GENERATOR MATRIX ELEMENTS FOR NONCOMPACT SP(6) IN A SP(2) X O(3) BASIS

by

Mihaela Niculescu-Sanielevici

A thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the
Ph. D. degree.

Department of Physics
McGill University
Montréal, Québec
Canada.

February 1986

GENERATOR MATRIX ELEMENTS FOR NONCOMPACT SP(6) IN A SP(2) X O(3) BASIS

ABSTRACT

This thesis finds the basis states of the symplectic model for nuclear collective behaviour. We are able to find a complete set of states for Sp(6), the most important group of collective motions in nuclei. From the multitude of possible eigenstates we are interested in vibrational type ones, or Bohr-Mottelson type, which are at the same time a basis in the subgroup $Sp(2) \times O(3)$ of Sp(6). The basis we find is not orthonormal but complete and nonredundant and we show how to use it to determine generator matrix elements between these states. Since the Hamiltonian of the problem is in the enveloping algebra of Sp(6) it can be expressed as a low rank polynomial in the Sp(6) generators and the actual vibrational spectrum of nuclei is then calculable. The results are applicable to all nuclei (any Z). We are able to exactly solve the difficult problem of finding the basis states using generating function methods. We first obtain the generating function giving branching rules for the chain $Sp(6) \supset Sp(2) \times O(3)$ and we interpret the terms appearing in the generating function as stretched products of a finite set of elementary permissible diagrams which form the integrity basis. The basis states of the group and subgroup are defined in terms of this integrity basis.

RESUME

Cette thèse trouve les états de base du modèle symplectique pour le comportement collectif nucléaire. Nous trouvons une base complète d'états pour Sp(6), le plus important groupe de mouvements collectifs dans les noyaux. Parmi la multitude d'états propres possibles, nous ne sommes intéressés que dans ceux de type vibrationnel, dits de Bohr-Mottelson, qui forment en meme temps une base dans le sous-groupe $Sp(2)\times O(3)$ de Sp(6). La base que nous trouvons n'est pas orthonormale mais elle est complète et nonredondante et nous montrons comment l'utiliser pour calculer des éléments de matrices des générateurs entre ces états. Puisque le Hamiltonien du problème obéit à l'algèbre enveloppante de Sp(6), il peut s'exprimer comme un polynôme de bas rang dans les générateurs de Sp(6) et le spectre vibrationnel des noyaux peut donc être calculé. Ces résultats peuvent être appliqués pour tout noyau (tout Z). Nous parvenons à résoudre le problème compliqué de trouver les états de base en utilisant les méthodes de la fonction génératrice. Nous obtenons premièrement la fonction génératrice donnant des règles d'embranchement pour la chaîne $Sp(6)\supset Sp(2)\times O(3)$ et nous interprétons les termes apparaissant dans la fonction génératrice comme des produits allongés d'un ensemble fini de diagrammes élémentaires permissibles qui forment une base d'intégrité. Les états de base de ce groupe et de ce sousgroupe sont définis en termes de cette base d'intégrité.

ACKNOWLEDGMENTS

First of all I wish to thank Professor R.T. Sharp, for accepting me as a graduate student and for doing his best in helping me graduate. Due to his contibutions to Group Theory over more than thirty years, he was able to talk to me about problems in the field and many of the authors with the freshness one talks about lived experiences and close friends. I thank for the time and patience he took to clarify many points to me and to properly guide me through the jungle of different notations and conventions. Many times I took advantage of his genuine kindness and I preferred to talk to him instead of going down to the library. And he always knew everything!

Even if the research done together is unrelated to this thesis, I take this opportunity to thank Prof. J.Patera of CRMA and Prof. R.V.Moody for the joyful experience I had working in such a great team. They taught me by personal example how rewarding it is to put all the enthusiasm and energy into a work well done and to impose to oneself the highest standards of quality and scientific honesty.

I thank Dr. Pierre Valin for allowing and helping me to use his personal computer as a word processor and for the many enjoyable physics related discussions we had.

Finally, I thank Dr. Sergiu Sanielevici for the many evenings he spent baby-sitting and for encouraging me not to give up being a woman scientist.

STATEMENT OF ORIGINALITY

Apart from the Introduction and Chapter 2 which places the present work in the general framework of the group theoretical approach to Nuclear Collective Models, most of the material presented in this thesis is original. Chapter 3 discusses the needed algebra and the problems we encounter. The solutions are found in the literature but their application to our specific problem is original. The most important contribution is the generating function for $Sp(6) \supset Sp(2) \times O(3)$ given in Chapter 4. The Sp(6) generators are known but we treat them in the light of the $Sp(2) \times O(3)$ subgroup. We calculate the Clebsch-Gordan couplings for Sp(2) which we were unable to find in the literature. The research presented in Chapters 5 and 6 is also original as it uses as a starting point the basis states given by the original generating function.

TABLE OF CONTENTS

PAGE

CHAPTER 1. INTRODUCTION

1.1. Brief exposition of the problem

Group theoretical methods have been proven very useful in solving physical problems. Here we are interested in the collective behaviour of many - nucleon systems which is associated with the symplectic geometry of a system of A nucleons. The goal is to determine complete sets of states which can further be used in nuclear structure calculations.

The first successful introduction of collective degrees of freedom in nuclear theory was in the framework of the liquid drop model of Niels Bohr [Bo36] and what became later the Bohr-Mottelson model [BM53]. In this model the surface of the drop is parametrized by

$$r=r(\theta,\varphi)=r_0\left(1+\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\alpha_{\ell m}Y_{\ell m}(\theta,\varphi)\right) \tag{1-1}$$

where the right hand side is just some function in spherical coordinates expanded in spherical harmonics and the coefficients can be considered as collective coordinates. Such a surface equation is certainly not the most general one but is widely used and quite successful in nuclear structure problems. The quadrupole deformations correspond to $\ell=2$, the octupole ones to $\ell=3$, etc. Since in almost all nuclei the quadrupole degree of freedom plays a fundamental role, we restrict ourselves to the case $\ell=2$ and we have five parameters α_{2m} as collective coordinates which we now call α_m, m=2,1,0,-1,-2. The equation of the surface becomes [BM53]

$$r=r_0\,(1+\sum_{m=-2}^{+2}\alpha_m Y_{2m}) \qquad (1\text{-}2)$$

with $\alpha_m=(-1)^m\alpha_{-m}$. The deformation parameters α_m are related to the mass quadrupole of the nucleon system which is a scalar with respect to the O(n) group of rotations in the nucleon index space (n is essentially the number of nucleons) and thus can be used to describe collective behaviour.

In the group theoretical approach we use the concept of a dynamical group as opposed to a symmetry group. A symmetry group of transformations leaves the Hamiltonian invariant and gives rise to degenerate multiplets of states which carry representations of the group. Examples are the rotation group SO(3) and the isospin group SU(2). A dynamical group, on the other hand, requires only that energy eigenstates belong to a single irreducible representation of the group but does not require that all states of an irreducible representation be degenerate. A familiar example is the simple 3-dimensional harmonic oscillator for which the dynamical group is Sp(6) (or Sp(3,R)) while the symmetry group is its SU(3) subgroup. Another example is Elliott's model in which SU(3) and its SO(3) subgroup are both dynamical groups but only SO(3) is a symmetry group. Thus in Elliott's model, the different angular momentum states of a given SU(3) representation are separated in energy but the multiplets of a given angular momentum remain degenerate. The Lie algebra of a dynamical group is also called a "spectrum generating algebra".

For a system of A nucleons in ordinary 3-dimensional space the problem has 3n degrees of freedom, where n=A-1 (the center of mass of the system of nucleons is eliminated). Assuming the basis states to be the energy eigenstates of an isotropic 3n-dimensional harmonic oscillator, the symmetry group is SU(3n) which is contained in Sp(6n), the dynamical group of a 3n-dimensional harmonic oscillator. We must retain only the states of symmetric representations. This means all representation labels of SU(3n) are zero except the first one. This corresponds to the metaplectic irreducible representations of Sp(6n) $[(1/2)^{3n}]$ and $[(1/2)^{3n-1},(3/2)]$ which are spanned by the SU(3n) states of even and odd representation labels respectively.

Historically the first authors to develop the point of view in which we are interested here are Goshen and Lipkin [GL59,GL68] who considered an n-body system in one dimension described by Sp(2n), the symplectic group in 2n dimensions. When considering the subgroup $Sp(2)\times O(n)$, Sp(2) describes the collective excitations and the Hamiltonian is in the O(2) subgroup of the latter. A few years later they extended their analysis to two dimensions, i.e. $Sp(4n)\supset Sp(4)\times O(n)$. The works of Goshen and Lipkin provide the foundations for the whole branch of nuclear physics and group theory called microscopic collective models.

In the early seventies Zickendraht [Zi71] and Dzublik et al. [DO72] showed how to transform the coordinates from the 3n degrees of freedom of the problem (in the usual 3 dimensional space) to six collective degrees of freedom and 3n-6 single-particle excitation degrees of freedom. It was shown by Morinigo [Mo72], Filippov [Fi73] and Vanagas [VK73] that, whereas collective rotational coordinates are associated with the rotational group SO(3), intrinsic coordinates can be associated with the group O(n) of orthogonal transformations in particle index space. Hence Filippov [Fi73,Fi78] introduced the method of generalized hyperspherical functions exploiting the subgroup structure

$$O(3n) \supset SO(3) \times O(n)$$

Instead of expanding the nuclear wave functions on O(3n) spherical harmonics, Filippov expanded them on O(n) spherical functions with coefficients (collective wave functions) depending on six collective coordinates. A similar approach was used by Neudachin and Smirnov and Vanagas [Va71] in their translationally invariant shell model when they introduced a basis of states for shell model calculations in the subgroup chain

$$U(3n) \supset U(3) \times U(n) \supset SO(3) \times O(n)$$

Major progress was made with the complementarity theorem of Moshinsky and Quesne [MQ71]. They showed that in the chain

$$Sp(6n) \supset Sp(6) \times O(n)$$

keeping all shell model states of a given hyperspherical function is equivalent to restricting to a single Sp(6) irreducible representation (IR). Note that restricting to a given hypersherical function is not the same as restricting to a single O(n) representation any more than states of fixed angular momentum quantum numbers LM belong to a single SO(3) representation. In fact one can think of the O(n) hyperspherical functions as the intrinsic component of the many-particle wave function and the complementary Sp(6) wave function as the collective component.

Authors like Rosensteel and Rowe [RR76,RR80], Biedenharn, Buck, Cusson and Weaver [WB73,WC76,BB79] worked with groups related to Sp(6). Studying the collective motions and the operator algebra they introduced cm(3), the collective motion Lie algebra and the vortex spin operator which couples the rotational motion to the internal dynamics. It was soon realized that CM(3), the group associated with the Lie algebra cm(3), is a subgroup of Sp(6), which these authors call Sp(3,R). Rowe and Rosensteel then proceeded in finding a basis for the IRs of $Sp(6) \supset CM(3)$ using both group-theoretical and shell model considerations and carried out calculations of nuclear structure using these bases. Sp(6) contains also U(3) as a natural subgroup and Sp(6) $\supset$ U(3) $\supset$ SO(3) was considered to obtain bases for nuclear collective motions. After the universal acceptance of Sp(6) as describing the collective behaviour of many nucleon systems, many authors tried to find the solution to the problem, that is to find the matrix elements of the generators of Sp(6) between basis states. This is highly nontrivial because it implies finding a solution to the commutation relations of the generators' algebra. Realizing the difficulty of this task, approximate solutions were tried. For instance Rosensteel and Rowe [RR80] took the liquid (i.e. many nucleon) limit of the Sp(6) Lie algebra and obtained the u(3)-boson Lie algebra which consists of the unitary algebra u(3) plus a bosonic (Heisenberg) algebra as the ideal and they were able to solve exactly this liquid limit. In order to solve the Sp(6) problem numerically they suggested the steepest descent method using the u(3)-boson matrix elements as a first approximation and they carried out the computer calculations only for very special cases (representations) which do not arise for real nuclei. It is interesting to note here that their boson algebra is mathematically isomorphic to the interacting boson algebra of Arima and Iachello [AI76].

The Sp(6) collective model, called also the symplectic nuclear model combines the features of both Elliott and Bohr-Mottelson models. When interested in rotational or Elliott [El58] type states one has to consider the subgroup U(3) the unitary group in three dimensions which is a natural subgroup of Sp(6) while for vibrational motions one has to find Bohr-Mottelson type states which form a basis in the chain $Sp(6) \supset Sp(2) \times O(3)$.

Gaskell, Rosensteel and Sharp [GR81] used generating functions to obtain a basis in the chain $Sp(6) \supset SU(3) \times U(1)$ which is of interest when studying collective rotations (remember Elliott's U(3) group). Their method enables one to get analytically a complete set of basis states for the group-subgroup $Sp(6) \supset U(3)$ (for the compact version Sp(6) as well as for noncompact Sp(6,R)). In this group-subgroup chain the subgroup does not provide enough labels to specify the states uniquely - the same U(3) may occur many times in one Sp(6,R) (or Sp(6)) representation. One solution to this missing labels problem is to define a complete linearly independent but nonorthonormal basis. The generating function obtained implies a set of polynomial bases. First, one evaluates the highest states of the elementary multiplets as polynomials in the states of the fundamental irreducible representations (100), (010), and (001). Compatible products of powers of these highest states correspond one-to-one to highest states of all U(3) multiplets contained in Sp(6) representations. The analytic basis states in the subgroup $SU(3) \times U(1)$ obtained by Gaskell et al. could be used to derive analytic generator matrix elements for nuclear structure calculations.

For Bohr-Mottelson or vibrational type states one has to find a basis in the chain $Sp(6) \supset Sp(2) \times O(3)$. After the pioneering work of

Goshen and Lipkin, Moshinsky [Mo84] studied the problem in d=1,2 dimensions, i.e. the chains $Sp(2d) \supset Sp(2) \times O(d)$. Their effort to solve the three dimensional problem proved unsuccessful except for the very simple case of closed shells.

1.2 Summary of this work

The present work deals with the vibrational collective nuclear behaviour as described by the symplectic model. We use the generating function approach to derive analytically basis states for the chain $Sp(6) \supset Sp(2) \times O(3)$, i.e. for the real three dimensional world. We encounter the missing labels problem and in this case there are five missing labels (after Racah [Ra51] Sp(6) has $1/2(r-\ell)=1/2(21-3)=9$ internal labels where r is the rank and ℓ the order of the Lie algebra and the subgroup $Sp(2) \times O(3)$ provides 2+2=4 labels; missing labels=9-4=5). Like in the $Sp(6) \supset U(3)$ case (with 3 missing labels) one approach is to use a nonorthonormal basis, but Which is normally [illegible] complete and nonredundant. We explicitly construct these states in terms of the elementary multiplets suggested by the generating function and we calculate analytically representative matrix elements of Sp(6) generators between the basis states thus obtained.

In the next two chapters, we introduce the theoretical background and we discuss in more detail the related nuclear models as well as the group theoretical notions we are using. Chapter 4 contains a description of the generating function method and the actual way of obtaining the desired $Sp(6) \supset Sp(2) \times O(3)$ generating function. We start with the known $Sp(6) \supset U(3)$ generating function and we convert the SU(3) subgroup to SO(3) using the $SU(3) \supset SO(3)$ generating function and also convert U(1) to noncompact Sp(2). We also obtain some additional generating functions which we present at the end of chapter 4. Knowing the $Sp(6) \supset U(3)$ generating function we obtain the $SU(n) \supset O(n)$ one, first three labels of SU(n) non-zero. Also we convert the $Sp(6) \supset Sp(2) \times O(3)$ generating function to $O(3n) \supset O(3) \times O(n)$, first label of O(3n) non-zero. The respective generating functions (and the branching rules) are related by complementarity conditions. The generating function for our problem suggests an integrity basis, that is, a set of elementary permissible diagrams (epd's). In chapter 5, we define the vibrational states in the nuclear symplectic model in terms of products of powers of the epd's. Certain combinations of epd's are forbidden and we call the incompatible products syzygies ; the syzygies can be read directly from the generating function and they correspond to epd's which never appear together in the binomial expansion of the generating function. We give explicit expressions for the epd's and show by explicit examples how they are constructed. Chapter 6 discusses the generating matrix elements of Sp(6) in our basis. We start with the Sp(6) algebra and its generators; we then show how use of the Wigner-Eckart theorem reduces the number of matrix elements to be calculated and we give examples of generator matrix elements. The last chapter gives a summary of the results as well as presenting conclusions and outlook for future work.

CHAPTER 2. THE SYMPLECTIC MODEL AS A UNIFIED MODEL

2.1. The collective model and the vortex spin

In the Bohr-Mottelson model (also known as the Bohr-Mottelson-Frankfurt or BMF model), one assumes that the collective configuration of the nucleus is given by the shape and the orientation of its surface. It is natural then to describe the surface by an expansion in spherical harmonics of the form given in Eq.(1-1). The deformation parameters $\alpha_{\ell m}$ are the collective coordinates. The coefficient α_{00} describes changes in the nuclear volume. Since the nuclear fluid is highly incompressible, we require the volume to be kept fixed at $V=4/3\pi r_0^3$ for all deformations. This defines the constant

$$\alpha_{00} = -\frac{1}{4} \sum_{\ell \geq 1} \sum_{m} |\alpha_{\ell m}|^2 \tag{2-1}$$

The term $\ell=1$ describes mainly (at least for small deformations) a translation of the whole system and the three parameters α_{1m} can be fixed by the condition that the origin coincides with the center of mass

$$\int \vec{r}\, d^3 r = 0 \tag{2-2}$$

In the following we omit both $\ell=0$ and $\ell=1$ since they are not relevant for small collective oscillations of the surface. The $\ell=2$ case corresponds to quadrupole deformations which give a good description of the collective oscillations for the purpose of further calculations. Octupole (and higher) deformations are known to be small.

For the quadrupole deformations we have five parameters α_m . Not all of them describe the shape of the drop. Three determine the orientation of the drop in space and correspond to the three Euler angles. Requiring that the surface equation is invariant under a rotation of the coordinate system, the collective coordinates must behave like $Y_{\ell m}$ under a rotation of the coordinate system (characterized by the Euler angles $\Omega = \alpha, \beta, \gamma$) [Ed57, Eq.(5.2.1)], i.e.,

$$(Y_{\ell m})\text{new} = \sum_{m'} D^{\ell}_{mm'}(\Omega)\,(Y_{\ell m'})\text{old}$$

$$a_{\ell m} = \sum_{m'} D^{\ell}_{mm'}(\Omega)\,\alpha_{\ell m'} \tag{2-3}$$

where the $D^{\ell}_{mm'}(\Omega)$ are rotation matrix elements and the $a_{\ell m}$ are the deformation parameters in the new system. For $\ell=2$ the rotation is

$$a_{2m} = \sum_{m'} D^{2}_{mm'}(\Omega)\,\alpha_{2m'} \tag{2-3'}$$

and we choose the rotation which brings us to the body-fixed system whose axes coincide with the principal axes of the mass distribution of the drop. In this system of coordinates the five coefficients a_{2m} reduce to two real independent variables a_{20} and $a_{22} = a_{2\bar{2}}$ ($a_{21} = a_{2\bar{1}} = 0$), which, together with the three Euler angles, give a complete description of the system. It is usual to introduce for convenience instead of a_{20} and a_{22} the so-called Hill-Wheeler [HW53] coordinates β, γ ($\beta > 0$) through the relations

$$a_{20} = \beta \cos\gamma \qquad (2\text{-}4)$$

$$a_{22} = \beta \sin\gamma/\sqrt{2}$$

from which we have

$$\sum_m |\alpha_{2m}|^2 = a_{20}^2 + 2a_{22}^2 = \beta^2 \qquad (2\text{-}5)$$

and

$$r(\theta,\varphi) = r_o[1 + \beta\sqrt{\tfrac{5}{16\pi}}(\cos\gamma(3\cos^2\theta - 1) + 3\sin\gamma\sin^2\theta\cos2\varphi)] \qquad (2\text{-}6)$$

The coordinates β, γ are related to the rotational invariants $(Q_2 x Q_2)^0$ and $(Q_2 x Q_2 x Q_2)^0$ as follows

$$\beta^2 \sim (Q_2 \times Q_2)^0$$

$$\beta^3 \cos 3\gamma \sim (Q_2 \times Q_2 \times Q_2)^0$$

and they are used in writing down phenomelogical potentials such as $A\beta^2$ for a spherical nucleus or $A\beta^2 + B\beta^3\cos3\gamma + C\beta^4$ for a nucleus with an axially deformed equilibrium shape. Hess et al.[HS80,HM81] consider more general potentials in β and γ with application to BMF calculations.

The excitations in the BMF model are the surface oscillations. The surface parameters $\alpha_{\ell m}$ ($\ell \geq 2$) of Eq.(2-1) are the dynamical variables. They are considered to be functions of time : $\alpha_{\ell m}(t)$. For low-lying excitations one can expect that they produce small oscillations around the spherical equilibrium shape with $\alpha_{\ell m}=0$, and that the classical collective Hamiltonian H_{coll} that describes this process has the harmonic oscillator form [Bo52]

$$H_{coll} = T + V = \frac{1}{2} \sum_{\ell} \sum_{m} [A_{\ell} |\dot{\alpha}_{\ell m}|^2 + B_{\ell} |\alpha_{\ell m}|^2] \tag{2-7}$$

Here the parameters of inertia A_{ℓ} and of stiffness B_{ℓ} are real constants. The Eq.(2-7) is, in fact, the only quadratic form which is invariant under rotation and time reversal.

The constants A_{ℓ} and B_{ℓ} can be calculated within the fluid picture; they depend on the flow associated with the surface oscillations. Now it becomes important to look at the nucleus from a hydrodynamic perspective. For practical purposes the nucleus is a zero temperature system so that its equilibrium state is its ground state. In considering its collective properties, it was natural therefore, at the beginning, to expect it to behave like familiar macroscopic quantum fluids, eg. liquid helium at low temperature.

Consequently, the liquid drop model portrayed the nucleus as a superfluid (vortex free) droplet [BK37]. However, it turned out that experimentally observed collective mass parameters and moments of inertia for low-lying states were several times larger than superfluid values. If we look closer at why a low temperature quantum fluid exhibits superfluidity we can see if the nucleus is or not vortex free.

Consider, for example, an ideal gas of non-interacting Bose particles at sufficiently low temperature that all particles are in their ground state. If the container is rotated slowly, the flow can be calculated using perturbation theory and one finds that the velocity field, $\vec{v}(r)$, is irrotational, i.e., $\nabla \times \vec{v}(r)=0$. Starting from this idea, Inglis [In54,In55] proposed the cranking model for the nucleus in which the nuclear container is an ellipsoidally deformed harmonic oscillator potential. The ground state of a system of non-interacting nucleons in a slowly rotating harmonic oscillator potential exhibits some interesting properties [BM55]. If the nucleon number corresponds to a closed spherical harmonic oscillator shell, the cranking returns an irrotational flow value for the moment of inertia. On the other hand, for an open shell nucleus, the moment of inertia turns out to be the rigid body value for flow rotations. Including spin-orbit forces and pairing correlations the crancking model yields moments of inertia between the two limits and in good accord with experimentally observed values.

To see why the collective model needs vorticity we make the following point (from a 1968 paper by Cusson [Cu68]). Consider a general linear model in which the collective flow lines satisfy the equation

$$x_i(t) = \sum_j g_{ij}(t)\ x_j(0) \tag{2-8}$$

where $g_{ij}(t)$ is a time-dependent 3x3 matrix. The velocity field is given by

$$v_i(t) = \sum_j \dot{g}_{ij}(t)\ x_j(0) \qquad \text{where} \quad \dot{g}_{ij} = (\partial/\partial t)\ g_{ij} \tag{2-9}$$

For rigid rotations the matrix g_{ij} should be an orthogonal (rotation) matrix, implying that its time-derivative is an antisymmetric matrix and that $\vec{v}$ can be expressed as

$$\vec{v} = \vec{\omega} \times \vec{r} \tag{2-10}$$

On the other hand for irrotational flow, one requires

$$\nabla \times \vec{v} = 0 \tag{2-11}$$

which means that the time-derivative of the matrix g_{ij} has to be symmetric. Allowing the matrix $\dot{g}_{ij}$ to have both symmetric and antisymmetric parts, the model admits rotations with and without vorticity. Altogether, we get 9 collective degrees of freedom. In this work, we restrict to monopole (ℓ=0) and quadrupole (ℓ=2) collective motions. The model has then 6 collective degrees of freedom. With the assumption that quadrupole collective motions are volume-conserving, one decouples the monopole and quadrupole modes and restricts to 5 quadrupole degrees of freedom. We include both $\ell=0$ and $\ell=2$ because they are coupled by the algebraic structure. For a rigid nucleus, it is possible to suppress the vibrational degrees of freedom and consider a collective model with only rotational degrees of freedom. This is the rigid rotor model as opposed to the soft rotor model with vibrational shape fluctuations. As we discussed before, if we allow both rigid rotations and irrotational flow, we have to augment the six-dimensional collective model with three additional rotational degrees of freedom corresponding to rigid flow. The total angular momentum of the augmented model then has two components

$$\vec{J} = \vec{L} + \vec{S} \tag{2-12}$$

This nine-dimensional model reverts to the six-dimensional model with irrotational flows only if restricted to states of $\vec{S}=0$. Thus, loosely speaking, we can regard $\vec{S}$ as a vortex spin angular momentum which takes zero value for irrotational flow. However very recently Le Blanc et al.[LC85] showed that it is possible to take into account the vortex spin degrees of freedom by a renormalization of the collective parameters. If this is correct we can use the six vibrational degrees of freedom in the frame of the Sp(6) symplectic nuclear model without fear of neglecting the vortex spin coordinates. Also this is in accord with the remarkable successes of the BMF model with adjustable parameters in nuclear phenomenology.

2.2 The shell model and the CM(3) collective model

While the liquid drop model generically, and BMF in particular, study global properties of the nucleus, the shell model is needed to describe many nuclear properties, including the similarities with atomic physics such as the occurrence of so-called magic numbers. In the shell model the nucleons are considered as independent particles moving on almost unperturbed single particle orbits. This is possible because the nucleus is not a very dense system as a consequence of the Pauli and uncertainty principles. This model takes into account individual nucleons and thus provides a quantum-mechanical many-body non-relativistic microscopic description of the nucleus with two-body interactions.

In this single-particle model the nucleons move in an average potential which is assumed to be of harmonic oscillator type. For A nucleons in three dimensions we write the harmonic oscillator Hamiltonian

$$H_0 = 1/(2m)\sum_{i=1}^{3}\sum_{s=1}^{A} p_{is}^2 + (1/2)\, m\omega^2 \sum_{i=1}^{3}\sum_{s=1}^{A} x_{is}^2 \tag{2-13}$$

Of course much better zero-order independent Hamiltonians can de devised, like Hartree-Fock, but the harmonic oscillator is an irresistible choice because of its rich group theoretical structure [KM68,KJ80]. Thus, even if one uses Hartree-Fock wave functions in a shell model calculation, it is usually convenient to expand them in a harmonic oscillator basis.

This particular Hamiltonian in (2-13) has the nice property that it easily separates the center-of-mass component. By an orthogonal transformation of coordinates the harmonic Hamiltonian separates

$$H_0 = 1/(2m)\sum_{i=1}^{3}\sum_{s=1}^{A-1} P_{is}^2 + (1/2)\ m\omega^2\sum_{i=1}^{3}\sum_{s=1}^{A-1} X_{is}^2 + H_{c.m.} \qquad (2\text{-}14)$$

where X_{is} and P_{is} are the Jacobi coordinates and momenta and n=A-1 is the number of relative Jacobi coordinates. The Jacobi coordinates are defined by

$$X_{is} = [s(s+1)]^{-1/2}\ [\sum_{t=1}^{s} x_{it} - s\ x_{is+1}\ ]\ , \qquad s=1,...n \qquad (2\text{-}15)$$

with their conjugated momenta $P_{is}=-i(\partial/\partial X_{is})$. Thus, to remove the center-of-mass, one simply replaces in (2-13) A by n=A-1 and reinterprets the coordinates. To explain detailed properties one must include two-body interactions. To be able to use the conventional shell model to describe the properties of highly collective states, it is necessary to include effective interactions and charges. If specifically interested in collective states, one needs to diagonalize a collective Hamiltonian, like the one in Eq.(2-7) in shell model space. For many years the search for a shell model expression for the collective kinetic energy remained unsuccessful. Looking for momenta conjugate to the mass quadrupole moments, Weaver and Biedenharn [WB70,WB72] were able to construct generators of quadrupole deformations. To close the algebra under commutation, they were forced to include the three angular momentum operators. In this way, they ended up with the eight generators of the special linear group SL(3,R).

Thus, Weaver, Biedenharn and Cusson [WB73] proposed the CM(3) (Collective Motion in 3 dimensions) model with six monopole and quadrupole operators and eight momentum operators. This model has the algebraic structure of a semi-direct sum Lie algebra $cm(3) \sim [R^6]sl(3,R)$. Adding a generator for monopole deformation, the algebra extends to $cm+(3) \sim [R^6]gl(3,R)$.

The six generators of monopole and quadrupole deformations are the six components of the tensor

$$S_{ij} = \sum_{s=1}^{n} (X_{is} P_{js} + P_{is} X_{js}) \tag{2-16}$$

fully symmetrized and the angular momenta are the three components of the antisymmetric tensor

$$L_{ij} = \sum_{s=1}^{n} (X_{is} P_{js} - X_{js} P_{is}) \tag{2-17}$$

These nine generators span the Lie algebra gl(3,R). The six monopole-quadrupole moments

$$Q_{ij} = \sum_{s=1}^{m} X_{is} X_{js} \qquad (2\text{-}18)$$

span the abelian R^6 algebra.

2.3 The symplectic model

The unified symplectic model is based on the sp(6) (known also as sp(3,R)) algebraic structure and embraces both the microscopic collective model and the harmonic oscillator shell model. A simple Hamiltonian would be

$$H = H_0 + V(Q) \qquad (2\text{-}19)$$

where H_0 is the harmonic oscillator Hamiltonian of Eq.(2-13) and $V(Q)$ is a collective potential like $A\beta^2$. The eigenvectors and eigenvalues can be calculated analytically even for a huge number of nucleons because H is a simple polynomial in the generators of the Sp(6) symplectic group.

The Sp(6) generators are given by the 6 Cartesian quadrupole moments (Q_{ij} of (2-18)), the 9 gl(3,R) generators of deformations (S_{ij} of (2-16)) and rotations (L_{ij} of (2-17)) and the 6 components of the quadrupole flow tensor (K_{ij})

$$K_{ij} = \sum_{s=1}^{m} P_{is} P_{js} \qquad (2\text{-}20)$$

Note that the generators of Sp(6) are invariant under permutation of nucleons (rotations in O(n) space) and thus working with Sp(6) does not involve taking into account the Pauli exclusion principle. This is quite a nice feature of the Sp(6) collective model since some authors, trying to find shell model states in O(n) representation spaces, were forced to conclude that "we have no choice but to violate the Pauli principle"[Va80] in order to realize microscopically the Bohr-Mottelson model. This algebra contains the collective motion algebra cm(3) as a subalgebra. It also contains Elliott's su(3) algebra as a subalgebra. Since H_0 is an element of the Lie algebra, one can choose basis states for an irreducible representation which are eigenstates of H_0 and work in the shell model. These basis states can also be chosen to belong to Elliott's SU(3) subgroup and to have definite angular momentum. This is equivalent to working with the group-subgroup chain

$$Sp(6) \supset U(3) \supseteq U(1) \times SU(3) \supset SO(3)$$

For this chain Gaskell et al.[GR81] gave the generating function for the branching rules which enables one to write down the basis states in terms of epd's. However they did not calculate the matrix elements of the generators of Sp(6) in this basis.

When interested in quadrupole vibrations one has to find a basis in Sp(6) which is at the same time a basis in the subgroup $Sp(2) \times O(3)$; this is what concerns us in this thesis.

In the next chapter, we review the notions of symplectic geometry and the properties of the dynamical group Sp(6n), emphasizing the subgroup Sp(6)$\times$O(n). We discuss the complementarity principle showing the relation between the IR's of O(n) and Sp(6). We give explicitly the generators and the weight operators of Sp(6) and of its subgroups O(3) and Sp(2). Looking for states of definite angular momentum we use polynomials in the raising Sp(6) generators and to get states belonging to a specific O(3n) representation we have to use traceless versions of these generators.

CHAPTER 3.THE SYMPLECTIC ALGEBRA

3.1 Sp(6n) and its subgroups

We start with A=n+1 nucleons in three dimensions as in equation (2-14), eliminate the center-of-mass and work with the Jacobi coordinates X_{is} and their canonically conjugate momenta P_{is}. They form a 3n-dimensional Weyl-Lie algebra

$$[\ X_{is}\ ,\ P_{jt}\ ] = i\,\delta_{ij}\,\delta_{st} \tag{3-1}$$

(we work in a system of units where $\hbar$, the nucleon mass and the classical oscillator frequency are all equal to unity)

The Hermitian quadratic expressions in X_{is} and P_{jt}

$$X_{is}\,X_{jt}\ ,\quad X_{is}\,P_{jt} + P_{jt}\,X_{is}\ ,\quad P_{is}\,P_{jt} \tag{3-2}$$

close under commutation [Mo73] and provide the 3n(6n+1) generators of the symplectic group Sp(6n). In Sp(6n) we consider the following Hamiltonian of the shell model type (2-14)

$$H_o = \sum_{i=1}^{3}\sum_{s=1}^{n} H_{is} \quad \text{with } H_{is} = (1/2)\ (\ P_{is}^2 + X_{is}^2\ ) \tag{3-3}$$

The 3n operators H_{is} are particular combinations of the Sp(6n) generators in (3-2) that commute among themselves, thus giving the weight generators of this group [Mo67].

Observe that the generators in (3-2) are quadratic expressions and thus are invariant under space reflections and that a general Sp(6n) Hamiltonian will be a function of them rather than linear in X_{is}, P_{jt}. Thus, if we have the matrix elements of the generators of Sp(6n) in the basis of eigenstates of H_0, we know in principle the matrix representation of an arbitrary Hamiltonian and diagonalizing it provides us with the energy levels.

The subgroups of Sp(6n) allow us to classify further the eigenstates of H_0. The relevant subgroup is

$$Sp(6n) \supset Sp(6) \times O(n) \tag{3-4}$$

For O(n) the generators are obtained from the (3-2) generators of Sp(6n) by contracting with respect to i while the Sp(6) ones are obtained by contracting with respect to s. For O(n) these generators have the well known form

$$\mathcal{L}_{st} = \sum_{i=1}^{3} (X_{is} P_{it} - X_{it} P_{is}) \ . \tag{3-5}$$

There are n(n-1)/2 of them with the quadratic Casimir operator

$$\mathcal{L}^2 = (1/2) \sum_{st} \mathcal{L}_{st}^2 \ . \tag{3-6}$$

For Sp(6) there are 6(6+1)/2=21 of them and they are the S_{ij}, L_{ij},

Q_{ij} and K_{ij} given in (2-16), (2-17), (2-18) and (2-20). We observe that S_{ij} and K_{ij} can be expressed in terms of commutators of Q_{ij} with H_0

$$S_{ij} = i\,[\,H_0\,,\,Q_{ij}\,]$$
$$K_{ij} = Q_{ij} - (1/2)\,[\,H_0\,,\,[\,H_0\,,\,Q_{ij}\,]] \tag{3-7}$$

and that the L_{ij} are the generators of the O(3) subgroup of Sp(6). When we need the matrix elements of the Sp(6) generators, it is sufficient to calculate only the matrix elements of Q_{ij} in the specific basis and those of S_{ij} and K_{ij} are easily obtained from the commutation relations (3-7).

Note that the Hamiltonian H_0 given in (3-3) is the Hamiltonian of the 3n-dimensional harmonic oscillator. Its dynamical group is Sp(6n) and its physical states of even number of quanta belong to the IR $[(1/2)^{3n}]$ while the states of odd number of quanta belong to the IR $[(1/2)^{3n-1}(3/2)]$ of this group.

If we deal only with the metaplectic IR's of Sp(6n), the IR's of Sp(6) and O(n) are complementary, i.e., if we fix the IR of O(n), the IR of Sp(6) is given and vice versa. This is related to the fact that there are polynomial relations between the Casimir operators of O(n) and of Sp(6). Collectivity means restricting to a specific IR of O(n) and thus to a specific IR of Sp(6). Then the collective Hamiltonian is defined in the enveloping algebra of Sp(6) rather than in that of Sp(6n) so that H_{coll} is a function of the Sp(6) generators S_{ij}, L_{ij}, Q_{ij} and K_{ij}. Also one has to impose for the collective Hamiltonians invariance under the subgroup O(3) of Sp(6) (i.e. invariance under space rotations) and invariance under O(n) (rotations in the nucleon index space). These restrictions drastically reduce the dimension of the Hilbert space. From a problem with 3n quantum numbers (the occupation number for 3n harmonic oscillators) we reduce it to one with only 9 quantum numbers (the number of internal labels of Sp(6)). This is a spectacular reduction if we consider that the number of nucleons can be of the order of one hundred. However this is a trading off situation because we deal now with the more complex case of a problem with constraints.

The symmetry group of the 3n-dimensional harmonic oscillator is U(3n) and the physical states are symmetric irreducible representations (N) of N quanta. This corresponds to the metaplectic IR's of the dynamical group Sp(6n) of even or odd N. In the chain

$$SU(3n) \supset SU(3) \times SU(n) \quad (3\text{-}8)$$
$$\cup$$
$$O(n)$$

the IR's of U(3) and U(n) are also related by complementarity when we fix the IR of U(3n) to be (N). Both IR's of U(3) and U(n) are partitions of N (number of quanta) in 3 numbers. If the U(n) IR is

$[h_1, h_2, h_3]$, the IR of O(n) as a subgroup of U(n) is also a partition in three numbers $(\omega_1, \omega_2, \omega_3)$, where $\omega_i \leq h_i$, i=1,2,3; all the other O(n) labels are zero. When the representation labels of O(n) are chosen, the Sp(6) labels are determined [Mo84]

$$[(n/2)+\omega_1\ ,\ (n/2)+\omega_2\ , (n/2)+\omega_3\ ]$$

These are in fact the weight labels of Sp(6), i.e. the eigenvalues of the weight operators of the group; the weight operators of Sp(6) are the weights of Sp(6n) H_{is} summed over the particle index to obtain the three $H_i = \sum_s H_{is}$, as we discuss in the next section.

The O(n) IR's $(\omega_1, \omega_2, \omega_3)$ are directly related to the actual nucleus we are interested in, its (Z,A) values. The scalar representation of O(n) $(\omega_1, \omega_2, \omega_3)=(0,0,0)$ corresponds to doubly closed shell nuclei and was studied by the Mexican group [CH84] in a two dimensional space, that is in the chain $Sp(4n) \supset Sp(4) \times O(n)$, with the Sp(4) representation given by (n/2, n/2). But most nuclei do not fit in this category and the actual IR has to be found. Sabaliauskas [Sa79] cosidered that the ground state band is the one which corresponds to the maximal eigenvalue of the quadratic Casimir operator of the SU(3). Thus he gave a table of O(n) IR 's for different nuclei. His table is reproduced by Castanos et al. [CF82]. In our work we consider the general case with open shells.

To be consistent we use in this thesis Dynkin representation labels λ_i for the compact groups O(n), SU(n) :

$$\lambda_i = \frac{2\langle M_\lambda | \alpha_i \rangle}{\langle \alpha_i | \alpha_i \rangle} \tag{3-9}$$

Here M_λ is the highest weight of the IR (λ) and α_i are the simple roots of the group. The O(n) partition labels given above $(\omega_1, \omega_2, \omega_3)$ are related to the Dynkin labels by :

$$\omega_1 = \lambda_1 + \lambda_2 + \lambda_3 \;,\; \omega_2 = \lambda_2 + \lambda_3 \;,\; \omega_3 = \lambda_3 \quad \text{for } n \geq 9 \;; \tag{3-10}$$

for O(8) $\omega_3 = \lambda_3 = \lambda_4$ and for O(7) $\omega_3 = \lambda_3/2$. The exception is SO(3) for which we use $\lambda/2 = \ell$ as the IR label. For non-compact Sp(6) we use the labels (p,q,d) of the "bottom" U(3) IR : (p,q) are its SU(3) labels and (d) its "vertical" weight component. Given the complementarity, for an IR $(\omega_1, \omega_2, \omega_3)$ of O(n) (or $(\lambda_1, \lambda_2, \lambda_3)$ in Dynkin notation) the Sp(6) IR labels (p,q,d) are:

$$p = \lambda_1 = \omega_1 - \omega_2 \;,\quad q = \lambda_2 = \omega_2 - \omega_3 \;,\quad d = (\omega_1 + \omega_2 + \omega_3)/2 \tag{3-11}$$
$$= (\lambda_1 + 2\lambda_2 + 3\lambda_3)/2$$

Observe here that d is half the number of quanta , $d = (\omega_1 + \omega_2 + \omega_3)/2$ so that integer d means even metaplectic Sp(6n) IR $[(1/2)^{3n}]$ while half-odd d corresponds to odd metaplectic IR of Sp(6n) $[(1/2)^{3n-1}(3/2)]$. Vice-versa, for a given Sp(6) IR of (p,q,d) labels, the corresponding O(n) IR are given by

$$(\lambda_1, \lambda_2, \lambda_3) = [\, p,\, q,\, (2d-p-2q)/3 \,] \tag{3-12}$$

in Dynkin notation, or $\omega_1=(2p+q)/3$, $\omega_2=(p-q)/3$, $\omega_3=(2d-p-2q)/3$ in partition language.

We characterize our states by an IR of O(n) and thus by an IR of Sp(6) and also by an IR of a subgroup of the latter. We are interested in the chain $Sp(6) \supset Sp(2) \times O(3)$. The generators of O(3) are the L_{ij} of (2-17) while those of Sp(2) are the O(3) scalars that we can form from the Sp(6) generators. There are three of them given by

$$
\begin{aligned}
I_1 &= (1/4) \sum_{i=1}^{3} (K_{ii} - Q_{ii}) \\
I_2 &= (1/4) \sum_{i=1}^{3} (S_{ii}) \\
I_3 &= (1/4) \sum_{i=1}^{3} (K_{ii} + Q_{ii})
\end{aligned}
\qquad (3\text{-}13)
$$

which satisfy [Mo73] the commutation rules

$$
[I_1, I_2] = -iI_3 \;, \quad [I_3, I_1] = iI_2 \;, \quad [I_2, I_3] = iI_1 \quad . \qquad (3\text{-}14)
$$

The Sp(2) Casimir operator is $I^2 = I_3^2 - I_1^2 - I_2^2$ and is related to the Casimir operator of O(3n) because of the complementarity in the chain $Sp(6) \supset Sp(2) \times O(3n)$. We discuss this relation in Section 3.3 when we need states of a definite O(3n) IR, or equivalently, of a definite Sp(2) IR.

3.2 Polynomials in the raising generators of Sp(6)

Until now we discussed the algebra of Sp(6n) and its subgroups in terms of the Jacobi coordinates of Eq.(2-15) and their conjugated momenta $P_{is} = -i\partial/\partial X_{is}$, which obey the Heisenberg-Weyl algebra (3-1). For convenience we transform to the creation and annihilation operators in Sp(6n)

$$\eta_{is} = (X_{is} - iP_{is})/\sqrt{2}\ , \qquad \xi_{is} = (X_{is} + iP_{is})/\sqrt{2} \tag{3-15}$$

which satisfy the commutation relations

$$[\ \xi_{is}\ ,\ \eta_{jt}\ ] = \delta_{ij}\,\delta_{st} \tag{3-16}$$

In this new basis the generators of Sp(6) in (2-16), (2-17), (2-18) and (2-20) become

$$B^{+}_{ij} = \sum_{s=1}^{n} \eta_{is}\eta_{js} \tag{3-17a}$$

$$C_{ij} = (1/2)\sum_{s=1}^{n} (\eta_{is}\xi_{js} + \xi_{js}\eta_{is}) \tag{3-17b}$$

$$B_{ij} = \sum_{s=1}^{n} \xi_{is}\xi_{js} \tag{3-17c}$$

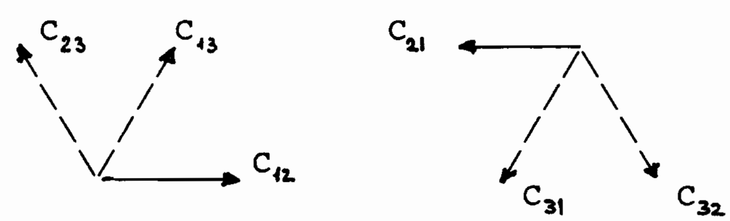
The B^{+}_{ij} and B_{ij} are respectively raising and lowering generators of Sp(6) and there are six of each (the η_{is} and ξ_{is} commute with themselves so that these operators are symmetric under the interchange i <--> j).

The C_{ij} coincide for $i \neq j$ with the generators of the U(3) subgroup of Sp(6) $\mathcal{C}_{ij}$ given below

$$\mathcal{C}_{ij} = \sum_{s=1}^{n} \eta_{is} \xi_{js} , \quad C_{ij} = \mathcal{C}_{ij} + (n/2)\delta_{ij} \tag{3-18}$$

and they can be subdivided into raising, lowering and weight generators

$$\begin{array}{lll} C_{ij} & i<j & \text{raising operators} \\ C_{ij} & i>j & \text{lowering operators} \\ C_{ii} & & \text{weight operators} \end{array} \tag{3-19}$$



(a) (b)

The above figures (a) and (b) diagrammatically represent the raising U(3) generators C_{12}, C_{13}, C_{23} and the lowering generators C_{21}, C_{31} and C_{32} respectively.

From the commutation relations (3-16) one can verify the commutation relations of the operators in (3-17). We give them in

Chapter 6 when we discuss the Sp(6) generators and their matrix elements in our basis.

We can see that the weight operators of Sp(6) C_{ii} are the H_i we get after summing the Sp(6n) weight generators H_{is} of Eq.(3-3) over the particle index s. Thus the weight generators of Sp(6) give the energy associated with the Hamiltonians H_i in units of $\hbar\omega=1$ (ω_i plus the ground state energy n/2) while the weight operators of the U(3) subgroup of Sp(6) $\mathcal{C}_{ii}$ count the number of quanta ω_i of H_i.

$$C_{ii} = (1/2)\sum_{s=1}^{n} (\eta_{is}\xi_{is}+\xi_{is}\eta_{is}) = \sum_{s=1}^{n}\eta_{is}\xi_{is} + (n/2) \quad (3\text{-}20)$$

$$H_i = (1/2)\sum_{s=1}^{n} (P_{is}^2 + X_{is}^2) = C_{ii} \quad (3\text{-}21)$$

The total number of quanta is $(\omega_1+\omega_2+\omega_3)$ and corresponds to the U(1) label in Elliott notation and to (2d) where d labels the "vertical" component of the Sp(6). The SU(3) labels in Elliott notation are $\omega_1-\omega_2$ and $\omega_2-\omega_3$, the same with our (p, q) label of the "bottom" SU(3). It follows that the Moshinsky way of labelling the Sp(6) IR by $[(n/2)+\omega_1, (n/2)+\omega_2, (n/2)+\omega_3]$ (in [Mo84], [CC84]) refers to the eigenvalues of the three Sp(6) weight generators $H_i = C_{ii}$.

Consider now the oscillator Hamiltonian in (3-3)

$$H_0 = (1/2)\sum_{i=1}^{3}\sum_{s=1}^{n}(X_{is}^2 + P_{is}^2) = \sum_{i=1}^{3} C_{ii} = \sum_{i=1}^{3}\left[\sum_{s}\eta_{is}\xi_{is} + (n/2)\right] = \sum_{i=1}^{3}\mathcal{C}_{ii} + (3n/2) \quad (3\text{-}22)$$

and let us call its ground state $|0\rangle$. From the expression of H_0 we see that $|0\rangle$ is a state of lowest Sp(6) and lowest U(3) weight. This means that applying the lowering Sp(6) generators B_{ij} and C_{ij} $i>j$ one must get zero. Applying the weight generators C_{ii} one gets the U(3) weights of the state $[\omega_1+(n/2),\omega_2+(n/2),\omega_3+(n/2)]$.

$$B_{ij}|0\rangle=0 \quad , \quad C_{ij}|0\rangle=0 \text{ for } i>j \tag{3-23}$$

$$C_{ii}|0\rangle=(\omega_i+n/2)|0\rangle \quad i=1,2,3 \tag{3-24}$$

If the nucleus is open shells-type, the ground state energy is $(\omega_1+\omega_2+\omega_3+3n/2)$, where we can consider $(\omega_1+\omega_2+\omega_3)$ as "intrinsic" quanta, due to the internal structure of the nucleus in question. The excited states with 2ν "extra" quanta are obtained by applying all possible homogeneous polynomials of degree (ν) in B^+_{ij} (B^+_{ij} is quadratic in the creation operator η_{is} so that applying one B^+_{ij} we create two quanta). We get states of the form

$$P_\nu\ (B^+_{ij})|0\rangle \tag{3-25}$$

which belong to the same O(n) IR $(\omega_1,\omega_2,\omega_3)$. As we see in the next Section they form an Sp(6) basis with the O(2) subgroup of Sp(2) labelled by (2ν).

In principle one would consider in (3-25) polynomials in all Sp(6) raising generators, i.e., both B^+_{ij} and C_{ij} , $i<j$. Using the

commutation relation of B^{+}_{ij} and C_{ij} (see Chapter 6)

$$[\, C_{ij} \,,\, B^{+}_{kl} \,] = \delta_{jk}\, B^{+}_{il} + \delta_{jl}\, B^{+}_{ik} \tag{3-26}$$

one pulls all the C_{ij} at right in front of $|0\rangle$. The C_{ij} are essentially the $\mathcal{C}_{ij}$ of U(3) (see (3-18)) and the annihilation operator ξ_{js} when applied to the ground state gives zero. In this way all terms containing C_{ij} vanish, leaving only terms of the form (3-25).

The states in (3-25) are characterized by the angular momentum L and we consider highest O(3) states of angular momentum projection M=L, which implies

$$L_{+} P_{\nu}\ (B^{+}_{ij})\,|0\rangle=0, \qquad L_{3} P_{\nu}\ (B^{+}_{ij})\,|0\rangle=LP_{\nu}\ (B^{+}_{ij})\,|0\rangle \tag{3-27}$$

where L_i are the components of the angular momentum (the 3 independent components of the antisymmetric tensor L_{ij} of (2-17)) and $L_{\pm}=(L_1 \pm iL_2)$. To get the states of arbitrary M one applies (L_{-}) repeatedly (L-M) times to the polynomials (3-25).

The B^{+}_{ij} , i,j=1,2,3 is a symmetric tensor of rank two. It has therefore six independent components and we prefer to transform from Cartesian components to spherical-type ones $B^{+}_{qq'}$, q,q'= 1,0,-1 and then to the irreducible tensor form

$$B^{+}_{\ell m} = \sum_{q\,q'} \langle 1q,\ 1q' | \ell m\rangle\, B^{+}_{qq'} \quad , \ \ell = 0,2 \tag{3-28}$$

Here $\langle 1q, 1q' | \ell m\rangle$ is the Clebsch-Gordan coefficient for the coupling (1q) with (1q') to form a (00) or a (2m), m=2,1,0,-1,-2 state. In spherical notation (q,q'=-1,0,1) the raising generators are given by $B^{+}_{qq'} = \sum_{s} \eta_{qs}\eta_{q's}$, i.e.

$$\begin{aligned}
B^{+}_{11} &= (1/\sqrt{2}) \sum_{s} \eta^{2}_{1s}\ , & B^{+}_{10} &= B^{+}_{01} = \sum_{s} \eta_{1s}\eta_{0s}\ ,\\
B^{+}_{00} &= (1/\sqrt{2}) \sum_{s} \eta^{2}_{0s}\ , & B^{+}_{\bar{1}0} &= B^{+}_{0\bar{1}} = \sum_{s} \eta_{\bar{1}s}\eta_{0s}\ ,\\
B^{+}_{\bar{1}\bar{1}} &= (1/\sqrt{2}) \sum_{s} \eta^{2}_{\bar{1}s}\ , & B^{+}_{1\bar{1}} &= B^{+}_{\bar{1}1} = \sum_{s} \eta_{1s}\eta_{\bar{1}s}\ ,
\end{aligned} \tag{3-29}$$

and the irreducible tensor form of B^{+}_{ij} is easily obtained if we take the ℓ=m=2 component to be B^{+}_{11} and we crank down with L_{-}. Then the B^{+}_{m} (ℓ=2) are given by:

$$\begin{aligned}
B^{+}_{2} &= (1/\sqrt{2}) \sum_{s} \eta^{2}_{1s}\\
B^{+}_{1} &= (1/2)[L_{-}, B^{+}_{2}] = \sum_{s} \eta_{1s}\eta_{0s}\\
B^{+}_{0} &= (1/\sqrt{6})[L_{-}, B^{+}_{1}] = (1/\sqrt{3}) \sum_{s} (\eta^{2}_{0s} + \eta_{1s}\eta_{\bar{1}s})\\
B^{+}_{\bar{1}} &= (1/\sqrt{6})[L_{-}, B^{+}_{0}] = \sum_{s} \eta_{\bar{1}s}\eta_{0s}\\
B^{+}_{\bar{2}} &= (1/2)[L_{-}, B^{+}_{\bar{1}}] = (1/\sqrt{2}) \sum_{s} \eta^{2}_{\bar{1}s}
\end{aligned} \tag{3-30}$$

The ℓ=0 part of the B^{+}_{ij} is orthogonal to the ℓ=2, m=0:

$$B^{+}_{00} = \sum_{s} (\eta^{2}_{0s} - 2\eta_{1s}\eta_{\bar{1}s})/\sqrt{6}$$

Now the polynomial in B^{+}_{ij} became a polynomial in B^{+}_{m} and B^{+}_{00}. We note that B^{+}_{00} is an O(3) scalar and the Sp(2) raising generator.

When applied to an Sp(6) state it increases only the vertical component, adding two quanta. When we construct the basis states by coupling the $B^{+}_{\ell m}$ to the bottom Sp(6) IR we are interested in states of lowest Sp(2) weight and we use only the $\ell=2$ part of the raising generators B^{+}_{m} m=2,1,0,-1,-2. Note here that the B^{+}_{m} being only functions of η_{is} commute among themselves and are the components of a Racah tensor of order 2.

To determine the states of the form (3-25) and of angular momentum L one has to find some basic homogeneous polynomials of the type $(2\nu,L)$, of degree ν in B^{+}_{m}. This is the procedure used by Chacón, Moshinsky and Sharp [CM76] for the states characterized by the IR of the chain $U(5)\supset O(3)$. Their problem was solved by introducing three such basic polynomials and the states were products of these simple polynomials called elementary permissible diagrams (epd's) [MS69,SL69]. The same type of polynomials will solve our problem and Castaños, Chacón and Moshinsky [CC84] found five of them which characterize the scalar IR of O(n). Their epd's contain only polynomials in the raising generators since the ground state is trivial.

In our case, when the ground state we start with is nontrivial (has nonzero SU(3) labels), we have to couple the "bottom" SU(3) (p,q) of the Sp(6) IR (p,q,d) of the given nucleus (given $(\omega_1,\omega_2,\omega_3)$ O(n) IR) to the $B^{+}_{\ell m}$. The $B^{+}_{\ell m}$ form the U(3) multiplet (2,0,1), that is, the SU(3) sextet (2,0). The O(3) projection of the sextet provides the quintet ℓ=2 (the B^{+}_{m}) and the ℓ=0 part B^{+}_{00} . To create two quanta states one applies a polynomial of first degreee in $B^{+}_{\ell m}$ to the ground state, that is one couples the U(3) multiplet (2, 0, 1) to the "bottom" (ground state) U(3) multiplet (p,q,d) of the Sp(6) IR. This translates in terms of SU(3)-coupled products

$$(2,0) \times (p,q) = (p+2,\ q) + (p,\ q+1) + (p+1,\ q-1) + \\ + (p-2,\ q+2) + (p-1,\ q) + (p,\ q-2)$$

provided p$\geq$2 and q$\geq$2. For states of four quanta one applies a second degree polynomial in $B^{+}_{\ell m}$ and these states span a reducible U(3) representation which decomposes into the U(3) IR's contained in the symmetric part of the product

$$[(2,0,1) \times (2,0,1)]_{s} \times (p,q,d)$$

or, in SU(3) projection $[(2,0) \times (2,0)]_{s} \times (p,q)$. To obtain states of 2ν quanta one uses the U(3) multiplet $(a_1,\ b_1,\ a_1/2+b_1)$ corresponding to polynomials of degree $\nu = a_1/2+b_1$ in B^{+}_{m} . To apply these polynomials to the ground state means to couple their U(3) IR to the "bottom" U(3) IR (p,q,d) and obtain the excited Sp(6) states labelled $(a,\ b,\ a_1/2+b_1+d)$. The "vertical" component is the sum of the original Sp(6) IR (d equals half the number of "intrinsic" quanta) and the degree in the raising generators B^{+}_{m} (ν equals half the number of "created" quanta). Then the "vertical" component of the "final" Sp(6) , $d+\nu$, is half the total number of quanta. Coupling the B^{+}_{m} to SU(3) representations considerably complicates the problem, and the number of epd's in our problem is nearly sixty.

Of course the closed shell ones found by Castaños et al. [CC84] are in our list. This problem could not be solved without the use of generating functions methods.

3.3 The traceless Sp(6) generators

We require the basis states of our problem to belong to a given IR of Sp(2) of label (z). The total number of quanta N is related to the IR of the O(2) subgroup of Sp(2) whose generator is I_3 of (3-13); it is easy to verify that $I_3 = H_0/2$ and this means that z is half the eigenvalue of H_0. We introduce the number operator $\hat{N}$ which counts the quanta of a given state and the eigenvalue of H_0 is $N+3n/2$ if N is the eigenvalue of the operator $\hat{N}$.

$$\hat{N} = \sum_i \sum_s \eta_{is} \xi_{is} = \sum_{i=1}^{3} C_{ii} - (3n)/2 \tag{3-31}$$

$$\hat{N}\, P_\nu(B^+_m)|0\rangle = N\, P_\nu(B^+_m)|0\rangle \tag{3-32}$$

As discussed before if the ground state has 2d quanta and we apply P_ν which creates 2ν additional quanta, $N=2d+2\nu$.

As stressed many times, in the chain $Sp(6n) \supset Sp(2) \times O(3n)$, because of the metaplectic Sp(6n) IR, there is complementarity between the IR's of Sp(2) and O(3n). Requiring a certain Sp(2) label

(z) uniquely determines the O(3n) representation label (λ) (only the first one is nonzero). To see how the two labels are related we consider the IR's labelled by the eigenvalues of their Casimir operators. The Casimir operator of Sp(2) is given by

$$I^2 = I_3^2 - I_1^2 - I_2^2 \tag{3-33}$$

with I_1, I_2, I_3 given in Eq. (3-13). For O(3n) we use the Casimir operator of (3-6) and it is simple to verify that the two are related [Mo84]

$$I^2 = (1/4)[\mathcal{L}^2 + (3n)^2/4 - (3n)] \tag{3-34}$$

We denote the eigenvalues of I^2 by z(z-1) and those of $\mathcal{L}^2$ by $\lambda(\lambda+3n-2)$; we obtain from (3-34) the relation between the eigenvalues z and λ:

$$z = \lambda/2 + (3n)/4 \tag{3-35}$$

This equation implies $2z = \lambda + (3n)/2$ which is transparent from $I_3 = H_0/2$ and $H_0 = \hat{N} + (3n)/2$ if z is the eigenvalue of I_3 and if the eigenvalue λ equals the eigenvalue N of the number operator $\hat{N}$. But λ equals N only for special states.

To see the relation between the Casimir operator of O(3n) and the number operator $\hat{N}$ we write out explicitly $\mathcal{L}^2$ [EG70]

$$\mathcal{L}^2 = (1/2)\sum_{s,t} [\eta_{is}\xi_{it} - \eta_{it}\xi_{is}]\,[\eta_{it}\xi_{is} - \eta_{is}\xi_{it}] \tag{3-36}$$

This is the same Casimir operator given in (3-6) but expressed now in terms of creation and annihilation operators. After some rearrangement of factors it can be written as

$$\mathcal{L}^2 = \hat{N}(\hat{N}+3n-2)-(\sum_t \eta_{it}\eta_{it})(\sum_s \xi_{is}\xi_{is}) \tag{3-37}$$

From (3-37) we see that the polynomials in (3-25) correspond to a definite IR of Sp(2) of label z and at the same time to the IR of O(3n) of label λ=N if and only if the polynomials are "harmonic", i.e. they satisfy

$$(\sum_s \xi_{is}\xi_{is})\ P_\nu(B^+_{m})|0\rangle = 0 \tag{3-38}$$

Our polynomials in (3-25) do not satisfy (3-38) as they stand and thus they do not correspond to a definite IR of O(3n). There is however a method originated by Vilenkin [Vi68] and further developed by Lohe [Lo74] by means of which we can enforce the harmonicity in a relatively simple way.

Like these authors we introduce "traceless boson operators" defined by

$$a^+_{is} = \eta_{is} - (2,0)(2\hat{N}+3n)^{-1}\xi_{is} \tag{3-39}$$

where (2,0) is the O(3) scalar of the B^+_{lm} sextet in (3-30)

$$(2,0) = \sqrt{6}B^+_{00} = \sum_s (\eta^2_{0s} - 2\eta_{1s}\eta_{\bar{1}s})$$

Replacing in (3-17a) η_{is} by their traceless versions a^+_{is} we get the traceless partners $\beta^{+}_{\ell m}$ of the raising generators $B^{+}_{\ell m}$. We use the definitions (3-17) and the commutation relations

$$(2\hat{N}+3n)^{-1}\eta_{is}=\eta_{is}(2\hat{N}+3n+2)^{-1} \tag{3-40}$$

$$(2\hat{N}+3n)^{-1}\xi_{is}=\xi_{is}(2\hat{N}+3n-2)^{-1} \tag{3-41}$$

to get the following form for the traceless raising generators

$$\begin{aligned}\beta^{+}_{ij}= B^{+}_{ij}-(2,0)(2\hat{N}+3n)^{-1}(C_{ij}+C_{ji})+\\ +(2,0)^{2}[(2\hat{N}+3n+4)(2\hat{N}+3n+2)]^{-1}B_{ij}\end{aligned} \tag{3-42}$$

As we did in the paragraph with (3-28) we can convert the traceless versions of the raising generators to the irreducible tensor form. The $\ell=2$ part is given by:

$$\beta^{+}_{m}=B^{+}_{m}-2(2,0)(2\hat{N}+3n)^{-1}Q_{m}+(2,0)^{2}[(2\hat{N}+3n+4)(2\hat{N}+3n+2)]^{-1}B_{m} \tag{3-43}$$

Here we introduced the notation $Q_{ij}=(C_{ij}+C_{ji})/2$ and Q_m and B_m are the $\ell=2$, $m=-2,-1,0,1,2$ components of the tensors Q_{ij} and B_{ij} in the same way as B^{+}_{m} is the $\ell=2$ part of B^{+}_{ij} in (3-30). Using the traceless versions of the raising generators, the polynomials become harmonic, and the states (3-25) obey (3-38) if we make sure that the ground state $|0\rangle$ is also harmonic.

To make the ground state harmonic we remember that it is essentially an SU(3) IR (p,q), where p and q are the degrees in the unbarred η_i and barred $\bar{\eta}_i$ variables which form respectively the triplets (1,0) and (0,1). To make the variables traceless we make the replacements [DS82] :

$$\begin{aligned} \eta_i &\dashrightarrow a_i = \eta_i - B\ (\hat{N}+3)^{-1}\ \partial_{\bar{\eta}_i} \\ \bar{\eta}_i &\dashrightarrow \bar{a}_i = \bar{\eta}_i - B\ (\hat{N}+3)^{-1}\ \partial_{\eta_i} \end{aligned} \tag{3-44}$$

Here $B=\sum_i \eta_i \bar{\eta}_i$ is an SU(3) scalar and $\hat{N}=\sum_i (\eta_i \partial_{\eta_i} + \bar{\eta}_i \partial_{\bar{\eta}_i})=p+q$ is the total degree. The proof that the polynomials in a_i and $\bar{a}_i$ are traceless, i.e., orthogonal to any state containing B as a factor, is similar to that for traceless raising operator given by Lohe and Hurst [LH71].

With these the eigenstates are "harmonic", meaning they obey (3-38) and thus they correspond to the O(3n) IR of label λ=N and, by complementarity, to the Sp(2) IR of label z. In Chapter 5 when we explicitly construct the basis states we use the traceless versions of the raising generators (3-43) and of the creation operators (3-44).

CHAPTER 4. THE SP(6) $\supset$ SP(2) $\times$ O(3) GENERATING FUNCTION

4.1. Generating function method in group theory

The concept of generating function was introduced in 1897 by Molien [Mo97] in connection with the invariants of a finite group of matrices. The generating function proved to be a very useful tool in the representation theory of finite [DS79,JB77,Mc74,Me54,PS78,PS79,PS80,Sl77] and continuous groups and, more recently, spacegroups [PS85], supergroups [SV85] and Kac-Moody algebras.

There are several types of generating functions named after the information they carry; a good account and examples of use are given in a recent Ph.D thesis by Couture [Co80]. We name here the most used categories of generating functions :

- generating function for polynomial irreducible tensors;

- generating function for Clebsch-Gordan series;

- generating function for weights;

- generating function for group-subgroup branching rules.

As one can see the generating functions may be used in many different group theoretical calculations; they are a compact and convenient way of storing, carrying and manipulating information. All generating functions have the same structure : they are infinite sums of monomials in several variables, say v_j ,

$$F(v_1, v_2, \ldots, v_k) = \sum_{i_1 \ldots i_k} C_{i_1 \ldots i_k} v_1^{i_1} v_2^{i_2} \ldots v_k^{i_k} \tag{4-1}$$

containing only positive terms, $C_{i_1 \ldots i_k} \geq 0$. They can be written as fractions, or sums of several fractions, whose denominator factors are of the form (1-X). The X's in the denominators and the Y's in the numerators are of the form

$$v_1^{p_1} v_2^{p_2} \ldots v_k^{p_k}$$

where $p_1, \ldots, p_k$ are integers. When expanded, one obtains the infinite power series in (4-1). As an example consider the following function

$$G(v_1, v_2, v_3) = (1+v_1 v_2 v_3^2)[(1-v_1^2)(1-v_1 v_2)(1-v_1 v_3^2)(1-v_2 v_3^2)]^{-1} \tag{4-2}$$

which meets the requirements of a would-be generating function. The generating functions are not only a neat, convenient and compact way of presenting results but they make possible the manipulation of a large amount of information; they may be added, subtracted, and in certain cases, coupled (multiplied) and substituted one into another.

In this work we are concerned with the generating functions for group-subgroup branching rules. It has been shown [SL69,Wy72] that the reduction of the irreducible representations (IR's) of a group into the irreducible representations of a subgroup can be given in terms of powers of a finite set of elementary factors called elementary permissible diagrams (epd's). The name epd was first introduced by Bargman and Moshinsky [BM61] and was generally adopted; the term elementary multiplet is also used. The epd's are denoted by $(\lambda_1,\ldots,\lambda_\ell;n_1,\ldots,n_m)$ where the λ_i and n_j are respectively the Cartan labels for the IR's of the group and the subgroup. This means that the subgroup multiplet $(n_1,\ldots,n_m)$ is contained in the low irreducible representation $(\lambda_1,\ldots\lambda_\ell)$ of the group. For the multiplet content of higher IR's of the group one uses stretched products of epd's. Weitzenböck [We32] proved that the set of epd's is finite for all semisimple Lie algebras. We note here that for noncompact group-subgroup chains, the epd's need not have any λ.

The connection with generating functions is immediate if we consider the X and Y defined before as epd's or stretched products of powers of them. To make things clear we take as a simple example the chain $SU(3)\supset SO(3)$ for which the branching rules are given by the following stretched product of elementary factors

$$(1,0;2)^a\ (0,1;2)^b\ (2,0;0)^c\ (0,2;0)^d\ (1,1;2)^f \qquad (4\text{-}3)$$

where a,b,c,d are integers from 0 to ∞ and f=0,1 only. The corresponding generating function is [BM61,Ga78]

$$F(P,Q;N)=(1+PQN^2)[(1-PN^2)(1-QN^2)(1-P^2)(1-Q^2)]^{-1} \qquad (4\text{-}4)$$

where P, Q and N carry respectively the SU(3) and SO(3) representation labels as exponents. If we want to see the SO(3) content of the SU(3) IR of Cartan labels (2,2) we look for the term P^2Q^2 in our generating function; it is

$$P^2Q^2(N^8+N^6+2N^4+1) \qquad (4\text{-}5)$$

This is to be read as

$$(2,2)\supset(8)+(6)+2(4)+(0)$$

Note that the infinite powers arise from the denominator factors when expanded while the numerator corresponds to the power f=0,1 in (4-4).

4.2. Obtaining the desired generating function

In this section we describe in detail how to obtain the generating function for Sp(6)$\supset$ SO(3)XSp(2) branching rules. The strategy is first to determine what subgroup multiplets are generated by the raising generators of Sp(6). By raising generators we mean the B^{+}_{m} ; there are six of them (five after making them traceless) and then keep all subgroup multiplets obtained by coupling them to the "bottom states" which comprise a single SU(3)XU(1) representation. We use Dynkin or Cartan representation labels for compact groups (e.g. the SU(3) octet is (1,1)) and for Sp(6) we use (p,q,d) where (p,q) is the "bottom" SU(3) representation and d the U(1) value of the bottom states.

The coupling of "raising generator multiplets" to "bottom multiplet states" can be done directly in an SO(3) basis or in an SU(3) basis followed by the replacement of SU(3) by SO(3) with the help of the SU(3)$\supset$ SO(3) branching rules generating function. In the end the two forms would be equivalent. Since the coupling is already done in the SU(3) basis [GR81] we are going to follow that route. But first we indicate how it could also be done directly in SO(3).

For the raising generators we can use the SO(3) quintet B^{+}_{m} . The use of B^{+}_{00} is not necessary since we are interested in traceless states, that is bottom states of Sp(2) multiplets.

The SO(3) multiplets which are polynomials in the components of a quintet are described by the generating function

$$\frac{1+Z^3L^3}{(1-Z^2)(1-Z^3)(1-ZL^2)(1-Z^2L^2)} \qquad (4\text{-}6)$$

When expanded in a power series

$$\sum_{z,\ell} C_{z\ell} Z^{z} L^{\ell}$$

the coefficient $C_{z\ell}$ gives the members of linearly independent ℓ-multiplets of degree z. One can recognize in (4-6) the elementary multiplets (epd's) (2,0), (3,0), (1,2), (2,2) . Since (3,3) appears in the numerator, it can be used at most linearly (no higher powers).

Similarly the SO(3) states in the bottom SU(3) multiplet (p,q) are described by the generating function

$$\frac{1+PQL}{(1-P^2)(1-PL)(1-Q^2)(1-QL)} \qquad (4\text{-}7)$$

which gives branching rules for SU(3)$\supset$ SO(3). Note that this is the same equation as (4-4) with $L=N^2$, or $\ell=(1/2)n$. When we expand (4-7) in a power series

$$\sum_{p,q,\ell} P^{p} Q^{q} L^{\ell} C_{pq\ell} \, ,$$

the coefficient $C_{pq\ell}$ gives the multiplicity of ℓ in (p,q).

Now we want to couple ℓ's from (4-6) to ℓ's from (4-7) and keep all direct products. This is done by computing

$$\frac{1+Z^3 L_1^3 L_1'^3}{(1-Z^2)(1-Z^3)(1-ZL_1L_1')(1-Z^2L_1^2L_1'^2)} \times$$

$$\times \frac{1+PQL_2L_2'}{(1-P^2)(1-PL_2L_2')(1-Q^2)(1-QL_2L_2')} \times \tag{4-8}$$

$$\times \left. \frac{1+L_1'^{-1}L_2'^{-1}L}{(1-L_1'^{-1}L)(1-L_2'^{-1}L)(1-L_1'^{-1}L_2'^{-1})} \right|_{L_1'^0 L_2'^0}$$

The first factor in (4-8) is just (4-6) with $L \rightarrow L_1 L_1'$, the second is (4-7) with $L \rightarrow L_2 L_2'$, the third is the Clebsch-Gordan generating function for SO(3), integral ℓ's :

$$\frac{1+L_1L_2L}{(1-L_1L)(1-L_2L)(1-L_1L_2)} \tag{4-9}$$

(here the coefficient of $L_1^{\ell_1} L_2^{\ell_2} L^{\ell}$ is the multiplicity of (ℓ) in $(\ell_1)\times(\ell_2)$) with $L_1 \rightarrow L_1'^{-1}$, $L_2 \rightarrow L_2'^{-1}$.

The subscript $L_1'^0 L_2'^0$ is an instruction to retain only the coefficient of $L_1' L_2'$ at power zero.

The result of (4-8) will be called here

$$F(P,Q;L_1,L_2,L,Z) \tag{4-10}$$

and has to be divided by (1 - DZ) to become the desired generating function for $Sp(6) \supset SO(3) X Sp(2)$ branching rules. When expanded

$$\sum_{pqd} \sum_{\ell_1 \ell_2 \ell z} C_{pqd\ell_1\ell_2\ell z} P^p Q^q D^d L_1^{\ell_1} L_2^{\ell_2} L^{\ell} Z^z \qquad (4\text{-}11)$$

the coefficient $C_{pqd\ell_1\ell_2\ell z}$ is the multiplicity of the $SO(3)XSp(2)$ representation (ℓ,z) in the $Sp(6)$ representation (p,q,d). From (4-10) all the epd's for the problem can be read off. The labels ℓ_1, ℓ_2 which are the ℓ's values of the multiplet from the bottom states and from the raising generators respectively, which are coupled to get the resultant ℓ in (4-11), are not essential but are useful in sorting out the syzygies and in seeing the structure of the epd's. The extraction of the $L'^0_1 L'^0_2$ term in (4-8) involves somewhat messy algebra, but the rules are known and the procedure is straightforward.

Now let's start over, proceeding via SU(3) and later going to SO(3). This is the route we followed.

The raising generators form an U(3) sextet (2,0;1). The SU(3) multiplets which are polynomials in its components are described by the generating function (see [De70]) :

$$\frac{1}{(1-Z^3)(1-ZA^2)(1-Z^2B^2)} \qquad (4\text{-}12)$$

The coefficient of $Z^z A^a B^b$ is the multiplicity of the SU(3) representation (a,b) of degree, or U(1) label, z. Similarly the SU(3)XU(1) representation (a,b,z) at the bottom of the Sp(6) representation is described by the generating function

$$\frac{1}{(1-PA)(1-QB)(1-DZ)} \tag{4-13}$$

Now we must couple the SU(3) representations described by (4-12) and (4-13) using the SU(3) Clebsch-Gordan generating function which is

$$\frac{1}{(1-A_1A)(1-A_2A)(1-B_1B)(1-B_2B)(1-A_1B_2)(1-B_1A_2)} \times$$

$$\times\left[\frac{1}{1-A_1A_2B}+\frac{B_1B_2A}{1-B_1B_2B}\right] \tag{4-14}$$

or keeping the part even in A_1, B_1 (which will be the A,B of (4-12))

$$\frac{1}{(1-A_1^2A^2)(1-A_2A)(1-B_1^2B^2)(1-B_2B)(1-A_1^2B_2^2)(1-B_1^2A_2^2)} \times$$

$$\times\left[\frac{(1+A_1^2B_2A+A_1^2A_2AB+A_1^2A_2B_2B)(1+B_1^2A_2B)}{1-A_1^2A_2^2B^2}+\right. \tag{4-15}$$

$$\left.+\frac{(1+A_1^2AB_2)(B_1^2B_2^2A^2+B_1^2A_2B_2A+B_1^2B_2AB+B_1^4A_2B_2^2A^2B)}{1-B_1^2B_2^2A^2}\right]$$

We multiply (4-12) , with the replacements $A \to A_1 A_1'^{-1}$, $B \to B_1 B_1'^{-1}$, by (4-13) with $A \to A_2'^{-1}$, $B \to B_2'^{-1}$, and (4-15) with $A_1 \to A_1'$, $B_1 \to B_1'$, $A_2 \to A_2'$, $B_2 \to B_2'$ and keep the $A_1'^0\, A_2'^0\, B_1'^0\, B_2'^0$ part. Because of the simple form of (4-12) and (4-13) this is simple to do. The result is just (4-15) with the replacements $A_1^2 \to ZA_1^2$, $B_1^2 \to Z^2 B_1^2$, $A_2 \to P$, $B_2 \to Q$, with of course the additional denominator factors $(1 - Z^3)(1 - DZ)$. We get the $Sp(6) \supset SU(3) \times U(1)$ branching rules generating function

$$\frac{1}{(1-Z^3)(1-DZ)(1-ZA_1^2A^2)(1-PA)(1-Z^2B_1^2B^2)(1-QB)(1-ZA_1^2Q^2)(1-Z^2B_1^2P^2)} \times$$

$$\times \left[\frac{(1+ZA_1^2QA+ZA_1^2PAB+ZA_1^2PQB)(1+Z^2B_1^2PB)}{1-ZA_1^2P^2B^2} + \right. \tag{4-16}$$

$$\left. + \frac{(1+ZA_1^2QA)(Z^2B_1^2Q^2A^2+Z^2B_1^2PQA+Z^2B_1^2QAB+Z^4B_1^4PQ^2A^2B)}{1-Z^2B_1^2Q^2A^2} \right]$$

Of course this is just equation (4-6) of [GR81], except that they did not keep A_1 and B_1 (they were set = 1). It is better to keep A_1, B_1 because the labels a_1, b_1 which they carry as exponents describe the SU(3) representation formed from the raising generators. They help sort out the syzygies especially when we have gone to $SO(3) \times U(1)$. Without them in the second term of (4-16) one is tempted to divide the numerator factor $(1 + ZA_1^2QA) \to (1 + ZQA)$ into the denominator factor $(1 - Z^2B_1^2Q^2A^2) \to (1 - Z^2Q^2A^2)$.

We can rewrite (4-16) in a more compact form if we observe that nothing changes when interchanging

$$P \longleftrightarrow Q \tag{4-17a}$$

$$A \longleftrightarrow B \tag{4-17b}$$

$$A_1^2 Z \longleftrightarrow B_1^2 Z^2. \tag{4-17c}$$

This is due to the form of the equations used to obtain (4-16). The first two replacement leave (4-13) unchanged and the last one does not affect (4-12). One can also verify directly that (4-16) is invariant under these changes. This observation is helpful in simplifying the generating function in (4-16). We can now separate the epd's into pairs defining the "conjugate" of an epd as the one obtained by the replacements (4-17a,b,c). With this we rewrite (4-16) as

$$\begin{aligned} & F(P,\ Q,\ D;\ A\ ,\ B\ ;\ A,\ B,\ Z) = \\ & = [(1-\alpha)(1-\beta)(1-\gamma)(1-\gamma^*)(1-\delta)(1-\delta^*)(1-\varepsilon)(1-\varepsilon^*)]^{-1}\ \mathrm{X} \\ & \quad \mathrm{X}[(1+\eta+\theta+\varkappa+\zeta\eta^*+\eta\theta+\eta\varkappa+\eta\eta^*)(1-\zeta)^{-1}\ + \\ & +(\zeta^*+\eta^*+\theta^*+\varkappa^*+\zeta^*\eta+\eta^*\theta^*+\eta^*\varkappa^*+\zeta^*\eta\eta^*)(1-\zeta^*)^{-1}] \end{aligned} \tag{4-18}$$

The letters on the right hand side stand for the epd's :

$\alpha = Z^3$, $\beta = DZ$, $\gamma = PA$, $\delta = Q^2 A_1^2 Z$, $\varepsilon = A_1^2 A^2 Z$,

$\zeta = P^2 A_1^2 B^2 Z$, $\eta = PB_1^2 BZ^2$, $\theta = PA_1^2 ABZ$, $\varkappa = PQA_1^2 BZ$.

The asterisk denotes the "conjugate" epd, obtained by the replacements (4-17). As we just discussed, the generating function (4-18) is conjugation-symmetric, i.e., is unaffected by these interchanges. When (4-18) is expanded in a power series,

$$F = \sum P^{p} Q^{q} D^{d} A_1^{a_1} B_1^{b_1} A^{a} B^{b} Z^{z} C_{pqda_1b_1abz} \tag{4-19}$$

the coefficient C, summed over a_1, b_1, gives the multiplicity of the U(3) multiplet (a,b,z) in the Sp(6) IR (p,q,d).

The exponents in (4-19) (or in the epd's) provide instructions for constructing the basis states (or the epd's): couple the U(3) multiplet $(a_1, b_1, a_1/2+b_1)$, whose components are polynomials of degree $a_1/2+b_1$ in the Sp(6) raising generators (they form the U(3) multiplet (2,0,1) and are the B^{+}_{ij} of Eq. (3.2a) or B^{+}_{lm} of Eq.(3.6) of [Mo84]), to the bottom U(3) multiplet (p,q,d) of the Sp(6) IR to obtain the U(3) multiplet $(a, b, a_1/2+b_1+d)$. The U(1) label is greater than $a_1/2+b_1+d$ by three times the degree in α, the SU(3) scalar of third degree in the raising generators.

The subgroup SU(3) of Sp(6) is converted to SO(3) by substituting into Eq.(4-17) the SU(3)$\supset$O(3) branching rules generating function

$$G(A,B,L) = [(1-A^2)(1-B^2)(1-AL)(1-BL)]^{-1}(1+ABL) \tag{4-20}$$

The substitution is accomplished [PS80] by evaluating

$$F(P,Q,D;A_1,B_1;A',B',Z)\cdot G(A'^{-1}A,B'^{-1}B,L)\Big|_{A'^0B'^0} \tag{4-21}$$

The subscript $A'^0B'^0$ is an instruction to retain only the term in A' and B' of degree zero. The variables A,B are inserted to retain the SU(3) representation labels; as noted above, we will need all the labels we can get. The U(1) label z now becomes the weight label of the Sp(2) subgroup. The U(1) group is converted to (noncompact) Sp(2) simply by multiplying by 1-Z, or, more precisely, 1-M*, where M* is the epd $A_1{}^2A^2Z$ defined below. Then z, the exponent of Z, is the Sp(2) representation label, the lowest weight of the Sp(2) multiplet.

To evaluate the expression in Eq.(4-21) we used a procedure for separating positive and negative powers in a product communicated by Richard Stanley of M.I.T. The starting point is the identity:

$$\frac{1}{(1-AX)(1-BX^{-1})} = \frac{1}{1-AB}\left[\frac{1}{1-AX} + \frac{BX^{-1}}{1-BX^{-1}}\right]$$

$$= \frac{1}{1-AB}\left[\frac{AX}{1-AX} + \frac{1}{1-BX^{-1}}\right]$$

If one wants to separate the more complicated expression

$$\frac{1}{(1-AX^{m})(1-BX^{-n})}$$

(m and n positive integers), then one can still use this formula by multiplying both numerator and denominator by

$$(1+AX^{m}+A^{2}X^{2m}+\ldots+A^{n-1}X^{(n-1)m})(1+BX^{-n}+B^{2}X^{-2n}+\ldots B^{m-1}X^{-(m-1)n})$$

The result is the following formula

$$\frac{1}{(1-AX^{m})(1-BX^{-n})}=\frac{1}{1-A^{n}B^{m}}\left[\frac{1+BX^{-n}+\ldots+B^{m-1}X^{-(m-1)n}}{1-AX^{m}}+\right.$$

$$\left.+\frac{B^{m}X^{-m}(1+AX^{m}+\ldots.A^{n-1}X^{(n-1)m})}{1-BX^{-n}}\right]$$

$$=\frac{1}{1-A^{n}B^{m}}\left[\frac{A^{n}X^{mn}(1+BX^{-n}+\ldots+B^{m-1}X^{-(m-1)n})}{1-AX^{m}}+\right.$$

$$\left.+\frac{1+AX^{m}+\ldots+A^{n-1}X^{(n-1)m}}{1-BX^{-n}}\right]$$

Using these repeatedly one is able to separate any product with positive and negative degree parts.

To separate the term of degree zero in Eq.(4-21) we analyse the type of terms we encounter and, for each type, get a formula which gives directly the zero power part in one variable. We extract first the zero power part in A' and we repeat the procedure to get also zero power in B'. We list below the types of terms which appear and their zero power parts. In these formulae a, b, c, d, e do not contain the variable A.

$$\left.\frac{A^{-1}}{(1-aA)(1-bA^2)(1-cA^{-1})(1-dA^{-2})}\right|_{A^0} = \frac{1}{(1-ac)(1-bd)}\left[\frac{a}{1-a^2d} + \frac{bc}{1-bc^2}\right]$$

$$\left.\frac{1}{(1-aA)(1-bA^2)(1-cA^{-1})(1-dA^{-2})}\right|_{A^0} = \frac{1}{(1-ac)(1-bd)}\left[\frac{1}{1-a^2d} + \frac{bc}{1-bc^2}\right]$$

$$\left.\frac{A}{(1-aA)(1-bA^2)(1-cA^{-1})(1-dA^{-2})}\right|_{A^0} = \frac{1}{(1-ac)(1-bd)}\left[\frac{ad}{1-a^2d} + \frac{c}{1-bc}\right]$$

$$\left.\frac{A^{-1}}{(1-aA)(1-bA)(1-cA^2)(1-dA^{-1})(1-eA^{-2})}\right|_{A^0} = \frac{1}{(1-ad)(1-ce)}\left[\frac{a+b}{(1-a^2e)(1-b^2e)} + \frac{b^2d}{(1-b^2e)(1-bd)} + \frac{cd}{(1-bd)(1-cd^2)}\right]$$

$$\left.\frac{1}{(1-aA)(1-bA)(1-cA^2)(1-dA^{-1})(1-eA^{-2})}\right|_{A^0} = \frac{1}{(1-ad)(1-ce)}\left[\frac{1+abe}{(1-a^2e)(1-b^2e)} + \right.$$

$$+\frac{bd}{(1-b^2e)(1-bd)}+\frac{cd^2}{(1-bd)(1-cd^2)}\Bigg]$$

$$\frac{A}{(1-aA)(1-bA)(1-cA^2)(1-dA^{-1})(1-eA^{-2})}\Bigg|_{A^0}=\frac{1}{(1-ad)(1-ce)}\Bigg[\frac{ae+be}{(1-a^2e)(1-b^2e)}+$$

$$+\frac{b^2de}{(1-b^2e)(1-bd)}+\frac{cd^3}{(1-bd)(1-cd^2)}\Bigg]$$

$$\frac{A^2}{(1-aA)(1-bA)(1-cA^2)(1-dA^{-1})(1-eA^{-2})}\Bigg|_{A^0}=\frac{1}{(1-ad)(1-ce)}\Bigg[\frac{e+abe^2}{(1-a^2e)(1-b^2e)}+$$

$$+\frac{bde}{(1-b^2e)(1-bd)}+\frac{d^2}{(1-bd)(1-cd^2)}\Bigg]$$

$$\frac{1}{(1-aA)(1-bA^2)(1-cA^2)(1-dA^{-1})(1-eA^{-2})}\Bigg|_{A^0}=\frac{1}{(1-ad)(1-be)}\Bigg[\frac{a^2e}{(1-a^2e)(1-ce)}+$$

$$+\frac{ce}{(1-ce)(1-cd^2)}+\frac{1}{(1-cd^2)(1-bd^2)}\Bigg]$$

$$\frac{A}{(1-aA)(1-bA^2)(1-cA^2)(1-dA^{-1})(1-eA^{-2})}\Bigg|_{A^0}=\frac{1}{(1-ad)(1-be)}\Bigg[\frac{ae}{(1-a^2e)(1-ce)}+$$

$$+\frac{cde}{(1-ce)(1-cd^2)}+\frac{d}{(1-cd^2)(1-bd^2)}\Bigg]$$

$$\left.\frac{A^2}{(1-aA)(1-bA^2)(1-cA^2)(1-dA^{-1})(1-eA^{-2})}\right|_{A^0} = \frac{1}{(1-ad)(1-be)}\left[\frac{e}{(1-a^2e)(1-ce)} + \frac{cd^2e}{(1-ce)(1-cd^2)} + \frac{d^2}{(1-cd^2)(1-bd^2)}\right]$$

The result of the above calculation is the desired $Sp(6) \supset Sp(2) \times O(3)$ generating function. It is given in (4-22), in terms of epd's. The epd's α, β in (4-22) are the same as in (4-18); the others are as follows (the notation is $(pq, a_1 b_1, ab, \ell)$ which stands for

$P^p Q^q A_1^{a_1} B_1^{b_1} A^a B^b Z^{\frac{1}{2}a_1+b_1} L^\ell$):

J=(02,20,00,0), K=(10,00,10,1), M=(00,02,02,0),

a=(20,00,20,0), b=(00,20,20,2), c=(20,20,02,0),

d=(20,20,02,2), e=(11,00,11,1), f=(10,02,01,1),

g=(11,20,01,1), h=(10,20,11,1), i=(11,20,20,0),

j=(10,02,12,2), k=(20,02,11,1), ℓ=(12,20,02,0),

m=(10,20,11,2), n=(21,20,11,1), p=(20,20,21,1),

q=(21,22,02,0), r=(21,20,22,0), s=(11,22,11,1),

t=(11,20,12,1), u=(30,22,22,0), v=(30,20,12,2),

w=(20,22,12,1), x=(10,22,21,2), y=(00,22,22,3).

$$
\begin{aligned}
&H\,(\,P,\,Q,\,D;\,A_1,\,B_1;\,A,\,B;\,Z,\,L\,)=\\
&=[(1-\alpha)(1-\beta)(1-J)(1-J^*)(1-K)(1-K^*)(1-M)]^{-1}\times\\
&\times\{[(1-a)(1-a^*)(1-c)]^{-1}[a^*+e+r+a^*h+i^*+a^*k+\ell+a^*n+ci+ck^*+\\
&\qquad +u+ch^*+q+eq+ii^*+a^*s]+\\
&+[(1-a)(1-a^*)(1-c^*)]^{-1}[c^*+c^*e+r^*+ah^*+i+ak^*+\ell^*+an^*+c^*i^*+\\
&\qquad +c^*k+u^*+c^*h+q^*+eq^*+c^*ii^*+c^*s]+\\
&+[(1-a)(1-c)(1-d)]^{-1}[c+v+cp+ch+cf+ck+cg+cn+cdi+cgh+cfp+\\
&\qquad +cfh+cfg+cfn+cfi+cs]+\\
&+[(1-a^*)(1-c^*)(1-d^*)]^{-1}[c^*d^*+v^*+c^*p^*+c^*h^*+c^*f^*+c^*k^*+c^*g^*+c^*n^*+\\
&\qquad +c^*d^*i^*+c^*g^*h^*+c^*f^*p^*+c^*f^*h^*+c^*f^*g^*+c^*f^*n^*+\\
&\qquad +c^*f^*i^*+c^*d^*s]+\\
&+[(1-a)(1-b^*)(1-d)]^{-1}[1+j+p+h+f+k+g+n+di+gh+fp+fh+fg+fn+fi+s]+\\
&+[(1-a^*)(1-b)(1-d^*)]^{-1}[d^*+d^*j^*+p^*+h^*+f^*+k^*+g^*+n^*+d^*i^*+g^*h^*+\\
&\qquad +f^*p^*+f^*h^*+f^*g^*+f^*n^*+d^*f^*i^*+d^*s]+\\
&+[(1-a^*)(1-b)(1-c)]^{-1}[a^*b+j^*+t+a^*bh+bi^*+a^*x+b\ell+a^*f^*h+\\
&\qquad +cf^*+bck^*+w+bch^*+bq+cf^*h^*+f^*i^*+a^*bs]+\\
&+[(1-a)(1-b^*)(1-c^*)]^{-1}[b^*c^*+c^*j+t^*+ab^*h^*+b^*i+ax^*+b^*\ell^*+afh^*+c^*f+\\
&\qquad +b^*c^*k+w^*+b^*c^*h+b^*q^*+c^*fh+c^*fi+b^*c^*s]+\\
&+[(1-b)(1-c)(1-d)]^{-1}[bc+hm+cm+bch+bcf+cx+bcg+cf^*h+cdf^*+\\
&\qquad +bcgh+cfm+bcfh+bcfg+cms+cff^*+bcs]+\\
&+[(1-b^*)(1-c^*)(1-d^*)]^{-1}[b^*c^*d^*+h^*m^*+c^*m^*+b^*c^*h^*+b^*c^*f^*+c^*x^*+b^*c^*g^*+\\
&\qquad +c^*fh^*+c^*d^*f+b^*c^*g^*h^*+c^*f^*m^*+b^*c^*f^*h^*+\\
&\qquad +b^*c^*f^*g^*+c^*m^*s+c^*ff^*+b^*c^*d^*s]+\\
&+[(1-b)(1-b^*)(1-d)]^{-1}[b+y+m+bh+bf+x+bg+f^*h+df^*+\\
&\qquad +bgh+fm+bfh+bfg+ms+ff^*+bs]+\\
&+[(1-b)(1-b^*)(1-d^*)]^{-1}[b^*d^*+d^*y+m^*+b^*h^*+b^*f^*+x^*+b^*g^*+fh^*+\\
&\qquad +d^*f+b^*g^*h^*+f^*m^*+b^*f^*h^*+b^*f^*g^*+\\
&\qquad +m^*s+d^*ff^*+b^*d^*s]\}
\end{aligned}
$$

Eq.(4-22)

The conjugate of an epd, denoted by an asterisk in Eq(4-22), has the same meaning as in (4-18): interchange the SU(3) labels in each of the three pairs, i.e. $(pq,a_1b_1,ab,\ell)^*=(qp,b_1a_1,ba,\ell)$. The epd's α, β, e, s, y are self-conjugate. The generating function in (4-22), apart from the missing denominator factor 1-M*, removed in converting from U(1) to Sp(2), is conjugation symmetric, a fact which was helpful in sorting out the epd's and giving a first indication about it's being correct.

To make sure you get only bottom states of Sp(2) multiplets from the other epd's you must render them traceless by the procedure described in [Mo84]. Those substitutions of course mutilate the SU(3) properties, as expected; nevertheless the SU(3) labels a_1b_1 and ab are useful as instructions on how to construct the epd's and were invaluable to us (we had to rederive the generating function with them included) in writing the generating function with the different terms interpreted consistently in terms of epd's.

4.3. Checking the generating function

As mentioned above, the generating function in (4-22) is conjugation symmetric, a fact which was considered as a first positive indication about its correctness. The generating function was also subjected to what may be called consistency checks. For example the coefficient of i in (4-22) is

$$[(1-a)(1-a^*)(1-c^*)]^{-1} + c[(1-a)(1-a^*)(1-c)]^{-1} +$$
$$d[(1-a)(1-b^*)(1-d)]^{-1} + b^*[(1-a)(1-b^*)(1-c^*)]^{-1} + cd[(1-a)(1-c)(1-d)]^{-1}.$$

It may be verified that each product of powers of three denominators epd's which appear in the same fraction (including those in which one or more exponents are zero) appears just once in the above expression: this check was made separately for each numerator epd and for each product of numerator epd's.

As a final check we converted the expression in (4-22) into a generating function for SO(3) weights instead of SO(3) multiplets; it was then compared with the corresponding weight generating function obtained by converting (4-18) directly; since an analytic comparison would be prohibitively laborious, the necessary substitutions were made by a computer program and the two generating functions compared for random values of their arguments. In what follows we explain this checking test and we give in Appendix A a listing of the program used and the actual result. We start with the known [GR81] generating function for $Sp(6) \supset SU(3) \times U(1)$ generating function

$$\frac{1}{(1-Z^3)(1-DZ)(1-ZA^2)(1-PA)(1-Z^2B^2)(1-QB)(1-ZQ^2)(1-Z^2P^2)} \times$$

$$\times \left[\frac{(1+ZQA+ZPAB+ZPQB)(1+Z^2PB)}{1-ZP^2B^2} + \right. \tag{4-23}$$

$$\left. + \frac{(1+ZQA)(Z^2Q^2A^2+Z^2PQA+Z^2QAB+Z^4PQ^2A^2B)}{1-Z^2Q^2A^2}\right]$$

with Sp(6) labels (pqd) and the SU(3) labels (ab). This is (4-16) with $A_1=B_1=1$. Since the factor (1-DZ) appears both in the generating function (4-22) to be checked and in (4-23) considered correct and used for numerical comparison we drop this factor in both generating functions. We get the function F(P, Q, A, B, Z). We want now to convert the SU(3) group to SU(2)$\times$U(1) and to do this we couple our function to

$$G(A, B, T, Y) = [(1-ATY^{1/3})(1-AY^{-2/3})(1-BTY^{-1/3})(1-BY^{2/3})]^{-1}. \tag{4-24}$$

This is the SU(3)$\supset$SU(2)$\times$U(1) generating function where we use the same (AB) labels for SU(3) representations, T is the SU(2) label and Y stands for U(1). Since we are not interested in the U(1) subgroup here, we drop the Y in this equation and get the simpler function

$$G(A, B, T) = [(1-AT)(1-A)(1-BT)(1-B)]^{-1} \tag{4-25}$$

We want to go from Sp(6) to SU(2) via SU(3). To do this we substitute the generating function for $SU(3) \supset SU(2) \times U(1)$ in the generating function for $Sp(6) \supset SU(3) \times U(1)$. We have to make sure that the "transient" SU(3) representation is the same in both generating functions and to enforce this we couple (i.e. multiply together)

$$F(P, Q, A, B, Z) \cdot G(A^{-1}, B^{-1}, T) \tag{4-26}$$

and retain only the $A^0 B^0$ part. In this way the SU(3) labels from the two generating functions are equal and what we get is a $Sp(6) \supset SU(2) \times U(1)$ generating function. Since the labels P, Q, Z are not affected in these calculations we write F(A, B) and understand by it the complete F(P, Q, A, B, Z). To get the $A^0 B^0$ part of (4-26) involves some algebra and we prefer to use the residue calculation method. It can be easily shown that taking the $A^0 B^0$ part of our expression

$$\frac{F(A, B)}{(1-A^{-1}T)(1-A^{-1})(1-B^{-1}T)(1-B^{-1})}$$

is equivalent with taking the residue of

$$\operatorname*{Res}_{A,B} \frac{A^{-1}B^{-1}F(A, B)}{(1-A^{-1}T)(1-A^{-1})(1-B^{-1}T)(1-B^{-1})} = \operatorname*{Res}_{A,B} \frac{ABF(A, B)}{(A-T)(A-1)(B-T)(B-1)}$$

with respect to both A and B. To perform the residue we use the following formula for the residue of a function with two simple poles

$$\operatorname*{Res}_{A} \frac{AF(A)}{(A-T)(A-1)} = \frac{TF(T)}{T-1} + \frac{F(1)}{1-T}$$

Applying this formula twice we get

$$\operatorname*{Res}_{AB} \frac{ABF(A,\ B)}{(A-T)(A-1)(B-T)(B-1)} = \frac{T^2F(T,T)-TF(T,1)-TF(1,T)+F(1,1)}{(T-1)^2} \quad (4\text{-}27)$$

We want to keep track of the SU(3) labels (A,B) since the generating function to be checked contains these labels. Call this H(P,Q,A,B,Z,T).

$$H(P,Q,A,B,Z,T) = \quad (4\text{-}28)$$

$$= \frac{T^2F(P,Q,AT,BT,Z)-TF(P,Q,AT,B,Z)-TF(P,Q,A,BT,Z)+F(P,Q,A,B,Z)}{(T-1)^2}$$

F(P,Q,A,B,Z) is given by (4-23) without the denominator factor (1-DZ). The expression in (4-28) can be used as a generating function for $Sp(6) \supset SU(2) \times U(1)$ with Sp(6) labels (P, Q, D=Z) and SU(2) label T. We want to convert this generating function for multiplets into the corresponding generating function for weights.

We know that the SU(2) weights are given by

$$\frac{1}{(1-T\eta)(1-T\eta^{-1})} \tag{4-29}$$

such that we have to multiply H(T) of (4-28) (we disregard here the labels P,Q,A,B,Z which are not affected) by (4-29) with $T \to T^{-1}$, and take the T° part of it or, equivalently, the residue with respect with T.

$$\left.\frac{H(T)}{(1-T^{-1}\eta)(1-T^{-1}\eta^{-1})}\right|_{T^{\circ}} = \underset{T}{\mathrm{Res}} \frac{TH(T)}{(T-\eta)(T-\eta^{-1})} = \frac{\eta H(\eta) - \eta^{-1} H(\eta^{-1})}{\eta - \eta^{-1}} \tag{4-30}$$

This gives us the SU(2) weights. To compare with the SO(3) weights that we get directly from our $Sp(6) \supset Sp(2) \times O(3)$ generating function we double the weights in (4-30), i.e. replace η by η^2. Now we are ready to compare the weights given by

$$\frac{\eta^2 H(P,Q,A,B,Z,\eta^2) - \eta^{-2} H(P,Q,A,B,Z,\eta^{-2})}{\eta^2 - \eta^{-2}} \tag{4-31}$$

with the weights given by the $Sp(6) \supset Sp(2) \times O(3)$ generating function. For the actual comparison we randomly generated positive real values for the labels P,Q,A,B,Z,η and evaluated the generating function given by (4-22) with $A_1 = B_1 = 1$. We compared the value with the result of (4-31). In Appendix A we give the listing of the program used and an output with the two functions and their ratios for randomly generated values for the labels.

4.4. Related branching rules

Complementarity relations in group-subgroup chains imply connections between apparently unrelated branching rules. Thus the generating functions of Eq.(4-18) and (4-22), for $Sp(6) \supset U(3)$ and $Sp(6) \supset Sp(2) \times O(3)$ respectively, imply branching rules generating functions for $SU(n) \supset SO(n)$, with all but the first three $SU(n)$ labels zero and for $O(3n) \supset O(3) \times O(n)$, with all but the first $O(3n)$ label zero, respectively.

Although not needed for the theory of nuclear collective motions, we present the results here since we get them at not much extra effort.

For the chains of subgroups

$$Sp(6n) \supset Sp(6) \times O(n) \tag{4-32a}$$

$$Sp(6n) \supset Sp(2) \times O(3n) \tag{4-32b}$$

complementarity relations hold (see [CC84], Section II). This means that, since the representation of $Sp(6n)$ is $[(1/2)^{3n}]$ or $[(1/2)^{3n-1},(3/2)]$ (the metaplectic ones), the IR of $Sp(6)$ determines the IR of $O(n)$ and vice-versa in (4-32a) and the same holds for $Sp(2)$ and $O(3n)$ in (4-32b).

We can give the generating function for metaplectic $Sp(6n) \supset Sp(6) \times SO(n)$ (pqd labels for Sp(6) and hjk for the first three SO(n) labels and all the next labels zero).

$$\frac{1}{(1-PD^{1/2}H)(1-QDJ)(1-D^{3/2}K)} \qquad (4\text{-}33)$$

Here the integer powers of D correspond to even metaplectic Sp(6n) and half-odd powers of D to odd metaplectic Sp(6n). Substituting into this the generating function (4-18) for $Sp(6) \supset U(3)$ we get the generating function for metaplectic Sp(6n) into $U(3) \times SO(n)$ (ABZ stand for U(3) and HJK for SO(n)). Looking at (4-33) we see that this is done replacing in (4-18) $P \to Z^{1/2}H$, $Q \to ZJ$, $D \to Z^{1/2}K$. This means that the generating function for metaplectic $Sp(6n) \supset U(3) \times O(n)$ is also given by (4-18) but the letters now stand for:

$\alpha = Z^3 \to Z^3$, $\beta = DZ \to Z^{3/2}K$, $\gamma = PA \to Z^{1/2}AH$, $\gamma^* = QB \to ZBJ$,
$\delta = Q^2Z \to Z^3J^2$, $\delta^* = P^2Z^2 \to Z^3H^2$, $\varepsilon = A^2Z \to ZA^2$, $\varepsilon^* = B^2Z^2 \to Z^2B^2$,
$\zeta = P^2B^2Z \to Z^2B^2H^2$, $\zeta^* = Q^2A^2Z^2 \to Z^4A^2J^2$, $\eta = PBZ^2 \to Z^{5/2}BH$,
$\eta^* = QAZ \to Z^2AJ$, $\kappa = PQBZ \to Z^{5/2}BHJ$, $\kappa^* = PQAZ^2 \to Z^{7/2}AHJ$.

Now we can convert further to a generating function for the chain $SU(n) \supset O(n)$ (from ABZ as labels for U(3) to EFG as the first three labels of SU(n) and the rest of them zero). We use the generating function for $SU(3n) \supset SU(3) \times SU(n)$ (the first SU(3n) label is Z, the SU(3) labels are AB and SU(n) labels EFG) :

$$\frac{1}{(1-Z^{1/2}AE)(1-ZBF)(1-Z^{3/2}G)} \qquad (4\text{-}34)$$

From this generating function we see that we have to make the replacements $Z \dashrightarrow G^{2/3}$, $A \dashrightarrow EG^{1/3}$, $B \dashrightarrow FG^{2/3}$ in the previous generating function or replacing directly in (4-18) :

$$P \dashrightarrow G^{1/3}H \ ,\ Q \dashrightarrow G^{2/3}J \ ,\ D \dashrightarrow G^{1/3}K \ ,\ A \dashrightarrow EG^{-1/3}$$
$$B \dashrightarrow FG^{-2/3},\ Z \dashrightarrow G^{2/3},\ A_1 \dashrightarrow 1,\ \ B_1 \dashrightarrow 1 \ .$$

We can say now that (4-18) is the generating function for $SU(n)\supset O(n)$ branching rules ($SU(n)$ labels EFG and $O(n)$ labels HJK) if the letters on the right hand side stand for:

$$\alpha=Z^3 \dashrightarrow G^2,\ \beta=Z^{3/2}K \dashrightarrow GK,\ \gamma=Z^{1/2}AH \dashrightarrow EH,\ \gamma^*=ZJB \dashrightarrow FJ,$$
$$\delta=Z^3J^2 \dashrightarrow G^2J^2,\ \delta^*=Z^3H^2 \dashrightarrow G^2H^2,\ \varepsilon=ZA^2 \dashrightarrow E^2,\ \varepsilon^*=Z^2B^2 \dashrightarrow F^2,$$
$$\zeta=Z^2B^2H^2 \dashrightarrow F^2H^2,\ \zeta^*=Z^4A^2J^2 \dashrightarrow E^2G^2J^2,\ \eta=Z^{5/2}BH \dashrightarrow FGH,$$
$$\eta^*=Z^2AJ \dashrightarrow EGJ,\ \varkappa=Z^{5/2}BHJ \dashrightarrow FGHJ,\ \varkappa^*=Z^{7/2}AHJ \dashrightarrow EG^2HJ.$$

The above substitutions are valid when $n \geq 9$. For $n = 8$ the substitution for D changes to $D \dashrightarrow G^{1/3}KK'$ where the nonzero $O(8)$ labels are (h,j,k,k'), with $k = k'$, and for $n = 7$ the substitution for D is $D \dashrightarrow G^{1/3}K^2$ (here the three $O(7)$ labels are (h,j,k)). We do not consider the case $n \leq 6$.

Similarly, starting with the generating function for $Sp(6)\supset Sp(2)\times O(3)$ given in (4-22) ($Sp(6)$ labels PQD, $Sp(2)$ label z and

O(3) label L), substituting (4-33) and (4-34) generating functions we get the branching rules generating function for the chain $O(3n) \supset O(3) \times O(n)$, all but the first label of O(3n) zero. The labels are U for O(3n), L for O(3) and HJK for O(n) (all but the first three labels zero).

All what we have to do is to substitute in (4-22)

$$P \to UH \ , \ Q \to U^2 J \ , \ D \to UK \ , \ Z \to U^2 \ , \ L \to L$$

$$A_1 \to 1 \ , \ B_1 \to 1 \ , \ A \to 1 \ , \ B \to 1 \ ;$$

These substitutions hold for $n \geq 9$. The substitutions for D become $D \to UKK'$ (n=8) and $D \to UK^2$ (n=7).

CHAPTER 5. THE SP(6)$\supset$SP(2)$\times$O(3) BASIS STATES

5.1. The Basis States for Our Problem

The general basis states for the group-subgroup chain Sp(6)$\supset$ Sp(2)xO(3) can be read from the generating function given in Eq.(4-22). Expanding the generating function (4-22) in a power series, we get an infinite number of terms (infinite powers from the denominator factors) and we interpret each term as a basis state.

In fact, the energy eigenstates are linear combinations of the basis states we use here. As stated in Chapter 1 the subgroup Sp(2)$\times$O(3) does not provide enough labels for Sp(6). After Racah [Ra51], the group Sp(6) needs $(r-\ell)/2 = 9$ ($r=21$ is the order of Sp(6) and $\ell=3$ is the rank) internal labels while the subgroup provides only four (two from Sp(2) and two from O(3)). Thus in the branching Sp(6)$\supset$ Sp(2)$\times$O(3) there are five missing labels and this leads to a nonorthonormal basis. However the basis is complete and the states are linearly independent. One could apply an orthogonalization procedure (such as Gramm-Schmidt) but the new states would be cumbersome to work with. There is no need to orthogonalize because the non-orthonormal basis is as good to work with as an orthogonal one. One has to define matrix elements of an operator Ω by the coefficients Ω_{ji} below

$$\Omega|i\rangle = \sum_j |j\rangle \Omega_{ji}$$

and not by $\Omega_{ji} = \langle j|\Omega|i\rangle$. To find the eigenvalues and the eigenstates of the operator Ω (e.g. Ω=H) one just diagonalizes the matrix Ω_{ji}.

All the terms (i.e. all the basis states) in the expansion of the generating function have the same structure. Firstly, all contain the factors from the common denominator in front,

$$F = \alpha^{\alpha'} \beta^{\beta'} J^{J'} J^{*J^{*'}} K^{K'} K^{*K^{*'}} M^{M'}$$

where the powers α', β', J', J*', K', K*', M' are positive integers or zero. Secondly, all terms contain the three denominators in front of each square bracket. There are twelve different ones of the form

$$G_1 = a^{a'} a^{*a^{*'}} c^{c'}, \quad G_2 = a^{a'} a^{*a^{*'}} c^{*c^{*'}}, \ldots, G_{12} = b^{b'} b^{*b^{*'}} d^{*d^{*'}}$$

And finally the terms inside each square bracket (there are 16 of them) differ by their numerator factor. Let us call this numerator factor H^i_j; i ($1 \le i \le 12$) stands for the square bracket to which H^i_j belongs and j ($1 \le j \le 16$) shows its position inside the square bracket i. Note that the numerator factors appear just at the power one and in each bracket we write in the first position (j=1) the term which contains only epd's which occur in the denominator G_i. We call this first term the denominator term.

To summarize, the basis states are of the form

$$F \quad G_i \quad H^i{}_j \tag{5-1}$$

(no sum over i); there are 12x16=192 types of terms. Of course there is an infinity of basis states (Sp(6) is non-compact) corresponding to the infinity of powers in F and G .

The basis states involve products of epd's. The products are stretched (all labels additive except the SU(3) ones). Since each epd is the highest SO(3) and lowest Sp(2) state of an SO(3)XSp(2) multiplet, a general basis state which is a product of such epd's is also highest SO(3) and lowest Sp(2) of the subgroup multiplet which it represents.

Of course in the stretched products with all labels additive we refer only to the Sp(6), Sp(2) and SO(3) labels. We ignore now the labels a, b, a_1 , b_1 which were helpful in sorting out the epd's. They are not additive under the multiplication of epd's. We could have kept them as actual labels but then we would need projection procedures to retain only the stretched part in them. It is simpler to ignore them.

Looking at the generating function given by (4-22), one can see that certain epd's never appear multiplied together. We call these combinations incompatible products or syzygies.

5.2. The epd's and their Syzygies

We mean by a syzygy an incompatible product of epd's. The syzygies can be read from the generating function because the incompatible epd's do not appear multiplied together in the series expansion of the generating function. The incompatible products are superfluous because they can be expressed as linear combinations of products of compatible epd's, i.e., which are present in the generating function and provide the same group-subgroup labels. Strictly, the word syzygy means just the equation which relates the incompatible product to the allowed products of epd's ; we use the word in a looser sense to designate the incompatible products.

For example the pair "be" does not appear in the generating function; thus b and e are incompatible (see also Figure 5-3 below). One can verify that be is expressible as

$$be = (Kj^* - K^*m) / \sqrt{2}$$

where Kj* and K*m are compatible products.

From the generating function we can extract all the syzygies. The common denominator epd's (α, β, J, J*, K, K*, M) are all compatible among themselves and with all the other epd's. The epd's in the sets of three denominator factors in front of the twelve square brackets are not all compatible. We give below in Figure 5-1 the compatibility among the denominator epd's a, a*, b, b*, c, c*, d, d*. The star (*) stands for an incompatible product.

	a*	b	b*	c	c*	d	d*
a	>	*	>	>	>	>	*
a*		>	*	>	>	*	>
b			>	>	*	>	>
b*				*	>	>	>
c					*	>	*
c*						*	>
d							*

Fig 5-1. Compatibility table for denominator epd's in the generating function for $Sp(6) \supset Sp(2) \times O(3)$ branching rules.

All the other epd's appear only in the numerators and we call them numerator epd's. We list below in Figure 5-2 all numerator epd's together with the denominator factors G_i with which they are compatible. The generating function is conjugation symmetric, i.e. is not affected if all epd's are replaced by their conjugates (see Chapter 4, Eqs. (4-17a,b,c)). This implies that a compatibility relation (or an incompatibility one) between two epd's holds for their conjugate partners. Using this we reduce the size of our table almost to half (the epd's e, s, y are self-conjugate).

	G_1 aa*c	G_2 aa*c*	G_3 acd	G_4 a*c*d*	G_5 ab*d	G_6 a*bd*	G_7 a*bc	G_8 ab*c*	G_9 bcd	G_{10} b*c*d*	G_{11} bb*d	G_{12} bb*d*
e	>	>	*	*	*	*	*	*	*	*	*	*
f	*	*	>	*	>	*	*	>	>	>	>	>
g	*	*	>	*	>	*	*	*	>	*	>	*
h	>	>	>	*	>	*	>	>	>	*	>	*
i	>	>	>	*	>	*	*	>	*	*	*	*
j	*	*	*	*	>	*	*	>	*	*	*	*
k	>	>	>	*	>	*	*	>	*	*	*	*
l	>	*	*	*	*	*	>	*	*	*	*	*
m	*	*	*	*	*	*	*	*	>	*	>	*
n	>	*	>	*	>	*	*	*	*	*	*	*
p	*	*	>	*	>	*	*	*	*	*	*	*
q	>	*	*	*	*	*	>	*	*	*	*	*
r	>	*	*	*	*	*	*	*	*	*	*	*
s	>	>	>	>	>	>	>	>	>	>	>	>
t	*	*	*	*	*	*	>	*	*	*	*	*
u	>	*	*	*	*	*	*	*	*	*	*	*
v	*	*	>	*	*	*	*	*	*	*	*	*
w	*	*	*	*	*	*	>	*	*	*	*	*
x	*	*	*	*	*	*	>	*	>	*	>	*
y	*	*	*	*	*	*	*	*	*	*	>	>

Fig 5-2. The numerator epd's and their compatibility with the denominator factors G_i.

In Fig.5-2 the (>) stands for compatible and (*) for incompatible. If interested in the compatibility of the conjugate of a numerator epd, say g*, we first observe that the twelve G_i can be organized into six pairs under conjugacy, such that $G_1^*=G_2$, $G_3^*=G_4,\ldots, G_{11}^*=G_{12}$ and if, for instance, g is compatible with G_3, G_5, G_9 and G_{11} and incompatible with the rest, then g* is compatible with G_4, G_5, G_{10} and G_{12} (the conjugates of G_3, G_5, G_9 and G_{11}) and incompatible with the rest. This is a consequence of the fact that the entire generating function is conjugation symmetric and, since the denominators are in conjugacy pairs ($G_{2k}=G^*_{2k-1}$, k=1,..6), the corresponding numerator parentheses are also paired under conjugacy. From Figure 5-2 we extract the compatibility between numerator epd's and individual denominator epd's. When a numerator epd is compatible with a three factors denominator, say G_1=aa*c, it is compatible with each of a, a*, c. When the numerator epd is incompatible with the denominator G_i, it is incompatible with at least one of its three factors.

	e	f	g	h	i	j	k	l	m	n	p	q	r	s	t	u	v	w	x	y
a	>	>	>	>	>	>	>	>	*	>	>	>	>	>	*	>	>	*	*	*
a*	>	*	*	>	>	*	>	>	*	>	*	>	>	>	>	>	*	>	>	*
b	*	>	>	>	*	*	*	>	>	*	*	>	*	>	>	*	*	>	>	>
b*	*	>	>	>	>	>	>	*	>	>	>	*	*	>	*	*	*	*	>	>
c	>	>	>	>	>	*	>	>	>	>	>	>	>	>	>	>	>	>	>	*
c*	>	>	*	>	>	>	>	*	*	*	*	*	*	>	*	*	*	*	*	*
d	*	>	>	>	>	>	>	*	>	>	>	*	*	>	*	*	>	*	>	>
d*	*	>	*	*	*	*	*	*	*	*	*	*	*	>	*	*	*	*	*	>

Fig 5-3. Compatibility of numerator epd's with denominator epd's

For the conjugate numerator epd's (f*, g*, h*,...etc.) which are not listed in Figure 5-3 one has to take the "conjugate" of this table, i.e. if f is incompatible only with a*, then f* is incompatible with a and compatible with everything else in the list. Note that the self-conjugate epd's (e, s, y) are compatible (or incompatible) with both members of a conjugacy pair (e is compatible with a, a* and c, c*).

What about compatibility of the numerator epd's among themselves? Observe that in the generating function, they appear in pairs at most and even the appearing pairs are exceptions (just a few in the list of all possible ones). All combinations of three or more numerator epd's are forbiddden and the only allowed pairs of numerator epd's are the following ones:

ff*, ee*, eq, eq*, fg, f*g*, fh, f*h*, fh*, f*h, fi, f*i*, fm, f*m*, fn, f*n*, fp, f*p*, gh, g*h*, hm, h*m*, ms, m*s.

All other pairs are forbidden. The allowed pairs are incompatible with all numerator epd's but they are compatible with the denominator factors they appear with. We give below in Figure 5-4 the compatibility table of the allowed numerator pairs with the denominators G_i. Again we use the conjugacy property to reduce the size of our table. For compatibility of f*m*, for example, one sees that fm is compatible with G_9 and G_{11} ; hence f*m* is compatible with G^*_9 and G^*_{11} , that is, with G_{10} and G_{12}.

	G_1 aa*c	G_2 aa*c*	G_3 acd	G_4 a*c*d*	G_5 ab*d	G_6 a*bd*	G_7 a*bc	G_8 ab*c*	G_9 bcd	G_{10} b*c*d*	G_{11} bb*d	G_{12} bb*d*
eq	>	*	*	*	*	*	*	*	*	*	*	*
ff*	*	*	*	*	*	*	*	*	>	>	>	>
ii*	>	>	*	*	*	*	*	*	*	*	*	*
fg	*	*	>	*	>	*	*	*	>	*	>	*
fh	*	*	>	*	>	*	*	*	>	*	>	*
fh*	*	*	*	*	*	*	*	>	*	>	*	>
fi	*	*	>	*	>	*	*	>	*	*	*	*
fm	*	*	*	*	*	*	*	*	>	*	>	*
fn	*	*	>	*	>	*	*	*	*	*	*	*
fp	*	*	>	*	>	*	*	*	*	*	*	*
gh	*	*	>	*	>	*	*	*	>	*	>	*
hm	*	*	*	*	*	*	*	*	>	*	*	*
ms	*	*	*	*	*	*	*	*	>	*	>	*

Fig 5-4. The compatibility of allowed pairs of numerator epd's with the denominator factors G_t.

As we did for individual numerator epd's, we can extract from Figure 5-4 the compatibility of numerator pairs with individual denominator epd's. Since the denominator epd's appear in the binomial expansion of the generating function to all powers, the compatibility with a, a*, b, b*, c, c*, d, d* is in fact compatibility with these to any power. In the table below we give the compatibility of pairs of numerator epd's with denominator epd's.

	eq	ff*	ii*	fg	fh	fh*	fi	fm	fn	fp	gh	hm	ms
a	>	*	>	>	>	>	>	*	>	>	>	*	*
a*	>	*	>	*	*	*	*	*	*	*	*	*	*
b	*	>	*	>	>	>	*	>	*	*	>	>	>
b*	*	>	*	>	>	>	>	>	>	>	>	*	>
c	>	>	>	>	>	*	>	>	>	>	>	>	>
c*	*	>	>	*	*	>	>	*	*	*	*	*	*
d	*	>	*	>	>	*	>	>	>	>	>	>	>
d*	*	>	*	*	*	>	*	*	*	*	*	*	*

Fig 5-5. Compatibility of allowed pairs of numerator epd's with denominator epd's

This analysis was helpful in checking the consistency of our generating function. For instance, an allowed pair of numerator epd's is expected to appear together with the denominator G_i if both epd's of the pair appear by themselves with G_i. Checking this we had to change the interpretation of a few terms in order to comply with this consistency requirement. This explains why the version given in Equation (4-22) is slightly different from the one given in the published paper [MN85] (see Appendix B). We made the following five (times two) changes:

du --> cfp	and	d*u* --> c*f*p*	in G_3 and G_4	resp.
k*w --> cf*h*	and	kw* --> c*fh	in G_7 and G_8	resp.
hs --> bq	and	h*s* --> b*q*	in G_7 and G_8	resp.
dhs --> bcfg	and	d*h*s* --> b*c*f*g*	in G_9 and G_{10}	resp.
dw --> cfm	and	d*w* --> c*f*m*	in G_9 and G_{10}	resp.

For example, fp was established as a compatible pair of numerator epd's (it appeared in G_5). Checking the compatibility of f and p separately, we see that both f and p appear in G_3, so the pair fp was expected to be there too, and this is why we reinterpreted du --> cfp. The changes of du and of dw into cfp and cfm respectively (and their conjugates, of course) involve giving up single numerator epd's u and w (the epd "d" is denominator one and it will appear anyways multiplied at an arbitrary power from the G_3 and G_9 in front), and the changes affect only the table in Fig. 5-2 and 5-3, everything remaining consistent. The other changes involve pairs of numerator epd's into pairs of numerator epd's and we have to be careful not to give up a pair which is expected to be there. Fortunately everything comes out nicely by making hs and k*w (and their conjugates) incompatible; and this is possible since they do not occur anywhere else in the generating function. In the end we have the minimum number of allowed pairs of numerator epd's and these pairs are present with the denominators compatible with both members of the pair.

5.3. How to Construct the epd's Explicitly

The epd's are specified by the set of labels (p,q,a_1,b_1,a,b,ℓ) which stands for

$$P^p Q^q A_1^{a_1} B_1^{b_1} A^a B^b Z^{a_1/2+b_1} L^{\ell} \tag{5-2}$$

in the binomial expansion of the generating function in Eq. (4-22). Here pq are the first two Sp(6) labels (or the SU(3) labels of the "bottom" states since we consider the lowest Sp(2) states). The labels a_1, b_1 are the SU(3) labels of the SU(3) tensor made from the raising generators while a, b are the "final" SU(3) labels and are found in the Clebsch-Gordan series $(pq)\times(a_1 b_1)$. The label ℓ is the SO(3) label and we consider highest $(m=\ell)$ states. The label z is the U(1) label which gives the degree in the raising generators. The third Sp(6) label d appears only in the epd $\beta = DZ$ so we did not reserve space for it in the notation. If we omit $\alpha = Z^3$ and $\beta = DZ$, the label z is redundant, for it always equals $a_1/2 + b_1$.

To construct explicitly an epd given by its labels $(pq, a_1 b_1, ab, \ell)$ one has to multiply the SU(3) representation (pq) by the SU(3) representation $(a_1 b_1)$ to obtain the final (ab), project in SO(3) and take the $m = \ell$ component.

The Sp(6) in our problem originated from Sp(6n), the dynamical group of a 3n-dimensional harmonic oscillator. For details on symplectic geometry see Chapter 3. As discussed there, we have n particles (in fact A = n+1 nucleons but one eliminates the center-of-mass) in 3 dimensions with the Jacobi coordinates and momenta x_{is} and p_{is}, i = 1, 2, 3 and s = 1, 2,...n satisfying the commutation relations of Weyl type. The creation and annihilation operators are given by the usual

$$\eta_{is} = (x_{is} - ip_{is}) / \sqrt{2} \qquad (5\text{-}3)$$
$$\xi_{is} = (x_{is} + ip_{is}) / \sqrt{2}$$

Without losing any generality we may think of n as being equal to 9 (A=10 nucleons). We get a 27-dimensional harmonic oscillator and the chain is $Sp(54) \supset Sp(6) \times O(9)$. Instead of the components i = 1, 2, 3 we prefer to use polar-type -1, 0, 1 components defined as

$$x_0 = x_3 \quad , \quad x_{\pm} = (x_1 \pm ix_2) / \sqrt{2} \qquad (5\text{-}4)$$

Also in the particle-index space, we prefer "hyperspherical" components. For this purpose we define j by n = 2j+1 and replace the index s = 1, 2,...n by the index m = j, j-1,...,-j. In our case (n=9) the new components are given by

$$
\begin{aligned}
x_{\pm 4} &= (x_1 \pm i\, x_2) / \sqrt{2} \\
x_{\pm 3} &= (x_3 \pm i\, x_4) / \sqrt{2} \\
x_{\pm 2} &= (x_5 \pm i\, x_6) / \sqrt{2} \\
x_{\pm 1} &= (x_7 \pm i\, x_8) / \sqrt{2} \\
x_0 &= x_9
\end{aligned}
\tag{5-5}
$$

The highest state in SO(3) and SO(9) is the i = +1 and m = +4 state, η_{14}. To construct the SU(3) representations (pq) we use the highest SO(3) and SO(9) states. For (pq)=(10) triplet

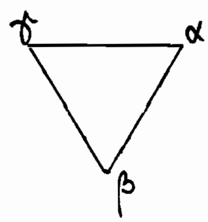


Fig. 5-6a

we use

$$\alpha = \eta_{14}\ , \quad \beta = \eta_{04} \quad \text{and} \quad \gamma = \eta_{\bar{1}4}\ . \tag{5-6}$$

The SO(3) projection is

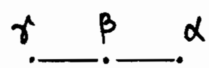


Fig. 5-6b

For the anti-triplet (pq)=(01) we have two choices: either the particle-antiparticle scheme where the (01) represents an antiparticle (and the states have an asterisk "*") as opposed to (10) in Fig 5-6a which stands for a particle,

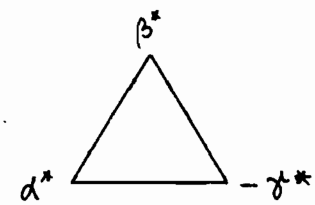
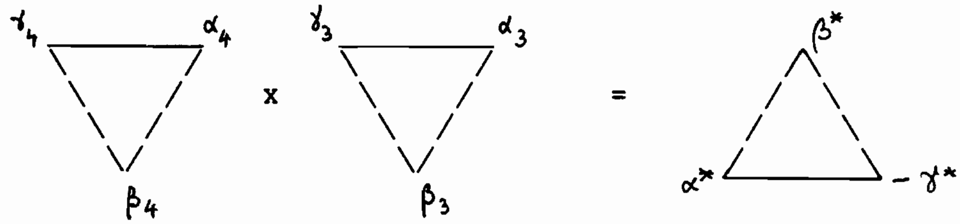


Fig 5-7a

either the two-particles scheme where the (01) is obtained by multiplying two triplet (10) representations of two different particles, say of m=4 and m=3 in SO(9).



In the two-particles scheme the product is antisymmetrized as follows:

$$
\begin{aligned}
-\gamma^* &= \alpha_4 \beta_3 - \alpha_3 \beta_4 = \begin{vmatrix} \eta_{14} & \eta_{13} \\ \eta_{04} & \eta_{03} \end{vmatrix} \\
\beta^* &= \alpha_4 \gamma_3 - \gamma_4 \alpha_3 = \begin{vmatrix} \eta_{14} & \eta_{13} \\ \eta_{\bar{1}4} & \eta_{\bar{1}3} \end{vmatrix} \\
\alpha^* &= \gamma_4 \beta_3 - \beta_4 \gamma_3 = \begin{vmatrix} \eta_{\bar{1}4} & \eta_{\bar{1}3} \\ \eta_{04} & \eta_{03} \end{vmatrix}
\end{aligned}
\tag{5-7}
$$

And the SO(3) projection is the following:

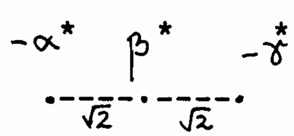


Fig 5-7b

The $\sqrt{2}$ stands for the matrix elements of $L_+(L_-)$. We use α, β, γ for (pq)=(10) and $-\gamma^*$, β^*, α^* for (pq)=(01) understanding that they stand for (5-6) and (5-7). As discussed in Chapter 3 p.51 the "ground state" bottom Sp(6) IR (p,q) has to be "harmonic" and, for this purpose, we have to use traceless variables for the basic SU(3) IR's (1,0) and (0,1). When the antitriplet (0,1) is given in the particle-antiparticle scheme one replaces the triplet variables (α,β,γ), i.e. the unbarred (η_{im}) ones and the antitriplet states $(-\gamma^*,\beta^*,\alpha^*)$, i.e. the barred ($\bar{\eta}_{im}$) ones by their traceless versions given in chapter 3 eq. (3-44).

$$\begin{aligned} \eta_{im} &\dashrightarrow \eta_{im} - B(\hat{N}+3)^{-1}\partial_{\bar{\eta}_{im}} \\ \bar{\eta}_{im} &\dashrightarrow \bar{\eta}_{im} - B(\hat{N}+3)^{-1}\partial_{\eta_{im}} \end{aligned} \tag{5-8}$$

Here B is the SU(3) scalar

$$B = \alpha\alpha^* + \beta\beta^* + \gamma\gamma^* \tag{5-9}$$

and $\hat{N}$ is the total degree operator which gives p+q when acting on an SU(3) IR (p,q). By these replacements we insure that the states do not contain the scalar B; they are in fact orthogonal to B.

The problem of tracelessness does not arise in the two-particles scheme for the antitriplet (0,1). In this scheme the scalar B vanishes when we multiply the antitriplet $(-\gamma^*,\beta^*,\alpha^*)$ of (5-7) with the triplet (α,β,γ) of (5-6).

$$B = \alpha\alpha^* + \beta\beta^* + \gamma\gamma^* = \eta_{14}\begin{vmatrix} \eta_{\bar{1}4} & \eta_{\bar{1}3} \\ \eta_{04} & \eta_{03} \end{vmatrix} + \eta_{04}\begin{vmatrix} \eta_{14} & \eta_{13} \\ \eta_{\bar{1}4} & \eta_{\bar{1}3} \end{vmatrix} - \eta_{\bar{1}4}\begin{vmatrix} \eta_{14} & \eta_{13} \\ \eta_{04} & \eta_{03} \end{vmatrix}$$

$$= \eta_{14}\begin{vmatrix} \eta_{03} & \eta_{04} \\ \eta_{\bar{1}3} & \eta_{\bar{1}4} \end{vmatrix} - \eta_{04}\begin{vmatrix} \eta_{13} & \eta_{14} \\ \eta_{\bar{1}3} & \eta_{\bar{1}4} \end{vmatrix} + \eta_{\bar{1}4}\begin{vmatrix} \eta_{13} & \eta_{14} \\ \eta_{03} & \eta_{04} \end{vmatrix}$$

which is the expansion after the first column of the determinant:

$$\begin{vmatrix} \eta_{14} & \eta_{13} & \eta_{14} \\ \eta_{04} & \eta_{03} & \eta_{04} \\ \eta_{\bar{1}4} & \eta_{\bar{1}3} & \eta_{\bar{1}4} \end{vmatrix} .$$

We just proved that the unwanted SU(3) scalar vanishes identically in the two-particles scheme, in which case the starting Sp(6) states are automatically traceless.

The higher IR's are obtained by taking products of the triplet and antitriplet states given above.

The (pq)=(20) representation is given by

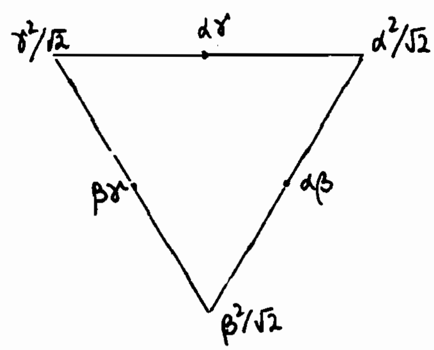


Fig 5-8a

with its SO(3) projection

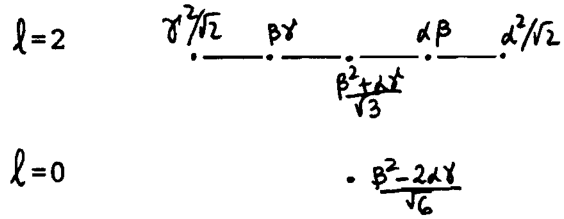


Fig 5-8b

while the (02) representation is

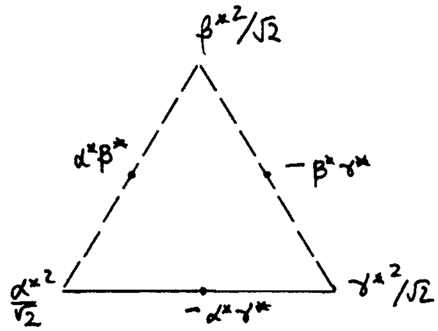


Fig 5-9a

with the SO(3) projection

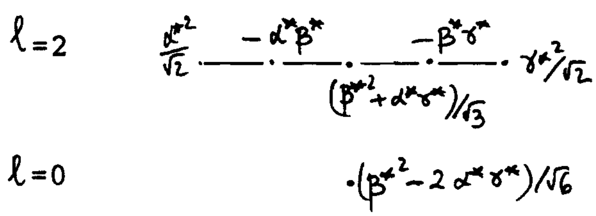


Fig 5-9b

The (pq) = (11) representation is

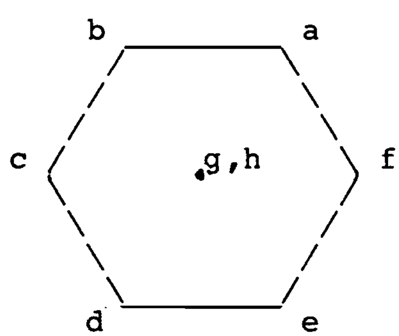


Fig 5-10a

where the letters stand for

$$
\begin{aligned}
a &= \alpha\beta^* \\
b &= \gamma\beta^* \\
c &= \alpha^*\gamma \\
d &= \alpha^*\beta \\
e &= -\beta\gamma^* \\
f &= -\alpha\gamma^* \\
g &= (\alpha\alpha^* - \gamma\gamma^*)/\sqrt{2} \qquad \text{is the } \ell=1,\ m=0 \text{ state} \\
h &= (2\beta\beta^* - \alpha\alpha^* - \gamma\gamma^*)/\sqrt{6} \qquad \text{is the } \ell=2,\ m=0 \text{ state}
\end{aligned}
\tag{5-10}
$$

with the SO(3) content $(11) \supset (2)+(1)$.

$\ell=2$ $\quad -c$. ______ . ______ h ______ . ______ . f

$\qquad\qquad\qquad (b-d)/\sqrt{2} \qquad (a+e)/\ 2$

$\ell=1$ $\quad (b+d)/\sqrt{2}$. ______ . ______ . $(-a+e)/\sqrt{2}$ $\qquad$ Fig 5-10b

$\qquad\qquad\qquad\qquad\qquad g$

To construct the epd's we also need the representations $(a_1 b_1)$ which couple to (pq) to get the final (ab) representations. The $(a_1 b_1)$ are made of the raising generators $B^{\dagger}_{\ell m}$ (see Chapter 3). The raising generators are quadratic in the creation operators such that the $(a_1 b_1)=(20)$ representation is linear in $B^{\dagger}_{\ell m}$ and $(a_1 b_1)=(02)$ is quadratic in $B^{\dagger}_{\ell m}$. The degree in $B^{\dagger}_{\ell m}$ is given by $(a_1/2+b_1)$. Here also we use the traceless versions of $B^{\dagger}_{m}$ called $\beta^{\dagger}_{m}$ in (3-42) and (3-43).

For $(a_1 b_1) = (20)$ we use the sextet

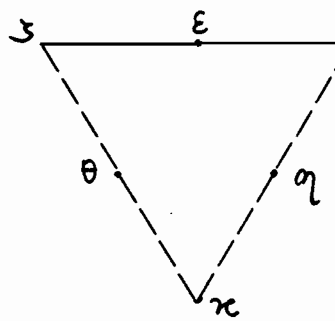


Fig 5-11a

The SO(3) projection $(20) \supset (2) + (0)$

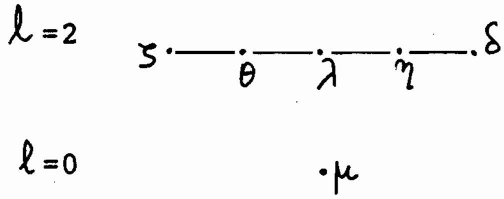


Fig 5-11b

corresponds to the raising generators B^{+}_{2m} $(=B^{+}_{m})$ and B^{+}_{00}. We can identify

$$\delta = B^{+}_{2}\,, \quad \eta = B^{+}_{1}\,, \quad \zeta = B^{+}_{\bar{2}}\,, \quad \theta = B^{+}_{\bar{1}}$$

and (5-11)

$$\lambda = (\sqrt{2}\varkappa + \varepsilon)/\sqrt{3} = B^{+}_{20}\,, \quad \mu = (\varkappa - \sqrt{2}\varepsilon)/\sqrt{3} = B^{+}_{00}\,.$$

Since we are interested in "bottom" Sp(6) states (lowest Sp(2)) we do not use the O(3) scalar B^{+}_{00}, which is the Sp(2) raising generator, to construct our states. Then $\varkappa$ and ε are given by

$$\varepsilon = B^{+}_{20}/\sqrt{3} \quad \text{and} \quad \varkappa = \sqrt{2/3}\; B^{+}_{20} \tag{5-12}$$

To construct the $(02) = (a_1 b_1)$ representation which is quadratic in

B^{+}_{lm}, we take the square of the (20) representation. The highest state (l=m=2) is a linear combination of η^2 and $\delta\varkappa$. The coefficients are determined by requesting that L_+ applied to the state give zero. L_+ is $\sqrt{2}(E_{13}+E_{32})$ (where E_{13}, E_{32} are SU(3) generators, see below the diagram)

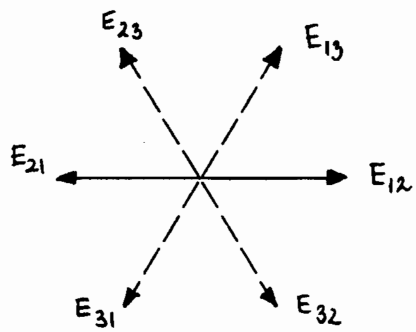


Fig 5-12

With this we get the highest weight state of $(a_1 b_1)=(02)$ proportional to $(\eta^2-2\delta\varkappa)$ and we normalize the state deviding by $\sqrt{6}$. Applying the generators in Fig 5-12 we get the following anti-sextet

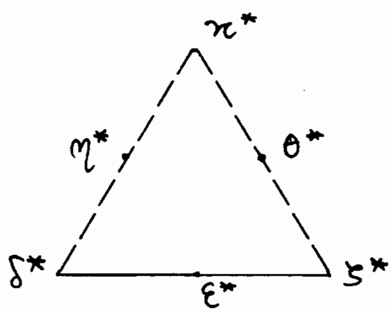


Fig 5-13a

and the corresponding SO(3) projection.

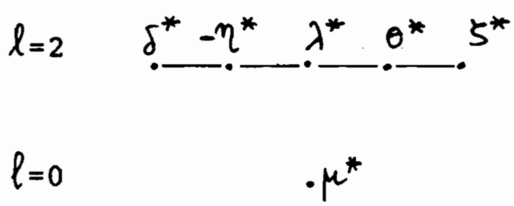


Fig 5-13b

The anti-sextet states are the following second degree combinations in the sextet states

$$
\begin{aligned}
\zeta^* &= (\eta^2 - 2\delta\kappa)/\sqrt{6} \\
\theta^* &= (\eta\varepsilon - \sqrt{2}\,\delta\theta)/\sqrt{3} \\
\kappa^* &= (\varepsilon^2 - 2\delta\zeta)/\sqrt{6} \\
\eta^* &= -(\varepsilon\theta - \sqrt{2}\,\eta\zeta)/\sqrt{3} \\
\delta^* &= (\theta^2 - 2\kappa\zeta)/\sqrt{6} \\
\varepsilon^* &= (\eta\theta - \sqrt{2}\,\varepsilon\kappa)/\sqrt{3} \\
\lambda^* &= (\sqrt{2}\,\kappa^* - \varepsilon^*)/\sqrt{3} = (\varepsilon^2 - 2\delta\zeta - \eta\theta + \sqrt{2}\,\varepsilon\kappa)/3 \\
\mu^* &= (\kappa^* + \sqrt{2}\,\varepsilon^*)/\sqrt{3} = (\varepsilon^2 - 2\delta\zeta + 2\eta\theta - 2\sqrt{2}\,\varepsilon\kappa)/(3\sqrt{2})
\end{aligned}
\tag{5-13}
$$

Using these (pq) and $(a_1 b_1)$ representations we can begin constructing the epd's.

5.4. Examples of epd's

The simplest to construct are those epd's for which either (pq) or $(a_1 b_1)$ labels equal zero, such that no coupling of SU(3) representations is required. Looking in our list we see that K, K*, M, a, a*, b, b*, e, y are in this simple category. For the rest of them one has to couple the (pq) with $(a_1 b_1)$, get the (ab) representation (first the highest state) and take the required ℓ=m part of it. When multiplying two representations $(p_1 q_1)$ and $(p_2 q_2)$ we obtain the representations of the following two types:

I. $(p_1+p_2-a-b-2c,\ q_1+q_2-a-b+c)$

with $0 \leq a \leq p_1,\ q_2$

$0 \leq b \leq q_1,\ p_2$

$0 \leq c \leq p_1-a,\ p_2-b$

II. $(p_1+p_2-a-b+c,\ q_1+q_2-a-b-2c)$

with $0 \leq a \leq p_1,\ q_2$

$0 \leq b \leq q_1,\ p_2$

$1 \leq c \leq q_2-a,\ q_1-b$

To verify if the content of a product of two representations is correct one can check the dimensions on both sides of the multiplication. The dimension of an SU(3) IR (pq) is given by $(p+q+2)(p+1)(q+1)/2$. The rest of this chapter contains examples of explicitly written epd's.

5.4.1. $K=(10,00,10,1)$

Here $(pq)=(10)$ is by itself and one takes the $\ell=m=1$ part. It is clear that $K=\alpha$, i.e. η_{14} like in (5-6).

5.4.2. $K^*=(01,00,01,1)$

This is $(pq)=(01)$ by itself and from (5-7) we see $K^*=-\gamma^*=\begin{vmatrix}\eta_{14} & \eta_{13}\\ \eta_{04} & \eta_{03}\end{vmatrix}$

5.4.3. $M=(00,02,02,0)$

Here $(a_1b_1)=(02)$, the degree in B^{+}_{m} is $z=2$ $(=a_1/2+b_1)$. So we need the $\ell=m=0$ part of the antisextet. One takes a linear combination of $\varkappa^*$ and ε^* as $M=\varkappa^*+A\varepsilon^*$ and determines A imposing

$L_+M=0$. Alternatively one starts with the $\ell=m=2$ state $\zeta^*=(\eta^2-2\delta\kappa)/\sqrt{6}$, applies L_- twice to get the $\ell=2$, m=0 state the λ^* in Fig 4-13b and take M ($\ell=m=0$) as the combination perpendicular to it, i.e. $M=\mu^*$. We get $M=2\delta\zeta-2\eta\theta+3\varepsilon^2$.

5.4.4. a=(20,00,20,0)

To get the epd a we consider the (pq)=(20) and a is the linear combination $\beta^2+A\alpha\gamma$. Impose $L_+(\beta^2+A\alpha\gamma)=2\beta\sqrt{2}\alpha+A\alpha\sqrt{2}\beta=0$, then A=-2 such that $a=\beta^2-2\alpha\gamma$.

5.4.5. a*=(02,00,02,0)

Now we start with (pq)=(02) which is the antisextet in Fig 5-9 with α^*, β^* and $-\gamma^*$ given by (5-7). The epd a* is a linear combination of $\beta^{*2}/\sqrt{2}$ and $-\gamma^*\alpha^*$ such that $L_+a^*=0$. After some simple algebra we get $a^*=\beta^{*2}-2\alpha^*\gamma^*$

5.4.6. J=(02,20,00,0)

To construct J we have to couple (pq)=(02) from Fig 5-9 to $(a_1 b_1)=(20)$ from Fig 5-11. The things are quite simple if we observe that all we need is a scalar ($\ell=m=0$). Projected in SO(3) both (02) and (20) consist of a quintet ($\ell=2$) and a singlet ($\ell=0$). For the $(a_1 b_1)=(20)$ we do not use the singlet, so the only way to get a scalar is to use the two $\ell=2$ projections (from Fig 5-9b and 5-11b), coupling m=2 from one to m=-2 from the other, and so on. We have for J the linear combination:

$$J = \delta\,\alpha^{*2}/\sqrt{2} + \zeta\,\gamma^{*2}/\sqrt{2} + A[\theta(-\beta^*\gamma^*) + \eta(-\alpha^*\beta^*)] + \\ + B\lambda(\beta^{*2} + \alpha^*\gamma^*)/\sqrt{3}$$

The constants A,B are determined requiring $L_+J=0$. We obtain A=-1 and B=+1.

5.4.7. $J^*=(20,02,00,0)$

We couple to (pq)=(20) the $(a_1b_1)=(02)$ quadratic in the raising generators. We make a scalar from the two quintets.

$$J^* = \delta^*\alpha^2/\sqrt{2} + \zeta^*\gamma^2/\sqrt{2} + A[\theta^*\beta\gamma + (-\eta^*)\alpha\beta] + \\ + B\lambda^*(\beta^2 + \alpha\gamma)/\sqrt{3}$$

We require $L_+J^*=0$ and obtain A=-1 and B=+1.

5.4.8. $\alpha = Z^3$

The epd α is a scalar ($\ell=0$) cubic in the raising generators. It is of the form

$$\alpha = \delta\theta^2 + A\delta\lambda\zeta + B\zeta\eta^2 + C\eta\lambda\theta + D\lambda^3$$

We require $L_+\alpha=0$ and obtain $A=-2\sqrt{2/3}$, B=1, $C=-\sqrt{2/3}$ and $D=(1/3)\sqrt{2/3}$.

5.4.9. β=DZ

This is an SO(3) scalar (ℓ=m=0). We take it to be the determinant

$$\begin{vmatrix} \eta_{14} & \eta_{13} & \eta_{12} \\ \eta_{04} & \eta_{03} & \eta_{02} \\ \eta_{\bar{1}4} & \eta_{\bar{1}3} & \eta_{\bar{1}2} \end{vmatrix}$$

5.4.10. b=(00,20,20,2)

This is simply δ of $(a_1 b_1)=(20)$ in Fig 5-11.

5.4.11. b*=(00,02,02,2)

This is ζ^* of Fig 5-13. We take $b^* = \eta^2 - 2\delta\varkappa$.

5.4.12. c=(20,20,02,0)

We have to construct a scalar out of (pq)=(20) in Fig 5-8 and the quintet of the raising generators in Fig 5-11. The scalar is

$$c = \delta\,\gamma^2/\sqrt{2} + \zeta\,\alpha^2/\sqrt{2} + A(\eta\beta\gamma + \theta\alpha\beta) + B\lambda(\beta^2 + \alpha\gamma)/\sqrt{3}$$

Requiring $L_+ c = 0$ we get A=-1, B=1.

5.4.13. d=(20,20,02,2)

We combine the sextets in Fig 5-8 and Fig 5-11 to get an antisextet and d is its highest state. Then d is of the form:

$$d = \varkappa\,\alpha^2/\sqrt{2} + A\,\delta\,\beta^2/\sqrt{2} + B\eta\alpha\beta$$

By requiring $L_+ d = 0$ we get A=+1, B=-1. Observe that if we start with d and apply L_- twice we obtain the |20> state of the antisextet. The epd c is the combination orthogonal to this |20> state.

5.4.14. d*=(02,02,20,2)

Similarly, we construct d* as the highest state of the sextet (20) obtained by multiplying the antisextets in Fig 5-9 and Fig 5-13. Then d* is the combination:

$$d^* = \zeta^*\beta^{*2}/\sqrt{2} + A\varkappa^*\gamma^{*2}/\sqrt{2} + B\theta^*(-\beta^*\gamma^*)$$

By requiring $L_+ d^* = 0$ we get A=+1, B=-1. From d* we get c* if we apply

L_- twice and take the combination orthogonal to this.

5.4.15. e=(11,00,11,1)

This epd is the $m=\ell=1$ state of the (pq)=(11) octet in Fig 5-10. The wanted projection is $(-a+e)=-\alpha\beta^* - \beta\gamma^*$ (here a,e are those given in (5-10)).

5.4.16. f=(10,02,01,1)

We combine the triplet (pq)=(10) with the antisextet (02) in Fig 5-13. The highest state of the resulting triplet (01) is the combination

$$f= \alpha\varepsilon^* + A\beta\theta^* + B\gamma\zeta^*$$

By requiring $E_{12}f=E_{32}f=0$ we get $A=1$, $B=-\sqrt{2}$

5.4.17. f*=(01,20,10,1)

Similar to f we couple (01) of (5-7) to (20) of (5-11) to get a triplet (10).

$$f^*= \delta\alpha^* + A\eta\beta^* + B\,\varepsilon(-\gamma^*)$$

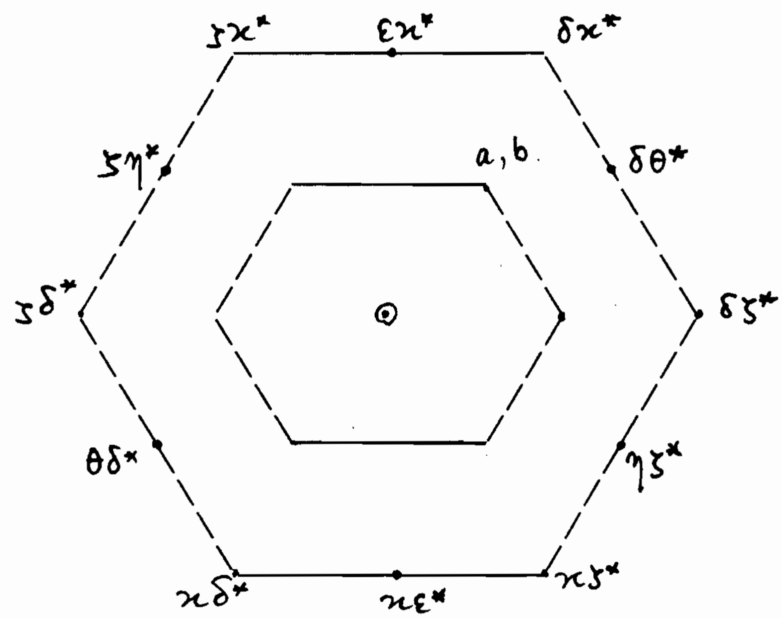
We require $E_{12}f^*=E_{32}f^*=0$ and we obtain $A=1/\sqrt{2}$, $B=-1/\sqrt{2}$. We take

$$f^*= \varepsilon\gamma^* + \eta\beta^* + \sqrt{2}\,\delta\alpha^*$$

We analyze in this way all the epd's in our list and we get their explicit expressions. Some of the epd's are not as simple to construct as the examples given above. Since we do not wish to bore the reader we conclude with an epd which is more complicated to construct.

5.4.18. $x^*=(01,22,12,2)$

We combine the antitriplet (01) of (5-7) with the $(a_1b_1)=(22)$ representation which is the product of the sextet (20) of (5-11) with the antisextet (02) of (5-13). The first step is to explicit the (22) representation of the raising generators. We give it below



$a=\delta\eta^*+\sqrt{2}\,\varepsilon\theta^*$
$b=\eta\varkappa^*+\sqrt{2}\,\varepsilon\theta^*$

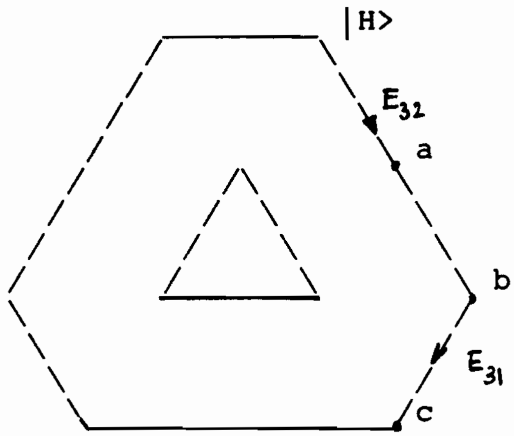
We couple this to the antitriplet (01) to get the representation (12). The highest state is the following combination

$$|H\rangle=\beta^*(\delta\eta^*+\sqrt{2}\,\varepsilon\theta^*)+A\beta^*(\eta\varkappa^*+\sqrt{2}\,\varepsilon\theta^*)+B(-\gamma^*)\varepsilon\varkappa^*+C\alpha^*\delta\varkappa^*$$

We impose $E_{12}|H\rangle=E_{23}|H\rangle=0$ and get $A=-3/2$, $B=5/2$ and $C=-5/\sqrt{2}$. With these

$$|H\rangle=\beta^*(2\delta\eta^*-3\eta\varkappa^*)-\sqrt{2}\,\beta^*\varepsilon\theta^*-5\gamma^*\varepsilon\varkappa^*-5\sqrt{2}\,\alpha^*\delta\varkappa^*.$$

The (21) representation looks like



The SO(3) projection of the (12) is (3)+(2)+(1). The epd x^* is the $m=\ell=2$ state, a linear combination of a and c in the figure above such that $L_+ x^*=0$. The state a is obtained from $|H\rangle$ by applying E_{32}.

$$a=-5\,\alpha^*\delta\theta^*+\beta^*(\delta\varepsilon^*-\varepsilon\zeta^*-2\sqrt{2}\,\eta\theta^*)-\gamma^*(\delta\eta^*+\eta\varkappa^*+2\sqrt{2}\,\varepsilon\theta^*)$$

One applies again E_{32} to get $b=|3,3\rangle$. We then calculate $c=E_{31}\,b$

$$c=-5\,\alpha^*\eta\zeta^*-5\sqrt{2}\,\beta^*\varkappa\zeta^*-\gamma^*(3\,\theta\zeta^*+2\varkappa\theta^*+\sqrt{2}\,\eta\varepsilon^*)$$

Finally $x^*=a+Ac$ such that $L_+ x^*=0$. The result is

$$x^*=5\,\alpha^*(\eta\zeta^*-\delta\theta^*)+\beta^*(5\sqrt{2}\,\varkappa\zeta^*+\delta\varepsilon^*-\varepsilon\zeta^*-2\sqrt{2}\,\eta\theta^*)-$$
$$-\gamma^*(\delta\eta^*+\eta\varkappa^*+2\sqrt{2}\,\varepsilon\theta^*-3\,\theta\zeta^*-2\varkappa\theta^*-\sqrt{2}\,\eta\varepsilon^*)$$

CHAPTER 6. GENERATOR MATRIX ELEMENTS

6.1. The Generators of Sp(6)

The problem we discuss here involves a system of n particles in the 3-dimensional physical space. The phase-space is then 6n-dimensional and the group of linear canonical transformations (i.e. the transformations which leave the Hamiltonian and the Poisson brackets of coordinates and momenta unchanged) is the real symplectic group Sp(6n). This is also the dynamical group of the 3n-dimensional harmonic oscillator.

In the Cartan classification of classical Lie groups Sp(6n) is called C_{3n} of rank 3n and order (number of generators) 3n(6n+1). The Sp(6n) generators are given in (3-2) in terms of Jacobi momenta and coordinates. The 3(6+1)=21 generators of the Sp(6) subgroup are obtained by contracting the Sp(6n) generators of (3-2) over the particle index and one gets as generators the Q_{ij}, L_{ij}, S_{ij} and K_{ij} in Eqs.(2-16) to (2-20). In terms of creation and annihilation operators the Sp(6) generators are the B^{+}_{ij}, C_{ij} and B_{ij} of Eqs.(3-17). They satisfy the following commutation relations

$$[C_{ij}, C_{kl}] = C_{il}\delta_{jk} - C_{jk}\delta_{il}$$

$$[C_{ij}, B^{+}_{kl}] = B_{il}\delta_{jk} + B^{+}_{ik}\delta_{jl}$$

$$[C_{ij}, B_{kl}] = -B_{jl}\delta_{ik} - B_{jk}\delta_{il} \tag{6-1}$$

$$[B^{+}_{ij}, B^{+}_{kl}] = [B_{ij}, B_{kl}] = 0$$

$$[B_{ij}, B^{+}_{kl}] = C_{lj}\delta_{ik} + C_{kj}\delta_{il} + C_{li}\delta_{jk} + C_{ki}\delta_{jl}$$

The B^{+}_{ij}, C_{ij}, B_{ij} relate to the Q_{ij}, L_{ij}, S_{ij}, K_{ij} in the following way:

$$C_{ij} = (Q_{ij} + K_{ij} + iL_{ij})/2$$

$$B^{+}_{ij} = [Q_{ij} - K_{ij} - n\delta_{ij} - i\sum_{s}(X_{is}P_{js} + X_{js}P_{is})]/2 \tag{6-2}$$

$$B_{ij} = [Q_{ij} - K_{ij} + n\delta_{ij} + i\sum_{s}(X_{is}P_{js} + X_{js}P_{is})]/2$$

As we work in a Sp(2)x SO(3) basis of Sp(6) it is very useful to look at the Sp(6) generators in a Sp(2)x SO(3) scheme. The raising B^{+}_{ij} and the lowering B_{ij} generators project in SO(3) as a quintet $\ell=2$ and a singlet $\ell=0$. The $\ell=0$ raising B^{+}_{00} and the $\ell=0$ lowering B_{00} generators are respectively the raising and lowering Sp(2) generators called I_{+} and I_{-}.

$$B^{\dagger}_{00} \sim I_{+} = I_1 + iI_2$$

$$B_{00} \sim I_{-} = I_1 - iI_2$$

where I_1, I_2, I_3 are the Sp(2) generators like those in (3-13) but now expressed in terms of creation and annihilation operators. They obey the commutation rules (3-14). From the Sp(2) point of view the raising generators $B^{\dagger}_{\ell m}$ and the lowering ones $B_{\ell m}$ are respectively the M=+1 and M=-1 of an Sp(2) triplet $|I=1,M\rangle$. The M=0 part of the Sp(2) triplet is the SO(3) quintet (ℓ=2) made out of the SU(3) generators E_{ij} shown in Fig. 5-12.

$$\begin{array}{lcccccc}
 & & B^{\dagger}_{\bar{2}} & B^{\dagger}_{\bar{1}} & B^{\dagger}_{0} & B^{\dagger}_{1} & B^{\dagger}_{2} \\
 & M=+1 & * & * & * & * & * \\
 & & Q_{\bar{2}} & Q_{\bar{1}} & Q_{0} & Q_{1} & Q_{2} \\
Sp(2) & M=0 & * & * & * & * & * \\
(I=1) & & B_{\bar{2}} & B_{\bar{1}} & B_{0} & B_{1} & B_{2} \\
 & M=-1 & * & * & * & * & * \\
 & & m=-2 & m=-1 & m=0 & m=+1 & m=+2
\end{array}$$

$$\begin{array}{c} SO(3) \\ (\ell=2) \end{array}$$

Fig 6-1 The Sp(6) generators under Sp(2)XSO(3)

The SU(3) generators form an octet (1,1) in Figure 5-12 which projects in SO(3) as a quintet Q_m which is the middle row in the above figure and a triplet, the usual angular momentum operators L_1,

L_{-1}, L_0. Thus we organize the 21 Sp(6) generators into the Sp(2)×SO(3) fifteenplet in Figure 6-1 and two triplets shown below in Figure 6-2.

$$\begin{array}{lllccc}
M=+1 & * \; I_1 & & & & \\
M=0 & * \; I_0 & ; & \underset{L_{-1}}{*} & \underset{L_0}{*} & \underset{L_{+1}}{*} \\
M=-1 & * \; I_{-1} & & m=-1 & m=0 & m=1
\end{array}$$

Figure 6-2 The Sp(2) and the SO(3) generators

For the purpose of our calculation we write explicitly the generators in the fifteenplet in a consistent way. This is done by requiring that the middle row of the SU(3) generators have their usual expressions. The members of the fifteenplet are related by I_+, I_- in the Sp(2) direction and L_+, L_- in the SO(3) direction. We take as usual L_+ and L_- to have the following matrix elements

$$L_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \qquad L_- = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \tag{6-3}$$

In terms of the creation operators we write:

$$L_+ = \sqrt{2}\,(E_{13} + E_{32}) = \sqrt{2}\sum_s (\eta_{1s}\partial_{0s} + \eta_{0s}\partial_{\bar{1}s})$$
$$L_- = \sqrt{2}\,(E_{31} + E_{23}) = \sqrt{2}\sum_s (\eta_{0s}\partial_{1s} + \eta_{\bar{1}s}\partial_{0s}) \qquad (6\text{-}4)$$
and
$$L_0 = \sum_s (\eta_{1s}\partial_{1s} - \eta_{\bar{1}s}\partial_{\bar{1}s})$$

They obey the commutation rules

$$[L_0, L_\pm] = \pm L_\pm\,, \quad [L_+, L_-] = 2L_0 \qquad (6\text{-}5)$$

or, for L_1, L_0, L_{-1}

$$[L_m, L_n] = -\sqrt{2}\left\langle \begin{matrix} 1 & 1 \\ m & n \end{matrix} \right| \left. \begin{matrix} 1 \\ m+n \end{matrix} \right\rangle L_{m+n} \qquad (6\text{-}6)$$

with the SO(3) Clebsch-Gordan coefficients:

$$\left\langle \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix} \right| \left. \begin{matrix} 1 \\ 1 \end{matrix} \right\rangle = \left\langle \begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix} \right| \left. \begin{matrix} 1 \\ 0 \end{matrix} \right\rangle = \left\langle \begin{matrix} 1 & 1 \\ 0 & -1 \end{matrix} \right| \left. \begin{matrix} 1 \\ -1 \end{matrix} \right\rangle = \frac{1}{\sqrt{2}} \qquad (6\text{-}7a)$$

$$\left\langle \begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix} \right| \left. \begin{matrix} 1 \\ 1 \end{matrix} \right\rangle = \left\langle \begin{matrix} 1 & 1 \\ -1 & 1 \end{matrix} \right| \left. \begin{matrix} 1 \\ 0 \end{matrix} \right\rangle = \left\langle \begin{matrix} 1 & 1 \\ -1 & 0 \end{matrix} \right| \left. \begin{matrix} 1 \\ -1 \end{matrix} \right\rangle = -\frac{1}{\sqrt{2}} \qquad (6\text{-}7b)$$

The angular momentum components are given by $L_1 = -L_+/\sqrt{2}$, $L_{-1} = L_-/\sqrt{2}$ and $L_0 = E_{11} - E_{22}$. In a similar way, the Sp(2) raising and lowering generators have the following matrix elements

$$I_+ = i\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \qquad I_- = i\begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \qquad (6\text{-}8)$$

and the triplet components in Fig 6-2: $I_1 = iI_+/\sqrt{2}$, $I_{-1} = -iI_-/\sqrt{2}$. The I_+, I_- of (6-8) have the following expressions in terms of creation operators

$$\begin{aligned} I_+ &= (1/2)\sum_s (\eta_{0s}^2 - 2\eta_{1s}\eta_{\bar{1}s}) \\ I_- &= (1/2)\sum_s (\partial_{0s}^2 - 2\partial_{1s}\partial_{\bar{1}s}) \qquad (6\text{-}9) \\ \text{and} \quad I_0 &= (1/2)\sum_s (\eta_{0s}\partial_{0s} + \eta_{1s}\partial_{1s} + \eta_{\bar{1}s}\partial_{\bar{1}s}) + (3/4)n = (1/2)\hat{N} + (3/4)n \end{aligned}$$

They obey the commutation rules

$$[I_0, I_\pm] = \pm I_\pm, \quad [I_-, I_+] = 2I_0 \qquad (6\text{-}10)$$

or, in terms of I_y, I_z $(y,z=-1,0,1)$:

$$[I_y, I_z] = -\sqrt{2}\left\langle \begin{matrix} 1 & 1 \\ y & z \end{matrix} \middle| \begin{matrix} 1 \\ y+z \end{matrix} \right\rangle I_{y+z} \qquad (6\text{-}11)$$

The Sp(2) Clebsch-Gordan coefficients in (6-11) happens to be the same with the SO(3) ones in (6-7a,b).

Now we turn our attention towards the fifteenplet in Fig 6-1 and we want the middle row, Q_m to be the $\ell=2$ projection of the known SU(3) generators:

$$Q_2 = E_{12}$$
$$Q_1 = (1/2)[L_-, Q_2] = (E_{32} - E_{13})/\sqrt{2}$$
$$Q_0 = (1/\sqrt{6})[L_-, Q_1] = (1/\sqrt{6})(E_{11} + E_{22} - 2E_{33}) \tag{6-12}$$
$$Q_{\bar{1}} = (1/\sqrt{6})[L_-, Q_0] = (E_{31} - E_{23})/\sqrt{2}$$
$$Q_{\bar{2}} = E_{21}$$

In terms of the creation and annihilation operators we rewrite the middle row of the fifteenplet:

$$Q_2 = E_{12} = \sum_s \eta_{1s} \partial_{\bar{1}s}$$
$$Q_1 = (1/\sqrt{2})(E_{32} - E_{13}) = (1/\sqrt{2}) \sum_s (\eta_{0s}\partial_{\bar{1}s} - \eta_{1s}\partial_{0s})$$
$$Q_0 = (1/\sqrt{6})(E_{11} + E_{22} - 2E_{33}) = (1/\sqrt{6}) \sum_s (\eta_{1s}\partial_{1s} + \eta_{\bar{1}s}\partial_{\bar{1}s} - 2\eta_{0s}\partial_{0s})$$
$$Q_{\bar{1}} = (1/\sqrt{2})(E_{31} - E_{23}) = (1/\sqrt{2}) \sum_s (\eta_{0s}\partial_{1s} - \eta_{\bar{1}s}\partial_{0s}) \tag{6-13}$$
$$Q_{\bar{2}} = E_{21} = \sum_s \eta_{\bar{1}s}\partial_{1s}$$

As stressed here, the members of the fifteenplet are related by the Sp(2) raising and lowering generators in the "vertical" Sp(2) direction and by the L_+, L_- in the SO(3) direction. Then the B_2^+ has to be

$$B_2^+ = [I_+, Q_2]/(i\sqrt{2}) = (-i/\sqrt{2}) \sum_s \eta_{1s}^2$$

The rest of the quintet follows either by applying L_- repeatedly to B_2^+ or, applying I_+ to the quintet Q_m:

$$
\begin{aligned}
B_2^{+} &= [I_+,\ Q_2]/(i\sqrt{2}) = (-i/\sqrt{2})\sum_s \eta_{1s}^2 \\
B_1^{+} &= [I_+,\ Q_1]/(i\sqrt{2}) = (1/2)[L_-,\ B_2^{+}] = -i\sum_s \eta_{1s}\eta_{0s} \\
B_0^{+} &= [I_+,\ Q_0]/(i\sqrt{2}) = (1/\sqrt{6})[L_-,\ B_1^{+}] = (-i/\sqrt{3})\sum_s (\eta_{0s}^2 + \eta_{1s}\eta_{\bar{1}s}) \\
B_{\bar{1}}^{+} &= [I_+,\ Q_{\bar{1}}]/(i\sqrt{2}) = (1/\sqrt{6})[L_-,\ B_0^{+}] = -i\sum_s \eta_{0s}\eta_{\bar{1}s} \\
B_{\bar{2}}^{+} &= [I_+,\ Q_{\bar{2}}]/(i\sqrt{2}) = (1/2)[L_-,\ B_{\bar{1}}^{+}] = (-i/\sqrt{2})\sum_s \eta_{\bar{1}s}^2
\end{aligned}
\tag{6-14}
$$

The quintet of the lowering generators B_m are also related to Q_m and they are obtained taking the commutator with I_- :

$$
\begin{aligned}
B_2 &= [I_-,\ Q_2]/(i\sqrt{2}) = (i/\sqrt{2})\sum_s \partial_{\bar{1}s}^2 \\
B_1 &= [I_-,\ Q_1]/(i\sqrt{2}) = (1/2)[L_-,\ B_2] = -i\sum_s \partial_{0s}\partial_{\bar{1}s} \\
B_0 &= [I_-,\ Q_0]/(i\sqrt{2}) = (1/\sqrt{6})[L_-,\ B_1] = (i/\sqrt{3})\sum_s (\partial_{0s}^2 + \partial_{1s}\partial_{\bar{1}s}) \\
B_{\bar{1}} &= [I_-,\ Q_{\bar{1}}]/(i\sqrt{2}) = (1/\sqrt{6})[L_-,\ B_0] = -i\sum_s \partial_{1s}\partial_{0s} \\
B_{\bar{2}} &= [I_-,\ Q_{\bar{2}}]/(i\sqrt{2}) = (1/2)[L_-,\ B_{\bar{1}}] = (i/\sqrt{2})\sum_s \partial_{1s}^2
\end{aligned}
\tag{6-15}
$$

We got now a consistent picture of the Sp(6) generators under the Sp(2)xSO(3) subgroup. We look now for the commutation relations between the Sp(6) generators. Of course the commutation rules given in (6-1) are perfectly valid but they refer to the B_{ij}^{+} , B_{ij} and C_{ij} , the Sp(6) generators in Cartesian components (i,j=1,2,3). The commutation relations inside the SO(3) triplet L_m are given in (6-6) while those between the Sp(2) triplet I_z are given in (6-11). For a more compact writing we call the members of the fifteenplet in Fig 6-1 generically B_{zm}. The first subscript is the Sp(2) projection (z=-1,0,1) and the second subscript refers to the SO(3) projection (m=-2,-1,0,1,2). Thus the Q_m of (6-13) are the B_{0m}, the B_m^{+} of

(6-14) the new B_{1m} and the lowering B_m are now called $B_{\bar{1}m}$. The I_y have the SO(3) label zero while the L_m are Sp(2) scalars. Thus when we commute I_y with B_{zm} the coupling is done only in Sp(2) while when we commute L_m with B_{zm} the coupling is an SO(3) one. We deduce the following commutation rules

$$[I_y, B_{zm}] = -\sqrt{2} \left\langle \begin{matrix} 1 & 1 \\ y & z \end{matrix} \middle| \begin{matrix} 1 \\ y+z \end{matrix} \right\rangle B_{y+z,m} \tag{6-16}$$

$$[L_m, B_{yn}] = -\sqrt{6} \left\langle \begin{matrix} 1 & 2 \\ m & n \end{matrix} \middle| \begin{matrix} 2 \\ m+n \end{matrix} \right\rangle B_{y,m+n} \tag{6-17}$$

The Sp(2) Clebsch-Gordan (CG) coefficients in (6-16) are those already given in (6-7a,b). The CG coefficients which appear in (6-17) are given below:

$$\left\langle \begin{matrix} 1 & 2 \\ 1 & 1 \end{matrix} \middle| \begin{matrix} 2 \\ 2 \end{matrix} \right\rangle = \frac{1}{\sqrt{3}} \quad \left\langle \begin{matrix} 1 & 2 \\ 0 & 2 \end{matrix} \middle| \begin{matrix} 2 \\ 2 \end{matrix} \right\rangle = -\frac{\sqrt{2}}{\sqrt{3}} \quad \left\langle \begin{matrix} 1 & 2 \\ 1 & 0 \end{matrix} \middle| \begin{matrix} 2 \\ 1 \end{matrix} \right\rangle = \frac{1}{\sqrt{2}} \tag{6-18a}$$

$$\left\langle \begin{matrix} 1 & 2 \\ 0 & 1 \end{matrix} \middle| \begin{matrix} 2 \\ 1 \end{matrix} \right\rangle = -\frac{1}{\sqrt{6}} \quad \left\langle \begin{matrix} 1 & 2 \\ -1 & 2 \end{matrix} \middle| \begin{matrix} 2 \\ 1 \end{matrix} \right\rangle = -\frac{1}{\sqrt{3}} \quad \left\langle \begin{matrix} 1 & 2 \\ 1 & -1 \end{matrix} \middle| \begin{matrix} 2 \\ 0 \end{matrix} \right\rangle = \frac{1}{\sqrt{2}} \tag{6-18b}$$

$$\left\langle \begin{matrix} 1 & 2 \\ -1 & 1 \end{matrix} \middle| \begin{matrix} 2 \\ 0 \end{matrix} \right\rangle = -\frac{1}{\sqrt{2}} \quad \left\langle \begin{matrix} 1 & 2 \\ 0 & -1 \end{matrix} \middle| \begin{matrix} 2 \\ -1 \end{matrix} \right\rangle = \frac{1}{\sqrt{6}} \quad \left\langle \begin{matrix} 1 & 2 \\ 1 & -2 \end{matrix} \middle| \begin{matrix} 2 \\ -1 \end{matrix} \right\rangle = \frac{1}{\sqrt{3}} \tag{6-18c}$$

$$\left\langle\begin{matrix}1 & 2\\ -1 & 0\end{matrix}\middle|\begin{matrix}2\\ -1\end{matrix}\right\rangle=-\frac{1}{\sqrt{2}}\quad\left\langle\begin{matrix}1 & 2\\ 0 & -2\end{matrix}\middle|\begin{matrix}2\\ -2\end{matrix}\right\rangle=\frac{\sqrt{2}}{\sqrt{3}}\quad\left\langle\begin{matrix}1 & 2\\ -1 & -1\end{matrix}\middle|\begin{matrix}2\\ -2\end{matrix}\right\rangle=-\frac{1}{\sqrt{3}} \tag{6-18d}$$

The commutator of two B_{ym} involves coupling in both Sp(2) and SO(3) directions. As a result one gets three terms containing B_{ym}, I_y and L_m:

$$\begin{aligned}[B_{ym}, B_{zn}] = &-\frac{\sqrt{42}}{3}\left\langle\begin{matrix}1 & 1\\ y & z\end{matrix}\middle|\begin{matrix}1\\ y+z\end{matrix}\right\rangle\left\langle\begin{matrix}2 & 2\\ m & n\end{matrix}\middle|\begin{matrix}2\\ m+n\end{matrix}\right\rangle B_{y+z,\, m+n}\\ &-\frac{4\sqrt{10}}{3}\left\langle\begin{matrix}1 & 1\\ y & z\end{matrix}\middle|\begin{matrix}1\\ y+z\end{matrix}\right\rangle\left\langle\begin{matrix}2 & 2\\ m & n\end{matrix}\middle|\begin{matrix}0\\ 0\end{matrix}\right\rangle I_{y+z}\\ &-\frac{\sqrt{30}}{2}\left\langle\begin{matrix}1 & 1\\ y & z\end{matrix}\middle|\begin{matrix}0\\ 0\end{matrix}\right\rangle\left\langle\begin{matrix}2 & 2\\ m & n\end{matrix}\middle|\begin{matrix}1\\ m+n\end{matrix}\right\rangle L_{m+n}\end{aligned} \tag{6-19}$$

Of course $[I_y, L_m]=0$. The Sp(2) CG coefficients in (6-19) are given below

$$\left\langle\begin{matrix}1 & 1 & 0\\ 1 & -1 & 0\end{matrix}\right\rangle=\left\langle\begin{matrix}1 & 1 & 0\\ -1 & 1 & 0\end{matrix}\right\rangle=\frac{1}{\sqrt{3}}\qquad\left\langle\begin{matrix}1 & 1 & 0\\ 0 & 0 & 0\end{matrix}\right\rangle=-\frac{1}{\sqrt{3}} \tag{6-20}$$

We give below the SO(3) CG coefficients which appear in (6-19). For the coefficient of $B_{\zeta n}$ we need

$$\left\langle \begin{matrix} 2 & 2 \\ 2 & -2 \end{matrix} \middle| \begin{matrix} 2 \\ 0 \end{matrix} \right\rangle = \left\langle \begin{matrix} 2 & 2 \\ -2 & 2 \end{matrix} \middle| \begin{matrix} 2 \\ 0 \end{matrix} \right\rangle = \frac{2}{\sqrt{14}} \qquad \left\langle \begin{matrix} 2 & 2 \\ 0 & 0 \end{matrix} \middle| \begin{matrix} 2 \\ 0 \end{matrix} \right\rangle = -\frac{2}{\sqrt{14}} \tag{6-21a}$$

$$\left\langle \begin{matrix} 2 & 2 \\ -1 & 1 \end{matrix} \middle| \begin{matrix} 2 \\ 0 \end{matrix} \right\rangle = \left\langle \begin{matrix} 2 & 2 \\ 1 & -1 \end{matrix} \middle| \begin{matrix} 2 \\ 0 \end{matrix} \right\rangle = \frac{1}{\sqrt{14}}$$

while for the one of $I_{\gamma+\gamma}$

$$\left\langle \begin{matrix} 2 & 2 \\ 2 & -2 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \end{matrix} \right\rangle = \left\langle \begin{matrix} 2 & 2 \\ -2 & 2 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \end{matrix} \right\rangle = -\left\langle \begin{matrix} 2 & 2 \\ 1 & -1 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \end{matrix} \right\rangle = -\left\langle \begin{matrix} 2 & 2 \\ -1 & 1 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \end{matrix} \right\rangle = \left\langle \begin{matrix} 2 & 2 \\ 0 & 0 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \end{matrix} \right\rangle = \frac{1}{\sqrt{5}}$$

and the one of L_{m+n}

$$\left\langle \begin{matrix} 2 & 2 \\ 2 & -2 \end{matrix} \middle| \begin{matrix} 1 \\ 0 \end{matrix} \right\rangle = -\left\langle \begin{matrix} 2 & 2 \\ -2 & 2 \end{matrix} \middle| \begin{matrix} 1 \\ 0 \end{matrix} \right\rangle = \frac{2}{\sqrt{10}} \quad , \quad \left\langle \begin{matrix} 2 & 2 \\ 0 & 0 \end{matrix} \middle| \begin{matrix} 1 \\ 0 \end{matrix} \right\rangle = 0 \tag{6-21c}$$

$$\left\langle \begin{matrix} 2 & 2 \\ 1 & -1 \end{matrix} \middle| \begin{matrix} 1 \\ 0 \end{matrix} \right\rangle = -\left\langle \begin{matrix} 2 & 2 \\ -1 & 1 \end{matrix} \middle| \begin{matrix} 1 \\ 0 \end{matrix} \right\rangle = -\frac{1}{\sqrt{10}} .$$

With this we exhausted the commutation relations among the Sp(6) generators. When we construct the basis states as shown in Chapter 5 we use the quintet of the raising generators $B^{\dagger}_{m}$. To ensure that the basis states are bottom Sp(2) states the raising generators we use to construct them have to be traceless. Thus the raising generators we call in Chapter 5 $\delta, \eta, \lambda, \theta$ and ζ are the traceless part of the Sp(6) raising generators $B^{\dagger}_{m}$ in (6-14) and are given by

(3-43) on page 50.

$$\beta^{+}_{m} = B^{+}_{m} - (2,0)(2\hat{N}+3n)^{-1} Q_{m} + (2,0)^{2} [(2\hat{N}+3n+2)(2\hat{N}+3n+4)]^{-1} B_{m}$$

The traceless part of the raising generators (the β^{+}_{m} above) are obtained from the Sp(6) raising generators $B^{+}_{m}=B_{1m}$ given in (6-14) by subtracting a term proportional to $Q_{m}=B_{0m}$ given in (6-13) and one proportional to $B_{m}=B_{\bar{1}m}$ given in (6-15). The SO(3) scalar (2,0) in the equation above is

$$(2,0) = \sum_{s} (\eta_{0s}^{2} - 2\eta_{1s}\eta_{\bar{1}s}) = 2I_{+}$$

6.2. Reduced Matrix Elements of the Sp(6) Generators.

In the collective symplectic model collectivity means restricting to a definite IR of O(n) which, by complementarity, is equivalent to restricting to an IR of Sp(6) and the collective Hamiltonian is in the enveloping algebra of Sp(6), i.e., it is a function of the Sp(6) generators. The main problem is then to calculate the matrix elements of the Sp(6) generators in the chosen basis.

The Sp(6) basis states we are working with are products of powers of epd's as given by the generating function in (4-22). The epd's, and consequently the basis states, are taken as highest SO(3) (m=L) and lowest Sp(2) ("bottom" states, M=I) in their multiplet. To obtain the rest of the multiplet one applies the lowering SO(3) operator L_- and the Sp(2) raising one $B^{\dagger}_{00} \sim I_+$. Clearly, the raising SO(3) L_+ and the lowering Sp(2) $B_{00} \sim I_-$ generators give zero when applied. Diagrammatically, our states are in the lowest right corner of their Sp(6) IR.

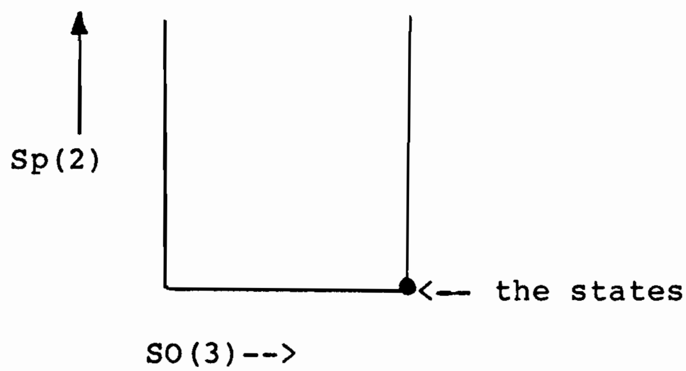


Fig 6-3 The basis states as lowest Sp(2) and highest SO(3) states

For the members of the same SO(3) multiplet, we can use the Wigner-Eckart theorem to calculate the reduced matrix elements for the generators. As stated at the beginning of Chapter 5 we use non-orthonormal states and define the matrix elements of an operator Ω as the coefficient Ω_{ji} in $\Omega|i\rangle = \sum_j |j\rangle \Omega_{ji}$. This does not affect the validity of the Wigner-Eckart theorem. Let us assume that we apply the operator A^{L}_{m} (in our case $L=2$ and m refers to any component) to a state characterized by L_1, m_1 and other quantum numbers α_1

$$A^{l}_{m}\left|\begin{matrix}L_1\\ m_1\end{matrix}\;\alpha_1\right\rangle$$

Multiplying with the CG coefficient

$$\left\langle\begin{matrix}L & L_1\\ m & m_1\end{matrix}\,\middle|\,\begin{matrix}L_2\\ m_2\end{matrix}\right\rangle$$

the whole transforms like $|L_2, m_2\rangle$, i.e.,

$$\sum_{\substack{m_1\\ (m_2=m+m_1)}} A^{L}_{m}\left|\begin{matrix}L_1\\ m_1\end{matrix}\;\alpha_1\right\rangle\left\langle\begin{matrix}L & L_1\\ m & m_1\end{matrix}\,\middle|\,\begin{matrix}L_2\\ m_2\end{matrix}\right\rangle = \sum_{\alpha_2}\left|\begin{matrix}L_2\\ m_2\end{matrix}\;\alpha_2\right\rangle C^{L_2}_{\alpha_2}$$

We multiply both sides by the CG coefficient

$$\left\langle\begin{matrix}L_2\\ m_2\end{matrix}\,\middle|\,\begin{matrix}L & L_1\\ m' & m_1'\end{matrix}\right\rangle$$

and sum over L_2, m_2. Using the property

$$\sum_{L_2, m_2}\left\langle\begin{matrix}L & L_1\\ m & m_1\end{matrix}\,\middle|\,\begin{matrix}L_2\\ m_2\end{matrix}\right\rangle\left\langle\begin{matrix}L_2\\ m_2\end{matrix}\,\middle|\,\begin{matrix}L & L_1\\ m' & m_1'\end{matrix}\right\rangle = \delta_{mm'}\,\delta_{m_1m_1'}$$

we obtain

$$\sum_{m_1} A^{L}_{m}\left|\begin{matrix}L_1\\ m_1\end{matrix}\;\alpha_1\right\rangle\delta_{mm'}\delta_{m_1m_1'} = \sum_{L_2,\alpha_2}\left|\begin{matrix}L_2\\ m_2\end{matrix}\;\alpha_2\right\rangle\left\langle\begin{matrix}L & L_1\\ m' & m_1'\end{matrix}\,\middle|\,\begin{matrix}L_2\\ m_2\end{matrix}\right\rangle C^{L_2}_{\alpha_2}$$

and dropping the primes (with of course $m_2=m+m_1$)

$$A^{L}_{m}\left|\begin{matrix}L_1\\ m_1\end{matrix}\;\alpha_1\right\rangle = \sum_{L_2,\alpha_2}\left|\begin{matrix}L_2\\ m_2\end{matrix}\;\alpha_2\right\rangle\left\langle\begin{matrix}L & L_1\\ m & m_1\end{matrix}\,\middle|\,\begin{matrix}L_2\\ m_2\end{matrix}\right\rangle C^{L_2}_{\alpha_2}$$

On the right-hand side we obtain a linear combination of states

$$\Omega \mid i > = \sum_j \mid j > \Omega_{ji}$$

and the coefficients

$$\Omega_{ji} = \left\langle \begin{matrix} L & L_1 \\ m & m_1 \end{matrix} \middle| \begin{matrix} L_2 \\ m_2 \end{matrix} \right\rangle C^{L_2}_{\alpha_2} = (L_2 \; m_2 \; \alpha_2 \mid A^L_m \mid L_1 \; m_1 \; \alpha_1)$$

are the matrix elements of the operator A^L_m between the states $|L_1, m_1, \alpha_1>$ and $|L_2, m_2, \alpha_2>$. We use round parentheses to stress the fact that our basis is not orthonormal. This is essentially the Wigner-Eckart theorem: the m-dependence of the matrix element of an arbitrary tensor operator is given by a CG coefficient. The m-independent part $C^{L_2}_{\alpha_2}$ is called "reduced matrix element" (or "double-barred") and is obtained by dividing the matrix element by the corresponding CG coefficient.

$$(L_2 \, m_2 \, \alpha_2 \,| A^L_m |\, L_1 \, m_1 \, \alpha_1) = \frac{(L_2 \, \alpha_2 \,||A^L||\, L_1 \, \alpha_1)}{\sqrt{2L_2 + 1}} \left\langle \begin{matrix} L & L_1 \\ m & m_1 \end{matrix} \middle| \begin{matrix} L_2 \\ m_2 \end{matrix} \right\rangle \quad (6\text{-}22)$$

To render the reduced matrix elements more symmetric for different L_2 values one defines them with a supplementary factor $\sqrt{2L_2+1}$ [Go66]. If the basis were orthonormal (which is not our case) we would multiply both sides by the bra $<L_2, m_2, \alpha_2|$ and get the usual form of the theorem

$$\langle L_2\, m_2\, \alpha_2 |\, A^L_m \,|\, L_1\, m_1\, \alpha_1 \rangle = \frac{\langle L_2 \alpha_2 \| A^L \| L_1 \alpha_1 \rangle}{\sqrt{2L_2+1}} \left\langle \begin{matrix} L & L_1 \\ m & m_1 \end{matrix} \middle| \begin{matrix} L_2 \\ m_2 \end{matrix} \right\rangle$$

The Wigner-Eckart theorem shows that, independent of the operator component we use and the m-values of the states, we can calculate the reduced matrix elements which depend on the nature of the operator and the L_1, L_2 of the states involved and not on the orientation of the coordinate frame (m values). Thus for the operators in an SO(3) multiplet B^+_m, we need to calculate the matrix elements for only one component, say B^+_2. Dividing by the appropriate Clebsch-Gordan coefficient, we get the reduced matrix elements which are the same for all m-components of the tensor operator. We use the Wigner-Eckart theorem also in the Sp(2) direction and we need the Sp(2) CG coefficients. In this way, we characterize the whole fifteenplet in Fig 6-1 by one reduced matrix element.

6.3. Clebsch-Gordan Coefficients for SO(3) and Sp(2)

We need CG coefficients for both SO(3) and Sp(2) in order to calculate the matrix elements of the fifteen generators. For SO(3) we relate first the CG coefficients to the Wigner coefficients

$$\left\langle \begin{matrix} T_1 & T_2 \\ M_1 & M_2 \end{matrix} \middle| \begin{matrix} T_3 \\ -M_3 \end{matrix} \right\rangle = (-1)^{T_1-T_2-T_3} \sqrt{2T_3+1} \begin{pmatrix} T_1 & T_2 & T_3 \\ M_1 & M_2 & M_3 \end{pmatrix} \qquad (6\text{-}23)$$

Since we apply the Sp(6) generators which are members of an SO(3) quintet (L=2) to the initial state $|i\rangle = |L_1, m_1\rangle$ and get the final state $|L_2, m_2\rangle$, the L_2 is greater than L_1 by at most 2. We further transform the Wigner array in (6-23) into the so-called Regge form :

$$\begin{pmatrix} T_1 & T_2 & T_3 \\ & & \\ M_1 & M_2 & M_3 \end{pmatrix} = \begin{pmatrix} T_2+T_3-T_1 & T_1+T_3-T_2 & T_1+T_2-T_3 \\ T_1-M_1 & T_2-M_2 & T_3-M_3 \\ T_1+M_1 & T_2+M_2 & T_3+M_3 \end{pmatrix} \tag{6-24}$$

Since $L_2 \leq L_1+2$, the Regge form in (6-24) has at least one element on the first row smaller than 2 and we use the formulae (6-25), (6-26) and (6-27) to calculate them:

$$\begin{pmatrix} a & b & 0 \\ c & d & e \\ f & g & h \end{pmatrix} = (-1)^{d+f} \sqrt{\frac{a!\ b!\ e!\ h!}{c!\ d!\ f!\ g!\ (a+b+1)!}} \tag{6-25}$$

$$\begin{pmatrix} a & b & 1 \\ c & d & e \\ f & g & h \end{pmatrix} = (-1)^{d+f} (fd-cg) \sqrt{\frac{a!\ b!\ e!\ h!}{c!\ d!\ f!\ g!\ (a+b+2)!}} \tag{6-26}$$

$$\begin{pmatrix} a & b & 2 \\ c & d & e \\ f & g & h \end{pmatrix} = (-1)^{d+f} [d(d-1)f(f-1)-2dfcg+c(c-1)g(g-1)] \times \tag{6-27}$$

$$\times \sqrt{\frac{a!\ b!\ e!\ h!}{2c!\ d!\ f!\ g!\ (a+b+3)!}}$$

For Sp(2) the CG coefficients were not calculated before. The Sp(6) generators form an Sp(2) triplet and thus are a finite IR of label I=1. The states on which we apply the generators (the basis states of our problem) are SO(3) Sp(2) multiplets labelled by their highest SO(3) and lowest Sp(2) member as shown in Fig 6-3. These multiplets extend to infinity in the "vertical" Sp(2) direction. They form infinite Sp(2) IR's of label I and the projection is $M=I,I+1,\ldots\infty$ $(I\geq 1)$. We need the CG coefficients which couple the finite Sp(2) IR of the generators to the infinite Sp(2) IR of the states. The generators can change the Sp(2) label of the state by -1,0 or +1. Starting with the state $|I,M\rangle$ we end up with $|I_f,M_f\rangle$, where I_f can be only I+1, I or I-1, depending on whether the generator is raising, SU(3) (middle row) or lowering one. Thus we need to calculate CG coefficients of the form

$$\left\langle \begin{matrix} 1 & I \\ -1 & M+1 \end{matrix} \middle| \begin{matrix} I_f \\ M \end{matrix} \right\rangle , \left\langle \begin{matrix} 1 & I \\ 0 & M \end{matrix} \middle| \begin{matrix} I_f \\ M \end{matrix} \right\rangle , \left\langle \begin{matrix} 1 & I \\ 1 & M-1 \end{matrix} \middle| \begin{matrix} I_f \\ M \end{matrix} \right\rangle \qquad (6\text{-}28)$$

with $I_f=I+1$, I, I-1. We show in Appendix C the detailed calculation of these 9 Clebsch-Gordan coefficients.

Now we can use the Wigner-Eckart theorem in both SO(3) and Sp(2) directions. We generalize (6-22):

$$(L_2\ I_2\ m_2\ M_2\ \alpha_2|A^{L,I}_{m,M}|\ L_1\ I_1\ m_1\ M_1\ \alpha_1) = \frac{(L_2\ I_2\ \alpha_2||A^{LI}||L_1\ I_1\ \alpha_1)}{\sqrt{(2L_2+1)(2I_2+1)}} \tag{6-29}$$

$$\left\langle \begin{matrix} L & L_1 & | L_2 \\ m & m_1 & | m_2 \end{matrix} \right\rangle \left\langle \begin{matrix} I & I_1 & | I_2 \\ M & M_1 & | M_2 \end{matrix} \right\rangle$$

The double-reduced matrix elements are obtained from the matrix element by division with the CG coefficients for the SO(3) and Sp(2) couplings and multiplication by the two numerical factors $\sqrt{2L_2+1}$ and $\sqrt{2I_2+1}$. The Sp(6) generators in Fig 6-1 are characterized by L=2, I=1. L_1 and M_1 depend on the initial state we apply the generator to and, since we consider highest SO(3) and lowest Sp(2) states, $m_1=L_1$ and $M_1=I_1$.

6.4. How to Calculate the Matrix Elements

The states are given by the power expansion of the generating function in Eq.(4-22) as discussed in Chapter 5. The "denominator" part of the generating function consists of the terms in the first position of each bracket and which contain only denominator epd's. The denominator part of the generating function is

$$\frac{a^*}{(1-a)(1-a^*)(1-c)} + \frac{c^*}{(1-a)(1-a^*)(1-c^*)} + \frac{c}{(1-a)(1-c)(1-d)} +$$

$$+ \frac{c^*d^*}{(1-a^*)(1-c^*)(1-d^*)} + \frac{1}{(1-a)(1-b^*)(1-d)} + \frac{d^*}{(1-a^*)(1-b)(1-d^*)} +$$

$$+ \frac{a^*b}{(1-a^*)(1-b)(1-c)} + \frac{b^*c^*}{(1-a)(1-b^*)(1-c^*)} + \frac{bc}{(1-b)(1-c)(1-d)} \tag{6-30}$$

$$+ \frac{b^*c^*d^*}{(1-b^*)(1-c^*)(1-d^*)} + \frac{b}{(1-b)(1-b^*)(1-d)} + \frac{b^*d^*}{(1-b)(1-b^*)(1-d^*)}$$

It can be easily verified that this is the same as the following:

$$\frac{aa^*c}{(1-a)(1-a^*)(1-c)} + \frac{aa^*}{(1-a)(1-a^*)(1-c^*)} + \frac{acd}{(1-a)(1-c)(1-d)} +$$

$$\frac{c^*}{(1-a^*)(1-c^*)(1-d^*)} + \frac{a}{(1-a)(1-b^*)(1-d)} + \frac{d^*}{(1-a^*)(1-b)(1-d^*)} +$$

$$\frac{1}{(1-a^*)(1-b)(1-c)} + \frac{ac^*}{(1-a)(1-b^*)(1-c^*)} + \frac{cd}{(1-b)(1-c)(1-d)} + \tag{6-31}$$

$$\frac{b^*}{(1-b^*)(1-c^*)(1-d^*)} + \frac{d}{(1-b)(1-b^*)(1-d)} \quad \frac{bb^*}{(1-b)(1-b^*)(1-d^*)} +$$

All the numerator terms remain the same. We choose as initial state the one given by

$$[(1-a^*)(1-b)(1-c)]^{-1} = a^{*a^{*\prime}} b^{b'} c^{c'}$$

with a*', b', c' arbitrary non-negative powers. Of course we include the factors from the common denominator in front:

$$|i\rangle = \alpha^{\alpha'}\beta^{\beta'}J^{J'}J^{*J^{*'}}K^{K'}K^{*K^{*'}}M^{M'}a^{*a^{*'}}b^{b'}c^{c'} \qquad (6\text{-}32)$$

To this state we apply the raising generators B^{+}_{m} of Fig 6-1. The epd's α, β, J, J*, K, K*, M, a*, b and c are given in Chapter 4 in the list on page 68 and they are constructed explicitly as shown in Section 5.4. The state $|i\rangle$ is a stretched product of epd's such that the Sp(2) label $z=a_1/2+b_1$ and the SO(3) label L are additive. The Sp(2) label I and the SO(3) label L of $|i\rangle$ are given by

$$I = 3\alpha'+\beta'+J'+2J^{*'}+2M'+b'+c' \qquad (6\text{-}33)$$
$$L = K'+K^{*'}+2b'$$

We consider $|i\rangle$ to be in the lower right corner of the Sp(2)$\times$SO(3) multiplet it represents as in Fig 6-3 such that M=I and m=L. When we apply the Sp(6) generators of the fifteenplet B_{zm} to our initial state $|i\rangle$ we obtain a linear combination of states of our basis. We obtain the simplest combination on the right-hand side when we apply the lower right corner of the fifteenplet, i.e. $B_{\bar{1}2}$. This is due to the fact that we decrease the Sp(2) label by 1 and, at the same time, increase the L value by 2. The generator $B_{\bar{1}2}$ is essentially a double derivative with respect to the creation operators $\partial_{\eta_{2TS}}$ (see first equation in (6-15)). When we operate on the states (6-33) with this double derivative we obtain the following two types of terms:

a) terms in which $B_{\bar{1}2}$ operates on each epd. These are listed below.

$$\alpha'\,\alpha^{\alpha'-1}\,\beta^{\beta'}\,J^{J'}\,J^{*\,J^{*\prime}}\,K^{K'}\,K^{*\,K^{*\prime}}\,M^{M'}\,a^{*\,a^{*\prime}}\,b^{b'}\,c^{c'}\,B_{\bar{1}2}\,\alpha \tag{6-34a}$$

$$J'\,\alpha^{\alpha'}\,\beta^{\beta'}\,J^{J'-1}\,J^{*\,J^{*\prime}}\,K^{K'}\,K^{*\,K^{*\prime}}\,M^{M'}\,a^{*\,a^{*\prime}}\,b^{b'}\,c^{c'}\,B_{\bar{1}2}\,J \tag{6-34b}$$

$$J^{*\prime}\,\alpha^{\alpha'}\,\beta^{\beta'}\,J^{J'}\,J^{*\,J^{*\prime}-1}\,K^{K'}\,K^{*\,K^{*\prime}}\,M^{M'}\,a^{*\,a^{*\prime}}\,b^{b'}\,c^{c'}\,B_{\bar{1}2}\,J^{*} \tag{6-34c}$$

$$M'\,\alpha^{\alpha'}\,\beta^{\beta'}\,J^{J'}\,J^{*\,J^{*\prime}}\,K^{K'}\,K^{*\,K^{*\prime}}\,M^{M'-1}\,a^{*\,a^{*\prime}}\,b^{b'}\,c^{c'}\,B_{\bar{1}2}\,M \tag{6-34d}$$

$$c'\,\alpha^{\alpha'}\,\beta^{\beta'}\,J^{J'}\,J^{*\,J^{*\prime}}\,K^{K'}\,K^{*\,K^{*\prime}}\,M^{M'}\,a^{*\,a^{*\prime}}\,b^{b'}\,c^{c'-1}\,B_{\bar{1}2}\,c \tag{6-34e}$$

$B_{\bar{1}2}$ applied to all other individual epd's in $|i\rangle$ give zero.

b) terms in which $B_{\bar{1}2}$ operates on pairs of epd's like

$$M'a^{*\prime}\,\alpha^{\alpha'}\,\beta^{\beta'}\,J^{J'}\,J^{*\,J^{*\prime}}\,K^{K'}\,K^{*\,K^{*\prime}}\,M^{M'-1}\,a^{*\,a^{*\prime}-1}\,b^{b'}\,c^{c'}\,\sum_{s}(\partial_{\bar{1}s}M)(\partial_{\bar{1}s}a^{*})$$

There are 45 terms in this category which we do not list here.

The terms of the form (6-34) are relatively easy to identify. We used the program REDUCE [He85] to obtain $B_{\bar{1}2}$(epd). The result is an explicit expression in the variables η_{is}. Since we know that $B_{\bar{1}2}$ decreases z by 1 and increases L by 2 we can expect certain products of epd's to appear, which we also expand in the variables η_{is} using REDUCE. The exact coefficient is given by the comparison of the two expressions. For example $B_{\bar{1}2}\alpha$ contains b*, $B_{\bar{1}2}J$ is proportional to K^{*2}, $B_{\bar{1}2}M$ to b and $B_{\bar{1}2}c$ to K^2; more complicated is $B_{\bar{1}2}J^*$ which is a combination of d, Kh and ab. The matrix element of $B_{\bar{1}2}$ is the coefficient in $B_{\bar{1}2}(\text{epd}\,i)=\sum_f A^{if}_{\bar{1}2}(\text{epd}\,f)$. This coefficient (matrix element) depends on the definition of the two epd's, more precisely on their normalization constant. Since we work with non-orthonormal states there is no point in an overall normalization of the individual epd's or of the states made out of them. We only need to be consistent, that is, to define the epd's once for all in the simplest way to work with and look for their coefficients. When we diagonalize the matrix of generators the result will not depend on the actual normalization we use in these calculations.

We continue on a specific example for the purpose of giving the complete algorithm. When we operate with $B_{\bar{1}2}$ on $|i\rangle$ in (6-32) one of the final states is (6-34b). Let us identify the states by the exponents of the epd and not repeat the exponents which remain unchanged such that $|i\rangle=|J',K^{*\prime}\rangle$ and $|f\rangle=|J'-1,K^{*\prime}+2\rangle$. The coefficient is $A^{if}_{\bar{1}2} = J'\ 19\sqrt{2}/3$: this is the matrix element of $B_{\bar{1}2}$.

$$B_{\bar{1}2}\,|i\rangle=A^{if}_{\bar{1}2}\,|f\rangle \quad \text{or} \quad (J'-1,\ K^{*\prime}+2|B_{\bar{1}2}|\ J',\ K^{*\prime})=\ A^{if}_{\bar{1}2}$$

To obtain the reduced matrix element between $|i\rangle$ and $|f\rangle$ which is the same for the whole fifteenplet we divide $A^{if}_{\bar{1}2}$ by the two CG coefficients and multiply by $\sqrt{2(L+2)+1}$ and by $\sqrt{2(I-1)+1}$ as prescribed by (6-29).

$$A^{ij}_{\bar{1}2} = (L+2,L+2,I-1,I-1|B_{\bar{1}2}|\ L,L,I,I) = \frac{(L+2,I-1||B||L,I)}{\sqrt{(2L+5)(2I-1)}} \qquad (6\text{-}35)$$

$$\left\langle \begin{matrix} 2 & L \\ 2 & L \end{matrix} \middle| \begin{matrix} L+2 \\ L+2 \end{matrix} \right\rangle \left\langle \begin{matrix} 1 & I \\ -1 & I \end{matrix} \middle| \begin{matrix} I-1 \\ I-1 \end{matrix} \right\rangle$$

The two CG coefficients are equal to one. In terms of the Sp(2) and SO(3) couplings we can write

$$B_{\bar{1}2}|i\rangle = \left| \begin{matrix} 1 & 2 \\ -1 & 2 \end{matrix} \right\rangle \left| \begin{matrix} I & L \\ I & L \end{matrix} \right\rangle = \left| \begin{matrix} I-1 & L+2 \\ I-1 & L+2 \end{matrix} \right\rangle \qquad (6\text{-}36)$$

Now we are ready to apply another generator of the fifteenplet. If we move in the Sp(2) direction and choose B_{02}, the SO(3) CG remains unchanged and we get two different Sp(2) components. We apply B_{02} to the same initial state and we get:

$$B_{02}|i\rangle = \left| \begin{matrix} 1 & 2 \\ 0 & 2 \end{matrix} \right\rangle \left| \begin{matrix} I & L \\ I & L \end{matrix} \right\rangle = \left\langle \begin{matrix} 2 & L \\ 2 & L \end{matrix} \middle| \begin{matrix} L+2 \\ L+2 \end{matrix} \right\rangle \left\langle \begin{matrix} 1 & I \\ 0 & I \end{matrix} \middle| \begin{matrix} I-1 \\ I \end{matrix} \right\rangle \left| \begin{matrix} I-1 & L+2 \\ I & L+2 \end{matrix} \right\rangle +$$

$$+ \left\langle \begin{matrix} 2 & L \\ 2 & L \end{matrix} \middle| \begin{matrix} L+2 \\ L+2 \end{matrix} \right\rangle \left\langle \begin{matrix} 1 & I \\ 0 & I \end{matrix} \middle| \begin{matrix} I \\ I \end{matrix} \right\rangle \left| \begin{matrix} I & L+2 \\ I & L+2 \end{matrix} \right\rangle \qquad (6\text{-}37)$$

The SO(3) CG is the same and still equal to one; the Sp(2) CG are given in Appendix C:

$$\left\langle \begin{matrix} 1 & I \\ 0 & I \end{matrix} \middle| \begin{matrix} I-1 \\ I \end{matrix} \right\rangle = i/\sqrt{I-1} \quad \text{and} \quad \left\langle \begin{matrix} 1 & I \\ 0 & I \end{matrix} \middle| \begin{matrix} I \\ I \end{matrix} \right\rangle = i\sqrt{I/(I-1)}$$

The new information is contained in the second term, the $|L+2,I\rangle$

state. We have to isolate it by subtracting from $B_{02}|i\rangle$ the first term which is nothing else but I_+ applied to $B_{\bar{1}2}|i\rangle$ of (6-36) divided by the matrix element of I_+.

$$I_+ \mid I-1,\ I-1\rangle = \sqrt{(2I-2)} \mid I-1,\ I\rangle$$

Then (6-37) becomes :

$$B_{02}|i\rangle = \frac{(i/\sqrt{I-1}) I_+ B_{\bar{1}2}|i\rangle}{\sqrt{2I-2}} + i\sqrt{I/(I-1)} \left| \begin{matrix} I & L+2 \\ I & L+2 \end{matrix} \right\rangle \tag{6-38}$$

and we can now extract $|I,I,L+2,L+2\rangle$ and decide which particular combination of basis states it represents. Since we already separated the CG coefficients in (6-37) and (6-38) the coefficients we find need only to be multiplied by $\sqrt{(2L+5)}\cdot\sqrt{(2I+1)}$ to become the reduced matrix elements between $|i\rangle$ and $|j\rangle = |\ I,\ L+2\rangle$. We can go one more step in the Sp(2) direction by applying now B_{12}. We obtain three terms. Since the SO(3) CG coefficient does not change we do not write it anymore.

$$B_{12}|i\rangle = \left\langle \begin{matrix} 1 & I \\ 1 & I \end{matrix} \middle| \begin{matrix} I-1 \\ I+1 \end{matrix} \right\rangle \left| \begin{matrix} I-1 \\ I+1 \end{matrix} \right\rangle + \left\langle \begin{matrix} 1 & I \\ 1 & I \end{matrix} \middle| \begin{matrix} I \\ I+1 \end{matrix} \right\rangle \left| \begin{matrix} I \\ I+1 \end{matrix} \right\rangle + \left\langle \begin{matrix} 1 & I \\ 1 & I \end{matrix} \middle| \begin{matrix} I+1 \\ I+1 \end{matrix} \right\rangle \left| \begin{matrix} I+1 \\ I+1 \end{matrix} \right\rangle$$

As before the new combinations of basis states are in the last term. We subtract the first one as being $I_+ I_+ B_{\bar{1}2}|i\rangle$ and the second one as $I_+ B_{02}|i\rangle$ (of course after division by the matrix elements).

We apply the same strategy in the SO(3) direction. After we apply $B_{\bar{1}2}$ as given by (6-36) we apply $B_{\bar{1}1}$ to $|i\rangle$. Since nothing changes in the Sp(2) direction we do not repeat the Sp(2) CG coefficients and we write :

$$B_{11}\,|i\rangle = \begin{vmatrix}2\\1\end{vmatrix}\!\!\Big\rangle \begin{vmatrix}L\\L\end{vmatrix}\!\!\Big\rangle = \left\langle \begin{matrix}2 & L\\ 1 & L\end{matrix} \middle| \begin{matrix}L+2\\ L+1\end{matrix}\right\rangle \begin{vmatrix}L+2\\L+1\end{vmatrix}\!\!\Big\rangle + \left\langle \begin{matrix}2 & L\\ 1 & L\end{matrix} \middle| \begin{matrix}L+1\\ L+1\end{matrix}\right\rangle \begin{vmatrix}L+1\\L+1\end{vmatrix}\!\!\Big\rangle$$

The $|L+2,L+1\rangle$ is just L_- applied to $|L+2,L+2\rangle$ of (6-36) divided by the matrix element. The SO(3) CG coefficients are easy to calculate and we extract the $|L+1,L+1\rangle$ state which provides new matrix elements. We can go on in the SO(3) direction in this way until we sweep the whole fifteenplet. The strategy exposed here gives at each step only one new type of final states $|I_f,L_f\rangle$ which are easier to identify. If we try to apply directly the upper left corner of the fifteenplet of generators $(B_{1\bar{2}})$ to the initial state $|i\rangle=|I,L\rangle$, the final state contains all possible combinations of basis states with $I-1\leq I_f\leq I+1$ and $L-2\leq L_f\leq L+2$. Since the basis states are rather complicated we believe that the interpretation of the result would be impossible.

SUMMARY AND OUTLOOK

In the group theoretical description of Nuclear Collective Models, there is an unanimous consensus upon the central role of the symplectic Sp(6) group. As we tried to make it clear in Chapter 2, all previous algebraic models were proposing different groups which turn out to be subgroups of Sp(6).

One can choose to work with Sp(6) states which are at the same time basis states in the U(3) subgroup of Sp(6) and these correspond to Elliott-type states (well suited for the study of the rotational nuclear spectra) or with Bohr-Mottelson (vibrational) states which form a basis in the $Sp(2) \times O(3)$ subgroup of Sp(6). Since the generating function for $Sp(6) \supset U(3)$ was known [GR81] (see equation (4-16) on p.61 or (4-18) on p.62) we felt the challenge to try and solve the $Sp(6) \supset Sp(2) \times SO(3)$.

The problem of finding the basis states was not simple and, due to the missing labels, the basis is non-orthonormal. However it is complete and non-redundant. The basis states are easily identifiable and can be written explicitly in terms of products of epd's. All the information needed to construct them is contained in a condensed form in the generating function on page 69 which is a major result of our research.

Once the basis states are obtained, the next step is to calculate the generator (we use the singular on account of the Wigner-Eckart theorem) matrix element between our basis states. We use the algorithm described in the last chapter. Due to the arbitrary normalization of our epd's (and, consequently, states) the only interesting results are the eigenvalues of the matrix, i.e. the diagonal elements after diagonalization of the matrix. This can be done only after we complete the entire list of generator matrix elements.

The problem is quite complex: the basis states are products of powers of at least 10 different epd's from a list of over 50. Even for the simpler case of $Sp(6) \supset U(3)$ with a total of less than 20 epd's, the generator matrix elements are not yet published. It will take us some extra effort and time to complete the list of the generator matrix elements in the $Sp(6) \supset Sp(2) \times O(3)$ basis and to diagonalize it. Only afterwards can we apply the generator matrix elements to phenomenological models for calculating vibrational nuclear spectra.

APPENDIX A. Computer Program to Check the Generating Function

In this Appendix we give the listing of the program used to check the generating function followed by an actual output which compares the SO(3) weights derived from the generating function (4-22) with the SO(3) weights obtained directly from the previously known $Sp(6) \supset U(3)$ generating function [GR81]. The known generating function appears as the function F(P,Q,A,B,Z) in the program. We convert SU(3) to SO(3) and one recognizes in the function FG(P,Q,A,B,Z,T) the H(P,Q,A,B,Z,T) of eq.(4-28) (see p.74). Finally, the SO(3) weights of eq.(4-31) are calculated by the function FH.

On the other hand, we input our generating function in the subroutine FJ(P,Q,A_1,B_1,A,B,Z,L). The function H6 converts FJ to SO(3) weights. We calculate numerically FH and H6 (with A_1=B_1=1) for randomly generated values of the labels which are now real numbers between 0 and 2. In the output we give in the first 6 columns the actual random values of P, Q, A, B, Z,η; columns 7 and 8 contain respectively FH and H6 corresponding to these values of the labels. The last column is their ratio which is seen to be exactly 1 to double precision accuracy for 50 different sets of labels.

```
/INFO MVS JOB(PZ35CHCK) C(PZ35) S(000) TI(5) PA(30) CL(11) N(MIKI)
/INFO MSGL(0,0)

//  EXEC FORTRAN
//FORT.SYSIN DD *

      IMPLICIT REAL*8 (A-Z,$)
      INTEGER J
      COMMON $
      $=1.D0
      CALL RSTART (1,7)
      PRINT 8
8     FORMAT('0'//)
      DO 7 J=1,50
      P=UNI(0)*2.D0
      Q=UNI(0)*2.D0
      A=UNI(0)*2.D0
      B=UNI(0)*2.D0
      Z=UNI(0)*2.D0
      ALPHA=UNI(0)*2.D0
      RFH=FH(P,Q,A,B,Z,ALPHA)
      RH6=H6(P,Q,$,$,A,B,Z,ALPHA)
      RAT=RFH/RH6
      PRINT 6,P,Q,A,B,Z,ALPHA,RFH,RH6,RAT
6     FORMAT(' ',10X,6(F9.6,1X),2(G12.5,1X),F9.6)
7     CONTINUE
      STOP
      END

      DOUBLE PRECISION FUNCTION F(P,Q,A,B,Z)
      IMPLICIT REAL*8 (A-Z,$)
      COMMON $
      F=$/($-Z*A**2)/($-Z**2*B**2)/($-P*A)/($-Q*B)
     $/($-Z*Q**2)/($-Z**2*P**2)/
     $($-Z**3)*(($+Q*A*Z+P*A*B*Z+P*Q*B*Z)*($+P*B*Z**2)/
     $($-P**2*B**2*Z)+(Q*A*B*Z**2+P*Q*A*Z**2+Q**2*A**2
     $*Z**2+P*Q**2*A**2*B*Z**4)*($+Q*A*Z)/
     $($-Q**2*A**2*Z**2))
      RETURN
      END

      DOUBLE PRECISION FUNCTION FG(P,Q,A,B,Z,T)
      IMPLICIT REAL*8 (A-Z,$)
      COMMON $
      FG=T**2*F(P,Q,A*T,B*T,Z)
      FG=FG-T*F(P,Q,A,B*T,Z)
      FG=FG-T*F(P,Q,A*T,B,Z)
      FG=FG+F(P,Q,A,B,Z)
      FG=FG/(T-$)**2
      RETURN
      END

      DOUBLE PRECISION FUNCTION FH(P,Q,A,B,Z,ALPHA)
```

```
      IMPLICIT REAL*8 (A-Z,$)
      COMMON $
      FH=ALPHA**2*FG(P,Q,A,B,Z,ALPHA**2)
      FH=FH-FG(P,Q,A,B,Z,$/ALPHA**2)/ALPHA**2
      FH=FH/(ALPHA**2-$/ALPHA**2)
      RETURN
      END

      DOUBLE PRECISION FUNCTION H6(P,Q,A1,B1,A,B,Z,ALPHA)
      IMPLICIT REAL*8 (A-Z,$)
      COMMON $
      H6=ALPHA*FJ(P,Q,A1,B1,A,B,Z,ALPHA**2)
      H6=H6-FJ(P,Q,A1,B1,A,B,Z,$/ALPHA**2)/ALPHA
      H6=H6/(ALPHA-$/ALPHA)
      RETURN
      END

      DOUBLE PRECISION FUNCTION FJ(P,Q,A1,B1,A,B,Z,L)
      IMPLICIT REAL*8 (A-Z,$)
      COMMON $
      E=Z**3
      G=Q**2*A1**2*Z
      H=P**2*B1**2*Z**2
      J=P*A*L
      K=Q*B*L
      M=A1**2*A**2*Z
      N=B1**2*B**2*Z**2
      A2=P**2*A**2
      B2=Q**2*B**2
      C=A1**2*A**2*Z*L**2
      D=P**2*A1**2*B**2*Z
      E1=B1**2*B**2*Z**2*L**2
      F1=Q**2*B1**2*A**2*Z**2
      G1=P**2*A1**2*B**2*Z*L**2
      H1=Q**2*B1**2*A**2*Z**2*L**2
      I=P*Q*A*B*L
      J1=P*B1**2*B*Z**2*L
      K1=Q*A1**2*A*Z*L
      L1=P*Q*B1**2*A*Z**2*L
      M1=Q*B1**2*A*B*Z**2*L
      N1=P*Q*B1**2*B**2*Z**2
      P1=P*A1**2*A*B*Z*L
      Q1=P*Q*A1**2*A**2*Z
      R=P*Q*A1**2*B*Z*L
      S=Q*B1**2*A*B*Z**2*L**2
      T1=Q**2*A1**2*A*B*Z*L
      U=P*Q**2*A1**2*B**2*Z
      V=P**2*B1**2*A*B*Z**2*L
      W=P*A1**2*A*B*Z*L**2
      X=P**2*Q*B1**2*A**2*Z**2
      Y=P*Q*B1**2*A**2*B*Z**2*L
      Z1=P*Q**2*B1**2*A*B*Z**2*L
      A3=Q**2*B1**2*A*B**2*Z**2*L
```

```
      B3=P*B1**2*A*B**2*Z**2*L**2
      C1=P**2*A1**2*A**2*B*Z*L
      D1=P**2*Q*A1**2*A*B*Z*L
      E2=Q*A1**2*A**2*B*Z*L**2
      F2=P*Q*A1**2*A*B**2*Z*L
      I1=P*Q**2*A1**2*B1**2*A**2*Z**3
      T2=P*Q**2*B1**2*A**2*B**2*Z**2
      K2=P**2*Q*A1**2*A**2*B**2*Z
      L2=P**2*Q*A1**2*B1**2*B**2*Z**3
      M2=P*Q*A1**2*B1**2*A*B*Z**3*L
      N2=P*A1**2*B1**2*A**2*B*Z**3*L**2
      O=P**3*A1**2*A*B**2*Z*L**2
      P2=Q**2*A1**2*B1**2*A**2*B*Z**3*L
      R1=Q*A1**2*B1**2*A*B**2*Z**3*L**2
      S1=P**2*A1**2*B1**2*A*B**2*Z**3*L
      T3=Q**3*B1**2*A**2*B*Z**2*L**2
      F3=Q**3*A1**2*B1**2*A**2*B**2*Z**3
      C2=P**3*A1**2*B1**2*A**2*B**2*Z**3
      P3=A1**2*B1**2*A**2*B**2*Z**3*L**3
      FJ=$/($-E)/($-G)/($-H)/($-J)/($-K)/($-M)/($-N)
      JJ=(B2+I+K2+B2*P1+B2*Q1+T1+U+B2*D1+N1+B2*V+C2+D*M1+N1*Q1+
     #B2*M2+L2+I*L2)/($-A2)/($-B2)/($-D)
      JJ=JJ+(D+O+D*C1+D*P1+D*Q1+P1*R+D*R+D*D1+D*J1+D*V+G1*C2+
     #D*J1*P1+D*J1*Q1+D*M2+D*J1*R+D*J1*D1)/($-A2)/($-D)/($-G1)
      JJ=JJ+($+B3+C1+P1+J1*Q1+M2+Q1+R1+R+D1+J1+V+J1*C1+J1*P1+J1*R+
     #J1*D1)/($-A2)/($-G1)/($-E1)
      JJ=JJ+(B2*C+E2+F2+B2*C*P1+K1*N1+B2*C*M2+B2*K1+C*T1+C*U+B2*K1*P1
     #+C*N1+B2*N2+S1+C*D*M1+P1*M2+D*K1*M1)/($-B2)/($-C)/($-D)
      JJ=JJ+(C*D+P1*W+D*W+C*D*P1+D*J1*K1+C*D*M2+D*K1+C*P1*R+C*D*R+
     #D*K1*P1+C*D*J1+D*N2+G1*S1+C*D*J1*P1+G1*P1*M2+D*W*M2)/($-C)/
     #($-D)/($-G1)
      JJ=JJ+(C+P3+W+C*P1+J1*K1+C*M2+K1+C*R1+C*R+K1*P1+C*J1+N2+J1*W+
     #C*J1*P1+J1*K1*W+W*M2)/($-C)/($-G1)/($-E1)
      JJ=JJ+(F1+F1*I+I1+F1*D1+X+A2*Z1+T2+A2*M1+F1*N1+F1*V+F3+F1*P1+
     #F1*Q1+F1*T1+X*F3+F1*M2)/($-A2)/($-B2)/($-F1)
      JJ=JJ+(E1*F1+F1*B3+E1*I1+F1*J1*P1+E1*X+X*B3+Y+A2*E1*M1+F1*J1+E1*F1
     #*V+P2+E1*F1*P1+E1*F1*Q1+F1*R1+F1*J1*Q1+E1*F1*M2)/($-A2)/($-E1)/
     #($-F1)
      JJ=JJ+(F1*H1+F1*T3+H1*I1+F1*K1*Z1+F1*L1+F1*Z1+F1*A3+F1*M1+F1*H1*N1
     #+F1*L1*M1+F1*K1*A3+F1*K1*M1+F1*K1+F1*H1*T1+F1*K1*N1+F1*H1*M2)/
     #($-B2)/($-F1)/($-H1)
      JJ=JJ+(F1*H1*J1+E1*F1*L1*M1+F1*K1*S+E1*F1*K1*M1+E1*F1*K1+F1*H1*R1
     #+F1*J1*K1+E1*F1*H1*M2+E1*F1*H1+F1*M1*S+E1*H1*I1+F1*L1*R1+E1*F1*L1
     #+F1*J1*M1+F1*S+E1*F1*M1)/($-E1)/($-F1)/($-H1)
      JJ=JJ+(H1+T3+K1*L1+K1*Z1+L1+Z1+A3+M1+H1*N1+L1*M1+K1*A3+K1*M1+
     #H1*K1+H1*T1+H1*K1*N1+H1*M2)/($-B2)/($-C)/($-H1)
      JJ=JJ+(E1*H1+M1*S+J1*K1*S+L1*R1+E1*L1+J1*M1+S+E1*M1+H1*J1+
     #E1*L1*M1+K1*S+E1*K1*M1+E1*H1*K1+H1*R1+H1*J1*K1+E1*H1*M2)/
     #($-H1)/($-C)/($-E1)
      FJ=FJ*JJ
      RETURN
      END
//
```

P	Q	A	B	Z	η	F_1	F_2	F_1/F_2
0.000459	0.221694	1.297415	0.822077	1.436584	1.535693	14.164	14.164	1.000000
1.135040	1.177902	1.047140	0.918596	0.980348	1.276054	.13220D+08	.13220D+08	1.000000
1.971756	1.944533	0.757175	0.367130	0.699804	1.962421	8049.8	8049.8	1.000000
0.601940	1.525586	1.003661	0.581995	0.887661	0.934614	.22529D+08	.22529D+08	1.000000
1.430254	0.076828	1.261462	1.791177	1.158016	0.244873	.43901D-03	.43901D-03	1.000000
0.009669	1.210900	0.145959	1.881301	0.150257	1.633643	.75772	.75772	1.000000
0.047873	0.741471	1.709974	0.959166	1.040268	0.546467	-2484.9	-2484.9	1.000000
1.805962	1.685408	0.501650	1.563967	0.151056	0.066433	-.33864D-05	-.33864D-05	1.000000
1.572353	1.232394	1.645727	1.558941	1.963697	1.592436	-125.19	-125.19	1.000000
1.909813	1.630780	1.773096	0.796705	1.988783	1.746838	-.83465	-.83465	1.000000
1.546157	1.239308	1.840005	0.677037	1.762801	1.568070	1925.1	1925.1	1.000000
0.222222	1.035767	0.767435	1.348718	1.908256	0.620327	222.64	222.64	1.000000
1.026392	1.474789	1.644814	1.637753	1.922702	0.140062	.96967D-07	.96967D-07	1.000000
0.400874	0.954825	1.960293	0.459655	1.220445	0.373698	-3.9305	-3.9305	1.000000
0.271555	0.565906	1.554153	1.683072	0.141867	0.855750	-5242.8	-5242.8	1.000000
1.716203	0.410425	1.772381	1.201397	0.204514	1.926657	7.5486	7.5486	1.000000
0.317504	0.445534	0.608693	1.678077	1.144411	1.249753	54345.	54345.	1.000000
1.193642	0.598208	1.863295	0.291843	1.421131	0.562731	-10.334	-10.334	1.000000
0.588792	0.546096	0.787771	0.299117	1.324464	0.187408	.49959D-01	.49959D-01	1.000000
1.776605	1.155727	0.809218	1.338594	0.640748	1.690601	755.58	755.58	1.000000
0.781782	1.914993	1.210487	1.235534	0.695370	1.420359	25217.	25217.	1.000000
0.202421	1.752978	1.599658	1.642977	0.995868	0.543128	77.901	77.901	1.000000
1.615463	0.476879	0.745899	1.047686	1.775581	1.306762	-3887.9	-3887.9	1.000000
1.581306	1.154875	0.932250	1.829645	0.515767	0.507025	-38.831	-38.831	1.000000
1.275217	1.854159	0.697078	1.496784	0.130381	0.416502	16.553	16.553	1.000000
0.478036	1.978344	1.835894	0.980006	1.951315	0.007223	.56855D-13	.56855D-13	1.000000
0.033157	1.884003	0.802539	1.782185	1.046358	0.542689	-203.50	-203.50	1.000000
0.618734	0.718397	1.093470	1.676111	0.795966	0.236923	.10084	.10084	1.000000
1.512956	1.509039	1.041242	0.287329	0.238093	1.945702	-45.109	-45.109	1.000000
0.398667	1.228148	1.475894	0.230843	0.678184	1.741603	21304.	21304.	1.000000
0.864957	0.712703	0.102195	1.483264	1.272729	0.074754	.14884D-01	.14884D-01	1.000000
0.678279	0.003187	0.884910	0.295084	1.663754	1.275446	-1291.3	-1291.3	1.000000
1.120410	0.792751	0.047238	1.855003	0.751821	0.180165	-51.425	-51.425	1.000000
0.035523	0.095103	1.872037	1.012225	1.603445	1.882125	-.16500	-.16500	1.000000
1.302186	1.934626	1.291583	1.381969	0.382225	1.871175	75.152	75.152	1.000000
1.026663	0.824637	0.730684	0.850584	0.744154	1.660219	-26.492	-26.492	1.000000
1.352895	1.850341	1.572836	1.824553	0.279391	1.297724	-1823.2	-1823.2	1.000000
1.843660	1.290257	1.680005	1.586140	0.930434	1.945558	-7.0567	-7.0567	1.000000
1.910596	1.948910	1.966413	0.439224	1.974227	0.588896	-24.641	-24.641	1.000000
1.781687	0.144010	1.907967	0.780016	0.937523	1.674794	-50.798	-50.798	1.000000
0.269177	0.927063	1.956819	0.217645	0.834554	1.840278	-109.29	-109.29	1.000000
1.318984	1.829834	1.420482	1.412868	1.663358	1.312061	-159.02	-159.02	1.000000
1.434597	0.104780	0.653684	1.365076	0.538370	1.143124	41593.	41593.	1.000000
1.430540	1.405555	0.480291	1.806896	1.277766	1.575267	344.88	344.88	1.000000
1.932617	0.547423	1.173811	1.747985	1.710906	0.139139	-.79541D-05	-.79541D-05	1.000000
0.498141	0.060232	0.003837	1.133856	0.470686	0.626207	-196.58	-196.58	1.000000
0.170384	1.350023	1.545989	0.808012	1.281601	0.013490	.66927D-09	.66927D-09	1.000000
1.114359	0.148786	1.454223	1.206511	1.625811	0.136372	.57420D-06	.57420D-06	1.000000
1.805589	1.464592	0.154234	1.166813	0.763391	1.580126	-133.95	-133.95	1.000000
0.123227	0.995802	1.025773	0.220693	1.813234	1.877050	4.0208	4.0208	1.000000

APPENDIX B. Published Version of the Generating Function

We reproduce here the published version of the generating function (4-22) which differs from the version on page 69 by the changes given on page 91. These changes arose from consistency considerations. The new generating function has the minimal number of compatible numerator epd pairs and these pairs are found where they are expected to appear, i.e. with the denominator factors with which both pair members are compatible.

Collectivity and geometry. IV. Sp(6) ⊃ Sp(2)×O(3) basis states for open shells

M. Moshinsky[a)]
Instituto de Física, Universidad Nacional Autonoma de México, Apartado Postal 20364, México, D. F. 01000, Mexico

M. Nicolescu and R. T. Sharp
Physics Department, McGill University, Montréal, PQ H3A 2T8, Canada

(Received 27 March 1985; accepted for publication 18 June 1985)

Using generating function methods, branching rules for Sp(6) ⊃ Sp(2)×O(3) are derived. The branching rules suggest an integrity basis, or set of elementary permissible diagrams, in terms of which the subgroup basis states are defined; they correspond to vibrational, or Bohr–Mottelson type, states in the nuclear symplectic model.

I. INTRODUCTION

We refer the reader to the earlier papers in this series[1-3] for general literature references and for a historical and physical introduction to the subject. The nuclear symplectic model combines the features of the Bohr–Mottelson and Elliott models.

The basis states of a nucleus of $n+1$ nucleons are taken to be the energy eigenstates of an isotropic $3n$-dimensional harmonic oscillator. The symmetry group is then SU($3n$) and the states are those of symmetric representations (all representation labels zero except the first one). The metaplectic irreducible representations (IR's) $[(1/2)^{3n}]$, $[(1/2)^{3n-1},(3/2)]$ of Sp($6n$) are spanned by the SU($3n$) states of even, odd representation labels, respectively.

The physically significant subgroup of Sp($6n$) is Sp(6)×O(n), and for the IR's of Sp($6n$) under consideration the Sp(6) and O(n) IR's are correlated; see Eq. (3.1) below. The Hamiltonian for nuclear collective motion is assumed to be in the enveloping algebra of Sp(6).

The Sp(6) basis states may be classified according to the U(3) subgroup, yielding Elliott or rotational type states, or according to the subgroup Sp(2)×O(3), yielding Bohr–Mottelson, or vibrational type states. It is our purpose in this paper to derive the integrity basis, or elementary permissible diagrams (epd's) with their syzygies (incompatible products); they define vibrational, or Sp(2)×O(3), type basis states for general IR's of Sp(6), corresponding to open shells.

In Sec. II we derive the generating function for Sp(6) ⊃ Sp(2)×O(3) branching rules and interpret it in terms of a finite set of epd's. The basis states are defined in terms of products of powers of the epd's, with certain combinations forbidden because of syzygies (polynomial identities). The basis states obtained are not orthonormal, but are complete, nonredundant, and analytic.

In Sec. III we discuss briefly the problem of computing generator matrix elements of Sp(6) between our states; it is in terms of them that the Hamiltonian operator for collective nuclear motions is defined.

Section IV shows how to convert the Sp(6) ⊃ U(3) generating function to that for Su(n) ⊃ O(n), all but the first three labels of SU(n) zero, and how to convert the Sp(6) ⊃ Sp(2)×O(3) to O($3n$) ⊃ O(3)×O(n), all but the first label of O($3n$) zero; the respective generating functions (and the branching rules) are related because of complementarity conditions.

We use Dynkin representation labels λ_i for the compact groups O(n), SU(n):

$$\lambda_i = 2\langle M_\lambda|\alpha_i\rangle/\langle\alpha_i|\alpha_i\rangle,$$

where M_λ is the highest weight of the IR (λ), and α_i are the simple roots; the exception is SO(3), where we use $\lambda/2 = l$ as the IR label. For (noncompact) Sp(6) we use the labels (p,q,d) of the "bottom" U(3) IR; (p,q) are its SU(3) labels and (d) its "vertical" weight component. The O(n) labels $(\omega_1,\omega_2,\omega_3)$ of Ref. 3 are related to the Dynkin labels used here by

$$\omega_1=\lambda_1+\lambda_2+\lambda_3,\quad \omega_2=\lambda_2+\lambda_3,\quad \omega_3=\lambda_3 \quad \text{for} \quad n\geqslant 9;$$

for O(8) we have $\omega_3=\lambda_3=\lambda_4$; and for O(7), $\omega_3=\lambda_3/2$.

II. Sp(6) ⊃ Sp(2)×O(3) BRANCHING RULES

We begin our derivation of Sp(6) ⊃ Sp(2)×O(3) branching rules with the known[4] generating function for Sp(6) ⊃ U(3) branching rules

$$\begin{aligned}F(P,Q,D;A_1,B_1;A,B;Z) = {} & [(1-\alpha)(1-\beta)(1-\gamma)(1-\gamma^*)(1-\delta)(1-\delta^*)(1-\epsilon)(1-\epsilon^*)]^{-1}\\ & \times[(1+\eta+\theta+\kappa+\zeta\eta^*+\eta\theta+\eta\kappa+\eta\eta^*)(1-\zeta)^{-1}\\ & +(\zeta^*+\eta^*+\theta^*+\kappa^*+\zeta^*\eta+\eta^*\theta^*+\eta^*\kappa^*+\zeta^*\eta\eta^*)(1-\zeta^*)^{-1}].\end{aligned} \tag{2.1}$$

[a)] Member of Instituto Nacional de Investigaciones Nucleares and El Colegio Nacional.

The letters on the right stand for the epd's:

$$\alpha = Z^3,\quad \beta = DZ,\quad \gamma = PA,\quad \delta = Q^2A_1{}^2Z,$$
$$\epsilon = A_1{}^2A^2Z,\quad \zeta = P^2A_1{}^2B^2Z,\quad \eta = PB_1{}^2BZ^2,$$
$$\theta = PA_1{}^2ABZ,\quad \kappa = PQA_1{}^2BZ.$$

The "conjugate" of an epd, denoted above by an asterisk, is obtained by the replacements $P \leftrightarrow Q$, $A_1{}^2Z \leftrightarrow B_1{}^2Z^2$, $A \leftrightarrow B$; the generating function (2.1) is conjugation symmetric, i.e., is unaffected by these interchanges. Equation (2.1) with A_1 and B_1 set equal to unity is just Eq. (3.6) of Ref. 4. Strictly, the labels a_1,b_1, carried as exponents by the dummies A_1,B_1, are not necessary; we comment below on their usefulness. When (2.1) is expanded in a power series

$$F = \sum P^pQ^qD^dA_1{}^{a_1}B_1{}^{b_1}A^aB^bZ^zC_{pqd,a_1b_1,abz}, \tag{2.2}$$

the coefficient C, summed over a_1,b_1, gives the multiplicity of the U(3) multiplet (a,b,z) in the Sp(6) IR (p,q,d).

Throughout this paper we follow the convention that representation labels, denoted by lowercase letters, are carried as exponents by the corresponding uppercase letters.

The exponents in (2.2) (or in the epd's) provide instructions for constructing the basis states (or the epd's): couple the U(3) multiplet $(a_1,b_1,a_1/2+b_1)$, whose components are polynomials of degree $a_1/2+b_1$ in the Sp(6) raising generators [they form the U(3) multiplet (2,0,1) and are the B^+_{ij} of Eq. (3.2a) or B^+_{lm} of Eq. (3.6) of Ref. 3], to the bottom U(3) multiplet (p,q,d) of the Sp(6) IR to obtain the U(3) multiplet $(a,b,a_1/2+b_1+d)$. The U(1) label is greater than $a_1/2+b_1+d$ by three times the degree in α, the SU(3) scalar of third degree in the raising generators.

The labels a_1,b_1 help in the interpretation of the epd's and of the basis states. For example, without them, one might, erroneously, think that ζ^* is the square of η^*. When (2.1) has been converted to give $\mathrm{Sp}(6) \supset \mathrm{Sp}(2)\times\mathrm{O}(3)$ branching rules, the difficulties and ambiguities in interpreting it in terms of epd's are greatly increased. It is important to keep labels like a_1,b_1.

The subgroup SU(3) of Sp(6) is converted to SO(3) by substituting into Eq. (2.1) the $\mathrm{SU}(3) \supset \mathrm{O}(3)$ branching rules generating function

$$G(A,B,L) = [(1-A^2)(1-B^2)(1-AL)(1-BL)]^{-1}(1+ABL). \tag{2.3}$$

The substitution is accomplished[5] by evaluating

$$F(P,Q,D;A_1B_1;A',B',Z)G(A'^{-1}A,B'^{-1}B,L)|_{A'^0B'^0}.$$

The subscript $A'^0B'^0$ is an instruction to retain only the term in A' and B' of degree zero. The variables A,B are inserted to retain the SU(3) representation labels; as noted above, we will need all the labels we can get. The U(1) label z now becomes the weight label of the Sp(2) subgroup. The U(1) group is converted to (noncompact) Sp(2) simply by multiplying by $1\text{-}Z$, or, more precisely, $1\text{-}M^*$, where M^* is the epd $A_1{}^2A^2Z$ defined below. Then z, the exponent of Z, is the Sp(2) representation label, the lowest weight of the Sp(2) multiplet.

The result of the above operation is the desired $\mathrm{Sp}(6) \supset \mathrm{Sp}(2)\times\mathrm{O}(3)$ generating function. It is given as follows:

$$\begin{aligned}
&H(P,Q,D;\ A_1,B_1;\ A,B;\ Z,L)\\
&\quad = [(1-\alpha)(1-\beta)(1-J)(1-J^*)(1-K)(1-K^*)(1-M)]^{-1}\\
&\qquad \times\{[(1-a)(1-a^*)(1-c)]^{-1}[a^*+e+r+a^*h+i^*+a^*k+l+a^*n+ci+ck^*\\
&\qquad +u+ch^*+q+eq+ii^*+a^*s]+[(1-a)(1-a^*)(1-c^*)]^{-1}[c^*+c^*e+r^*+ah^*+i+ak^*\\
&\qquad +l^*+an^*+c^*i^*+c^*k+u^*+c^*h+q^*+eq^*+c^*ii^*+c^*s]+[(1-a)(1-c)(1-d)]^{-1}[c+v+cp+ch\\
&\qquad +cf+ck+cg+cn+cdi+cgh+du+cfh+cfg+cfn+cfi+cs]+[(1-a^*)(1-c^*)(1-d^*)]^{-1}[c^*d^*\\
&\qquad +v^*+c^*p^*+c^*h^*+c^*f^*+c^*k^*+c^*g^*+c^*n^*+c^*d^*i^*+c^*g^*h^*+d^*u^*+c^*f^*h^*\\
&\qquad +c^*f^*g^*+c^*f^*n^*+c^*f^*i^*+c^*d^*s]+[(1-a)(1-b^*)(1-d)]^{-1}[1+j+p+h+f+k+g+n+di\\
&\qquad +gh+fp+fh+fg+fn+fi+s]+[(1-a^*)(1-b)(1-d^*)]^{-1}[d^*+d^*j^*+p^*+h^*+f^*+k^*\\
&\qquad +g^*+n^*+d^*i^*+g^*h^*+f^*p^*+f^*h^*+f^*g^*+f^*n^*+d^*f^*i^*+d^*s]\\
&\qquad +[(1-a^*)(1-b)(1-c)]^{-1}[a^*b+j^*+t+a^*bh+bi^*+a^*x+bl+a^*f^*h+cf^*+bck^*+w\\
&\qquad +bch^*+hs+k^*w+f^*i^*+a^*bs]+[(1-a)(1-b^*)(1-c^*)]^{-1}[b^*c^*+c^*j+t^*+ab^*h^*+b^*i\\
&\qquad +ax^*+b^*l^*+afh^*+c^*f+b^*c^*k+w^*+b^*c^*h+h^*s+kw^*+c^*fi+b^*c^*s]\\
&\qquad +[(1-b)(1-c)(1-d)]^{-1}[bc+hm+cm+bch+bcf+cx+bcg+cf^*h+cdf^*\\
&\qquad +bcgh+dw+bcfh+dhs+cms+cff^*+bcs]+[(1-b^*)(1-c^*)(1-d^*)]^{-1}[b^*c^*d^*\\
&\qquad +h^*m^*+c^*m^*+b^*c^*h^*+b^*c^*f^*+c^*x^*+b^*c^*g^*+c^*fh^*+c^*d^*f+b^*c^*g^*h^*+d^*w^*\\
&\qquad +b^*c^*f^*h^*+d^*h^*s+c^*m^*s+c^*ff^*+b^*c^*d^*s]+[(1-b)(1-b^*)(1-d)]^{-1}[b+y\\
&\qquad +m+bh+bf+x+bg+f^*h+df^*+bfg+fm+bfh+bfg+ms+ff^*+bs]\\
&\qquad +[(1-b)(1-b^*)(1-d^*)]^{-1}[b^*d^*+d^*y+m^*+b^*h^*+b^*f^*+x^*+b^*g^*+fh^*+d^*f+b^*g^*h^*\\
&\qquad +f^*m^*+b^*f^*h^*+b^*f^*g^*+m^*s+d^*ff^*+b^*d^*s]\}.
\end{aligned} \tag{2.4}$$

The epd's α,β in Eq. (2.4) are the same as in (2.1); the others are as follows [the notation is (pq,a_1b_1,ab,l) which stands for $P^pQ^qA_1{}^{a_1}B_1{}^{b_1}A^aB^bZ^{(1/2)a_1+b_1}L^l$]:

$J=(02,20,00,0)$, $K=(10,00,10,1)$, $M=(00,02,02,0)$,
$a=(20,00,20,0)$, $b=(00,20,20,2)$, $c=(20,20,02,0)$,
$d=(20,20,02,2)$, $e=(11,00,11,1)$, $f=(10,02,01,1)$,
$g=(11,20,01,1)$, $h=(10,20,11,1)$, $i=(11,20,20,0)$,
$j=(10,02,12,2)$, $k=(20,02,11,1)$, $l=(12,20,02,0)$,
$m=(10,20,11,2)$, $n=(21,20,11,1)$, $p=(20,20,21,1)$,
$q=(21,22,02,0)$, $r=(21,20,22,0)$, $s=(11,22,11,1)$,
$t=(11,20,12,1)$, $u=(30,22,22,0)$, $v=(30,20,12,2)$,
$w=(20,22,12,1)$, $x=(10,22,21,2)$, $y=(00,22,22,3)$.

The conjugate of an epd, denoted by an asterisk in Eq. (2.4) obtained by interchanging the SU(3) labels in each of the three pairs: $(pq,a_1b_1,ab,)^* = (qp,b_1a_1,ba,)$; the epd's α,β,e,s,y are self-conjugate. The generating function in (2.4) apart from the missing denominator factor $1-M^*$, removed in converting from U(1) to Sp(2), is conjugation symmetric, a fact which was helpful in writing it in terms of the epd's. The generating function was also subjected to what may be called consistency checks. For example, the coefficient of i in Eq. (2.4)

$$\begin{aligned}&[(1-a)(1-a^*)(1-c^*)]^{-1}+c[(1-a)(1-a^*)(1-c)]^{-1}\\&\quad+d\,[(1-a)(1-b^*)(1-d)]^{-1}+b^*[(1-a)\\&\quad\times(1-b^*)(1-c^*)]^{-1}\\&\quad+cd\,[(1-a)(1-c)(1-d)]^{-1}.\end{aligned}$$

It may be verified that each product of powers of three denominator epd's which appear in the same fraction (including those in which one or more exponents are zero) appears just once in the above expression; this check was made separately for each numerator epd and for each product of numerator epd's. As a final check we converted the expression in Eq. (2.4), by appropriate substitutions, into a generating function for SO(3) weights instead of SO(3) multiplets; it was then compared with the corresponding weight generating function obtained by converting (2.1) directly; since an analytic comparison would be prohibitively laborious, the necessary substitutions were made by a computer program and the two generating functions compared for random values of their arguments.

III. CONSTRUCTING THE BASIS STATES

The $\mathrm{Sp}(6)\supset\mathrm{Sp}(2)\times\mathrm{O}(3)$ generating function, as given in Eq. (2.4) defines the epd's (integrity basis) in terms of which all subgroup representations are given as stretched (all representation labels additive) products; epd's which are incompatible because of syzygies may be read from the generating function: they never appear multiplied together.

It is straightforward to construct the epd's, using their labels $(p,q;a_1,b_1;a,b;l)$: couple the SU(3) IR(a_1,b_1), of degree $a_1/2+b_1$ in the raising Sp(6) generators, to the bottom SU(3) multiplet (p,q) to obtain the IR (a,b); in every case the coupling is nondegenerate, i.e., unique. Next, choose the SO(3) multiplet contained in the SU(3) multiplet (a,b); again, the multiplet l is always nondegenerate. Apart from their usefulness in constructing the epd's, the labels a_1,b_1,a,b were invaluable in sorting out the epd's and their syzygies. Finally, to ensure that the states we are constructing are bottom states of Sp(2) multiplets, they must be rendered traceless (harmonic) by the use of Eq. (4.7b) of Ref. 3.

The bottom [lowest U(1)] multiplet of the Sp(6) IR (pqd) is best visualized in terms of the epd's for $\mathrm{Sp}(6n)\supset\mathrm{Sp}(6)\times\mathrm{O}(n)$.

The branching rules generating function

$$\begin{aligned}&[(1-PD^{1/2}H)(1-QDJ)(1-D^{3/2}K)]^{-1}\\&\quad=\sum_{hjk}H^hJ^jK^kP^hQ^jD^{(1/2)(h+2j+3k)}\end{aligned}\tag{3.1}$$

shows that the O(n) IR (hjk) is correlated with the Sp(6) IR (pqd) with $p=h$, $q=j$, $d=(h+2j+3k)/2$; integer values of d belong to even metaplectic Sp($6n$), $[(\frac{1}{2})^{3n}]$, half-odd values of d to odd metaplectic Sp($6n$), $[(\frac{1}{2})^{3n-1},\frac{3}{2}]$. Each of the three epd's stands for an elementary $\mathrm{Sp}(6)\times\mathrm{O}(n)$ multiplet, and is conveniently represented by the state of the multiplet which has the highest O(n) weight and, for Sp(6), the lowest U(1) and highest SU(3) [or SO(3)] weight. The exponent of D is one half the number of quanta in the state in question. Then $PD^{1/2}H$ is represented by η_{11}, where the first subscript denotes the highest state of the SU(3) [or SO(3)] triplet while the second one implies the highest state of the O(n) multiplet $(100\cdots0)$. The epd QDJ, represented by

$$\begin{vmatrix}\eta_{11} & \eta_{12}\\ \eta_{21} & \eta_{22}\end{vmatrix},$$

is the highest state of an SU(3) antitriplet [or O(3) triplet] and the highest state of the O(n) multiplet $(010\cdots0)$. The third epd $D^{3/2}K$ is represented by

$$\begin{vmatrix}\eta_{11} & \eta_{12} & \eta_{13}\\ \eta_{21} & \eta_{22} & \eta_{23}\\ \eta_{31} & \eta_{32} & \eta_{33}\end{vmatrix};$$

it is an SU(3) [or O(3)] scalar and is the highest state of the O(n) multiplet $(0010\cdots0)$. Thus the bottom states of the Sp(6) IR(pqd) are defined by the product of powers of the three epd's with respective exponents p, q, $(2d\text{-}p\text{-}2q)/3$. These states are annihilated by the Sp(6) lowering (annihilation) generators, and no steps are needed to render them traceless.

The 21 generators of Sp(6) decompose under the subgroup $\mathrm{Sp}(2)\times\mathrm{O}(3)$ into three irreducible tensors which can be denoted by (1,0), (0,1), (1,2). The first two triplets are just the generators of Sp(2) and O(3); their matrix elements are well known. The matrix elements of only the (1,2) 15-plet need to be computed between our basis states. For that purpose it is necessary to compute only its reduced matrix elements between pairs of subgroup multiplets; although straightforward, that task is made laborious by the size of the integrity basis and the consequent large number of types of subgroup multiplet. We hope to complete it in a future publication.

IV. RELATED BRANCHING RULES

Complementarity relations in group–subgroup chains imply connections between apparently unrelated branching rules. Thus the generating functions of Eqs. (2.1) and (2.4), for Sp(6) ⊃ U(3) and Sp(6) ⊃ Sp(2)×O(3), respectively, imply branching rules generating functions for SU(n) ⊃ SO(n), all but the first three SU(n) labels zero, and for O($3n$) ⊃ O(3)×O(n), all but the first O($3n$) label zero, respectively.

Although not needed for the theory of nuclear collective motions, we present the results here since we get them at no extra cost.

For the chains of subgroups

$$\mathrm{Sp}(6n) \supset \mathrm{Sp}(6)\times \mathrm{O}(n), \quad (4.1\mathrm{a})$$

$$\mathrm{Sp}(6n) \supset \mathrm{Sp}(2)\times \mathrm{O}(3n), \quad (4.1\mathrm{b})$$

complementarity relations hold (see Ref. 3, Sec. II). This means that, since the representation of Sp($6n$) is $[(\frac{1}{2})^{3n}]$ or $[(\frac{1}{2})^{3n-1},(\frac{3}{2})]$ (the metaplectic ones), the IR of Sp(6) determines the IR of O(n) and vice versa in (4.1a) and the same holds for Sp(2) and O($3n$) in (4.1b).

Because of the complementarity in the chain in Eq. (4.1a) we can convert the generating function (2.1) giving the branching rules for Sp(6) ⊃ U(3) to a generating function for the chain SU(n) ⊃ O(n) [all but the first three labels of SU(n) zero] by the substitutions

$$P\rightarrow G^{1/3}H,\quad Q\rightarrow G^{2/3}J,\quad D\rightarrow G^{1/3}K,\quad A\rightarrow EG^{-1/3},$$
$$B\rightarrow FG^{-2/3},\quad Z\rightarrow G^{2/3},\quad A_1\rightarrow 1,\quad B_1\rightarrow 1;$$

the SU(n) nonzero labels are denoted by (e,f,g) and the O(n) ones by (h,j,k).

The above substitutions are valid when $n\geqslant 9$. For $n=8$ the substitution for D changes to $D\rightarrow G^{1/3}KK'$, where the nonzero O(8) labels are (h,j,k,k') with $k=k'$, and for $n=7$ the substitution for D is $D\rightarrow G^{1/3}K^2$ [here the three O(7) labels are (h,j,k)]. We do not consider the case $n<6$.

Similarly, starting with the generating function for Sp(6) ⊃ Sp(2)×O(3) given in Eq. (2.4) we get the branching rules generating function for the chain O($3n$) ⊃ O(3)×O(n), all but the first label of O($3n$) zero, by the substitutions

$$P\rightarrow UH,\quad Q\rightarrow U^2J,\quad D\rightarrow UK,\quad Z\rightarrow U^2,\quad L\rightarrow L,$$
$$A_1\rightarrow 1,\quad B_1\rightarrow 1,\quad A\rightarrow 1,\quad B\rightarrow 1;$$

(u) labels O($3n$) IR's, (l) labels O(3) IR's, and (h,j,k) are the O(n) labels (all but the first three zero). These substitutions hold for $n\geqslant 9$. The substitutions for D become $D\rightarrow UKK'$ ($n=8$) and $D\rightarrow UK^2$ ($n=7$).

ACKNOWLEDGMENTS

We are grateful to E. Chacón and O. Castaños for helpful discussions.

This work was supported by the National Sciences and Engineering Research Council of Canada and by the Ministère de l'Education du Québec.

[1] M. Moshinsky, J. Math. Phys. 25, 1555 (1984).
[2] E. Chacón, P. Hess, and M. Moshinsky, J. Math. Phys. 25, 1565 (1984).
[3] O. Castaños, E. Chacón, and M. Moshinsky, J. Math. Phys. 25, 2815 (1984).
[4] R. Gaskell, G. Rosensteel, and R. T. Sharp, J. Math. Phys. 22, 2732 (1981).
[5] J. Patera and R. T. Sharp, J. Phys. A: Math. Gen. 13, 397 (1980).

APPENDIX C. Clebsch-Gordan Coefficients for Sp(2)

On account of the Wigner-Eckart theorem used in Section 6.2 the Sp(2) Clebsch-Gordan coefficients figure together with the SO(3) ones in the expression for the generator matrix elements (6-). Hence we need Clebsch-Gordan coefficients of the form (6-13) which couple the generators which form a finite Sp(2) IR of label I=1 to the infinite IR of the basis states. The Sp(2) raising and lowering operators are as usual

$$I_+ = I_1 + iI_2 \ , \qquad I_- = I_1 - iI_2 \tag{C-1}$$

with I_1, I_2 given in Chapter 3 by eqs (3-13) and they satisfy the commutation rules in (3-14) where I_3 is the weight generator of Sp(2). They act on triplet states $|1,m\rangle$ (m=-1,0,1) in the following way:

$$\begin{array}{ll} I_+|1,-1\rangle = i\sqrt{2}|1,\ 0\rangle & I_+|1,\ 0\rangle = i\sqrt{2}|1,\ 1\rangle \\ I_-|1,\ 0\rangle = i\sqrt{2}|1,-1\rangle & I_-|1,\ 1\rangle = i\sqrt{2}|1,\ 0\rangle \\ I_-|1,-1\rangle = I_+|1,1\rangle = 0 & \end{array} \tag{C-2}$$

Note that because of the finite IR of a noncompact group the matrix elements are complex. The scalar product is obtained by multiplying a state by its dual, and, to be consistent, we choose $\langle 1,1|1,1\rangle = \langle 1,-1|1,-1\rangle = 1$ and $\langle 1,0|1,0\rangle = -1$. These matrix elements in

(C-2) obey Gel'fand phase conventions for Sp(2) SU(1,1) [HB66]; matrix elements of I_+, I_- have the same phase, positive imaginary. Another possible choice, which avoids imaginary matrix elements, is to ask that I_+ have positive matrix elements; then those of I_- are negative.

The infinite Sp(2) IR of the basis states are labelled by the lowest Sp(2) weight I. The states belonging to the IR (I) extend from M=I, I+1, I+2,...to infinity, $I \geq 1$. When we crank with I_+ and I_- on an arbitrary state $|I, M\rangle$ we get (Gel'fand [GZ50],[GG65],[HB66]):

$$\begin{aligned} I_+|I, M\rangle &= \sqrt{(M+I)(M-I+1)}\ |I, M+1\rangle \\ I_-|I, M\rangle &= \sqrt{(M-I)(M+I-1)}\ |I, M-1\rangle \end{aligned} \qquad \text{(C-3)}$$

Obviously, $I_-|I,I\rangle=0$.

Let us begin with the couplings

$$\left\langle \begin{matrix} 1 & I \\ -1 & M+1 \end{matrix} \middle| \begin{matrix} I+1 \\ M \end{matrix} \right\rangle, \left\langle \begin{matrix} 1 & I \\ 0 & M \end{matrix} \middle| \begin{matrix} I+1 \\ M \end{matrix} \right\rangle, \left\langle \begin{matrix} 1 & I \\ 1 & M-1 \end{matrix} \middle| \begin{matrix} I+1 \\ M \end{matrix} \right\rangle \qquad \text{(C-4)}$$

For this we form the "composite" $|I+1,I+1\rangle$ as the linear combination:

$$\left| \begin{matrix} I+1 \\ I+1 \end{matrix} \right\rangle = A \left| \begin{matrix} 1 \\ 1 \end{matrix} \right\rangle \left| \begin{matrix} I \\ I \end{matrix} \right\rangle + B \left| \begin{matrix} 1 \\ 0 \end{matrix} \right\rangle \left| \begin{matrix} I \\ I+1 \end{matrix} \right\rangle + C \left| \begin{matrix} 1 \\ -1 \end{matrix} \right\rangle \left| \begin{matrix} I \\ I+2 \end{matrix} \right\rangle \qquad \text{(C-5)}$$

We determine the constants A, B, C requiring I_- to give zero on this state. Using (C-2) and (C-3) we get

$$B=i\sqrt{2I+1}\,C, \quad A=i\sqrt{I}\,B = -\sqrt{I(2I+1)}\,C$$

We normalize to 1

$$\left\langle \begin{matrix} I+1 \\ I+1 \end{matrix} \middle| \begin{matrix} I+1 \\ I+1 \end{matrix} \right\rangle = |A|^2 - |B|^2 + |C|^2 = 1 \text{ , then } C=-\left(\sqrt{I(2I-1)}\right)^{-1}$$

and (C-5) becomes

$$\left| \begin{matrix} I+1 \\ I+1 \end{matrix} \right\rangle = \sqrt{\frac{2I+1}{2I-1}} \left| \begin{matrix} 1 \\ 1 \end{matrix} \right\rangle \left| \begin{matrix} I \\ I \end{matrix} \right\rangle - i\sqrt{\frac{2I+1}{I(2I-1)}} \left| \begin{matrix} 1 \\ 0 \end{matrix} \right\rangle \left| \begin{matrix} I \\ I+1 \end{matrix} \right\rangle - \frac{1}{\sqrt{I(2I-1)}} \left| \begin{matrix} 1 \\ -1 \end{matrix} \right\rangle \left| \begin{matrix} I \\ I+2 \end{matrix} \right\rangle \qquad \text{(C-6)}$$

We apply I_+ to (C-6) repeatedly (M-I-1) times. The left-hand side becomes

$$I_+^{M-I-1} \left| \begin{matrix} I+1 \\ I+1 \end{matrix} \right\rangle = \sqrt{(2I+2)(2I+3)\ldots(M+I).1.2\ldots(M-I-1)} \left| \begin{matrix} I+1 \\ M \end{matrix} \right\rangle$$

$$= \sqrt{\frac{(M+I)!\,(M-I-1)!}{(2I+1)!}} \left| \begin{matrix} I+1 \\ M \end{matrix} \right\rangle \qquad \text{(C-7)}$$

On the right-hand side, we work separately on the three terms. The first term gives

$$I_+^{M-I-1}\left|\begin{matrix}1\\1\end{matrix}\right\rangle\left|\begin{matrix}I\\I\end{matrix}\right\rangle = \left|\begin{matrix}1\\1\end{matrix}\right\rangle I_+^{M-I-1}\left|\begin{matrix}I\\I\end{matrix}\right\rangle = \left|\begin{matrix}1\\1\end{matrix}\right\rangle\left|\begin{matrix}I\\M-1\end{matrix}\right\rangle\sqrt{2I(2I+1)\ldots(M+I-2).1.2\ldots(M-I-1)}$$

$$= \left|\begin{matrix}1\\1\end{matrix}\right\rangle\left|\begin{matrix}I\\M-1\end{matrix}\right\rangle\sqrt{\frac{(M+I-2)!\,(M-I-1)!}{(2I-1)!}} \qquad \text{(C-8)}$$

while the second term is

$$I_+^{M-I-1}\left|\begin{matrix}1\\0\end{matrix}\right\rangle\left|\begin{matrix}I\\I+1\end{matrix}\right\rangle = \left|\begin{matrix}1\\0\end{matrix}\right\rangle I_+^{M-I-1}\left|\begin{matrix}I\\I+1\end{matrix}\right\rangle + (M-I-1)\ I_+\left|\begin{matrix}1\\0\end{matrix}\right\rangle I_+^{M-I-2}\left|\begin{matrix}I\\I+1\end{matrix}\right\rangle =$$

$$= \left|\begin{matrix}1\\0\end{matrix}\right\rangle\left|\begin{matrix}I\\M\end{matrix}\right\rangle\sqrt{\frac{(M+I-1)!\,(M-I)!}{(2I)!}} + (M-I-1)\sqrt{2}\,i\sqrt{\frac{(M+I-2)!\,(M-I-1)!}{(2I)!}}\left|\begin{matrix}1\\1\end{matrix}\right\rangle\left|\begin{matrix}I\\M-1\end{matrix}\right\rangle \qquad \text{(C-9)}$$

and the third term

$$I_+^{M-I-1}\left|\begin{matrix}1\\-1\end{matrix}\right\rangle\left|\begin{matrix}I\\I+2\end{matrix}\right\rangle = \left|\begin{matrix}1\\-1\end{matrix}\right\rangle I_+^{M-I-1}\left|\begin{matrix}I\\I+2\end{matrix}\right\rangle + (M-I-1)\ I_+\left|\begin{matrix}1\\-1\end{matrix}\right\rangle I_+^{M-I-2}\left|\begin{matrix}I\\I+2\end{matrix}\right\rangle$$

$$+ \frac{(M-I-1)(M-I-2)}{2}\ I_+^2\left|\begin{matrix}1\\-1\end{matrix}\right\rangle I_+^{M-I-3}\left|\begin{matrix}I\\I+2\end{matrix}\right\rangle =$$

$$= \left|\begin{matrix}1\\-1\end{matrix}\right\rangle\left|\begin{matrix}I\\M+1\end{matrix}\right\rangle\sqrt{\frac{(M+I)!\,(M-I+1)!}{(2I+1)!}} + (M-I-1)\,i\sqrt{2}\left|\begin{matrix}1\\0\end{matrix}\right\rangle\left|\begin{matrix}I\\M\end{matrix}\right\rangle\sqrt{\frac{(M+I-1)!\,(M-I)!}{(2I+1)!}} +$$

$$+ \frac{(M-I-1)(M-I-2)}{2} (-2) \left|\begin{matrix}1\\1\end{matrix}\right\rangle \left|\begin{matrix}I\\M-1\end{matrix}\right\rangle \sqrt{\frac{(M+I-2)!(M-I-1)!}{(2I+1)!}} \tag{C-10}$$

We go back to (C-6) with the left-hand side given by (C-7) and the right-hand side by the three terms in (C-8), (C-9) and (C-10) and we obtain the Clebsch-Gordan coefficients

$$\left\langle \begin{matrix}1 & I\\ -1 & M+1\end{matrix} \middle| \begin{matrix}I+1\\M\end{matrix}\right\rangle = \frac{-1}{\sqrt{I(2I-1)}} \times \sqrt{\frac{(2I+1)!}{(M+I)!(M-I-1)!}} \times \sqrt{\frac{(M+I)!(M-I+1)!}{(2I+1)!}}$$

$$= - \sqrt{\frac{(M-I)(M-I+1)}{I(2I-1)}} \tag{C-11}$$

$$\left\langle \begin{matrix}1 & I\\ 0 & M\end{matrix} \middle| \begin{matrix}I+1\\M\end{matrix}\right\rangle = \sqrt{\frac{(2I+1)!}{(M+I)!(M-I-1)!}} \left\{ \frac{-(M-I-1)i\sqrt{2}}{\sqrt{I(2I-1)}} \sqrt{\frac{(M+I-1)!(M-I)!}{(2I+1)!}} - \right.$$

$$\left. - i\sqrt{\frac{2I+1}{I(2I-1)}} \sqrt{\frac{(M+I-1)!(M-I)!}{(2I)!}} \right\} = -i \sqrt{\frac{(M+I)(M-I)}{I(2I-1)}} \tag{C-12}$$

$$\left\langle \begin{matrix}1 & I\\ 1 & M-1\end{matrix} \middle| \begin{matrix}I+1\\M\end{matrix}\right\rangle = \sqrt{\frac{(2I+1)!}{(M+I)!(M-I-1)!}} \left\{ \sqrt{\frac{2I+1}{2I-1}} \sqrt{\frac{(M+I-2)!(M-I-1)!}{(2I-1)!}} - \right.$$

$$- i\sqrt{\frac{2I+1}{I(2I-1)}} (M-I-1) i\sqrt{2} \sqrt{\frac{(M+I-2)!(M-I-1)!}{(2I)!}} -$$

$$- \frac{1}{\sqrt{I(2I-1)}} (-2) \; \frac{(M-I-1)(M-I-2)}{2} \sqrt{\frac{(M+I-2)!(M-I-1)!}{2(2I+1)!}} \Bigg\} =$$

$$= \sqrt{\frac{2I+1}{(M+I-1)(M+I)}} \left\{ \sqrt{\frac{(2I+1)2I}{2I-1}} + \sqrt{\frac{(2I+1)}{I(2I-1)}} \; (M-I-1) + \frac{(M-I-1)(M-I-2)}{\sqrt{2I(2I+1)(2I-1)}} \right\} =$$

$$= \sqrt{\frac{(M+I)(M+I-1)}{2I(2I-1)}} \qquad \text{(C-13)}$$

To calculate the couplings

$$\left\langle \begin{matrix} 1 & I \\ -1 & M+1 \end{matrix} \middle| \begin{matrix} I \\ M \end{matrix} \right\rangle , \left\langle \begin{matrix} 1 & I \\ 0 & M \end{matrix} \middle| \begin{matrix} I \\ M \end{matrix} \right\rangle , \left\langle \begin{matrix} 1 & I \\ 1 & M-1 \end{matrix} \middle| \begin{matrix} I \\ M \end{matrix} \right\rangle \qquad \text{(C-14)}$$

we start with the state $|I,I\rangle$ as a linear combination

$$\left| \begin{matrix} I \\ I \end{matrix} \right\rangle = A \left| \begin{matrix} 1 \\ 0 \end{matrix} \right\rangle \left| \begin{matrix} I \\ I \end{matrix} \right\rangle + B \left| \begin{matrix} 1 \\ -1 \end{matrix} \right\rangle \left| \begin{matrix} I \\ I+1 \end{matrix} \right\rangle \qquad \text{(C-15)}$$

and, by imposing $I_- |I,I\rangle = 0$, we obtain $A = i\sqrt{I}\, B$. The normalization condition $-|A|^2+|B|^2=|B|^2(1-I)=-1$ then yields $B=1/\sqrt{I-1}$ and $A=i\sqrt{I/(I-1)}$. Now we apply I_+ (M-I) times to (C-15): the left-hand side becomes

$$I_+^{M-I} \left| \begin{matrix} I \\ I \end{matrix} \right\rangle = \left| \begin{matrix} I \\ M \end{matrix} \right\rangle \sqrt{\frac{(M+I-1)!(M-I)!}{(2I-1)!}} \qquad \text{(C-16)}$$

The two terms on the right-hand side give

$$I_+^{M-I}\left|\begin{matrix}1\\0\end{matrix}\right\rangle\left|\begin{matrix}I\\I\end{matrix}\right\rangle = \left|\begin{matrix}1\\0\end{matrix}\right\rangle\left|\begin{matrix}I\\M\end{matrix}\right\rangle\sqrt{\frac{(M+I-1)!(M-I)!}{(2I-1)!}} +$$

$$+\ (M-I)i\sqrt{2}\sqrt{\frac{(M+I-2)!(M-I-1)!}{(2I-1)!}}\left|\begin{matrix}1\\1\end{matrix}\right\rangle\left|\begin{matrix}I\\M-1\end{matrix}\right\rangle \qquad \text{(C-17)}$$

$$I_+^{M-I}\left|\begin{matrix}1\\-1\end{matrix}\right\rangle\left|\begin{matrix}I\\I+1\end{matrix}\right\rangle = \left|\begin{matrix}1\\-1\end{matrix}\right\rangle\left|\begin{matrix}I\\M+1\end{matrix}\right\rangle\sqrt{\frac{(M+I)!(M-I+1)!}{(2I)!}} +$$

$$+\ (M-I)i\sqrt{2}\left|\begin{matrix}1\\0\end{matrix}\right\rangle\left|\begin{matrix}I\\M\end{matrix}\right\rangle\sqrt{\frac{(M+I-1)!(M-I)!}{(2I)!}}$$

$$+\ \frac{(M-I)(M-I-1)}{2}(-2)\left|\begin{matrix}1\\1\end{matrix}\right\rangle\left|\begin{matrix}I\\M-1\end{matrix}\right\rangle\sqrt{\frac{(M+I-2)!(M-I-1)!}{(2I)!}}$$

$$\text{(C-18)}$$

With (C-16) for the left-hand side of (C-15) and (C-17)-(C-18) for its right-hand side, we obtain the Clebsch-Gordan coefficients

$$\left\langle\begin{matrix}1 & I\\-1 & M+1\end{matrix}\middle|\begin{matrix}I\\M\end{matrix}\right\rangle = \sqrt{\frac{(2I-1)!(M+I)!(M-I+1)!}{(M+I-1)!(M-I)!(2I)!(I-1)}} = \sqrt{\frac{(M+I)(M-I+1)}{2I(I-1)}} \qquad \text{(C-19)}$$

$$\left\langle\begin{matrix}1 & I\\0 & M\end{matrix}\middle|\begin{matrix}I\\M\end{matrix}\right\rangle = \sqrt{\frac{(2I-1)!}{(M+I-1)!(M-I)!}}\left\{\frac{(M-I)}{\sqrt{I-1}}\, i\sqrt{2}\sqrt{\frac{(M+I-1)!(M-I)!}{(2I)!}} +\right.$$

$$+ i\sqrt{\frac{I}{I-1}}\sqrt{\frac{(M+I-1)!(M-I)!}{(2I-1)!}}\Bigg\} =$$

$$= i\frac{(M-I)}{\sqrt{I(I-1)}} + i\frac{I}{\sqrt{I(I-1)}} = \frac{iM}{\sqrt{I(I-1)}} \tag{C-20}$$

$$\left\langle \begin{matrix} 1 & I \\ 1 & M-1 \end{matrix} \middle| \begin{matrix} I \\ M \end{matrix} \right\rangle = \sqrt{\frac{(2I-1)!}{(M+I-1)!(M-I)!}} \Bigg\{ \frac{-(M-I)(M-I-1)}{\sqrt{I-1}} \times$$

$$\times \sqrt{\frac{(M+I-2)!(M-I-1)!}{(2I)!}} + i\sqrt{\frac{I}{I-1}}(M-I)i\sqrt{2}\sqrt{\frac{(M+I-2)!(M-I-1)!}{(2I-1)!}}\Bigg\} =$$

$$=-\sqrt{\frac{M-I}{(M+I-1)(I+1)2I}}\left[(M-I-1)+2I\right] = -\sqrt{\frac{(M-I)(M+I-1)}{2I(I-1)}} \tag{C-21}$$

Finally, to the state $|I-1,I-1\rangle = |1,-1\rangle|I,I\rangle$ we apply I_+ $(M-I+1)$ times to obtain

$$I_+^{M-I+1}\left|\begin{matrix} I-1 \\ I-1 \end{matrix}\right\rangle = \sqrt{\frac{(M+I-2)!(M-I+1)!}{(2I-3)!}}\left|\begin{matrix} I-1 \\ M \end{matrix}\right\rangle \tag{C-22}$$

$$= \sqrt{\frac{(M+I)!(M-I+1)!}{(2I-1)!}}\left|\begin{matrix} 1 \\ -1 \end{matrix}\right\rangle\left|\begin{matrix} I \\ M+1 \end{matrix}\right\rangle + i\sqrt{2}(M-I+1)\sqrt{\frac{(M+I-1)!(M-I)!}{(2I-1)!}}\left|\begin{matrix} 1 \\ 0 \end{matrix}\right\rangle\left|\begin{matrix} I \\ M \end{matrix}\right\rangle$$

$$- (M-I+1)(M-I)\sqrt{\frac{(M+I-2)!(M-I-1)!}{(2I-1)!}}\left|\begin{matrix} 1 \\ 1 \end{matrix}\right\rangle\left|\begin{matrix} I \\ M-1 \end{matrix}\right\rangle \tag{C-23}$$

From (C-22) and (C-23), we obtain after some simplification the last three Clebsch-Gordan coefficients

$$\left\langle \begin{matrix} 1 & I \\ -1 & M+1 \end{matrix} \middle| \begin{matrix} I-1 \\ M \end{matrix} \right\rangle = \sqrt{\frac{(M+I-1)(M+I)}{(2I-2)(2I-1)}} \tag{C-24}$$

$$\left\langle \begin{matrix} 1 & I \\ 0 & M \end{matrix} \middle| \begin{matrix} I-1 \\ M \end{matrix} \right\rangle = i\sqrt{\frac{2(M-I+1)(M+I-1)}{(2I-2)(2I-1)}} \tag{C-25}$$

$$\left\langle \begin{matrix} 1 & I \\ 1 & M-1 \end{matrix} \middle| \begin{matrix} I-1 \\ M \end{matrix} \right\rangle = -\sqrt{\frac{(M-I)(M-I+1)}{(2I-2)(2I-1)}} \tag{C-26}$$

BIBLIOGRAPHY

[AI76] A.Arima and F.Iachello, Ann. Phys. (N.Y.) 94, 253 (1976); Ann. Phys. (N.Y.) 111, 201 (1978)

[BB79] B.Buck, L.C.Biedenharn and R.Y.Cusson, Nucl. Phys. A317, 205 (1979)

[BK37] N.Bohr and F.Kalckar, Mat. Fys. Medd. Dan. Vid. Selks. 14, No.10 (1937)

[BM53] A.Bohr and B.R.Mottelson, Mat. Fys. Medd. Dan. Vid. Selks. 27, No.16 (1953)

[BM55] A.Bohr and B.R.Mottelson, Mat. Fys. Medd. Dan. Vid. Selks. 32, No.16 (1955)

[BM61] V.Bargman and M.Moshinsky, Nucl. Phys. 23, 177 (1961)

[Bo36] N.Bohr, Nature 137, 344 (1936)

[Bo52] A.Bohr, Mat. Fys. Medd. Dan. Vid. Selsk. 26, No.14 (1952)

[CC84] O.Castaños, E.Chacón and M.Moshinsky, J. Math. Phys. 25, 2815 (1984)

[CH84] E.Chacón, P.Hess and M.Moshinsky, J. Math. Phys. 25, 1565 (1984)

[CM76] E.Chacón, M.Moshinsky and R.T.Sharp, J. Math. Phys. 17, 668 (1976)

[Co80] M.Couture, Reduction of Enveloping Algebras of Low Rank Groups, Ph.D. Thesis, McGill University (1980)

[Cu68] R.Y.Cusson, Nucl. Phys. A114, 289 (1968)

[De70] V.Syamala Devi, Nuovo Cimento A 68, 1 (1970)

[DO72] A.Y.Dzublik, V.I.Ovcharenko, A.I.Steshenko and G.F.Filippov, Yad. Fiz. 15, 869 (1972) [Sov. J. Nucl. Phys. 15, 487 (1972)]

[DS79] P.E.Desmier and R.T.Sharp, J. Math. Phys. 20, 74 (1979)

[DS82] P.E.Desmier, R.T.Sharp and J.Patera, J. Math. Phys. 23, 1393 (1982)

[Ed57] A.R.Edmonds, Angular Momentum in Quantum Mechanics, Princeton University Press (1957)

[EG70] J.M.Eisenberg and W.Greiner, Nuclear Models (North Holland, Amsterdam, 1970) Appendix A p.431

[El58] J.P.Elliott, Proc. Roy. Soc A245, 128 (1958); ibid. 562 (1958)

[Fi73] G.F.Filippov, Fiz. Elem. Chastits At. Yadra 4, 992 (1973) [Sov. J. Part. Nucl. 4, 405 (1973)]

[Fi78] G.F.Filippov, Fiz. Elem. Chastits At. Yadra 9, 1241 (1978) [Sov. J. Part. Nucl. 9, 486 (1978)]

[GG65] I.M.Gel'fand and M.I.Graev, Izv. Akad. Nauk SSSR Ser. Math. 29, 1329 (1965)

[GL59] S.Goshen and H.J.Lipkin, Ann. Phys. (N.Y.) 6, 301 (1959)

[GL68] S.Goshen and H.J.Lipkin, "On the Application of the Group Sp(4) or R(5) to Nuclear Structure" in "Spectroscopic and Group Theoretical Methods in Physics" Racah Memorial Volume, edited by F.Bloch, S.G.Cohen, A.deShalit, S.Sambursky and I.Talmi, North Holland, Amsterdam, 1968, pp.245

[Go66] K.Gottfried, Quantum mechanics, W.A.Benjamin Inc., N.Y. (1966), vol I, p.303

[GP78] R.Gaskell, A.Peccia and R.T.Sharp, J. Math. Phys. 19, 727 (1978)

[GR81] R.Gaskell, G.Rosensteel and R.T.Sharp, J. Math. Phys. 22, 2732 (1981)

[GZ50] I.M.Gel'fand and M.L.Zetlin, Dokl. Akad. Nauk SSSR 71, 825 (1950)

[He85] A.C.Hearn, REDUCE User's Manual, Version 3.2, Rand Publication CP78 (April 1985)

[HM81] P.O.Hess, J.Maruhn and W.Greiner, J. Phys. G : Nucl. Phys. 7, 737 (1981)

[HB66] W.J.Holman III and L.C.Biedenharn Jr., Ann. Phys. 39, 39 (1966)

[HS80] P.O.Hess, M.Seiwert, J.Maruhn and W.Greiner, Z. Phys. A296, 147 (1980)

[HW53] D.L.Hill and J.A.Wheeler, Phys. Rev. 89, 1102 (1953)

[In54] D.R.Inglis, Phys. Rev. 96, 1059 (1954)

[In55] D.R.Inglis, Phys. Rev. 97, 701 (1955)

[JB77] M.W.Jarič and J.L.Berman, J. Math. Phys. 18, 1456, 1459 and 2085 (erratum) (1977)

[KJ80] P.Kramer, G.John and D.Schenzle in "Group Theory and the Interaction of Composite Nucleon Systems" (Braunschweig, Wiesbaden, 1980)

[KM68] P.Kramer and M.Moshinsky in "Group Theory and its Applications" (ed. E.M.Loebl, Acad. Press, N.Y., 1968), p.339

[LC85] R.LeBlanc, J.Carvalho, M.G.Vassanji and D.J.Rowe, An Effective Shell Model Theory of Collective States, University of Toronto Preprint (1985)

[LH71] M.A.Lohe and C.A.Hurst, J. Math. Phys. 12, 1882 (1971)

[Lo74] M.A.Lohe, The Development of the Boson Calculus for the Orthogonal and Symplectic Groups, Thesis, University of Adelaide (1974)

[Mc74] A.G.McLellan, J. Phys. C7, 3326 (1974)

[Me54] B.Meyer, Can. J. Math. 6, 135 (1954)

[MN85] M.Moshinsky, M.Nicolescu and R.T.Sharp, J. Math. Phys. 26, 2995 (1985)

[Mo97] T.Molien, Sitzungber. König. Preuss. Akad. Wiss., 1152 (1897)

[Mo67] M.Moshinsky, "Group Theory and the Many Body Problem" (Gordon and Breach, N.Y., 1967)

[Mo72] F.B.Morinigo, Nucl. Phys. A192, 209 (1972)

[Mo73] M.Moshinsky, SIAM, J. Appl. Math. 25, 193 (1973)

[Mo84] M.Moshinsky, J. Math. Phys. 25, 1555 (1984);

[MQ69] M.Moshinsky and C.Quesne, J. Math. Phys. 11, 1631 (1969)

[MQ71] M.Moshinsky and C.Quesne, J. Math. Phys. 12, 1772 (1971)

[PS78] J.Patera, R.T.Sharp and P.Winternitz, J. Math. Phys. 19, 2362 (1978)

[PS79] J.Patera and R.T.Sharp, Recent Advances in Group Theory and their Applications to Spectroscopy, ed. J.S.Donini (New York, Plenum, 1979)

[PS80] J.Patera and R.T.Sharp, J.Phys.A : Math. Gen. 13, 397 (1980)

[PS85] D.Phaneuf and R.T.Sharp, J. Math. Phys. 26, 901 (1985)

[Ra51] G.Racah, Lecture Notes on "Group Theory and Spectroscopy", Institute for Advanced Study, Princeton, N.J., 1951

[Ro85] D.J.Rowe, Microscopic Theory of the Nuclear Collective Model, Reports on Progress in Physics 48, 1419 (1985)

[RR76] G.Rosensteel and D.J.Rowe, Ann. Phys. (N.Y.) 96, 1 (1976); Int. J. Theor. Phys. 15, 453 (1976)

[RR80] G.Rosensteel and D.J.Rowe, J. Math. Phys. 21, 924 (1980); Ann. Phys. 126, 343 (1980)

[Sa79] L.Sabaliauskas, Liet. Fiz. Rink. 19, 5 (1979)

[Sh70] R.T.Sharp, Proc. Cambridge Phil. Soc. 68, 571 (1970)

[Sh72] R.T.Sharp, J. Math. Phys. 13, 183 (1972)

[SL69] R.T.Sharp and C.S.Lam, J. Math. Phys. 10, 2033 (1969)

[Sl77] N.J.A.Sloane, Amer. Math. Month. 84, 82 (1977)

[SV85] R.T.Sharp, J. Van der Jeugt and B.J.W.Hughes, J. Math. Phys. 26, 901 (1985)

[Va71] V.V.Vanagas, "Algebraic Methods in Nuclear Theory", Mintis, Vilnius, 1971

[Va80] V.V.Vanagas, Fiz. Elem. Chastits At. Yadra 11, 454 [Sov. J. Part. Nucl. 11, 172 (1980)]

[Vi68] N.Y.Vilenkin, Special Functions and Theory of Groups Representations (A.M.S. Transl., Providence, R.I. 1968)

[VK73] V.V.Vanagas and R.K.Kalinauskas, Yad. Fiz. 18, 768 (1973) [Sov. J. Nucl. Phys. 18, 395 (1973)]

[WB70] L.Weaver and L.C.Biedenharn, Phys. Lett. 32B, 326 (1970)

[WB72] L.Weaver and L.C.Biedenharn, Nucl. Phys. A185, 1 (1972)

[WB73] L.Weaver, L.C. Biedenharn and R.Y.Cusson, Ann. Phys. (N.Y.) 77, 250 (1973)

[WC76] L.Weaver, R.Y.Cusson and L.C.Biedenharn, Ann. Phys. (N.Y.) 102, 493 (1976)

[We32] R.Weitzenboeck, Acta Math. 58, 231 (1932)

[Wy72] B.G.Wybourne, J. Math. Phys 13, 455 (1972)

[Zi71] W.Zickendraht, J. Math. Phys. 12, 1663 (1971)