PRIMITIVE GROUP RINGS

by J. Lawrence

ABSTRACT

This thesis concerns some problems in primitive group rings and related problems in primitive rings. The first two chapters are an exposition of recently published work on primitive group rings and the related problem, due to Kaplansky; does prime and regular imply primitive?

Chapters III and IV extend results, due to E. Formanek, on group rings of free products of groups. In chapter IV primitive group rings, where the coefficient ring is not assumed to be zero-divisor-free, are considered.

The final chapter consists of an example of a primitive ring with nonzero singular ideal. This answers a conjecture, due to C. Faith and B. Osofsky, on the existence of such rings.

Department of Mathematics
McGill University

June 1973

ANNEAUX DE GROUPES PRIMITIFS

par J. Lawrence

RÉSUMÉ

Cette thèse traite de problèmes dans les anneaux de groupes primitifs et de problèmes connexes dans les anneaux primitifs. Les deux premiers chapitres sont une exposition des travaux récemment publiés sur les anneaux de groupes primitifs et le problème connexe, dû à Kaplansky; est-ce que premier et régulier implique primitif?

Les chapitres III et IV étendent des résultats, dus à E. Formanek, sur les anneaux de groupes de produit libre de groupes. Dans le chapitre IV on considère des anneaux de groupes primitifs, où on n'assume pas que le anneau coefficient est avec diviseurs de zéro.

Le dernier chapitre consiste en un example d'anneau primitif avec un idéal singulier et non nul. Cela répond à la conjecture émise par C. Faith et B. Osofsky, sur l'existence de tels anneaux.

Département de Mathématiques
Université McGill

Juin 1973

PRIMITIVE GROUP RINGS

J. W. Lawrence

Submitted to the Faculty of Graduate Studies and Research, in partial fulfillment of the requirements for the degree of Master of Science.

Department of Mathematics
McGill University
Montreal, Canada

June 1973

PREFACE

Primitive rings have been studied for several decades. Recently, new techniques have evolved for dealing with problems in primitive rings. Examples of this technique are to be found in the Formanek and Snider proofs of the existence of primitive group rings and the Fisher-Snider proof that countable prime regular rings are primitive. The aim of this thesis is to illustrate this technique - especially its applications to the study of primitive group rings, and to give an exposition of the work that has been done on primitive group rings.

Chapter I consists of basic definitions and theorems on rings, groups and group rings, to be used in the thesis.

The second chapter consists of a summary of recent results on Kaplansky's problem (does prime and regular imply primitive?), and results on primitive group rings, due mainly to Formanek, Passman and Snider. In this chapter we look at primitive group rings of locally finite groups [5], [8], [17], and solvable groups [17].

Chapter III introduces a new class of prime rings, 'strongly prime', which are prime rings with a finiteness condition on zero-divisors. Properties of these rings are discussed, for use in chapter IV. An extension of this chapter will appear in a paper by D. Handelman and the author [22].

In chapter IV we extend results, due to Formanek [7], on group rings of free products. Formanek showed that group rings

of free products, with domains as coefficient rings, are primitive. We exhibit a larger class of prime rings which are coefficient rings of such primitive group rings. We also look at the more general question of which prime rings are the coefficient ring of some primitive group ring.

Chapter V answers a conjecture due to Faith and Osofsky [4], [15]; an example of a primitive ring with nonzero singular ideal is constructed.

In chapters III - V all theorems are believed original to the author, unless stated otherwise.

I would like to express my appreciation to my adviser Professor I. Connell, to Professor J. Lambek at McGill and to Dr. E. Formanek at Carleton University. Thanks to my colleagues Michael Josephy, Abdoul Rahman and especially David Handelman. Thanks also to Helene Massam for her aid in translating the abstract.

John W. Lawrence
McGill University
June 1973

CONTENTS

Preface		1
I	Introduction	1
II	Primitive Group Rings	6
III	Strongly Prime Rings	14
IV	Group Rings of Free Products	21
V	Singular Primitive Rings	34
Bibliography		39

NOTATION

R	an associative ring with identity
F	a commutative field
G	a group
R[G]	the group ring of a group G, with coefficient ring R
Module	a unitary module
J(R)	the Jacobson radical of R
R_n	the ring of n x n matrices with entries from R
C(R)	the centre of the ring R
G*H	the free product of the groups G and H
Z(R)	the left singular ideal of R
N	the set of positive integers
Z_2	the two-element field, or the two-element group (In each case, the meaning will either be explicitly stated or will be obvious.)
(r)	the two sided ideal generated by r
|S|	the cardinality of the set S

INTRODUCTION

All rings are rings with unity.

THEOREM 1.1. In a ring R, the following are equivalent.

1. For any elements a and b of R, $aRb = 0 \Rightarrow a = 0$ or $b = 0$.
2. For any ideals A and B of R, $AB = (0) \Rightarrow A = (0)$ or $B = (0)$
3. The left annihilator of any nonzero left ideal is zero.

Proof. See [12] p. 54.

DEFINITION A ring with the above properties is said to be prime.

THEOREM 1.2. In a ring R, the following are equivalent.

1. R has a faithful irreducible left module.
2. R has a maximal left ideal in which (0) is maximal as a two-sided ideal.
3. R has a proper left ideal I comaximal with every nonzero two-sided ideal of R, i.e. if J is a nonzero ideal of R, then $I + J = R$.

Proof. See [12] p. 52.

DEFINITION A ring with the above properties is said to be left primitive.

REMARK G. Bergman [1] has shown that primitivity is a one-sided property. From now on we shall use the term 'primitive' to mean left primitive.

A simple but important result is the following.

THEOREM 1.3. Every primitive ring is prime.

Proof. See [12], 54.

The ring of rational integers is an example of a ring which is prime but not primitive. In fact, a commutative ring is primitive if and only if it is a field.

THEOREM 1.4. Suppose R is a prime ring with minimal left ideal I. Then the following hold:

1. $I = Re$, where $e^2 = e$, and eRe is a division ring.
2. R is primitive.

Proof. 1. See [12], 62-63.

2. I is a faithful irreducible left R-module.

THEOREM 1.5. Suppose R is a prime ring with centre C(R). Then

1. C(R) is a domain,
2. R can be embedded in an algebra A over the quotient field F of C(R).

Proof. 1. Let a be a nonzero element of C(R), and let b be a nonzero element of R. Then $0 \neq aRb = Rab$, hence $ab \neq 0$.

2. Localize R at the multiplicatively closed set $C(R)-\{0\}$. This is possible since $C(R)-\{0\}$ is zero-divisor-free. If A is the localization, we see that there is a natural embedding of R into A.

DEFINITION An algebra A over a field F is said to satisfy a

polynomial identity if there is an $f \neq 0$ in $F[X_1, X_2, \cdots, X_n]$, the free algebra over F in the noncommuting variables $X_1, \cdots, X_n$ for some n, such that $f(a_1, \cdots, a_n) = 0$ for all $a_1, \cdots, a_n$ in A. Every prime ring can be embedded in an algebra over the quotient ring of its centre (thm. 1.5). We say that a prime ring R satisfies a polynomial identity if the algebra satisfies the identity.

DEFINITIONS Given a group G we define

$$\Delta(G) = \{g \in G: g \text{ has finitely many conjugates}\},$$

$$\Delta^+(G) = \{g \in \Delta(G): g \text{ has finite order}\}.$$

Both $\Delta(G)$ and $\Delta^+(G)$ are normal subgroups of G, and G has no finite normal subgroups if and only if $\Delta^+(G) = \langle 1 \rangle$.

DEFINITIONS A group is said to be polycyclic if it has a finite series of subgroups $\langle 1 \rangle = G_0 \subset G_1 \subset \cdots \subset G_n = G$, where G_i is a normal subgroup of G_{i+1} and G_{i+1}/G_i is cyclic, for each $i = 0, \cdots, n-1$. If the latter condition is replaced by the condition that G_{i+1}/G_i is abelian, we say that G is solvable. The rank of a polycyclic group is the number of infinite cyclic quotients in the series . This number is independent of the particular series and is thus well defined.

DEFINITION Suppose V is a vector space over the field F. The group of automorphisms of V over F is called the general linear group, denoted by $GL_F(V)$.

We now look at a particular group. Let B be an infinite

countable set, and let S_∞ be the group of permutations on B which leave all but a finite number of elements fixed. S_∞ is called the infinite symetric group. It is locally finite and has no (non-trivial) finite normal subgroups.

DEFINITIONS Suppose R is a ring and G is a group. The group ring $R[G]$ is the R-algebra with basis $\{g: g \in G\}$ and multiplication defined distributively using the group multiplication of G. There is an embedding of R into $R[G]$, $r \mapsto r1$, and an epimorphism

$$\alpha: R[G] \longrightarrow R$$

defined by

$$\alpha(r_1 g_1 + \cdots + r_n g_n) = r_1 + \cdots + r_n.$$

The kernel of α is the augmentation ideal of $R[G]$.

We wish to descibe the ring properties of $R[G]$ in terms of the group properties of G and the ring properties of R. The following four theorems will be used later in this thesis.

THEOREM 1.6. The group ring $R[G]$ is simple if and only if R is simple and $G = \langle 1 \rangle$.

Proof. The augmentation ideal is trivial if and only if $G = \langle 1 \rangle$, and then $R[G]$ is isomorphic to R.

THEOREM 1.7. If R is a commutative domain and G is a torsion-free abelian group, then $R[G]$ is a commutative domain.

Proof. See [16], 110-114.

THEOREM 1.8. The group ring $R[G]$ is regular if and only if R is regular, G is locally finite, and the order of any finite subgroup of G is a unit in R.

Proof. See [12], 155.

THEOREM 1.9. The group ring $R[G]$ is completely reducible (Artinian and $J(R[G]) = (0)$) if and only if R is completely reducible, G is finite, and the order of G is a unit in R.

Proof. See [12], 156.

PRIMITIVE GROUP RINGS

In this chapter we survey the rather short history of primitive group rings.

Until recently no examples of primitive group rings were known (except for the trivial case $|G| = \langle 1 \rangle$), and it had even been conjectured that none existed. For this reason, the reader may be surprised when he sees how easily their existence can be proved. To some extent, this reflects the nature of primitive rings. It has usually been difficult to link primitivity with other properties of rings, and this has meant that in proving the primitivity or nonprimitivity of certain rings, one has had to use the Density Theorem (which is rather difficult to apply) or go straight back to the definition. For this reason many problems in primitive rings have remained open for several years and then been solved in a rather easy manner.

The problem of the existence of primitive rings has been linked closely to other recent problems in primitive rings. An example of such a problem (and the main example) is a problem of Kaplansky: Is every prime regular ring necessarily primitive? No counterexamples are known, although the implication has only been proved in special cases. One of these is the following.

THEOREM 2.1. (Fisher-Snider [5]) Let R be a prime regular ring

with a countable ideal base (i.e. a countable set of nonzero ideals such that every nonzero ideal of R contains an ideal in the set). Then R is primitive.

Proof. Let $I_1, I_2, \cdots$ be the ideal base. By taking $J_k = \bigcap_{i=1}^{k} I_i$, if necessary, we may assume that $I_{k+1} \subset I_k$. Since R is regular, every right ideal contains a nonzero idempotent. Suppose $0 \neq e_1 = e_1^2 \in I_1$ and let $J_2 = e_1R \cap I_2$. Since R is prime, J_2 is a nonzero right ideal and so we can choose $0 \neq e_2 = e_2^2 \in J_2$. Let $J_3 = e_2R \cap I_3$. As before, J_3 is a nonzero right ideal. Continuing in this manner, we obtain a sequence of idempotents $e_1, e_2, \cdots$, with $e_1R \supset e_2R \supset \cdots$, and so $R(1-e_1) \subset R(1-e_2) \subset \cdots$. Let $M = \bigcup R(1-e_i)$. If $1 = \sum_{i=1}^{n} r_i(1-e_i)$, then we have $e_{n+1} = \sum r_i(1-e_i)e_{n+1} = 0$, a contradiction. Hence M is proper as a left ideal. If J is any nonzero two sided ideal of R, then $I_k \subset J$, for some k, and so $e_k \in J$. Since $1-e_k \in M$, we get $J + M = R$, thus M is comaximal with ideals. Since R has a proper left ideal comaximal with nonzero two sided ideals, R is primitive.

COROLLARY A countable prime regular ring is primitive.

Proof. The principal ideals form an ideal base.

Attempts to extend the above proof to the uncountable case have been unsuccessful.

Another interesting case of Kaplansky's problem is the following.

THEOREM 2.2. (Goodearl [9]) If R is a prime regular self-injective ring, then R is primitive.

The theorem is a corollary to some general structure theorems for regular self-injective rings, discovered by Goodearl.

Since primitive rings are both prime and semiprimitive, it would be useful to have necessary and sufficient conditions for a group ring to have the above two properties, before we go on to look at primitive group rings. In the case of semiprimitive, the problem is unsolved and is one of the classic problems in group rings. In the case of prime group rings we have the following.

THEOREM 2.3.(Connell [3]) R[G] is prime if and only if R is prime and $\Delta^{+}(G) = \langle 1 \rangle$.

Before primitive group rings had been discovered, A. Rosenberg gave several conditions which the group G must satisfy in order for the group algebra F[G]to be primitive. Perhaps the most interesting of these is:

THEOREM 2.4. (Rosenberg [21]) If F is a field and G is a finite extension of an abelian group, then F[G] is not primitive.

Rosenberg proved this by showing that F[G] satisfies a polynomial identity. A theorem of Kaplansky states that all primitive rings satisf ying a polynomial identity are simple.

Since $F[G]$ is simple only when G is trivial, the theorem follows.

In 1972 E. Formanek and R. Snider showed that primitive group rings do in fact exist, with the following pair of theorems.

THEOREM 2.5. (Formanek-Snider [8]) Suppose G is a group and F is a field. Then there exists a group H containing G such that $F[H]$ is primitive.

THEOREM 2.6. (Formanek-Snider [8]) Suppose G is a countable locally finite group and F is a field of characteristic 0, or characteristic p if G has no elements of order p. Then $F[G]$ is primitive if and only if it is prime.

Proof of theorem 2.5. Define a sequence $\{G_i\}$ of groups and a sequence $\{M_i\}$ of modules inductively by

$$G_1 = G \qquad M_1 = F[G_1]$$
$$\cdots \qquad \cdots$$
$$G_{n+1} = GL_F(M_n) \qquad M_{n+1} = F[G_{n+1}] \oplus M_n .$$

We have $G_1 \subset G_2 \subset \cdots$ and $M_1 \subset M_2 \subset \cdots$. Let $H = \cup G_i$ and $M = \cup M_i$. Since each M_i is an $F[G_i]$-module, M is an $F[H]$-module. Each M_i is a faithful $F[G_i]$-module, hence M is a faithful $F[H]$-module. Each M_i is an irreducible $F[G_{i+1}]$-module, hence M is an irreducible $F[H]$-module. Since $F[H]$ has a faithful irreducible module, it is primitive.

Proof of theorem 2.6. Let $\{G_i\}$ be a sequence of finite groups, $G_1 \subset G_2 \subset \cdots$, $\cup G_i = G$. For each G_i, $F[G_i]$ is completely reducible. If we let $e_1, e_2, \cdots$ be an enumeration of the central

irreducible idempotents of all the $F[G_1]$, then the set $\{F[G]e_1F[G]\}$ is a countable ideal base of $F[G]$. Since $F[G]$ is regular, if it is prime, then it is primitive, by theorem 2.1.

COROLLARY Suppose F is a field of characteristic 0 and S_∞ is the countable symmetric group. Then $F[S_\infty]$ is primitive.

Fisher and Snider extended the above results to certain uncountable locally finite groups [5].

We now look at several theorems of Passman on primitive group rings. First, Passman gave necessary and sufficient conditions for the group ring of a countable locally finite group to be primitive.

THEOREM 2.7. (Passman [17]) Let G be a countable locally finite group and let F be any field. Then $F[G]$ is primitive if and only if $J(F[G]) = (0)$ and $\Delta(G) = \langle 1 \rangle$.

Passman also found two interesting theorems dealing with arbitrary group algebras.

THEOREM 2.8. (Passman [17]) Suppose that $F[G]$ is primitive and let K be a field extension of F. Suppose that either K/F is algebraic or $\Delta(G) = \langle 1 \rangle$. Then $K[G]$ is primitive.

THEOREM 2.9. (Passman [17]) Suppose $F[G]$ is primitive and $|F| > |G|$. Then $\Delta(G) = \langle 1 \rangle$.

The latter theorem is interesting because the condition

$\Delta(G) = \langle 1 \rangle$ is similar to the condition, $\Delta^+(G) = \langle 1 \rangle$, for prime group rings. It does, however, depend on the cardinality condition $|F| > |G|$. Formanek has shown that there do exist primitive group algebras $F[G]$ in which $\Delta(G) \neq \langle 1 \rangle$.

Proof of theorem 2.9. By theorem 2.8, we may assume that F is algebraically closed. Suppose $F[G]$ is primitive with M as a faithful irreducible module. Let $D = \mathrm{End}_{F[G]}(M)$ be the commuting ring, which, by Schur's lemma, is a division ring. By the Density Theorem there exists a surjection from a subring of $F[G]$ onto D, hence

$$\dim_F D \leq \dim_F F[G] = |G| < |F|.$$

We show that $D = F$. Suppose $d \in D \setminus F$. The set $\{(d-f)^{-1}, f \in F\}$ has cardinality greater than $\dim_F D$, and so we must have linear dependence, say

$$a_1(d-f_1)^{-1} + \cdots + a_k(d-f_k)^{-1} = 0 .$$

This gives a non-trivial equation which d satisfies over F, and since F is algebraically closed, we conclude that $d \in F$. Hence $D = F$. Suppose $x \in \Delta(G)$, and let y be the class sum of the conjugacy class of x in $F[G]$. We have $y \in D = F$, and so $x = 1$. Thus $\Delta(G) = \langle 1 \rangle$.

In his paper, Passman also looked at primitive group rings of solvable groups. Suppose F is a field and A is a torsion-free abelian group. Then $F[A]$ is an integral domain. We denote the quotient ring of $F[A]$ by $F[A]^{-1}F[A]$.

THEOREM 2.10. (Passman [17]) Let F be a field and let G be a group with a normal torsion-free abelian subgroup A. Let K be isomorphic to $F[A]^{-1}F[A]$. If $A \cap \Delta(G) = \langle 1 \rangle$, then K[G] has an irreducible module M on which F[A] acts faithfully.

Combining this theorem with a theorem, due to A. Zalesskey, on group rings of solvable groups, Passman was able to prove:

THEOREM 2.11. (Passman [17]) Let G be a polycyclic group with $\Delta(G) = \langle 1 \rangle$ and let F be a field with

transcendence degree F > rank G .

Then F[G] is primitive. (The transcendence degree is over the prime subfield.)

Thus if a field is sufficiently 'large', the group ring is primitive. On the other hand, J. Roseblade has shown that not all group rings of polycyclic groups are primitive.

THEOREM 2.12. (Roseblade [20]) Suppose G is a polycyclic group and F is an algebraic extension of a finite field. Then F[G] is not primitive.

The result for polycyclic groups can be extended to solvable groups. In this case we get a new embedding theorem.

THEOREM 2.13. (Passman [17]) Suppose G is a torsion-free solvable group with $\Delta(G) = \langle 1 \rangle$ and F is any field. Then there exists a field K containing F, such that K[G] is primitive.

E. Formanek continued the investigation of primitive group rings in [7]. In this paper he showed that if $G = A*B$ is the free product of non-trivial groups A and B, not both of order two, then $R[G]$ is primitive, where R is a domain such that $|R| \leq |G|$. In chapter 4 we generalize these results.

STRONGLY PRIME RINGS

We now look at a subclass of the class of prime rings. The results of this section will be used later in the chapter on group rings of free products.

DEFINITION. A ring R is said to be (left) strongly prime (denoted by S.P.) if $\forall\, 0 \neq r \in R$, there exists a finite subset $S(r) \subset R$ such that $\forall\, 0 \neq t \in R$, we have $tS(r)r \neq \{0\}$.

Obviously all left S.P. rings are prime and all domains are left S.P. A less trivial example of a left S.P. ring is the matrix ring F_n over a field F. In this case the set S(r) is the set of matrix units.

In example (1) (below) we show that left S.P. does not imply right S.P. Henceforth, whenever we do not explicitly state the side (left or right) for the property S.P., we shall assume it to be the left.

DEFINITION A class α of rings is said to be inductive if it has the following property: Let I be a totally ordered set and suppose $\{R_i\}_{i \in I} \subset \alpha$. Suppose further that $R_i \subset R_j$ if $i < j$. Then $R = \bigcup R_i \in \alpha$.

One can quickly verify that the classes of prime rings, domains and fields are all inductive. We will later show that the class of primitive rings is not inductive.

EXAMPLE 1. We show that left S.P. does not imply right S.P. Let $A = Z_2\langle X_1, X_2, \cdots\rangle$ be the free Z_2-algebra in noncommuting variables $X_1, X_2, \cdots$. Let I be the ideal generated by monomials of the form $X_i X_j X_k$, where $i < j < k$. Set $R = A/I$. We first show that R is left S.P.. If $0 \neq m = X_{j_1} X_{j_2} \cdots X_{j_k}$ is a monomial in R, let $S(m) = \{X_1 X_{j_1}\}$. It is easily checked that if $0 \neq m'$ is any monomial in R, then $m'S(m)m \neq \{0\}$. Suppose $r = m_1 + \cdots + m_n$ is a sum of nonzero monomials in R and let m_1 be a monomial of maximal degree in this sum. We then set $S(r) = S(m_1)$. This completes the proof that R is left S.P. If $\{r_i\}$ is a finite set of elements of R, then $\{r_i\}X_n X_{n+1} = \{0\}$ for sufficiently large n. Hence R is not right S.P.

EXAMPLE 2. We show that the class of S.P. rings is not inductive. Take the set $S_n = \{X_1, \cdots, X_{2^n}\}$ of variables and let $A_n = Z_2\langle S_n\rangle$ be the free Z_2-algebra in noncommuting variables. Let I_n be the ideal generated by the set $\{I_{n-1}\} \cup \{X_i X_j \,;\, i, j < 2^n \text{ and } 2^{n-1} < i \text{ or } 2^{n-1} < j\}$, where $I_1 = (0)$. This defines a sequence of ideals, inductively. Let $R_n = A_n/I_n$. R_n is a S.P. ring with $S(r) = \{X_{2^n}\}$, for all nonzero elements r. Also, by construction, $R_n \subset R_{n+1}$ for all n. However, $R = \cup R_n$ is not S.P., since there exists no finite set $S \subset R$ such that $X_n S X_1 \neq \{0\}$ for all n.

We will now prove several theorems which show that the class of S.P. rings is rather large.

THEOREM 3.1.

1. If R is S.P., then R_n is S.P.

2. If R is S.P. and e is a nonzero idempotent of R, then eRe is S.P.

Proof. 1. Let $\{e_{ij}\}$ be a set of matrix units of R_n. Suppose that $r = \Sigma\, r_{ij}e_{ij}$, $r_{ij} \in R$, is a nonzero element of R_n, with $r_{km} \neq 0$. Let $S(r_{km})$ be the finite set corresponding to r_{km} in R, (see the definition of S.P.). We define a new set, $S' = \{s_1 e_{ij} : s_1 \in S(r_{km})\}$. Assume that $t = \Sigma\, t_{ij}e_{ij}$ is any nonzero element of R_n, with $t_{uv} \neq 0$. Then $t_{uv}s_1 r_{km}$ occurs as a component in $ts_1 e_{vk} r$, and for some 1, this product is nonzero.

2. Let $r = ere$ be a nonzero element of eRe, and let $S(r)$ be the finite set corresponding to r in R. If $0 \neq t = ete \in eRe$, then $teS(r)er = tS(r)r \neq \{0\}$. Hence eRe is S.P.

DEFINITION A ring R is said to be a (left) Goldie ring if:

1. R satisfies the ascending chain condition on left annihilators.

2. R contains no infinite direct sum of left ideals.

Clearly a left Noetherian ring is a left Goldie ring.

THEOREM 3.2. A prime left Goldie ring is a S.P. ring.

Proof. The proof uses two well known theorems:

1. Goldie's theorem. This states that every prime left Goldie

ring can be embedded as a left order in a matrix ring D_n, where D is a division ring. (See [10], 169-179 and [12], 108-113.)

2. The Faith-Utumi theorem. This theorem characterizes left orders in D_n. (See [12], 114-116.)

Let R be a prime left Goldie ring. By the above theorems, there exists a positive integer n, a division ring D, a left order C of D, and a complete set of matrix units $\{e_{ij}\}$, such that

$$\Sigma\, Ce_{ij} \subset R \subset \Sigma\, De_{ij}\ .$$

Let c be a nonzero element of C and let $r = \Sigma\, r_{ij}e_{ij} \in R$ and $t = \Sigma\, t_{ij}e_{ij} \in R$ be nonzero elements, with $r_{uv} \neq 0$ and $t_{xy} \neq 0$. Then $t_{xy}cr_{uv} \neq 0$ occurs as a component in $tce_{yu}r$. We complete the proof by letting $S(r) = \{ce_{ij}\}$.

COROLLARY A prime left Noetherian ring is S.P.

Proof. A left Noetherian ring is a left Goldie ring.

COROLLARY A prime ring which satisfies a polynomial identity is S.P.

Proof. A theorem of Posner ([10], 179-186) shows that such a ring can be embedded as an order in a matrix ring D_n, where D is a division ring.

The proof of the following theorem depends on several theorems dealing with the free product of rings. The theorem

has several interesting consequences, although no subsequent theorems in this thesis depend on it. We outline the main ideas.

Let R be a prime ring whose centre is a field F and let $F[X]$ be the polynomial ring with variable X. Take the set of formal words $\{ r_1 X r_2 X \cdots X r_n : r_i \in R \}$ and define an equivalence relation $\approx$ on these words:

$$r_1 X r_2 \cdots X f r_i X \cdots r_n \approx f r_1 X r_2 \cdots X r_i X \cdots r_n ,$$

where $f \in F$. Let $R*F[X]$ denote the set of equivalence classes. We can define a ring structure on this in an obvious way: multiplication of formal words being $(r_1 X \cdots r_k)(s_1 X \cdots s_n) = r_1 X \cdots r_k s_1 X \cdots s_n$. This ring is called the free product of R and $F[X]$ over F. There is an embedding of R into this free product.

Remark. The free product of two rings does not necessarily exist, although it does in the above case. For information on this, one should see [2].

THEOREM 3.3. Every prime ring can be embedded in a S.P. ring.

Proof. Let R be a prime ring. We have proved that a prime ring can be embedded in an algebra; hence we may assume that the centre of R is a field F. Let $S = \{a_i\}_{i \in I}$ be a basis for R over F. Then $\{a_{i_1} X a_{i_2} X \cdots X a_{i_k} : a_{i_j} \in S\}$ is a basis for $R*F[X]$ over F (see [2] and [14]). Thus if u and v are nonzero elements of $R*F[X]$, then $uXv \neq 0$, and the theorem follows by taking $S(v) = \{X\}$.

We have shown that the class of S.P. rings is 'much larger' than the classes of domains or prime Noetherian rings. Neither of these classes has the above embedding property.

We now prove two theorems which extend to S.P. rings results known for domains and prime Noetherian rings.

DEFINITIONS A left ideal M of a ring R is said to be <u>essential</u> (or large) if the intersection with every nonzero left ideal is non-trivial; i.e. if $J \neq (0)$ is a left ideal, then $M \cap J \neq (0)$. The (left) <u>singular ideal</u>, denoted by Z(R), is the set of elements of R which annihilate essential left ideals of R on the right:

$$Z(R) = \{x \in R : Ex = (0), \text{ for some essential left ideal } E\}.$$

This is a two-sided ideal of R ([12]; 106).

The following theorem is due to David Handelman.

THEOREM 3.4. If R is a S.P. ring, then $Z(R) = (0)$.

Proof. If not, suppose $0 \neq x \in Z(R)$ and let $S(x) = \{y_1, \cdots y_n\}$. Since Z(R) is a two sided ideal, there exist essential left ideals $E_1, \cdots, E_n$, such that $E_i y_i x = (0)$, $i = 1, \cdots, n$. Now the intersection of a finite number of essential ideals is essential ([12], 62), and hence, $E = \cap E_i \neq (0)$. However, $ES(x)x = \{0\}$, a contradiction. We conclude that $Z(R) = (0)$.

THEOREM 3.5. If R is a regular S.P. ring, then R is simple.

Proof. Let R be such a ring and suppose $0 \neq r \in R$. Let $S(r) = \{s_1, \cdots, s_n\}$, and let I be the right ideal generated by all the $s_i r$. Since I is finitely generated and R is regular, $(r) \supset I = eR$, for some idempotent e. As $(1-e)I = (0)$, we have $(1-e)S(r)r = \{0\}$. This implies that $1-e = 0$, hence $(r) = R$, completing the proof of the theorem.

As a final remark, we look at two interesting connections between simple and S.P. rings, pointed out to the author by David Handelman. To begin with, theorem 3.5 gives us a complete solution to Kaplansky's problem in the case where the ring is in fact simple. It is trivial to see that all simple rings are S.P. Also, we can show that every prime ring can be embedded in a simple ring. We sketch the proof.

If R is a prime ring with centre F, then R is a subring of $S = R*F[X]$, by theorem 3.3. Since $Z(S) = (0)$, the complete ring of quotients of S satisfies the conditions of theorem 3.5 ([12], 94-107).

GROUP RINGS OF FREE PRODUCTS

While several papers have been written on primitive group algebras, very little seems to have been done in looking at the more general case, where the coefficient ring is not assumed to be a field. In this section we prove a theorem due to E. Formanek: group algebras of free products are primitive; we also look at several generalizations. In order to facilitate further discussion, we give two definitions.

DEFINITIONS Let α denote the class of rings R such that if $G = A*B$ is a free product of non-trivial groups A and B (except $|A| = |B| = 2$), and $|G| \geq |R|$, then $R[G]$ is primitive.

Let β denote the class of rings R for which there exists a group G, such that $R[G]$ is primitive.

Clearly, $\alpha \subset \beta$ and β contains all primitive rings. Formanek showed that the class of domains is a subclass of α, and so, α contains certain nonprimitive prime rings. We will show that α contains a larger class of prime rings, and, in fact, every prime ring is a subring of some element of α. Theorem 2.3 shows that the elements of β are prime. Examples will be give to show, however, that β is a proper subclass of the class of prime rings.

THEOREM 4.1.(Formanek [7]). If R is a domain (not necessarily commutative), then $R \in \alpha$.

We postpone the proof of this, for the moment. A more general result will be proved in theorem 4.3. We will now look at several interesting corollaries, all due to Formanek.

COROLLARY 1. Let R be a domain and G a nonabelian free group such that $|G| \geq |R|$. Then $R[G]$ is primitive.

Proof. $G = A*Z$, where $|A| \geq |R|$. (Z = infinite cyclic group).

COROLLARY 2. Suppose F is a field and G is a group. Then there exists a group H, containing G, such that $F[H]$ is primitive.

Proof. Let $H = G*A$, where A is a nonabelian free group such that $|A| \geq |F|$.

COROLLARY 3. $R[G]$ can be primitive even if R is not primitive.

Proof. Let $R = Z$ and $G = Z*Z$.

COROLLARY 4. If F is a field and $G = A*B$ is a free product of non-trivial groups not both of order two, then $F[G]$ is primitive.

Proof. Let K be the prime subfield of F. Since $\Delta(G) = \langle 1 \rangle$ (except for $G = Z_2*Z_2$, the free product two groups of order 2), theorem 4.1 shows that $K[G]$ is primitive, and theorem 2.8 allows us to conclude that $F[G]$ is primitive.

Remark. Let A and B be two-element groups generated by a and b respectively. Let H be the subgroup of $A*B$, consisting of those elements of the form ab $\cdots$ b or bab $\cdots$ ba. Then H is an

abelian group, and the sequence

$$1 \longrightarrow H \xrightarrow{\alpha} A*B \xrightarrow{\beta} Z_2 \longrightarrow 1$$

is exact, where α is the inclusion map and β is defined by letting $a = b$. Since $A*B$ is a finite extension of an abelian group, by theorem 2.4 , $F[A*B]$ is not primitive.

Remark. The cardinality condition, $|G| \geq |R|$, is not necessary if R is a field. Example 1 will show, however, that it is necessary in the case where R is a commutative domain.

COROLLARY 5. $F[G]$ can be primitive even if G has a non-trivial centre.

Proof. Let F be a countable field, and let $G = Z \times (Z*Z)$. $F[Z]$ is a domain, so $(F[Z])[Z*Z]$ is primitive. Since

$$(F[Z])[Z*Z] = F[Z \times (Z*Z)],$$

we see that $F[Z \times (Z*Z)]$ is primitive even though

$$\Delta(Z \times (Z*Z)) = Z = \text{centre}\,(Z \times (Z*Z)).$$

COROLLARY 6. There exist groups such that $F[G]$ is primitive if and only if F is countable.

Proof. Let $G = Z \times (Z*Z)$. If F is countable, then $F[G]$ is primitive. On the other hand, if $|F| > |G|$, then $F[G]$ is not primitive, by theorem 2.9.

THEOREM 4.2. The class of primitive rings is not inductive.

Proof. Let d be a countable ordinal, and let $S(d) = \{X_t : t \leq d\}$.

Let $R_d = Z_2[S_d]$. For each countable ordinal d, R_d is countable, and hence, $R_d[Z*Z]$ is primitive. If T is the set of countable ordinals, then $R = \bigcup R_d$ is a polynomial ring in uncountablely many variables. We will show, in example 1 (p. 30), that

$$R[Z*Z] = \bigcup R_d[Z*Z]$$

is not primitive, and this gives the necessary counterexample.

In proving theorems 4.3 and 4.4, we draw heavily from the methods used in [7]. We start with some definitions.

DEFINITIONS If G is a group, $G' = G - \{1\}$. Let $G = A*B$. We say that $g \in G$ is of <u>type AA</u> and has <u>length</u> 2n+1 if it has the form

$$g = a_1 b_1 a_2 b_2 \cdots a_n b_n a_{n+1}, \quad a_i \in A', \ b_i \in B'.$$

We define elements of types AB, BA and BB, and their lengths, in a similar way.

THEOREM 4.3. If R is a strongly prime ring, then $R \in \alpha$.

Proof. Let $G = A*B$. We may assume, without loss of generality, that $|A| \geq |B| > 1$.

Case 1. A is infinite. We obviously have $|R[G]| = |G| = |A| = |A'|$. Let

$$W : A' \longrightarrow (R[G] - \{0\}) \times N$$

be a bijection, and let $A^* = W^{-1}[(R[G] - \{0\}) \times 1]$. For each $a^{(1)} \in A^*$, there corresponds a unique element in $R[G] - \{0\}$, which we denote by $r_1(a)$. Let $a^{(n)}$ denote the element mapped

onto $r_1(a) \times n$. For $a^{(1)} \in A^*$, let $g_1(a)$ be an element of maximal length in the support of $r_1(a)$, and suppose that in $r_1(a)$, $g_1(a)$ has coefficient r. Let b be a fixed element of B', and set $h_1(a)$ equal to

$$\sum_i [s_i b r_1(a) a^{(i)} + s_i r_1(a) a^{(i)} b] + 1,$$

if $g_1(a)$ is of type AB or $g_1(a) = 1$, and where the sum is taken through all $s_i \in S(r)$. Since R is S.P., this sum is finite. Similarly we define $h_1(a)$ equal to

$$\sum_i [s_i b r_1(a) b a^{(i)} + s_i r_1(a) b a^{(i)} b] + 1, \text{ if } g_1(a) \text{ is of type AA,}$$

$$\sum_i [s_i r_1(a) b a^{(i)} b + s_i a^{(i)} r_1(a) b a^{(i)}] + 1, \text{ if } g_1(a) \text{ is of type BA, and}$$

$$\sum_i [s_i r_1(a) a^{(i)} b + s_i a^{(i)} r_1(a) a^{(i)}] + 1, \text{ if } g_1(a) \text{ is of type BB.}$$

If $\alpha \in R[G]$, then an element of maximal length in the support of $\alpha h_1(a)$ ends in either $a^{(n)}b$ or $a^{(n)}$, in its reduced form. Hence, an equation of the form

$$\alpha_1 h_1(a_1) + \alpha_2 h_1(a_2) + \cdots + \alpha_k h_1(a_k) = 1$$

is impossible, since group elements of maximal length in the support of $\alpha_i h_1(a_i)$, end in either $a_i^{(n)}b$ or $a_i^{(n)}$, for some n.

Let M be the left ideal of R[G], generated by all the $h_1(a)$. The above result shows that M is a proper left ideal. Also, by the definition of $h_1(a)$, it is clear that M is comaximal with every two sided nonzero ideal of R[G]. Hence R[G] is left primitive.

Case 2. A is finite. In this case, R[G] is countable, thus, we

have a bijection

$$W: N \longrightarrow (R[G] - \{0\}) \times N.$$

Let $N^* = W^{-1}[(R[G] - \{0\}) \times 1]$. To each $n = n(1) \in N^*$, there corresponds a unique element in $R[G] - \{0\}$, which we denote by $r(n)$. Let $n(k)$ denote the element of N mapped onto $r(n) \times k$. For $n(1) \in N^*$, let $g(n)$ be an element of maximal length in the support of $r(n)$, and suppose that in $r(n)$, $g(n)$ has coefficient r.

Let a and c be fixed distinct elements of A', and let b be a fixed element of B'. Given $n = n(1) \in N^*$, set $h(n)$ equal to $\sum_i [s_i r(n)c(ba)^{n(i)}b + s_i br(n)c(ba)^{n(i)}] + 1$, if $g(n)$ is of type AB, and where the sum is taken through all $s_i \in S(R)$. Similarly, we define $h(n)$ equal to

$\sum_i [s_i r(n)bc(ba)^{n(i)}b + s_i br(n)bc(ba)^{n(i)}] + 1$, if $g(n)$ is of type AA,

$\sum_i [s_i ar(n)bc(ba)^{n(i)} + s_i r(n)bc(ba)^{n(i)}b] + 1$, if $g(n)$ is of type BA,

$\sum_i [s_i ar(n)c(ba)^{n(i)} + s_i r(n)c(ba)^{n(i)}b] + 1$, if $g(n)$ is of type BB.

As in case 1, the elements of maximal length in the support of $h(n)$ can be identified, since they end in $(ba)^{n(i)}$ or $(ba)^{n(i)}b$, for some i. Thus, if M is the left ideal of $R[G]$, generated by all the $h(n)$, then M is a proper left ideal comaximal with every nonzero two sided ideal of $R[G]$. Hence $R[G]$ is primitive.

This completes the proof of the theorem.

This theorem has several corollaries which follow from our knowledge of S.P. rings.

COROLLARY If R is a domain, a prime Goldie ring, or a prime ring satisfying a polynomial identity, then $R \in \alpha$.

Proof. Trivially, all domains are S.P. By theorem 3.2 and corollaries, prime left Goldie rings and prime P.I. rings are S.P.

The converse of the above theorem is not true. The following theorem proves that another class of prime rings makes up part of α.

THEOREM 4.4. If R is a prime ring with a minimal left ideal, then $R \in \alpha$.

Proof. Let $G = A*B$. We may assume, without loss of generality, that $|A| \geq |B| > 1$. In proving this theorem, we will deal only with the case where A is infinite. The case where A is finite will follow easily, by modifying the proof, as we did in proving case 2 of the previous theorem.

Since R is semiprime, the minimal left ideal is of the form Re, where $e^2 = e \neq 0$, and $D = eRe$ is a division ring ([12], 63). Thus, if $b = be$ and $a = eae$ are nonzero elements of Re and eRe, respectively, then $ba \neq 0$.

Let

$$r : A' \longrightarrow D[G] - \{0\}$$

be a bijection. Fix $b \in B'$. For $a \in A'$, let $g(a)$ be an element of maximal length in the support of $r(a)$, and put $h(a)$ equal to

$br(a)a + r(a)ab + 1$, if $g(a)$ is of type AB or $g(a) = 1$,

$br(a)ba + r(a)bab + 1$, if $g(a)$ is of type AA,

$r(a)bab + ar(a)ba + 1$, if $g(a)$ is of type BA,

$r(a)ab + ar(a)a + 1$, if $g(a)$ is of type BB.

If $\alpha' = \alpha e$ is a nonzero element of $R[G]$, then an element of maximal length in the support of $\alpha' h(a)$ ends in either ab or a. Hence, an equation of the form

$$\alpha_1 eh(a_1) + \alpha_2 eh(a_2) + \cdots + \alpha_k eh(a_k) = e, \quad 0 \neq \alpha_i = \alpha_i e,$$

is impossible, since an element of maximal length in the support of $\alpha_i h(a_i)$ ends in either $a_i b$ or a_i. Suppose we have an equation of the form

$$\alpha_1 h(a_1) + \alpha_2 h(a_2) + \cdots + \alpha_k h(a_k) = e,$$

where $\alpha_i \in R[G]$. Then

$$e = \Sigma\, \alpha_i h(a_i) = \Sigma\, \alpha_i h(a_i)e = \Sigma\, [\alpha_i e + \alpha_i(1-e)]h(a_i)e$$
$$= \Sigma\, \alpha_i eh(a_i)e,$$

which we have shown to be impossible.

Let M be the left ideal of $R[G]$, generated by all the $h(a)$. We have shown that M is proper; we claim that it is comaximal with every nonzero two sided ideal of $R[G]$. Suppose $r = r_1 g_1 + \cdots + r_k g_k \in J$, $r_1 \neq 0$, $r_i \in R$, $g_i \in G$, J a nonzero two sided ideal. Since R is prime, we can choose $u, v \in R$ such that $eur_1 ve \neq 0$, so $0 \neq eurve \in J \cap D[R]$. Thus $h(a) - 1 \in J$, for some a, and so M is comaximal with J.

Since $R[G]$ has a proper left ideal comaximal with every

ideal, $R[G]$ is primitive.

Embedding theorems are of interest in the study of primitive group rings. We have already seen two: the Formanek-Snider theorem for groups, and the Passman theorem for fields. We now give a third embedding theorem- one for prime rings.

THEOREM 4.5. Let R be a prime ring. Then there exists a prime ring S containing R such that $S \in \alpha$.

Proof. We may assume, without loss of generality, that the centre of R is a field F. R is a vector space over F and it can be embedded into $S = End_F(R)$ by defining $T_r \in S$ by

$$T_r(v) = rv$$

and then mapping

$$r \longmapsto T_r .$$

S is a prime ring with a minimal left ideal, so, by the previous theorem, $S \in \alpha$.

In examples 2 and 3 we will show that not every prime ring is the coefficient ring of some primitive group ring, so the embedding theorem is the most we can hope for.

We will now look at some necessary conditions which R and G must satisfy if $R[G]$ is primitive. From now on, we will assume that G is arbitrary and not necessarily a free product.

DEFINITION An ideal I is said to be <u>nilpotent on finite sets</u> if, given a finite subset $S \subset I$, there exists an integer n, such that for any sequence $s_1, s_2, \cdots, s_n$ of n terms from S

(repetitions allowed), $s_1 s_2 \cdots s_n = 0$.

THEOREM 4.6. Let G be a group and R be a ring with a nonzero ideal nilpotent on finite sets. Then R[G] is not semiprimitive.

Proof. It is enough to show that R[G] has a nonzero nil ideal. Let I be an ideal of R which is nilpotent on finite sets. Using the above definition, we see that I[G] is a nil ideal of R[G].

EXAMPLE 1. We give an example of a domain R such that R[G] primitive $\Rightarrow |G| \geq |R|$.

Let $R = Z_2[X_i]$, $i \in I$, where I is an uncountable set. Thus, R is a commutative polynomial ring in uncountablely many variables. Suppose G is countable. We will derive a contradiction by assuming that R[G] is primitive. Let M be a left ideal comaximal with every nonzero two sided ideal of R[G]. If R[G] is primitive, we may assume that M is proper. For every X_i, there exists $a_i \in (X_i)$ such that $a_i + 1 \in M$. Since the set $\{a_i\}$ is uncountable, while the set $\bigcup \text{supp.}(a_i)$ is only countable, there exists an infinite sequence $a_{i_1}, a_{i_2}, \cdots$, such that

(1) $a_{i_j} \in (X_{i_j}), a_{i_j} + 1 \in M$, $j = 1, 2, \cdots$,

(2) X_{i_k} does not occur in any coefficient of a_{i_j}, $j < k$, $k = 1, \cdots$,

(3) $\text{supp.}(a_{i_1}) = \text{supp.}(a_{i_2}) = \cdots$.

By linear dependence, there exists $b_i \in R$, $i = 1, \cdots, k$, for some k, $b_k \neq 0$, such that $b_1 a_{i_1} + \cdots + b_k a_{i_k} = 0$. Choose k minimal with respect to these properties. Using (1) and (2) we see

that $b_j \in (X_{i_k})$, $j<k$, and therefore, if we assume G.C.D. $(b_1,\cdots,b_k) = 1$, $b = b_1 + \cdots + b_k \neq 0$. Then

$$b = b_1(a_{i_1} + 1) + \cdots + b_k(a_{i_k} + 1) \in M,$$

so $(b) \subset M$, a contradiction, since M cannot be comaximal with (b).

EXAMPLE 2. We give an example of a prime ring R such that $R[G]$ is not semiprimitive, where G is any group.

Let $F = Z_2[X,Y_i]$, $i = 1, 2, \cdots$, be the free Z_2-algebra in noncommuting variables. Let

$$m = X^{i_1} Y_{j_1} X^{i_2} \cdots Y_{j_{n-1}} X^{i_n}, \quad i_k \geq 0, \ j_k \geq 1,$$

repetitions allowed, be an arbitrary monomial in F.
Let I be the ideal of F generated by all monomials containing at least one Y term such that

$$\Sigma\, i_k > \max\{j_k\}. \qquad (*)$$

Let $R = F/I$. We claim that this is the desired example.

If f_1 and f_2 are nonzero elements of R, then, for sufficiently large n, $f_1 Y_n f_2 \neq 0$. Hence R is prime. However, (XY_1) is an ideal nilpotent on finite sets, so, by theorem 4.6, $R[G]$ is not semiprimitive.

In the above example, we seem to be proving too much. For this reason we give another, perhaps more interesting, example.

EXAMPLE 3. We give an example of a prime semiprimitve ring R such that $R[G]$ is not primitive, where G is any group.

Let us modify example 2 by replacing (*) by

$\Sigma\, i_k > (\max\{j_k\})$(the number of times $Y_{\max j_k}$ appears),

everything else remaining the same. Let the resulting ring be R. Osofsky constructed this ring as an example of a semiprimitive ring with nonzero singular ideal.

THEOREM 4.7. (Osofsky [15]) R is a prime semiprimitive ring.

Proof. If f_1 and f_2 are nonzero elements of R, then, for sufficiently large n, $f_1 Y_n f_2 \neq 0$. Thus R is prime. Suppose $0 \neq f = m_1 + \cdots + m_k \in R$, where the m_i are nonzero distinct monomials of R. Choose h large enough so that

(1) if Y_i occurs in f, then $h > i$,

(2) $h > 2$ degree (f).

We claim that $1 - fY_h$ is not right invertible, and hence, $f \notin \overline{J(R)}$.

Suppose

$$[1 - (m_1 + \cdots + m_k)Y_h][1 + (M_1 + \cdots + M_n)] = 1,$$

where the M_i are distinct nonzero monomials of R, all nonconstant. In the expansion of the product on the right, all the nonconstant monomials will be zero or they will cancel trivially. Thus $0 \neq m_1 Y_h$ must cancel. It cannot cancel with a term of the form $m_i Y_h M_j$, since Y_h occurs in the middle of the latter monomial; therefore $m_1 Y_h = M_{i_1}$, for some i_1. Since $0 \neq m_1 Y_h m_1 Y_h$ (by our choice of h), $m_1 Y_h M_{i_1} = M_{i_2}$, for some i_2. Continuing in this way, we get an unending sequence of distinct nonzero monomials in the product, giving a contradiction.

We conclude that $J(R) = (0)$.

THEOREM 4.8. Let G be a group. Then $R[G]$ is not primitive.

Proof. Proof by contradiction. Assume that $R[G]$ is primitive, and let M be a proper left ideal of $R[G]$ comaximal with every nonzero two sided ideal. By hypothesis, there exists $a \in (X)$ such that $a-1 \in M$. Choose h so that if Y_i occurs in a, $h > i$. Then there exists $b \in (Y_h)$ such that $b-1 \in M$. Let n be any positive integer and consider $a^n b$. By our choice of h, $(\max\{j_k\})$(the number of times $Y_{\max j_k}$ appears) is independent of n in any monomial of $a^n b$. However, in such a monomial, X occurs at least n times, and so for sufficiently large n, $a^n b = 0$. Then

$$-1 = a^n(b-1) + [\sum_{i=0}^{n-1} a^i](a-1) \in M,$$

contradicting the fact that M is proper. In a similar way we can show that $R[G]$ is not right primitive.

In examples 2 and 3 the ring R has a nonzero singular ideal. We might therefore ask if this is a criterion for the nonprimitivity of $R[G]$. In the next chapter, we answer this in the negative.

SINGULAR PRIMITIVE RINGS

There are few connections between this chapter and the previous chapters on primitive group rings. Rather, the continuity is provided by the method used in proving the primitivity of rings.

DEFINITION A ring with nonzero singular ideal is said to be singular.

The existence of singular primitive rings has been an open problem for several years. In [15], Osofsky gave an example of a singular semiprimitive ($J(R) = (0)$) ring; however, as we have shown in example 3 of the previous chapter, this ring is not primitive. In [4], Faith listed the problem in the final chapter on open problems. Since then, the problem has remained open. We give an affirmative answer to the problem, in this chapter.

Let $F = Z_2[X, Y_j]$, $j = 1, 2, \cdots$, be the free Z_2-algebra in noncommuting variables. Let

$$m = X^{i_1} Y_{j_1} X^{i_2} \cdots Y_{j_{n-1}} X^{i_n}, \quad i_k \geq 0, \ j_k \geq 1,$$

repetitions allowed, be an arbitrary monomial in F.

Define:

$$c(m) = \Sigma\, i_k = \text{degree of } X \text{ in } m$$

$$d(m) = \Sigma\, i_k + n-1 = \text{degree of } m$$

$$e(m) = \max \{j_k \text{ times the number of times } Y_{j_k} \text{ appears}\},$$
if m has a Y term,
$$= 0 \text{ , otherwise .}$$

Let I be the ideal generated by monomials m such that $c(m) > e(m) \geq 1$. I is "homogeneous" in the sense that a sum of distinct monomials is in the ideal if and only if each monomial is. This allows us to speak of monomials in F/I.

Let $R = F/I$. We claim that R is a singular primitive ring. This ring is very similar to Osofsky's ring, which we constructed in the last chapter.

If $0 \neq f = m_1 + m_2 + \cdots + m_n \in R$ is a sum of distinct nonzero monomials, we define:

$$c(f) = \max \{c(m_k)\}$$

$$d(f) = \max \{d(m_k)\}$$

$$e(f) = \max \{e(m_k)\}.$$

THEOREM 5.1 $Z(R) \neq (0)$.

Proof. Let $0 \neq f = m_1 + \cdots + m_n \in R$ be a sum of distinct nonzero monomials, and let $s = \min \{c(m_k)\}$. Choose $t > \max \{d(f), e(f)\}$. Then $g = X^{t-s} Y_t f \neq 0$ and $gX = 0$. Let E be the left ideal of R, generated by all the g, as f runs through all nonzero elements of R. We claim that E is an essential

left ideal. If $(0) \neq J$ is a left ideal, then let f be a nonzero element of J. Then $0 \neq g \in J \cap E$, hence, E is essential. But $EX = (0)$, and so, $X \in Z(R)$. This completes the proof of the theorem.

Since R is countable, we can order the nonzero elements $f_2, f_3, \cdots$. We start the numbering at 2 to simplify the notation in certain subsequent statements. Given f_n in this sequence, choose $j_n > 1$ large enough so that $Y_{j_n} f_n Y_{j_n} \neq 0$, and let

$$q_n = Y_1^{i_n} Y_{j_n} f_n Y_{j_n} Y_1^{i_n} Y_n \neq 0,$$

where $i_n > 2 \max \{d(f_n), e(f_n)\}$. For convenience choose $j_{n+1} > j_n$ and $i_{n+1} > i_n$. Suppose $q_n = a_1^{(n)} + \cdots + a_{k_n}^{(n)}$, where $a_i^{(n)}$ is a nonzero monomial in R, $i = 1, 2, \cdots k_n$. Let A be the Z_2-subalgebra of R generated by all the $a_i^{(n)}$.

LEMMA The following hold in A:

(1) If m_1, m_2 are any monomials of R and $a_i^{(n)}$, $a_j^{(m)}$ are generators of A, then $0 \neq m_1 a_i^{(n)} = m_2 a_j^{(m)} \Rightarrow i = j$, $m = n$.

(2) If $a_1, \cdots, a_n$ are generators of A, then $\pi a_i \neq 0$.

(3) A is a free Z_2-algebra on the given generators.

(4) If $a \in A$ and $b \in R$ and b is a sum of monomials, none of which is in A, then $ba \in A \Rightarrow ba = 0$.

Proof. (1) Since $a_i^{(n)}$ ends with Y_n and $a_j^{(m)}$ ends with Y_m, we conclude that $m = n$. Since the products are nonzero, they are equal if and only if they are identical. Now

$$m_1 a_1^{(n)} = \cdots Y_1^{i_n} Y_{j_n} b Y_{j_n} Y_1^{i_n} Y_n \,, \quad n \neq 1, \; j_n \neq 1.$$

As $i_n > d(b)$, we can 'decide' which generator occurs at the end of the product.

(2) $\pi a_k = \pi Y_1^{i_k} Y_{j_k} b_k Y_{j_k} Y_1^{i_k} Y_k$,

where $i_k > 2 \max \{d(b_k), e(b_k)\}$. If this product is zero, some segment of it must be a generator of I, say

$$m = Y_{j_s} b_s Y_{j_s} Y_1^{i_s} Y_s \cdots Y_1^{i_t} Y_{j_t} b_t Y_{j_t} .$$

If $s \neq t$, then $c(m) \leq \Sigma\, c(b_k)$ and $e(m) \geq \Sigma\, i_k$ and so $e(m) > c(m)$, a contradiction. If the generator contains only one b_k (or part of one b_k), the result is obvious.

(3) This follows from (1) and (2). If $a_{i_1} \cdots a_{i_k} = a_{j_1} \cdots a_{j_n}$, then (2) implies that this product is nonzero and using (1) inductively, we obtain $a_{i_k} = a_{j_n}$ etc.

(4) Since both I and A are generated by monomials, we will have finished if we can prove the result assuming both a and b are monomials. Suppose $a = a_{i_1} \cdots a_{i_k}$ and $ba = a_{j_1} \cdots a_{j_n} \neq 0$. Now $ba = ba_{i_1} \cdots a_{i_k}$, and so using (1) inductively, we get $a_{i_k} = a_{j_n}$, $a_{i_1} = a_{j_s}$, for some $1 \leq s \leq n$. We conclude that $b \in A$, a contradiction. If $a = 1$, the result is obvious.

THEOREM 5.2 R is left primitive.

Proof. Let M be the left ideal of R generated by $\{q_n + 1\}$, $n = 2, 3, \cdots$. We claim that M is a proper left ideal comaximal with every nonzero two sided ideal of R. If J is a nonzero

ideal of R, then $f_n \in J$, for some n, and hence, $q_n \in J$. Thus, $1 = q_n + 1 - q_n \in M + J$. If M is not proper, then for some $\{r_i\} \subset R$ and some integer n, we have

$$r_2(q_2 + 1) + \cdots + r_n(q_n + 1) = 1.$$

Let $r_i = b_i + c_i$, where each monomial in b_i is in A and c_i is the sum of monomials not in A. Then

$$\Sigma\ c_i(q_i + 1) = 0\ ,$$

by the previous lemma, and

$$\Sigma\ b_i(q_i + 1) = 1.$$

The latter equation gives a non-trivial relation in the free algebra A and hence is impossible. Since we arrive at a contradiction by assuming that M is not proper, we conclude that M is a proper left ideal. We now use theorem 1.2 to complete the proof of the primitivity of R.

BIBLIOGRAPHY

1. Bergman, G.: A ring primitive on the right but not on the left, Proc. Amer. Math. Soc. 15 , (1964), 473-475.
2. Cohn, P. M.: On the free product of associative rings, Math. Zeit. 71, (1959), 380-398.
3. Connell, I.: On the group ring, Canadian J. Math. 15, (1963), 650-685.
4. Faith, C.: Lectures on Injective Modules and Quotient Rings, Lecture Notes in Mathematics #49, Springer-Verlag.
5. Fisher, J. and Snider, R. L.: Prime von Neumann regular rings and primitive group algebras, Proc. Amer. Math. Soc. (to appear).
6. Formanek, E.: A problem of Passman on semisimplicity, Bull. London Math. Soc. 4, (1972).
7. _____ : Group rings of free products are primitive, J. Algebra (to appear).
8. _____ and Snider, R. L.: Primitive group rings, Proc. Amer. Math. Soc. 36, (1972), 650-685.
9. Goodearl, K.: Prime ideals in regular self-injective rings, Canadian J. Math. (to appear).
10. Herstein, I. N.: Noncommutative Rings, Carus Monograph, Math. Assoc. of Amer. (1971).

11. Kaplansky, I.: Algebraic and Analytic Aspects of Operator Algebras, C.B.M.S. Regional Conference Series in Mathematics, Amer. Math. Soc. (1970).

12. Lambek, J.: Lectures on Rings and Modules, Ginn and Blaisdell (1966).

13. Lawrence, J.: A primitive ring with nonzero singular ideal.

14. Martindale, W.: A note on generalized polynomial identities, Canad. Math. Bull. 15, (1972), 605-606.

15. Osofsky, B.: A non-trivial ring with non-rational injective hull, Canad. Math. Bull. 10, (1967), 275-282.

16. Passman, D.: Infinite Group Rings, Marcel Dekker, (1971).

17. _____ : Primitive group rings, Pac. J. Math. (to appear).

18. _____ : The semisimplicity problem for group rings.

19. _____ : Advances in group rings, Israel J. Math. (to appear).

20. Roseblade, J.: Group rings of polycyclic groups, J. Pure and Appl. Alg. (to appear).

21. Rosenberg, A.: On the primitivity of the group algebra, Canadian J. Math. 23, (1971), 536-540.

The following references were added after the main body of the thesis had been completed.

22. Handelman, D. and Lawrence, J.: Strongly prime rings, in preparation.

23. Lawrence, J.: The coefficent ring of a primitive group ring, in preparation.