optimal control problem for a minor acquirer trader $\mathcal{A}_i,\ 1 \leq i \leq N_a$, is given by the set of dynamical equations

$$d\nu_i(t) = u_i(t)dt, \tag{3.17}$$

$$dY_i(t) = -\nu_i(t)dt + \sigma^Q dw_i^Q(t), \tag{3.18}$$

$$dS_i(t) = \Big(\lambda_0\nu_0(t) + \frac{\lambda}{N}\sum_{i=1}^{N}\nu_i(t)\Big)dt + a u_i(t)dt + \sigma dw_i^F(t), \tag{3.19}$$

where $Y_i(t) = \mathcal{N}_a - Q_i(t)$ is the remaining shares at t to be acquired until the end of the trading horizon. Accordingly, the cost function for acquisition is given by

$$J_i(u_i, u_{-i}) = \mathbb{E}\Big[\psi_a Z_i(T) + \mu_a Y_i(T)\big(F_i(T) + \alpha_a Y_i(T)\big) + \xi_a S_i^2(T) + \gamma_a \nu_i^2(T) + \int_0^T \big(\phi_a Y_i(s)^2 + \delta_a S_i^2(s) + \theta_a \nu_i^2(s) + R_a u_i^2(s)\big)ds\Big], \quad 1 \leq i \leq N_a.$$

We define a generic minor acquirer trader's state vector as

$$x_i = \begin{bmatrix} \nu_i \\ Y_i \\ S_i \end{bmatrix},$$

and its quadratic cost function is given by

$$J_i(u_i, u_{-i}) = \mathbb{E}\Big[\|x_i(T)\|^2_{\bar{P}_a} + \int_0^T \big(\|x_i(s)\|^2_{P_a} + \|u_i(s)\|^2_{R_a}\big) ds\Big], \tag{3.20}$$

where

$$\bar{P}_a = \begin{bmatrix} \gamma_a & -\frac{1}{2}\mu_a a & 0 \\ -\frac{1}{2}\mu_a a & \mu_a \alpha_a & \frac{1}{2}\mu_a \\ 0 & \frac{1}{2}\mu_a & \xi_a \end{bmatrix}, \quad P_a = \begin{bmatrix} \theta_a & 0 & -\frac{1}{2}\psi_a \\ 0 & \phi_a & 0 \\ -\frac{1}{2}\psi_a & 0 & \delta_a \end{bmatrix}, \quad R_a > 0.$$

We denote by $w = \{w_i, 0 \leq i \leq N\}$ the set of $(N+1)$ independent $\mathbb{R}^r$-valued standard Wiener processes on the probability space $(\Omega, \mathcal{F}, P)$, where w is progressively measurable with respect to the filtration $\mathcal{F}^w = \{\mathcal{F}^w_t \subset \mathcal{F}; t \geq 0\}$.

Assumption 3.1. *The initial states* $\{x_i(0),\ 0 \leq i \leq N\}$ *defined on* $(\Omega, \mathcal{F}, P)$ *are identically distributed, mutually independent and also independent of* $\mathcal{F}^w_\infty$*, with* $\mathbb{E}x_i(0) = 0$*. Moreover,* $\sup_i \mathbb{E}\|x_i(0)\|^2 \leq c < \infty$*,* $0 \leq i \leq N < \infty$*, with c independent of* N*.*

3.3.2 Mean Field Evolution

Minor agents are categorized in two distinct types. The notation $\mathcal{I}_k$ is defined as

$$\mathcal{I}_k = \{i : \theta_i = k,\ 1 \leq i \leq N\}, \quad k \triangleq a, l$$

where the cardinality of $\mathcal{I}_k$ is denoted by $N_k = |\mathcal{I}_k|$. Then, $\pi^N = (\pi^N_a, \pi^N_l)$, $\pi^N_k = \frac{N_k}{N}$, $k \triangleq a, l$, denotes the empirical distribution of the parameters $(\theta_1, ..., \theta_N)$ sampled independently of the initial conditions and Wiener processes of the agents $A_i, 1 \leq i \leq N$. The first assumption is as follows.

Assumption 3.2. *There exists* π *such that* $lim_{N\to\infty}\pi^N = \pi = (\pi_a, \pi_l)$ *a.s.*

Following the LQG MFG methodology [16], the mean field, $\bar{x}$, is defined as the L^2 limit, when it exists, of the average of minor agents' states when the population size goes to infinity

$$\bar{x}^k(t) = \lim_{N_k\to\infty} x^{N_k}(t) = \lim_{N_k\to\infty} \frac{1}{N_k}\sum_{i=1}^{N_k} x_i(t), \quad q.m., \quad k \triangleq a, l \tag{3.21}$$

Now, if the control strategy for each minor agent is considered to have the general feedback form

$$u_i = L_1^k x_i + L_2^k x_0 + \sum_{j\neq i, j=1}^{N} L_4^k x_j + L_3^k, \quad 1 \leq i \leq N, \quad k \triangleq a, l \tag{3.22}$$

then the mean field dynamics can be obtained by substituting (3.22) in the minor agents' dynamics (3.13)-(3.15), (3.17)-(3.19) and taking the average and then its L^2 limit as $N_k \to \infty$.
The dynamical equation of the mean field $\bar{x} = [(\bar{x}^a)^T, (\bar{x}^l)^T]^T$ for the optimal execution problem can be written as

$$d\bar{x} = \bar{A}\bar{x}dt + \bar{G}x_0 dt + \bar{m}dt, \tag{3.23}$$

where

$$\bar{A} = \begin{bmatrix} \bar{A}^a \\ \bar{A}^l \end{bmatrix}, \quad \bar{G} = \begin{bmatrix} \bar{G}^a \\ \bar{G}^l \end{bmatrix}, \quad \bar{m} = \begin{bmatrix} \bar{m}^a \\ \bar{m}^l \end{bmatrix}, \tag{3.24}$$

which can be determined from the consistency equations.

3.3.3 Infinite Populations

Following the mean field game methodology with a major agent [42], the optimal execution problem is first solved in the infinite population case where the average term in the finite population dynamics and cost function of each agent is replaced with its infinite population limit, i.e. the mean field. Then specializing to MFG linear systems [16], the major agent's state is extended with the mean field, while the minor agent's state is extended with the mean field and the major agent's state; this yields LQG problems for each trader linked only through the mean field and the major agent's state. Finally the infinite population best response strategies are applied to the finite population system which yields an ϵ-Nash equilibria (see Theorem 3.1).

In this chapter we address the optimal execution problem in the MFG framework when the traders have, first, complete observations and, second, partial observations of their state and the major trader's state in Sections 3.4 and 3.5, respectively.

The stochastic optimal control problem for each agent in the infinite population case is given below.

Major Liquidator Agent

The major trader's stochastic optimal control problem in the infinite population case is given by

$$dx_0 = A_0 x_0 dt + B_0 u_0 dt + E_0^{\pi} \bar{x} dt + D_0 dw_0, \tag{3.25}$$

where $E_0^{\pi} = \pi \otimes E_0 \triangleq [\pi_a E_0, \pi_l E_0]$, and

$$A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \lambda_0 & 0 & 0 \end{bmatrix}, \; B_0 = \begin{bmatrix} 1 \\ 0 \\ a_0 \end{bmatrix}, \; E_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix}, \; D_0 = \begin{bmatrix} 0 & 0 \\ \sigma_0^Q & 0 \\ 0 & \sigma \end{bmatrix}, \; w_0 = \begin{bmatrix} w_0^Q \\ w_0^F \end{bmatrix},$$

together with the cost function (3.11).

Minor Liquidator Agent

The stochastic optimal control problem for a minor liquidator agent in the infinite population case is given by

$$dx_i = A_l x_i dt + E_l \bar{x} dt + B_l u_i dt + G_l x_0 dt + D_l dw_i, \quad 1 \leq i \leq N_l, \tag{3.26}$$

with $E_l^{\pi} = \pi \otimes E_l \triangleq [\pi_a E_l, \pi_l E_l]$, and the matrices

$$A_l = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_l = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix}, \quad B_l = \begin{bmatrix} 1 \\ 0 \\ a \end{bmatrix},$$

$$G_l = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda_0 & 0 & 0 \end{bmatrix}, \quad D_l = \begin{bmatrix} 0 & 0 \\ \sigma^Q & 0 \\ 0 & \sigma \end{bmatrix}, \quad w_i = \begin{bmatrix} w_i^Q \\ w_i^F \end{bmatrix},$$

together with the cost function (3.16).

Minor Acquirer Agent

The stochastic optimal control problem for an acquirer agent in the infinite population case is given by

$$dx_i = A_a x_i dt + E_a \bar{x} dt + B_a u_i dt + G_a x_0 dt + D_a dw_i, \quad 1 \le i \le N_a, \tag{3.27}$$

where $E_a^\pi = \pi \otimes E_a \triangleq [\pi_a E_a, \pi_l E_a]$, and

$$A_a = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix}, \quad B_a = \begin{bmatrix} 1 \\ 0 \\ a \end{bmatrix},$$

$$G_a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda_0 & 0 & 0 \end{bmatrix}, \quad D_a = \begin{bmatrix} 0 & 0 \\ \sigma^Q & 0 \\ 0 & \sigma \end{bmatrix}, \quad w_i = \begin{bmatrix} w_i^Q \\ w_i^F \end{bmatrix},$$

together with the cost function (3.20).

3.4 Completely Observed Optimal Execution Problems

In the completely observed (CO) optimal execution problem it is assumed that the major trader completely observes its own state, and each generic minor trader completely observes its own state and the major trader's state. In the following we introduce the admissible sets of controls for each agent. The null set augmented σ-field $F_{i,t}, 1 \le i \le N$, is defined to be the increasing family of null set augmented σ-fields generated by $(x_i(\tau); 0 \le \tau \le t)$, and by definition $F_{0,t}$ is the increasing family of σ-fields generated by $(x_0(\tau); 0 \le \tau \le t)$. F_t^N is the increasing family of σ-fields generated by the set $\{x_j(\tau), x_0(\tau); 0 \le \tau \le t, 1 \le j \le N\}$. The set of control actions $\mathcal{U}_g^{N,L}$ consists of linear feedback control actions adapted to $\{F_t^N, t \ge 0\}, 1 \le N < \infty$.

Assumption 3.3 (Major Agent σ-Fields and Linear Controls)**.** *For the major agent $\mathcal{A}_0$ the set of control inputs $\mathcal{U}_0^L$ is defined to be the collection of linear feedback controls adapted to the filteration $\{\mathcal{F}_{0,t}, t \ge 0\}$.*

Assumption 3.4 (Minor Agent σ-Fields and Linear Controls)**.** *For the minor agent $\mathcal{A}_i, 1 \le i \le N$, the set of control inputs $\mathcal{U}_i^L$ is defined to be the collection of linear feedback controls adapted to the filtration $\{\mathcal{F}_{i,t}, t \ge 0\}, 1 \le i \le N$.*

The best response MFG trading strategies which are obtained later in this section yield an ϵ-Nash equilibria for the market by the following theorem.

Theorem 3.1 (ϵ-Nash Equilibria for CO MM-MF Systems)**.** *Subject to Assumptions 3.1-3.5, the system equations* (3.8)-(3.20) *together with the mean field equations* (3.38) *generate the set of control laws* $\mathcal{U}_{MF}^{N} \triangleq \{u_i^\circ; 0 \leq i \leq N\}$, $1 \leq N < \infty$, *given by* (3.29) *and* (3.34) *such that*

(i) All agent systems $\mathcal{A}_i,\ 0 \leq i \leq N$*, are second order stable.*

(ii) $\{\mathcal{U}_{MF}^{N}; 1 \leq N < \infty\}$ *yields an* ϵ*-Nash equilibrium for all* ϵ*, i.e. for all* $\epsilon > 0$*, there exists* $N(\epsilon)$ *such that for all* $N \geq N(\epsilon)$*;*

$$J_i^{s,N}(u_i^\circ, u_{-i}^\circ) - \epsilon \leq \inf_{u_i \in \mathcal{U}_g^{N,L}} J_i^{s,N}(u_i, u_{-i}^\circ) \leq J_i^{s,N}(u_i^\circ, u_{-i}^\circ).$$

After applying the mean field methodology to decouple the agents, the problem of obtaining the best response trading strategy is transformed to a stochastic indefinite LQ problem that is solved for using the *Theorem 2.1* which is a restriction to the constant matrix parameter case of the general result in [36]. Henceforth we discuss the stochastic optimal control problem for the major trader, and a generic minor trader.

3.4.1 Major Liquidator Agent

The dynamics for the major agent's extended state $x_0^{ex} = [x_0^T, \bar{x}^T]^T$ in the infinite population is given by

$$\begin{bmatrix} dx_0 \\ d\bar{x} \end{bmatrix} = \begin{bmatrix} A_0 & E_0^\pi \\ \bar{G} & \bar{A} \end{bmatrix} \begin{bmatrix} x_0 \\ \bar{x} \end{bmatrix} dt + \begin{bmatrix} 0_{3\times 1} \\ \bar{m} \end{bmatrix} dt + \begin{bmatrix} B_0 \\ 0_{3\times 1} \end{bmatrix} u_0(t) dt + \begin{bmatrix} D_0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} dw_0 \\ 0 \end{bmatrix}. \tag{3.28}$$

Accordingly, the following matrices are defined

$$\mathbb{A}_0 = \begin{bmatrix} A_0 & E_0^\pi \\ \bar{G} & \bar{A} \end{bmatrix}, \quad \mathbb{M}_0 = \begin{bmatrix} 0_{3\times 1} \\ \bar{m} \end{bmatrix}, \quad \mathbb{B}_0 = \begin{bmatrix} B_0 \\ 0_{3\times 1} \end{bmatrix}, \quad \mathbb{D}_0 = \begin{bmatrix} D_0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Consequently, using *Theorem 2.1*, the infinite population best response control is given by

$$u_0^\circ(t) = -R_0^{-1}\mathbb{B}_0^T\Pi_0\big(x_0^T, \bar{x}^T\big)^T, \tag{3.29}$$
$$-\frac{d\Pi_0}{dt} =\Pi_0\mathbb{A}_0 + \mathbb{A}_0^T\Pi_0 - \Pi_0\mathbb{B}_0R_0^{-1}\mathbb{B}_0^T\Pi_0 + \mathbb{P}_0, \quad \Pi_0(T) = \bar{\mathbb{P}}_0,$$

where in the above Riccati equation

$$\mathbb{P}_0 = [I_{3\times 3}, 0_{3\times 3}]^T P_0[I_{3\times 3}, 0_{3\times 3}], \tag{3.30}$$
$$\bar{\mathbb{P}}_0 = [I_{3\times 3}, 0_{3\times 3}]^T \bar{P}_0[I_{3\times 3}, 0_{3\times 3}]. \tag{3.31}$$

3.4.2 Minor Acquirer/Liquidator Agent

For brevity, the notation $(.)_{a/l}$ is used in the rest of this chapter to denote the matrices and parameters correspoding to a generic acquirer or a liquidator agent, respectively. Accordingly, a generic minor (acquirer/liquidator) agent $\mathcal{A}_i$'s extended dynamics with the extended state $x_i^{ex} = [x_i^T, x_0^T, \bar{x}^T]^T$ is

$$\begin{bmatrix} dx_i \\ dx_0 \\ d\bar{x} \end{bmatrix} = \begin{bmatrix} A_{a/l} & \begin{bmatrix} G_{a/l} & E^\pi_{a/l} \end{bmatrix} \\ 0_{6\times 3} & \mathbb{A}_0 \end{bmatrix} \begin{bmatrix} x_i \\ x_0 \\ \bar{x} \end{bmatrix} dt + \begin{bmatrix} 0_{3\times 1} \\ \mathbb{M}_0 \end{bmatrix} dt + \begin{bmatrix} 0_{3\times 1} \\ \mathbb{B}_0 \end{bmatrix} u_0(t)dt$$
$$+ \begin{bmatrix} B_{a/l} \\ 0_{6\times 1} \end{bmatrix} u_i(t)dt + \begin{bmatrix} D_{a/l} & 0_{3\times 6} \\ 0_{6\times 3} & \mathbb{D}_0 \end{bmatrix} \begin{bmatrix} dw_i \\ dw_0 \\ 0 \end{bmatrix}. \tag{3.32}$$

Substituting the major agent's control action (3.29) into (3.32) yields

$$dx_i^{ex} = \mathbb{A}_{a/l}x_i^{ex}dt + \mathbb{M}_{a/l}dt + \mathbb{B}_{a/l}dt + \mathbb{D}_{a/l}dW_i, \tag{3.33}$$

where

$$\mathbb{A}_{a/l} = \begin{bmatrix} A_{a/l} & \begin{bmatrix} G_{a/l} & E^\pi_{a/l} \end{bmatrix} \\ 0_{6\times 3} & \mathbb{A}_0 - \mathbb{B}_0R_0^{-1}\mathbb{B}_0^T\Pi_0 \end{bmatrix}, \quad \mathbb{M}_{a/l} = \begin{bmatrix} 0_{3\times 1}, \\ \mathbb{M}_0 \end{bmatrix},$$

$$\mathbb{B}_{a/l} = \begin{bmatrix} B_{a/l} \\ 0_{6\times 1} \end{bmatrix}, \quad \mathbb{D}_{a/l} = \begin{bmatrix} D_{a/l} & 0_{3\times 6} \\ 0_{6\times 3} & \mathbb{D}_0 \end{bmatrix}, \quad W_i = \begin{bmatrix} w_i \\ w_0 \\ 0 \end{bmatrix}.$$

We utilize *Theorem 2.1* again to obtain the best response control for a generic minor agent as

$$u_i^\circ(t) = -R_{a/l}^{-1}\mathbb{B}_{a/l}^T \Pi_{a/l}\big(x_i^T, x_0^T, \bar{x}^T\big)^T, \tag{3.34}$$

where $\Pi_{a/l}$ is calculated by the following Riccati equation

$$-\frac{d\Pi_{a/l}}{dt} = \Pi_l \mathbb{A}_{a/l} + \mathbb{A}_{a/l}^T \Pi_{a/l} - \Pi_{a/l}\mathbb{B}_{a/l} R_{a/l}^{-1}\mathbb{B}_{a/l}^T \Pi_{a/l} + \mathbb{P}_{a/l}, \quad \Pi_{a/l}(T) = \bar{\mathbb{P}}_{a/l},$$

with the matrices

$$\mathbb{P}_{a/l} = [I_{3\times 3}, 0_{3\times 6}]^T P_{a/l}[I_{3\times 3}, 0_{3\times 6}],$$
$$\bar{\mathbb{P}}_{a/l} = [I_{3\times 3}, 0_{3\times 6}]^T \bar{P}_{a/l}[I_{3\times 3}, 0_{3\times 6}].$$

3.4.3 Mean Field Consistency Equations

The closed loop trading dynamics of a generic minor agent $\mathcal{A}_i,\ 1 \le i \le N$ applying (3.34) is given by

$$dx_i = \big(A_{a/l}x_i + E_{a/l}^\pi \bar{x} - B_{a/l}R_k^{-1}\mathbb{B}_{a/l}^T(\Pi_{a/l}[(x_i)^T, (x_0)^T, \bar{x}^T]^T) + G_{a/l}x_0\big)dt + D_{a/l}dw_i. \tag{3.35}$$

Let us define

$$\Pi_{a/l} = \begin{bmatrix} \Pi_{a/l,11} & \Pi_{a/l,12} & \Pi_{a/l,13} \\ \Pi_{a/l,21} & \Pi_{a/l,22} & \Pi_{a/l,23} \\ \Pi_{a/l,31} & \Pi_{a/l,32} & \Pi_{a/l,33} \end{bmatrix},$$
$$\mathbf{e}_a = [I_n, 0_{n\times n}], \quad \mathbf{e}_l = [0_{n\times n}, I_n]. \tag{3.36}$$

If we average out (3.35) over subpopulation $\mathcal{A}_{a/l}$, and then take the L^2 limit as the number $N_{a/l}$ of agents within the subpopulation goes to infinity (i.e. $N_{a/l} \to \infty$), we get

$$d\bar{x}^{a/l} = \big(E^{\pi}_{a/l} + [A_{a/l} - B_{a/l}R^{-1}_{a/l}B^{T}_{a/l}\Pi_{a/l,11}]\mathbf{e}_{a/l} - B_{a/l}R^{-1}_{a/l}B^{T}_{a/l}\Pi_{a/l,13}\big)\bar{x}dt \\ + (G_{a/l} - B_{a/l}R^{-1}_{a/l}B^{T}_{k}\Pi_{a/l,12})x_0 dt. \quad (3.37)$$

If we equate (3.37) with (3.23), then by consistency requirement a compact description of the major minor mean field equations determining $\bar{A}$, $\bar{G}$, $\bar{m}$ is given by

$$\begin{aligned} &\dot{\Pi}_0 + \Pi_0\mathbb{A}_0 + \mathbb{A}_0^T\Pi_0 - \Pi_0\mathbb{B}_0R_0^{-1}\mathbb{B}_0^T\Pi_0 + \mathbb{P}_0 = 0, \quad \Pi_0(T) = \bar{\mathbb{P}}_0, \\ &\dot{\Pi}_{a/l} + \Pi_{a/l}\mathbb{A}_{a/l} + \mathbb{A}^T_{a/l}\Pi_{a/l} - \Pi_{a/l}\mathbb{B}_{a/l}R^{-1}_{a/l}\mathbb{B}^T_{a/l}\Pi_{a/l} + \mathbb{P}_{a/l} = 0, \quad \Pi_{a/l}(T) = \bar{\mathbb{P}}_{a/l}, \\ &\bar{A}_{a/l} = E^{\pi}_{a/l} + [A_{a/l} - B_{a/l}R^{-1}_{a/l}B^T_{a/l}\Pi_{a/l,11}]\mathbf{e}_{a/l} - B_{a/l}R^{-1}_{a/l}B^T_{a/l}\Pi_{a/l,13}, \\ &\bar{G}_{a/l} \ = G_{a/l} - B_{a/l}R^{-1}_{a/l}B^T_{a/l}\Pi_{a/l,12}, \\ &\bar{m}_{a/l} = 0. \end{aligned} \quad (3.38)$$

Assumption 3.5. *There exists a stabilizing solution Π_0, Π_k, $\bar{A}_{a/l}$, $\bar{G}_{a/l}$ to the major-minor mean field equations (3.38) in the sense that the matrices*

$$\mathbb{A}_0 - \mathbb{B}_0R_0^{-1}\mathbb{B}_0^T\Pi_0, \\ \mathbb{A}_{a/l} - \mathbb{B}_{a/l}R^{-1}\mathbb{B}^T_{a/l}\Pi_{a/l},$$

are asymptotically stable.

3.5 Partially Observed Optimal Execution Problems

In this section it is assumed that the major trader has partial observations of its own state. This can happen for example in the foreign exchange (Forex) market, where an electronic communication network (ECN) Forex broker as a major agent trades on behalf of banks, high net worth (HNW) traders, and other brokers, and hence needs to estimate the trades, amount of exchanges and prices of each agent reqularly. Fig 3.1 depicts this scenario.

It is also assumed that each minor trader has partial observations of its own state and the major trader's state. A justification for the partial observations assumption on the minor agents'

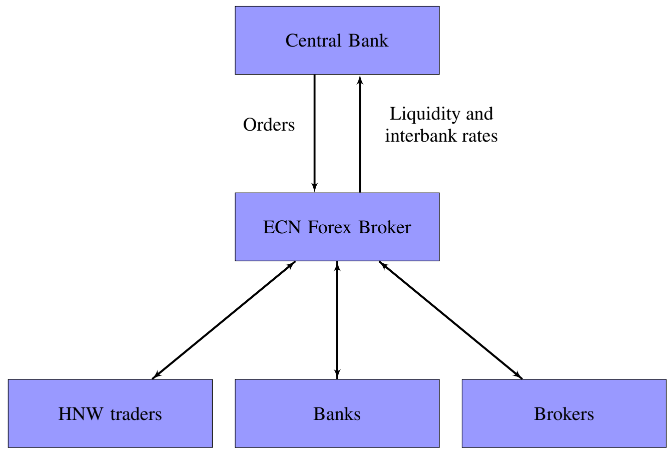

Figure 3.1: ECN broker (major trader) in the Forex market

own state is similar to that of the major trader but at a smaller scale. However one may also argue that the minor agents have complete observations of their states because they carry out smaller trades which they manage individually. We note that the latter special case may be obtained from the former more general one by setting the corresponding part of the filter equations to zero; this will not cause any singularities because the observability and noise controllability conditions will still hold.

We now follow the general development in [31,33] for PO MM LQG MFG systems to address the partially observed optimal execution problem for the major trader and a generic minor trader.

3.5.1 Observation Processes

The major agent's partial observations y_0 is given by

$$dy_0 = L_0[x_0^T,\, (x^{(N)})^T]^T dt + R_{v_0}^{\frac{1}{2}} dv_0, \tag{3.39}$$

where v_0 is a standard Wiener process in $\mathbb{R}^\ell$ with $\mathbb{E}[v_0 v_0^T] = R_{v_0}$ and matrix L_0 is given by

$$L_0 = \left[\begin{array}{cc} l_0^1 & l_0^2 \end{array} \right], \tag{3.40}$$

with $l_0^1,\, l_0^2 \in \mathbb{R}^{\ell\times n}$. Now, assume the partial observations for a minor agent $\mathcal{A}_i,\ 1\le i\le N$, of type $k \triangleq a, l$, is given by

$$dy_i = L_{a/l}[x_i^T,\, x_0^T,\, (x^{(N)})^T]^T dt + R_v^{\frac{1}{2}} dv_i, \tag{3.41}$$

where $\{v_i,\ 1\le i\le N\}$ denote N independent standard Wiener processes in $\mathbb{R}^\ell$ with $\mathbb{E}[v_i v_i^T] = R_v$, and matrix $L_{a/l}$ is defined as in

$$L_{a/l} = \begin{bmatrix} l_{a/l}^1 & l_{a/l}^2 & l_{a/l}^3 \end{bmatrix}, \tag{3.42}$$

where $l_{a/l}^1,\, l_{a/l}^2,\, l_{a/l}^3 \in \mathbb{R}^{\ell\times n}$.

We note that in contrast to the analysis of the partially observed major agent case in [21], where the major agent has complete observations on its own state, in the case studied in this chapter the minor and major agents are equipped with partial observations on the empirical (i.e. finite population) mean field, denoted $x^{(N)}$, and the limiting (i.e. infinite population) mean field, denoted $\bar{x}$. This turns out to be necessary in order that detectability conditions may be imposed which imply the convergence of the solutions to the associated filter Riccati equations to positive definite limits which necessarily yield asymptotic stable filters.

Control σ-Fields

The family of partial observation information sets $\mathcal{F}_0^y$ is defined to be the increasing family of σ-fields of partial observations $\{\mathcal{F}_{0,t}^y; 0\le t\}$ generated by the major agent $\mathcal{A}_0$'s partial observations $(y_0(\tau);\ 0\le\tau\le t)$ on its own state as given in (3.39).

Assumption 3.6 (Major Agent σ-Fields and Linear Controls)**.** *For the major agent $\mathcal{A}_0$ the set of control inputs $\mathcal{U}_{0,y}^L$ is defined to be the collection of linear feedback controls adapted to the increasing σ-fields of partial observations $\{\mathcal{F}_{0,t}^y, t\ge 0\}$.*

We recall that the family of partial observation information sets $\mathcal{F}_i^y, 1\le i\le N$, is defined to be the increasing σ-fields $\{\mathcal{F}_{i,t}^y;\ 0\le t<\infty\}$ generated by the minor agent $\mathcal{A}_i$'s partial observations $(y_i(\tau);\ 0\le\tau\le t)$, on its own state and the major agent's state, as given in (3.41).

Assumption 3.7 (Minor Agent σ-Fields and Linear Controls)**.** *For each minor agent $\mathcal{A}_i,\ 1\le i\le N$, the set of control inputs $\mathcal{U}_{i,y}^{N,L}$ is defined to be the collection of linear feedback controls adapted to the increasing σ-fields of partial observations $\{\mathcal{F}_{i,t}^y; t\ge 0\}$.*

Moreover, the set of control inputs $\mathcal{U}_y^{N,L}$ is defined to be the collection of linear feedback control laws adapted to $\mathcal{F}_t^{N,y} = \{\bigvee_{i=0}^{N} \mathcal{F}_i^y\}$.

3.5.2 Mean Field Evolution

If we consider for each minor agent $\mathcal{A}_i$ of type $k \triangleq a, l$, a uniform (with respect to i) feedback control $u_i^{a/l} \in \mathcal{U}_{i,L} \subset \mathcal{U}_i$, then it can be shown that the L^2 limit $\bar{x}$ of x^N, i.e. the mean field satisfies

$$d\bar{x} = \bar{A}\bar{x}dt + \bar{G}x_0 dt + \bar{H}\hat{x}_{0|\mathcal{F}_0^y}dt + \bar{L}\hat{\bar{x}}_{|\mathcal{F}_0^y}dt + \bar{J}\bar{\tilde{x}}^{ex}dt + \bar{m}dt, \tag{3.43}$$

where $\hat{x}_{0|\mathcal{F}_0^y}$ and $\hat{\bar{x}}_{|\mathcal{F}_0^y}$, respectively, denote the conditional expectation of x_0 and $\bar{x}$ with respect to the observation σ-field $\mathcal{F}_{0,t}^y$ of the major agent $\mathcal{A}_0$ at the instant $t \geq 0$, i.e.

$$\hat{x}_{0|\mathcal{F}_0^y} \triangleq \mathbb{E}_{|\mathcal{F}_0^y} x_0 = \mathbb{E}\{x_0|\mathcal{F}_0^y\}, \tag{3.44}$$

$$\hat{\bar{x}}_{|\mathcal{F}_0^y} \triangleq \mathbb{E}_{|\mathcal{F}_0^y} \bar{x} = \mathbb{E}\{\bar{x}|\mathcal{F}_0^y\}. \tag{3.45}$$

Moreover, $(\bar{\tilde{x}}^{ex})^T = [(\bar{\tilde{x}}^{1,ex})^T, \ldots, (\bar{\tilde{x}}^{K,ex})^T]$, where we denote by $\bar{\tilde{x}}^{k,ex}$ the average of the estimation errors of the minor agents of subpopulation k as $N_k \to \infty$, and which satisfies the dynamical equation (3.67). Finally, the matrices $\bar{A}$, $\bar{G}$, $\bar{H}$, $\bar{L}$, $\bar{m}$, and $\bar{J}$ in (3.43) may be represented as

$$\bar{A} = \begin{bmatrix} \bar{A}_a \\ \bar{A}_l \end{bmatrix}, \quad \bar{G} = \begin{bmatrix} \bar{G}_a \\ \bar{G}_l \end{bmatrix}, \quad \bar{H} = \begin{bmatrix} \bar{H}_a \\ \bar{H}_l \end{bmatrix},$$

$$\bar{L} = \begin{bmatrix} \bar{L}_a \\ \bar{L}_l \end{bmatrix}, \quad \bar{m} = \begin{bmatrix} \bar{m}_a \\ \bar{m}_l \end{bmatrix}, \quad \bar{J} = \begin{bmatrix} \bar{J}_a & 0 \\ 0 & \bar{J}_l \end{bmatrix}. \tag{3.46}$$

and are to be solved for in the tracking solution. By abuse of language, the mean value of the system's Gaussian mean field given by the state process $\bar{x} = [\bar{x}^a,\ \bar{x}^l]$ shall also be termed the system's mean field (The derivation of the properties above may performed using the methods of [21], [19] and [16]).

3.5.3 Major Liquidator Agent: Infinite Population

The major agent's observation process in the infinite population is given by

$$dy_0 = \mathbb{L}_0[x_0^T, \bar{x}^T]^T dt + \sigma_{v_0} dv_0, \tag{3.47}$$

$$\mathbb{L}_0 = \begin{bmatrix} l_0^1 & \bar{l}_0^2 \end{bmatrix}, \tag{3.48}$$

$$\bar{l}_0^2 = \pi \otimes l_0^2 \triangleq [\pi_a l_0^2,\ \pi_l l_0^2]. \tag{3.49}$$

where $\mathbb{L}_0$ is a constant matrix with appropriate dimension. Then the corresponding Kalman filter equation to generate the estimates of the major agent's state and the mean field based on its own observations are, respectively, given by

$$d\hat{x}_{0|\mathcal{F}_0^y} = A_0\hat{x}_{0|\mathcal{F}_0^y}dt + B_0\hat{u}_0 dt + K_0^1 d\nu_0, \tag{3.50}$$

and

$$d\hat{\bar{x}}_{|\mathcal{F}_0^y} = (\bar{G} + \bar{H})\hat{x}_{0|\mathcal{F}_0^y}dt + (\bar{A} + \bar{L})\hat{\bar{x}}_{|\mathcal{F}_0^y}dt + \bar{m}dt + K_0^2 d\nu_0, \tag{3.51}$$

where $\hat{\tilde{\bar{x}}}_{|\mathcal{F}_0^y} = 0$ is used (see (3.73)). Moreover, $\bar{m}$ is a deterministic process, K_0^1 and K_0^2 are the Kalman filter gains, and ν_0 is the innovation process. Henceforth, the Kalman filter which generates the estimates of the major agent's extended state is given by

$$d\hat{x}^{ex}_{0|\mathcal{F}_0^y} = \mathbf{A}_0\hat{x}^{ex}_{0|\mathcal{F}_0^y}dt + \mathbf{M}_0 dt + \mathbf{B}_0\hat{u}_{0|\mathcal{F}_0^y}dt + K_0(t)[dy_0 - \mathbb{L}_0\hat{x}_{0|\mathcal{F}_0^y}dt], \tag{3.52}$$

where

$$\mathbf{A}_0 = \begin{bmatrix} A_0 & 0_{n\times nK} \\ \bar{G} + \bar{H} & \bar{A} + \bar{L} \end{bmatrix}, \quad \mathbf{B}_0 = \begin{bmatrix} B_0 \\ 0_{nK\times m} \end{bmatrix},$$

$$\mathbf{M}_0 = \begin{bmatrix} 0_{n\times 1} \\ \bar{m} \end{bmatrix}, \quad \mathbf{D}_0 = \begin{bmatrix} D_0 & 0_{n\times rK} \\ 0_{nK\times r} & 0_{nK\times rK} \end{bmatrix}. \tag{3.53}$$

Moreover, the corresponding Kalman filter gain $K_0 = [(K_0^1)^T, (K_0^2)^T]^T$, and the innovation process ν_0 are given by

$$K_0 = V_0 \mathbb{L}_0^T R_{v_0}^{-1}, \tag{3.54}$$

$$d\nu_0 = dy_0 - \mathbb{L}_0 \left[\hat{x}^T_{0|\mathcal{F}_0^y}, \hat{\bar{x}}^T_{|\mathcal{F}_0^y}\right]^T dt, \tag{3.55}$$

where $R_{v_0} = \sigma_{v_0}\sigma_{v_0}^T$, and $V_0(t)$ is the solution to the corresponding Riccati equation

$$\dot{V}_0(t) = \mathbf{A}_0 V_0(t) + V_0(t)\mathbf{A}_0^T - K_0(t) R_{v_0} K_0(t)^T + \mathbf{J}_0 \bar{V}(t) \mathbf{J}_0^T + Q_{w_0}, \tag{3.56}$$

where $Q_{w_0} = \mathbf{D}_0\mathbf{D}_0^T$, $\mathbf{J}_0^T = \big[0_{(3nK+2nK^2)\times n}, \bar{J}^T\big]$, $\bar{V}(t) = \mathbb{E}\big[\tilde{\bar{x}}^{ex}(t)\big(\tilde{\bar{x}}^{ex}(t)\big)^T\big]$ satisfies (3.69), and $V(0) = \mathbb{E}\big[\big(x_0^{ex}(0) - (\widehat{x_0^{ex}(0)})_{|\mathcal{F}_0^y}\big)\big(x_0^{ex}(0) - (\widehat{x_0^{ex}(0)})_{|\mathcal{F}_0^y}\big)^T\big]$.

Following the methodology in *Chapter 2* ([31, 33]), the cost function (3.11) can be decomposed as

$$J_0 = \mathbb{E}\Big[\|\hat{x}_{0|\mathcal{F}_0^y}(T)\|^2_{\bar{P}_0} + \int_0^T \big(\|\hat{x}_{0|\mathcal{F}_0^y}(s)\|^2_{P_0} + \|u_0(s)\|^2_{R_0}\big) ds$$
$$+ \|x_0(T) - \hat{x}_{0|\mathcal{F}_i^y}(T)\|^2_{\bar{P}_0} + \int_0^T \big(\|x_0(s) - \hat{x}_{0|\mathcal{F}_0^y}(s)\|^2_{P_0}\big) ds\Big],$$

and thence employing the Separation Principle of LQG stochastic control the corresponding infinite population best response control action is given by

$$\hat{u}_0^\circ = -R_0^{-1}\mathbf{B}_0^T \big[\Pi_0 \big(\hat{x}^T_{0|\mathcal{F}_0^y}, \hat{\bar{x}}^T_{|\mathcal{F}_0^y}\big)^T\big]. \tag{3.57}$$

3.5.4 Minor (Acquirer/Liquidator) Agent

The extended state shall be denoted by

$$X_i = [x_i^T, x_0^T, \bar{x}^T, \hat{x}^T_{0|\mathcal{F}_0^y}, \hat{\bar{x}}^T_{|\mathcal{F}_0^y}]^T, \tag{3.58}$$

then the minor agent's observation process in the infinite population is given by

$$dy_i(t) = \mathbb{L}_{a/l}[x_i^T, x_0^T, \bar{x}^T, \hat{x}^T_{0|\mathcal{F}_0^y}, \hat{\bar{x}}^T_{0|\mathcal{F}_0^y}]^T dt + \sigma_{v_i} dv_i, \tag{3.59}$$

with the constant matrix $\mathbb{L}_{a/l}$ given by

$$\mathbb{L}_{a/l} = \left[\begin{array}{cccc} l^1_{a/l} & l^2_{a/l} & \bar{l}^3_{a/l} & 0_{n\times(n+nK)} \end{array} \right], \tag{3.60}$$

$$\bar{l}^3_{a/l} = \pi \otimes l^3_{a/l} \triangleq [\pi_a l^3_{a/l},\ \pi_l l^3_{a/l}]. \tag{3.61}$$

Then the extended dynamics of the minor agent is given by

$$\begin{aligned} \left[\begin{array}{c} dx_i \\ dx_0 \\ d\bar{x} \\ d\hat{x}_{0|\mathcal{F}^y_0} \\ d\hat{\bar{x}}_{|\mathcal{F}^y_0} \end{array} \right] &= \left[\begin{array}{ccc} A_{a/l} & \left[\begin{array}{cc} G_{a/l} & E_{a/l} \end{array} \right] & 0_{3\times 6} \\ 0_{6\times 3} & \left[\begin{array}{cc} A_0 & 0_{3\times 6} \\ \bar{G} & \bar{A} \end{array} \right] & \left[\begin{array}{c} -\mathbb{B}_0 R_0^{-1} \mathbb{B}_0^T \Pi_0 \\ \left[\begin{array}{cc} \bar{H} & \bar{L} \end{array} \right] \end{array} \right] \\ 0_{6\times 3} & K_0 \mathbb{L}_0 & \mathbb{A}_0 - \mathbb{B}_0 R_0^{-1} \mathbb{B}_0^T \Pi_0 - K_0 \mathbb{L}_0 \end{array} \right] \left[\begin{array}{c} x_i \\ x_0 \\ \bar{x} \\ \hat{x}_{0|\mathcal{F}^y_0} \\ \hat{\bar{x}}_{|\mathcal{F}^y_0} \end{array} \right] dt + \left[\begin{array}{c} 0_{3\times 1} \\ \mathbb{M}_0 \\ \mathbb{M}_0 \end{array} \right] dt \\ &+ \left[\begin{array}{c} \mathbb{B}_{a/l} \\ 0_{6\times 1} \end{array} \right] u_i(t) dt + \left[\begin{array}{cc} \mathbb{D}_{a/l} & 0 \\ 0 & K_0 \end{array} \right] \left[\begin{array}{c} dW_i \\ dW_0 \\ 0_{rK\times 1} \\ dv_0 \end{array} \right], \end{aligned} \tag{3.62}$$

or equivalently

$$dX_i = \mathbf{A}_{a/l} X_i dt + \mathbf{M}_{a/l} dt + \mathbf{B}_{a/l} u_i dt + \Sigma_{a/l} \left[\begin{array}{cccc} dW_i^T, & dW_0^T, & 0_{1\times rK}, & dv_0 \end{array} \right]^T.$$

The Kalman filter which generates the estimates of the minor (liquidator/acquirer) agent's states is

$$d\hat{X}_{i|\mathcal{F}^y_i} = \mathbf{A}_{a/l} \hat{X}_{i|\mathcal{F}^y_i} dt + \mathbf{M}_{a/l} dt + \mathbf{B}_{a/l} \hat{u}_{i|\mathcal{F}^y_i} dt + K_{a/l}(t) \big[dy_i - \mathbb{L}_{a/l} \hat{X}_{i|\mathcal{F}^y_i} dt \big], \tag{3.63}$$

where the filter gain is given as

$$K_{a/l}(t) = V_{a/l}(t) \mathbb{L}^T_{a/l} R^{-1}_{v_i}, \tag{3.64}$$

with $R_{v_i} = \sigma_{v_i} \sigma^T_{v_i}$, and where $\hat{\bar{x}}^{ex}_{|\mathcal{F}^y_i} = 0$ (see (3.73)) is used. The corresponding Riccati equation is

$$\dot{V}_{a/l}(t) = \mathbf{A}_{a/l} V_{a/l}(t) + V_{a/l}(t) \mathbf{A}^T_{a/l} - K_{a/l}(t) R_v K_{a/l}(t)^T + \mathbf{J} \bar{V}(t) \mathbf{J}^T + Q^{a/l}_w, \tag{3.65}$$

where $Q_w^{a/l} = \Sigma_{a/l}\Sigma_{a/l}^T$, $\bar{V}(t) = \mathbb{E}\big[\bar{\tilde{x}}^{ex}(t)\big(\bar{\tilde{x}}^{ex}(t)\big)^T\big]$ satisfies (3.69), and $V_k(0) = \mathbb{E}\big[\big(x_i^{ex}(0) - (\widehat{x_i^{ex}(0)})_{|\mathcal{F}_i^y}\big)\big(x_i^{ex}(0) - (\widehat{x_i^{ex}(0)})_{|\mathcal{F}_i^y}\big)^T\big]$.

Then the same procedure as in *Chapter 2* ([31,33]) can be used to decompose the cost function (3.16) or (3.20) as

$$\begin{aligned} J_i =& \mathbb{E}\Big[\|\hat{x}_{i|\mathcal{F}_i^y}(T)\|^2_{\bar{P}_{a/l}} + \int_0^T \big(\|\hat{x}_{i|\mathcal{F}_i^y}(s)\|^2_{P_{a/l}} + \|u_i(s)\|^2_{R_{a/l}}\big)ds \\ &+ \|x_i(T) - \hat{x}_{i|\mathcal{F}_i^y}(T)\|^2_{\bar{P}_{a/l}} + \int_0^T \|x_i(s) - \hat{x}_{i|\mathcal{F}_i^y}(s)\|^2_{P_{a/l}} ds\Big]. \end{aligned}$$

So employing the Separation Principle the corresponding infinite population best response control for a generic minor trader is seen to be

$$\hat{u}_i^\circ = -R_{a/l}^{-1}\mathbf{B}_{a/l}^T\big[\Pi_{a/l}\big(\hat{x}^T_{i|\mathcal{F}_i^y}, \hat{x}^T_{0|\mathcal{F}_i^y}, \hat{\bar{x}}^T_{|\mathcal{F}_i^y}, (\hat{x}_{0|\mathcal{F}_0^y})^T_{|\mathcal{F}_i^y}, (\hat{\bar{x}}_{|\mathcal{F}_0^y})^T_{|\mathcal{F}_i^y}\big)^T\big]. \tag{3.66}$$

From *Chapter 2*, $(\bar{\tilde{x}}^{ex})^T = [(\bar{\tilde{x}}^{a,ex})^T, (\bar{\tilde{x}}^{l,ex})^T]$ satisfies

$$d\bar{\tilde{x}}^{ex} = \begin{bmatrix} (\mathbf{A}_a - K_a\mathbb{L}_a)\tilde{\mathbf{e}}_a + \mathbf{J} \\ (\mathbf{A}_l - K_l\mathbb{L}_l)\tilde{\mathbf{e}}_b + \mathbf{J} \end{bmatrix} \bar{\tilde{x}}^{ex} dt + \begin{bmatrix} -\Sigma_a \\ -\Sigma_b \end{bmatrix} \begin{bmatrix} 0_{r\times 1} \\ dW_0 \\ 0_{rK\times 1} \\ dv_0, \end{bmatrix}, \tag{3.67}$$

or equivalently in the compact form

$$d\bar{\tilde{x}}^{ex} = \tilde{\mathbf{A}}\bar{\tilde{x}}^{ex}dt + \tilde{\mathbf{D}}[0_{1\times r}, dw_0^T, 0_{1\times rK}, dv_0^T]^T. \tag{3.68}$$

Subsequently, $\bar{V}(t) = \mathbb{E}\big[\bar{\tilde{x}}^{ex}(t)\big(\bar{\tilde{x}}^{ex}(t)\big)^T\big]$ satisfies

$$\dot{\bar{V}} = \tilde{\mathbf{A}}\bar{V} + \bar{V}\tilde{\mathbf{A}}^T + \tilde{\mathbf{Q}}\tilde{\mathbf{Q}}^T, \tag{3.69}$$

where

$$\tilde{\mathbf{Q}}\tilde{\mathbf{Q}}^T = \tilde{\mathbf{1}}^T \begin{bmatrix} 0_{n\times n} & & \\ & Q_{w_0} & \\ & & K_0 R_{v_0} K_0^T \end{bmatrix} \tilde{\mathbf{1}}, \tag{3.70}$$

$$\tilde{\mathbf{1}} = [I_{21}, I_{21}]. \tag{3.71}$$

To guarantee the convergence of the solution to the corresponding Lyapunov equation to a unique, symmetric and positive definite solution, we assume:

Assumption 3.8. *The pair* $[\tilde{\mathbf{A}}, \tilde{\mathbf{Q}}]$ *is controllable.*

The solution to (3.68) is given by

$$\bar{\tilde{x}}^{ex}(t) = \Phi(t,0)\bar{\tilde{x}}^{ex}(0) + \int_0^t \Phi(t,\tau)\tilde{\mathbf{D}}[0_{1\times r}, dw_0^T, 0_{1\times rK}, dv_0^T]^T d\tau, \tag{3.72}$$

where $\Phi(t,\tau) = \exp\big(\tilde{\mathbf{A}}(t-\tau)\big)$. Then the conditional expectation of $\bar{\tilde{x}}^{ex}(t)$ with respect to $\mathcal{F}_i^y, 0 \le i \le N$, i.e. $\hat{\bar{\tilde{x}}}^{ex}_{|\mathcal{F}_i^y}(t)$, is given by

$$\begin{aligned} \hat{\bar{\tilde{x}}}^{ex}_{|\mathcal{F}_i^y}(t) &\triangleq \mathbb{E}[\bar{\tilde{x}}^{ex}(t)|\mathcal{F}_i^y] \\ &= \Phi(t,0)\mathbb{E}[\bar{\tilde{x}}^{ex}(0)|\mathcal{F}_i^y] + \mathbb{E}\Big[\int_0^t \Phi(t,\tau)\tilde{\mathbf{D}} \begin{bmatrix} 0_{r\times 1} \\ dw_0 \\ 0_{nr\times 1} \\ dv_0 \end{bmatrix} d\tau \Big| \mathcal{F}_i^y\Big] = 0, \end{aligned} \tag{3.73}$$

where the first term is zero due to *Assumption 3.1*, and the second term is zero due to the independence of $\{w_i, 0 \le i \le N\}$ and $\{v_i, 0 \le i \le N\}$.

Finally the set of mean field consistency equations (see *Chapter 2*) is given by

$$\begin{aligned} -\dot{\Pi}_0 &= \Pi_0\mathbf{A}_0 + \mathbf{A}_0^T\Pi_0 - \Pi_0\mathbf{B}_0R_0^{-1}\mathbf{B}_0^T\Pi_0 + Q_0^\pi, \\ -\dot{\Pi}_{a/l} &= \Pi_{a/l}\mathbf{A}_{a/l} + \mathbf{A}_{a/l}^T\Pi_{a/l} - \Pi_{a/l}\mathbf{B}_{a/l}R^{-1}\mathbf{B}_{a/l}^T\Pi_{a/l} + Q^\pi, \\ \bar{A}_{a/l} &= [A_{a/l} - B_{a/l}R^{-1}B_{a/l}^T\Pi_{a/l,11}]\mathbf{e}_{a/l} - B_{a/l}R^{-1}B_{a/l}^T\Pi_{a/l,13}, \\ \bar{G}_{a/l} &= G - B_{a/l}R^{-1}B_{a/l}^T\Pi_{a/l,12}, \\ \bar{H}_{a/l} &= -B_{a/l}R^{-1}B_{a/l}^T\Pi_{a/l,14}, \end{aligned}$$

$$\bar{L}_{a/l} = -B_{a/l}R^{-1}B_{a/l}^T\Pi_{a/l,15},$$
$$\bar{J}_{a/l} = -B_{a/l}R^{-1}\mathbb{B}_{a/l}^T\Pi_{a/l}. \tag{3.74}$$

which forms a fixed point problem which should be solved by each agent to compute the matrices in the mean field equation (3.43).

Assumption 3.9. *There exists a stabilizing solution* Π_0, $\Pi_{a/l}$, $\bar{A}_{a/l}$, $\bar{G}_{a/l}$, $\bar{H}_{a/l}$, $\bar{L}_{a/l}$ *to the Major-Minor MF equations* (3.74) *in the sense that the matrices*

$$\mathbf{A}_0 - \mathbf{B}_0 R_0^{-1}\mathbf{B}_0^T\Pi_0,$$
$$\mathbf{A}_{a/l} - \mathbf{B}_{a/l}R^{-1}\mathbf{B}_{a/l}^T\Pi_{a/l},$$

are asymptotically stable.

Moreover, one may show (see [31,33]) that the infinite population best response control laws applied to a finite population system yield the following ϵ-Nash equilibrium.

Theorem 3.2 (ϵ-Nash Equilibria for PO MM-MF Systems)**.** *Subject to Assumptions 3.1-3.2, and Assumptions 3.6-3.9, the KF-MF state estimation scheme* (3.52)-(3.56) *and* (3.63)-(3.65) *together with the MM-MFG equation scheme* (3.74) *generate the set of control laws* $\hat{\mathcal{U}}_{MF}^N \triangleq \{\hat{u}_i^\circ; 0 \le i \le N\}$, $1 \le N < \infty$, *given by*

$$\hat{u}_0^\circ = -R_0^{-1}\mathbb{B}_0^T\Pi_0(\hat{x}_{0|\mathcal{F}_0^y}^T, \hat{\bar{x}}_{|\mathcal{F}_0^y}^T)^T,$$
$$\hat{u}_i^\circ = -R^{-1}\mathbb{B}^T\Pi(\hat{x}_{i|\mathcal{F}_i^y}^T, \hat{x}_{0|\mathcal{F}_i^y}^T, \hat{\bar{x}}_{|\mathcal{F}_i^y}^T, (\hat{x}_{0|\mathcal{F}_0^y})_{|\mathcal{F}_i^y}^T, (\hat{\bar{x}}_{|\mathcal{F}_0^y})_{|\mathcal{F}_i^y}^T)^T,\ 1 \le i \le N$$

such that

(i) All agent systems $\mathcal{A}_i,\ 0 \le i \le N$, *are second order stable.*

(ii) $\{\hat{\mathcal{U}}_{MF}^N; 1 \le N < \infty\}$ *yields an* ϵ*-Nash equilibrium for all* ϵ*, i.e. for all* $\epsilon > 0$*, there exists* $N(\epsilon)$ *such that for all* $N \ge N(\epsilon)$*;*

$$J_i^{s,N}(\hat{u}_i^\circ, \hat{u}_{-i}^\circ) - \epsilon \le \inf_{u_i \in \mathcal{U}_y^{N,L}} J_i^{s,N}(u_i, \hat{u}_{-i}^\circ) \le J_i^{s,N}(\hat{u}_i^\circ, \hat{u}_{-i}^\circ).$$

3.6 Simulations

In the numerical experiments it is assumed that the trading action takes place within $T = 1$. The temporary impact strength of the major agent's trading and a generic minor agent's trading on the market are $a_0 = a = 5.43 \times 10^{-6}$, while their permanent impact strengths are taken to be $\lambda_0 = \lambda = 2 \times 10^{-8}$. The diffusion coefficients in the trading dynamics are selected as $\sigma_0^Q = 0.05$, and $\sigma_i^Q = 0.02$. The weights in the cost function for the major trader are: $\psi_0 = 100$, $\mu_0 = 100$, $\alpha_0 = 5a_0 \times 10^5$, $\phi_0 = 10^{-6}a_0$, $\delta_0 = 1/(2a0)$, $\xi_0 = 1/(2\alpha_0)$, $\theta_0 = 1/(2\delta_0)$, $\gamma_0 = 10$; and those of a generic minor (liquidator/acquirer) trader are: $\psi_l = \psi_a = 1$, $\mu_l = \mu_a = 1000$, $\alpha_l = \alpha_a = 5a \times 10^5$, $\phi_l = \phi_a = 10^{-1}a$, $\xi_l = 1/(2\alpha_l)$, $\xi_a = 1/(2\alpha_a)$, $\delta_l = 1/(2a_l)$, $\delta_a = 1/(2a_a)$, $\theta_l = 1/(2\delta_l)$, $\theta_a = 1/(2\delta_a)$, $\gamma_l = \gamma_a = 10$. Furthermore, the market volatility is $\sigma = 0.6565$, the initial asset price is taken to be $F_0(0) = F_i(0) = \$35$, and the initial inventory stock of the major trader to be liquidated is set to $Q_0(0) = 5 \times 10^6$, while the minor liquidator HFT aims to sell $Q_i(0) = 5000$ shares and the acquirer HFT wishes to buy $Q_i(0) = 5000$ shares. In the estimation part, the measurement noise standard deviation for the major trader is $\sigma_0 = 0.05$, and for the HFT is $\sigma = 0.05$.

The resulting ϵ-Nash equilibria trajectories of the major agent and generic acquirer/liquidator HFTs for the complete observation case are displayed in Figure 3.2, and the corresponding estimated trajectories in the partial observation case are depicted in Figure 3.3. As can be seen in Figure 3.2, the major trader liquidates its shares gradually during the trading interval and comes up with 28520 shares at the end of trading horizon. The minor acquirer buys 5004 shares and the minor liquidator sells 4974 shares during the trading horizon T. Moreover, in the partial observation case shown in Fig. 3.3, the estimated trajectories generated by the Kalman filter closely follow the real ones.

3.7 Conclusions

In this chapter, an execution problem in finance with major and minor traders having liquidation or acquisition objectives was formulated and addressed in the mean field game framework by application of the Separation Principle of stochastic optimal control theory extended to indefinite partially observed LQG problems. Our future work will include parameter estimation of dynamic models of real market data employing methodologies including those in [26].

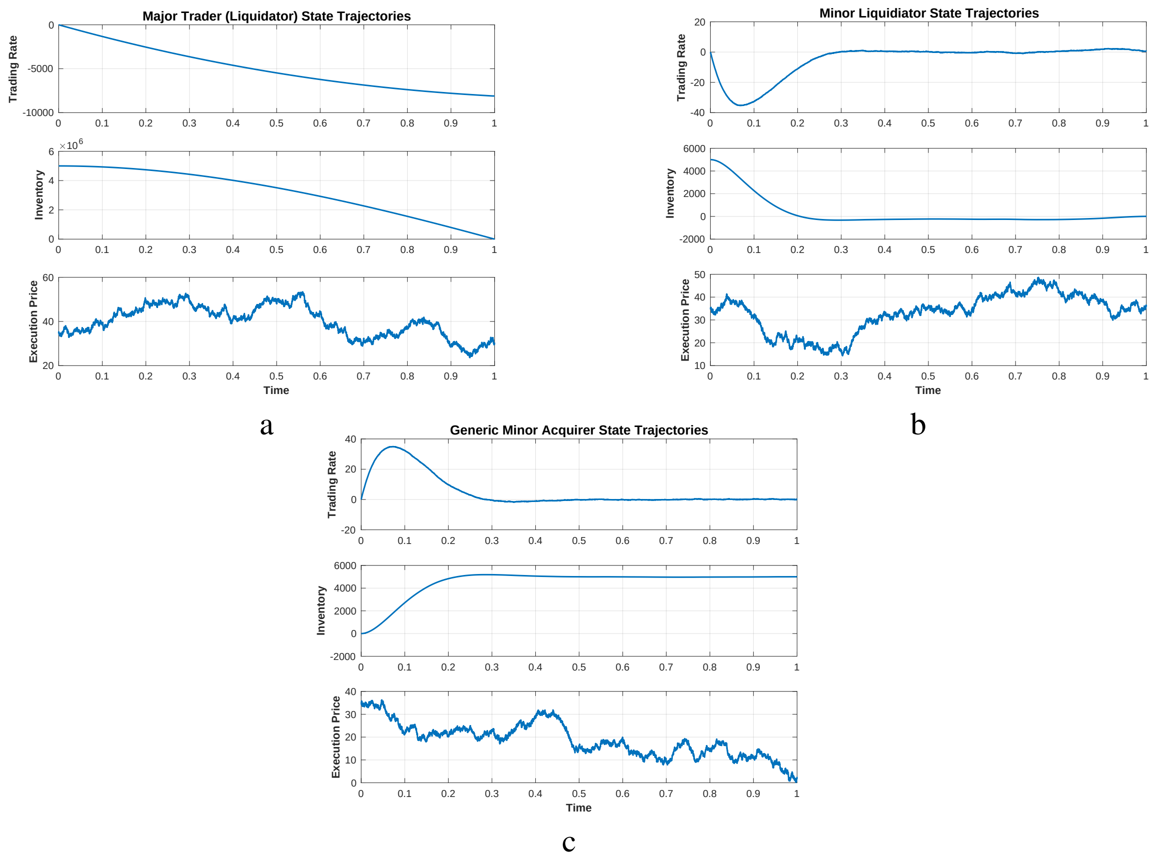

Figure 3.2: The trading rate, inventory, and execution price trajectories of (a) the major liquidator trader, (b) a generic minor liquidator, and (c) a generic minor acquirer trader in the market

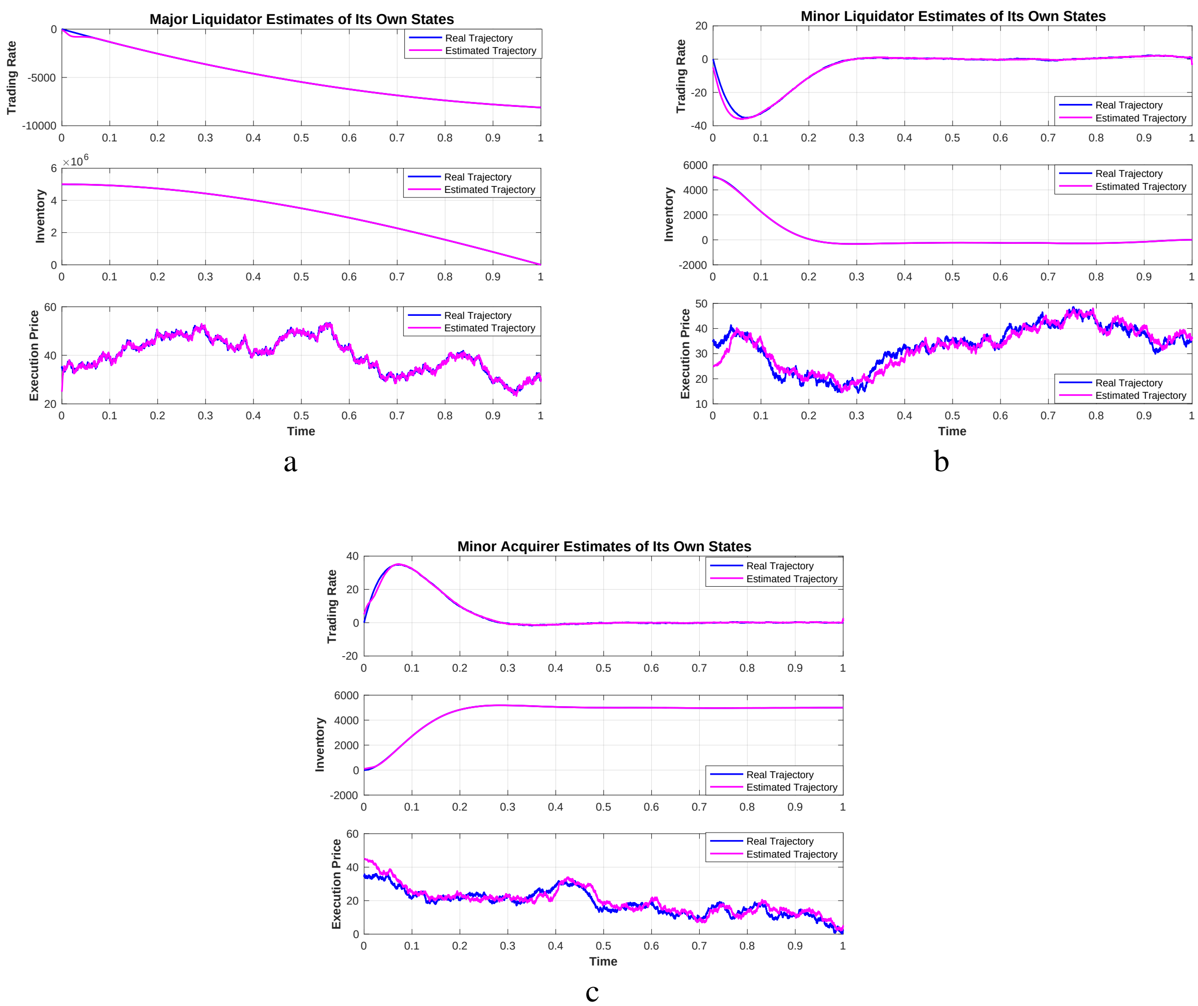

Figure 3.3: The trading rate, inventory, and execution price trajectories and the corresponding estimated trajectories based on its own observations of (a) the major liquidator trader, (b) a generic minor liquidator, and (c) a generic minor acquirer trader in the market

Part II

Major Minor LQG Hybrid Mean Field Game Systems

Chapter 4

A Hybrid Optimal Control Approach to LQG Mean Field Games with Switching and Stopping Strategies

4.1 Introduction

In several situations in stochastic dynamic games, such as in mathematical finance [43], agents wish to find the best time at which to enter or exit a given strategy. In order to determine the optimal stopping time strategies together with best response policies for the agents one is required to invoke the necessary optimality conditions of stochastic hybrid optimal control theory [44–47]. These optimality conditions are an extension of deterministic optimal control theory [48–53] for systems interacting with stochastic diffusions. In [45], in particular, the Stochastic Hybrid Minimum Principle (SHMP) is established for a general class of stochastic hybrid systems with both autonomous and controlled switchings and jumps possibly accompanied by dimension changes. Given the computational difficulty of the generally nonlinear forward-backward stochastic differential equations (FB-SDE) and the associated boundary conditions in the SHMP, a class of linear quadratic Gaussian (LQG) hybrid optimal control (HOC) problems are presented in [44] for which the corresponding Riccati equations are independent from realizations of stochastic diffusion terms.

The first combination of MFG theory and HOC theory appeared in [43] (see Chapter 5) in a non-cooperative game formulation of the financial market where high frequency trading (HFT) minor agents may leave the market before the final time. The best response policies for the agents are further shown to yield an ϵ-Nash equilibrium for the the market. In this chapter, we further extend the results and develop a hybrid systems MFG (HS-MFG) framework for a general class of LQG mean field game systems with a major agent permitted to switch between different dynamics and several subpopulations of minor agents provided with the option to stop at some optimal time. Each agent has stochastic linear dynamics with quadratic costs, and the agents are coupled in their dynamics by the average state of minor agents. Since the governing stochastic differential equations for the system change with the switching of the major agent or cessation of one or both subpopulations of minor agents, a hybrid systems formulation of the problem is presented with indexing these modes by discrete states. Optimal switching time and stopping time strategies together with best response control actions for, respectively, the major agent and all minor agents are established with respect to their individual cost criteria by an application of LQG HOC theory. The results of this chapter appear in [54].

We note that the following terms are used interchangeably throughout the chapter: optimal and best response, quit and stop.

The chapter organization is as follows. Section 4.2 presents LQG hybrid-MFG problems where the class of the problems under study is described briefly in Section 4.2.1 and it is argued that due to the presence and interactions of discrete and continuous states and dynamics, one needs to formulate the problem within hybrid optimal control framework. Discrete states and transitions are introduced in Section 4.2.2 and the underlying continuous dynamics and costs in the finite population case are presented in Section 4.2.3. Then, Section 4.3 presents hybrid-MFG approach, where following the MFG methodology, with the introduction of the mean field's hybrid evolution in Section 4.3.1, major agent's and minor agents' extended hybrid optimal control problems are, respectively, formulated in Sections 4.3.2 and 4.3.3, and best response policies for the infinite population case are determined. Then, subject to the consistency conditions in Section 4.3.4, the existence and uniqueness of the Nash equilibrium for the infinite population system, and ϵ-Nash equilibrium for the finite population system are established where the latter is obtained by the implementation of the infinite population best response strategies. Next, Section 4.4 depicts simulation results. Finally, Section 4.5 presents concluding remarks.

4.2 Major Minor LQG Hybrid Mean Field Game Systems

4.2.1 Problem Description

It is assumed that there exist one major agent and N minor agents interacting with each other through the mean field coupling in their dynamics over the time interval $[0, T]$. Two types of minor agents are considered: type $\mathcal{A}^a$ with the population of N_a and type $\mathcal{A}^b$ with the population of N_b, such that $N_a + N_b = N$. The dynamics of the major agent and a generic minor agent are described by the linear time evolution of their states and a quadratic performance function. However, the two populations of minor agents have different linear dynamics and quadratic performance objectives. We study the interaction of agents over the interval $[0, T]$, where the major agent $\mathcal{A}_0$ is permitted to switch from one set of dynamics to another at time t_s^0 if optimal, while a generic minor agent $\mathcal{A}_i$, $1 \leq i \leq N$, is permitted to stop at an optimal time t_s^i. With abuse of notation, the superscript k in $\mathcal{A}_0^k$, $k = 1, 2$, denotes that the major agent is subject to the dynamics k, and in $\mathcal{A}_i^k$, $1 \leq i \leq N$, $k \triangleq a, b$, denotes that minor agent $\mathcal{A}_i$, $1 \leq i \leq N$ is of type k, $k \triangleq a, b$. As it will be discussed in Section 4.2.2, the optimal switching or stopping time policy for each agent is trajectory and state independent, and depends only on its dynamical parameters (i.e. the agent's type). Since the dynamical parameters for all minor agents in their respective types are the same, it follows that the stopping times are the same for all agents of each subpopulation. The distinct nature of the switching (stopping) events, together with the continuous evolution of the state processes between switchings, result in the stochastic hybrid form of the problem analyzed in this chapter. Moreover, the fact that the minor agents are modeled as members of large populations gives rise to our use of the LQG mean field games framework. The system has several distinct combinatoric alternatives; this is because there are various distinct sequences wherein one minor population or another drops out first, or the major agent switches to one particular discrete state before or after a minor agent stopping event. It is to be emphasized that the discrete state sequence that actually occurs for any given system depends upon the solution of the complete (initial to terminal) MFG equations for the system, and in particular is not prescribed. We note that a key condition which yields the collective switching of the entire subpopulations is given by (4.92) and while this is reasonable in a class of LQG problems, the corresponding condition is most unlikely to hold in a nonlinear framework.

4.2.2 Discrete State Association

In order to present the dynamics of the system in the stochastic hybrid systems framework of [44, 45], the discrete states $q_{0^k\bullet}$ are assigned (see Figure 4.1) where $k \triangleq a, b$ refers to the mode in the dynamics of the major agent and $\bullet$ represents the active populations of minor agents. For instance, the discrete state q_{0^1ab} indicates that the major agent is subject to its first dynamics and both subpopulations $\mathcal{A}^a$ and $\mathcal{A}^b$ are present, and the discrete state q_{0^2a} indicates that the major agent is subject to its second dynamics, subpopulation $\mathcal{A}^a$ is present and subpopulation $\mathcal{A}^b$ has already quit the system. Furthermore, in order to refer to the temporal mode of the system, the multivalued discrete states $Q_j,\ 0 \leq j \leq 3$, are introduced (see Figure 4.1), which correspond to the evolution of the system within the intervals $[t_j, t_{j+1})$, where $t_0 = 0$ is the initial time, t_1, t_2, t_3 correspond to the times of the events of stopping of a subpopulation or switching of the major agent, in the order of occurance, and $t_4 = T$ is the terminal time. This corresponds to the scenario in which all the possible discrete changes in the system occur before the terminal time, i.e. $Q_3 = q_{0^2}$. Other scenarios where the discrete state at terminal time is different from the case considered here are possible with minor variations over the results presented in this chapter.

We remark that the HS-MFG problems studied in this chapter lie within the class of hybrid LQG problems for which optimal switching strategies are $\mathcal{F}_t$-independent, where $\mathcal{F}_t$ is the natural filtration associated with the sigma-algebra generated by the corresponding Wiener process (see appendix A). Therefore optimal switching or stopping strategies depend only on the dynamical parameters of the major agent and those of each subpopulation, respectively. In particular, an individual's optimal stopping decision coincides with stopping time of all agents in its subpopulation since the dynamical parameters are the same across a subpopulation.

Now, we describe the evolution of the system over the sequence of generic discrete states Q_j, $0 \leq j \leq 3$. The discrete state Q_0, as indicated in Figure 4.1, associates with the system evolution over the interval $[0, t_1)$ in the system's initial setting where both subpopulations of minor agents are interacting together and with the major agent which is subject to its first dynamics $\mathcal{A}_0^1$.

The multivalued discrete state Q_1 corresponds to the evolution of the system over $[t_1, t_2)$ with one change relative to the initial setting; this consists of three possible situations: (i) the major agent subject to its second dynamics $\mathcal{A}_0^2$ is interacting with both subpopulations $\mathcal{A}^a$, $\mathcal{A}^b$ present in the system; this corresponds to the centre node inside Q_1 in Figure 4.1 and is denoted by $Q_1 = q_{0^2ab}$, (ii) the major agent subject to its first dynamics $\mathcal{A}_0^1$ is interacting with the subpopulation $\mathcal{A}^a$ while the subpopulation $\mathcal{A}^b$ has quit the system; this corresponds to the left-

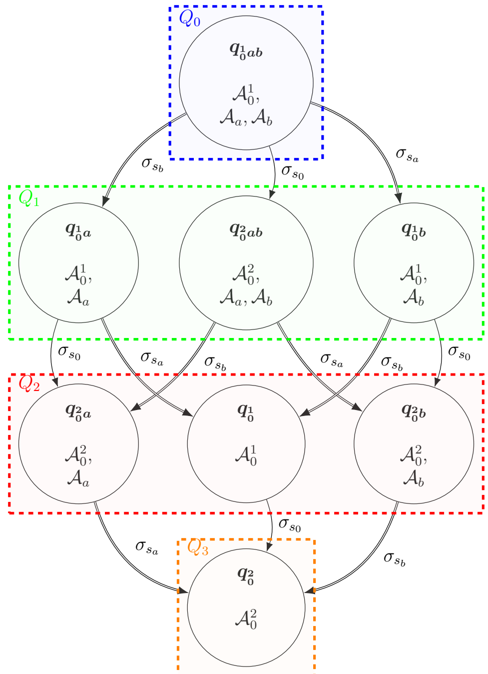

Figure 4.1: Hybrid Automata Diagram with a single major player and two populations of minor players with stopping times. Transitions accompanied by dimension changes are identified with double-line arrows.

most node inside Q_1 in Figure 4.1 and is denoted by $Q_1 = q_0^1 a$, and (iii) the major agent subject to its first dynamics $\mathcal{A}_0^1$ is interacting with $\mathcal{A}^b$ while $\mathcal{A}^a$ has quit, corresponding to the right-most node inside Q_1 in Figure 4.1, denoted by $Q_1 = q_0^1 b$.

The multivalued discrete state Q_2 represents the evolution of the system over $[t_2, t_3)$ with two changes relative to the initial setting for which three situations can be considered: (I) the major agent subject to its second dynamics $\mathcal{A}_0^2$ is interacting with the subpopulation $\mathcal{A}^a$, and the subpopulation $\mathcal{A}^b$ have already quit, which corresponds to the left-most node inside Q_2 in Figure 4.1 denoted as $Q_2 = q_0^2 a$, (II) the major agent subject to its second dynamics $\mathcal{A}_0^2$ is interacting with $\mathcal{A}^b$, and the subpopulation $\mathcal{A}^a$ has already quit, which corresponds to the right-most node inside Q_2 in Figure 4.1 denoted by $Q_2 = q_0^2 b$, (III) the major agent is subject to its first dynamics $\mathcal{A}_0^1$ and both subpopulations $\mathcal{A}^a$, $\mathcal{A}^b$ have already quit, which corresponds to the centre node inside Q_2 in Figure 4.1, denoted by $Q_2 = q_0^1$.

The discrete state Q_3 corresponds to the evolution of the major agent subject to its second dynamics $\mathcal{A}_0^2$ over $[t_3, T]$ which corresponds to $Q_3 = q_0^2$.

In this work it is assumed that each of the time periods $[t_j, t_{j+1})$ associated with the multivalued discrete state Q_j, $0 \leq j \leq 3$, is non-empty. This assumption is tenable since it will be shown that the switching times $t_1,\ t_2,\ t_3$ are deterministic and depend only on the system parameters.

4.2.3 Dynamics and Costs: Finite Population

Major Agent

Let the evolution of the major agent $\mathcal{A}_0^k,\ k = 1, 2$, be expressed as

$$dx_0 = A_0^k x_0 dt + B_0^k u_0 dt + F_0^k x^{(N_t)} dt + D_0^k dw_0, \tag{4.1}$$

where $x_0 \in \mathbb{R}^n$ is the state, $u_0 \in \mathbb{R}^m$ is the control input, and $w_0 \in \mathbb{R}^r$ is a standard Wiener process. The matrices A_0^k, B_0^k, F_0^k, and D_0^k, $k = 1, 2$, are of appropriate dimension. We note once again that the superscript k in $\mathcal{A}_0^k$ denotes that the major agent is in dynamics k.

From (4.1), the major agent is coupled with the minor agents by the average term $x^{(N_t)} = \frac{1}{N_t}\sum_{i=1}^{N_t} x_i$. Note that in (4.1), N_t may take the following values.

$$N_t = \begin{cases} N_a + N_b & \text{for } Q_0 = q_{0ab}^1, Q_1 = q_{0ab}^2 \\ N_a & \text{for } Q_1 = q_{0a}^1, Q_2 = q_{0a}^2 \\ N_b & \text{for } Q_1 = q_{0b}^1, Q_2 = q_{0b}^2 \\ 0 & \text{for } Q_2 = q_0^1, Q_3 = q_0^2. \end{cases} \tag{4.2}$$

The major agent $\mathcal{A}_0^k, k = 1, 2$, aims to minimize the following cost functional

$$J_0^k(u_0, u_{-0}) = \mathbb{E}\big[\|x_0(T)\|^2_{\bar{P}_0^k} + \int_0^T (\|x_0 - \Phi(x^{(N_t)})\|^2_{P_0^k} + \|u_0\|^2_{R_0^k})dt\big], \tag{4.3}$$

$$\Phi(.) := H_0^k x^{(N_t)}, \tag{4.4}$$

with $R_0^k > 0$, $\bar{P}_0^k \geq 0$, $P_0^k \geq 0$, and H_0^k of appropriate dimension.

Equation (4.1) together with the cost functional (4.3) form the stochastic LQG problem for the major agent.

Generic $\mathcal{A}^a$-Type Minor Agent

The dynamics for a minor agent $\mathcal{A}_i^a$, is given by

$$dx_i = A_a x_i dt + B_a u_i dt + G_a x_0 dt + F_a x^{(N_t)} dt + D_a dw_i, \tag{4.5}$$

where $x_i \in \mathbb{R}^n$ is the state of agent $\mathcal{A}_i^a$, $u_i \in \mathbb{R}^m$ is the control input, $w_i \in \mathbb{R}^r$ is a standard Wiener process, and A_a, B_a, G_a, F_a, D_a are constant matrices of appropriate dimension. Note that N_t in (4.5) again takes values as in (4.2) over the horizon T. The cost for a type $\mathcal{A}^a$ minor agent is given by

$$J_i^a(u_i, u_{-i}) = \mathbb{E}\big[\|x_i(t_s^i) - \Psi_a(x^{(N_{t_s^i})})\|^2_{\bar{P}_a} + \int_0^{t_s^i} (\|x_i - \Psi_a(x^{(N_t)})\|^2_{P_a} + \|u_i\|^2_{R_a})dt\big], \tag{4.6}$$

$$\Psi_a(.) \coloneqq H_1^a x_0(.) + H_2^a x^{(N.)}, \tag{4.7}$$

where the weight matrices $\bar{P}_a \geq 0$, $P_a \geq 0$, $R_a > 0$, H_1^a, and H_2^a have appropriate dimensions.

The set of equations (4.5) and (4.6) constitute the stochastic optimal control problem for a minor agent of type $\mathcal{A}^a$. It can be seen that a generic $\mathcal{A}^a$-type minor agent interacts with the major agent's state as well as the average state of all existing minor agents through its dynamics and cost functional.

Generic $\mathcal{A}^b$-Type Minor Agent

Similarly, we define the state vector x_i of a generic minor agent $\mathcal{A}_i^b$ whose evolution can be written as

$$dx_i = A_b x_i dt + B_b u_i dt + G_b x_0 dt + F_b x^{(N_t)} dt + D_b dw_i, \tag{4.8}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $w_i \in \mathbb{R}^r$ is a standard Wiener process, and A_b, B_b, G_b, F_b, D_b are matrices of appropriate dimension.

The cost functional for a generic minor agent of type $\mathcal{A}^b$ is given by

$$J_i^b(u_i, u_{-i}) = \mathbb{E}\big[\|x_i(t_s^i) - \Psi_b(x^{(N_{t_s^i})})\|^2_{\bar{P}_b} + \int_0^{t_s^i} (\|x_i - \Psi_b(x^{(N_t)})\|^2_{P_b} + \|u_i\|^2_{R_b})dt\big], \tag{4.9}$$

$$\Psi_b(.) \coloneqq H_1^b x_0(.) + H_2^b x^{(N.)}, \tag{4.10}$$

with matrices $\bar{P}_b \geq 0, P_b \geq 0, R_b > 0, H_1^b$, and H_2^b having appropriate dimensions.

Equations (4.8) and (4.9) form the stochastic LQG problem for a generic minor agent of type $\mathcal{A}^b$. Additionally, they show that a $\mathcal{A}^b$-type minor agent is coupled with the major agent's state and the average state of all existing minor agents in its dynamics.

We denote by $w = \{w_i, 0 \leq i \leq N\}$ the set of $(N+1)$ independent $\mathbb{R}^r$-valued standard Wiener processes on the probability space $(\Omega, \mathcal{F}, P)$, where w is progressively measurable with respect to the filtration $\mathcal{F}^w = \{\mathcal{F}^w_t \subset \mathcal{F}; t \geq 0\}$.

Assumption 4.1. *The initial states* $\{x_i(0),\ 0 \leq i \leq N\}$ *defined on* $(\Omega, \mathcal{F}, P)$ *are identically distributed, mutually independent and also independent of* $\mathcal{F}^w_\infty$*, with* $\mathbb{E}x_i(0) = 0$*. Moreover,* $\sup_i \mathbb{E}\|x_i(0)\|^2 \leq c < \infty$*,* $0 \leq i \leq N < \infty$*, with c independent of* N*.*

The empirical distribution of the agents sampled independently of the initial conditions and Wiener processes within populations $\mathcal{A}^a$ and $\mathcal{A}^b$ at time t_0 is denoted by $\pi^N = (\pi^N_a, \pi^N_b)$, where $\pi^N_a = \frac{N_a}{N}$ and $\pi^N_b = \frac{N_b}{N}$.

Assumption 4.2. *There exists* $\pi = (\pi_a, \pi_b)$ *such that* $lim_{N\to\infty}\pi^N \overset{a.s.}{=} \pi$.

In the following we introduce the admissible sets of controls for each agent. The null set augmented σ-field $\mathcal{F}_{i,t}, 1 \leq i \leq N$, is defined to be the increasing family of null set augmented σ-fields generated by $(x_i(\tau); 0 \leq \tau \leq t)$, and by definition $\mathcal{F}_{0,t}$ is the increasing family of σ-fields generated by $(x_0(\tau); 0 \leq \tau \leq t)$. $\mathcal{F}^N_t$ is the increasing family of σ-fields generated by the set $\{x_j(\tau), x_0(\tau); 0 \leq \tau \leq t, 1 \leq j \leq N\}$. The set of control actions $\mathcal{U}^{N,L}_g$ consists of linear feedback control actions adapted to $\{\mathcal{F}^N_t, t \geq 0\}, 1 \leq N < \infty$.

Assumption 4.3 (Major Agent σ-Fields and Linear Controls)**.** *For the major agent* $\mathcal{A}_0$ *the set of control inputs* $\mathcal{U}^L_0$ *is defined to be the collection of linear feedback controls adapted to the filteration* $\{\mathcal{F}_{0,t}, t \geq 0\}$.

Assumption 4.4 (Minor Agent σ-Fields and Linear Controls)**.** *For the minor agent* $\mathcal{A}_i, 1 \leq i \leq N$*, the set of control inputs* $\mathcal{U}^L_i$ *is defined to be the collection of linear feedback controls adapted to the filtration* $\{\mathcal{F}_{i,t}, t \geq 0\}, 1 \leq i \leq N$.

4.3 Hybrid Mean Field Game Approach

Following the mean field game methodology with a major agent [16, 42], the hybrid MFG problem is first solved in the infinite population limit where the average term in the finite population dynamics and cost functional of each agent is replaced by its infinite population limit, i.e. the mean field. Then specializing to linear systems (see e.g. [16]), the major agent's state is extended with the mean field, while the minor agent's state is extended with the mean field and the major agent's state; this yields LQG hybrid optimal control problems (see appendix A) for each agent linked only through the mean field and the major agent's state. Then the main results of [16], [42] are (i) the existence of infinite population best response strategies which yield the Nash equilibria, and (ii) the infinite population best response strategies applied to the finite population system yield an ϵ-Nash equilibrium (see *Theorem 4.1*).

In this section, first, the hybrid evolution of the mean field is derived. Then the extended hybrid optimal control problems for the major agent and minor agents are formed and addressed in the infinite population case. Finally, *Theorem 4.1* is presented which links the infinite population and finite population LQG Hybrid-MFG problem solutions.

4.3.1 Hybrid Evolution of Mean Field

Following the LQG MFG methodology [16], the mean field is defined as the limit (in quadratic mean), when it exists, of the average of minor agents' states when the population size goes to infinity

$$\bar{x}^k(t) = \lim_{N_k \to \infty} x^{N_k}(t) = \lim_{N_k \to \infty} \frac{1}{N_k} \sum_{i=1}^{N_k} x_i(t), \quad q.m.$$

where $k \triangleq a, b$, for the case considered in this chapter. Now, if the control strategy for each minor agent is considered to have the general feedback form

$$u_i = L_1^k x_i + L_2^k x_0 + \sum_{j \neq i, j=1}^{N_t} L_4^k x_j + m_k, \quad 1 \leq i \leq N_k, \tag{4.11}$$

then the mean field dynamics is obtained by substituting (4.11) in the minor agents' dynamics (4.8) (respectively, (4.5)), and taking the average over population $\mathcal{A}^k$, $k \triangleq a, b$, and then its L^2 limit as $N_k \to \infty$.

With the assignment of discrete states Q_j introduced in Section 4.2.2, the set of the mean field equations is given by

$$d\bar{x}^{Q_j} = \bar{A}^{Q_j} \bar{x}^{Q_j} dt + \bar{G}^{Q_j} x_0^{Q_j} dt + \bar{m}^{Q_j} dt, \quad j = 0, 1, 2, 3. \tag{4.12}$$

For $Q_0 = q_{\substack{1\\0}ab}$, $\bar{x}^{Q_0} = [\bar{x}_a^T, \bar{x}_b^T]^T$ consists of the mean field $\bar{x}_a$ of the population $\mathcal{A}^a$, and the mean field $\bar{x}_b$ of the population $\mathcal{A}^b$ with $\pi^{Q_0} = \pi$. The matrices in (4.12) are then

$$\bar{A}^{Q_0} = \begin{bmatrix} \bar{A}_a \\ \bar{A}_b \end{bmatrix}, \quad \bar{G}^{Q_0} = \begin{bmatrix} \bar{G}_a \\ \bar{G}_b \end{bmatrix}, \quad \bar{m}^{Q_0} = \begin{bmatrix} \bar{m}_a \\ \bar{m}_b \end{bmatrix}, \tag{4.13}$$

where $\bar{A}_a, \bar{A}_b \in \mathbb{R}^{n\times 2n}$, $\bar{G}_a, \bar{G}_b \in \mathbb{R}^{n\times n}$, $\bar{m}_a, \bar{m}_b \in \mathbb{R}^n$. The above matrices shall be determined from the consistency equations discussed in Section 4.3.4.

In case (i) in Section 4.2.2 where $Q_1 = q_{\substack{2\\0}ab}$, the mean field is defined as $\bar{x}^{q_{\substack{2\\0}ab}} = [\bar{x}_a^T, \bar{x}_b^T]^T$, hence $\pi^{q_{\substack{2\\0}ab}} = \pi$, and

$$\bar{A}^{q_{\substack{2\\0}ab}} = \begin{bmatrix} \bar{A}_a \\ \bar{A}_b \end{bmatrix}, \quad \bar{G}^{q_{\substack{2\\0}ab}} = \begin{bmatrix} \bar{G}_a \\ \bar{G}_b \end{bmatrix}, \quad \bar{m}^{q_{\substack{2\\0}ab}} = \begin{bmatrix} \bar{m}_a \\ \bar{m}_b \end{bmatrix}. \tag{4.14}$$

For case (ii) where $Q_1 = q_{\substack{1\\0}a}$, $\bar{x}^{q_{\substack{1\\0}a}} = \bar{x}_a$, and hence $\pi^{q_{\substack{1\\0}a}} = (1,0)$, and the matrices in (4.12) are given as

$$\bar{A}^{q_{\substack{1\\0}a}} = \bar{A}_a, \quad \bar{G}^{q_{\substack{1\\0}a}} = \bar{G}_a, \quad \bar{m}^{q_{\substack{1\\0}a}} = \bar{m}_a, \tag{4.15}$$

where $\bar{A}_a \in \mathbb{R}^{n\times n}$, $\bar{G}_a \in \mathbb{R}^{n\times n}$, $\bar{m}_a \in \mathbb{R}^n$.

For case (iii) where $Q_1 = q_{\substack{1\\0}b}$, $\bar{x}^{q_{\substack{1\\0}b}} = \bar{x}_b$, and hence $\pi^{q_{\substack{1\\0}b}} = (0,1)$, and the matrices in (4.12) are given by

$$\bar{A}^{q_{\substack{1\\0}b}} = \bar{A}_b, \quad \bar{G}^{q_{\substack{1\\0}b}} = \bar{G}_b, \quad \bar{m}^{q_{\substack{1\\0}b}} = \bar{m}_b. \tag{4.16}$$

For case (I) in Section 4.2.2 where $Q_2 = q_{\substack{2\\0}a}$, the mean field is defined as $\bar{x}^{q_{\substack{2\\0}a}} = \bar{x}_a$, and hence $\pi^{q_{\substack{2\\0}a}} = (1,0)$, and the matrices in (4.12) are given as

$$\bar{A}^{q_{\substack{2\\0}a}} = \bar{A}_a, \quad \bar{G}^{q_{\substack{2\\0}a}} = \bar{G}_a, \quad \bar{m}^{q_{\substack{2\\0}a}} = \bar{m}_a. \tag{4.17}$$

For case (II) where $Q_2 = q_{\substack{2\\0}b}$, $\bar{x}^{q_{\substack{2\\0}b}} = \bar{x}_b$, and hence $\pi^{q_{\substack{2\\0}b}} = (0,1)$, and the matrices in (4.12) are given by

$$\bar{A}^{q_{\substack{2\\0}b}} = \bar{A}_b, \quad \bar{G}^{q_{\substack{2\\0}b}} = \bar{G}_b, \quad \bar{m}^{q_{\substack{2\\0}b}} = \bar{m}_b. \tag{4.18}$$

For case (III) where $Q_2 = q_{_0}^1$, $\bar{x}^{q_{_0}^1} = 0$, hence $\pi^{q_{_0}^1} = (0,0)$.

Finally, for $Q_3 = q_{_0}^2$, $\bar{x}^{Q_3} = 0$, and as a result $\pi^{Q_3} = (0,0)$.

4.3.2 Major Agent: Infinite Populations

Hybrid Dynamics and Cost

The extended hybrid dynamics of the major agent in the infinite population, i.e. the dynamics for x_0^{ex,Q_j} is given by

$$dx_0^{ex,Q_j} = (\mathbb{A}_0^{Q_j} x_0^{ex,Q_j} + \mathbb{M}_0^{Q_j} + \mathbb{B}_0^{Q_j} u_0^{Q_j})dt + \mathbb{D}_0^{Q_j} dW_0^{Q_j}, \quad 0 \le j \le 3, \tag{4.19}$$

where the dynamical matrices are given by

$$\mathbb{A}_0^{Q_j} = \begin{bmatrix} A_0^{Q_j} & \pi^{Q_j} \otimes F_0^{Q_j} \\ \bar{G}^{Q_j} & \bar{A}^{Q_j} \end{bmatrix}, \quad \mathbb{M}_0^{Q_j} = \begin{bmatrix} 0_{n\times 1} \\ \bar{m}^{Q_j} \end{bmatrix}, \quad \mathbb{B}_0^{Q_j} = \begin{bmatrix} B_0^{Q_j} \\ 0_{\bullet\times\bullet} \end{bmatrix},$$
$$\mathbb{D}_0^{Q_j} = \begin{bmatrix} D_0^{Q_j} & 0_{\bullet\times\bullet} \\ 0_{\bullet\times\bullet} & 0_{\bullet\times\bullet} \end{bmatrix}, \quad W_0^{Q_j} = \begin{bmatrix} w_0 \\ 0_{\bullet\times\bullet} \end{bmatrix}. \tag{4.20}$$

In (4.20), $0_{\bullet\times\bullet}$ denotes a zero matrix of appropriate dimension, and $\pi^{Q_j} \otimes F_0^{Q_j}$ denotes the Kronecker product of π^{Q_j} and $F_0^{Q_j}$.

The cost functional for the extended major agent's hybrid system would be given by

$$J_0(u_0, u_{-0}) = \mathbb{E}\Big[\|x_0^{ex,Q_3}(T)\|^2_{\bar{\mathbb{P}}_0^{Q_3}} + \sum_{j=1}^{3} \|x_0^{ex,Q_j}(t_j^-)\|^2_{\mathbb{C}_{0,j}}$$
$$+ \sum_{j=0}^{3} \int_{t_j}^{t_{j+1}} \big(\|x_0^{ex,Q_j}(s)\|^2_{\mathbb{P}_0^{Q_j}} + \|u_0^{Q_j}(s)\|^2_{R_0^{Q_j}}\big) ds\Big], \tag{4.21}$$

where $t_0 = 0$, $t_4 = T$. In (4.21), the first term denotes terminal cost and the third term denotes running cost where the corresponding weight matrices are defined as

$$\bar{\mathbb{P}}_0^{Q_3} = \bar{P}_0^2,$$
$$\mathbb{P}_0^{Q_j} = [I_{n\times n}, -\pi^{Q_j} \otimes H_0^{Q_j}]^T P_0^{Q_j} [I_{n\times n}, -\pi^{Q_j} \otimes H_0^{Q_j}]. \tag{4.22}$$

Moreover, the second term in (4.21) denotes switching cost where the corresponding weight matrix $\mathbb{C}_{0,j}$ shall be identified for each switching in Section 4.3.2.

Now the dynamical and weight matrices introduced in their general form, respectively, in (4.20) and (4.22) are specified for each discrete state Q_j, $0 \leq j \leq 3$.

Over the interval $[t_0, t_1)$, and in discrete state Q_0, the dynamics of the continuous state $x_0^{ex,Q_0} = [x_0^T, \bar{x}_a^T, \bar{x}_b^T]^T$ is determined by (4.19) with

$$\mathbb{A}_0^{Q_0} = \begin{bmatrix} A_0^1 & \pi \otimes F_0^1 \\ \begin{bmatrix} \bar{G}_a \\ \bar{G}_b \end{bmatrix} & \begin{bmatrix} \bar{A}_a \\ \bar{A}_b \end{bmatrix} \end{bmatrix}, \quad \mathbb{M}_0^{Q_0} = \begin{bmatrix} 0_{n\times 1} \\ \begin{bmatrix} \bar{m}_a \\ \bar{m}_b \end{bmatrix} \end{bmatrix},$$

$$\mathbb{B}_0^{Q_0} = \begin{bmatrix} B_0^1 \\ 0_{2n\times m} \end{bmatrix}, \quad \mathbb{D}_0^{Q_0} = \begin{bmatrix} D_0^1 & 0_{n\times 2r} \\ 0_{2n\times r} & 0_{2n\times 2r} \end{bmatrix}, \quad W_0^{Q_0} = \begin{bmatrix} w_0 \\ 0_{2r\times 1} \end{bmatrix}, \tag{4.23}$$

where $\pi \otimes F_0^1 = [\pi_a F_0^1, \pi_b F_0^1]$, and $\mathbb{P}_0^{Q_0}$ in (4.21) is given by

$$\mathbb{P}_0^{Q_0} = [I_{n\times n}, -\pi_a H_0^1, -\pi_b H_0^1]^T P_0^1 [I_{n\times n}, -\pi_a H_0^1, -\pi_b H_0^1]. \tag{4.24}$$

We also define

$$\bar{\mathbb{P}}_0^{Q_0} = [I_{n\times n}, -\pi_a H_0^1, -\pi_b H_0^1]^T \bar{P}_0^1 [I_{n\times n}, -\pi_a H_0^1, -\pi_b H_0^1], \tag{4.25}$$

which will be used in section 4.3.2 to specify the switching cost at t_1.

Over the interval $[t_1, t_2)$, in case (i) where $Q_1 = q_{0}^{2}{}_{ab}$ holds over the interval $[t_1, t_2)$, the dynamics of $x_0^{ex,q_{0}^{2}{}_{ab}} = [x_0^T, \bar{x}_a^T, \bar{x}_b^T]^T$ is governed by (4.19) with

$$\mathbb{A}_0^{q_{0}^{2}{}_{ab}} = \begin{bmatrix} A_0^2 & \pi \otimes F_0^2 \\ \begin{bmatrix} \bar{G}_a \\ \bar{G}_b \end{bmatrix} & \begin{bmatrix} \bar{A}_a \\ \bar{A}_b \end{bmatrix} \end{bmatrix}, \quad \mathbb{M}_0^{q_{0}^{2}{}_{ab}} = \begin{bmatrix} 0_{n\times 1} \\ \begin{bmatrix} \bar{m}_a \\ \bar{m}_b \end{bmatrix} \end{bmatrix},$$

$$\mathbb{B}_0^{q_{0}^{2}{}_{ab}} = \begin{bmatrix} B_0^2 \\ 0_{2n\times m} \end{bmatrix}, \quad \mathbb{D}_0^{q_{0}^{2}{}_{ab}} = \begin{bmatrix} D_0^2 & 0_{n\times 2r} \\ 0_{2n\times r} & 0_{2n\times 2r} \end{bmatrix}, \quad W_0^{q_{0}^{2}{}_{ab}} = \begin{bmatrix} w_0 \\ 0_{2r\times 1} \end{bmatrix}, \tag{4.26}$$

and $\mathbb{P}_0^{q_{0}^{2}{}_{ab}}$ in (4.21) is given by

$$\mathbb{P}_0^{q_{0}^{2}{}_{ab}} = [I_{n\times n}, -\pi_a H_0^2, -\pi_b H_0^2]^T P_0^2 [I_{n\times n}, -\pi_a H_0^2, -\pi_b H_0^2]. \tag{4.27}$$

Moreover,

$$\bar{\mathbb{P}}_0^{q^2_{0ab}} = [I_{n\times n}, -\pi_a H_0^2, -\pi_b H_0^2]^T \bar{P}_0^2 [I_{n\times n}, -\pi_a H_0^2, -\pi_b H_0^2], \tag{4.28}$$

which will be used in section 4.3.2 to specify the switching cost at t_2.

Over the interval $[t_1, t_2)$, in case (ii) where $Q_1 = q^1_{0a}$ holds, the dynamics for $x_0^{ex,q^1_{0a}} = [x_0^T, \bar{x}_a^T]^T$ is determined by (4.19) with

$$\mathbb{A}_0^{q^1_{0a}} = \begin{bmatrix} A_0^1 & F_0^1 \\ \bar{G}_a & \bar{A}_a \end{bmatrix}, \quad \mathbb{M}_0^{q^1_{0a}} = \begin{bmatrix} 0_{n\times 1} \\ \bar{m}_a \end{bmatrix}, \quad \mathbb{B}_0^{q^1_{0a}} = \begin{bmatrix} B_0^1 \\ 0_{n\times m} \end{bmatrix},$$
$$\mathbb{D}_0^{q^1_{0a}} = \begin{bmatrix} D_0^1 & 0_{n\times r} \\ 0_{n\times r} & 0_{n\times r} \end{bmatrix}, \quad W_0^{q^1_{0a}} = \begin{bmatrix} w_0 \\ 0_{r\times 1} \end{bmatrix}, \tag{4.29}$$

and the cost functional is determined by (4.21) with $\mathbb{P}_0^{q^1_{0a}} = [I_{n\times n}, -H_0^1]^T P_0^1 [I_{n\times n}, -H_0^1]$. In addition, matrix $\bar{\mathbb{P}}_0^{q^1_{0a}}$ which shall be used in Section 4.3.2 to identify the switching cost at t_2 is defined as

$$\bar{\mathbb{P}}_0^{q^1_{0a}} = [I_{n\times n}, -H_0^1]^T \bar{P}_0^1 [I_{n\times n}, -H_0^1]. \tag{4.30}$$

Over the interval $[t_1, t_2)$, in case (iii) where $Q_1 = q^1_{0b}$ holds, $x^{ex,q^1_{0b}} = [x_0^T, \bar{x}_b^T]^T$ and

$$\mathbb{A}_0^{q^1_{0b}} = \begin{bmatrix} A_0^1 & F_0^1 \\ \bar{G}_b & \bar{A}_b \end{bmatrix}, \quad \mathbb{M}_0^{q^1_{0b}} = \begin{bmatrix} 0_{n\times m} \\ \bar{m}_b \end{bmatrix}, \quad \mathbb{B}_0^{q^1_{0b}} = \begin{bmatrix} B_0^1 \\ 0_{n\times m} \end{bmatrix},$$
$$\mathbb{D}_0^{q^1_{0b}} = \begin{bmatrix} D_0^1 & 0_{n\times r} \\ 0_{n\times r} & 0_{n\times r} \end{bmatrix}, \quad W_0^{q^1_{0b}} = \begin{bmatrix} w_0 \\ 0_{r\times 1} \end{bmatrix}, \tag{4.31}$$
$$\mathbb{P}_0^{q^1_{0b}} = [I_{n\times n}, -H_0^1]^T P_0^1 [I_{n\times n}, -H_0^1], \tag{4.32}$$
$$\bar{\mathbb{P}}_0^{q^1_{0b}} = [I_{n\times n}, -H_0^1]^T \bar{P}_0^1 [I_{n\times n}, -H_0^1]. \tag{4.33}$$

Over the interval $[t_2, t_3)$, in case (I) where $Q_2 = q^2_{0a}$ holds, $x^{ex,q^2_{0a}} = [x_0^T, \bar{x}_a^T]^T$ and

$$\mathbb{A}_0^{q^2_{0a}} = \begin{bmatrix} A_0^2 & F_0^2 \\ \bar{G}_a & \bar{A}_a \end{bmatrix}, \quad \mathbb{M}_0^{q^2_{0a}} = \begin{bmatrix} 0_{n\times 1} \\ \bar{m}_a \end{bmatrix}, \quad \mathbb{B}_0^{q^2_{0a}} = \begin{bmatrix} B_0^2 \\ 0_{n\times m} \end{bmatrix},$$

$$\mathbb{D}_0^{q_0^{2a}} = \begin{bmatrix} D_0^2 & 0_{n\times r} \\ 0_{n\times r} & 0_{n\times r} \end{bmatrix}, \quad W_0^{q_0^{2a}} = \begin{bmatrix} w_0 \\ 0_{r\times 1} \end{bmatrix}, \tag{4.34}$$

$$\mathbb{P}_0^{q_0^{2a}} = [I_{n\times n}, -H_0^2]^T P_0^2 [I_{n\times n}, -H_0^2], \tag{4.35}$$

$$\bar{\mathbb{P}}_0^{q_0^{2a}} = [I_{n\times n}, -H_0^2]^T \bar{P}_0^2 [I_{n\times n}, -H_0^2]. \tag{4.36}$$

Over the interval $[t_2, t_3)$, in case (II) where $Q_2 = q_0^{2b}$ holds, $x^{ex, q_0^{2b}} = [x_0^T, \bar{x}_b^T]^T$ and

$$\mathbb{A}_0^{q_0^{2b}} = \begin{bmatrix} A_0^2 & F_0^2 \\ \bar{G}_b & \bar{A}_b \end{bmatrix}, \quad \mathbb{M}_0^{q_0^{2b}} = \begin{bmatrix} 0_{n\times 1} \\ \bar{m}_b \end{bmatrix}, \quad \mathbb{B}_0^{q_0^{2b}} = \begin{bmatrix} B_0^2 \\ 0_{n\times m} \end{bmatrix},$$

$$\mathbb{D}_0^{q_0^{2b}} = \begin{bmatrix} D_0^2 & 0_{n\times r} \\ 0_{n\times r} & 0_{n\times r} \end{bmatrix}, \quad W_0^{q_0^{2b}} = \begin{bmatrix} w_0 \\ 0_{r\times 1} \end{bmatrix}, \tag{4.37}$$

$$\mathbb{P}_0^{q_0^{2b}} = [I_{n\times n}, -H_0^2]^T P_0^2 [I_{n\times n}, -H_0^2], \tag{4.38}$$

$$\bar{\mathbb{P}}_0^{q_0^{2b}} = [I_{n\times n}, -H_0^2]^T \bar{P}_0^2 [I_{n\times n}, -H_0^2]. \tag{4.39}$$

Over the interval $[t_2, t_3)$, in case (III) where $Q_2 = q_0^1$ holds, $x^{ex, q_0^1} = x_0$ and

$$\mathbb{A}_0^{q_0^1} = A_0^1, \quad \mathbb{M}_0^{q_0^1} = 0_{n\times 1}, \quad \mathbb{B}_0^{q_0^1} = B_0^1, \quad \mathbb{D}_0^{q_0^1} = D_0^1, \quad W_0^{q_0^1} = w_0, \quad \mathbb{P}_0^{q_0^1} = P_0^1, \quad \bar{\mathbb{P}}_0^{q_0^1} = \bar{P}_0^1.$$

Finally, over the interval $[t_3, T]$, in discrete state Q_3, $x^{ex, Q_3} = x_0$ and

$$\mathbb{A}_0^{Q_3} = A_0^2, \quad \mathbb{M}_0^{Q_3} = 0_{n\times 1}, \quad \mathbb{B}_0^{Q_3} = B_0^2, \quad \mathbb{D}_0^{Q_3} = D_0^2, \quad W_0^{Q_3} = w_0, \quad \mathbb{P}_0^{Q_3} = P_0^2, \quad \bar{\mathbb{P}}_0^{Q_3} = \bar{P}_0^2.$$

Jump Transition Maps and Switching Costs

We first define the notation $M_0^{Q_j}(l:m)$, $0 \le j \le 3$, which shall be used to identify the switching cost associated with switching time t_j, $1 \le j \le 3$, for the major agent $\mathcal{A}_0$. Matrix $M_0^{Q_j}(l:m)$ is formed by using matrix $\bar{\mathbb{P}}_0^{Q_j}$ wherein all the entires are made zero except those associated with its l-th to m-th columns and rows. Hence it has the same dimension (size) as $\bar{\mathbb{P}}_0^{Q_j}$, i.e.

$$M_0^{Q_j}(l:m) = \begin{array}{c} \overbrace{}^{\bar{\mathbb{P}}_0^{Q_j}(:,l:m)} \\ \left[\begin{array}{c|c|c} 0 & & 0 \\ \hline & & \\ \hline 0 & & 0 \end{array}\right]_{\mathrm{size}(\bar{\mathbb{P}}_0^{Q_j})} \end{array} \left.\vphantom{\begin{array}{c} 0 \\ 0 \\ 0 \end{array}}\right\} \bar{\mathbb{P}}_0^{Q_j}(l:m,:) \tag{4.40}$$

where $\bar{\mathbb{P}}_0^{Q_j}(:, l:m)$ and $\bar{\mathbb{P}}_0^{Q_j}(l:m, :)$, respectively, denote l-th to m-th columns and l-th to m-th rows of $\bar{\mathbb{P}}_0^{Q_j}$.

The values of the major agent's continuous state before and after switching at t_1 satisfy the following jump map

$$x_0^{ex,Q_1}(t_1) = \Psi_{0,1} x_0^{ex,Q_0}(t_1-). \tag{4.41}$$

For the transition between Q_0 and case (i) for Q_1 where $Q_1 = q_{0ab}^{2}$ the map $\Psi_{0,1}$ is the identity matrix, i.e.

$$\Psi_{0,1} = \Psi_{0,q_{0ab}^{1} q_{0ab}^{2}} = I_{3n\times 3n}. \tag{4.42}$$

This transition is not accompanied by change in the dimension of the major agent's extended state. Furthermore, the weight matrix for the corresponding switching cost is given by

$$\mathbb{C}_{0,1} = \mathbb{C}_{0,q_{0ab}^{1} q_{0ab}^{2}} = 0_{3n\times 3n}. \tag{4.43}$$

For the transition between Q_0 and case (ii) where $Q_1 = q_{0a}^{1}$

$$\Psi_{0,1} = \Psi_{0,q_{0ab}^{1} q_{0a}^{1}} = \begin{bmatrix} I_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & I_{n\times n} & 0_{n\times n} \end{bmatrix}, \tag{4.44}$$

$$\mathbb{C}_{0,1} = \mathbb{C}_{0,q_{0ab}^{1} q_{0a}^{1}} = M_0^{q_{0ab}^{1}}(2n+1:3n). \tag{4.45}$$

For the transition between Q_0 and case (iii) where $Q_1 = q_{0b}^{1}$

$$\Psi_{0,1} = \Psi_{0,q_{0ab}^{1} q_{0b}^{1}} = \begin{bmatrix} I_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & I_{n\times n} \end{bmatrix}, \tag{4.46}$$

$$\mathbb{C}_{0,1} = \mathbb{C}_{0,q_{0ab}^{1} q_{0b}^{1}} = M_0^{q_{0ab}^{1}}(n+1:2n). \tag{4.47}$$

The values of the major agent's continuous state before and after the switching at t_2 satisfy the following jump transition map

$$x_0^{ex,Q_2}(t_2) = \Psi_{0,2} x_0^{ex,Q_1}(t_2-), \tag{4.48}$$

where

$$
\Psi_{0,2}=\begin{cases}
\Psi_{0,q^1_{0}a q^2_{0}a}=I_{2n\times 2n}, & \text{for transition from } Q_1=q^1_{0}a \text{ to } Q_2=q^2_{0}a,\\
\Psi_{0,q^1_{0}a q^1_{0}}=\begin{bmatrix} I_{n\times n} & 0_{n\times n}\end{bmatrix}, & \text{for transition from } Q_1=q^1_{0}a \text{ to } Q_2=q^1_{0},\\
\Psi_{0,q^2_{0}ab q^2_{0}a}=\begin{bmatrix} I_{n\times n} & 0_{n\times n} & 0_{n\times n}\\ 0_{n\times n} & I_{n\times n} & 0_{n\times n}\end{bmatrix}, & \text{for transition from } Q_1=q^2_{0}ab \text{ to } Q_2=q^2_{0}a,\\
\Psi_{0,q^2_{0}ab q^2_{0}b}=\begin{bmatrix} I_{n\times n} & 0_{n\times n} & 0_{n\times n}\\ 0_{n\times n} & 0_{n\times n} & I_{n\times n}\end{bmatrix}, & \text{for transition from } Q_1=q^2_{0}ab \text{ to } Q_2=q^2_{0}b,\\
\Psi_{0,q^1_{0}b q^2_{0}b}=I_{2n\times 2n}, & \text{for transition from } Q_1=q^1_{0}b \text{ to } Q_2=q^2_{0}b,\\
\Psi_{0,q^1_{0}b q^1_{0}}=\begin{bmatrix} I_{n\times n} & 0_{n\times n}\end{bmatrix}, & \text{for transition from } Q_1=q^1_{0}b \text{ to } Q_2=q^1_{0}.
\end{cases}
\tag{4.49}
$$

Furthermore, the matrix coefficient $\mathbb{C}_{0,2}$ of the switching cost at t_2 for each case is defined as

$$
\mathbb{C}_{0,2}=\begin{cases}
\mathbb{C}_{0,q^1_{0}a q^2_{0}a}=0_{2n\times 2n}, & \text{for transition from } Q_1=q^1_{0}a \text{ to } Q_2=q^2_{0}a,\\
\mathbb{C}_{0,q^1_{0}a q^1_{0}}=M_0^{q^1_{0}a}(n+1:2n), & \text{for transition from } Q_1=q^1_{0}a \text{ to } Q_2=q^1_{0},\\
\mathbb{C}_{0,q^2_{0}ab q^2_{0}a}=M_0^{q^2_{0}ab}(2n+1:3n), & \text{for transition from } Q_1=q^2_{0}ab \text{ to } Q_2=q^2_{0}a,\\
\mathbb{C}_{0,q^2_{0}ab q^2_{0}b}=M_0^{q^2_{0}ab}(n+1:2n), & \text{for transition from } Q_1=q^2_{0}ab \text{ to } Q_2=q^2_{0}b,\\
\mathbb{C}_{0,q^1_{0}b q^2_{0}b}=0_{2n\times 2n}, & \text{for transition from } Q_1=q^1_{0}b \text{ to } Q_2=q^2_{0}b,\\
\mathbb{C}_{0,q^1_{0}b q^1_{0}}=M_0^{q^1_{0}b}(n+1:2n), & \text{for transition from } Q_1=q^1_{0}b \text{ to } Q_2=q^1_{0}.
\end{cases}
\tag{4.50}
$$

The values of the major agent's continuous state before and after the switching at t_3 satisfy the following jump map

$$
x_0^{ex,Q_3}(t_3)=\Psi_{0,3}x_0^{ex,Q_2}(t_3-), \tag{4.51}
$$

where

$$
\Psi_{0,3}=\begin{cases}
\Psi_{0,q^2_{0}a q^2_{0}}=\begin{bmatrix} I_{n\times n} & 0_{n\times n}\end{bmatrix}, & \text{for transition from } Q_2=q^2_{0}a \text{ to } Q_3,\\
\Psi_{0,q^1_{0} q^2_{0}}=I_{n\times n}, & \text{for transition from } Q_2=q^1_{0} \text{ to } Q_3,\\
\Psi_{0,q^2_{0}b q^2_{0}}=\begin{bmatrix} I_{n\times n} & 0_{n\times n}\end{bmatrix}, & \text{for transition from } Q_1=q^2_{0}b \text{ to } Q_3.
\end{cases}
\tag{4.52}
$$

Accordingly, the matrix coefficient $\mathbb{C}_{0,3}$ of the switching cost at t_3 for each case is given by

$$\mathbb{C}_{0,3} = \begin{cases} \mathbb{C}_{0,q_0^2{}_a q_0^2} = M_0^{q_0^2{}_a}(n+1:2n), & \text{for transition from } Q_2 = q_0^2{}_a \text{ to } Q_3, \\ \mathbb{C}_{0,q_0^1 q_0^2} = 0_{n\times n}, & \text{for transition from } Q_2 = q_0^1 \text{ to } Q_3, \\ \mathbb{C}_{0,q_0^2{}_b q_0^2} = M_0^{q_0^2{}_b}(n+1:2n), & \text{for transition from } Q_1 = q_0^2{}_b \text{ to } Q_3, \end{cases} \tag{4.53}$$

Notice that some of the transitions of (4.41), (4.71), (4.51) are between the spaces of the same dimension such as (4.42) while other transitions may be accompanied by changes in the dimension of the state space, e.g. (4.44) is a mapping from $\mathbb{R}^{3n}$ into $\mathbb{R}^{2n}$. These dimension changes are permitted in the stochastic hybrid systems framework of [44,45] (see [55] for another motivating example for change of dimension at switching).

Best Response Hybrid Control Action

To obtain the best response hybrid control action for the major agent in the infinite population, we utilize *Theorem* 4.2 in Appendix A developed for single agent LQG hybrid optimal control problems.

By the definition of the terms $\mathbb{D}_0^{Q_j}$, they automatically satisfy the condition (4.92) (see appendix A), or equivalently condition A1 in [45, Eq. (3)] as

$$\mathbb{D}_0^{Q_j} = \Psi_{0,j}\mathbb{D}_0^{Q_{j-1}}, \qquad j = 1, 2, 3, \tag{4.54}$$

holds for all the jump transition maps introduced in this section. Moreover, it is assumed conditions (4.97)- (4.99) (in Appendix A) hold. Therefore, the optimal controlled switching times for the major agent are $\mathcal{F}_t$-independent. Then an application of the LQG hybrid optimal control theory (i.e. *Theorem* 4.2) yields the infinite population best response hybrid control action for discrete states $\{Q_0, \dots, Q_3\}$ as in

$$u_0^{Q_j}(t) = -[R_0^{Q_j}]^{-1}[\mathbb{B}_0^{Q_j}]^T \Pi_0^{Q_j}(t)\, x_0^{ex,Q_j}(t), \tag{4.55}$$

$$-\dot{\Pi}_0^{Q_j} = \Pi_0^{Q_j}\mathbb{A}_0^{Q_j} + [\mathbb{A}_0^{Q_j}]^T\Pi_0^{Q_j} - \Pi_0^{Q_j}\mathbb{B}_0^{Q_j}[R_0^{Q_j}]^{-1}[\mathbb{B}_0^{Q_j}]^T\Pi_0^{Q_j} + \mathbb{P}_0^{Q_j}, \tag{4.56}$$

subject to the terminal and boundary conditions

$$\Pi_0^{Q_3}(T) = \bar{\mathbb{P}}_0^{Q_3}, \tag{4.57}$$
$$\Pi_0^{Q_{j-1}}(t_j) = \Psi_{0,j}^T \Pi_0^{Q_j}(t_j)\Psi_{0,j} + \mathbb{C}_{0,j}, \tag{4.58}$$

$$\begin{aligned}
&\mathbb{P}_0^{Q_{j-1}} + \Pi_0^{Q_{j-1}}(t_j)\mathbb{A}_0^{Q_{j-1}} + [\mathbb{A}_0^{Q_{j-1}}]^T \Pi_0^{Q_{j-1}}(t_j) - \Pi_0^{Q_{j-1}}(t_j)\mathbb{B}_0^{Q_{j-1}}[R_0^{Q_{j-1}}]^{-1}[\mathbb{B}_0^{Q_{j-1}}]^T \Pi_0^{Q_{j-1}}(t_j) \\
&= \Psi_{0,j}^T \Big(\mathbb{P}_0^{Q_j} + \Pi_0^{Q_j}(t_j)\mathbb{A}_0^{Q_j} + [\mathbb{A}_0^{Q_j}]^T \Pi_0^{Q_j}(t_j) - \Pi_0^{Q_j}(t_j)\mathbb{B}_0^{Q_j}[R_0^{Q_j}]^{-1}[\mathbb{B}_0^{Q_j}]^T \Pi_0^{Q_j}(t_j)\Big)\Psi_{0,j} + \frac{\partial \mathbb{C}_{0,j}}{\partial t}\Big|_{t=t_j},
\end{aligned} \tag{4.59}$$

where t_j, $j = 1, 2, 3$, indicate the times of change in the system due to the major agent's switching of dynamics or cessation of subpopulations of minor agents.

4.3.3 Minor Agents: Infinite Population

Hybrid Dynamics and Costs

The extended dynamics for a generic minor agent $\mathcal{A}_i^k$, $1 \le i \le N$, in the population $k \triangleq a, b$, with the extended state x_i^{ex,Q_j} has a general form as in

$$dx_i^{ex,Q_j} = (\mathbb{A}_k^{Q_j} x_i^{ex,Q_j} + \mathbb{M}_k^{Q_j} + \mathbb{B}_k^{Q_j} u_i^{Q_i})dt + \mathbb{D}_k^{Q_j} dW_i^{Q_j}, \tag{4.60}$$

where

$$\mathbb{A}_k^{Q_j} = \begin{bmatrix} A_k & \begin{bmatrix} G_k & \pi^{Q_j} \otimes F_k \end{bmatrix} \\ 0_{\bullet\times\bullet} & \mathbb{A}_0^{Q_j} - \mathbb{B}_0^{Q_j} R_{0,Q_j}^{-1} \mathbb{B}_{0,Q_j}^T \Pi_0^{Q_j} \end{bmatrix}, \quad \mathbb{M}_k^{Q_j} = \begin{bmatrix} 0_{n\times 1}, \\ \mathbb{M}_0^{Q_j} \end{bmatrix},$$
$$\mathbb{B}_k^{Q_j} = \begin{bmatrix} B_k \\ 0_{\bullet\times\bullet} \end{bmatrix}, \quad \mathbb{D}_k^{Q_j} = \begin{bmatrix} D_k & 0_{\bullet\times\bullet} \\ 0_{\bullet\times\bullet} & \mathbb{D}_0^{Q_j} \end{bmatrix}, \quad W_i^{Q_j} = \begin{bmatrix} w_i \\ w_0^{Q_j} \end{bmatrix}. \tag{4.61}$$

Notice that in (4.60) the major agent's closed-loop dynamics at discrete state Q_j, $0 \le j \le 3$, given by (4.19) is used to derive the extended dynamics for minor agent $\mathcal{A}_i^k$ at discrete state Q_j, $0 \le j \le 3$. Similar to the major agent's case, $0_{\bullet\times\bullet}$ in (4.61) denotes a zero matrix of appropriate dimensions.

The cost functional for the extended minor agent $\mathcal{A}_i^k$'s hybrid system is given by

$$J_i^k(u_i, u_{-i}) = \mathbb{E}\Big[\|x_i^{ex,Q_*}(t_s^i)\|^2_{\bar{\mathbb{P}}_k^{Q_*}} + \sum_{j=1}^{*} \|x_i^{ex,Q_j}(t_j^-)\|^2_{\mathbb{C}_{i,j}^k} + \sum_{j=0}^{*} \int_{t_j}^{t_{j+1}} \big(\|x_i^{ex,Q_j}(s)\|^2_{\mathbb{P}_k^{Q_j}} + \|u_i^{Q_j}(s)\|^2_{R_k}\big) ds\Big], \quad (4.62)$$

where Q_* denotes the discrete state at which minor agent $\mathcal{A}_i^k$ quits the system at time t_s^i and $* \in \{1, 2\}$ denotes the index of the associate discrete state. The weight matrices associated with the terminal cost (first term) and the running cost (third term) in (4.62) are, respectively, given by

$$\bar{\mathbb{P}}_k^{Q_*} = \bar{P}_k,$$

$$\mathbb{P}_k^{Q_j} = [I_{n\times n}, -H_1^k, -\pi^{Q_j} \otimes H_2^k]^T P_k [I_{n\times n}, -H_1^k, -\pi^{Q_j} \otimes H_2^k], \quad (4.63)$$

$$\bar{\mathbb{P}}_k^{Q_j} = [I_{n\times n}, -H_1^k, -\pi^{Q_j} \otimes H_2^k]^T \bar{P}_k [I_{n\times n}, -H_1^k, -\pi^{Q_j} \otimes H_2^k], \quad (4.64)$$

where $\bar{\mathbb{P}}_k^{Q_j}$ shall be used in Section 4.3.3 to specify the weight matrix $\mathbb{C}_{i,j}^k$ associated with the switching cost (second term) in (4.62) .

Jump Transition Maps and Switching Costs

We first define the notation $M_k^{Q_j}(l : m)$, $k \triangleq a, b$, $0 \le j \le 3$, which shall be used to identify the switching cost associated with switching time t_j, $1 \le j \le 3$. Matrix $M_k^{Q_j}(l : m)$ is made by making all the entires of $\bar{\mathbb{P}}_k^{Q_j}$ zero except those associated with its l-th to m-th columns and rows, hence it has the same size as $\bar{\mathbb{P}}_k^{Q_j}$, i.e.

$$M_k^{Q_j}(l : m) = \begin{array}{c} \overbrace{\quad\quad}^{\bar{\mathbb{P}}_k^{Q_j}(:, l:m)} \\ \left[\begin{array}{c|c|c} 0 & & 0 \\ \hline & & \\ \hline 0 & & 0 \end{array}\right]_{\text{size}(\bar{\mathbb{P}}_k^{Q_j})} \end{array} \left.\vphantom{\begin{array}{c}0\\ \\0\end{array}}\right\} \bar{\mathbb{P}}_k^{Q_j}(l:m, :) \quad (4.65)$$

where $\bar{\mathbb{P}}_k^{Q_j}(:, l : m)$ and $\bar{\mathbb{P}}_k^{Q_j}(l : m, :)$, respectively, denote l-th to m-th columns and l-th to m-th rows of $\bar{\mathbb{P}}_k^{Q_j}$.

The values of minor agent $\mathcal{A}_i^k$ continuous state before and after the switching at switching time t_1 satisfy the following jump transition map

$$x_i^{ex,Q_1}(t_1) = \Psi_{i,1}^k x_i^{ex,Q_0}(t_1-), \quad (4.66)$$

where for $k \triangleq a$

$$\Psi^a_{i,1} = \begin{cases} \Psi^a_{i,q^1_{0ab}q^2_{0ab}} = I_{3n\times 3n}, & \text{for transition from } Q_0 = q^1_{0ab} \text{ to } Q_1 = q^2_{0ab}, \\ \Psi^a_{i,q^1_{0ab}q^1_{0a}} = \begin{bmatrix} I_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & I_{n\times n} & 0_{n\times n} \end{bmatrix}, & \text{for transition from } Q_0 = q^1_{0ab} \text{ to } Q_1 = q^1_{0a}, \\ \Psi^a_{i,q^1_{0ab}q^1_{0b}} = \begin{bmatrix} 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \end{bmatrix}, & \text{for transition from } Q_0 = q^1_{0ab} \text{ to } Q_1 = q^1_{0b}. \end{cases} \tag{4.67}$$

Moreover, the weight matrix $\mathbb{C}^a_{i,1}$ associated with the switching cost in (4.62) at time t_1 is specified as

$$\mathbb{C}^a_{i,1} = \begin{cases} \mathbb{C}^a_{i,q^1_{0ab}q^2_{0ab}} = 0_{3n\times 3n}, & \text{for transition from } Q_0 = q^1_{0ab} \text{ to } Q_1 = q^2_{0ab}, \\ \mathbb{C}^a_{i,q^1_{0ab}q^1_{0a}} = M_a^{q^1_{0ab}}(3n+1:4n), & \text{for transition from } Q_0 = q^1_{0ab} \text{ to } Q_1 = q^1_{0a}, \\ \mathbb{C}^a_{i,q^1_{0ab}q^1_{0b}} = \bar{\mathbb{P}}_a^{q^1_{0ab}}, & \text{for transition from } Q_0 = q^1_{0ab} \text{ to } Q_1 = q^1_{0b}. \end{cases} \tag{4.68}$$

For $k \triangleq b$, the jump transition map (4.66) at t_1 is given by

$$\Psi^b_{i,1} = \begin{cases} \Psi^b_{i,q^1_{0ab}q^2_{0ab}} = I_{3n\times 3n}, & \text{for transition from } Q_0 = q^1_{0ab} \text{ to } Q_1 = q^2_{0ab}, \\ \Psi^b_{i,q^1_{0ab}q^1_{0a}} = \begin{bmatrix} 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \end{bmatrix}, & \text{for transition from } Q_0 = q^1_{0ab} \text{ to } Q_1 = q^1_{0a}, \\ \Psi^b_{i,q^1_{0ab}q^1_{0b}} = \begin{bmatrix} I_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & I_{n\times n} \end{bmatrix}, & \text{for transition from } Q_0 = q^1_{0ab} \text{ to } Q_1 = q^1_{0b}, \end{cases} \tag{4.69}$$

and the corresponding switching cost weight matrix is given by

$$\mathbb{C}^b_{i,1} = \begin{cases} \mathbb{C}^b_{i,q^1_{0ab}q^2_{0ab}} = I_{3n\times 3n}, & \text{for transition from } Q_0 = q^1_{0ab} \text{ to } Q_1 = q^2_{0ab}, \\ \mathbb{C}^b_{i,q^1_{0ab}q^1_{0a}} = \bar{\mathbb{P}}_b^{q^1_{0ab}}, & \text{for transition from } Q_0 = q^1_{0ab} \text{ to } Q_1 = q^1_{0a}, \\ \mathbb{C}^b_{i,q^1_{0ab}q^1_{0b}} = M_b^{q^1_{0ab}q^1_{0b}}(2n+1:3n), & \text{for transition from } Q_0 = q^1_{0ab} \text{ to } Q_1 = q^1_{0b}. \end{cases} \tag{4.70}$$

The values of the minor agent's continuous state before and after the switching at t_2 satisfy the following jump map

$$x_i^{ex,Q_2}(t_2) = \Psi_{i,2}^k x_i^{ex,Q_1}(t_2-), \tag{4.71}$$

where $\Psi_{i,2}^k,\ k \triangleq a$, is given by

$$\Psi_{i,2}^a = \begin{cases} \Psi^a_{i,q^1_0 a q^2_0 a} = I_{2n\times 2n}, & \text{for transition from } Q_1 = q^1_0 a \text{ to } Q_2 = q^2_0 a, \\ \Psi^a_{i,q^1_0 a q^1_0} = \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \end{bmatrix}, & \text{for transition from } Q_1 = q^1_0 a \text{ to } Q_2 = q^1_0, \\ \Psi^a_{i,q^2_0 ab q^2_0 a} = \begin{bmatrix} I_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & I_{n\times n} & 0_{n\times n} \end{bmatrix}, & \text{for transition from } Q_1 = q^2_0 ab \text{ to } Q_2 = q^2_0 a, \\ \Psi^a_{i,q^2_0 ab q^2_0 b} = \begin{bmatrix} 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \end{bmatrix}, & \text{for transition from } Q_1 = q^2_0 ab \text{ to } Q_2 = q^2_0 b. \end{cases} \tag{4.72}$$

Furthermore, the weight matrix $\mathbb{C}^a_{i,2}$ associated with the switching cost at time t_2 is specified by

$$\mathbb{C}_{i,2}^a = \begin{cases} \mathbb{C}^a_{i,q^1_0 a q^2_0 a} = 0_{2n\times 2n}, & \text{for transition from } Q_1 = q^1_0 a \text{ to } Q_2 = q^2_0 a, \\ \mathbb{C}^a_{i,q^1_0 a q^1_0} = \bar{\mathbb{P}}_a^{q^1_0 a}, & \text{for transition from } Q_1 = q^1_0 a \text{ to } Q_2 = q^1_0, \\ \mathbb{C}^a_{i,q^2_0 ab q^2_0 a} = M_a^{q^2_0 ab}(3n+1:4n), & \text{for transition from } Q_1 = q^2_0 ab \text{ to } Q_2 = q^2_0 a, \\ \mathbb{C}^a_{i,q^2_0 ab q^2_0 b} = \bar{\mathbb{P}}_a^{q^2_0 ab}, & \text{for transition from } Q_1 = q^2_0 ab \text{ to } Q_2 = q^2_0 b. \end{cases} \tag{4.73}$$

In (4.71), the jump transition map $\Psi_{i,2}^k,\ k \triangleq b$, is given by

$$\Psi_{i,2}^b = \begin{cases} \Psi^b_{i,q^1_0 b q^2_0 b} = I_{2n\times 2n}, & \text{for transition from } Q_1 = q^1_0 a \text{ to } Q_2 = q^2_0 a, \\ \Psi^b_{i,q^1_0 b q^1_0} = \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \end{bmatrix}, & \text{for transition from } Q_1 = q^1_0 a \text{ to } Q_2 = q^1_0, \\ \Psi^b_{i,q^2_0 ab q^2_0 a} = \begin{bmatrix} 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \end{bmatrix}, & \text{for transition from } Q_1 = q^2_0 ab \text{ to } Q_2 = q^2_0 a, \\ \Psi^b_{i,q^2_0 ab q^2_0 b} = \begin{bmatrix} I_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & I_{n\times n} & 0_{n\times n} \end{bmatrix}, & \text{for transition from } Q_1 = q^2_0 ab \text{ to } Q_2 = q^2_0 b, \end{cases} \tag{4.74}$$

and the corresponding switching cost weight matrix $\mathbb{C}^b_{i,2}$ is given by

$$\mathbb{C}^b_{i,2} = \begin{cases} \mathbb{C}^b_{i,q^1_{0}{}_b q^2_{0}{}_b} = 0_{2n\times 2n}, & \text{for transition from } Q_1 = q^1_{0}{}_a \text{ to } Q_2 = q^2_{0}{}_a, \\ \mathbb{C}^b_{i,q^1_{0}{}_b q^1_0} = \bar{\mathbb{P}}^{q^1_{0}{}_b}_b, & \text{for transition from } Q_1 = q^1_{0}{}_a \text{ to } Q_2 = q^1_0, \\ \mathbb{C}^b_{i,q^2_{0}{}_{ab} q^2_{0}{}_a} = \bar{\mathbb{P}}^{q^2_{0}{}_{ab}}_b, & \text{for transition from } Q_1 = q^2_{0}{}_{ab} \text{ to } Q_2 = q^2_{0}{}_a, \\ \mathbb{C}^b_{i,q^2_{0}{}_{ab} q^2_{0}{}_b} = M^{q^2_{0}{}_{ab}}_b(2n+1:3n), & \text{for transition from } Q_1 = q^2_{0}{}_{ab} \text{ to } Q_2 = q^2_{0}{}_b. \end{cases} \tag{4.75}$$

The values of the minor agent's continuous state before and after the switching at t_3 satisfy the following jump transition map

$$x^{ex,Q_3}_i(t_3) = \Psi^k_{i,3} x^{ex,Q_2}_i(t_3-), \tag{4.76}$$

where for $k \triangleq a$

$$\Psi^a_{i,3} = \Psi^a_{i,q^2_{0}{}_a q^2_0} = \left[\begin{array}{ccc} 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \end{array} \right], \tag{4.77}$$

$$\mathbb{C}^a_{i,3} = \mathbb{C}^a_{i,q^2_{0}{}_a q^2_0} = \bar{\mathbb{P}}^{q^2_{0}{}_a}_a, \tag{4.78}$$

and for $k \triangleq b$

$$\Psi^b_{i,3} = \Psi^b_{i,q^2_{0}{}_b q^2_0} = \left[\begin{array}{ccc} 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \end{array} \right], \tag{4.79}$$

$$\mathbb{C}^b_{i,3} = \mathbb{C}^b_{i,q^2_{0}{}_b q^2_0} = \bar{\mathbb{P}}^{q^2_{0}{}_b}_b. \tag{4.80}$$

Best Response Hybrid Control Actions

The optimal stopping problem for a minor agent is equivalent to a hybrid optimal control problem in which the dynamics and costs become zero after stopping. Let us assume that minor agent $\mathcal{A}^k_i$ stops at time t^k_s after the discrete state Q_*, $* \in \{0,1,2\}$. The definitions for $\mathbb{D}^{Q_j}_k$ directly result in the satisfaction of condition (4.92) (see Appendix A), or equivalently condition A1 in [45, Eq. (3)], i.e.

$$\mathbb{D}^{Q_j}_k = \Psi^k_{i,j}\mathbb{D}^{Q_{j-1}}_k, \quad j \in \{1,\ldots,*\}, \quad k \triangleq a,b. \tag{4.81}$$

Furthermore, it is assumed that conditions (4.97)-(4.99), and (4.111)-(4.113), respectively, hold for the stopping time t^k_s and the switching times $t_j < t^k_s$. Hence, the optimal stopping time for each minor agent is $\mathcal{F}_t$-independent and only depends on its dynamical parameters which implies that all minor agents of the same type stop at the same time. Then the application of the results of *Theorem 4.2* and *Corollary 4.3* yield the infinite population best response strategies for the discrete states $\{Q_0, \dots, Q_*\}$ given by

$$u_i^{Q_j}(t) = -R_k^{-1}[\mathbb{B}_k^{Q_j}]^T \Pi_k^{Q_j}(t)\, x_i^{ex,Q_j}(t), \tag{4.82}$$

with

$$-\dot{\Pi}_k^{Q_j} = \Pi_k^{Q_j}\mathbb{A}_k^{Q_j} + \mathbb{A}_{k,Q_j}^T \Pi_k^{Q_j} - \Pi_k^{Q_j}\mathbb{B}_k^{Q_j} R_k^{-1}[\mathbb{B}_k^{Q_j}]^T \Pi_k^{Q_j} + \mathbb{P}_k^{Q_j}, \tag{4.83}$$

subject to the terminal conditions

$$\Pi_k^{Q_*}(t_s^k) = \bar{\mathbb{P}}_k^{Q_*}, \tag{4.84}$$

$$\left(\mathbb{P}_k^{Q_*} + \bar{\mathbb{P}}_k^{Q_*}\mathbb{A}_k^{Q_*} + \mathbb{A}_{k,Q_*}^T \bar{\mathbb{P}}_k^{Q_*} - \bar{\mathbb{P}}_k^{Q_*}\mathbb{B}_k^{Q_*} R_k^{-1}[\mathbb{B}_k^{Q_*}]^T \bar{\mathbb{P}}_k^{Q_*}\right)_{t=t_s^k} = \frac{\partial \mathbb{C}_{i,*}^k}{\partial t}\Big|_{t=t_s^k}, \tag{4.85}$$

and the boundary conditions

$$\Pi_k^{Q_{j-1}}(t_j) = \Psi_{i,k}^T \Pi_k^{Q_j}(t_j)\Psi_{i,k} + \mathbb{C}_{i,j}^k, \tag{4.86}$$

$$\begin{aligned} &\mathbb{P}_k^{Q_{j-1}} + \Pi_k^{Q_{j-1}}(t_j)\mathbb{A}_k^{Q_{j-1}} + [\mathbb{A}_k^{Q_{j-1}}]^T \Pi_k^{Q_{j-1}}(t_j) - \Pi_k^{Q_{j-1}}(t_j)\mathbb{B}_k^{Q_{j-1}} R_k^{-1}[\mathbb{B}_k^{Q_{j-1}}]^T \Pi_k^{Q_{j-1}}(t_j) \\ &= \Psi_{i,k}^T\left(\mathbb{P}_k^{Q_j} + \Pi_k^{Q_j}(t_j)\mathbb{A}_k^{Q_j} + [\mathbb{A}_k^{Q_j}]^T \Pi_k^{Q_j}(t_j) - \Pi_k^{Q_j}(t_j)\mathbb{B}_k^{Q_j} R_k^{-1}[\mathbb{B}_k^{Q_j}]^T \Pi_k^{Q_j}(t_j)\right)\Psi_{i,k} + \frac{\partial \mathbb{C}_{i,j}^k}{\partial t}\Big|_{t=t_j}, \end{aligned} \tag{4.87}$$

where $\{t_j,\, j \in \{1, \dots, *\}\}$ indicate the times of change in the system due to the major agent's switching of dynamics or cessation of the other subpopulation of minor agents. We observe that for the case where subpopulation $k,\ k \triangleq a, b$, stops at time t_1, there is not boundary condition associated with the Riccati equation (4.83).

4.3.4 Hybrid Mean Field Consistency Equations

Let us define

$$\Pi_k^{Q_j} = \begin{bmatrix} \Pi_{k,11}^{Q_j} & \Pi_{k,12}^{Q_j} & \Pi_{k,13}^{Q_j} \\ \Pi_{k,21}^{Q_j} & \Pi_{k,22}^{Q_j} & \Pi_{k,23}^{Q_j} \\ \Pi_{k,31}^{Q_j} & \Pi_{k,32}^{Q_j} & \Pi_{k,33}^{Q_j} \end{bmatrix}, \quad k \triangleq a, b,$$

$$\mathbf{e}_k^{Q_j} = \begin{cases} I_n & \text{if } \bar{x}^{Q_j} = \bar{x}_k, \\ [I_n, 0_{n\times n}] & \text{if } \bar{x}^{Q_j} \neq \bar{x}_k \wedge k = a, \\ [0_{n\times n}, I_n] & \text{if } \bar{x}^{Q_j} \neq \bar{x}_k \wedge k = b, \end{cases} \tag{4.88}$$

where I_n is an $n \times n$ identity matrix.

Then, by consistency requirement, a compact description of the hybrid major minor mean field equations determining $\bar{A}^{Q_j}$, $\bar{G}^{Q_j}$, $\bar{m}^{Q_j}$ is given by

$$\begin{aligned}
-\dot{\Pi}_0^{Q_j} &= \Pi_0^{Q_j}\mathbb{A}_0^{Q_j} + \left(\mathbb{A}_0^{Q_j}\right)^T\Pi_0^{Q_j} - \Pi_0\mathbb{B}_0^{Q_j}R_0^{-1}\left(\mathbb{B}_0^{Q_j}\right)^T\Pi_0^{Q_j} + \mathbb{P}_0^{Q_j},\\
-\dot{\Pi}_k^{Q_j} &= \Pi_k^{Q_j}\mathbb{A}_k^{Q_j} + \left(\mathbb{A}_k^{Q_j}\right)^T\Pi_k^{Q_j} - \Pi_k^{Q_j}\mathbb{B}_k^{Q_j}R_k^{-1}\left(\mathbb{B}_k^{Q_j}\right)^T\Pi_k^{Q_j} + \mathbb{P}_k^{Q_j},\\
\bar{A}_k^{Q_j} &= [A_k - B_kR_k^{-1}B_k^T\Pi_{k,11}^{Q_j}]\mathbf{e}_k^{Q_j} + F_k \otimes \pi^{Q_j} - B_kR_k^{-1}B_k^T\Pi_{k,13}^{Q_j},\\
\bar{G}_k^{Q_j} &= G_k - B_kR_k^{-1}B_k^T\Pi_{k,12}^{Q_j},\\
\bar{m}_k^{Q_j} &= 0,
\end{aligned} \tag{4.89}$$

for each discrete state Q_j, $0 \leq j \leq 3$, and the corresponding population k, $k \triangleq a, b$. The set of equations (4.89) forms a fixed point problem for each discrete state $Q_j, 1 \leq j \leq 3$, that should be solved by each minor agent in order to compute the matrices in the mean field dynamics.

Assumption 4.5. *There exists a stabilizing solution* $\Pi_0^{Q_j}$, $\Pi_k^{Q_j}$, $\bar{A}_k^{Q_j}$, $\bar{G}_k^{Q_j}$, $1 \leq j \leq 3$, $k \triangleq a, b$, *to the major-minor mean field equations (4.89) in the sense that the matrices*

$$\begin{aligned}
&\mathbb{A}_0^{Q_j} - \mathbb{B}_0^{Q_j}[R_0^{Q_j}]^{-1}[\mathbb{B}_0^{Q_j}]^T\Pi_0^{Q_j},\\
&\mathbb{A}_k^{Q_j} - \mathbb{B}_k^{Q_j}R_k^{-1}[\mathbb{B}_k^{Q_j}]^T\Pi_k^{Q_j},
\end{aligned}$$

are asymptotically stable.

The following theorem links the infinite population equilibria to the finite population case.

Theorem 4.1 (ϵ-Nash Equilibrium for LQG Hybrid-MFG Systems)**.** *Subject to Assumptions 4.1-4.5, the system equations* (4.19)*,* (4.60) *together with the mean field equations* (4.89) *generate a set of control laws which yields the infinite population Nash equilibrium. When the set of infinite population control laws* $\mathcal{U}_{MF}^{N_t} \triangleq \{u_i^{Q_j}; 0 \leq i \leq N_t\}$, $1 \leq N_t \leq N < \infty$, *given by* (4.55)*,* (4.82) *is applied to the finite population system* (4.1)*,* (4.5)*,* (4.8)*, it results in the following properties:*

(i) All agent systems $\mathcal{A}_i$, $0 \leq i \leq N$, *are second order stable.*

(ii) $\mathcal{U}_{MF}^{N_t}, 1 \leq N_t < \infty$ *yields an* ϵ*-Nash equilibrium for all* ϵ*, i.e. for all* $\epsilon > 0$*, there exists* $N(\epsilon)$ *such that for all* $N \geq N(\epsilon)$*;*

$$J_i^{s,N}(u_i^\circ, u_{-i}^\circ) - \epsilon \leq \inf_{u_i \in \mathcal{U}_g^{N,L}} J_i^{s,N}(u_i, u_{-i}^\circ) \leq J_i^{s,N}(u_i^\circ, u_{-i}^\circ).$$

□

Proof. Applying the approach of [16] backwards from T along the optimal realization of the sequence Q_0, Q_1, Q_2, Q_3 establishes the existence and uniqueness of the Nash equilibrium and ϵ-Nash equilibrium for the infinite population system and finite population system, respectively. □

4.3.5 Hybrid Dynamic Programming Methodology

The order of the switching and stopping events Q_0, Q_1, Q_2, Q_3, if all of them occur, is assumed to be fixed. As depicted in Fig. 4.1 and explained in Section 4.2.2, there are three possible realizations for each of the discrete states Q_1 and Q_2. The optimal sequence of switching, that is to say the discrete trajectory of the system, is determined via dynamic programming backward propagation. For this purpose, the steps below are followed.

Step 1. (Solving backwards for transitions from Q_3 to Q_2). Equation (4.56) is solved for $\Pi_0^{Q_3}(t)$ backward in time, subject to the terminal condition (4.57). Then the values for $\Pi_0^{Q_3}(t)$ are substituted in the right hand side of (4.58) to obtain $\Pi_0^{Q_2}(t)$ for all three realizations of $\Psi_{0,3}$ and $\mathbb{C}_{0,3}$ given by (4.52) and (4.53), respectively. Next, we substitute $\Pi_0^{Q_2}(t)$ and the corresponding $\Psi_{0,3}$ and $\mathbb{C}_{0,3}$ in (4.59). Then the time instant at which (4.59) holds determines t_3 for the transition from the corresponding realization of Q_2 to Q_3. The transitions from $Q_2 \triangleq q^2_{0b}$ to Q_3 or from $Q_2 \triangleq q^2_{0a}$ to Q_3 are equivalent to the stopping of subpopulation $\mathcal{A}_b$ or $\mathcal{A}_a$, respectively, at the obtained switching time t_3. Hence equation (4.85) must also hold at the associated t_3 for each of the mentioned cases. Similarly, for the transition from $Q_2 \triangleq q^1_0$ to Q_3 both (4.59) and (4.87) must hold at the same time.

We observe that if (4.59) does not hold for any of the realizations of $Q_2 = \{q^2_{0a}, q^1_0, q^2_{0b}\}$, then we conclude that Q_3 is not the final discrete state of the system. Subsequently, we start from *Step 2* solving the dynamic programming backward in time from $t = T$.

Step 2. (Solving backwards for transitions from Q_2 to Q_1). Starting from the obtained realizations of Q_2 in *Step 1* and the corresponding switching times t_3, we follow a similar approach as in *Step1* to determine the realizations of Q_1 which may take place and their corresponding switching times t_2. More specifically, equation (4.56) is solved with the boundary (terminal) condition (4.58) with $j = 3$ at t_3. Then, for example, to determine from $Q_2 \triangleq q^2_{0b}$ which of (either of or neither of) the transitions to $Q_1 \triangleq q^2_{0ab}$ and $Q_1 \triangleq q^1_{0b}$ may occur, equations (4.59), (4.85) and (4.59), (4.87) are checked, respectively.

Step 3. (Solving backwards for transitions from Q_1 to Q_0). Similar to previous steps, starting from the determined cases for Q_1 and the determined t_1 in *Step 2*, it is investigated whether the transition to Q_0 may occur or not using equations (4.59), (4.85) and (4.87).

Step 4. (Specifying the optimal discrete sequence). If *Steps 1-3* yield more than one discrete trajectory for the system, the optimal one is determined by comparing the value functions along the obtained discrete state sequences with the value function for the case where no switching or stopping event happens. Finally it should be noted that if *Steps 1-3* result in no realized discrete trajectory, then the system may remain in the discrete state Q_0 over the interval $[0, T]$.

4.4 Simulations

Consider a system of 100 minor agents with two types $\mathcal{A}^a$ and $\mathcal{A}^b$ and a single major agent $\mathcal{A}_0$. The system matrices for minor subpopulation $\mathcal{A}^a$ with $N_a = 50$ are defined as

$$A_a \triangleq \begin{bmatrix} 2e^{-t} & e^{-0.5t} \\ e^{-0.5t} & 2e^{-t} \end{bmatrix}, \quad B_a \triangleq \begin{bmatrix} 1 \\ 0.1 \end{bmatrix},$$

and for minor subpopulation $\mathcal{A}^b$ with $N_b = 50$ are given by

$$A_b \triangleq \begin{bmatrix} 5e^{-1.5t}cos(t) & 5e^{-2t} \\ 5e^{-2t}sin(t) & 5e^{-1.5t} \end{bmatrix}, \quad B_b \triangleq \begin{bmatrix} 0 \\ 0.1 \end{bmatrix},$$

and for the major agent is given by

$$A_0 \triangleq \begin{bmatrix} 2e^{-t} & e^{-t} \\ e^{-0.5t} & 2e^{-0.5t} \end{bmatrix}, \quad B_0 \triangleq \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}.$$

The parameters used in the simulation are: $t_{final} = 18\,\text{sec}$, $\Delta t = 0.01\,\text{sec}$, $\sigma_0 = 0.015$, $\sigma_a = \sigma_b = 0.05$, $H_0 = 0.6 \times I_{2\times2}$, $H_1^a = H_1^b = 0.2 \times I_{2\times2}$, $H_2^a = H_2^b = 0.02 \times I_{2\times2}$, $G_a = G_b = 0_{2\times2}$. The control actions and state trajectories for a single realization in discrete states Q_0, Q_1, Q_2 can be displayed for the entire population of 101 agents together, but in Figure 4.2 and Figure 4.3 only 10 minor agents are shown for the sake of clarity.

4.5 Conclusions

A class of hybrid LQG mean field game problems was introduced where there exists one major agent together with a large number of minor agents within two subpopulations, each agent with stochastic linear dynamics and quadratic cost. The agents are coupled in their dynamics and cost

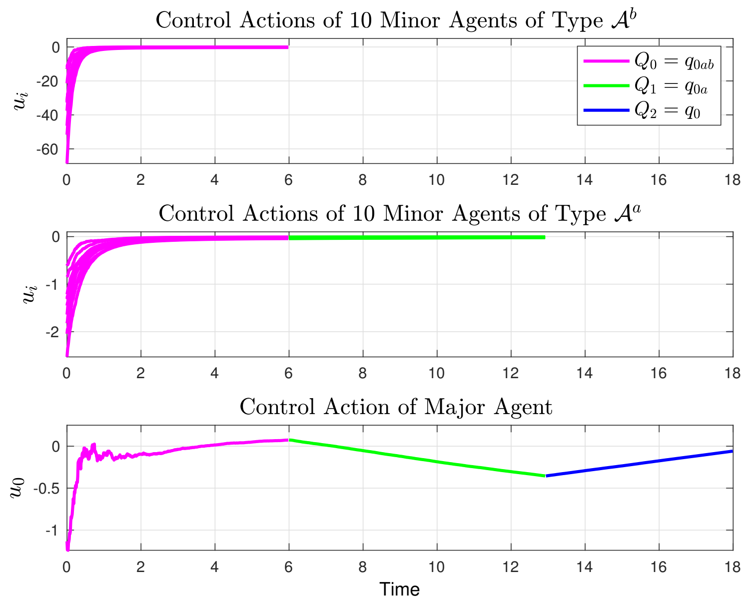

Figure 4.2: The control actions for a single realization of the major agent, 10 sample minor agents of type $\mathcal{A}^a$, and 10 sample minor agents of type $\mathcal{A}^b$ in discrete states Q_0, Q_1, Q_2.

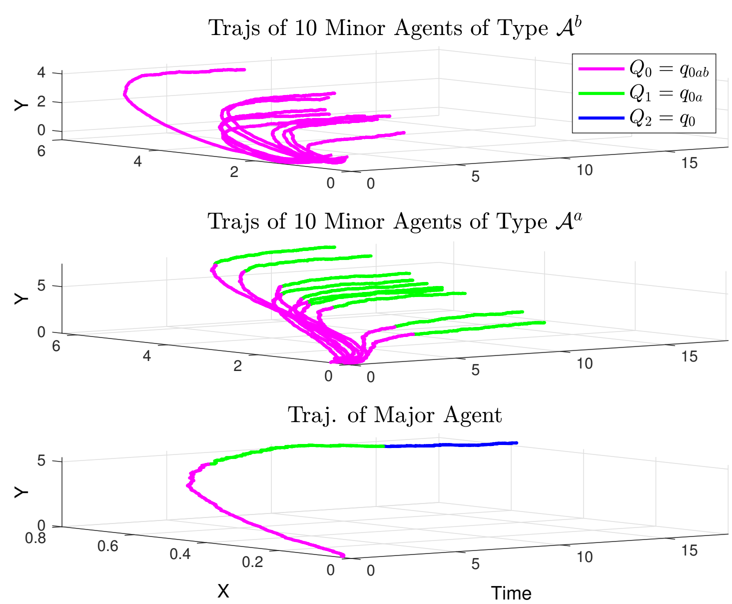

Figure 4.3: The state trajectories for a single realization of the major agent, 10 sample minor agents of type $\mathcal{A}^a$, and 10 sample minor agents of type $\mathcal{A}^b$ in discrete states Q_0, Q_1, Q_2.

functionals by the average state of minor agents (i.e. the empirical mean field). In addition, the major agent is provided with the option to switch to another dynamics, and each minor agent is provided with the option to quit if it is optimal for them to do so. It was shown that for this class of problems the stopping and switching times are realization independent, and only depend on the dynamical parameters of each agent. Hence, all the minor agents within the same subpopulation stop at the same time. Therefore, the hybrid feature of the system was formulated via the indexing by discrete states: (i) the switching of the major agent or (ii) the cessation of one or both subpopulations of minor agents. Finally, by developing and then utilizing hybrid LQG mean field game theory, optimal switching and stopping time strategies for, respectively, the major agent and all minor agents, together with their best response control actions which yield a unique ϵ-Nash equilibrium were established.

4.6 Appendix

4.6.1 $\mathcal{F}_t$-Independent State-Invariant Optimal Switchings and Stopping Strategies

The following exposition is an elaboration of the results of [44] that presents a set of conditions under which the optimal switching and stopping times for LQG systems are $\mathcal{F}_t$-independent and state-invariant and therefore, to be almost surely equal for all agents within a subpopulation.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a probability space such that $\mathcal{F}_0$ contains the P-null sets, $\mathcal{F}_{t_f} = \mathcal{F}$ for a fixed final time t_f, and let $\mathcal{F}_t = \sigma\left\{w\left(s\right) : 0 \leq s \leq t\right\}$ be the natural filtration associated with the sigma-algebra generated by the Wiener process.

Consider a stochastic hybrid system governed by the family of linear Itô differential equations of the form

$$dx^{Q_j}(t) = \left(A^{Q_j}\left(t\right)x^{Q_j}\left(t\right) + B^{Q_j}\left(t\right)u^{Q_j}\left(t\right)\right)dt + D^{Q_j}\left(t\right)dw(t), \quad t \in \left[t_j^{\omega}, t_{j+1}^{\omega}\right), \tag{4.90}$$

where $Q_j \in \mathbb{Q}$, with $\mathbb{Q}$ denoting the sequence of the discrete states of the system and having finite cardinality , $x^{Q_j}\left(t\right) \in \mathbb{R}^{n_{Q_j}}$, $u^{Q_j}\left(t\right) \in \mathbb{R}^{m_{Q_j}}$, $A^{Q_j}\left(t\right) \in \mathbb{R}^{n_{Q_j}\times n_{Q_j}}$, $B^{Q_j}\left(t\right) \in \mathbb{R}^{n_{Q_j}\times m_{Q_j}}$, $D^{Q_j}\left(t\right) \in \mathbb{R}^{n_{Q_j}}$, $0 \leq j \leq L$, $t_{L+1} := t_f$.

Switching from a discrete state $Q_{j-1} = q \in \mathbb{Q}$ to another discrete state $Q_j = q' \in \mathbb{Q}$ is considered to be a controlled switchings, that is the direct result of a discrete input $\sigma_j \in \Sigma$ at an arbitrary $\mathcal{F}_t$-adapted switching time t_j^{ω}. Upon switching, the continuous component of the state is reinitialized according to a jump map provided as

$$x^{Q_j}\left(t_j\right) = \Psi_{\sigma_j} x^{Q_{j-1}}\left(t_j-\right) \equiv \Psi_{Q_{j-1}Q_j}\, x^{Q_{j-1}}\left(t_j-\right). \tag{4.91}$$

It is further assumed that

$$D^{Q_j}(t_j) = \Psi_{Q_{j-1}Q_j} D^{Q_{j-1}}(t_j), \tag{4.92}$$

for all $1 \leq j \leq L$, which implies equivalent diffusion fields before and after switching events.

Over a fixed time horizon $[t_0, t_f]$ and for a given initial condition $\left(Q(t_0), x^{Q_0}(t_0)\right) = \left(Q_0, x_0^{Q_0}\right)$, consider the hybrid optimal control problem associated with the cost

$$\begin{aligned} J(u^{Q_0}, ..., u^{Q_L}) = \frac{1}{2}\,\mathbb{E}\Bigg\{ & \left\|x^{Q_L}\left(t_f\right)\right\|^2_{\bar{P}^{Q_L}(t_f)} + \sum_{j=1}^{L} \left\|x^{Q_{j-1}}\left(t_j^{\omega}-\right)\right\|^2_{C_{\sigma_j}(t_j^{\omega})} \\ & + \sum_{i=0}^{L} \int_{t_i^{\omega}}^{t_{i+1}^{\omega}} \left(\left\|x^{Q_i}\left(t\right)\right\|^2_{P^{Q_i}(t)} + \left\|u^{Q_i}\left(t\right)\right\|^2_{R^{Q_i}(t)} \right) dt \Bigg\}, \end{aligned} \tag{4.93}$$

where $0 \leq \left[\bar{P}^{Q_L}\left(t\right)\right]^T = \bar{P}^{Q_L}\left(t\right) \in \mathbb{R}^{n_{Q_L} \times n_{Q_L}}$, $0 \leq \left[C_{\sigma_j}\left(t\right)\right]^T = C_{\sigma_j}\left(t\right) \in \mathbb{R}^{n_{Q_{j-1}} \times n_{Q_{j-1}}}$, $0 \leq \left[P^{Q_i}\left(t\right)\right]^T = P^{Q_i}\left(t\right) \in \mathbb{R}^{n_{Q_i} \times n_{Q_i}}$, $0 < \left[R^{Q_i}\left(t\right)\right]^T = R^{Q_i}\left(t\right) \in \mathbb{R}^{m_{Q_i} \times m_{Q_i}}$.

Theorem 4.2 (Switching Policies for LQG Hybrid Systems)**.** *For the system governed by* (4.90)-(4.93)*, assume that a family of matrices* $\left\{\Pi^{Q_j}\left(t\right); j = 0, 1, \cdots, L\right\}$ *exists such that*

$$\Pi^{Q_L}\left(t_f\right) = \bar{P}^{Q_L}, \tag{4.94}$$

and $\Pi^{Q_j} \equiv \Pi^{Q_j}\left(t\right)$ *satisfy the following family of Riccati equations (for simplicity of notation, the explicit time dependence* (t) *is dropped whenever it is clear from the context)*

$$\dot{\Pi}^{Q_j} = \Pi^{Q_j} B^{Q_j} \left[R^{Q_j}\right]^{-1} \left[B^{Q_j}\right]^T \Pi^{Q_j} - \Pi^{Q_j} A^{Q_j} - [A^{Q_j}]^T \Pi^{Q_j} - P^{Q_j}, \tag{4.95}$$

where

$$\Pi^{Q_{j-1}}\left(t_j\right) = \Psi_{\sigma_j}^T \Pi^{Q_j}\left(t_j\right) \Psi_{\sigma_j} + C_{\sigma_j}(t_j), \tag{4.96}$$

and for every $j = L, L-1, \cdots, 1$ *(i.e. determined from a backward sequence), there exist* $t_j \in [0, t_{j+1})$ *satisfying the following algebraic matrix relations (equality, strict positive definiteness,* *and strict negative definiteness):*

$$\overline{H}_{\sigma_j}(s) = 0, \qquad s = t_j, \tag{4.97}$$
$$\overline{H}_{\sigma_j}(s) > 0, \qquad s > t_j, \tag{4.98}$$
$$\overline{H}_{\sigma_j}(s) < 0, \qquad s < t_j, \tag{4.99}$$

where

$$\begin{aligned}\overline{H}_{\sigma_j}(s) := {}& \Psi_{\sigma_j}^T \Pi^{Q_j}(s)\Big[B^{Q_j}[R^{Q_j}]^{-1}[B^{Q_j}]^T - \Psi_{\sigma_j} B^{Q_{j-1}}[R^{Q_{j-1}}]^{-1}[B^{Q_{j-1}}]^T \Psi_{\sigma_j}^T\Big]\Pi^{Q_j}(s)\Psi_{\sigma_j} \\ & + \Psi_{\sigma_j}^T \Pi^{Q_j}(s)\Big[\Psi_{\sigma_j} A^{Q_{j-1}} - A^{Q_j}\Psi_{\sigma_j} - \Psi_{\sigma_j} B^{Q_{j-1}}[R^{Q_{j-1}}]^{-1}[B^{Q_{j-1}}]^T C_{\sigma_j}\Big] \\ & + \Big[[A^{Q_{j-1}}]^T \Psi_{\sigma_j} - \Psi_{\sigma_j}^T [A^{Q_j}]^T - C_{\sigma_j} B^{Q_{j-1}}[R^{Q_{j-1}}]^{-1}[B^{Q_{j-1}}]^T \Psi_{\sigma_j}^T\Big]\Pi^{Q_j}(s)\Psi_{\sigma_j} \\ & + P^{Q_{j-1}} - C_{\sigma_j} B^{Q_{j-1}}[R^{Q_{j-1}}]^{-1}[B^{Q_{j-1}}]^T C_{\sigma_j} + C_{\sigma_j} A^{Q_{j-1}} + [A^{Q_{j-1}}]^T C_{\sigma_j} \\ & - \Psi_{\sigma_j}^T P^{Q_j}\Psi_{\sigma_j} - \left.\frac{\partial C_{\sigma_j}(t)}{\partial t}\right|_{t=s}. \end{aligned} \tag{4.100}$$

Then switching times are $\mathcal{F}_t$-independent (almost surely deterministic) independent of the initial condition, and optimal control actions are determined by

$$u^{Q_j,\circ}(t,x) = -\left[R^{Q_j}(t)\right]^{-1}\left[B^{Q_j}(t)\right]^T \Pi^{Q_j}(t)\, x^{Q_j,\circ}(t). \tag{4.101}$$

□

Proof. We invoke the Stochastic Hybrid Minimum Principle [45] and form the family of system Hamiltonians as

$$\begin{aligned} H^{Q_j}\left(x^{Q_j}, u^{Q_j}, \lambda^{Q_j}, K^{Q_j}\right) = {}& \frac{1}{2}\left(\left\|x^{Q_j}(t)\right\|^2_{P^{Q_j}(t)} + \left\|u^{Q_j}(t)\right\|^2_{R^{Q_j}(t)}\right) \\ & + \left[\lambda^{Q_j}\right]^T\left(A^{Q_j}x^{Q_j} + B^{Q_j}u^{Q_j}\right) + \left[K^{Q_j}\right]^T D^{Q_j}, \end{aligned} \tag{4.102}$$

It immediately follows that

$$\underset{u^Q \in \mathbb{R}^m}{\operatorname{argmin}}\, H^{Q_j}\left(x^{Q_j}, u^{Q_j}, \lambda^{Q_j}, K^{Q_j}\right) = -\left[R^{Q_j}\right]^{-1}\left[B^{Q_j}\right]^T \lambda^{Q_j}, \tag{4.103}$$

and therefore, it remains to be shown that along a trajectory $x^{Q_j}(t)$ associated with the input (4.101) and switchings at t_j's satisfying (4.97)–(4.99), the processes defined as $\lambda^{Q_j}(t) := \Pi^{Q_j}(t)\, x^{Q_j}(t)$ are adjoint processes of the associated optimal control problem.

Beginning with the final discrete state Q_L, similar arguments as those in the classical LQG theory (see e.g. [36]) show that

$$\lambda^{Q_L}(t_f) = \Pi^{Q_L}(t)\, x^{Q_L}(t_f) = \frac{1}{2}\frac{\partial}{\partial x}\left\|x(t_f)\right\|^2_{\bar{P}^{Q_L}(t_f)}, \tag{4.104}$$

$$\begin{aligned} d\lambda^{Q_L} &= -\frac{\partial H^{Q_L}}{\partial x}\left(x^{Q_L}, u^{Q_L}, \lambda^{Q_L}, K^{Q_L}\right) dt + K^{Q_L} dw \\ &= -\left(P^{Q} x^{Q_L} + \left[A^{Q_L}\right]^T \lambda^{Q_L}\right) dt + K^{Q_L} dw, \end{aligned} \tag{4.105}$$

with $K^{Q_L}(t) = \Pi^{Q_L}(t)\, D^{Q_L}$.

As the (backward) induction hypothesis, assume that $\lambda^{Q_{j+1}}(t) = \Pi^{Q_{j+1}}(t)\, x^{Q_{j+1}}(t)$ holds. We need to show that $\lambda^{Q_j}(t) = \Pi^{Q_j}(t)\, x^{Q_j}(t)$ follows. To this end, we note that from [45] (see also [44]) adjoint processes and Hamiltonians must satisfy

$$\lambda^{Q_j}(t_{j+1}) = \left[\Psi_{\sigma_{Q_j,Q_{j+1}}}\right]^T \lambda^{Q_{j+1}}(t_{j+1}+) + C_{\sigma_{Q_j,Q_{j+1}}} x^{Q_j}(t_{j+1}), \tag{4.106}$$

$$\begin{aligned} &H^{Q_j}_{\left(x^{Q_j}, u^{o,Q_j}, \lambda^{Q_j}, K^{Q_j}\right)} - \left[K^{Q_j}\right]^T D^{Q_j} + \frac{\partial}{\partial t}\left\|x^{Q_j}\right\|^2_{C^{(t)}_{\sigma_{Q_j,Q_{j+1}}}}\Bigg|_{t^{\omega}_{j+1}-} \\ &\qquad = H^{Q_{j+1}}_{\left(x^{Q_{j+1}}, u^{o,Q_{j+1}}, \lambda^{Q_{j+1}}, K^{Q_{j+1}}\right)} - \left[K^{Q_{j+1}}\right]^T D^{Q_{j+1}}\Bigg|_{t^{\omega}_{j+1}}. \end{aligned} \tag{4.107}$$

One can easily verify by substitution that (4.96) and (4.97) lead to the satisfaction of (4.106) and (4.107) with $\mathcal{F}_t$-independence. Moreover, (4.98) and (4.99) ensure that such a switching instant is uniqune for all values of state and therefore the associated Riccati equations and switching conditions golbaly represent a unique optimal strategy.

□

As an important result of Theorem 4.2, one can obtain $\mathcal{F}_t$-independence and state-invariance of optimal stopping times for controlled LQG systems. Consider a system governed by

$$dx(t) = (A(t)\,x(t) + B(t)\,u(t))\,dt + D(t)\,dw(t), \qquad t \in [0, t^{\omega}_s), \tag{4.108}$$

where t^{ω}_s is an $\mathcal{F}_t$-adapted stopping time, to be determined together with a continuous input in order to infimize (minimize) the cost

$$J(u) = \frac{1}{2}\mathbb{E}\left\{\left\|x(t^{\omega}_s)\right\|^2_{C(t^{\omega}_s)} + \int_{t_0}^{t^{\omega}_s} \left\|x(t)\right\|^2_{P(t)} + \left\|u(t)\right\|^2_{R(t)} dt\right\}, \tag{4.109}$$

Define

$$\overline{H}\,(s) := P\,(s) + C\,(s)\,B\,(s)\,R^{-1}\,(s)\,B^{T}\,(s)\,C\,(s) + C\,(s)\,A\,(s) + A^{T}\,(s)\,C\,(s) - \left.\frac{\partial C(t)}{\partial t}\right|_{t=s}. \tag{4.110}$$

Corollary 4.3 (Stopping Policies for LQG Systems)**.** *Consider the (deterministic) algebraic matrix expression* (4.110)*. If there exists a finite time $t_s \in [0, \infty)$ for which*

$$\overline{H}\,(s) = 0, \qquad s = t_s, \tag{4.111}$$

$$\overline{H}\,(s) > 0, \qquad s > t_s, \tag{4.112}$$

$$\overline{H}\,(s) < 0, \qquad s < t_s, \tag{4.113}$$

then $t_s^{\omega} = t_s$ for all $\omega \in \Omega$, that is the optimal stopping time for the system (4.108) *with the cost* (4.109) *is $\mathcal{F}_t$-independent state-invariant and is equal to t_s almost surely, and the optimal input is determined by*

$$u\,(t, x) = -R^{-1}\,(t)\,B^{T}\,(t)\,\Pi\,(t)\,x\,(t)\,, \tag{4.114}$$

where $\Pi\,(t)$ is the solution to

$$\dot{\Pi}(t) = \Pi(t)B(t)R^{-1}(t)B^{T}(t)\Pi(t) - \Pi(t)A(t) - A^{T}(t)\Pi(t) - P(t), \tag{4.115}$$

subject to the terminal (stopping) condition

$$\Pi\,(t_s) = C\,(t_s)\,. \tag{4.116}$$

□

Chapter 5

A Mean Field Game - Hybrid Systems Approach to Optimal Execution Problems in Finance with Stopping Times

5.1 Introduction

In this chapter, the considered financial market consists of an institutional investor, interpreted as the major agent, who aims to liquidate a specific amount of shares, and a large population high frequency traders (HFTs), interpreted as minor agents, who wish to liquidate or acquire a certain amount of shares within a specific time horizon. The traders are coupled in their dynamics and cost functions by the market's average trading rate (a component of the system mean field) and the hybrid feature enters via the indexing of the cessation of trading by one or both subpopulations of minor traders by discrete states. This work combines two contemporary systems and control techniques: MFG theory and hybrid optimal control (HOC) theory to establish optimal stopping time strategies together with best response trading policies for all agents with respect to their individual cost criteria which yield a unique ϵ-Nash equilibria for the market.

We note major trader (respectively, minor trader), and institutional trader (respectively, HFT) are used interchangeably in this chapter.

The rest of the chapter is organized as follows. Section 5.2 presents the trading dynamics and performance functions in the market. Optimal execution problems in the market are then formulated in the Hybrid MFG framework in Section 5.3. Finally, concluding remarks are made in Section 5.4.

5.2 Trading Dynamics of Agents in Market

As stated in the Introduction, the institutional investor is considered as a major agent in the mean field model of the market which liquidates its shares and the HFTs are considered as minor agents, where two types of them are considered: acquirers $\mathcal{A}_a$ with the population of N_a and liquidators $\mathcal{A}_l$ with the population of N_l, such that $N_a + N_l = N$. All agents trade over the interval $[0, T]$, and minor agents are allowed to stop trading at an optimal time $t_s^i \leq T$. It will be shown in Section 5.3 that the optimal stopping time policy for each agent is $\mathcal{F}_t$-independent, and depends only on its dynamical parameters. In this chapter, for simplicity of exposition the dynamical parameters for all minor traders in their respective type are the same, and hence the stopping times are the same for all agents of each population. Employing the trading model in [25], the trading dynamics of the major agent and any generic minor agent in the market are described by the linear time evolution of the (i) inventories, (ii) trading rates and (iii) prices while the bilinear cash process appears in the quadratic performance function for each agent.

5.2.1 Inventory Dynamics

It is assumed that the institutional investor liquidates its inventory of shares, $q_0(t)$, by trading at a rate $\nu_0(t)$ during the trading period $[0, T]$. Hence the major agent's inventory dynamics is given by

$$dq_0(t) = \nu_0(t)dt + \sigma_0^q dw_0^q, \quad 0 \leq t \leq T,$$

where w_0^q is a Wiener process modeling the noise in the inventory information that the institutional trader collects from its branches in different locations; σ_0^q is a positive scalar and we assume that $q_0(0) \gg 1$. The same dynamical model is adopted for the trading dynamics of a generic HFT

$$dq_i(t) = \nu_i(t)dt + \sigma_i^q dw_i^q,$$

where for a minor acquirer trader $\mathcal{A}_i \in \mathcal{A}_a$, $0 \leq t \leq t_s^a$, and correspondingly for a minor liquidator $\mathcal{A}_i \in \mathcal{A}_l$, $0 \leq t \leq t_s^l$. The Wiener process w_i^q models the HFT's information noise, σ_i^q is a positive scalar, $\nu_i(t)$ is the agent's rate of trading which can be positive or negative depending on whether the agent is acquirer or liquidator, respectively; $q_i(t)$ is the minor liquidator's remaining shares at time t, or the shares the minor acquirer has bought until time t. However, the initial

share stock of the HFTs, $\{q_i(0), 1 \leq i \leq N_a + N_l\}$, are not considered to be large, furthermore they are not motivated to retain shares and are assumed to trade them quickly.
We assume that the trading rate of the major agent is controlled via $u_0(t)$ as

$$d\nu_0(t) = u_0(t)dt, \quad 0 \leq t \leq T,$$

where the trading strategy $u_0(t)$ can be seen to be the trading acceleration of the major trader. Correspondingly, $u_i(t)$ controls the trading rate of minor agent, $\mathcal{A}_i$, by

$$d\nu_i(t) = u_i(t)dt,$$

where again for a minor acquirer trader $\mathcal{A}_i \in \mathcal{A}_a$, $0 \leq t \leq t_s^a$, and correspondingly for a minor liquidator $\mathcal{A}_i \in \mathcal{A}_l$, $0 \leq t \leq t_s^l$, and $u_i(t)$ is the trading acceleration of the minor acquirer or liquidator.

5.2.2 Price Dynamics

The trading rate of the major agent and the average trading rate of the minor agents give rise to the fundamental asset price which models the permanent effect of agents' trading rates on the market price. Further, each agent has a temporary effect on the asset price which only persists during the action of the trade and which determines the execution price, that is to say the price at which each agent can trade.

Fundamental Asset Price

We model the dynamics of the fundamental asset price, as seen from the major agent's viewpoint, by

$$dF_0(t) = \big(\lambda_0\nu_0(t) + \lambda\nu^{N_t}(t)\big)dt + \sigma dw_0^F(t), \quad 0 \leq t \leq T,$$

where N_t is the number of minor agents trading at time t, $\nu^{N_t}(t) = \frac{1}{N_t}\sum_{i=1}^{N_t} \nu_i(t)$ is the average trading rate of the minor agents trading at time t. The Wiener process $w_0^F(t)$ models the aggregate effect of all traders in the market which - unlike the major and minor agents $\mathcal{A}_0$, $\mathcal{A}_i$, - have no complete or partial observations on any of the state variables appearing in the dynamical market model (these are termed uninformed traders). Further, σ denotes the intensity of the market volatility and $\lambda_0, \lambda \geq 0$ denote the strength of the linear permanent impact of the major and minor agents' trading on the fundamental asset price, respectively. Similarly, we model the fundamental asset price dynamics, as seen by a minor agent $\mathcal{A}_i$, by

$$dF_i(t) = \big(\lambda_0 \nu_0(t) + \lambda \nu^{N_t}(t)\big)dt + \sigma dw_i^F(t),$$

where $0 \le t \le t_s^a$, for $\mathcal{A}_i \in \mathcal{A}_a$, and $0 \le t \le t_s^l$, for $\mathcal{A}_i \in \mathcal{A}_l$, $\nu^{N_t}(t) = \frac{1}{N_t}\sum_{i=1}^{N_t} \nu_i(t)$ is again the average trading rate of the N_t minor agents trading at t, and the Wiener process, $w_i^F(t)$, represents the mass effect of all uninformed traders in the market.

Execution Price

The major agent's execution price $S_0(t)$ evolution is assumed to be given by

$$dS_0(t) = dF_0(t) + a_0 d\nu_0(t), \quad 0 \le t \le T, \tag{5.1}$$

where $a_0 \ge 0$ is the temporary impact strength of the major agent on fundamental asset price. Likewise, a minor agent's execution price, $S_i(t)$, is assumed to evolve by

$$dS_i(t) = dF_i(t) + a d\nu_i(t), \tag{5.2}$$

where $0 \le t \le t_s^a$, for $\mathcal{A}_i \in \mathcal{A}_a$, and $0 \le t \le t_s^l$, for $\mathcal{A}_i \in \mathcal{A}_l$, and a models the temporary impact of a minor agent's trading on its execution price.

5.2.3 Cash Process

The cash processes for the major agent and a generic minor agent, $Z_0(t)$, $Z_i(t)$, respectively, are given by

$$dZ_0(t) = -S_0(t)dq_0(t), \quad 0 \le t \le T, \tag{5.3}$$

$$\begin{cases} dZ_i(t) = -S_i(t)dq_i(t), & \text{for } \mathcal{A}_i \in \mathcal{A}_a,\ 0 \le t \le t_s^a \\ dZ_i(t) = -S_i(t)dq_i(t), & \text{for } \mathcal{A}_i \in \mathcal{A}_l,\ 0 \le t \le t_s^l, \end{cases} \tag{5.4}$$

where $Z_0(t)$, and $Z_i(t)$ for $\mathcal{A}_i \in \mathcal{A}_l$ are the cash obtained through liquidation of shares, and $Z_i(t)$, for $\mathcal{A}_i \in \mathcal{A}_a$ is the cash paid for acquisition of shares up to time t. We note that the value of $dq_0(t)$ in a stock sale (respectively, buy) is negative (respectively, positive) and hence for positive $S_0(t)$, $Z_0(t)$ increases (respectively, decreases).

5.2.4 Performance Function

Major Liquidator

The objective for the major trader is to liquidate $\mathcal{N}_0$ shares and maximize the cash it holds at the end of the trading horizon, i.e. maximize $Z_0(T)$, and if the remaining inventory at the final time T is $q_0(T)$, it can liquidate it at a lower price than the market asset price reflected at cost function by $q_0(T)(F_0(T) - \alpha q_0(T))$. Further, the major trader's utility in minimizing the inventory over the period $[0, T]$ is modeled by including the penalty $\phi \int_0^T q_0^2(s)ds$ in its objective function, and the utility of avoiding very high execution prices, large trading intensities and large trading accelerations by including the terms $\epsilon S_0^2(T)$, $\int_0^T \delta S_0^2(s)ds$, $\beta \nu_0^2(T)$, $\int_0^T \theta \nu_0^2(s)ds$ and $\int_0^T R_0 u_0^2(s)ds$ in the objective function. Therefore, its cost function to be minimized is given by

$$J_0(u_0, u_{-0}) = \mathbb{E}\Big[- rZ_0(T) - pq_0(T)\big(F_0(T) - \alpha q_0(T)\big) + \epsilon S_0^2(T) + \beta \nu_0^2(T) + \int_0^T \big(\phi q_0^2(s) + \delta S_0^2(s) + \theta \nu_0^2(s) + R_0 u_0^2(s)\big)ds\Big], \quad (5.5)$$

where r, p, α, ϵ, β, ϕ, δ, θ, and R_0 are positive scalars, and $u_{-0} := (u_1, u_2, ..., u_N)$ are trading strategies of the minor traders. Note that for larger values of ϕ the trader attempts to liquidate its inventory more quickly.

Minor Liquidator

In a similar way, the objective function to be minimized for a liquidator HFT who wants to liquidate $\mathcal{N}_l$ shares over the interval $[0, T]$ with the stopping time $0 \leq t_s^l \leq T$ is given by

$$J_i(u_i, u_{-i}) = \mathbb{E}\Big[- r_l Z_i(t_s^l) - p_l q_i(t_s^l)\Big(F_i(t_s^l) - \psi_l q_i(t_s^l)\Big) + \xi_l S_i^2(t_s^l) + \mu_l \nu_i^2(t_s^l) + \int_0^{t_s^l} \Big(\kappa_l q_i^2(s) + \gamma_l S_i^2(s) + \varrho_l \nu_i^2(s) + R_l u_i^2(s)\Big)ds\Big], \text{ for } \mathcal{A}_i \in \mathcal{A}_l \quad (5.6)$$

where r_l, p_l, ψ_l, ξ_l, μ_l, κ_l, γ_l, ϱ_l and R_l are positive scalars, and $u_{-i} := (u_0, u_1, ..., u_{i-1}, u_{i+1}, ..., u_N)$. Note that $\mathcal{N}_l \ll \mathcal{N}_0$.

Minor Acquirer

The objective for a minor acquirer is to buy $\mathcal{N}$ shares during the trading horizon $[0, T]$. Given that it stops trading at $t_s^a \leq T$, it also wishes to minimize the execution cost including the cash $Z_i(t_s^a)$ paid up to time t_s^a, and the cash must be paid at time t_s^a to buy the remaining shares at once at a higher price than the market's asset price, i.e. $\big(\mathcal{N} - q_i(t_s^a)\big)\big(F_i(t_s^a) + \psi_a(\mathcal{N} - q_i(t_s^a)\big)$. It is also intended to avoid high execution prices, large trading intensities and large trading accelerations modeled by including $\xi_a S_i^2(t_s^a) + \mu_a \nu_i^2(t_s^a) + \int_0^{t_s^a} \big(\gamma_a S_i^2(s) + \varrho_a \nu_i^2(s) + R_A u_i^2(s)\big)ds$ in its objective function

$$J_i(u_i, u_{-i}) = \mathbb{E}\Big[p_a(\mathcal{N} - q_i(t_s^a))\big(F_i(t_s^a) + \psi_a(\mathcal{N} - q_i(t_s^a))\big) + r_a Z_i(t_s^a) + \xi_a S_i^2(t_s^a) + \mu_a \nu_i^2(t_s^a) + \int_0^{t_s^a} \big(\kappa_a(\mathcal{N} - q_i(s))^2 + \gamma_a S_i^2(s) + \varrho_a \nu_i^2(s) + R_a u_i^2(s)\big) ds\Big], \ \mathcal{A}_i \in \mathcal{A}_a, \quad (5.7)$$

where $\int_0^{t_s^a} \kappa_a(\mathcal{N} - q_i(s))^2 ds$ is to penalize the agent for the remaining shares to be bought up to t_s^a and to expedite the acquisition. The parameters p_a, ψ_a, r_a, ξ_a, μ_a, κ_a, γ_a, ϱ_a, and R_a are positive scalars and $u_{-i} := (u_0, u_1, ..., u_{i-1}, u_{i+1}, ..., u_N)$.

5.3 Hybrid Mean Field Game Formulation of Optimal Execution Problems

In this section we formulate optimal execution problems in the Hybrid MM LQG MFG framework.

5.3.1 Discrete State Association

In order to present the trading dynamics of the stock market in the stochastic hybrid systems framework of [44,45], the discrete states $Q_j,\ j = 0, 1, 2$ are introduced, which correspond to the evolution of the market in the intervals $[t_j, t_{j+1})$, where $t_0 = 0$ is the initial time, t_1 and t_2 denote the stopping times of the first population and the second population respectively, and $t_3 = T$ is the terminal time.

We remark that the HS-MFG problems studied in this chapter lie within the class of hybrid LQG problems in [44] for which optimal switching strategies are $\mathcal{F}_t$-independent, and therefore, optimal stopping strategies depend only on the dynamical parameters of each population.

We associate the discrete state Q_0 to the initial case where both the liquidator and acquirer populations are trading together with the major agent over the interval $[0, t_1)$.

The discrete state Q_1 corresponds to the interval $[t_1, t_2)$ for which two situations can be considered: (i) the liquidator population stops at t_1 while the acquirer population is still trading, in which case $Q_1 = q_{0a}$, and (ii) the acquirer population stops at t_1 while the liquidator population is trading, which corresponds to $Q_1 = q_{0l}$.

The discrete state Q_2 represents the system over the interval $[t_2, T]$ after the second population of HFTs stops at t_2, and hence the major agent is trading in the absence of both populations.

The above discrete state association is summarized in the following table.

Table 5.1: Discrete State Association

Discrete State		$\mathcal{A}_0$	$\mathcal{A}_a$	$\mathcal{A}_l$
Q_0		✓	✓	✓
Q_1	q_{0a}	✓	✓	×
	q_{0l}	✓	×	✓
Q_2		✓	×	×

5.3.2 Finite Populations

Major Agent

The dynamics of the major trader in the market can be modeled as

$$\begin{aligned} d\nu_0(t) &= u_0(t)dt, \\ dq_0(t) &= \nu_0(t)dt + \sigma_0^q dw_0^q, \\ dS_0(t) &= \big(\lambda_0\nu_0(t) + \lambda\nu^{N_t}(t)\big)dt + a_0u_0(t)dt + \sigma dw_0^F(t). \end{aligned}$$

Let the major agent's state be denoted by $x_0 = [\nu_0, q_0, S_0]^T$, then its dynamics can be expressed as

$$dx_0 = A_0x_0dt + B_0u_0dt + E_0x^{N_t}dt + D_0dw_0 \tag{5.8}$$

with the matrices

$$A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \lambda_0 & 0 & 0 \end{bmatrix},\ B_0 = \begin{bmatrix} 1 \\ 0 \\ a_0 \end{bmatrix},\ w_0 = \begin{bmatrix} w_0^q \\ w_0^F \end{bmatrix},\ E_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix},\ D_0 = \begin{bmatrix} 0 & 0 \\ \sigma_0^q & 0 \\ 0 & \sigma \end{bmatrix}.$$

Note that in (5.8), N_t takes the following values.

$$N_t = \begin{cases} N_a + N_l & \text{for } Q_0, \\ N_a & \text{for } Q_1 = q_{0a}, \\ N_l & \text{for } Q_1 = q_{0l}, \\ 0 & \text{for } Q_2. \end{cases} \tag{5.9}$$

The major trader's cost function (5.5) can also be described in terms of its states with replacing the final cash process by $\mathbb{E}[Z_0(T)] = -\mathbb{E}[\int_0^T S_0(s)\nu_0(s)ds]$, and the fundamental asset price

$F_0(T)$ using (5.1). The equation (5.8) together with the cost function (5.5) form the stochastic LQG problem for the major trader. Note that the major trader is involved with the market's average trading rate in its dynamics while involved with the market's average selling rate in its cost function.

Minor Liquidator

Similarly, the stochastic optimal control problem for a minor liquidator $\mathcal{A}_i \in \mathcal{A}_l$, is given by the set of dynamical equations

$$\begin{aligned} d\nu_i(t) &= u_i(t)dt, \\ dq_i(t) &= \nu_i(t)dt + \sigma_i^q dw_i^q, \\ dS_i(t) &= \big(\lambda_0\nu_0(t) + \lambda\nu^{N_t}(t)\big)dt + au_i(t)dt + \sigma dw_i^F. \end{aligned}$$

Similar to the major trader, we define a generic minor trader's state vector as $x_i = [\nu_i, q_i, S_i]^T$, and its dynamics can be written as

$$dx_i = A_l x_i dt + B_l u_i dt + E_l x^{N_t} dt + D_l dw_{l_i} \tag{5.10}$$

with

$$A_l = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_l = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix}, \quad B_l = \begin{bmatrix} 1 \\ 0 \\ a \end{bmatrix},$$

$$G_l = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda_0 & 0 & 0 \end{bmatrix}, \; D_l = \begin{bmatrix} 0 & 0 \\ \sigma_i^q & 0 \\ 0 & \sigma \end{bmatrix}, \; w_{li} = \begin{bmatrix} w_i^q \\ w_i^F \end{bmatrix}.$$

The quadratic cost function (5.6) can also be expressed in terms of the minor agent's state when the final cash process in (5.6) is replaced by $\mathbb{E}[Z_i(t_s^l)] = -\mathbb{E}[\int_0^{t_s^l} S_i(s)\nu_i(s)ds]$ using (5.4), and the fundamental asset price $F_i(t_s^l)$ is replaced using (5.2).

The equations (5.10) and (5.6) form the stochastic LQG problem for a generic minor liquidator. Additionally, they show that a minor liquidator is coupled with the major agent's trading rate and the market's average trading rate in its dynamics while coupled with the market's average selling rate in its cost function.

Minor Acquirer Agent

The stochastic optimal control problem for a minor acquirer $\mathcal{A}_i \in \mathcal{A}_a$, is given by the set of dynamical equations

$$\begin{aligned}
d\nu_i(t) &= u_i(t)dt,\\
dY_i(t) &= -\nu_i(t)dt + \sigma_i^q dw_i^q,\\
dS_i(t) &= \big(\lambda_0\nu_0(t) + \lambda\nu^{N_t}(t)\big)dt + au_i(t)dt + \sigma dw_i^F,
\end{aligned}$$

where $Y_i(t) = \mathcal{N}_a - q_i(t)$ is the remaining shares at t to be acquired until the end of trading horizon. We define a generic minor acquirer's state vector as $x_i = [\nu_i, Y_i, S_i]$, hence its dynamics in compact form would be

$$dx_i = A_a x_i dt + B_a u_i dt + E_a x^{N_t} dt + D_a dw_{a_i}, \tag{5.11}$$

where

$$A_a = \begin{bmatrix} 0 & 0 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix},\ E_a = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ \lambda & 0 & 0 \end{bmatrix},\ B_a = \begin{bmatrix} 1\\ 0\\ a \end{bmatrix}$$

$$G_a = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ \lambda_0 & 0 & 0 \end{bmatrix},\ D_a = \begin{bmatrix} 0 & 0\\ \sigma_i^q & 0\\ 0 & \sigma \end{bmatrix},\ w_{ai} = \begin{bmatrix} w_i^q\\ w_i^F \end{bmatrix}.$$

Note that N_t in (5.11) again takes values as in (5.9) over the trading horizon. Accordingly, the cost function for acquisition is given by

$$\begin{aligned}
J_i(u_i, u_{-i}) = \mathbb{E}\Big[&p_a Y_i(t_s^a)\big(S_i(t_s^a) - a\nu_i(t_s^a) + \psi_a Y_i(t_s^a)\big) + \xi_a S_i^2(t_s^a) + \mu_a \nu_i^2(t_s^a)\\
&+ \int_0^{t_s^a} \Big(\kappa_a Y_i^2(s) + \gamma_a S_i^2(s) + \varrho_a \nu_i^2(s) - r_a S_i(s)\nu_i(s) + R_a u_i^2(s)\Big)ds\Big], \text{ for } \mathcal{A}_i \in \mathcal{A}_a.
\end{aligned} \tag{5.12}$$

The set of equations (5.11)-(5.12) constitute the standard stochastic LQG problem for a minor acquirer. It can be seen that a generic minor acquirer interacts with the major agent's trading rate as well as the market's average trading rate through it dynamics, and with the market's average buying rate through its cost function.

5.3.3 Mean Field Evolution

Following the LQG MFG methodology [16], the mean field, $\bar{x}$, is defined as the L^2 limit, when it exists, of the average of minor agents' states when population size goes to infinity

$$\bar{x}(t) = \lim_{N_t \to \infty} x^{N_t}(t) = \lim_{N \to \infty} \frac{1}{N_t} \sum_{i=1}^{N_t} x_i(t), \ a.s.$$

Now, if the control strategy for each minor agent is considered to have the general feedback form

$$u_i = L_1 x_i + L_2 x_0 + \sum_{j \neq i, j=1}^{N_t} L_4 x_j + L_3, \quad 1 \leq i \leq N_t, \tag{5.13}$$

then the mean field dynamics can be obtained by substituting (5.13) in the minor liquidator (respectively, acquirer) agents' dynamics (5.10) (respectively, (5.11)), and taking the average and then its L^2 limit as $N \to \infty$.

The set of mean field equations for the optimal execution problem can be written as

$$d\bar{x} = \bar{A}\bar{x}dt + \bar{G}x_0 dt + \bar{m}dt. \tag{5.14}$$

For Q_0, $\bar{x} = [\bar{x}_a^T, \bar{x}_l^T]^T$ consists of the mean field $\bar{x}_l$ of the liquidator population, and the mean field $\bar{x}_a$ of the acquirer population. The matrices in (5.14) are defined as

$$\bar{A} = \begin{bmatrix} \bar{A}_a & \bar{A}_{al} \\ \bar{A}_{la} & \bar{A}_l \end{bmatrix}, \quad \bar{G} = \begin{bmatrix} \bar{G}_a \\ \bar{G}_l \end{bmatrix}, \quad \bar{m} = \begin{bmatrix} \bar{m}_a \\ \bar{m}_l \end{bmatrix}, \tag{5.15}$$

which shall be determined from consistency equations discussed in section 5.3.5.

For q_{0a}, $\bar{x} = \bar{x}_a$, and the matrices in (5.14) are given as

$$\bar{A} = \bar{A}_a, \quad \bar{G} = \bar{G}_a, \quad \bar{m} = \bar{m}_a. \tag{5.16}$$

For q_{0l}, $\bar{x} = \bar{x}_l$, and the matrices in (5.14) are given by

$$\bar{A} = \bar{A}_l, \quad \bar{G} = \bar{G}_l, \quad \bar{m} = \bar{m}_l. \tag{5.17}$$

Finally, for Q_2, $\bar{x} = 0$.

The empirical distribution of the minor traders is denoted by $\pi^N = (\pi_a^N, \pi_l^N), \ \pi_k^N = \frac{N_k}{N}, \ k \triangleq a, l$. The first assumption is as follows.

Assumption 5.1. *There exists π such that $lim_{N \to \infty} \pi^N = (\pi_a, \pi_l)$ a.s.*

5.3.4 Infinite Populations

Following the mean field game methodology with a major agent [16, 42] the hybrid optimal execution problem is first solved in the infinite population case where the average term in the finite population dynamics and cost function of each agent is replaced by its infinite population limit, i.e. the mean field. Then specializing to linear systems [16], the major agent's state is extended with the mean field, while the minor agent's state is extended with the mean field and the major agent's state; this yields LQG problems for each trader linked only through the mean field and the major agent's state. Then the main results of [16], [42] are (i) the existence of infinite population best response strategies which yield the Nash equilibria, and (ii) the infinite population best response strategies applied to the finite population system yield an ϵ-Nash equilibria (see Theorem 5.1).

Major Liquidator Agent

The extended dynamics of the major agent in the infinite population, i.e. the dynamic for the x_0^{ex,Q_j} is given by

$$dx_0^{ex,Q_j} = (\mathbb{A}_0^{Q_j} x_0^{ex,Q_j} + \mathbb{M}_0^{Q_j} + \mathbb{B}_0^{Q_j} u_0^{Q_j})dt + \mathbb{D}_0^{Q_j} dW_0, \tag{5.18}$$

$0 \leq j \leq 2$, and the cost function for the extended major agent's system would be

$$J_0(u_0, u_{-0}) = \mathbb{E}\Big[\|x_0^{ex,Q_2}(T)\|^2_{\bar{\mathbb{P}}_0^{Q_2}} + \sum_{j=0}^{2} \int_{t_j}^{t_{j+1}} \big(\|x_0^{ex,Q_j}(s)\|^2_{\mathbb{P}_0^{Q_j}} + \|u_0^{Q_j}(s)\|^2_{R_0^{Q_j}}\big) ds\Big], \tag{5.19}$$

where $t_0 = 0, t_3 = T$. Let matrix coefficients P_0, $\bar{P}_0$, respectively, associated with the running and final costs in (5.5) be given by

$$\bar{P}_0 = \begin{bmatrix} \beta & \frac{1}{2}pa_0 & 0 \\ \frac{1}{2}pa_0 & p\alpha & -\frac{1}{2}p \\ 0 & -\frac{1}{2}p & \epsilon \end{bmatrix}, \quad P_0 = \begin{bmatrix} \theta & 0 & \frac{1}{2}r \\ 0 & \phi & 0 \\ \frac{1}{2}r & 0 & \delta \end{bmatrix},$$

then over the interval $[t_0, t_1)$, and in the discrete state Q_0, the dynamics of the continuous state $x_0^{ex,Q_0} = [x_0^T, \bar{x}_a^T, \bar{x}_l^T]^T$ is determined from (5.18) with

$$\mathbb{A}_0^{Q_0} = \begin{bmatrix} A_0 & [\pi_a E_0, \pi_l E_0] \\ \bar{G} & \bar{A} \end{bmatrix}, \ \mathbb{M}_0^{Q_0} = \begin{bmatrix} 0_{3\times 1} \\ \bar{m} \end{bmatrix}, \ \mathbb{B}_0^{Q_0} = \begin{bmatrix} B_0 \\ 0_{6\times 1} \end{bmatrix}, \ \mathbb{D}_0^{Q_0} = \begin{bmatrix} D_0 & 0_{3\times 6} \\ 0_{6\times 3} & 0_{6\times 6} \end{bmatrix}.$$

and $\mathbb{P}_0^{Q_0}$ in (5.19) is given by

$$\mathbb{P}_0^{Q_0} = [I_{3\times 3}, 0_{3\times 3}, 0_{3\times 3}]^T P_0 [I_{3\times 3}, 0_{3\times 3}, 0_{3\times 3}].$$

In case (i) where $Q_1 = q_{0a}$ over the interval $[t_1, t_2)$, the dynamics for $x_0^{ex,q_{0a}} = [x_0^T, \bar{x}_a^T]^T$ is determined from (5.18) with

$$\mathbb{A}_0^{q_{0a}} = \begin{bmatrix} A_0 & E_0 \\ \bar{G}_a & \bar{A}_a \end{bmatrix}, \quad \mathbb{M}_0^{q_{0a}} = \begin{bmatrix} 0_{3\times 1} \\ \bar{m}_a \end{bmatrix}, \quad \mathbb{B}_0^{q_{0a}} = \begin{bmatrix} B_0 \\ 0_{3\times 1} \end{bmatrix}, \quad \mathbb{D}_0^{q_{0a}} = \begin{bmatrix} D_0 & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \end{bmatrix}.$$

and $\mathbb{P}_0^{q_{0a}}$ is given by

$$\mathbb{P}_0^{q_{0a}} = [I_{3\times 3}, 0_{3\times 3}]^T P_0 [I_{3\times 3}, 0_{3\times 3}].$$

In this case, the values of the continuous state before and after t_1 are related by the jump map

$$x_0^{ex,q_{0a}}(t_1) = \Psi_{0,a} x_0^{ex,Q_0}(t_1-) \tag{5.20}$$

where

$$\Psi_{0,a} = \begin{bmatrix} I_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & I_{3\times 3} & 0_{3\times 3} \end{bmatrix}. \tag{5.21}$$

In case (ii) where $Q_1 = q_{0l}$ holds, $x^{ex,q_{0l}} = [x_0^T, \bar{x}_l^T]^T$ and

$$\mathbb{A}_0^{q_{0l}} = \begin{bmatrix} A_0 & E_0 \\ \bar{G}_l & \bar{A}_l \end{bmatrix}, \quad \mathbb{B}_0^{q_{0l}} = \begin{bmatrix} B_0 \\ 0_{3\times 1} \end{bmatrix}, \quad \mathbb{M}_0^{q_{0l}} = \begin{bmatrix} 0_{3\times 1} \\ \bar{m}_l \end{bmatrix}, \quad \mathbb{D}_0^{q_{0l}} = \begin{bmatrix} D_0 & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \end{bmatrix}.$$

$$\mathbb{P}_0^{q_{0l}} = [I_{3\times 3}, 0_{3\times 3}]^T P_0 [I_{3\times 3}, 0_{3\times 3}, 0_{3\times 3}].$$

In this case, the values of the continuous state of the major trader before and after t_1 are related by the jump map

$$x_0^{ex,q_{0l}}(t_1) = \Psi_{0,l} x_0^{ex,Q_0}(t_1-) \tag{5.22}$$

where

$$\Psi_{0,l} = \begin{bmatrix} I_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} & I_{3\times 3} \end{bmatrix}. \tag{5.23}$$

For the discrete state Q_2, the continuous state of the major trader is $x_0^{ex,q_{0a}} \equiv x_0$, and

$$\mathbb{A}_0^{Q_2} = A_0, \quad \mathbb{M}_0^{Q_2} = 0_{3\times 1}, \quad \mathbb{B}_0^{Q_2} = B_0, \quad \mathbb{D}_0^{Q_2} = D_0$$
$$\bar{\mathbb{P}}_0^{Q_2} = \bar{P}_0, \quad \mathbb{P}_0^{Q_2} = P_0$$

The values continuous state of the major trader before and after t_2 are related by the the jump map

$$x_0^{ex,Q_2}(t_2) = \Psi_{0,2} x_0^{ex,Q_1}(t_2-) \tag{5.24}$$

where $\Psi_{0,2} = \begin{bmatrix} I_{3\times 3} & 0_{3\times 3} \end{bmatrix}$.

By the definition of the terms $\mathbb{D}_0^{Q_j}$ necessarily satisfy the condition A1 in [45], which in the LQG takes the following form

$$\mathbb{D}_0^{Q_j} = \Psi_{0,j} \mathbb{D}_0^{Q_{j-1}}, \qquad j = 1, 2. \tag{5.25}$$

An application of the stochastic hybrid control theory of [45], specialized to the LQG case in [44], yield the infinite population best response hybrid control action as

$$u_0^{Q_j}(t) = -R_{0,Q_j}^{-1} \mathbb{B}_{0,Q_j}^T \Pi_0^{Q_j}(t)\, x_0^{ex,Q_j}(t), \tag{5.26}$$

where $\Pi_0^{Q_j}(t)$ is the solution of

$$-\dot{\Pi}_0^{Q_j} = \Pi_0^{Q_j} \mathbb{A}_0^{Q_j} + \mathbb{A}_{0,Q_j}^T \Pi_0^{Q_j} - \Pi_0^{Q_j} \mathbb{B}_0^{Q_j} R_{0,Q_j}^{-1} \mathbb{B}_{0,Q_j}^T \Pi_0^{Q_j} + \mathbb{P}_0, \tag{5.27}$$

subject to the terminal and boundary conditions

$$\Pi_0^{Q_2}(T) = \bar{\mathbb{P}}_0, \tag{5.28}$$

$$\Pi_0^{Q_{j-1}}(t_j) = \Psi_{0,j}^T \Pi_0^{Q_j}(t_j) \Psi_{0,j}, \tag{5.29}$$

$$\begin{aligned} &\mathbb{P}_0^{Q_{j-1}} + \Psi_{0,j}^T \Pi_0^{Q_j}(t_j) \Psi_{0,j} \mathbb{A}_0^{Q_{j-1}} + \mathbb{A}_{0,Q_{j-1}}^T \Psi_{0,j}^T \Pi_0^{Q_j}(t_j) \Psi_{0,j} \\ &\qquad - \Psi_{0,j}^T \Pi_0^{Q_j}(t_j) \Psi_{0,j} \mathbb{B}_0^{Q_{j-1}} R_{0,Q_{j-1}}^{-1} \mathbb{B}_{0,Q_{j-1}}^T \Psi_{0,j}^T \Pi_0^{Q_j}(t_j) \Psi_{0,j} \\ &= \Psi_{0,j}^T \Big(\mathbb{P}_0^{Q_j} + \Pi_0^{Q_j}(t_j) \mathbb{A}_0^{Q_j} + \mathbb{A}_{0,Q_j}^T \Pi_0^{Q_j}(t_j) - \Pi_0^{Q_j}(t_j) \mathbb{B}_0^{Q_j} R_{0,Q_j}^{-1} \mathbb{B}_{0,Q_j}^T \Pi_0^{Q_j}(t_j) \Big) \Psi_{0,j}, \text{ for } j = 1, 2. \end{aligned} \tag{5.30}$$

Minor Acquirer

A generic minor agent $\mathcal{A}_i$'s extended dynamics in the acquirer population with the extended state x_i^{ex,Q_j} is

$$dx_i^{ex,Q_j} = (\mathbb{A}_a^{Q_j} x_i^{ex,Q_j} + \mathbb{M}_a^{Q_j} + \mathbb{B}_0^{Q_j} u_0^{Q_j} + \mathbb{B}_a^{Q_j} u_i^{Q_j}) dt + \mathbb{D}_a^{Q_j} dW_i, \tag{5.31}$$

where for Q_0, $x_i^{ex,Q_0} = [x_i^T, x_0^T, \bar{x}_a^T, \bar{x}_l^T]^T$, and

$$\mathbb{A}_a^{Q_0} = \begin{bmatrix} A_a & [G_a, \pi_a E_a, \pi_l E_a] \\ 0_{9\times 3} & \mathbb{A}_0^{Q_0} - \mathbb{B}_0^{Q_0} R_{0,Q_0}^{-1} \mathbb{B}_{0,Q_0}^T \Pi_0^{Q_0} \end{bmatrix},$$

$$\mathbb{M}_a^{Q_0}=\begin{bmatrix} 0_{3\times 1}, \\ \mathbb{M}_0 \end{bmatrix},\ \mathbb{B}_a^{Q_0}=\begin{bmatrix} B_a \\ 0_{9\times 1} \end{bmatrix},\ \mathbb{D}_a^{Q_0}=\begin{bmatrix} D_a & 0_{3\times 9} \\ 0_{9\times 3} & \mathbb{D}_0^{Q_0} \end{bmatrix},$$

and for q_{0a}, $x_i^{ex,q_{0a}} = [x_i^T, x_0^T, \bar{x}_a^T]^T$, and

$$\mathbb{A}_a^{q_{0a}} = \begin{bmatrix} A_a & [G_a, E_a] \\ 0_{6\times 3} & \mathbb{A}_0^{q_{0a}} - \mathbb{B}_0^{q_{0a}} R_{0,q_{0a}}^{-1} \mathbb{B}_{0,q_{0a}}^T \Pi_0^{q_{0a}} \end{bmatrix},$$

$$\mathbb{M}_a^{q_{0a}}=\begin{bmatrix} 0_{3\times 1} \\ \mathbb{M}_0 \end{bmatrix},\ \mathbb{B}_a^{q_{0a}}=\begin{bmatrix} B_a \\ 0_{6\times 1} \end{bmatrix},\ \mathbb{D}_a^{q_{0a}}=\begin{bmatrix} D_a & 0_{3\times 6} \\ 0_{6\times 3} & \mathbb{D}_0^{q_{0a}} \end{bmatrix}.$$

In case (i) where the acquirer population is trading over $[t_1, t_2)$, i.e. $Q_1 = q_{0a}$, the total hybrid cost for a minor acquirer is given by

$$J_i^a(u_i, u_{-i}) = \mathbb{E}\Big[\|x_i^{ex,q_{0a}}(t_2)\|^2_{\mathbb{P}_a^{q_{0a}}} + \sum_{j=0}^{1} \int_{t_j}^{t_{j+1}} \big(\|x_i^{ex,Q_j}(s)\|^2_{\mathbb{P}_a^{Q_j}} + \|u_i^{Q_j}(s)\|^2_{R_a^{Q_j}}\big) ds\Big], \tag{5.32}$$

with

$$\bar{\mathbb{P}}_a^{q_{0a}} = [I_{3\times 3}, 0_{3\times 6}]^T \bar{P}_a [I_{3\times 3}, 0_{3\times 6}] \tag{5.33}$$

$$\mathbb{P}_a^{q_{0a}} = [I_{3\times 3}, 0_{3\times 6}]^T P_a [I_{3\times 3}, 0_{3\times 6}] \tag{5.34}$$

$$\mathbb{P}_a^{Q_0} = [I_{3\times 3}, 0_{3\times 9}]^T P_a [I_{3\times 3}, 0_{3\times 9}], \tag{5.35}$$

where $\bar{P}_a, P_a$ are, respectively, associated with the running and final costs in (5.7) are given by

$$\bar{P}_a = \begin{bmatrix} \mu_a & -\frac{1}{2}p_a a & 0 \\ -\frac{1}{2}p_a a & p_a\psi_a & \frac{1}{2}p_a \\ 0 & \frac{1}{2}p_a & \xi_a \end{bmatrix}, \quad P_a = \begin{bmatrix} \varrho_a & 0 & -\frac{1}{2}r_a \\ 0 & \kappa_a & 0 \\ -\frac{1}{2}r_a & 0 & \gamma_a \end{bmatrix}. \tag{5.36}$$

In this case, the extended state for a generic minor agent in the acquirer population at t_1 satisfies the jump transition map

$$x^{ex,q_{0a}}(t_1) = \Psi_{i,a} x^{ex,Q_0}(t_1-)$$

with

$$\Psi_{i,a} = \begin{bmatrix} I_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & I_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} & I_{3\times 3} & 0_{3\times 3} \end{bmatrix}.$$

In case (ii) where $Q_1 = q_{0l}$ holds over the interval $[t_1, t_2)$, the cost for the minor acquirer agent $\mathcal{A}_i$ is given by

$$J_i^a(u_i,u_{-i}) = \mathbb{E}\Big[\|x_i^{ex,Q_0}(t_1)\|^2_{\bar{\mathbb{P}}_a^{Q_0}} + \int_{t_0}^{t_1} \big(\|x_i^{ex,Q_0}(s)\|^2_{\mathbb{P}_a^{Q_0}} + \|u_i^{Q_0}(s)\|^2_{R_a^{Q_0}}\big)ds\Big], \tag{5.37}$$

with $\bar{\mathbb{P}}^{Q_0} = [I_{3\times 3}, 0_{3\times 9}]^T \bar{P}[I_{3\times 3}, 0_{3\times 9}]$.

The optimal stopping problem for a minor acquirer is equivalent to a hybrid optimal control problem in which the dynamics and costs become zero after stopping. By the definition of the terms $\mathbb{D}_a^{Q_j}$ necessarily satisfy the condition A1 in [45]. To be specific, for the case (i) the diffusion coefficients in (5.31) satisfy

$$\mathbb{D}_a^{q_{0a}} = \Psi_{i,a}\mathbb{D}_a^{Q_0}, \tag{5.38}$$

$$\mathbb{D}_a^{Q_2} = \Psi_{i,\sigma_{q_{0a},q_{\text{stop}}}}\mathbb{D}_a^{q_{0a}} \equiv 0, \tag{5.39}$$

where $\sigma_{Q_j,q_{\text{stop}}}$ denotes the stopping event in the discrete state Q_j. Both conditions in (5.39) are satisfied since $\mathbb{D}_a^{Q_2} = 0$ due to the zero dynamics after stopping and $\Psi_{i,\sigma_{q_{0a},q_{\text{stop}}}} = 0$ due to removal of the minor acquirer trader's state from the market dynamics. For the case (ii) we also have

$$\mathbb{D}_a^{Q_1} = \Psi_{i,\sigma_{Q_0,q_{\text{stop}}}}\mathbb{D}_a^{Q_0} \equiv 0, \tag{5.40}$$

which holds due to the stopping decision at t_1. The results of [44, 45] yield

$$u_i^{Q_j}(t) = -R_{Q_j}^{-1}\mathbb{B}_{a,Q_j}^T \Pi_a^{Q_j}(t)\, x_i^{ex,Q_j}(t), \tag{5.41}$$

with

$$-\dot{\Pi}_a^{Q_j} = \Pi_a^{Q_j}\mathbb{A}_a^{Q_j} + \mathbb{A}_{a,Q_j}^T\Pi_a^{Q_j} - \Pi_a^{Q_j}\mathbb{B}_a^{Q_j}R_{a,Q_j}^{-1}\mathbb{B}_{a,Q_j}^T\Pi_a^{Q_j} + \mathbb{P}_a, \tag{5.42}$$

where for the case (i), in which $Q_1 = q_{0a}$, $\Pi_a^{Q_j}(t)$ is the solution of (5.42) subject to the terminal conditions

$$\Pi_a^{q_{0a}}(t_2) = \bar{\mathbb{P}}_a^{q_{0a}},$$
$$\Big(\mathbb{P}_a^{q_{0a}} + \bar{\mathbb{P}}_a^{q_{0a}}\mathbb{A}_a^{q_{0a}} + \mathbb{A}_{a,q_{0a}}^T\bar{\mathbb{P}}_a^{q_{0a}} - \bar{\mathbb{P}}_a^{q_{0a}}\mathbb{B}_a^{q_{0a}}R_{a,q_{0a}}^{-1}\mathbb{B}_{a,q_{0a}}^T\bar{\mathbb{P}}_a^{q_{0a}}\Big)_{t=t_2} = 0,$$

and the boundary conditions

$$\Pi_a^{Q_0}(t_1) = \Psi_{i,a}^T\Pi_a^{q_{0a}}(t_1)\Psi_{i,a}, \tag{5.43}$$

$$\begin{aligned}
&\mathbb{P}_a^{Q_0} + \Psi_{i,a}^T\Pi_a^{q_{0a}}(t_1)\Psi_{i,a}\mathbb{A}_a^{Q_0} + \mathbb{A}_{a,Q_0}^T\Psi_{i,a}^T\Pi_a^{q_{0a}}(t_1)\Psi_{i,a} \\
&\qquad - \Psi_{i,a}^T\Pi_a^{q_{0a}}(t_1)\Psi_{i,a}\mathbb{B}_a^{Q_0}R_{a,Q_0}^{-1}\mathbb{B}_{a,Q_0}^T\Psi_{i,a}^T\Pi_a^{q_{0a}}(t_1)\Psi_{i,a} \\
&= \Psi_{i,a}^T\Big(\mathbb{P}_a^{q_{0a}} + \Pi_a^{q_{0a}}(t_1)\mathbb{A}_a^{q_{0a}} + \mathbb{A}_{a,q_{0a}}^T\Pi_a^{q_{0a}}(t_1) - \Pi_a^{q_{0a}}(t_1)\mathbb{B}_a^{q_{0a}}R_{a,q_{0a}}^{-1}\mathbb{B}_{a,q_{0a}}^T\Pi_a^{q_{0a}}(t_1)\Big)\Psi_{i,a},
\end{aligned} \tag{5.44}$$

and in case (ii) where $Q_1 = q_{0l}$ holds, $\Pi_a^{Q_0}(t)$ is the solution of (5.42) subject to the terminal conditions

$$\Pi_a^{Q_0}(t_1) = \bar{\mathbb{P}}_a^{Q_0}, \tag{5.45}$$

$$\Big(\mathbb{P}_a^{Q_0} + \bar{\mathbb{P}}_a^{Q_0}\mathbb{A}_a^{Q_0} + \mathbb{A}_{a,Q_0}^T\bar{\mathbb{P}}_a^{Q_0} - \bar{\mathbb{P}}_a^{Q_0}\mathbb{B}_a^{Q_0}R_{a,Q_0}^{-1}\mathbb{B}_{a,Q_0}^T\bar{\mathbb{P}}_a^{Q_0}\Big)_{t=t_1} = 0. \tag{5.46}$$

Minor Liquidator

The hybrid dynamics, jump maps and performance measures for a minor liquidator are presented in a similar form as the minor acquirer, and therefore, due to space limitations, are not presented here. The infinite population best response hybrid control action as

$$u_i^{Q_j}(t) = -R_{Q_j}^{-1}\mathbb{B}_{l,Q_j}^T\Pi_l^{Q_j}(t)\,x_i^{ex,Q_j}(t), \tag{5.47}$$

with

$$-\dot{\Pi}_l^{Q_j} = \Pi_l^{Q_j}\mathbb{A}_l^{Q_j} + \mathbb{A}_{l,Q_j}^T\Pi_l^{Q_j} - \Pi_l^{Q_j}\mathbb{B}_l^{Q_j}R_{l,Q_j}^{-1}\mathbb{B}_{l,Q_j}^T\Pi_l^{Q_j} + \mathbb{P}_l, \tag{5.48}$$

where for the case (i), in which $Q_1 = q_{0a}$, $\Pi_l^{Q_j}(t)$ is the solution of (5.48) subject to the terminal conditions

$$\Pi_l^{Q_0}(t_1) = \bar{\mathbb{P}}_l^{Q_0},$$

$$\Big(\mathbb{P}_l^{Q_0} + \bar{\mathbb{P}}_l^{Q_0}\mathbb{A}_l^{Q_0} + \mathbb{A}_{l,Q_0}^T\bar{\mathbb{P}}_l^{Q_0} - \bar{\mathbb{P}}_l^{Q_0}\mathbb{B}_l^{Q_0}R_{l,Q_0}^{-1}\mathbb{B}_{l,Q_0}^T\bar{\mathbb{P}}_l^{Q_0}\Big)_{t=t_1} = 0.$$

and in case (ii) where $Q_1 = q_{0l}$ holds, $\Pi_l^{Q_0}(t)$ is the solution of (5.42) subject to the terminal conditions

$$\Pi_l^{q_{0l}}(t_2) = \bar{\mathbb{P}}_l^{q_{0l}},$$

$$\Big(\mathbb{P}_l^{q_{0l}} + \bar{\mathbb{P}}_l^{q_{0l}}\mathbb{A}_l^{q_{0l}} + \mathbb{A}_{l,q_{0l}}^T\bar{\mathbb{P}}_l^{q_{0l}} - \bar{\mathbb{P}}_l^{q_{0l}}\mathbb{B}_l^{q_{0l}}R_{l,q_{0l}}^{-1}\mathbb{B}_{l,q_{0l}}^T\bar{\mathbb{P}}_l^{q_{0l}}\Big)_{t=t_2} = 0,$$

and the boundary conditions

$$\Pi_l^{Q_0}(t_1) = \Psi_{i,l}^T\Pi_l^{q_{0l}}(t_1)\Psi_{i,l},$$

$$\begin{aligned}&\mathbb{P}_l^{Q_0}+\Psi_{i,l}^T\Pi_l^{q_{0l}}(t_1)\Psi_{i,l}\mathbb{A}_l^{Q_0}+\mathbb{A}_{l,Q_0}^T\Psi_{i,l}^T\Pi_l^{q_{0l}}(t_1)\Psi_{i,l}-\Psi_{i,l}^T\Pi_l^{q_{0l}}(t_1)\Psi_{i,l}\mathbb{B}_l^{Q_0}R_{l,Q_0}^{-1}\mathbb{B}_{l,Q_0}^T\Psi_{i,l}^T\Pi_l^{q_{0l}}(t_1)\Psi_{i,l}\\ &= \Psi_{i,l}^T\Big(\mathbb{P}_l^{q_{0l}} + \Pi_l^{q_{0l}}(t_1)\mathbb{A}_l^{q_{0l}} + \mathbb{A}_{l,q_{0l}}^T\Pi_l^{q_{0l}}(t_1) - \Pi_l^{q_{0l}}(t_1)\mathbb{B}_l^{q_{0l}}R_{l,q_{0l}}^{-1}\mathbb{B}_{l,q_{0l}}^T\Pi_l^{q_{0l}}(t_1)\Big)\Psi_{i,l}.\end{aligned} \tag{5.49}$$

The infinite population equilibria is linked to the finite population equilibria by the following theorem.

Theorem 5.1 (ϵ-Nash Equilibria for Hybrid MM LQG MFG Systems)**.** *Subject to Assumptions*

4.1-4.5 in Chapter 4, the system equations (5.8)*,* (5.10)*,* (5.11) *together with the mean field equations* (5.51) *generate the set of control laws* $\mathcal{U}_{MF}^N \triangleq \{u_i^{Q_j}; 0 \leq i \leq N_t\}$, $1 \leq N_t \leq N < \infty$, *given by* (5.26)*,* (5.41)*, and* (5.47) *such that*

(i) *All agent systems* $\mathcal{A}_i$, $0 \leq i \leq N$*, are second order stable.*

(ii) $\{\mathcal{U}_{MF}^N; 1 \leq N < \infty\}$ *yields an* ϵ*-Nash equilibrium for all* ϵ*, i.e. for all* $\epsilon > 0$*, there exists* $N(\epsilon)$ *such that for all* $N \geq N(\epsilon)$*;*

$$J_i^{s,N}(u_i^\circ, u_{-i}^\circ) - \epsilon \leq \inf_{u_i \in \mathcal{U}_{i,y}^N} J_i^{s,N}(u_i, u_{-i}^\circ) \leq J_i^{s,N}(u_i^\circ, u_{-i}^\circ).$$

□

Proof. Applying the approach of [16] backwards from T along the optimal realization of the sequence Q_0, Q_1, Q_2, establishes the existence and uniqueness of the Nash equilibrium and ϵ-Nash equilibrium for the infinite population system and finite population system, respectively. □

5.3.5 Mean Field Consistency Equations

The closed loop trading dynamics of a minor acquirer $\mathcal{A}_i \in \mathcal{A}_a$ applying (5.41), or correspondingly a minor liquidator $\mathcal{A}_i \in \mathcal{A}_l$ applying (5.47) is consequently

$$d\nu_i = -R_{a/l}^{-1}\mathbb{B}_{a/l}^T \Pi_{a/l}\big(x_i^T, x_0^T, \bar{x}^T\big)^T dt - R_{a/l}^{-1}\mathbb{B}_{a/l}^T s_{a/l}(t)dt,$$

then the average of the closed loop trading dynamics over the acquirer or liquidator population is obtained as

$$\frac{1}{N_{a/l}}\sum_{i=1}^{N_{a/l}} d\nu_i = -\frac{1}{N_{a/l}}\sum_{i=1}^{N_{a/l}} R_{a/l}^{-1}\mathbb{B}_{a/l}^T \Pi_{a/l}\big(x_i^T, x_0^T, \bar{x}^T\big)^T dt - \frac{1}{N_{a/l}}\sum_{i=1}^{N_{a/l}} R_{a/l}^{-1}\mathbb{B}_{a/l}^T s_{a/l}(t)dt, \quad (5.50)$$

where $\bar{x} = [\bar{x}_a^T, \bar{x}_l^T]^T$. Then taking the L^2 limit of (5.50) as the population size $N_{a/l}$ goes to infinity yields the trading rate mean field dynamics

$$d\bar{\nu}_{a/l} = \lim_{N_{a/l}\to\infty} d\nu^{N_{a/l}} = -R_{a/l}^{-1}\mathbb{B}_{a/l}^T \Pi_{a/l} \times \lim_{N_{a/l}\to\infty} \big((x^{N_{a/l}})^T, x_0^T, \bar{x}_a^T, \bar{x}_l^T\big)^T dt - R_{a/l}^{-1}\mathbb{B}_{a/l}^T s_{a/l} dt,$$

and hence the consistency equations are given by

$$\bar{A}_{a,11} = -R_a^{-1}(\Pi_{a,11} + \Pi_{a,17}) - aR_a^{-1}(\Pi_{a,31} + \Pi_{a,37}),$$

$$
\begin{aligned}
\bar{A}_{a,12} &= -R_a^{-1}(\Pi_{a,12}+\Pi_{a,18}) - aR_a^{-1}(\Pi_{a,32}+\Pi_{a,38}),\\
\bar{A}_{a,13} &= -R_a^{-1}(\Pi_{a,13}+\Pi_{a,19}) - aR_a^{-1}(\Pi_{a,33}+\Pi_{a,39}),\\
\bar{A}_{al,11} &= -R_a^{-1}(\Pi_{a,110}+a\Pi_{a,310}),\\
\bar{A}_{al,12} &= -R_a^{-1}(\Pi_{a,111}+a\Pi_{a,311}),\\
\bar{A}_{al,13} &= -R_a^{-1}(\Pi_{a,112}+a\Pi_{a,312}),\\
\bar{A}_{l,11} &= -R_l^{-1}(\Pi_{l,11}+\Pi_{l,110}) - aR_l^{-1}(\Pi_{l,31}+\Pi_{l,310}),\\
\bar{A}_{l,12} &= -R_l^{-1}(\Pi_{l,12}+\Pi_{l,111} - aR_l^{-1}(\Pi_{l,32}+\Pi_{l,311}),\\
\bar{A}_{l,13} &= -R_l^{-1}(\Pi_{l,13}+\Pi_{l,112}) - aR_l^{-1}(\Pi_{l,33}+\Pi_{l,312}),\\
\bar{A}_{la,11} &= -R_l^{-1}\Pi_{l,17} - aR_l^{-1}\Pi_{l,37},\\
\bar{A}_{la,12} &= -R_l^{-1}\Pi_{l,18} - aR_l^{-1}\Pi_{l,38},\\
\bar{A}_{la,13} &= -R_l^{-1}\Pi_{l,19} - aR_l^{-1}\Pi_{l,39},\\
\bar{G}_{a/l,11} &= -R_{a/l}^{-1}(\Pi_{a/l,14}+a\Pi_{a/l,34}),\\
\bar{G}_{a/l,12} &= -R_{a/l}^{-1}(\Pi_{a/l,15}+a\Pi_{a/l,35}),\\
\bar{G}_{a/l,13} &= -R_{a/l}^{-1}(\Pi_{a/l,16}+a\Pi_{a/l,36}),\\
\bar{m}_{a/l,1} &= 0,
\end{aligned}
\tag{5.51}
$$

where $\Pi_{a/l,ij} = \Pi_{a/l}(i,j)$ for $i=\{1,3\},\ j=\{1,2,3,...,12\}$. Hence the matrices in (5.15) are given as

$$
\bar{A}_{a/l} = \begin{bmatrix} \bar{A}_{a/l,11} & \bar{A}_{a/l,12} & \bar{A}_{a/l,13} \\ 1 & 0 & 0 \\ (\pi_{a/l}\lambda + a\bar{A}_{a/l,11}) & a\bar{A}_{a/l,12} & a\bar{A}_{a/l,13} \end{bmatrix}, \quad \bar{A}_{al} = \begin{bmatrix} \bar{A}_{al,11} & \bar{A}_{al,12} & \bar{A}_{al,13} \\ 0 & 0 & 0 \\ \pi_l\lambda + a\bar{A}_{al,11} & a\bar{A}_{al,12} & a\bar{A}_{al,13} \end{bmatrix},
$$

$$
\bar{m}_{a/l} = \begin{bmatrix} \bar{m}_{a/l,1} \\ 0 \\ a\bar{m}_{a/l,1} \end{bmatrix}, \quad \bar{A}_{la} = \begin{bmatrix} \bar{A}_{la,11} & \bar{A}_{la,12} & \bar{A}_{la,13} \\ 0 & 0 & 0 \\ \pi_a\lambda + \bar{A}_{la,11} & a\bar{A}_{la,12} & a\bar{A}_{la,13} \end{bmatrix},
$$

$$
\bar{G}_{a/l} = \begin{bmatrix} \bar{G}_{a/l,12} & \bar{G}_{a/l,22} & \bar{G}_{a/l,23} \\ 0 & 0 & 0 \\ (\lambda_0 + a\bar{G}_{a/l,21}) & a\bar{G}_{a/l,22} & a\bar{G}_{a/l,23} \end{bmatrix}.
$$

5.4 Conclusions

Hybrid MFG theory was utilized in a non-cooperative game formulation of the financial market where HFTs (minor agents) may leave the market before the final time. The best response trading and stopping policies for the agents are further shown to yield an ϵ-Nash equilibrium for the the market.

Part III

Mean Field Game Systems with Common noise and Latent Processes

Chapter 6

Convex Analysis for LQG Systems with Applications to Major Minor LQG Mean Field Game Systems

6.1 Introduction

In the literature, various approaches such as calculus of variations, (stochastic) maximum principle, dynamic programming, and change of functional have been used to address deterministic linear quadratic (LQ) and stochastic linear quadratic (LQG) optimal control problems [36, 56–58].

In a convex analysis approach to optimization for static systems, the Gâteaux derivative of the functional to be optimized is used to solve the problem (see e.g., [59], [60]). In [61], the relationship between the Gâteaux derivative of the cost functional of a dynamic system and its Hamiltonian is established. A stochastic tracking problem in finance is studied in [62] using the convex analysis approach, while an algorithmic trading problem is investigated in [63] and the best response trading strategies are obtained for a large number of heterogeneous traders using the convex analysis approach.

In this work, a convex analysis method is used to rederive the solutions to LQG optimal control problems. Then the methodology is applied to major minor LQG mean field game (MM LQG MFG) systems to retrieve the best response strategies for the major agent and each individual minor agent addressed in [16].

6.2 Convex Analysis

Let V be a reflexive Banach space with the dual space V^* and $\mathcal{V}$ be a non-empty closed convex subset of V.

Definition 6.1 (Gâteaux Derivative)**.** The function J defined on a neighbourhood of $u \in V$ with values in $\mathbb{R}$ is differentiable in the sense of Gâteaux at u in the direction of ω, if there exists $J'(u) \in V^*$ such that

$$\langle J'(u), \omega \rangle = \lim_{\epsilon \to 0} \frac{J(u + \epsilon\omega) - J(u)}{\epsilon}. \tag{6.1}$$

The function $J'(u)$ is called the Gâteaux derivative of J at u.

Theorem 6.1 (Euler Inequality)**.** *Assume that the function J is convex, continuous, proper, and Gâteaux differentiable with continuous derivative $J'(u)$. Then*

$$J(u) = \inf_{v \in \mathcal{V}} J(v), \tag{6.2}$$

if and only if

$$\langle J'(u), v - u \rangle \geq 0, \quad \forall v \in \mathcal{V}. \tag{6.3}$$

□

Proof of *Theorem 6.1* may be found in [59] and [60].

Remark 6.1 (Euler Equality). In the case where $\mathcal{V} = V$, $\omega = v - u$ produces the whole space of V, and therefore (6.3) reduces to Euler equality

$$\langle J'(u), \omega \rangle = 0, \quad \forall \omega \in V, \tag{6.4}$$

which implies that

$$J'(u) = 0. \tag{6.5}$$

We note that the Banach space under consideration in this paper is the space of square-integrable $\mathbb{R}^m$-valued measurable functions which will be specified in more detail in the next sections.

6.3 Single-Agent LQG Problems

In this section, the solutions to single-agent LQG problems are rederived using a convex analysis method.

6.3.1 Dynamics

Consider single-agent LQG systems with governing dynamics

$$dx_t = (Ax_t + Bu_t + b(t))dt + \sigma(t)dw_t, \tag{6.6}$$

where $t \geq 0$, the continuous processes $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$, and $w_t \in \mathbb{R}^r$ denote, respectively, the state, the control action, and a standard Wiener process. Moreover, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $b(t) \in \mathbb{R}^n$, $\sigma(t) \in \mathbb{R}^{n\times r}$, are deterministic continuous functions of time.

Control σ-Fields

We denote by $\mathcal{F} := (\mathcal{F}_t)_{t\in[0,T]}$ the natural filtration generated by the agent's state $(x_t)_{t\in[0,T]}$. Then, we introduce the admissible control set $\mathcal{U}$ to be the set of feedback control laws $u = (u_t)_{t\in[0,T]}$ which are $\mathcal{F}$-adapted $\mathbb{R}^m$-valued continuous processes such that $\mathbb{E}[\int_0^T u_t^T u_t dt] < \infty$, for any finite T.

6.3.2 Cost Functional

The cost functional to be minimized is given by

$$J(u) = \frac{1}{2}\mathbb{E}\Big[e^{-\rho T} x_T^T G x_T + \int_0^T e^{-\rho t}\Big\{x_t^T Q x_t + 2x_t^T N u_t + u_t^T R u_t - 2x_t^T \eta - 2u_t^T n\Big\}dt\Big], \tag{6.7}$$

where ρ denotes the discount rate.

Assumption 6.1. *For the cost functional* (6.7) *to be convex, it is assumed that* $G \geq 0$, $R > 0$, *and* $Q - NR^{-1}N^T > 0$.

6.3.3 Optimal Control Action

The system dynamics (6.6) together with the cost functional (6.7) constitute an LQ stochastic optimal control problem, which is solved for using the following theorem.

Theorem 6.2. *(Gâteaux Derivative of Cost for LQG Systems) For system* (6.6)-(6.7)*, the Gâteaux* *derivative of the cost functional is given by*

$$\langle J'(u), \omega \rangle = \mathbb{E}\Big[\int_0^t \omega_t^T \Big\{ e^{-\rho t} N^T x_t^u + e^{-\rho t} R u_t - e^{-\rho t} n \\ + B^T \Big(e^{-A^T t} M_t - \int_0^t e^{-\rho s} e^{A^T(s-t)} (Q x_s^u + N u_s - \eta) ds \Big) \Big\} dt \Big], \tag{6.8}$$

where M_t is a martingale process given by

$$M_t = \mathbb{E}\Big[e^{-\rho T} e^{A^T T} G x_T^u + \int_0^T e^{-\rho s} e^{A^T s} (Q x_s^u + N u_s - \eta) ds \Big| \mathcal{F}_t \Big]. \tag{6.9}$$

□

Proof. The Gâteaux derivative $J'(u)$ of (6.7) is computed as follows.

The solution x_t^u to the state representation of the system (6.6) subject to the control action u_t is given by

$$x_t^u = e^{At} x_0 + \int_0^t e^{A(t-s)} \big(B u_s + b(s) \big) ds + \int_0^t e^{A(t-s)} \sigma(s) dw_s, \tag{6.10}$$

where $x_0 \in \mathbb{R}^n$ and $\phi(t,s) = e^{A(t-s)}, s \leq t \leq T$, denote, respectively, the initial state and the state transition matrix for the system (6.6).

Let $x_t^{u+\epsilon\omega}$ denote the solution to (6.6) subject to a perturbed control action $u_t + \epsilon\omega_t$ in the direction of $\omega_t \in \mathcal{U}$ given by

$$x_t^{u+\epsilon\omega} = e^{At} x_0 + \int_0^t e^{A(t-s)} \big(B u_s + b(s) \big) ds + \int_0^t e^{A(t-s)} \sigma(s) dw_s + \epsilon \int_0^t e^{A(t-s)} B \omega_s ds \tag{6.11}$$

To find the relation between x_t^u and $x_t^{u+\epsilon\omega}$, (6.10) is substituted in (6.11) which yields

$$x_t^{u+\epsilon\omega} = x_t^u + \epsilon \int_0^t e^{A(t-s)} B \omega_s ds. \tag{6.12}$$

Then by differentiating both sides of (6.12), the evolution of $x^{u+\epsilon\omega}(t)$ in terms of $x^u(t)$ is given by

$$dx_t^{u+\epsilon\omega} = dx_t^u + \epsilon B \omega_t dt + \epsilon A \int_0^t e^{A(t-s)} B \omega_s ds. \tag{6.13}$$

The cost induced by the perturbed control action $u_t + \epsilon\omega_t$ and, subsequently, the perturbed state

$x_t^{u+\epsilon\omega}$ is given by

$$J(u+\epsilon\omega) = \frac{1}{2}\mathbb{E}\Big[e^{-\rho T}(x_T^{u+\epsilon\omega})^T G x_T^{u+\epsilon\omega} + \int_0^T e^{-\rho s}\Big\{(x_s^{u+\epsilon\omega})^T Q x_s^{u+\epsilon\omega} \\ + 2(x_s^{u+\epsilon\omega})^T N(u_s+\epsilon\omega_s) + (u_s+\epsilon\omega_s)^T R(u_s+\epsilon\omega_s) - 2(x_s^{u+\epsilon\omega})^T\eta - 2(u_s+\epsilon\omega_s)^T n\Big\}ds\Big], \tag{6.14}$$

where the terminal cost, by utilizing the integration by parts technique for Itô processes [64], can be presented in integral form as

$$e^{-\rho T}(x_T^{u+\epsilon\omega})^T G x_T^{u+\epsilon\omega} = (x_0)^T G x_0 + \int_0^T d\big(e^{-\rho s}(x_s^{u+\epsilon\omega})^T G x_s^{u+\epsilon\omega}\big) \\ = (x_0)^T G x_0 - \rho\int_0^T e^{-\rho s}(x_s^{u+\epsilon\omega})^T G x_s^{u+\epsilon\omega} ds + 2\int_0^T e^{-\rho s}(x_s^{u+\epsilon\omega})^T G dx_s^{u+\epsilon\omega} \\ + \int_0^T e^{-\rho s}\sigma(s)^T G\sigma(s) ds. \tag{6.15}$$

To write $J(u+\epsilon\omega)$ in terms of $J(u)$, u_t and x_t^u, first (6.15), and then (6.12)-(6.13) are substituted in (6.14) which gives rise to

$$J(u+\epsilon\omega) = J(u) + \mathbb{E}\Big[\epsilon\int_0^T e^{-\rho s}\Big\{\big(\int_0^s e^{A(s-t)}B\omega_t dt\big)^T\big(G dx_s^u \\ + (Qx_s^u + Nu_s + A^T G X_s^u - \rho G x_s^u - \eta)ds\big) + \big((x_s^u)^T N\omega_s + (x_s^u)^T G B\omega_s \\ + (u_s)^T R\omega_s - n^T\omega_s\big)ds\Big\} + \epsilon^2\int_0^T e^{-\rho s}\Big\{\big(\int_0^s e^{A(s-t)}B\omega_t dt\big)^T\big(GA\int_0^s e^{A(s-t)}B\omega_t dt \\ - \rho G\int_0^s e^{A(s-t)}B\omega_t dt + GB\omega_s + Q\int_0^s e^{A(s-t)}B\omega_t dt + N\omega_s\big) + (\omega_s)^T R\omega_s\big)\Big\}ds\Big]. \tag{6.16}$$

Then the Gâteaux derivative of $J'(u)$ in the direction of ω is obtained by first taking $J(u)$ to the left hand side of (6.16), then dividing both sides of the equation by ϵ, and finally taking the limit as $\epsilon \to 0$, which yields

$$\langle J'(u), \omega\rangle = \mathbb{E}\Big[\int_0^T e^{-\rho s}\Big\{\big(\int_0^s e^{A(s-t)}B\omega_t dt\big)^T\big(G dx_s^u + (Qx_s^u + Nu_s + A^T G x_s^u \\ - \rho G x_s^u - \eta)ds\big) + \big((x_s^u)^T N\omega_s + (x_s^u)^T G B\omega_s + (u_s)^T R\omega_s - n^T\omega_s\big)ds\Big\}\Big]. \tag{6.17}$$

Given that the processes in (6.17) are $\mathcal{F}$-measurable, continuous and bounded on the interval $[0,T]$, the conditions of the stochastic Fubini's theorem hold [65]. Subsequently an application of Fubini's theorem to change the order of integration in (6.17) results in

$$\langle J'(u),\omega\rangle = \mathbb{E}\bigg[\int_0^T \omega_t^T\bigg\{e^{-\rho t}B^T G x_t^u + e^{-\rho t}N^T x_t^u + e^{-\rho t}Ru_t - e^{-\rho t}n \\ + B^T\int_t^T e^{-\rho s}e^{A^T(s-t)}\Big(Gdx_s^u + \big(A^T G x_s^u - \rho G x_s^u + Qx_s^u + Nu_s - \eta\big)ds\Big)\bigg\}\bigg]dt. \tag{6.18}$$

By using integration by parts again, we have

$$\int_t^T e^{-\rho s}e^{A^T(s-t)}(A^T G x_s ds + G dx_s - \rho G x_s^u ds) = \int_t^T d(e^{-\rho s}e^{A^T(s-t)}Gx_s) \\ = e^{-\rho T}e^{A^T(T-t)}Gx_T - e^{-\rho t}Gx_t, \tag{6.19}$$

whose substitution in (6.18) yields

$$\langle J'(u),\omega\rangle = \mathbb{E}\bigg[\int_0^T \omega_t^T\bigg\{e^{-\rho T}B^T e^{A^T(T-t)}Gx_T^u + e^{-\rho t}N^T x_t^u + e^{-\rho t}Ru_t - e^{-\rho t}n \\ + B^T\int_t^T e^{-\rho s}e^{A^T(s-t)}(Qx_s^u + Nu_s - \eta)ds\bigg\}dt\bigg]. \tag{6.20}$$

Using the smoothing property of conditional expectations [38], the Gâteaux derivative (6.20) may be rewritten as

$$\langle J'(u),\omega\rangle = \mathbb{E}\bigg[\int_0^T \omega_t^T\bigg\{e^{-\rho t}N^T x_t^u + e^{-\rho t}Ru_t - e^{-\rho t}n \\ + B^T\mathbb{E}\Big[e^{-\rho T}e^{A^T(T-t)}Gx_T^u + \int_t^T e^{-\rho s}e^{A^T(s-t)}(Qx_s^u + Nu_s - \eta)ds\Big|\mathcal{F}_t\Big]\bigg\}dt\bigg]. \tag{6.21}$$

Then the following martingale is defined

$$M_t = \mathbb{E}\Big[e^{-\rho T}e^{A^T T}Gx_T^u + \int_0^T e^{-\rho s}e^{A^T s}(Qx_s^u + Nu_s - \eta)ds\Big|\mathcal{F}_t\Big], \tag{6.22}$$

and is substituted in (6.21) to give

$$\langle J'(u),\omega\rangle = \mathbb{E}\bigg[\int_0^T \omega_t^T\bigg\{e^{-\rho t}N^T x_t^u + e^{-\rho t}Ru_t - e^{-\rho t}n \\ + B^T\Big(e^{-A^T t}M_t - \int_0^t e^{-\rho s}e^{A^T(s-t)}(Qx_s^u + Nu_s - \eta)ds\Big)\bigg\}dt\bigg]. \tag{6.23}$$

□

Theorem 6.3 (LQG Optimal Control Action)**.** *Given Assumption 6.1, the optimal control action for LQG systems given by* (6.6)-(6.7) *is specified by*

$$u_t^* = -R^{-1}\left[N^T x_t^* - n + B^T e^{\rho t}\Big(e^{-A^T t}M_t - \int_0^t e^{-\rho s}e^{A^T(s-t)}(Qx_s^* + Nu_s^* - \eta)ds\Big)\right]. \quad (6.24)$$

□

Proof. As per *Theorem 6.1* and *Remark 6.1*, the necessary condition for $u^*(t)$ to be the optimal control is given by

$$\langle J'(u^*), \omega\rangle = 0, \quad a.s. \text{ for all possible paths of } \omega(t) \in \mathcal{U}. \quad (6.25)$$

Moreover, since *Assumption 6.1* holds, (6.25) is the sufficient condition of optimality as well.

According to (6.8), equation (6.25) holds if and only if

$$u_t^* = -R^{-1}\left[N^T x_t^* - n + B^T e^{\rho t}\Big(e^{-A^T t}M_t - \int_0^t e^{-\rho s}e^{A^T(s-t)}(Qx_s^* + Nu_s^* - \eta)ds\Big)\right]. \quad (6.26)$$

All the processes in the right hand side of (6.26) are $\mathcal{F}$-measurable. Moreover, using the triangle inequality and Cauchy-Schwarz inequality it can be shown that $\mathbb{E}[\int_0^T u_t^{*T} u_t^* dt] < \infty$, and hence $u_t^* \in \mathcal{U}$.

Then the sufficiency condition can be shown to hold by the direct substitution of (6.26) in (6.8). The necessity condition is proved by contradiction. Let us choose $\omega_t \in \mathcal{U}$ as

$$\omega_t = e^{-\rho t}N^T x_t^* + e^{-\rho t}Ru_t^* - e^{-\rho t}n + B^T\Big(e^{-A^T t}M_t - \int_0^t e^{-\rho s}e^{A^T(s-t)}(Qx_s^* + Nu_s^* - \eta)ds\Big).$$

If we substitute (6.27) in (6.8), we have

$$\langle J'(u), \omega\rangle > 0, \quad (6.27)$$

which contradicts (6.25). □

Theorem 6.4 (LQG State Feedback Optimal Control)**.** *For LQG systems governed by* (6.6)-(6.7)*, the optimal control action is given by the linear state feedback control*

$$u_t^* = -R^{-1}\big(N^T x_t^* - n + B^T[\Pi(t)x_t^* + s(t)]\big), \quad (6.28)$$

where $\Pi(t)$ *and* $s(t)$ *are given by*

$$\dot{\Pi}(t) + \Pi(t)A + A^T\Pi(t) - \big(B^T\Pi(t) + N\big)^T R^{-1}\big(B^T\Pi(t) + N\big) + Q = 0, \quad (6.29)$$

$$\dot{s}(t) + [(A - BR^{-1}N)^T - \Pi BR^{-1}B^T]s(t) + \Pi(t)b(t) = 0. \quad (6.30)$$

with terminal conditions $\Pi(T) = G$ *and* $s(T) = 0$.

□

Proof. Let us define $p(t)$ as

$$p_t = e^{\rho t}\big(e^{-A^T t}M_t - \int_0^t e^{-\rho s}e^{A^T(s-t)}(Qx_s^* + Nu_s^* - \eta)ds\big), \tag{6.31}$$

which is the adjoint process for the system (6.6)-(6.7) in the framework of stochastic maximum principle. Then the ansatz for p_t^0 is adopted to be

$$p_t = \Pi(t)x_t^* + s(t), \tag{6.32}$$

and is substituted in (6.24) to give

$$u_t^* = -R^{-1}\big[N^T x_t^* - n + B^T\big(\Pi(t)x_t^* + s(t)\big)\big]. \tag{6.33}$$

To find $\Pi(t)$ and $s(t)$, both sides of (6.32) are first differentiated, and then (6.6) and (6.33) are substituted to yield

$$\begin{aligned} dp_t = \Big[&\big(\dot{\Pi}(t) + \Pi(t)A - \Pi(t)BR^{-1}N^T - \Pi(t)BR^{-1}B^T\Pi(t)\big)x_t^* - \Pi(t)BR^{-1}B^T s(t) \\ &+ \Pi(t)b + \Pi(t)BR^{-1}n + \dot{s}(t)\Big]dt + \Pi(t)\sigma(t)dw_t. \end{aligned} \tag{6.34}$$

Next, both sides of (6.31) are differentiated to give

$$dp_t = (\rho p_t - A^T p_t - Qx_t^* - Nu_t^* + \eta)dt + e^{\rho t}e^{-A^T t}dM_t, \tag{6.35}$$

where according to the martingale representation theorem, the martingale M_t may be written as

$$M_t = M_0 + \int_0^t Z_s dw_s, \tag{6.36}$$

and hence

$$dM_t = Z_t dw_t. \tag{6.37}$$

with Z_t being an $\mathcal{F}_t$-adapted process.

Then, equations (6.32), (6.33) and (6.37) are substituted in (6.35) to get

$$\begin{aligned} dp_t = \Big[&\big(\rho\Pi(t) - Q + NR^{-1}N^T + NR^{-1}B^T\Pi(t) - A^T\Pi(t)\big)x_t^* + \rho s(t) \\ &+ (NR^{-1}B^T - A^T)s(t) + \eta - NR^{-1}n\Big]dt + q_t dw_t, \end{aligned} \tag{6.38}$$

where $q_t = e^{\rho t}e^{-A^T t}Z_t$.

Finally, for (6.34) and (6.38) to be equal, the corresponding drifts and diffusions must be equal. Hence the following equations must hold

$$q_t = \Pi(t)\sigma(t), \tag{6.39}$$

$$\begin{cases} \rho\Pi(t) = \dot{\Pi}(t) + \Pi(t)A + A^T\Pi(t) - (\Pi(t)B + N)R^{-1}(B^T\Pi(t) + N^T) + Q, \\ \Pi(T) = G, \end{cases} \tag{6.40}$$

$$\begin{cases} \rho s(t) = \dot{s}(t) + \big[(A - BR^{-1}N^T)^T - \Pi(t)BR^{-1}B^T\big]s(t) \\ \qquad\qquad\qquad\qquad\qquad +\Pi(t)(b(t) + BR^{-1}n) + NR^{-1}n - \eta, \\ s(T) = 0, \end{cases} \tag{6.41}$$

□

Remark 6.2 (Finite Horizon LQG Systems). Typically, the cost functional for finite horizon LQG systems is not discounted, i.e. $\rho = 0$, and hence the Riccati and offset equations (6.29)-(6.30) reduce to

$$\begin{cases} -\dot{\Pi}(t) = \Pi(t)A + A^T\Pi(t) - (\Pi(t)B + N)R^{-1}(B^T\Pi(t) + N^T) + Q, \\ -\dot{s}(t) = [(A - BR^{-1}N^T)^T - \Pi(t)BR^{-1}B^T]s(t) + \Pi(t)(b(t) + BR^{-1}n) + NR^{-1}n - \eta, \end{cases} \tag{6.42}$$

subject to the terminal conditions $\Pi(T) = G$, $s(T) = 0$.

Remark 6.3 (Infinite Horizon LQG Systems). For Infinite horizon LQG systems where the terminal time T in (6.7) is set to infinity, the terminal cost becomes zero. Hence, the infinite horizon cost functional is given by

$$J(u) = \frac{1}{2}\mathbb{E}\Big[\int_0^\infty e^{-\rho t}\Big\{x_t^TQx_t + 2x_t^TNu_t + u_t^TRu_t - 2x_t^T\eta - 2u_t^Tn\Big\}dt\Big], \tag{6.43}$$

The dynamics (6.6) remains the same in the infinite horizon LQG systems.

Assumption 6.2. *The pair* $(L, A - (\rho/2)I)$ *is detectable where* $L = Q^{1/2}$.

Assumption 6.3. *The pair* $(A - (\rho/2)I, B)$ *is stabilizable.*

Given that *Assumptions 6.7-6.8* hold, for infinite horizon LQG systems governed by (6.6) and (6.43), the optimal control action is given by (6.28), where the steady state Riccati matrix Π

satisfies an algebraic Riccati equation given by

$$\rho\Pi = \Pi A + A^T\Pi - (\Pi B + N)R^{-1}(B^T\Pi + N^T) + Q, \tag{6.44}$$

and the steady state offset vector s_0 satisfies the differential equation

$$\rho s(t) = \dot{s}(t) + [(A - BR^{-1}N^T)^T - \Pi BR^{-1}B^T]s(t) + \Pi(M(t) + BR^{-1}n) + NR^{-1}n - \eta. \tag{6.45}$$

6.4 Major Minor LQG Mean Field Game Systems

In this section, the convex analysis method introduced in Section 6.3 is utilized to rederive the best response strategies for major minor LQG MFG problems addressed in [16]. A large population N of minor agents with a major agent, where agents are subject to stochastic linear dynamics and quadratic cost functionals are considered. Each agent is coupled with other agents through their dynamics and cost functional with the average state of minor agents, i.e. the empirical mean field.

6.4.1 Dynamics

The dynamics of the major and minor agents are assumed to be given, respectively, by

$$dx_t^0 = [A_0x_t^0 + F_0x_t^{(N)} + B_0u_t^0 + b_0(t)]dt + \sigma_0 dw_t^0, \tag{6.46}$$
$$dx_t^i = [A_kx_t^i + F_kx_t^{(N)} + B_ku_t^i + b_k(t)]dt + \sigma_k dw_t^i, \tag{6.47}$$

where $t \geq 0$, $i \in \mathfrak{N}$, $\mathfrak{N} = \{1, \dots, N\}$, $N < \infty$, and the subscript k, $k \in \mathfrak{K}$, $\mathfrak{K} = \{1, \dots, K\}$, $K \leq N$, denotes the type of a minor agent. Here $x_t^i \in \mathbb{R}^n$, $i \in \mathfrak{N}_0$, $\mathfrak{N}_0 = \{0, \dots, N\}$, are the states, $u_t^i \in \mathbb{R}^m$, $i \in \mathfrak{N}_0$ are the control inputs, $\{w_t^i,\ i \in \mathfrak{N}_0\}$ denotes $(N+1)$ independent standard Wiener processes in $\mathbb{R}^r$, where w_i is progressively measurable with respect to the filtration $\mathcal{F}^w := (\mathcal{F}_t^w)_{t\in[0,T]}$. All matrices in (6.46) and (6.47) are constant and of appropriate dimension; vectors $b_0(t)$, and $b_k(t)$ are deterministic functions of time.

Agents types

Minor agents are given in K distinct types with $1 \leq K < \infty$. The notation

$$\Psi_k \triangleq \Psi(\theta_i), \quad \theta_i = k$$

is introduced where $\theta_i \in \Theta$, with Θ being the parameter set, and Ψ may be any dynamical parameter in (6.47) or wight matrix in the cost functional (6.50). The symbol $\mathcal{I}_k$ denotes

$$\mathcal{I}_k = \{i : \theta_i = k,\ i \in \mathfrak{N}\}, \quad k \in \mathfrak{K}$$

where the cardinality of $\mathcal{I}_k$ is denoted by $N_k = |\mathcal{I}_k|$. Then, $\pi^N = (\pi_1^N, ..., \pi_K^N)$, $\pi_k^N = \frac{N_k}{N}$, $k \in \mathfrak{K}$, denotes the empirical distribution of the parameters $(\theta_1, ..., \theta_N)$ sampled independently of the initial conditions and Wiener processes of the agents $\mathcal{A}_i, i \in \mathfrak{N}$. The first assumption is as follows.

Assumption 6.4. *There exists* π *such that* $\lim_{N\to\infty} \pi^N = \pi$ *a.s.*

Control σ-Fields

We denote by $\mathcal{F}^i := (\mathcal{F}_t^i)_{t\in[0,T]}$, $i \in \mathfrak{N}$, the natural filtration generated by the i-th minor agent's state $(x_t^i)_{t\in[0,T]}$, by $\mathcal{F}^0 := (\mathcal{F}_t^0)_{t\in[0,T]}$ the natural filtration generated by the major agent's state $(x_t^0)_{t\in[0,T]}$, and $\mathcal{F}^g := (\mathcal{F}_t^g)_{t\in[0,T]}$ the natural filtration generated by the states of all agents $((x_t^i)_{i\in\mathfrak{N}}, x_t^0)_{t\in[0,T]}$.

Next, we introduce three admissible control sets. Let $\mathcal{U}^0$ denote the set of feedback control laws $u_{(.)}^0$ which are adapted to the local information set of the major agent $\mathcal{A}_0$, i.e. $\mathcal{F}^0$ such that $\mathbb{E}[\int_0^T (u_t^0)^T u_t^0 dt] < \infty$, for any finite T. The set of control inputs $\mathcal{U}^i$, $i \in \mathfrak{N}$, based upon the local information set of the minor agent $\mathcal{A}_i$, $i \in \mathfrak{N}$, consists of the feedback control laws adapted to the filtration $\mathcal{F}^{i,r} := (\mathcal{F}_t^{i,r})_{t\in[0,T]}$, where $\mathcal{F}^{i,r} := \mathcal{F}^i \vee \mathcal{F}^0$, $i \in \mathfrak{N}$, and $\mathbb{E}[\int_0^T (u_t^i)^T u_t^i dt] < \infty$, $i \in \mathfrak{N}$, for any finite T. The set of control inputs $\mathcal{U}_g^N$ consists of feedback control laws $u_{(.)}$ which are adapted to the general filtration $\mathcal{F}^g := (\mathcal{F}_t^g)_{t\in[0,T]}$, $\mathcal{F}^g := \vee_{i\in\mathfrak{N}_0}\mathcal{F}^i$, such that $\mathbb{E}[\int_0^T u_t^T u_t dt] < \infty$, for any finite T.

6.4.2 Cost functionals

The individual (finite) large population finite horizon cost functional for the major agent is specified by

$$J_0^N(u^0, u^{-0}) = \frac{1}{2}\mathbb{E}\Big[\|x_T^0 - \Phi(x_T^{(N)})\|_{G_0}^2 + \int_0^T \Big\{\|x_t^0 - \Phi(x_t^{(N)})\|_{Q_0}^2 + 2\big(x_t^0 - \Phi(x_t^{(N)})\big)^T N_0 u_t^0 + \|u_t^0\|_{R_0}^2\Big\}dt\Big], \quad (6.48)$$

where

$$\Phi(.) := H_0 x_t^{(N)} + \eta_0. \tag{6.49}$$

Assumption 6.5. *For the cost functional* (6.48) *to be convex, we assume that* $G_0 \geq 0$, $R_0 > 0$, *and* $Q_0 - N_0 R_0^{-1} N_0^T > 0$.

The individual (finite) large population finite horizon cost functional for a minor agent $\mathcal{A}_i, i \in \mathfrak{N}$, is specified as

$$J_i^N(u^i, u^{-i}) = \frac{1}{2}\mathbb{E}\Big[\|x_T^i - \Psi(x_T^{(N)})\|_{G_k}^2 + \int_0^T \Big\{\|x_t^i - \Psi(x_t^{(N)})\|_{Q_k}^2 + 2\big(x_t^i - \Psi(x_t^{(N)})\big)^T N_k u_t^i + \|u_t^i\|_{R_k}^2\Big\}dt\Big], \tag{6.50}$$

where

$$\Psi(.) \ := \ H_k x_t^0 + \hat{H}_k^\pi x_t^{(N)} + \eta_k. \tag{6.51}$$

Assumption 6.6. *For the cost functional* (6.50) *to be convex, we assume that* $G_k \geq 0$, $R_k > 0$, *and* $Q_k - N_k R_k^{-1} N_k^T > 0$ *for* $k \in \mathfrak{K}$.

We note that the major agent $\mathcal{A}_0$ and minor agents $\mathcal{A}_i$, $i \in \mathfrak{N}$ are coupled with each other through the average term $x_t^{(N)} = \frac{1}{N}\sum_{i=1}^N x_t^i$ in their dynamics and cost functionals given by, respectively, (6.46)-(6.47) and (6.48)-(6.50).

6.4.3 Solutions to Major Minor LQG MFG Problems

Following the mean field game methodology with a major agent [42], [16], the problem is first solved in the infinite population case where the average terms in the finite population dynamics and cost functional of each agent are replaced with their infinite population limit, i.e. the mean field. Then specializing to LQG MFG systems, the major agent's state is extended with the mean field, while the minor agent's state is extended with the major agent's state, and mean field; this yields stochastic optimal control problems for each agent linked only through the major agent's state and mean field. Finally the infinite population best response strategies are applied to the finite population system which yields an ϵ-Nash equilibrium [16]. The following theorem (a more general version of the theorem in [16]) specifies the control laws which yield the infinite population Nash equilibrium and their relation with the finite population ϵ-Nash equilibrium.

Theorem 6.5 (ϵ-Nash Equilibrium for LQG MFG Systems)**.** *Assume that the conditions of [16] for the existence and uniqueness of Nash equilibrium hold, then the system equations*

(6.46)-(6.50) *together with the mean field equations* (6.77)-(6.78) *generate a set of control laws* $\mathcal{U}_{MF}^{\infty} \triangleq \{u^{i,*}; i \geq 0\}$ *where* $u_t^{i,*}$ *is given by*

$$u_t^{0,*} = -R_0^{-1}\Big[\big(\mathbb{N}_0^T + \mathbb{B}_0^T\Pi_0(t)\big)\big[(x_t^0)^T, (\bar{x}_t)^T\big]^T + \mathbb{B}_0^T s_0(t) - \bar{n}_0\Big], \tag{6.52}$$

$$u_t^{i,*} = -R_k^{-1}\Big[\big(\mathbb{N}_k^T + \mathbb{B}_k^T\Pi_k(t)\big)\big[(x_t^i)^T, (x_t^0)^T, (\bar{x}_t)^T\big]^T + \mathbb{B}_k^T s_k(t) - \bar{n}_k\Big], \tag{6.53}$$

such that

(i) the set of infinite population control laws $\mathcal{U}_{MF}^{\infty} \triangleq \{u^{i,*}; i \geq 0\}$ *yields the infinite population Nash equilibrium.*

$$J_i^{\infty}(u^{i,*}, u^{-i,*}) = \inf_{u^i \in \mathcal{U}_i^{\infty,L}} J_i^{\infty}(u^i, u^{-i,*});$$

(ii) All agent systems $\mathcal{A}_i,\ i \in \mathfrak{N}_0$, *are second order stable.*

(iii) the set of control laws $\mathcal{U}_{MF}^{N} \triangleq \{u^{i,*}; i \in \mathfrak{N}_0\}$, $1 \leq N < \infty$, *yields an* ϵ*-Nash equilibrium for all* ϵ, *i.e. for all* $\epsilon > 0$, *there exists* $N(\epsilon)$ *such that for all* $N \geq N(\epsilon)$*;*

$$J_i^N(u^{i,*}, u^{-i,*}) - \epsilon \leq \inf_{u^i \in \mathcal{U}_i^{N,L}} J_i^N(u^i, u^{-i,*}) \leq J_i^N(u^{i,*}, u^{-i,*}).$$

$\square$

The proof of *Theorem 6.5* consists of two parts: (i) the set of control laws $\mathcal{U}_{MF}^{\infty}$ yields the Nash equilibrium for the infinite population system, (ii) when a finite subset of the control laws $\mathcal{U}_{MF}^{N}$ is applied to the finite population system, all agent systems are second order stable and it yields an ϵ-Nash equilibrium. In this section, a novel convex analysis approach is presented to retrieve the set of best response strategies $\mathcal{U}_{MF}^{\infty}$ which yields the Nash equilibrium.

Mean Field Evolution

We introduce the empirical state average as

$$x_k^{(N_k)} = \frac{1}{N_k}\sum_{j=1}^{N_k} x_j^k, \qquad k \in \mathfrak{K},$$

and write $x^{(N)} = [x_1^{(N_1)}, x_2^{(N_2)}, ..., x_K^{(N_K)}]$, where the pointwise in time L^2 limit of $x^{(N)}$, if it exists, is called the mean field of the system and is denoted by $\bar{x} = [\bar{x}^1, ..., \bar{x}^K]$. We consider for each minor agent $\mathcal{A}_i$ of type k, $k \in \mathfrak{K}$, a uniform (with respect to i) feedback control $u_i^k \in \mathcal{U}_{i,L} \subset \mathcal{U}_i$, where $U_{i,L}$ consists of linear time invariant controls, as

$$u_t^{i,k} = L_1^k x_t^{i,k} + \Sigma_{l=1}^K \Sigma_{j=1}^{N_l} L_2^{k,l} x_t^{j,l} + L_3^k x_t^0 + m^k(t),$$

where $0 \le t \le \infty,\ L_1^k,\ L_2^{k,l}$ and L_3^k are constant matrices, and $m^k(t)$ is a continuous bounded function of time. If we substitute $u_t^{i,k}$ in (6.47) for $i \in \mathfrak{N}$, and take the average of the states of closed loop systems of type $k,\ k \in \mathfrak{K}$, and hence calculate $x_t^{(N)}$, it can be shown that the L^2 limit $\bar{x}_t$ of $x_t^{(N)}$, i.e. the mean field satisfies

$$d\bar{x}_t = \bar{A}\bar{x}_t dt + \bar{G}x_t^0 dt + \bar{m}(t)dt, \tag{6.54}$$

where $\bar{A}$, $\bar{G}$, and $\bar{m}$ are to be solved for in the tracking solution. By abuse of language, the mean value of the system's Gaussian mean field given by the state process $\bar{x}_t = [\bar{x}_t^1, ..., \ \bar{x}_t^K]$ shall also be termed the system's mean field.

Major Agent: Infinite Population

To solve the infinite population tracking problem for the major agent $\mathcal{A}_0$, first, its state is extended with the mean field process $\bar{x}_t$, where this is assumed to exist. Then the dynamics of major agent's extended state $X_t^0 \ \triangleq \ \left[(x_t^0)^T,\ (\bar{x}_t)^T\right]^T$ is given as (see [16])

$$dX_t^0 = \mathbb{A}_0 X_t^0 dt + \mathbb{B}_0 u_t^0 dt + \mathbb{M}_0(t)dt + \Sigma_0 dW_t^0, \tag{6.55}$$

where

$$\mathbb{A}_0 = \begin{bmatrix} A_0 & F_0 \\ \bar{G} & \bar{A} \end{bmatrix}, \mathbb{B}_0 = \begin{bmatrix} B_0 \\ 0 \end{bmatrix}, \mathbb{M}_0(t) = \begin{bmatrix} b_0(t) \\ \bar{m}(t) \end{bmatrix}, \Sigma_0 = \begin{bmatrix} \sigma_0 & 0 \\ 0 & 0 \end{bmatrix}, W_t^0 = \begin{bmatrix} w_t^0 \\ 0 \end{bmatrix}. \tag{6.56}$$

The infinite population individual cost functional for the major agent is given by

$$J_0^\infty(u^0) = \frac{1}{2}\mathbb{E}\Bigg[(X_T^0)^T \mathbb{G}_0 X_T^0 + \int_0^T \Big\{(X_s^0)^T \mathbb{Q}_0 X_s^0 + 2(X_s^0)^T \mathbb{N}_0 u_s^0 + (u_s^0)^T R_0 u_s^0 \\ - 2(X_s^0)^T \bar{\eta}_0 - 2(u_s^0)^T \bar{n}_0\Big\} ds\Bigg], \tag{6.57}$$

where the corresponding weight matrices are specified by

$$\mathbb{G}_0 = [I_n, -H_0^\pi]^T G_0 [I_n, -H_0^\pi], \quad \mathbb{Q}_0 = [I_n, -H_0^\pi]^T Q_0 [I_n, -H_0^\pi], \\ \mathbb{N}_0 = [I_n, -H_0^\pi]^T N_0, \quad \bar{\eta}_0 = [I_n, -H_0^\pi]^T Q_0 \eta_0, \quad \bar{n}_0 = N_0^T \eta_0. \tag{6.58}$$

The dynamics (6.55) together with the cost functional (6.57) constitute a stochastic LQ optimal control problem for the major agent $\mathcal{A}_0$'s extended system in the infinite population limit. To determine the optimal control $u_t^{0,*}$, first *Theorem 6.2* (with $\rho = 0$) is utilized to compute the Gâteaux derivative $J_0^{\infty\prime}(u^0)$ of (6.57) in the direction of $\omega_t^0 \in \mathcal{U}^0$ as in

$$\begin{aligned}\langle J_0^{\infty\prime}(u^0), \omega^0\rangle = \mathbb{E}\Big[\int_0^T (\omega_t^0)^T \Big\{ \mathbb{N}_0^T X_t^{0,u} + R_0 u_t^0 - \bar{n}_0 \\ + \mathbb{B}_0^T \Big(e^{-\mathbb{A}_0^T t} M_t^0 - \int_0^t e^{\mathbb{A}_0^T (s-t)} (\mathbb{Q}_0 X_s^{0,u} + \mathbb{N}_0 u_s^0 - \bar{\eta}_0) ds \Big) \Big\} dt \Big], \quad (6.59)\end{aligned}$$

where

$$M_t^0 = \mathbb{E}\Big[e^{\mathbb{A}_0^T T} \mathbb{G}_0 X_T^{0,u} + \int_0^T e^{\mathbb{A}_0^T s} (\mathbb{Q}_0 X_s^{0,u} + \mathbb{N}_0 u_s^0 - \bar{\eta}_0) ds \Big| \mathcal{F}_t^0 \Big]. \quad (6.60)$$

Then, as per *Theorem 6.3*, the optimal control action for the major agent's extended system (6.55)-(6.58) in the infinite population limit is given by

$$u_t^{0,*} = -R_0^{-1} \Big[\mathbb{N}_0^T X_t^{0,*} - \bar{n}_0 + \mathbb{B}_0^T \Big(e^{-\mathbb{A}_0^T t} M_t^0 - \int_0^t e^{\mathbb{A}_0^T (s-t)} (\mathbb{Q}_0 X_s^{0,*} + \mathbb{N}_0 u_s^{0,*} - \bar{\eta}_0) ds \Big) \Big], \quad (6.61)$$

Finally, using *Theorem 6.4*, (6.61) can be written in the state feedback form as

$$u_t^{0,*} = -R_0^{-1} \big[\mathbb{N}_0^T X_t^0 - \bar{n}_0 + \mathbb{B}_0^T \big(\Pi_0(t) X_t^0 + s_0(t) \big) \big], \quad (6.62)$$

where

$$\begin{cases} -\dot{\Pi}_0(t) = \Pi_0(t)\mathbb{A}_0 + \mathbb{A}_0^T \Pi_0(t) - (\Pi_0(t)\mathbb{B}_0 + \mathbb{N}_0) R_0^{-1} (\mathbb{B}_0^T \Pi_0(t) + \mathbb{N}_0^T) + \mathbb{Q}_0, \\ \Pi_0(T) = \mathbb{G}_0, \end{cases} \quad (6.63)$$

$$\begin{cases} -\dot{s}_0(t) = [(\mathbb{A}_0 - \mathbb{B}_0 R_0^{-1} \mathbb{N}_0^T)^T - \Pi_0(t)\mathbb{B}_0 R_0^{-1} \mathbb{B}_0^T] s_0(t) \\ \qquad\qquad + \Pi_0(t)(\mathbb{M}_0(t) + \mathbb{B}_0 R_0^{-1} \bar{n}_0) + \mathbb{N}_0 R_0^{-1} \bar{n}_0 - \bar{\eta}_0, \\ s_0(T) = 0. \end{cases} \quad (6.64)$$

Minor Agent: Infinite Population

To solve the infinite population tracking problem for a minor agent $\mathcal{A}_i$, $i \in \mathfrak{N}$, first, its state is extended with the major agent's state and the mean field process $\bar{x}_t$, where this is assumed to exist. Then the dynamics of minor agent $\mathcal{A}_i$'s extended state $X_t^i \triangleq \big[(x_t^i)^T,\ (x_t^0)^T,\ (\bar{x}_t)^T \big]^T$ is given as (see [16])

$$dX_t^i = \mathbb{A}_k X_t^i dt + \mathbb{B}_k u_t^i dt + \mathbb{M}_k(t)dt + \Sigma_k dW_t^i, \tag{6.65}$$

where

$$\mathbb{A}_k = \begin{bmatrix} A_k & [H_k, F_k^\pi] \\ 0 & \mathbb{A}_0 - \mathbb{B}_0 R_0^{-1}\mathbb{N}_0 - \mathbb{B}_0 R_0^{-1}\mathbb{B}_0^T \Pi_0 \end{bmatrix}, \quad \mathbb{B}_k = \begin{bmatrix} B_k \\ 0 \end{bmatrix},$$
$$\mathbb{M}_k(t) = \begin{bmatrix} b_k(t) \\ \mathbb{M}_0(t) - \mathbb{B}_0 R_0^{-1}\mathbb{B}_0^T s_0(t) \end{bmatrix}, \quad \Sigma_k = \begin{bmatrix} \sigma_k & 0 \\ 0 & \Sigma_0 \end{bmatrix}, \quad W_t^i = \begin{bmatrix} w_t^i \\ W_t^0 \end{bmatrix}. \tag{6.66}$$

The infinite population individual cost functional for minor agent $\mathcal{A}_i,\ 1 \le i \le N$, is given by

$$J_i^\infty(u^i) = \frac{1}{2}\mathbb{E}\Big[(X_T^i)^T \mathbb{G}_k X_T^i + \int_0^T \Big\{(X_s^i)^T \mathbb{Q}_k X_s^i + 2(X_s^i)^T \mathbb{N}_k u_s^i + (u_s^i)^T R_k u_s^i - 2(X_s^i)^T \bar{\eta}_k - 2(u_s^i)^T \bar{n}_k\Big\}ds\Big], \tag{6.67}$$

where the corresponding weight matrices are specified by

$$\mathbb{G}_k = [I_n, -H_k, -\hat{H}_k^\pi]^T G_k [I_n, -H_k, -\hat{H}_k^\pi], \quad \mathbb{Q}_k = [I_n, -H_k, -\hat{H}_k^\pi]^T Q_k [I_n, -H_k, -\hat{H}_k^\pi],$$
$$\mathbb{N}_k = [I_n, -H_k, -\hat{H}_k^\pi]^T N_0, \quad \bar{\eta}_k = [I_n, -H_k, \hat{H}_k^\pi]^T Q_k \eta_k, \quad \bar{n}_k = N_k^T \eta_k. \tag{6.68}$$

The dynamics (6.65) together with the cost functional (6.67) constitute a stochastic LQ optimal control problem for the minor agent $\mathcal{A}_i$'s extended system in the infinite population limit. To determine the optimal control $u_t^{i,*}$ for minor agent $\mathcal{A}_i, 1 \le i \le N$, first, using *Theorem 6.2*, the Gâteaux derivative $J_i^{\infty\prime}(u^i)$ of (6.67) in the direction of ω_t^i, where $\omega_t^i \in \mathcal{U}^i$, is computed as

$$\langle J_i^{\infty\prime}(u^i), \omega^i\rangle = \mathbb{E}\Big[\int_0^T (\omega_t^i)^T \Big\{\mathbb{N}_k^T X_t^{i,u} + R_k u_t^i - \bar{n}_k + \mathbb{B}_k^T\Big(e^{-\mathbb{A}_k^T t} M_t^i - \int_0^t e^{\mathbb{A}_k^T (s-t)}(\mathbb{Q}_k X_s^{i,u} + \mathbb{N}_k u_s^i - \bar{\eta}_k)ds\Big)\Big\}dt\Big], \tag{6.69}$$

where

$$M_t^i = \mathbb{E}\Big[e^{\mathbb{A}_k^T T}\mathbb{G}_k X_T^{i,u} + \int_0^T e^{\mathbb{A}_k^T s}(\mathbb{Q}_k X_s^{i,u} + \mathbb{N}_k u_s^i - \bar{\eta}_k)ds\Big|\mathcal{F}_t^i\Big]. \tag{6.70}$$

Then according to *Theorem 6.3*, the optimal control action for minor agent $\mathcal{A}_i,\ i \in \mathfrak{N}$, is given by

$$u_t^{i,*} = -R_k^{-1}\Big[\mathbb{N}_k^T X_t^{i,*} - \bar{n}_k + \mathbb{B}_k^T\Big(e^{-\mathbb{A}_k^T t}M_t^i - \int_0^t e^{\mathbb{A}_k^T(s-t)}(\mathbb{Q}_k X_s^{i,*} + \mathbb{N}_k u_s^{i,*} - \bar{\eta}_k)ds\Big)\Big]. \tag{6.71}$$

Finally, using *Theorem 6.4*, the control action (6.71) can be presented in linear state feedback form as

$$u_t^{i,*} = -R_k^{-1}\big[\mathbb{N}_k^T X_t^i - \bar{n}_k + \mathbb{B}_k^T\big(\Pi_k(t)X_t^i + s_k(t)\big)\big], \tag{6.72}$$

where

$$\begin{cases} -\dot{\Pi}_k(t) = \Pi_k(t)\mathbb{A}_k + \mathbb{A}_k^T\Pi_k(t) - (\Pi_k(t)\mathbb{B}_k + \mathbb{N}_k)R_k^{-1}(\mathbb{B}_k^T\Pi_k(t) + \mathbb{N}_k^T) + \mathbb{Q}_k, \\ \Pi_k(T) = \mathbb{G}_k, \end{cases} \tag{6.73}$$

$$\begin{cases} -\dot{s}_k(t) = [(\mathbb{A}_k - \mathbb{B}_k R_k^{-1}\mathbb{N}_k^T)^T - \Pi_k(t)\mathbb{B}_k R_k^{-1}\mathbb{B}_k^T]s_k(t) \\ \qquad\qquad + \Pi_k(t)(\mathbb{M}_k(t) + \mathbb{B}_k R_k^{-1}\bar{n}_k) + \mathbb{N}_k R_k^{-1}\bar{n}_k - \bar{\eta}_k, \\ s_k(T) = 0. \end{cases} \tag{6.74}$$

Mean Field Consistency Conditions

To obtain the consistency conditions, we substitute (6.72) into (6.47) which results in

$$\begin{aligned} dx_t^i = \Big(A_k x_t^i - B_k R_k^{-1}\Big[\mathbb{N}_k^T[(x_t^i)^T, (x_t^0)^T, \bar{x}_t^T)]^T - \bar{n}_k + \mathbb{B}_k^T\big(\Pi_k[(x_t^i)^T, (x_t^0)^T, \bar{x}_t^T]^T + s_k\big)\Big] \\ + H_k x_t^0 + F_k^\pi \bar{x}_t + b_k\Big)dt + \sigma_k dw_t^i. \end{aligned} \tag{6.75}$$

Let define

$$\Pi_k = \begin{bmatrix} \Pi_{k,11} & \Pi_{k,12} & \Pi_{k,13} \\ \Pi_{k,21} & \Pi_{k,22} & \Pi_{k,23} \\ \Pi_{k,31} & \Pi_{k,32} & \Pi_{k,33} \end{bmatrix}, \quad k \in \mathfrak{K},$$

and $\mathbf{e}_k = [0_{n\times n}, ..., 0_{n\times n}, I_n, 0_{n\times n}, ..., 0_{n\times n}]$, where the $n \times n$ identity matrix I_n is at the kth block.

If we take the average of (6.75) over subpopulation $\mathcal{A}_k,\ k \in \mathfrak{K}$, and then take the L^2 limit as the number N_k of agents within the subpopulation goes to infinity (i.e. $N_k \to \infty$), we get

$$d\bar{x}_t^k = \Big(F_k^\pi + [A_k - B_k R_k^{-1}(\mathbb{N}_{k,1}^T + B_k^T \Pi_{k,11})]\mathbf{e}_k - B_k R_k^{-1} B_k^T \Pi_{k,13}\Big)\bar{x}_t dt \\ + (H_k - B_k R_k^{-1} B_k^T \Pi_{k,12}) x_t^0 dt + (b_k + B_k R_k^{-1}\bar{n}_k - B_k R_k^{-1}\mathbb{B}_k^T s_k)dt. \quad (6.76)$$

If we equate (6.76) with (6.54), then by consistency requirement a compact description of the major minor mean field equations determining $\bar{A}$, $\bar{G}$, $\bar{m}$ is given by

$$\begin{cases} -\dot{\Pi}_0 = \Pi_0 \mathbb{A}_0 + \mathbb{A}_0^T \Pi_0 - (\Pi_0 \mathbb{B}_0 + \mathbb{N}_0) R_0^{-1} (\mathbb{B}_0^T \Pi_0 + \mathbb{N}_0^T) + \mathbb{Q}_0, \quad \Pi_0(T) = \mathbb{G}_0, \\ -\dot{\Pi}_k = \Pi_k \mathbb{A}_k + \mathbb{A}_k^T \Pi_k - (\Pi_k \mathbb{B}_k + \mathbb{N}_k) R_k^{-1} (\mathbb{B}_k^T \Pi_k + \mathbb{N}_k^T) + \mathbb{Q}_k, \quad \Pi_k(T) = \mathbb{G}_k, \ \forall k, \\ \bar{A}_k = F_k^\pi + [A_k - B_k R_k^{-1}(\mathbb{N}_{k,1}^T + B_k^T \Pi_{k,11})]\mathbf{e}_k - B_k R_k^{-1} B_k^T \Pi_{k,13}, \ \forall k, \\ \bar{G}_k \ = H_k - B_k R_k^{-1} B_k^T \Pi_{k,12}, \ \forall k, \end{cases} \quad (6.77)$$

$$\begin{cases} -\dot{s}_0(t) = [(\mathbb{A}_0 - \mathbb{B}_0 R_0^{-1} \mathbb{N}_0^T)^T - \Pi_0 \mathbb{B}_0 R_0^{-1} \mathbb{B}_0^T] s_0(t) \\ \qquad\qquad + \Pi_0(\mathbb{M}_0(t) + \mathbb{B}_0 R_0^{-1} \bar{n}_0) + \mathbb{N}_0 R_0^{-1} \bar{n}_0 - \bar{\eta}_0, \quad s_0(T) = 0, \\ -\dot{s}_k(t) = [(\mathbb{A}_k - \mathbb{B}_k R_k^{-1} \mathbb{N}_k^T)^T - \Pi_k \mathbb{B}_k R_k^{-1} \mathbb{B}_k^T] s_k(t) \\ \qquad\qquad + \Pi_k(\mathbb{M}_k(t) + \mathbb{B}_k R_k^{-1} \bar{n}_k) + \mathbb{N}_k R_k^{-1} \bar{n}_k - \bar{\eta}_k, \quad s_k(T) = 0, \ \forall k, \\ \bar{m}_k \ = b_k + B_k R_k^{-1} \bar{n}_k - B_k R_k^{-1} \mathbb{B}_k^T s_k, \ \forall k. \end{cases} \quad (6.78)$$

Remark 6.4 (Infinite Horizon LQG MFG Systems). For Infinite horizon LQG MFG systems where the terminal time is set to infinity, the terminal cost becomes zero. Hence, the major agent's infinite horizon cost functionals is given by

$$J_0^N(u^0, u^{-0}) = \frac{1}{2}\mathbb{E}\Big[\int_0^\infty e^{-\rho t}\Big\{\|x_t^0 - \Phi(x_t^{(N)})\|_{Q_0}^2 + 2\big(x_t^0 - \Phi(x_t^{(N)})\big)^T N_0 u_t^0 + \|u_t^0\|_{R_0}^2\Big\}dt\Big], \quad (6.79)$$

Similarly, the discounted infinite horizon cost functional for minor agent $\mathcal{A}_i$, $1 \le i \le N$ is given by

$$J_i^N(u^i, u^{-i}) = \frac{1}{2}\mathbb{E}\Big[\int_0^\infty e^{-\rho t}\Big\{\|x_t^i - \Psi(x_t^{(N)})\|_{Q_k}^2 + 2\big(x_t^i - \Psi(x_t^{(N)})\big)^T N_k u_t^i + \|u_t^i\|_{R_k}^2\Big\}dt\Big]. \quad (6.80)$$

The dynamics (6.46)-(6.47) for the major agent and minor agents remain the same in the infinite horizon LQG MFG systems.

Assumption 6.7. *The pair* $(L_a, \mathbb{A}_0 - (\rho/2)I)$ *is detectable, and for each* $k \in \mathfrak{K}$*, the pair* $(L_b, \mathbb{A}_k - (\rho/2)I)$ *is detectable, where* $L_a = Q_0^{1/2}[I, -H_0^\pi]$ *and* $L_b = Q_k^{1/2}[I, -H_k, -\hat{H}_k^\pi]$.

Assumption 6.8. *The pair* $(\mathbb{A}_0 - (\rho/2)I, \mathbb{B}_0)$ *is stabilizable and* $(\mathbb{A}_k - (\rho/2)I, \mathbb{B}_k)$ *is stabilizable for each* $k \in \mathfrak{K}$.

Given that *Assumptions 6.7-6.8* hold, for the major agent's system (6.46), (6.79), the best response strategy is given by (6.62), where the steady state Riccati matrix Π_0 satisfies an algebraic Riccati equation given by

$$\rho\Pi_0 = \Pi_0\mathbb{A}_0 + \mathbb{A}_0^T\Pi_0 - (\Pi_0\mathbb{B}_0 + \mathbb{N}_0)R_0^{-1}(\mathbb{B}_0^T\Pi_0 + \mathbb{N}_0^T) + \mathbb{Q}_0, \tag{6.81}$$

and the steady state offset vector s_0 satisfies the differential equation

$$\rho s_0(t) = \dot{s}_0(t)+[(\mathbb{A}_0-\mathbb{B}_0R_0^{-1}\mathbb{N}_0^T)^T-\Pi_0\mathbb{B}_0R_0^{-1}\mathbb{B}_0^T]s_0(t)+\Pi_0(\mathbb{M}_0(t)+\mathbb{B}_0R_0^{-1}\bar{n}_0)+\mathbb{N}_0R_0^{-1}\bar{n}_0-\bar{\eta}_0. \tag{6.82}$$

Similarly, for minor agent $\mathcal{A}_i$'s system (6.47), (6.80), $i \in \mathfrak{N}$, the best response strategy is given by (6.72), where the steady state Riccati matrix Π_k and offset matrix s_k satisfy the following algebraic Riccati equation and differential offset equation.

$$\begin{cases} \rho\Pi_k = \Pi_k\mathbb{A}_k + \mathbb{A}_k^T\Pi_k - (\Pi_k\mathbb{B}_k + \mathbb{N}_k)R_k^{-1}(\mathbb{B}_k^T\Pi_k + \mathbb{N}_k^T) + \mathbb{Q}_k, \quad \forall k, \\ \rho s_k(t) = \dot{s}_k(t) + [(\mathbb{A}_k - \mathbb{B}_kR_k^{-1}\mathbb{N}_k^T)^T - \Pi_k\mathbb{B}_kR_k^{-1}\mathbb{B}_k^T]s_k(t) + \Pi_k(\mathbb{M}_k(t) + \mathbb{B}_kR_k^{-1}\bar{n}_k) \\ \qquad\qquad\qquad +\mathbb{N}_kR_k^{-1}\bar{n}_k - \bar{\eta}_k, \quad s_k(T) = 0, \ \forall k. \end{cases} \tag{6.83}$$

6.5 Conclusions

A convex analysis method was used to rederive the solutions to LQG optimal control problems. Then the methodology was applied to major minor LQG mean field game (MM LQG MFG) systems to retrieve the best response strategies for the major agent and each individual minor agent which yield an ϵ-Nash equilibrium for the entire system.

Chapter 7

Mean Field Game Systems including Common Noise and Markovian Latent Processes

7.1 Introduction

In this chapter, an MFG framework is considered where there exist one major agent and a large number of minor agents which are subject to linear dynamics and quadratic cost functionals. Each agent interacts with other agents in the system through the coupling in their cost functional with a common process. The common process is modulated by a latent Markov chain process and a latent Wiener process, which are not directly observed by the agents but rather are inferred from the agents' observation processes. We refer to the latent Wiener process as the common noise process. Moreover, the common process is impacted by the major agent's state, the major agent's control action, the average state of all the minor agents, and the average control action of all the minor agents. We obtain the best response strategies for the major agent and each individual minor agent in the infinite population limit which collectively yield an ϵ-Nash equilibrium for the finite population system.

Motivation: Financial and economic systems (among others) are often driven by latent factors, and these latent factors also affect the cost (profit) functional of the traders involved. Moreover, the agents in these system are often acting in a non-cooperative manner, and hence playing a large stochastic game with one another; while they may control aspects of the system, they are also at the whim of factors they cannot control or observe. For example, in optimal execution problems (where traders aim to sell or buy shares of an asset), all traders are subject to the same asset price process and must make their trading decisions based on the observed price. The asset price dynamics may be driven by a common Wiener process, which accounts for so-called noise (uninformed) traders. In addition, the effect of unobserved factors on the price dynamics, other than the major agent's trading action and the aggregate impact of minor agents' trading actions, are important factors to incorporate (see e.g. [66], [63]) in specifying the best response trading strategies and ϵ-Nash equilibrium.

Methodology: Although latent processes are not directly observable, the information provided from the realized trajectories of the common process and the evolution of system's aggregate state (mean field) can be used to obtain posteriori estimates, and to subsequently partially predict future behavior of the common process [66]. Certain versions of such problems can then be recast as MFG systems with a common noise. A variation of this type of MFG system has been investigated in [67], where the case of correlated randomness in a nonlinear setting is analyzed. Here we utilize a different approach in order to address the existence of a latent process together with the common noise. Specifically, we treat the common process as a major agent and further extend the Major - Minor LQG MFG analysis of [16] to incorporate such a latent process in the dynamics. Then, we utilize the convex analysis approach in Chapter 6 ([68]) to obtain the best response strategies for all agents that yield an ϵ-Nash equilibrium.

The rest of the chapter is organized as follows. Section 7.2 introduces a class of major minor MFG problems with a common process as well as a latent process. The MFG formulation of the problem is then presented in Section 7.3. Concluding remarks are made in Section 7.4.

7.2 Major Minor Mean Field Game Systems with a Common Process

7.2.1 Dynamics: Finite Population

We consider a large population of N minor agents and a major agent, where the agents are coupled through their individual cost functionals with a common process.

Major and Minor agents

The underlying dynamics of the major and minor agents are assumed to be given, respectively, by

$$dx_t^0 = [A_0 x_t^0 + B_0 u_t^0 + b_0(t)]dt + \sigma_0 dw_t^0, \tag{7.1}$$

$$dx_t^i = [A_k x_t^i + B_k u_t^i + b_k(t)]dt + \sigma_k dw_t^i, \tag{7.2}$$

where $t \in [0,T]$, $i \in \mathfrak{N}$, $\mathfrak{N} = \{1,\dots,N\}$, $N < \infty$, and the subscript $k \in \mathfrak{K}$, $\mathfrak{K} = \{1,\dots,K\}$, $K \leq N$, denotes the type of a minor agent. Here, $x_t^i \in \mathbb{R}^n$, $i \in \mathfrak{N}_0, \mathfrak{N}_0 = \{0,\dots,N\}$, are the states, $u_t^i \in \mathbb{R}^m$, $i \in \mathfrak{N}_0$ are the control inputs, $\{w_t^i,\ i \in \mathfrak{N}_0\}$ denotes $(N+1)$ independent standard Wiener processes in $\mathbb{R}^r$, where w_i is progressively measurable with respect to the filtration $\mathcal{F}^w := (\mathcal{F}_t^w)_{t\in[0,T]}$. All matrices in (7.1) and (7.2) are constant and of appropriate dimension; the vector processes $b_0(t)$, and $b_k(t)$ are deterministic functions of time.

Assumption 7.1. *The initial states $\{x_0^i,\ i \in \mathfrak{N}_0\}$ are identically distributed and mutually independent and also independent of $\mathcal{F}^w$; $\mathbb{E}[w_t^i (w_t^i)^T] = \Sigma$, $i \in \mathfrak{N}_0$. Moreover, $\mathbb{E}x_0^i = 0$, and $\mathbb{E}\|x_0^i\|^2 \leq C < \infty$, $i \in \mathfrak{N}_0$, with Σ and C independent of N.*

Minor Agents Types:

Minor agents are given in K distinct types with $1 \leq K < \infty$. The notation

$$\Psi_k \triangleq \Psi(\theta_i), \quad \theta_i = k$$

is introduced where $\theta_i \in \Theta$, with Θ being the parameter set, and Ψ may be any dynamical parameter in (7.2) or weight matrix in the cost functional (7.6). The symbol $\mathcal{I}_k$ denotes

$$\mathcal{I}_k = \{i : \theta_i = k,\ i \in \mathfrak{N}\}, \quad k \in \mathfrak{K}$$

where the cardinality of $\mathcal{I}_k$ is denoted by $N_k = |\mathcal{I}_k|$. Then, $\pi^N = (\pi_1^N, ..., \pi_K^N)$, $\pi_k^N = \frac{N_k}{N}$, $k \in \mathfrak{K}$, denotes the empirical distribution of the parameters $(\theta_1, ..., \theta_N)$ sampled independently of the initial conditions and Wiener processes of the agents $\mathcal{A}_i, i \in \mathfrak{N}$. The first assumption is as follows.

Assumption 7.2. *There exists π such that $lim_{N\to\infty}\pi^N = \pi$ a.s.*

Common Process: Finite Population

We consider the systems where the major agent and any minor agent $\mathcal{A}_i,\ i \in \mathfrak{N}$, observe a common stochastic process y_t, where both the state and common process y_t appear in an agent's cost functional as introduced in Section 7.2.2. The common process $y_t \in \mathbb{R}^n$ is governed by

$$dy_t = dy_t^L + (Fu_t^{(N)}dt + F_0u_t^0 + Hx_t^{(N)} + H_0x_t^0)dt, \tag{7.3}$$

where y_t^L evolves as in

$$dy_t^L = f(t, y_t^L, \Gamma_t)dt + \sigma dw_t. \tag{7.4}$$

In (7.4), the process $\Gamma := (\Gamma_t)_{t\in[0,T]}$ denotes a latent continuous Markov chain process with $\Gamma_t \in \{\gamma_j,\ j \in \mathfrak{M}\}$, $\mathfrak{M} = \{1, \dots, M\}$, $M < \infty$; the vector $f(t, y_t^L, \Gamma_t)$ denotes a deterministic nonlinear function of t, y^L, and Γ; $w_t \in \mathbb{R}^r$ denotes a latent Wiener process independent of $\{w_t^i,\ i \in \mathfrak{N}_0\}$, and the matrices F, F_0, H, H_0, and σ are deterministic, constant and of appropriate dimension. Moreover, by substituting (7.4) in (7.3), it is evident that the common process y_t is impacted by

1) a latent Markov chain process Γ_t,
2) the major agent's state x_t^0,
3) the major agent's control action u_t^0,
4) the average state of minor agents, i.e. $x_t^{(N)} = \frac{1}{N}\sum_{i=1}^N x_t^i$,
5) the average control action of minor agents, i.e. $u_t^{(N)} = \frac{1}{N}\sum_{i=1}^N u_t^i$,
6) a latent Wiener (common noise) process $w_t \in \mathbb{R}^r$ independent of w_t^0, $w_t^i,\ i \in \mathfrak{N}$.

Assumption 7.3. *The major agent $\mathcal{A}_0$ completely observes its own state and the common process y_t.*

Assumption 7.4. *Each minor agent $\mathcal{A}_i,\ i \in \mathfrak{N}$ completely observes its own state, the major agent's state and the common process y_t.*

We again emphasize that the latent processes Γ_t and w_t are not directly observed by the agents $\mathcal{A}_i,\ i \in \mathfrak{N}_0$. However, each agent may obtain their posteriori estimates based on its complete observations on the common process y_t. We refer to the latent Wiener process as the common noise process in this work.

Control σ-Fields

We denote by $\mathcal{F}^i := (\mathcal{F}^i_t)_{t\in[0,T]}$, $i \in \mathfrak{N}$, the natural filtration generated by the i-th minor agent's state $(x^i_t)_{t\in[0,T]}$, by $\mathcal{F}^0 := (\mathcal{F}^0_t)_{t\in[0,T]}$ the natural filtration generated by the major agent's state $(x^0_t)_{t\in[0,T]}$, and $\mathcal{F} := (\mathcal{F}_t)_{t\in[0,T]}$ the natural filtration generated by the states of all agents $((x^i_t)_{i\in\mathfrak{N}}, x^0_t)_{t\in[0,T]}$.

Moreover, we denote by $\mathcal{G} := (\mathcal{G}_t)_{t\in[0,T]}$ the natural filtration generated by $(\Gamma_t, w_t)_{t\in[0,T]}$, and $\mathcal{F}^y := (\mathcal{F}^y_t)_{t\in[0,T]}$ the natural filtration generated by $(y_t)_{t\in[0,T]}$.

Next, we introduce two admissible control sets. Let $\mathcal{U}^0$ denote the set of feedback control laws with second moment lying in $L^1[0,T]$, for any finite T, which are adapted to the smaller filtration $\mathcal{F}^{0,r} := (\mathcal{F}^{0,r}_t)_{t\in[0,T]}$, where $\mathcal{F}^{0,r} := \mathcal{F}^0 \vee \mathcal{F}^y$. The set of control inputs $\mathcal{U}^i$, $i \in \mathfrak{N}$, based upon the local information set of the minor agent $\mathcal{A}_i$, $i \in \mathfrak{N}$, consists of the feedback control laws adapted to the smaller filtration $\mathcal{F}^{i,r} := (\mathcal{F}^{i,r}_t)_{t\in[0,T]}$, where $\mathcal{F}^{i,r} := \mathcal{F}^i \vee \mathcal{F}^0 \vee \mathcal{F}^y$, $i \in \mathfrak{N}$, while $\mathcal{U}^N_g$ is adapted to the general filtration $\mathcal{F}^g := (\mathcal{F}^g_t)_{t\in[0,T]}$, where $\mathcal{F}^g := \mathcal{F} \vee \mathcal{F}^y \vee \mathcal{G}$, $1 \le N \le \infty$, and the $L^1[0,T]$ constraint on second moments applies in each case. We note in passing the significant differences between the information structures specified here and those in the team theory literature [69].

Assumption 7.5 (Major Agent's Linear Control Laws)**.** *For major agent $\mathcal{A}_0$, the set of control laws $\mathcal{U}^{0,L} \in \mathcal{U}^0$, is defined to be the collection of linear feedback control laws adapted to $\mathcal{F}^{0,r}$.*

Assumption 7.6 (Minor Agent's Linear Control Laws)**.** *For each minor agent $\mathcal{A}_i$, $i \in \mathfrak{N}$, the set of control laws $\mathcal{U}^{i,L} \in \mathcal{U}^i$, $i \in \mathfrak{N}$, is defined to be the collection of linear feedback control laws adapted to $\mathcal{F}^{i,r}$, $i \in \mathfrak{N}$.*

7.2.2 Cost Functionals: Finite Population

Given the vector z^0_t as

$$z^0_t = \begin{bmatrix} y_t \\ x^0_t \end{bmatrix},$$

the major agent's cost functional to be minimized is formulated by

$$J_0(u^0, u^{-0}) = \frac{1}{2}\mathbb{E}\Big[(z^0_T)^T G_0 z^0_T + \int_0^T \Big\{(z^0_s)^T Q_0 z^0_s + 2(z^0_s)^T N_0 u^0_s + (u^0_s)^T R_0 u^0_s\Big\} ds\Big], \quad (7.5)$$

where $u^{-0} = (u^1, u^2, ..., u^N)$.

Assumption 7.7. *For the cost functional* (7.5) *to be convex, we assume that* $G_0 \geq 0$*,* $R_0 > 0$*, and* $Q_0 - N_0 R_0^{-1} N_0^T > 0$.

Similarly, given the vector $z_t^i,\ i \in \mathfrak{N}$, as

$$z_t^i = \begin{bmatrix} x_t^i \\ y_t \end{bmatrix},$$

the cost functional to be minimized for minor agent $\mathcal{A}_i,\ i \in \mathfrak{N}$, is formulated by

$$J_i(u^i, u^{-i}) = \frac{1}{2}\mathbb{E}\bigg[(z_T^i)^T G_k z_T^i + \int_0^T \Big\{(z_s^i)^T Q_k z_s^i + 2(z_s^i)^T N_k u_s^i + (u_s^i)^T R_k u_s^i\Big\} ds\bigg], \qquad (7.6)$$

$1 \leq k \leq K$, where $u^{-i} = (u^0, ..., u^{i-1}, u^{i+1}, ..., u^N)$.

Assumption 7.8. *For the cost functional* (7.6) *to be convex, we assume that* $G_k \geq 0$*,* $R_k > 0$*, and* $Q_k - N_k R_k^{-1} N_k^T > 0$ *for* $k \in \mathfrak{K}$.

7.3 Major Minor LQG Mean Field Games Approach

In the mean field game methodology with a major agent [42], [16], the problem is first solved in the infinite population case where the average terms in the finite population dynamics and cost functional of each agent are replaced with their infinite population limit, i.e. the mean field. For this purpose, the major agent's state is extended with the mean field, while the minor agent's state is extended with the major agent's state, and the mean field; this yields stochastic optimal control problems for each agent linked only through the major agent's state and mean field. Finally the infinite population best response strategies are applied to the finite population system which yields an ϵ-Nash equilibrium.

To address major minor mean field game systems involving a common process and a latent Markov chain process, the following steps are followed. We first note that the common process in this work represents an extended form of common noise in [67]. However, a different approach is followed to incorporate the common process in the major minor LQG mean field game framework. First in Section 7.3.1, the evolution of the state mean field and the control mean field in the infinite population case are derived. Then, an $\mathcal{F}^{0,r}$-adapted and $\mathcal{F}^{i,r}$-adapted, $i \in \mathfrak{N}$, forms of the common process in the infinite population case are presented in Section 7.3.2. Next in Sections 7.3.3 and 7.3.4, the common process is perceived as a major agent in the major minor LQG MFG framework. Subsequently, the major minor LQG analysis described above is further extended where the major agent's state is extended with the mean field and the $\mathcal{F}^{0,r}$-adapted common process, while a minor agent's state is extended with the major agent's state, the mean field, and the $\mathcal{F}^{i,r}$-adapted common process. Finally, a convex analysis method is performed in Section 7.3.5 to obtain the best response strategies which yield the infinite population Nash equilibrium and finite population ϵ-Nash equilibrium.

7.3.1 Mean Field Evolution

The common process y_t governed by (7.3) is involved with the empirical average of the minor agents' states, i.e. $x_t^{(N)}$, as well as the empirical average of the minor agents' control actions, i.e. $u_t^{(N)}$. To attain the infinite population limit $\bar{y}_t$ of y_t, the state mean field $\bar{x}_t$ and the control mean field $\bar{u}_t$ are introduced as the infinite population limits of $x_t^{(N)}$ and $u_t^{(N)}$, respectively.

Control Mean Field

The empirical average of minor agents' control actions is introduced as

$$u_t^{(N_k)} = \frac{1}{N_k}\sum_{j=1}^{N_k} u_t^{j,k}, \qquad k \in \mathfrak{K}, \tag{7.7}$$

and the vector $u_t^{(N)} = [u_t^{(N_1)}, u_t^{(N_2)}, ..., u_t^{(N_K)}]$ is defined, where the pointwise in time limit (in quadratic mean) of $u_t^{(N)}$, if it exists, is called the control mean field of the system and is denoted by $\bar{u}_t = [\bar{u}_t^1, ..., \bar{u}_t^K]$. We consider for each minor agent $\mathcal{A}_i$, $i \in \mathfrak{N}$, of type k, $k \in \mathfrak{K}$, a uniform (with respect to i) state feedback control $u_t^{i,k} \in \mathcal{U}^{i,L}$ as in

$$u_t^{i,k} = L_1^k x_t^{i,k} + \Sigma_{l=1}^{K}\Sigma_{j=1}^{N_l} L_2^{k,l} x_t^{j,l} + L_3^k x_t^0 + L_4^k y_t + m_t^k, \tag{7.8}$$

where $t \in [0, T]$, L_1^k, $L_2^{k,l}$, L_3^k and L_4^k are constant matrices of appropriate dimension, $L_2^{k,l}$ is assumed to depend upon N_l and satisfy $N_l L_2^{k,l} \to \bar{L}_2^{k,l}$ as $N_l \to \infty$ for all k, $1 \le k \le K$, and m_t^k is a $\mathcal{F}_t^{0,r}$-measurable process. If we take the average of the control actions $u_t^{i,k}$ over the population of the agents of type k, $k \in \mathfrak{K}$, and hence calculate $u_t^{(N)}$, then it can be shown that $u_t^{(N)}$ as $N \to \infty$, converges in quadratic mean to the control mean field $\bar{u}_t$ given by

$$\bar{u}_t = \bar{C}\bar{x}_t + \bar{D}x_t^0 + \bar{E}\bar{y}_t + \bar{r}_t, \tag{7.9}$$

where $\bar{x}_t$, if it exists, denotes the state mean field introduced in Section 7.3.1, $\bar{y}_t$ denotes the limiting process associated with the common process y_t as $N \to \infty$ (see Section 7.3.2), and $\bar{r}_t$ is a $\mathcal{F}_t^{0,r}$-measurable process. Furthermore, the matrices in (7.9), i.e.

$$\bar{C} = \begin{bmatrix} \bar{C}_1 \\ \vdots \\ \bar{C}_K \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} \bar{D}_1 \\ \vdots \\ \bar{D}_K \end{bmatrix}, \quad \bar{E} = \begin{bmatrix} \bar{E}_1 \\ \vdots \\ \bar{E}_K \end{bmatrix}, \quad \bar{r}_t = \begin{bmatrix} \bar{r}_t^1 \\ \vdots \\ \bar{r}_t^K \end{bmatrix}, \tag{7.10}$$

are to be solved for using the mean field consistency equations (7.47)-(7.48) derived in Section 7.3.5.

State Mean Field

Similarly, the empirical state average is introduced as

$$x_t^{(N_k)} = \frac{1}{N_k} \sum_{j=1}^{N_k} x_t^{j,k}, \qquad k \in \mathfrak{K}, \tag{7.11}$$

and the vector $x_t^{(N)} = [x_t^{(N_1)}, x_t^{(N_2)}, ..., x_t^{(N_K)}]$ is defined, where the pointwise in time limit (in quadratic mean) of $x_t^{(N)}$, if it exists, is called the state mean field of the system and is denoted by $\bar{x}_t = [\bar{x}_t^1, ..., \bar{x}_t^K]$.

If we substitute (7.8) in (7.2) for $i \in \mathfrak{N}$, and take the average of the states of the minor agents' closed loop systems of type $k,\ k \in \mathfrak{K}$, and hence calculate $x_t^{(N)}$, it can be shown that $x_t^{(N)}$ as $N \to \infty$ converges in quadratic mean to the state mean field $\bar{x}_t$ which satisfies

$$d\bar{x}_t = \bar{A}\bar{x}_t dt + \bar{G}x_t^0 dt + \bar{L}\bar{y}_t dt + \bar{m}_t dt, \tag{7.12}$$

where $\bar{y}_t$ denotes the infinite population limit of the common process y_t (see Section 7.3.2), $\bar{m}_t$ is a $\mathcal{F}_t^{0,r}$-measurable process, and the matrices

$$\bar{A} = \begin{bmatrix} \bar{A}_1 \\ \vdots \\ \bar{A}_K \end{bmatrix}, \quad \bar{G} = \begin{bmatrix} \bar{G}_1 \\ \vdots \\ \bar{G}_K \end{bmatrix}, \quad \bar{L} = \begin{bmatrix} \bar{L}_1 \\ \vdots \\ \bar{L}_K \end{bmatrix}, \quad \bar{m}_t = \begin{bmatrix} \bar{m}_t^1 \\ \vdots \\ \bar{m}_t^K \end{bmatrix}, \tag{7.13}$$

are again to be solved for using the mean field consistency conditions (7.47)-(7.48) derived in Section 7.3.5.

By abuse of language, the mean value of the system's Gaussian mean field given by the state process $\bar{x}_t = [\bar{x}_t^1, ..., \bar{x}_t^K]$ shall also be termed the system's mean field (The derivation of the state mean field equation above may be performed using the methods of [21], [19] and [16]).

7.3.2 Common Process: Infinite Population

Each agent completely observes the common process y_t but has no observations on the latent Markov chain process Γ_t. In order to resolve the issue of the unobserved latent process Γ_t, Wonham filtering method is used to estimate the distribution of Γ_t based on the observations of each agent on y_t, i.e. $\mathcal{F}^y_t$. Subsequently, $f(t, y^L_t, \Gamma_t)$ and w_t in (7.4) are presented in their $\mathcal{F}^y_t$-adapted forms (see e.g. [66], [70]).

Denote the transition probabilities for the continuous time Markov chain process Γ by

$$p_{ij} = P(\Gamma_{t+h} = \gamma_j | \Gamma_t = \gamma_i), \quad 1 \leq i, j \leq M \tag{7.14}$$

and the corresponding transition rates by $v_{ij} \geq 0$, and

$$v_i = \sum_{j=1,\, j\neq i}^{M} v_{ij}, \quad i \in \mathfrak{M}. \tag{7.15}$$

The posterior distribution of Γ_t conditional on $\mathcal{F}^y_t$ is denoted by $\Pi = \{\pi^j_t,\ j \in \mathfrak{M},\ t \in [0, T]\}$, where

$$\pi^j_t = \mathbb{E}[\mathbb{1}_{\{\Gamma_t = \gamma_j\}} | \mathcal{F}^y_t], \quad j \in \mathfrak{M}, \quad t \in [0, T], \tag{7.16}$$

with initial distribution $\{\pi^j_0, j \in \mathfrak{M}\}$.

Remark 7.1. As a result of *Assumptions 7.3-7.4*, the major agent $\mathcal{A}_0$, and each minor agent $\mathcal{A}_i$, $i \in \mathfrak{N}$, completely observe the unaffected common process y^L_t given by (7.3) in the infinite population limit. Consequently

$$\pi^j_t = \mathbb{E}[\mathbb{1}_{\{\Gamma_t = \gamma_j\}} | \mathcal{F}^y_t] \triangleq \mathbb{E}[\mathbb{1}_{\{\Gamma_t = \gamma_j\}} | \mathcal{F}^{y^L}_t]. \tag{7.17}$$

Lemma 7.1 (Wonham Filter). *[70] If $\sigma > 0$, the posterior distribution Π of Γ_t is given by*

$$\begin{aligned} d\pi^j_t = \Big(- v_j \pi^j_t + \sum_{i=1,\, i\neq j}^{M} v_{ij} \pi^i_t \Big) dt - \sigma^{-2} \Big(\sum_{i=1}^{M} \pi^i_t \gamma_i \Big) \Big[f(t, y^L_t, \gamma_j) - \sum_{i=1}^{M} \pi^i_t \gamma_i \Big] \pi^j_t dt \\ + \sigma^{-2} \Big[f(t, y^L_t, \gamma_j) - \sum_{i=1}^{M} \pi^i_t \gamma_i \Big] \pi^j_t dy^L_t, \end{aligned} \tag{7.18}$$

$i \in \mathfrak{M}$.

□

Lemma 7.2. *[66] Define the process $\widehat{w} = (\widehat{w}_t, t \in [0,T])$ as*

$$\widehat{w}_t = w_t + \sigma^{-1} \int_0^t (f_\tau - \widehat{f}_\tau) d\tau, \tag{7.19}$$

where $\widehat{f} = (\widehat{f}_t,\ t \in [0,T])$ is an $\mathcal{F}_t^y$-adapted process defined as

$$\widehat{f}_t = \mathbb{E}[f(t, y_t^L, \Gamma_t) | \mathcal{F}_t^{y_L}], \tag{7.20}$$

and is computed by

$$\widehat{f}_t = \widehat{f}(t, y_t^L, \Pi) = \sum_{j=1}^{M} \pi_t^j f(t, y_t^L, \gamma_j). \tag{7.21}$$

Then the process $\widehat{w}_t$ is an $\mathcal{F}_t^y$-adapted Wiener process.

□

According to *Lemma 7.1* and *Lemma 7.2*, equation (7.4) can be rewritten as

$$dy_t^L = \widehat{f}_t dt + \sigma d\widehat{w}_t, \tag{7.22}$$

and by substituting (7.22) in (7.3), the $\mathcal{F}_t^{0,r}$-adapted dynamics of the common process for the infinite population case, i.e. $\bar{y}_t$, is given by

$$d\bar{y}_t = [\widehat{f}_t + F^\pi \bar{u}_t dt + F_0 u_t^0 + H^\pi \bar{x}_t + H_0 x_t^0] dt + \sigma d\widehat{w}_t, \tag{7.23}$$

where the average terms $x_t^{(N)}$ and $u_t^{(N)}$ in (7.3) have been replaced with their (quadratic mean) limit as $N \to \infty$, i.e. the state mean field $\bar{x}_t$ and the control mean field $\bar{u}_t$, respectively. Moreover, $F^\pi = \pi \otimes F$ and $H^\pi = \pi \otimes H$, where $\otimes$ denotes the Kronecker product of the corresponding matrices.

Remark 7.2. Since the state and the control action of each individual minor agent $\mathcal{A}_i$, $i \in \mathfrak{N}$, do not affect the infinite population evolution of the common process, i.e. $\bar{y}_t$, the $\mathcal{F}_t^{i,r}$-adapted and $\mathcal{F}_t^{0,r}$-adapted dynamics of the common process $\bar{y}_t$ in the infinite population limit are identical and given by (7.23).

7.3.3 Major Agent's Regulation Problem : Infinite Population

First, the major agent's state x_t^0 is extended with the state mean field $\bar{x}_t$ and the infinite population common process $\bar{y}_t$ to form the major agent's extended state $X_t^0 = [(\bar{y}_t)^T, (x_t^0)^T, (\bar{x}_t)^T]^T$ which is governed by

$$dX_t^0 = \mathbb{A}_0 X_t^0 dt + \mathbb{B}_0 u_t^0 dt + \mathbb{M}_t^0 dt + \Sigma_0 dW_t^0, \tag{7.24}$$

By substituting (7.9) into (7.23), the matrices in the extended major agent's dynamics (7.24) are given by

$$\mathbb{A}_0 = \begin{bmatrix} F^\pi \bar{E} & F^\pi \bar{D} + H_0 & F^\pi \bar{C} + H^\pi \\ 0_{n\times n} & A_0 & 0_{n\times nK} \\ \bar{L} & \bar{G} & \bar{A} \end{bmatrix}, \quad \mathbb{B}_0 = \begin{bmatrix} F_0 \\ B_0 \\ 0_{nK\times m} \end{bmatrix},$$
$$\mathbb{M}_t^0 = \begin{bmatrix} \widehat{f}_t + F^\pi \bar{r}_t \\ b_0(t) \\ \bar{m}_t \end{bmatrix}, \quad \Sigma_0 = \begin{bmatrix} \sigma & 0_{n\times r} & 0_{n\times rK} \\ 0_{n\times r} & \sigma_0 & 0_{n\times rK} \\ 0_{nK\times r} & 0_{nK\times r} & 0_{nK\times rK} \end{bmatrix}, \quad W_t^0 = \begin{bmatrix} \widehat{w}_t \\ w_t^0 \\ 0_{rK\times 1} \end{bmatrix}. \tag{7.25}$$

Next, the major agent's extended cost functional is given as

$$J_0^{ex}(u^0) = \frac{1}{2}\mathbb{E}\Big[(X_T^0)^T \mathbb{G}_0 X_T^0 + \int_0^T \Big\{(X_s^0)^T \mathbb{Q}_0 X_s^0 + 2(X_s^0)^T \mathbb{N}_0 u_s^0 + (u_s^0)^T R_0 u_s^0\Big\} ds\Big], \tag{7.26}$$

where the corresponding weight matrices are given by

$$\mathbb{G}_0 = [I_{2n}, 0_{2n\times nK}]^T G_0 [I_{2n}, 0_{2n\times nK}], \tag{7.27}$$
$$\mathbb{Q}_0 = [I_{2n}, 0_{2n\times nK}]^T Q_0 [I_{2n}, 0_{2n\times nK}], \tag{7.28}$$
$$\mathbb{N}_0 = \begin{bmatrix} N_0 \\ 0_{nK\times m} \end{bmatrix}. \tag{7.29}$$

The minimization of the extended cost functional (7.26) subject to the extended dynamics (7.24) constitutes a stochastic optimal control problem for the major agent in the infinite population limit. Then, according to *Theorem 7.3* the major agent's optimal control action is given by

$$u_t^{0,*} = -R_0^{-1}\Big[\mathbb{N}_0^T X_t^0 + \mathbb{B}_0^T\big(\Pi_0(t) X_t^0 + s_t^0\big)\Big], \tag{7.30}$$

where $\Pi_0(t)$ and s_t^0 are to be solved for using

$$\dot{\Pi}_0 + \Pi_0 \mathbb{A}_0 + \mathbb{A}_0^T \Pi_0 - (\mathbb{B}_0^T \Pi_0 + \mathbb{N}_0^T)^T R_0^{-1} (\mathbb{B}_0^T \Pi_0 + \mathbb{N}_0^T) + \mathbb{Q}_0 = 0, \quad \Pi_0(T) = \mathbb{G}_0, \tag{7.31}$$

and the BSDE

$$ds_t^0 + [(\mathbb{A}_0 - \mathbb{B}_0 R_0^{-1} \mathbb{N}_0)^T - \Pi_0 \mathbb{B}_0 R_0^{-1} \mathbb{B}_0^T] s_t^0 dt + \Pi_0 \mathbb{M}_t^0 dt + (\Pi_0 \Sigma_0 - q_t^0) dW_t^0 = 0, \quad s_T^0 = 0. \tag{7.32}$$

The Riccati eqution (7.31) and the offset equation (7.32) shall be derived in Section 7.3.5.

Finally, the closed-loop dynamics of the major agent $\mathcal{A}_0$ when the control action (7.30) is substituted in (7.1) is given by

$$dX_t^0 = \Big(A_0 X_t^0 - B_0 R_0^{-1}\Big[\mathbb{N}_0^T X_t^0 + \mathbb{B}_0^T\big(\Pi_0(t)X_t^0 + s_t^0\big)\Big] + b_0(t)\Big)\, dt + \sigma_0 dw_t^0. \tag{7.33}$$

7.3.4 Minor Agent's Regulation Problem: Infinite Population

First, minor agent $\mathcal{A}_i$'s, $i \in \mathfrak{N}$, state is extended with the infinite population common process $\bar{y}_t$, the major agent's state x_t^0, and the state mean field $\bar{x}_t$ to form the minor agent's extended state $X_t^i = [(x_t^i)^T, (\bar{y}_t)^T, (x_t^0)^T, (\bar{x}_t)^T]^T$ which satisfies

$$dX_t^i = \mathbb{A}_k X_t^i dt + \mathbb{B}_k u_i dt + \mathbb{M}_t^k dt + \Sigma_k dW_t^i. \tag{7.34}$$

To attain the extended matrices in (7.34), the joint dynamics of (i) minor agent $\mathcal{A}_i$'s system given by (7.2), (ii) the common process $\bar{y}_t$ given by (7.23) where (7.9) and (7.30) are substituted , (iii) the major agent $\mathcal{A}_0$'s closed loop system given by (7.33), and (iv) the state mean field $\bar{x}_t$ given by (7.12) are utilized which results in

$$\mathbb{A}_k = \begin{bmatrix} A_k & 0_{n\times(2n+nK)} \\ 0_{(2n+nK)\times n} & \mathbb{A}_0 - \mathbb{B}_0 R_0^{-1}\mathbb{N}_0 - \mathbb{B}_0 R_0^{-1}\mathbb{B}_0^T \Pi \end{bmatrix}, \quad \mathbb{B}_k = \begin{bmatrix} B_k \\ 0_{(2n+nK)\times m} \end{bmatrix},$$
$$\mathbb{M}_t^k = \begin{bmatrix} b_k(t) \\ \mathbb{M}_t^0 - \mathbb{B}_0 R_0^{-1}\mathbb{B}_0^T s_0(t) \end{bmatrix}, \quad \Sigma_k = \begin{bmatrix} \sigma_k & 0_{n\times(2r+rK)} \\ 0_{(2n+nK)\times r} & \Sigma_0 \end{bmatrix}, \quad W_t^i = \begin{bmatrix} w_t^i, \\ W_t^0 \end{bmatrix}. \tag{7.35}$$

Next, the minor agent $\mathcal{A}_i$'s extended cost functional is formed as

$$J_i^{ex}(u^i) = \frac{1}{2}\mathbb{E}\Big[(X_T^i)^T \mathbb{G}_k X_T^i + \int_0^T \Big\{(X_s^i)^T \mathbb{Q}_k X_s^i + 2(X_s^i)^T \mathbb{N}_k u_s^i + (u_s^i)^T R_k u_s^i\Big\} ds\Big], \tag{7.36}$$

where the corresponding weight matrices are given by

$$\mathbb{G}_k = \left[I_{2n}, 0_{2n\times(n+nK)}\right]^T G_k \left[I_{2n}, 0_{2n\times(n+nK)}\right],$$
$$\mathbb{Q}_k = \left[I_{2n}, 0_{2n\times(n+nK)}\right]^T Q_k \left[I_{2n}, 0_{2n\times(n+nK)}\right],$$
$$\mathbb{N}_k = \begin{bmatrix} N_k \\ 0_{(n+nK)\times m} \end{bmatrix}. \tag{7.37}$$

The dynamics (7.34) together with the cost functional (7.36) constitute a stochastic optimal control problem for minor agent $\mathcal{A}_i$, $i \in \mathfrak{N}$, in the infinite population limit. Then, according to *Theorem 7.3*, the minor agent $\mathcal{A}_i$'s optimal control action for the infinite population case is given by

$$u_t^{i,*} = -R_k^{-1}\Big[\mathbb{N}_k^T X_t^i + \mathbb{B}_k^T\big(\Pi_k(t)X_t^i + s_t^{i,k}\big)\Big], \tag{7.38}$$

where $\Pi_k(t),\ k \in \mathfrak{K}$, is the solutions to the following deterministic Riccati equation

$$\dot{\Pi}_k + \Pi_k\mathbb{A}_k + \mathbb{A}_k^T\Pi_k - (\mathbb{B}_k^T\Pi_k + \mathbb{N}_k^T)^T R_k^{-1}(\mathbb{B}_k^T\Pi_k + \mathbb{N}_k^T) + \mathbb{Q}_k = 0, \quad \Pi_k(T) = \mathbb{G}_k, \tag{7.39}$$

and $s_t^{i,k},\ k \in \mathfrak{K}$, is the solution to the following BSDE

$$ds_t^{i,k} + \Big(\big[(\mathbb{A}_k - \mathbb{B}_k R_k^{-1}\mathbb{N}_k)^T - \Pi_k\mathbb{B}_k R_k^{-1}\mathbb{B}_k^T\big]s_t^{i,k} + \Pi_k\mathbb{M}_t^k\Big)dt + (\Pi_k\Sigma_k - q_t^i)dW_t^i = 0, \quad s_T^{i,k} = 0. \tag{7.40}$$

The complete derivation of (7.39)-(7.40) will be discussed in Section 7.3.5.

Finally, control action (7.68) is substituted in (7.2) which gives minor agent $\mathcal{A}_i$'s, $i \in \mathfrak{N}$, closed loop system as

$$dX_t^i = \Big(A_kX_t^i - B_kR_k^{-1}\Big[\mathbb{N}_k^T X_t^i + \mathbb{B}_k^T\big(\Pi_kX_t^i + s_t^{i,k}\big)\Big] + b_k\Big)dt + \sigma_k dw_t^i. \tag{7.41}$$

Remark 7.3. We note that for the case where there exists no latent process, i.e. $y_t^L = 0,\ t \in [0,T]$, the diffusion terms of (7.32) and (7.40) become zero and they reduce to the deterministic offset equations of classical major minor LQG mean field games in [16].

7.3.5 Nash and ϵ-Nash Equilibria

To derive the mean field consistency equations which specify the matrices in the control and state mean field equations, respectively, (7.9) and (7.12), the closed loop system (7.41) of minor agent $\mathcal{A}_i$ is rewritten as

$$dx_t^i = \Big(A_kx_t^i - B_kR_k^{-1}\big(\mathbb{N}_k^T + \mathbb{B}_k^T\Pi_k\big)\big[(x_t^i)^T, \bar{y}_t^T, (x_t^0)^T, \bar{x}_t^T)\big]^T - B_kR_k^{-1}\mathbb{B}_k^T s_t^{i,k} + b_k\Big)dt + \sigma_k dw_t^i, \tag{7.42}$$

where $i \in \mathfrak{N},\ k \in \mathfrak{K}$.

Then the block matrices

$$\Pi_k = \begin{bmatrix} \Pi_{k,11} & \Pi_{k,12} & \Pi_{k,13} & \Pi_{k,14} \\ \Pi_{k,21} & \Pi_{k,22} & \Pi_{k,23} & \Pi_{k,24} \\ \Pi_{k,31} & \Pi_{k,32} & \Pi_{k,33} & \Pi_{k,34} \\ \Pi_{k,41} & \Pi_{k,42} & \Pi_{k,43} & \Pi_{k,44} \end{bmatrix}, \quad \mathbb{N}_k = \begin{bmatrix} \mathbb{N}_{k,1} \\ \mathbb{N}_{k,2} \\ \mathbb{N}_{k,3} \\ \mathbb{N}_{k,4} \end{bmatrix},$$

$$\mathbf{e}_k = [0_{n\times n}, ..., 0_{n\times n}, I_n, 0_{n\times n}, ..., 0_{n\times n}], \tag{7.43}$$

are defined, where $\Pi_{k,11}, \Pi_{k,22}, \Pi_{k,33} \in \mathbb{R}^{n\times n}$, $\Pi_{k,44} \in \mathbb{R}^{nK\times nK}$; $\mathbb{N}_{k,1}, \mathbb{N}_{k,2}, \mathbb{N}_{k,3} \in \mathbb{R}^{n\times m}$, $\mathbb{N}_{k,4} \in \mathbb{R}^{nK\times m}$; and $\mathbf{e}_k \in \mathbb{R}^{n\times nK}$, $k \in \mathfrak{K}$, denotes a matrix which has the identity matrix I_n in its kth block and zero matrix $0_{n\times n}$ in other $(K-1)$ blocks.

Now, if the average of (7.42) over N_k minor agents of type k, $k \in \mathfrak{K}$, and then its L^2 limit as the number N_k of agents within the subpopulation k goes to infinity (i.e. $N_k \to \infty$) be taken, it yields

$$\begin{aligned} d\bar{x}_t^k = &\Big[\Big(A_k - B_k R_k^{-1}(\mathbb{N}_{k,1}^T + B_k^T\Pi_{k,11})\Big)\mathbf{e}_k - B_k R_k^{-1}(\mathbb{N}_{k,4}^T + B_k^T\Pi_{k,14})\Big]\bar{x}_t dt \\ &- B_k R_k^{-1}(\mathbb{N}_{k,3}^T + B_k^T\Pi_{k,13})x_t^0 dt - B_k R_k^{-1}(\mathbb{N}_{k,2}^T + B_k^T\Pi_{k,12})\bar{y}_t dt + (b_k - B_k R_k^{-1}\mathbb{B}_k^T\bar{s}_t^k)dt. \end{aligned} \tag{7.44}$$

In (7.44), $\bar{s}_t^k$ is obtained by taking the average and then the L^2 limit of (7.40) over the subpopulation $k \in \mathfrak{K}$ as $N_k \to \infty$, and is given by

$$d\bar{s}_t^k + \Big(\big[(\mathbb{A}_k - \mathbb{B}_k R_k^{-1}\mathbb{N}_k^T)^T - \Pi_k\mathbb{B}_k R_k^{-1}\mathbb{B}_k^T\big]\bar{s}_t^k + \Pi_k\mathbb{M}_t^k\Big)dt + (\Pi_k\Sigma_k - \bar{q}_t)d\bar{W}_t = 0, \quad \bar{s}_T^k = 0, \tag{7.45}$$

where

$$\bar{W}_t = \begin{bmatrix} 0_{r\times 1}, \\ W_t^0 \end{bmatrix}, \tag{7.46}$$

since $\lim_{N_k\to\infty}\frac{1}{N_k}\sum_{i=1}^{N_k} w_t^i = 0$; and hence $\bar{q}_t$ is an $\mathcal{F}_t^{0,r}$-adapted process.

Then, equating (7.44) with (7.12) results in the following sets of equations.

$$\begin{cases} \dot{\Pi}_0 + \Pi_0\mathbb{A}_0 + \mathbb{A}_0^T\Pi_0 - (\mathbb{N}_0^T + \mathbb{B}_0^T\Pi_0)^T R_0^{-1}(\mathbb{N}_0^T + \mathbb{B}_0^T\Pi_0) + \mathbb{Q}_0 = 0, \quad \Pi_0(T) = \mathbb{G}_0, \\ \dot{\Pi}_k + \Pi_k\mathbb{A}_k + \mathbb{A}_k^T\Pi_k - (\mathbb{N}_k^T + \mathbb{B}_k^T\Pi_k)^T R_k^{-1}(\mathbb{N}_k^T + \mathbb{B}_k^T\Pi_k) + \mathbb{Q}_k = 0, \quad \Pi_k(T) = \mathbb{G}_k, \\ \bar{C}_k = -R_k^{-1}(\mathbb{N}_{k,1}^T + B_k^T\Pi_{k,11})\mathbf{e}_k - R_k^{-1}(\mathbb{N}_{k,4}^T + B_k^T\Pi_{k,14}), \\ \bar{D}_k = -R_k^{-1}(\mathbb{N}_{k,3}^T + B_k^T\Pi_{k,13}), \\ \bar{E}_k = -R_k^{-1}(\mathbb{N}_{k,2}^T + B_k^T\Pi_{k,12}), \\ \bar{A}_k = A_k\mathbf{e}_k + B_k\bar{C}_k, \\ \bar{G}_k = B_k\bar{D}_k, \\ \bar{L}_k = B_k\bar{E}_k, \end{cases} \tag{7.47}$$

$$\begin{cases} ds_t^0 + \Big(\big[(\mathbb{A}_0 - \mathbb{B}_0 R_0^{-1}\mathbb{N}_0^T)^T - \Pi_0\mathbb{B}_0 R_0^{-1}\mathbb{B}_0^T\big] s_t^0 + \Pi_0\mathbb{M}_t^0\Big)dt + (\Pi_0\Sigma_0 - q_t^0)dW_t^0 = 0, \quad s_T^0 = 0, \\ d\bar{s}_t^k + \Big(\big[(\mathbb{A}_k - \mathbb{B}_k R_k^{-1}\mathbb{N}_k^T)^T - \Pi_k\mathbb{B}_k R_k^{-1}\mathbb{B}_k^T\big] \bar{s}_t^k + \Pi_k\mathbb{M}_t^k\Big)dt + (\Pi_k\Sigma_k - \bar{q}_t)d\bar{W}_t = 0, \quad \bar{s}_T^k = 0, \\ \bar{r}_t^k = -R_k^{-1}\mathbb{B}_k^T\bar{s}_t^k, \\ \bar{m}_t^k = B_k\bar{r}_t^k + b_k. \end{cases} \tag{7.48}$$

Equations (7.47)-(7.48) are called the mean field consistency equations (see [16]) from which the matrices in (7.9) and (7.12) can be calculated.

Now, according to the asymptotic equilibrium analysis performed in [16], the following matrices are defined.

$$M_1 = \begin{bmatrix} A_1 - B_1R_1^{-1}(\mathbb{N}_{1,1}^T + B_1^T\Pi_{1,11}) & & 0 \\ & \ddots & \\ 0 & & A_K - B_KR_K^{-1}(\mathbb{N}_{K,1}^T + B_K^T\Pi_{K,11}) \end{bmatrix},$$

$$M_1' = \begin{bmatrix} -\pi_1 FR_1^{-1}(\mathbb{N}_{1,1}^T + B_1^T\Pi_{1,11}) & & 0 \\ & \ddots & \\ 0 & & -\pi_K FR_K^{-1}(\mathbb{N}_{K,1}^T + B_K^T\Pi_{K,11}) \end{bmatrix},$$

$$M_2 = \begin{bmatrix} -B_1R_1^{-1}(\mathbb{N}_{1,4}^T + B_1^T\Pi_{1,14}) \\ \vdots \\ -B_KR_K^{-1}(\mathbb{N}_{K,4}^T + B_K^T\Pi_{K,14}) \end{bmatrix}, \quad M_2' = \begin{bmatrix} -\pi_1 FR_1^{-1}(\mathbb{N}_{1,4}^T + B_1^T\Pi_{1,14}) \\ \vdots \\ -\pi_K FR_K^{-1}(\mathbb{N}_{K,4}^T + B_K^T\Pi_{K,14}) \end{bmatrix},$$

$$M_3 = \begin{bmatrix} F^\pi\bar{E} & F^\pi\bar{D} + H_0 & F^\pi\bar{C} + H^\pi & 0_{n\times nK} & 0_{n\times n} \\ 0_{n\times n} & A_0 & 0_{n\times nK} & 0_{n\times nK} & 0_{n\times n} \\ \bar{L} & \bar{G} & \bar{A} & 0_{nK\times nK} & 0_{nK\times n} \\ \bar{L} & \bar{G} & M_2 & M_1 & 0_{nK\times n} \\ F^\pi\bar{E} & F^\pi\bar{D} + H_0 & M_2' & M_1' & 0_{n\times n} \end{bmatrix},$$

$$L_{0,H} = Q_0^{\frac{1}{2}} \begin{bmatrix} 0_{n\times n} & I_n & 0_{n\times nK} & 0_{n\times nK} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & 0_{n+nK} & 0_{n+nK} & I_n \end{bmatrix},$$

$$L_a = Q_0^{\frac{1}{2}}\left[I_{2n}, 0_{2n\times nK}\right], \quad L_b = Q_k^{\frac{1}{2}}\left[I_{2n}, 0_{2n\times(n+nK)}\right]. \tag{7.49}$$

Assumption 7.9. *The matrix M_1 is Hurwitz.*

Assumption 7.10. *The pair $(L_{0,H}, M_3)$ is observable.*

The analysis above leads to the following theorem where convex analysis and asymptotic

MFG equilibrium analysis are utilized to establish the infinite population Nash equilibrium and finite population ϵ-Nash equilibrium.

Theorem 7.3. *Subject to Assumptions 7.1-7.10, the mean field equations* (7.47)-(7.48) *together with the system equations* (7.1)-(7.3) *and* (7.5)-(7.6)*, generate an infinite family of stochastic control laws $\mathcal{U}_{MF}^{\infty,*}$, with finite sub-families $\mathcal{U}_{MF}^{N,*} \triangleq \{u_t^{i,*};\ i \in \mathfrak{N}\}$, $1 \leq N < \infty$, given by* (7.30)-(7.32) *and* (7.68)-(7.40)*, such that*

(i) $\mathcal{U}_{MF}^{\infty,*}$ *yields a unique Nash equilibrium within the set of linear control laws $\mathcal{U}_L^\infty$ such that*

$$J_i^\infty(u^{i,*}, u^{-i,*}) = \inf_{u^i \in \mathcal{U}_L^\infty} J_i^\infty(u^i, u^{-i,*}),$$

(ii) All agent systems $i \in \mathfrak{N}_0$, are second order stable in the sense that $\sup_{t\in[0,T], i\in\mathfrak{N}_0} \mathbb{E}\Big\{(\|x_t^i\|^2 + \|x_t^{(N)}\|^2 + \|\bar{x}_t\|^2 + \|y_t\|^2)\Big\} < C$ *with C independent of N.*

(iii) $\{\mathcal{U}_{MF}^N;\ 1 \leq N < \infty\}$ *yields a unique ϵ-Nash equilibrium within the set of linear control laws $\mathcal{U}_L^N$ for all $\epsilon > 0$, i.e. for all $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $N \geq N(\epsilon)$*

$$J_i^N(u^{i,*}, u^{-i,*}) - \epsilon \leq \inf_{u^i \in \mathcal{U}_L^N} J_i^N(u^i, u^{-i,*}) \leq J_i^N(u^{i,*}, u^{-i,*}),$$

where $J_i^N(u^{i,*}, u^{-i,*}) \to J_i^\infty(u^{i,*}, u^{-i,*})$, $i \in \mathfrak{N}_0$, *as* $N \to \infty$.

□

Proof. We use the convex analysis method developed in [68] to obtain the best response strategies (17)-(19) and (23)-(25); this proves parts (i) and (ii) of the theorem. Then following the asymptotic equilibrium analysis of [10], the set of infinite population control actions yields an ϵ-Nash equilibrium for the large population system which proves part (iii) of the theorem.

First, the convex analysis is performed for the major agent $\mathcal{A}_0$'s extended system to derive the major agent's optimal control action in the infinite population limit. Using *Theorem 6.2* in Chapter 6 ([68]), the Gâteaux derivative of the major agent's extended cost $J_0^{ex'}(u^0)$ in the direction of $\omega_t^0 \in \mathcal{U}^0$ is given by

$$\langle J_0^{ex'}(u^0), \omega^0 \rangle = \mathbb{E}\Bigg[\int_0^T (\omega_t^0)^T \Big\{ \mathbb{N}_0^T X_t^{0,u} + R_0 u_t^0 + \mathbb{B}_0^T \Big(e^{-\mathbb{A}_0^T t} M_t^0 - \int_0^t e^{\mathbb{A}_0^T (s-t)} \big(\mathbb{Q}_0 X_s^{0,u} + \mathbb{N}_0 u_s^0\big) ds \Big) \Big\} dt \Bigg], \quad (7.50)$$

where the martingale M_t^0 is specified by

$$M_t^0 = \mathbb{E}\Big[e^{\mathbb{A}_0^T T}\mathbb{G}_0 X_T^{0,u} + \int_0^T e^{\mathbb{A}_0^T s}(\mathbb{Q}_0 X_s^{0,u} + \mathbb{N}_0 u_s^0)ds\Big|\mathcal{F}_t^{0,r}\Big]. \tag{7.51}$$

Given that *Assumption 7.7* holds, according to *Theorem 6.2* in Chapter 6 ([68]), the optimal control action $u_t^{0,*}$ for the major agent $\mathcal{A}_0$ in the infinite population limit is given by

$$u_t^{0,*} = -R_0^{-1}\Big[\mathbb{N}_0^T X_t^{0,*} + \mathbb{B}_0^T\Big(e^{-\mathbb{A}_0^T t}M_t^0 - \int_0^t e^{\mathbb{A}_0^T(s-t)}\big(\mathbb{Q}_0 X_s^{0,*} + \mathbb{N}_0 u_s^{0,*}\big)ds\Big)\Big], \tag{7.52}$$

which is obtained by setting (7.50) to zero for all possible paths of $\omega_t^0 \in \mathcal{U}^0$.

Now, Let us define p_t^0 as in

$$p_t^0 = e^{-\mathbb{A}_0^T t}M_t^0 - \int_0^t e^{\mathbb{A}_0^T(s-t)}\big(\mathbb{Q}_0 X_s^{0,*} + \mathbb{N}_0 u_s^{0,*}\big)ds, \tag{7.53}$$

which is the adjoint process for the major agent's system in the stochastic maximum principle framework. Next, we adopt an ansatz for p_t^0 given by

$$p_t^0 = \Pi_0(t)X_t^{0,*} + s_t^0, \tag{7.54}$$

whose substitution in (7.52) yields a linear state feedback form for the major agent's optimal control action, i.e.

$$u_t^{0,*} = -R_0^{-1}\big[\mathbb{N}_0^T X_t^{0,*} + \mathbb{B}_0^T\big(\Pi_0(t)X_t^{0,*} + s_t^0)\big)\big]. \tag{7.55}$$

To find $\Pi_0(t) \in \mathbb{R}^{(2+K)n\times(2+K)n}$ and $s_t^0 \in \mathbb{R}^{(2+K)n}$, first both sides of (7.54) are differentiated and then (7.24) and (7.55) are substituted, which gives

$$\begin{aligned} dp_t^0 = \Big[&\big(\dot{\Pi}_0 + \Pi_0\mathbb{A}_0 - \Pi_0\mathbb{B}_0R_0^{-1}\mathbb{N}_0^T - \Pi_0\mathbb{B}_0R_0^{-1}\mathbb{B}_0^T\Pi_0\big)X_t^0 dt \\ &+ \big(-\Pi_0\mathbb{B}_0R_0^{-1}\mathbb{B}_0^T s_t^0 + \Pi_0\mathbb{M}_t^0\big)dt + ds_t^0\Big] + \Pi_0\Sigma_0(t)dW_t^0. \end{aligned} \tag{7.56}$$

Next, both sides of (7.53) are differentiated to yield

$$dp_t^0 = (-\mathbb{A}_0^T p_t^0 - \mathbb{Q}_0 X_t^0 - \mathbb{N}_0 u_t^0)dt + e^{-\mathbb{A}_0^T t}dM_t^0. \tag{7.57}$$

According to the martingale representation theorem, the martingale M_t^0 can be written as

$$M_t^0 = M_0^0 + \int_0^t Z_s^0 dW_s^0, \tag{7.58}$$

where Z_t^0 is an $F_t^{0,r}$-adapted process. Differentiating both sides of (7.58) yields

$$dM_t^0 = Z_t^0 dW_t^0. \tag{7.59}$$

Then, (7.55) and (7.59) are substituted in (7.57) which gives rise to

$$dp_t^0 = \Big[(-\mathbb{Q}_0 + \mathbb{N}_0 R_0^{-1}\mathbb{N}_0^T + \mathbb{N}_0 R_0^{-1}\mathbb{B}_0^T\Pi_0 - \mathbb{A}_0^T\Pi_0)X_t^{0,*} + (\mathbb{N}_0 R_0^{-1}\mathbb{B}_0^T - \mathbb{A}_0^T)s_t^0\Big]dt + q_t^0 dW_t^0, \quad (7.60)$$

where $q_t^0 = e^{-\mathbb{A}_0^T t}Z_t^0$.

Finally, (7.56) and (7.60) are equated which results in a deterministic Riccati equation as

$$\dot{\Pi}_0 + \Pi_0\mathbb{A}_0 + \mathbb{A}_0^T\Pi_0 - (\mathbb{B}_0^T\Pi_0 + \mathbb{N}_0^T)^T R_0^{-1}(\mathbb{B}_0^T\Pi_0 + \mathbb{N}_0^T) + \mathbb{Q}_0 = 0, \quad \Pi_0(T) = \mathbb{G}_0, \quad (7.61)$$

and a stochastic offset equation as

$$ds_t^0 + \Big(\big[(\mathbb{A}_0 - \mathbb{B}_0 R_0^{-1}\mathbb{N}_0^T)^T - \Pi_0\mathbb{B}_0 R_0^{-1}\mathbb{B}_0^T\big]s_t^0 + \Pi_0\mathbb{M}_t^0\Big)dt + (\Pi_0\Sigma_0 - q_t^0)dW_t^0 = 0, \quad s_T^0 = 0. \quad (7.62)$$

To derive the optimal control action for minor agent $\mathcal{A}_i$, $i \in \mathfrak{N}$, as well as the corresponding Riccati and offset equations, a similar approach is followed. Utilizing *Theorem 6.2* in Chapter 6 ([68]), the Gâteaux derivative of the extended cost functional $J_k^{ex'}(u^i)$, $k \in \mathfrak{K}$, for minor agent $\mathcal{A}_i, i \in \mathfrak{N}$, is computed as

$$\langle J_k^{ex'}(u^i), \omega^i\rangle = \mathbb{E}\bigg[\int_0^T (\omega_t^i)^T\Big\{\mathbb{N}_k^T X_t^{i,u} + R_k u_t^i + \mathbb{B}_k^T\Big(e^{-\mathbb{A}_k^T t}M_t^i - \int_0^t e^{\mathbb{A}_k^T(s-t)}(\mathbb{Q}_k X_s^{i,u} + \mathbb{N}_k u_s^i)ds\Big)\Big\}dt\bigg]. \quad (7.63)$$

where the martingale M_t^i is defined by

$$M_t^i = \mathbb{E}\Big[e^{\mathbb{A}_k^T T}\mathbb{G}_k X_T^{i,u} + \int_0^T e^{\mathbb{A}_k^T s}(\mathbb{Q}_k X_s^{i,u} + \mathbb{N}_k u_s^i)ds\Big)\Big|\mathcal{F}_t^{i,r}\Big]. \quad (7.64)$$

Given *Assumption 7.8*, as per *Theorem 3*, the optimal control action $u_t^{i,*}$ for minor agent $\mathcal{A}_i, i \in \mathfrak{N}$, in the infinite population limit is given by

$$u_t^{i,*} = -R_k^{-1}\Big[\mathbb{N}_k^T X_t^{i,*} + \mathbb{B}_k^T\Big(e^{-\mathbb{A}_k^T t}M_t^i - \int_0^t e^{\mathbb{A}_k^T(s-t)}(\mathbb{Q}_k X_s^{i,*} + \mathbb{N}_k u_s^{i,*})ds\Big)\Big], \quad (7.65)$$

which is obtained by setting (7.63) to zero for all possible paths of $\omega_t^i \in \mathcal{U}^i$.

Let us define p_t^i as

$$p_t^i = e^{-\mathbb{A}_k^T t}M_t^i - \int_0^t e^{\mathbb{A}_0^T(s-t)}(\mathbb{Q}_k X_s^{i,*} + \mathbb{N}_k u_s^{i,*})ds, \quad (7.66)$$

which is in fact the adjoint process for the minor agent $\mathcal{A}_i$'s system in the stochastic maximum principle framework. Then we adopt an ansatz for p_t^i given by

$$p_t^i = \Pi_k(t) X_t^{i,*} + s_t^{i,k}, \tag{7.67}$$

whose substitution in (7.65) results in a linear state feedback form for $u_t^{i,*}$ as

$$u_t^{i,*} = -R_k^{-1}\big[\mathbb{N}_k^T X_t^{i,*} + \mathbb{B}_k^T\big(\Pi_k(t)X_t^{i,*} + s_t^{i,k}\big)\big]. \tag{7.68}$$

To find $\Pi_k(t) \in \mathbb{R}^{(3+K)n\times(3+K)n}$ and $s_t^{i,k} \in \mathbb{R}^{(3+K)n}$, first both sides of (7.67) are differentiated and then (7.34) and (7.68) are substituted which yields

$$\begin{aligned} dp_t^i = \Big[\big(\dot{\Pi}_k + \Pi_k\mathbb{A}_k - \Pi_k\mathbb{B}_kR_k^{-1}\mathbb{N}_k^T - \Pi_k\mathbb{B}_kR_k^{-1}\mathbb{B}_k^T\Pi_k\big)X_t^{i,*} - \Pi_k\mathbb{B}_kR_k^{-1}\mathbb{B}_k^T s_t^{i,k} \\ + \Pi_k\mathbb{M}_t^k + ds_t^{i,k}\Big]dt + \Pi_k\Sigma_k(t)dW_t^i. \end{aligned} \tag{7.69}$$

Next, both sides of (7.66) are differentiated

$$dp_t^i = (-\mathbb{A}_k^T p_t^i - \mathbb{Q}_k X_t^{i,*} - \mathbb{N}_k^T u_t^{i,*})dt + e^{-\mathbb{A}_k^T t}dM_t^i. \tag{7.70}$$

According to the martingale representation theorem, the martingale M_t^i shall be written as

$$M_t^i = M_0^i + \int_0^t Z_s^i dW_s^i, \tag{7.71}$$

or equivalently, when both sides of (7.71) are differentiated, as

$$dM_t^i = Z_t^i dW_t^i, \tag{7.72}$$

where Z_t^i is an $\mathcal{F}_t^{i,r}$-adapted process.

Then, (7.68) and (7.72) are substituted in (7.70) which gives

$$\begin{aligned} dp_t^i = \Big[(-\mathbb{Q}_k + \mathbb{N}_kR_k^{-1}\mathbb{N}_k^T + \mathbb{N}_kR_k^{-1}\mathbb{B}_k^T\Pi_k - \mathbb{A}_k^T\Pi_k)X_t^{i,*} \\ + (\mathbb{N}_kR_k^{-1}\mathbb{B}_k^T - \mathbb{A}_k^T)s_t^{i,k}\Big]dt + q_t^i dW_t^i, \end{aligned} \tag{7.73}$$

where $q_t^i = e^{-\mathbb{A}_k^T t}Z_t^i$. Finally, (7.69) is equated with (7.73) which yields

$$\dot{\Pi}_k + \Pi_k\mathbb{A}_k + \mathbb{A}_k^T\Pi_k - (\mathbb{B}_k^T\Pi_k + \mathbb{N}_k^T)^T R_k^{-1}(\mathbb{B}_k^T\Pi_k + \mathbb{N}_k^T) + \mathbb{Q}_k = 0, \quad \Pi_k(T) = \mathbb{G}_k, \tag{7.74}$$

$$ds_t^{i,k} + \Big(\big[(\mathbb{A}_k - \mathbb{B}_kR_k^{-1}\mathbb{N}_k^T)^T - \Pi_k\mathbb{B}_kR_k^{-1}\mathbb{B}_k^T\big]s_t^{i,k} + \Pi_k\mathbb{M}_t^k\Big)dt + (\Pi_k\Sigma_k - q_t^i)dW_t^i = 0, \quad s_T^{i,k} = 0, \tag{7.75}$$

$i \in \mathfrak{N}, k \in \mathfrak{K}$.

Finally, following the asymptotic equilibrium analysis of [16], the set of control actions

$\mathcal{U}_{MF}^{N,*} \triangleq \{u_t^{i,*};\ i \in \mathfrak{N}\}$, $1 \leq N < \infty$, yields an ϵ-Nash equilibrium for the large population system given by (7.1)-(7.3) and (7.5)-(7.6). $\square$

7.4 Conclusions

In this chapter, we introduced and formulated a new class of major minor MFG systems motivated from financial and economic systems. In this novel setup, the major agent and each of the mass of minor agents interact with a common process, and this process also affects their cost functionals. The common process is influenced by (i) a latent process which is not observed, (ii) a common Wiener process, (iii) the major agent's state and control action, and (iv) the average state and control action of all minor agents. Then, we used the convex analysis method to establish the best trading strategies for all agents which yield an ϵ-Nash equilibrium. Our framework can be easily extended to the case where each agent's dynamics also is influenced by the common process.

Chapter 8

Future Research Directions

LQG Hybrid Mean Field Game Theory

In this thesis, the hybrid mean field game theory has been established for a class of MM LQG MFG systems for which controlled switching and stopping times are state and trajectory independent, and only depend on the dynamical and cost functional parameters of each agent. As a result, all agents of the same type would stop or switch at the same time. and state jumps subject to possible changes in the dimension of the state space. It is of significant interest to develop and extend the hybrid MFG theory in the following directions.

- Switchings and stoppings upon arrival on switching manifolds where individuals in subpopulations may quit or switch to alternative dynamics at different times. This is of particular importance in the modelling of optimal execution problems where traders stop or switch after reaching a specific number of shares.
- Tractable formulation for several subpopulations, including a systematic methodology for treating more complex discrete state sequence lattices.
- Extend model so that subpopulations and individuals can rejoin game after quitting or switch back and forth between specific modes of operation.

Mean Field Game Systems with Multiple Major Agents

All the theorems in this thesis are established for the LQG MFG systems with one major agent and a large population of minor agents. This is not necessarily the case as for example in markets there are usually several institutional investors whose trading actions move the asset price significantely. Hence, the extension of MM MFG theory to incorporate multiple major agents surely merits study.

Mean Field Game Systems with Latent and Common Processes

In this thesis, MM MFG theory is extended to incorporate a common process in the dynamics of all agents. The common process is driven by a latent process which is not directly observed by agents. Given that the realized trajectories of the common process are completely observed, the posteriori estimates of the latent process are generated and subsequently the future movements in the common process are predicted. The generalization of this setup to the systems influenced by more than one latent processes, and where the nested information patterns on latent processes are available to agents, and studying the value of information is of interest. Moreover, generalizing the setup and the utilized convex analysis method to accommodate jump processes, as well as correlated Wiener processes, present interesting and important extensions.

Partially Observed Nonlinear Major Minor MFG Theory

In this thesis, partially observed LQG major minor MFG (PO LQG MM MFG) systems are formulated where (i) major agent has partial observations on its own state, and (ii) each minor agent has partial observations on its own state and the major agent's state. Partially observed nonlinear major minor mean field game (PO NL MM MFG) problems in the case where the major agent completely observes its own state and each minor agent partially observes its own state and the major agent's state are studied in [22–24]. An extension of the problem tackled in this thesis, where the major agent also partially observes its own state, to the PO NL MM MFG case could be a future research direction.

Numerical Experiments

Performing simulations and analysis using real market data in all three parts of the thesis would be of particular interest. Furthermore, the sensitivity analysis of the terms in the cost functional of each trader and its impact on the market equilibrium would be another future direction.

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