OPTIMAL CONTROL OF HYBRID SYSTEMS: THEORY AND ALGORITHMS

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May 2004

A thesis submitted to McGill University
in partial fulfilment of the requirements of the degree of
Doctor of Philosophy

Your file Votre référence
ISBN: 0-494-06340-8
Our file Notre référence
ISBN: 0-494-06340-8

Abstract

Many complex systems are hybrid in the sense that: (i) the state set possesses continuous and discrete components and (ii) system evolution may occur in both continuous and discrete time. One important class of hybrid systems is that characterized by a feedback configuration of a set of continuous controlled low level systems and a high level discrete controller; such systems appear frequently in engineering and are particularly evident when a system is required to operate in a number of distinct modes. Other classes of hybrid systems are found in such diverse areas as (i) air traffic management systems, (ii) chemical process control, (iii) automotive engine-transmission systems, and (iv) intelligent vehicle-highway systems.

In this thesis we first formulate a class of hybrid optimal control problems (HOCPs) for systems with controlled and autonomous location transitions and then present necessary conditions for hybrid system trajectory optimality. These necessary conditions constitute generalizations of the standard Minimum Principle (MP) and are presented for the cases of open bounded control value sets and compact control value sets. These conditions give information about the behaviour of the Hamiltonian and the adjoint process at both autonomous and controlled switching times.

Such proofs of the necessary conditions for hybrid systems optimality which can be found in the literature are sufficiently complex that they are difficult to verify and use; in contrast, the formulation of the HOCP given in Chapter 2 of this thesis, together with the use of (i) classical variational methods and more recent needle variation techniques, and (ii) a local controllability condition, called the small time tubular fountain (STTF) condition, make the proofs in that chapter comparatively accessible. We note that the STTF condition is used to establish the adjoint and Hamiltonian jump conditions in the autonomous switchings case.

A hybrid Dynamic Programming Principle (HDPP) generalizing the standard dynamic programming principle to hybrid systems is also derived and this leads to hybrid Hamilton-Jacobi-Bellman (HJB) equation which is then used to establish a verification theorem within this framework.

The necessary conditions for optimality expressed in the Hybrid Minimum Principle (HMP) form the foundation for the general, effective hybrid system optimization algorithms called HMP algorithms. HMP algorithms appear to be more efficient than any of the recently proposed hybrid optimization algorithms and have been implemented on complex non-linear systems. Using results from the theory of penalty function methods and Ekeland's variational principle we show the convergence of these algorithms under reasonable assumptions. Furthermore, we show that the HMP algorithm class can be extended combinatorically with discrete search algorithms which, by repeated runs of HMP, find locally optimal location sequence and their associated switching times. This combinatoric search is possible due to the efficacy of HMP algorithms but faces the intrinsic complexity of an exponential explosion of alternate possible cases.

The problem of finding the optimal location switching sequence from among all the permutations of distinct modes of operation gives rise to the notion of optimality zones. These zones are regions in the switching time-state space and are associated with switching sequences. They have a simple topological and geometric structure, and once they have been computed (or approximated) they may be employed in the algorithm HMP[Z] which is essentially a zone dependent version of the HMP algorithm. The algorithm HMP[Z] permits one to reach the global optimum in a single run of the (zone dependent version of) HMP algorithm. The complexity cost of the method is essentially that of the cost of (approximately) mapping the optimality zones in $\mathbb{R}^{n+1}$, where n is the dimension of the state space, plus the cost of one run of the standard HMP algorithm. The efficacy of the proposed algorithms is illustrated via computational examples.

Résumé

Beaucoup de systèmes complexes sont hybride dans le sens: (i) qu'ils possèdent des composants continus et discrets et (ii) que l'évolution du système peut se passer en temps continu et en temps discret. Une classe importante de systèmes hybrides est caractérisée par une configuration asservie d'une série de systèmes continus de bas niveau et d'un contrôleur discret de niveau supérieur. De tels systèmes apparaissent fréquemment en ingénierie et sont particulièrement évidents quand un système doit opérer en plusieurs modes distincts. Les autres classes de systèmes hybrides sont trouvées dans divers domaines tels que (i) les systèmes de gestion de trafic aérien, (ii) le contrôle de procédés chimiques, (iii) les systèmes de transmission-moteur d'automobiles, et (iv) les systèmes d'autoroutes intelligentes pour véhicules.

Dans cette thèse nous formulons, dans un premier temps, la classe de problèmes associés aux commandes optimales hybrides (HOCP) pour les systèmes avec des transitions d'emplacement contrôlées et autonomes, puis nous présentons des conditions nécessaires pour optimiser la trajectoire des systèmes hybrides. Ces conditions nécessaires constituent des généralisations du Principe Minimum (MP) standard. Elles sont présentées pour le cas d'un ensemble borné ouvert de la commande et pour un ensemble compact de la commande. Ces conditions donnent de l'information sur le comportement de l'hamiltonien et du procédé adjoint pour deux cas possibles: lorsque les temps de commutation sont autonomes et lorsqu'ils sont contrôlés.

De telles preuves des conditions nécessaires pour l'optimalité des systèmes hybrides qui peuvent être trouvées dans la littérature sont suffisamment complexes qu'elles sont difficiles à vérifier et à utiliser; par opposition, la formulation de HOCP donnée au Chapitre 2 de cette thèse, avec l'usage (i) des méthodes de variations classiques et plus récentes techniques de variation d'aiguille, et (ii) d'une condition de contrôlabilité locale, soit la condition "Small Time Tubular Fountain (STTF)" rendent nos preuves comparativement accessibles.

Nous notons que la condition STTF est utilisée pour établir les conditions adjointes et les conditions hamiltoniennes de sauts dans les cas de commutations autonomes.

Un Principe de Programmation Dynamique Hybride (HDPP) généralisant le principe de programmation dynamique standard aux systèmes hybrides est aussi dérivé et ceci mène à l'équation Hamilton-Jacobi-Bellman (HJB) hybride qui est utilisée pour établir un théorème de vérification.

Les conditions nécessaires d'optimisation exprimées comme le Principe Minimum Hybride (HMP) forment la fondation pour les algorithmes d'optimisation de systèmes hybrides appelés algorithmes HMP. Les algorithmes HMP semblent être plus efficaces que n'importe quel autre algorithme hybride d'optimisation récemment proposé. De plus, ils sont implémentés pour des systèmes complexes non-linéaires. Utilisant les résultats de la théorie pour la méthode de la fonction de pénalité et du principe de variation d'Ekeland, nous montrons la convergence de ces algorithmes sous des hypothèses raisonnables. De plus, nous montrons que la classe d'algorithmes HMP peut être étendue de manière combinatoire avec des algorithmes discrets de recherche qui, par passes répétées du HMP, trouve la séquence d'emplacement localement optimale et les temps de commutation associés. Cette recherche combinatoire est possible grâce à l'efficacité des algorithmes HMPs mais elle fait face à la complexité intrinsèque d'une explosion exponentielle des cas possibles alternatifs.

Le problème de trouver la séquence d'emplacement localement optimale parmi toutes les permutations de modes distincts d'opération engendre la notion de zones d'optimisation. Ces zones sont des régions dans l'espace de temps-état de commutation et sont associées à des séquences de commutation. Elles ont une structure topologique et géométrique bien simple, et une fois qu'elles ont été calculées (ou approximées), elles peuvent être employées dans l'algorithme HMP[Z] qui est essentiellement une version zonée de l'algorithme HMP. L'algorithme HMP[Z] permet d'atteindre le maximum global en une seule passe de la version zonée de l'algorithme HMP. Le coût de complexité de la méthode est essentiellement celui d'associer (approximativement) la zone d'optimisation à $\mathbb{R}^{n+1}$, où n est la dimension de l'espace d'état, plus le coût d'une seule éxécution de l'algorithme HMP standard. L'efficacité des algorithmes proposés est illustrée à l'aide d'exemples numériques.

Claims of Originality

The following original results are reported in this thesis:

- *Rigorous formulation of the hybrid optimal control problem (HOCP).*
  We formulate a general optimal control problem for hybrid systems with nonlinear dynamics in each location (i.e. discrete state value) and with (i) controlled switchings, and (ii) autonomous switchings occurring on switching manifolds.

- *Development of necessary conditions, or Minimum Principle, for hybrid optimality in the case of compact control value sets.*
  We use the needle variation technique to establish the Minimum Principle in the case where the continuous control takes values in a compact set and there are no autonomous switchings.

- *Development of necessary conditions, or Minimum Principle, for hybrid optimality in the case of open bounded control value sets.*
  We give a proof using the classical smooth variations technique for the case of hybrid systems where both controlled switchings and autonomous switchings on time invariant switching manifolds are permitted. The proof of the result employs a controllability condition called the *small time tubular fountain (STTF)* condition.

- *Derivation of the Hybrid Dynamic Programming Principle (HDPP) and the Hybrid Verification Theorem.*
  Within the hybrid systems framework of this thesis, we develop a Dynamic Programming Principle for hybrid systems. The HDPP leads to a hybrid Hamilton-Jacobi-Bellman (HJB) equation which is used to establish a Hybrid Verification Theorem.

- *Development of efficient hybrid optimization algorithms along with their proofs of convergence.*
  We propose and analyze a new class of so-called Hybrid Minimum Principle (HMP) algorithms for the solution of hybrid optimal control problems. We provide convergence results for the optimization algorithms in the case of hybrid systems with both autonomous switchings on switching manifolds and controlled switchings.

- *Introduction of optimality zones and development of associated algorithms.*
  We define the notion of optimality zones and investigate some of their topological properties. Based on the optimality zones construction we develop optimization algorithms whose complexity is linear in the number of locations N at the cost of an initial computational investment.

N.B. Almost all of the work above appears in articles which have been published or submitted for publication; see page x. In particular, Chapters 2 and 3, and Chapters 4 and 5, respectively, correspond to the submitted papers S1 and S2.

Published and Submitted Articles

[P1] M. S. Shaikh and P. E. Caines. Suboptimal control of automotive powertrain systems via hybrid partition machines. In *Abstracts Volume, Optimization Days*. Montreal, Canada, May 2000.

[P2] M. S. Shaikh and P. E. Caines. An application of hierarchical hybrid control theory to automotive powertrain systems. In *Proc. Chinese Control Conference*, pages 33–37, Hong Kong, December 2000.

[P3] M. S. Shaikh and P. E. Caines. On trajectory optimization for hybrid systems: Theory and algorithms for fixed schedules. In *Proc. 41st IEEE Int. Conf. Decision and Control*, pages 1997–1998, Las Vegas, NV, 2002.

[P4] M. S. Shaikh and P. E. Caines. On the hybrid dynamic programming principle. In *Proc. IEEE Int. Multitopic Conf.*, Karachi, Pakistan, December 2002.

[P5] M. S. Shaikh and P. E. Caines. On the optimal control of hybrid systems: Optimization of trajectories, switching times, and location schedules. In O. Maler and A. Pnueli, editors, *Proc. sixth international workshop, Hybrid Systems: Computation and Control, LNCS 2623*, pages 466–481, Berlin, Germany, 2003. Springer-Verlag.

[P6] M. S. Shaikh and P. E. Caines. On the optimal control of hybrid systems: Optimization of switching times and combinatoric location schedules. In *Proc. American Control Conference*, pages 2773–2778, Denver, CO, 2003.

[P7] M. S. Shaikh and P. E. Caines. On the optimal control of hybrid systems: Analysis and algorithms for trajectory and schedule optimization. In *Proc. 42nd IEEE Int. Conf. Decision and Control*, pages 2144–2149, Maui, Hawaii, 2003.

[S1] M. S. Shaikh and P. E. Caines. On the hybrid optimal control problem: The hybrid minimum principle and hybrid dynamic programming. IEEE Trans. Automat. Contr., 2004. submitted.

[S2] M. S. Shaikh and P. E. Caines. Hybrid optimization algorithms: Convergence, combinatoric search and optimality zones. IEEE Trans. Automat. Contr., 2004. submitted.

Acknowledgments

It gives me great pleasure to acknowledge a debt of gratitude to my thesis supervisor Professor Peter E. Caines. His expert guidance, depth of knowledge, keen insight and patience were very valuable throughout my doctoral studies.

I must also thank my first PhD supervisor at McGill, Professor Pierre R. Bélanger; without his initial offer of research assistantship it would not have been possible for me to attend McGill.

I would like to express my gratitude to the following past and present members of the systems and control group for their encouragement and support: Professors Benoit Boulet, Charalambos D. Charalambous and Hannah Michalska.

Fellow students who made my stay at CIM a pleasurable experience are too numerous to name but Byung No Min, Miguel Torres-Torriti, Iakov Romanovski, Nabil Aouf, Minyi Huang, Ammar Haurani, Dany Dionne and Guy Gauthier must be singled out. They were a constant source of intellectual and moral support.

Following past and present staff helped provide an excellent research environment at the Centre for Intelligent Machines (CIM): Cynthia E. Davidson, Daniel "Danny" Chouinard, Kathleen VanderNoot, Jan Binder and Marlene Gray. Jerome Pasquero was kind enough to correct my French translation of the abstract.

The financial support of National Science and Engineering Research Council (NSERC grant number ERC 1329-00) is gratefully acknowledged.

Finally, thanks to my parents, and brothers and sister for their love and support and thanks to my wife, Almas, for everything.

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CHAPTER 1

Introduction

Over the last few years, the notion of a hybrid control system with continuous and discrete states and dynamics has crystallized and classes of optimal control problems for such systems have been formalized (see for example [**48, 47, 44, 45, 46, 43, 7, 6, 5, 13, 14, 24, 25, 27, 28, 33, 32, 36, 37, 30, 29, 53, 55, 40, 58, 59, 60, 20, 62**]). In particular, Sussmann [**55**] and Riedinger et al. [**40**], among other authors, have given versions of the Hybrid Minimum Principle (HMP) with indications of proof methods. The Hybrid Minimum Principle constitutes a set of necessary conditions for optimality; these conditions give information about the behaviour of the Hamiltonian and the adjoint trajectories at the switching time and are referred to in the classical literature as "jump conditions".

Jump conditions have been studied at least since the time of Weierstrass and in the calculus of variations they are known as Weierstrass-Erdmann corner conditions [**31**], corners being the points of non-differentiability of extremals. In the context of optimal control theory, similar conditions arose in the study of problems with bounded state values. These problems were studied, for example, by Pontryagin et al. [**38**] and Berkovitz [**10**]. The necessary conditions given in [**38**, pp. 311] and [**10**] at the state and time instant where an optimal trajectory passes from the interior to the boundary of a state constraint set, are similar to those derived in this thesis in the case of autonomous switchings. In [**38**], Pontryagin et al. also give necessary conditions satisfied by the adjoint trajectory at an autonomous switching time. They use the terms "junction point" and "junction time" for switching state and switching time respectively. Witsenhausen [**56**] also gives the HMP necessary conditions and outlines a proof of these conditions. He framework is close in spirit to the modern hybrid control systems literature and his proof uses a system of needle variations and geometric arguments similar to those in employed in [**38**]. Bryson and Ho [**16**] obtain jump conditions using classical variational methods; they mention that a controllability condition is required for the derivation but do not specify the exact nature of the controllability. Recently, Xu and Antsaklis [**57, 60**] have derived the jump conditions using the standard calculus of variations techniques without the controllability hypothesis.

Several authors have proposed modeling frameworks for hybrid control systems. In [**13**] Branicky et al. proposed a unified framework for modeling hybrid control systems with controlled and autonomous vector field switchings as well as controlled and autonomous impulses, impulse being a discontinuity (jump) in the continuous state. Their model subsumes many of the previously proposed hybrid system models. In Chapter 2 we first present a general hybrid system model which is similar to but less general than the Branicky's model in the sense that we take the continuous state to be continuous at switching times. We then give the assumptions which allow us to establish the conditions for the existence and uniqueness of hybrid execution. A related results on the existence and uniqueness of hybrid execution for hybrid automata is given in [**34**].

We then formulate a class of optimal control problems for general hybrid systems with nonlinear dynamics in each location and with autonomous and controlled switchings and present a set of necessary conditions for hybrid system optimality in two cases: (i) where the optimal control takes values in a compact control value set, and (ii) where the control value set is an open bounded set. The proofs of the necessary conditions found in the literature are so complicated that they are difficult to verify and use. The accessible proofs in this thesis use recent needle variation techniques and classical variational methods. We give the precise controllability condition, called the small time tubular fountain (STTF) property, that is required to establish the adjoint and Hamiltonian jump conditions in the autonomous switching case. We follow the approach of de la Barrière [**22**] and Zabczyk [**61**] who obtained the necessary conditions of the MP using a single needle variation rather than a complicated system of variations and geometric arguments as in [**38**]. It is seen that the conditions involving controlled switchings can be obtained using mild assumptions whereas those involving autonomous switchings require stronger assumptions.

Since the Minimum Principle is a collection of necessary conditions for optimality, a control function obtained by using this principle is not necessarily optimal. In the standard optimal control theory one uses the Minimum Principle to obtain a candidate control function and then one tests (or verifies) the optimality of this candidate using verification theorem. In Chapter 3 we extend this approach of verification of candidate optimal controls to hybrid systems by presenting a hybrid Dynamic Programming Principle (HDPP). HDPP is a generalization of the standard Dynamic Programming Principle for differentiable control systems [**4, 61, 52**] and leads to the hybrid Hamilton-Jacobi-Bellman (HJB) equation which is a generalization of the standard HJB equation. The hybrid HJB equation is then used to establish a verification theorem for hybrid systems. In this context, Branicky et al. [**13**] have generalized the *quasi-variational inequalities* of impulse control framework [**8**] to the hybrid systems case. They also give a verification theorem for optimal control problems with discounted cost on semi-infinite intervals. Hedlund and Rantzer [**29, 30**] have applied the Dynamic Programming Principle to a class of hybrid systems to obtain a "hybrid Bellman inequality" which gives a lower bound on the value function. This inequality is then used, via discretization and convex optimization, to compute and approximate feedback control laws.

In Chapter 4 we employ the necessary conditions developed in Chapter 2 to propose and analyze a new class of so-called Hybrid Minimum Principle (HMP) algorithms for the solution of optimal control problems. We provide convergence results for an optimization algorithm (denoted HMP[MAS]) with multiple autonomous switchings (MAS) on switching manifolds. Our convergence proofs are based on results from the theory of the penalty function methods [**3**] and Ekeland's variational principle [**26**]. We then present an algorithm for the case of multiple controlled switching (MCS) times (denoted HMP[MCS]) which invokes (i.e. calls) HMP[MAS] and computes optimal switching times for a given location sequence. The efficacy of these algorithms is illustrated via several computational examples.

The HMP algorithms class is then embedded in the so-called HMP[Comb] (see [**45, 46**]) algorithms class; this extends the HMP class with combinatoric search algorithms which find (combinatorially local) optimal location sequences and their associated locally optimal switching times and control inputs. HMP[Comb] algorithms generate a list of Hamming distance ($\leq k$) sequences from an initial sequence, execute the multiple controlled switchings algorithm (HMP[MCS]; see [**44, 45, 46**]) on each one of them, and finds the best locally k-optimal sequence from among them. We note that other recently proposed hybrid optimization algorithms, for example those of [**58, 60, 24, 25, 54**], can be extended to perform combinatorially k-optimal sequence from among them. We note that other recently proposed hybrid optimization algorithms, for example those of [**58, 60, 24, 25, 54**], can be extended to perform combinatoric search, with an associated exponential increase in the computational cost.

The computational examples of Chapter 2, in addition to illustrating the efficacy of Algorithm HMP[Comb], also serve to show that a global optimization of location sequences and the associated HOCPs will be overwhelmed by the combinatorial complexity engendered by even moderate problem sizes. In Chapter 5 we introduce a new notion of *optimality zones*, give their precise definition and present some of their topological properties. The properties of optimality zones lead to a new class of algorithms (called HMP[Z] algorithms) whose complexity is linear in the number of locations N at the cost of an initial computational investment. In particular, we give examples for the case of linear dynamics with quadratic cost criteria where these zones have a geometrically simple form. The notion of optimality zones must be distinguished from the so-called "switching regions" presented in [**27, 28, 7, 6, 5**]. The switching regions partition the continuous state space of the hybrid system where as the optimality zones partition the switching time-state space. The optimal control problem considered in those papers have autonomous linear time-invariant dynamics, quadratic loss function in each location and a fixed sequence of finite number of location. We consider non-linear dynamics and the general hybrid optimal control problem (HOCP) described in Chapter 2.

Finally, suggestions for possible future research, related to the topics treated in this thesis, are given in Chapter 6.

CHAPTER 2

Necessary Conditions for Hybrid Optimality

2.1. Optimal Control of Hybrid Systems

2.1.1. Hybrid System. Within the standard overall framework (see e.g. [**13**], [**15**]) we define a hybrid control system as:

DEFINITION 2.1. *A hybrid control system* $\mathbb{H}$ *is a 5-tuple*

$$\mathbb{H} \triangleq \{H \triangleq Q \times \mathbb{R}^n, I \triangleq \Sigma \times U, F, \Gamma, \mathcal{M}\}, \tag{2.1}$$

where the symbols in the expression above are defined below. □

2.1.2. Standing Assumptions (A0–A3).

A0 $Q = \{1, 2, \ldots, |Q|\}$ is a finite set of *discrete states* (called *control locations*);

H is the *(hybrid) state space* of $\mathbb{H}$;

$U \subset \mathbb{R}^u$ is the set of *admissible input control values* where U is an open (respectively compact) set in $\mathbb{R}^u$. The set of *admissible input control functions* is either $\mathcal{U}^0 \triangleq \mathcal{U}(\overset{o}{U}, L_\infty([0, T_*))$, (respectively $\mathcal{U}^{\text{cpt}} \triangleq \mathcal{U}(U^{\text{cpt}}, L_\infty([0, T_*)))$, the set of all bounded measurable functions on some interval $[0, T_*)$, $T_* \leq \infty$, taking values in the open set $\overset{o}{U}$ (respectively the compact set U^{cpt});

$\Sigma = \Sigma_u \dot{\cup} \Sigma_c \dot{\cup} \{id\}$ is a finite set of distinct autonomous (uncontrolled) and controlled *transition labels* extended with the identity element $\{id\}$;

$I \triangleq \Sigma \times U$ is the *set of system input values*;

F is an indexed collection of *vector fields* $\{f_j\}_{j \in Q}$, such that $f_j : \mathbb{R}^n \times U \to \mathbb{R}^n$ is a vector field assigned to each control location and $f_j \in C^k(\mathbb{R}^n \times U; \mathbb{R}^n), k \geq 1$, and such that a uniform Lipschitz condition holds, i.e. there exists $L_f < \infty$ such that $\|f_j(x_1, u) - f_j(x_2, u)\| \leq L_f \|x_1 - x_2\|$, $x_1, x_2 \in \mathbb{R}^n$, $u \in U$, $j \in Q$;

$\Gamma : H \times \Sigma \to Q$ is a time independent (partially defined) *discrete transition map*;

$\mathcal{M} = \{\tilde{m}^k_\alpha : \alpha \in Q \times Q, k \in \mathbb{Z}_+\}$ is a collection of *switching manifold subcomponents*, also called *guard subcomponents,* such that, for the ordered pair $\alpha = (p, q)$, $\tilde{m}^k_\alpha$ is a smooth codimension 1 submanifold of $\mathbb{R}^n$, possibly with boundary $\partial \tilde{m}^k_\alpha$, described locally by $\tilde{m}^k_\alpha = \{x : \tilde{m}^k_\alpha(x) = 0\}$. It is assumed that $\tilde{m}^k_\alpha \cap \tilde{m}^l_\beta = \emptyset$, for all $\alpha, \beta \in Q \times Q$, $\alpha \neq \beta$, $k, l \in \mathbb{Z}_+$, except in those cases where, for all k, l, α is identified with its reverse ordered version β giving $\tilde{m}^k_\alpha = \tilde{m}^k_\beta$.

A *switching manifold (component)* $m^k_{p,q} = \bigcup_{k_i; 1 \leq i \leq n(k)} \tilde{m}^{k_i}_{p,q}$ is the union of subcomponents $\tilde{m}^{k_i}_{p,q}$, possibly with boundary, where,

(i) $\tilde{m}^{k_i}_{p,q}$ is a manifold (i.e. guard) subcomponent (as defined above).

(ii) $x \in \tilde{m}^k_{p,q}$ is such that $x \in \tilde{m}^{k_i}_{p,q} \bigcap \tilde{m}^{k_j}_{p,q}$, $k_i \neq k_j$, if and only if $x \in \partial \tilde{m}^{k_i}_{p,q} \bigcap \partial \tilde{m}^{k_j}_{p,q}$, $k_i \neq k_j$.

(iii) If $\partial \tilde{m}^{k_i}_{p,q} \bigcap \partial \tilde{m}^{k_j}_{p,q} \neq \emptyset$ then $\partial \tilde{m}^{k_i}_{p,q} \bigcap \partial \tilde{m}^{k_j}_{p,q}$ is a piecewise C^1 codimension 2 submanifold of $\mathbb{R}^n$ (possibly with boundary).

A1 The initial state $(h_0 \triangleq x(t_0), q_0) \in H$ is such that $\tilde{m}^k_{q_0 q_j}(x_0) \neq 0$, for all $k \in \mathbb{Z}_+$, $q_j \in Q$.

Remark: Switching manifolds will function as follows: whenever a trajectory governed by the controlled vector field f_p (respectively f_q) meets the guard $\tilde{m}_{p,q}$

transversally there is an autonomous switching to the controlled vector field f_q (respectively f_p) as described in Definition 2.4 below. We note that in our formulation switching manifolds do not depend upon time since the proof techniques we employ only permit the derivation of the Hamiltonian continuity property at time independent manifolds for (i) controlled switchings in Theorem 2.2, and (ii) controlled and autonomous switchings in Theorem 2.3. In particular, the Hamiltonian discontinuity property for autonomous switchings at time dependent manifolds ($H(t_j+) = H(t_j-) + p_j \nabla_t m_{j,j+1}|_{t=t_j}$) found in [**16, 40**] is not derived here; this property is assumed in Chapter 4 on hybrid system optimization algorithms.

Remark:

(i) We note that the autonomous and controlled discrete dynamics (2.3) and (2.4) with inputs σ_{ij}, $1 \leq i, j \leq |Q|$, consist of necessarily discontinuous (in time) transitions of the discrete state component $q(t) \in Q$, $t \in [t_0, T]$; moreover, for controlled transitions, for *any* pair q_i, q_j, $1 \leq i, j \leq |Q|$, there exists σ_{ij} such that $\Gamma(q_i, \sigma_{ij}) = q_j$.

(ii) We note that no transition of the continuous state component $x(\cdot)$ of the hybrid state $h(\cdot) = (x(\cdot), q(\cdot))$ occurs at the instant of a discrete state transition; furthermore, the hybrid system axioms (A0–A3) imply that the trajectory $x(\cdot)$ of the continuous state component of a hybrid state execution (i.e. trajectory) is continuous in t for all $t \in [t_0, T]$.

DEFINITION 2.2. *A* hybrid (system) time trajectory *is a strictly increasing (finite or infinite) sequence of times* $\tau = (t_0, t_1, t_2, \dots)$ *or equivalently, a sequence of non-empty half open intervals* $\tau = ([t_0, t_1), [t_1, t_2), \dots)$.

A hybrid (system) switching (event) sequence *is a (finite or infinite) sequence* $S \triangleq (\tau, \sigma) = ((t_0, \sigma_0), (t_1, \sigma_1), \dots)$ *of pairs of times and discrete input events where* τ *is a hybrid time trajectory,* $\sigma_0 = id$, $\sigma_i \in \Sigma$, $i \geq 1$, *and where* σ *is called a* location schedule.

A hybrid (system) input *is a triple* $I \triangleq (\tau, \sigma, u)$ *defined on a half open interval* $[t_0, T), T \leq \infty$, (τ, σ) *is a hybrid* switching sequence *and* $u \in \mathcal{U}$. □

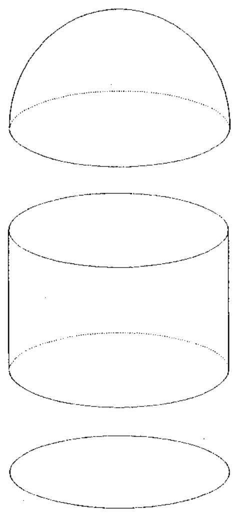

FIGURE 2.1. Switching manifold subcomponents.

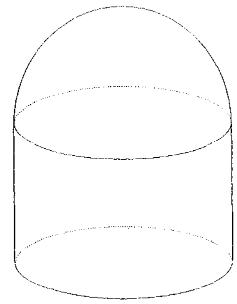

FIGURE 2.2. A switching manifold.

DEFINITION 2.3. *A* hybrid state trajectory *is a triple* (τ, q, x) *consisting of a hybrid time trajectory* $\tau = (t_0, t_1, t_2, \dots)$, *an associated sequence of discrete states* $q = (q_0, q_1, q_2, \dots)$, *and a sequence* $x(\cdot) = (x_{q_0}(\cdot), x_{q_1}(\cdot), x_{q_2}(\cdot), \dots)$ *of absolutely continuous functions* $x_{q_j} : [t_j, t_{j+1}) \longrightarrow \mathbb{R}^n$. □

DEFINITION 2.4. *A* hybrid system execution $e_{\mathbb{H}} = (\tau, \sigma, u, q, x)$ *for the hybrid system* $\mathbb{H}$ *satisfying A0 and A1 is a hybrid input* (τ, σ, u), *together with a hybrid state trajectory* (τ, q, x) *defined over* $[t_0, T)$, $T \leq \infty$, *such that* (τ, σ, u, q, x) *satisfies*

(i) (continuous dynamics)

x is a continuous function satisfying

$$CS: \quad \frac{d}{dt} x_{q_j}(t) = f_{q_j}(x_{q_j}(t), u(t)), \quad a.e.\ t \in [t_j, t_{j+1}),\ x_{q_0}(t_0) = x_0,\ j \in \mathbb{Z}_+, \tag{2.2}$$

i.e. x is an absolutely continuous function satisfying the initial condition, $x(t_0) = x_{q_0}(t_0) = x_0$, *and such that each* $x_{q_j}(\cdot)$ *satisfies*

$$x_{q_j}(t) = x_{q_j}(t_j) + \int_{t_j}^{t} f_{q_j}(x_{q_j}(s), u(s))ds, \quad t \in [t_j, t_{j+1}), \quad j \in \mathbb{Z}_+,$$
$$\lim_{t \uparrow t_{j+1}} x_{q_j}(t) = x_{q_{j+1}}(t_{j+1}), \quad j \in \mathbb{Z}_+.$$

(ii) (autonomous discrete dynamics)
An autonomous discrete transition from q_{j-1} *to* q_j *occurs at the* autonomous switching time t_j, $j \in \mathbb{Z}_1$, *if* $x^*(t_j) \triangleq \lim_{t \uparrow t_j} x_{q_{j-1}}(t)$ *and* t_j *satisfy*

$$DSU: \qquad m_{q_{j-1},q_j}(x^*(t_j)) = 0, \tag{2.3}$$

where $m_{q_{j-1},q_j}(x) = 0$ *defines a* (q_{j-1}, q_j) *switching manifold. Such a transition, denoted by* Γ_u, *corresponds to an element* $\sigma_j \in \Sigma_u$.

(iii) (controlled discrete dynamics)
A controlled discrete transition occurs at the controlled switching time t_j, $j \in \mathbb{Z}_1$, *if* t_j *is not an autonomous switching time and if there exists a discrete control input* $\sigma_j \in \Sigma_c$ *for which*

$$DSC: \quad \Gamma_c(q_{j-1}, \sigma_j(t_j)) \equiv \Gamma_c(q_{j-1}, \sigma_j) = q_j, \quad (t_j, \sigma_j) \in (\tau, \sigma), \quad q_{j-1} \neq q_j. \tag{2.4}$$

□

We note that the hybrid time trajectory as defined in Definition 2.2 may depend upon the continuous state since an autonomous switching time occurs when the continuous state satisfies a switching manifold constraint (2.3).

THEOREM 2.1. *([**17**], see also [**34**]) Given an initial hybrid state* (q_0, x_0) *a hybrid system* $\mathbb{H}$ *satisfying Assumptions A0 and A1 possesses a unique hybrid execution, passing through* (q_0, x_0), *up to the least of:*
(i) $T_* \leq \infty$, *where* $[t_0, T_*)$ *is the temporal domain of the definition of the hybrid system,*

(ii) the instant of a tangential meeting of the continuous trajectory x_{q_i} with a switching manifold subcomponent boundary $\partial\tilde{m}^k_{q_i q_j}$,

(iii) a Zeno time, i.e. an accumulation point of autonomous or controlled switching times.

PROOF. The proof consists in the iterative construction of an execution along the hybrid input trajectory $I = (\tau, \sigma, u)$ over $[t_0, T_*), T_* \geq 0$, through the hybrid initial state.

For simplicity we assume that $T_* = \infty$ and hence exclude a priori the alternative of solutions being undefined due to the absence of defined inputs.

Initial trajectory segment: initial condition

Consider a trajectory starting at the admissible hybrid initial condition (q_0, x_0) at the instant t_0.

By assumption A1, the initial state x_0 lies in the complement of $\tilde{m} = \bigcup \tilde{m}^k_{p,q}$. Since $\tilde{m}$ is relatively closed in $\mathbb{R}^{n+1}$ there is a neighbourhood $N_{(t_0,x_0)} \in \mathbb{R}^{n+1}$ of (t_0, x_0) such that $\tilde{m} \cap N_{(t_0,x_0)} = \emptyset$.

Hence, by continuity, if a solution to the controlled ODE exits it must not intersect any switching manifold component over some non-empty time interval $[t_0, t')$.

Initial trajectory segment: existence and uniqueness

We now utilize the standard results on existence of solutions in the sense of Carathéodory [**21**, pp. 43–49] and the on existence and uniqueness of solutions to ODEs with inputs [**52**, pp. 347–354].

Since $f_{q_0}(x, u)$ is continuously differentiable (and hence continuous) in (x, u) and since $u(\cdot) \in \mathcal{U}$ is measurable with respect to t, $f_{q_0}(x, u(t))$ is measurable in t for fixed x. By Assumption A0, $f_{q_j}(x, u)$, $q_j \in Q$ is globally Lipschitz in x for fixed $u \in U$. These two conditions guarantee the existence of a unique solution to the ODE $\dot{x} = f_{q_0}(x, u)$, through the given hybrid initial condition (q_0, x_{t_0}) over the interval $[t_0, T)$ [**52**, pp. 347–354]. Furthermore, since by the assumptions on vector fields (A0) the solution is bounded on bounded sets in $\mathbb{R}$, either (a) $T = T_* = \infty$, or (b) T is the first controlled or autonomous discontinuous transition event time in (τ, σ, u), i.e. $T = t_1$.

Continuation of execution through t_1, et. seq.

Case (a) above is conclusion (i) of the theorem and hence only case (b) is of interest.

By the definition of controlled switching at t_1 we must have exactly one of either a controlled or an uncontrolled discontinuous transition.

Consider the controlled jump alternative first; in this case the discontinuous transition

$$q_1 = \Gamma_c(q_0, \sigma_1(t_1)), \quad (t_1, \sigma_1) \in (\tau, \sigma),$$

is defined with $q_0 \neq q_1$.

In the second case of an uncontrolled hybrid state jump at t_1, with $x^*_{t_1} \triangleq \lim_{t \uparrow t_1} x_{q_0}(t)$ satisfying $m_{q_0,q_1}(t_1, x^*_{t_1}) = 0$ the same arguments show that the trajectory may be continued from the new hybrid state over a positive half open time interval. The uncontrolled discrete state jump is well defined since by Assumption A0 the switching manifolds are non-intersecting.

Hence in either controlled or uncontrolled discrete state transition a hybrid state $(q_1, x_{q_1}(t_1))$ results such that, together, the hybrid trajectory segment $\tau = (t_0, t_1)$, an associated location sequence $q = (q_0, q_1)$ and a sequence $x = (x_{q_0}, x_{q_1})$ of piecewise C^1 functions of time $x_{q_j} : [t_j, t_{j+1}) \to \mathbb{R}^n$, $j = 0, 1$, all exist and satisfy the definition of a hybrid execution passing through the initial hybrid state.

Evidently the argument on the first and second intervals of the hybrid switching sequence (τ, σ) may be iterated a countable number of times with the only method of termination on the assumed finite intervals $[t_k, t_{k+1})$, $0 \leq k$, being (i) or (iii) specified by the theorem.

Zeno times

If (i) or (ii) do not occur, then there exists a finite upper limit T to the overall period of existence and uniqueness. This case corresponds to the existence of a finite accumulation point of discrete autonomous switching times, in other words a Zeno time at the instant T.

In addition, it may happen that a sequence of controlled jump times $\{t_n\}$ is such that for a Zeno time T: $\lim_{n\to\infty} t_n = T$, $\lim_{n\to\infty} x(t_n) = x^*(T)$ and $m(T, x^*(T)) = 0$, for some $m(\cdot,\cdot)$. In any case, there exists a unique hybrid execution over the interval $[t_0, T)$. □

2.1.3. The Hybrid Optimal Control Problem (HOCP). Let $\{l_j\}_{j\in Q}$, $l_j \in C^k(\mathbb{R}^n \times U; \mathbb{R}_+)$, $k \geq 1$, be a family of *loss functions* satisfying:

A2 There exist $K_l < \infty$ and $1 \leq \gamma < \infty$ such that $|l_j(x,u)| \leq K_l(1 + \|x\|^\gamma)$, $x \in \mathbb{R}^n$, $u \in U$, $j \in Q$.

Let $g \in C^k(\mathbb{R}^n; \mathbb{R}_+)$, $k \geq 1$ denote a terminal cost function satisfying the following assumption.

A3 There exist $K_g < \infty$ and $1 \leq \delta < \infty$ such that $|g(x)| \leq K_g(1 + \|x\|^\delta)$, $x \in \mathbb{R}^n$.

Consider the initial time t_0, final time $t_f < \infty$, initial hybrid state $h_0 = (q_0, x_0)$, and $\bar{L} \leq \infty$. Let

$$S_L = ((t_0, \sigma_0), (t_1, \sigma_1), \ldots, (t_L, \sigma_L)),$$

or equivalently

$$S_L = ((t_0, q_0), (t_1, q_1), \ldots, (t_L, q_L)),$$

be a hybrid switching sequence and let $I_{L(\bar{L})} \triangleq (S_L, u)$, $u \in \mathcal{U}$, where $\mathcal{U} = \mathcal{U}^0$ or $\mathcal{U} = \mathcal{U}^{\text{cpt}}$, $\bar{L} \leq \infty$, be a hybrid input trajectory which subject to A0 and A1 results in a (necessarily unique) hybrid execution and is such that $L \leq \bar{L}$ controlled and autonomous switchings occur on the time interval $[t_0, T(I_L)]$, where $T(I_L) \geq t_f$. Further let the collection of such inputs be denoted $\{I_{L(\bar{L})}\}$. We define the hybrid cost function as:

$$J: \qquad J(t_0, t_f, h_0; I_L, \bar{L}, \mathcal{U}) \triangleq \sum_{i=0}^{L(\bar{L})} \int_{t_i}^{t_{i+1}} l_{q_i}(x_{q_i}(s), u(s))\, ds + g(x_{q_L}(t_f)), \tag{2.5}$$

where

$$\dot{x}_{q_i}(t) = f_{q_i}(x_{q_i}(t), u(t)), \quad a.e.\, t \in [t_i, t_{i+1}), \quad i = 0, 1, \ldots, L,$$
$$u(t) \in U \subset \mathbb{R}^n, \text{ where in case } \mathcal{U} = \mathcal{U}^{\mathrm{cpt}}, \quad U = \overset{\circ}{\bar{U}},$$
$$u(\cdot) \in L_\infty(U),$$
$$h_0 = (q_0, x_{q_0}(t_0)) = (q_0, x_0),$$
$$x_{q_{i+1}}(t_{i+1}) = \lim_{t \uparrow t_{i+1}} x_{q_i}(t), \quad i = 0, 1, \ldots, L, \text{ and}$$
$$t_{L+1} = t_f < \infty, \qquad L \leq \bar{L} \leq \infty.$$

DEFINITION 2.5. *(Hybrid Optimal Control Problem (HOCP)) Given the system* $\mathbb{H}$ *with continuous dynamics (CS) and discrete dynamics (DSU, DSC), loss functions* $\{l_q, q \in Q\}$, *initial and final times,* t_0, t_f, *the initial hybrid state* $h_0 = (q_0, x_0)$, *and an upper bound on the number of switchings* $\bar{L} \leq \infty$, *the* hybrid optimal control problem *(HOCP($t_0, t_f, x_0, \bar{L}, \mathcal{U}$)), where* $\mathcal{U} = \mathcal{U}^0$ *or* $\mathcal{U} = \mathcal{U}^{\mathrm{cpt}}$, *is to find the infimum* $J^0(t_0, t_f, h_0, \bar{L}, \mathcal{U})$ *of the hybrid cost function* $J(t_0, t_f, h_0; I_L, \bar{L}, \mathcal{U})$ *over the family of input trajectories* $\{I_{L(\bar{L})}\}$. *If a hybrid input trajectory* $I_{L^0(\bar{L})}$ *exists which realizes* $J^0(t_0, t_f, h_0, \bar{L}, \mathcal{U})$ *it is called a* hybrid optimal control *for the HOCP($t_0, t_f, x_0, \bar{L}, \mathcal{U}$).*
□

The HOCP can have the usual variations of fixed or free initial or terminal state, fixed or free initial or terminal location, free terminal time etc. We adopt the notation HOCP(X) to indicate a HOCP where X is the given data.

DEFINITION 2.6. *([**18**], Small Time Tubular Fountain (STTF) Condition) A continuous state* $x(t) \in \mathbb{R}^n$, $t \in [t_0, t_f)$, $x(t) \in \phi(u^\phi) \triangleq \{x(s, x_0, u^\phi_{[t_0,s]}); t_0 \leq s \leq t_f\}$, *is said to be a* positive small time tubular fountain with respect to the trajectory $\phi(u^\phi)$ *of the system (CS) if for all sufficiently small* $\gamma > 0$, $\gamma' > 0$ *and* $\gamma'' > 0$ *there exists* $t < t^* < t_f$, *such that for all* t', $t < t' < t^*$, *there exist* $0 < \sigma(t, t', \gamma, \gamma', \gamma'') < \gamma$ *and* $0 < \epsilon(t, t', \gamma, \gamma', \gamma'') < \gamma''$ *such that: (i) for all* $\tau \in (t' - \epsilon, t' + \epsilon)$, *and (ii) for all* $z \in B_\sigma(x(t'))$, *where* $x(t') = \phi(t', x(t), u^\phi_{[t,t']})$, *there exists* $u_{[t,\tau]}(\cdot) \in \mathcal{U}$ *such that* *(a)* $\sup_{s\in[t,\tau]} \|u(s) - u^{\phi}(s)\| < \gamma'$;

(b) $\|\phi(s, x(t), u_{[t,s]}) - \phi(s, x(t), u^{\phi}_{[t,s]})\| < \gamma, \quad s \in [t, \tau]$;

(c) $\phi(\tau, x(t), u_{[t,\tau]}) = z \in B_{\sigma}(x(t'))$; *and*

(d) σ *and* ϵ *are continuous in their arguments* t, t', γ, γ', *and* γ''.

Furthermore, a continuous state $x(t) \in \mathbb{R}^n$, $t \in (t_0, t_f]$, $x(t) \in \phi(u^{\phi})$, $\phi(u^{\phi}) \triangleq \{x(s, x_0, u^{\phi}_{[t_0,s]}); t_0 \leq s \leq t_f\}$ *is said to be a* negative small time tubular fountain with respect to the trajectory $\phi_x(u^{\phi})$ *of the system (CS) if for all sufficiently small* $\gamma > 0$, $\gamma' > 0$ *and* $\gamma'' > 0$ *there exists* $t_0 < t_* < t$ *such that for all* t'', $t_* < t'' < t$, *there exist* $0 < \sigma(t, t'', \gamma, \gamma', \gamma'') < \gamma$ *and* $0 < \epsilon(t, t'', \gamma, \gamma', \gamma'') < \gamma''$ *such that: (i) for all* $\tau \in (t'' - \epsilon, t'' + \epsilon) \subset (t_0, t_f)$, *and (ii) for all* $z \in B_{\sigma}(x(t''))$, $x(t) = \phi(t, x(t''), u^{\phi}_{[t'',t]})$, *there exists* $u_{[\tau,t]}(\cdot) \in \mathcal{U}$ *such that*

(a′) $\sup_{s\in[\tau,t]} \|u(s) - u^{\phi}(s)\| < \gamma'$;

(b′) $\|\phi(s, x(\tau), u_{[\tau,s]}) - \phi(s, x(\tau), u^{\phi}_{[\tau,s]})\| < \gamma, \quad s \in [\tau, t]$;

(c′) $\phi(t, z, u_{[\tau,t]}) = x(t)$; *and*

(d′) σ *and* ϵ *are continuous in their arguments* t, t'', γ, γ' *and* γ''.

A continuous state x *which is negative and positive small time tubular fountain with respect to a trajectory is called a* small time tubular fountain (STTF) with respect to that trajectory; *if all the states on a trajectory* ϕ *are STTFs with respect to* ϕ *we say that the STTF condition is satisfied on* ϕ. □

We observe that for the negative STTF condition one considers the set of points coaccessible to the state $x(t)$ as opposed to the positive STTF condition where the one is interested in the set of points accessible from $x(t)$. We also observe that using the standard techniques (employed for instance in [**18**]) we may verify that the states on a trajectory ϕ which has a uniformly controllable linearization along any segment of ϕ satisfies the STTF condition there.

2.2. Controls in Compact Value Sets

In this section we establish Theorem 2.2, the first of the two HMP results of this chapter. In Chapter 4 we use these results to devise a set of conceptual algorithms for computing the switching times, states and the associated control functions which satisfy (at least) these necessary conditions in a fixed location sequence.

The proof technique follows an approach first used by de la Barrière in [**22**] (see also [**61**]) where it was shown that when the control values are allowed to lie in a compact set, the Hamiltonian maximization condition of the maximum principle can be obtained by using single needle variations. This methodology is more suitable for the HOCP in the Mayer formulation, i.e. with only a terminal cost. This involves no loss of generality since it is well known that under Assumptions A0, A2, A3 the Bolza problem is equivalent to the Mayer problem in the sense that (i) a problem in Bolza form can be transformed into the Mayer form and vice-versa, and (ii) if a solution of problem in one form exists then a solution of the transformed problem also exists [**11**]. Using the usual state augmentation technique this equivalence can be shown to exist between the hybrid Bolza and the hybrid Mayer problems as well.

The statement and the proof in the case of open control value sets is given in the second form in Theorem 2.3 below. A shortcoming of the open value set condition is that the optimal control is required to take values in the interior of the value set. This rules out, for example, "bang-bang" optimal controls taking values on the boundary of the value set. Furthermore, the small time tubular fountain condition of Theorem 2.3 is not required for Theorem 2.2. However, the stronger set of conditions permits us to obtain, in addition to the results of Theorem 2.2, the Hamiltonian continuity conditions at autonomous switching times; beyond their intrinsic interest, these are crucial for the computational algorithms of Chapter 4.

We say that a continuous state trajectory $x(\cdot)$ resulting from a continuous control $u(\cdot)$ meets a switching manifold subcomponent $\tilde{m}_{p,q}$ transversally at $x^* = x(t^*)$ if (taking limit through Lebesgue points) $\lim_{t\uparrow t^*} f_p(x(t), u(t))$ is transversal to the tangent space $T\tilde{m}_{p,q}$ and $\lim_{t\downarrow t^*} f_q(x(t), u(t))$ is transversal to the tangent space $T\tilde{m}_{p,q}$.

THEOREM 2.2. *Consider a hybrid system* $\mathbb{H}$ *satisfying Assumptions A0, A1, and the*

HOCP$(t_0, t_f, x_0, \bar{L}, \mathcal{U}^{\mathrm{cpt}})$ *satisfying A3, and define*

$$H_q(x,\lambda,u) = \lambda^T f_q(x,u), \quad x, \lambda \in \mathbb{R}^n,\ u \in \overset{o}{U},\ q \in Q.$$

1) Let $J^0(t_0, t_f, h_0, \mathcal{U}^{\mathrm{cpt}}) = \inf_{\{I_{L(\bar{L})}\}} J^0(t_0, t_f, h_0, I_L, \bar{L}, \mathcal{U}^{\mathrm{cpt}})$ *be realized at a minimizing control and trajectory* $I^0_{L^0(\bar{L})}$, (x^0, q^0).

2) Let $I^0_{L^0(\bar{L})}$ *have* L_c *controlled switchings and* L_a *autonomous switchings, and let* $L_a + L_c = L^0(\bar{L})$.

3) Let $t_1,\ t_2,\ \ldots,\ t_{L^0}$, *denote the switching times along the optimal trajectory* (x^0, q^0).

4) Assume that x^0 *meets* $\tilde{m} = \bigcup \tilde{m}^k_{p,q}$ *transversally and does not meet* $\partial \tilde{m}^{k_i}_{p,q} \bigcap \partial \tilde{m}^{k_j}_{p,q}$ *for any* k_i, k_j, p, q.

5) Assume that either (a) $\bar{L} < \infty$ *and* $L^0(\bar{L}) = L_a + L_c + 2 \le \bar{L}$, *or (b)* $\bar{L} = \infty$ *and* $L^0(\bar{L}) < \infty$.

Then

(i) There exists a (continuous to the right), piecewise absolutely continuous adjoint process λ^0 *satisfying*

$$\dot{\lambda}^0 = -\frac{\partial H_{q(j)}}{\partial x}(x^0, \lambda^0, u^0), \quad \text{a.e. } t \in (t_j, t_{j+1}), \quad j \in \{0, 1, 2, \ldots, L^0\}, \tag{2.6}$$

where $t_{L^0+1} = t_f$ *and where the following boundary value conditions hold with* $\lambda^0(t_0)$ *free:*

(a) $\lambda^0(t_f) = \nabla_x g(x^0(t_f))$.

(b) If t_j *is a controlled switching time, then*

$$\lambda^0(t_j-) \equiv \lambda^0(t_j) = \lambda^0(t_j+), \quad j \in \{0, 1, 2, \ldots, L^0\}. \tag{2.7}$$

(c) If t_j is an autonomous switching time satisfying $m_{j,j+1}(x(t_j)) = 0$, then

$$\lambda^0(t_j-) \equiv \lambda^0(t_j) = \lambda^0(t_j+) + p_j \nabla_x m_{j,j+1}\big|_{t=t_j}, \qquad p_j \in \mathbb{R}. \tag{2.8}$$

(ii) The Hamiltonian minimization conditions are satisfied, i.e.

(a)

$$H_{q(j)}(x^0(t), \lambda^0(t), u^0(t)) \leq H_{q(j)}(x^0(t), \lambda^0(t), v), \tag{2.9}$$
$$\text{a.e. } t \in [t_j, t_{j+1}), \quad \forall v \in U, \quad j \in \{0, 1, 2, \ldots, L^0\}.$$

(b1) If $\bar{L} < \infty$ and $L^0(\bar{L}) = L_a^0 + L_c^0 + 2 \leq \bar{L}$, then

$$H_{q(j)}(x^0(t), \lambda^0(t), u^0(t)) \leq H_k(x^0(t), \lambda^0(t), u^0(t)), \tag{2.10}$$
$$\text{a.e. } t \in [t_j, t_{j+1}), \quad j \in \{0, 1, 2, \ldots, L^0\}, \quad \forall k \in Q.$$

(b2) If $\bar{L} = \infty$ and $L^0(\bar{L}) < \infty$, then

$$H_{q(j)}(x^0(t), \lambda^0(t), u^0(t)) \leq H_k(x^0(t), \lambda^0(t), u^0(t)), \tag{2.11}$$
$$\text{a.e. } t \in [t_j, t_{j+1}), \quad j \in \{0, 1, 2, \ldots, L^0\}, \quad \forall k \in Q.$$

(iii) If t_j is a controlled switching time then the following Hamiltonian continuity *condition holds at $t = t_j$*

$$H(t_j-) \equiv H_{q(j-1)}(t_j-) = H_{q(j-1)}(t_j) = H_{q(j)}(t_j) = H_{q(j)}(t_j+) \equiv H(t_j+),$$
$$j \in \{1, 2, \ldots, L^0\}.$$

□

Remark: In order to prove the Hamiltonian minimization w.r.t. the discrete location we will need to perform "Q-needle variation" where the systems switches to location $k \in Q$ at time $t - \epsilon_i$ and switches back to location j at the Lebesgue point t

such that $[t-\epsilon_i, t] \subset (t_{j-1}, t_j)$. This is possible in case (ii)(b1) since, by assumption, $L^0 + 2 \leq \bar{L}$, and it clearly holds in (ii)(b2) with $\bar{L} = \infty$.

PROOF. The theorem will be proved in several steps. Let the optimal location sequence have $m = L^0 + 1$ locations labeled $1, 2, \cdots, m$ and let the optimal switching times be $t_1, t_2, \cdots, t_{m-1} = t_{L^0}$. We first do a needle variation in the m-th location to derive the adjoint equation, adjoint boundary condition and Hamiltonian minimization condition in that location. Next we will do a variation in the $m-1$-st location to derive an equation satisfied by λ_{m-1}, to obtain the adjoint transversality condition at the switching time t_{m-1} and to obtain the Hamiltonian minimization condition there. This derivation is then extended to the location j. In the next step we will do variations of the optimal switching times in either direction to obtain the Hamiltonian continuity condition in the controlled switchings case. Finally, the Hamiltonian minimization condition with respect to discrete locations will be established by inserting two controlled switchings in the time period of any existing location.

In the rest of the proof we let $\{\epsilon_i\}_{i=1}^{\infty}$ be a monotonically decreasing sequence of real numbers such that $\epsilon_1 < \infty$, $\epsilon_i > 0$, for all i and $\lim_{i\to\infty} \epsilon_i = 0$.

Step 1: We first derive the Hamiltonian minimization condition in the last location and show the existence of an adjoint process there by deriving a differential equation and boundary condition satisfied by it.

For some $v \in U$ and $(t-\epsilon_1, t) \subset (t_{m-1}, t_f]$ consider a needle variation of the optimal input in the interval $[t_{m-1}, t_f]$,

$$u_i(\tau) = \begin{cases} u^0(\tau) & \text{if} \quad 0 \leq \tau < t - \epsilon_i \\ v & \text{if} \quad t - \epsilon_i \leq \tau < t \\ u^0(\tau) & \text{if} \quad t \leq \tau \leq t_f \end{cases}$$

with the corresponding state response $x_i(\tau), \tau \in [0, t_f]$, The variation and the perturbed trajectories are shown in Figure 2.3.

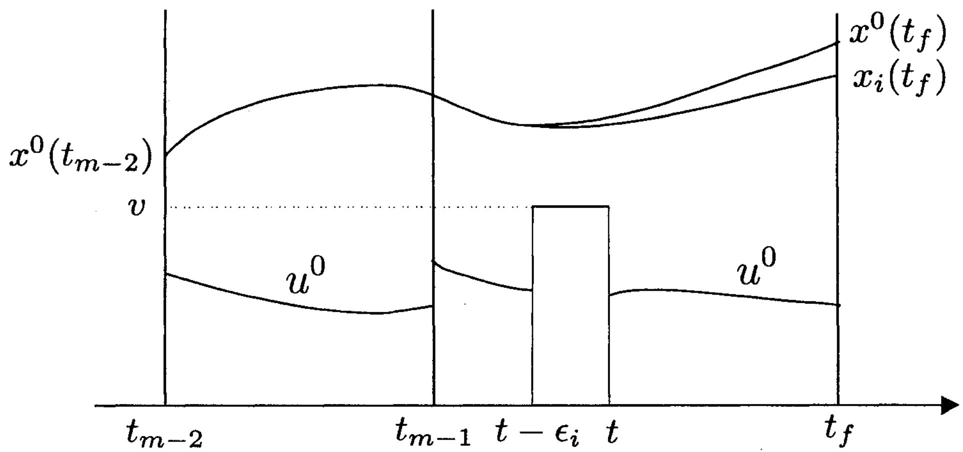

FIGURE 2.3. Variation in u^0 causes variation in x^0.

The difference $\delta x_i(\tau) \triangleq x_i(\tau) - x^0(\tau)$ is caused by the following two types of perturbations:

(i) *Input perturbation:* We note that $\delta u(\tau) \triangleq u_i(\tau) - u^0(\tau) = 0$, $\tau \in [0, t-\epsilon_i) \cup [t, t_f]$, and $\delta x_i(\tau) = 0$, $\tau \in [0, t-\epsilon_i]$; while

$$\begin{aligned}\delta x_i(t) = x_i(t) - x^0(t) &= x_i(t-\epsilon_i) + \int_{t-\epsilon_i}^{t} f_m(x_i(\tau), v)\, d\tau \\ &\quad - x^0(t-\epsilon_i) - \int_{t-\epsilon_i}^{t} f_m(x^0(\tau), u^0(\tau))\, d\tau \\ &= \int_{t-\epsilon_i}^{t} [f_m(x_i(\tau), v) - f_m(x^0(\tau), u^0(\tau))]\, d\tau. \qquad (2.12)\end{aligned}$$

Equation (2.12) gives $\delta x_i(t)$ exactly. However, for small ϵ_1, the following approximation to $\delta x_i(\tau)$, $\tau \in [t-\epsilon_i, t)$, holds,

$$\begin{aligned}\frac{d}{d\tau}\delta x_i(\tau) &= \frac{\partial f_m}{\partial x}(x^0(\tau), u^0(\tau))\delta x_i(\tau) \\ &\quad + \frac{\partial f_m}{\partial u}(x^0(\tau), u^0(\tau))\delta u(\tau) + o(\epsilon_i), \quad \tau \in [t-\epsilon_i, t), \qquad (2.13)\\ \delta x_i(t-\epsilon_i) &= 0.\end{aligned}$$

In the rest of the proof $\Phi_j(\tau,\tau_0)$ will denote the the state transition matrix corresponding to the system
$\frac{d}{d\tau}z(\tau) \;=\; \frac{\partial f_j}{\partial x}(x^0(\tau),u^0(\tau))z(\tau)$, i.e. $\Phi_j(\tau,\tau_0)$ satisfies

$$\frac{d}{d\tau}\Phi_j(\tau,\tau_0) = \frac{\partial f_j}{\partial x}(x^0(\tau),u^0(\tau))\Phi_j(\tau,\tau_0), \quad \Phi_j(\tau_0,\tau_0) = I. \tag{2.14}$$

Hence

$$\delta x_i(\tau) = \int_{t-\epsilon_i}^{\tau} \Phi(\tau,s)\frac{\partial f_m}{\partial u}(x^0(s),u^0(s))\delta u(s)\,ds + o(\epsilon_i), \quad \tau \in [t-\epsilon_i, t). \tag{2.15}$$

(ii) *Initial condition perturbation:* Since $\delta u(\tau) = 0$, $\tau \in [t, t_f)$, a linearization similar to the previous case yields

$$\delta x_i(\tau) = \Phi(\tau,t)\delta x_i(t) + o(\epsilon_i), \quad \tau \in [t, t_f]. \tag{2.16}$$

Now dividing (2.12) by ϵ_i and adding and subtracting $f_m(x^0(\tau),v)$ from the integrand we obtain

$$\begin{aligned}\frac{1}{\epsilon_i}\delta x_i(t) \;&=\; \frac{1}{\epsilon_i}\int_{t-\epsilon_i}^{t} [f_m(x^0(\tau),v) - f_m(x^0(\tau),u^0(\tau))]\,d\tau \\ &\quad +\frac{1}{\epsilon_i}\int_{t-\epsilon_i}^{t} [f_m(x_i(\tau),v) - f_m(x^0(\tau),v]\,d\tau\end{aligned}$$

The following lemma establishes that in the limit as $i \to \infty$ the second term on the right hand side vanishes. Then, since almost every point of a bounded measurable function is a Lebesgue point, see [**41**, pp. 158], we obtain:

$$\lim_{i\to\infty} \frac{1}{\epsilon_i}\delta x_i(t) = f_m(x^0(t),v) - f_m(x^0(t),u^0(t)), \quad \text{a.e. } t \in (t_{m-1}, t_f).$$

LEMMA 2.1. *Under Assumption A0,*

$$\lim_{i\to\infty} \frac{1}{\epsilon_i}\int_{t-\epsilon_i}^{t} [f_m(x_i(\tau),v) - f_m(x^0(\tau),v)]\,d\tau = 0,$$

for $t \in (t_{m-1}, t_f)$ and all $v \in U$. □

PROOF. According to Assumption A0,

$$\|f_m(x_i(\tau),v) - f_m(x^0(\tau),v)\| \le L_f\|x_i(\tau) - x^0(\tau)\|, \quad 0 \le L_f < \infty.$$

Hence we have

$$\begin{aligned}\left\|\int_{t-\epsilon_i}^{t} [f_m(x_i(\tau),v) - f_m(x^0(\tau),v)]\,d\tau\right\| &\le \int_{t-\epsilon_i}^{t} \|f_m(x_i(\tau),v) - f_m(x^0(\tau),v)]\|\,d\tau \\ &\le \int_{t-\epsilon_i}^{t} L_f\|x_i(\tau) - x^0(\tau)\|\,d\tau \\ &= \int_{t-\epsilon_i}^{t} L_f\|\delta x_i(\tau)\|\,d\tau. \end{aligned} \tag{2.17}$$

Let

$$\sup_{\tau\in[t-\epsilon_i,t]} \|\Phi(\tau,s)\| \cdot \sup_{\tau\in[t-\epsilon_i,t]} \left\|\frac{\partial f_m}{\partial u}(x^0(\tau),u^0(\tau))\right\| \cdot \sup_{\tau\in[t-\epsilon_i,t]} \|\delta u(\tau)\| \le K_1K_2K_3 = K_x.$$

Then (2.15) yields

$$\|\delta x_i(\tau)\| \le K_x(\tau - (t_\tau - \epsilon_i)) + o(\epsilon_i), \quad \tau \in [t-\epsilon_i, t].$$

Hence (2.17) gives

$$\begin{aligned}\left\|\int_{t-\epsilon_i}^{t} [f_m(x_i(\tau),v) - f_m(x^0(\tau),v)]\,d\tau\right\| &\le \int_{t-\epsilon_i}^{t} L_fK_x(\tau - (t-\epsilon_i)\,d\tau + o(\epsilon_i) \\ &= \frac{1}{2}L_fK_x\epsilon_i^2 + o(\epsilon_i).\end{aligned}$$

Dividing by ϵ_i and letting $i \to \infty$ the desired result follows from the properties of $\{\epsilon_i\}$. □

Now let $y_i(t) = \frac{1}{\epsilon_i}\delta x_i(t)$ so that $x_i(t) = x^0(t) + \epsilon_i y_i(t)$ and $\lim_{i\to\infty} y_i(t) \triangleq y(t) = f_m(x^0(t),v) - f_m(x^0(t),u^0(t))$.

By (2.16),

$$\delta x_i(t_f) = \Phi_m(t_f,t)\delta x_i(t) + o(\epsilon_i).$$

Dividing by ϵ_i and letting $i \to \infty$ we obtain

$$\begin{aligned}\lim_{i\to\infty} \frac{1}{\epsilon_i}\delta x_i(t_f) &= \Phi_m(t_f, t) \lim_{i\to\infty} \frac{1}{\epsilon_i}\delta x_i(t) \\ &= \Phi_m(t_f, t)y(t) \\ &= \Phi_m(t_f, t)[f_m(x^0(t), v) - f_m(x^0(t), u^0(t))].\end{aligned}$$

Setting $\frac{1}{\epsilon_i}[x_i(t_f) - x^0(t_f)] = y_i(t_f)$ so that $x_i(t_f) = x^0(t_f) + \epsilon_i y_i(t_f)$, we see that

$$y(t_f) \triangleq \lim_{i\to\infty} y_i(t_f) = \Phi_m(t_f, t)[f_m(x^0(t), v) - f_m(x^0(t), u^0(t))]$$

Now since x^0 is an optimal trajectory $g(x_i(t_f)) \geq g(x^0(t_f))$ or equivalently $g(x^0(t_f) + \epsilon_i y_i(t_f)) - g(x^0(t_f)) \geq 0$. Dividing by ϵ_i and passing to the limit we obtain

$$\lim_{i\to\infty} \frac{1}{\epsilon_i}[g(x^0(t_f) + \epsilon_i y_i(t_f)) - g(x^0(t_f))] \geq 0. \tag{2.18}$$

We use the following fact to simplify the above expression. Let $\{b_i\}_{i=1}^{\infty}$ be a sequence such that $b_i \in \mathbb{R}^n$ and $b_i \to b \in \mathbb{R}^n$ as $i \to \infty$. Let $a \in \mathbb{R}^n$ be a fixed vector. Then by the continuous differentiability of g we can define the *directional derivative of g at a in the direction of b* as

$$\lim_{i\to\infty} \frac{1}{\epsilon_i} \left(g(a + \epsilon_i b_i) - g(a)\right) = \left(\nabla_x g(a)\right)^T b.$$

Using this fact (2.18) turns into

$$\begin{aligned}&\left(\nabla_x g(x^0(t_f))\right)^T y(t_f) \\ &\quad = \left(\nabla_x g(x^0(t_f))\right)^T \Phi_m(t_f, t)[f_m(x^0(t), v) - f_m(x^0(t), u^0(t))] \geq 0\end{aligned} \tag{2.19}$$

or

$$\left(\nabla_x g(x^0(t_f))\right)^T \Phi_m(t_f, t) f_m(x^0(t), v) \geq \left(\nabla_x g(x^0(t_f))\right)^T \Phi_m(t_f, t) f_m(x^0(t), u^0(t)). \tag{2.20}$$

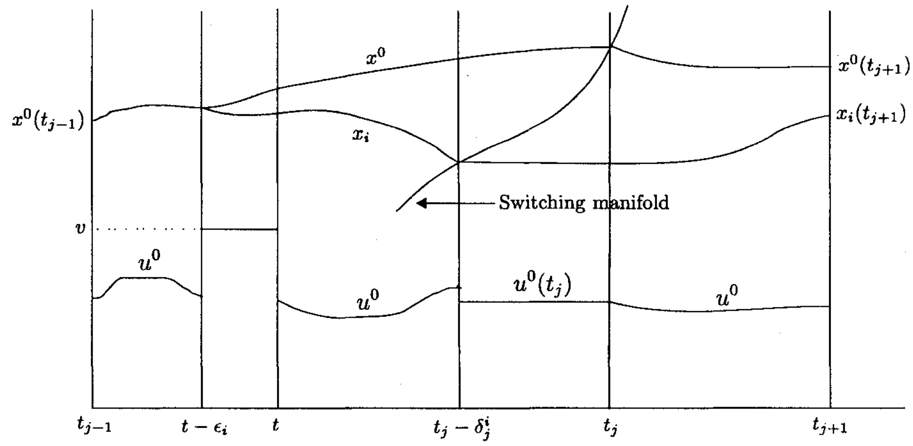

FIGURE 2.4. Autonomous switching case: variation in u^0 causes change in switching time and state.

Setting $\lambda_m^T(t) = \left(\nabla_x g(x^0(t_f))\right)^T \Phi_m(t_f,t)$, $t \in [t_{m-1}, t_f]$ we obtain

$$\begin{aligned}
\lambda_m(t_f) &= \nabla_x g(x^0(t_f)), && (2.21)\\
\dot{\lambda}_m(t) &= -\left(\frac{\partial f_m}{\partial x}(x^0(t))\right)^T \Phi_m^T(t_f,t)\nabla_x g(x^0(t_f)) \\
&= -\left(\frac{\partial f_m}{\partial x}(x^0(t))\right)^T \lambda_m(t). && (2.22)
\end{aligned}$$

Noting that in this case the Hamiltonian $H_m(x,\lambda,u) = \lambda_m^T f_m(x,u)$, the inequality (2.20) is seen to be equivalent to the minimization of the Hamiltonian in location m.

Step 2: The next step is to determine how the variation in location j is propagated to the switching time t_{j+1}. The result obtained here will be used later to derive general expressions for propagation of perturbation in location j to the final time t_f.

As in Step 1 we define a "needle" perturbation in the optimal control in location j. This causes a change in the switching time as the perturbed trajectory does not necessarily intersect the switching manifold at time t_j. Let the control values be held constant at an arbitrary value $v \in U$ over the interval $[t-\epsilon_i, t)$ and at $u^0(t_j)$ over the interval $[t_j - \delta_j^i, t_j)$.

$$u_i(t) = \begin{cases} u^0(\tau) & \text{if} \quad t_{j-1} \le \tau < t - \epsilon_i \\ v & \text{if} \quad t - \epsilon_i \le \tau < t \\ u^0(\tau) & \text{if} \quad t \le \tau < t_j - \delta_j^i \\ u^0(t_j) & \text{if} \quad t_j - \delta_j^i \le \tau < t_j \\ u^0(\tau) & \text{if} \quad t_j \le \tau \le t_{j+1}, \end{cases}$$

The new switching time is $t_{m-1} - \delta_{m-1}^i$ where $\{\delta_{m-1}^i\}_{i=i}^{\infty}$ is a sequence of positive numbers (the case of $\delta_{m-1}^i \le 0$ can be handled similarly).

Since $\delta x_i(t - \epsilon_i) = 0$, we have

$$\delta x_i(t) = \int_{t-\epsilon_i}^{t} \left(f_j(x_i(\tau), v) - f_j(x^0(\tau), u^0(\tau)) \right) d\tau. \tag{2.23}$$

We use the same procedure as in Step 1 to deduce that

$$y(t) \triangleq \lim_{i\to\infty} y_i(t) = \lim_{i\to\infty} \frac{1}{\epsilon_i} \delta x_i(t) = f_j(x^0(t), v) - f_j(x^0(t), u^0(t)).$$

Notice that in this case $\delta u(\tau) = 0$, $\tau \in [t_{j-1}, t - \epsilon_i) \cup [t, t_j - \delta_j^i) \cup [t_j, t_{j+1}]$. Then the following sequence of equations is clear

$$\begin{aligned} \delta x_i(t_j - \delta_j^i) &= \Phi_j(t_j - \delta_j^i, t)\delta x_i(t) + o(\epsilon_i), \\ \delta x_i(t_j) &= \delta x_i(t_j - \delta_j^i) + \int_{t_j-\delta_j^i}^{t_j} [f_{j+1}(x_i(\tau), u^0(t_j)) - f_j(x^0(\tau), u^0(\tau))]\, d\tau, \\ \delta x_i(t_{j+1}) &= \Phi_{j+1}(t_{j+1}, t_j)\Phi_j(t_j - \delta_j^i, t)\delta x_i(t) \\ &\quad + \Phi_{j+1}(t_{j+1}, t_j) \int_{t_j-\delta_j^i}^{t_j} [f_{j+1}(x_i(\tau), u^0(t_j)) - f_j(x^0(\tau), u^0(\tau))]\, d\tau + o(\epsilon_i), \end{aligned}$$

where we recall that Φ_k is the state transition matrix corresponding to the location k as defined in Equation (2.14).

By the definition of the switching manifold, $m(t_j, x^0(t_j)) = 0$ and $m(t_j - \delta_j^i, x_i(t_j - \delta_j^i)) = 0$, but $m(t_j, x_i(t_j))$ does not necessarily vanish unless the perturbed trajectory undergoes an autonomous switch again at time t_j. In either case, there exists $p_j^i \in \mathbb{R}$

such that

$$p_j^i[m(t_j, x_i(t_j)) - m(t_j, x^0(t_j))] = p_j^i m(t_j, x_i(t_j)) \geq 0.$$

or

$$p_j^i m(t_j, x^0(t_j) + \epsilon_i y_i(t_j)) \geq 0$$

Let $\lim_{i\to\infty} p_j^i = p_j \in \mathbb{R}$ (by picking a subsequence, if necessary). Dividing by ϵ_i and letting $i \to \infty$ we obtain

$$p_j \left(\nabla_x m(t_j, x^0(t_j))\right)^T y(t_j-) \geq 0. \tag{2.24}$$

The vectors $y(t_j-)$ and $y(t_{j+1}-)$ are computed next.

$$\begin{aligned}
y(t_j-) &= \lim_{i\to\infty} \frac{1}{\epsilon_i}\delta x_i(t_j) \\
&= \lim_{i\to\infty} \frac{1}{\epsilon_i}\Phi_j(t_j - \delta_j^i, t)\delta x_i(t) \\
&\quad + \lim_{i\to\infty} \frac{1}{\epsilon_i}\int_{t_j-\delta_j^i}^{t_j} [f_{j+1}(x_i(\tau), u^0(t_j)) - f_j(x^0(\tau), u^0(\tau))]\, d\tau \\
&= \Phi_j(t_j, t) \lim_{i\to\infty} \frac{1}{\epsilon_i}\delta x_i(t) \\
&\quad + \lim_{i\to\infty} \frac{1}{\epsilon_i}\int_{t_j-\delta_j^i}^{t_j} [f_{j+1}(x_i(\tau), u^0(t_j)) - f_j(x^0(\tau), u^0(\tau))]\, d\tau \\
&= \Phi_j(t_j, t)y(t) \\
&\quad + \lim_{i\to\infty} \frac{1}{\epsilon_i}\int_{t_j-\delta_j^i}^{t_j} [f_{j+1}(x_i(\tau), u^0(t_j)) - f_j(x^0(\tau), u^0(\tau))]\, d\tau.
\end{aligned}$$

The second term on the RHS above is independent of t. The optimal control $u^0(\cdot)$ is measurable by assumption. But $u^0(t)$, $t \in [t_{j-1}, t_j)$ can be replaced by $\hat{u}^0(t)$, $t \in [t_{j-1}, t_j)$ such that $\hat{u}^0(\cdot)$ is left continuous and $\mu\{t \in [t_{j-1}, t_j) : u^0(t) \neq \hat{u}^0(t)\} = 0$, where μ is Lebesgue measure. This leaves the optimal cost $\int_{t_{j-1}}^{t_j} l_j(x^0(s), u^0(s))\, ds$ unchanged. Assuming that $u^0(\cdot)$ has been replaced by $\hat{u}^0(\cdot)$, if we let $t \to t_j$, we see that since $\Phi_j(t_j, t_j) = I$,

$$\lim_{i\to\infty} \frac{1}{\epsilon_i}\int_{t_j-\delta_j^i}^{t_j} [f_{j+1}(x_i(\tau), u^0(t_j)) - f_j(x^0(\tau), u^0(\tau))]\, d\tau = 0. \tag{2.25}$$

Hence

$$y(t_j-) = \Phi_j(t_j, t)y(t) = \Phi_j(t_j, t)[f_j(x^0(t), v) - f_j(x^0(t), u^0(t))]. \tag{2.26}$$

Similarly we compute $y(t_{j+1}-)$ as follows,

$$\begin{aligned} y(t_{j+1}-) &= \lim_{i\to\infty} \frac{1}{\epsilon_i}\delta x_i(t_{j+1}) \\ &= \lim_{i\to\infty} \frac{1}{\epsilon_i}\Phi_{j+1}(t_{j+1}, t_j)\Phi_j(t_j - \delta_j^i, t)\delta x_i(t) \\ &\quad + \Phi_{j+1}(t_{j+1}, t_j) \lim_{i\to\infty} \frac{1}{\epsilon_i} \int_{t_j-\delta_j^i}^{t_j} [f_{j+1}(x_i(\tau), u^0(t_j)) - f_j(x^0(x), u^0(\tau))]\, d\tau \\ &= \Phi_{j+1}(t_{j+1}, t_j)\Phi_j(t_j, t)y(t) \\ &= \Phi_{j+1}(t_{j+1}, t_j)\Phi_j(t_j, t)[f_j(x^0(t), v) - f_j(x^0(t), u^0(t))]. \end{aligned} \tag{2.27}$$

Before proceeding to the general case, we show how the equations describing λ and the minimization of Hamiltonian in each location can be obtained in a single switching time two locations case.

Set $j = 1$ and $t_{j+1} = t_2 = t_f$ in (2.26) and (2.27) above. Then

$$y(t_1-) = \Phi_1(t_1, t)y(t) = \Phi_1(t_1, t)[f_1(x^0(t), v) - f_1(x^0(t), u^0(t))],$$

and

$$y(t_f) = y(t_2) = \Phi_2(t_2, t_1)\Phi_1(t_1, t)[f_1(x^0(t), v) - f_1(x^0(t), u^0(t))].$$

Combining (2.19) and (2.24) yields

$$\left(\nabla_x g(x^0(t_f))\right)^T y(t_f) + p\left(\nabla_x m_{1,2}(t_1, x^0(t_1))\right)^T y(t_1-) \geq 0, \tag{2.28}$$

where $p = 0$ if t_1 is a controlled switching time.

Substituting the values of $y(t_1-)$ and $y(t_f)$, inequality (2.28) becomes

$$\begin{aligned} &\left(\nabla_x g(x^0(t_f))\right)^T \Phi_2(t_f, t_1)\Phi_1(t_1, t)[f_1(x^0(t), v) - f_1(x^0(t), u^0(t))] \\ &\quad + p\left(\nabla_x m(t_1, x^0(t_1))\right)^T \Phi_1(t_1, t)[f_1(x^0(t), v) - f_1(x^0(t), u^0(t))] \geq 0. \end{aligned}$$

or

$$\begin{aligned}&\left(\Phi_1^T(t_1,t)\left[\Phi_2^T(t_f,t_1)\nabla_x g(x^0(t_f))+p\,\nabla_x m(t_1,x^0(t_1))\right]\right)^T f_1(x^0(t),v)\\&\geq\left(\Phi_1^T(t_1,t)\left[\Phi_2^T(t_f,t_1)\nabla_x g(x^0(t_f))+p\,\nabla_x m(t_1,x^0(t_1))\right]\right)^T f_1(x^0(t),u^0(t)).\end{aligned} \tag{2.29}$$

Setting $\lambda_1(t)=\Phi_1^T(t_1,t)\left[\Phi_2^T(t_f,t_1)\nabla_x g(x^0(t_f))+p\,\nabla_x m(t_1,x^0(t_1))\right]$, $t\in[0,t_1)$ we obtain

$$\begin{aligned}\lambda_1(t_1-) &= \Phi_2^T(t_f,t_1)\nabla_x g(x^0(t_f))+p\,\nabla_x m(t_1,x^0(t_1))\\&= \lambda_2(t_1)+p\,\nabla_x m(t_1,x^0(t_1)),\end{aligned} \tag{2.30}$$

$$\begin{aligned}\dot{\lambda}_1(t) &= -\left(\frac{\partial f_1}{\partial x}(x^0(t))\right)^T\Phi_1^T(t_1,t)\left[\Phi_2^T(t_f,t_1)\nabla_x g(x^0(t_f))+p\,\nabla_x m(t_1,x^0(t_1))\right]\\&= -\left(\frac{\partial f_1}{\partial x}(x^0(t))\right)^T\lambda_1(t).\end{aligned} \tag{2.31}$$

Noting that in this case $H_1(x,\lambda,u)=\lambda_1^T f_1(x,u)$ the inequality (2.29) is equivalent to the Hamiltonian minimization in location 1.

2.2.1. Detailed Calculation of the Adjoint Variable Evolution. The derivation of the equations describing λ and the minimization of Hamiltonian in each location is now generalized to more than two locations case. In order to derive the adjoint equation in a location $j\in\{1,2,\cdots,m\}$, we perform a needle variation at a Lebesgue point $t\in(t_{j-1},t_j)$, where $t_m=t_f$, and compute the the deviation of the perturbed trajectory from the optimal trajectory at the final time, i.e. $\delta x_i(t_f)$ (see Figure 2.5).

$$\delta x_i(t)=\int_{t-\epsilon_i}^{t}\left(f_j(x_i(s),u^0(t))-f_j(x^0(s),u^0(s))\right)ds,$$

$$\delta x_i(t_j)=\Phi_j(t_j-\delta_j^i,t)\delta x_i(t)+\int_{t_j-\delta_j^i}^{t_j}\left(f_{j+1}(x_i(s),u^0(s))-f_j(x^0(s),u^0(s))\right)ds+o(\epsilon_i),$$

For brevity we write $f^{(i)}$ for $f(x_i(s), u^0(s))$ and $f^{(0)}$ for $f(x^0(s), u^0(s))$. Then

$$\begin{aligned}
\delta x_i(t_{j+1}) &= \Phi_{j+1}(t_{j+1} - \delta^i_{j+1}, t_j)\delta x_i(t_j) + \int_{t_{j+1}-\delta^i_{j+1}}^{t_{j+1}} \left(f^{(i)}_{j+2} - f^{(0)}_{j+1}\right) ds + o(\epsilon_i) \\
&= \Phi_{j+1}(t_{j+1} - \delta^i_{j+1}, t_j)\Phi_j(t_j - \delta^i_j, t)\delta x_i(t) \\
&\quad + \Phi_{j+1}(t_{j+1} - \delta^i_{j+1}, t_j) \int_{t_j-\delta^i_j}^{t_j} \left(f^{(i)}_{j+1} - f^{(0)}_j\right) ds \\
&\quad + \int_{t_{j+1}-\delta^i_{j+1}}^{t_{j+1}} \left(f^{(i)}_{j+2} - f^{(0)}_{j+1}\right) ds + o(\epsilon_i).
\end{aligned}$$

In general, if $j + k \neq m$ then

$$\begin{aligned}
\delta x_i(t_{j+k}) &= \left[\prod_{l=0}^{k-1} \Phi_{j+k-l}(t_{j+k-l} - \delta^i_{j+k-l}, t_{j+k-l-1})\right] \Phi_j(t_j - \delta^i_j, t)\delta x_i(t) \\
&\quad + \left[\prod_{l=0}^{k-1} \Phi_{j+k-l}(t_{j+k-l} - \delta^i_{j+k-l}, t_{j+k-l-1})\right] \int_{t_j-\delta^i_j}^{t_j} \left(f^{(i)}_{j+1} - f^{(0)}_j\right) ds \\
&\quad + \left[\prod_{l=0}^{k-2} \Phi_{j+k-l}(t_{j+k-l} - \delta^i_{j+k-l}, t_{j+k-l-1})\right] \int_{t_{j+1}-\delta^i_{j+1}}^{t_{j+1}} \left(f^{(i)}_{j+2} - f^{(0)}_{j+1}\right) ds \\
&\quad \vdots \\
&\quad + \int_{t_{j+k}-\delta^i_{j+k}}^{t_{j+k}} \left(f^{(i)}_{j+k+1} - f^{(0)}_{j+k}\right) ds + o(\epsilon_i)
\end{aligned}$$

or

$$\begin{aligned}
\delta x_i(t_{j+k}) &= \left[\prod_{l=0}^{k-1} \Phi_{j+k-l}(t_{j+k-l} - \delta^i_{j+k-l}, t_{j+k-l-1})\right] \Phi_j(t_j - \delta^i_j, t)\delta x_i(t) \\
&\quad + \sum_{r=1}^{k} \left(\left[\prod_{l=0}^{k-r} \Phi_{j+k-l}(t_{j+k-l} - \delta^i_{j+k-l}, t_{j+k-l-1})\right] \right. \\
&\qquad\qquad \left. \times \int_{t_{j+r-1}-\delta^i_{j+r-1}}^{t_{j+r-1}} \left(f^{(i)}_{j+r} - f^{(0)}_{j+r-1}\right) ds \right) \\
&\quad + \int_{t_{j+k}-\delta^i_{j+k}}^{t_{j+k}} \left(f^{(i)}_{j+k+1} - f^{(0)}_{j+k}\right) ds + o(\epsilon_i).
\end{aligned}$$

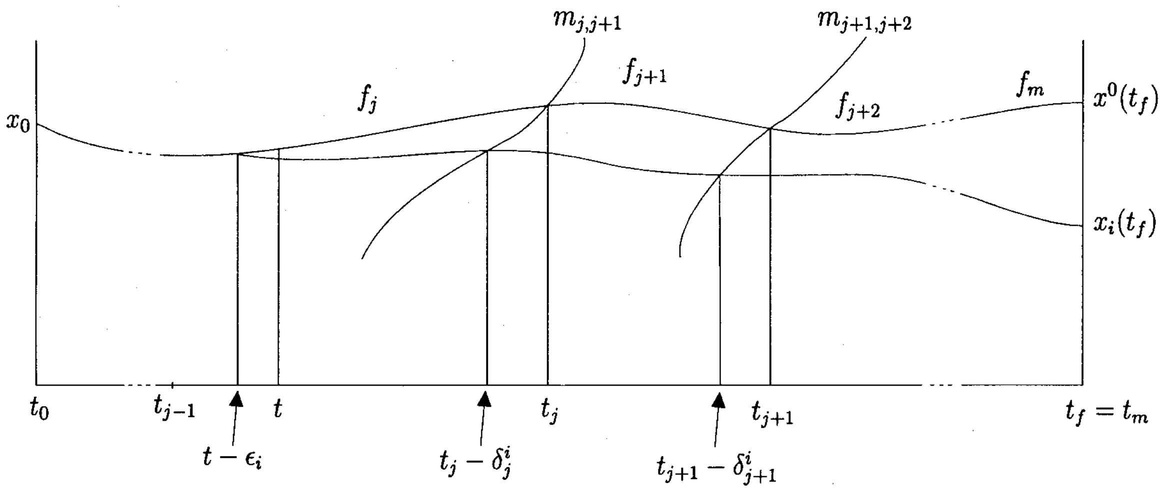

FIGURE 2.5. Autonomous switching case: variation in u^0 causes change in switching times and states and is carried forward to t_f.

In case $j + k = m$, the last integral above is zero and

$$\begin{aligned}
\delta x_i(t_f) = \delta x_i(t_m) \;&=\; \left[\prod_{l=0}^{m-j-1} \Phi_{m-l}(t_{m-l} - \delta_{m-l}^i, t_{m-l-1})\right] \Phi_j(t_j - \delta_j^i, t)\delta x_i(t) \\
&\quad + \sum_{r=1}^{m-j} \left(\left[\prod_{l=0}^{m-j-r} \Phi_{m-l}(t_{m-l} - \delta_{m-l}^i, t_{m-l-1})\right] \right. \\
&\qquad\qquad \left. \times \int_{t_{j+r-1}-\delta_{j+r-1}^i}^{t_{j+r-1}} \left(f_{j+r}^{(i)} - f_{j+r-1}^{(0)}\right) ds \right) + o(\epsilon_i).
\end{aligned}$$

As before, we now divide both sides by ϵ_i and let $i \to \infty$.

$$
\begin{aligned}
y(t_f) = y(t_m) \quad \triangleq \quad & \lim_{i\to\infty} \frac{1}{\epsilon_i} \delta x_i(t_m) \\
= \quad & \left[\prod_{l=0}^{m-j-1} \Phi_{m-l}(t_{m-l}, t_{m-l-1}) \right] \Phi_j(t_j, t) \lim_{i\to\infty} \frac{1}{\epsilon_i} \delta x_i(t) \\
& + \lim_{i\to\infty} \frac{1}{\epsilon_i} \left(\sum_{r=1}^{m-j} \left[\prod_{l=0}^{m-j-r} \Phi_{m-l}(t_{m-l} - \delta^i_{m-l}, t_{m-l-1}) \right] \right. \\
& \left. \int_{t_{j+r-1}-\delta^i_{j+r-1}}^{t_{j+r-1}} \left(f^{(i)}_{j+r} - f^{(0)}_{j+r-1} \right) ds \right). \qquad (2.32)
\end{aligned}
$$

or

$$
\begin{aligned}
y(t_m) \quad = \quad & \left[\prod_{l=0}^{m-j-1} \Phi_{m-l}(t_{m-l}, t_{m-l-1}) \right] \Phi_j(t_j, t) y(t) \\
& + \sum_{r=1}^{m-j} \left(\left[\prod_{l=0}^{m-j-r} \Phi_{m-l}(t_{m-l} - \delta^i_{m-l}, t_{m-l-1}) \right] \right. \\
& \left. \times \lim_{i\to\infty} \frac{1}{\epsilon_i} \int_{t_{j+r-1}-\delta^i_{j+r-1}}^{t_{j+r-1}} \left(f^{(i)}_{j+r} - f^{(0)}_{j+r-1} \right) ds. \right) \qquad (2.33)
\end{aligned}
$$

As we saw in equation (2.25) above,

$$
\lim_{i\to\infty} \frac{1}{\epsilon_i} \int_{t_{j+r-1}-\delta^i_{j+r-1}}^{t_{j+r-1}} \left(f^{(i)}_{j+r} - f^{(0)}_{j+r-1} \right) ds = 0.
$$

Hence we have

$$
y(t_f) = y(t_m) = \left[\prod_{l=0}^{m-j-1} \Phi_{m-l}(t_{m-l}, t_{m-l-1}) \right] \Phi_j(t_j, t) y(t). \qquad (2.34)
$$

$$
\left(\nabla_x g(x^0(t_f)) \right)^T y(t_f) + \sum_{r=1}^{m-j} p_{m-r} \left(\nabla_x m_{m-r,m-r+1}(t_{m-r}, x^0(t_{m-r})) \right)^T y(t_{m-r}-) \geq 0. \qquad (2.35)
$$

Substituting (2.34) and a similar expression for $y(t_{m-r}-)$ into (2.35), we obtain

$$\begin{aligned}
&\left(\nabla_x g(x^0(t_f))\right)^T \left[\prod_{l=0}^{m-j-1} \Phi_{m-l}(t_{m-l}, t_{m-l-1})\right] \Phi_j(t_j, t) y(t) \\
&+\sum_{r=1}^{m-j} \Bigg(p_{m-r} \left(\nabla_x m_{m-r,m-r+1}(t_{m-r}, x^0(t_{m-r}))\right)^T \\
&\qquad\qquad \times \left[\prod_{l=0}^{m-r-j-1} \Phi_{m-r-l}(t_{m-r-l}, t_{m-r-l-1})\right] \Bigg) \Phi_j(t_j, t) y(t) \geq 0.
\end{aligned}$$

If we set

$$\begin{aligned}
\Psi(m,j) \;\triangleq\; &\left(\nabla_x g(x^0(t_f))\right)^T \left[\prod_{l=0}^{m-j-1} \Phi_{m-l}(t_{m-l}, t_{m-l-1})\right] \\
&+\sum_{r=1}^{m-j} \Bigg(p_{m-r} \left(\nabla_x m_{m-r,m-r+1}(t_{m-r}, x^0(t_{m-r}))\right)^T \\
&\qquad\qquad \times \left[\prod_{l=0}^{m-r-j-1} \Phi_{m-r-l}(t_{m-r-l}, t_{m-r-l-1})\right] \Bigg),
\end{aligned}$$

then the above inequality can be written compactly as

$$\Psi(m,j)\Phi_j(t_j,t)y(t) = \Psi(m,j)\Phi_j(t_j,t)[f_j(x^0(t),v) - f_j(x^0(t),u^0(t))] \geq 0. \qquad (2.36)$$

Now setting $\lambda_j^T(t) = \Psi(m,j)\Phi_j(t_j,t)$, $t \in [t_j, t_{j+1})$, and $H_j(x,\lambda,u) = \lambda_j^T f_j(x,u)$, $t \in [t_{j-1}, t_j)$, $j = 1, 2, \cdots, m$, yields the following relationships

(i)

$$\lambda_j^T(t)[f_j(x^0(t),v) - f_j(x^0(t),u^0(t)] \geq 0,$$

or

$$\lambda_j^T(t) f_j(x^0(t),v) \geq \lambda_j^T(t) f_j(x^0(t),u^0(t)],$$

which shows the minimization of Hamiltonian in location j.

(ii) Differentiating $\lambda_j(t) = \Phi_j^T(t_j,t)\Psi^T(m,j)$ yields

$$\begin{aligned}
\dot{\lambda}_j(t) &= -\left(\frac{\partial f_j}{\partial x}(x^0(t),u^0(t)\right)^T \Phi_j^T(t_j,t)\Psi^T(m,j) \\
&= -\left(\frac{\partial f_j}{\partial x}(x^0(t),u^0(t)\right)^T \lambda_j(t) \\
&= -\frac{\partial}{\partial x}\left(\lambda_j^T(t) f_j(x^0(t),u^0(t)\right) \\
&= -\frac{\partial H_j}{\partial x}(x^0(t),\lambda(t),u^0(t), \quad t \in [t_j,t_{j+1}).
\end{aligned}$$

(iii) In the final location m,

$$\begin{aligned}
\lambda_m(t) &= \Phi_m^T(t_f,t)\nabla_x g(x^0(t_f)), \quad t \in [t_{m-1},t_f], \\
\dot{\lambda}_m(t) &= -\left(\frac{\partial f_m}{\partial x}(x^0(t))\right)^T \Phi_m^T(t_f,t)\nabla_x g(x^0(t_f)) \\
&= -\left(\frac{\partial f_m}{\partial x}(x^0(t))\right)^T \lambda_m(t), \quad t \in [t_{m-1},t_f], \\
\lambda_m(t_f) &= \nabla_x g(x^0(t_f)).
\end{aligned}$$

(iv) In order to derive the λ-transversality at the switching time t_j, $j = 1,2,\cdots,m-1$, we compute the following expressions

$$\lambda_j^T(t_j-) - \lambda_{j+1}^T(t_j) = \Psi(m,j) - \Psi(m,j+1)\,\Phi_{j+1}(t_{j+1},t_j).$$

$$
\begin{aligned}
&\Psi(m, j+1)\Phi_{j+1}(t_{j+1}, t_j) \\
&\quad = \left(\nabla_x g(x^0(t_f))\right)^T \left[\prod_{l=0}^{m-j-2} \Phi_{m-l}(t_{m-l}, t_{m-l-1})\right] \Phi_{j+1}(t_{j+1}, t_j) \\
&\quad + \sum_{r=1}^{m-j-1} \Bigg(p_{m-r} \left(\nabla_x m_{m-r,m-r+1}(t_{m-r}, x^0(t_{m-r}))\right)^T \\
&\qquad \times \left[\prod_{l=0}^{m-r-j-2} \Phi_{m-r-l}(t_{m-r-l}, t_{m-r-l-1})\right] \Bigg) \Phi_{j+1}(t_{j+1}, t_j)
\end{aligned}
$$

or

$$
\begin{aligned}
&\Psi(m, j+1)\Phi_{j+1}(t_{j+1}, t_j) \\
&\quad = \left(\nabla_x g(x^0(t_f))\right)^T \left[\prod_{l=0}^{m-j-1} \Phi_{m-l}(t_{m-l}, t_{m-l-1})\right] \\
&\qquad + \sum_{r=1}^{m-j-1} p_{m-r} \Bigg(\left(\nabla_x m_{m-r,m-r+1}(t_{m-r}, x^0(t_{m-r}))\right)^T \\
&\qquad\qquad \times \left[\prod_{l=0}^{m-r-j-1} \Phi_{m-r-l}(t_{m-r-l}, t_{m-r-l-1})\right] \Bigg) \\
&\qquad + p_j \left(\nabla_x m_{j,j+1}(t_j, x^0(t_j))\right)^T - p_j \left(\nabla_x m_{j,j+1}(t_j, x^0(t_j))\right)^T,
\end{aligned}
$$

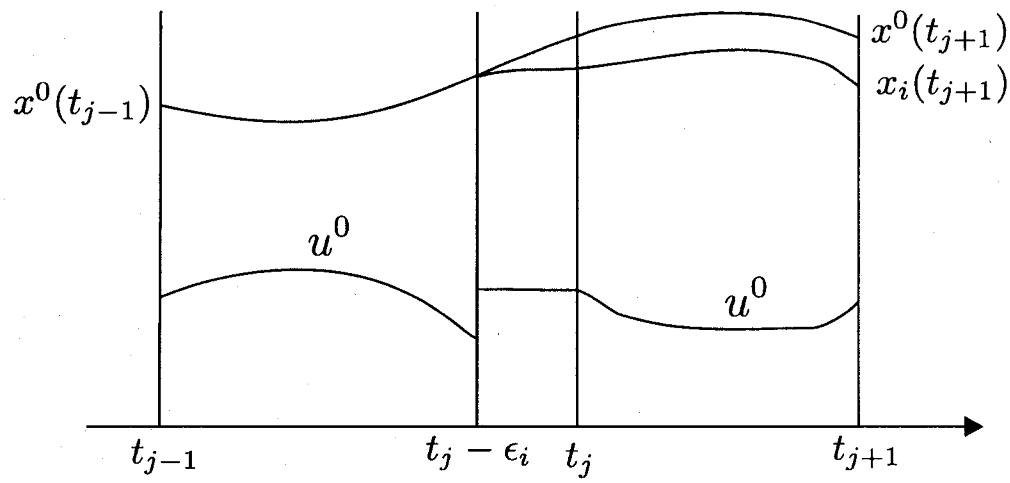

FIGURE 2.6. Controlled switching case: variation in the switching time.

or

$$\begin{aligned}
\Psi(m, j+1)\Phi_{j+1}(t_{j+1}, t_j) &= \left(\nabla_x g(x^0(t_f))\right)^T \left[\prod_{l=0}^{m-j-1} \Phi_{m-l}(t_{m-l}, t_{m-l-1})\right] \\
&\quad + \sum_{r=1}^{m-j} p_{m-r}\Bigg(\left(\nabla_x m_{m-r,m-r+1}(t_{m-r}, x^0(t_{m-r}))\right)^T \\
&\quad \times \left[\prod_{l=0}^{m-r-j-1} \Phi_{m-r-l}(t_{m-r-l}, t_{m-r-l-1})\right]\Bigg) \\
&\quad - p_j \left(\nabla_x m_{j,j+1}(t_j, x^0(t_j))\right)^T \\
&= \Psi(m, j) - p_j \left(\nabla_x m_{j,j+1}(t_j, x^0(t_j))\right)^T.
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_j^T(t_j-) - \lambda_{j+1}^T(t_j) &= \Psi(m, j) - \Psi(m, j) + p_j \left(\nabla_x m_{j,j+1}(t_j, x^0(t_j))\right)^T \\
&= p_j \left(\nabla_x m_{j,j+1}(t_j, x^0(t_j))\right)^T.
\end{aligned}$$

If t_j is a controlled switching time then $p_j = 0$.

Step 3: In this step we show the continuity of Hamiltonian in the controlled switching case. Let the optimal switching time t_j be shifted to $t_j - \epsilon_i$ (as shown in

Figure 2.6) and let the control input over the interval $[0, t_f]$ be given by

$$u_i(\tau) = \begin{cases} u^0(\tau) & \text{if} \quad 0 \le \tau < t_j - \epsilon_i \\ u^0(t_j) & \text{if} \quad t_j - \epsilon_i \le \tau < t_j \\ u^0(\tau) & \text{if} \quad t_j \le \tau \le t_f. \end{cases}$$

Then

$$x_i(t_j) = x_i(t_j - \epsilon_i) + \int_{t_j-\epsilon_i}^{t_j} f_j(x_i(\tau), v)\, d\tau,$$

and

$$x^0(t_j) = x^0(t_j - \epsilon_i) + \int_{t_j-\epsilon_i}^{t_j} f_{j-1}(x^0(\tau), u^0(\tau))\, d\tau.$$

Then

$$\frac{1}{\epsilon_i}\delta x_i(t_j) = \frac{1}{\epsilon_i}\int_{t_j-\epsilon_i}^{t_j} [f_j(x_i(\tau), v) - f_{j-1}(x^0(\tau), u^0(\tau))]\, d\tau.$$

Adding and subtracting $f_j(x^0(\tau), v)$ and then using Lemma 2.1 we obtain, as in Step 1,

$$y(t_j-) \triangleq \lim_{i\to\infty} y_i(t_j) = f_j(t_j, x^0(t_j), v) - f_{j-1}(t_j, x^0(t_j), u^0(t_j)).$$

Then $\delta x_i(t_f) = \Phi_j(t_f, t_j)\delta x_i(t_j)$ and

$$y(t_f) = \lim_{i\to\infty} \frac{1}{\epsilon_i}\delta x_i(t_f) = \Phi_j(t_f, t_j)[f_j(t_j, x^0(t_j), v) - f_{j-1}(t_j, x^0(t_j), u^0(t_j))].$$

Again since x^0 is optimal $g(x_i(t_f)) - g(x_i(t_f)) = g(x^0(t_f) + \epsilon_i y_i(t_f)) - g(x_i(t_f)) \ge 0$. Dividing by ϵ_i and taking the limit as before we obtain

$$\begin{aligned} &\left(\nabla_x g(x^0(t_f))\right)^T y(t_f) \\ &\quad = \left(\nabla_x g(x^0(t_f))\right)^T \Phi_j(t_f, t_j)[f_j(t_j, x^0(t_j), v) - f_{j-1}(t_j, x^0(t_j), u^0(t_j))] \ge 0. \end{aligned}$$

Using the definition of λ_j from Step 1 and setting $v = u^0(t_j)$ the above inequality can be written as

$$\lambda_j^T(t_j) f_j(t_j, x^0(t_j), u^0(t_j)) \ge \lambda_j^T(t_j) f_{j-1}(t_j, x^0(t_j), u^0(t_j)).$$

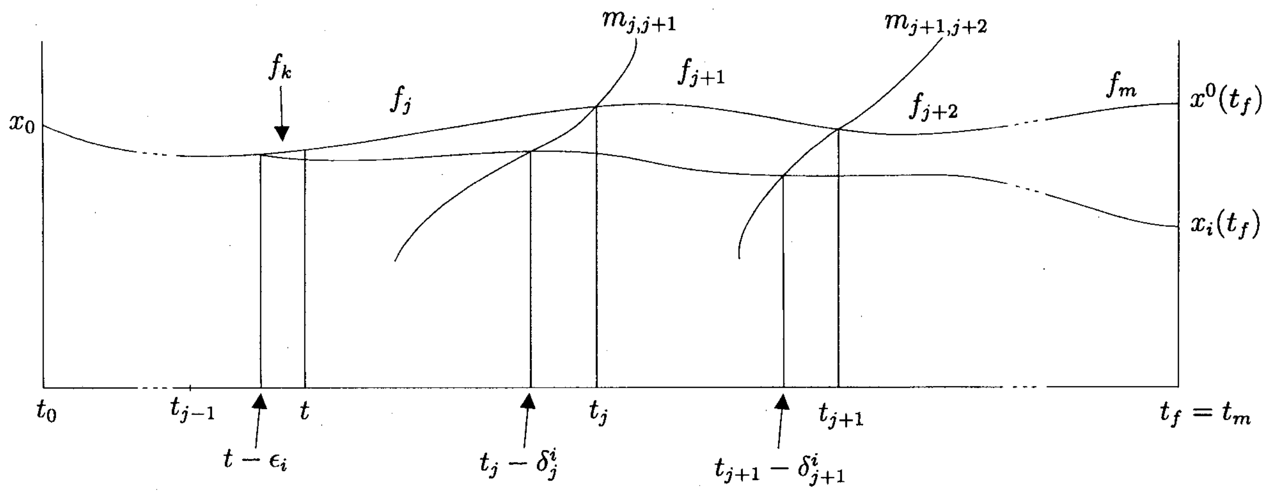

FIGURE 2.7. Q-needle variation.

If we specialize the result of Step 2 to the controlled switching case we get $\lambda_{j-1}(t_j) = \lambda_j(t_j)$. so that the above inequality can be written as

$$\lambda_j^T(t_j)f_j(t_j, x^0(t_j), u^0(t_j)) \geq \lambda_{j-1}^T(t_j)f_{j-1}(t_j, x^0(t_j), u^0(t_j)).$$

A similar derivation with the switching time shifted to other side i.e. to $t_j + \epsilon_i$ gives the inequality in the opposite direction. Hence at a controlled switching time we must have

$$\lambda_j^T(t_j)f_j(t_j, x^0(t_j), u^0(t_j)) = \lambda_{j-1}^T(t_j)f_{j-1}(t_j, x^0(t_j), u^0(t_j)).$$

Step 4: We now establish the minimization of the Hamiltonian w.r.t. the discrete locations.

Consider a Lebesgue point $t \in (t_{j-1}, t_j)$ in location j. We introduce two controlled switchings so that the systems switches to location $k \in Q$ at time $t - \epsilon_i$ and switches back to location j at time t such that $[t - \epsilon_i, t] \subset (t_{j-1}, t_j)$. This is possible since $L^0 + 2 \leq \bar{L}$.

Let the control input be held constant during the interval $[t - \epsilon_i, t]$. Then

$$u_i(\tau) = \begin{cases} u^0(\tau) & \text{if} \quad t_{j-1} \leq \tau < t - \epsilon_i \\ u^0(t) & \text{if} \quad t - \epsilon_i \leq \tau < t \\ u^0(\tau) & \text{if} \quad t \leq \tau \leq t_j. \end{cases}$$

Then

$$x_i(t) = x_i(t - \epsilon_i) + \int_{t-\epsilon_i}^{t} f_k(x_i(\tau), u^0(t))\, d\tau,$$

and

$$x^0(t) = x^0(t - \epsilon_i) + \int_{t-\epsilon_i}^{t} f_j(x^0(\tau), u^0(\tau))\, d\tau.$$

Then

$$y_i(t) \triangleq \frac{1}{\epsilon_i}\delta x_i(t) = \frac{1}{\epsilon_i}\int_{t-\epsilon_i}^{t} [f_k(x_i(\tau), u^0(t)) - f_j(x^0(\tau), u^0(\tau))]\, d\tau.$$

Adding and subtracting $f_k(x^0(\tau), u^0(t))$ to the integrand above we obtain

$$\begin{aligned} y_i(t) &= \frac{1}{\epsilon_i}\int_{t-\epsilon_i}^{t} \Big(f_k(x_i(\tau), u^0(t)) - f_j(x^0(\tau), u^0(\tau)) \\ &\qquad + f_k(x^0(\tau), u^0(t)) - f_k(x^0(\tau), u^0(t)) \Big)\, d\tau \\ &= \frac{1}{\epsilon_i}\int_{t-\epsilon_i}^{t} \left(f_k(x^0(\tau), u^0(t)) - f_j(x^0(\tau), u^0(\tau)) \right) d\tau \\ &\quad + \frac{1}{\epsilon_i}\int_{t-\epsilon_i}^{t} \left(f_k(x_i(\tau), u^0(t)) - f_k(x^0(\tau), u^0(t)) \right) d\tau. \end{aligned}$$

By Lemma 2.1 the second integral vanishes as $i \to 0$, as in Step 1, yielding

$$y(t) \triangleq \lim_{i\to\infty} y_i(t) = f_k(x^0(t), u^0(t)) - f_j(x^0(t), u^0(t)).$$

Next we notice that if we mimic the derivation in Step 2 we see that Equation (2.36) holds in this case as well, with the appropriate expression for $y(t)$ substituted. Hence

$$\Psi(m, j)\Phi_j(t_j, t)[f_k(x^0(t), v) - f_j(x^0(t), u^0(t)] \geq 0.$$

Or, as before, if we set $\lambda_j^T(t) = \Psi(m,j)\Phi_j(t_j,t)$, $t \in [t_j, t_{j+1})$, and $H_j(x,\lambda,u) = \lambda_j^T f_j(x,u)$, $t \in [t_{j-1}, t_j)$, $j = 1, 2, \cdots, m$, then

$$\lambda_j^T(t) f_k(x^0(t), u^0(t)) \geq \lambda_j^T(t) f_j(x^0(t), u^0(t)),$$

or

$$H_k(x^0(t), \lambda_j(t), u^0(t)) \geq H_j(x^0(t), \lambda_j(t), u^0(t)).$$

$\square$

2.3. Controls in Open Value Sets

THEOREM 2.3. *Consider a hybrid system $\mathbb{H}$ satisfying Assumptions A0–A3, and the HOCP$(t_0, t_f, x_0, \bar{L}, \mathcal{U}^0)$ and define*

$$H_q(x,\lambda,u) = \lambda^T f_q(x,u) + l_q(x,u), \quad x, \lambda \in \mathbb{R}^n,\ u \in \overset{o}{U},\ q \in Q.$$

1) Let $J^0(t_0, t_f, h_0, \mathcal{U}^0) = \inf_{\{I_{L(\bar{L})}\}} J^0(t_0, t_f, h_0, I_L, \bar{L}, \mathcal{U}^0)$ be realized at a minimizing control $I_{\bar{L}}^0$ and trajectory (x^0, q^0).

2) Let $I_{L^0(\bar{L})}^0$ have L_c controlled switchings and L_a autonomous switchings, and let $L_a + L_c = L^0(\bar{L})$.

3) Let t_1, t_2, $\ldots$, t_{L^0} denote the switching times along the optimal trajectory (x^0, q^0).

4) Assume that x^0 meets $\tilde{m} = \bigcup \tilde{m}_{p,q}^k$ transversally and does not meet $\partial\tilde{m}_{p,q}^{k_i} \bigcap \partial\tilde{m}_{p,q}^{k_j}$ for any k_i, k_j, p, q.

5) Assume that either (a) $\bar{L} < \infty$ and $L^0(\bar{L}) = L_a + L_c + 2 \leq \bar{L}$ or, (b) $\bar{L} = \infty$ and $L^0(\bar{L}) < \infty$.

Then the conclusions of Theorem 2.2 hold.

If, in addition,

6) almost every continuous state x on the optimal trajectory $x^0(\cdot)$ is a small time tubular fountain with respect to $x^0(\cdot)$.

Then the following Hamiltonian continuity *condition holds at a (controlled or autonomous) switching time* $t = t_j$

$$H(t_j-) \equiv H_{q(j-1)}(t_j-) = H_{q(j-1)}(t_j) = H_{q(j)}(t_j) = H_{q(j)}(t_j+) \equiv H(t_j+), \qquad j \in \{1, 2, \ldots, L^0\}.$$

□

Remark: The difference between Theorem 2.2 and Theorem 2.3 is that subject to the stronger condition of U open and subject to the small time tubular fountain condition Theorem 2.3 gives the Hamiltonian continuity condition at autonomous switching times.

Remark: When (i) the control value set U is compact but has nonempty interior $\overset{\circ}{U}$, (ii) the continuous optimal control takes values in $\overset{\circ}{U}$, and (iii) the small time tubular fountain condition holds, then all the results of Theorem 2.2 hold together with the Hamiltonian continuity at an autonomous switching time. To obtain this necessary condition result it is sufficient to consider control input variations (around the hypothesized bounded measurable optimal control u^0) which themselves fall in the more restricted class of piecewise continuous controls.

PROOF. To simplify the notation consider the case of two locations, 1 and 2, and a single autonomous switch at t_s. In an obvious notation consider the optimal cost function J^0

$$J^0 = J(u^0, t_s) = \int_{t_0}^{t_s} l_1(x^0, u^0)\, dt + \int_{t_s}^{t_f} l_2(x^0, u^0)\, dt + g(x^0(t_f)).$$

Let the function $\lambda(\cdot)$ be defined to be any absolutely continuous function which satisfies

$$\dot{\lambda}^0 = -\frac{\partial H_i}{\partial x}(x^0, \lambda^0, u^0), \quad \text{a.e. } t \in [t_0, t_f], \quad i \in \{1, 2\}, \tag{2.37}$$

$$\lambda(t_f) = \nabla_x g(x^0(t_f)), \tag{2.38}$$

$$\lambda^0(t_s-) \equiv \lambda^0(t_s) = \lambda^0(t_s+) + p\, \nabla_x m_{1,2}\big|_{t=t_s}, \tag{2.39}$$

along the optimal trajectory. Also let $\lambda_1(t) \triangleq \lambda(t), t < t_s$ and $\lambda_2(t) \triangleq \lambda(t), t \geq t_s$. Since x satisfies the respective ODEs in the respective locations over the intervals (t_0, t_s) and (t_s, t_f) and since the switching constraint is satisfied at $t = t_s$ we have

$$\begin{aligned} J(u^0, t_s) &= p\, m(x^0(t_s)) + \int_{t_0}^{t_s} \left(l_1(x^0, u^0) + {\lambda_1}^T (f_1(x^0, u^0) - \dot{x}^0)\right) dt \\ &\quad + \int_{t_s}^{t_f} \left(l_2(x^0, u^0) + {\lambda_2}^T (f_2(x^0, u^0) - \dot{x}^0)\right) dt + g(x^0(t_f)), \end{aligned}$$

for $p \in \mathbb{R}$, where in the rest of the proof $m_{1,2}$ will be written simply as m.

We now perturb the optimal input $u^0 \in \mathcal{U}$ to $u^0 + \delta u^0 \in \mathcal{U}$ such that $\operatorname{ess\,sup}_{[0,T]} |\delta u^0| < \delta$, where δ will later be taken to be sufficiently small for the required estimates to hold.

Denote the shifted switching time resulting from the new space-time intersection point with the manifold m by $t_s + \delta t_s$ and the associated state values by $x^0 + \delta x^0$. We note that by (i) the differentiability of m and (ii) the continuity of the solutions to the system equations with respect to the initial (respectively, terminal) conditions and with respect to perturbations in u (with respect to the L_∞ norm), it follows that $\delta t_s = O(\delta)$ and $\delta x^0 = O(\delta)$. We obtain:

$$\begin{aligned} & J(u^0 + \delta u^0, t_s + \delta t_s) = p\, m((x^0 + \delta x^0)(t_s + \delta t_s)) \\ & + \int_{t_0}^{t_s + \delta t_s} \Big(l_1(x^0 + \delta x^0, u^0 + \delta u^0) + {\lambda_1}^T (f_1(x^0 + \delta x^0, u^0 + \delta u^0) - \dot{x}^0 - \delta \dot{x}^0)\Big) dt \\ & + \int_{t_s + \delta t_s}^{t_f} \Big(l_2(x^0 + \delta x^0, u^0 + \delta u^0) + {\lambda_2}^T (f_2(x^0 + \delta x^0, u^0 + \delta u^0) - \dot{x}^0 - \delta \dot{x}^0)\Big) dt \\ & + g((x^0 + \delta x^0)(t_f)), \end{aligned}$$

where we note that the definition of $\lambda_1(\cdot)$ on $[t_s, t_s + \delta t_s)$, the definition of $\lambda_2(\cdot)$ only on $[t_s + \delta t_s, t_f]$ is obtained by redefining $\lambda(\cdot)$ with the ODE (2.37) and the boundary conditions (2.38) and (2.39) given on the adjusted intervals.

In the above equation $\delta \dot{x}^0$ has the following precise definition: with $x(\cdot)$ generated by the ODE

$$\dot{x} = f(x, u), \qquad x(0) = x_0, \tag{2.40}$$

let $x^\delta(\cdot)$ be generated by the ODE

$$\dot{x}^\delta = f(x^\delta, u + \delta u) \equiv f(x + \delta x, u + \delta u), \qquad x^\delta(0) = x_0^\delta = x_0 + \delta x(0),$$

where $\delta x(t) \triangleq x^\delta(t) - x(t)$ and correspondingly $\delta\dot{x}(t) \triangleq \dot{x}^\delta(t) - \dot{x}(t)$. This yields

$$\begin{aligned} \frac{d}{dt}\delta x(t) &= \frac{d}{dt}x^\delta(t) - \frac{d}{dt}x(t) \\ &\equiv \dot{x}^\delta(t) - \dot{x}(t) \equiv \delta\dot{x}(t) \\ &= f(x^\delta, u + \delta u) - f(x, u), \qquad \delta x(0) = x_0^\delta - x_0. \end{aligned} \tag{2.41}$$

Hence with $\delta x(\cdot)$ generated by (2.41) above:

$$\begin{aligned} \dot{x}^0 + \delta\dot{x}^0 &= f((x^0)^\delta, u^0 + \delta u^0) \\ &= f(x^0 + \delta x^0, u^0 + \delta u^0). \end{aligned}$$

If we linearize (2.41) about $(x(\cdot), u(\cdot))$ as in Equation (2.40) then we have

$$\delta\dot{x} = \frac{d}{dt}\delta x(t) = \frac{\partial f}{\partial x}(x, u)\delta x + \frac{\partial f}{\partial u}(x, u)\delta u + O(\delta^2),$$

with the initial condition $\delta x(0) = x_0^\delta - x_0$. Clearly, in general, it is not the case that $\delta\dot{x} \equiv \frac{d}{dt}\delta x(t) = \frac{\partial f}{\partial x}(x, u)\delta x + \frac{\partial f}{\partial u}(x, u)\delta u$.

By the assumption of the optimality of u^0 we have the following classical variational inequalities:

$$\begin{aligned}
0 \le \Delta J^0 &= J(u^0 + \delta u^0, t_s + \delta t_s) - J(u^0, t_s) \\
&= p\, m(t_s + \delta t_s, (x^0 + \delta x^0)(t_s + \delta t_s)) \\
&\quad + \int_{t_0}^{t_s + \delta t_s} \big(l_1(x^0 + \delta x^0, u^0 + \delta u^0) \\
&\qquad + \lambda_1^T (f_1(x^0 + \delta x^0, u^0 + \delta u^0) - \dot{x}^0 - \delta \dot{x}^0)\big) dt \\
&\quad + \int_{t_s + \delta t_s}^{t_f} \big(l_2(x^0 + \delta x^0, u^0 + \delta u^0) \\
&\qquad + \lambda_2^T (f_2(x^0 + \delta x^0, u^0 + \delta u^0) - \dot{x}^0 - \delta \dot{x}^0)\big) dt \\
&\quad - p\, m(t_s, x^0(t_s)) - \int_{t_0}^{t_s} \big(l_1(x^0, u^0) + {\lambda_1}^T (f_1(x^0, u^0) - \dot{x}^0)\big)\, dt \\
&\quad - \int_{t_s}^{t_f} \big(l_2(x^0, u^0) + {\lambda_2}^T (f_2(x^0, u^0) - \dot{x}^0)\big)\, dt.
\end{aligned}$$

$$
\begin{aligned}
0 \leq \Delta J^0 \;=\;& p\,\big(m(t_s+\delta t_s,(x^0+\delta x^0)(t_s+\delta t_s)) - m(t_s,x^0(t_s))\big) \\
&+ \int_{t_0}^{t_s} \big(l_1(x^0+\delta x^0,u^0+\delta u^0) \\
&\qquad +\lambda_1^T(f_1(x^0+\delta x^0,u^0+\delta u^0) - \dot{x}^0 - \delta\dot{x}^0)\big)dt \\
&- \int_{t_0}^{t_s} \big(l_1(x^0,u^0) + \lambda_1^T(f_1(x^0,u^0) - \dot{x}^0)\big)dt \\
&+ \int_{t_s}^{t_s+\delta t_s} \big(l_1(x^0+\delta x^0,u^0+\delta u^0) \\
&\qquad +\lambda_1^T(f_1(x^0+\delta x^0,u^0+\delta u^0) - \dot{x}^0 - \delta\dot{x}^0)\big)dt \\
&+ \int_{t_s+\delta t_s}^{t_f} \big(l_2(x^0+\delta x^0,u^0+\delta u^0) \\
&\qquad +{\lambda_2}^T(f_2(x^0+\delta x^0,u^0+\delta u^0) - \dot{x}^0 - \delta\dot{x}^0)\big)dt \\
&- \int_{t_s+\delta t_s}^{t_f} \big(l_2(x^0,u^0) + {\lambda_2}^T(f_2(x^0,u^0) - \dot{x}^0)\big)dt \\
&- \int_{t_s}^{t_s+\delta t_s} \big(l_2(x^0,u^0) + {\lambda_2}^T(f_2(x^0,u^0) - \dot{x}^0)\big)dt \\
&+ g((x^0+\delta x^0)(t_f)) - g(x^0(t_f)).
\end{aligned}
$$

Next we use the Taylor series expansion of $m(t_s+\delta t_s,(x^0+\delta x^0)(t_s+\delta t_s))$, $l_i(x^0+\delta x^0,u^0+\delta u^0)$, $f_i(x^0+\delta x^0,u^0+\delta u^0)$ and $g((x^0+\delta x^0)(t_f))$ about $m(x^0(t_s))$, $l_i(x^0,u^0)$, $f_i(x^0,u^0)$, $i=1,2$, and $g(x^0(t_f))$, respectively, to obtain

$$
\begin{aligned}
0 \leq \Delta J^0 \;=\;& p\,\nabla_{x^0} m\big|_{t=t_s} \delta x^0 + p\,\nabla_t m\big|_{t=t_s} \delta t_s \\
&+ \int_{t_0}^{t_s} \left(\frac{\partial l_1}{\partial x^0}\delta x_1^0 + \frac{\partial l_1}{\partial u^0}\delta u_1^0 + {\lambda_1}^T \left(\frac{\partial f_1}{\partial x^0}\delta x_1^0 + \frac{\partial f_1}{\partial u^0}\delta u_1^0 - \delta \dot{x}_1^0 \right) \right) dt \\
&+ \int_{t_s}^{t_s+\delta t_s} \Bigg(l_1(x,u) + \frac{\partial l_1}{\partial x^0}\delta x_1^0 + \frac{\partial l_1}{\partial u^0}\delta u_1^0 \\
&\qquad + {\lambda_1}^T \left(f_1(x,u) + \frac{\partial f_1}{\partial x^0}\delta x_1^0 + \frac{\partial f_1}{\partial u^0}\delta u_1^0 - \dot{x}_1^0 - \delta \dot{x}_1^0 \right) \Bigg) dt \\
&+ \int_{t_s+\delta t_s}^{t_f} \left(\frac{\partial l_2}{\partial x^0}\delta x_2^0 + \frac{\partial l_2}{\partial u^0}\delta u_2^0 + {\lambda_2}^T \left(\frac{\partial f_2}{\partial x^0}\delta x_2^0 + \frac{\partial f_2}{\partial u^0}\delta u_2^0 - \delta \dot{x}_2^0 \right) \right) dt \\
&- \int_{t_s}^{t_s+\delta t_s} \left(l_2(x^0,u^0) + {\lambda_2}^T (f_2(x^0,u^0) - \dot{x}^0) \right) dt \\
&+ \left(\nabla_x g(x_2^0(t_f)) \right)^T \delta x_2^0(t_f) + O(\delta^2).
\end{aligned}
$$

Rearranging terms we obtain,

$$
\begin{aligned}
0 \leq \Delta J^0 \;=\;& p\,\nabla_{x^0} m\big|_{t=t_s} \delta x^0 + p\,\nabla_t m\big|_{t=t_s} \delta t_s + \int_{t_0}^{t_s} \left(\frac{\partial l_1}{\partial x^0} + {\lambda_1}^T \frac{\partial f_1}{\partial x^0} \right) \delta x_1^0 \, dt \\
&+ \int_{t_0}^{t_s} \left(\frac{\partial l_1}{\partial u^0} + {\lambda_1}^T \frac{\partial f_1}{\partial u^0} \right) \delta u_1^0 \, dt + \int_{t_s+\delta t_s}^{t_f} \left(\frac{\partial l_2}{\partial x^0} + {\lambda_2}^T \frac{\partial f_2}{\partial x^0} \right) \delta x_2^0 \, dt \\
&+ \int_{t_s+\delta t_s}^{t_f} \left(\frac{\partial l_2}{\partial u^0} + {\lambda_2}^T \frac{\partial f_2}{\partial u^0} \right) \delta u_2^0 \, dt - \int_{t_0}^{t_s} {\lambda_1}^T \delta \dot{x}_1^0 \, dt - \int_{t_s+\delta t_s}^{t_f} {\lambda_2}^T \delta \dot{x}_2^0 \, dt \\
&+ \int_{t_s}^{t_s+\delta t_s} \left(l_1(x^0,u^0) + {\lambda_1}^T (f_1(x^0,u^0) - \dot{x}^0) \right) dt \\
&- \int_{t_s}^{t_s+\delta t_s} \left(l_2(x^0,u^0) + {\lambda_2}^T (f_2(x^0,u^0) - \dot{x}^0) \right) dt \\
&+ \int_{t_s}^{t_s+\delta t_s} \left(\frac{\partial l_1}{\partial x^0}\delta x_1^0 + \frac{\partial l_1}{\partial u^0}\delta u_1^0 + {\lambda_1}^T \left(\frac{\partial f_1}{\partial x^0}\delta x_1^0 + \frac{\partial f_1}{\partial u^0}\delta u_1^0 \right) - {\lambda_1}^T \delta \dot{x}_1^0 \right) dt \\
&+ \left(\nabla_x g(x_2^0(t_f)) \right)^T \delta x_2^0(t_f) + O(\delta^2).
\end{aligned}
$$

Now recognizing that $\delta \dot{x}^0 = \frac{d}{dt}(\delta x^0), \delta x^0(t_0) = 0$, integrating $\int_{t_0}^{t_s} {\lambda_1}^T \delta \dot{x}_1^0 \, dt$ and $\int_{t_s+\delta t_s}^{t_f} {\lambda_2}^T \delta \dot{x}_2^0 \, dt$ by parts, rearranging terms and noting that the last integral is $O(\delta^2)$, we obtain

$$\begin{aligned}
0 \le \Delta J^0 \;=\;& p\,\nabla_{x^0} m\big|_{t=t_s}\delta x^0 + p\,\nabla_t m\big|_{t=t_s}\delta t_s + \int_{t_0}^{t_s}\left(\frac{\partial l_1}{\partial x^0} + \lambda_1{}^T\frac{\partial f_1}{\partial x^0}\right)\delta x_1^0\,dt \\
&+ \int_{t_0}^{t_s}\left(\frac{\partial l_1}{\partial u^0} + \lambda_1{}^T\frac{\partial f_1}{\partial u^0}\right)\delta u_1^0\,dt + \int_{t_s+\delta t_s}^{t_f}\left(\frac{\partial l_2}{\partial x^0} + \lambda_2{}^T\frac{\partial f_2}{\partial x^0}\right)\delta x_2^0\,dt \\
&+ \int_{t_s+\delta t_s}^{t_f}\left(\frac{\partial l_2}{\partial u^0} + \lambda_2{}^T\frac{\partial f_2}{\partial u^0}\right)\delta u_2^0\,dt - (\lambda_1{}^T\delta x_1^0)\big|_{t=t_s} + \int_{t_0}^{t_s}\dot{\lambda}_1{}^T\delta x_1^0\,dt \\
&- (\lambda_2{}^T\delta x_2^0)\big|_{t=t_f} + (\lambda_2{}^T\delta x_2^0)\big|_{t=t_s+\delta t_s} - \int_{t_s+\delta t_s}^{t_f}\dot{\lambda}_2{}^T\delta x_2^0\,dt \\
&+ \int_{t_s}^{t_s+\delta t_s}\left(l_1(x^0,u^0) + \lambda_1{}^T(f_1(x^0,u^0) - \dot{x}^0)\right)dt \\
&- \int_{t_s}^{t_s+\delta t_s}\left(l_2(x^0,u^0) + \lambda_2{}^T(f_2(x^0,u^0) - \dot{x}^0)\right)dt \\
&+ \left(\nabla_x g(x_2^0(t_f))\right)^T\delta x_2^0(t_f) + O(\delta^2).
\end{aligned}$$

Setting $H_i = l_i + \lambda_i^{0T} f_i, i = 1,2$, approximating the last two integrals and rearranging terms we obtain

$$\begin{aligned}
0 \le \Delta J^0 \;=\;& \int_{t_0}^{t_s}\left(\frac{\partial H_1}{\partial x^0} + \dot{\lambda}_1{}^T\right)\delta x_1^0\,dt + \int_{t_s+\delta t_s}^{t_f}\left(\frac{\partial H_2}{\partial x^0} + \dot{\lambda}_2{}^T\right)\delta x_2^0\,dt \\
&+ \left(\lambda_2(t_f) - \nabla_x g(x_2^0(t_f))\right)^T\delta x_2^0(t_f) + \int_{t_0}^{t_s}\left(\frac{\partial H_1}{\partial u_1^0}\right)\delta u^0\,dt \\
&+ \int_{t_s+\delta t_s}^{t_f}\left(\frac{\partial H_2}{\partial u^0}\right)\delta u_2^0\,dt - (\lambda_1{}^T\delta x_1^0)\big|_{t=t_s} + (\lambda_2{}^T\delta x_2^0)\big|_{t=t_s+\delta t_s} \\
&+ p\,\nabla_{x^0} m\big|_{t=t_s}\delta x^0 + (H_1 - \lambda_1{}^T\dot{x}_1)\big|_{t=t_s}\delta t_s - (H_2 - \lambda_2{}^T\dot{x}_2)\big|_{t=t_s+\delta t_s}\delta t_s \\
&+ O(\delta^2).
\end{aligned}$$

Since $\lambda_i, i = 1,2$ is generated by the ODE $\dot{\lambda}_i = -\frac{\partial H_i}{\partial x^0}$ and since $\lambda_2(t_f) = \nabla_x g(x_2^0(t_f))$, the first three terms in the above expression vanish.

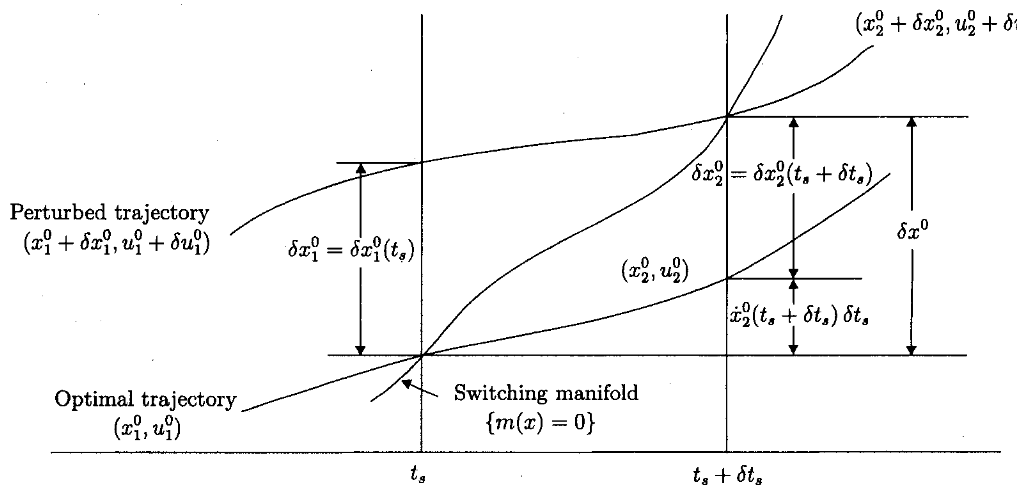

FIGURE 2.8. Autonomous switching case: variation in u^0causes change x^0 and t_s.

$$
\begin{aligned}
0 \leq \Delta J^0 \;=\; & \int_{t_0}^{t_s} \left(\frac{\partial H_1}{\partial u^0}\right) \delta u_1^0 \, dt + \int_{t_s+\delta t_s}^{t_f} \left(\frac{\partial H_2}{\partial u^0}\right) \delta u_2^0 \, dt + \left(H_1|_{t=t_s} - H_2|_{t=t_s+\delta t_s}\right) \delta t_s \\
& -\lambda_1^T(t_s) \left(\delta x_1^0(t_s) + \dot{x}_1^0(t_s)\, \delta t_s\right) \\
& +\lambda_2{}^T(t_s+\delta t_s) \left(\delta x_2^0(t_s+\delta t_s) + \dot{x}_2^0(t_s+\delta t_s)\, \delta t_s\right) \\
& + p \nabla_x m|_{t=t_s} \, \delta x^0 + O(\delta^2).
\end{aligned}
$$

Since δx^0 is the total space-time variation in x, and since

$$
\begin{aligned}
& x_1^0(t_s) = x_2^0(t_s), \\
& (x_1^0 + \delta x_1^0)(t_s + \delta t_s) = (x_2^0 + \delta x_2^0)(t_s + \delta t_s), \\
& \delta x^0 = (x_2^0 + \delta x_2^0)(t_s + \delta t_s) - x_1^0(t_s), \\
& \delta x_1^0(t_s) = (x_1^0 + \delta x_1^0)(t_s) - x_1^0(t_s), \\
& \delta x_2^0(t_s + \delta t_s) = (x_2^0 + \delta x_2^0)(t_s + \delta t_s) - x_2^0(t_s + \delta t_s),
\end{aligned}
$$

we have the following relationships: (i) $\delta x^0 = \delta x_1^0(t_s) + \dot{x}_1^0(t_s)\, \delta t_s + O(\delta^2)$, and (ii) $\delta x^0 = \delta x_2^0(t_s + \delta t_s) + \dot{x}_2^0(t_s + \delta t_s)\, \delta t_s + O(\delta^2)$, as shown in Figure 2.8.

Using these relations the expression for ΔJ^0 becomes

$$\begin{aligned} 0 \leq \Delta J^0 &= \int_{t_0}^{t_s} \left(\frac{\partial H_1}{\partial u^0}\right) \delta u_1^0 \, dt + \int_{t_s+\delta t_s}^{t_f} \left(\frac{\partial H_2}{\partial u^0}\right) \delta u_2^0 \, dt + \left(H_1|_{t=t_s} - H_2|_{t=t_s+\delta t_s}\right) \delta t_s \\ &\quad + \left(-{\lambda_1}^T(t_s) + {\lambda_2}^T(t_s+\delta t_s) + p \, \nabla_x m\right) \delta x^0 + O(\delta^2). \end{aligned}$$

Since $\delta t_s = O(\delta)$ the second to last term in the above expression is $O(\delta^2)$ by the boundary conditions on λ_1 and λ_2, and so the expression reduces to

$$0 \leq \Delta J^0 = \int_{t_0}^{t_s} \left(\frac{\partial H_1}{\partial u^0}\right) \delta u^0 \, dt + \int_{t_s}^{t_f} \left(\frac{\partial H_2}{\partial u^0}\right) \delta u^0 \, dt + \left(H_1|_{t=t_s} - H_2|_{t=t_s+}\right) \delta t_s + O(\delta^2).$$

The variations δu^0 and δt_s are not in general independent since δt_s depends upon δu^0 via the equations describing the switching surface. We now invoke the small time tubular fountain (STTF) property (Definition 2.6) to generate a variation δu^0 which shows that the coefficient of δt_s in the above expression vanishes whenever u^0 and t_s are optimal. To see this we split each of the two integrals over the intervals $[t_0, t_s]$ and $[t_s, t_f]$ as follows (see Figure 2.8):

$$\begin{aligned} 0 \leq \Delta J^0 &= \int_{t_0}^{t_a} \left(\frac{\partial H_1}{\partial u^0}\right) \delta u^0 \, dt + \int_{t_a}^{t_s} \left(\frac{\partial H_1}{\partial u^0}\right) \delta u^0 \, dt + \int_{t_s}^{t_c} \left(\frac{\partial H_2}{\partial u^0}\right) \delta u^0 \, dt \\ &+ \int_{t_c}^{t_f} \left(\frac{\partial H_2}{\partial u^0}\right) \delta u^0 \, dt + \left(H_1|_{t=t_s} - H_2|_{t=t_s+}\right) \delta t_s + O(\delta^2), \end{aligned} \tag{2.42}$$

for any t_a, t_c such that $t_0 \leq t_a < t_s < t_c \leq t_f$.

Control Variations via the Small Time Tubular Fountain (STTF) Condition

For economy of notation define $y = x^0 + \delta x^0$ and $v = u^0 + \delta u^0$. Suppose the coefficient of δt_s in (2.42) above is negative. We wish to show that there exists a control perturbation δu^0 such that $\delta u^0(t) = 0$, $t \in [t_0, t_a] \cup [t_c, t_f]$, where $t_a = t_s - \delta$ and $t_c = t_s + \delta$, and such that $u^0 + \delta u^0 \in \mathcal{U}$ steers the system trajectory so that it intersects the switching manifold m for the first time in $[t_a, t_c]$ at some $t_s + \delta t_s$ where $0 < \delta t_s < \delta$, and then steers it back to the unperturbed trajectory at time t_c (see Figure 2.9).

Identify γ' in Definition 2.6 with δ and henceforth take δ sufficiently small that $x^0(\cdot)$ intersects m only once in $[t_a, t_c]$. This is possible by (i) the continuity of $x^0(\cdot)$,

(ii) the smoothness of $m(\cdot,\cdot)$, and (iii) the transversality hypothesis (Hypothesis 4). Consider the set $T^l_\mu \triangleq \cup_{t_a \le \tau < t_l} B_\mu(x^0(\tau))$ where μ and t_l are such that $T^l_\mu \cap \{x : m(t_s, x) = 0\} = \emptyset$. Let $\gamma = \frac{\mu}{2}$. By the positive STTF conditions (b), (c) and (d) there exist $t_{a'}$, $\epsilon > 0$ and $v^l(\cdot)$ such that:

(i) $t_a \le t_{a'} < t_l < t_s$,

(ii) $\|\phi(s, x^0(t_a), v^l_{[t_{a'},s]}) - \phi(s, x^0(t_a), u^0_{[t_{a'},s]})\| < \gamma, \quad s \in [t_a, t_l]$, and

(iii) $\phi(t_l - \epsilon, x^0(t_a), v^l_{[t_a, t_l - \epsilon]}) = \phi(t_l, x^0(t_a), u^0_{[t_a, t_l]})$.

Now consider the set $T^r_\mu \triangleq \cup_{t_r < \tau \le t_c} B_\mu(x^0(\tau))$ where μ and t_r are such that $T^r_\mu \cap \{x : m(t_s, x) = 0\} = \emptyset$. Let $\gamma = \frac{\mu}{2}$. By the negative STTF (b′), (c′) and (d′) there exist $t_{c'}$, $\epsilon > 0$ and $v^r(\cdot)$ such that:

(i) $t_s < t_r < t_{c'} \le t_c$,

(ii) $\|\phi(s, x^0(t_r), v^r_{[t,s]}) - \phi(s, x^0(t_r), u^0_{[t,s]})\| < \gamma, \quad s \in [t_r, t_c]$, and

(iii) $\phi(t_c, x^0(t_r - \epsilon), v^r_{[t_r - \epsilon, t_c]}) = \phi(t_c, x^0(t_r - \epsilon), u^0_{[t_r - \epsilon, t_c]})$.

Early arrival case

The continuous dynamics in each location are time-invariant. Consequently, if the perturbed control input over the interval $[t_l - \epsilon, t_r - \epsilon] \subset [t_{a'}, t_{c'}]$ is chosen so that $v(s) = u^0(s + \epsilon)$, $t_l - \epsilon \le s \le t_r - \epsilon$, then: (i) $\{y(s) : t_l - \epsilon \le s \le t_r - \epsilon\} = \{x^0(s) : t_l \le s \le t_r\}$ and (ii) $y(\cdot)$ takes the same amount of time to traverse the optimal path from $x^0(t_l)$ to $x^0(t_r)$ as does $x^0(\cdot)$. Furthermore, $y(\cdot)$ intersects the switching manifold only once in the interval $[t_a, t_c]$ (see Figure 2.9). Set $t_1 = t_l - \epsilon, t_2 = t_r - \epsilon$; then $|t_2 - t_1| = |t_r - t_l| < |t_{c'} - t_{a'}| \le |t_c - t_a| < 2\delta$.

In (2.42), by the construction of δu^0, the first and the fourth integrals are zero and the second and third integrals are $O(\delta^2)$ since $|t_c - t_a| = O(\delta)$. This causes the integral terms to be of second order ($O(\delta^2)$) but renders ΔJ^0 negative to first order ($O(\delta)$). Since this is impossible by the optimality of u^0 and t_s, the term $\left(H_1|_{t=t_s} - H_2|_{t=t_s+}\right)$ must be zero.

Late arrival case

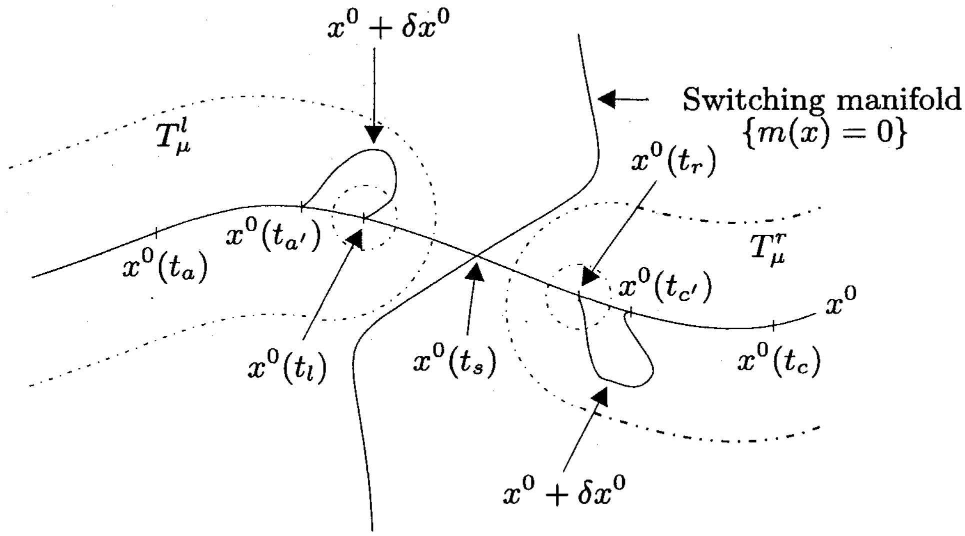

FIGURE 2.9. Early and late arrival variations using STTF.

Now suppose the coefficient of δt_s in (2.42) is positive. It is clear from Definition 2.6 that a similar construction can be done where $v(\cdot)$ and the resulting trajectory $y(\cdot)$ are such that: (i) $v(s) = u^0(s-\epsilon)$, $t_l+\epsilon \le s \le t_r+\epsilon$, (ii) $\{y(s) : t_l+\epsilon \le s \le t_r+\epsilon\} = \{x^0(s) : t_l \le s \le t_r\}$, (iii) $y(\cdot)$ takes the same amount of time to traverse the optimal path from $x^0(t_l)$ to $x^0(t_r)$ as does $x^0(\cdot)$, and (iv) $y(\cdot)$ intersects the switching manifold only once in the interval $[t_a, t_c]$ (see Figure 2.9). As in the negative coefficient case, this again leads to the conclusion that the term $\left(H_1|_{t=t_s} - H_2|_{t=t_s+}\right)$ must be zero.

An appropriate variation $\{\delta u^0, t_0 \le t \le t_f\}$ now shows that $\frac{\partial H_1}{\partial u^0} = 0$ and $\frac{\partial H_2}{\partial u^0} = 0$ on $[t_0, ts)$ and $[t_s, t_f]$ respectively.

The proof of Hamiltonian minimization with respect to the discrete locations is similar to that in Theorem 2.2. □

2.4. Examples

EXAMPLE 2.1. *In this example we show how the Weierstrass-Erdmann corner conditions of calculus of variations* [**31**] *can be recovered from the hybrid maximum* *principle. First we briefly recall these conditions. Consider the class $\mathcal{E}$ of all continuous functions, $x : [a,b] \to \mathbb{R}^n$, with fixed end points, $x(a) = x(b) = 0$, possessing piecewise continuous derivative $\dot{x}(t)$, $t \in [a,b]$ in a bounded open set. Let $f : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be of class C^2. Then the integral*

$$I(x) = \int_a^b f(x(s), \dot{x}(s))\, ds,$$

is well defined for $x \in \mathcal{E}$.

Suppose that for some $x^0 \in \mathcal{E}$, $I(x^0) \leq I(x)$, for all $x \in \mathcal{E}$. Then the following necessary conditions are satisfied:

(i) x^0 satisfies the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}(x^0(t), \dot{x}^0(t))\right) = \frac{\partial f}{\partial x}(x^0(t), \dot{x}^0(t)).$$

(ii) The Weierstrass E-function*:*

$$E(x, \dot{x}, w) \triangleq f(x(t), w) - f(x(t), \dot{x}(t)) - (w - \dot{x}(t))^T \frac{\partial f}{\partial \dot{x}}(x(t), \dot{x}(t)), \quad w \in \mathbb{R}^n,$$

is positive along x^0.

(iii) The functions

$$f(x(t), \dot{x}(t)) - \dot{x}^T(t)\,\frac{\partial f}{\partial \dot{x}}(x(t), \dot{x}(t)) \qquad \text{and} \qquad \frac{\partial f}{\partial \dot{x}}(x(t), \dot{x}(t)),$$

are continuous along x^0 and hence at each corner point (point of non-differentiability) of x^0.

Conditions (i) and (ii) can be obtained as consequences of the Minimum Principle (see **[11]***, for example). Here we demonstrate how condition (iii) can be recovered from the Hybrid Minimum Principle. We transform the above problem into an optimal control problem as follows. Set $\dot{x} = u$ so that $I(x) = \int_a^b f(x(s), u(s))\, ds$. Let x^0 have one corner at $t_1 \in (a,b)$ (the generalization to a finite number of corners is similar).*

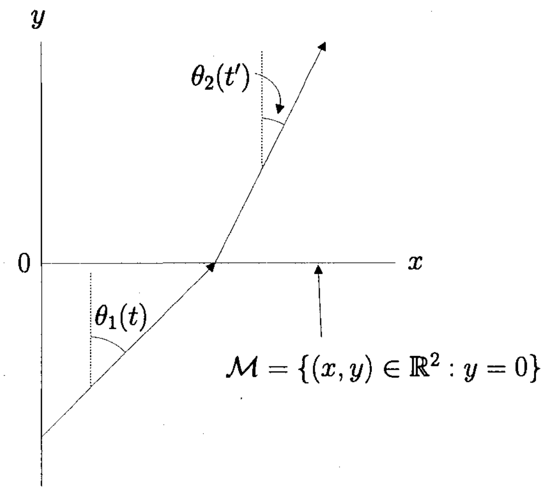

FIGURE 2.10. Snell's Law.

Then by the HMP the Hamiltonian and the adjoint process are continuous at t_1, *i.e.*

$$H(x,\lambda,u) = \lambda^T u + f(x,u).$$

Minimization of H *with respect to* u *implies*

$$\frac{\partial H}{\partial u} = \lambda + \frac{\partial f}{\partial u} = 0 \quad \Rightarrow \quad \lambda(t) = -\frac{\partial f}{\partial u}(x(t),u(t)) = -\frac{\partial f}{\partial \dot{x}}(x(t),\dot{x}(t)).$$

Hence

$$\begin{aligned} H(t) = H(x(t),\lambda(t),u(t)) &= f(x(t),u(t)) - u^T(t)\,\frac{\partial f}{\partial u}(x(t),u(t)) \\ &= f(x(t),\dot{x}(t)) - \dot{x}^T(t)\,\frac{\partial f}{\partial \dot{x}}(x(t),\dot{x}(t)). \end{aligned}$$

And we get W-E corner conditions as a result of the continuity of $H(t)$ *and* $\lambda(t)$ *along* $x^0(t)$. *Notice that the positivity of the Weierstrass E-function is equivalent to minimization of Hamiltonian.* □

EXAMPLE 2.2. *Let a ray of light pass from a medium, in which its velocity is* v_1, *to another medium, in which its velocity is* v_2, *in such a way as to minimize the time of travel (Fermat's Principle), as shown in Figure 2.10. The initial and final times are* 0 *and* t_f *respectively and the time at which it enters medium 2 is* $0 < t_s < t_f$. *The* *boundary (switching manifold) between two media is* $\mathcal{M} = \{(x,y) \in \mathbb{R}^2 : m(t,x,y) = y = 0\}$. *In the rest of this example the subscripts* 1 *and* 2 *refer to media* 1 *and* 2 *respectively. At each time along its trajectory let the ray form angles* $\theta_1(t)$ *and* $\theta_2(t)$ *with the normal to* $\Updownarrow$. *Further, let* $-\frac{\pi}{2} < \theta_1(t) < \frac{\pi}{2}$, $0 \leq t < t_1$, *and* $-\frac{\pi}{2} < \theta_2(t) < \frac{\pi}{2}$, $t_1 \leq t \leq t_f$, *Then the horizontal and vertical components of velocities* v_1 v_2 *can be written as:*

$$\dot{x}_1 = v_1 \sin\theta_1(t), \qquad \dot{x}_2 = v_2 \sin\theta_2(t),$$

$$\dot{y}_1 = v_1 \cos\theta_1(t), \qquad \dot{y}_2 = v_2 \cos\theta_2(t).$$

The objective is to minimize:

$$J(\theta_1, \theta_2) = \int_0^{t_s} dt + \int_{t_s}^{t_f} dt.$$

The Hamiltonian in each medium is given by:

$$H_i = 1 + \lambda_i v_i \sin\theta_i(t) + \mu_i v_i \cos\theta_i(t), \qquad i = 1, 2,$$

where λ_i *and* μ_i *are variables adjoint to* x_i *and* y_i, $i = 1, 2$, *respectively. Since* θ_1 *and* θ_2 *take values in an open set* $(-\frac{\pi}{2}, \frac{\pi}{2})$ *if* θ_i^0 *is optimal then*

$$\left.\frac{\partial H_i}{\partial \theta_i}\right|_{\theta_i = \theta_i^0} = \lambda_i v_i \cos\theta_i^0 - \mu_i v_i \sin\theta_i^0 = 0,$$

and so $\tan\theta_i^0 = \frac{\lambda_i}{\mu_i}$.

In the sequel we drop the superscript on θ_i, $i = 1, 2$, *as all* θ_i *will be optimal. Along the optimal trajectory,*

$$\dot{\lambda}_i = -\frac{\partial H_i}{\partial x_i} = 0 \quad \textit{and} \quad \dot{\mu}_i = -\frac{\partial H_i}{\partial y_i} = 0, \quad i = 1, 2.$$

which shows that λ_i *and* μ_i *are constant and hence* θ_i, $i = 1, 2$, *are constant. So the the path of light ray in each medium is a straight line.*

Next, applying the Hamiltonian and adjoint transversality conditions at the switching time t_s we obtain:

$$\begin{aligned} H_1(t_s-) &= 1 + \lambda_1 v_1 \sin\theta_1 + \mu_1 v_1 \cos\theta_1 \\ &= 1 + \lambda_2 v_2 \sin\theta_2 + \mu_2 v_2 \cos\theta_2 + 0 = H_2(t_s) + p\,\nabla_t m, \end{aligned}$$

or

$$\lambda_1 v_1 \sin\theta_1 + \mu_1 v_1 \cos\theta_1 = \lambda_2 v_2 \sin\theta_2 + \mu_2 v_2 \cos\theta_2, \tag{2.43}$$

$$\lambda_1 = \lambda_2 + p\,\nabla_x m$$

or

$$\lambda_1 = \lambda_2 \qquad \textit{and} \qquad \mu_1 = \mu_2 + p\,\nabla_y m = \mu_2 + p.$$

In light of the above equations (2.43) can be written as

$$\lambda_2 v_1 \sin\theta_1 + (\mu_2 + p) v_1 \cos\theta_1 = \lambda_2 v_2 \sin\theta_2 + \mu_2 v_2 \cos\theta_2.$$

But since $\lambda_i = \mu_i \tan\theta_i$, we have

$$\mu_2 = \lambda_2 \cot\theta_2 \qquad \textit{and} \qquad \mu_2 + p = \mu_1 = \lambda_1 \cot\theta_1 = \lambda_2 \cot\theta_1.$$

Hence if $\lambda_2 \neq 0$ then

$$\begin{aligned} v_1 (\sin\theta_1 + \cot\theta_1 \cos\theta_1) &= v_2 (\sin\theta_2 + \cot\theta_2 \cos\theta_2) \\ \Rightarrow \quad \frac{v_1}{v_2} &= \frac{\sin\theta_1}{\sin\theta_2}. \end{aligned}$$

□

CHAPTER 3

Sufficient Conditions for Hybrid Optimality

3.1. Hybrid Dynamic Programming

In this section we present a hybrid Dynamic Programming Principle (HDPP) which is a generalization of the standard Dynamic Programming Principle for differentiable control systems [**4, 61, 52**]. This results in a generalization of the standard Hamilton-Jacobi-Bellman (HJB) equation. This hybrid HJB equation is then used to establish a sufficient condition for the optimality of a candidate hybrid control. In contrast, Branicky et al. [**13**] generalize the quasi-variational inequalities (QVIs) of impulse control framework [**8**] to hybrid systems. They consider optimization of discounted cost over semi-infinite intervals and use the QVIs to establish a verification theorem.

Let $\{I\}$ be the set of all possible hybrid input trajectories with fixed initial time and hybrid state (t_0, h_0), having finite number of switchings and satisfying Assumptions A0 and A1. Then the value function, at (t_0, h_0), for HOCP$(t_0, t_f, h_0, \mathcal{U})$, such that A2 and A3 hold, is defined to be:

$$v(t_0,h_0) = \inf_{I_L \in \{I\}} J(t_0,h_0;I_L) = \inf_{I_L \in \{I\}} \left(\sum_{i=0}^{L} \int_{t_i}^{t_{i+1}} l_{q_i}(x_{q_i}(s),u(s))\,ds + g(x_{q_L}(t_f)) \right), \tag{3.1}$$

where L is the number of switchings and $L < \bar{L} = \infty$.

If $I = (\tau, \sigma, u) = ((t_0, \sigma_0, u[t_0, t_1)), (t_1, \sigma_1, u[t_1, t_2)), \ldots)$, where $u[t_i, t_{i+1})$ denotes the restriction of $u(\cdot)$ to the half open interval $[t_i, t_{i+1})$, then the restriction of I to the interval $[t', t'') \subset [t_0, T)$, where $t_k \leq t' < t_{k+1}$, $t_l \leq t'' < t_{l+1}$, $k \leq l$, is denoted $I[t', t'')$ and defined by:

$$I[t', t'') \triangleq \begin{cases} ((t', \sigma_k, u[t', t_{k+1})), (t_{k+1}, \sigma_{k+1}, u[t_{k+1}, t_{k+2}), \ldots, (t_l, \sigma_l, u[t_l, t'')) & \text{if} \quad t' = t_k, \\ ((t', id, u[t', t_{k+1})), (t_{k+1}, \sigma_{k+1}, u[t_{k+1}, t_{k+2}), \ldots, (t_l, \sigma_l, u[t_l, t'')) & \text{if} \quad t' > t_k. \end{cases}$$

Let $h(t; h_0, I_L[t_0, t])$ denote the hybrid state at time t in a hybrid execution $e_{\mathbb{H}}$, resulting from the hybrid input $I_L[t_0, t]$, where $I_L[t_0, t]$ is the restriction of I_L to the interval $[t_0, t]$.

THEOREM 3.1. *([42] after [2]) (Hybrid Dynamic Programming Principle) Consider the*

$HOCP(t_0, t_f, h_0, \bar{L}, \mathcal{U})$ with $\bar{L} = \infty$ and $L^0(\bar{L}) < \infty$. Let $t_0 \leq t' < t \leq t_f$ and let t_k and t_l be switching times such that $t_k \leq t' < t_{k+1}$, $t_l \leq t < t_{l+1}$ and $0 \leq k \leq l$ where, $\{t_i\}_{i=k}^{l}$, and k and l all depend upon I. Let $\{I[t', t_f]\}$ denote the input sequences in the class $\{I\}$ restricted to the time interval $[t', t_f]$. Then under Assumptions A0–A3,

$$\begin{aligned} v(t', h') = \inf_{I \in \{I[t', t_f]\}} \Bigg(\int_{t'}^{t_{k+1}} l_{q_k}(x_{q_k}(s), u(s))\, ds + \sum_{i=k+1}^{l-1} \int_{t_i}^{t_{i+1}} l_{q_i}(x_{q_i}(s), u(s))\, ds \\ + \int_{t_l}^{t} l_{q_l}(x_{q_l}(s), u(s))\, ds + v(t, h(t; h', I[t', t])) \Bigg), \\ v(t_f, h) = g(x), \quad \forall h = (q, x) \in Q \times \mathbb{R}^n, \end{aligned}$$

If either of l or k equals L then in the intervals of definition of t' and t the right end point is taken to be t_f. □

PROOF. We prove the theorem by showing the inequality in both directions. Fix $I_L \in \{I\}$. Then as in equation (2.5)

$$\begin{aligned} J(t',h';I_L) &= \int_{t'}^{t_{k+1}} l_{q_k}(x_{q_k}(s),u(s))\,ds + \sum_{i=k+1}^{L} \int_{t_i}^{t_{i+1}} l_{q_i}(x_{q_i}(s),u(s))\,ds + g(x_{q_L}(t_f)) \\ &= \int_{t'}^{t_{k+1}} l_{q_k}(x_{q_k}(s),u(s))\,ds + \sum_{i=k+1}^{l-1} \int_{t_i}^{t_{i+1}} l_{q_i}(x_{q_i}(s),u(s))\,ds \\ &\quad + \int_{t_l}^{t} l_{q_l}(x_{q_l}(s),u(s))\,ds + \int_{t}^{t_{l+1}} l_{q_l}(x_{q_l}(s),u(s))\,ds \\ &\quad + \sum_{i=l+1}^{L} \int_{t_i}^{t_{i+1}} l_{q_i}(x_{q_i}(s),u(s))\,ds + g(x_{q_L}(t_f)). \end{aligned}$$

or

$$\begin{aligned} J(t',h';I_L) &\geq \int_{t'}^{t_{k+1}} l_{q_k}(x_{q_k}(s),u(s))\,ds + \sum_{i=k+1}^{l-1} \int_{t_i}^{t_{i+1}} l_{q_i}(x_{q_i}(s),u(s))\,ds \\ &\quad + \int_{t_l}^{t} l_{q_l}(x_{q_l}(s),u(s))\,ds + v(t,h(t;h',I_L[t',t])). \end{aligned}$$

Taking infimum over $\{I\}$ on both sides we get the inequality "$\geq$".

In order to obtain the inequality in the other direction we fix $I_{L'} \in \{I[t',t)\}$ such that t is not a switching time, fix $\epsilon > 0$, and set $h = h(t;h',I_{L'})$. By the definition of infimum there is $I'[t,t_f] \in \{I[t,t_f]\}$ such that $v(t,h) + \epsilon \geq J(t,h,I'[t,t_f])$. Let $\bar{I}_L$ be the hybrid input sequence formed by the concatenation of the input sequences $I_{L'}[t',t)$ and $I'[t,t_f]$. Then we have

$$\begin{aligned}
v(t',h') &\leq J(t',h';\bar{I}_L) \\
&= \int_{t'}^{t_{k+1}} l_{q_k}(x_{q_k}(s),u(s))\,ds + \sum_{i=k+1}^{l-1}\int_{t_i}^{t_{i+1}} l_{q_i}(x_{q_i}(s),u(s))\,ds \\
&\quad + \int_{t_l}^{t} l_{q_l}(x_{q_l}(s),u(s))\,ds + \int_{t}^{t_{l+1}} l_{q_l}(x_{q_l}(s),u(s))\,ds \\
&\quad + \sum_{i=l+1}^{L}\int_{t_i}^{t_{i+1}} l_{q_i}(x_{q_i}(s),u(s))\,ds + g(x_{q_L}(t_f)) \\
&= \int_{t'}^{t_{k+1}} l_{q_k}(x_{q_k}(s),u(s))\,ds + \sum_{i=k+1}^{l-1}\int_{t_i}^{t_{i+1}} l_{q_i}(x_{q_i}(s),u(s))\,ds \\
&\quad + \int_{t_l}^{t} l_{q_l}(x_{q_l}(s),u(s))\,ds \\
+J(t,h,I') & \\
&\leq \int_{t'}^{t_{k+1}} l_{q_k}(x_{q_k}(s),u(s))\,ds + \sum_{i=k+1}^{l-1}\int_{t_i}^{t_{i+1}} l_{q_i}(x_{q_i}(s),u(s))\,ds \\
&\quad + \int_{t_l}^{t} l_{q_l}(x_{q_l}(s),u(s))\,ds + v(t,h) + \epsilon.
\end{aligned}$$

Hence

$$\begin{aligned}
v(t',h') \leq \inf_{\bar{I}_L\in\{I\}} \Bigg(&\int_{t'}^{t_{k+1}} l_{q_k}(x_{q_k}(s),u(s))\,ds + \sum_{i=k+1}^{l-1}\int_{t_i}^{t_{i+1}} l_{q_i}(x_{q_i}(s),u(s))\,ds \\
&+ \int_{t_l}^{t} l_{q_l}(x_{q_l}(s),u(s))\,ds + v(t,h(t;h',I_{L'}))\Bigg) + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary we have the desired inequality.

Finally, $v(t_f, h \,\underline{\Delta}\, (q,x)) = g(x)$ follows from the definition of v (Equation (3.1)).

□

Before presenting Theorem 3.2 we quote the following results on the continuity of executions of hybrid systems with respect to initial conditions and times. We assume that there exists $K_f < \infty$ such that $\max_{q\in Q}\sup_{u\in U}\|f_q(0,u)\| \leq K_f$.

LEMMA 3.1. *([**2**]) (*Dependence on Initial Conditions*) For the ODE* $\dot{x}(t) = f(x(t), u(t))$*, let* $x_t^1 \triangleq x(t; x_1, t_1, u)$ *and* $x_t^2 \triangleq x(t; x_2, t_2, u)$ *denote the states at time* t *with initial conditions* x_1 *and* x_2 *at times* t_1 *and* t_2 *respectively and with input* u*. Then under Assumption A0*

$$\|x^1 - x^2\| \leq (\|x_1 - x_2\| + K_f|t_1 - t_2|)\, e^{L_f(t - t_1 \vee t_2)}.$$

□

PROOF. Assume, without loss of generality, that $t_1 \leq t_2$.
Since $x(t; x_0, t_0, u) = x_0 + \int_{t_0}^t f(x(s; x_0, t_0, u), u(s))\, ds$

$$\begin{aligned}
\|x^1 - x^2\| &= \|x(t; x_1, t_1, u) - x(t; x_2, t_2, u)\| \\
&= \left\| x_1 - x_2 + \int_{t_1}^t f(x(s; x_1, t_1, u), u(s))\, ds - \int_{t_2}^t f(x(s; x_2, t_2, u), u(s))\, ds \right\| \\
&\leq \|x_1 - x_2\| + \left\| \int_{t_1}^{t_2} f(x(s; x_1, t_1, u), u(s))\, ds \right\| \\
&\quad + \left\| \int_{t_2}^t (f(x(s; x_1, t_1, u), u(s)) - f(x(s; x_2, t_2, u), u(s)))\, ds \right\| \\
&\leq \|x_1 - x_2\| + \int_{t_1}^{t_2} \|f(x(s; x_1, t_1, u), u(s))\|\, ds \\
&\quad + \int_{t_2}^t \|f(x(s; x_1, t_1, u), u(s)) - f(x(s; x_2, t_2, u), u(s))\|\, ds \\
&\leq \|x_1 - x_2\| + \int_{t_1}^{t_2} K_f\, ds \\
&\quad\quad + \int_{t_2}^t L_f\, \|x(s; x_1, t_1, u), u(s) - x(s; x_2, t_2, u), u(s)\|\, ds \\
&= \|x_1 - x_2\| + K_f|t_1 - t_2| \\
&\quad\quad + L_f \int_{t_2}^t \|x(s; x_1, t_1, u), u(s) - x(s; x_2, t_2, u), u(s)\|\, ds.
\end{aligned}$$

An application of the Bellman-Gronwall Lemma now yields the desired inequality. □

Lemma 3.1 extends to hybrid executions using the results of Theorem 2.1. Assume that t_1 is not a switching time in a hybrid execution $e_{\mathbb{H}}$ and that $(q_1(t_1), x_1(t_1))$ is the hybrid state at t_1. Then as in the proof of Theorem 2.1, there exists $t' > 0$ such that no switching is defined over the interval $[t_1, t')$ in the execution $e_{\mathbb{H}}$. In this case Lemma 3.1 generalizes as follows.

LEMMA 3.2. *(Dependence on Initial Conditions (Hybrid)) Suppose that the time t_1 along a hybrid execution $e_{\mathbb{H}}$ is not a switching time and that $(q_1(t_1), x_1(t_1))$ is the initial hybrid state at t_1. Let $(q_1(t), x_t^1)$ be the hybrid state in the execution $e_{\mathbb{H}}$, with input I, at time $t > t_1$ where t is not a switching time either. Then there exist a neighbourhood $N_{(t_1,x_1)}$ of $(t_1, x_1) \in \mathbb{R} \times \mathbb{R}^n$ such that if $(t_2, x_2) \in N_{(t_1,x_1)}$ (assume without loss of generality $t_2 \geq t_1$) then $(q_2(t), x_t^2) = h(t; (t_2, x_2), I)$ satisfies $q_2(t) = q_1(t) \in Q$ and*

$$\|x_t^1 - x_t^2\| \leq (\|x_1 - x_2\| + K_f|t_1 - t_2|)\, e^{L_f(t - t_1 \vee t_2)}.$$

□

THEOREM 3.2. *([**42**] after [**2**])*

(i) $v(t, h)$ is bounded on $[t_0, t_f] \times Q \times \mathbb{R}^n$,

(ii) if there exists $L_l < \infty$ such that $|l_j(x_1, u) - l_j(x_2, u)| \leq L_l\|x_1 - x_2\|$, $x_1, x_2 \in \mathbb{R}^n$, $u \in U$, $j \in Q$ and there exists $L_g < \infty$ such that $|g(x_1) - g(x_2)| \leq L_g\|x_1 - x_2\|$, $x_1, x_2 \in \mathbb{R}^n$, and if t is not a switching time, then $v(t, h)$ is continuous at $(t, h) \in [t_0, t_f] \times Q \times \mathbb{R}^n$.

□

PROOF. *(i) Boundedness*

Fix $I_L \in \{I\}$, with $L < \infty$. Then

$$J(t, h; I) \leq \int_t^{t_{k+1}} |l_{q_k}(x_{q_k}(s), u(s))|\, ds + \sum_{i=k+1}^{L} \int_{t_i}^{t_{i+1}} |l_{q_i}(x_{q_i}(s), u(s))|\, ds + |g(x_{q_L}(t_f))|. \tag{3.2}$$

Take, without loss of generality, λ and δ in Assumptions A2 and A3 to be integers. Let $x_0 \triangleq x_{q_0}(t_0)$, $K_f \triangleq \max\{L_f, \max_{q\in Q}\sup_{u\in U} f_q(0,u)\}$. Then using Assumptions A0, A2 and A3 and employing Bellman-Gronwall Lemma we obtain following bounds on the terms in (3.2)

(i)

$$\int_t^{t_{k+1}} |l_{q_k}(x_{q_k}(s), u(s))|\, ds \leq K_l(t_{k+1}-t)$$
$$+K_l \sum_{j=0}^{\gamma} \binom{\gamma}{j} K_f^j \|x_0\|^{\gamma-j}\left(p_j(t_{k+1}-t)e^{\gamma K_f(t_{k+1}-t)} + (-1)^{j+1}\frac{j!}{(\gamma K_f)^{j+1}}\right),$$

where $p_j(\cdot)$ is a j-th order polynomial such that $p_j(0) = (-1)^j \frac{j!}{(\gamma K_f)^{j+1}}$. Let us define

$$\beta_j(\tau) \triangleq p_j(\tau)e^{\gamma K_f \tau} + (-1)^{j+1}\frac{j!}{(\gamma K_f)^{j+1}}.$$

Then $\beta(0) = 0$.

(ii)

$$\int_{t_i}^{t_{i+1}} |l_{q_i}(x_{q_i}(s), u(s))|\, ds \leq K_l(t_{i+1}-t_i) + K_l \sum_{j=0}^{\gamma} \binom{\gamma}{j} K_f^j \|x_0\|^{\gamma-j}\beta(t_{i+1}-t_i)$$

(iii)

$$|g(x_{q_L}(t_f))| \leq K_g + K_g \sum_{j=0}^{\delta} \binom{\delta}{j} K_f^j \|x_0\|^{\delta-j} K_f (t_f - t_0)^j e^{\delta K_f(t_f-t_0)}.$$

Using the above bounds and taking the infimum over $\{I\}$ in (3.2) we get an upper bound on $v(t,h)$.

(ii) Continuity

Fix $q \in Q$, $I \in \{I\}$, $\epsilon > 0$ such that $v(t_2,h_2) \geq J(t_2,h_2,I) - \epsilon$. Then

$$v(t_1,h_1) - v(t_2,h_2) \leq J(t_1,h_1,I) - J(t_2,h_2,I) + \epsilon,$$

or

$$\begin{aligned} v(t_1, h_1) - v(t_2, h_2) \;\le\; & \int_{t_1}^{t_{k+1}} l_{q_k}(x'_{q_k}(s), u(s))\, ds \\ & + \sum_{i=k+1}^{L} \int_{t_i}^{t_{i+1}} l_{q_i}(x'_{q_i}(s), u(s))\, ds + g(x'_{q_L}(t_f)) \\ & - \int_{t_2}^{t_{k+1}} l_{q_k}(x''_{q_k}(s), u(s))\, ds \\ & - \sum_{i=k+1}^{L} \int_{t_i}^{t_{i+1}} l_{q_i}(x''_{q_i}(s), u(s))\, ds - g(x''_{q_L}(t_f)) + \epsilon. \end{aligned}$$

Assume again without loss of generality that $t_1 \le t_2$. Then, dropping the subscripts on x for brevity, we obtain

$$\begin{aligned} |v(t_1, h_1) - v(t_2, h_2)| \;\le\; & \int_{t_1}^{t_2} |l_{q_k}(x'(s), u(s))|\, ds + \int_{t_2}^{t_{k+1}} |l_{q_k}(x'(s), u(s)) \\ & - l_{q_k}(x''(s), u(s))|\, ds + \sum_{i=k+1}^{L} \int_{t_i}^{t_{i+1}} |l_{q_i}(x'(s), u(s)) \\ & - l_{q_i}(x''(s), u(s))|\, ds + |g(x'(t_f)) - g(x''(t_f))| + \epsilon \\ \le\; & K_l(t_2 - t_1) + K_l \sum_{j=0}^{\gamma} \binom{\gamma}{j} K_f^j \|x_0\|^{\gamma-j} \beta(t_2 - t_1) \\ & + L_l \int_{t_2}^{t_{k+1}} \|x'(s) - x''(s)\|\, ds \\ & + L_l \sum_{i=k+1}^{L} \int_{t_i}^{t_{i+1}} \|(x'(s) - x''(s))\|\, ds + L_g \|x'(t_f) - x''(t_f)\| + \epsilon. \end{aligned}$$

Next, we use Lemma 3.1, define

$$\alpha(t_2 - t_1) \;\triangleq\; K_l(t_2 - t_1) + K_l \sum_{j=0}^{\gamma} \binom{\gamma}{j} K_f^j \|x(t_1)\|^{\gamma-j} \beta(t_2 - t_1),$$

and note that $\alpha(0) = 0$. This yields

$$
\begin{aligned}
|v(t_1,h_1) - v(t_2,h_2)| \;\leq\;& \alpha(t_2 - t_1) + L_l \int_{t_2}^{t_{k+1}} (\|x_1 - x_2\| + K_f|t_1 - t_2|)\, e^{L_f(s-t_2)}\, ds \\
& + L_l \sum_{i=k+1}^{L} \int_{t_i}^{t_{i+1}} (\|x_1 - x_2\| + K_f|t_1 - t_2|)\, e^{L_f(s-t_2)}\, ds \\
& + L_g\, (\|x_1 - x_2\| + K_f|t_1 - t_2|)\, e^{L_f(t_f-t_2)} + \epsilon \\
=\;& \alpha(t_2 - t_1) + (\|x_1 - x_2\| + K_f|t_1 - t_2|) \\
& \times \left(\frac{L_l}{L_f} \left(e^{L_f(t_{k+1}-t_2)} - 1 + \sum_{i=k+1}^{L} \left(e^{L_f(t_{i+1}-t_2)} - 1 \right)\right)\right. \\
& \qquad \left. + L_g\, e^{L_f(t_f-t_2)} \right) + \epsilon \\
=\;& \alpha(t_2 - t_1) + \frac{L_l}{L_f} (\|x_1 - x_2\| + K_f|t_1 - t_2|) \\
& \times \left(\sum_{i=k}^{L} \left(e^{L_f(t_{i+1}-t_2)} - 1 \right) + L_g L_l\, e^{L_f(t_f-t_2)} \right) + \epsilon.
\end{aligned}
$$

□

THEOREM 3.3. *([42] after [2]) Assume A0–A3 hold. Let $v(t,x)$ be a value function for the HOCP($t_0, t_f, h_0, \bar{L}, \mathcal{U}$) with $\bar{L} = \infty$ and $L^0(\bar{L}) < \infty$. If $v(t,x)$ is continuously differentiable on $[t_0, t_f] \times \mathbb{R}^n$ for each fixed $q \in Q$ and if t is not a switching time then v satisfies the following HJB equation:*

$$
\begin{aligned}
\frac{\partial v}{\partial t}(t,x) \;&+\; \inf_{q \in Q, u \in U} \left(\left(\frac{\partial v}{\partial x}(t,x) \right)^T f_q(x,u) + l_q(x,u) \right) = 0, \\
v(t_f, x) \;&=\; g(x), \quad \forall x \in \mathbb{R}^n.
\end{aligned}
$$

□

Proof. The boundary condition follows immediately from the Hybrid Dynamic Programming Principle. In order to derive the HJB equation we write the HDPP as

$$\begin{aligned} 0 &= \inf_{I\in\hat{I}[t',t_f]} \left(\int_{t'}^{t_{k+1}} l_{q_k}(x_{q_k}(s),u(s))\,ds + \sum_{i=k+1}^{l-1} \int_{t_i}^{t_{i+1}} l_{q_i}(x_{q_i}(s),u(s))\,ds \right. \\ &\qquad \left. + \int_{t_l}^{t} l_{q_l}(x_{q_l}(s),u(s))\,ds + v(t,h(t;h',I[t',t])) - v(t',h') \right). \end{aligned}$$

Since t is not a switching time, there exits a $t_* > t$ such that there is no switching time in the interval $[t,t_*)$. Let $t' \in (t,t_*)$ so that $t'-t>0$. In this case the above equation becomes

$$0 = \inf_{q\in Q,u\in U} \left(\int_{t'}^{t} l_q(x(s),u(s))\,ds + v(t,x) - v(t',x') \right).$$

Next, we divide both sides by $t-t'$ and let t' approach t to obtain

$$\begin{aligned} 0 &= \inf_{q\in Q,u\in U} \left(\lim_{t'\to t} \frac{1}{t-t'} \int_{t'}^{t} l_q(x(s),u(s))\,ds + \lim_{t'\to t}\frac{1}{t-t'}\left(v(t,x)-v(t',x')\right)\right) \\ &= \inf_{q\in Q,u\in U} \left(l_q(x(t),u(t)) + \frac{dv}{dt}(t,x) \right) \\ &= \inf_{q\in Q,u\in U} \left(l_q(x(t),u(t)) + \frac{\partial v}{\partial t}(t,x) + \left(\frac{\partial v}{\partial x}(t,x)\right)^T \frac{dx_q}{dt} \right) \\ &= \inf_{q\in Q,u\in U} \left(l_q(x(t),u(t)) + \frac{\partial v}{\partial t}(t,x) + \left(\frac{\partial v}{\partial x}(t,x)\right)^T f_q(x_q(t),u(t)) \right), \end{aligned}$$

or, since $\frac{\partial v}{\partial t}$ is independent of (u,q),

$$\frac{\partial v}{\partial t}(t,x) + \inf_{q\in Q,u\in U} \left(l_q(x(t),u(t)) + \left(\frac{\partial v}{\partial x}(t,x)\right)^T f_q(x_q(t),u(t)) \right) = 0, \tag{3.3}$$

which is the required HJB equation. □

The HJB equation can be used to establish a sufficiency condition for a candidate control (we use the term control to mean continuous control function and switching sequence; control value at time t would be the pair $(u(t),q(t))$) to be optimal. It is a generalization of well known result in optimal control theory and calculus of variations known as the *verification theorem.*

THEOREM 3.4. (Hybrid Verification Theorem)

(i) Let $(\hat{u}(x,w),\hat{q})$ *be such that*

$$l_{\hat{q}}(x,\hat{u}(x,w)) + w^T f_{\hat{q}}(x,\hat{u}(x,w)) = \inf_{q\in Q, u\in U}\left(l_q(x,u) + w^T f_q(x,u)\right), \quad \forall w \in \mathbb{R}^n.$$

(ii) Suppose there is an admissible control $(u^0(\cdot), q^0(\cdot))$ *resulting in the hybrid trajectory* $h^0(\cdot) = (x^0(\cdot), q^0(\cdot))$ *such that* $h_0^0(t_0) = (x^0_{q^0_0}(t_0), q^0(t_0)) = (x_0, q_0)$.

(iii) Assume that there is a solution $v(t,x)$ *of the HJB equation (3.3) such that*

(a) $v(t_f, x) = g(x), \quad \forall h \in Q \times \mathbb{R}^n$ *and*

(b) $u^0(t) = \hat{u}\big(x^0(t), \frac{\partial v}{\partial x}(t, x^0(t))\big)$, $u^0(\cdot) \in \mathcal{U}^0$ *or* $u^0(\cdot) \in \mathcal{U}^{\mathrm{cpt}}$.

Then $v(t,x)$ *is the value function for the HOCP and the control* (u^0, q^0) *is an optimal control.*

PROOF. By the hypotheses of the theorem we have

$$\frac{\partial v}{\partial t}(t,x) + \left(\frac{\partial v}{\partial x}(t,x)\right)^T f_{q^0}(x_{q^0}(t), u^0(t)) + l_{q^0}(x^0_{q^0}(t), u^0(t)) = 0.$$

Hence for a.e. $t \in [t_0, t_f]$

$$\frac{dv}{dt}(t,x) + l_{q^0}(x^0_{q^0}(t), u^0(t)) = 0.$$

We integrate the above equation to obtain

$$\int_t^{t_f} \frac{dv}{ds}(s,x)\, ds + \int_t^{t_f} l_{q^0}(x^0_{q^0}(s), u^0(s))\, ds = 0,$$

or

$$v(t_f, x(t_f)) - v(t,x) = -\int_t^{t_f} l_{q^0}(x^0_{q^0}(s), u^0(s))\, ds.$$

Now since by boundary condition (iii) (a) $v(t_f, x(t_f)) = g(x(t_f))$

$$v(t,x) = \int_t^{t_f} l_{q^0}(x^0_{q^0}(s), u^0(s))\, ds + g(x(t_f)),$$

which shows that $v(t,x)$ is the cost of transferring (t,x) to $(t_f, x(t_f))$ along the trajectory generated by (u^0, q^0). The following analysis will show that (u^0, q^0) achieves this transfer optimally, i.e. (u^0, q^0) is an optimal control and $v(t,x)$ is the value function.

Let $(\tilde{u}(\cdot), \tilde{q}(\cdot))$ be an arbitrary control resulting in the hybrid trajectory $\tilde{h}(\cdot) = (\tilde{x}(\cdot), \tilde{q}(\cdot))$. Then, by assumption (i) of the theorem, if $(t,x) \in \mathbb{R} \times \mathbb{R}^n$ is not a switching time-state

$$\begin{aligned} l_{\tilde{q}}(\tilde{x}(t), \tilde{u}(t)) + \left(\frac{\partial v}{\partial x}(t,x)\right)^T f_{\tilde{q}}(\tilde{x}(t), \tilde{u}(t)) \;&\geq\; l_{q^0}(x^0(t), u^0(t)) + \\ &\qquad \left(\frac{\partial v}{\partial x}(t,x)\right)^T f_{q^0}(x^0(t), u^0(t)) \\ &= -\frac{\partial v}{\partial t}(t,x), \end{aligned}$$

or

$$l_{\tilde{q}}(\tilde{x}(t), \tilde{u}(t)) + \left(\frac{\partial v}{\partial x}(t,x)\right)^T f_{\tilde{q}}(\tilde{x}(t), \tilde{u}(t)) + \frac{\partial v}{\partial t}(t,x) \geq 0. \tag{3.4}$$

Notice that along the two trajectories generated by the controls $(\tilde{u}(\cdot), \tilde{q}(\cdot))$ and $(u^0(\cdot), q^0(\cdot))$ we have respectively

$$\dot{\tilde{x}} = f_{\tilde{q}}(\tilde{x}, \tilde{u}), \qquad \tilde{x}(t) = x, \qquad \text{and} \qquad \dot{x}^0 = f_{q^0}(x^0, u^0), \qquad x^0(t) = x.$$

Now inequality (3.4) can be written as

$$l_{\tilde{q}}(\tilde{x}(t), \tilde{u}(t)) + \frac{dv}{dt}(t,x) \geq 0.$$

Integrating both sides from t to t_f yields

$$\int_t^{t_f} l_{\tilde{q}}(\tilde{x}(s), \tilde{u}(s))\, ds + \int_t^{t_f} \frac{dv}{ds}(s,x)\, ds \geq 0,$$

or

$$\int_t^{t_f} l_{\tilde{q}}(\tilde{x}(s), \tilde{u}(s))\, ds + v(t_f, \tilde{x}(t_f)) - v(t,x) \geq 0,$$

or

$$v(t,x) \leq \int_t^{t_f} l_{\tilde{q}}(\tilde{x}(s), \tilde{u}(s))\, ds + g(\tilde{x}(t_f)),$$

which shows that $(u^0(\cdot), q^0(\cdot))$ is the optimal control and $v(\cdot,\cdot)$ is the value function.

$\square$

CHAPTER 4

Hybrid Optimality Algorithms

4.1. Introduction

4.1.1. General Introduction. In Chapter 2 a class of hybrid optimal control problems is formulated for general hybrid systems with nonlinear dynamics in each location (i.e. discrete state) and with autonomous and controlled switchings. Employing these conditions, we propose and analyze a new class of so-called Hybrid Minimum Principle (HMP) algorithms for the solution of hybrid optimal control problems (HOCP). We provide convergence results for the optimization algorithm called HMP[MAS] which treats HOCPs where multiple autonomous switchings (MAS) occur, that is to say, where the location switches whenever the continuous state passes through specified switching manifolds. Our convergence proofs are based on results from the theory of penalty function methods [**3**] and Ekeland's variational principle [**26**]. The HMP[MAS] algorithm extends directly to multiple controlled switchings (MCSs) when the location sequence is fixed; for this case, we present an algorithm called HMP[MCS] which invokes (i.e. calls) HMP[MAS] and hence computes a sequence of optimal switching times and states for a given HOCP. The efficacy of these algorithms is illustrated via several computational examples.

The class of HMP algorithms may be embedded in the so-called HMP[Comb] (see [**45, 46**]) algorithm class; this extends the HMP class with combinatoric search algorithms which find (combinatorially local) optimal location sequences and their associated locally optimal switching times and control inputs. HMP[Comb] algorithms generate a list of Hamming distance ($\leq k$) sequences from an initial sequence, and finds the optimal sequence by executing the HMP[MCS] on each one of them (see [**44, 45, 46**]). Other recently proposed hybrid optimization algorithms, for example those of [**58, 60, 24, 25, 54**], can be extended to perform combinatoric search, with an associated exponential increase in the computational cost.

Next we present the notion of optimality zones and the associated HMP[Z] algorithms; it is shown that knowledge of these zones permits one to reach the global optimum in a single run of the HMP[MCS] algorithm. In particular, we give examples for the case of linear dynamics with quadratic cost criteria where these zones have a geometrically simple form. The notion of optimality zones must be distinguished from the so-called "switching regions" presented in [**27, 28, 7, 6, 5**]; switching regions partition the continuous state space of autonomous (steady state) hybrid systems whereas optimality zones partition the time-state space of finite horizon HOCPs. The optimal control problems considered in [**27, 28, 7, 6, 5**] have autonomous linear time-invariant dynamics, quadratic loss function in each location and a fixed finite sequence of locations.

4.1.2. Theoretical Framework for the Optimization Algorithms. The results of this chapter derive from the inter-relationship of the following:

(i) *The Hybrid Optimal Control Problem (HOCP).* The HOCP is formulated in Chapter 2 where necessary conditions are presented which are satisfied by locally (and hence also by globally) optimal solutions to HOCP.

(ii) *The Hybrid Optimization Algorithm HMP[MAS].* An optimization algorithm for multiple autonomous switchings called HMP[MAS] is presented in Section 4.2.1. This algorithm uses an iteration based upon the necessary conditions and the associated boundary conditions presented in Chapter 2 to compute a sequence of control inputs and switching time-state pairs $\{u^k, t_s^k, x_s^k\}_{k\in\mathbb{Z}_+}$.

(iii) *Penalty Function Convergence Analysis.* Let us assume the global solution to HOCP is unique and no distinct local optimal solutions exist. We shall denote the necessary conditions of Chapter 2 by the highly abbreviated equations $\eta = 0$, $\dot{\lambda} = -H_x$, $H = H_q(x, \lambda, u)$, where $\eta = 0$ represents the necessary conditions satisfied by the Hamiltonian, the adjoint variable and the switching manifold constraint at the optimal switching time, λ is the adjoint variable whose left and right limits at a switching time t_s are denoted $\lambda_s^{+/-}$, and H is the Hamiltonian of the system. The global solution (which must satisfy the necessary conditions, $\eta = 0$, $\dot{\lambda} = -H_x$, and the associated boundary conditions of Chapter 2 is identical to the (t_s, x_s, u) projection of the (assumed unique) (t_s, x_s, u, λ) solution to:

$$J^* = \inf_{t_s, x_s} \left(\inf_u J(x(u), u; t_s, x_s) : \eta(t_s, x_s, \lambda_s^{+/-}) = 0; \dot{x} - H_\lambda = 0; \dot{\lambda} + H_x = 0 \right). \quad (4.1)$$

By applying the penalty function methods of Section 4.2.2, any $\{(t_s, x_s)_{r_k}\}$ solution sequence to (4.2) below (where the differential constraints on x and λ are not displayed) is shown to have a sub-sequential limit which is a solution $(t_s, x_s)^*$ to (4.1), where

$$J_p^* = \lim_{r \to \infty} \left(\inf_{t_s, x_s} \left(\inf_u J(x(u), u; t_s, x_s) + r\eta(t_s, x_s, \lambda_s^{+/-}) \right) \right), \quad (4.2)$$

for which moreover it is proven that $J^* = J_p^*$.

(iv) *Association of the Algorithm HMP[MAS] with (4.2).* Now let us assume that the constraints and iterations of Algorithm HMP[MAS] correspond to the infimization of (4.2). This is in the sense that (see [A5] below) for any required level of accuracy $\epsilon > 0$ and corresponding choices of parameters in the Algorithm HMP[MAS], each of the limit (i.e. halting) points (t_s^K, x_s^K) of the Algorithm and the generated values $(\mathrm{Alg})^K$ are within ϵ of the limit point of a sub-sequence $\{(t_s, x_s)_{r_k}\}$ and its associated values $(J_p^*)_{r_k}$ respectively, where the latter are generated by (4.2) for all sufficiently large r_k where r_k is a multiplier of the penalty term $\eta(t_s, x_s, \lambda_s^{+/-})$.

(v) *Resulting Convergence of the Algorithm.* Hence, if we assume that the entire set of sub-sequential limits of solutions to (4.2) consists exactly of the solutions to (4.1), we may conclude that (for suitably chosen parameters) the halting points and value of the Algorithm are within the arbitrary specified ϵ of the global solution to HOCP.

(In Section 4.2.3 below, the solutions to (4.2) for increasing $\{r_k\}$ are simply identified with the values generated by the Algorithm HMP[MAS] at successive iterations.)

4.2. Hybrid Trajectory Optimization Algorithms

Based on the necessary conditions of Chapter 2 (and the assumed generalization with respect to time varying switching manifolds) we formulate the following algorithm for optimizing the location switching times τ and associated continuous controls u for a given location location sequence σ.

4.2.1. HMP[MAS] (Multiple Autonomous Switchings) Conceptual Algorithm. For simplicity we present the single autonomous switching case (see Comment 2 below).

0. `Algorithm Initialization:` `Fix` $0 < \epsilon_1 \ll 1$, $0 < \epsilon_2 \ll 1$, $0 < \epsilon_f \ll 1$ `and` $0 \le \mu \le 1$. `Let` (t_s, x_s) `be a nominal switching time-state pair such that` $t_0 < t_s < t_f$. `Set the iteration counter` $k = 0$. `Set` $t_s^k = t_s$ `and.` $x_s^k = x_s$. `Compute the optimal control functions` $u_1^k(t)$, $0 \le t < t_s$ `and` $u_2^k(t)$, $t_s \le t \le t_f$. `Compute the associated state and costate trajectories and Hamiltonians over the two intervals` $[0, t_s^k]$ `and` $[t_s^k, t_f]$, `with the terminal state pairs` (x_0, x_s^k) `and` (x_s^k, x_f) `respectively. Also compute the new total cost` $J^k(t_s^k, x_s^k)$.

1. `Increment` k `by 1. Compute` $\nabla_x m(t_s^{k-1}, x_s^{k-1})$ `and` $\nabla_t m(t_s^{k-1}, x_s^{k-1})$.

2. Set

$$\eta_k \triangleq \left\| \begin{pmatrix} H_1^k(t_s^{k-1}) - H_2^k(t_s^{k-1}) \\ \lambda_2^k(t_s^{k-1}) - \lambda_1^k(t_s^{k-1}) \end{pmatrix} - \begin{pmatrix} \nabla_t m(t_s^{k-1}, x_s^{k-1}) \\ \nabla_x m(t_s^{k-1}, x_s^{k-1}) \end{pmatrix} p^k \right\|^2 + \left\| m(t_s^{k-1}, x_s^{k-1}) \right\|^2 .$$

Set

$$Q_k \triangleq \begin{pmatrix} \nabla_t m(t_s^{k-1}, x_s^{k-1}) \\ \nabla_x m(t_s^{k-1}, x_s^{k-1}) \end{pmatrix}$$

and compute the unique minimizing argument $p^k \in \mathbb{R}$ of η_k given by

$$p^k = (Q_k^T Q_k)^{-1} Q_k^T \begin{pmatrix} H_1^k(t_s^{k-1}) - H_2^k(t_s^{k-1}) \\ \lambda_2^k(t_s^{k-1}) - \lambda_1^k(t_s^{k-1}) \end{pmatrix}.$$

3. Set

$$\begin{aligned} t_s^k = t_s^{k-1} \quad &- \quad \epsilon_1 \left(H_1^k(t_s^{k-1}) - H_2^k(t_s^{k-1}) - \nabla_t m(t_s^{k-1}, x_s^{k-1}) p^k \right) \\ &- \quad \epsilon_1 \, \nabla_t m(t_s^{k-1}, x_s^{k-1}) \, m(t_s^{k-1}, x_s^{k-1}). \end{aligned}$$

4. Set

$$\begin{aligned} x_s^k = x_s^{k-1} \quad &- \quad \epsilon_2 \left(\lambda_2^k(t_s^{k-1}) - \lambda_1^k(t_s^{k-1}) - \nabla_x m(t_s^{k-1}, x_s^{k-1}) p^k \right) \\ &- \quad \epsilon_2 \, \nabla_x m(t_s^{k-1}, x_s^{k-1}) \, m(t_s^{k-1}, x_s^{k-1}). \end{aligned}$$

5. Compute the optimal control functions $u_1^k(t)$, $0 \le t < t_s$ and $u_2^k(t)$, $t_s \le t \le t_f$. Compute the associated state and costate trajectories and Hamiltonians over the two intervals $[0, t_s^k]$ and $[t_s^k, t_f]$ with the terminal state pairs (x_0, x_s^k) and (x_s^k, x_f) respectively. Next, compute the new total cost $J^k(t_s^k, x_s^k)$.

6. `If` $\mu(J^{k-1}(t_s^{k-1}, x_s^{k-1}) - J^k(t_s^k, x_s^k)) + (1-\mu)\eta_k < \epsilon_f$ `then STOP;`
`else go to Step 1.`

Comments on HMP[MAS].

1. Since there is no switching cost, the function $J(t_s, x_s)$ is continuous but not necessarily differentiable at $(t_s^{\text{opt}}, x_s^{\text{opt}})$ (see Chapter 3).

   Concerning the relationship between the various necessary conditions, we note that (see [**9**])

$$\begin{aligned}&\eta(t_s^{\text{opt}}, x_s^{\text{opt}}) = 0\\ &\iff \left(m(t_s^{\text{opt}}, x_s^{\text{opt}}) = 0\right) \bigwedge \left(\frac{\partial J}{\partial x_s}(t_s^{\text{opt}}, x_s^{\text{opt}}) = 0\right) \bigwedge \left(\frac{\partial J}{\partial t_s}(t_s^{\text{opt}}, x_s^{\text{opt}}) = 0\right).\end{aligned}$$

   The expressions

$$\left(H_1^k(t_s^{k-1}) - H_2^k(t_s^{k-1}) - \nabla_t m(t_s^{k-1}, x_s^{k-1}) p^k\right)$$

   and

$$\left(\lambda_2^k(t_s^{k-1}) - \lambda_1^k(t_s^{k-1}) - \nabla_x m(t_s^{k-1}, x_s^{k-1}) p^k\right)$$

   approximate $\frac{\partial J^{\text{opt}}}{\partial t}(t_s^{k-1}, x_s^{k-1})$ and $\frac{\partial J^{\text{opt}}}{\partial x}(t_s^{k-1}, x_s^{k-1})$, respectively, in the neighbourhood of $(t_s^{\text{opt}}, x_s^{\text{opt}})$ where $m(t_s^{\text{opt}}, x_s^{\text{opt}}) = 0$ (see [**9**]) Hence their use in Steps 3 and 4 where we note that on $\{m(t,x) = 0\}$ their approximation of $\frac{\partial J^{\text{opt}}}{\partial x}$ and $\frac{\partial J^{\text{opt}}}{\partial t}$ guarantees that the steps in 3, 4 are in the correct direction.

2. Algorithm 4.2.1 can be generalized to the multiple autonomous switchings case in a straightforward manner. This is possible because of assumption A0 of Chapter 2 that switching manifolds never intersect. The algorithm can also be specialized to the controlled switchings case by skipping Step 2 and setting $p^k = 0$ in Steps 3 and 4.

3. We observe that, as for HMP[MAS], the recently proposed hybrid control algorithms of [**58, 59, 60**] repeatedly compute the optimal control functions in each location.

4. Let $\delta z_s^k \triangleq \begin{pmatrix} \delta t_s^k \\ \\ \delta x_s^k \end{pmatrix} = \begin{pmatrix} t_s^k - t_s^{k-1} \\ \\ x_s^k - x_s^{k-1} \end{pmatrix}$ and $\gamma_k \triangleq \begin{pmatrix} H_1^k(t_s^{k-1}) - H_2^k(t_s^{k-1}) \\ \\ \lambda_2^k(t_s^{k-1}) - \lambda_1^k(t_s^{k-1}) \end{pmatrix}$. Assume that at iteration k (t_s^{k-1}, x_s^{k-1}) is in the switching manifold, i.e. $m(t_s^{k-1}, x_s^{k-1}) = 0$. Then by use of a Taylor series expansion:

$$\begin{aligned} m(t_s^k, x_s^k) &= \nabla_t m(t_s^{k-1}, x_s^{k-1})\delta t_s^k + \left(\nabla_x m(t_s^{k-1}, x_s^{k-1})\right)^T \delta x_s^k + o(\|\delta z_s^k\|) \\ &= Q_k^T \delta z_s^k + o(\|\delta z_s^k\|). \end{aligned} \tag{4.3}$$

We assume that Q_k is nonzero. The update equations in Steps 2, 3, 4 (setting $\epsilon_1 = \epsilon_2 = \epsilon$) are $\delta z_k = -\epsilon(\gamma_k - Q_k p^k)$. The analytic minimization of η_k with respect to p^k yields $p^k = (Q_k^T Q_k)^{-1} Q_k^T \gamma_k$ and this implies that $Q_k^T \delta z_s^k = -\epsilon Q_k^T(\gamma_k - Q_k p^k) = 0$. (This corresponds to the fact that the vector difference of γ_k and its orthogonal projection on the span $\mathrm{Sp}(Q_k)$ of the columns of Q_k, i.e. the normals to the switching manifold $\{m = 0\}$, is orthogonal to $\mathrm{Sp}(Q_k)$.) Hence Equation 4.3 implies:

LEMMA 4.1. *Let $\epsilon_1 = \epsilon_2 = \epsilon$. At any iteration k in the Algorithm HMP[MAS], if (t_s^{k-1}, x_s^{k-1}) is such that $m(t_s^{k-1}, x_s^{k-1}) = 0$ then $m(t_s^k, x_s^k)$ is of $o(\|\delta z_s^k\|)$, i.e. to first order, δz_s^k lies in the tangent space $Q_k^{\perp}$ to the switching manifold $\{m = 0\}$.* □

6. One of the objectives of Algorithm HMP[MAS] is to drive the vector η_k in Step 2 to zero so as to satisfy the conditions of the Hybrid Minimum Principle of Chapter 2 This is not sufficient to ensure optimality but in the following we will give conditions under which (at least a reasonable mathematical model of) the conceptual algorithm generates values which converge to values satisfying sufficient conditions for optimality.

We state the convergence properties of the algorithm HMP[MAS] in Proposition 4.1 below whose proof mimics that of the convergence of the interior penalty method for equality constrained finite dimensional optimization problems [**3**].

4.2.2. General Results for Penalty Function Methods. Penalty Function Methods can be used to convert an optimization problem involving a cost function and equality constraints into a sequence of unconstrained optimization problems. We note the following associations:

(i) The cost function J^k, computed in Steps 0 and 5 of Algorithm HMP[MAS], is generally a function of the input trajectories $u_{[0,t_s)}$ and $u_{[t_s,t_f]}$ over the two time intervals $[0, t_s)$ and $[t_s, t_f]$ respectively and the switching time-state pair z_s. Since $u_{[0,t_s)}$ and $u_{[t_s,t_f]}$ are computed to be optimal in their respective locations

$$J^k(z_s) = \inf_{u_{[0,t_s)}, u_{[t_s,t_f]}} J^k(z_s, u_{[0,t_s)}, u_{[t_s,t_f]}).$$

Hence J^k is associated with the cost function of a finite-dimensional optimization problem.

(ii) Similarly, η_k in Step 2 of Algorithm is a function of z_s only and is associated to the equality constraints in a finite-dimensional optimization problem.

Given a fixed initial state $x(0) = x_0$, assumption A0 implies that $\|x(t)\| \leq (\|x_0\| + K_f t_f)\, e^{L_f t}$, $0 \leq t \leq t_f$. Hence if t_f is finite then the set $S \triangleq \{(t, x) \in [0, t_f] \times \mathbb{R}^n : \|x\| \leq (\|x_0\| + K_f t_f)\, e^{L_f t}\} = [0, t_f] \times \{x \in \mathbb{R}^n : \|x\| \leq (\|x_0\| + K_f t_f)\, e^{L_f t}\}$ is compact and convex. Let us set $z \triangleq (t, x)$ and let $S' \triangleq cl\{z \in S : m(z) \equiv m(t, x) = 0\}$. Notice that if $(t_s, x_s) \in S'$ then $x(x_0, t_s) = x_s$. Clearly all switching time-state pairs lie in S'; so assuming a switching time exists in the interval $[0, t_f]$, S' is a nonempty compact set and S' can be made a complete metric space by endowing it with the metric $d(z_1, z_2) \triangleq \|z_1 - z_2\|$.

Define a penalty function

$$\begin{aligned} R(z_s, r) &\triangleq J(z_s) + r\eta(z_s) \\ &= J(z_s) + r(\|\gamma(z_s) - Q(z_s)p(z_s)\|^2 + \|m(z_s)\|^2) \\ &= J(z_s) + r(\|\gamma(z_s) - (Q^T(z_s)Q(z_s))^{-1}Q^T(z_s)\gamma(z_s)\|^2 + \|m(z_s)\|^2), \end{aligned}$$

where $z_s \triangleq (t_s, x_s)$, γ is defined above and J, γ and Q are computed at z_s as in Steps 2 and 3 of the Algorithm HMP[MAS].

To discuss general penalty function methods we consider the continuous functions $J, \eta : \mathbb{R}^n \to \mathbb{R}$ and the compact set S. (J, η, S in this section need not be identified with J, η, S in Section 4.2.1.)

Define a function $\psi(r) \triangleq \inf_{z \in S}\{J(z) + r\eta(z)\}$. Then in view of the compactness of S and the continuity of $J(\cdot)$, and $\eta(\cdot)$ it is clear that for each r there exists z_r in S such that $\psi(r) = \inf_{z \in S}\{J(z) + r\eta(z)\} = J(z_r) + r\eta(z_r)$.

LEMMA 4.2. *([3]) For the continuous functions $J, \eta : \mathbb{R}^n \to \mathbb{R}$ defined over a compact set S and $\psi(r) \triangleq \inf_{z \in S}\{J(z) + r\eta(z)\} = J(z_r) + r\eta(z_r)$: (i) $\inf_{z \in S}\{J(z) : \eta(z) = 0\} \geq \sup_{r \geq 0} \psi(r)$;*
(ii) (a) $\eta(z_r)$ is a decreasing function of r, (b) $J(z_r)$ is an increasing function of r, (c) $\psi(r)$ is an increasing function of r. □

THEOREM 4.1. *([3]) For the continuous functions $J, \eta : \mathbb{R}^n \to \mathbb{R}$ defined over a compact set S and $\psi(r) = \inf_{z \in S}\{J(z) + r\eta(z)\}$:*

$$\inf_{z \in S}\{J(z) : \eta(z) = 0\} = \sup_{r \geq 0} \psi(r) = \lim_{r \to \infty} \psi(r).$$

□

4.2.3. Convergence Analysis of Algorithm HMP[MAS]. Theorem 4.1 above is not immediately applicable to the convergence analysis of the Algorithm HMP[MAS]. In order to make such an application, we first recall that

$$J^k(z_s) = \inf_{u_{[0,t_s)}, u_{[t_s,t_f]}} J^k(z_s, u_{[0,t_s)}, u_{[t_s,t_f]});$$

then the missing link is provided by the following key assumptions, where we associate $J(z_k)$ here with $J^k(z_s^k)$ in Step 5 of the algorithm.

A4 There is a unique optimizing switching time-state pair $z_s^0 \triangleq (t_s^0, x_s^0) \in S'$.

A5′ At the kth iteration, at Steps 3, 4 and 5, the Algorithm HMP[MAS] computes $z_k = \arg\min_{z \in S}\{J(z) + r_k \eta(z)\}$, where the sequence $\{r_k\}_{k=1}^{\infty}$ is a strictly increasing sequence of positive numbers such that $\lim_{k \to \infty} r_k = \infty$.

The assumptions A4 and A5′ permit us to apply Theorem 4.1 to the Algorithm HMP[MAS] to obtain:

PROPOSITION 4.1. *Under the standing assumptions and assumptions A4, A5′, the Algorithm HMP[MAS] generates a sequence $\{z_k : k \geq 1\}$ which converges to z_s^0.*

□

It is seen that assumption A5′ may be too restrictive for applications since it is unlikely that HMP[MAS] minimizes $J(z_k) + r_k \eta(z_k)$ at iteration k. We replace A5′ by the weaker assumption A5 as follows:

A5 z_k, $J(z_k)$ computed by Steps 3, 4 and 5 of Algorithm HMP[MAS] at iteration k are such that: $J(z_k) + r_k \eta(z_k) \leq \inf_{z \in S}\{J(z) + r_k \eta(z)\} + \alpha_k$, where $\{\alpha_k\}_{k \in \mathbb{Z}}$ is a sequence of positive numbers such that $\lim_{k \to \infty} \alpha_k = 0$.

PROPOSITION 4.2. *Under the standing assumptions and assumptions A4, A5, Algorithm HMP[MAS] generates a sequence $\{z_k : k \geq 1\}$ which converges to z_s^0 satisfying:*

$$J(z_s^0) = \inf_{z \in S}\{J(z) : \eta(z) = 0\} = \inf_{z \in S'} J(z) = \sup_{r \geq 0} \psi(r) = \lim_{r \to \infty} \psi(r).$$

□

PROOF. The first equality follows by the continuity of $J(\cdot)$ and assumption A. By assumption A5 the point computed at iteration k, z_k in S, is such that

$$J(z_k) + r_k \eta(z_k) \leq \inf_{z \in S}\{J(z) + r_k \eta(z)\} + \alpha_k.$$

Hence the hypotheses of Ekeland's theorem [**26**] are satisfied and so there is z_k^* in S such that at each iteration k, $\|z_k - z_k^*\| \leq \sqrt{\alpha_k}$ and z_k^* minimizes $G(z_k) = J(z_k) + r_k \eta(z_k) + \sqrt{\alpha_k}\,\|z_k - z_{r_k}^*\|$. Hence as k goes to infinity, the sequences $\sqrt{\alpha_k}$ and $\|z_k - z_k^*\|$ approach zero and by Theorem 4.1 the desired convergence result follows. □

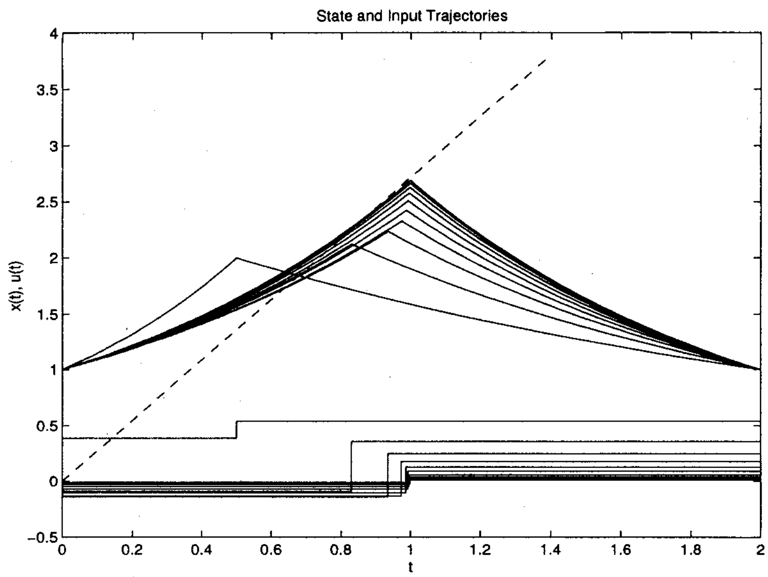

FIGURE 4.1. Convergence to the Optimal Trajectory in Example 4.1: A Single Autonomous Switching Case

EXAMPLE 4.1. *To illustrate the Algorithm HMP[MAS] we consider a system which successively occupies the locations* q_1 *and* q_2 *and which has one switching manifold defined by* $m(x,t) = x - et = 0$ *(*$e = 2.7183...$*). The dynamics in* q_1, q_2 *are given by:* $\dot{x} = x + xu, \quad \dot{x} = -x + xu, \quad t_0 = 0,\ t_f = 2, \quad x(0) = 1,\ x(2) = 1$. *The cost function to be minimized is:* $J(u) = \frac{1}{2}\int_0^2 u^2(s)\,ds$.

Starting with the initial guess $t_s = 0.5$, $x_s = 2$, *Figure 4.1 shows the convergence of* $\{t_s^k, x_{s_1}^k\}_{k=1}^{\infty}$, *to the unique optimal switching time* $t_s = 1$ *and state* $x_s = e$. *The unique optimal control in this case is* $u^0 \equiv 0$ *resulting in optimal cost* $J^0 = 0$. *The computation was performed using Matlab 6.0 on a Pentium III 550MHz machine with 128MB of SDRAM running a Redhat Linux 6.2 operating system. It took 105.98 seconds of CPU time.* □

EXAMPLE 4.2. *In this example we apply Algorithm HMP[MAS] to the system of Example 3 in* [**60**]. *The system consists of two discrete modes* q_1 *and* q_2 *and switches from* q_1 *to* q_2 *when the state trajectory intersects the linear manifold defined by* $m(x) = x_1 + x_2 - 7 = 0$. *The linear dynamics in each location are given by:*

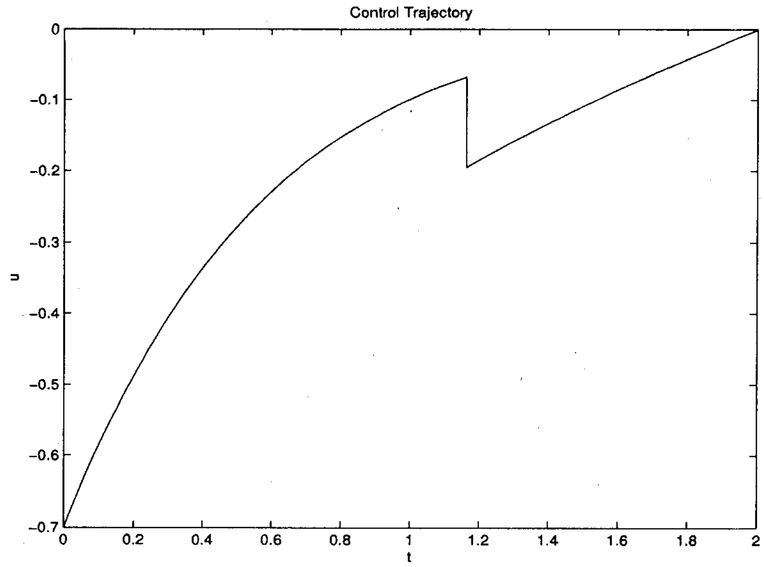

FIGURE 4.2. Example 4.2: Optimal Input trajectory

$$q_1: \quad \dot{x} = \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u,$$

$$q_2: \quad \dot{x} = \begin{bmatrix} 0.5 & 0.866 \\ 0.866 & 0.5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u,$$

$t_0 = 0$, $t_f = 2$, $x_0 = [1 \quad 1]^T$.

The cost function to be minimized is:

$$J(u) = \frac{1}{2}(x_1(2) - 10)^2 + \frac{1}{2}(x_2(2) - 6)^2 + \frac{1}{2}\int_0^2 u^2(s)\,ds.$$

Starting with the initial guess $t_s = 1.5$, $x_s = [4.5 \quad 2.5]^T$, the algorithm converges to the switching time $t_s = 1.1621$ and state $x_s = [4.5556 \quad 2.4444]^T$ yielding the optimal cost $J = 0.1132$. The corresponding figures obtained in [**60**] *are: switching time $t_s = 1.1624$, optimal cost $J = 0.1130$. The control and state trajectories are shown in Figures 4.2 and 4.3 respectively, and are essentially identical to those in* [**60**].

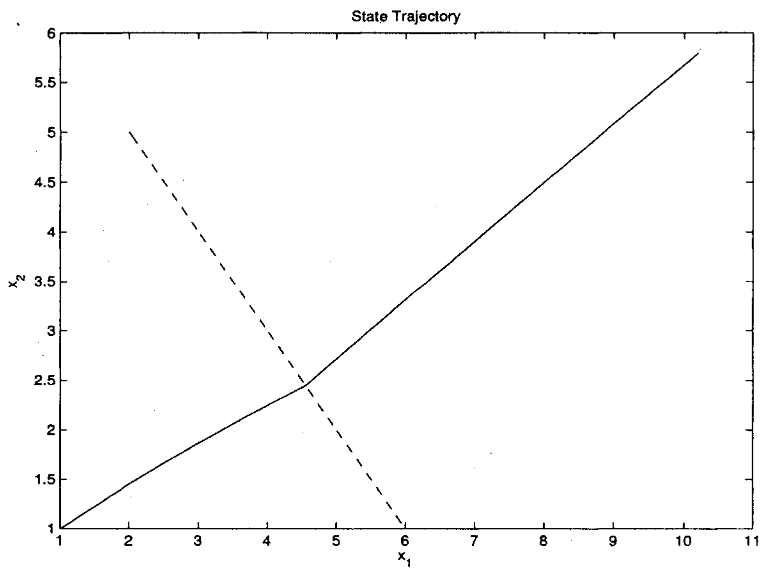

FIGURE 4.3. Example 4.2: Optimal State trajectory

The computation was performed using Matlab 6.5.1 on a Pentium III 550MHz machine with 128MB of SDRAM running a Redhat Linux 6.2 operating system; it took 29.58 seconds of CPU time. The computation time in [**60**] *was 34 minutes on a faster machine.* □

4.2.4. HMP[MCS] (Multiple Controlled Switchings, Fixed Schedule) Conceptual Algorithm.

0. Algorithm Initialization: Fix the H-tolerance $0 < h \ll 1$, λ-tolerance $0 < l \ll 1$, minimum location residence time $0 < T_r \ll 1$, $0 < \epsilon_1 \ll 1$, $0 < \epsilon_2 \ll 1$. Set the iteration counter k to 0. Let S^{N^0} be

   a given location sequence and $\overline{u}^0$ a nominal control resulting in a nominal trajectory $\overline{x}^0$. Set the total cost $J^0(S^{N^0}, \overline{x}^0) = \infty$.

1. Increment k by 1. Compute the optimal controls $u_{q_i}^k(\cdot)$, the resulting state $x_{q_i}^k(\cdot)$, the resulting costate $\lambda_{q_i}^k(\cdot)$ and the Hamiltonian functions $H_{q_i}^k(t) = \lambda_{q_i}^k(t) f_{q_i}(x^k(t), u^k(t)) + l_{q_i}(x^k(t))$, in each location for

$i = 0, 1, \ldots, |S^{N^0}|$, where $|S^{N^0}|$ denotes the cardinality of the set S^{N^0}. Also compute the cost in each location $J^k_{q_i}$. If $J^k = \sum_{i=0}^{|S^{N^0}|} J^k_{q_i} \geq J^{k-1}$ then STOP.

2. Compute the difference of Hamiltonians $H^k_{q_i}(t^k_{s_i}) - H^k_{q_{i+1}}(t^k_{s_i})$ at each switching time $t^k_{s_i}$.

3. If $k > 1$ then

   For $i \in \{1, 2, \ldots, |S^{N^0}| - 1\}$

   If $|H^k_{q_i}(t^k_{s_i}) - H^k_{q_{i+1}}(t^k_{s_i})| > h$ and

   $(H^k_{q_i}(t^k_{s_i}) - H^k_{q_{i+1}}(t^k_{s_i}))(H^{k-1}_{q_i}(t^{k-1}_{s_i}) - H^{k-1}_{q_{i+1}}(t^{k-1}_{s_i})) > 0$ then

   $t^k_{s_i} = t^{k-1}_{s_i} - \epsilon_1(H^k_{q_i}(t^k_{s_i}) - H^k_{q_{i+1}}(t^k_{s_i}))$;

   else $t^k_{s_i} = t^{k-1}_{s_i}$.

4. If $k > 1$ then

   For each $i \in \{1, 2, \ldots, |S^{N^0}| - 1\}$

   If $\|\lambda^k_{q_{i+1}}(t^k_{s_i}) - \lambda^k_{q_i}(t^k_{s_i})\| > l$ and

   $\left(\lambda^k_{q_{i+1}}(t^k_{s_i}) - \lambda^k_{q_i}(t^k_{s_i})\right)^T \left(\lambda^k_{q_{i+1}}(t^k_{s_i}) - \lambda^k_{q_i}(t^k_{s_i})\right) > 0$ then

   $x^k(t^k_{s_i}) = x^{k-1}(t^{k-1}_{s_i}) - \epsilon_2 \left(\lambda^k_{q_{i+1}}(t^k_{s_i}) - \lambda^k_{q_i}(t^k_{s_i})\right)$;

   else $x^k(t^k_{s_i}) = x^{k-1}(t^{k-1}_{s_i})$.

5. If all the switching time differences satisfy $t^k_{s_{i+1}} - t^k_{s_i} > T_r$ then accept the new switching times; else, if $t^k_{s_i}$ is such that $t^k_{s_{i+1}} - t^k_{s_i} \leq T_r$, then compute the costs $\int_{t_{s_i}}^{t_{s_{i+1}}} l_{q_{i-1}}(x^k(t), u^k(t))dt$ and $\int_{t_{s_i}}^{t_{s_{i+1}}} l_{q_{i+1}}(x^k(t), u^k(t))dt$, $i \in \{1, 2, \ldots, |S^{N^0}| - 1\}$, in locations q_{i-1} and q_{i+1} respectively and replace the location q_i by whichever location on

```
   either side gives lower cost and by either location whenever they
   give an equal cost.

6. If the criteria in Steps 3, 4, 5 all fail then STOP;
   else go to Step 1.
```

4.2.5. Convergence Analysis of Algorithm HMP[MCS]. An analysis very similar to the one for algorithm HMP[MAS] can be performed to show the convergence of algorithm HMP[MCS]. Assume a location sequence with N controlled switching times. Let the sequence of switching time-state pairs be denoted

$$z_{s^N} \triangleq \{z_{s_1}, z_{s_2}, \ldots, z_{s_N}\} = \{(t_{s_1}, x_{s_1}), (t_{s_2}, x_{s_2}), \ldots, (t_{s_N}, x_{s_N})\}.$$

Then z_{s^N} can be thought of as a vector in $N \times (n+1)$ dimensional Euclidean space. If we identify the vector z_{s^N} with the vector z_s in Section 4.2.1 and define a compact set $S_N \in \mathbb{R}^{N(n+1)}$ similarly to the set $S \in \mathbb{R}^{n+1}$ in Section 4.2.1 then we see that the analysis of Sections 4.2.1-4.2.3 carries over to the case of multiple controlled switchings without much difficulty. Hence under assumptions similar to Assumptions A4 and A5 of Sections 4.2.2-4.2.3 we have the following result

PROPOSITION 4.3. *Let the sequence of points generated by the Algorithm HMP[MCS] be denoted* $\{z_s^k\}_{k=0}^{\infty} \triangleq \{(t_{s_1}^k, x_{s_1}^k), (t_{s_2}^k, x_{s_2}^k), \ldots (t_{s_N}^k, x_{s_N}^k)\}_{k=0}^{\infty}$. *Then* $\{z_s^k\}_{k=0}^{\infty}$ *converges to the optimal point* $z_s^0 \triangleq \{(t_{s_1}^0, x_{s_1}^0), (t_{s_2}^0, x_{s_2}^0), \ldots (t_{s_N}^0, x_{s_N}^0)\}$. □

EXAMPLE 4.3. *To illustrate the Algorithm HMP[MCS] we consider a system which successively occupies the locations* q_1, q_2 *and* q_3 *and which has two controlled switchings. The dynamics in* q_1, q_2, q_3 *are given by:*
$\dot{x} = x + xu$, $\dot{x} = -x + xu$, $\dot{x} = x + u$, $t_0 = 0$, $t_f = 3$, $x(0) = 1$, $x(3) = e$. *The cost function to be minimized is:* $J(u) = \frac{1}{2}\int_0^3 u^2(s)\,ds$.

Starting with the initial guess $t_{s_1} = 0.8$, $x(t_{s_1}) = 2.5$, $t_{s_2} = 2.2$, $x(t_{s_2}) = 0.8$, *Figure 4.4 shows the convergence of* $\{z_{s^2}^k\}_{k=1}^{\infty} = \{(t_{s_1}^k, x_{s_1}^k), (t_{s_2}^k, x_{s_2}^k)\}_{k=1}^{\infty}$ *to the unique optimal switching times* $t_{s_1} = 1$, $t_{s_2} = 2$ *and states* $x_{s_1} = e$ *and* $x_{s_1} = 1$. *The*

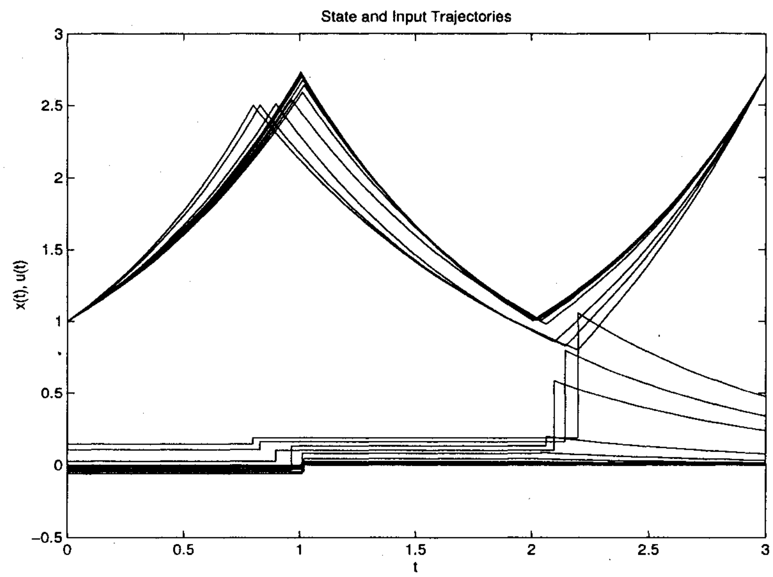

FIGURE 4.4. Convergence to the Optimal Trajectory in Example 4.3: Multiple Controlled Switchings Case

unique optimal control in this case is $u^0 \equiv 0$ *resulting in optimal cost* $J^0 = 0$. *The computation was performed using Matlab 6.0 on a Pentium III 550MHz machine with 128MB of SDRAM running a Redhat Linux 6.2 operating system. It took 152.43 seconds of CPU time.* □

EXAMPLE 4.4. *In this example we apply Algorithm HMP[MCS] to the system of Example 2 in* [**60**]. *The system consists of two discrete modes* q_1 *and* q_2 *with one controlled switching time* $t_s \in (0, 2)$. *The linear dynamics in each location are given by:*

$$q_1: \quad \dot{x} = \begin{bmatrix} 0.6 & 1.2 \\ -0.8 & 3.4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u,$$

$$q_2: \quad \dot{x} = \begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ -1 \end{bmatrix} u,$$

$t_0 = 0$, $t_f = 2$, $x_0 = [0 \quad 2]^T$.

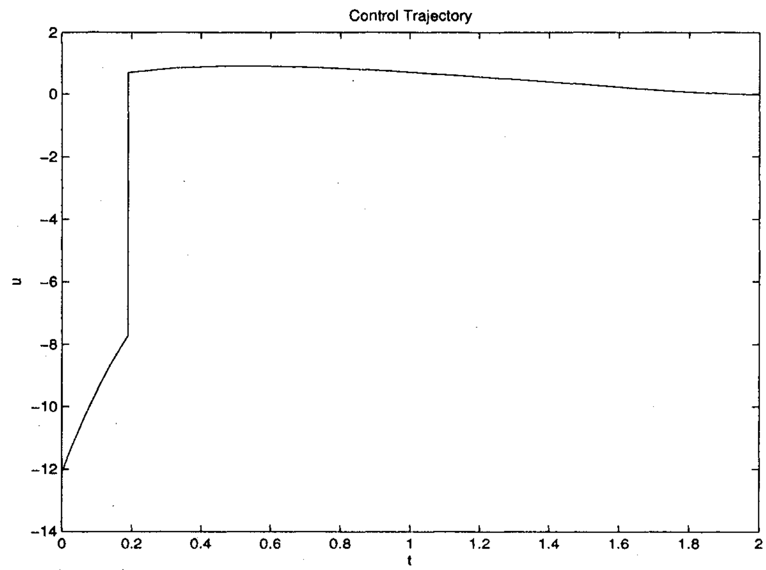

FIGURE 4.5. Example 4.4: Optimal Input trajectory

The cost function to be minimized is:

$$J(u) = \frac{1}{2}(x_1(2) - 4)^2 + \frac{1}{2}(x_2(2) - 2)^2 + \frac{1}{2}\int_0^2 \left((x_2(2) - 2)^2 + u^2(s)\right) ds.$$

Starting with the initial guess $t_s = 1$, *the algorithm converges to the switching time* $t_s = 0.18876$ *and state* $x_s = [-1.5626 \quad 1.3231]^T$ *yielding the optimal cost* $J = 9.7666$. *The corresponding figures obtained in* [**60**] *are: switching time* $t_s = 0.1897$, *optimal cost* $J = 9.7667$. *The control and state trajectories are shown in Figures 4.5 and 4.6 respectively, and are essentially identical to those in* [**60**].

The computation was performed using Matlab 6.5.1 on a Pentium III 550MHz machine with 128MB of SDRAM running a Redhat Linux 6.2 operating system; it took 69.52 seconds of CPU time. The computation time in [**60**] *was 30.75 seconds on a faster machine.* □

4.3. Combinatoric Search for Locally Optimal Schedules

In the worst case, combinatoric search has exponential time complexity (see e.g. [**35**]); however, the efficiency of HMP[MCS] permits us to define *local* heuristic search

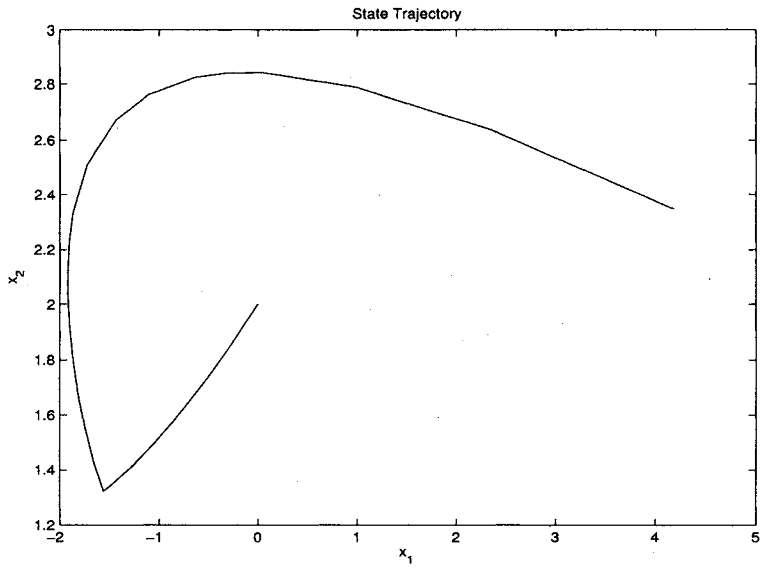

FIGURE 4.6. Example 4.4: Optimal State trajectory

methods which can find a (combinatorially local) hybrid optimal control law and its associated cost which are not achievable by other available methods. This is achieved by embedding the Algorithm HMP[MCS] in the so-called HMP[Comb] class which extends the HMP class with a combinatorial search algorithm.

Let $Q = \{1, 2, \ldots, M\}$ be a list of locations. Let $s_0 = (q_0, q_1, \ldots, q_{N-1})$, where $q_0, q_1, \ldots, q_{N-1}$ are not necessarily distinct, be an ordered list of locations from Q. Then a *k-neighbourhood boundary* of s_0, denoted $N_k(s_0)$, for some $k \leq N$, is defined to be the set of all lists $S \triangleq \{(p_0, p_1, \ldots, p_{N-1}) : p_i \in Q, i = 0, 1, \ldots, N-1\}$ which differ from s_0 in exactly k places, i.e. they have a *Hamming distance* k from s_0.

Define a *k-neighbourhood* of s_0, $N_{\leq k}(s_0) \triangleq \bigcup_{i=0}^{k} N_i(s_0)$. Then given an initial sequence s_0, a *locally k-optimal solution* is the one that is best among all $N_{\leq k}(s_0)$ sequences. We note that the following results are evident facts,

(i) $card(N_k(s_0)) = \binom{N}{k}(M-1)^k, \quad card(N_{\leq k}(s_0)) = \sum_{i=0}^{k}\binom{N}{i}(M-1)^i,$

(ii) given a list s_0 of length N, all lists of length N from alphabet of locations Q are given by $\bigcup_{k=0}^{N} N_k(s_0)$, and clearly, $card(N_{\leq N}(s_0)) = \sum_{i=0}^{N}\binom{N}{i}(M-1)^i = M^N$.

The algorithm (HMP[Comb]) repeatedly calls the algorithm HMP[MCS] to find a locally optimal solution $N_{\leq 2}(s_0)$, where s_0 is some initial sequence; it clearly extends to the case of $N_{\leq k}(s_0)$ for $k > 2$.

4.3.1. HMP[Comb] (HMP Combinatoric Search) Conceptual Algorithm.

We first give a brief description of the algorithm: Given an initial location sequence s_0 and a size of the discrete neighbourhood N, we define a matrix S of appropriate dimension such that each row of S holds: (i) the sequence of locations $s_i \in N_{\leq N}(s_0)$, (ii) initial switching times τ_i, and (iii) the cost J_i associated with (τ_i, s_i) which is initially set to zero. The Algorithm progresses by executing HMP[MCS] on each row and finally picking the row with the lowest J_i and the corresponding (τ_i, s_i).

0. Algorithm Initialization: Let $s_0 = \{(t_0, q_0), (t_1, q_1), \ldots, (t_{N-1}, q_{N-1})\}$ be an initial switching sequence. Set $iter$ to 1. Let S be a matrix of dimension

$$\left(1 + N(M-1) + \frac{N(N-1)}{2}(M-1)^2\right) \times (2N+1),$$

each of whose rows has the form

$$(s, J_s) = ((t_0, q_0), (t_1, q_1), \ldots, (t_{N-1}, q_{N-1}), J_s^0).$$

Initialize S to a matrix of zeros.

1. Execute Algorithm HMP[MCS] on s_0. Store s_0 and the cost returned by HMP[MCS] in the first row of the matrix S.

2. For each $i = 1, 2, \ldots, N$

   3. For each q in $\{Q\} - \{q_i\}$

      4. Increment $iter$ by 1. Obtain a new list $s(iter)$ by replacing q_i by q in s_0. Execute Algorithm HMP[MCS] on $s(iter)$; store $s(iter)$ and the cost returned by HMP[MCS] in row number $iter$ of the matrix S.

5. For each $j = i+1, i+2, \ldots, N$

6. For each q' in $\{Q\} - \{q_i\}$

7. For each q in $\{Q\} - \{q_j\}$

8. Increment $iter$ by 1. Obtain a new list $s(iter)$ by replacing q_i by q' and q_j by q in s_0. Execute Algorithm HMP[MCS] on $s(iter)$; store $s(iter)$ and the cost returned by HMP[MCS] in row number $iter$ of the matrix S.

9. Find the the row of matrix S with minimum cost. Return the cost and the corresponding switching sequence.

4.3.2. Illustrative Application of HMP[Comb].

EXAMPLE 4.5. *In an application of HMP[Comb] to a combinatoric extension of Example 4.3 we consider the case of three locations, i.e.* $Q = \{1,2,3\}$, *and modify the cost function to include a terminal cost* g *(to illustrate the case* $\lambda = \mathrm{grad}(g)$ *at the terminal time).*

$$J(u) = \frac{1}{2}(x(5) - e)^2 + \frac{1}{2}\int_0^5 u^2(s)\,ds.$$

Now consider the following two cases:

(i) Starting with

$s_0^1 = \{(t_0, q_0), (t_1, q_1), \ldots, (t_{N-1}, q_{N-1})\} = \{(0,1), (1,1), (2,1), (3,1), (4,1)\}$

the algorithm generates $\sum_{i=0}^{2} \binom{5}{i} 2^i = 51$ *sequences in* $N_2(s_0^1)$ *and executes HMP[MCS] on each one of them. Figure 4.7 shows the execution of Algorithm HMP[MCS] on the locally optimal sequence. The initial state trajectory is the top one and the final state trajectory is the bottom one.*

(ii) In this case, starting with

$s_0^2 = \{(0,1), (0.5,1), (1,1), (1.5,1), (2,1), (2.5,1), (3,1), (3.5,1), (4,1), (4.5,1)\}$

the algorithm generates $\sum_{i=0}^{2} \binom{10}{i} 2^i = 201$ *sequences in* $N_2(s_0^2)$ *and executes HMP[MCS] on each one of them as shown in Figure 4.8.*

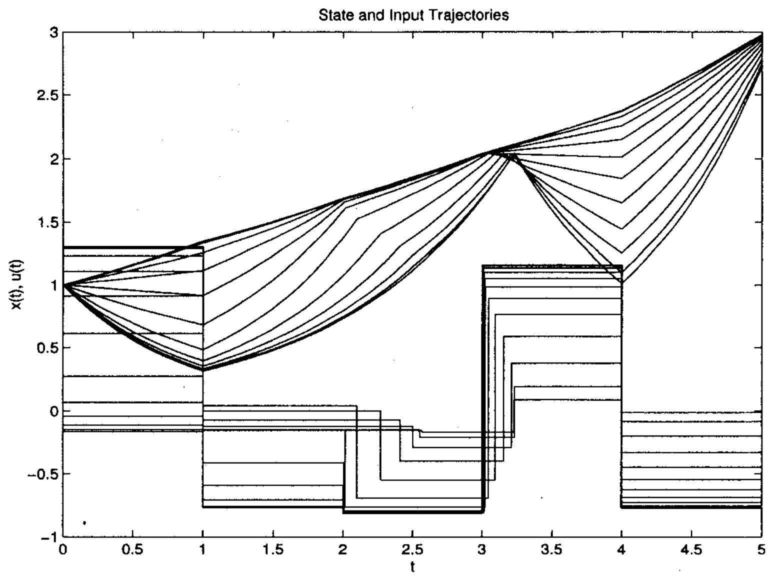

FIGURE 4.7. Example 4.5: State and Input Trajectories for Locally 2-opt Sequence (2,1,1,2,1).

For the purpose of illustration, each string in $N_2(s_0^i)$, $i = 1, 2$, *is converted to a unique corresponding ternary number as follows: if* $s = (q_0, q_1, \ldots, q_{N-1})$ *where* $q_i \in \{1, 2, 3\}$, $i = 0, 1, \ldots N-1$, *then its ternary representation is* $\sum_{j=0}^{N-1}(q_j - 1)3^{(N-1-j)}$. *For this example these numbers have been normalized and are plotted against the corresponding costs as shown in Figures 4.9 and 4.10.*

We note the stability of the overall procedure in the sense that locally optimal 10-time slot location sequence (Figure 4.7) consists simply of lengthened segments of the locally optimal 5-time slot sequence (Figure 4.8). □

In addition to illustrating the efficacy of Algorithm HMP[Comb], the examples above also serve to show that a global optimization of location sequences and the associated HOCPs will be overwhelmed by the combinatorial complexity engendered by even moderate values of $|Q|$ and N. The next chapter introduces a method whose complexity is linear in N at the cost of an initial computational investment.

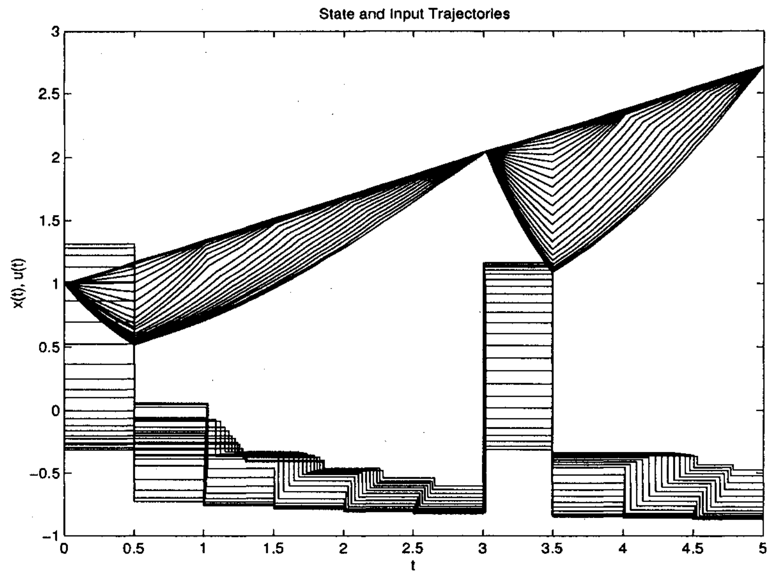

FIGURE 4.8. Example 4.5: State and Input Trajectories for Locally 2-opt Sequence (2,1,1,1,1,1,2,1,1,1).

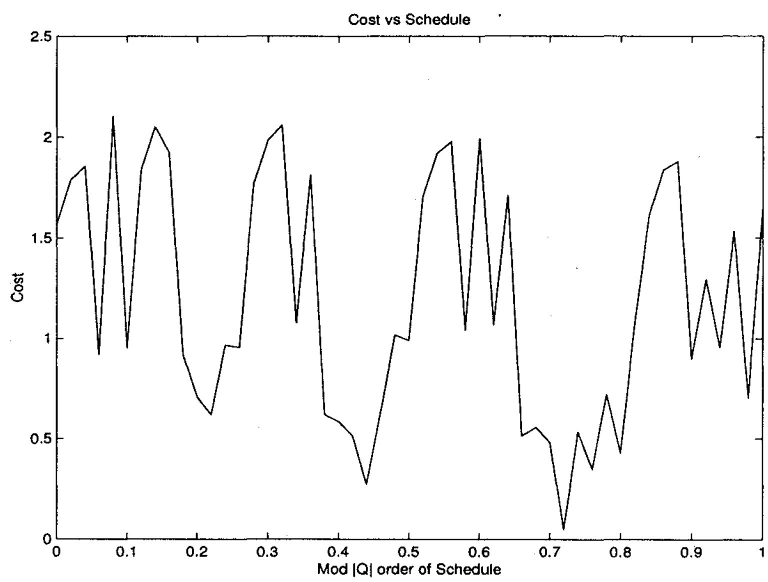

FIGURE 4.9. Example 4.5: Costs in $N_2(1,1,1,1,1)$.

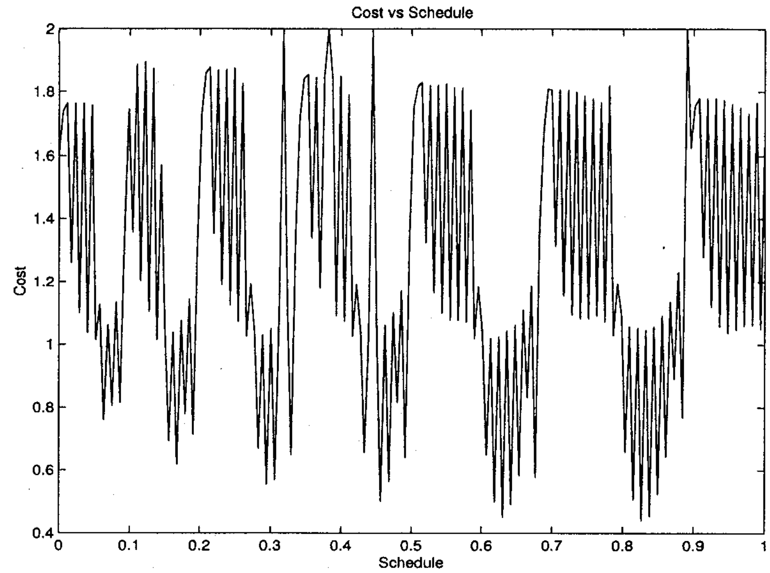

FIGURE 4.10. Example 4.5: Costs in $N_2(1,1,1,1,1,1,1,1,1,1)$.

CHAPTER 5

Optimality Zones

5.1. Optimality Zones and Location Sequences

Fundamental Implication of the Dynamic Programming Principle for Optimal Location Sequences. Along an optimal hybrid trajectory (I_L^0, x^0) the Dynamic Programming Principle implies that the part of the continuous and discrete input I_L^0 (and correspondingly the trajectory (q^0, x^0)) from the j-th switching time-state pair to the $j+1$-st switching time-state pair, $(t_s^0, x_s^0)_j \to (t_s^0, x_s^0)_{j+1}$, is optimal.

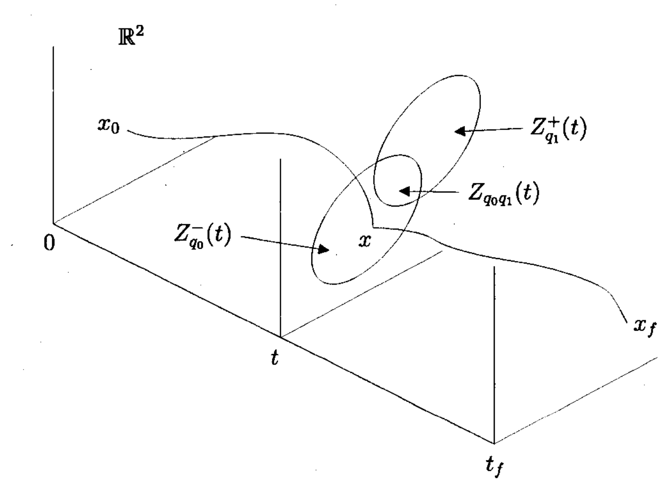

FIGURE 5.1. Relationship between inbound/outbound zones and optimality zones.

Hence, in particular, q_j^0 must be an optimal location for the trajectory from $(t_s^0, x_s^0)_j$ to $(t_s^0, x_s^0)_{j+1}$.

The implication of this for location sequence HOCP computation is that $(u_{[t_j,t_{j+1}]}, q_j)$ generates a candidate optimal HOCP trajectory segment between any candidate pair of switching time-states $\{(t_s^0, x_s^0)_j,\ (t_s^0, x_s^0)_{j+1}\}$ only if $(u_{[t_j,t_{j+1}]}, q_j)$ is optimal between $(t_s^0, x_s^0)_j$ and $(t_s^0, x_s^0)_{j+1}$. For each $q = q((t_s^0, x_s^0)_j, (t_s^0, x_s^0)_{j+1}) \in Q$ this optimization is a standard (non-hybrid) optimal control problem; hence the set-up cost of a search over a set Q of such optimizations to find the optimal $q^0((t_s^0, x_s^0)_j, (t_s^0, x_s^0)_{j+1})$ is proportional to $|Q|$ and is not linked to an analogous optimization over any other interval.

Let the executions of HMP[MCS] be modified so that, after each iterative shift of the vector of switching time-state pairs $(t_s, x_s)^{[k]}$ to $(t_s, x_s)^{[k+1]}$ in $\mathbb{R}^{L(n+1)}$, the location $q_j^{[k+1]}$ on the interval $[t_j^{[k+1]}, t_{j+1}^{[k+1]})$ is optimal for trajectories from $x_{t_{s_j}}^{[k+1]}$ to $x_{t_{s_{j+1}}}^{[k+1]}$ and is assumed to have been generated in the set-up computations.

Then the modified HMP[MCS] algorithm (which shall be called the HMP[Z] algorithm) reaches an equilibrium with respect to all possible iterative shifts of the $\mathbb{R}^{L(n+1)}$ vector (t_s, x_s) and of the associated control inputs $I(t_s, x_s)$ only if the necessary conditions for HOCP optimality are satisfied.

Let us adopt the assumptions that (i) the switching times may be restricted to a set of L intervals $\ldots I_j, I_{j+1} \ldots$ and (ii) for any $t < t'$ in adjacent intervals, the continuity property holds that space-time grids with $|G|^2$ points in $\mathbb{R}^{2(n+1)}$ permit computation of the dependence of the optimal $q^0 = q^0((t, x), (t', x'))$ with respect to $((t, x), (t', x'))$ varying over $I_j \times \mathbb{R}^n \times I_{j+1} \times \mathbb{R}^n$.

Then the set-up cost for the determination of the optimality zones (see below) in $\mathbb{R}^{2(n+1)}$ specifying the optimal locations is $O(|G|^2 \cdot |Q| \cdot (L+1))$; hence the resulting HMP[Z] algorithm computes (i) the optimal continuous variables and controls, and (ii) the optimal discrete location sequence for the HOCP with an overall complexity cost of $O(|G|^2 \cdot |Q| \cdot (L+1))$ + const. This general methodology will be analyzed in future work (following from this thesis), while in this chapter the optimality zones are precisely defined and the HMP[Z] is applied to certain specific cases.

Definition of the Optimality Zones. Throughout this section the HOCPs under consideration do not have any autonomous switchings defined, in other words the collection of autonomous switching manifold sets is empty.

According to Theorem 3.2, under the assumptions generating the class of hybrid systems $\mathbb{H}$ (and the associated HOCP (see Definition 2.5) the value function $J^0(t,x,q)$ of HOCP is bounded and continuous in (t,x) for each $q \in Q$. Then it is possible to define regions Z_q^+, $q \in Q$, of points (t,x) in the space $\mathbb{R} \times \mathbb{R}^n$ for which a specific location $q \in Q$ corresponds to the optimal hybrid system trajectory starting at (t,x) and terminating at (t_f, x_f), when no controlled switchings are permitted and when autonomous switchings are undefined for the HOCP in question.

Similarly regions Z_q^- can be defined for analogous HOCP with initial condition $(0,x_0)$ and terminal condition (t,x) and, furthermore, regions $Z_{qq'}$ in $(\mathbb{R} \times \mathbb{R}^n)^2$ for the corresponding HOCP with two free end points in (time ordered) $(\mathbb{R} \times \mathbb{R}^n)^2$.

The regions Z_q^-, Z_q^+ and $Z_{qq'}$ have a well defined geometrical structure and once they have been computed (or approximated) they permit the exponential $O(|Q|^N)$ complexity search for optimal location sequences (of length N) to be reduced to the complexity of a single run of the Algorithm HMP[MCS]. The algorithm HMP[Z] which performs this optimization is essentially a minor modification of the HMP algorithm.

We adopt the convention that if (t,x) is not accessible from $(0,x_0)$ and similarly if (t,x) is not co-accessible to $(t_f,x_f))$ then $J_q^0((0,x_0),(t,x)) = \infty$ and $J_q^0((t,x),(t_f,x_f)) = \infty$.

Definition 5.1. *Consider the case of one switching time and state (t,x). The* inbound optimality zone *Z_q^-, corresponding to a location $q \in Q$, is the subset of $(0,t_f) \times \mathbb{R}^n$ given by*

$$Z_q^- \triangleq \{(t,x) \in (0,t_f) \times \mathbb{R}^n : J_q^0((0,x_0),(t,x)) \leq J_{q'}^0((0,x_0),(t,x)), \quad \forall q' \in Q\}.$$

Similarly, the outbound optimality zone Z_q^+ *is given by*

$$Z_q^+ \triangleq \{(t,x) \in (0,t_f) \times \mathbb{R}^n : J_q^0((t,x),(t_f,x_f)) \leq J_{q'}^0((t,x),(t_f,x_f)), \quad \forall q' \in Q\}.$$

Further, we have subsets of Z_q^- and Z_q^+ given by

$$\begin{aligned} I(Z_q^-) \triangleq \{(t,x) \in (0,t_f) \times \mathbb{R}^n \quad &: \quad J_q^0((0,x_0),(t,x)) < J_{q'}^0((0,x_0),(t,x)), \\ & \quad\ \forall q' \in Q,\ q' \neq q\}, \end{aligned}$$

$$\begin{aligned} B(Z_q^-) \triangleq \{(t,x) \in (0,t_f) \times \mathbb{R}^n \quad &: \quad \exists q' \in Q,\ q' \neq q, \\ & \quad\ J_q^0((0,x_0),(t,x)) = J_{q'}^0((0,x_0),(t,x))\}. \end{aligned}$$

Similarly $I(Z_q^+)$ and $B(Z_q^+)$ can be defined with respect to $((t,x),(t_f,x_f))$.

(The subscript denotes location, whereas the superscripts, $+$, $-$, denote respectively whether the final or initial point is fixed.) □

Notice that the sets $I(Z_q^+)$ and $B(Z_q^+)$ (respectively $I(Z_q^-)$ and $B(Z_q^-)$) are not the topological interior and topological boundary of Z_q^+ (respectively Z_q^-) unless additional conditions are satisfied (see below).

In the case of two switchings, the space of switching times and states is the product $(0,t_f) \times \mathbb{R}^n \times (0,t_f) \times \mathbb{R}^n$ and the middle location in the switching sequence has variable end points at both ends; this gives rise to the so-called *internal zones* defined as follows.

DEFINITION 5.2. *The* internal zone Z_q^{int} *and the subsets* $I(Z_q^{\text{int}})$ *and* $B(Z_q^{\text{int}})$ *are given by*

$$
\begin{aligned}
Z_q^{\text{int}} \;&\triangleq\; \{((t_1,x_1),(t_2,x_2)) \in ((0,t_f)\times\mathbb{R}^n)^2 : \\
&\qquad J_q^0((t_1,x_1),(t_2,x_2)) \le J_{q'}^0((t_1,x_1),(t_2,x_2)),\; t_1<t_2,\; \forall q' \in Q\}, \\
I(Z_q^{\text{int}}) \;&\triangleq\; \{((t_1,x_1),(t_2,x_2)) \in ((0,t_f)\times\mathbb{R}^n)^2 : \\
&\qquad J_q^0((t_1,x_1),(t_2,x_2)) < J_{q'}^0((t_1,x_1),(t_2,x_2)),\; t_1<t_2,\; \forall q' \in Q,\; q' \neq q\}, \\
B(Z_q^{\text{int}}) \;&\triangleq\; \{((t_1,x_1),(t_2,x_2)) \in ((0,t_f)\times\mathbb{R}^n)^2 : \\
&\qquad \exists q' \in Q,\; q' \neq q,\, J_q^0((t_1,x_1),(t_2,x_2)) = J_{q'}^0((t_1,x_1),(t_2,x_2)),\; t_1<t_2\}.
\end{aligned}
$$

□

Definitions 5.1 and 5.2 generalize to the case of M switchings where there are inbound and outbound zones corresponding to the two end locations and $M-1$ internal zones corresponding to the intermediate locations.

Consider a location sequences with M switchings where the sequence of (not necessarily distinct) locations and the switching times and states are $\{q_0^M\} \triangleq (q_0, q_1, ..., q_M)$ and $z \triangleq \{(t_i,x_i)\}_{i=1}^M$, $0<t_1<t_2<\cdots<t_M<t_f$, respectively. Let $\Pi(q_0^M)$ denote the $(M+1)!$ possible permutations of any $\{q_0^M\} \in Q^M$.

DEFINITION 5.3. *An* optimality zone, $Z_{q_0^M}$ *corresponding to the location sequence* $\{q_0^M\}$ *is a region in the space of M-fold product* $((0,t_f)\times\mathbb{R}^n)^M$

$$
\begin{aligned}
Z_{q_0^M} \;&=\; Z_{q_0q_1\cdots q_M} \triangleq \{z \in ((0,t_f)\times\mathbb{R}^n)^M : (t_1,x_1) \in Z_{q_0}^- \\
&\qquad \bigwedge ((t_1,x_1),(t_2,x_2)) \in Z_{q_1}^{\text{int}} \bigwedge \cdots \bigwedge ((t_{M-1},x_{M-1}),(t_M,x_M)) \in Z_{q_{M-1}}^{\text{int}} \\
&\qquad \bigwedge (t_M,x_M) \in Z_{q_M}^+, \quad 0<t_1<t_2<\cdots<t_M<t_f\}.
\end{aligned}
$$

Or, equivalently,

$$
\begin{aligned}
Z_{q_0^M} \triangleq \{z \in ((0,t_f)\times\mathbb{R}^n)^M \;:\;& J_{q_0^M}^0(z) \le J_{p_0^M}^0(z), p_0^M \in \Pi(q_0^M), \\
& z = \{(t_i,x_i)\}_{i=1}^M, 0<t_1<t_2<\cdots<t_M<t_f\},
\end{aligned}
$$

where $J^0_{p_0^M}(z)$ denotes the optimal cost incurred with respect to all choices of control inputs subject to the constraint that the location sequence is fixed at the value p_0^M and the switching time and state sequence is fixed at $z = \{(t_i, x_i)\}_{i=1}^M$ (i.e. $J^0_{p_0^M}(z)$ is the optimal value for HOCP subject to the constraints $\{q_0^M, z\}$). □

We note that the chaining condition of common initial and end point switching times and states for consecutive internal zones Z_q^{int} in the definition of $Z_{q_0^M}$ above is necessarily satisfied for $J^0_{p_0^M}(z)$.

Figure 5.1 shows the relationship between inbound/outbound zones, $Z^+_{q_1}$, $Z^-_{q_0}$, and the optimality zone $Z_{q_0 q_1}$ at a fixed switching time t, where $Q = \{q_0, q_1\}$. Notice that for any $q \in Q$, $\mathbf{P}_{[1]}(Z_q^{\text{int}}) = Z_q^+$ and $\mathbf{P}_{[2]}(Z_q^{\text{int}}) = Z_q^-$, where $\mathbf{P}_{[1]}$ denotes projection of a set in $((0, t_f) \times \mathbb{R}^n)^2$ on the first $n+1$ coordinates corresponding to $(0, t_f) \times \mathbb{R}^n$, and analogously for $\mathbf{P}_{[2]}$.

In the case of two locations the optimality zones can be simply related to the level sets of the optimal value functions as follows. Let L_1^c and L_2^c denote the c-level sets of the optimal value functions $J_1^0((t,x),(t_f,x_f))$ and $J_2^0((t,x),(t_f,x_f))$, for some $c \in \mathbb{R}_+$, respectively, i.e.

$$L_1^c = \{(t,x) : J_1^0((t,x),(t_f,x_f)) = c)\}$$

and

$$L_2^c = \{(t,x) : J_2^0((t,x),(t_f,x_f)) = c)\}.$$

Then

$$\bigcup_{c \in \mathbb{R}_+} \{L_1^c \cap L_2^c\} = \{(t,x) : J_1^0((t,x),(t_f,x_f)) = J_2^0((t,x),(t_f,x_f))\} = Z_1^+ \cap Z_2^+.$$

The Topology of Optimality Zones. We now introduce the basic assumptions concerning the geometry of the zones associated with an HOCP.

A6 Nonempty optimality zones corresponding to two distinct location sequences have disjoint topological interiors.

A7 An optimality zone corresponding to a location sequence $\{q_0, q_1, ..., q_M\}$ is either empty or is a finite union of $C^k, 1 \leq k \leq \infty$ or $k = \omega$, $M(n+1)$-dimensional manifolds with boundary.

A8 Zonal topological boundaries are either (i) empty or (ii) piecewise $C^k, 1 \leq k \leq \infty$ or $k = \omega$, $M(n+1) - 1$-dimensional manifolds with boundary.

Optimality zones satisfying assumptions A6, A7 and A8 are called *regular zones.*

LEMMA 5.1. *If an optimality zone $Z_{q_0^M}$ satisfies A6 and A7, then*
(i) $Z_{q_0^M}$ is a closed set, and
(ii) $Z_{q_0^M} = I(Z_{q_0^M}) \uplus B(Z_{q_0^M})$,
where $\uplus$ denotes disjoint union, and

$$\begin{aligned} I(Z_{q_0^M}) \;\triangleq\; & \{z \in ((0,t_f) \times \mathbb{R}^n)^M : J^0_{q_0^M}(z) < J^0_{p_0^M}(z), \\ & \forall p_0^M \in \Pi(q_0^M) - \{q_0^M\}, \\ & z = \{(t_i, x_i)\}_{i=1}^M, 0 < t_1 < t_2 < \cdots < t_M < t_f\}, \end{aligned}$$

$$\begin{aligned} B(Z_{q_0^M}) \;\triangleq\; & \{z \in ((0,t_f) \times \mathbb{R}^n)^M : \exists p_0^M \in \Pi(q_0^M) - \{q_0^M\}, \\ & J^0_{q_0^M}(z) = J^0_{p_0^M}(z), \\ & z = \{(t_i, x_i)\}_{i=1}^M, 0 < t_1 < t_2 < \cdots < t_M < t_f\}. \end{aligned}$$

Furthermore,

(iii) $B(Z_{q_0^M}) = \partial Z_{q_0^M}$, the topological boundary of $Z_{q_0^M}$, and
(iv) $I(Z_{q_0^M}) = \overset{\circ}{Z}_{q_0^M}$, the topological interior of $Z_{q_0^M}$.
In case $M = 1$,

$$\begin{aligned} \overset{\circ}{Z}_{q_0 q_1} \;=\; & \{z \in (0,t_f) \times \mathbb{R}^n : (t_1, x_1) \in \overset{\circ}{Z}{}^{-}_{q_0} \\ & \bigwedge (t_1, x_1) \in \overset{\circ}{Z}{}^{+}_{q_1}, \; z = (t_1, x_1), \; 0 < t_1 < t_f\}, \end{aligned}$$

$$\partial Z_{q_0 q_1} = \{(\exists p \in Q : J^-_{q_0} = J^-_p) \bigwedge (J^+_{q_1} < J^+_q, \forall q \in Q, q \neq q_1)\}$$
$$\bigcup \{(J^-_{q_1} < J^-_q, \forall q \in Q, q \neq q_1) \bigwedge (\exists p \in Q : J^+_{q_0} = J^+_p)\},$$

where

$J^-_q = J^0_q((0, x_0), (t, x))$, $J^+_p = J^0_p((t, x), (t_f, x_f))$, *etc.*

Equivalently, $\partial Z_{q_0 q_1} = (Z^-_{q_0} \cap \partial Z^+_{q_1}) \bigcup (\partial Z^-_{q_0} \cap Z^+_{q_1})$, *where* $\partial Z^+_{q_1}$ *(respectively* $\partial Z^-_{q_0}$*) is the topological boundary of* $Z^+_{q_1}$ *(respectively* $Z^-_{q_0}$*).* □

PROOF. First, $Z_{q_0^M}$ is closed by assumption A7.

Second, by definition, $I(Z_{q_0^M}) \subset Z_{q_0^M}$, $B(Z_{q_0^M}) \subset Z_{q_0^M}$, $I(Z_{q_0^M}) \cap B(Z_{q_0^M}) = \emptyset$ and $I(Z_{q_0^M}) \cup B(Z_{q_0^M}) = Z_{q_0^M}$. Hence $Z_{q_0^M} = I(Z_{q_0^M}) \uplus B(Z_{q_0^M})$.

Third, let $z \in I(Z_{q_0^M}) \subset Z_{q_0^M}$. Then since $h_{q_0^M p_0^M}(z) \triangleq J^0_{q_0^M}(z) - J^0_{p_0^M}(z)$ for any $p_0^M \in \Pi(S_Q) - q_0^M$ is continuous and $|\Pi(S_Q)| < \infty$, z has an open neighbourhood contained in $I(Z_{q_0^M})$. Hence $I(Z_{q_0^M})$ is open and $I(Z_{q_0^M}) \subset \overset{\circ}{Z}_{q_0^M}$. Now let $z \in B(Z_{q_0^M}) \subset Z_{q_0^M}$ be such that $J^0_{q_0^M}(z) = J^0_{p_0^M}(z)$ for some $p_0^m(z) \in \Pi(q_0^M)$, $p_0^M \neq q_0^M$. Then $z \in Z_{q_0^M} \cap Z_{p_0^M}$ and by assumption A6 z can only be in the topological boundary of $Z_{q_0^M}$. Hence $B(Z_{q_0^M}) \subset \partial Z_{q_0^M}$. Now let $z \in \partial Z_{q_0^M}$; then $z \notin \overset{\circ}{Z}_{q_0^M}$ and, since $Z_{q_0^M}$ is closed, $z \in Z_{q_0^M}$. Since $I(Z_{q_0^M}) \subset \overset{\circ}{Z}_{q_0^M}$, $z \notin I(Z_{q_0^M})$. Hence by the second claim, $z \in B(Z_{q_0^M})$. This shows that $\partial Z_{q_0^M} \subset B(Z_{q_0^M})$ and so $\partial Z_{q_0^M} = B(Z_{q_0^M})$, and further, $I(Z_{q_0^M}) = Z_{q_0^M} - B(Z_{q_0^M}) = \bar{Z}_{q_0^M} - \partial Z_{q_0^M} = \overset{\circ}{Z}_{q_0^M}$, which are the third and fourth claims respectively.

The $M = 1$ case follows from the Definitions 5.1 and 5.3. □

In the rest of this chapter all optimality zones will be assumed to be regular.

5.2. Single Pass Schedule Optimization: The Algorithms HMP[Z]

We recall the fundamental implication of the Dynamic Programming Principle as given in the beginning to Section 5.1: Given that the zonal boundaries divide the (t, x)-space into a finite number of pre-computed connected zones, one may run the

Algorithm HMP[MCS] with the modification that location switchings occur as the values of (t_s, x_s) cross the boundaries $\partial Z_{p_0^M}$; this is the key feature which makes the class of HMP[Z] algorithms linear in the number of locations.

We now present the conceptual algorithms called HMP[Z]S and HMP[Z]V (S: scalar, V: vector). The particular case of this Algorithm HMP[Z]S uses the fact that corresponding to each switching time-state pair there is an optimality zone as shown in Figure 5.1. Hence there is a sequence of zones $\{Z_{q_{i-1}q_i}\}_{i=1}^{M}$ corresponding to a location sequence with M switchings. Algorithm HMP[Z]V executes on the zones as given in Definition 5.3.

We assume that HMP[MCS] returns the updated values (t_s^k, x_s^k) so that the Algorithm may employ these values at each iteration.

Zonal Algorithm HMP[Z]S.

1. Initialization: Set $k = 0$, $switch = 0$. Let $\bar{S}$ be an initial switching sequence, $\{(\bar{t}_{s_i}, \bar{x}_{s_i})\}_{i=1}^{M}$ be initial switching time-state pairs and $\{\bar{Z}_{q_{i-1}q_i}\}_{i=1}^{M}$ be the initial sequence of zones corresponding to $\bar{S}$.

2. Increment k by 1.
   For each $j \in \{1, 2, \ldots, M\}$

   Execute a single switch version of HMP[MCS] to adjust $(t_{s_j}^k, x_{s_j}^k)$.
   If $(t_{s_j}^{k+1}, x_{s_j}^{k+1})$ lies in a zone distinct from $\{\bar{Z}_{q_{j-1}q_j}\}$ then

   set $\{\bar{Z}_{q_{j-1}q_j}\}$ equal to the zone that $(t_{s_j}^{k+1}, x_{s_j}^{k+1})$ belongs to and set $\bar{S}$ equal to the sequence corresponding to the zonal sequence $\{\bar{Z}_{q_{i-1}q_i}\}_{i=1}^{M}$ and set $switch = 1$.

3. If $switch = 1$ then go to Step 2
   else execute Algorithm HMP[MCS] with $\bar{S}$, $\{(\bar{t}_{s_i}, \bar{x}_{s_i})\}_{i=1}^{M}$ as initial data.

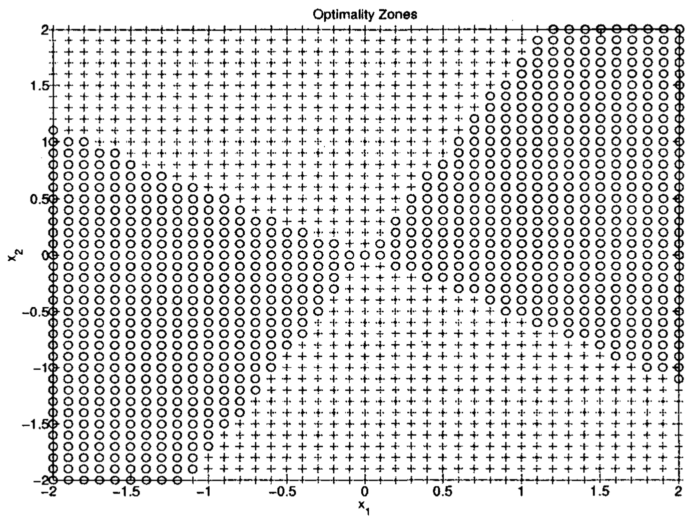

FIGURE 5.2. Optimality Zones in Example 5.1: LQ Case with Free Terminal State. o:(q_2, q_1), +:(q_2, q_2).

Zonal Algorithm HMP[Z]V.

1. Initialization: Set $k = 0$. Let $\bar{S}$ be an initial switching sequence, $\{(\bar{t}_{s_i}, \bar{x}_{s_i})\}_{i=1}^{M}$ be initial switching time-state pairs.

2. Increment k by 1. Execute Algorithm HMP[MCS] with $\bar{S}$, $\{(\bar{t}_{s_i}, \bar{x}_{s_i})\}_{i=1}^{M}$ as initial data.

3. If HMP[MCS] stops then STOP; else check whether $\{(t_{s_i}^{k+1}, x_{s_i}^{k+1})\}$ lies in a zone distinct from $Z_{\bar{S}}$. If yes, then set $\bar{S}$ equal to the location sequence corresponding to the zone to which $\{(t_{s_i}^{k+1}, x_{s_i}^{k+1})\}$ belongs. Set $\{(\bar{t}_{s_i}, \bar{x}_{s_i})\}_{i=1}^{M} = \{(t_{s_i}^{k+1}, x_{s_i}^{k+1})\}_{i=1}^{M}$ and go to Step 2.

A convergence analysis of the HMP[Z] algorithms can be performed by considering the fact that HMP[Z] (S or V) executes HMP[MCS] a finite number of times in the inner loop.

5.3. Linear Quadratic Regulator (LQR) Case

5.3.1. Free Terminal State Case. Again for simplicity we consider the case of two locations, $Q = \{q_1, q_2\}$, with a single controlled switch at (t_s, x_s), fixed initial and final times, 0 and t_f respectively, and fixed initial state x_0. The final state x_f is free. Let the dynamics in each location be of the form:

$$\dot{x} = A_i x + B_i u, \quad i \in \{1, 2\},$$

and let the costs associated with the locations $Q = \{q_1, q_2\}$ in the intervals $[0, t_s]$, $[t_s, t_f]$ be:

$$J_{1,i}((0, x_0), (t_s, x_s)) = \frac{1}{2} \int_0^{t_s} (x^T Q_i x + u^T R_i u)\, dt,$$

and

$$J_{2,i}((t_s, x_s), (t_f, x_f)) = \frac{1}{2} \int_{t_s}^{t_f} (x^T Q_i x + u^T R_i u)\, dt,$$

where A_i, B_i, Q_i, R_i, $i \in \{1, 2\}$, are constant matrices of appropriate dimensions and R_i, $i \in \{1, 2\}$, are positive and Q_i, $i \in \{1, 2\}$, are nonnegative.

The optimal location costs on the time interval $[t_s, t_f]$ are given by

$$J_i^0((t_s, x_s), (t_f, x_f)) = \frac{1}{2} x_s^T P_i(t_s) x_s,$$

where the matrices $P_i(t)$ are solutions to the corresponding matrix Riccati differential equations with $P_i(t_f) = 0$. Hence, in this case, the optimality zones are separated by boundaries defined by:

$$\{(t, x) : x^T P_1(t) x = x^T P_2(t) x\}.$$

Assuming t fixed at an arbitrary t_s these boundaries are given by:

$$\{x : x^T (P_1(t_s) - P_2(t_s)) x = 0\}.$$

In particular, in the case of a 2-dimensional system where $P_1(t_s) - P_2(t_s)$ can be diagonalized by a coordinate transformation to a matrix of the form $\begin{bmatrix} \lambda_1^2 & 0 \\ 0 & -\lambda_2^2 \end{bmatrix}$, the boundaries are two straight lines, $\lambda_1 x_1 = \lambda_2 x_2$ and $\lambda_1 x_1 = -\lambda_2 x_2$ intersecting at the origin. Example 5.1 illustrates this situation via numerical computation. Here, due to the nature of solutions to the Riccati equation, the optimal (outbound, inbound and internal) performance indices are quadratic functions of x and are smooth transcendental functions of t; hence the zones are regular, that is to say conditions A6, A7 and A8 hold, and the zonal boundaries are in principle computable.

5.3.2. Terminal State Cost Case. The LQ case becomes more complicated when either there is a terminal cost term in the cost index or the terminal state is fixed.

We first consider the case where the cost indices are modified to include a terminal cost term:

$$\begin{aligned} J_{1,i}((0,x_0),(t_s,x_s)) &= \frac{1}{2}(x(t_s)-x_s)^T N_i (x(t_s)-x_s) \\ &\quad + \frac{1}{2}\int_0^{t_s} (x^T Q_i x + u^T R_i u)\, dt, \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} J_{2,i}((t_s,x_s),(t_f,x_f)) &= \frac{1}{2}(x(t_f)-x_f)^T N_i (x(t_f)-x_f) \\ &\quad + \frac{1}{2}\int_{t_s}^{t_f} (x^T Q_i x + u^T R_i u)\, dt, \end{aligned} \tag{5.2}$$

where N_i is positive.

For definiteness, we consider the second time interval $[t_s, t_f]$ and drop the subscript i from A_i, B_i, Q_i, R_i, N_i, in the following derivation.

Next, we introduce the augmented system:

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x - x_f \\ x_f \end{bmatrix} = \begin{bmatrix} Ax + Bu \\ 0 \end{bmatrix} &= \begin{bmatrix} A(x-x_f) + Ax_f + Bu \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} A & A \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x - x_f \\ x_f \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u. \end{aligned}$$

Correspondingly, we write the cost index (5.2) as:

$$\begin{aligned} J_{2,i}((t_s,x_s),(t_f,x_f)) &= \frac{1}{2}((x(t_f)-x_f)^T x_f^T) \begin{bmatrix} N & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x - x_f \\ x_f \end{bmatrix} \\ &+\frac{1}{2}\int_{t_s}^{t_f} \left((x-x_f)^T x_f^T) \begin{bmatrix} Q & Q \\ Q & Q \end{bmatrix} \begin{bmatrix} x - x_f \\ x_f \end{bmatrix} \right) + u^T R u)\, dt. \end{aligned}$$

Employing the well-known LQR state feedback solution [1] in this case yields:

$$u^0(t) = -R^{-1}\tilde{B}^T\tilde{P}(t) \begin{bmatrix} x(t) - x_f \\ x_f \end{bmatrix}, \qquad t \in [t_s, t_f],$$

where˜denotes the augmented matrices and $\tilde{P}$ satisfies the matrix Riccati differential equation:

$$-\dot{\tilde{P}} = \tilde{A}^T\tilde{P} + \tilde{P}\tilde{A} + \tilde{Q} - \tilde{P}\tilde{B}R^{-1}\tilde{B}^T\tilde{P}, \qquad \tilde{P}(t_f) = \tilde{N}.$$

If we partition $\tilde{P}$ as $\tilde{P} = \begin{bmatrix} P_1 & P_2^T \\ P_2 & P_3 \end{bmatrix}$ we can write the optimal control and the value function as:

$$u^0(t) = -R^{-1}B^T P_1(t)(x(t)-x_f) - R^{-1}B^T P_2(t) x_f, \qquad t \in [t_s, t_f],$$

and

$$\begin{aligned} J^0((t,x),(t_f,x_f)) &= \frac{1}{2}(x-x_f)^T P_1(t)(x-x_f) \\ &+\frac{1}{2}(x-x_f)^T P_2(t) x_f + \frac{1}{2} x_f^T P_3(t) x_f, \quad t \in [t_s, t_f]. \end{aligned} \tag{5.3}$$

Hence the zonal boundaries in this case are defined by expressions of the form:

$$\{(t,x) : J_1^0((t,x),(t_f,x_f)) = J_2^0((t,x),(t_f,x_f))\}.$$

In general, the solutions to these equations for the zonal boundaries are not linear manifolds as they are in the case of a free terminal state. In fact, the case of fixed terminal states is obtained by letting the matrix N tend to infinity in the cost index (5.2). In the limit, the time dependent matrices determining the regular zones in this case are given by:

$$\frac{d}{dt}P_1^{-1} = P_1^{-1}A^T + AP_1^{-1} + P_1^{-1}QP_1^{-1} - BR^{-1}B^T, \qquad P_1^{-1}(t_f) = 0.$$

while P_2 and P_3 are given by the following differential equations:

$$-\dot{P}_2 = A^T P_1 + P_2 A + Q - P_2 B R^{-1} B^T P_1, \qquad P_2(t_f) = 0,$$

and

$$-\dot{P}_3 = A^T P_2^T + P_2 A + Q - P_2 B R^{-1} B^T P_2^T, \qquad P_3(t_f) = 0.$$

EXAMPLE 5.1. *We consider a system with two locations q_1 and q_2, linear dynamics and quadratic cost criteria in each location. The system and weighting matrices are:*

$$A_1 = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}, B_1 = B_2 = Q_1 = Q_2 = R_1 = R_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The initial and final times and states are: $t_0 = 0$, $t_f = 2$, $x_0 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, x_f free.

For any given fixed t_s the corresponding optimality zones in Figure 5.2 are, naturally, infinite triangular regions whose vertices lie at the origin and hence they give rise to regular optimality zones. □

EXAMPLE 5.2. *We consider a system with two locations q_1 and q_2, linear dynamics and quadratic cost criteria in each location. The system and weighting matrices are:*

$$A_1 = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}, B_1 = B_2 = Q_1 = Q_2 = R_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$R_2 = 1.6 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

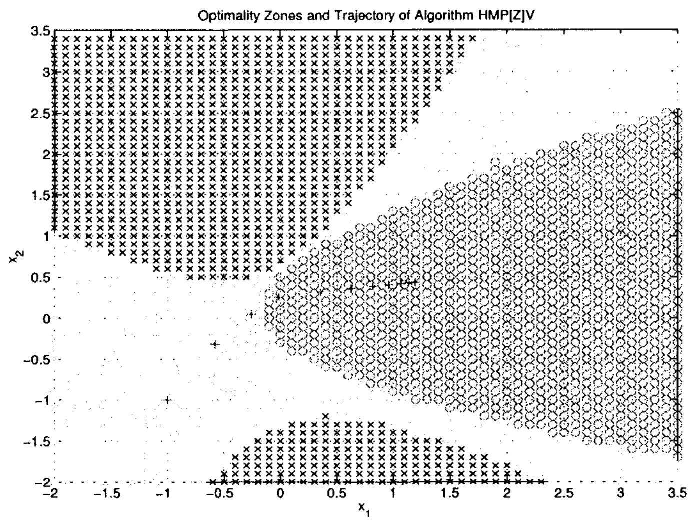

FIGURE 5.3. Optimality Zones in Example 5.2: LQ Case with Fixed Terminal State. $\times$:(q_1, q_1), $*$:(q_1, q_2), o:(q_2, q_1).

The initial and final times and states are: $t_0 = 0$, $t_f = 2$, $x_0 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $x_f = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. □

Figure 5.3 shows an $\mathbb{R}^2$ (space) slice of the three dimensional space-time $\mathbb{R}^2 \times \mathbb{R}$ which corresponds to the HOCP in Example 5.2 at the particular switching time $t_s = 1$. It also displays a sequence of values $\{x_{t_s}^k\}$ generated by an execution of the Algorithm HMP[Z]V (with t_s fixed at the value 1). Figure 5.4 shows a set of $\mathbb{R}^2$ (space) slices of $\mathbb{R}^2 \times \mathbb{R}$ corresponding to these same example at the fixed switching times: $t_s = 0.9, 1.0, 1.1$; each slice intersects the (q_i, q_j)-indexed regular optimality zones in the indicated connected regions.

EXAMPLE 5.3. *Consider system with two locations q_1 and q_2, linear dynamics and quadratic cost criteria in each location. The system and weighting matrices are:*

$$A_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

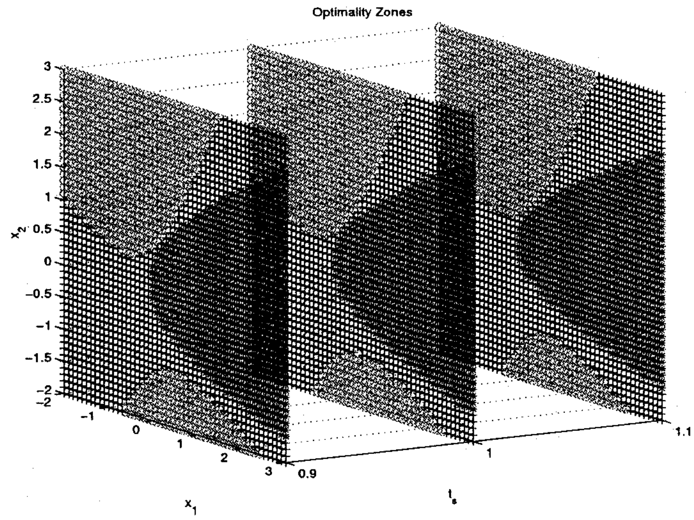

FIGURE 5.4. Optimality Zones in Example 5.2: LQ Case with Fixed Terminal State. 2-D Slices of the Zones at the Set of Switching Times: $t_s = 0.9, 1, 1.1$. $\times$:(q_1, q_1), $*$:(q_1, q_2), o:(q_2, q_1).

$$B_1 = B_2 = Q_1 = Q_2 = R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$R_2 = 1.6R1$

The initial and final times and states are:

$$t_0 = 0,\ t_f = 2,\ x_0 = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix},\ x_f = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The corresponding regular optimality zones for fixed $t_s = 1$ are display in Figure 5.5. □

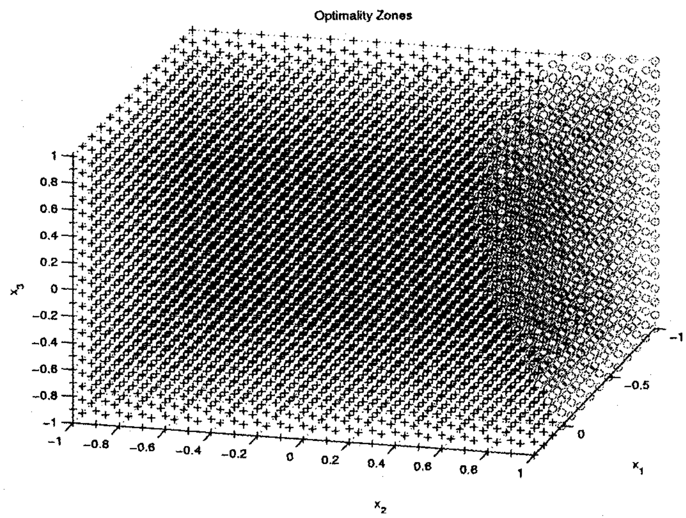

FIGURE 5.5. Optimality Zones in Example 5.3: LQ Case with Fixed Terminal State. $+$:(q_1, q_1), o:(q_1, q_2).

CHAPTER 6

Conclusion and Future Research

The work reported in this thesis can be extended in several directions. Suggestions for research related to topics treated here are outlines below.

6.1. Hybrid Minimum Principle

- *The Extension to time varying dynamics and time varying manifolds.*

  In Chapter 2 we presented the necessary conditions for hybrid system optimality in two cases: (i) where the optimal control takes values in a compact control value set, and (ii) where the control value set is an open bounded set. It was seen that the needle variations technique furnished the adjoint transversality conditions in case of controlled and autonomous switching, and Hamiltonian continuity condition in case of controlled switching. The switching manifold was taken to be time invariant. In this context, it is of immediate interest to extend the results to time varying switching manifolds. The Hamiltonian transversality property for autonomous switchings at time varying manifolds found in [**16, 40**] has the form:

$$H(t_j+) = H(t_j-) + p_j \, \nabla_t m_{j,j+1} \big|_{t=t_j} ,$$

  and its derivation may require a suitable local controllability condition.

  The above remark about time varying manifolds applies in the case of smooth variations technique as well, and it is conjectured that the small time tubular fountain (STTF) condition along with an assumption about a bound on $||\nabla_t m_{j,j+1}||$ should furnish the proof.

- *Extension to jumps in continuous state (impulses).*

  Many hybrid systems modeling frameworks found in the literature (see, for example [**12, 13, 39**]) provide for discrete jumps in the continuous state at switching times. This is generally achieved by defining a set of state reset functions $n_j \in C^k(\mathbb{R}^{n+1}; \mathbb{R}^n), k \geq 1$, so that if t_j is a switching time then $x(t_j+) = n_j(t_j, x(t_j-))$; necessary conditions in this case are given in [**39**] without proof. Inspiration for further investigations in this direction can also be found in the extensive work on *impulse control* [**8, 61**].

6.2. Hybrid Dynamic Programming

- *Relationship between the Hybrid Minimum Principle and Hybrid Dynamic Programming.*

  It is well known that under suitable differentiability conditions on the value function of a standard optimal control problem, the adjoint variable vector (λ) of the Minimum Principle is the same as gradient of the value function ($\nabla_x v$) [**9**]. This can be extended to the case of hybrid optimal control problems by showing that:

  (i) $\nabla_x v$ satisfies the same differential equation as λ in each location,

  (ii) $\nabla_x v$ satisfies the same boundary condition as λ, and

  (iii) $\nabla_x v$ satisfies the same transversality conditions as λ at switching times.

  Dreyfus [**23**] gives a heuristic derivation of the transversality conditions $\nabla_x v$ and they are similar to the λ-transversality conditions derived in Chapter 2 of this thesis.

6.3. Numerical Algorithms

- *Extension to other gradient based search methods.*

  Within the HMP Algorithms of Chapter 4 we employed a simple gradient based search with fixed step size to find the optimal switching time-state pair. In finite-dimensional optimization problems the most commonly used steepest descent algorithm solves a subproblem at each iteration step to determine the optimal step size. Further investigations need to be carried out to devise efficient methods for determining optimal step sizes for HMP Algorithms.

- *Hybrid Hierarchically Accelerated Dynamic Programming (HHADP).*

  It is proposed to construct an algorithm called the Hybrid Hierarchically Accelerated Dynamic Programming (HHADP) Algorithm which would constitute an effective procedure for finding suboptimal solutions, with estimates of suboptimality, for the standard optimal control problems appearing in the HMP Algorithms of Chapter 4; this algorithm extends the Hierarchically Accelerated Dynamic Programming (HADP) methodology of [**50, 49, 51**] to the continuous control systems case. A preliminary investigation of HHADP can be found in [**19**].

6.4. Optimality Zones

- *Topological and geometric properties of optimality zones.*

  Some elementary topological properties of optimality zones were presented in Chapter 5, but in order to develop a more complete theory of optimality zones and their use in hybrid optimization algorithms further topological and geometric investigations need to be carried out.

- *Efficient computation (or estimation) of zones.*

  A major component of the overall computational cost of Algorithm HMP[Z] of Chapter 5 is the computation of optimality zones. Hence it is of intrinsic interest to find efficient methods of computing (or estimating) the optimality zones for a given HOCP. These zones have a simple topological structure in the case of linear quadratic (LQ) problems as was pointed out in Chapter 5; but more work needs to be carried out to characterize the optimality zones in the case of general HOCPs.

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APPENDIX A

Bellman-Gronwall Lemma

LEMMA A.1. *([2]) Let $r, k \in C([a,b], \mathbb{R}_+)$. Let $h \in C^1([a,b], \mathbb{R}_+)$ with $\dot{h}(t) \geq 0$, $t \in [a,b]$. If*

$$r(t) \leq h(t) + \int_a^t k(s)r(s)\,ds, \qquad t \in [a,b], \tag{A.1}$$

then

$$r(t) \leq h(t)e^{\int_a^t k(s)\,ds} \qquad t \in [a,b]. \tag{A.2}$$

□

PROOF. Let $R(t) = h(t) + \int_a^t k(s)r(s)\,ds$, $t \in [a,b]$. Then $R(a) = h(a)$ and $\dot{R}(t) = \dot{h}(t) + k(t)r(t)$. By (A.1), $0 \leq r(t) \leq R(t)$. Hence $\dot{R}(t) \leq \dot{h}(t) + k(t)R(t)$, or $\dot{R}(t) - k(t)R(t) \leq \dot{h}(t)$. Multiplying both sides by the integrating factor $0 \leq e^{-\int_a^t k(s)\,ds} \leq 1$ we obtain

$$\dot{R}(t)e^{-\int_a^t k(s)\,ds} - k(t)R(t)e^{-\int_a^t k(s)\,ds} \leq \dot{h}(t)e^{-\int_a^t k(s)\,ds},$$

or, since $\dot{h}(t) \geq 0$ and $0 \leq e^{-\int_a^t k(s)\,ds} \leq 1$,

$$\dot{R}(t)e^{-\int_a^t k(s)\,ds} - k(t)R(t)e^{-\int_a^t k(s)\,ds} \leq \dot{h}(t),$$

or

$$\frac{d}{dt}\left(R(t)e^{-\int_a^t k(s)\,ds}\right) \leq \dot{h}(t).$$

Integrating both sides from a to t we obtain

$$R(t)e^{-\int_a^t k(s)\,ds} - R(a) \leq h(t) - h(a),$$

or

$$r(t) \leq R(t) \leq h(t)e^{\int_a^t k(s)\,ds} \qquad t \in [a, b].$$

$\square$

APPENDIX B

Standard Results from the Theory of Penalty Function Methods

LEMMA B.1. *([3]) Let $J, \eta : \mathbb{R}^n \to \mathbb{R}$ be continuous functions defined over a compact set S and let $\psi(r) \triangleq \inf_{z \in S}\{J(z) + r\eta(z)\} = J(z_r) + r\eta(z_r)$. Then*

(i) $\inf_{z \in S}\{J(z) : \eta(z) = 0\} \geq \sup_{r \geq 0} \psi(r)$;

(ii)(a) $\eta(z_r)$ is a decreasing function of r,

(b) $J(z_r)$ is an increasing function of r,

(c) $\psi(r)$ is an increasing function of r.

PROOF. Let $w \in S$ be such that $P(w) = 0$. Then

$$J(w) = J(w) + rP(w) \geq \inf_{z \in S}\{J(z) + rP(z)\} = \psi(r),$$

which implies (i).

To prove (ii) let $0 \leq r_1 < r_2$. Then by definition of z_{r_1} and z_{r_2} we have

$$J(z_{r_2}) + r_1 P(z_{r_2}) \geq J(z_{r_1}) + r_1 P(z_{r_1}) \tag{B.1}$$

$$J(z_{r_1}) + r_2 P(z_{r_1}) \geq J(z_{r_2}) + r_2 P(z_{r_2}) \tag{B.2}$$

Adding inequalities (B.1) and (B.2), we get $(r_2 - r_1)[P(z_{r_1}) - P(z_{r_2})] \geq 0$, which implies $P(z_{r_1}) \geq P(z_{r_2})$. Hence $P(z_r)$ is a decreasing function of r, i.e. (ii)(a) holds.

In view of the inequality $P(z_{r_1}) \geq P(z_{r_2})$, inequality (B.1) yields

$$J(z_{r_2}) + r_1 P(z_{r_2}) \geq J(z_{r_1}) + r_1 P(z_{r_1}) \geq J(z_{r_1}) + r_1 P(z_{r_2}) \Longrightarrow J(z_{r_2}) \geq J(z_{r_1}).$$

Hence J is an increasing function of r and so (ii)(b) holds.

Next, adding and subtracting $r_2 P(z_{r_2})$ on the left hand side of (B.1) we obtain

$$J(z_{r_2}) + r_1 P(z_{r_2}) = J(z_{r_2}) + r_1 P(z_{r_2}) + r_2 P(z_{r_2}) - r_2 P(z_{r_2}) \geq J(z_{r_1}) + r_1 P(z_{r_1}) = \psi(r_1)$$

or

$$\psi(r_2) + (r_1 - r_2) P(z_{r_2}) = (J(z_{r_2}) + r_2 P(z_{r_2})) + (r_1 - r_2) P(z_{r_2}) \geq J(z_{r_1}) + r_1 P(z_{r_1}) = \psi(r_1),$$

since $(r_1 - r_2) P(z_{r_2}) \leq 0$, $\psi(r_2) \geq \psi(r_1)$ as required. □

THEOREM B.1. *([3]) Let $J, \eta : \mathbb{R}^n \to \mathbb{R}$ be continuous functions defined over a compact set S and let $\psi(r) \triangleq \inf_{z \in S}\{J(z) + r\eta(z)\}$. Then*

$$\inf_{z \in S}\{J(z) : \eta(z) = 0\} = \sup_{r \geq 0} \psi(r) = \lim_{r \to \infty} \psi(r).$$

PROOF. Since $\psi(r)$ is an increasing function of r, $\sup_{r \geq 0} \psi(r) = \lim_{r \to \infty} \psi(r)$.

We next show that $\lim_{r \to \infty} \eta(z_r) = 0$ by showing that for every $\epsilon > 0$ there is $r_\epsilon \in \Re$ such that, for all $r \geq r_\epsilon$, $\eta(z_r) \leq \epsilon$. Let $z_0 \in S$ be a feasible point, i.e. $\eta(z_0) = 0$ and let $\epsilon > 0$. Further, for $r = 1$, let $z_1 = \operatorname{argmin}_{z \in S}\{J(z) + \eta(z)\}$ (i.e. in this case $r = 1$). Finally take r to be such that $r \geq \frac{1}{\epsilon}|J(z_1) - J(z_0)| + 2$. Then, since $r > 1$, by Lemma B.1 (ii)(b), $J(z_r) \geq J(z_1)$. In order to obtain a contradiction, assume that $\eta(z_r) > \epsilon$. Then $r\eta(z_r) > r\epsilon \geq |J(z_1) - J(z_0)| + 2\epsilon$. Then by Lemma

B.1,

$$\begin{aligned}
\inf_{z\in S}\{J(z):\eta(z)=0\} &\geq \psi(r)=J(z_r)+r\eta(z_r)\\
&\geq J(z_1)+r\eta(z_r)\\
&> J(z_1)+|J(z_1)-J(z_0)|+2\epsilon\\
&> J(z_1)+J(z_0)-J(z_1)+2\epsilon\\
&> J(z_0)
\end{aligned}$$

which is a contradiction. Hence $\eta(z_r)\leq\epsilon$ for such an r. Since ϵ is arbitrary, this shows that $\lim_{r\to\infty}\eta(z_r)=0$.

Let $\{r_l : l\in \mathbb{Z}_1\}$ be a sequence tending to ∞, and let $\{z_{r_k}\}$ be a convergent subsequence of the infinite sequence $\{z_{r_l}\}\subset S$. Let $\{z_{r_k}\}$ converge to $\bar{z}\in S$. Then

$$\sup_{r\geq 0}\psi(r)\geq\psi(r_k)=J(z_{r_k})+r_k\eta(z_{r_k})\geq J(z_{r_k}),$$

which by continuity of J implies that $\sup_{r\geq 0}\psi(r)\geq J(\bar{z})$.

Similarly, continuity of η implies that $\lim_{r_k\to\infty}\eta(z_{r_k})=\eta(\bar{z})=0$. This shows that $\bar{z}$ is feasible. Again from Lemma B.1

$$\inf_{z\in S}\{J(z):\eta(z)=0\}\geq\sup_{r\geq 0}\psi(r)\geq\sup_{r_k\to\infty}(J(z_{r_k})+r_k\eta(z_{r_k}))\geq J(\bar{z}).$$

However, since $\inf_{z\in S}\{J(z):\eta(z)=0\}\leq J(\bar{z})$ we must have equality throughout in the above expression and hence $J(\bar{z})=\sup_{r\geq 0}\psi(r)=\lim_{r\to\infty}\psi(r)$. Also since $0\leq r_k\eta(z_{r_k})\leq\sup_{r\geq 0}\psi(r)-J(z_{r_k})$ and since $\lim_{r_k\to\infty}(\sup_{r\geq 0}\psi(r)-J(z_{r_k}))=J(\bar{z})-J(\bar{z})=0$ we must have $\lim_{r_k\to\infty}r_k\eta(z_{r_k})=0$. □

APPENDIX C

Ekeland's Variational Principle

THEOREM C.1. *([**26**]) Let V be a complete metric space with metric $d(\cdot,\cdot)$. Let $F : V \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function which is lower bounded and not identically $+\infty$. If $u \in V$ is such that for $\epsilon > 0$:*

$$F(u) \leq \inf_{x \in V} F(x) + \epsilon$$

then there exists $v \in V$ such that

$$d(u,v) \leq \sqrt{\epsilon}$$

and v minimizes $G(w) = F(w) + \sqrt{\epsilon}\, d(v,w)$. □

PROOF. See [**26**]. □