

**MODELLING POLYMORPHISM
WITH CATEGORIES**

by

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*Les ignorants ont un mot pour décrire les
autres ignorants, spécialisés : ils les
appellent des savants*

Alfred Jerry

ABSTRACT

In this work we describe a category of domains, whose objects are in general categories instead of posets, such that J.-Y. Girard's category of qualitative domains and stable functions is contained in it as a full subcategory. We describe two ways of interpreting Martin-Löf type theory in this category, the first one allowing Σ and Π , the second one only Π . Finally we show how to extend the second interpretation to a model of the theory of constructions.

RESUME

Nous présentons dans ce travail une catégorie de domaines, dont les objets sont en général des catégories et non des ensembles ordonnés, telle que la catégorie des domaines qualitatifs et fonctions stables de Girard en est une sous-catégorie pleine. Nous donnons deux façons d'y interpréter la théorie des types de Martin-Löf : la première permet d'interpréter les quantificateurs Σ et Π , la seconde seulement Π . Finalement nous étendons la seconde interprétation à un modèle de la théorie des constructions.

TABLE OF CONTENTS

Abstract	iii
Résumé	iv
Table of contents	v
Acknowledgements	vi
Introduction	vii
 Chapter 0	 1
Chapter 1	19
Chapter 2	31
Chapter 3	40
Chapter 4	71
 Conclusion	 128
References	129

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INTRODUCTION

The lambda calculus, which embodies a certain formal property of naive sets (the application of a function to an argument) has given rise to syntactic systems whose behavior is definitely impossible to model in the category of sets. The simplest example is the untyped lambda calculus, which forces one to consider the idea of a set which is isomorphic to its own set of endofunctions. In order to find models of this theory that "live in the real world", Dana Scott invented a notion of "limited set", that allows one to escape the Cantor diagonalization argument. In their original presentation [Sc72] these "sets" could be defined in two ways: as a class of topological spaces, or as a class of complete lattices. In the first case the "functions" between them were simply the continuous functions. In the second case, the "functions" were morphisms of posets that preserved filtered colimits. This pioneering work, due in a major part to its applications in theoretical computer science, gave rise to a large amount of activity, which usually goes under the name of domain theory. The topological point of view quickly receded in the background in favor of the lattice-theoretical approach (poset-theoretical would be more fitting since Scott lattices were soon generalized to posets that were not lattices).

These domains can furnish models of versions of lambda calculus which have a property intuitively quite contrary to the accepted notion of set: the naive interpretation of these theories allows one to take a large indexed family of "sets" (meaning the indexing "set" is "large", in the order of size of the class of all "sets" of the theory, or a finite power thereof) and still get a "set". Clearly, this goes against the Russel paradox, and cannot be modeled in the category of ordinary sets. In fact, the proof that any one of these systems is consistent is always quite non-trivial. This general phenomenon has been called polymorphism; the simplest polymorphic system is second-order lambda calculus, called F by J.-Y. Girard in his thesis [Gi72], and discovered independently by Reynolds [Re]. A different

class of polymorphic systems is given by systems with dependent types, à la Martin-Löf. The first and simplest polymorphic system with dependent types is the theory constructions of Coquand-Huet [Co85].

Two years ago Girard constructed a model of F using a version of domain theory slightly different from Scott's [Gi86]. The domains themselves were typical of ordinary domain theory. But the morphisms of domains (the "functions") were required to preserve pullbacks in addition to filtered colimits (it turns out Girard had been preceded in this direction by G. Berry) [Be]. The advantage of this approach is that it makes everything "leaner" and easier to calculate. In fact Girard's model was the first model of a polymorphic theory where anything at all could be calculated! The essential reason for this computational simplicity is the property of *atomicity* which makes the posets under consideration more than abstract, general posets, but *concrete* ones, (as subposets of the power set of a given set) and thus much easier to describe or present.

To the author, it was obvious from the start that Girard's posets (he called them qualitative domains) could be generalized to categories. In fact the category qD of qualitative domains and their "strong morphisms" of [Gi86] could be taken as a paradigm of the sort of category that would generalize the concept of a qualitative domain. There was interest in doing this generalization, since the modelling of a system like the theory of constructions forces one to consider a "domain of all domains", which for example could very well be (and in fact will be in our case) the category qD . There already are in the literature models of polymorphism that use categories as domains, for instance [H-P], which is basically in the tradition of Scott.

In this work we describe in detail a category *Aggr* of domains whose objects are categories we call aggregates, and which contains as a full subcategory Girard's category of domains and "functions". This category contains many models of polymorphism, and we will describe the "most natural" model of the theory of constructions

living in it, which was announced in [La1] . Another model contained in *Aggr* is described in [La2] . Our choice of *Aggr* as a subject of study was motivated by conflicting considerations of generality and simplicity. By experience, the author has found that finding models of polymorphism with dependent types is a two-step procedure: when you meet a potential model the first thing you should do is to try to interpret classical, predicative Martin-Löf type theory in it. An interpretation is essentially given by a class of morphisms which are stable under pullbacks; a category may contain several related interpretations. Only after this should you look for the specifically polymorphic features of the model, say a "notion of small set", along with an "object of all small sets" which in this particular case will allow you to interpret the theory of constructions (a more standard terminology would be "notion of type"). Therefore it would be a good thing to interpret classical Martin-Löf theory in the largest possible category that generalizes Girard's. A candidate for this is Thierry Coquand's category of all "stable categories of embeddings" and all stable (which we call entire) functors [Co88] . It contains *Aggr* as a full subcategory, and there obviously are in it many different "notions of small set", each one giving rise to a model of constructions. Interestingly, the gap of generality between Coquand's approach and ours can be stated exactly: with all other axioms the same, an aggregate is required to have all connected limits, while a category of embeddings is only required to have "small pullbacks" (see below). By slightly generalizing the work of Yves Diers, these conditions on limits can be translated to conditions on the colimit structure of the categories involved; by "colimit" we mean a generalized notion of colimit which applies to e.g. fields or algebraically closed fields. In this light categories of embeddings have features which sharply deviate from aggregates: for example, a generalized initial object in a category of embeddings can start having nontrivial automorphisms.

We decided to work in a more limited context for practical reasons. Our proof of the main theorem is already quite intricate, and we feel an added layer of generality (going from "multi" to

"poly") would simply add bulk to the proof without much insight. Maybe we are wrong in this respect. But also, what is the use of having a model if you can't calculate things in it ? The trouble is that a general notion of presentation for a category of embeddings (e.g. Coquand's "categories of information") seems a very unwieldy tool, just as complicated as the category it defines. Things are already much too complicated with aggregates. Fortunately there is a full subcategory $Sem \subset Aggr$, whose objects we call semigranular categories, for which there is a simple representation theorem: semigranular categories are concrete categories, and their structure can be described in detail; they are exactly the aggregates that obey Girard's magic property of atomicity. It is unfortunate that due to time constraints we cannot include the chapter about their structure theory in this work; it will be published independently. Using only semigranular categories, one can interpret Martin-Löf theory with Π but not Σ ; aggregates appear as auxiliaries but do not model types. This is already something since the original theory of constructions had only Π , which is by far the more important connective from a polymorphic point of view. Thus, the model of constructions we will give will be "essentially semigranular". If one adds in aggregates, it is possible to interpret Martin-Löf type theory with both Σ and Π , but not the equality predicate (which would immediately kill the possibility of polymorphism). This larger universe also possesses a "notion of small set", and therefore another model of the theory of constructions. This latter model will only be described briefly.

Thus our work centers around the description of the category of aggregates and that of semigranulars, and the Martin-Löf structure(s) they contain. It is the first time that the latter is done in full for such a class of domains, although [Co88] contains partial results. We have been faithful to Girard's approach, and generalize his notion of trace by the definition of generic and ultrageneric arrows. They are instances of the general concept of a universal family, discovered by Diers. This allows us to describe the Π operator in much more detail than if we were only using, say, limits

and filtered colimits. We should say that the result falls short (but, we think, not by much) of our hope of making everything computable; examples of how complicated things can be will be given (4.6). Still, the model that we get is way simpler than any model previously constructed that uses categories as domains; compare ours with [H-P].

We have been talking about interpreting Martin-Löf type theory in categories without saying much on how this is done. During the past year, a general categorical framework, built on the work of Cartmell [Ca], has appeared. It is characterized by a remarkable simplicity, in comparison with the formal system (judgements, and the like) interpreted in it. A detailed account is in [H-P], which we will use as a reference, although our terminology will not coincide perfectly. Regrettably, this paper contains a gap, in that the actual interpretation of the formal system into the categorical framework is not given. This is a long multiple induction (triple, at the least), which Andy Pitts tells me he will cover in a subsequent paper. We will nevertheless defer to the authority of [H-P] in these matters; in particular we will not describe any formal system, but rely entirely on categorical properties.

The plan of the work is as follows. Chapter 0 contains all the definitions of categorical concepts which go beyond Mac Lane [CWM]. An exception is the notion of Grothendieck fibration, which is only cursorily treated there, since it has recently become quite referred to in theoretical computer science [CGW],[Eh],[Pi]. The proofs are not given in full, since they are easy and well documented elsewhere. Chapter 1 is the definition of aggregates and semigranular categories, along with their basic properties. Chapter 2 is the general theory of morphisms (stable, or entire, functors) between aggregates. Chapter 3 is the theory of variable aggregates, that is, display map. Chapter 4 is the proof of the main theorem on the Martin-Löf structures that can be defined on the category of aggregates, and the description of the model(s) of constructions.

CHAPTER 0

This is a preliminary chapter in which we give some more or less well-known definitions and standardize the notation. We will only give sketches of the proofs.

Unless it is said otherwise, a category is always locally small, i.e. has small hom-sets. For variety, we will also call the morphisms of a category arrows. A small set will be called simply a set, and a potentially large one a class. We will denote the category of small sets and functions by Set . Let $\mathcal{C}$ be a category, and X, Y, Z objects of it. The set of morphisms from X to Y is denoted $\mathcal{C}(X, Y)$. If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ we denote their composition by either gf or $g \circ f$, depending on readability. If $\mathcal{D}$ is another category, $F, G: \mathcal{C} \rightarrow \mathcal{D}$ functors and $\varphi: F \rightarrow G$ a natural transformation, for $X \in \mathcal{C}$ we denote the components of φ by $\varphi_X: FX \rightarrow GX$. We will use the subscript notation for diagrams, that is, a small diagram $(X_d)_{d \in \mathcal{D}}$ in $\mathcal{C}$ is composed of a small category $\mathcal{D}$ and a functor $X: \mathcal{D} \rightarrow \mathcal{C}$, and for $q: d \rightarrow d'$ in $\mathcal{D}$ we denote $Xq: Xd \rightarrow Xd'$ by $X_q: X_d \rightarrow X_{d'}$. We will also sometimes say things like $\varinjlim X$. Given $X \in \mathcal{C}$ a subobject of X is as usual an equivalence class of monomorphisms into X , but we will be notationally quite abusive and say things like "let $y: Y \rightarrow X$ be a subobject of X " or even "let Y be a subobject of X ". We will use the term "essential" in the standard meaning of "modulo equivalence"; for example, an essentially small category is one which is equivalent to a small one.

Given a category $\mathcal{C}$ and an object $X \in \mathcal{C}$, the slice category $\mathcal{C}/X$ is defined as usual: an object is a pair (A, a) where $A \in \mathcal{C}$ and $a: A \rightarrow X$ and a morphism $f: (A, a) \rightarrow (B, b)$ is an $f: A \rightarrow B$ with $bf = a$. Thus we will use the same name to describe morphisms in different categories; there is no danger to this.

We will assume the notion of filtered category ([CWM IX,1]) and filtered diagram is known to the reader. We will also assume the

reader knows a bit about Grothendieck fibrations. The next section is dedicated to a review of $\aleph_0$ -accessible categories [M-P].

0.1 Definition

Let a category $\mathcal{C}$ have filtered colimits, that is, all diagrams $D \rightarrow \mathcal{C}$ where D is small and filtered have a colimit. An object $x \in \mathcal{C}$ is said to be finitely presentable (abbreviated as f.p.) if the functor $\mathcal{C}(x, -): \mathcal{C} \rightarrow \mathbf{Set}$ preserves filtered colimits. In the category of groups, an f.p. object is the same as a group that can be described by a finite number of generators and relations. If $\mathcal{C}$ is a poset, an f.p. object is also called an isolated (or compact) point.

In a category $\mathcal{C}$ we say that idempotents split if given any endomorphism f of $\mathcal{C}$ such that $f \circ f = f$ the equalizer of f and identity exists (equivalently (exercise), the coequalizer of f and identity).

0.2 Definition

Let $\mathcal{C}$ be a small category. A presheaf $F: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ is said to be flat if in the associated discrete fibration $\text{el}(F) \rightarrow \mathcal{C}$ the category $\text{el}(F)$ is a filtered category.

0.3 Theorem

Let $\mathcal{A}$ be a small category. Let $\text{Flat}(\mathcal{A}) \subset \mathbf{Set}^{\mathcal{A}^{\text{op}}}$ be the full subcategory of flat presheaves and $\text{Flat}_f(\mathcal{A})$ its full subcategory of finitely presented flat presheaves. Then idempotents split in $\text{Flat}_f(\mathcal{A})$ and

$$\begin{array}{ccc} \text{Flat}_f(\mathcal{A}) & \subset & \text{Flat}(\mathcal{A}) \subset \mathbf{Set}^{\mathcal{A}^{\text{op}}} \\ & \nwarrow & \nearrow \text{Yoneda} \\ & \mathcal{A} & \end{array}$$

the Yoneda inclusion factors through $\text{Flat}_f(\mathcal{A})$. If idempotents split in $\mathcal{A}$ then the arrow to the left of the diagram is an equivalence of categories. If not, it is a universal arrow (in the "up to equivalence"

sense of the solution of universal problems in the category of categories) to the "free idempotent-splitting completion of $\mathbf{A}$ ".

0.4 Definition

A (locally small) category $\mathbf{C}$ is said to be finitely accessible or $\aleph_0$ -accessible if

- i) It has filtered colimits.
- ii) The full subcategory $\mathbf{C}_f$ of f.p. objects is essentially small.
- iii) For every X in $\mathbf{C}$ the category $\mathbf{C}_f/X$ (which has for objects pairs (A,a) , where $a:A \rightarrow X$, $A \in \mathbf{C}_f$, and where a morphism $f:(A,a) \rightarrow (B,b)$ is a $f:A \rightarrow B$ with $bf = a$) is filtered. The forgetful functor $U_X: (\mathbf{C}_f/X) \rightarrow \mathbf{C}$ sending $f:a \rightarrow b$ above to $f:A \rightarrow B$ has an obvious cocone to X and that cocone is a colimit cocone.

The reader can show as an exercise (he will need the help of the concept of (co-)final functor, [M-P 1.1.2, CWM IX,3]) that condition iii) is equivalent to saying that every object of $\mathbf{C}$ is the colimit of a filtered diagram of f.p. objects.

0.5 Theorem

For every small $\mathbf{A}$, $\text{Flat}(\mathbf{A})$ is $\aleph_0$ -accessible. If $\mathbf{C}$ is $\aleph_0$ -accessible then idempotents split in $\mathbf{C}_f$ and the functor $\mathbf{C} \rightarrow \text{Flat}(\mathbf{C}_f)$ which sends $X \in \mathbf{C}$ to $\mathbf{C}(-, X)$ is an equivalence.

These two very important facts are easy to prove, and are a recommended exercise for the inexperienced reader. Another interesting related fact is that $\text{Flat}(\mathbf{A})$ is the universal completion of the small category $\mathbf{A}$ with respect to filtered colimits.

The class of $\aleph_0$ -accessible categories has a number of subclasses, determined by additional structure on the categories. A very important (historically and technically) subclass is the following, which we will make no use of, but mention for the sake of completeness.

0.6 Theorem (Gabriel-Ulmer duality, Makkai-Paré version)

Let $\mathcal{C}$ be an $\aleph_0$ -accessible category. Then the following are equivalent.

- i) $\mathcal{C}$ is complete.
- ii) $\mathcal{C}$ is cocomplete.
- iii) $\mathcal{C}_f$ has finite colimits.

Such a category is called a locally $\aleph_0$ -presentable category. Notice that a flat presheaf on $\mathcal{C}_f$ is the same as a functor $\mathcal{C}_f^{\text{op}} \rightarrow \text{Set}$ which preserves finite limits. Thus a locally $\aleph_0$ -presentable category is precisely the category of finite-limit preserving functors from a (essentially uniquely determined) small category with finite limits to sets (with morphisms all natural transformations). This algebraic result has a logical (i.e. syntactic) counterpart. The $\aleph_0$ -locally presentable categories are the categories of models of first-order theories where the only connectives used are conjunction and the quantifier "there exists a unique" [Cs, chapter III]. This logical facet is also true for other classes of $\aleph_0$ -accessible categories.

Gabriel-Ulmer duality in its original version did not use the notion of $\aleph_0$ -accessibility, but instead that of a strong generating set, which has some technical advantages.

0.7 Definition

A set Q of objects of a category $\mathcal{C}$ is called a strong generating set if the functor

$$\coprod_{A \in Q} \mathcal{C}(A, -) : \mathcal{C} \rightarrow \text{Set}$$

reflects isomorphisms ($\coprod$ is the disjoint sum). This is a fancy way of saying the following: a morphism $f: X \rightarrow Y$ is an isomorphism iff for every $a \in A$ the morphism of sets $\mathcal{C}(A, f) : \mathcal{C}(A, X) \rightarrow \mathcal{C}(A, Y)$ is an isomorphism. The element of a strong generating set which is a singleton is called a strong generator. For example in the category of groups $\mathbb{Z}$ is a strong generator.

0.8 Theorem (Gabriel-Ulmer duality, original version)

A category $\mathcal{C}$ is locally $\aleph_0$ -presentable iff it is cocomplete and has a strong generating set of f.p. objects.

We are interested in two classes of $\aleph_0$ -accessible categories which are a generalization of the one discovered by Gabriel-Ulmer. One is due to Diers [Di80] and the other is a slight generalization of it, due to the author.

0.9 Definition

Let $\mathcal{C}$ be a category. An object $A \in \mathcal{C}$ is said to be generic if the following holds: given any diagram of the form

$$\begin{array}{ccc} & Y & \\ & \downarrow f & \\ A & \longrightarrow & X \\ & a & \end{array}$$

in $\mathcal{C}$, there exists a unique $b: A \rightarrow Y$ with $fb = a$.

0.10 Proposition

If A is a generic object of $\mathcal{C}$ then every morphism $b: B \rightarrow A$ is a split epi. If B is also generic, then b is an iso.

Proof

Looking at

$$\begin{array}{ccc} & B & \\ & \downarrow b & \\ A & \longrightarrow & A \\ & 1_A & \end{array}$$

by genericity of A there is a unique $f: A \rightarrow B$ with $bf = 1_A$. If B is also generic, then f is not only a split mono, but a split epi, too, which forces it to be an iso.

0.11 Proposition

Let $\mathcal{C}$ have pullbacks and suppose all its morphisms are mono. Then $A \in \mathcal{C}$ is generic iff it has no proper subobject (i.e. its only subobject is itself).

Proof

Suppose A has no proper subobject. Then given

$$\begin{array}{ccc} & Y & \\ & \downarrow f & \\ A & \longrightarrow & X \\ & a & \end{array}$$

let $p: P \rightarrow A$ and $q: P \rightarrow Y$ be the pullback. By assumption p is an isomorphism and qp^{-1} is the required filler. It is unique since all morphisms are mono. For the converse, if A is generic, then any mono to it is a split epi, i.e. an iso.

0.12 Definition

A poly-initial family in a category $\mathcal{C}$ is a family $(A_j)_{j \in J}$ of generic objects such that given any $X \in \mathcal{C}$, there exists a unique $j \in J$ and a morphism $a: A_j \rightarrow X$. a is not necessarily unique, but since A is generic, defined up to unique automorphism of A_j . To say that $\mathcal{C}$ has a poly-initial family is to say that every one of its components has an "initial object with possibly nontrivial automorphisms".

Example

Take $\mathcal{C}$ to be the category of algebraically closed fields. Take J to be the set $\{0\} \cup \{\text{all primes}\}$. Then if for $j \in J$ A_j is the algebraic closure of the prime field of characteristic j the family $(A_j)_{j \in J}$ is poly-initial, as the reader may verify.

We say a category has small pullbacks if all diagrams of the form $(a_i: A_i \rightarrow X)_{i \in I}$ (where I is a small set!), have a limit. Here is a generalization of a famous theorem of Freyd [CWM V,6]:

0.13 Lemma

Let $\mathcal{C}$ have small pullbacks. Then $\mathcal{C}$ has a poly-initial family if and only if it satisfies the solution set condition; that is, there is a family $(C_i)_{i \in I}$ such that every object of $\mathcal{C}$ admits an arrow from a C_i .

Proof

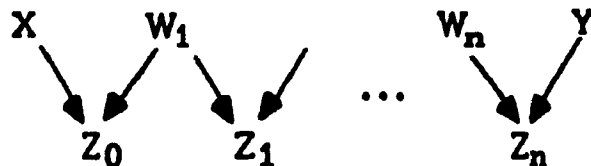
Assume there is a solution set. Choose a C_1 . The category $\mathcal{C}/C_1$ has all products and pullbacks, and is therefore complete. Since it obviously has a solution set, it has an initial object $a_1: A_1 \rightarrow C_1$ [op. cit.]. Let $a: A_1 \rightarrow X$, $f: Y \rightarrow X$. Form the pullback:

$$\begin{array}{ccc} P & \longrightarrow & Y \\ p \downarrow & & \downarrow f \\ C_1 & \longleftarrow A_1 \longrightarrow & X \\ a_1 & & a \end{array} .$$

A morphism $b: A_1 \rightarrow Y$ with $fb = a$ is the same as a splitting of p . Considering such a splitting as a morphism $a_1 \rightarrow a_1 p$ in $\mathcal{C}/C_1$ one easily sees that because a_1 is initial therein the splitting exists and is unique. To get a poly-initial family one just has to take a subset $J \subset I$ such that the family $(A_j)_{j \in J}$ is a minimal representative choice of isomorphism classes. The converse is trivial.

0.14 Definition

A category $\mathcal{D}$ is connected if it is nonempty and any pair of objects $X, Y \in \mathcal{D}$ can be joined by a "finite zig-zag" of morphisms; i.e. there are $Z_0, \dots, Z_n, W_1, \dots, W_n \in \mathcal{D}$ and morphisms



The zig-zag property between X and Y defines an equivalence relation on the objects of $\mathcal{C}$. The full subcategory determined by such an equivalence class is called a connected component of $\mathcal{C}$ (for short, a component). A component is obviously connected. A

connected limit diagram in some category $\mathcal{C}$ is a diagram $D \rightarrow \mathcal{C}$ which has a limit and where D is a small connected category. Examples of (finite) connected limits are pullbacks and equalizers. An example of a category which has all connected limits but is not complete is the category of fields.

0.15 Definition

An object A of a category $\mathcal{C}$ is said to be an initial candidate if it is initial in its connected component of $\mathcal{C}$.

0.16 Proposition

An object $A \in \mathcal{C}$ is an initial candidate iff it is generic and has a trivial group of automorphisms.

Proof

Suppose A is an initial candidate. Then obviously, it is generic, and its only automorphism is identity. For the converse, if A is generic, it is easy to see that for any X in its component, there is a morphism $A \rightarrow X$: just "climb the steps of the zig-zag". If $f, g: A \rightarrow X$ are two morphisms, by genericity of A there exists $\alpha: A \rightarrow A$ with $f\alpha = g$. But by assumption α is identity and so $f = g$.

0.17 Definitions (Diers)

A family $(0_j)_{j \in J}$ of objects of a category $\mathcal{C}$ is called a multi-initial family if the following holds: given any $X \in \mathcal{C}$, the disjoint sum $\coprod_{j \in J} \mathcal{C}(0_j, X)$ has a unique element. Let $X, Y \in \mathcal{C}$, and $x: 0_{j'} \rightarrow X$ and $y: 0_{j''} \rightarrow Y$ be the morphisms X and Y determine. If there is $f: X \rightarrow Y$ then having both fx and y going to Y implies that $j = j''$ and $fx = y$. More generally if X and Y can be connected by a zig-zag as in 0.14, we can prove easily that $0_j = 0_{j'}$. Hence a multi-initial family is exactly a choice, for every connected component of $\mathcal{C}$, of an initial candidate in that component (so $\mathcal{C}$ must have only a set of components). For example, in the category of fields the family $\{\mathbb{Q}\} \cup (\mathbb{Z}/(p))_p$ prime is a multi-initial family.

Let $A = (A_i)_{i \in I}$ be a family of objects of some category $\mathcal{C}$. A can be thought of as a diagram from the discrete category I . A cocone (sometimes called an A -cocone) is then a pair $(X, (x_i)_{i \in I})$ where $X \in \mathcal{C}$ and $x_i: A_i \rightarrow X$ is a family of morphisms. A morphism of cocones $f: (X, (x_i)_i) \rightarrow (Y, (y_i)_i)$ is an $f: X \rightarrow Y$ such that $fx_i = y_i$. A multicoproduct family for A is a multi-initial family in the category of cocones and morphisms of cocones; that is, it is a family $(X^s, (x^s_i)_i)_{s \in S}$, $x^s_i: A_i \rightarrow X^s$ of cocones such that given any cocone $(Y, (y_i)_i)$ there exists a unique $s \in S$ and a unique $f: X^s \rightarrow Y$ which is a morphism of A -cocones $(X^s, (x^s_i)_i) \rightarrow (Y, (y_i)_i)$. We say that $\mathcal{C}$ has multicoproducts if any family $(A_i)_{i \in I}$ admits a multicoproduct family. More generally we say $\mathcal{C}$ is multicomplete (or has all multicolimits) if for any small category D , any diagram $D: D \rightarrow \mathcal{C}$, the category of cocones from D has a multi-initial family. We will not say much here about the general notion of multicolimit since we can define everything we need in terms of multicoproducts.

Examples

A category which has all coproducts has all multicoproducts. In this case multicoproduct families are one-element families, with coproduct cocones as unique elements.

Let $\mathcal{C}$ be the category of total orderings and injective order preserving functions. Let $(A_i)_{i \in I}$ be a family of objects of $\mathcal{C}$. Let a covering cocone be a $(x_i: A_i \rightarrow X)_{i \in I}$ such that $\bigcup_i x_i(A_i) = X$ where $x_i(A_i) \subset X$ is the image of A_i by x_i . It is easy to prove (exercise) that between any two covering cocones there exists at most one morphism of cocones, and that it has to be an isomorphism. For every isomorphism class of covering cocones choose one representative cocone, and call the resulting family $(X^s, (x^s_i)_i)_{s \in S}$. It is easy to see that this is a small family: every isomorphism class corresponds to a unique pair (E, O) , where E is an equivalence relation on the disjoint sum

$$\bigsqcup_{i \in I} A_i$$

of underlying sets and O an ordering on the set of classes $\coprod_1 |A_i|/E$ (the correspondence is bijective if one puts simple conditions on E and O). The family $(X^s, (x_i^s)_i)_s$ is a multicoproduct family : given a cocone $(y_i: A_i \rightarrow Y)$, if one takes the image $Y' \subset Y$

$$Y' = \bigcup_i y_i(A_i)$$

then the obvious cocone $(y'_i: A_i \rightarrow Y')_i$ is a covering cocone and therefore there exists a unique $s \in S$ and a unique isomorphism from $(x_i^s)_i$ to it.

Let us show what it means for a poset P to have multicoproducts (or equivalently, to be multicocomplete) as a category: for every family $(a_i)_{i \in I}$ of elements of P there is a family $(x_s)_{s \in S}$ with $a_i \leq x_s$ such that given any upper bound y of $(a_i)_i$ there is a unique s with $x_s \leq y$. An important case of this is when P is consistently cocomplete, meaning that every bounded set has a least upper bound. In fact to say that a poset is consistently cocomplete is just to say that it is multicocomplete and that multicoproduct families are either empty or singletons.

Notice that for $\mathcal{C}$ having multicoproducts includes the case of the empty family and that means $\mathcal{C}$ has a multi-initial family: it has a small set of connected components and every one of them has an initial object.

Let $\mathcal{C}$ have multicoproducts. We say a cocone $(y_i: A_i \rightarrow Y)_{i \in I}$ is a coproduct candidate (or simply a candidate) if it belongs to a multicoproduct family; that is, if it is an initial candidate in the category of cocones. We have the following terminology: a given cocone $(y_i: A_i \rightarrow Y)_{i \in I}$ determines a candidate $(x_i: A_i \rightarrow X)_{i \in I}$, unique up to unique isomorphism of cocones, and a unique $f: X \rightarrow Y$, the factoring determined by $(y_i)_i$ such that

$$f x_i = y_i \quad .$$

Obviously, if two $(A_i)_i$ -cocones belong to the same component of the category of cocones they determine the same candidate. Arguments of this sort will appear repeatedly. Let $(x_i: A_i \rightarrow X)_{i \in I}$ be a candidate. The family $(x_i)_i$ is obviously jointly epimorphic; that is, if $f, g: X \rightarrow Y$ and $fx_i = gx_i$ for all i then $f = g$. Also, if $(a_i: B_i \rightarrow A_i)_i$ is a family of isomorphisms and $h: X \rightarrow Z$ another iso, then the family $(hx_i a_i: B_i \rightarrow A_i)_i$ is also a candidate.

Here is the equivalent of 0.13 for multi-initial families.

0.18 Lemma

Let $\mathcal{C}$ have connected limits. Then $\mathcal{C}$ has a multi-initial family iff it satisfies the solution set condition.

Proof

We already know $\mathcal{C}$ has a poly-initial family. It suffices to show (0.16) generic objects have trivial automorphism groups. So let A be generic, $\alpha: A \rightarrow A$. Let $e: E \rightarrow A$ be the equalizer of α and the identity. e is a split epi as well as a regular mono, forcing it to be an iso, and $\alpha = 1_A$.

We can now describe the generalizations of Gabriel-Ulmer duality which were announced earlier.

0.19 Theorem [Di80]

Let $\mathcal{C}$ be a category. Then the following are equivalent.

- i) $\mathcal{C}$ is $\aleph_0$ -accessible and has all connected limits.
- ii) $\mathcal{C}$ is $\aleph_0$ -accessible and is multicocomplete.
- iii) $\mathcal{C}$ has filtered colimits, a small strongly generating set of f.p. objects, and is multicocomplete.

Y. Diers calls such a category an $\aleph_0$ -locally multipresentable category, but we will call it a Diers category.

Sketch of proof

To show $i) \Rightarrow ii)$ use the fact that for any small diagram in $\mathbf{C}$ the category of cocones is $\aleph_0$ -accessible, and therefore satisfies the solution set condition, and apply the previous lemma. $ii) \Rightarrow iii)$ is obvious. To prove $iii) \Rightarrow i)$ show that the full subcategory $\mathbf{C}_f$ of f.p. objects is essentially small (transfinite recursion has to be used (*op. cit.* 4.2)) and finitely multicocomplete (i.e. has multicolimits for finite diagrams), and that if $\mathbf{D}$ is a small finitely multicocomplete category the category $\mathbf{Flat}(\mathbf{D})$ of flat presheaves over $\mathbf{D}$ has connected limits. The result follows from the natural comparison functor $\mathbf{C} \rightarrow \mathbf{Flat}(\mathbf{C}_f)$ which is an equivalence whenever $\mathbf{C}$ is $\aleph_0$ -accessible. The syntactical aspect of Diers categories has been studied by P. Johnstone [Jn79]. A Diers category is precisely the category of models for a first-order theory which only uses the connectors of finite conjunction, unique existence, and (potentially infinite) disjoint disjunction.

Example

We can easily prove that the category of total orderings is a Diers category. Other standard examples include the category of fields, and the category of local rings and local homomorphisms.

It is a natural step to generalize from "multi" to "poly". The whole terminology of multicolimits, etc. can be adapted. We will not bother to give the details since in this work the more general situation plays a secondary role.

A small discrete category is a Diers category of a degenerate nature: all its objects are initial candidates!

0.20 Theorem

Let $\mathbf{C}$ be a category. Then the following are equivalent.

- i) $\mathbf{C}$ is $\aleph_0$ -accessible and has all small pullbacks.
- ii) $\mathbf{C}$ is $\aleph_0$ -accessible and is polycocomplete.

- iii) $\mathcal{C}$ has filtered colimits, a small strongly generating set of f.p. objects, and is polycocomplete.

We will call such a category (we won't have to call it very often, fortunately) a locally $\aleph_0$ -polypresentable category. The proof is the same as for 0.19, replacing every mention of "multi" by "poly", and every mention of "connected limit" by "small pullbacks".

Examples

The paradigm of such categories is the category of algebraically closed fields and morphisms of fields. We leave the necessary verifications to the reader.

A small groupoid (hence, just a group will do) is such a category too, since groupoids have small pullbacks and filtered colimits and they obviously have enough f.p. objects. Groupoids are to $\aleph_0$ -locally polypresentable categories what discrete categories are to Diers categories.

The syntactical presentation of locally $\aleph_0$ -polypresentable categories is not known at this time.

In the sections that follow we will describe the necessary categorical framework needed to interpret Martin-Löf type theory. Remember that its essential characteristic is the presence of *types which depend on another type*; that is, if S is a type of the theory, and s a variable of type S , then there may arise types of the form $T[s]$, in which the variable s appears. The naive interpretation is that T is an S -indexed family of sets. Now there is an already venerable tradition in category theory of interpreting indexed families by morphisms: that is, one represents the above in a category as a morphism $w:T \rightarrow S$, where S models the type S , and T should be thought of as the disjoint union $\coprod_s T[s]$ and w the obvious projection. Then if $f:V \rightarrow S$ is a morphism of the given category such that the pullback $f^*w:f^*T \rightarrow V$ exists, this latter morphism is interpreted as the variable type $T[f(v)]$, where v is of type V and the meaning of f, v, V should be clear. Now

substitution in syntax is rigid: if Q and B are two expressions of a formal language, x a variable appearing in Q , then the substitution $Q[x|B]$ is a uniquely defined entity. In contrast, pullback, the counterpart of type substitution in categories, is defined as a universal property, that is, only up to unique isomorphism. This prompted Cartmell [Ca] to equip his categories with the added structure of predetermined, functorial pullbacks, to make categories look more like languages. There is something to be said for this approach, which gives more rigidity and in some cases could be technically advantageous. We have opted for the opposite tack of adapting the semantics to the more common categorical practice of having things determined only up to isomorphism. This is already common practice in categorical logic. In other words, whenever you need a pullback (or a right adjoint, or whatever universal construction) to interpret a term, just take the first one at hand; the universal properties will ensure that the final outcome will be independent of the chosen pullback. Whence the following definition:

0.21 Definition

A display category (C, D) is made of a not-necessarily-locally-small category C with finite products, along with a distinguished class D of arrows, the display maps, such that

- i) If $E: E \rightarrow C$ is a display map and $F: D \rightarrow C$ any arrow then the pullback $F^*E: F^*E \rightarrow D$ exists and is a display map
- ii) If E is as above and $\alpha: C \rightarrow C'$, $\beta: E' \rightarrow E$ isomorphisms then $\alpha E \beta$ is a display map.
- iii) every isomorphism is a display map.

Some comments are in order. First notice that our notation for pullback is ambiguous, but this poses no problem in practice. Condition i) is called Stability in [H-P], and condition iii) Unit. Condition ii) is a natural complement to condition i), and comes from our desire of making things as close as possible to categorical practice. It says that the property of being a display map is intrinsic

to the map, and not an arbitrary whim; it is always met in practice. The Unit condition gives us the possibility of modelling the (type corresponding to the) one-element set, and all its constant indexed families. This simplifies life in many instances, but actually in all the known models a stronger condition is met: if for $X \in \mathcal{C}$ we denote by $\mathcal{D}_X$ the full subcategory of $\mathcal{C}/X$ whose objects are display maps, then we always have that $\mathcal{D}_X$ has finite products. In this case we will say that $(\mathcal{C}, \mathcal{D})$ admits discrete products.

0.22 Definition

Let $(\mathcal{C}, \mathcal{D})$ be a display category. We say it admits sums if $\mathcal{D}$ is closed under composition.

This condition entails the following well-known consequence: for every $F: F \rightarrow Y$ in $\mathcal{D}$ the pullback functor $F^*: \mathcal{D}_Y \rightarrow \mathcal{D}_F$ has a left adjoint Σ_F , and the Beck condition holds for pullback diagrams with two parallel display maps: if

$$\begin{array}{ccc} & S & \\ E \longrightarrow & F & \\ E \downarrow & & \downarrow F \\ X \longrightarrow & Y & \\ & X & \end{array}$$

is a pullback where $E, F \in \mathcal{D}$ and $G: G \rightarrow F$ is also a display map, then the natural morphism $\Sigma_E S^* G \rightarrow X^* \Sigma_F G$ in $\mathcal{D}_X$ is an iso. Clearly, Σ_F is left composition by F , i.e. $\Sigma_F G = FG$. Admitting sums means that we can interpret the Martin-Löf connective Σ in $(\mathcal{C}, \mathcal{D})$ [H-P].

0.23 Proposition (Streicher [St])

Let $(\mathcal{C}, \mathcal{D})$ be a display category. Then the following are equivalent:

- i) For every display map $F:F \rightarrow Y$ the pullback functor $F^*: \mathcal{C}/Y \rightarrow \mathcal{C}/F$ has a partial right adjoint Π_F which is defined for every object of $\mathcal{D}_F \subset \mathcal{C}/F$ and lands in $\mathcal{D}_Y$. In other words, for every display map $G:G \rightarrow F$ there is a display map $\Pi_F G: \Pi_F G \rightarrow Y$ and $\varepsilon_G: F^* \Pi_F G \rightarrow G$ in $\mathcal{D}_F$ such that ε_G is (co)universal: for every $H:H \rightarrow Y$ in $\mathcal{C}$ we have the isomorphism $\mathcal{C}/Y(H, \Pi_F G) \cong \mathcal{C}/F(F^* H, G)$, mediated in the usual way.
- ii) For every display map $F:F \rightarrow Y$ the pullback functor $F^*: \mathcal{D}_Y \rightarrow \mathcal{D}_F$ has a right adjoint Π_F and the Beck condition holds for pullback diagrams with parallel display maps.

Proof

For i) $\Rightarrow$ ii) we obviously only have to prove the Beck condition. Let E, F, S, X be the same pullback square as above, and let $G:G \rightarrow F$ and $A:A \rightarrow X$ be display maps. Let $E^*A: B \rightarrow E$ be the pullback.

$$\begin{array}{ccccc}
 E^*A & S & G & & \\
 B \longrightarrow & E & \longrightarrow & F & \longleftarrow G \\
 A^*E \downarrow & E \downarrow & & \downarrow F & \\
 A \longrightarrow & X & \longrightarrow & Y & \\
 A & X & & &
 \end{array}$$

We get

$$\begin{aligned}
 \mathcal{D}_X(A, X^* \Pi_F G) &\cong \mathcal{C}/Y(XA, \Pi_F G) && \text{by pullbackness} \\
 &\cong \mathcal{C}/F(S \circ E^*A, G) && \text{by assumption} \\
 &\cong \mathcal{D}_E(E^*A, S^*G) && \text{By pullbackness} \\
 &\cong \mathcal{D}_X(A, \Pi_E S^*G) && ,
 \end{aligned}$$

and this being true for any A , proves the claim. For the converse, let $F:F \rightarrow Y$, $G:G \rightarrow F$ be display maps. We have to show that for any $X:X \rightarrow Y$ in $\mathcal{C}$ $\mathcal{C}/Y(X, \Pi_F G) \cong \mathcal{C}/F(F^*X, G)$. Let $S:E \rightarrow F$, $E:E \rightarrow X$ be the pullback of F and X . Then

$$\begin{aligned}
\mathcal{C}/Y(X, \Pi_F G) &\cong \mathcal{C}/X(1_X, X^* \Pi_F G) \text{ by pullbackness} \\
&= \mathcal{D}_X(1_X, X^* \Pi_F G) \text{ by Unit} \\
&\cong \mathcal{D}_X(1_X, \Pi_E S^* G) \text{ by Beck} \\
&\cong \mathcal{D}_Y(1_E, S^* G) \text{ by } \Pi_E\text{-adjunction} \\
&\cong \mathcal{C}/F(S, G) \text{ by pullbackness} \\
&= \mathcal{C}/F(F^* X, G) \quad \text{QED.}
\end{aligned}$$

When the conditions above are satisfied, we say that $(\mathcal{C}, \mathcal{D})$ admits products. This allows us to interpret the Martin-Löf connective Π . The unexperienced reader should try the following as an exercise.

0.24 Proposition

Let $(\mathcal{C}, \mathcal{D})$ be a display category which admits products. Then $\mathcal{D}_1$ is cartesian-closed (1 the terminal object). If in addition $(\mathcal{C}, \mathcal{D})$ admits discrete products, then for any $X \in \mathcal{C}$ the category $\mathcal{D}_X$ is cartesian-closed, and for any $F: X \rightarrow Y$ the pullback functor $\mathcal{D}_Y \rightarrow \mathcal{D}_X$ preserves the full cartesian-closed structure.

We can now say what to add to a display category in order to be able to interpret the theory of constructions.

0.25 Definition

Let $(\mathcal{C}, \mathcal{D})$ be a display category which admits products. A notion of smallness is a pair $(\mathcal{S}, S)$, where $\mathcal{S}$ is a subclass of $\mathcal{D}$ whose elements are called the small maps, and $S: A \rightarrow S$ is a classifier of small maps in the sense that $S \in \mathcal{S}$ and the elements of $\mathcal{S}$ are exactly the morphisms obtained by pulling back S by an arbitrary arrow of $\mathcal{C}$. We require that for any $F: X \rightarrow Y$ in $\mathcal{C}$ the functor $\Pi_F: \mathcal{D}_X \rightarrow \mathcal{D}_Y$ send objects of $\mathcal{D}_X$ which are small maps to small maps of $\mathcal{D}_Y$ (this last condition is polymorphism proper).

The formal systems we will interpret are simply those of [H-P] that apply to our models. That is, we will construct two display categories using aggregates. Both of them admit products, and the first one also admits sums. Therefore the "purely Martin-Löf" part

of type theory (the restriction of the formal system to Orders) will apply to both models, with both "quantifiers" for the first model and only Π for the second. This is not apparent at first reading of [H-P], but the two quantifiers are independent in both syntax and categorical interpretation (in other words, the system is quite modular, and can be customized by paring down). Then we will describe a notion of smallness for the second model, which will allow the addition of an Order of Types to its formal language. Finally we will give a short description of an Order of Types for the first model.

We end this chapter with a short review of Grothendieck fibrations. The reader who wants more information can consult [CGW], [Eh] or [Pi].

0.26 Definition

Let $E: \mathcal{C} \rightarrow \mathcal{C}$ be a functor between categories. If, for $X \in E$ $EX = S$ we will often say that X is above S , and samewise for morphisms. Let $f: X \rightarrow Y$ be above $s: S \rightarrow T$. We say f is cartesian if for every $g: Z \rightarrow Y$ such that Eg is of the form st , for $t: EZ \rightarrow S$ there is a unique $h: Z \rightarrow X$ above t such that $fh = g$. We say f is cocartesian if f is cartesian for the dual functor $E^{op}: E^{op} \rightarrow \mathcal{C}^{op}$. A

(Grothendieck) fibration is an $E: \mathcal{C} \rightarrow \mathcal{C}$ such that for every $Y \in E$ above T and every $s: S \rightarrow T$ there exists a cartesian $f: X \rightarrow Y$ above s . An opfibration is an E such that E^{op} is a fibration. For $S \in \mathcal{C}$ we denote by E^S the fiber above S , i.e. the category of all objects above S and all morphisms above 1_S . If E is a fibration and $s: S \rightarrow T$ in $\mathcal{C}$ then "the" usual functor $E^T \rightarrow E^S$ will be denoted by s^* . If s^* has a left adjoint we will denote it by $\exists_s$. The following will be left to the reader as an exercise.

0.27 Proposition

Let $E: \mathcal{C} \rightarrow \mathcal{C}$ be a fibration. Then the following are equivalent.

- i) For every morphism s of $\mathcal{C}$ the functor s^* has a left adjoint $\exists_s$.
- ii) E is a bifibration, that is, it is also an opfibration.

CHAPTER 1

In this chapter we introduce the two classes of categories which will be our main subject of study. Aggregates and semigranular categories are Diers categories with conditions added, an important one being that all their morphisms are mono. This permits us a more specialized, simpler presentation than the full general theory of Diers categories, and in particular we only have to consider multicoproducts instead of multicolimits in general. But first the example that gave birth to the theory.

1.0 Definition

A qualitative domain X is composed of an underlying set $|X|$, and a subset $X \subset \mathcal{P}|X|$ with the following properties

- i) $\emptyset \in X$
- ii) for any $x \in X$, $\{x\} \in |X|$
- iii) $a \in X, b \subset a \Rightarrow b \in X$
- iv) X is closed under directed (filtered) unions.

If X is a qualitative domain, the poset of elements of X ordered by inclusion is denoted $\tilde{X}$. It is quite easy to see that $\tilde{X}$ is consistently (co)complete (0.16), i.e. that any bounded subset of X has a least upper bound, and that $\tilde{X}$ as a category has pullbacks. Notice that a nonempty consistently cocomplete poset always has a bottom element: the l.u.b of the empty set. A stable function between qualitative domains X, Y is a morphism of posets $\tilde{X} \rightarrow \tilde{Y}$ that preserves filtered sups and pullbacks. Girard proves that the category of qualitative domains and stable functions is cartesian-closed.

A morphism of qualitative domains $X \rightarrow Y$ is an injective function $f: |X| \rightarrow |Y|$ such that $f(a) \in Y$ iff $a \in X$, $f(a)$ being the direct

image of a by f . We denote the category of qualitative domains and morphisms by qD .

1.2 Proposition

Let $\mathcal{C}$ have pullbacks and a strong generating set Q . Then to check that a diagram

$$\begin{array}{ccc} X & \xrightarrow{k} & Y \\ h \downarrow & & \downarrow g \\ Z & \xrightarrow{f} & W \end{array}$$

is a pullback it suffices to show the pullback property for all $A \in Q$, i.e. that for all $A \in Q$, $t: A \rightarrow Y$, $s: A \rightarrow Z$ such that $fs = gt$ there is a unique $x: A \rightarrow X$ making $hx = s$, $kx = t$.

Proof

Let P be a pullback object. There exists $x: X \rightarrow P$. Now the property above will show that $\mathcal{C}(A, x)$ is an isomorphism for all $A \in Q$ and therefore that x is an isomorphism.

1.3 Definition

Let $\mathcal{C}$ have multicoproducts, and let all its morphisms be monomorphisms. This implies that for any object $X \in \mathcal{C}$ the set of subobjects $Sub(X)$ is a (co-)complete lattice. We say that X is prime if $Sub(X)$ is finite, and whenever we have a coproduct candidate $(a_i: A_i \rightarrow X)_{i \in I}$ then there is $i \in I$ such that a_i is an isomorphism. X is said to be atomic if it has exactly two subobjects: itself and the initial candidate it determines. It is easy to see that an atom is always prime.

1.4 Definition

A category $\mathcal{C}$ is an aggregate (resp. a semigranular category) if the following five axioms hold.

- 1- It has filtered colimits.
- 2- All morphisms are mono.
- 3- It has all multicoproducts.
- 4- It has a strong generating set $\mathcal{G}$ of f.p. prime (resp. f.p. atomic) objects.

1.4.1 Remarks

Having only monomorphisms in $\mathcal{C}$ simplifies our life considerably. For example, to prove that $f: X \rightarrow Y$ is an iso it is sufficient to show that there is $g: Y \rightarrow X$ such that $gf = 1_X$. Or if we use the generating set all we have to do to show that f is an iso is to prove that for all $A \in \mathcal{G}$ and $y: A \rightarrow Y$ there is $x: A \rightarrow X$ with $fx = y$. Also an object $C \in \mathcal{C}$ is f.p. iff the following holds :

Given a filtered diagram $(X_h)_{h \in H}$ in $\mathcal{C}$ with colimit X and coprojections $\alpha_h: X_h \rightarrow X$, and a morphism $f: C \rightarrow X$ there exist $h \in H$ and $f': C \rightarrow X_h$ with $\alpha_h f' = f$.

Finally a morphism $f: A \rightarrow B$ between atoms is always an iso : f is a representative of either the full or the minimum subobject. In the first case there is $g: B \rightarrow A$ with

$$\begin{array}{ccc} & g & \\ B & \xrightarrow{\quad} & A \\ 1 \swarrow & & \searrow f \\ & B & \end{array}$$

commuting and f is an isomorphism. The second case is impossible, since it forces A to be an initial candidate, contradicting the atomicity of A . And now the last axiom :

- 5- If $(A_i)_{i \in I}$ is a family of objects of $\mathcal{C}$, $(x_i: A_i \rightarrow X)_{i \in I}$ a candidate, $f: B \rightarrow X$ some morphism where B is prime then there is $i \in I$, and $g: B \rightarrow A_i$ with $x_i g = f$.

We will soon show that Axioms 1-4 imply that $\mathcal{C}$ always has pullbacks, and then that Axiom 5 is equivalent to saying that coproduct candidates (a notion of covering family) are stable under pullbacks.

1.4.2 Example

Let $\mathcal{C}$ be the category of totally ordered sets and injective morphisms of orders. It is easy to show that the underlying set functor $|-|: \mathcal{C} \rightarrow \mathbf{Set}$ creates filtered colimits (That is, given a filtered diagram $(X_h)_{h \in H}$ in $\mathcal{C}$ there is a unique total ordering on the set $\varinjlim_h |X_h|$ that will make it a colimit in $\mathcal{C}$). We have already shown that $\mathcal{C}$ has multicoproducts, and it is trivial to prove that the one-element ordered set 1 is a strong generating set and that it satisfies Axiom 5. Therefore $\mathcal{C}$ is semigranular.

Until the end of this chapter $\mathcal{C}$ is an aggregate with G a generating set of primes.

1.5 Proposition

Let $(x_i: X_i \rightarrow X)_{i \in I}$ be a discrete cocone in $\mathcal{C}$. Then it is a candidate iff any $A \rightarrow X$ where A is prime factors through some x_i .

Proof

Necessity is just Axiom 5. For sufficiency let $(y_i: X_i \rightarrow Y)_i$ be the candidate determined by $(x_i)_i$ and $y: Y \rightarrow X$ the factoring. We just show y is an iso by verifying that every $A \rightarrow Y$ where A is prime factors through y .

1.6 Definition

Let $X \in \mathcal{C}$. A lean cocone for X is a discrete cocone $(x_i: A_i \rightarrow X)_{i \in I}$ where A_i is prime such that given any prime B and $f: B \rightarrow X$ there exists exactly one $i \in I$ such that f factors through x_i by an isomorphism.

$$\begin{array}{ccc} B & \xrightarrow{\quad} & A_i \\ & \searrow f & \nearrow x_i \\ & X & \end{array}$$

By the previous proposition a lean cocone is a coproduct candidate. An object always admits a lean cocone to it: if $X \in \mathcal{C}$ let $\mathcal{Q}/X$ be the full subcategory of $\mathcal{C}/X$ whose objects are arrows with domain in $\mathcal{Q}$. Choosing one representative for every isomorphism class of objects of $\mathcal{Q}/X$ will give a lean cocone.

1.6.1 Corollary

Every prime of $\mathcal{C}$ is isomorphic to an object of $\mathcal{Q}$.

If X is a prime of $\mathcal{C}$ we know it has a lean cocone where all the domains of the arrows are in $\mathcal{Q}$. But one of the components of that cocone has to be an isomorphism by the definition of "prime".

1.7 Proposition

Let $X \in \mathcal{C}$. Then X is f.p. iff $\text{Sub}(X)$ is finite.

Proof

Suppose $\text{Sub}(X)$ is finite. Then obviously there is a lean cocone $(x_i: A_i \rightarrow X)_{i \in I}$ with I finite. Let $(Y_d)_{d \in D}$ be a filtered diagram $D \rightarrow \mathcal{C}$ with colimit Y and coprojections $\alpha_d: Y_d \rightarrow Y$. Let $f: X \rightarrow Y$. Since the primes are f.p., for every $i \in I$ there is $d_i \in D$ and $g_i: A_i \rightarrow Y_{d_i}$ with $\alpha_{d_i} \circ g_i = f x_i$. Since I is finite and D filtered there is a discrete cocone $(s_i: d_i \rightarrow e)_i$ in D . This determines a cocone $(Y_{s_i} \circ g_i: A_i \rightarrow Y_e)_{i \in I}$ in $\mathcal{C}$.

$$\begin{array}{ccccc}
 & g_i & & Y_{s_i} & \\
 A_i & \longrightarrow & Y_{d_i} & \longrightarrow & Y_e \\
 x_i \downarrow & & \searrow \alpha_{d_i} & & \nearrow \alpha_e \\
 X & \xrightarrow{f} & Y & &
 \end{array}$$

We claim $(x_i)_i$ is the candidate that cocone determines. This is because $\alpha_e \circ Y_{s_i} \circ g_i = f x_i$, as the diagram above shows, so $(x_i)_i$ and $(Y_{s_i} \circ g_i)_i$ belong to the same component in the category of $(A_i)_i$ -cocones. Therefore there is $f': X \rightarrow Y_e$ with $f' x_i = Y_{s_i} \circ g_i$. But then

$$\alpha_e \circ f' \circ x_i = \alpha_e \circ Y_{s_i} \circ g_i = f x_i$$

and by joint epiness of $(x_i)_i$ we get that $\alpha_e \circ f' = f$ and so X is f.p. by 1.4.1.

Now suppose that X is f.p.. Let $(x_i: A_i \rightarrow X)_{i \in I}$ be a lean cocone. As before, for $J \subset I$, X_J is the subobject determined by the cocone $(x_j)_{j \in J}$, and $x_J: X_J \rightarrow X$ the factoring. Obviously, for $J \subset J'$ we have

$$\begin{array}{ccc}
 X_J & \longrightarrow & X_{J'} \\
 x_J \searrow & & \nearrow x_{J'} \\
 & X &
 \end{array}$$

So look at the filtered system $(X_J)_{J \text{ finite}}$. Let X' be its colimit, with coprojections $\alpha_J: X_J \rightarrow X'$. The family $(x_J)_J$ forms a cocone. So there is $f: X' \rightarrow X$. We will prove it is an isomorphism by testing with primes. If $A \in \mathcal{Q}$ and $a: A \rightarrow X$, then since there is $i \in I$ and

$$\begin{array}{ccc}
 A & \longrightarrow & A_i \\
 a \searrow & & \nearrow x_i \\
 & X &
 \end{array}$$

and there is J finite with $i \in J$ (take $\{i\}$), so

$$\begin{array}{ccc}
 A_i & \longrightarrow & X_J \\
 x_i \searrow & & \nearrow x_J \\
 & X &
 \end{array}$$

and by composing the two and following with α_J , $A \rightarrow A_1 \rightarrow X_J \rightarrow X'$ we get a morphism $a': A \rightarrow X'$ such that $fa' = a$ and therefore f is an isomorphism.

Hence X is the sup of its f.p. subobjects. But since X is f.p there is J such that f^{-1} factors through α_J and that forces $J=1$. Therefore I is finite. QED.

1.8 Corollary

Every object of $\mathcal{C}$ is the sup of its f.p subobjects.

Since f.p. objects in $\mathcal{C}$ have such strong finiteness properties it is fitting to call them finite objects. There is only a small set of isomorphism types of finite objects: to every finite X one can associate essentially one lean cocone $(A_i \rightarrow X)_{i \in I}$. But the class of all multicoproduct families from all finite families $(A_i)_i$ of objects of $\mathcal{G}$ forms a set. This shows $\mathcal{C}$ is $\aleph_0$ -accessible. In the same vein of thought, since the initial candidates are exactly the objects of $\mathcal{C}$ whose subobject lattice is a singleton, we will call them empty objects.

1.9 Proposition

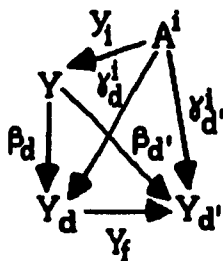
$\mathcal{C}$ has connected limits.

Let $(Y_d)_{d \in D}$ be a connected diagram $D \rightarrow \mathcal{C}$. Let $\mathcal{B}$ be the following category: an object of $\mathcal{B}$ is a cone $(A, (\gamma_d)_d)$, $\gamma_d: A \rightarrow Y_d$ where $A \in \mathcal{G}$ and a morphism $(A, (\gamma_d)_d) \rightarrow (A', (\gamma'_d)_d)$ is a morphism of cones. Such a morphism forces

$$\begin{array}{ccc} A & \xrightarrow{\quad} & A' \\ \gamma_d \searrow & & \nearrow \gamma'_d \\ & Y_d & \end{array}$$

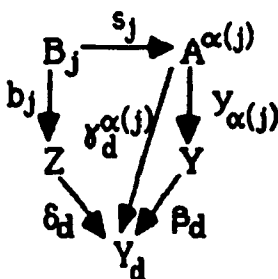
to commute and there is therefore at most one morphism between objects of $\mathcal{B}$. Let I be the set of isomorphism classes of $\mathcal{B}$ and for every $i \in I$ let $(A^i, (\gamma_d^i)_d)$ be a choice of a representative. Now for

every $d \in D$ there is a discrete cocone $(\gamma_d^1: A^1 \rightarrow Y_d)_{d \in D}$. Since for $f: d \rightarrow d'$ Y_f is a morphism of cocones $(\gamma_d^1)_1 \rightarrow (\gamma_{d'}^1)_1$ and since D is connected, for any $d, d' \in D$ the cocones $(\gamma_d^1)_1$ and $(\gamma_{d'}^1)_1$ determine the same candidate. Hence there is a candidate $(y_1: A^1 \rightarrow Y)_1$ and for every d there is $\beta_d: Y \rightarrow Y_d$ with $\beta_d \circ y_1 = \gamma_d^1$, for all d, i . $(\beta_d)_d$ is a cone: if $f: d \rightarrow d'$ then



$$\begin{aligned} Y_f \circ \beta_d \circ y_1 &= Y_f \circ \gamma_d^1 &= \gamma_{d'}^1 \\ &= \beta_{d'} \circ y_1 \end{aligned}$$

and by joint epiness of $(y_1)_1$ we get $Y_f \circ \beta_d = \beta_{d'}$. We claim $(Y, (\beta_d)_d)$ is a limit cone: let $(\delta_d: Z \rightarrow Y_d)_{d \in D}$ be some cone. Let $(b_j: B_j \rightarrow Z)_{j \in J}$ be a lean cocone to Z . For every $j \in J$ we get a cone $(\delta_d \circ b_j)_{d \in D}$ and since B_j is prime there is a unique $\alpha(j)$ and a unique morphism of D -cocones $s_j: (B_j, (\delta_d \circ b_j)_d) \rightarrow (A^{\alpha(j)}, (\gamma_d^{\alpha(j)})_d)$.



But now the family $(y_{\alpha(j)} \circ s_j: B_j \rightarrow Y)_{j \in J}$ is a discrete cocone from $(B_j)_j$. The candidate it determines is $(b_j)_j$. This is because

$$\begin{aligned} \beta_d \circ y_{\alpha(j)} \circ s_j &= \gamma_d^{\alpha(j)} \circ s_j \\ &= \delta_d \circ b_j \end{aligned}$$

and so the usual connectedness argument holds. So there is $g: Z \rightarrow Y$ with $g \circ b_j = \gamma_{\alpha(j)} \circ s_j$ and therefore

$$\begin{aligned} \beta_d \circ g \circ b_j &= \beta_d \circ \gamma_{\alpha(j)} \circ s_j \\ &= \delta_d \circ b_j \end{aligned}$$

So $\beta_d \circ g = \delta_d$ for any d , and this shows $(X, (\beta_d))$ is a limit.

1.10 Remark

We do end up proving that aggregates and semigranular categories are Diers categories. In fact, an aggregate is just a Diers category where axioms 2,4 and 5 hold in addition. We could have chosen other paths than the rather elementary (and explicit) one we took. For example, it is not hard to show that a category with multicoproducts all whose morphisms are mono is multicocomplete.

1.11 Examples

In the category qD the only qualitative domain on the one point set is a strong generator, and obviously atomic. It is easy to see that qD has multicoproducts ; it is done just as for total orders, and the same holds for Axiom 5.

The same happens with the category $\mathbb{M}$ of sets and monomorphisms.

The three examples above have two things in common: they are semigranular and the only automorphisms of atoms are trivial. A semigranular category is said to be granular if this is the case.

Here is an example of a non-granular semigranular category. Let G be a group. Let $\hat{G}$ be the category whose objects are (right) actions $X \times G \rightarrow X$ where X is a set, such that for any $x \in X$, $g \in G$ with $g \neq 1$ we have $xg \neq x$ (i.e. the orbits are all isomorphic to G). The morphisms are injective morphisms of group actions. We leave to the reader the verification that $\hat{G}$ is semigranular.

Let X be a qualitative domain. Then $\tilde{X}$ is a granular poset: we already know $\tilde{X}$ has multicoproducts; the singletons of X are obviously a strong generating set of atoms, and Axiom 5 is trivial to check. For the converse:

1.11.1 Proposition

Let G be a connected semigranular poset which is consistently cocomplete. Then there is a qualitative domain X such that G is isomorphic to $\tilde{X}$.

Proof

Take $|X|$ to be the set of atoms of G (since we are in a poset there is exactly one atom per isomorphism type and G is granular). Define a function τ from the underlying set of G to the power set of $|X|$ by $\tau(g) = \{x \in |X| \mid x \leq g\}$. Since G is connected the unique $g \in G$ such that $\tau(g) = \emptyset$ is the bottom element 0 . Since G is consistently cocomplete τ is injective on $G - \{0\}$, and therefore on the whole of G . If we take $X \subset \mathcal{P}(|X|)$ to be the image of τ it is trivial to check that X is a qualitative domain and that τ preserves order.

Here is an example of an aggregate which is not semigranular: let C be semigranular, and let I be a finite partially ordered set. It is easy to check that the functor category C^I has filtered colimits and multicoproducts (multicoproducts are calculated pointwise in a functor category; i.e. a coproduct candidate in C^I is a discrete cocone $(\varphi_x: F_x \rightarrow F)$ such that for every $i \in I$ $(\varphi_x(i))_x$ is a coproduct candidate). If A is an atom of C and $i \in I$ let $[A, i] \in C^I$ be defined by:

$$[A, i](j) = \begin{cases} A & \text{if } j \geq i \\ \text{the initial candidate } 0_A \rightarrow A & \text{if not.} \end{cases}$$

An easy exercise will show that $[A, i]$ is prime, and that the family

$$([A, i])_{i \in I, A \in \mathcal{Q}}$$

obeys the necessary requirements to make $\mathcal{C}^1$ an aggregate.

A small discrete category is a semigranular category whose generating set of f.p. atoms is empty.

1.12 Proposition

Coproduct candidates in $\mathcal{C}$ are stable under pullbacks.

Proof

Let $(x_i: X_i \rightarrow X)_{i \in I}$ be a coproduct candidate, $f: Y \rightarrow X$ some morphism. If $y_i: Y_i \rightarrow Y$ is the pullback of x_i by f to show $(y_i)_i$ is a candidate we just have to use 1.5 and test with primes.

1.13 Proposition

If $\mathcal{C}$ is semigranular, then for any $X \in \mathcal{C}$ $Sub(X)$ is a complete atomic boolean algebra.

Proof

Let $(x_i: A_i \rightarrow X)_{i \in I}$ be a lean cocone. All A_i are atomic. We will prove that $Sub(X)$ is lattice-isomorphic to $\mathcal{P}(I)$. Define functions

$$\begin{array}{c} (-)^{\#} \\ \mathcal{P}(I) \xrightleftharpoons{(-)^{\flat}} Sub(X) \end{array}$$

where for $J \subseteq I$,

$$J^{\#} = \text{The factoring } X_J \rightarrow X \text{ determined by the cocone } (x_j)_{j \in J}$$

and for a subobject $Y \rightarrow X$

$$Y^{\flat} = \{i \in I \mid x_i \text{ factors through } Y\}$$

We first show that $(Y^b)^* = Y$. Let $y: Y \rightarrow X$ be a subobject. If $j \in Y^b$, let $x'_j: A_j \rightarrow Y$ be such that $yx'_j = x_j$. $(Y^b)^*$ comes from a factoring $z: Z \rightarrow X$ and a coproduct candidate $(z_j: A_j \rightarrow Z)_{j \in Y^b}$. But by definition $(x'_j)_{j \in Y^b}$ is a lean cocone, and therefore is a candidate. So we have two candidates $(z_j)_j$ and $(x'_j)_j$ that are sent by z and y respectively to $(x_j)_j$. Therefore y and z determine the same subobject.

To show that $(J^*)^b = J$, let $J \subset I$, and $x: X_J \rightarrow X$ be the factoring determined by the cocone $(x_j)_j$ with candidate $(x'_j: A_j \rightarrow X_j)_{j \in J}$. Now

$$(J^*)^b = \{i \in I \mid x_i \text{ factors through } x\}.$$

Obviously $J \subset (J^*)^b$ since for each $j \in J$ there is x'_j with $xx'_j = x_j$. Let $x_i: A_i \rightarrow X$ be such that there is $f: A_i \rightarrow X_j$ with $xf = x_i$. By axiom 5 f is isomorphic in $\mathcal{Q}/X_j$ to a x'_j , $j \in J$, but since $(x'_j)_j$ is lean this forces $i = j$.

CHAPTER 2

In this chapter we study the "right notion" (for our purposes) of morphism between aggregates.

2.1 Definition

Let $\mathcal{C}, \mathcal{D}$ be categories and $F: \mathcal{C} \rightarrow \mathcal{D}$ a functor between them. For $A \in \mathcal{D}$ the category A/F is defined as usual: an object is a pair (X, x) where $X \in \mathcal{C}$ and $x: A \rightarrow FX$. A morphism $(X, x) \rightarrow (Y, y)$ is an $f: X \rightarrow Y$ such that $Ff \circ x = y$. f will be thought of as a morphism living both in $\mathcal{C}$ and A/F . We say that (X, x) is a generic morphism, or a generic arrow if it is a generic object in A/F : given any diagram of the form

$$\begin{array}{ccc} (X, x) & & (Y, y) \\ & \searrow f & \swarrow g \\ & (Z, z) \end{array}$$

in A/F there is a unique $h: (X, x) \rightarrow (Y, y)$ such that $gh = f$.

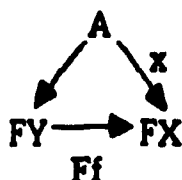
This notion is related to, but stronger than the Joyal notion of genericity [Jo] and so should be called strong genericity. But both notions are the same in a category all whose morphisms are mono, in particular for aggregates. Note that if $x: A \rightarrow FX$ is generic and $a: B \rightarrow A$, $s: X \rightarrow Y$ are isomorphisms, then $Fs \circ xa$ will be a generic arrow.

Let us recall that if (X, x) is generic, then any morphism $f: (Y, y) \rightarrow (X, x)$ is a split epi. If in addition (Y, y) is generic then f is an isomorphism.

2.2 Proposition

Let $\mathcal{C}$ and $\mathcal{D}$ have pullbacks. Let all the morphisms of $\mathcal{C}$ be monos and let $F: \mathcal{C} \rightarrow \mathcal{D}$ preserve pullbacks (and therefore monomorphisms). Let $x: A \rightarrow FX$ be an object of A/F . Then TFAE :

- i) x is generic
- ii) if $f: Y \rightarrow X$ is such that x factors through Ff



then f is an isomorphism.

Proof

This is just 0.11 over again, if one proves that under the conditions given A/F has pullbacks. But this is quite easy and will be left to the reader.

2.3 Proposition

Let $\mathcal{C}$ and $\mathcal{D}$ be aggregates and let $F: \mathcal{C} \rightarrow \mathcal{D}$ be some functor. Then TFAE

- i) F preserves filtered colimits and pullbacks.
- ii) For any prime $A \in \mathcal{D}$, $y: A \rightarrow FY$ there is a generic $x: A \rightarrow FX$ where X is finite and a morphism $(X, x) \rightarrow (Y, y)$ in A/F .
- iii) For any finite $A \in \mathcal{D}$, $y: A \rightarrow FY$ there is a generic $x: A \rightarrow FX$ where X is finite and a morphism $(X, x) \rightarrow (Y, y)$ in A/F .

Proof

iii) $\Rightarrow$ ii) is obvious.

i) $\Rightarrow$ iii)

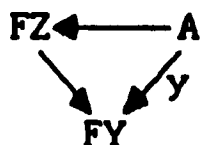
Just as in [Gi86]. Let A be finite and $y: A \rightarrow FY$. We know that Y is the sup of its finite subobjects.

$$Y = \varinjlim_{Z \subset Y, Z \text{ finite}} Z$$

This a filtered colimit diagram, and since F preserves filtered colimits

$$FY = \varinjlim_{Z \subset Y, Z \text{ finite}} FZ$$

Since A is finite there is $Z \subset Y$ finite such that



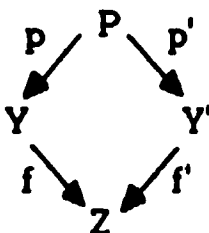
y factors through FZ . But $Sub(Z)$ is finite. Let

$$X = \bigcap \{ Z' \subset Z \mid y \text{ factors through } FZ' \}$$

where intersection of subobjects means pullback as usual. There will be $x:A \rightarrow FX$ and it is easy to prove that $x \in A/F$ does not have any proper subobject. Therefore x is generic. This proof is just a slightly modified version of Freyd's adjoint functor theorem.

ii) $\Rightarrow$ i)

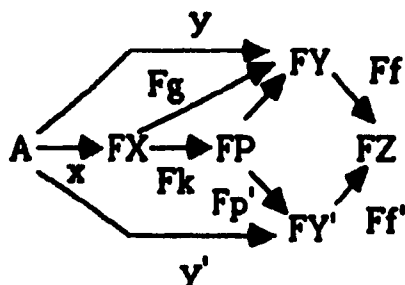
Let every $y:A \rightarrow FY$ where A is prime admit a generic arrow to it. Let



be a pullback diagram. Let A be prime, and $y:A \rightarrow FY$, $y':A \rightarrow FY'$ with $Ff \circ y = Ff' \circ y'$, and let $z = Ff \circ y$. Let (X,x) be generic with $g:(X,x) \rightarrow (Y,y)$. Then we have



in A/F . So by genericity there is $h:X \rightarrow Y'$ with $f'h = fg$ and $Fh \circ x = y'$. But then there is $k:X \rightarrow P$, with $pk = g$ and $p'k = h$.



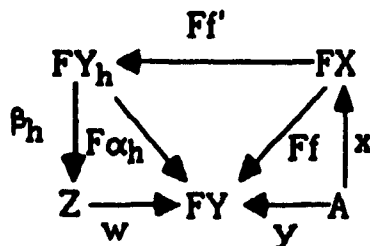
Since we have

$$\begin{aligned} Fp \circ Fk \circ x &= Fg \circ x = y \\ Fp' \circ Fk \circ x &= Fh \circ x = y' \end{aligned}$$

we have shown that F preserves pullbacks by testing with primes.

Now let $(Y_h)_{h \in H}$ be a filtered diagram $H \rightarrow \mathcal{C}$ with colimit Y and coprojections $\alpha_h: Y_h \rightarrow Y$. Let $Z = \varinjlim_h FY_h$ with coprojections $\beta_h: FY_h \rightarrow Z$. There is a canonical $w: Z \rightarrow FY$. We will prove w is an iso by testing with primes. Let A be prime and $y: A \rightarrow FY$. There is $x: A \rightarrow FX$ generic with X finite and $f: X \rightarrow Y$ with $y = Ff \circ x$. Since X is finite there is $h \in H$ and $f': X \rightarrow Y_h$ with $\alpha_h \circ f' = f$. But then

$$\begin{aligned} w \circ \beta_h \circ Ff' \circ x &= F\alpha_h \circ Ff' \circ x = Ff \circ x \\ &= y \end{aligned}$$



so $\mathcal{C}(A, w)$ is bijective and this proves F preserves filtered colimits.

We will call a functor between aggregates that fills the conditions above an entire functor. A functor which is constant over each

component of $\mathcal{C}$ is entire, since pullbacks and filtered diagrams are connected. Another example of an entire functor is a stable function between qualitative domains. The concept of entire functor is related to some very classical concepts. Given any two categories $\mathcal{C}$ and $\mathcal{D}$ say a functor $F:\mathcal{C}\rightarrow\mathcal{D}$ "has a polyadjoint" if for every $A\in\mathcal{D}$ the category A/F has a poly-initial family. This is a generalization of the property of having a left adjoint: this latter case is equivalent to A/F having an initial object for every A . For example the inclusion of the category of algebraically closed fields into the category of fields has a polyadjoint; in this case every category of the form A/F is in fact connected. If A is a field a generic object of A/F is an inclusion $A\rightarrow\tilde{A}$ where $\tilde{A}$ is the algebraic closure of A . An entire functor between aggregates is the same as a functor which preserves filtered colimits and has a polyadjoint.

In the same way we can define what it means for a functor to have a "multiadjoint". This case has been studied by Diers [Di81]. It does not hold much interest for us since the categories it gives rise to are not cartesian-closed.

2.4 Proposition

Let $\mathcal{C}, \mathcal{D}$ aggregates, $F, G:\mathcal{C}\rightarrow\mathcal{D}$ entire functors and $\alpha:F\rightarrow G$ a natural transformation. Then TFAE

- i) For every $A\in\mathcal{D}$, every generic arrow $x:A\rightarrow FX$, $\alpha X \cdot x$ is generic.
- ii) For every prime $A\in\mathcal{D}$, every generic $x:A\rightarrow FX$, $\alpha X \cdot x$ is generic.
- iii) For every $f:Z\rightarrow Y$ in $\mathcal{C}$, the square

$$\begin{array}{ccc}
 & Ff & \\
 FZ & \longrightarrow & FY \\
 \alpha Z \downarrow & & \downarrow \alpha Y \\
 GZ & \longrightarrow & GY \\
 & Gf &
 \end{array}$$

is a pullback.

We will call such a natural transformation a cartesian transformation.

Proof

i) $\Rightarrow$ ii) is obvious.

ii) $\Rightarrow$ iii)

We just have to prove "pullbackness" for prime objects. So let A be a prime and $s: A \rightarrow GZ$, $t: A \rightarrow FY$ with $Gf \circ s = \alpha Y \circ t (= u)$, by definition). Let $x: A \rightarrow FX$ be generic and $g: (X, x) \rightarrow (Y, u)$. Since $\alpha X \circ x$ is generic and we have a diagram

$$\begin{array}{ccc}
 (X, \alpha X \circ x) & & (Z, s) \\
 & \searrow g & \swarrow f \\
 & (Y, u) &
 \end{array}$$

in A/G there exists $k: X \rightarrow Z$ with $Gk \circ \alpha X \circ x = s$ and $fk = g$.

$$\begin{array}{ccccc}
 & & & t & \\
 & & & \searrow & \\
 A & & & & FY \\
 \downarrow x & & Fg & & \downarrow \alpha Y \\
 FX & & & & \\
 \downarrow \alpha X & \searrow Fk & FZ & \longrightarrow & FY \\
 GX & & \downarrow \alpha Z & & \\
 \downarrow Gk & & Gg & & \\
 GZ & \longrightarrow & GY & & \\
 & Gf & & &
 \end{array}$$

Then looking at the diagram above will show $Fk \circ x$ fills in the square.

iii) $\Rightarrow$ ii)

Let $x:A \rightarrow FX$ be generic. Let there be $m:W \rightarrow X$ and $w:A \rightarrow GW$ with $Gm \circ w = \alpha X \circ x$. By assumption there is $w':A \rightarrow FW$ with $\alpha W \circ w' = w$ and $Fm \circ w' = x$. But the last equation forces m to be an isomorphism, so $\alpha X \circ x$ is generic.

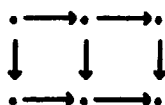
2.5 Proposition

Let F, G, H be entire functors, and $\theta:F \rightarrow G$, $\varphi:G \rightarrow H$ natural transformations. Then

θ, φ cartesian $\Rightarrow \varphi \theta$ cartesian.
 φ and $\varphi \theta$ cartesian $\Rightarrow \theta$ cartesian.

Proof

One just uses the fact that in a composite of squares



If the left and the right squares are pullbacks then the outer square is.

If the right and the outer square are pullbacks then the left square is.

The following proposition uses in an essential way the fact that we are dealing only with monomorphisms. That is, while 2.3 and 2.4 generalize to categories with non-monomorphisms, this one does not.

2.6 Proposition

Let $\mathcal{C}, \mathcal{D}$ be aggregates. Let $F, G: \mathcal{C} \rightarrow \mathcal{D}$ be any two functors, $\theta: F \rightarrow G$ a natural transformation. Let $x: A \rightarrow FX$ be such that $\theta X \circ x$ is generic. Then x is generic.

Proof

Let

$$\begin{array}{ccc} (X, x) & & (Z, z) \\ f \searrow & & \swarrow g \\ & (Y, y) & \end{array}$$

be a diagram in A/F . Then

$$\begin{array}{ccc} (X, \theta X \circ x) & & (Z, \theta Z \circ z) \\ f \searrow & & \swarrow g \\ & (Y, \theta Y \circ y) & \end{array}$$

is a diagram in A/G . So there exists $h: (X, \theta X \circ x) \rightarrow (Z, \theta Z \circ z)$ with $gh = f$. But then

$$\theta Z \circ Fh \circ x = Gh \circ \theta X \circ x = \theta Z \circ z$$

$$\begin{array}{ccccc} & & \theta X & & \\ & & FX \longrightarrow GX & & \\ x \nearrow & & \downarrow & & \downarrow \\ A & & Fh & & Gh \\ z \searrow & & \downarrow & & \downarrow \\ & & FZ \longrightarrow GZ & & \\ & & \theta Z & & \end{array}$$

and since θZ is mono we get that $Fh \circ x = z$. QED.

2.7 Proposition

Let *Aggr* be the not-locally-small category of aggregates and entire functors. Then *Aggr* has all small products and coproducts. The same is true of *Sem*, the full subcategory of semigranular categories and entire functors.

Proof

Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ be aggregates (semigranular). It is easy to see that $\mathcal{C} \times \mathcal{D}$, the usual product of categories, is an aggregate (semigranular). First, connected limits and filtered colimits are taken pointwise, e.g. if $(X_i, Y_i)_{i \in I}$ is a connected diagram then its limit in $\mathcal{C} \times \mathcal{D}$ is $(\varprojlim_i X_i, \varprojlim_i Y_i)$. An object (X, Y) is finitely presented iff X, Y are f.p. in their respective categories. Primes (atoms) are either of the form (I, A) or (A', I') where $I \in \mathcal{C}, I' \in \mathcal{D}$ are initial candidates and $A \in \mathcal{C}, A' \in \mathcal{D}$ primes (atoms). The projections $\mathcal{C} \times \mathcal{D} \rightarrow \mathcal{C}$ and $\mathcal{C} \times \mathcal{D} \rightarrow \mathcal{D}$ are obviously entire functors. Hence to show that $\mathcal{E}$ has finite products we just have to show that if $f: \mathcal{E} \rightarrow \mathcal{C}, g: \mathcal{E} \rightarrow \mathcal{D}$ are entire then $\langle f, g \rangle: \mathcal{E} \rightarrow \mathcal{C} \times \mathcal{D}$ is entire. This is also quite trivial. The terminal object of both *Aggr* and *Sem* is the one-object discrete category **1**.

The proof for binary products generalizes trivially to arbitrary products: if $(\mathcal{C}_i)_{i \in I}$ is a family of aggregates (semigranulars) then the usual product $\prod_i \mathcal{C}_i$ is an aggregate (semigranular) and the product in *Aggr* (*Sem*). The proof is just as above. Primes (atoms) are families $(A_i)_{i \in I}, A_i \in \mathcal{C}_i$ such that there is exactly one i with A_i prime (atomic), A_j an initial candidate, for $j \neq i$.

The disjoint sum $\coprod_i \mathcal{C}_i$ is an aggregate (semigranular). This is trivial to prove since all the limit, colimit and candidate diagrams always stay in the same component. The coprojections $j_i: \mathcal{C}_i \rightarrow \coprod_i \mathcal{C}_i$ are surely entire and if $F_i: \mathcal{C}_i \rightarrow \mathcal{D}$ is a family of entire functors then $[F_i]_i: \coprod_i \mathcal{C}_i \rightarrow \mathcal{D}$ is entire and this shows we have coproducts.

CHAPTER 3

This chapter generalizes the definitions of the previous two chapters to the case of a "variable aggregate", i.e. an indexed family of aggregates indexed in an "entire fashion".

3.1 Proposition

Let $\mathcal{C}, \mathcal{D}$ be categories with $F: \mathcal{D} \rightarrow \mathcal{C}$ a functor admitting a right adjoint G . Then if $(x_i^s: X_i \rightarrow Y^s)_{i \in I, s \in S}$ is a multicoproduct family in $\mathcal{D}$, $(Fx_i^s: FX_i \rightarrow FY^s)_{i, s}$ is a muticoproduct family in $\mathcal{C}$.

Proof

Just as in the classical case: for $Z \in \mathcal{C}$

$$\begin{aligned} \prod_i \mathcal{C}(FX_i, Z) &\cong \prod_i \mathcal{D}(X_i, GZ) &\cong \coprod_s \mathcal{D}(Y^s, GZ) \\ &\cong \coprod_i \mathcal{C}(FY^s, Z) \end{aligned}$$

Let $\mathcal{C}$ be an aggregate, with $\mathcal{G}$ a strongly generating set of primes. Let $\mathcal{B} \subset \mathcal{G}$. We are interested in the full subcategory $\mathcal{D}$ of $\mathcal{C}$ generated by $\mathcal{B}$. An object of $\mathcal{D}$ will simply be an $X \in \mathcal{C}$ admitting a coproduct candidate $(A_i \rightarrow X)_{i \in I}$ where $A_i \in \mathcal{B}$ for all i . It makes sense to require that $\mathcal{B}$ be "closed under prime subobjects": if $A \in \mathcal{B}$, and there is $B \in \mathcal{G}$ with $B \rightarrow A$, then $B \in \mathcal{B}$. Then $\mathcal{B}$ is essentially determined by $\mathcal{D}$ as its strongly generating set of primes, and an object of $\mathcal{D}$ is an X admitting a lean cocone all whose domains are (essentially) in $\mathcal{B}$. The inclusion $\mathcal{D} \rightarrow \mathcal{C}$ has a right adjoint G such that the counit $\epsilon_X: GX \rightarrow X$ is the factoring determined by the sub-cocone of the lean cocone of X all whose domains are in $\mathcal{B}$. It is now easy to see that $\mathcal{D}$ is an aggregate: as a coreflexive subcategory $\mathcal{D}$ has all filtered colimits. A very predictable argument will show that it also has all multicoproducts, and $\mathcal{B}$ will be a generating set of primes with all the required properties.

3.2 Proposition

Let $\mathcal{C}, \mathcal{D}$ be aggregates, $F: \mathcal{D} \rightarrow \mathcal{C}$ a full and faithful functor with right adjoint G . Then TFAE:

- i) F is powerful, that is, if $X \in \mathcal{D}$ any subobject of FX is in the essential image of F .
- ii) If $A \in \mathcal{D}$ is a prime and $B \in \mathcal{C}$ is such that there is $b: B \rightarrow FA$, then B is in the essential image of F (and so is b , obviously).
- iii) If $A \in \mathcal{D}$ is a prime and $B \in \mathcal{C}$ is prime and there is $b: B \rightarrow FA$, then B is in the essential image of F .
- iv) F sends lean cocones to lean cocones. In particular F sends primes to primes.
- v) $\epsilon: FG \rightarrow 1_{\mathcal{D}}$ is cartesian, i.e. has the pullback property for natural squares
- vi) Both F and G are entire and ϵ is cartesian.

We will call F in such a situation a stiff embedding.

Example

It is easy to see that if $\mathcal{C}, \mathcal{D}$ are qualitative domains, then $F: \mathcal{D} \rightarrow \mathcal{C}$ is a stiff embedding iff it is of the form $F = f^*$ for a morphism of qualitative domains $f: \mathcal{D} \rightarrow \mathcal{C}$, where f^* is the direct image functor.

Proof

i) $\Rightarrow$ ii) $\Rightarrow$ iii) is obvious.

iii) $\Rightarrow$ iv)

Let $(a_i: A_i \rightarrow X)_{i \in I}$ be a lean cocone. Let $B \in \mathcal{C}$ be prime, and $b: B \rightarrow FX$. By assumption there is $i \in I$ such that Fa_i is isomorphic to b in $\mathcal{C}/FX$. Such an i is unique since a full and faithful functor with a right adjoint reflects isos.

iv) $\Rightarrow$ v)Let $f: Y \rightarrow X$ be a morphism of $\mathcal{C}$. To show

$$\begin{array}{ccc}
 & \epsilon Y & \\
 FG Y & \longrightarrow & Y \\
 FG f \downarrow & & \downarrow f \\
 FG X & \longrightarrow & X \\
 & \epsilon X &
 \end{array}$$

is a pullback we will test with primes. Let B be one and let $s: B \rightarrow Y$ and $t: B \rightarrow FG Y$ with $\epsilon X \circ t = fs$. Let $(a_i: A_i \rightarrow GX)_{i \in I}$ be a lean cocone for GX in $\mathcal{D}$. By assumption $(Fa_i)_i$ is lean, so there is i and an iso $B \rightarrow Fa_i$. We can suppose without loss of generality that $B = Fa_i$ and $t = Fa_i$. Therefore

$$\begin{array}{ccc}
 & s & \\
 Fa_i & \xrightarrow{\quad} & Y \\
 Fa_i \downarrow & & \downarrow f \\
 FG X & \longrightarrow & X \\
 & \epsilon X &
 \end{array}$$

commutes. Applying the adjunction on both s and Fa_i gives

$$\begin{array}{ccc}
 & s^* & \\
 A_i & \longrightarrow & GY \\
 Fa_i^* \downarrow & & \downarrow Gf \\
 GFGX & \longrightarrow & GX \\
 & G\epsilon X &
 \end{array}$$

But both $G\epsilon X$ and $G\epsilon Y$ are isomorphisms by the "triangular identities" [CWM p.83], and the fact everything is monic.

$$\begin{array}{ccccc}
 & & GFGY & & \\
 GFGf & \swarrow & & \searrow & G\epsilon Y \\
 & GFGX \leftarrow A_i \rightarrow GY & & & \\
 G\epsilon X & \searrow & & \swarrow & Gf \\
 & & GX & &
 \end{array}$$

Hence there is $g: A_1 \rightarrow GFGY$ with $G\varepsilon_Y \circ g = s^*$ and $GFGf \circ g = Fa_1^*$. Applying the adjunction to g will give the required $FA_1 \rightarrow FGY$.

$v) \Rightarrow vi)$

It is now easy to show that F and G are entire. We already know that G preserves pullbacks and F preserves filtered colimits. Let $(X_h)_{h \in H}$ be a filtered diagram in $\mathcal{C}$ with colimit X and coprojections $\varphi_h: X_h \rightarrow X$. By the previous argument we know that $FG\varphi_h: FGX_h \rightarrow FGX$ is obtained by pulling back φ_h by ε_X . Then the cocone $(FG\varphi_h)_h$ seen as a discrete cocone is a coproduct candidate, and so is $(G\varphi_h)_h$: if $(z_h: GX_h \rightarrow Z)_h$ is a candidate for $(G\varphi_h)_h$, and $a: Z \rightarrow GX$ the factoring, then 3.1 shows a is an iso and we know F reflects isos. It is now only a formality to show that $(G\varphi_h)_h$ is a colimit cocone, and we leave the proof that F preserves pullbacks as an exercise. It is also easy to prove (exercise) that an adjoint pair among aggregates with cartesian counit always has the left adjoint full and faithful.

$vi) \Rightarrow i)$

Let $b: B \rightarrow FX$ be a subobject of FX . Since by hypothesis

$$\begin{array}{ccc} & \varepsilon_B & \\ & FGB \longrightarrow B & \\ FGb \downarrow & & \downarrow b \\ & FGFX \longrightarrow FX & \\ & \varepsilon_{FX} & \end{array}$$

is a pullback and ε_{FX} is an isomorphism, ε_B is an iso.

3.2.1 Corollary

Let $F: \mathcal{D} \rightarrow \mathcal{C}$ be a functor between semigranular categories. Then F is a stiff embedding iff it is full and faithful, has a right adjoint and sends atoms to atoms.

This is just because a prime subobject of an atomic object has to be the full object.

We are interested in Grothendieck fibrations $E \rightarrow \mathcal{C}$ corresponding to functors $X: \mathcal{C}^{op} \rightarrow \mathcal{Cat}$ where for every $f: X \rightarrow Y$ in $\mathcal{C}$, XX and XY are semigranular, and the transition functor $Xf: XY \rightarrow XX$ is the right adjoint to a stiff embedding. This boils down to the following:

3.3 Definition

Let $\mathcal{C}$ be an aggregate. A stiff fibration is a diagram $E: \mathcal{C} \rightarrow \mathcal{Cat}$ such that

- a) E is a bifibration: it is a Grothendieck fibration and $E^{op}: \mathcal{C}^{op} \rightarrow \mathcal{Cat}$ is also a Grothendieck fibration. As is customary we call cartesian arrows in the dual fibration cocartesian morphisms in E . A bifibration corresponds exactly to a (pseudo-)functor $X: \mathcal{C}^{op} \rightarrow \mathcal{Cat}$ where for every f in $\mathcal{C}$ Xf has a left adjoint.
- b) For every $S \in \mathcal{C}$ the fiber E^S is an aggregate. We will call an object X of E which is prime (atomic) in its fiber E^{EX} a local prime (atom). In the same way, we will say of an object which is empty (finite) in its fiber that it is locally empty (finite).
- c) (corresponding to full- and faithfulness of left adjoints) Every cocartesian arrow is also cartesian.
- d) (corresponding to stiffness of left adjoints). If $f: X \rightarrow Y$ is a cocartesian arrow and $b: B \rightarrow Y$ above identity then the pullback $P \rightarrow B$ of f by b is cocartesian.

A word of explanation about conditions c) and d) is in order. The first one just a way of formulating that the unit of the adjunction is an iso, making the left adjoint full and faithful. The second one is a formulation of 3.2 i), using condition c) and the well-known fact about fibrations that if $g: W \rightarrow Z$ is in the fiber E^T and $s: S \rightarrow T$ in $\mathcal{C}$ then the diagram

$$\begin{array}{ccc}
 s^*W & \longrightarrow & W \\
 s^*g \downarrow & & \downarrow g \\
 s^*Z & \longrightarrow & Z
 \end{array}$$

where the horizontal arrows are cartesian above s is a pullback (see 3.10). Condition d) could be formulated differently, in view of 3.2. For example we could give seemingly weaker conditions by requiring that X , or X and Y be local primes.

From now on until otherwise stated $E: E \rightarrow C$ is a stiff fibration. We will split our fibrations whenever we want, using a standard notation: if $s: S \rightarrow T$ is in C , then choosing for every $X \in E^T$ a cartesian arrow $s^*X \rightarrow X$ defines a functor $s^*: E^T \rightarrow E^S$, and choosing for every $Y \in E^S$ a cocartesian arrow $Y \rightarrow \exists_s Y$ defines a functor $\exists_s: E^S \rightarrow E^T$ and $\exists_s$ is left adjoint to s^* . When we split things we use the following notation for cartesian and cocartesian arrows: a cartesian arrow $s^*X \rightarrow X$ is called $\gamma_s X$, or γ_s , or γX , or simply γ . A cocartesian arrow $Y \rightarrow \exists_s Y$ is called $\iota_s Y$, ι_s , ιY , or ι .

The following proposition is just a translation in this language of results already proven, and therefore the proofs are omitted.

3.5 Proposition

Let $s: S \rightarrow T$ in C , $(x_i: X_i \rightarrow X)_{i \in I}$ a coproduct candidate in E^S . Then $(\exists_s x_i: \exists_s X_i \rightarrow \exists_s X)_{i \in I}$ is a coproduct candidate in E^T . If $(x_i)_i$ is a lean cocone, then $(\exists_s x_i)_i$ is lean too. Also, if 0 is empty in E^S , $\exists_s 0$ is empty in E^T . Finally, if A is prime in E^S then $\exists_s A$ is prime in E^T .

3.6 Proposition

Let $s: S \rightarrow T$ in C , $h: X \rightarrow Y$ a cocartesian arrow above s .

- i) If X is finite in E^S then Y is finite in E^T .
- ii) If Y is prime in E^T then X is prime in E^S .
- ii.) If $f: Z \rightarrow W$ is any arrow above s where W is empty in E^T then f is cartesian and Z is empty.

Proof

i)

Choose $\exists_s: E^S \rightarrow E^T$ such that $\exists_s X = Y$. Let $(x_i: A_i \rightarrow X)_{i \in I}$ be a lean cocone. I is finite, and $(\exists_s A_i \rightarrow Y)_i$ is a candidate cocone too. Since Y admits a finite candidate to it whose vertices are local primes, it is finite in E^T .

ii)

This is a simple argument using the fact that $\exists_s$ preserves candidates and reflects isos.

iii)

Let f factor as $z: Z \rightarrow Z'$, $g: Z' \rightarrow W$ where g is cartesian and z above identity. By 3.2 iv) Z' is empty in E^S and so z is an iso and f cartesian.

The following is folklore, and we will prove it since we could not find a reference.

3.7 Proposition

Let $\mathcal{C}$ be a class of limit diagrams, say all finite limits, or all products, all connected limits... Let $\mathcal{C}$ be any category, and $E: E \rightarrow \mathcal{C}$ a Grothendieck fibration such that every fiber of E has all $\mathcal{C}$ -limits, that for any $s: S \rightarrow T$ in $\mathcal{C}$ "the" functor $f^*: E^T \rightarrow E^S$ preserves them and that $\mathcal{C}$ has all $\mathcal{C}$ -limits. Then E has all $\mathcal{C}$ -limits, E preserves them, and so does the inclusion functor $E^S \rightarrow E$ for any fiber.

3.7.1 Corollary

If $\mathcal{C}$ is an aggregate and E a stiff fibration then E has connected limits (a right adjoint preserves all limits that exist). Also the inclusion of any fiber $E^S \rightarrow \mathcal{C}$ preserves all connected limits.

3.7.2 Corollary

If $\mathcal{C}$ is an aggregate and E a stiff fibration then E has all filtered colimits (the dual fibration $E^{op} \rightarrow \mathcal{C}^{op}$ will satisfy the conditions if $\mathcal{C}$ is the class of all (co-)filtered limits). Also, the inclusion of any fiber $E^S \rightarrow E$ preserves all filtered colimits.

Proof of 3.7

Since the "colimit" version of the result is more important for us, we will prove it, assuming $E: \mathcal{E} \rightarrow \mathcal{C}$ is an opfibration and $\mathcal{C}$ a class of colimit diagrams. Let $(Y_h)_{h \in H}$ be a $\mathcal{C}$ -diagram in E . Let $(\alpha_h: EY_h \rightarrow T)_h$ be a colimit cocone in $\mathcal{C}$. For every h extend a cocartesian $y_h: Y_h \rightarrow Z_h$ above α_h . We obviously get a diagram $(Z_h)_h$ in E^T . Let $(\gamma_h: Z_h \rightarrow Y)_h$ be a colimit cocone therein. We claim $(\gamma_h y_h)_h$ is a colimit cocone for $(Y_h)_h$ in E . Let $(\delta_h: Y_h \rightarrow X)_h$ be a cocone. There is $s: T \rightarrow EX$ with $s\alpha_h = E\delta_h$. By the cocartesianness of y_h for every h there is an arrow $z_h: Z_h \rightarrow X$ with $z_h y_h = \delta_h$. Let z_h factor as $w_h: Z_h \rightarrow \bar{Z}_h$ and $x_h: \bar{Z}_h \rightarrow X$, where w_h is cocartesian above s and x_h above 1_{EX} ; in the obvious manner we get an H -diagram $(\bar{Z}_h)_h$ and a cocone $(x_h)_h$. If $w: Y \rightarrow \bar{Y}$ is cocartesian above s , by pushing $(\gamma_h)_h$ we get another cocone $(\bar{\gamma}_h: \bar{Z}_h \rightarrow \bar{Y})_h$ above 1_{EX} . By assumption, this last

$$\begin{array}{ccccc}
 & y_h & & \gamma_h & \\
 Y_h & \xrightarrow{\quad} & Z_h & \xrightarrow{\quad} & Y \\
 \delta_h \downarrow & & w_h \downarrow & & \downarrow \\
 & x_h & \bar{Z}_h & \bar{\gamma}_h & w \\
 & \swarrow & & \searrow & \\
 X & \xleftarrow{\quad} & & & \bar{Y} \\
 & f & & &
 \end{array}$$

cocone is a colimit cocone, so there is $f: \bar{Y} \rightarrow X$ above identity with $f\bar{\gamma}_h = x_h$. Then we get $(fw)(\gamma_h y_h) = \delta_h$. To prove the uniqueness, if $g: Y \rightarrow X$ is such that $g(\gamma_h y_h) = \delta_h$ then g is necessarily above s ; by looking at a cocartesian-above identity factorization $g = f'w'$ and repeating the argument with f' , w' it is easy to see $g = fw$.

We revert back to the assumption that $E: \mathcal{E} \rightarrow \mathcal{C}$ is a stiff fibration above an aggregate. We will prove the analogue of the above for multicoproducts.

3.8 Proposition

E has multicoproducts and E preserves coproduct candidates.

Proof

To avoid an orgy of indices we will first describe coproduct candidates in E . Let $(X_i)_{i \in I}$ be a family of objects of E . It will turn out that a coproduct candidate in E can be described a triple $((s_i)_{i \in I}, (x_i)_{i \in I}, (a_i)_{i \in I})$ where

$(s_i: EX_i \rightarrow S)_i$ is a coproduct candidate in $\mathcal{C}$,

$(x_i: X_i \rightarrow A_i)_i$ is a choice of cocartesian arrows in E , with $Ex_i = s_i$,

$(a_i: A_i \rightarrow A)_i$ is a coproduct candidate in E^S .

The candidate itself is given by $(a_i x_i: X_i \rightarrow A)_i$. To get a multicoproduct family for $(X_i)_i$ choose a multicoproduct family $(s_i^j: EX_i \rightarrow S^j)_{i \in I, j \in J}$ for $(EX_i)_i$. Then for each s_i^j choose an $x_i^j: X_i \rightarrow A_i^j$ cocartesian above s_i^j , and for all j choose a multicoproduct family for the discrete cocone $(A_i^j)_i$. Then given $(y_i: X_i \rightarrow Y)_i$ there is a unique $j \in J, t: S^j \rightarrow EY$ with $ts_i^j = Ey_i$. Drop the superscripts for s_i^j, S^j, A_i^j, x_i^j . Let $(a_i^k: A_i \rightarrow A^k)_{k \in K}$ be the chosen multicoproduct family in E^S . Let $f: Z \rightarrow Y$ be a cartesian arrow above t . Since x_i is cocartesian above s_i and $Ey_i = ts_i$, there is a unique $g_i: A_i \rightarrow Y$ above t with $g_i x_i = y_i$. By cartesianness of f there is $h_i: A_i \rightarrow Z$ in E^S with $fh_i = g_i$. This gives a cocone $(h_i)_i$. There is a unique

$$\begin{array}{ccccc}
 & & X_i & & \\
 & & \swarrow x_i & \downarrow & \\
 & & A_i & \xrightarrow{g_i} & Y \\
 a_i^k \swarrow & & \downarrow h_i & \searrow & \\
 A^k & \xrightarrow{h} & Z & \xrightarrow{f} & Y
 \end{array}$$

k and a unique $h: A^k \rightarrow Z$ with $ha_1^k = h_1$. It is easy to see that k is independent of the choice of the cartesian arrow f , and that if $f': Z' \rightarrow Y$ is another such arrow, $h': A^k \rightarrow Z'$ the factoring it determines, that $fh = f'h'$. So $fha_1^k x_1 = y_1$ and $(y_1)_1$ determines a unique j , a unique k among the j -candidates, and a unique factoring $A^k \rightarrow Y$.

3.8.1 Corollary

The initial candidates in E are exactly the objects which are initial candidates in their fiber and above an initial candidate of C .

3.9 Proposition

Let X be a finite object of E^{EX} , where EX is finite. Then X is f.p. in E . Also, $Sub(X)$ is finite.

Proof

Let $(Y_h)_{h \in H}$ be filtered in E , with colimit cocone $\beta_h: Y_h \rightarrow Y$. Let $f: X \rightarrow Y$. Factor every β_h as

$$\begin{array}{ccccc} & y_h & & \gamma_h & \\ & \downarrow & & \downarrow & \\ Y_h & \longrightarrow & Z_h & \longrightarrow & Y \end{array}$$

where y_h is cocartesian and γ_h above identity. We get an H -diagram $(Z_h)_h$ and from the proof of 3.7 we know $(\gamma_h)_h$ is a colimit cocone for it. Let f factor as

$$\begin{array}{ccccc} & f' & & x & \\ & \downarrow & & \downarrow & \\ X & \longrightarrow & X' & \longrightarrow & Y \end{array}$$

where f' is cocartesian and x above identity. We know (3.6 i)) X' is finite in E^{EY} . Choose $h \in H$ such that there is both $m: X' \rightarrow Z_h$ in E^{EY} with $\gamma_h m = x$ and $t: EX \rightarrow EY_h$ in C with $E\beta_h \circ t = Ef$. Let $w: X \rightarrow W$ be a cocartesian arrow above t and $w': W \rightarrow X'$ the unique arrow above $E\beta_h$ such that $w'w = f'$. Since mw' and y_h are both above $E\beta_h$ and y_h is cartesian there is

$$\begin{array}{ccccc}
 & & k & & \\
 & & W \longrightarrow & Y_h & \\
 w \nearrow & & \downarrow w' & \downarrow y_h & \\
 & f' & X \longrightarrow & X' \longrightarrow & Z_h \\
 f \searrow & & \downarrow x & \nearrow \gamma_h & \\
 & & Y & &
 \end{array}$$

$k: W \rightarrow Y_h$ above identity with $y_h k = m w'$. But then $k w$ is such that

$$\beta_h(kw) = (\gamma_h y_h)(kw) = f.$$

The proof that $Sub(X)$ is finite is rather obvious.

Here is a collection of results about cartesian arrows in fibrations. The first three are standard and the proof is left to the reader.

3.10 Proposition

i) $E: \mathcal{E} \rightarrow \mathcal{C}$ any fibration. Let

$$\begin{array}{ccc}
 & f & \\
 X & \longrightarrow & Y \\
 h \searrow & & \downarrow g \\
 & Z &
 \end{array}$$

be such that h and g are cartesian. Then f is cartesian.

ii) Let

$$\begin{array}{ccc}
 & f & \\
 X & \longrightarrow & Y \\
 g' \downarrow & & \downarrow g \\
 Y' & \longrightarrow & Z \\
 & f' &
 \end{array}$$

be above a pullback, and such that g, g' are cartesian. Then the square is a pullback.

iii) If E is a stiff fibration, cartesian arrows are stable under pullback.

iv) E a stiff fibration. Let $f: X \rightarrow Y$, $g: Y \rightarrow Z$ be such that gf is cocartesian. Then both f and g are cocartesian.

Proof of iv)

Let $f': X \rightarrow Y'$ be cocartesian above Ef , $\alpha: Y' \rightarrow Y$ above identity, $\alpha f' = f$. By cocartesianness there is a unique $g': Y' \rightarrow Z$ above Eg with $g'f' = gf$. Since gf is cocartesian, g' is cocartesian, by the dual of i), and therefore also cartesian. So there is a unique $\beta: Y \rightarrow Y'$ with $g'\beta = g$. Now $g'\beta\alpha f' = g'f' = gf = g'f'$.

$$\begin{array}{ccccc} & f' & & g' & \\ X & \longrightarrow & Y' & \longrightarrow & Z \\ & f \searrow & \alpha \downarrow & \beta \nearrow & g \\ & & Y & & \end{array}$$

We get $\beta\alpha f' = f'$, and since $\beta\alpha$ is above identity, f' cocartesian, $\beta\alpha = 1_{Y'}$ and β, α are isomorphisms.

We want our bifibrations to satisfy one more condition: their left adjoint parts should correspond to functors $\mathcal{C} \rightarrow \mathcal{Cat}$ that are "entire" in some way. This is done as follows.

3.11 Definition

We say $X \in E$ is E-generic if given cocartesian arrows $f: X \rightarrow Y$ and $g: Z \rightarrow Y$ there exists $h: X \rightarrow Z$ (necessarily cocartesian) such that $gh = f$. Notice this implies that any cocartesian arrow $w: W \rightarrow X$ is an isomorphism (just take $f = 1_X$ and $g = w$). We will call an X fulfilling this weaker condition an E-minimal object of E . We say E is an entire fibration if for any local prime $B \in E$ there is an E -generic A such that EA is *finite* in $\mathcal{C}$ and a cocartesian arrow $A \rightarrow B$. A is necessarily a local prime by 3.6, ii).

3.12 Proposition

Let $E: \mathcal{E} \rightarrow \mathcal{C}$ be a stiff fibration. Then a local prime $X \in \mathcal{E}$ is E -generic iff given any $f: X \rightarrow Y$ and any cocartesian $g: Z \rightarrow Y$ there exists $h: X \rightarrow Z$ filling the triangle.

Proof

The second condition is obviously stronger. So let X be locally prime and generic. Let f, g be given as above. Assume f is above $s: S \rightarrow T$ and g is above $t: V \rightarrow T$. Let $(a_i: A_i \rightarrow Z)_{i \in I}$ be lean cocone for Z in E^V . By the usual nonsense there is a cocone $(y_i: \exists_t A_i \rightarrow Y)_i$ in E^T and since g is cocartesian this cocone is *lean*, by 3.5. Also there is $x: \exists_s X \rightarrow Y$ above identity with $x \circ \iota_s X = f$.

$$\begin{array}{ccc}
 & X & \\
 \iota_s X \swarrow & & \searrow k \\
 & \exists_s X \rightarrow \exists_t A_i \leftarrow A_i & \\
 x \downarrow & \swarrow y_i & \downarrow a_i \\
 Y & \xleftarrow{\quad} & Z \\
 & g &
 \end{array}$$

Since $(y_i)_i$ is a lean cocone in E^T and $\exists_s X$ is prime therein there exists $i \in I$ and an iso $\rho: \exists_s X \rightarrow \exists_t Y$ with $y_i \rho = x$. But then $\rho \circ \iota_s X$ is cocartesian too and since X is E -generic and $\iota_t A_i$ cocartesian there exists $k: X \rightarrow A_i$ with $\iota_t A_i \circ k = \rho \circ \iota_s X$. Then $h = a_i k$ is the arrow sought.

3.13 Proposition

Let $E: \mathcal{E} \rightarrow \mathcal{C}$ be an entire fibration. Then E is an aggregate. It follows from 3.7.1, 3.7.2 that E is an entire functor.

First we prove

3.13.1 Proposition

- i) Let X be empty in E^S , S prime in $\mathcal{C}$. Then X is prime in E .

- ii) Let X be prime in E^S and E -generic, with S finite. Then X is prime in E .

Proof

i)

If $(x_i: X_i \rightarrow X)_{i \in I}$ is a candidate in E , we know $(Ex_i)_i$ is a candidate in C . Then for some i Ex_i is an isomorphism. But x_i is cartesian (3.6 ii)) so it is an isomorphism.

ii)

Let $(x_i: X_i \rightarrow X)_{i \in I}$ be a candidate in E . Let $x'_i: X'_i \rightarrow X'_i$ be cocartesian, $a_i: X'_i \rightarrow X$ above 1_S such that $x_i = a_i x'_i$. By 3.8 $(a_i)_i$ is a candidate in E^S so there is i with a_i an isomorphism. Then x_i is cocartesian, and since X is E -generic, and therefore E -minimal, x_i is an isomorphism.

We will say X is a prime of the first (or second) kind if it satisfies the corresponding condition above. We can now prove 3.13 :

Since both kinds of primes are finite in their fibers and above finite objects of C , they are f.p. and have a finite subobject lattice. We will show that the collection of all primes of either kind forms a strong generating set. It is easy to see that it is an essentially small set. Let $f: X \rightarrow Y$ be a morphism of E above $s: S \rightarrow T$ such that every $A \rightarrow Y$ where A is one of the two kinds of primes factors through f . Let f factor as $f'x$ where $f': X' \rightarrow Y$ is cartesian and x above 1_S . We first prove s is an isomorphism by testing with primes: let P be a prime of C and $p: P \rightarrow T$. There is $k: K \rightarrow Y$ above p where k factors as $K \rightarrow p^*Y \rightarrow Y$ and K is the empty object of E^P determined by p^*Y . K is a prime of the first kind and by hypothesis there is $m: K \rightarrow X$ with $fm = k$. Then $s \circ Em = p$ and this proves our claim. Therefore f' , being cartesian, is an isomorphism, and we are left to prove that x is an iso too. We just

have to test with the primes in E^S . Let B be such a prime, and $b:B \rightarrow X'$ above identity. By hypothesis there is A E -generic and $a:A \rightarrow B$ cocartesian, EA finite. A is a prime of the second kind and so there is $n:A \rightarrow X$ with $f'xn = fn = f'ba$.

$$\begin{array}{ccc}
 & a & \\
 & A \longrightarrow B & \\
 n \downarrow & & \downarrow b \\
 X \longrightarrow X' \longrightarrow Y & & \\
 x & & f'
 \end{array}$$

It ensues that $ba = xn$, and since a is cocartesian and x above identity there is $c:B \rightarrow X$ with $ca = n$. We get that $xc = b$ by the uniqueness property associated with cocartesian arrows ($Ec = Eb$).

It is left to show that Axiom 5 holds to complete the proof. Suppose $(x_1: X_1 \rightarrow X)$ is a candidate in E and let $a: A \rightarrow X$ with A prime (a prime of E has to be of the first or the second kind: 1.6.1). If A is of the first kind, EA is prime in C , so there is $i \in I$ and $f: EA \rightarrow EX_1$ with $Ex_1 \circ f = Ea$. Let $y: Y \rightarrow X$ be a cartesian arrow above Ex_1 . There is $m: X_1 \rightarrow Y$ above identity with $ym = x_1$. Let $\gamma Y: f^*Y \rightarrow Y$ and $\gamma X_1: f^*X_1 \rightarrow X$ be cartesian above f . There is $p: A \rightarrow f^*Y$ above 1_{EA} with $y \circ \gamma Y \circ p = a$ since $y \circ \gamma Y$ is cartesian and above Ea .

$$\begin{array}{ccccc}
 & & \gamma X_1 & & \\
 & & f^*X_1 \longrightarrow X_1 & & \\
 q \nearrow & \downarrow f^*m & m \downarrow & \searrow x_1 & \\
 A \longrightarrow f^*Y & \longrightarrow & Y & \longrightarrow & X \\
 p & & \gamma Y & & y
 \end{array}$$

f^*m and p are in E^{EA} , and since A is an initial candidate therein there is $q: A \rightarrow f^*X_1$ with $f^*m \circ q = p$. But then $\gamma X_1 \circ q$ is the morphism we are looking for.

If A is of the second kind, let $a': A \rightarrow A'$ be cocartesian above Ea and $s: A' \rightarrow X$ above identity, $sa' = a$. If

$$\begin{array}{c} x_1' \quad t_1 \\ X_1 \rightarrow X_1' \rightarrow X \end{array}$$

is now the cocartesian-above-identity factorization of x_1 , we know $(t_1)_1$ is a candidate. A' is prime in E^{EX} by 3.2 iv) and therefore there is $i \in I$ and $q: A' \rightarrow X_1'$ with

$$\begin{array}{ccccc} & & X_1 & & \\ & & \downarrow x_1' & & \\ A & \xrightarrow{a'} & A' & \xrightarrow{q} & X_1' \\ & s \searrow & & \nearrow t_1 & \\ & & X & & \end{array}$$

$t_1 q = s$. Since A is E -generic and x_1' cocartesian, by 3.12 there is $h: A \rightarrow X_1$ with $x_1' h = q a'$. Then $x_1 h = a$.

3.13.2 Corollary

The finite objects of an entire fibration $E: E \rightarrow C$ are exactly those which are finite in their fiber and above a finite object of C .

This is because a finite object $X \in E$ will have a finite lean cocone $(a_i: A_i \rightarrow X)_{i \in I}$ in E . Since E preserves candidates cocones, and every EA_i is finite in C , it follows easily that X is above a finite object. By factoring every a_i as

$$\begin{array}{c} b_i \quad x_i \\ A_i \rightarrow X_i \rightarrow X \end{array}$$

where b_i is cocartesian and x_i above identity, we get a finite candidate $(x_i)_1$, and since X_1 is locally prime, X is finite in E^{EX} . The converse is 3.9.

3.14 Proposition

Let $E: E \rightarrow C$ be a stiff fibration. Then the following are equivalent:

- i) $\mathcal{E}$ satisfies the Beck condition for all pullback squares in $\mathcal{C}$.
- ii) The pullback of a cocartesian arrow by a cartesian arrow is cocartesian.
- iii) The pullback of a cocartesian arrow by an arbitrary arrow is cocartesian.

When the above hold, for brevity we say that $\mathcal{E}$ satisfies the Beck condition, or that Beck holds.

Proof

First we leave it to the reader to verify the following fact, which is true in more general circumstances than stiff fibrations (and can be used as a definition of the Beck condition by the inexperienced reader).

3.14.1 The Beck condition holds for a square

$$\begin{array}{ccc} & P & \\ R & \longrightarrow & S \\ q \downarrow & & \downarrow s \\ T & \longrightarrow & V \\ & t & \end{array}$$

in $\mathcal{C}$ iff for any $X \in \mathcal{E}^S$ the unique $v: \exists_q p^*X \rightarrow \exists_s X$ above t making the square below commute is a cartesian arrow

$$\begin{array}{ccc} & \gamma_p & \\ p^*X & \longrightarrow & X \\ \wr_q \downarrow & & \downarrow \wr_s \\ \exists_q p^*X & \longrightarrow & \exists_s X \end{array}$$

(v exists since $\wr_q$ is cocartesian).

We can now prove 3.14

i) $\Rightarrow$ ii)

Let

$$\begin{array}{ccc}
 & f & \\
 X & \longrightarrow & Y \\
 h \downarrow & & \downarrow g \\
 W & \longrightarrow & Z \\
 & k &
 \end{array}$$

be a pullback square in $\mathcal{E}$ where g is cocartesian and k cartesian. let $h': X \rightarrow W'$ be cocartesian above Eh , and $k': Y' \rightarrow Z$ the unique arrow above Eh making $k'h' = gf$. There is $\alpha: W' \rightarrow W$ above identity with $k\alpha = k'$. 3.14.1 applies and by hypothesis k' is cartesian. Therefore α is an isomorphism. By the cartesianness of k , $\alpha h'$ is the unique morphism above Eh making $k\alpha h' = gf$ and so $\alpha h' = h$. Therefore h is cocartesian.

ii) $\Rightarrow$ iii)

Let now the square above be a pullback where f and k are not necessarily cartesian. Let k factor as

$$\begin{array}{ccccc}
 & w' & & k' & \\
 W & \longrightarrow & Z' & \longrightarrow & Z
 \end{array}$$

where k' is cartesian and w' above identity. Let

$$\begin{array}{ccc}
 & f' & \\
 Y' & \longrightarrow & Y \\
 h'' \downarrow & & \downarrow g \\
 W' & \longrightarrow & Z \\
 & k' &
 \end{array}$$

be a pullback. f' is cartesian above Ef , (3.10) so there is $x: X \rightarrow Y'$ above identity with $f'x = f$. By hypothesis, h'' is cocartesian. By 3.3 d), h is cocartesian.

iii) $\Rightarrow$ i)

Let $p, q, s, \dots$ be as in 3.14.1. We want to prove $v: \exists_q p^* X \rightarrow \exists_s X$ is cartesian. Let

$$\begin{array}{ccc}
 & a & \\
 A & \longrightarrow & X \\
 b \downarrow & & \downarrow \iota_s X \\
 t^* \exists_s X & \longrightarrow & \exists_s X \\
 & \downarrow & \\
 & \gamma_t \exists_s X &
 \end{array}$$

be a pullback. By cartesianess, there is $z: \exists_q p^* X \rightarrow t^* \exists_s X$ above identity with $\gamma_t \exists_s X \circ z = v$. By pullbackness there is $c: p^* X \rightarrow A$ above identity with $ac = \gamma_p$ and $bc = z \circ \iota_q p^* X$.

$$\begin{array}{ccccc}
 & & \gamma_p X & & \\
 p^* X & \xrightarrow{\quad} & X & & \\
 \downarrow c \searrow & & a \nearrow & & \downarrow \iota_s X \\
 & A & & & \\
 \downarrow \iota_q p^* X & & b \downarrow & & \\
 & t^* \exists_s X & & & \\
 \downarrow z \nearrow & & \searrow & & \downarrow \\
 \exists_q p^* X & \xrightarrow{\quad} & \exists_s X & & \\
 & v & & &
 \end{array}$$

By 3.10 a is cartesian, and so c is an isomorphism. But by hypothesis b is cocartesian and so z is an iso, and this makes v cartesian.

3.14.2 Lemma

Let $E: \mathcal{E} \rightarrow \mathcal{C}$ be a fibration satisfying the conditions of 3.14. Let

$$\begin{array}{ccccc}
 & g' & & h' & \\
 X' & \longrightarrow & Z' & \longrightarrow & Y' \\
 x \downarrow & & \downarrow z & & \downarrow y \\
 X & \longrightarrow & Z & \longrightarrow & Y \\
 & g & & h &
 \end{array} \quad (*)$$

be a diagram in $\mathcal{E}$ such that the outer and left squares are pullbacks, g, g' are cocartesian, and the right square is above a pullback. Then the right square is a pullback.

Proof

Let $m:A \rightarrow Y'$, $n:A \rightarrow Z$ be such that $ym = hn$. Let

$$\begin{array}{ccc} & a & \\ & B \rightarrow A & \\ p \downarrow & & \downarrow n \\ & X \rightarrow Z & \\ & g & \end{array}$$

be a pullback. By hypothesis a is cocartesian. Since the outer square of $(*)$ is a pullback there is $q:B \rightarrow X'$ with $h'g'q = ma$ and $xq = p$. Since the right square is above a pullback there is $s:EA \rightarrow EZ'$ with $Ez \circ s = En$ and $Eh' \circ s = Em$. Then it is easy to show $s \circ Ea = Eg' \circ Eq$, since $Eh' \circ s \circ Ea = Em \circ Ea = Eh' \circ Eg' \circ Eq$. Since a is cocartesian there is $f:A \rightarrow Z'$ above s with $fa = g'q$. Then, using the "relative epiness" of a one can check that $h'f = m$ and $zf = n$.

3.15 Proposition

Let $E:E \rightarrow \mathcal{C}$ be a stiff fibration. Then the following are equivalent:

- i) E is entire
- ii) E satisfies the Beck condition (for all pullbacks) and every local prime $B \in E$ admits a cocartesian $A \rightarrow B$ where A is E -minimal and above a finite object of $\mathcal{C}$.
- iii) E satisfies the Beck condition and every $B \in E$ which is locally finite admits a cocartesian $A \rightarrow B$ where A is E -minimal and above a finite object of $\mathcal{C}$.

Proof

i) $\Rightarrow$ ii)

Suppose E is an entire fibration. Obviously we only have to show that Beck holds. Let

$$\begin{array}{ccc}
 & f & \\
 X & \longrightarrow & Y \\
 k \downarrow & & \downarrow g \\
 W & \longrightarrow & Z \\
 & h &
 \end{array}$$

be a pullback where g is cocartesian. Let k split as

$$\begin{array}{ccccc}
 & k' & & w & \\
 X & \longrightarrow & W' & \longrightarrow & W
 \end{array}$$

where k' is cocartesian above Ek and w above identity. To show k is cocartesian it suffices to show w is an isomorphism by testing with primes of $\mathbf{E}^{EW}$. Let $b:B \rightarrow W$ be above identity with B prime in its fiber. There is $a:A \rightarrow B$ cocartesian with A $\mathbf{E}$ -generic. Since g is cocartesian there is $y:A \rightarrow Y$ with $gy = hba$. Therefore there is $x:A \rightarrow X$ with $kx = ba$. The cocartesianness of a will give the required $B \rightarrow W'$.

ii) $\Rightarrow$ iii)

Let B be finite in its fiber and let $(b_i: B_i \rightarrow B)_{i \in I}$ be a lean cocone in $\mathbf{E}^{EB}$. For every i let $f_i: A_i \rightarrow B_i$ be a cocartesian arrow where A_i is $\mathbf{E}$ -minimal. Let $(a_i: A_i \rightarrow A)_i$ be the candidate determined by $(b_i f_i)_i$ and $f: A \rightarrow B$ the factoring. It is easy to see that f is cocartesian: if

$$\begin{array}{ccc}
 a'_i & x_i & \\
 A_i & \rightarrow A'_i & \rightarrow A
 \end{array}$$

is a cocartesian-above identity factoring for $(a_i)_i$, then there is $g_i: A'_i \rightarrow B_i$ cocartesian above Ef with $g_i a'_i = f_i$, and the fact that $\exists_{Ef}$ preserves candidates and both $(x_i)_i$ and $(b_i)_i$ are will force the cocartesianness of f . We just have to show A is $\mathbf{E}$ -generic: let $c: C \rightarrow A$ be some cocartesian arrow. By 3.12 for every i there is $c_i: A_i \rightarrow C$ with $cc_i = a_i$ and this shows c is an isomorphism.

iii) $\Rightarrow$ i)

Let B be a local prime in E . Since it is locally finite, there is $a: A \rightarrow B$ where A is E -minimal (and A is above a finite object of C). We want to prove A is E -generic. Let

$$\begin{array}{ccc} & f & g \\ A & \longrightarrow Y & \longleftarrow Z \end{array}$$

where g is cocartesian. Let $c: C \rightarrow A$ be the pullback. By assumption (Beck), c is cocartesian, and therefore an isomorphism, since A is E -minimal. We get $A \rightarrow Z$ making the triangle commute in an obvious manner.

3.15.1 Corollary

Let C be an aggregate. An entire fibration above C is the same as a bifibration $E: E \rightarrow C$ where all the fibers are aggregates, cocartesian arrows are cartesian and stable under all pullbacks, and where for any locally finite A there is an E -minimal object above a finite object and a cocartesian morphism to A (it suffices to assume A is locally prime).

This is just the fact that condition 3.3 d) and Beck are covered by the assumption that cocartesians are stable under arbitrary pullbacks.

3.15.2 Corollary

In a stiff fibration with the Beck condition, E -minimal objects are E -generic.

3.16 Definition

Let $E: E \rightarrow C$ be an entire fibration above an aggregate. If for every $S \in C$ the fiber E^S is semigranular we say E is a semigranular fibration. We denote by $Fib(C)$ the category whose objects are entire fibrations $E: E \rightarrow C$ and morphisms entire morphisms of fibrations, i.e. where a morphism $E \rightarrow F$ ($F: F \rightarrow C$) is an entire functor $H: E \rightarrow F$ such that $FH = E$. We denote by $Sem(C)$

the full subcategory of $Fib(\mathbf{C})$ whose objects are the semigranular fibrations. Notice that $Fib(\mathbf{1}) = Aggr$ (2.7), and $Sem(\mathbf{1}) = Sem$. By Fib and Sgr we denote the class of all entire fibrations, and the class of all semigranular fibrations.

3.17 Proposition

- i) Let $\mathbf{C}, \mathbf{D}$ be aggregates. Then the projection $\pi: \mathbf{C} \times \mathbf{D} \rightarrow \mathbf{C}$ is an entire fibration. If $\mathbf{D}$ is semigranular it is a semigranular fibration. In particular identity functors are semigranular fibrations.
- ii) Let $E: \mathbf{E} \rightarrow \mathbf{C}$ be an entire fibration, $F: \mathbf{F} \rightarrow \mathbf{C}$ an entire functor. Then the pullback $P: \mathbf{P} \rightarrow \mathbf{F}$ is an entire fibration. If E is semigranular, P is semigranular.
- iii) the composite of two entire fibrations is an entire fibration. In particular, if $E: \mathbf{E} \rightarrow \mathbf{C}$ is an entire fibration, the pullback functor $E^*: Fib(\mathbf{C}) \rightarrow Fib(\mathbf{E})$ has a left adjoint Σ_E .
- iv) The categories $Ent(\mathbf{C})$ and $Fib(\mathbf{C})$ have products.

3.17.1 Corollary

Both $(Aggr, Fib)$ and $(Aggr, Sgr)$ form display categories. The first one also admits sums, and satisfies the condition Display of [H-P]: every morphism to the terminal object is a display map. Both display categories admit discrete products.

Proof of 3.17

i)

We know that the projection as a fibration corresponds to the constant functor $D: \mathbf{C}^{op} \rightarrow Cat$; therefore, π is a stiff fibration. It is easy to check that a π -generic object of $\mathbf{C} \times \mathbf{D}$ is of the form (K, X) , where K is empty in $\mathbf{C}$ and X any object of $\mathbf{D}$. It follows that π is an entire fibration. Trivially, π is semigranular when $\mathbf{D}$ is.

ii)

If E is a fibration corresponding to the pseudo-functor $\tilde{E}: \mathcal{C}^{op} \rightarrow \mathcal{Cat}$, then we know that P corresponds to $\tilde{E} \circ F^{op}$. Hence, P is a stiff fibration, and its fibers are semigranular when those of E are. We have to show P is an entire fibration. Remember that an object of P above $X \in F$ is a pair (X, A) where $A \in E^{FX}$. A very standard argument shows that a (co)cartesian arrow $(X, A) \rightarrow (Y, B)$ is a pair (f, a) where a is (co)cartesian above Ff . Therefore, cocartesian arrows are cartesian and stable under arbitrary pullbacks. Let (X, A) be any object of P . Let $a: A' \rightarrow A$ be cocartesian, above $s: S \rightarrow FX$ with A' E -generic. Since F is entire there is $t: S \rightarrow FX'$ generic along with $x: (X', t) \rightarrow (X, s)$ in S/F . $\exists_t A'$ is above FX' , so $(X', \exists_t A')$ is an object of P and there is $(x, w): (X', \exists_t A') \rightarrow (X, A)$ where w is the unique morphism $\exists_t A' \rightarrow A$ above Fx such that $w \circ \iota_t A' = a$. (x, w) is cocartesian by 3.10 iv).

$$\begin{array}{ccccc}
 & & t & & \iota_t A' \\
 & & S \longrightarrow & FX' & A' \longrightarrow \exists_t A' \\
 X' & & & & \\
 \swarrow x & s \searrow & \swarrow Fx & a \searrow & \swarrow w \\
 X & & FX & & A
 \end{array}$$

We claim $(X', \exists_t A')$ is P -minimal, which will prove P is an entire fibration, by 3.15.1. Let $(f, b): (Y, B) \rightarrow (X', \exists_t A')$ be P -cocartesian, that is, b is E -cocartesian. Then, since A' is E -generic there is $c: A' \rightarrow B$ with $bc = \iota_t A'$. Then it is easy to see $Ec: S \rightarrow FY$ is such that f is a morphism $(Y, Ec) \rightarrow (X', t)$ in S/F , forcing f to be an iso, and then b is one too since it is cocartesian above Ff .

iii)

It is well known that if $F: F \rightarrow E$ and $E: E \rightarrow \mathcal{C}$ are two ordinary (bi-)fibrations then EF is a (bi-)fibration, where a (co-)cartesian arrow $f: X \rightarrow Y$ for EF is a morphism such that f is (co-)cartesian for F and Ff is (co-)cartesian for E . It is then easy to see then that in our case cocartesian arrows are cartesian and stable under pullbacks. If $S \in \mathcal{C}$ then the EF -fiber F^S is the composite pullback

$$\begin{array}{ccc}
 F^S & \longrightarrow & F \\
 \downarrow & & \downarrow F \\
 E^S & \longrightarrow & E \\
 \downarrow & & \downarrow E \\
 1 & \longrightarrow & C \\
 S & &
 \end{array}$$

and by the previous proposition F^S is an aggregate. In view of 3.15.1 it now suffices to show that if B is finite in F^S (as above) then there is an EF -minimal object and a cocartesian arrow to B . By 3.13.2 FB is finite in E^S and since E is an entire fibration there is X E -minimal and $x:X \rightarrow FB$ cocartesian (EX is finite). But also, since F is an entire fibration there is C F -minimal and $c:C \rightarrow B$ cocartesian for F , and FC is finite in E . Let f, g be the candidate determined by Fc, x and y the factoring:

$$\begin{array}{ccc}
 & FC & \\
 f \downarrow & & \searrow Fc \\
 & Y & \longrightarrow FB \\
 g \uparrow & & \nearrow x \\
 & X &
 \end{array}$$

by 3.10, iv) both g and y are cocartesian. We can split c as a pair

$$\begin{array}{ccccc}
 & h & & a & \\
 C & \longrightarrow & A & \longrightarrow & B
 \end{array}$$

of cocartesian arrows above f and y . Then a is cocartesian for EF , and EFA is obviously finite. We are left to show A is EF -minimal. Let $k:K \rightarrow A$ be EF -cocartesian. This means k is F -cocartesian and $Fk:FK \rightarrow Y$ is E -cocartesian. By the former, there is $m:C \rightarrow K$ with $km = h$ (F -minimal objects are F -generic, by 3.15.2). By the latter there is $z:X \rightarrow FK$ with $Fk \circ z = g$. But then both f and g factor through Fk and this shows Fk is an isomorphism. But since k is cocartesian above Fk , it is an iso too, QED.

Notice that if F and E are both *semigranular* fibrations, it is not necessarily the case that EF is a semigranular fibration. This is because given an entire fibration $E:E \rightarrow C$, a prime of the second kind in E is atomic in E if and only if it is atomic in its fiber and is above an empty object of C , as the reader may verify.

iv)

In view of the preceding proposition, the proof that $Ent(C)$ has products is trivial: if $D:D \rightarrow C$ and $E:E \rightarrow C$ are entire fibrations then their product in $Ent(C)$ is the composite, say $P \rightarrow D \rightarrow C$ where $P \rightarrow D$ is the pullback of E by D . But $P \rightarrow D$ is an entire fibration by ii), and then $P \rightarrow C$ is an entire fibration by iii). Now if both D, E are semigranular fibrations, it is well known that for $S \in C$ the fiber P^S is the product $D^S \times E^S$, and this is semigranular.

The right notion of genericity for morphisms of entire fibrations is the following.

3.18 Definition

Let $E:E \rightarrow C$, $F:F \rightarrow C$ be entire fibrations, $H:E \rightarrow F$ with $FH = E$. We say an arrow $x:A \rightarrow HX$ of F is ultrageranic if:

- a) x is above identity.
- b) For any cocartesian $b:B \rightarrow A$, xb is generic.

Note this implies x is generic itself.

3.19 Proposition

Let $E:E \rightarrow C$ be a stiff fibration. Let A above S be locally empty, that is, be empty in E^S . Let $s:S \rightarrow T$. For any $X \in E^T$ there is at most one morphism $A \rightarrow X$ above s . If there exists $f:X \rightarrow Y$ and $y:A \rightarrow Y$ such that $Ef \circ s = Ey$, then that morphism exists.

Proof

The first part is just the observation that any arrow x above s decomposes in an essentially unique way as one in E^S followed by a cartesian arrow, and that if A is the source of x , the arrow in E^S is unique. For the second part, let $g:Z \rightarrow X$ be cocartesian above s . Then there is $a:A \rightarrow Z$ with $fga = y$.

3.20 Proposition

E, F as above. Let $H:E \rightarrow F$ be a functor such that $FH = E$. H is entire iff for any local prime $B \in F$, any $y:B \rightarrow HY$ above identity, there is a cocartesian arrow $a:A \rightarrow B$ (A will necessarily be locally prime, cf. 3.6), an ultrageneric $x:A \rightarrow HX$ and a morphism $f:(X, x) \rightarrow (Y, ya)$ in A/H .

Proof

Suppose H is entire. Let B be locally prime, $y:B \rightarrow HY$ above identity. Let B' be E -generic, $b:B' \rightarrow B$ cocartesian. There is $y':B' \rightarrow HX$ generic and $f:X \rightarrow Y$ with $Hf \circ y' = yb$. Let y' split as

$$\begin{array}{c} m \quad x \\ B' \longrightarrow A \longrightarrow HX \end{array}$$

where m is cocartesian above Ey' and x above identity. We have

$$\begin{aligned} Fb &= Fy \circ Fb = F(Hf \circ y') = F(Hf \circ xm) \\ &= F(Hf) \circ Fm = Ef \circ Fm \end{aligned}$$

$$\begin{array}{ccccc} & & b & & \\ & & \longrightarrow & & \\ & & B' & \longrightarrow & B \\ & y' \downarrow & \searrow m & \downarrow & \downarrow y \\ & & A & & \\ & \downarrow & \nearrow x & \downarrow & \\ & & HX & \longrightarrow & HY \\ & & Hf & & \end{array}$$

and since m is cocartesian there is $a:A \rightarrow B$ above Ef with $am = b$. We claim x is ultrageneric. Since H preserves pullbacks it suffices to prove that if $c:C \rightarrow A$ is cocartesian and there are $z:C \rightarrow HZ$, $g:Z \rightarrow X$ such that $Hg \circ z = xc$ then g is an iso.

$$\begin{array}{ccc}
 & m & c \\
 B' & \longrightarrow & A \longleftarrow C \\
 y' \searrow & \downarrow x & \downarrow z \\
 & HX \longleftarrow HZ \\
 & Hg &
 \end{array}$$

Since B' is E -generic there is $n:B' \rightarrow C$ with $cn = m$. Then we get $g:(Z, zn) \rightarrow (X, y')$ in B'/H and since y' is generic g is an iso.

For the converse, let A be prime in F , $y:A \rightarrow HY$ above $s:S \rightarrow T$. We want to find a generic arrow and a morphism to y in A/H . Two cases may happen:

i) A is empty in E^S (and S is prime).

Let X be the initial candidate of s^*Y in E^S . There is a unique $f:X \rightarrow Y$ above s . Hf and y are both above s so there are $HX \rightarrow s^*HY$ and $A \rightarrow s^*HY$ above 1_S .

$$\begin{array}{ccc}
 HX \rightarrow s^*HY \longleftarrow A \\
 Hf \searrow \quad \downarrow \quad \swarrow y \\
 \quad \quad HY
 \end{array}$$

But since A is empty in E^S , there is $x:A \rightarrow HX$, and obviously $Hf \circ x = y$. Let us prove x is a generic arrow: let

$$\begin{array}{ccc}
 (X, x) & & (Z, z) \\
 g \searrow & & \swarrow g' \\
 & (Z', z') &
 \end{array}$$

be morphisms in A/H . Since we have $Hg \circ x = Hg' \circ z$ and x is above identity, we have $FHg \circ Fx = FHg' \circ Fz$, i.e. $Eg = Eg' \circ Fz$.

$$\begin{array}{ccc}
 & Fz & \\
 EX & \longrightarrow & EZ \\
 Eg \searrow & & \nearrow Eg' \\
 & EZ' &
 \end{array}$$

Since X is locally empty, by 3.19 there is $h:X \rightarrow Z$ above Fz with $g'h = g$. It is trivial to show that $Hh \circ x = z$.

ii) A is E -generic and atomic in E^S .

Let y factor as

$$\begin{array}{ccccc}
 & a & & y' & \\
 A & \longrightarrow & A' & \longrightarrow & HY
 \end{array}$$

where a is cocartesian above s and y' in E^T . There is $x:B \rightarrow HX$ ultrageneric, with $b:B \rightarrow A'$ cocartesian and $f:(X,x) \rightarrow (Y,y'b)$ in B/H .

$$\begin{array}{ccccc}
 & b & & a & \\
 B & \longrightarrow & A' & \longleftarrow & A \\
 x \downarrow & & y' \downarrow & & \nearrow y \\
 HX & \longrightarrow & HY & & \\
 & Hf & & &
 \end{array}$$

Since A is E -generic and B cocartesian there is $c:A \rightarrow B$ with $bc = a$. But then xc is generic, since x is ultrageneric. QED

3.20.1 Corollary

Let E, F be as above, $H:E \rightarrow F$ an entire morphism of fibrations. Let A be a prime of the first kind, $z:A \rightarrow HZ$ a generic arrow. Then Z is initial in its fiber, and z is above an isomorphism.

Proof

The proof of the theorem in the case " A is of the first kind" constructs a generic $x:A \rightarrow HX$ where X is initial in its fiber and x above identity. The conclusion follows from the fact that x and z are initial in A/H .

3.21 Proposition

a)

Let E, F be as above. Let $H, K: E \rightarrow F$ be entire morphisms of fibrations, $\varphi: H \rightarrow K$ a natural transformation above identity, i.e. such that for every $X \in E$, $\varphi X: HX \rightarrow KX$ is above identity. The following are equivalent:

- i) For every local prime A of F , every ultrageneric $x: A \rightarrow HX$, $\varphi X \circ x$ is ultrageneric.
- ii) For every ultrageneric $x: A \rightarrow HX$ in F , $\varphi X \circ x$ is ultrageneric.
- iii) φ is cartesian.

b)

E, F as above. $H, K: E \rightarrow F$ not necessarily entire, but such that $FH = FK = E$. Let $\varphi: H \rightarrow K$ be a natural transformation above identity. Let $x: A \rightarrow HX$ above identity be such that $\varphi X \circ x$ is ultrageneric. Then x is ultrageneric.

Proof

a)

ii) $\Rightarrow$ i) is obvious.

i) $\Rightarrow$ iii)

Let $y: A \rightarrow HY$ be generic, with A prime. We want to prove $\varphi X \circ y$ is generic. If A is of the first kind, we know from 3.20.1 that Y is empty in its fiber. Without loss of generality we can suppose that y is above identity. Then $\varphi Y \circ y$ is a morphism $A \rightarrow KY$ above identity. Using the fact that Y is locally empty, argument i) in the converse of 3.20 will show that $\varphi Y \circ y$ is generic.

If A is of the second kind it is locally prime, and therefore argument ii) in 3.20 shows there is $x: B \rightarrow HX$ ultrageneric,

$c:A \rightarrow B$ cocartesian and $f:X \rightarrow Y$ with $Hf \circ xc = y$. Since A is F -generic, c is an isomorphism, and then both xc and y are generic, showing f is an isomorphism. But by hypothesis, $\varphi X \circ x$ is ultrageneric. Hence $\varphi X \circ c$ is generic, and since f is an iso $(X, \varphi X \circ xc) \rightarrow (Y, \varphi Y \circ y)$ in A/K , $\varphi Y \circ y$ is generic.

iii) $\Rightarrow$ ii)

Let $x:A \rightarrow HX$ be ultrageneric. To show $\varphi X \circ x$ is ultrageneric, let there be $b:B \rightarrow A$ cocartesian, $y:B \rightarrow KY$ and $f:(Y, y) \rightarrow (X, \varphi X \circ xb)$ in A/K . We want to show f is an isomorphism. Since φ is cartesian, there is $x':B \rightarrow HY$ with $Hf \circ x' = xb$ (and $\varphi Y \circ x' = y$).

$$\begin{array}{ccc}
 & B & \\
 b \swarrow & & \searrow x' \\
 A & HY \rightarrow KY & \\
 x \searrow & \downarrow Hf & \downarrow Kf \\
 & HX \rightarrow KX & \\
 & \varphi X &
 \end{array}$$

But x is ultrageneric, and therefore f is an isomorphism.

The proof of b) is trivial, using the definition of an ultrageneric arrow and 2.6

CHAPTER 4

The main theorem 4.7 asserts that our display categories admit products. We can then construct the model of the theory of constructions. As in the last few pages, $E:E \rightarrow \mathcal{C}$ and $F:F \rightarrow \mathcal{C}$ are entire fibrations above aggregate $\mathcal{C}$.

4.1 Definition

We denote by $Ent(E,F)$, or $Ent_{\mathcal{C}}(E,F)$ the category whose objects are entire morphisms of fibrations, and morphisms cartesian transformations above identity.

In order to simplify the arguments in what follows, if $H:E \rightarrow F$ is a functor with $FH = E$ we will denote by $F//H$ the category whose objects are triples (A,X,x) , where $x:A \rightarrow HX$ is above identity, and where a morphism $(a,f):(A,X,x) \rightarrow (B,Y,y)$ is a pair $a:A \rightarrow B$, $f:X \rightarrow Y$, where a is cocartesian, $Fa = Ef$, and $Hf \circ x = ya$. We will say (A,X,x) is generic if x is generic, and ultrageric if x is ultrageric. The statement of 3.20 can now be rewritten as: for any (B,Y,y) in $F//H$ with B locally prime there is an ultrageric (A,X,x) and a morphism $(A,X,x) \rightarrow (B,Y,y)$. When the context is clear we will simply use x to denote the full object (A,X,x) .

4.2 Theorem

Let $E:E \rightarrow \mathcal{C}$ and $F:F \rightarrow \mathcal{C}$ be entire fibrations above aggregate $\mathcal{C}$. Then $Ent(E,F)$ is an aggregate. If F is a semigranular fibration, $Ent(E,F)$ is semigranular.

In order to prove this, we will embed $Ent(E,F)$ in the category $\mathcal{C}$ whose objects are all functors $H:E \rightarrow F$ with $FH = E$ and whose morphisms are all natural transformations above identity.

4.2.1 Filtered colimits are calculated pointwise in $\mathcal{C}$ and $Ent(E,F)$.

Since the proof for $\mathcal{C}$ is a subset of the proof for $Ent(E,F)$ we will only prove the latter. Let $(H_h)_{h \in H}$ be a filtered diagram in $Ent(E,F)$. Let H be its pointwise colimit: $HX = \varinjlim_h H_h X$. Since the diagram $(H_h X)_h$ lays entirely in the fiber F^{EX} , one can choose HX and the coprojections $\varphi_h: H_h X \rightarrow HX$ to be in F^{EX} for every X (remember, the inclusion of fibers preserves filtered colimits). H is entire since in a Diers category filtered colimits commute with pullbacks [Di80, 5.0]. We will first show that if $A \in F$ is locally prime and $x: A \rightarrow H_h X$ ultrageneric, then $\varphi_h X \circ x$ is ultrageneric too, proving φ_h is cartesian. So let $b: B \rightarrow A$ be cocartesian, $y: B \rightarrow HY$ above identity, and $f: Y \rightarrow X$ such that $Hf \circ y = \varphi_h X \circ xb$.

$$\begin{array}{ccccc}
 & & b & & \\
 & & \longrightarrow & & \\
 B & \xrightarrow{\quad} & A & & \\
 \downarrow y & \searrow y' & H_k f & H_q X \downarrow x & \\
 & H_k Y & \longrightarrow & H_k X & \longleftarrow & H_h X \\
 \downarrow & \swarrow \varphi_k Y & & \varphi_k X \searrow & \downarrow \varphi_h X \\
 HY & \xrightarrow{\quad} & HX & & \\
 & & Hf & &
 \end{array}$$

It suffices to show f is an isomorphism. Since y is above identity, B locally prime and therefore finite in its fiber, and $(H_h Y)_h$ a filtered diagram in that fiber, there exists $k \in H$, $y': B \rightarrow H_k Y$ above identity such that $\varphi_k Y \circ y' = y$; k can be chosen with $q: h \rightarrow k$. Then $H_q X \circ x$ is ultrageneric by hypothesis, and since

$$\varphi_k X \circ H_k f \circ y' = \varphi_k X \circ H_q X \circ xb$$

and $\varphi_k X$ is mono, we get that f is an iso.

We still have to prove $(\varphi_h)_h$ is a colimit cocone in $Ent(E,F)$. Let $(\psi_h: H_h \rightarrow K)_h$ be some cocone therein. There exists a unique natural transformation $\theta: H \rightarrow K$ such that $\theta \circ \varphi_h = \psi_h$, and θ can be shown easily to be above identity. Let us show θ is cartesian: let $x: A \rightarrow HX$ be ultrageneric with A locally prime. There exists h

and $x': A \rightarrow H_h X$ with $\psi_h X \circ x' = x$, all this happening in the same fiber of F . x' is ultrageneric, too by 3.21 b).

$$\begin{array}{ccc} & H_h X & \\ x' \nearrow & \downarrow & \searrow \psi_h X \\ A & \rightarrow HX \rightarrow KX & \\ x & \theta_X & \end{array}$$

Then $\psi_h X \circ x' = \theta_X \circ x$ is ultrageneric.

Remark

This proposition can be proven without using the fact that filtered colimits commute with pullbacks. One then has to use the original definition of generic arrow, which gives a more intricate argument.

4.2.2 $\mathcal{C}$ has "weak multicoproducts", in the sense that multicoproduct families may not be small. $Ent(E, F)$ has multicoproducts.

Say a discrete cocone $(\varphi_i: H_i \rightarrow H)_{i \in I}$ in $\mathcal{C}$ or $Ent(E, F)$ is correct if for every $X \in E$, $(\varphi_i X: H_i X \rightarrow HX)_{i \in I}$ is a coproduct candidate in F^{EX} . One sees easily that between two correct cocones there can be at most one morphism of cocones, and that that morphism must be an iso. Given a family $(H_i)_{i \in I}$ in either $\mathcal{C}$ or $Ent(E, F)$ we will show that a choice of one representative for each isomorphism type of correct cocone in the corresponding category constitutes a multicoproduct family therein. Let us first work in $\mathcal{C}$: let $(\psi_i: H_i \rightarrow K)_{i \in I}$ be some cocone in $\mathcal{C}$. For every $X \in E$ look at $(\psi_i X: H_i X \rightarrow KX)_{i \in I}$ in F^{EX} and let $(\varphi_i X: H_i X \rightarrow HX)_{i \in I}$ be the candidate in that fiber determined by $(\psi_i X)_{i \in I}$, and $\theta X: HX \rightarrow LX$ the factoring. We claim $H(-)$ is the object part of a functor, and $\varphi_i(-)$, $\theta(-)$ components of natural transformations.

Let $f: X \rightarrow Y$ in E . We will show there is a unique $Hf: HX \rightarrow HY$ such that $Hf \circ \varphi_i X = \varphi_i Y \circ H_i f$ for all $i \in I$.

$$\begin{array}{ccc}
& \varphi_i X & \\
& H_i X \rightarrow HX & \\
H_i f \downarrow & & \downarrow Hf \\
& H_i Y \rightarrow HY & \\
& \varphi_i Y &
\end{array}$$

Clearly, this will ensure that H thus defined is a functor and the φ_i are natural transformations. Let $v_i: H_i X \rightarrow V_i$, $v: HX \rightarrow V$, and $w: KX \rightarrow W$ be cocartesian arrows above Ef . There are the usual $m_i: V_i \rightarrow H_i Y$ and $n: V \rightarrow W$ corresponding to the image of $\varphi_i X$ and θX by $\exists_i$ respectively. Let $H_i f$ factor as $y_i v_i$ and Kf as zw , where y_i and z are in F^{EY} .

$$\begin{array}{ccccc}
& \varphi_i X & \theta X & & \\
& H_i X \rightarrow HX & \rightarrow KX & & \\
v_i \downarrow & & \downarrow v & \downarrow w & \\
& V_i & \rightarrow V & \rightarrow W & \\
y_i \downarrow & m_i & & n & \downarrow z \\
& H_i Y \rightarrow HY & \rightarrow KY & & \\
& \varphi_i Y & \theta Y & &
\end{array}$$

By 3.1 $(m_i)_i$ is a candidate in F^{EY} . We claim it is the candidate determined by $(\varphi_i Y \circ y_i)_i$. This is because composing $(m_i)_i$ with zn yields the same cocone as composing $(\varphi_i Y \circ y_i)_i$ with θY . Hence there is $c: V \rightarrow HY$ with cv fulfilling the requirements for Hf . To prove uniqueness of Hf let $r: HX \rightarrow HY$ be any morphism of F with $r \circ \varphi_i X = \varphi_i Y \circ H_i f$ for all i . There is $c': V \rightarrow HY$ with $c'v = r$. It follows that $c'm_i v_i = \varphi_i Y \circ y_i v_i$, and by the cocartesianness of v_i , that $c'm_i = \varphi_i Y \circ y_i$. Therefore $c' = c$ and Hf is unique.

Suppose now that the original diagram $(\psi_i: H_i \rightarrow K)_i$ was in $Ent(E, F)$. Let us show the φ_i are "cartesian", in the sense that they send ultragenerics to ultragenerics: if $x: A \rightarrow H_i X$ is ultrageneric, then $\theta X \circ \varphi_i X \circ x = \psi_i X \circ x$ is ultrageneric, and 3.21 b) shows $\varphi_i X \circ x$ is ultrageneric. To show θX is "cartesian", let $y: B \rightarrow HY$ be ultrageneric, with B locally prime. By Axiom 5 there will be i , and $y': B \rightarrow H_i Y$ with $\varphi_i Y \circ y' = y$. But then $\theta Y \circ y = \psi_i Y \circ y'$ will be

ultrageneric. To show H is entire, let $(B, Y, y) \in \mathbf{F} \parallel H$, with B locally prime. Again, there is i , and $y': B \rightarrow H_i Y$ with $\varphi_i Y \circ y' = y$. Find (A, X, x) ultrageneric and $(a, f): (A, X, x) \rightarrow (B, Y, y')$ in $\mathbf{F} \parallel H_i$. $\varphi_i X \circ x$ will be the required ultrageneric arrow for y , i.e. we have $(a, f): (A, X, \varphi_i X \circ x) \rightarrow (B, Y, y)$ in $\mathbf{F} \parallel H$.

We are left to show multicoproduct families in $\mathbf{Ent}(E, F)$ are small. This is a standard argument, using the fact that a natural transformation between filtered-colimit-preserving functors is determined by its components on the finite objects.

4.2.3 *Connected limits are calculated pointwise (and fiberwise) in $\mathbf{Ent}(E, F)$.*

That is, if $(H_h)_{h \in H}$ is a diagram with H connected, and its limit is called H , then $HX = \varprojlim_h H_h X$. We have that $(H_h X)_h$ is a diagram above X , and since both the inclusion $\mathbf{F}^X \rightarrow \mathbf{F}$ and F preserve connected limits, HX can (and should) be taken in $\mathbf{F}^X$. Let us give the details in the case the diagram above is a pullback, which is the only case we will need. Let

$$\begin{array}{ccc} & \varphi & \psi \\ H & \longrightarrow & L \longleftarrow K \end{array}$$

be the diagram and let

$$\begin{array}{ccc} & \sigma & \\ M & \longrightarrow & K \\ \theta \downarrow & & \downarrow \psi \\ H & \longrightarrow & L \\ & \varphi & \end{array}$$

be the pointwise pullback, that is, M is the pullback in $\mathbf{E}^F$ and has been chosen such that θ, σ are above identity. M is entire, since filtered colimits and pullbacks commute with each other. Let us show θ is cartesian. Let $f: X \rightarrow Y$ in $\mathbf{E}$, and let

$$\begin{array}{ccc}
 & y & \\
 & A \longrightarrow MY & \\
 x \downarrow & & \downarrow \theta Y \\
 & HX \longrightarrow HY & \\
 & Hf &
 \end{array}$$

commute. An easy diagram chase will show $\psi Y \circ \sigma Y \circ y = Lf \circ \varphi X \circ x$, and by the cartesianness of ψ there is $z: A \rightarrow KX$ with $Kf \circ z = \sigma Y \circ y$ and $\psi X \circ z = \varphi X \circ x$. But then there is $w: A \rightarrow MX$ with $\sigma X \circ w = z$, $\theta X \circ w = x$. Then, trivially $Mf \circ w = y$, $\theta X \circ w = x$ and that proves the claim.

So if $\alpha: N \rightarrow H$, $\beta: N \rightarrow K$ are in $Ent(E, F)$ and such that $\varphi\alpha = \psi\beta$, there is a unique $\rho: N \rightarrow M$ in E^F with $\theta\rho = \alpha$, $\sigma\rho = \beta$. These equations force ρ to be above identity, and an easy use of 3.21 b) will show ρ is cartesian.

We now have to describe the prime generators of $Ent(E, F)$. It is natural to use the fact that ultrageneric arrows are an "invariant" property of entire morphisms of fibrations, stable under cartesian transformations. That is, if $x: A \rightarrow HX$ is an ultrageneric arrow, it makes sense to ask whether there is a "smallest" subobject $K \rightarrow H$ in $Ent(E, F)$ through which x will factor. It could turn out to be that whenever A is locally prime, K would be prime in $Ent(E, F)$, and that there would be only a small set of K 's thus obtained; they certainly look as if they would form a strong generating set. With the goal of proving these facts in mind, for any f.p. $A \in F$, $X \in E$, we define the categories $Arr(A, X)$ and $Ult(A, X)$. The objects in the first one are pairs (H, x) where H is an object of $\mathcal{C}$ and $x: A \rightarrow HX$ an arrow above identity. A morphism $(H, x) \rightarrow (H', x')$ is a transformation $\theta: H \rightarrow H'$ above identity such that $\theta X \circ x = x'$. We take $Ult(A, X)$ to be the subcategory of $Arr(A, X)$ whose objects are pairs (H, x) where H is entire and where a morphism θ is a cartesian transformation.

4.2.4 Definition

Let $A \in \mathcal{F}$, $X \in \mathcal{E}$ be finite objects such that $EX = FA$. An A, X -representable functor is an object (H, x) of $Arr(A, X)$ such that for every $Y \in \mathcal{E}$ the family

$$(Hk \circ x: A \rightarrow HY)_{k \in E(X, Y)}$$

has the following property: if $Hk \circ x$ is factored as

$$\begin{array}{ccc} & a_k & b_k \\ A & \longrightarrow & A_k \longrightarrow HY \end{array}$$

where a_k is cocartesian above E_k and b_k above 1_{EY} then the family $(b_k)_{k \in E(X, Y)}$ is a coproduct candidate in $\mathcal{F}^{EY}$. Such an x will be called a representor for H . Sometimes we will just say "let H be A, X -representable", meaning that there exists an $x: A \rightarrow HX$ with the desired properties, or even "let H be representable", meaning there exist unspecified A, X and x . Notice that if $E(X, Y)$ is empty, then HY is an initial candidate. The definition entails that if (H, x) and (K, y) are A, X -representable there exists at most one natural transformation $\theta: (H, x) \rightarrow (K, y)$ and θ is an isomorphism if it exists.

Let (H, x) be A, X -representable. For any $Y, k: X \rightarrow Y$ let $x_k = Hk \circ x$. Let x_k factor as $b_k \circ a_k$ as above. Then for $f: Y \rightarrow Z$ we have

$$\begin{aligned} Hf \circ x_k &= Hf \circ Hk \circ x = H(fk) \circ x \\ &= x_{fk} \end{aligned}$$

Therefore, since $(\exists_{E_f} b_k)_{k \in C(X, Y)}$ is a candidate in $\mathcal{F}^{EZ}$, and we can put $A_{fk} = \exists_{E_f} A_k$, by the usual properties of cocartesian arrows, we find that...

$$\begin{array}{ccc}
 a_k & b_k & \\
 A \rightarrow A_k & \longrightarrow & HY \\
 a_{fk} \searrow & \downarrow \exists_{E_f} b_k \downarrow \wr_{E_f} HY & \\
 & A_{fk} \rightarrow \exists_{E_f} HY & \\
 b_{fk} \searrow & \downarrow v & (v \in F^Z \text{ s.t. } v \cdot \wr_{E_f} HY = Hf) \\
 & HZ &
 \end{array}$$

4.2.5 ... an A, X -representable (H, x) is completely determined by the following data: for every $Y \in E$, an object $HY \in F^{EY}$ along with a candidate $(b_k: \exists_{E_k} A \rightarrow HY)_{k \in E(X, Y)}$ in F^{EY} , such that for every $f: Y \rightarrow Z$ the cocone $(b_{fk}: \exists_{E(fk)} A \rightarrow HZ)_{k \in E(X, Y)}$ in F^Z determines $(\exists_{E_f} b_k)_k$ for candidate.

Hence the value of H on morphisms is determined by the factorings and one takes $x = b_{1_X}$. We will always use the notation $a_k: A \rightarrow A_k$ for $\wr_{E_k} A$.

4.2.6 Proposition

A, X as before. $Arr(A, X)$ has a "weak multi-initial family", in the sense that every one of its connected components has an initial object, but there may be a proper class of connected components.

Proof

We know there is at most one morphism in $Arr(A, X)$ between A, X -representables, and it must be an isomorphism. So we have to show that given any (K, y) in $Arr(A, X)$ there exists a unique (up to iso) A, X -representable (H, x) and a unique morphism to (K, y) in $Arr(A, X)$. Choose $Y \in E$. For $k: X \rightarrow Y$ let $Kk \cdot y$ factor as

$$\begin{array}{ccc}
 a_k & c_k & \\
 A \rightarrow A_k & \rightarrow & KY
 \end{array}$$

where a_k is cocartesian and c_k in F^Y . The cocone $(c_k)_{k \in E(X, Y)}$ determines a candidate in F^Y , call it $(b_k: A_k \rightarrow HY)_k$, and a factoring $\varphi Y: HY \rightarrow KY$.

$$\begin{array}{ccc}
 & Y & \\
 A & \xrightarrow{\quad} & KX \\
 a_k \downarrow & c_k & \downarrow Kk \\
 A_k & \xrightarrow{\quad} & KY \\
 b_k \searrow & & \nearrow \varphi Y \\
 & HY &
 \end{array}$$

Do this construction for any Y in $\mathbf{E}$. Let $f: Y \rightarrow Z$ in $\mathbf{E}$. The discrete cocone $(b_{fk}: A_{fk} \rightarrow HZ)_{k \in \mathbf{E}(X, Y)}$ determines $(\exists_{\mathbf{E}f} b_k)_k$ for candidate because if we factor Kf as $w \circ \iota_{\mathbf{E}f} KY$ where $w \in \mathbf{F}^{\mathbf{E}Z}$ and if we call $d: A_k \rightarrow A_{fk}$ the unique (cocartesian) arrow above $\mathbf{E}f$ such that $da_k = a_{fk}$

$$\begin{array}{ccccccc}
 & a_k & & b_k & & \varphi Y & \\
 A & \rightarrow & A_k & \xrightarrow{\quad} & HY & \xrightarrow{\quad} & KY \\
 a_{fk} \searrow & \downarrow d & & & \downarrow \iota_{\mathbf{E}f} HY & & \downarrow \iota_{\mathbf{E}f} KY \\
 & A_{fk} & \xrightarrow{\quad} & \exists_{\mathbf{E}f} HY & \xrightarrow{\quad} & \exists_{\mathbf{E}f} KY & \\
 b_{fk} \searrow & & \exists_{\mathbf{E}f} b_k & & \exists_{\mathbf{E}f} \varphi Y & \downarrow w & \\
 & HZ & \xrightarrow{\quad} & KZ & & & \\
 & & \varphi Z & & & &
 \end{array}$$

then chasing the diagram above shows

$$w \circ \exists_{\mathbf{E}f} \varphi Y \circ \exists_{\mathbf{E}f} b_k \circ d = \varphi Z \circ b_{fk} \circ d$$

and the cocartesianness of d allows it to be removed from the equation. Hence, by 4.2.5 we have defined an A, X -representable (H, x) , where $x = b_{1_X}$ and it is easy to see we have a natural transformation above identity $\varphi: H \rightarrow K$ such that $\varphi X \circ x = y$. Such a φ is unique, since for any $Y \in \mathbf{E}$ the family $(b_k)_{k \in \mathbf{E}(X, Y)}$ is jointly epi in its fiber.

If (K, y) is initial in its component, then the morphism $(H, x) \rightarrow (K, y)$ it determines, where (H, x) is A, X -representable, must be an iso, and therefore (K, y) is already A, X -representable.

We are now ready to describe the generating set of $\text{Ent}(\mathbf{E}, \mathbf{F})$ in the case $\mathbf{F}$ is a semigranular fibration. The general case is more complicated.

4.2.7 Proposition

Let (H, x) be A, X -representable with A locally atomic and $x: A \rightarrow HX$ ultrageneric. Then H is entire. If $y: B \rightarrow HY$ is an ultrageneric arrow with B locally atomic then H is B, Y -representable, y is a representor, and $(B, Y, y), (A, X, x)$ are isomorphic in $\mathbf{F}||H$.

Proof

Let $b: B \rightarrow HY$ be some arrow above identity with B locally prime. By Axiom 5, and since H is A, X -representable, there exists $k: X \rightarrow Y$ along with $c: B \rightarrow A_k$ above identity with $b_k c = b$. Since A_k is atomic in $\mathbf{F}^{EY}$ by 3.2.1 c is an isomorphism.

$$\begin{array}{ccc}
 & a_k & c \\
 A & \longrightarrow & A_k \longleftarrow B \\
 x \downarrow & & b_k \searrow \downarrow b \\
 HX & \longrightarrow & HY \\
 & Hk &
 \end{array}$$

Therefore $c^{-1}a_k$ is cocartesian and H is entire, by 3.20. If now b is ultrageneric, since $c^{-1}a_k$ is cocartesian, $bc^{-1}a_k$ is generic, by definition, and since then k is a morphism in A/H between generic arrows it is an isomorphism. This shows $c^{-1}a_k$, being cocartesian above an iso, is also an isomorphism. Then a simple translation argument shows b is a representor for H .

4.2.8 Corollary

i)

If A is locally atomic then $Ult(A, X)$ has a (small) multi-initial family.

ii)

If F is a semigranular fibration, then $Ent(E, F)$ is semigranular.

Proof

i)

We already know that if $(K,y) \in \mathcal{U}lt(A,X)$ then there exists an essentially unique representable (H,x) and a unique $\varphi:(H,x) \rightarrow (K,y)$ in $\mathcal{A}rr(A,X)$. By 3.21 b) x is ultrageneric and by the preceding proposition H is entire, so $(X,x) \in \mathcal{U}lt(A,X)$. So we are left to prove φ is cartesian. But if $z:B \rightarrow HZ$ is an ultrageneric arrow we know (4.2.7) z is isomorphic to x in $\mathcal{F}||H$, hence $\varphi Z \circ z$ is isomorphic to y in $\mathcal{F}||K$ and therefore ultrageneric. To show there is only a small number of representables $(H,x) \in \mathcal{U}lt(A,X)$, use the fact that such an H must preserve filtered colimits, and therefore is entirely defined by its value on finite objects of $\mathcal{E}$. The description of representables given by 4.2.5 shows there can be only a small number of them.

Let us now show that if A is atomic, (H,x) A,X -representable, then H is atomic in $\mathcal{E}nt(\mathcal{E},F)$. We know its bottom subobject is the functor L , where LX is the initial candidate determined by HX in $\mathcal{F}^{EX}$. So let $\varphi:K \rightarrow H$ be a morphism in $\mathcal{E}nt(\mathcal{E},F)$ where $K \neq L$. Then for some $Y \in \mathcal{E}$ KY is nonempty in its fiber, and so there is $y:B \rightarrow KY$ in $\mathcal{F}^{EY}$ where B , by the same argument as in 4.2.7, is locally atomic. There is an ultrageneric $x:A \rightarrow KX$ along with $(a,f):(A,X,x) \rightarrow (B,Y,y)$ in $\mathcal{F}||K$. By 4.2.7 $\varphi X \circ x$ is a representer for H since it is ultrageneric. Let $Z \in \mathcal{E}$. We will show φZ is an isomorphism by testing with the primes of $\mathcal{F}^{EZ}$. Let C be one and $c:C \rightarrow HZ$ in $\mathcal{F}^{EZ}$. Since H is A,X -representable, by Axiom 5 there is $k:X \rightarrow Z$ and $\alpha:C \rightarrow A_k$ with $b_k \alpha = c$ ($b_k a_k$ being a cocartesian-above identity factorization of $Hk \circ \varphi X \circ x$).

$$\begin{array}{ccccc}
 & & A & & \\
 & x \swarrow & & \searrow a_k & \\
 & KX \rightarrow HX & & A_k & \\
 Kk \downarrow & Hk \downarrow & b_k \swarrow & \uparrow \alpha & \\
 KZ \rightarrow HZ & \leftarrow C & & & \\
 \varphi Z & & c & &
 \end{array}$$

Again as in 4.2.7, C and A_k are locally atomic and α is an iso. Since a_k is cocartesian there is $d: A_k \rightarrow KZ$ in F^{EZ} with $da_k = Kk \circ x \cdot d\alpha^{-1}$ will be the required arrow.

ii)

Now suppose F is a semigranular fibration. Take a representative class $\mathcal{G}$ of functors H such that there is $A \in F$ locally atomic, $X \in E$ finite and $x: A \rightarrow HX$ ultrageneric making (H, x) representable. Such a class is obviously small (4.2.5, essentially). It is a strong generating set of finitely presented objects: let us prove the strong generation property; the proof for finite presentedness will be postponed a little. Let $\theta: K \rightarrow L$ be a morphism of $Ent(E, F)$ such that for every $H \in \mathcal{G}$ $Hom(H, \theta)$ is an isomorphism. Let B be some local atom of F and $y: B \rightarrow LY$ some arrow above identity. There is the usual $x: A \rightarrow LX$ ultrageneric along with $(a, f): x \rightarrow y$ in $F \parallel L$. Let (H, x') be the initial candidate of $Ult(A, X)$ determined by (L, x) and $\rho: (H, x') \rightarrow (L, x)$ the unique morphism. (H, x') is isomorphic to some object of $\mathcal{G}$. By assumption there is $\psi: H \rightarrow K$ with $\theta\psi = \rho$.

$$\begin{array}{ccccc}
 & & HX & & \\
 & \psi X \swarrow & \downarrow \rho X & \searrow x' & \\
 KX & \longrightarrow & LX & \longleftarrow & A \\
 Kf \downarrow & \theta X & \downarrow Lf^x & \downarrow a & \\
 KY & \longrightarrow & LY & \longleftarrow & B \\
 & \theta Y & & y &
 \end{array}$$

Since a is cocartesian there is $z: B \rightarrow KY$ above identity such that $za = Kf \circ \psi X \circ x'$, and a simple diagram chase shows $\theta Y \circ za = ya$, and a can be factored out by cocartesianness.

Let us now prove Axiom 5 for $Ent(E, F)$. Let $(\varphi_i: H_i \rightarrow L)_{i \in I}$ be a candidate in $Ent(E, F)$, and $\theta: K \rightarrow L$ where K is in our set of generators. There is $a: A \rightarrow KX$ ultrageneric with A atomic making (K, a) an initial candidate in $Ult(A, X)$. Since $(\varphi_i X)_i$ is a candidate in F^X (4.2.2) by Axiom 5 there is $i \in I$ and $b: A \rightarrow H_i X$ with $\varphi_i X \circ b = \theta X \circ a$. Since $\theta X \circ a$ is ultrageneric b is (3.21 b)). Then,

since (K, a) is an initial candidate in $Ult(A, X)$ there is $\psi: K \rightarrow H_1$ making the triangle commute. The proof that the elements of $\mathcal{Q}$ are finitely presented is essentially the same. This completes the proof.

We can now tackle the general case of entire fibrations. The concept of A, X -representable functor is not sufficient in this case, since given an ultrageneric arrow, the representable it determines may not necessarily be an entire functor. We need a broader concept:

4.2.9 Definition

Let $A \in \mathcal{F}$ be locally prime, $H \in \mathcal{C}$, $x: A \rightarrow HX$ an ultrageneric arrow where X is finite. Let $(a_i: A_i \rightarrow A)_{i \in Sub(A)}$ be a choice of representatives of the subobjects of A in $\mathcal{F}^A$. If $A_i \subset A_j$ let $a_{ij}: A_i \rightarrow A_j$ be the morphism that realizes the inclusion. We say (H, x) is A, X -generated if

- i) For every $i \in Sub(A)$ there exists a $y_i: B_i \rightarrow HY_i$ ultrageneric, where Y_i is finite, and $(b_i, f_i): y_i \rightarrow xa_i$ in $\mathcal{F} \parallel H$.
- ii) If $\varphi_i: H_i \rightarrow H$ is the morphism in $\mathcal{C}$ where H_i is the B_i, Y_i -representable determined by y_i , then $(\varphi_i)_i$ is a candidate cocone in $\mathcal{C}$, that is, a correct cocone.

We will call x a generator for H .

4.2.10 If (H, x) is A, X -generated, then H is entire.

First we show (all notation as above) that for any i, j such that $A_i \subset A_j$ there is $f_{ij}: Y_i \rightarrow Y_j$ with $f_j f_{ij} = f_i$. Let $p: P \rightarrow B_i$, $q: P \rightarrow B_j$ be the pullback of $b_j: B_j \rightarrow A_j$ by $a_{ij} b_i: B_i \rightarrow A_j$. p is cocartesian since b_j is. Defining $m = xa_i b_i p = xa_j b_j q$ it is easy to see we get morphisms

$$\begin{array}{ccccc}
 & p & y_1 & & \\
 P & \longrightarrow & B_1 & \longrightarrow & HY_1 \\
 \downarrow q & & \downarrow b_1 & & \downarrow Hf_1 \\
 & & A_1 & & \\
 & & \downarrow a_{1j} & \searrow a_1 & \\
 B_j & \longrightarrow & A_j & \longrightarrow & A \\
 \downarrow y_j & & \downarrow b_j & \searrow a_j & \downarrow x \\
 HY_j & \longrightarrow & & & HX \\
 & & Hf_j & &
 \end{array}$$

$f_1: (Y_1, y_1 p) \rightarrow (X, m)$ and $f_j: (Y_j, y_j q) \rightarrow (X, m)$ in P/H . Since y_1 is ultrageneric and p cocartesian, $y_1 p$ is generic and there is $f_{1j}: (Y_1, y_1 p) \rightarrow (Y_j, y_j q)$ with $f_j f_{1j} = f_1$.

Now let $z: C \rightarrow HZ$ be an arbitrary arrow above identity. Clearly, by the definition of A, X -generated and Axiom 5 there is j , $g: Y_j \rightarrow Z$ such that z factors through $\exists_{Eg} B_j$, by a morphism above

$$\begin{array}{ccc}
 \downarrow_{Eg} B_j & \xrightarrow{c} & C \\
 B_j & \longrightarrow & \exists_{Eg} B_j \longleftarrow C \\
 \downarrow y_j & & \searrow \downarrow z \\
 HY_j & \longrightarrow & HZ \\
 & & Hg
 \end{array}$$

identity we will call c . Let $c': C' \rightarrow B_j$, $n: C' \rightarrow C$ be the pullback of c by $\downarrow_{Eg} B_j$. n is cocartesian. There is $d: \exists_{f_j} C' \rightarrow A_j$ above identity with $d \circ \downarrow_{Ef_j} C' = b_j c'$. There is i such that $a_j d$ as a subobject is equivalent to A_1 , and so $A_1 \subset A_j$. Identify $\exists_{Ef_j} C'$ with A_1 . We can then restate the above by saying there is $r: C' \rightarrow A_1$ cocartesian such that $a_{1j} r = b_j c'$. By the cartesianness of r , there is $l: B_1 \rightarrow C'$ above f_{1j} with $rl = b_1$. l is cocartesian by 3.10 iv). Then $nl: B_1 \rightarrow C$ is cocartesian, and $gf_{1j}: Y_1 \rightarrow Z$ is such that $F(nl) = E(gf_{1j})$ and $H(gf_{1j}) \circ y_1 = znl$, and y_1 is ultrageneric. QED.

Corollary

If $z: C \rightarrow HZ$ is ultrageneric then it is isomorphic in $F||H$ to a b_i for some i .

We already know there is $i, d: B_1 \rightarrow C$ cocartesian and $g: Y_1 \rightarrow X$ with $Hg \circ y_1 = zd$, d and g being above a common arrow. But if z is ultrageneric g is an iso and therefore d too.

4.2.11 Proposition

If (H, x) is A, X -generated, $h: W \rightarrow X$ such that Hh is cocartesian, then h is an isomorphism.

Proof

We assume Y_1, B_1, y_1 are defined as above. Let $w: D \rightarrow HW$, $d: D \rightarrow A$ be the pullback of x by Hh . d is cocartesian and w above identity. By assumption, there is $i, k: Y_1 \rightarrow W$, along with

$$\begin{array}{ccccc}
 & & P & & \\
 & p \swarrow & & \searrow q & \\
 B_1 & & & & D \xrightarrow{d} A \\
 \downarrow y_1 & c \searrow & & \swarrow i & \downarrow \\
 & C & & & w \\
 & \downarrow e & & & \downarrow x \\
 HY_1 & \xrightarrow{Hk} & HW & \xrightarrow{Hh} & HX
 \end{array}$$

a cocartesian arrow $c: B_1 \rightarrow C$ above Ek such that w factors through the unique $e: C \rightarrow HW$ above identity with $Hk \circ y_1 = ec$. Calling this factoring $i: D \rightarrow C$, let $p: P \rightarrow B_1$ and $q: P \rightarrow D$ be the pullback of c and i . q is cocartesian since c is, and we get a morphism $hk: (Y_1, y_1 p) \rightarrow (X, x dq)$ in P/H . But $x dq$ is generic since dq is cocartesian, and therefore hk, h are isomorphisms.

4.2.12 For any A locally prime, X appropriate, the category $Ult(A, X)$ has a poly-initial family. The category $Ent_C(E, F)$ is an aggregate.

This is 4.2.8 over again. Let $(K, x) \in Ult(A, X)$. Let $(a_i: A_i \rightarrow A)_{i \in I}$ be a choice of representatives for the subobjects of A above identity, as in 4.2.9, and let $a_0: A_0 \rightarrow A$ denote the identity subobject, so $0 \in I$. Let $(b_i, f_i): (B_i, Y_i, y_i) \rightarrow (A, X, x a_i)$ be a choice of

ultragerenic y_i 's and arrows in $\mathbf{Fll}K$. We have $y_0 = x$. For any $i \in I$ there is an essentially unique B_i, Y_i -representable $(H_i, \bar{y}_i)$ and a unique $\varphi_i: (H_i, \bar{y}_i) \rightarrow (K, y_i)$ in $\mathbf{Arr}(B_i, Y_i)$. This gives a discrete cocone $(\varphi_i: H_i \rightarrow K)_i$ in $\mathbf{C}$. Call $\psi_i: H_i \rightarrow H$ the coproduct candidate in $\mathbf{C}$ it determines and $\varphi: H \rightarrow K$ the factoring. Let z_i denote $\psi_i Y_i \circ y_i$ and call $z = z_0$. Since every z_i is sent by φ to y_i , all the z_i are ultragerenic. It is now easy to see that (H, z) is A, Y -generated, and therefore entire (4.2.10). We can now show (H, z) is generic in $\mathbf{Ult}(A, X)$. First, 4.2.3 along with a trivial argument show that $\mathbf{Ult}(A, X)$ has pullbacks. Therefore, by 0.11 we only have to show (H, z) does not have any subobject but itself. But this is obvious, since H is the entire functor generated by the ultragerenic arrow z : any entire subfunctor of H through z factors will have all the z_i 's factor through it too, and therefore will be H . The poly-initial family of $\mathbf{Ult}(A, X)$ is small by the usual counting argument (cf. 4.2.8). A generic object in $\mathbf{Ult}(A, X)$ is obviously always A, X -generated since it admits a morphism from an A, X -generated object to it.

Now suppose (K, x) as above is such that it is A, X -generated. We want to prove K is prime in $\mathbf{Ent}(E, F)$; let us first show $\mathbf{Sub}(K)$ is finite. For every $i \in I$ let (K_i, v_i) be the generic object of $\mathbf{Ult}(B_i, Y_i)$ determined by (K, y_i) , and $v_i: (K_i, v_i) \rightarrow (K, y_i)$ the arrow. If $\tau: L \rightarrow K$ is a subobject, let $J \subseteq I$ be the set of i such that v_i factors through τ . This happens iff y_i factors through $\tau Y_i: LY_i \rightarrow KY_i$, and determines a discrete cocone $(\lambda_j: K_j \rightarrow L)_{j \in J}$. We claim it is a coproduct candidate, thus showing $\mathbf{Sub}(K)$ has a smaller cardinal than the powerset of I , and proving our claim: it suffices to show (by 1.5 + 4.2.2) that for any object W of E , any $w: C \rightarrow LW$ above identity where C is locally prime, there is a $j \in J$ such that w factors through $\lambda_j W$. Since, by the corollary to 4.2.10 the y_i 's are a representative set of ultragerenic arrows for K , there is $i \in I$ and $(c, g): (B_i, Y_i, y_i) \rightarrow (C, W, \tau W \circ w)$ in $\mathbf{Fll}K$. By the pullbackness of the square

$$\begin{array}{ccc}
 & \tau Y_1 & \\
 & \downarrow & \\
 LY_1 & \xrightarrow{\quad} & KY_1 \\
 \downarrow Lg & & \downarrow Kg \\
 LW & \xrightarrow{\quad} & KW \\
 & \tau W &
 \end{array}$$

there is $u: B_1 \rightarrow LY_1$ such that $\tau Y_1 \circ u = y_1$ and $Lg \circ u = wc$. The first equation says that $i \in J$, and so $u = \lambda_i Y_1 \circ v_i$. Since c is cocartesian above g there is $s: C \rightarrow K_1 W$ with $sc = K_1 g \circ v_i$.

$$\begin{array}{ccccc}
 & & Lg & & \\
 & & \downarrow & & \\
 LY_1 & \xrightarrow{\quad} & LW & & \\
 \uparrow \lambda_i Y_1 & \searrow u & \swarrow c & \nearrow w & \uparrow \lambda_i W \\
 & B_1 \rightarrow C & & & \\
 \downarrow & \nearrow v_i & \searrow s & \downarrow & \\
 K_1 Y_1 & \xrightarrow{\quad} & K_1 W & & \\
 & K_1 g & & &
 \end{array}$$

Then chasing the diagram above shows $\lambda_i W \circ sc = wc$, and since c is cocartesian it can be factored out, so w does indeed factor through a $\lambda_j W$.

We will now show K has the other property of primes: let $(\varphi_s: H_s \rightarrow H)_{s \in S}$ be a candidate cocone and $\theta: K \rightarrow H$. By Axiom 5 there is s such that $\theta X \circ x$ factors through $\varphi_s X$. But since (K, x) is generic, there is $K \rightarrow H_s$ making the triangle commute.

We now know what a strong generating set for $Ent(E, F)$ should look like: take a representative class $\mathcal{Q}$ of functors H such that there are $A \in F$ locally prime, $X \in E$ finite, and a generator x making H A, X -generated. The same old argument shows $\mathcal{Q}$ is essentially small. The proof that $\mathcal{Q}$ is a strong generating set is exactly the same as in 4.2.8 ii); one simply has to substitute "generic object" for "initial candidate". The same substitution will yield proofs that Axiom 5 holds and that the objects of $\mathcal{Q}$ are finitely presented. This completes the proof of 4.2.

Let us show some special cases of that theorem where the prime generators are more tractable than in general. First we will need some more information about initial candidates in functor categories. Let $\pi_0: \mathbf{Aggr} \rightarrow \mathbf{Set}$ be the functor that sends every aggregate to its set of connected components. Notice that π_0 preserves small products; that is, if $(A_i)_{i \in I}$ is a family of aggregates, then $\pi_0(\prod_i A_i) \cong \prod_i (\pi_0(A_i))$. If $A: \mathbf{A} \rightarrow \mathbf{D}$ is a morphism in $\mathbf{Aggr}$, for $x \in \pi_0(A)$ define $A|_x: x \rightarrow \mathbf{D}$ to be the restriction of A to $x \subset A$. x is an aggregate by itself, and $A|_x$ is also an entire functor. We assume $\mathbf{C}, \mathbf{E}, \mathbf{F}$ are still the same as above. If $z \in \mathbf{C}$, let $E^z: E^z \rightarrow z$ be the entire fibration obtained by pulling back E by the inclusion $z \rightarrow \mathbf{C}$. Quite clearly, we have

$$\text{Ent}(E, F) \cong \prod_{z \in \pi_0(\mathbf{C})} \text{Ent}(E^z, F^z) .$$

4.3 Proposition

Let $\mathbf{C}$ be connected. Then for any $S \in \mathbf{C}$ $\pi_0(I^S): \pi_0(E^S) \rightarrow \pi_0(E)$ is a bijection (I^S being the inclusion). It follows that if $\mathbf{C}$ is any aggregate, then $\pi_0(\text{Ent}(E, F))$ is isomorphic to $\mathbf{Set}/\pi_0(\mathbf{C})(\pi_0(E), \pi_0(F))$, that is, the set of functions $f: \pi_0(E) \rightarrow \pi_0(F)$ with $\pi_0(F) \circ f = \pi_0(E)$.

Proof

Remember that in an aggregate connected components are in bijective correspondence with isomorphism classes of initial candidates. Let V be "the" initial object of $\mathbf{C}$. Then by 3.8.1 the initial candidates of E^V are the initial candidates of E . There is a unique $!: V \rightarrow S$ and since $\exists_!, !^*$ form an adjoint pair they determine isomorphisms between the connected components of E^V and those of E^S , and this proves the first claim. Still assuming $\mathbf{C}$ is connected, let $y \in \pi_0(F)$. We can define a section $\langle y \rangle: \mathbf{C} \rightarrow F$, $F \circ \langle y \rangle = 1_{\mathbf{C}}$ as follows: for $S \in \mathbf{C}$ $\langle y \rangle S$ is (a choice of) the initial candidate of F^S contained in y . The value on a morphism of $\mathbf{C}$ is uniquely defined, and the rules for calculating limits and colimits in

stiff fibrations (3.7) easily show that $\langle y \rangle$ is entire. We have $\langle y \rangle: \mathbf{C} \rightarrow y\mathbf{C}\mathbf{F}$, which can be restated by noting that $\pi_0(\mathbf{C}) = \{\mathbf{C}\}$, so $(\pi_0 \langle y \rangle)(\mathbf{C}) = y$. Now if $f: \pi_0(\mathbf{E}) \rightarrow \pi_0(\mathbf{F})$ is any function (we obviously have $\pi_0(\mathbf{F}) \circ f = \pi_0(\mathbf{E})$), define $\hat{f}: \mathbf{E} \rightarrow \mathbf{F}$ as follows: using the fact that $\mathbf{E}$ is the coproduct

$$\mathbf{E} = \coprod_{x \in \pi_0(\mathbf{E})} \mathbf{x}$$

we take $\hat{f}$ to be the unique functor whose restriction to $x \in \pi_0(\mathbf{E})$ is $\langle f(x) \rangle \circ E|_x$. Obviously, by its construction $\hat{f} \in \text{Ent}_{\mathbf{C}}(\mathbf{E}, \mathbf{F})$, and $\pi_0(\hat{f}) = f$. Notice that $\hat{f}$ is a functor that sends every object of $\mathbf{E}$ to an object which is an initial candidate in its fiber. Therefore $\hat{f}$ is an initial candidate in $\text{Ent}(\mathbf{E}, \mathbf{F})$. We can now prove the second claim; let $\mathbf{C}$ be any aggregate. Let

$$p: \text{Ent}(\mathbf{E}, \mathbf{F}) \longrightarrow \text{Set}/\pi_0(\mathbf{C})(\pi_0(\mathbf{E}), \pi_0(\mathbf{F}))$$

send H to $\pi_0(H)$. Notice that p is a functor to a discrete category. If H, K are in the same component of $\text{Ent}(\mathbf{E}, \mathbf{F})$ we have $p(H) = p(K)$ since H and K will send every component of $\mathbf{E}$ to the same component of $\mathbf{F}$. Therefore there is a natural

$$\pi: \pi_0(\text{Ent}(\mathbf{E}, \mathbf{F})) \longrightarrow \text{Set}/\pi_0(\mathbf{C})(\pi_0(\mathbf{E}), \pi_0(\mathbf{F}))$$

and our goal is to prove it is bijective. Notice that the argument above is just a proof that p is surjective when $\mathbf{C}$ is connected, and therefore that π is too. Let z range over $\pi_0(\mathbf{C})$.

$$\begin{array}{ccc} \pi_0(\text{Ent}(\mathbf{E}, \mathbf{F})) & \longrightarrow & \text{Set}/\pi_0(\mathbf{C})(\pi_0(\mathbf{E}), \pi_0(\mathbf{F})) \\ \downarrow & & \downarrow \\ \pi_0(\prod_z \text{Ent}(\mathbf{E}^z, \mathbf{F}^z)) & & \\ \downarrow & & \downarrow \\ \prod_z (\pi_0(\text{Ent}(\mathbf{E}^z, \mathbf{F}^z))) & \longrightarrow & \prod_z (\pi_0(\mathbf{E}^z), \pi_0(\mathbf{F}^z)) \end{array}$$

The nature of the vertical isomorphisms should be clear. If $f \in \text{Set}/\pi_0(\mathbf{C})(\pi_0(\mathbf{E}), \pi_0(\mathbf{F}))$ the iso to the right (which is just the remark that $\text{Set}/\pi_0(\mathbf{C}) \cong \text{Set}^{\pi_0(\mathbf{C})}$) transforms f into a family

$(f_z)_z$, where $f_z: \pi_0(E^z) \rightarrow \pi_0(F^z)$. Since every z is connected there exists $f_z: E^z \rightarrow F^z$ with $\pi_0(f_z) = f_z$, and therefore the morphism at the bottom is surjective, making the top morphism surjective. Notice that the vertical isos at the left piece the f_z 's into a single $\hat{f}: E \rightarrow F$ with $\pi_0(\hat{f}) = f$, and it is easy to see $\hat{f}$ is an initial candidate in $Ent(E, F)$, since $\hat{f}X = f_z X$, where z is the component of EX .

To prove the injectivity of π , let $H, K \in Ent(E, F)$ be such that $\pi_0(H) = \pi_0(K) (= f, \text{ by definition})$. For every $X \in E$ we have that $\hat{f}X$ is in the same component of F as both HX and KX , and therefore (by the first claim, applied to $F^z \rightarrow z$ where z is the component of EX) they all are in the same component of F^{EX} . Therefore, there are $\hat{f}X \rightarrow HX$, $\hat{f}X \rightarrow KX$, for every X , and so $\hat{f} \rightarrow H$, $\hat{f} \rightarrow K$, proving that H and K represent the same element of $\pi_0(Ent(E, F))$. This completes the proof.

4.3.1 Corollary

If F is a fibration all whose fibers are connected then $Ent(E, F)$ is a connected aggregate.

Proof

By the first claim, $\pi_0(F)$ is an isomorphism, and therefore in terminal in $Set/\pi_0(C)$, making $\pi_0(Ent(E, F))$ a one-element set by the second claim.

We can now explore some special cases of 4.2.

4.4 All the fibers of F are qualitative domains.

Since qualitative domains are (semi)granular, $Ent(E, F)$ is semigranular; it is obviously a poset, and the corollary above shows it is connected. We only have to show $Ent(E, F)$ is consistently cocomplete to get that it is a qualitative domain, by 1.11.1. But this is trivial to show, since multicoproduct families are calculated pointwise and fiberwise (4.2.2), and therefore have to be singletons.

Let us try to see what the "underlying" set of atoms looks like. Since the fibers are posets we know that for $s:S \rightarrow T$ in $\mathcal{C}$ the functor $\exists_s: \mathbf{F}^S \rightarrow \mathbf{F}^T$ is uniquely defined. Let $X \in E$ be finite and A be an atom of $\mathbf{F}^{EX}$. Since the coproduct candidates in the fibers of $\mathbf{F}$ are uniquely defined, it is easy to see that there exists at most one A, X -representable functor, call it $[X, A]$, and $[X, A]$ is defined iff for every $Y \in E$ the family $(\exists_{E_k} A)_{k \in E(X, Y)}$ has an upper bound in $\mathbf{F}^{EY}$; if $E(X, Y)$ is empty, $[X, A]Y$ is the bottom element of $\mathbf{F}^{EY}$. Let X, A be a pair satisfying the condition above. Let $(X, A), (Y, B)$ be pairs such that $[X, A]$ and $[Y, B]$ are defined and equal, say to H . By 4.2.7 we know the morphisms $A \rightarrow HX$ and $B \rightarrow HY$ (above identity) are isomorphic in $\mathbf{F} \parallel H$. This is equivalent to saying there is an iso $\alpha: X \rightarrow Y$ such that $B = \exists_{E\alpha} A$. Conversely, it is trivial to prove that if $\beta: X \rightarrow Z$ is an iso, then $[X, A]$ and $[Z, \exists_{E\beta} A]$ are the same functor. Therefore the set of atoms of $\text{Ent}(E, \mathbf{F})$ is isomorphic to the set $\mathcal{X}$ of all sets X such that

- The elements of $\mathcal{X}$ are pairs (X, A) where $X \in E$ is finite and $A \in \mathbf{F}$ a local atom above EX .
- If both (X, A) and (Y, B) are in $\mathcal{X}$ then there is an iso $\alpha: X \rightarrow Y$ (not necessarily unique) such that $B = \exists_{E\alpha} A$.
- If $(X, A) \in \mathcal{X}$ and $\beta: X \rightarrow Z$ is an iso then $(Z, \exists_{E\beta} A) \in \mathcal{X}$.

When does a subset $\mathcal{B} \subset \mathcal{X}$ have a sup? For every $X \in \mathcal{B}$ let (X_X, A_X) be a choice of an element of $\mathcal{X}$. Clearly, a necessary condition is that the family of functors $([X_X, A_X])_{X \in \mathcal{B}}$ have a sup in $\mathcal{C}$, i.e. that for every $Y \in E$ the family $(\exists_{E_k} A_X)_{X \in \mathcal{B}, k \in E(X_X, Y)}$ have a sup in $\mathbf{F}^{EY}$, call it H . Notice this will be the case if all the X_X are in different components of $\mathbf{F}$. Also, notice that H thus defined always preserves pullbacks (exercise; use the fact that for any $f: X \rightarrow Y$ in E Hf is cocartesian). To make this necessary condition sufficient, all we have to add (cf. proof of 4.2.2) is to require that the morphisms $[X_X, A_X] \rightarrow H$ be cartesian for all X . Since $[X_X, A_X]$ has an essentially unique ultrageneric morphism, namely $A_X \rightarrow [X_X, A_X]X_X$, we are simply asking that $A_X \rightarrow HX_X$ be

ultrageneric. Assume it is not the case for X . Then there is $a: B \rightarrow A_X$ cocartesian, $y: Y \rightarrow X_X$ not an iso (since H preserves pullbacks, we can use 2.2) such that

$$\begin{array}{ccc} & a & \\ B & \longrightarrow & A_X \\ \downarrow & & \downarrow \\ HY & \longrightarrow & HX_X \\ & Hy & \end{array}$$

the vertical arrow to the left is above identity and the square commutes. But since H is the sup of the $[X_X, A_X]$'s by Axiom 5 there is $y \in \mathcal{B}$, $x: X_y \rightarrow Y$ such that $B = \exists_{E_X} A_y$. By composing the cocartesian arrows $A_y \rightarrow B \rightarrow A_X$ we get that

There exists $y \in \mathcal{B}$ and $z: X_y \rightarrow X_X$ not an iso such that $A_X = \exists_{E_X} A_y$.

Conversely, if that condition arises, then the natural $[X_X, A_X] \rightarrow H$ will certainly not be cartesian. Hence we have found a condition of compatibility for the atoms of $\text{Ent}(E, F)$: a family $([X_X, A_X])_{X \in \mathcal{B}}$ of atomic functors has a sup iff for every $Y \in E$ the family $(\exists_{E_k} A_X)_{X \in \mathcal{B}, k \in E(X_X, Y)}$ has a sup in F^{EY} and for no pair $X, Y \in \mathcal{B}$ is there a $k: X_X \rightarrow X_Y$ not an iso such that $A_Y = \exists_{E_k} A_X$. This condition is a generalization of the one given in [Gi86], and should be compared with it.

4.5 $\mathcal{C}, E, F$ are posets.

Obviously the fibers of E and F are posets too. Since in this situation it is useless to name the morphisms, for $S \leq T$ in $\mathcal{C}$ and X above S we will use the notation $\exists_T X$ for the object obtained by pushing X above T by a cocartesian arrow. Let now $X \in E$ be finite, A a local prime of F above EX . Let $g: \pi_0(E) \rightarrow \pi_0(F)$ be such that $\pi_0(F) \circ g = \pi_0(E)$ and $g(\text{Component}(X)) = \text{Component}(A)$. Define a functor $[X, A, g]: E \rightarrow F$ as follows:

$$[X, A, g]Y = \begin{cases} \exists y A & \text{if } Y \geq X \\ \hat{g}Y & \text{if not (cf. 4.3)} \end{cases} .$$

It is quite easy to see $\hat{g}$ is indeed a functor, in fact, an A, X -representable one. What is more surprising is that $\hat{g}$ is entire, and therefore A, X -generated; the verification of this poses no problem, and will be left to the reader. Here is an instance of a fibration of aggregates where the unwieldy concept of A, X -generated functor can be avoided.

4.6 $\mathbb{C} = \mathbf{1}$. That is, $Ent(E, F) = Aggr(E, F)$.

Many definitions simplify considerably in this context. For instance, the notion of ultrageneric becomes identical with that of generic. Given $X \in E$ finite, $A \in F$, an A, X -representable (H, x) is a functor $H: E \rightarrow F$ along with $x: A \rightarrow HX$ such that for every $Y \in E$ the family $(Hk \circ x)_{k \in E(X, Y)}$ is a coproduct candidate. Let $F = \mathbb{M}$, the category of sets and monomorphisms, and let us determine the atomic entire functors $E \rightarrow \mathbb{M}$. Since $\mathbb{M}$ is granular and has essentially one atom, the one-element set $\mathbf{1}$, the atoms of the functor category form a generating set and they all are $\mathbf{1}, X$ -representable for some $X \in E$. Let H be such an atomic functor. Let $x: \mathbf{1} \rightarrow HX$ be generic. We can also write $x \in HX$, obviously. Let G be the automorphism group of x in $\mathbf{1}/H$; it is the set of all endomorphisms (they being automatically automorphisms) α of X such that $H\alpha \circ x = x$. G is obviously a subgroup of $Aut(X)$. For every $Y \in E$ G acts on the right on $E(X, Y)$ by $(f, \alpha) \mapsto f\alpha$. For a morphism $Y \rightarrow Z$ in E there is a morphism of actions $E(X, Y) \rightarrow E(X, Z)$ and so we can define a functor $K: E \rightarrow Set$ that sends Y to the set of G -orbits on $E(X, Y)$. By Yoneda there is a unique natural transformation $\varphi: h^X \rightarrow H$ sending 1_X to x , where h^X is the covariant representable functor associated to X (h^X can obviously be considered as a functor $E \rightarrow \mathbb{M}$). Since (H, x) is $\mathbf{1}, X$ -generated, φ is surjective in every component. Let $f, g \in E(X, Y)$ be

such that $\varphi_Y(f) = \varphi_Y(g) (= y)$, by definition). We can restate this by saying there are morphisms

$$(X, x) \xrightarrow{f} (Y, y) \xleftarrow{g} (X, x)$$

in $1/H$, and therefore by the genericity of x there is $\alpha \in G$ with $f\alpha = g$, and so f and g belong to the same orbit of the G -action.

The converse, that G -congruent morphisms of $Ent(X, Y)$ are identified by φ_Y , is trivial. Thus we have proven that $H \cong K$.

Conversely it is quite easy to show that given any $X \in E$ finite, any subgroup $G \subset Aut(X)$, the functor $h^X/G: E \rightarrow \mathcal{M}$ defined as K above is $1, X$ -representable, with the orbit of 1_X , an element $1 \rightarrow (h^X/G)X$ as a representor. This means that our notion of A, X -representable is a generalization of the classical notion of representable functor: if we restrict ourselves to $\mathcal{M}$ (*Sets*) as a target category, a $1, X$ -representable functor is the same thing as a multiple coequalizer of representable functors

$$h^X \xrightarrow{(h_g)_{g \in G}} h^X \longrightarrow h^X/G,$$

where all the g 's are isomorphisms. Let $X' \in E$, and G' be a subgroup of $Aut(X')$. Clearly, if there is a natural transformation $\psi: h^{X'}/G' \rightarrow h^X/G$, by Yoneda and the surjectivity of the projection

$$\begin{array}{ccc} h^{X'} & \longrightarrow & h^X \\ \downarrow & & \downarrow \\ h^{X'}/G' & \longrightarrow & h^X/G \end{array}$$

$h^X \rightarrow h^X/G'$ there is a natural $\theta: h^{X'} \rightarrow h^X$ making the square above commute, and then, a morphism $f: X \rightarrow X'$ with $\theta = h^f$. In order for the transformation to be cartesian, f has to be an isomorphism (since $1_{X'}: 1 \rightarrow h^{X'}X'$ is generic, and $\theta X' \circ 1_{X'} = f$ can be generic in $h^{X'}X$ only if f is an iso). We get that cartesian morphisms of $1, X$ -representable functors have to be isomorphisms and are mediated by isomorphisms in E (the first observation is not surprising since $1, X$ -representables are atomic). So let $f: X \rightarrow X'$ be an isomorphism

such that the lower horizontal (iso)morphism in the square above exists, making the square commute. For any $Y \in E$, any $h, k: X' \rightarrow Y$, we get

$$\begin{aligned} h \text{ G'-congruent to } k &\Rightarrow hf \text{ G-congruent to } kf \\ h^{-1}k \in G' &\Rightarrow (hf)^{-1}(kf) \in G \\ h^{-1}k \in G' &\Rightarrow f^{-1}(h^{-1}k)f \in G \end{aligned} ,$$

and therefore such an f is exactly a morphism such that for every $\alpha \in G'$ we have $f^{-1}\alpha f \in G$. In particular, by putting $X = X'$ we can describe the set I_X of isomorphism types of all $1, X$ -representable functors: if $\mathcal{W}_X$ is the set of subgroups of $\text{Aut}(X)$, then I_X is the set of orbits of the standard action of $\text{Aut}(X)$ on $\mathcal{W}_X$ by conjugacy. The automorphism groups of the $1, X$ -representable functors can be similarly characterized. This is not very good news: for example, if $E = \mathcal{M}$, then the category $\text{Ent}(\mathcal{M}, \mathcal{M})$ has its set of isomorphism types of atoms in bijective correspondence with the disjoint sum

$$\bigsqcup_{n \in \mathbb{N}} \text{Orbit}(\mathfrak{S}_n, \mathcal{W}_n)$$

where $\mathcal{W}_n$ is the set of subgroups of the symmetrical group $\mathfrak{S}_n$ and $\mathfrak{S}_n$ acts on $\mathcal{W}_n$ by conjugacy. Not a trivial computation.

Let us finish this remark by noting that if E is any aggregate, and if $\mathcal{Q}$ is the set of isomorphism types of atoms of $\text{Ent}(E, \mathcal{M})$, for every $A \in \mathcal{Q}$, denoting by G_A the automorphism group of (a representative of) A , then $\text{Ent}(E, \mathcal{M})$ is equivalent to the product category

$$\prod_{A \in \mathcal{Q}} \hat{G}_A$$

where $\hat{G}_A$ is as in 1.11. This is proven by noting that in $\text{Ent}(E, \mathcal{M})$, for every discrete family of objects there is a unique lean cocone, and that this property characterizes categories of the form $\prod_i \hat{G}_i$.

4.7 Theorem

Let $E: \mathbf{E} \rightarrow \mathbf{C}$ be an entire fibration. The pullback functor

$$E^*: \text{Fib}(\mathbf{C}) \longrightarrow \text{Fib}(\mathbf{E})$$

has a right adjoint Π_E . The Beck condition holds for any pullback diagram of entire functors where two parallel ones are entire fibrations. If $F: \mathbf{F} \rightarrow \mathbf{E}$ is a semigranular fibration, then $\Pi_E F$ is a semigranular fibration too.

Corollary

The display categories $(\text{Aggr}, \text{Fib})$ and $(\text{Aggr}, \text{Sgr})$ (3.16, 3.17.1) both admit products. Since in addition they both admit discrete products, for any aggregate $\mathbf{C}$, $\text{Fib}(\mathbf{C})$ and $\text{Sem}(\mathbf{C})$ are cartesian-closed. In particular, both the category of aggregates and entire functors and the category of semigranular categories and entire functors are cartesian-closed.

Proof

Let $F: \mathbf{F} \rightarrow \mathbf{E}$ be in $\text{Fib}(\mathbf{C})$. We will construct $\Pi_E F$ as a diagram $G: \mathbf{G} \rightarrow \mathbf{C}$ defined as follows.

An object of $\mathbf{G}$ above S is a pair (S, H) , where H is an entire functor $H: \mathbf{E}^S \rightarrow \mathbf{F}$ with $FH = 1_{\mathbf{F}}^S$. As is customary we will say things like "let H be above S " in lieu of "let $(S, H) \in \mathbf{G}$ ". A morphism $\chi: H \rightarrow K$ in $\mathbf{E}$ above $s: S \rightarrow T$ is a function that assigns to every arrow $f: X \rightarrow Y$ above s a morphism $\chi[f]: HX \rightarrow KY$ above f , i.e. such that $F(\chi[f]) = f$. χ is subject to the condition that if

$$\begin{array}{ccc} & f & \\ X & \longrightarrow & Y \\ x \downarrow & & \downarrow y \\ X' & \longrightarrow & Y' \\ & f' & \end{array}$$

is a pullback square where f, f' are above s , and x, x' above identity then the square

$$\begin{array}{ccc}
 & \chi[f] & \\
 & \text{HX} \longrightarrow \text{KY} & \\
 \text{Hx} \downarrow & & \downarrow \text{Ky} \\
 & \text{HX}' \longrightarrow \text{KY}' & \\
 & \chi[f'] &
 \end{array}$$

is a pullback too.

4.7.1 Proposition

Let $\chi: H \rightarrow K$ be above $s: S \rightarrow T$ in $\mathbb{G}$.

i) Let

$$\begin{array}{ccc}
 & Y' & \\
 f \nearrow & \downarrow y & \\
 X & \longrightarrow Y & \\
 & f' &
 \end{array}$$

be a commutative triangle in $\mathbb{E}$ where f, f' are above s and y is above identity. Then $Ky \circ \chi[f] = \chi[f']$.

ii) Let

$$\begin{array}{ccc}
 & W' & \\
 w \downarrow & \searrow g & \\
 & W \rightarrow Z & \\
 & g' &
 \end{array}$$

commute in $\mathbb{E}$ where g, g' are above s and w above identity. Then $\chi[g'] \circ Hw = \chi[g]$.

Proof

for i), let $x: X' \rightarrow X$ and $h: X' \rightarrow Y'$ complete the diagram to a pullback. There is $a: X \rightarrow X'$ with $xa = 1_X$ and $ha = f$. This forces x to be an isomorphism, and the statement follows from the pullback condition on χ . For ii), let $k: W' \rightarrow Z'$ be cocartesian above s and $z: Z' \rightarrow Z$ above identity with $zk = g'w$. By the above, we have $Kz \circ \chi[k] = \chi[g]$, and since w, z, g', k form a pullback (3.3 d) and a trivial argument) we have $\chi[g'] \circ Hw = Kz \circ \chi[k] = \chi[g]$. This shows

among others that χ is entirely determined by its values $\chi[f]$ where f is cocartesian:

4.7.2 Proposition

Let $s:S \rightarrow T$ in $\mathcal{C}$, $H \in \mathcal{G}^S$ and $K \in \mathcal{G}^T$. Let χ be a function which assigns to every cocartesian arrow $f:X \rightarrow Y$ above s an arrow $\chi[f]:HX \rightarrow KY$ above f such that the pullback condition holds for the squares that can be made in this context: for any commutative

$$\begin{array}{ccc} & f & \\ X & \longrightarrow & Y \\ x \downarrow & & \downarrow y \\ X' & \longrightarrow & Y' \\ & f' & \end{array}$$

square where f, f' are cocartesian above s , and x, x' above identity (any such square is a pullback) the square

$$\begin{array}{ccc} & \chi[f] & \\ HX & \longrightarrow & KY \\ Hx \downarrow & & \downarrow Ky \\ HX' & \longrightarrow & KY' \\ & \chi[f'] & \end{array}$$

is a pullback too. Then χ can be extended to a unique morphism $H \rightarrow K$ of $\mathcal{G}$ above s , which we will also call χ .

Proof

First notice that 4.7.1 i) holds in this context: the proof carries through since cocartesians are stable under pullbacks. Therefore given any two cocartesian arrows g, g' above s with common domain, the morphisms $\chi[g], \chi[g']$ will be isomorphic in the obvious sense. This allows us to extend χ to all arrows f above s : given such an f , let $f = hg$ where g is cocartesian above s and h above identity and define $\chi[f] = Kh \circ \chi[g]$. The previous remark ensures independence from the choice of g . To check the pullback condition, let

$$\begin{array}{ccc}
 & f' & \\
 X' & \longrightarrow & Y' \\
 x \downarrow & & \downarrow y \\
 X & \longrightarrow & Y \\
 & f &
 \end{array}$$

be an arbitrary pullback diagram. Factor f as hg just as above, with $g:X \rightarrow Z$, $h:Z \rightarrow Y$. Let $z:Z' \rightarrow Z$, $h':Z' \rightarrow Y'$ be the pullback of h, y and $g':X' \rightarrow Z'$ the unique arrow giving $zg' = gx$.

$$\begin{array}{ccccc}
 & g' & & h' & \\
 X' & \longrightarrow & Z' & \longrightarrow & Y' \\
 x \downarrow & & \downarrow z & & \downarrow y \\
 X & \longrightarrow & Z & \longrightarrow & Y \\
 & g & & h &
 \end{array}$$

The left square is a pullback (standard argument) and by 3.3 d) g' is cocartesian. Therefore

$$\begin{array}{ccccc}
 & \chi[g'] & & Kh' & \\
 HX' & \longrightarrow & KZ' & \longrightarrow & KY' \\
 Hx \downarrow & & \downarrow Kz & & \downarrow Ky \\
 HX & \longrightarrow & KZ & \longrightarrow & KY \\
 & \chi[g] & & Kh &
 \end{array}$$

the left square above is a pullback by assumption, the right square is one since K is entire, and this shows our claim since $\chi[f] = Kh \circ \chi[g]$, etc. The unicity of the extended χ is quite obvious.

We can now show how to compose morphisms of $\mathbf{G}$. If $\chi:H \rightarrow K$ is above $r:R \rightarrow S$ and $\xi:K \rightarrow L$ above $s:S \rightarrow T$ we want a value to $\xi\chi[f]$ for any $f:X \rightarrow Y$ above sr . If f can be decomposed as hg , g above r and h above s , then we could define $\xi\chi[f]$ as $\xi[h] \circ \chi[g]$, hoping that the final result is independent of the choice of g, h . Such g, h always exist: just take g to be a cocartesian above r with domain X , and h the unique arrow above s with $hg = f$. We leave it to the reader to verify that the value of $\xi\chi[f]$ thus obtained is indeed independent of the actual decomposition of f , and that the axioms of a category hold.

Let H, K be in $\mathcal{G}^S$. Let $\chi: H \rightarrow K$ be a morphism above 1_S . By definition, the family $(\chi[1_X]: HX \rightarrow KX)_{X \in \mathcal{E}^S}$ is such that for $f: X \rightarrow Y$ above identity

$$\begin{array}{ccc} & \chi[1_X] & \\ & HX \longrightarrow KX & \\ Hf \downarrow & & \downarrow Kf \\ & HY \longrightarrow KY & \\ & \chi[1_Y] & \end{array}$$

is a pullback. That is, if we define $\chi X = \chi[1_X]$ we get a cartesian natural transformation above identity. The converse is also true: given $\varphi: H \rightarrow K$ cartesian above identity, one gets a morphism of $\mathcal{G}$ by defining $\varphi[f] = Kf \circ \varphi X = \varphi Y \circ Hf$, and the verification of this is trivial. All this obviously defines a bijection between $\mathcal{G}^S(H, K)$ and the set of cartesian transformations $H \rightarrow K$ above identity. Hence the fiber $\mathcal{G}^S$ is the category of all entire splittings of $p^S: F^S \rightarrow E^S$, where p^S is the pullback

$$\begin{array}{ccc} & F^S \longrightarrow F & \\ p^S \downarrow & & \downarrow F \\ & E^S \longrightarrow E & \\ & I & \end{array}$$

and I the inclusion, with cartesian transformations above identity as morphisms. This shows $\mathcal{G}^S$ is an aggregate since it is isomorphic to $\text{Ent}_{\mathcal{E}^S}(1_{\mathcal{E}^S}, p^S)$. This also shows $\mathcal{G}^S$ is semigranular when F is a semigranular fibration. Let us show $\mathcal{G}$ is a bifibration.

4.7.3 Proposition

Let $\chi: H \rightarrow K$ in $\mathcal{G}$ above $s: S \rightarrow T$ be such that for every cocartesian $f: X \rightarrow Y$ above s , $\chi[f]: HX \rightarrow KY$ is cartesian (that is, F -cartesian). Then χ is cartesian.

Proof

Let $\xi: L \rightarrow K$ be above sv with $v: V \rightarrow S$. We are looking for a unique $\xi: L \rightarrow H$ above v with $\chi\xi = \xi$. Let $f: X \rightarrow Y$ be some arrow above v . If $g: Y \rightarrow Z$ is a cocartesian arrow above s , $\chi[g]$ is

cartesian, and there is a unique way to define $\xi[f]$ to have $\chi[g] \circ \xi[f] = \zeta[gf]$. If ξ is defined thus, we do have $\chi\xi = \zeta$; by the last proposition we just have to check this for a cocartesian arrow $h: X \rightarrow Z$ above sv . It decomposes as $h = gf$, where g and f are as above, and since both morphisms are cocartesian,

$$\chi\zeta[h] = \chi[g] \circ \zeta[f] = \zeta[gf] = \zeta[h],$$

as above. So we are left to check the pullback condition. Let

$$\begin{array}{ccc} & f & \\ X & \longrightarrow & Y \\ x \downarrow & & \downarrow y \\ X' & \longrightarrow & Y' \\ & f' & \end{array}$$

be a pullback square, with f, f' above v and x, y above identity. Let $g: Y \rightarrow Z$, $g': Y' \rightarrow Z'$ be cocartesian above s , and $z: Z \rightarrow Z'$ above identity with $zg = g'y$. We want to prove that the left square below

$$\begin{array}{ccccc} & \xi[f] & & \chi[g] & \\ LX & \longrightarrow & HY & \longrightarrow & KZ \\ Lx \downarrow & & \downarrow Hy & & \downarrow Kz \\ LX' & \longrightarrow & HY' & \longrightarrow & KZ' \\ & \xi[f'] & & \chi[g'] & \end{array}$$

is a pullback. The square $gf, g'f', x, z$ is a pullback, being a composite of pullbacks. Therefore the outer square above is a pullback by the definition of ζ . The square to the right is a pullback by the definition of χ . Therefore the result follows by the standard fact about composites of pullbacks.

4.7.4 Proposition

Let $K \in \mathbf{G}$ be above T , and $s: S \rightarrow T$. There is $H \in \mathbf{G}$ and a cartesian arrow $\chi: H \rightarrow K$ above s .

Proof

For every $X \in \mathbf{E}^S$ let $\iota_s X: X \rightarrow \exists_s X$ be a choice of a cocartesian arrow above s , and let $\chi X: HX \rightarrow K\exists_s X$ be a choice of a cartesian

arrow above $\iota_s X$. It follows that χX is EF-cartesian above s (cf. 3.17 iii). For $f: X \rightarrow Y$ in $\mathbf{E}^S$ there is a unique $Hf: HX \rightarrow HY$ above f such that

$$\begin{array}{ccc} & \chi X & \\ HX & \xrightarrow{\quad} & K\exists_s X \\ Hf \downarrow & & \downarrow K\exists_s f \\ & \chi Y & \\ HY & \xrightarrow{\quad} & K\exists_s Y \end{array}$$

commutes, since χY is cartesian. H is actually an entire functor, since it can be defined as a composite $f^* \circ K \circ \exists_s$, where $f^*: \mathbf{F}^T \rightarrow \mathbf{F}^S$ is obtained by choosing EF-cartesian arrows above s in the right manner. The family $(\chi X)_{X \in \mathbf{E}^S}$ can be extended to a unique $\chi: H \rightarrow K$ in $\mathbf{G}$ by 4.7.2.

Now for the cocartesian case

4.7.5 Proposition

Let $\chi: H \rightarrow K$ in $\mathbf{G}$ above $s: S \rightarrow T$ be such that for any cartesian $f: X \rightarrow Y$ above s $\chi[f]$ is F-cocartesian above f . Then χ is cocartesian.

Proof

Let $\zeta: H \rightarrow L$ be above ts , with $t: T \rightarrow V$. We are looking for $\xi: K \rightarrow L$ above t with $\xi\chi = \zeta$. Let $f: X \rightarrow Y$ be above t and extend $g: Z \rightarrow X$ cartesian above s . Since $\chi[g]$ is cocartesian there is a unique way to define $\xi[f]$ so that $\xi[f] \circ \chi[g] = \zeta[fg]$. Let us show ξ thus defined has the pullback condition. In the diagram below

$$\begin{array}{ccccc} & g' & & f' & \\ Z' & \xrightarrow{\quad} & X' & \xrightarrow{\quad} & Y' \\ z \downarrow & & \downarrow x & & \downarrow y \\ Z & \xrightarrow{\quad} & X & \xrightarrow{\quad} & Y \\ & g & & f & \end{array}$$

let the right square form a pullback, where f, f' are above t and x, y are above identity; let g, g' be cartesian arrows extended

above s , and z complete to a pullback. Let $m:A \rightarrow KX$, $n:A \rightarrow LY'$ be such that $\xi[f] \circ m = Ly \circ n$. By assumption there is a unique $h:FA \rightarrow X'$ with $x \circ h = Fm$ and $f' \circ h = Fn$. Let

$$\begin{array}{ccc} & a & \\ B & \longrightarrow & A \\ p \downarrow & & \downarrow m \\ HZ & \longrightarrow & KX \\ & \chi[g] & \end{array}$$

form a pullback. a is cocartesian. Since $\xi[f] \circ \chi[g] = \xi[fg]$, $\xi[f'] \circ \chi[g'] = \xi[f'g']$ and the latter along with HZ, Ly form a pullback there is a unique $q:B \rightarrow HZ'$ such that $HZ \circ q = p$ and $\xi[f'] \circ \chi[g'] \circ q = na$. Examination of the diagram below shows

$$\begin{array}{ccccc} & Fa & & & \\ FB & \longrightarrow & FA & & \\ Fq \downarrow & h \downarrow & \searrow & Fn & \\ Z' & \longrightarrow & X' & \longrightarrow & Y' \\ z \downarrow & g' \downarrow & x \downarrow & f' \downarrow & y \\ Z & \longrightarrow & X & \longrightarrow & Y \\ & g & & f & \end{array}$$

Fq is the unique morphism $FB \rightarrow Z'$ such that $h \circ Fa = g' \circ Fq$, $z \circ Fq = Fp$. Since a is cocartesian there is $b:A \rightarrow KX'$ with $\chi[g'] \circ q = ba$. Then a diagram chase using the cocartesianness of a will show $\xi[f'] \circ b = n$ and $Kx \circ b = m$.

4.7.6 Proposition

Let $H \in \mathcal{G}$ be above S and $s:S \rightarrow T$ in $\mathcal{C}$. There is $K \in \mathcal{G}$ and a cocartesian $\chi:H \rightarrow K$.

Proof

For any $Y \in \mathcal{E}^T$ choose a cartesian $\gamma_s Y:s^*Y \rightarrow Y$ above s , and a cocartesian $\chi Y:Hs^*Y \rightarrow KY$ above $\gamma_s Y$. $K(-)$ is obviously the object part of a functor: for $y:Y \rightarrow Y'$ in $\mathcal{E}^T$ take Ky to be the unique morphism $KY \rightarrow KY'$ above y such that $Ky \circ \chi Y = \chi Y' \circ Hs^*y$. We can define a partial $\bar{\chi}$, where $\bar{\chi}[f]$ exists for any cartesian

$f: X \rightarrow Y$ above s : if $a: X \rightarrow f^*Y$ is the unique iso above identity with $\chi_Y \circ a = f$, we take $\bar{\chi}[f]$ to be $\chi_Y \circ Ka$. It is quite easy to see that the partial $\bar{\chi}$ thus defined satisfies 4.7.1 i); therefore if we define a full χ by restricting $\bar{\chi}$ to the cocartesian arrows and using 4.7.2 we get that χ restricted to the cartesians is $\bar{\chi}$, and that it satisfies the condition of the previous proposition. To show χ satisfies the pullback condition we can restrict ourselves to cocartesian arrows, by 4.7.2. So let

$$\begin{array}{ccc} & f & \\ X & \longrightarrow & Y \\ x \downarrow & & \downarrow y \\ X' & \longrightarrow & Y' \\ & f' & \end{array}$$

be with x, y above identity and f, f' cocartesian above s . That square is a pullback. By 3.10 ii) the square $Kx, Ky, \chi[f], \chi[f']$ is a pullback, since the latter two arrows are F -(co)cartesian.

We are left to show K is entire. Let

$$\begin{array}{ccc} & x_2 & \\ X & \longrightarrow & Y_2 \\ x_1 \downarrow & & \downarrow w_2 \\ Y_1 & \longrightarrow & W \\ & w_1 & \end{array}$$

be a pullback diagram in E^T . Let $a_1: A \rightarrow KY_1$, $a_2: A \rightarrow KY_2$ in F^T with $Kw_1 \circ a_1 = Kw_2 \circ a_2$. There exists a unique $m: FA \rightarrow X$ with $x_1 m = Fa_1$, $x_2 m = Fa_2$. Let

$$\begin{array}{ccc} & b & \\ B & \longrightarrow & A \\ p \downarrow & & \downarrow Kw_1 \circ a_1 \\ Hs^*W & \longrightarrow & KW \\ & \chi W & \end{array}$$

be a pullback diagram. b is cocartesian. Since we know χ has the pullback property, the squares $\chi Y_i, \chi W, Hs^*w_i, Kw_i$ ($i=1,2$) are pullbacks. Therefore there are $b_i: B \rightarrow Hs^*Y_i$ with

$Hs^*w_1 \cdot b_1 = p = Hs^*w_2 \cdot b_2$. Since the image of the square x_1, w_1 by the entire functor Hs^* is a pullback, there is $q: B \rightarrow Hs^*X$ with $Hs^*x_1 \cdot q = b_1$. We have $\gamma_2 X \cdot Fq = m \cdot Fb$, obviously. Then we use the cocartesianness of b to get $n: A \rightarrow KX$ with $Kx_1 \cdot n = a_1$: take n to be the unique $A \rightarrow KX$ above m such that $\chi X \cdot q = nb$.

To show K preserves filtered colimits, let $(X_h)_{h \in H}$ be a filtered diagram in $\mathbf{E}^T$, with colimit cocone $(\alpha_h: X_h \rightarrow X)_h$. Let $(\beta_h: KX_h \rightarrow Y)_h$ be a colimit cocone in $\mathbf{F}^T$ (it also being a colimit cocone in $\mathbf{F}$) and let $y: Y \rightarrow KX$ be the factoring. Let

$$\begin{array}{ccc} & q & \\ & p \longrightarrow Y & \\ p \downarrow & & \downarrow y \\ Hs^*X & \longrightarrow & KX \\ & \chi X & \end{array}$$

be a pullback diagram. q is cocartesian. For every $h \in H$ the pair $\beta_h \cdot \chi X_h: Hs^*X_h \rightarrow Y$, $Hs^*\alpha_h: Hs^*X_h \rightarrow Hs^*X$ determines a unique $\delta_h: Hs^*X_h \rightarrow P$ with $p\delta_h = Hs^*\alpha_h$. But since $H \circ s^*$ is entire $(Hs^*\alpha_h)_h$ is a colimit cocone. Therefore p splits and is an iso. This forces y to be an iso too since $q, \chi X$ are cocartesian.

Note that it follows trivially that cocartesian arrows in $\mathbf{G}$ are cartesian. Therefore, in order to prove that $\mathbf{G}$ is a stiff fibration we only have to show cocartesian arrows are stable under pullback by morphisms above identity. So let $\chi: H \rightarrow K$ be cocartesian in $\mathbf{G}$, $\varphi: L \rightarrow K$ above identity. Let $\xi: M \rightarrow L$ be a cartesian arrow above $G\chi$, and $\psi: M \rightarrow H$ the unique morphism above identity completing the square, which is a pullback:

$$\begin{array}{ccc} & \xi & \\ & M \longrightarrow L & \\ \psi \downarrow & & \downarrow \varphi \\ H & \longrightarrow & K \\ & \chi & \end{array}$$

Let $f: X \rightarrow Y$ in $\mathbf{E}$ be cartesian above $G\chi = G\xi$. Our goal is to prove $\xi[f]$ is cocartesian. Let

$$\begin{array}{ccccc} & g & & y & \\ & \searrow & & \searrow & \\ X & \longrightarrow & Z & \longrightarrow & Y \end{array}$$

be a cocartesian-above identity factorization of f . Look at

$$\begin{array}{ccccc} & \xi[g] & & Ly & \\ & \searrow & & \searrow & \\ MX & \longrightarrow & LZ' & \longrightarrow & LY \\ \psi X \downarrow & & \downarrow \varphi Z & & \downarrow \varphi Y \\ HX & \longrightarrow & KZ & \longrightarrow & KY \\ & \chi[g] & & Ky & \end{array} .$$

Keeping in mind the remark just above 4.7.3 we denote morphisms of $\mathbb{G}$ above identity as natural transformations, i.e. φZ stands for $\varphi[1_Z]$. Since ξ is cartesian $\xi[g]$ is cartesian. By assumption $\chi[g]$ is cocartesian, and since ψX , φZ are above identity the left square is a pullback. The right square is also a pullback and since $\xi[f] = \chi[g] \circ Ky$ is cocartesian we get that $\xi[f] = Ly \circ \xi[g]$ is cocartesian, by 3.14.

4.7.7 The Beck condition holds in $\mathbb{G}$.

We will show cocartesian arrows are stable under pullback by cartesian arrows (3.14, ii)). Let $\zeta: K \rightarrow L$ in $\mathbb{G}$ be cartesian above $t: T \rightarrow V$, and let $\xi: M \rightarrow L$ be cocartesian above $v: W \rightarrow V$. By 3.10 ii), to form the pullback of ζ, ξ it suffices to get

$$\begin{array}{ccc} & w & \\ S & \longrightarrow & W \\ s \downarrow & & \downarrow v \\ T & \longrightarrow & V \\ & t & \end{array} \quad \begin{array}{ccc} & \mu & \\ H & \longrightarrow & M \\ \chi \downarrow & & \downarrow \xi \\ K & \longrightarrow & L \\ & \zeta & \end{array}$$

a pullback s, w in $\mathbb{C}$ and then extend cartesian arrows χ, μ above s, w respectively (3.10, ii) will be invoked repeatedly in the argument that follows). Let now $Y \in \mathbf{E}^T$. Let $f: X \rightarrow Y$ be cartesian above s . Our goal is to show $\chi[f]$ is cocartesian. Extend a cocartesian $y: Y \rightarrow Y'$ above t and let $f': X' \rightarrow Y'$ be cartesian above v . There is a unique $x: X \rightarrow X'$ above w making the square

commute, and that square is a pullback, and therefore x is cocartesian.

$$\begin{array}{ccc}
 & x & \\
 X & \longrightarrow & X' \\
 f \downarrow & & \downarrow f' \\
 Y & \longrightarrow & Y' \\
 & y &
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \mu[x] & \\
 HX & \longrightarrow & MX' \\
 \chi[f] \downarrow & & \downarrow \xi[f'] \\
 KY & \longrightarrow & LY' \\
 & \zeta[y] &
 \end{array}$$

$\mu[x]$ and $\zeta[y]$ are cartesian, and since the square to the left is a pullback, the one to the right is one too. Since ξ is cocartesian, $\xi[f']$ is cocartesian, and therefore $\chi[f]$ is cocartesian. QED.

4.7.8 Proposition

Let H be locally prime in $\mathbb{G}$, say above S . Let $x: A \rightarrow HX$ make (H, x) A, X -generated. Let $\chi: K \rightarrow H$ be above $s: T \rightarrow S$, and let $f: Y \rightarrow X$ be cartesian above s . Then TFAE

- i) The pullback $a: B \rightarrow A$ of $\chi[f]$ by x is cocartesian
- ii) a as above is cocartesian and f is cocartesian
- iii) χ is cocartesian.

If any of the above happens, then (K, y) is B, Y -generated, where $y: B \rightarrow KY$ is the pullback of x by $\chi[f]$.

Proof

i) $\Rightarrow$ ii)

Let

$$Y \xrightarrow{g} X' \xrightarrow{n} X$$

be a cocartesian-above-identity factorization of f . Since a is cocartesian there is $x': A \rightarrow HX'$ above identity with $\chi[g] \circ y = x' \circ a$. We have $Hn \circ x' \circ a = xa$ and since a is cocartesian, it can be factored out of the equation. Therefore n is an isomorphism, x being ultrageneric, and f is cocartesian.

ii) $\Rightarrow$ iii)

Let

$$K \xrightarrow{\xi} H' \xrightarrow{\varphi} H$$

be a cocartesian-above-identity factorization of χ . We have $\chi[f] = \varphi X \circ \xi[f]$ (keeping in mind the remark just above 4.7.3 we denote morphisms of $\mathcal{G}$ above identity as natural transformations, i.e. φX stands for $\varphi[1_X]$). Since a is cocartesian there is $w: A \rightarrow H'X$ with $wa = \xi[f] \circ x$. A similar argument as above will show $\varphi X \circ w = x$, and since (H, x) is generic in $\mathcal{U}t(A, X)$ φ is an isomorphism.

iii) $\Rightarrow$ i)

Since f is cartesian and χ cocartesian $\chi[f]$ is cocartesian. Therefore the pullback a is cocartesian.

We still have to prove (K, y) is B, Y -generated. Let us first show y is ultrageneric. Since K is entire there is (C, Z, z) ultrageneric in $\mathcal{F}||K$ (4.1) and $(b, h): (C, Z, z) \rightarrow (B, Y, y)$, where h is in E^T and b in F^T . We get a commutative square

$$\begin{array}{ccc} & ab & \\ C & \longrightarrow & A \\ z \downarrow & & \downarrow x \\ KZ & \longrightarrow & HX \\ & \chi[fh] & \end{array}$$

Since ab is cocartesian we can apply the argument of i) $\Rightarrow$ ii) again, replacing a by ab and f by fh . Therefore fh is (co)cartesian, and so h is too. But being above 1_T h is an isomorphism, so y is ultrageneric. Now let $\theta: (K', y') \rightarrow (K, y)$ be the generic object of $\mathcal{U}t(B, Y)$ determined by (K, y) . If $\chi': K' \rightarrow H'$ is a cocartesian arrow above s and $\psi: H' \rightarrow H$ above identity with $\psi\chi' = \chi\theta$ then by the cocartesianness of a there is $v: A \rightarrow K'X$ with $va = \chi'[f] \circ y'$. An oft-repeated argument will show $\psi X \circ v = x$, and this forces ψ to be an

isomorphism. But since the square $\chi, \chi', \theta, \psi$ is a pullback, θ is an iso too and (K, y) is B, Y -generated.

4.7.9 *For any $H \in \mathcal{G}$ locally prime there is a G -minimal K above a finite object of $\mathcal{C}$ and a cocartesian $\chi: K \rightarrow H$.*

Let $x: A \rightarrow HX$ make (H, x) A, X -generated. Notice that given any cocartesian $K \rightarrow H$, the proof above shows there is an F -cocartesian $a: B \rightarrow A$ above an E -cocartesian $f: Y \rightarrow X$. This is just saying a is EF -cocartesian. Therefore, to get a G -minimal K , first choose an EF -cocartesian $a: B \rightarrow A$ where B is EF -minimal. Denoting Efa by $s: T \rightarrow S$ and Fa by $f: Y \rightarrow X$, let $\chi: K \rightarrow H$ be a cartesian arrow in $\mathcal{G}$ above s . Since f is cocartesian by 4.7.3 $\chi[f]$ is cartesian above f . But then the pullback c

$$\begin{array}{ccc} & c & \\ & \downarrow & \\ C & \longrightarrow & A \\ z \downarrow & & \downarrow x \\ KY & \longrightarrow & HX \\ & \chi[f] & \end{array}$$

is a cartesian arrow above f , and is therefore isomorphic to a , so we can apply 4.7.8 and get that χ is cocartesian. Now the proof that K is G -minimal is just 4.7.8 ii).

We have proven G is an entire fibration.

Let $E^*G: E^*\mathcal{G} \rightarrow E$ be the pullback of G by E . An object of $E^*\mathcal{G}$ is a pair (X, H) where $X \in E$ and $H \in \mathcal{G}^{EX}$, i.e. $H: E^{EX} \rightarrow F^{EX}$. The fibration E^*G obviously sends (X, H) to X . A morphism $(X, H) \rightarrow (Y, K)$ is simply a pair (f, χ) , where $f: X \rightarrow Y$ and $\chi: H \rightarrow K$ is above Ef . We define a tentative counit for the adjunction to be the morphism $\varepsilon: E^*G \rightarrow F$ of fibrations that sends $(f, \chi): (X, H) \rightarrow (Y, K)$ to $\chi[f]: HX \rightarrow KY$. We have to show among other things that ε is entire. We will first show it preserves pullbacks. This amounts to proving (using the fact that in a pullback of fibrations pullbacks are calculated componentwise) that given any

$$\begin{array}{ccc}
 & f & \\
 X & \longrightarrow & Z \\
 h \downarrow & & \downarrow k \\
 W & \longrightarrow & Y \\
 & g &
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \chi & \\
 H & \longrightarrow & L \\
 v \downarrow & & \downarrow \epsilon \\
 M & \longrightarrow & K \\
 & \xi &
 \end{array}$$

pair of pullbacks, the left one in $\mathbf{E}$, the right one in $\mathbf{C}$, such that they are above a common pullback of $\mathbf{C}$, then the square

$$\begin{array}{ccc}
 & \chi[f] & \\
 HX & \longrightarrow & LZ \\
 v[h] \downarrow & & \downarrow \epsilon[k] \\
 MW & \longrightarrow & KY \\
 & \xi[g] &
 \end{array}
 \quad (*)$$

is a pullback too. We will not prove this directly, but first tackle a special case.

4.7.10 *If χ and ξ are cocartesian then the square above is a pullback*

Let

$$\begin{array}{ccccc}
 & f' & & z & \\
 X & \longrightarrow & Z' & \longrightarrow & Z \\
 h \downarrow & & \downarrow k' & & \downarrow k \\
 W & \longrightarrow & Y' & \longrightarrow & Y \\
 & g' & & y &
 \end{array}$$

be such that $zf' = f$, $yg' = g$, f' and g' are cocartesian and z, y above identity. Since the left square has two parallel cartesian arrows and is above a pullback, it is a pullback. The right square is above a trivial pullback, and we can apply 3.14.2, to conclude it is a pullback too. Now look at the following,

$$\begin{array}{ccccc}
 & \chi[f'] & & Lz & \\
 HX & \longrightarrow & LZ' & \longrightarrow & LZ \\
 v[h] \downarrow & & \downarrow \epsilon[k'] & & \downarrow \epsilon[k] \\
 KW & \longrightarrow & KY' & \longrightarrow & KY \\
 & \xi[g'] & & Ky &
 \end{array}$$

whose external square is $(*)$. Since χ, ξ, f', g' are cocartesian, $\chi[f']$ and $\xi[g']$ are cocartesian. Since the left square is above a pullback, it is a pullback. The right square is one too since it is the expression of the pullback condition for ξ .

We are now ready to prove ε preserves pullbacks in general.
Let

$$\begin{array}{ccccc} & \chi' & & \kappa & \\ & \downarrow & & \downarrow & \\ H & \longrightarrow & L' & \longrightarrow & L \\ \nu \downarrow & & \downarrow \varepsilon' & & \downarrow \varepsilon \\ M & \longrightarrow & K' & \longrightarrow & K \\ & \xi' & & \tau & \end{array}$$

be a cocartesian-above-identity decomposition of χ, ξ . That is, χ' is cocartesian, κ above 1_{LZ} , etc.. Let

$$\begin{array}{ccccc} & \kappa & & & \\ & \downarrow & & & \\ L' & \longrightarrow & L & & \\ \omega' \downarrow & & \tau' \downarrow & & \omega \\ N' & \longrightarrow & N & & \\ \sigma' \downarrow & & \downarrow \sigma & & \\ K' & \longrightarrow & K & & \\ & \tau & & & \end{array}$$

be a similar decomposition of $\varepsilon', \varepsilon$, where ω', ω are cocartesian and σ', σ above identity. By 3.14.2 the lower square is a pullback (it is above a trivial pullback). Now look at

$$\begin{array}{ccccccc} & \chi[f] & & \kappa Z & & & \\ & \downarrow & & \downarrow & & & \\ HX & \longrightarrow & L'Z & \longrightarrow & LZ & & \\ \nu[h] \downarrow & & \omega'[k] \downarrow & & \tau'Y \downarrow & & \omega[k] \\ & & N'Y & \longrightarrow & NY & & \\ \downarrow & & \sigma'Y \downarrow & & \downarrow \sigma Y & & \\ MW & \longrightarrow & K'Y & \longrightarrow & KY & & \\ & \xi[g] & & \tau Y & & & \end{array}$$

The outer square is $(*)$, and must be proven a pullback. The left square is one, because of 4.7.10. The same holds for the upper right

square, which is above a trivial pullback. But the lower right square is entirely above 1_Y . By 4.2.3 it is a pullback, and this proves our claim.

4.7.11 *Let $Y \in \mathcal{E}$, $H \in \mathcal{G}$ be above a common object of $\mathcal{C}$. Let $y: B \rightarrow KY$ above 1_Y be such that B is prime in $\mathcal{F}^Y$, EF -minimal, and (K, y) is B, Y -generated, as in 4.7.8. Thus K is G -minimal. Then $y: B \rightarrow \varepsilon(Y, K)$ is ultrageneric. As a consequence ε is entire.*

Since ε is already known to preserve pullbacks, it suffices to show that given $g: Z \rightarrow Y$ in $\mathcal{E}$ and $\xi: L \rightarrow K$ in $\mathcal{G}$ above a common arrow of $\mathcal{C}$, along with $b: C \rightarrow B$ F -cocartesian and $z: C \rightarrow LZ$ (not necessarily above identity) such that $\xi[g] \circ z = yb$, then both ξ and g are isomorphisms. Let

$$\begin{array}{ccccc} & p & & g' & \\ & \downarrow & & \downarrow & \\ Z & \longrightarrow & Z' & \longrightarrow & Y \end{array}$$

be a factorization of g where g' is cartesian and p above identity. Let

$$\begin{array}{ccc} & d & \\ & \downarrow & \\ D & \longrightarrow & B \\ v \downarrow & & \downarrow y \\ LZ' & \longrightarrow & KY \\ & \xi[g'] & \end{array}$$

be a pullback. There is $c: C \rightarrow D$ with $dc = b$ and $Hp \circ z = vc$. Since b is cocartesian by 3.10 iv) d and c are cocartesian.

exists a unique $\tilde{J}: D \rightarrow G$ with $\epsilon \circ E^* \tilde{J} = J$. So let $D: D \rightarrow C$ be an entire morphism of aggregates, $E^*D: E^*D \rightarrow E$ the pullback. As usual an object of E^*D is a pair (X, P) where $X \in E$, $P \in D$ and $EX = DP$. Let $J \in \text{Aggr}/E(E^*D, F)$. Let us see what are the requirements for a functor $\tilde{J}: D \rightarrow G$ with $GJ = D$ to make

$$\begin{array}{ccc} & E^* \tilde{J} & \\ E^*D & \longrightarrow & E^*G \\ J \searrow & & \nearrow \epsilon \\ & F & \end{array}$$

commute. Let $p: P \rightarrow Q$ in D , say above $s: S \rightarrow T$. For any $X \in E$ above S , $E^* \tilde{J}$ sends (X, P) to $(X, \tilde{J}P)$. $\tilde{J}P$ is a section $E^S \rightarrow F^S$ and since ϵ is evaluation and we want the triangle to commute, we have to have $(\tilde{J}P)X = J(X, P)$ for any $X \in E^S$. $(\tilde{J}P)X$ thus defined is an object of G^S since it is the composite $J \circ \hat{P}$, where $\hat{P}$ is the

$$\begin{array}{ccccc} E^*D & \xrightarrow{\quad} & D & & \\ \downarrow & \nearrow \hat{P} & \downarrow P & & \\ E^*D & & E^S \rightarrow 1 & & D \\ \downarrow & \nearrow I & \downarrow S & & \\ E & \xrightarrow{\quad} & C & & \\ & E & & & \end{array}$$

unique functor $E^S \rightarrow E^*D$ making the left triangle and the upper quadrangle commute (here we identify an object of say, C with the corresponding functor from the one-point category to C). Therefore $\tilde{J}$ is uniquely defined on objects, if it exists. $\tilde{J}p$ should be a morphism $\tilde{J}P \rightarrow \tilde{J}Q$ of G . But given $f: X \rightarrow Y$ above s the same considerations as above force $(\tilde{J}p)[f]$ to be $J(f, p)$. It is trivial to check that the family $(J(-, p))_{f \text{ above } s}$ satisfies the pullback property. All we have left to do is to prove $\tilde{J}$ is entire.

Let us first show it preserves pullbacks. Let

$$\begin{array}{ccc}
 & p_1 & \\
 0 & \longrightarrow & P_1 \\
 p_2 \downarrow & & \downarrow q_1 \\
 P_2 & \longrightarrow & Q \\
 & q_2 &
 \end{array}$$

be a pullback diagram in $\mathbf{D}$. Say this diagram is above the pullback

$$\begin{array}{ccc}
 & s_1 & \\
 R & \longrightarrow & S_1 \\
 s_2 \downarrow & & \downarrow t_1 \\
 S_2 & \longrightarrow & T \\
 & t_2 &
 \end{array}$$

in $\mathbf{C}$. Let $\chi_i: H \rightarrow \tilde{J}P_i$, $i = 1, 2$ with $\tilde{J}q_1 \circ \chi_1 = \tilde{J}q_2 \circ \chi_2$. Let χ_i be above $v_i: V \rightarrow S_i$. There is $v: V \rightarrow R$ with $s_1 v = v_1$. If there exists $\xi: H \rightarrow \tilde{J}O$ with $\tilde{J}p_i \circ \xi = \chi_i$, this ξ necessarily must be above v . We know the value of ξ is entirely determined by its value on cocartesian arrows. So let $X \in E^V$, and $f: X \rightarrow Y$ be cocartesian above v . Let

$$\begin{array}{ccc}
 & g_1 & \\
 Y & \longrightarrow & Z_1 \\
 g_2 \downarrow & & \downarrow h_1 \\
 Z_2 & \longrightarrow & W \\
 & h_2 &
 \end{array}$$

be a square of cocartesian arrows (necessarily a pullback) above s_1, t_1 . Since the combined square $(g_1, p_1), (h_1, q_1)$ is a pullback in $E^*\mathbf{D}$, we get that

$$\begin{array}{ccc}
 & \tilde{J}p_1[g_1] & \\
 (\tilde{J}O)Y & \longrightarrow & (\tilde{J}P_1)Z_1 \\
 \tilde{J}p_2[g_2] \downarrow & & \downarrow \tilde{J}q_1[h_1] \\
 (\tilde{J}P_2)Z_2 & \longrightarrow & (\tilde{J}Q)T \\
 & \tilde{J}q_2[h_2] &
 \end{array}$$

is a pullback in $\mathbf{F}$. The equation $(\tilde{J}p_1)[g_1] \circ \xi[f] = \chi_1[g_1 f]$ forces a unique value for $\xi[f]: HX \rightarrow (\tilde{J}O)Y$. We are left to check ξ thus defined satisfies the pullback condition. This is easy: given

$$\begin{array}{ccc}
 & f & \\
 & X \longrightarrow Y & \\
 x \downarrow & & \downarrow y \\
 & X' \longrightarrow Y' & \\
 & f' &
 \end{array}$$

where f, f' are cocartesian above v and x, y above identity, one just extends cocartesian arrows $k: Y \rightarrow W$, $k': Y' \rightarrow W'$ above $h_1 g_1 = h_2 g_2$ and then uses the standard proposition about pullbacks (2.5, second part).

In order to finish proving ε is entire we need yet another proposition.

4.7.13 Proposition

Let H be locally prime in $\mathbb{G}$, with $x: A \rightarrow HX$ a generator. Let $\varphi: H \rightarrow \tilde{J}P$ above identity be such that $\varphi X \circ x: A \rightarrow (\tilde{J}P)X = J(X, P)$ is ultrageneric for $J(-, -)$. Then φ is ultrageneric for $\tilde{J}$.

Proof

Let there be $\chi: K \rightarrow H$ cocartesian, $\xi: K \rightarrow \tilde{J}Q$ in $\mathbb{G}$, along with $q: Q \rightarrow P$ in $\mathbb{D}$ such that

$$\begin{array}{ccc}
 & \chi & \\
 & K \longrightarrow H & \\
 \xi \downarrow & & \downarrow \varphi \\
 & \tilde{J}Q \longrightarrow \tilde{J}P & \\
 & \tilde{J}q &
 \end{array}$$

commutes. Say ξ is above $r: R \rightarrow S$ and q above $s: S \rightarrow T$, so χ is above sr . We want to show q is an isomorphism. If $f: Y \rightarrow X$ is a cartesian arrow above sr and

$$\begin{array}{ccc}
 & a & \\
 & B \longrightarrow A & \\
 y \downarrow & & \downarrow x \\
 & KY \longrightarrow HX & \\
 & \chi[f] &
 \end{array}$$

the pullback we know by 4.7.8 that f, a are cocartesian, and y ultrageneric. Let f factor as hg , where $g: Y \rightarrow Z$ is cocartesian above r and $h: Z \rightarrow X$ cocartesian above s . Obviously the bottom square commutes;

$$\begin{array}{ccc}
 & a & \\
 B & \longrightarrow & A \\
 y \downarrow & & \downarrow x \\
 KY & \longrightarrow & HX \\
 \xi[g] \downarrow & \chi[f] & \downarrow \varphi X \\
 (\tilde{J}Z)Q & \longrightarrow & (\tilde{J}X)P \\
 & \tilde{J}q[h] &
 \end{array}$$

since $\tilde{J}q[h]$ is $J(h, q): J(Z, Q) \rightarrow J(Y, P)$ and $\varphi X \circ x$ is ultrageneric for J , we get that q, h are isomorphisms.

We can finally show $\tilde{J}$ is entire. Let L be locally prime in $\mathcal{G}$ and $z: C \rightarrow LZ$ be a generator. Let $\psi: L \rightarrow \tilde{J}Q$ be a morphism above identity. The object $(C, (Z, Q), \psi Z \circ z)$ of $F \parallel J$ determines an ultrageneric $(B, (Y, P), y)$ and a morphism $(b, (f, p))$ from the latter to the former. If $\xi: K \rightarrow L$ is a cocartesian morphism above Fb there is $\theta: K \rightarrow \tilde{J}P$ with $\tilde{J}p \circ \theta = \psi \xi$. Let f (which is above Fb) factor as

$$\begin{array}{ccccc}
 & m & f' & & \\
 Y & \longrightarrow & Y' & \longrightarrow & Z
 \end{array}$$

where f' is cartesian and m above identity. By 4.7.8 we know that the $y': B \rightarrow KY'$ with $\xi[f'] \circ y' = zb$ makes (K, y') B, Y' -generated.

$$\begin{array}{ccccc}
 & b & & & \\
 & B \longrightarrow C & & & \\
 y \swarrow & & \searrow y' & \downarrow z & \\
 (\tilde{J}P)Y & & KY' \longrightarrow HZ & & \\
 (\tilde{J}P)m \searrow & \swarrow \theta Y' & \downarrow \psi Z & & \\
 & (\tilde{J}P)Y' \longrightarrow (\tilde{J}Q)Z & & & \\
 & Jp[f'] & & &
 \end{array}$$

Since θ is cartesian there is $y'': B \rightarrow KY$ with $Km \circ y'' = y'$. But y' is generic, so m is an iso. Therefore $\theta Y' \circ y'$ is ultrageneric for J ,

and the preceding lemma shows θ is ultrageneric for $\tilde{J}$, QED. This completes the proof of 4.7.

4.8 Theorem

Let $\mathcal{A}$ be the class of all qualitative fibrations in $Aggr$, that is, all entire fibrations all whose fibers are qualitative domains. $\mathcal{A}$ is classified by the fibration $Q: \mathcal{Q} \rightarrow qD$, where qD is the category of qualitative domains and strong morphisms (1.1), and Q the Grothendieck fibration associated with the forgetful functor $qD \rightarrow Poset$ which to every $f: X \rightarrow Y$ in qD associates the stable function $f^+: \tilde{X} \rightarrow \tilde{Y}$. As a consequence of 4.4 the pair $(\mathcal{A}, Q)$ forms a notion of smallness.

Proof

Remember (1.1, 3.2: Example) that if X, Y are qualitative domains, the assignment $f \mapsto f^+$ defines a bijection between the set of strong morphisms $X \rightarrow Y$ and the set of stiff embeddings $\tilde{X} \rightarrow \tilde{Y}$. Therefore, if $E: \mathcal{C} \rightarrow \mathcal{C}$ is a stiff fibration all whose fibers are qualitative domains, it defines a functor $E^*: \mathcal{C} \rightarrow qD$, which sends $s: S \rightarrow T$ in $\mathcal{C}$ to the strong morphism corresponding to $\exists_s: E^S \rightarrow E^T$. Obviously, E will be the pullback of Q by E^* . Let 1 be the atomic qualitative domain, and $a: 1 \rightarrow E^*T$ a morphism in qD . This is the same thing as the choice of an atom $A \in E^T$. It is trivial to show that there exists a morphism $x \rightarrow a$ in A/E^* where x is generic iff A admits a cocartesian $B \rightarrow A$ in E where B is E -generic. Therefore E is an entire fibration iff E^* is an entire functor. The fibration Q is obviously entire, since it corresponds to the identity $qD \rightarrow qD$.

A natural question that arises now is: how does this model compare with Girard's original model of the second-order lambda calculus [Gi86]? A model of the theory of constructions can always be restricted to a model of F : let $(\mathcal{C}, D)$ be a display category, with $\lambda: \Lambda \rightarrow \mathcal{E}$ a notion of smallness. A variable type $\Phi(\alpha_1, \dots, \alpha_n)$ in F , where the α_i 's are type variables, is interpreted as a morphism

$S^n \rightarrow S$, giving rise by pulling back λ to a small display map $\Phi^*: \Sigma \Phi \rightarrow S^n$. Therefore if $t(x_1, \dots, x_m)$ is a term of type $\Phi(\alpha_1, \dots, \alpha_n)$, with x_i of type $\Psi_i(\alpha_1, \dots, \alpha_n)$, the natural interpretation of t is as a morphism

$$\Psi_1^* \times S^n \Psi_2^* \dots \times S^n \Psi_m^* \longrightarrow \Phi^*$$

in $\mathcal{D}S^n$, the category of display maps into S^n and commutative triangles. In our model, $S = qD$ and λ is the functor Q defined above. Now Girard in his model interprets types just as above, i.e. as entire functors $qD^n \rightarrow qD$. But now a term $t(x_1, \dots, x_m)$, with variables as above, is interpreted as a family $(\tau_X)_{X \in qD^n}$ indexed by the objects of qD^n , where τ_X is a stable function (entire functor)

$$\Psi_1 X \times \Psi_2 X \dots \times \Psi_m X \longrightarrow \Phi X,$$

subject to the condition that for any $f: X \rightarrow Y$ in qD^n the square

$$\begin{array}{ccc} & \Psi f^+ & \\ \Psi X & \longrightarrow & \Psi Y \\ \tau_X \downarrow & & \downarrow \tau_Y \\ \Phi X & \longleftarrow & \Phi Y \\ & \Psi f^- & \end{array} \quad (*)$$

of stable functions commutes; we use the same notation for a type, say Ψ_i , and its interpretation $\Psi_i: qD^n \rightarrow qD$. $\Psi: qD^n \rightarrow qD$ is the obvious product of functors $\Psi = \Psi_1 \times \Psi_2 \times \dots \times \Psi_m$. Notice that the directions of the arrows make this square a non-classical commutative square. In particular it does not follow by a formal argument that these modified natural transformations commute. The proof that they actually commute is rather non-trivial (if expressed in the original language of Girard, as he once explained it to the author. The proof becomes easier when translated in the language of this thesis, as we will see). It was unfortunately omitted in [Gi86]. A close reading of said paper shows that this way of interpreting terms was dictated by the choice of the interpretation of the operator Π . Let Θ be the interpretation of a one-variable type, i.e. $\Theta: qD \rightarrow qD$ is an entire functor. Define a qualitative

domain $\bar{\Pi}\Theta$ as follows: an atom $A \in |\bar{\Pi}\Theta|$ is an isomorphism class in $1/\Theta$ of generic arrows $a:1 \rightarrow \Theta X$, where a is such that for all $Y \in qD$ we have that the family $(\Theta k \circ a)_{k \in qD(X,Y)}$ (seen as a family of atoms of ΘY) is compatible, i.e. has a sup in ΘY . By filteredness, in order to define $\bar{\Pi}\Theta$ it is sufficient to say what are the finite compatible families of atoms. We say a family $a_1, \dots, a_n$ of (representatives of) atoms $a_i:1 \rightarrow \Theta X_i$ is compatible if for every $Y \in qD$, every family $(k_i: X_i \rightarrow Y)_{1 \leq i \leq n}$ of morphisms of qD the family $(\Theta k_i \circ a_i)_i$ is a compatible family of atoms of ΘY . Now let $Z \in qD$, and ΔZ be the constant functor $qD \rightarrow qD$ that sends everything to Z . We are looking for a set $\mathcal{T}(\Delta Z, \Theta)$ of morphisms of functors $\Delta Z \rightarrow \Theta$ that gives us an adjunction isomorphism $qD(Z, \bar{\Pi}\Theta) \cong \mathcal{T}(\Delta Z, \Theta)$. If one puts $Z = 1$ it is easy to see that $qD(Z, \bar{\Pi}\Theta)$ is isomorphic to the set of all modified natural transformations $\Delta Z \rightarrow \Theta$ as described above, and from this it is easy to infer that for any Z $\mathcal{T}(\Delta Z, \Theta)$ should be defined by using modified natural transformations, and therefore that the operation $\bar{\Pi}$ defines a right adjoint to the functor $\Delta: qD \rightarrow \mathcal{T}$, where the objects of $\mathcal{T}$ are stable endofunctors of qD and the morphisms modified natural transformations. This observation can be generalized to any arity $n \in \mathbb{N}$ by defining $\bar{\Pi}$ pointwise. That is, let $\mathcal{T}_n$ have for objects stable functors $qD^n \rightarrow qD$ and morphisms natural transformation. Then the operation that sends an object $\Phi \in \mathcal{T}_{n+1}$ to the object $\Psi \in \mathcal{T}_n$ defined by

$$\Psi(X_1, \dots, X_n) = \bar{\Pi}\Phi(X_1, \dots, X_n, -) \quad (X_1, \dots, X_n \in qD)$$

defines a right adjoint to the functor "compose to the right with the projection $qD^{n+1} \rightarrow qD^n$ ": $\mathcal{T}_n \rightarrow \mathcal{T}_{n+1}$. This, in a nutshell, is Girard's model.

Let $E: E \rightarrow qD^n$ be the Grothendieck fibration associated with a stable functor $\Phi: qD^n \rightarrow qD$. In other words, an object of E is a pair (X, A) where A is an element (object) of ΦX , and a morphism $f: (X, A) \rightarrow (Y, B)$ is a $f: X \rightarrow Y$ in qD^n such that $A \leq (\Phi f)^* B$, or equivalently $(\Phi f)^+ A \leq B$. Let $F: F \rightarrow qD^n$ be the same thing for

another stable Ψ . If $(\tau_X)_X$ is given as in $(*)$ it is easy to see (left to the reader) that the assignment

$$(X, A) \mapsto (X, \tau_X A)$$

uniquely defines a morphism $\tau^\square: F \rightarrow E$ in $Ent(F, E)$. It is also easy to see that the morphisms $F \rightarrow E$ of entire fibrations that come from modified natural transformations are exactly the ones that send cocartesian arrows to cartesian ones (this shows that the modified naturality condition can be generalized to any pair of entire fibrations, not just entire fibrations of qualitative domains).

Therefore the interpretation of any term in Girard's model belongs to our model. What comes as a surprise is the converse: if $H: F \rightarrow E$ is any entire morphism of fibrations, where F, E are the fibrations associated to any pair of entire functors $\Psi, \Phi: qD^n \rightarrow qD$, then H sends cocartesian arrows to cartesian ones. This observation, due to E. Moggi, was communicated to the author by Th. Coquand. We will work in the most general context that we know of:

4.9 Definition

Let $\mathcal{C}$ be an aggregate. We say $\mathcal{C}$ amalgamates weakly if given any morphism $f: X \rightarrow Y$ in $\mathcal{C}$ there exist $Z \in \mathcal{C}$, $g, h: Y \rightarrow Z$ making

$$\begin{array}{ccc} & f & \\ X & \longrightarrow & Y \\ f \downarrow & & \downarrow g \\ Y & \longrightarrow & Z \\ & h & \end{array}$$

a pullback.

Let us show that the category qD amalgamates weakly: let $f: X \rightarrow Y$ therein. Let Z have for set of atoms the amalgamated sum $|Y| + |x| |Y|$; That is, $|Z|$ is the pushout

$$\begin{array}{ccc}
 & |f| & \\
 |X| & \longrightarrow & |Y| \\
 |f| \downarrow & & \downarrow \\
 |Y| & \longrightarrow & |Z|
 \end{array}$$

in the category of sets (the meaning of $|f|$ should be obvious). $|Z|$ has two subsets W_1 and W_2 , corresponding to the horizontal and vertical inclusions of $|Y| \rightarrow |Z|$. If we define the domain structure of Z by requiring that a subset $A \subset |Z|$ be compatible iff A is either contained in W_1 or W_2 , it is trivial to prove that the resulting square in qD is a pullback. Note that the property of weak amalgamation is preserved by products.

4.10 Lemma

Let $\mathbf{C}$ be an aggregate that amalgamates weakly. Let $E: \mathbf{E} \rightarrow \mathbf{C}$, $F: \mathbf{F} \rightarrow \mathbf{C}$ be entire fibrations, such that all the fibers of $\mathbf{F}$ are posets. Let $H: \mathbf{E} \rightarrow \mathbf{F}$ be an entire morphism of fibrations. Let $x: A \rightarrow HX$ be an ultrageneric arrow (we do not have to name x since the fibers are posets, but this will help intelligibility). Let $b: B \rightarrow A$ be cocartesian in $\mathbf{E}$, say above $s: T \rightarrow S$. Then if $\gamma_s X: s^*X \rightarrow X$ is cocartesian, b (and therefore s , $\gamma_s X$) are isomorphisms.

Proof

Let $f: Y \rightarrow X$ denote $\gamma_s X$. Let $t, t': S \rightarrow V$ make s, t, s, t' a pullback. Let $g: X \rightarrow Z$, $g': X' \rightarrow Z'$ be cocartesian above t, t' respectively. Since gf and $g'f$ are cocartesian above $ts = t's$, we can put $Z = Z'$ and get $gf = g'f$, the square making a pullback (being formed of cartesian arrows, and above a pullback).

$$\begin{array}{ccc}
 & s & \\
 T & \longrightarrow & S \\
 s \downarrow & & \downarrow t \\
 S & \longrightarrow & V \\
 & t' &
 \end{array}
 \qquad
 \begin{array}{ccc}
 & f' & \\
 Y & \longrightarrow & X \\
 f \downarrow & & \downarrow g \\
 X' & \longrightarrow & Z \\
 & g' &
 \end{array}$$

In the same manner we extend cocartesian arrows $c: A \rightarrow C$ and $c': A \rightarrow C$ above t and t' respectively, and get that b, c, b', c form a

pullback. By the cocartesianness of c, c' there are $z: C \rightarrow HZ$ and $z': C \rightarrow HZ$ with $zc = Hg \circ x$ and $z'c' = Hg' \circ x$.

$$\begin{array}{ccc}
 B & \xrightarrow{b} & A \\
 \downarrow b & \begin{array}{c} \text{Hf} \quad x \swarrow \\ HY \rightarrow HX \\ \text{Hf} \downarrow \quad \downarrow Hg \\ HX \rightarrow HZ \\ \swarrow x \quad Hg' \searrow \end{array} & \downarrow c \\
 A & \xrightarrow{c'} & C
 \end{array}$$

But since the fibers are posets $z = z'$, and so $Hg \circ xb = Hg' \circ xb$. Therefore by pullbackness there is $y: B \rightarrow HY$ with $Hf \circ y = xb$ and since x is ultrageneric and b cocartesian f is an isomorphism. QED.

4.10.1 Corollary

Let $\mathbf{C}$ amalgamate weakly, $F: \mathbf{F} \rightarrow \mathbf{C}$ be an entire fibration all whose fibers are posets. Let H be an entire splitting of F , $x: A \rightarrow HX$ above identity. Then the following are equivalent:

- i) x is ultrageneric.
- ii) x is generic and A is F -generic.

Proof

The proof of $ii) \Rightarrow i)$ is trivial and left to the reader. It is obvious that to prove the converse we simply have to show that every cocartesian $b: B \rightarrow A$ is an isomorphism. Just apply the lemma by putting E equal to the identity on $\mathbf{C}$; the result follows because every arrow of $\mathbf{E} = \mathbf{C}$ is cocartesian.

Given any aggregate $\mathbf{C}$ we have shown in 4.8 how to identify entire functors $\mathbf{C} \rightarrow \mathbf{qD}$ with entire fibrations $\mathbf{E} \rightarrow \mathbf{C}$ all whose fibers are qualitative domains. Let $Qual(\mathbf{C})$ be the full subcategory of $Fib(\mathbf{C})$ whose objects have qualitative domains in every fiber.

Examination of 4.4 in the light of the corollary above will show that the functor $\Pi_1: \text{Qual}/(qD) \rightarrow \text{Ent}(\mathbf{1})$ will have the same value on objects as the operator $\bar{\Pi}$ seen just before 4.9. This can be proven in a more indirect and comprehensive fashion by the following

4.11 Theorem

Let $\mathbf{C}$ amalgamate weakly, $E: \mathbf{E} \rightarrow \mathbf{C}$, $F: \mathbf{F} \rightarrow \mathbf{C}$ be entire fibrations where the fibers of $\mathbf{F}$ are all posets, and $H: \mathbf{E} \rightarrow \mathbf{F}$ an entire morphism of fibrations. Then H sends cocartesian arrows of $\mathbf{E}$ to cartesian arrows of $\mathbf{F}$.

4.11.1 Corollary

Any term or type of the second-order lambda calculus has the same translation in our model as in Girard's.

This is because for any $n \in \mathbb{N}$ the category $\mathcal{T}_n$ used in Girard's model is equivalent to $\text{Qual}/(qD^n)$.

Proof of 4.11

Let $f: X \rightarrow Y$ be a cocartesian arrow of $\mathbf{E}$. We want to show Hf is cartesian. By a standard fact of fibration theory it is sufficient (the proof of sufficiency is left as an exercise) to prove that if $a: A \rightarrow HY$ is above Hf there is $x: A \rightarrow HX$ above identity with $Hf \circ w = a$. Take such an a and factor it as

$$\begin{array}{ccc} & b & y \\ A & \longrightarrow & B \longrightarrow HY \end{array},$$

b cocartesian, y above identity. There is $(g, c): (C, Z, z) \rightarrow (B, X, x)$ in $\mathbf{F} \parallel H$.

First let $f: X \rightarrow Y$ be a stiff embedding of DI-domains. By 3.2 we know f is an injective morphism of posets that preserves the sups that exist, and such that if $y \leq f(x)$ then y is in the image of f . Actually a slightly stronger property holds for f : if $A \subseteq X$ then A has a sup in X iff $f(A)$ has a sup in Y , and the sups are equal if they exist. This property is the expression of 3.1 in this specialized context: f preserves multicoproduct families. The converse is true: let W, Z be DI-domains, $g: W \rightarrow Z$ a powerful, injective morphism of posets that has the above property. Then g has a right adjoint g^- , and is therefore a stiff embedding. Given $z \in Z$ take $g^-(z)$ to be

$$g^-(z) = \sup_{w \in W, g(w) \leq z} w.$$

Armed with this knowledge, it is now quite easy to prove the two following facts: If X is a finite DI-domain then X is f.p. in DI. If X is finite and has a top element (i.e. X is a finite distributive lattice) then X is prime in DI. It is then a formality to prove that DI is an aggregate; it seems simpler to show DI is $\aleph_0$ -accessible and has connected limits than to show directly it has multicoproducts. Notice that by 1.6.1 the set of primes of DI is the (essential) set of finite distributive lattices. This gives a measure of the increase in complexity we have met by passing from qD to DI: from the set $\{2\}$, 2 being the 2-element lattice, to the set of all finite distributive lattices.

So if we take the class $\mathcal{S}$ of small maps to be DI-fibrations, we can construct a classifying map $S: \mathcal{S} \rightarrow DI$ for them, just as in 4.8. Take S to be the fibration associated with the forgetful functor $DI \rightarrow Poset$. Then $(\mathcal{S}, S)$ is a notion of smallness, and we have a new model of the theory of constructions. In this model, we can interpret Σ , but not with all the full generality of the model given in [K-P]. In this paper (2.12) is given a discussion of the different syntactic rules one can associate with Σ . The most powerful rule is called "Big Sums". Let $(\mathcal{C}, D)$ be a display category. If $(\mathcal{S}, S)$ is a

notion of smallness in it, the rule Big Sums is equivalent to saying that for every object $X \in \mathcal{C}$ the inclusion functor

$$\mathcal{S}_X \longrightarrow \mathcal{D}_X$$

has a left adjoint ($\mathcal{S}_X$ being the full subcategory of $\mathcal{D}_X$ whose objects are small maps). Our model falls short of having Big Sums by very little; what this means will be made precise in a forthcoming paper by Thierry Coquand.

CONCLUSION

We would like to give some remarks about the directions this research should take in the near future. First, we want to point out that the model of constructions we have given is the simplest conceptually, but not structurally. For instance, Girard's concept of coherent domain can be applied to semigranular categories, giving rise to coherent (or binary) semigranular categories, and the corresponding class of fibrations as display maps. In order to describe these categories we need the omitted chapter on the structure of semigranular fibrations. This further illustrates the slogan we gave in the introduction: in the category *Aggr* there lie many models of polymorphism, and the essence of every model lies in the choice of special morphisms, i.e. display and small maps.

The second remark concerns linear maps. They can be defined among semigranulars, but their rôle is at the present time quite mysterious. Are they significant? One essential technical problem is that the cotriple $!$ is much harder to define in this context, due to the 2-categorical nature of the structures involved. More should be done in order to clarify this situation.

Finally, we would like to mention that the topological point of view, which was important at the inception of domain theory and then was relegated to the background, is making a comeback. It turns out that Girard domains (and aggregates) can be given a geometric significance which is a generalization of the concept of topological space (and of Grothendieck topos). We made this realization very recently and intend to develop the necessary theoretical tools extensively.

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