De Sitter spacetime as a coherent state:

non-perturbative analysis

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Abstract

We discuss an explicit construction of a four-dimensional de Sitter spacetime as a coherent (Glauber-Sudarshan) state. The path integrals approach is used to calculate the scalar field expectation value on a coherent state. A special diagrammatic technique is introduced to simplify and visualize the calculations. The perturbative computations lead to an asymptotic series which are then analyzed by means of the Borel resummation technique. This procedure reveals the non-perturbative structure of the system. The analysis of our simplified toy model suggests the possible construction of a stable four-dimensional de Sitter spacetime as a Glauber-Sudarshan coherent state. The non-perturbative data gleaned from the Borel resummation of the asymptotic series strongly points towards a positive cosmological constant.

Abrègè

Nous discutons une construction explicite d'un espace de Sitter en quatre dimensions comme état cohérent (Glauber-Sudarshan). L'approche des intégrales de chemin est utilisée pour calculer la valeur d'attente du champ scalaire sur un état cohérent. Une technique graphique spéciale est introduite pour simplifier et visualiser les calculs. Les calculs perturbateurs conduisent à une série asymptotique qui est ensuite analysée au moyen de la technique de reprise de Borel. Cette procédure révèle la structure non-perturbative du système. L'analyse de notre modèle simplifié suggère la construction possible d'un espace de Sitter stable en quatre dimensions comme un état cohérent. Les données non-perturbatives glanées à partir de la reprise de Borel de la série asymptotique pointe fortement vers une constante cosmologique positive.

To all human beings and their futile search for meaning

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Chapter 1

Introduction: de Sitter space and coherent states

It is a truth universally acknowledged that any respected theory of physics audacious enough to claim to be a theory of everything must reproduce our Universe. Since its birth in the middle of the 1970s, string theory underwent a lightning development from a theory of strong interactions to a theory of quantum gravity, eventually developing, arguably, into a theory of everything, dealing with fundamental questions of the Universe and its underlying laws. According to our current knowledge, we live in de Sitter (dS) spacetime, and so string theory must contain de Sitter spacetime. The problem here is that not a single rigorous 4-dimensional de Sitter vacuum has been built in string theory so far [1].

The recent spectral and photometric observations of Ia supernovae suggest an eternally expanding Universe which is accelerated by a positive vacuum energy density [2]. These experimental results motivate the desire to find de Sitter vacua of supergravity and string theory and to construct models for late-time cosmology. The no-go theorem ensures that a solution with a positive cosmological constant cannot be obtained in string or M-theory by using only supergravity fluxes or branes and anti-branes [3] [4] [5]. The most recent findings point out that even the contributions from the O-planes cannot save the day, suggesting that the only hope comes from the quantum corrections in string theory [6] [7] [8] [9]. But the no-go theorem is only the tip of the iceberg, and the main challenges in building dS spacetime come from the very fundamental aspects of quantum gravity, such as trans-Planckian issues, which threaten the notion of Wilsonian effective action for accelerating backgrounds [10] [11] [11]. One of the promising solutions is the introduction of time-dependent degrees of freedom and to view de Sitter spacetime as a state instead of a vacuum [12] [13].

It turns out that it is possible to realize a quantum mechanically stable coherent state in the full string theory, which would replace the usual classical configuration. The coherent state representation of the dS vacuum solves the problems of zero point energy cancellations from the bosonic and the fermionic degrees of freedom, the spontaneous supersymmetry braking, and the finite entropy of the dS spacetime. The expectation value of the graviton operator, represented in our analysis by a scalar field, is calculated as an asymptotic series in powers of the coupling constant with a factorial growth. One of the fruitful techniques for extracting information from the asymptotic series is the Borel resummation method.

It is a well know fact, first pointed out by Dyson [14], that the perturbative expansions of quantum field theories are asymptotic. The asymptotic nature of the theory is due to the factorial growth of the number of Feynman diagrams. There is no observed cancellation from different diagrams of the same order, and the growth of the perturbative coefficients has the form $c_n \sim n!$, leading to the vanishing radius of convergence for the small expansion parameter [15] [16]. Despite its unsatisfying divergent nature, asymptotic perturbative expansions provide us with some impressive results, such as high-precision computation of the anomalous magnetic moment of the electron in quantum field theory. The reason is that the asymptotic nature becomes evident only at very high loop orders (around the 137th loop in the case of perturbative quantum electrodynamics).

At the same time, it can be argued that asymptotic series hide some information about the exact answer that they approximate. The Borel summation uses the analytic continuation of a divergent asymptotic series by means of the Borel transform with a subsequent contour integration in the complex plane [17]. This procedure has its limits, giving rise to ambiguities in the final result related to the presence of poles in the Borel transform and different choices of integration contours [18]. There is a conjecture that the poles of the Borel transform of the perturbative asymptotic series are associated with new non-perturbative physics and non-perturbative objects, such as instantons, D-branes etc. Although in certain cases, we are still lacking a decent physical explanation for the structures emerging from the resummation procedure [19].

1.1 Coherent states of the simple harmonic oscillator

Here we want to give a very brief introduction to the notion of a coherent state using a simple harmonic oscillator as a toy example. Coherent states are quantum systems that exhibit some sort of classical behaviour [20]. We start our discussion with the simplest example of a coherent state $|\hat{x}_0\rangle$ of a simple harmonic oscillator.

A coherent state in the real space $\mathbb{R}$ can be introduced via the notion of a translation operator T_{x_0}. Unitary translation operator act on quantum states by moving them by a distance x_0:

$$T_{x_0} = \exp\left(-\frac{i}{\hbar}\hat{p}x_0\right). \tag{1.1}$$

We note that the translation operator forms a group:

$$\begin{aligned}
&T_{x_0}\,T_{y_0} = T_{x_0+y_0},\\
&\text{there exist a unit element :}\quad T_0 = I,\\
&\text{the inverse is given by}\quad T_{-x_0}.
\end{aligned} \tag{1.2}$$

The coherent state is then defined as

$$|\hat{x}_0\rangle \equiv \exp\left(-\frac{i}{\hbar}\hat{p}x_0\right)|0\rangle\,, \tag{1.3}$$

where $|0\rangle$ is the ground state of the harmonic oscillator.

The physical interpretation of the coherent state is the translation of the ground state by a distance x_0.

To find the representation of the coherent states in the energy basis, we first rewrite the momentum operator in terms of creation and annihilation operators:

$$\hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}\left(a^\dagger - a\right) = i\frac{\hbar}{\sqrt{2}d}\left(a^\dagger - a\right), \tag{1.4}$$

where we defined $d = \sqrt{\frac{\hbar}{m\omega}}$. Now the coherent state $|x_0\rangle$ is given by

$$e^{-i\frac{i}{\hbar}\hat{p}x_0}|0\rangle = \exp\left(\frac{x_0}{\sqrt{2}d}\left(a^\dagger - a\right)\right)|0\rangle. \tag{1.5}$$

The previous analysis allows us to make further generalizations. We rewrite the coherent state in the following suggestive way:

$$|\alpha\rangle \equiv e^{\alpha(a^\dagger - a)}|0\rangle, \tag{1.6}$$

with $\alpha = \frac{x_0}{\sqrt{2}d} = x_0\sqrt{\frac{m\omega}{2\hbar}}$. As a next level of generalization, we take α to be a complex number, $\alpha \in \mathbb{C}$, keeping in mind that the operators in the exponential should remain anti-hermitian, making the exponential unitary. We than define

$$|\alpha\rangle \equiv D(\alpha)|0\rangle = \exp\left(\alpha a^\dagger - \alpha^* a\right)|0\rangle, \tag{1.7}$$

where we introduced the unitary displacement operator

$$D(\alpha) \equiv \exp\left(\alpha a^\dagger - \alpha^* a\right). \tag{1.8}$$

The unitarity of $D(\alpha)$ assures that $\langle\alpha|\alpha\rangle = 1$. We also note that the action of the annihilation operator a on the coherent states $|\alpha\rangle$ generates its eigenstates:

$$a\ |\alpha\rangle = a\ e^{\alpha a^\dagger - \alpha^* a}\ |0\rangle = \left[a, \alpha a^\dagger - \alpha^* a\right]|\alpha\rangle = \alpha|\alpha\rangle. \tag{1.9}$$

To prove these identities we can use some variation of Baker-Campbell-Hausdorff expansion for any matrices A and B:

$$e^B\ Ae^{-B} = A + [B, A] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + ... \tag{1.10}$$

In a special case, when $[A,[A,B]] = 0$ we have a simple version of this expansion which will prove to be useful in our case

$$e^B\ Ae^{-B} = A + [B, A]. \tag{1.11}$$

Rearranging some elements of the identity, we obtain very useful result

$$[A, e^B] = [A, B]e^B. \tag{1.12}$$

In our example we identify $A = a, B = \alpha a^\dagger - \alpha^* a$ and use all the previous results. First note

$$a\, e^{\alpha a^\dagger - \alpha^* a}\ |0\rangle = \left[a,\ e^{\alpha a^\dagger - \alpha^* a}\right]\ |0\rangle\,, \tag{1.13}$$

since a annihilates the vacuum $|0\rangle$. Then we prove that the simple version of identity can be used. Indeed,

$$[A, [A, B]] = \left[\left[a, \alpha a^\dagger - \alpha^* a\right]\right] = [a, \alpha] = 0. \tag{1.14}$$

Finally, we find

$$\left[a,\ e^{\alpha a^\dagger - \alpha^* a}\right]\ |0\rangle = \left[a, \alpha a^\dagger - \alpha^* a\right] e^{\alpha a^\dagger - \alpha^* a}\,|0\rangle = \left[a, \alpha a^\dagger - \alpha^* a\right] |\alpha\rangle = \alpha\,|\alpha\rangle\,. \tag{1.15}$$

Thus, we found the eigenstates of the non-hermitian operator a. Since operator a is non-hermitian, its eigenvalues can be complex, the eigenvectors cannot be orthogonal, and coherent states cannot generate a complete basis.

Here we introduce some comments related to generalization of this construction for the toy model discussed later. The problem arises when we move from a non-interacting theory, like simple harmonic oscillator to a highly interacting theory, like M-theory. In this last case, we have no choice, but shift the interacting vacuum $|\Omega\rangle$ by a displacement operator and check if the new state still has the features of the vacuum-shifted coherent state. If we denote the new coherent state by $|\sigma\rangle$ then it can be constructed as

$$|\sigma\rangle = D_{\text{int}}(\sigma)\,|\Omega\rangle\,, \tag{1.16}$$

where now $D_{\text{int}}(\sigma)$ is a displacement operator of an interacting theory. We can still formally construct it in terms of annihilation and creation operators

$$D_{\text{int}}(\sigma)\,|\Omega\rangle = \exp\left(\alpha a^\dagger_{\text{eff}} - \alpha^* a_{\text{eff}}\right)|\Omega\rangle\,, \tag{1.17}$$

where a_{eff} annihilates the interacting vacuum. We also note that the general form of the displacement operator D_{int} that fixes the form of a_{eff} and reproduces the background metric of our theory can be written down for any time t in terms of the non-unitary version of the same free-vacuum displacement operator $D_0^{\text{int}}(\sigma)$

$$D_{\text{int}}(\sigma, t) = D_0^{\text{int}}(\sigma) \exp\left(i \int_{-T}^{t} d^{11}x \, \mathbf{H}_{\text{int}} \right), \tag{1.18}$$

where $\mathbf{H}_{\text{int}}$ is a full interacting part of the M-theory Hamiltonian and $T \to \infty$ in a slightly imaginary direction. We can that now the norm of the coherent state in the interacting theory is not one, and the expectation value of any operator ϕ in this theory requires a division by a denominator of the form

$$\int [\mathcal{D}\phi] \, e^{i\mathbf{S}} D_{\text{int}}(\sigma)^{\dagger} D_{\text{int}}(\sigma), \tag{1.19}$$

where $\mathbf{S}$ is the total M-theory action.

To simplify the form of the coherent state α, we use the commutator identity

$$e^{X+Y} = e^X \; e^Y \; e^{-\frac{1}{2}[X,Y]}. \tag{1.20}$$

We then rearrange the exponential to obtain

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \; e^{\alpha a^{\dagger}} \, |0\rangle \,. \tag{1.21}$$

To gain some insights into the nature of the complex eigenvalue α, we note that real α is equivalent to the expectation value of the position operator $\hat{x}$ in the coherent state at time $t = 0$, that is the initial position x_0 of the coherent state. If we allow α to be complex, we find

$$\langle\alpha| \, \hat{x} \, |\alpha\rangle = \frac{d}{\sqrt{2}} \langle\alpha| \left(a + a^{\dagger}\right) = \sqrt{2} d \; \text{Re}\{\alpha\}, \tag{1.22}$$

and

$$\langle\alpha| \, \hat{p} \, |\alpha\rangle = \frac{i\hbar}{\sqrt{2} d} \langle\alpha| \left(a - a^{\dagger}\right) = \frac{\sqrt{2}\hbar}{d} d \; \text{Im}\{\alpha\}. \tag{1.23}$$

Thus we find the following insightful interpretation of the coherent state eigenvalues:

$$\alpha = \frac{\langle\hat{x}\rangle}{\sqrt{2} d} + i \frac{\langle\hat{p}\rangle d}{\sqrt{2}\hbar}. \tag{1.24}$$

In the next section, we will extend these fundamental ideas of the coherent states and displacement operator to the case of scalar fields.

Chapter 2

Path integral toolkit

2.1 Path integral formalism for the displaced vacuum: a toy model

Our first attempt to implement a path integral approach for the displaced vacuum will be based on free massive scalar field theory in 3+1 dimensions with the Lagrangian of the form:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2. \tag{2.1}$$

In particular, we want to compute the expectation value of the scalar field ϕ on a coherent state $|\alpha\rangle$. The general procedure invites us to find the vacuum solutions of the theory in the first place. Luckily, the vacuum of this theory is simply given by $\phi = 0$.

Using the most minus signature for the rest of this subsection, the off-shell Fourier representation of the fields could be written as

$$\phi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3\, 2\omega_{\mathbf{k}}} \left(a_{\mathbf{k}} e^{ik\cdot x} + a^\dagger_{\mathbf{k}} e^{-ik\cdot x}\right) = \int d^4k\ \phi_k\ e^{ik\cdot x}, \tag{2.2}$$

where k_0 and $\mathbf{k}$ are not related in any way since the fields are off-shell.

We need to do some extra work to find the written form for the creation and annihilation operators in Fourier modes. We are looking for a replacement for the original definition in terms of operators

$$
\begin{aligned}
a_{\mathbf{k}} &= \sqrt{\frac{\omega_{\mathbf{k}}}{2}} \left(\hat{\phi}_k + \frac{i\hat{\pi}_k}{\omega_{\mathbf{k}}} \right), \\
a^{\dagger}_{\mathbf{k}} &= \sqrt{\frac{\omega_{\mathbf{k}}}{2}} \left(\hat{\phi}^{*}_k - \frac{i\hat{\pi}^{*}_k}{\omega_{\mathbf{k}}} \right),
\end{aligned} \tag{2.3}
$$

by its analog in terms of fields. We invert the Fourier integrals for the field and its conjugate momenta $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$:

$$
\begin{aligned}
\int d^4x \; \phi(x) e^{ik\cdot x} &= \frac{1}{2\omega_{\mathbf{k}}} \left(a_{\mathbf{k}} + a^{\dagger}_{\mathbf{k}} e^{2ik_0 t} \right), \\
\int d^4x \; \pi(x) e^{ik\cdot x} &= -\frac{i}{2} \left(a_{\mathbf{k}} - a^{\dagger}_{\mathbf{k}} e^{2ik_0 t} \right).
\end{aligned} \tag{2.4}
$$

From the last set of equations, we find the definition of the creation and annihilation operators as fields:

$$
\begin{aligned}
a_{\mathbf{k}} &= \int d^4x \left(\omega_{\mathbf{k}} \phi(x) + i\pi(x) \right) e^{-ik\cdot x} = \left(\omega_{\mathbf{k}} + k_0 \right) \phi_k, \\
a^{\dagger}_{\mathbf{k}} &= \int d^4x \left(\omega_{\mathbf{k}} \phi(x) - i\pi(x) \right) e^{ik\cdot x} = \left(\omega_{\mathbf{k}} + k_0 \right) \phi^{*}_k,
\end{aligned} \tag{2.5}
$$

where the Fourier mode of the conjugate momentum $\pi_k = -ik_0\phi_k$, and $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$.

In the next step, we want to implement our creation/annihilation operators field representation into the definition of displacement operators. We recall that we defined the coherent state as an action of the displacement operator $\mathbb{D}_0(\alpha)$ over the free vacuum as

$$
\mathbb{D}_0(\alpha) \left| 0 \right\rangle = \exp\left(\alpha a^{\dagger} - \frac{|\alpha|^2}{2} \right) \left| 0 \right\rangle, \tag{2.6}
$$

where $\left| 0 \right\rangle$ is the free vacuum, and $\mathbb{D}_0(\alpha)$ is a non-unitary version of the displacement operator. Now we use the field representation of $a, a^{\dagger}$ to write down

$$
e^{\alpha a^{\dagger}} = \exp\left(\int d^4k \; a^{\dagger}_k \; \alpha_k \right) = \exp\left\{ \int d^4k \; (\omega_{\mathbf{k}} + k) \phi^{*}_k \tilde{\alpha}_k \right\}. \tag{2.7}
$$

where to make the last expression Lorentz invariant, we have redefined α_k to be

$$
\tilde{\alpha}_k = \frac{\alpha_k}{k_0 + \omega_{\mathbf{k}}}, \tag{2.8}
$$

with $\alpha_1(k) = 0$. Similarly, the conjugate operator

$$\left(e^{\alpha a^\dagger}\right)^\dagger = e^{\alpha^* a} = \exp\left\{\int d^4k\ (\omega_{\mathbf{k}} + k)\phi_k \tilde{\alpha}_k^*\right\}, \tag{2.9}$$

where $\tilde{\alpha}_k^* = \frac{\alpha_k^*}{k_0+\omega_{\mathbf{k}}}$.

Now we are almost ready to write down the expectation value of the scalar field ϕ on the coherent state $|\alpha\rangle$. We just note that since we chose the displacement operator to be non-unitary, we need to normalize the expectation value by the $\langle\alpha|\alpha\rangle$ factor:

$$\langle\phi(x)\rangle_\alpha = \frac{\langle\alpha|\,\phi(x)\,|\alpha\rangle}{\langle\alpha|\alpha\rangle} = \frac{\int \mathcal{D}\phi e^{iS_0}\hat{\mathbb{D}}_0^\dagger(\alpha)\ \phi(x)\ \hat{\mathbb{D}}_0(\alpha)}{\int \mathcal{D}\phi e^{iS_0}\hat{\mathbb{D}}_0^\dagger(\alpha)\ \hat{\mathbb{D}}_0(\alpha)}, \tag{2.10}$$

where S_0 is a free scalar field action:

$$S_0 = \int d^4x\mathcal{L} = \int d^4x\left[\frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}m^2\phi^2\right]. \tag{2.11}$$

In momentum space, this action becomes

$$S_0 = \int d^4k\left(k^2 - m^2\right)|\phi_k|^2. \tag{2.12}$$

The continual path integral can be approximated by a finite one. The integral measure is discretized as

$$\mathcal{D}\phi = \prod_i d\phi(x_i), \tag{2.13}$$

and the field values $\phi(x_i)$ can be represented by a discrete Fourier series:

$$\phi(x_i) = \frac{1}{V}\sum_n e^{-ik_n\cdot x_i}\phi(k_n), \tag{2.14}$$

where $k_n^\mu = \frac{2\pi n^\mu}{L}$, with n^μ an integer and $V = L^4$ a volume of 4 dimensional space. The Fourier coefficient $\phi(k)$ is complex with the constraint $\phi^*(k) = \phi(-k)$, which expresses the fact that $\phi(x)$ is real. We want to consider the real and imaginary parts of $\phi(k_n)$ with $k_n^0 > 0$ as independent variables. Since this is a unitary transformation, the integral measure becomes

$$\mathcal{D}\phi(x) = \prod_{k_0^n>0} d\mathbf{Re}\phi(k_n)\ d\mathbf{Im}\phi(k_n). \tag{2.15}$$

With these preliminary notes on the path integral democratization procedure, we are ready to write down the expression for the expectation value of the field ϕ. The discretized free theory action becomes

$$\begin{aligned} e^{iS_0} &= \exp\left\{\frac{i}{V}\sum_k \left(k^2 - m^2\right)|\phi_k|^2\right\} \\ &= \exp\left\{-\frac{i}{V}\sum_k \left(m^2 - k^2\right)\left[(\mathbf{Re}\phi_k)^2 + (\mathbf{Im}\phi_k)^2\right]\right\} \end{aligned} \tag{2.16}$$

The rest of the numerator integrand is

$$\begin{aligned} \hat{\mathbb{D}}_0^\dagger(\alpha)\ \phi(x)\ \hat{\mathbb{D}}_0(\alpha) &= e^{\alpha^* a} e^{-\frac{|\alpha|^2}{2}}\ \phi(x)\ e^{\alpha a^\dagger} e^{-\frac{|\alpha|^2}{2}} \\ &\to \exp\left\{\frac{1}{V}\sum_{k'} (\omega_{\mathbf{k}'} + k'_0)\, \tilde{\alpha}^*_{k'}\phi_{k'}\right\}\ \exp\left\{\frac{1}{V}\sum_{q'} (\omega_{\mathbf{q}'} + q'_0)\, \tilde{\alpha}_{q'}\phi^*_{q'}\right\} \\ &\times \frac{1}{V}\sum_{k''} (\mathbf{Re}\phi_{k''} + i\mathbf{Im}\phi_{k''})\ e^{ik''x}\ \exp\left(-\frac{1}{V}\sum_{k'} \alpha_{k'}\ \alpha^*_{k'}\right), \end{aligned} \tag{2.17}$$

where in the second and the third lines we used the Fourier representation and the definition of creation and annihilation operators in terms of fields ϕ_k. We note, that

$$\frac{1}{V}\sum_k (\tilde{\alpha}^*_k\phi_k + \tilde{\alpha}_k\phi^*_k) = \frac{2}{V}\sum_k (\mathbf{Re}\ \alpha_k\ \mathbf{Re}\ \phi_k + \mathbf{Im}\ \alpha_k\ \mathbf{Im}\ \phi_k)\,. \tag{2.18}$$

With all the aforementioned rearrangements, we obtain the following expression for the nominator of the expectation value

$$\begin{aligned} \int \mathcal{D}\phi e^{iS_0} \hat{\mathbb{D}}_0^\dagger(\alpha)\ \phi(x)\ \hat{\mathbb{D}}_0(\alpha) &= \prod_{k_0^n > 0} d\mathbf{Re}\phi_k\ d\mathbf{Im}\phi_k \exp\left\{\frac{i}{V}\sum_k \left(k^2 - m^2\right)|\phi_k|^2\right\} \\ &\times \exp\left\{\frac{2}{V}\sum_{k'} (\omega_{\mathbf{k}'} + k'_0)\,(\mathbf{Re}\ \alpha_{k'}\ \mathbf{Re}\ \phi_{k'} + \mathbf{Im}\ \alpha_{k'}\ \mathbf{Im}\ \phi_{k'})\right\} \\ &\times\ e^{ik''x}\,\frac{1}{V}\sum_{k''} (\mathbf{Re}\phi_{k''} + i\mathbf{Im}\phi_{k''})\ \ \exp\left(-\frac{1}{V}\sum_{k'} \alpha_{k'}\alpha^*_{k'}\right). \end{aligned} \tag{2.19}$$

From this analysis, we conclude that the displacement operator $\hat{\mathbb{D}}_0(\alpha)$ shifts the center of the Gaussian, giving a non-zero value for the integral.

The denominator can be also calculated to give us the similar expression

$$\langle\alpha|\alpha\rangle = \int \mathcal{D}\phi e^{iS_0} \hat{\mathbb{D}}_0^\dagger(\alpha)\ \hat{\mathbb{D}}_0(\alpha) = \prod_{k_0^n > 0} d\mathbf{Re}\phi_k\ d\mathbf{Im}\phi_k \exp\left\{\frac{i}{V}\sum_k \left(k^2 - m^2\right)|\phi_k|^2\right\}$$

$$\times \exp\left\{\frac{2}{V}\sum_{k'}(\omega_{\mathbf{k}'}+k_0')\left(\mathbf{Re}\ \alpha_{k'}\ \mathbf{Re}\ \phi_{k'}+\mathbf{Im}\ \alpha_{k'}\ \mathbf{Im}\ \phi_{k'}\right)\right\}$$
$$\exp\left(-\frac{1}{V}\sum_{k'}\alpha_{k'}\alpha^*_{k'}\right). \tag{2.20}$$

To simplify our calculations, we assume that all $\tilde{\alpha}_k$ are real. We also slightly change the notation to make the discrete nature of momenta more conspicuous. The denominator takes the form of a product of Gaussian integrals and can be easily computed. We only need to pay attention to the additional exponential piece linear in ϕ_k.

$$\langle\alpha|\alpha\rangle = \prod_{k_0^n>0}\int d\mathbf{Im}\phi_n\ \exp\left\{-\frac{i}{V}\sum_k\left(m^2-k_n^2\right)\left(\mathbf{Im}\phi_n\right)^2\right\}$$
$$d\mathbf{Re}\phi_n\ \exp\left\{-\frac{i}{V}\left(m^2-k_n^2\right)\left(\mathbf{Re}\phi_n\right)^2\right\}\exp\left\{\frac{2}{V}\left(\omega_n+k_n^0\right)\mathbf{Re}\ \alpha_n\ \mathbf{Re}\ \phi_n\right\}$$
$$=\prod_{k_0^n>0}\sqrt{\frac{-i\pi V}{m^2-k_n^2}}\sqrt{\frac{-i\pi V}{m^2-k_n^2}}\ \exp\left\{-\frac{i}{V}\frac{(\omega_n+k_n^0)^2\ \mathbf{Re}\alpha_n^2}{m^2-k_n^2}\right\}. \tag{2.21}$$

The nominator calculations reproduce the previous steps with only one very important difference: the presence of the source field in the integrand makes the shift in the field visible.

$$\langle\alpha|\,\phi(x)\,|\alpha\rangle =$$
$$\frac{1}{V}\sum_l e^{ik_lx}\prod_{k_0^n>0}\int d\mathbf{Re}\phi_n\ \mathbf{Re}\phi_l\ \exp\left\{-\frac{i\,(m^2-k_n^2)}{V}\left(\mathbf{Re}\phi_n+\frac{i(\omega_n+k_n^0)\mathbf{Re}\alpha_n}{m^2-k_n^2}\right)^2\right\}$$
$$\times\exp\left\{-\frac{i}{V}\frac{(\omega_n+k_n^0)^2\ \mathbf{Re}\alpha_n^2}{m^2-k_n^2}\right\}$$
$$\frac{1}{V}\sum_l e^{ik_lx}\ \prod_{k_0^n>0}\sqrt{\frac{-i\pi V}{m^2-k_n^2}}\ \frac{-i(\omega_n+k_n^0)\tilde{\mathbf{Re}}\alpha_n}{m^2-k_n^2}\ \times\exp\left\{-\frac{i}{V}\frac{(\omega_n+k_n^0)^2\ \mathbf{Re}\alpha_n^2}{m^2-k_n^2}\right\}. \tag{2.22}$$

We have almost reached our goals. We note that $\tilde{\alpha}_n = \frac{\alpha}{k_n^0+\omega_n}$. To convert discrete, finite sums over k_n to continuous integrals over k, we take the limit $L\to\infty$:

$$\frac{1}{V}\sum_n \to \int\frac{d^4k}{(2\pi)^4}. \tag{2.23}$$

We obtain the result for the field $\phi(x)$ expectation value

$$\langle\phi\rangle_\alpha = \int\frac{d^4k}{(2\pi)^4}\frac{-i\ \mathbf{Re}\alpha_k}{m^2-k^2}\ e^{ikx}. \tag{2.24}$$

The standard contour integration over the dk_0 gives us the final result:

$$\langle\phi\rangle_\alpha = \int \frac{d^3\mathbf{k}}{2\omega_\mathbf{k}} \mathbf{Re}\alpha_k \; e^{i\omega_\mathbf{k} t} e^{-i\mathbf{k}\cdot\mathbf{x}}. \tag{2.25}$$

This result suggests the form of the expectation value for the graviton field $g_{\mu\nu}$, which replaces the scalar field and introduces the general Schrodinger wave function $\phi_\mathbf{k}$ representing the solitonic solutions instead of the simple plane-wave $e^{-i\mathbf{k}\cdot\mathbf{x}}$.

2.2 Path integral approach in M-theory: nodal diagrams

We will attempt to include in our analysis of the Glauber-Sudarshan state the full interacting action with all the quantum corrections. At the same time, to make the following computations tractable, we restrict our attention to only scalar degrees of freedom. The three scalar fields (ϕ_1, ϕ_2, ϕ_3) will represent the three sets of degrees of freedom from the full M-theory description:

$$\{g_{\mu\nu}\}, \{C_{ABD}\}, \{\Psi_A\},$$

representing metric, fluxes, and gravitino fields. The path-integral structure that we are looking for here may be represented by:

$$\langle\varphi_1\rangle_{\overline{\sigma}} \equiv \frac{\int \mathcal{D}\varphi_1\mathcal{D}\varphi_2\mathcal{D}\varphi_3 \; e^{i\mathbf{S}_{\rm tot}} \; \mathbb{D}^\dagger(\overline{\alpha}, \overline{\beta}, \overline{\gamma})\varphi_1(x,y,z)\mathbb{D}(\overline{\alpha}, \overline{\beta}, \overline{\gamma})}{\int \mathcal{D}\varphi_1\mathcal{D}\varphi_2\mathcal{D}\varphi_3 \; e^{i\mathbf{S}_{\rm tot}} \; \mathbb{D}^\dagger(\overline{\alpha}, \overline{\beta}, \overline{\gamma})\mathbb{D}(\overline{\alpha}, \overline{\beta}, \overline{\gamma})}, \tag{2.26}$$

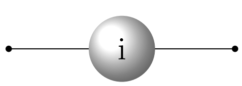
where $\overline{\sigma} = (\overline{\alpha}, \overline{\beta}, \overline{\gamma})$ is associated with $(\{\mathbf{g}_{\mu\nu}\}, \{\mathbf{C}_{\rm ABD}\}, \{\mathbf{\Psi}_{\rm A}\})$ degrees of freedom , $\mathbb{D}(\overline{\sigma})$ is a non-unitary displacement operator, *i.e.* $\mathbb{D}^\dagger(\overline{\sigma})\mathbb{D}(\overline{\sigma}) \neq \mathbb{D}(\overline{\sigma})\mathbb{D}^\dagger(\overline{\sigma}) \neq 1$ and the total action $\mathbf{S}_{\rm tot} \equiv \mathbf{S}_{\rm kin} + \mathbf{S}_{\rm int} + \mathbf{S}_{\rm ghost} + \mathbf{S}_{\rm gf}$ where the perturbative part of $\mathbf{S}_{\rm int}$ comes from an interaction term and $\mathbf{S}_{\rm gf}$ is the gauge-fixing term. As a reminder, we also mention here that $\overline{\alpha}, \overline{\beta}, \overline{\gamma}$ represent the coherent states of the three respective scalar fields of our toy model or, equivalently, the three sets of degrees of freedom from the full M-theory: metric, fluxes and gravitino fields.

Ignoring the complex parts of the various fields, we can write down the simplified version of the numerator of the path integral in the following form

$$
\begin{aligned}
\mathrm{Num}[\langle\varphi_1\rangle_{\overline{\sigma}}] = \mathbb{C}(\overline{\alpha}_i, a_j, \mathrm{V}, ..) \prod_j \int d\widetilde{\varphi}_1(k_j)\, \exp\Big[-a_j\Big(\widetilde{\varphi}_1(k_j) + \frac{\bar{\alpha}_j}{a_j}\Big)^2\Big] \\
\prod_u \int d\widetilde{\varphi}_1(l_u)\, \exp\Big[-b_u\Big(\widetilde{\varphi}_1(l_u) + \frac{\bar{\beta}_u}{a_u}\Big)^2\Big] \prod_v \int d\widetilde{\varphi}_1(f_v)\, \exp\Big[-c_v\Big(\widetilde{\varphi}_1(f_v) + \frac{\bar{\gamma}_v}{c_v}\Big)^2\Big] \\
\times \Big(\widetilde{\varphi}_1(k_1)\psi_{\mathbf{k}_1}(\mathbf{x}, y, z)e^{-ik_{0,1}t} + \widetilde{\varphi}_1(k_2)\psi_{\mathbf{k}_2}(\mathbf{x}, y, z)e^{-ik_{0,2}t} + \widetilde{\varphi}_1(k_3)\psi_{\mathbf{k}_3}(\mathbf{x}, y, z)e^{-ik_{0,3}t} + ..\Big) \\
\times \Big(1 + i\sum_{\mathcal{S}} \mathrm{V}^{-u} c_{nmpqrs}(-1)^v \begin{pmatrix} p \\ e_o \end{pmatrix} (k_{u_1} + k_{u_2} + ... + k_{u_q})^{n+e_o}(l_{v_1} + l_{v_2} + ... + l_{v_r})^{m+p-e_o} \\
\times \widetilde{\varphi}_1(k_{u_1})\widetilde{\varphi}_1(k_{u_2})...\widetilde{\varphi}_1(k_{u_q})\widetilde{\varphi}_2(l_{v_1})\widetilde{\varphi}_2(l_{v_2})...\widetilde{\varphi}_2(l_{v_r})\widetilde{\varphi}_3(f_{w_1})\widetilde{\varphi}_3(f_{w_2})...\widetilde{\varphi}_3(f_{w_{s-1}})\widetilde{\varphi}_3(f_{w_s}),
\end{aligned}
\tag{2.27}
$$

where the c_{mnpqrs} coefficients are the coupling constants for each contribution of the perturbative series. Note, that we have ignored the contributions coming from the local, non-local, and fermionic action.

Since it will be advantages for our computations to have some factors dimensionless, such as coefficients c_{nmpqrs}, Fourier transforms $\tilde{\phi}_i(k)$ and so on, we need to define the Fourier transforms in powers of M_p. As a result, the perturbative series of the field expectation value is ordered in powers of M_p. Keeping this in mind, we see that the tree-level contribution can be calculated in the limit $\mathrm{M}_p \to \infty$ when all the interaction terms become subdominant. The tree level contributions appear when we take vanishing coupling constants, *i.e.* we take $c_{nmpqrs} = 0$. In the limit $\mathrm{M}_p \to \infty$ and $g_s \ll 1$, the numerator has the following form



$$
\begin{aligned}
= \prod_{j\neq i} \int d\widetilde{\varphi}_1(k_j)\, \exp\Big[-a_j\Big(\widetilde{\varphi}_1(k_j) + \frac{\overline{\alpha}_j}{a_j}\Big)^2\Big] \\
\times \prod_u \int d\widetilde{\varphi}_2(l_u)\, \exp\Big[-b_u\Big(\widetilde{\varphi}_2(l_u) + \frac{\overline{\beta}_u}{b_u}\Big)^2\Big] \\
\times \prod_v \int d\widetilde{\varphi}_3(f_v)\, \exp\Big[-c_v\Big(\widetilde{\varphi}_3(f_v) + \frac{\overline{\gamma}_v}{c_v}\Big)^2\Big] \\
\times \int d\widetilde{\varphi}_1(k_i)\, \exp\Big[-a_i\Big(\widetilde{\varphi}_1(k_i) + \frac{\overline{\alpha}_i}{a_i}\Big)^2\Big]\widetilde{\varphi}_1(k_i)\psi_{\mathbf{k}_i}(\mathbf{x}, y, z)e^{-ik_{0,i}t} \\
= -\prod_j \Big(\frac{\pi_j}{a_j}\Big)^{1/2} \prod_u \Big(\frac{\pi_u}{b_u}\Big)^{1/2} \prod_v \Big(\frac{\pi_v}{c_v}\Big)^{1/2} \Big(\frac{\overline{\alpha}_i}{a_i}\Big)\psi_{\mathbf{k}_i}(\mathbf{x}, y, z)e^{-ik_{0,i}t}
\end{aligned}
\tag{2.28}
$$

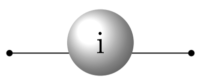
where $\pi_j = \pi_u = \pi_v = \pi$ and the overall minus sign is due to our choice of convention for the displacement operator $\mathbb{D}(\overline{\sigma})$. The above diagram is only for the momentum mode k_i.

An equivalent tree-level diagram for the denominator has the form

$$= \prod_j \left(\frac{\pi_j}{a_j}\right)^{1/2} \prod_u \left(\frac{\pi_u}{b_u}\right)^{1/2} \prod_v \left(\frac{\pi_v}{c_v}\right)^{1/2}. \tag{2.29}$$

In the continuum limit, V $\to \infty$, and therefore we can sum over all i from (2.28), to get the following result:

$$\frac{1}{V}\sum_{i=1}^{\infty}$$



$$=$$

$$\otimes \left(-\int d^{11}k\ \frac{\overline{\alpha}(k)}{a(k)}\ \psi_{\mathbf{k}}(\mathbf{x}, y, z)e^{-ik_0 t}\right). \tag{2.30}$$

It is time to move to the calculations involving interactions parameterized by non-zero c_{nmpqrs}.

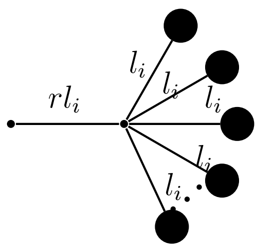
2.2.1 Contributions from the φ_2 fields

In the following computation we will introduce and extensively use the so called nodal diagrams. Nodal diagrams are a pictorial representation of the momenta distribution in the perturbative path integral calculations. They help to visualize a particular momenta configuration and also accentuate the presence or absence of the source field. Each nodal diagram is made up of small dot points group together a particular set of coinciding momenta. The bigger size nodes make this set of momenta more visual, representing one, two or greater number of the matching momenta. The presence of the source field is made conspicuous by a node with a letter i inside it.

The scalar field φ_2 is a representative field for the three-form flux components that have 84 massless degrees of freedom in M-theory. To make the following calculations manageable, we use only one representative component taken with arbitrary copies in r in the coupling constant c_{nmpqrs}. It means that each of these copies can have different momenta integrated over to provide the correct final result.

Case 1: $l_{v_1} = l_{v_2} = l_{v_3} = ... = l_{v_r} \equiv l_i$ **for** φ_2 **field**

Since the momentum modes k_{u_i}, l_{v_j}, and f_{w_t} are all independent, we can fix the k_{u_i} and f_{w_t} values and concentrate our attention for now on the l_{v_j} modes. This first interactive case is relatively simple and can be represented diagrammatically as



$$= \prod_{j\neq i} \int d\widetilde{\varphi}_2(l_j)\, \exp\Big[-b_j\Big(\widetilde{\varphi}_2(l_j) + \tfrac{\bar{\beta}_j}{b_j}\Big)^2\Big] \tag{2.31}$$

$$\times \int d\widetilde{\varphi}_2(l_i)\, \exp\Big[-b_i\Big(\widetilde{\varphi}_2(l_i) + \tfrac{\overline{\beta}_i}{b_i}\Big)^2\Big]\, (rl_i)^{m+p-e_o}\widetilde{\varphi}_2^r(l_i)$$

The value of this diagram is made up of the products of two distinct integrals. The first one is a simple variation of the Gaussian integral

$$\int d\widetilde{\varphi}_2(l_j)\, \exp\Big[-b_j\Big(\widetilde{\varphi}_2(l_j) + \frac{\bar{\beta}_j}{b_j}\Big)^2\Big] = \Big(\frac{\pi_j}{b_j}.\Big)^{1/2} \tag{2.32}$$

The second integral is more involved. To solve it we introduce a new integration variable $W = \widetilde{\varphi}_2(l_j) + \frac{\bar{\beta}_j}{b_j})^r$ and obtain the following integral

$$\int dW e^{-b_j W^2}(rl_i)^{m+p-e_0}\, \Big(W - \frac{\bar{\beta}_j}{b_j}\Big)^r \tag{2.33}$$

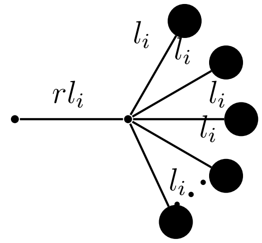
The value of this integral we can look up in the Table of integrals by Gradhstein and Ryzhik (integral 3.461.4, page 364). The result is

$$\sum_{e_p\in 2\mathbb{Z}_+}^{r} \begin{pmatrix} r \\ e_p \end{pmatrix} \Big(-\frac{\overline{\beta}_i}{b_i}\Big)^{r-e_p}\, \frac{(rl_i)^{m+p-e_o}(e_p-1)!!}{(2b_i)^{e_p/2}}. \tag{2.34}$$

Here the e_p is an even integer, otherwise, the integrals become zero.

In the next step, we some over the discrete momenta l_i:

$$\frac{1}{\mathrm{V}}\sum_{i=1}^{\infty} \tag{2.35}$$



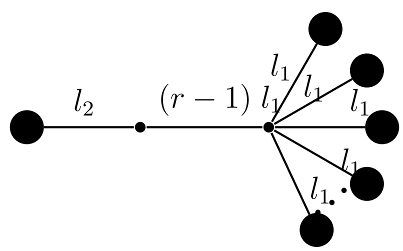
The sum is straightforward, and in the continuum limit, when $V \longleftarrow \infty$, it turns into an integral

$$\prod_j \left(\frac{\pi_j}{b_j}\right)^{1/2} \int d^{11}l \ (rl)^{m+p-e_o} \sum_{e_p \in 2\mathbb{Z}_+}^{r} \begin{pmatrix} r \\ e_p \end{pmatrix} \left(-\frac{\overline{\beta}(l)}{b(l)}\right)^{r-e_p} \frac{(e_p-1)!!}{(2b(l))^{e_p/2}}. \tag{2.36}$$

Case 2: $l_{v_1} = l_{v_2} = l_{v_3} = ... = l_{v_{r-1}} = l_i, l_{v_r} = l_j, i \neq j$ **for** φ_2 **field**

In this case, we have two different momenta, which vary, and the final result contains two sums over two different indices. This already suggests that the final result should contain the nested integrals.

Now our nodal diagram has an additional extension due to the presence of the $l_j, i \neq j$. To understand the general picture, we take a particular example of the case (l_1, l_2):



(2.37)

To understand how to write down the integral for the diagram, we need to come back to the initial formula for the Numerator. It contains the sum of the momenta $l_{v_1}, l_{v_2}, ...$:

$$(l_{v_1} + l_{v_2} + ... + l_{v_r})^{m+p-e_0} = [(r-1)l_1 + l_2]^{m+p-e_0} \tag{2.38}$$

$$= \sum_{e_1}^{m+p-e_0} \begin{pmatrix} m+p-e_0 \\ e_1 \end{pmatrix} [(r-1)l_1]^{e_1} \, l_2^{(m+p-e_0-e_1)} \tag{2.39}$$

Then the diagram represents the following integral

$$\prod_{j \neq i} \int d\widetilde{\varphi}_2(l_j) \ \exp\Big[-b_j\Big(\widetilde{\varphi}_2(l_j) + \frac{\bar{\beta}_j}{b_j}\Big)^2\Big] \tag{2.40}$$

$$\times \sum_{e_1}^{m+p-e_0} \begin{pmatrix} m+p-e_0 \\ e_1 \end{pmatrix} [(r-1)l_1]^{e_1} \int d\widetilde{\varphi}_2(l_1) \ \exp\Big[-b_1\Big(\widetilde{\varphi}_2(l_1) + \frac{\overline{\beta}_1}{b_1}\Big)^2\Big] \ \widetilde{\varphi}_2^{(r-1)}(l_1)$$

$$\times\, l_2^{(m+p-e_o-e_1)} \int d\widetilde{\varphi}_2(l_2)\ \exp\Big[-b_2\Big(\widetilde{\varphi}_2(l_2)+\frac{\overline{\beta}_2}{b_2}\Big)^2\Big]\ \ \widetilde{\varphi}_2(l_2)$$

The integration of the first line gives the standard answer $\left(\frac{\pi_j}{b_j}.\right)^{1/2}$. The second integral is very similar to case 2 with r power being replaced by $r-1$:

$$\int d\widetilde{\varphi}_2(l_1)\ \exp\Big[-b_1\Big(\widetilde{\varphi}_2(l_1)+\frac{\overline{\beta}_1}{b_1}\Big)^2\Big]\ \widetilde{\varphi}_2^{(r-1)}(l_1) \tag{2.41}$$

$$= \left(\frac{\pi}{b_1}.\right)^{1/2} \sum_{e_2\in 2\mathbb{Z}_+}^{r-1} \begin{pmatrix} r-1 \\ e_2 \end{pmatrix} \left(-\frac{\overline{\beta}_1}{b_1}\right)^{(r-1-e_2)} \frac{(e_2-1)!!}{(2b_1)^{e_2/2}}. \tag{2.42}$$

And the integral on the last line is simply calculated to be

$$\int d\widetilde{\varphi}_2(l_2)\ \exp\Big[-b_2\Big(\widetilde{\varphi}_2(l_2)+\frac{\overline{\beta}_2}{b_2}\Big)^2\Big]\ \ \widetilde{\varphi}_2(l_2) = \left(\frac{\pi}{b_2}.\right)^{1/2}\left(-\frac{\beta_2}{b_2}\right). \tag{2.43}$$

After collecting all the results, we obtain the final value of the case 3 diagram with $l_j = l_1$ and $l_i = l_2$:

$$\prod_j \left(\frac{\pi_j}{b_j}\right)^{1/2} \sum_{e_1,e_2} r\ (rl_1-l_1)^{e_1} l_2^{(m+p-e_0-e_1)} \begin{pmatrix} m+p-e_0 \\ e_1 \end{pmatrix} \begin{pmatrix} r-1 \\ e_2 \end{pmatrix} \left(-\frac{\overline{\beta}_1}{b_1}\right)^{(r-1-e_2)} \times \left(-\frac{\overline{\beta}_2}{b_2}\right) \frac{(e_2-1)!!}{(2b_1)^{e_2/2}}. \tag{2.44}$$

Now we want to generalize this particular case for all possible values (l_i, l_j) and then sum over all i and j. We begin our analysis with the case (l_1, l_j), summing over all values of j except $j = 1$

$$l_1^{e_1}\ \left(-\frac{\beta_1}{b_1}\right)^{r_1} \frac{1}{(b_1)^{e_2/2}} \left[l_2^{m_1}\ \left(-\frac{\beta_2}{b_2}\right) + l_3^{m_1}\ \left(-\frac{\beta_3}{b_3}\right) + l_4^{m_1}\ \left(-\frac{\beta_4}{b_4}\right) + ...\right] \tag{2.45}$$

The similar case $(l_2, l_j, j \neq 2)$ gives us

$$l_2^{e_1}\ \left(-\frac{\beta_2}{b_2}\right)^{r_1} \frac{1}{(b_2)^{e_2/2}} \left[l_1^{m_1}\ \left(-\frac{\beta_1}{b_1}\right) + l_3^{m_1}\ \left(-\frac{\beta_3}{b_3}\right) + l_4^{m_1}\ \left(-\frac{\beta_4}{b_4}\right) + ...\right] \tag{2.46}$$

And the next term $(l_3, l_j, j \neq 3)$ becomes

$$l_3^{e_1}\ \left(-\frac{\beta_3}{b_3}\right)^{r_1} \frac{1}{(b_3)^{e_2/2}} \left[l_1^{m_1}\ \left(-\frac{\beta_1}{b_1}\right) + l_2^{m_1}\ \left(-\frac{\beta_2}{b_2}\right) + l_4^{m_1}\ \left(-\frac{\beta_4}{b_4}\right) + ...\right] \tag{2.47}$$

Now we can observe a certain pattern by combining the summation terms in a particular order. First, we take all the sums over i and j with $j > i$:

$$\begin{aligned} & l_1^{e_1}\left(-\frac{\beta_1}{b_1}\right)^{r_1}\frac{1}{(b_1)^{e_2/2}}\left[l_2^{m_1}\left(-\frac{\beta_2}{b_2}\right)+l_3^{m_1}\left(-\frac{\beta_3}{b_3}\right)+...\right] \\ & +l_2^{e_1}\left(-\frac{\beta_2}{b_2}\right)^{r_1}\frac{1}{(b_2)^{e_2/2}}\left[l_3^{m_1}\left(-\frac{\beta_3}{b_3}\right)+l_4^{m_1}\left(-\frac{\beta_4}{b_4}\right)+...\right] \\ & +l_3^{e_1}\left(-\frac{\beta_3}{b_3}\right)^{r_1}\frac{1}{(b_3)^{e_2/2}}\left[l_4^{m_1}\left(-\frac{\beta_4}{b_4}\right)+...\right]+... \qquad (2.48) \\ & =\sum_{i=1}^{\infty} l_i^{e_1}\left(-\frac{\beta_i}{b_i}\right)\frac{1}{(b_i)^{e_2/2}}\sum_{j=i+1}^{\infty} l_j^{m_1}\left(-\frac{\beta_j}{b_j}\right) \qquad (2.49) \end{aligned}$$

The two last nested sums in the continuum limit are replaced by the nested integrals:

$$\begin{aligned} & \frac{1}{V^2}\sum_{i=1}^{\infty} l_i^{e_1}\left(-\frac{\beta_i}{b_i}\right)\frac{1}{(b_i)^{e_2/2}}\sum_{j=i+1}^{\infty} l_j^{m_1}\left(-\frac{\beta_j}{b_j}\right) \qquad (2.50) \\ & \longrightarrow_{V\to\infty}\int d^{11}l\frac{l^{e_1}}{b^{e_2/2}(l)}\left(-\frac{\beta(l)}{b(l)}\right)^{r_1}\int_l^{\infty} d^{11}\, l' l'^{m_1}\left(-\frac{\beta(l')}{b(l')}\right)^{r_1}. \qquad (2.51) \end{aligned}$$

It is evident that the previous nested integral structure does not contain all the elements of the double sums over (i,j). The terms left behind form a nested sum on their own with $j\in\mathbb{Z}_+, i>j$:

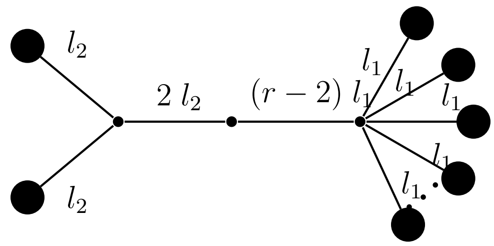
$$\begin{aligned} & l_1^{m_1}\left(-\frac{\beta_1}{b_1}\right)\left[\frac{l_2^{e_1}}{(b_2)^{e_2/2}}\left(-\frac{\beta_2}{b_2}\right)^{r_1}+\frac{l_3^{e_1}}{(b_3)^{e_2/2}}\left(-\frac{\beta_3}{b_3}\right)^{r_1}+...\right] \qquad (2.52) \\ & +l_2^{m_1}\left(-\frac{\beta_1}{b_1}\right)\left[\frac{l_3^{e_1}}{(b_3)^{e_2/2}}\left(-\frac{\beta_3}{b_3}\right)^{r_1}+\frac{l_4^{e_1}}{(b_4)^{e_2/2}}\left(-\frac{\beta_4}{b_4}\right)^{r_1}+...\right]+... \qquad (2.53) \\ & =\sum_{j=1}^{\infty} l_j^{m_1}\left(-\frac{\beta_j}{b_j}\right)\sum_{i=j+1}^{\infty}\frac{l_i^{e_1}}{(b_i)^{e_2/2}}\left(-\frac{\beta_i}{b_i}\right)^{r_1} \qquad (2.54) \end{aligned}$$

Again the last sum in the continuum limit translates into a nested integral structure:

$$\begin{aligned} & \frac{1}{V^2}\sum_{j=1}^{\infty} l_j^{m_1}\left(-\frac{\beta_j}{b_j}\right)\sum_{i=j+1}^{\infty}\frac{l_i^{e_1}}{(b_i)^{e_2/2}}\left(-\frac{\beta_i}{b_i}\right)^{r_1} \qquad (2.55) \\ & \longrightarrow_{V\to\infty}\int d^{11}\, l^{m_1}\left(-\frac{\beta(l)}{b(l)}\right)\int_l^{\infty} d^{11}l'\frac{l'^{e_1}}{(b(l'))^{e_2/2}}\left(-\frac{\beta(l')}{b(l')}\right)^{r_1}. \qquad (2.56) \end{aligned}$$

Case 3: $l_{v_1}=l_{v_2}=l_{v_3}=...l_{v_{r-2}}=l_i, l_{v_{r-1}}\neq l_i, l_{v_r}\neq l_i$ **for** φ_2 **field**

Our first task is to understand this more involved case in the simplest situation: $l_i = l_1$ and $l_{v_{r-2}} = l_{v_{r-1}} = l_2$. The respective diagram has the following form



We start the analysis from the original sum over all values of l_{v_i}:

$$(l_{v_1} + l_{v_2} + ... + l_{v_r})^{m+p-e_0} = [(r-2)l_1 + 2l_2]^{m+p-e_0} \tag{2.57}$$

$$= \sum_{e_3}^{m+p-e_0} \begin{pmatrix} m+p-e_0 \\ e_3 \end{pmatrix} ((rl_1 - 2l_1)^{e_3}\,(2l_2)^{(m+p-e_0-e_3)}. \tag{2.58}$$

Then the integral represented by the diagram looks like this one

$$\prod_{j \neq i} \int d\widetilde{\varphi}_2(l_j)\ \exp\Big[-b_j\Big(\widetilde{\varphi}_2(l_j) + \frac{\bar{\beta}_j}{b_j}\Big)^2\Big] \tag{2.59}$$

$$\times \sum_{e_3}^{m+p-e_0} \begin{pmatrix} m+p-e_0 \\ e_3 \end{pmatrix} (rl_1 - 2l_1)^{e_3} \int d\widetilde{\varphi}_2(l_1)\ \exp\Big[-b_1\Big(\widetilde{\varphi}_2(l_1) + \frac{\overline{\beta}_1}{b_1}\Big)^2\Big]\ \widetilde{\varphi}_2^{(r-2)}(l_1)$$

$$\times\ (2l_2)^{(m+p-e_o-e_3)} \int d\widetilde{\varphi}_2(l_2)\ \exp\Big[-b_2\Big(\widetilde{\varphi}_2(l_2) + \frac{\overline{\beta}_2}{b_2}\Big)^2\Big]\ \ \widetilde{\varphi}_2^2(l_2).$$

The first two integrals bring the expected results:

$$\int d\widetilde{\varphi}_2(l_j)\ \exp\Big[-b_j\Big(\widetilde{\varphi}_2(l_j) + \frac{\bar{\beta}_j}{b_j}\Big)^2\Big] = \Big(\frac{\pi}{b_j}\Big)^{1/2}, \tag{2.60}$$

and

$$\int d\widetilde{\varphi}_2(l_1)\ \exp\Big[-b_1\Big(\widetilde{\varphi}_2(l_1) + \frac{\overline{\beta}_1}{b_1}\Big)^2\Big]\ \widetilde{\varphi}_2^{(r-2)}(l_1) \tag{2.61}$$

$$= \Big(\frac{\pi}{b_1}\Big)^{1/2} \sum_{e_4 \in 2\mathbb{Z}_+}^{r-2} \begin{pmatrix} r-2 \\ e_4 \end{pmatrix} \Big(-\frac{\overline{\beta}_1}{b_1}\Big)^{(r-2-e_4)} \frac{(e_4-1)!!}{(2b_1)^{e_4/2}}. \tag{2.62}$$

The last integral from the diagram representation generates an additional term coming from the second power of the field variable. As usual we introduce a dummy variable $W = \widetilde{\varphi}_2 + \frac{\beta_2}{b_2}$.

Then

$$\widetilde{\varphi}_2^2 = W^2 - 2W\frac{\beta_2}{b_2} + \frac{\beta_2^2}{b_2^2}. \tag{2.63}$$

The Gaussian integral over the second term is zero, but the two terms give us slightly different non-trivial results:

$$\frac{\beta_2^2}{b_2^2}\int dW e^{-b_2 W} = \frac{\beta_2^2}{b_2^2}\left(\frac{\pi}{b_2}\right)^{1/2}, \tag{2.64}$$

and

$$\int dW e^{-b_2 W} W^2 = \frac{1}{b^2}\left(\frac{\pi}{b_2}\right)^{1/2}. \tag{2.65}$$

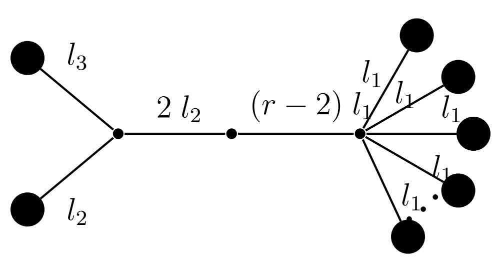
We are left only to collect all the pieces of our calculations to find the following value of the diagram

$$\prod_j \left(\frac{\pi}{b_j}\right)^{1/2} \sum_{e_3}^{m+p-e_0} \binom{m+p-e_0}{e_3} (rl_1 - 2l_1)^{e_3} (2l_1)^{(m+p-e_0-e_3)} \tag{2.66}$$

$$\times \sum_{e_4\in 2\mathbb{Z}_+}^{r-2} \binom{r-2}{e_4} \left(-\frac{\overline{\beta}_1}{b_1}\right)^{(r-2-e_4)} \frac{(e_4-1)!!}{(2b_1)^{e_4/2}}. \tag{2.67}$$

The next natural step would be to take different momenta values: $l_i = l_1, l_{v_{r-1}} = l_3, l_{v_r} = l_3$. In this case, sum over momenta gives additional combinatorial elements, but the integrals are simpler.

The diagram becomes



The sum over the momenta

$$(l_{v_1} + l_{v_2} + ... + l_{v_r})^{m+p-e_0} = [(r-2)l_1 + (l_2 + l_3)]^{m+p-e_0} \tag{2.68}$$

$$= \sum_{e_5}^{m+p-e_0} \begin{pmatrix} m+p-e_0 \\ e_5 \end{pmatrix} (rl_1 - 2l_1)^{e_5} (l_2+l_3)^{(m+p-e_0-e_3)}, \tag{2.69}$$

where

$$(l_2+l_3)^{(m+p-e_0-e_3)} = \sum_{e_6} \begin{pmatrix} m+p-e_0-e_5 \\ e_6 \end{pmatrix} l_2^{e_6}\, l_3^{(m+p-e_0-e_5-e_6)}. \tag{2.70}$$

The integral structure of the diagram is very similar to the previous one with the difference that now we have three different Gaussian integrals over the momenta l_1, l_2, l_3:

$$\prod_{j\neq i} \int d\widetilde{\varphi}_2(l_j)\, \exp\Big[-b_j\Big(\widetilde{\varphi}_2(l_j)+\frac{\bar{\beta}_j}{b_j}\Big)^2\Big] \tag{2.71}$$

$$\times \sum_{e_3}^{m+p-e_0} \begin{pmatrix} m+p-e_0-e_5 \\ e_6 \end{pmatrix} (rl_1-2l_1)^{e_5} \int d\widetilde{\varphi}_2(l_1)\, \exp\Big[-b_1\Big(\widetilde{\varphi}_2(l_1)+\frac{\overline{\beta}_1}{b_1}\Big)^2\Big]\, \widetilde{\varphi}_2^{(r-2)}(l_1)$$

$$\times (l_2)^{e_6} \int d\widetilde{\varphi}_2(l_2)\, \exp\Big[-b_2\Big(\widetilde{\varphi}_2(l_2)+\frac{\overline{\beta}_2}{b_2}\Big)^2\Big]\ \widetilde{\varphi}(l_2).$$

$$\times (l_3)^{m+p-e_0-e_5-e_6} \int d\widetilde{\varphi}_2(l_3)\, \exp\Big[-b_2\Big(\widetilde{\varphi}_2(l_3)+\frac{\overline{\beta}_3}{b_3}\Big)^2\Big]\ \widetilde{\varphi}(l_3).$$

It can be seen that all the present integrals have been calculated several times already, and so we can just easily state the final result:

$$\prod_j \Big(\frac{\pi}{b_j}\Big)^{1/2} \sum_{e_5}^{m+p-e_0} \begin{pmatrix} m+p-e_0 \\ e_5 \end{pmatrix} (rl_1-2l_1)^{e_5} \sum_{e_6} \begin{pmatrix} m+p-e_0-e_5 \\ e_6 \end{pmatrix} l_2^{e_6} \tag{2.72}$$

$$\times \sum_{e_7}^{r-2} \begin{pmatrix} r-2 \\ e_7 \end{pmatrix} \Big(-\frac{\overline{\beta}_1}{b_1}\Big)^{(r-2-e_7)} \frac{\beta_2}{b_2}\frac{\beta_3}{b_3}\, \frac{(e_7-1)!!}{(2b_1)^{e_7/2}}. \tag{2.73}$$

So far, this was the easy part of our analysis of case 4. In the next step, we need to sum over all possible values of i and j. The first diagram (l_i, l_i) can be summed relatively straightforwardly. The summation technique is exactly the same as we used for case 3. We divide the terms in the sum in such a way that they contain nested sums inside. The only difference comes from the presence of other variable coefficients

$$\frac{1}{V^2} \sum_{i=1}^{\infty} \frac{l^{e_3}}{(b_i)^{e_4/2}} \Big(-\frac{\overline{\beta}_i}{b_i}\Big)^{r_2} \sum_{j=i+1}^{\infty} l_j^{m_2} \Big(\frac{1}{2b_j}+\frac{\overline{\beta}_j^2}{b_j^2}\Big) \tag{2.74}$$

$$\longrightarrow \int d^{11}l \, \frac{l^{e_3}}{(b(l))^{e_4/2}} \left(-\frac{\overline{\beta}(l)}{b(l)}\right)^{r_2} \int_l^\infty d^{11}l' \, l'^{m_2} \left(\frac{1}{2b(l')} + \frac{\overline{\beta}^2(l')}{b^2(l')}\right), \tag{2.75}$$

and in the same manner

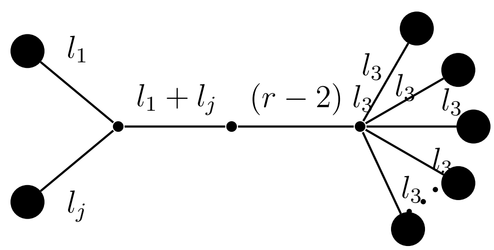
$$\frac{1}{V^2} \sum_{i=j}^{\infty} l_j^{m_2} \left(\frac{1}{2b_j} + \frac{\overline{\beta}_j^2}{b_j^2}\right) \sum_{i=j+1}^{\infty} \frac{l^{e_3}}{(b_i)^{e_4/2}} \left(-\frac{\overline{\beta}_i}{b_i}\right)^{r_2} \tag{2.76}$$

$$\longrightarrow \int d^{11}l \, l^{m_2} \left(\frac{1}{2b(l)} + \frac{\overline{\beta}^2(l)}{b^2(l)}\right) \int_l^\infty d^{11}l' \, \frac{l'^{e_3}}{(b(l'))^{e_4/2}} \left(-\frac{\overline{\beta}(l')}{b(l')}\right)^{r_2}. \tag{2.77}$$

To discover the final result, we just need to add these two different nested integral terms and multiply the sum by the combinatorial coefficients summing over all the permutation indices.

Now we turn to the case where both side legs can be different. Again we are going to move step by step, trying to figure out the general tendencies.

Let's take the case $(l_i = l_3, l_1, l_j)$ which is given by the following diagram



(2.78)

and sum over the j indices.

From our previous knowledge, we can write down the integral structure of this diagram without much ado:

$$\begin{aligned}
&\prod_{k \neq j} \int d\widetilde{\varphi}_2(l_k) \, \exp\Big[- b_j \Big(\widetilde{\varphi}_2(l_k) + \frac{\bar{\beta}_k}{b_k}\Big)^2 \Big] \\
&\times \sum_{e_3}^{m+p-e_0} \begin{pmatrix} m+p-e_0-e_5 \\ e_6 \end{pmatrix} (rl_3 - 2l_3)^{e_5} \int d\widetilde{\varphi}_2(l_3) \, \exp\Big[- b_1 \Big(\widetilde{\varphi}_2(l_3) + \frac{\overline{\beta}_3}{b_3}\Big)^2 \Big] \, \widetilde{\varphi}_2^{(r-2)}(l_3) \\
&\times (l_1)^{e_6} \int d\widetilde{\varphi}_2(l_1) \, \exp\Big[- b_2 \Big(\widetilde{\varphi}_2(l_1) + \frac{\overline{\beta}_1}{b_1}\Big)^2 \Big] \; \widetilde{\varphi}(l_1) \\
&\times (l_j)^{m+p-e_0-e_5-e_6} \int d\widetilde{\varphi}_2(l_j) \, \exp\Big[- b_2 \Big(\widetilde{\varphi}_2(l_j) + \frac{\overline{\beta}_j}{b_j}\Big)^2 \Big] \; \widetilde{\varphi}(l_j).
\end{aligned} \tag{2.79}$$

All the integrals present here have already been calculated many times, so we give the answer straight away, which is again very similar to the previous case:

$$\prod_k \left(\frac{\pi}{b_j}\right)^{1/2} \sum_{e_5}^{m+p-e_0} \begin{pmatrix} m+p-e_0 \\ e_5 \end{pmatrix} r \ (rl_3 - 2l_3)^{e_5} \sum_{e_6} \begin{pmatrix} m+p-e_0-e_5 \\ e_6 \end{pmatrix} l_1^{e_6} \tag{2.80}$$

$$\times \sum_{e_7}^{r-2} \begin{pmatrix} r-2 \\ e_7 \end{pmatrix} \left(-\frac{\overline{\beta}_3}{b_3}\right)^{(r-2-e_7)} \frac{\beta_1}{b_1}\frac{\beta_j}{b_j} \ \frac{(e_7-1)!!}{(2b_3)^{e_7/2}}. \tag{2.81}$$

Then we need to sum over all j indices. We will write the first several terms to figure out the general picture:

$$l_1^{e_6}\frac{\beta_1}{b_1}\left(l_2^{m_3}\frac{\beta_2}{b_2} + l_4^{m_3}\frac{\beta_4}{b_4} + l_5^{m_3}\frac{\beta_5}{b_5} + ...\right). \tag{2.82}$$

The main noticeable peculiarity of this sum is the absence of the $j = 3$ term, which is expected as the l_3 is already occupied in the different parts of the diagram. To restore the nested structure of the sum (and the subsequent integral), we add and subtract an l_3 containing the term:

$$l_1^{e_6}\frac{\beta_1}{b_1}\left(l_2^{m_3}\frac{\beta_2}{b_2} + l_3^{m_3}\frac{\beta_3}{b_3} + l_4^{m_3}\frac{\beta_4}{b_4} + l_5^{m_3}\frac{\beta_5}{b_5} + ...\right) - l_3^{m_3}\frac{\beta_3}{b_3}. \tag{2.83}$$

This trick will allow us to use the already standard technique to move from discrete summation to continuous integration:

$$l_1^{e_6}\frac{\beta_1}{b_1}\sum_{j=2}^{\infty}\left(l_j^{m_3}\frac{\beta_j}{b_j} - l_3^{m_3}\frac{\beta_3}{b_3}\right) \longrightarrow \frac{\overline{\beta}_1 l_1^{e_6}}{b_1}\left(\int_{l_1}^{\infty} d^{11}l' \ \frac{\overline{\beta}(l')l'^{m_3}}{b(l')} - \int_{l_2}^{l_3} d^{11}l' \ \frac{\overline{\beta}(l')l'^{m_3}}{b(l')}\right). \tag{2.84}$$

Since the step between different values of momenta in the infinite volume goes to zero, we can safely suggest that the last integral evaluates to zero:

$$\int_{l_2}^{l_3} d^{11}l' \ \frac{\overline{\beta}(l')l'^{m_3}}{b(l')} \longrightarrow 0, \tag{2.85}$$

as well as any other integral from a specific value of the momentum to its next allowed value.

The next step is to add the sum over the second branch of the side momenta, which is, for now, fixed to the l_1 value. We can write down the following formal sum

$$\frac{1}{V}\sum_j [l_1 + l_j, \ (r-2)l_3] + \frac{1}{V}\sum_j [l_2 + l_j, \ (r-2)l_3] + \frac{1}{V}\sum_j [l_4 + l_j, \ (r-2)l_3] + ... \tag{2.86}$$

where we sum over the different values of l_k, omitting, of course, the l_3 momentum which is already occupied.

We also note that as soon as we start summing over the second index $k \neq 3$, an additional term needs to be introduced into the general sum structure. Then the generic term of the total sum looks like this one

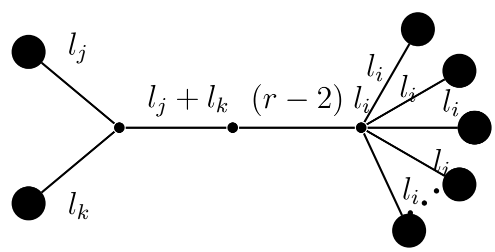
$$I(l_k) = \frac{\overline{\beta}_k}{b_k} \left(l_k^{e_6} \int_{l_k}^{\infty} d^{11}l' \, \frac{\overline{\beta}(l') l'^{m_3}}{b(l')} + l_k^{m_3} \int_{l_k}^{\infty} d^{11}l' \, \frac{\overline{\beta}(l') l'^{e_6}}{b(l')} \right). \tag{2.87}$$

Omitting the combinatorial coefficients and using the introduces conventions, we can write the double sum over j, k indices

$$\frac{l_3^{e_5}}{b_3^{e_7/2}} \left(-\frac{\beta_3}{b_3} \right)^{r_3} \left[I(l_1) + I(l_2) + I(l_4) + I(l_5) + ... \right]. \tag{2.88}$$

The final destination of our analysis of this case is the sum over all three indices i, j, k where each of them takes only the allowed values. The diagram for this case is given by

$$\frac{1}{\mathrm{V}^3} \sum_{i,j,k}$$



As expected, the total sum over the discrete momenta should include the nested structure, and the continuum variation of the sum contains the following nested integral structure:

$$\int d^{11}l \; I(l, l') \int_{l'}^{\infty} d^{11}l'' \, \frac{l''^{e_5}}{b^{e_7/2}(l'')} \left(-\frac{\overline{\beta}(l'')}{b(l'')} \right)^{r_3} \tag{2.89}$$

$$+ \int d^{11}l \, \frac{l^{e_5}}{b^{e_7/2}(l)} \left(-\frac{\overline{\beta}(l)}{b(l)} \right)^{r_3} \int_{l}^{\infty} d^{11}l' \; I(l', l''). \tag{2.90}$$

We can continue our analysis by introducing new, more complicated nodal diagrams. At the same time, it would be more and more difficult to find the values of these well-branched constructions without a possibility of gaining additional incites beyond what we've already discovered. It seems that now is a good time to introduce a structure of the $\widetilde{\varphi}_3$ field.

2.2.2 Contributions from the φ_3 fields

Before we start the analysis of each particular case for the φ_3 field, it would be instructive to understand how the momentum conservation affects the calculations. The first important fact is that the f_{ω_s} momentum is not independent and is fully determined by the values of all other momenta:

$$\sum_{k=1}^{q} k_{u_i} + \sum_{j=1}^{r} l_{v_j} + \sum_{t=1}^{s} f_{\omega_t} = 0, \tag{2.91}$$

and thus

$$-f_{\omega_s} = \sum_{k=1}^{q} k_{u_i} + \sum_{j=1}^{r} l_{v_j} + \sum_{t=1}^{s-1} f_{\omega_t}. \tag{2.92}$$

Since the f_{ω_s} is not independent, the result of the Gaussian integration over such a momentum depended on all the momenta is going to be slightly unusual. The general case looks something like this

$$\int d\widetilde{\varphi}_3(f_{\omega_s}) \exp\Big[-c_s\Big(\widetilde{\varphi}_3(f_{\omega_s}) + \frac{\bar{\gamma}_s}{c_s}\Big)^2\Big]\, \widetilde{\varphi}_3^{n_0}(f_{\omega_s}) \sim \left(\frac{\gamma_s}{c_s}\right)^{n_0} = \frac{\gamma_s}{f^2_{\omega_s}}, \tag{2.93}$$

where we suggested that $c_s \sim f^2_{\omega_s}$.

Now, we need to find a general form of the $f_{\omega_s}^{-2n_0}$, coming from the momenta conservation. We I taking several steps, involving some approximations:

$$\begin{aligned} \frac{1}{f^{2n_0}_{\omega_s}} &= \frac{1}{\sum_{k=1}^{q} k_{u_i} + \sum_{j=1}^{r} l_{v_j} + \sum_{t=1}^{s-1} f_{\omega_t}} \\ &= \frac{1}{\left(\sum_{t=1}^{s-1} f_{\omega_t}\right)^{2n_0}} \left(1 - \frac{\sum_{k=1}^{q} k_{u_i} + \sum_{j=1}^{r} l_{v_j}}{\sum_{t=1}^{s-1} f_{\omega_t}}\right)^{2n_0}, \end{aligned} \tag{2.94}$$

where we have assumed that $\sum_{t=1}^{s} f_{\omega_t} > \sum_{k=1}^{q} k_{u_i} + \sum_{j=1}^{r} l_{v_j}$, which is always possible in the limit when all the momenta f_{ω_t} are larger than k_{u_i} and l_{v_j}. Then we find the following general result:

$$\begin{aligned} \frac{1}{f^{2n_0}_{\omega_s}} &= \sum_{n_1} \binom{2n_0}{n_1} \left(\frac{\sum_{k=1}^{q} k_{u_i} + \sum_{j=1}^{r} l_{v_j}}{\left(\sum_{t=1}^{s-1} f_{\omega_t}\right)^2}\right)^{n_1} \frac{1}{\left(\sum_{t=1}^{s-1} f_{\omega_t}\right)^{2n_0-n_1}} \\ &= \sum_{n_1}\sum_{e_{n_1}} \binom{2n_0}{n_1}\binom{n_1}{e_{n_1}} \left(\sum_{k=1}^{q} k_{u_i}\right)^{n_1} \left(\sum_{j=1}^{r} l_{v_j}\right)^{n_1-e_{n_1}} \frac{1}{\left(\sum_{t=1}^{s-1} f_{\omega_t}\right)^{2n_0+n_1}} \end{aligned} \tag{2.95}$$

We also need to find the γ_s is affected by the momentum conservation

$$\gamma(f_{\omega_s}) = \gamma\left(-\sum_{k=1}^{q} k_{u_i} - \sum_{j=1}^{r} l_{v_j} - \sum_{t=1}^{s-1} f_{\omega_t}\right). \tag{2.96}$$

Using the previous assumption that all f_{ω_t} have the largest values of the moment, we can expand $\gamma(f_{\omega_s})$ into the Taylor series:

$$\gamma(f_{\omega_s}) = \gamma\left(-\sum_{t=1}^{s-1} f_{\omega_t}\left(1+\frac{\sum_{k=1}^{q} k_{u_i} + \sum_{j=1}^{r} l_{v_j}}{\sum_{t=1}^{s-1} f_{\omega_t}}\right)\right) \tag{2.97}$$

$$= \gamma(-\zeta) - \left(\sum_{k=1}^{q} k_{u_i} + \sum_{j=1}^{r} l_{v_j}\right)\frac{\partial\gamma}{\partial\zeta}(-\zeta) + \frac{(-1)^2}{2!}\left(\sum_{k=1}^{q} k_{u_i} + \sum_{j=1}^{r} l_{v_j}\right)^2 \frac{\partial^2\gamma}{\partial\zeta^2}(-\zeta) + \ldots \tag{2.98}$$

$$= \sum_{n_k} \frac{(-1)^{n_k}}{n_k!}\left(\sum_{k=1}^{q} k_{u_i} + \sum_{j=1}^{r} l_{v_j}\right)^{n_k} \frac{\partial^{n_k}\gamma}{\partial\zeta^{n_k}}(-\zeta) \tag{2.99}$$

$$= \sum_{n_k} \frac{(-1)^{n_k}}{n_k!} \sum_{e_{n_k}} \binom{n_k}{e_{n_k}} \left(\sum_{k=1}^{q} k_{u_i}\right)^{e_{n_k}} \left(\sum_{j=1}^{r} l_{v_j}\right)^{n_k - e_{n_k}} \frac{\partial^{n_k}\gamma}{\partial\zeta^{n_k}}(-\zeta), \tag{2.100}$$

where we introduced a new notation

$$\zeta = \sum_{t=1}^{s-1} f_{\omega_t}. \tag{2.101}$$

In the last step, we find the formula for the general case of any power n_0:

$$\gamma(f_{\omega_s})^{n_0} = \gamma(f_{\omega_s})\,\gamma(f_{\omega_s})\ldots\gamma(f_{\omega_s}) = \prod_{k=2}^{n_0+2} \sum_{n_k} \sum_{e_{n_k}} (\ldots)\,, \tag{2.102}$$

where the product starts from the case when the nodal diagrams have $k > 2$.

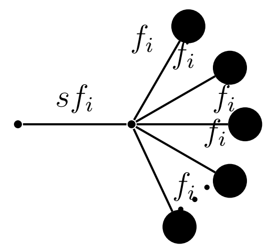
Combining all the accumulated knowledge about the case with dependent momentum f_{ω_s}, we come to the conclusion that the related Gaussian integral introduces an additional term of the form

$$\frac{1}{\zeta^{2n_0+n_1}}\,\frac{\partial^{n_k}\gamma(-\zeta)}{\partial\zeta^{n_k}}. \tag{2.103}$$

The integration over the rest of independent momenta f_{ω_t} follows the same rules as for all previous cases.

Case 1: $f_{w_1} = f_{w_2} = f_{w_3} = \ldots. = f_{w_{s-2}} = f_g, f_{w_{s-1}} = f_h, f_{w_s} \neq (f_g, f_h)$ for φ_3 field

We begin with a case similar to the diagram

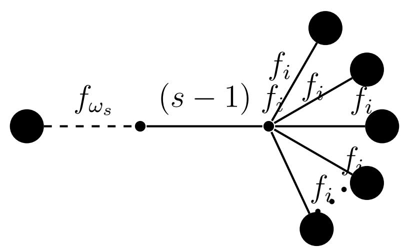
$$\frac{1}{\mathrm{V}}\sum_{i=1}^{\infty}$$ 

(2.104)

The simplest possible case for the φ_3 field is when all the momenta k_{u_i}, l_{v_j}, and f_{ω_t} are equally distributed:

$$-f_{\omega_s} = qk_i + rl_j + (s-1)f_k. \tag{2.105}$$

The diagram describing this case is the following



(2.106)

The integral representation of it is

$$\prod_{t\neq i}\int d\widetilde{\varphi}_3(f_t)\ \exp\Big[-c_t\Big(\widetilde{\varphi}_3(f_t)+\frac{\bar{\gamma}_t}{c_t}\Big)^2\Big]\int d\widetilde{\varphi}_3(f_i)\ \exp\Big[-c_i\Big(\widetilde{\varphi}_3(f_i)+\frac{\bar{\gamma}_i}{c_i}\Big)^2\Big]\ \widetilde{\varphi}_3^{(s-1)}(f_i)$$
$$\times\int d\widetilde{\varphi}_3(f_{\omega_s})\ \exp\Big[-c_t\Big(\widetilde{\varphi}_3(f_{\omega_s})+\frac{\bar{\gamma}_s}{c_s}\Big)^2\Big]. \tag{2.107}$$

The first two integrals were calculated previously and give the standard result

$$\prod_{t=1}^{s}\left(\frac{\pi_t}{c_t}\right)^{1/2}\sum_{e_s\in 2\mathbb{Z}_+}\begin{pmatrix} s-1 \\ e_s\end{pmatrix}\frac{(e_s-1)!!}{2^{e_s/2}}\frac{1}{c_i^{e_s/2}}\left(-\frac{\gamma_i}{c_i}\right)^{s-1-e_s}, \tag{2.108}$$

and the last integral will provide additional powers:

$$\frac{1}{\zeta^{2n_0+n_1}}\ \frac{\partial^{n_k}\gamma(-\zeta)}{\partial\zeta^{n_k}} = \frac{1}{(s-1)^{2n_0+n_1}}\frac{1}{f_i^{2n_0+n_1}}\ \frac{1}{(s-1)^{n_k}}\frac{\partial^{n_k}\gamma(-(s-1)f_i)}{\partial f_i^{n_k}}, \tag{2.109}$$

where we noted the fact that

$$\zeta = \sum_{t=1}^{s-1} f_{\omega_t} = (s-1)f_i. \tag{2.110}$$

Taking into account all the previous results and also noting that

$$n_0 = 1, \tag{2.111}$$

in this case, and also assuming that

$$\frac{1}{c_i} \sim \frac{1}{f_i^2}, \tag{2.112}$$

we come up with the following final result

$$\prod_{t=1}^{s} \left(\frac{\pi_t}{c_t}\right)^{1/2} \sum_{e_s \in 2\mathbb{Z}_+} \binom{s-1}{e_s} \frac{(e_s-1)!!}{2^{e_s/2}} \frac{1}{(s-1)^{n_1+n_k+2}} \frac{(-\gamma_i)^{s-1-e_s}}{f_i^{2s-e_s+n_1}} \frac{\partial^{n_k}\gamma(-(s-1)f_i)}{\partial f_i^{n_k}}. \tag{2.113}$$

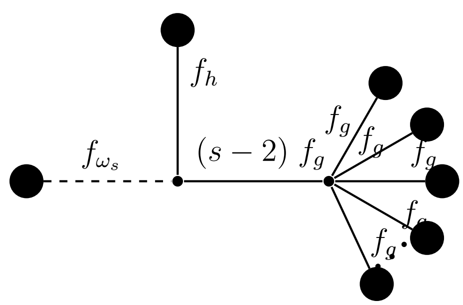
In the last step, we sum over all values of f_i and move to the continuum introducing integration instead of discrete summation:

$$\prod_{t=1}^{s} \left(\frac{\pi_t}{c_t}\right)^{1/2} \sum_{e_s \in 2\mathbb{Z}_+} \binom{s-1}{e_s} \frac{(e_s-1)!!}{2^{e_s/2}(s-1)^{n_1+n_k+2}} \int d^{11}f \, \frac{(-\overline{\gamma}(f))^{s-1-e_s}}{f^{2s-e_s+n_1}} \cdot \frac{\partial^{n_k}\overline{\gamma}(-(s-1)f)}{\partial f^{n_k}}. \tag{2.114}$$

Case 2: $f_{w_1} = f_{w_2} = f_{w_3} = ... = f_{w_{s-2}} = f_g, f_{w_{s-1}} = f_h, f_{w_s} \neq (f_g, f_h)$ for φ_3 field

This case is summarized by the nodal diagram

$$\frac{1}{\mathrm{V}^2} \sum_{g,h}$$



(2.115)

and the integral

$$\prod_{t \neq h,g} \int d\widetilde{\varphi}_3(f_t) \, \exp\Big[-c_t\Big(\widetilde{\varphi}_3(f_t) + \frac{\bar{\gamma}_t}{c_t}\Big)^2\Big] \int d\widetilde{\varphi}_3(f_g) \, \exp\Big[-c_g\Big(\widetilde{\varphi}_3(f_g) + \frac{\bar{\gamma}_g}{c_g}\Big)^2\Big] \, \widetilde{\varphi}_3^{(s-2)}(f_g)$$
$$\times \int d\widetilde{\varphi}_3(f_h) \, \exp\Big[-c_h\Big(\widetilde{\varphi}_3(f_g) + \frac{\bar{\gamma}_g}{c_g}\Big)^2\Big] \, \widetilde{\varphi}_3(f_h) \int d\widetilde{\varphi}_3(f_{\omega_s}) \, \exp\Big[-c_s\Big(\widetilde{\varphi}_3(f_{\omega_s}) + \frac{\bar{\gamma}_s}{c_s}\Big)^2\Big]. \tag{2.116}$$

The integral over the $\widetilde{\varphi}_3(f_g)$ generates the term

$$\sum_{e_s\in 2\mathbb{Z}_+}\begin{pmatrix} s-2 \\ e_s \end{pmatrix}\frac{(e_s-1)!!}{2^{e_s/2}}\frac{1}{c_g^{e_s/2}}\left(-\frac{\gamma_g}{c_g}\right)^{s-2-e_s}. \tag{2.117}$$

The integral over the $\widetilde{\varphi}_3(f_h)$ gives us the simple term

$$-\frac{\gamma_h}{c_h}. \tag{2.118}$$

The last integral is substituted with the following term

$$\frac{1}{\zeta^{2n_0+n_1}}\frac{\partial^{n_k}\gamma(-\zeta)}{\partial\zeta^{n_k}}=\frac{1}{((s-2)f_g+f_h)^{n_1+2}}\left(\frac{1}{(s-2)}\frac{\partial}{\partial f_g}+\frac{\partial}{\partial f_h}\right)^{n_k}\gamma(-(s-2)f_g-f_h). \tag{2.119}$$

The last identity is based on the fact that

$$\zeta=\sum_{t=1}^{s-1}f_{\omega_t}=(s-2)f_g+f_h, \tag{2.120}$$

and thus

$$\frac{\partial^{n_k}\gamma(-\zeta)}{\partial\zeta^{n_k}}=\left(\frac{1}{(s-2)}\frac{\partial}{\partial f_g}+\frac{\partial}{\partial f_h}\right)^{n_k}\gamma(-(s-2)f_g-f_h). \tag{2.121}$$

Again we collect all the results to obtain

$$\begin{aligned}&\sum_{e_s\in 2\mathbb{Z}_+}\begin{pmatrix} s-2 \\ e_s \end{pmatrix}\frac{(e_s-1)!!}{2^{e_s/2}}\frac{(-\gamma_g)^{s-2-e_s}}{c_g^{s-2-e_s/2}}\left(-\frac{\gamma_h}{c_h}\right)\frac{1}{((s-2)f_g+f_h)^{n_1+2}}\\ &\times\left(\frac{1}{(s-2)}\frac{\partial}{\partial f_g}+\frac{\partial}{\partial f_h}\right)^{n_k}\gamma(-(s-2)f_g-f_h).\end{aligned} \tag{2.122}$$

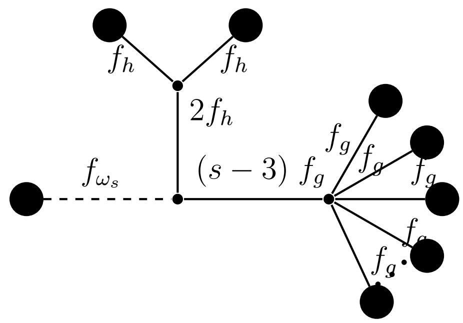
The summations procedure over the indices g and h is the same as described for the previous cases, including the nested structure of the sums and the subsequent integrals. The final result is expressed as the construction of the nested integrals

$$\begin{aligned}&\prod_{t=1}^{s}\left(\frac{\pi_t}{c_t}\right)^{1/2}\sum_{e_s\in 2\mathbb{Z}_+}\begin{pmatrix} s-1 \\ e_s \end{pmatrix}\frac{(e_s-1)!!}{2^{e_s/2}}\\ &\times\left[\int d^{11}f\,\frac{(-\overline{\gamma}(f))^{s-2-e_s}}{(c(f))^{s-2-e_s/2}}\int_f^\infty d^{11}f'\,\frac{\overline{\gamma}(f')}{c(f')((s-2)f+f')^{n_1+2}}\left(\frac{1}{s-2}\frac{\partial}{\partial f}+\frac{\partial}{\partial f'}\right)^{n_k}\overline{\gamma}\left(-(s-2)f-f'\right)\right.\\ &\left.+\int d^{11}f\,\frac{\overline{\gamma}(f)}{c(f)}\int_f^\infty d^{11}f'\,\frac{(-\overline{\gamma}(f'))^{s-2-e_s}}{(c(f'))^{s-2-e_s/2}((s-2)f'+f)^{n_1+2}}\left(\frac{1}{s-2}\frac{\partial}{\partial f'}+\frac{\partial}{\partial f}\right)^{n_k}\overline{\gamma}\left(-(s-2)f'-f\right)\right].\end{aligned} \tag{2.123}$$

Case 3: $f_{w_1} = .. = f_{w_{s-3}} = f_g, f_{w_{s-2}} = f_h, f_{w_{s-1}} = f_m, f_{w_s} \neq (f_g, f_h, f_m)$ **for** φ_3 **field**

There are two possible variations of this case. The first one is slightly simpler: $f_m = f_h \neq f_g$ and is represented by the diagram

$$\frac{1}{\mathrm{V}^2}\sum_{g,h} \qquad (2.124)$$



There are several differences here from the previous case, the major of which is the second power of the field $\widetilde{\varphi}_3(f_h)$. The Gaussian integral of this case was calculated in case 4, with the following result

$$\int d\widetilde{\varphi}_3(f_h)\ \exp\Big[-c_h\Big(\widetilde{\varphi}_3(f_g)+\frac{\bar{\gamma}_g}{c_g}\Big)^2\Big]\ \widetilde{\varphi}_3^2(f_h)=\frac{1}{c_h}+\frac{\gamma_h^2}{c_h^2}=\frac{c_h+\gamma_h^2}{c_h^2}. \qquad (2.125)$$

The rest of the calculations follow the same line of reasoning including the process of the summation with the final nested integration structure:

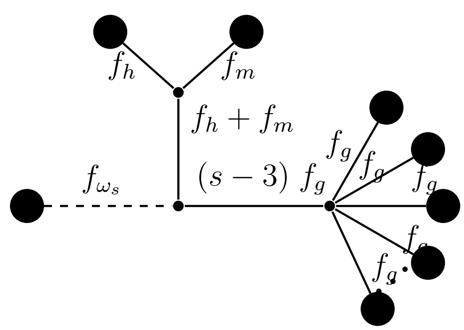
$$\prod_{t=1}^{s}\left(\frac{\pi_t}{c_t}\right)^{1/2}\sum_{e_s\in 2\mathbb{Z}_+}\binom{s-3}{e_s}\frac{(e_s-1)!!}{2^{e_s/2}} \qquad (2.126)$$
$$\times\Bigg[\int d^{11}f\ \frac{(-\overline{\gamma}(f))^{s_o}}{(c(f))^{s_o'}}\int_f^\infty d^{11}f'\ \frac{c(f')+2\overline{\gamma}^2(f')}{2c^2(f')((s-3)f+f')^{n_1+2}}\left(\frac{1}{s-3}\ \frac{\partial}{\partial f}+\frac{1}{2}\frac{\partial}{\partial f'}\right)^{n_k}\overline{\gamma}\left(-(s-3)f-f'\right)$$
$$+\int d^{11}f\ \left(\frac{c(f)+2\overline{\gamma}^2(f)}{2c^2(f)}\right)\int_f^\infty d^{11}f'\ \frac{(-\overline{\gamma}(f'))^{s_o}\,(c(f'))^{-s_o'}}{((s-3)f'+2f)^{n_1+2}}\left(\frac{1}{s-3}\ \frac{\partial}{\partial f'}+\frac{1}{2}\frac{\partial}{\partial f}\right)^{n_k}\overline{\gamma}\left(-(s-3)f'-f\right)\Bigg],$$

where we used the fact that

$$\zeta=\sum_{t=1}^{s-1}f_{\omega_t}=(s-3)f_g+2f_h, \qquad (2.127)$$

The second variation of this case is slightly more involved: $f_m \neq f_h \neq f_g$. The diagrammatic representation of the situation is given by

$$\frac{1}{\mathrm{V}^3}\sum_{g,h,m} \qquad (2.128)$$



The integral structure seems to be simpler, since we don't have a second powers, but the main difficulty is related to the presence of the third sum over the momenta indices. We expect to have the triple nested integral structure and the mechanism of how to built such sums and integrals were in much detail explored in the case 4. The derivatives and powers of the other terms remains the same except in places where the second powers are replaced by the first powers. The result is

$$\prod_{t=1}^{s}\left(\frac{\pi_t}{c_t}\right)^{1/2}\sum_{e_s\in 2\mathbb{Z}_+}\begin{pmatrix} s-3 \\ e_s \end{pmatrix}\frac{(e_s-1)!!}{2^{e_s/2}} \tag{2.129}$$

$$\times\Bigg[\int d^{11}f\,\frac{(-\overline{\gamma}(f))^{s_o}}{(c(f))^{s'_o}}\int_f^\infty d^{11}f'\,\frac{\overline{\gamma}(f')}{c(f')}\int_{f'}^\infty d^{11}f'\,\frac{\overline{\gamma}(f'')}{c(f'')\Big((s-3)f+f'+f''\Big)^{n_1+2}}\left(\frac{1}{s-3}\frac{\partial}{\partial f}+\frac{\partial}{\partial f'}+\frac{\partial}{\partial f''}\right)^{n_k}$$

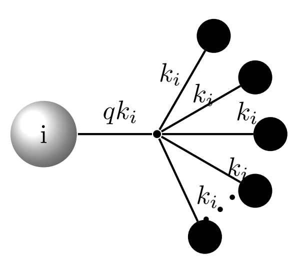
$$\times\ \overline{\gamma}(-(s-3)f-f'-f'')\ +\ \int d^{11}f\,\frac{\overline{\gamma}(f)}{c(f)}\int_f^\infty d^{11}f'\,\frac{\overline{\gamma}(f')}{c(f')}\int_{f'}^\infty d^{11}f''\,\frac{(-\overline{\gamma}(f''))^{s_o}}{(c(f''))^{s'_o}\Big((s-3)f''+f'+f\Big)^{n_1+2}}$$

$$\times\ \left(\frac{1}{s-3}\frac{\partial}{\partial f''}+\frac{\partial}{\partial f'}+\frac{\partial}{\partial f}\right)^{n_k}\overline{\gamma}(-(s-3)f''-f'-f)\Bigg] \tag{2.130}$$

2.2.3 Contributions from the φ_1 fields

Case 1: $k_{u_1}=k_{u_2}=k_{u_3}=..=k_{u_q}$ and k_i for the φ_1 fields

We will start with the simplest case where all the k_{u_i} momenta take the same value. The presence of the extra field φ_1 in the outer leg of the nodal diagram means that there are two possible cases now: one, with the field momenta aligned with the momenta of the q fields, and two, with the field momenta not aligned with the q fields. Note also that the powers of both the k_{u_i} momenta as well as the l_{v_j} momenta have to be changed to account for the momentum conservation.

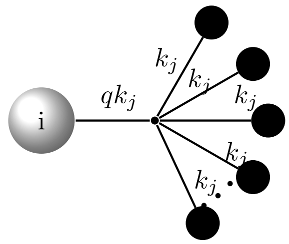
$\frac{1}{\mathrm{V}}\sum_i$ 

$$=\prod_{l=1}^{q}\left(\frac{\pi_l}{a_l}\right)^{1/2}\sum_{e_q\in 2\mathbb{Z}_+}\begin{pmatrix} q+1 \\ e_q \end{pmatrix}\frac{(e_q-1)!!q^{n_q}}{2^{e_q/2}}\int d^{11}k\,\frac{(-\overline{\alpha}(k))^{q+1-e_q}}{k^{2q+2-e_q-n_q}}\,\psi_{\mathbf{k}}(\mathbf{x},y,z)e^{-ik_0t}, \tag{2.131}$$

where we have taken $a(k)\propto k^2$ and $n_q\equiv n+e_o+e_{n_{q'}}$ with $q'\geq 1$.

On the other hand, for the non-aligned field momenta, the nodal diagram gives us:

$$\frac{1}{\mathrm{V}^2}\sum_{i,j}$$



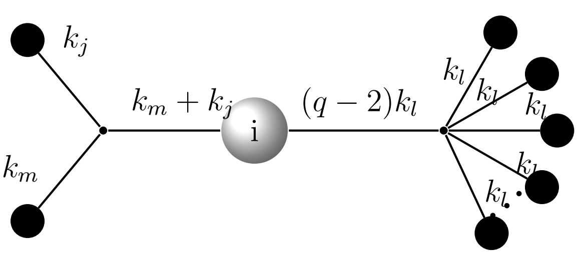
$$= -\prod_{l=1}^{q}\left(\frac{\pi_l}{a_l}\right)^{1/2}\sum_{e_q\in 2\mathbb{Z}_+}\binom{q}{e_q}\frac{(e_q-1)!!q^{n_q}}{2^{e_q/2}} \tag{2.132}$$
$$\times\left[\int d^{11}k\ \frac{\overline{\alpha}(k)}{k^2}\ \psi_{\mathbf{k}}(\mathrm{X})e^{-ik_0t}\int_k^\infty d^{11}k'\ \frac{(-\overline{\alpha}(k'))^{q_1}}{k'^{q_2}} + \int d^{11}k\ \frac{(-\overline{\alpha}(k))^{q_1}}{k^{q_2}}\int_k^\infty d^{11}k'\ \frac{\overline{\alpha}(k')}{k'^2}\ \psi_{\mathbf{k}'}(\mathrm{X})e^{-ik'_0t}\right],$$

where $q_1 \equiv q - e_q$, $q_2 \equiv 2q - e_q - n_q$ with n_q as defined above, and $\mathrm{X} \equiv (\mathbf{x}, y, z)$. It is important to note here that the nested integral structure of the second term involves an integral over the wave-function $\psi_{\mathbf{k}'}(\mathrm{X})e^{-ik'_0t}$ leading to possible temporal dependence.

Case 2: $k_{u_1} = k_{u_2} = ... = k_{u_{q-2}} = k_{u_i}, (k_{u_{q-1}}, k_{u_q}) \neq k_{u_i}$ **for** φ_1 **field**

We can use the same approach as in the previous cases and move from the simplest less general case to the most involved and the most general case. But it makes sense to take a slightly different approach here and start the computations from the most general case. The idea here is that the case with all the momenta differences does not depend on the way we re-assign the momenta indices. This general case should include all the cases with some of the momenta coinciding and all the possible momenta re-distributions.

The general case diagram for the case of the φ_1 field looks something like this one



(2.133)

In the end, we will need to sum over all the four indices i, j, k, l.

We will divide the total integral representing the diagram into four independent pieces and calculate them separately. The integrations over the external field momenta k_i of the

φ_1 field are given by

$$\int d\widetilde{\varphi}_1(k_i)\, \exp\Big[-a_i\Big(\widetilde{\varphi}_1(k_i)+\frac{\bar{\alpha}_i}{a_i}\Big)^2\Big]\, \widetilde{\varphi}_1(k_i)\, \psi_{k_i}(X)e^{-ik_i^0 t} \tag{2.134}$$

$$= \Big(-\frac{\alpha_i}{a_i}\Big)\, \psi_{k_i}(X)e^{-ik_i^0 t}, \tag{2.135}$$

where we ignore the terms coming from the Gaussian integration and are going to be included in the final total integral with all the three fields present.

The integral over the k_l internal momenta evaluates to

$$\int d\widetilde{\varphi}_1(k_l)\, \exp\Big[-a_l\Big(\widetilde{\varphi}_1(k_l)+\frac{\bar{\alpha}_l}{a_l}\Big)^2\Big]\, \widetilde{\varphi}_1^{q-2}(k_l) \tag{2.136}$$

$$= \sum_{e_q\in 2\mathbb{Z}_+} \begin{pmatrix} q-2 \\ e_q \end{pmatrix} \frac{(e_q-1)!!}{2^{e_q/2}} \Big(-\frac{\alpha_l}{a_l}\Big)^{(q-2-e_q)} \frac{1}{a_l^{e_q/2}} \tag{2.137}$$

The two last integrals give the same simple result:

$$\int d\widetilde{\varphi}_1(k_l)\, \exp\Big[-a_j\Big(\widetilde{\varphi}_1(k_j)+\frac{\bar{\alpha}_j}{a_j}\Big)^2\Big]\, \widetilde{\varphi}_1(k_j) = \Big(-\frac{\alpha_j}{a_j}\Big), \tag{2.138}$$

and

$$\int d\widetilde{\varphi}_1(k_m)\, \exp\Big[-a_m\Big(\widetilde{\varphi}_1(k_m)+\frac{\bar{\alpha}_m}{a_m}\Big)^2\Big]\, \widetilde{\varphi}_1(k_m) = \Big(-\frac{\alpha_m}{a_m}\Big). \tag{2.139}$$

We also need to remember the term with the momenta to the power n_q:

$$(k_1+k_2+...+k_q)^{n_q} = ((q-2)k_l+k_m+k_j)^{n_q}, \tag{2.140}$$

which we need to include in the result for the total integral.

Taking into account our previous experience with the summation of nodal diagrams of this kind, we expect to have here a sum structure with a total of 4 nested sums with all possible permutations in the summation indices. As an example of one of the nested sums, we will look at the following nested term:

$$\prod_{l=1}^{q}\Big(\frac{\pi_l}{a_l}\Big)^{1/2} \sum_{e_q\in 2\mathbb{Z}_+} \begin{pmatrix} q-2 \\ e_q \end{pmatrix} \frac{(e_q-1)!!}{2^{e_q/2}} \sum_{l}^{\infty} \Bigg(-\frac{\alpha_l^{(q-2-e_q)}}{a_l^{(q-2-e_q/2)}}\Bigg) \tag{2.141}$$
$$\times \sum_{j=l+1}^{\infty}\Big(-\frac{\alpha_j}{a_j}\Big) \sum_{m=j+1}^{\infty} ((q-2)k_l+k_m+k_j)^{n_q} \Big(-\frac{\alpha_m}{a_m}\Big) \sum_{i=m+1}^{\infty}\Big(-\frac{\alpha_i}{a_i}\Big)\, \psi_{k_i}(X)e^{-ik_i^0 t}$$

The other three terms are just the permutations of the terms and the indices. We also need to be aware of the problem related to the absence of one of the terms in each nested sum. This issue was encountered and discussed in one of the previous cases. It is common to all nested sums (and integrals) with a number of nested parted more than two.

Now, we can argue that the current general solutions should contain all the other variations, including coincided indices in all possible permutations.

2.2.4 Combining nodal diagrams

The next natural step is to combine all the nodal diagrams for the three scalar fields and calculate the expectation value $\langle\varphi_1\rangle_{\overline{\sigma}}$ from the path integral

$$\langle\varphi_1\rangle_{\overline{\sigma}} \equiv \frac{\int \mathcal{D}\varphi_1\mathcal{D}\varphi_2\mathcal{D}\varphi_3\ e^{i\mathbf{S}_{\text{tot}}}\ \mathbb{D}^\dagger(\overline{\alpha},\overline{\beta},\overline{\gamma})\varphi_1(x,y,z)\mathbb{D}(\overline{\alpha},\overline{\beta},\overline{\gamma})}{\int \mathcal{D}\varphi_1\mathcal{D}\varphi_2\mathcal{D}\varphi_3\ e^{i\mathbf{S}_{\text{tot}}}\ \mathbb{D}^\dagger(\overline{\alpha},\overline{\beta},\overline{\gamma})\mathbb{D}(\overline{\alpha},\overline{\beta},\overline{\gamma})}. \tag{2.142}$$

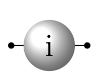
It is expected that the overall form of the nominator and the denominator of the path integral will have a very complicated structure due to the contributions from each scalar field sector. The complexity comes from the fact that, in general, changing the nodal diagrams in the (ϕ_1,ϕ_2) sectors changes the entire series of diagrams from the ϕ_3 sector, as some preliminary analysis shows.

On the other hand, we can ignore all these difficulties by using a product structure to the nodal diagrams as a first approximation. In our case, it means that we can independently sum the nodal diagrams of the tree fields in the limit

$$\zeta \equiv \sum_{t=1}^{s-1} f_{w_t} > \sum_{i=1}^{q} k_{u_i} + \sum_{j=1}^{r} l_{v_j}. \tag{2.143}$$

This simplifying condition is valid since all the momenta are lower than the M_p, and we are allowed to choose a limit when all f_{ω_t} are larger than both k_{u_i} and l_{v_j}.

Taking all these approximations into consideration, we write down the nominator and the denominator of the path integral in the following suggestive way:

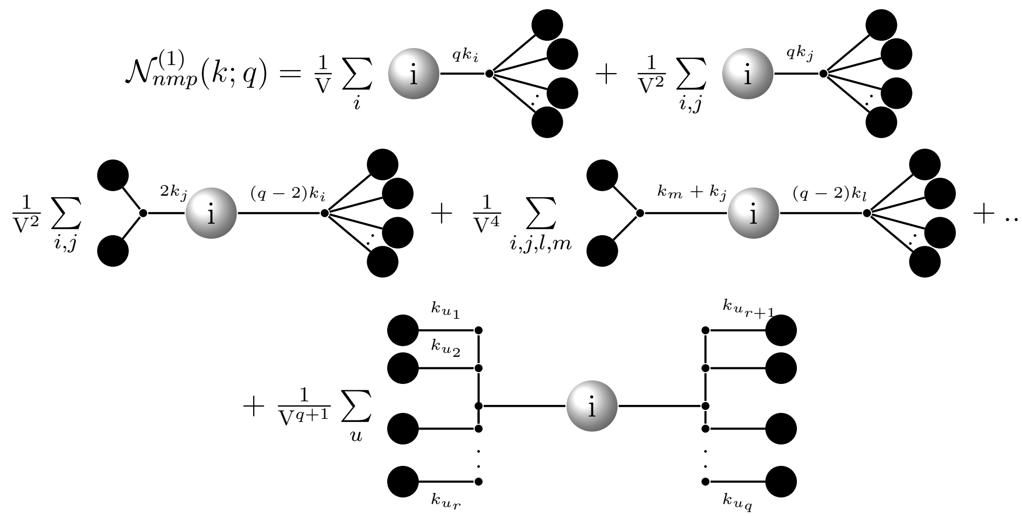
$\text{Num} =$ 

$$+ \sum_{n,\dots,s} c_{nmpqrs}\mathcal{N}^{(1)}_{nmp}(k;q)\otimes\mathcal{N}^{(2)}_{nmp}(\mu;r)\otimes\mathcal{N}^{(3)}_{nmp}(\mu;s) + \mathcal{O}(c^2_{nmpqrs}), \tag{2.144}$$

$$\text{Den} = \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet + \sum_{n,\ldots,s} c_{nmpqrs}\mathcal{N}^{(1)'}_{nmp}(k;q) \otimes \mathcal{N}^{(2)}_{nmp}(\mu;r) \otimes \mathcal{N}^{(3)}_{nmp}(\mu;s) + \mathcal{O}(c^2_{nmpqrs}). \tag{2.145}$$

In each case, the series is represented as a sum of the tree-level term and the quantum contributions in powers of the coupling constants c_{nmpqrs}. We see our approximate implementation of the quantum contributions as a product structure of the diagrammatic series $(\mathcal{N}^{(1)}_{nmp}(k;q), \mathcal{N}^{(1)'}_{nmp}(k;q))$ with momenta k and $(\mathcal{N}^{(2)}_{nmp}(\mu;r), \mathcal{N}^{(3)}_{nmp}(\mu;s))$ with the momenta integrated to the scale $\mu \propto \mathrm{M}_p$ related to the tree scalar fields (ϕ_1, ϕ_2, ϕ_3) respectively.

To better visualize contributions from each field sector, we explicitly write down several terms for each series which were in detail analyzed in the previous sections. The set of nodal diagrams contributing to $\mathcal{N}^{(1)}_{nmp}(k;q)$ is given by



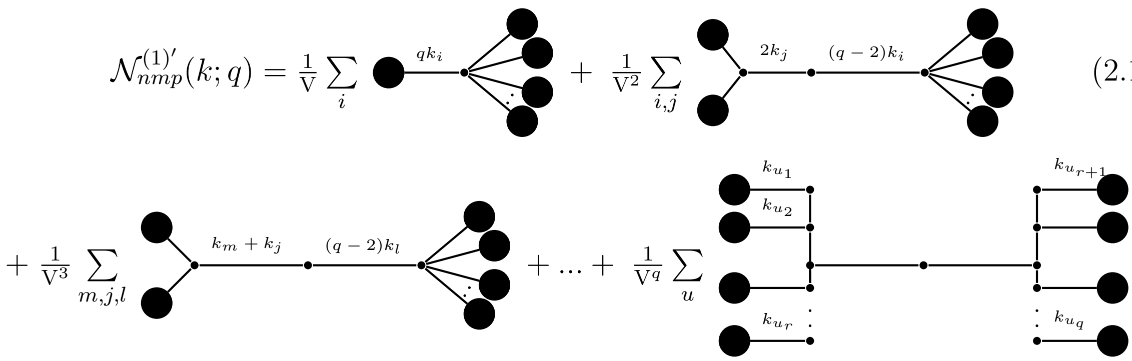
(2.146)

The last nodal diagram becomes the dominant one when we move to the continuous limit, replacing sums with integrals:

$$\frac{1}{\mathrm{V}^{q+1}}\sum_u = \int \tag{2.147}$$

Then we can easily see that the last nodal diagram is not suppressed by any volume factor and contributes $(q-1)!$ terms to the path integral.

The similar term in the denominator $\mathcal{N}^{(1)'}_{nmp}(k;q)$ differs by the absence of the source field ϕ_1:

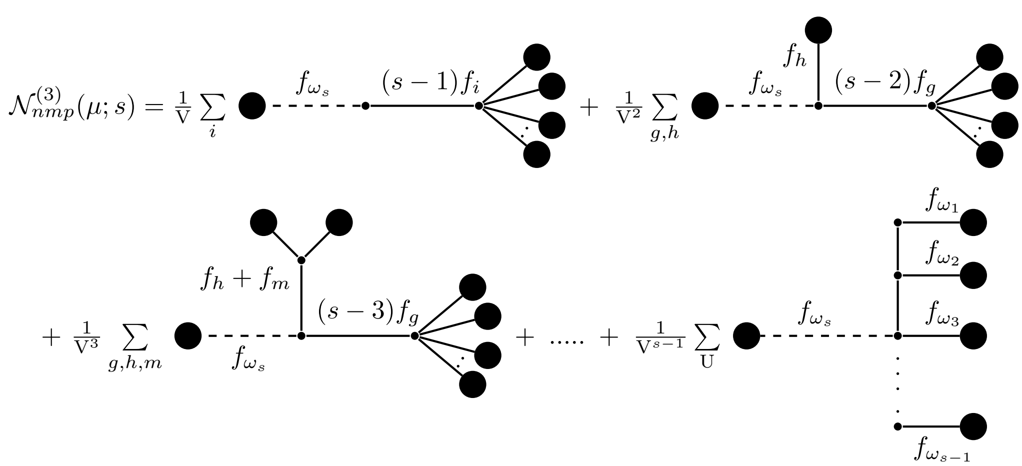


(2.148)

, where $\mathrm{U}' \equiv (k_{u_1}, k_{u_2}, ..., k_{u_q})$. The main contribution comes from the $q!$ terms, the last most dominant nodal diagram.

The $\mathcal{N}^{(2)}_{nmp}(\mu; r)$ sector is very like the last one, except that we need to keep in mind the l_{v_j} momenta have different powers and the outermost integrals in the nested integral structures are all integrated up to the energy scale $\mu \propto \mathrm{M}_p$.

The last $\mathcal{N}^{(3)}_{nmp}(\mu; s)$ sector is somewhat unique since it contains additional momentum constraints. It is appropriate to note that it was our initial choice to put the momentum constraints on the ϕ_3 field. The answer should not change if all previous calculations are rearranged with the momenta limits imposed on ϕ_2 of ϕ_3 fields. This been said, the nodal diagrams contributing to $\mathcal{N}^{(3)}_{nmp}(\mu; s)$ are the following:



(2.149)

Again, from this representation, we deduce that the most dominant diagram is with all the momenta different $f_{w_1} \neq f_{w_2} \neq .... \neq f_{w_{s_1}} \neq f_{w_s}$ and contributing $(s-1)!$ terms with $(s-1)$ nested integrals.

Chapter 3

Resurgence and positivity of cosmological constant

3.1 Introduction to Resurgence in Quantum theory

The great majority of problems in quantum field theory and string theory cannot be solved in closed form. As a result, the main tool of theoretical physicists for handling such problems is some form of approximation scheme in the powers of a small parameter of the theory. Quite often, the result of the approximate solution is some divergent power series.

The presence of the divergent series in physical theories demands a mathematical concept allowing making sense of the infinities. Such a tool was developed by mathematicians more than a century ago, long before physicists realized its usefulness in their own research. This technique is known as the theory of resurgence and was first introduced in the works of Emile Borel. The beauty and power of the theory of resurgence are that it allows us to deduce some knowledge about the non-perturbative effects of the theory using only information coming from the perturbative studies of the theory.

Before discussing the main toolkit of the resurgence theory, i.e. Borel resummation, we briefly review the definition and main features of the asymptotic series. In its most general definition, a power series $\sum_{n=0}^{\infty} a_n z^n$ can be seen as not a function but as an asymptotic approximation to some function $f(z)$, such that

$$\lim_{z\to 0}\frac{1}{z^N}\left(f(z)-\sum_{n=0}^{N}a_n z^n\right)=0. \tag{3.1}$$

In other words, for any $\mathbf{N}>0$, the remainder after (N+1) terms of the series is much smaller than the last controlled term. We also note that an asymptotic series remains well-defined even for cases when the remainder does not go to zero for large values of N and fixed values of z. The defining feature of an asymptotic expansion is that the N-dependent partial sums

$$\sum_{n=0}^{N}a_n z^n \tag{3.2}$$

first, converge to the value of its approximated function $f(z)$, but then they eventually diverge for sufficiently big values of N. To find the partial sum that is closest to the function value $f(z)$, we need to know the optimal value of N. The usual method is to keep all the convergent terms of the asymptotic series up to the smallest one and then eliminate the rest of the series. This technique is called the optimal truncation and can be easily understood in the case of the coefficients a_n with a factorial growth at large n,

$$a_n = cA^n n!, \quad n \gg 1. \tag{3.3}$$

We want to find the smallest term in the series, for a fixed value of z, by minimizing it with respect to N:

$$|a_N z^N| = cN!|Az|^N = c\exp\left\{N\log N - N - N\log\left|\frac{1}{Az}\right|\right\}, \tag{3.4}$$

where in the last step, we used the Stirling approximation. The saddle point at large N is

$$N_* = \left|\frac{1}{Az}\right|. \tag{3.5}$$

As we can see, for $|z|$ small, the truncated series contains many terms, but as the value of $|z|$ grows, the optimal truncation provides us with a meagre number of terms.

The next (N_*+1) term in the series represents the error made by the optimal truncation

$$\textbf{error} = C_{N_*+1}|z|^{N_*+1} \tag{3.6}$$

$$= C_{N_*+1} \exp\left\{(N_*+1)\log(N_*+1) - (N_*+1) - (N_*+1)\log\left|\frac{1}{z}\right|\right\} \sim e^{-\left|\frac{1}{Az}\right|}. \tag{3.7}$$

To summarize, an asymptotic expansion does not uniquely determine the function $f(z)$ since the optimal truncation simply ignores the remaining terms of the truncated series. On the contrary, Borel's resummation procedure considers the information contained in all the terms of the series.

The Borel transform $\mathcal{B}$ of the power series

$$\sum_{n\geqslant 0} a_n z^n \tag{3.8}$$

is a transformation in the complex plane $z^n \mapsto \frac{\zeta^n}{n!}$:

$$\mathcal{B}\sum_{n\geqslant 0} a_n z^n = \sum_{n\geqslant 0} a_n \frac{\zeta^n}{n!}. \tag{3.9}$$

We define the power series $\phi(z)$ to be Gevrey-1 if its coefficients have some factorial growth

$$|a_n| < M n! \rho^n, \tag{3.10}$$

for some constants $M, \rho > 0$. Then, the Borel transform of a Gevrey-1 series is analytic in a neighbourhood of $\zeta = 0$. The power of the Borel transform resides in the fact the singularities of the Borel transform contain information about other sectors of the theory.

The theory of Borel's resummation applies to the case of the resurgent functions. A Gevrey-1 series $\phi(z)$ is a resurgent function if, on any line starting from the origin, there is a finite number of the singularities of the Borel transform $\mathcal{B}\phi(z)$. The Borel transform can be analytically continued along the line, avoiding the singularity points.

In case of a resurgent function $\phi(z)$ with a logarithmic and a pole singularity at $\zeta = \zeta_\omega$, the local expansion of its Borel transform $\hat{\phi}(z) \equiv \mathcal{B}\phi(z)$ has the form

$$\hat{\phi}(\zeta_\omega + \xi) = -\frac{\mathtt{S}a}{2\pi} - \frac{\mathtt{S}a}{2\pi}\log(\xi)\sum_{n\geqslant 0}\hat{c}_n \xi^n + \text{regular}. \tag{3.11}$$

The series which appears in this expansion around the singularity point

$$\hat{\phi}_\omega(\xi) = \sum_{n\geqslant 0}\hat{c}_n \xi^n \tag{3.12}$$

has a finite radius of convergence. The explicit introduction of a so-called Stokes constant $\mathtt{S}$ allows us to choose a specific gauge for normalization of $\hat{\phi}_\omega(\xi)$. Now we can interpret $\hat{\phi}_\omega(\xi)$ as the Borel transform of another, divergent, series of the form

$$\phi_\omega(z) = \frac{a}{z} + \sum_{n\geqslant 0} c_n z^n, \tag{3.13}$$

with the new coefficients $c_n = n!\hat{c}_n$. We conclude that the expansion of the Borel transform $\mathcal{B}\phi(z)$ of a formal power series $\phi(z)$ around its singularities $\zeta_\omega, \omega \in \Omega\{\text{set of singular points}\}$:

$$\phi(z) \to \{\phi_\omega(z)\}_{\omega\in\Omega}\,. \tag{3.14}$$

We see that each singularity originating from the Borel transform generates a new power series. These new series reappear, "resurge" in the original series $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ due to the asymptotic behaviour of the a_k coefficients.

Now it is time to move from the Borel transform to the Borel resummation. Under certain conditions, Borel resummation produces an actual function reproducing the original power series. We define a Stokes ray, a ray in the Borel plane that starts at the origin and passes through a singularity ξ_ω of the Borel transform $\hat{\phi}(\xi)$. This ray in the complex plane can be formally represented as $e^{i\theta}\mathbb{R}_+$ with $\theta = \arg(\xi_\omega)$. The Borel resummation of the power series $\phi(z)$ is a Laplace transform along the ray $\mathcal{C}^\theta = e^{i\theta}\mathbb{R}_+$:

$$s(\phi)(z) = \int_0^\infty \hat{\phi}(z\zeta)e^{-\zeta}d\zeta = \frac{1}{z}\int_{\mathcal{C}_\theta} \hat{\phi}(\zeta)e^{-\frac{\zeta}{z}}d\zeta, \tag{3.15}$$

where $\hat{\phi}$ is analytically continued along the ray $\mathcal{C}_\theta$.

3.2 Borel resummation: real-life calculation

3.2.1 The beauty of Hermite polynomials

We introduce simplifying assumptions that will make the following calculations less messy but allow us to obtain some results. We do not introduce any IR cut-offs in the system and insert them later, ensuring that all the nodal diagrams in the three distinct sectors have different momenta. Next, eleven-dimensional momenta may be divided into the radial and the temporal parts as

$$k = (k_0, \mathbf{k}) = (k_0, |\mathbf{k}|, k_\Omega), \tag{3.16}$$

where k_Ω represents all the angular degrees of freedom.

And finally, to make progress in the calculations, we need to introduce explicit functional forms for $\overline{\alpha}(k_0, \mathbf{k}), \overline{\beta}(l_0, \mathbf{l})$, and $\overline{\gamma}(f_0, \mathbf{f})$:

$$\begin{aligned}
&\overline{\alpha}(k_0, \mathbf{k}) \equiv \sum_n \sum_m c_{nm} \mathbf{H}_n(\mathtt{k}) \mathbf{H}_m(k_0) \\
&= \sum_{n>0} c_{n0} \mathbf{H}_n(\mathtt{k}) \ + \sum_{m>0} c_{0m} \mathbf{H}_m(k_0) \ + \sum_{(n,m)>0} c_{nm} \mathbf{H}_n(\mathtt{k}) \ \mathbf{H}_m(k_0), \qquad (3.17) \\
&\overline{\beta}(l_0, \mathbf{l}) \equiv \sum_{n,m} b_{nm} \mathbf{H}_n(\mathtt{l}) \mathbf{H}_m(l_0), \\
&\overline{\gamma}(f_0, \mathbf{f}) \equiv \sum_{n,m} d_{nm} \mathbf{H}_n(\mathtt{f}) \mathbf{H}_m(f_0).
\end{aligned}$$

As a reminder, we also mention here that $\overline{\alpha}, \overline{\beta}, \overline{\gamma}$ represent the coherent states of the three respective scalar fields of our toy model or, equivalently, the three sets of degrees of freedom from the full M-theory: metric, fluxes and gravitino fields.

Since we are taking the momenta to be dimensionless, the Hermite polynomials are well-defined through the dimensionless parameters. The coefficients c_{nm}, b_{nm}, d_{nm} used in the definition of the coherent state eigenvalues incorporate all the angular dependencies coming from our initial decomposition of the momenta into the radial and the angular parts.

The Hermite polynomials' choice for the explicit calculations is not random and makes some sense. Hermite polynomials belong to the class of classical orthogonal polynomials and are defined by

$$\mathbf{H}_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \, e^{-x^2} = e^{\frac{x^2}{2}} \left(x - \frac{d}{dx} \right)^n e^{-\frac{x^2}{2}}. \tag{3.18}$$

They also satisfy the orthogonality condition:

$$\int_{-\infty}^{+\infty} dx \ \mathbf{H}_n(x) \ \mathbf{H}_m(x) \ e^{-x^2} = 2^n n! \sqrt{\pi} \delta_{mn}. \tag{3.19}$$

One of the most appreciated features of Hermite polynomials is that they form a complete orthogonal basis of the Hilbert space of functions satisfying

$$\int_{-\infty}^{+\infty} dx\, |f(x)|^2 e^{-x^2} < \infty, \tag{3.20}$$

and with the inner product defined by the integral

$$\langle f, g\rangle = \int_{-\infty}^{+\infty} \mathrm{d}x\, f(x) g^*(x)\; e^{-x^2}. \tag{3.21}$$

Another reason for making use of Hermite polynomials for these particular calculations is that they allow replacing derivatives over the powers of the arguments of the Hermite polynomials in terms of a series of Hermite polynomials themselves:

$$\partial_{\mathtt{k}}^m \mathbf{H}_n(\mathtt{k}) \equiv \mathbf{H}_n^{(m)}(\mathtt{k}) = 2^n m! \begin{pmatrix} n \\ m \end{pmatrix} \mathbf{H}_{n-m}(\mathtt{k}). \tag{3.22}$$

We also introduce here some useful identities which will be useful for the explicit calculations of the amplitudes:

$$\begin{aligned} \mathbf{H}_n(k^2) &= \mathbf{H}_n(-k_0^2 + |\mathbf{k}|^2) = 2^{-n/2} \sum_{p=0}^{n} \begin{pmatrix} n \\ p \end{pmatrix} \mathbf{H}_{n-p}(-\sqrt{2}k_0^2)\mathbf{H}_p(\sqrt{2}\mathtt{k}^2), \\ \mathtt{k}^n &\equiv \frac{n!}{2^n} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\mathbf{H}_{n-2m}(\mathtt{k})}{m!(n-2m)!}. \end{aligned} \tag{3.23}$$

Another interesting result to note is the value of the $\mathbf{H}_n^{(n)}$:

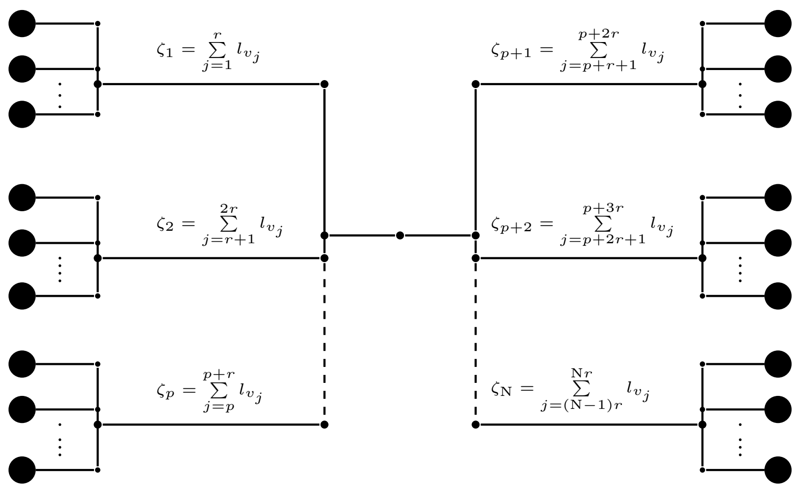
$$\mathbf{H}_n^{(n)}(\mathtt{k}) = 2^n m! \begin{pmatrix} n \\ n \end{pmatrix} \mathbf{H}_{n-n}(\mathtt{k}) = 2^n n! \tag{3.24}$$

Now we are ready to commence our total amplitude calculation. We will do it in several small steps, taking each sector at a time and then combining all the results. In all the calculations, we assume the slice over the angular directions and use only radial components of momenta.

3.2.2 Explicit calculations of the nodal diagrams' amplitudes

We start with the ϕ_2 sector with a set of momenta l_{v_j} and the dominant nodal diagram to N^{th} order in coupling. Note that the computations for ϕ_2 and ϕ_3 sectors will be very similar. The presence of the derivatives of $\overline{\gamma}(f)$ in the ϕ_3 sector is compensated by the fact that derivatives of the Hermite polynomials can be replaced by other Hermite polynomials with the final structure almost identical to the ϕ_2 sector.

The general form of the order g_2^N dominant nodal diagram is given by



(3.25)

We can see that each order in the coupling constant adds its own set of r momenta l_{v_j} to the total structure of the nodal diagram, eventually realizing N copies of $\{l_{v_j}\}$ sets containing r momenta. The amplitude of this complex diagram is given by the nested integral structure

$$g_2^{\mathrm{N}} \prod_{s=1}^{\mathrm{N}r} \left(\frac{\pi_s}{b_s}\right)^{1/2} \sum_{\{m_i\}} \mathbb{C}\left(m_1, ..., m_{\mathrm{N}r}\right) \int_{k_{\mathrm{IR}}}^{\mu} d^{11}l_1 \; \frac{\overline{\beta}(l_1) l_1^{m_1}}{b(l_1)} \quad ... \tag{3.26}$$

$$\times \int_{l_{\mathrm{N}r-2}}^{\mu} d^{11}l_{\mathrm{N}r-1} \; \frac{\overline{\beta}(l_{\mathrm{N}r-1}) l_{\mathrm{N}r-1}^{m_{\mathrm{N}r-1}}}{b(l_{\mathrm{N}r-1})} \int_{l_{\mathrm{N}r-1}}^{\mu} d^{11}l_{\mathrm{N}r} \; \frac{\overline{\beta}(l_{\mathrm{N}r}) l_{\mathrm{N}r}^{m_{\mathrm{N}r}}}{b(l_{\mathrm{N}r})} + ... \tag{3.27}$$

where the combinatoric coefficients are defined as $\mathbb{C}\left(m_1, ...m_{\mathrm{N}r}\right)$.

Taking into account the explicit form of the propagator

$$b(l) = l^2 = l_0^2 - \mathbf{l}^2 = -\mathbf{l}^2 \left(1 - \frac{l_0^2}{\mathbf{l}^2}\right). \tag{3.28}$$

and the Hermite polynomial in terms of its arguments

$$\mathbf{H}_n(\mathbf{l}) \equiv n! \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^m}{m!(n-2m)!} \; (2\mathbf{l})^{n-2m}, \tag{3.29}$$

we can integrate over the temporal and radial part of momenta.

It makes sense to make all the steps in the following computation very explicit with all the intermediate algebraic manipulations since we will need to perform similar calculations several times.

The integral we are interested in is the following one:

$$\int_{k_{\rm IR}}^{\mu} d^{11}l_r \; \frac{\overline{\beta}(l_r) l_r^{m_r}}{b(l_r)}. \tag{3.30}$$

We take into account the explicit form of all the integrand parts in terms of the Hermite polynomials:

$$\frac{l_r^{m_r}}{b(l_r)} = \frac{l_r^{m_r}}{l_r^2} = (l_r^2)^{\frac{m_r-2}{2}} = (-1)^{\frac{m_r-2}{2}} \mathbf{l}_r^{m_r-2} \left(1 - \frac{l_{0,r}^2}{\mathbf{l}_r^2}\right)^{\frac{m_r-2}{2}}, \tag{3.31}$$

$$\begin{aligned}
\bar{\beta}(l) &= \sum_n \sum_m b_{nm} \mathbf{H}_n(\mathtt{l}) \mathbf{H}_m(l_0) \\
&= \sum_n \sum_{p=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^p}{p!(n-2p)!} (2\mathtt{l})^{n-2p} \; \sum_m \sum_{q=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^q}{q!(m-2q)!} (2l_0)^{m-2q} \\
&= \sum_{nmpq} n!m! \frac{(-1)^{p+q}}{p!q!(n-2p)!(m-2q)!} (2\mathtt{l})^{n-2p} (2l_0)^{m-2q}.
\end{aligned} \tag{3.32}$$

We also can rewrite the integration measure in the temporal and the radial part as we mentioned before:

$$\int_{k_{\rm IR}}^{\mu} d^{11}l_r = \int_{k_{\rm IR}}^{\mu_0} dl_0 \int_{k_{\rm IR}}^{\mu} d^{10}\mathbf{l} = \int_{k_{\rm IR}}^{\mu_0} dl_0 \int_{k_{\rm IR}}^{|\mu|} d\mathtt{l} \; \mathtt{l}^9 \int d\Omega, \tag{3.33}$$

where $\mathtt{l}$ is the radial part of the momenta, and $\int d\Omega$ integration takes care of all the implicit angular dependencies.

We make a choice to do the $\int dl_0$ integration first:

$$\int_{k_{\rm IR}}^{\mu_0} dl_0 \; (2l_0)^{m-2q} \left(1 - \frac{l_{0,r}^2}{\mathbf{l}_r^2}\right)^{\frac{m_r-2}{2}}. \tag{3.34}$$

The value of this integral is given by some special functions, such as the gamma function, which will be difficult to analyze. Fortunately, we are interested only in the approximate results which can be obtained following only the highest powers of the energy scale. We open up brackets and evaluate only the first term in the resulting series:

$$\int_{k_{\rm IR}}^{\mu_0} dl_0 \; (2l_0)^{m-2q} = \frac{2^{m-2q}}{m-2q+1}\mu_0^{m-2q+1}. \tag{3.35}$$

Similarly, the integral over the radial part gives us the result:

$$\int_{k_{\rm IR}}^{|\mu|} d\mathtt{l} \; \mathtt{l}^9 \mathtt{l}^{m_r-2} \; (2\mathtt{l})^{n-2p} = \frac{2^{n-2p}}{n+m_r-2m+8}|\mu|^{n+m_r-2m+8}. \tag{3.36}$$

Combining all these nice simple results, we obtain the following expression for our generic term from the amplitude:

$$\begin{aligned}\int_{k_{\rm IR}}^{\mu} d^{11}l_r \; \frac{\overline{\beta}(l_r)l_r^{m_r}}{b(l_r)} &= \int_{k_{\rm IR}}^{\mu} dl_{r,0}\; dl_r^{10}\frac{l_r^{m_r}}{l_r^2}\sum_{n,m} b_{np}\mathbf{H}_n(\mathtt{l})\mathbf{H}_m(l_0) \\ = \sum_{n,m,p,q} &\frac{(-1)^{p+q+1}2^{n+m-2p-2q}n!m!}{p!q!(n-2p)!(m-2q)!}\cdot\frac{|\mu|^{n-2p+m_r+8}\mu_0^{m-2q+1}}{(m-2q+1)(n-2p+m_r+8)}\left(\int b_{nm}d\Omega\right),\end{aligned}$$

where the $\int b_{nm}d\Omega$ part takes into account the integration over the still unknown part of the b_{nm} angular dependence. It would also be interesting to see the behaviour over the energy scale $\mu \equiv (\mu_0, |\mu|)$ by isolating one term in the whole expansion for the values $m = 0, q = 0$

$$\begin{aligned}&\int_{k_{\rm IR}}^{\mu} d^{11}l_r \; \frac{\overline{\beta}(l_r)l_r^{m_r}}{b(l_r)} = \\ &\sum_{n=0}^{\infty}\sum_{p=0}^{\lfloor \frac{n}{2}\rfloor}\frac{2^{n-2p}(-1)^p n!}{p!(n-2p)!(n-2p+m_r+8)}\left(\int b_{n0}d\Omega\right)\; \mu_0|\mu|^{n-2p+m_r+8} + ...\end{aligned} \tag{3.37}$$

Since our main interest lies in the factorial growth of the particular amplitude configuration, we note that the individual term in the amplitude of the ϕ_2 sector does not have an expected $(Nr)!$ growth. From the previous analysis, we know that there are in total $(Nr)!$ terms contributing to the nodal diagram in the ϕ_2 sector. At the same time, our analysis shows that each term in this diagram is suppressed by at least $\frac{1}{(Nr)!}$ numerical coefficient such that the system as a whole does not contain an exponential growth.

The ϕ_3 sector, in many respects, reflects the previous case with one essential difference: the amplitudes involve expressions with the derivatives of $\bar{\gamma}(f)$. Potentially, it could make the computations more involved. But, at the very beginning of our analysis, we have made a smart choice for the orthogonal basis of the Hilbert space of functions. The Hermite polynomials allow replacing derivatives of the Hermite polynomials in terms of a series of Hermite polynomials themselves:

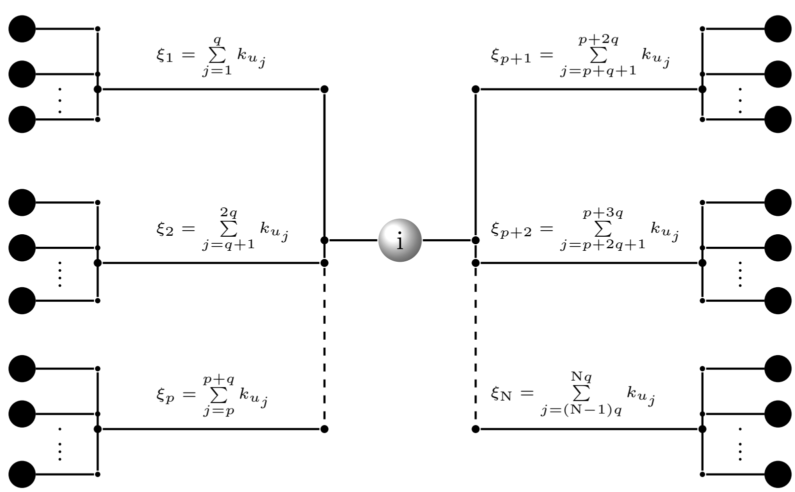
$$\partial_{\mathtt{k}}^{m}\mathbf{H}_n(\mathtt{k}) \equiv \mathbf{H}_n^{(m)}(\mathtt{k}) = 2^n m! \begin{pmatrix} n \\ m \end{pmatrix} \mathbf{H}_{n-m}(\mathtt{k}). \tag{3.38}$$

As a result, the full analysis of the ϕ_3 sector would take almost exactly the same steps as we made for the case of the ϕ_2 sector. The conclusions about the amplitude growth are the same: the number of diagrams increase as $(N(s-1))!$, the individual term does not show the $(N(s-1))!$ growth, confirming that the amplitudes of the nodal diagrams do not grow according to $(N(s-1))!$ law.

The presence of the source field in the ϕ_1 sector makes the computations here slightly more involved. And the most prominent difference between the two other sectors is the presence of the wave function of the source field. First, we define the amplitude of the most dominant nodal diagram as

$$\Gamma(\mathbf{k}_1, k_{1,0}) \equiv g_1^{\mathrm{N}} \prod_{s=1}^{\mathrm{N}q+1} \left(\frac{\pi_s}{b_s}\right)^{1/2} \sum_{\{m_i\}} \mathbb{D}\left(m_1, ..., m_{\mathrm{N}q}\right) \frac{\overline{\alpha}(k_1) k_1^{m_1}}{a(k_1)} \int_{k_1}^{\mu} d^{11}k_2 \, \frac{\overline{\alpha}(k_2) k_2^{m_2}}{a(k_2)} .... \int_{k_{\mathrm{N}q-1}}^{\mu} d^{11}k_{\mathrm{N}q} \, \frac{\overline{\alpha}(k_{\mathrm{N}q}) k_{\mathrm{N}q}^{m_{\mathrm{N}q}}}{a(k_{\mathrm{N}q})} \int_{k_{\mathrm{N}q}}^{\mu} d^{11}k' \psi_{\mathbf{k}'}(\mathrm{X}) e^{-ik'_0 t} \frac{\overline{\alpha}(k')}{a(k')}.$$

To make this expression more palpable, we also draw the respective nodal diagram, which, as expected, is very similar to the other cases:



(3.39)

Once again, one main goal is to prove or disprove the naive expectation that the amplitude grows as $(Nq+1)!$ since there are $(Nq+1)!$ different nested integrals.

To find the total value of the amplitude, we need to integrate over all k_1 momenta:

$$\int_{k_{\mathrm{IR}}}^{\mu} d^{11}k_1 \Gamma(\mathbf{k}_1, k_{1,0}).$$

The set of integrals without the source wave-function follows the same computation algorithm which has been done for other cases. There is nothing new here.

On the other hand, the integral with the source term present needs additional attention. First, we need to choose a more specific form of the wave function $\psi_k(X)$. The idea is to obtain only a temporal dependence at some specific limiting case. We make a definition

$$\psi_k(\mathrm{X})e^{-ik_0t} \equiv \frac{k^2}{\pi|\omega|\mathbf{k}^9} \exp\left(-\frac{(\mathbf{k}-\overline{\mathbf{k}}_{\mathrm{IR}})^2}{\omega^2} - ik_0t\right) \mathcal{F}\left(\mathbf{k}, \mathrm{X}\right),$$

where in the limit $\omega \leftarrow 0$ we have a delta function

$$\lim_{\omega\to 0} \frac{e^{-k^2/\omega^2}}{\pi\omega} = \delta(k),$$

which leads to the following simplification:

$$\begin{aligned}
&\lim_{\omega\to 0}\int_{k_{\mathrm{IR}}}^{\mu} d^{11}k\ \psi_k(\mathrm{X})e^{-ik_0t}\ \frac{\overline{\alpha}(\mathbf{k},k_0)}{a(k)} = \\
&= \lim_{\omega\to 0}\int_{k_{\mathrm{IR}}}^{\mu} dk_0\ dk\ k^9 \frac{\alpha(\bar{k})}{k^2}\frac{k^2}{\pi\omega k^9} \exp\left(\frac{-(\mathbf{k}-k_{IR})^2}{\omega^2}\right)\ e^{-ik_0t}\mathcal{F}\left(\mathbf{k},\mathrm{X}\right) \\
&= \int_{k_{\mathrm{IR}}}^{\mu} dk_0\ \delta(\mathbf{k}-k_{IR})e^{-ik_0t}\mathcal{F}\left(\mathbf{k},\mathrm{X}\right).
\end{aligned}$$

The other ingredient for the successful integration is the use of the Riemann-Lebesgue lemma:

$$\int_a^b d\mathtt{k}\ e^{i\mathtt{kx}}f(\mathtt{k}) = \sum_{n=0}^{\mathrm{N}-1}\ (-1)^n\ e^{i\mathtt{kx}}\ \frac{f^{(n)}(\mathtt{k})}{(i\mathtt{x})^{n+1}}\Bigg|_a^b\ +\ \mathcal{O}\left(\frac{1}{\mathtt{x}^{\mathrm{N}+1}}\right), \tag{3.40}$$

where $f^{(n)}(\mathtt{k}) = \frac{d^nf(\mathtt{k})}{d\mathtt{k}^n}$ and $f^{(0)}(\mathtt{k}) = f(\mathtt{k})$. A sufficient, but not necessary, condition is that $f(\mathtt{k})$ is continuously differentiable for $a \le \mathtt{k} \le b$ and $\int_a^b |f(\mathtt{k})|\ d\mathtt{k} < \infty$. In the limit of $\mathtt{x} \to \infty$, the factor $e^{i\mathtt{kx}}$ oscillates faster and faster such that $e^{i\mathtt{kx}}f(\mathtt{k})$ averages out to zero over any finite region of $\mathtt{k}$ inside the interval.

We move back to the most dominant diagram of the ϕ_1 sector and calculate the integral, which contains the wave-function part:

$$\int_{k_{\mathrm{N}q}}^{\mu} d^{11}k\psi_{\mathbf{k}}(\mathrm{X})e^{-ik_0t}\frac{\overline{\alpha}(k)}{a(k)} = \lim_{\omega\to 0}\int_{k_{\mathrm{N}q}}^{\mu} d^{11}k\ \mathbf{Re}\left(\psi_{\mathbf{k}}(\mathrm{X})e^{-ik_0t}\right)$$
$$\sum_{n,m}\int d\Omega\ c_{2n,m}\mathbf{H}_{2n}(k_{IR})\ \int_{k_{IR}}^{\mu} dk_0\ \mathbf{H}_m(k_0)\left(e^{-ik_0t}+e^{ik_0t}\right). \tag{3.41}$$

We note here that, since at the very beginning of our path-integral analysis, we decided to ignore the complex parts of the present fields, the Fourier representation of the real field components and only positive momenta is given by

$$\phi(x) = \int_0^{\infty} d^{11}k\ \mathbf{Re}\left(\psi_{\mathbf{k}}(\mathrm{X})e^{-ik_0t}\right). \tag{3.42}$$

To find the integral over the temporal domain, we use the Riemann-Lebesgue lemma:

$$\begin{aligned}
&\int_{k_{IR}}^{\mu} dk_0\ \mathbf{H}_m(k_0)\left(e^{-ik_0t}+e^{ik_0t}\right)\\
&=\sum_{p=0}^{m}(-1)^p e^{-ik_0t}\frac{1}{(-it)^{p+1}}\frac{d^p\mathbf{H}_m(k_0)}{dk_0^p}+\sum_{p=0}^{m}(-1)^p e^{ik_0t}\frac{1}{(it)^{p+1}}\frac{d^p\mathbf{H}_m(k_0)}{dk_0^p} \qquad (3.43)\\
&=\left[e^{-ik_0t}\sum_{p=0}^{m}(-1)^p2^pp!\begin{pmatrix}n\\ m\end{pmatrix}\frac{\mathbf{H}_{m-p}(k_0)}{(-it)^{p+1}}+e^{ik_0t}\sum_{p=0}^{m}(-1)^p2^pp!\begin{pmatrix}n\\ m\end{pmatrix}\frac{\mathbf{H}_{m-p}(k_0)}{(it)^{p+1}}\right]_{\kappa_{IR}}^{\mu_0}.
\end{aligned}$$

Now we can integrate the total amplitude of the ϕ_1 sector following the same technique used for the sectors without the source term. Since we are interested only in the energy-dependent part of the amplitude, we combine all the spacial dependencies into some coefficients:

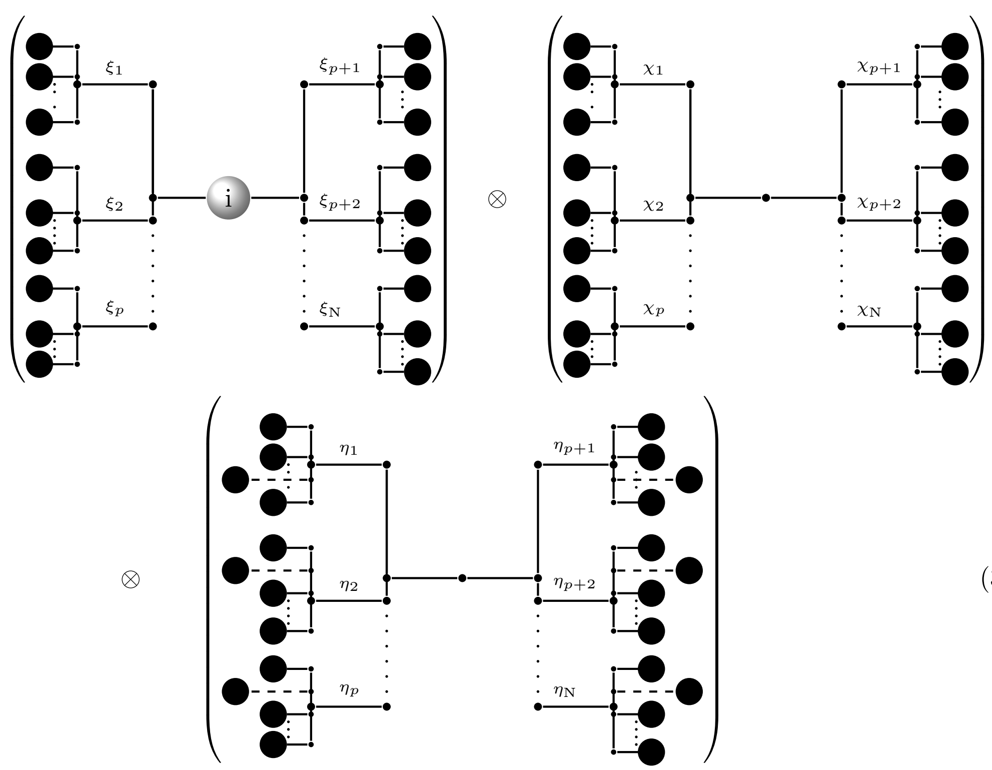
$$\begin{aligned}
\mathbb{C}_m^{(\mathrm{N})} \equiv\ & \prod_{t=1}^{\mathrm{N}q+1}\left(\frac{\pi_t}{b_t}\right)^{1/2}\sum_{\{m_i\}}\mathbb{D}\left(m_1,m_2,m_3,...,m_{\mathrm{N}q-1},m_{\mathrm{N}q}\right)\mathbb{D}_{\varphi_2}(\mathrm{N}r)\mathbb{D}_{\varphi_3}(\mathrm{N}(s-1)) \qquad (3.44)\\
\times & \int_{k_{\mathrm{IR}}}^{\mu} d^{11}k_1\frac{\overline{\alpha}(k_1)k_1^{m_1}}{a(k_1)}\int_{k_{\mathrm{IR}}}^{\mu} d^{11}k_2\ \frac{\overline{\alpha}(k_2)k_2^{m_2}}{a(k_2)}....\int_{k_{\mathrm{IR}}}^{\mu} d^{11}k_q\ \frac{\overline{\alpha}(k_q)k_q^{m_q}}{a(k_q)}....\int_{k_{\mathrm{IR}}}^{\mu} d^{11}k_{2q}\ \frac{\overline{\alpha}(k_{2q})k_{2q}^{m_{2q}}}{a(k_{2q})}....\\
\times & \int_{k_{\mathrm{IR}}}^{\mu} d^{11}k_{\mathrm{N}q-1}\ \frac{\overline{\alpha}(k_{\mathrm{N}q-1})k_{\mathrm{N}q-1}^{m_{\mathrm{N}q-1}}}{a(k_{\mathrm{N}q-1})}\int_{k_{\mathrm{IR}}}^{\mu} d^{11}k_{\mathrm{N}q}\ \frac{\overline{\alpha}(k_{\mathrm{N}q})k_{\mathrm{N}q}^{m_{\mathrm{N}q}}}{a(k_{\mathrm{N}q})}\sum_{n=1}^{\infty}\left(\int d\Omega\ c_{2n,m}\ \mathbf{H}_{2n}(\overline{\mathbf{k}}_{\mathrm{IR}})\right),
\end{aligned}$$

where $\mathbb{D}\left(m_1,m_2,m_3,...,m_{\mathrm{N}q-1},m_{\mathrm{N}q}\right)$ is a combinatorial factor that has been defined earlier. The coefficients coming from the two other sectors are incorporated into the $\mathbb{D}_{\varphi_2}(\mathrm{N}r)\mathbb{D}_{\varphi_3}(\mathrm{N}(s-1))$ combinatoric factors. Each of them contains an N! growth inside, which suggests at least $(\mathrm{N}!)^2$ growth for the $\mathbb{C}_m^{(\mathrm{N})}$ factor. We can see that our combinatoric factor $\mathbb{C}_m^{(\mathrm{N})}$ has only N and m as free parameters since the final result should not depend on which sector we impose the momentum conservation condition.

3.2.3 The amplitude of the nodal diagrams to Nth order

We begin our investigation of the nodal diagrams to the Nth order by taking a product of the three diagrams from the previous subsection representing each of the sectors. The overall coupling constant g^N combines all the coupling constants g_1^N, g_2^N, g_3^N for each sector. We define $\mathcal{S}^{(\mathrm{N})}(t)$ to be the amplitude of the nodal diagrams to $\mathcal{O}(\mathrm{N})$ order. The graphical representation of this combined amplitude is given by the following product

$\mathcal{S}^{(\mathrm{N})}(t) = g^{\mathrm{N}}$



(3.45)

The amplitude of each subbranch is given by the generic expression calculated in the previous subsection

$$g^{\mathrm{N}} \sum_{p=0}^{m} \frac{(-1)^p 2^p \mathbb{C}_m^{(\mathrm{N})} \mathbf{H}_{m-p}(k_0)}{(m-p)!(it)^{p+1}} e^{ik_0 t} \Bigg|_{k_{\mathrm{IR}}}^{\mu} + g^{\mathrm{N}} \sum_{p=0}^{m} \frac{(-1)^p 2^p \mathbb{C}_m^{(\mathrm{N})} \mathbf{H}_{m-p}(k_0)}{(m-p)!(-it)^{p+1}} e^{-ik_0 t} \Bigg|_{k_{\mathrm{IR}}}^{\mu} \tag{3.46}$$

To write down the expression for the total amplitude, we need to consider that we have an infinite series since all the Hermite polynomials contribute. At the same time, we have a divergent asymptotic series with at least $(\mathrm{N}^2)!$ growth. The defining feature of the asymptotic series is that the partial sums $\mathcal{S}^{(\mathrm{N})}(t)$ will first approach the true value of the function that they approximate, and then, for N sufficiently big, they will diverge. We arbitrarily divide the series into decreasing and increasing parts:

$$\begin{aligned}\mathcal{S}^{(\mathrm{N})}(t) = g^{\mathrm{N}}\Bigg[&\sum_{p=0}^{1}\frac{(-1)^p 2^p \mathbb{C}_1^{(\mathrm{N})}\mathbf{H}_{1-p}(k_0)}{(1-p)!(it)^{p+1}} + \sum_{p=0}^{2}\frac{(-1)^p 2^p 2!\mathbb{C}_2^{(\mathrm{N})}\mathbf{H}_{2-p}(k_0)}{(2-p)!(it)^{p+1}} \\ &+\sum_{p=0}^{3}\frac{(-1)^p 2^p 3!\mathbb{C}_3^{(\mathrm{N})}\mathbf{H}_{3-p}(k_0)}{(3-p)!(it)^{p+1}} + \sum_{p=0}^{4}\frac{(-1)^p 2^p 4!\mathbb{C}_4^{(\mathrm{N})}\mathbf{H}_{4-p}(k_0)}{(4-p)!(it)^{p+1}} + \ldots \\ &+\sum_{p=0}^{\mathrm{N}}\frac{(-1)^p 2^p \mathrm{N}!\mathbb{C}_{\mathrm{N}}^{(\mathrm{N})}\mathbf{H}_{\mathrm{N}-p}(k_0)}{(\mathrm{N}-p)!(it)^{p+1}} + \sum_{p=0}^{\mathrm{N}+1}\frac{(-1)^p 2^p (\mathrm{N}+1)!\mathbb{C}_{\mathrm{N}+1}^{(\mathrm{N})}\mathbf{H}_{\mathrm{N}+1-p}(k_0)}{(\mathrm{N}+1-p)!(it)^{p+1}} \\ &+\sum_{s=2}^{\infty}\sum_{p=0}^{\mathrm{N}+s}\frac{(-1)^p 2^p (\mathrm{N}+s)!\mathbb{C}_{\mathrm{N}+s}^{(\mathrm{N})}\mathbf{H}_{\mathrm{N}+s-p}(k_0)}{(\mathrm{N}+s-p)!(it)^{p+1}}\Bigg]\exp(ik_0t)\Bigg|_{k_{\mathrm{IR}}}^{\mu} + \mathtt{c.c.}\Bigg|_{k_{\mathrm{IR}}}^{\mu} \qquad (3.47)\end{aligned}$$

Next, we want to rearrange the terms in the infinite sum to make possible the Borel resummation. From the definition of the Hermite polynomial, we find that $\mathbf{H}_m(\mu) \sim \mu^m$ and thus $\mathbf{H}_m(\mu) > \mathbf{H}_{m-r}(\mu)$). The following form gives us more intuition about how to proceed:

$$\begin{aligned}\mathcal{S}^{(\mathrm{N})}(t) = \quad & g^{\mathrm{N}}\Bigg[\sum_{a=0}^{1}\frac{\mathbb{C}_a^{(\mathrm{N})}\mathbf{H}_a(k_0)}{it} - \frac{2\cdot 1!\mathbb{C}_1^{(\mathrm{N})}\mathbf{H}_0(k_0)}{(it)^2} \\ &+\frac{\mathbb{C}_2^{(\mathrm{N})}\mathbf{H}_2(k_0)}{it} - \frac{2\cdot 2!\mathbb{C}_2^{(\mathrm{N})}\mathbf{H}_1(k_0)}{(it)^2} + \frac{2^2\cdot 2!\mathbb{C}_2^{(\mathrm{N})}\mathbf{H}_0(k_0)}{(it)^3} \\ &+\frac{\mathbb{C}_3^{(\mathrm{N})}\mathbf{H}_3(k_0)}{it} - \frac{2\cdot 3!\mathbb{C}_3^{(\mathrm{N})}\mathbf{H}_2(k_0)}{2!(it)^2} + \frac{2^2\cdot 3!\mathbb{C}_3^{(\mathrm{N})}\mathbf{H}_1(k_0)}{(it)^3} - \frac{2^3\cdot 3!\mathbb{C}_3^{(\mathrm{N})}\mathbf{H}_0(k_0)}{(it)^4} + \ldots + \Bigg] \\ &\times e^{ik_0t}\Bigg|_{k_{\mathrm{IR}}}^{\mu} + g^{\mathrm{N}}e^{-ik_0t}\,[c.c.]\Bigg|_{k_{\mathrm{IR}}}^{\mu} \qquad (3.48)\end{aligned}$$

From this representation, it is easy to extract the *dominant* contributions at every $\mathcal{O}(\mathrm{N})$ for $\mu > 1$ and finite t. For instance, the dominant contribution for the N-th piece of the infinite sum

$$\begin{aligned}&\left[\frac{\mathbb{C}_{\mathrm{N}}^{(\mathrm{N})}\mathbf{H}_{\mathrm{N}}(k_0)}{it} - \frac{2\cdot \mathrm{N}!\mathbb{C}_{\mathrm{N}}^{(\mathrm{N})}\mathbf{H}_{\mathrm{N}-1}(k_0)}{(\mathrm{N}-1)!(it)^2} + \frac{2^2\cdot \mathrm{N}!\mathbb{C}_{\mathrm{N}}^{(\mathrm{N})}\mathbf{H}_{\mathrm{N}-2}(k_0)}{(\mathrm{N}-2)!(it)^3} + \ldots. + \frac{(-1)^{\mathrm{N}}2^{\mathrm{N}}\cdot \mathrm{N}!\mathbb{C}_{\mathrm{N}}^{(\mathrm{N})}\mathbf{H}_0(k_0)}{(it)^{\mathrm{N}+1}}\right]e^{ik_0t} \\ &+\left[\frac{\mathbb{C}_{\mathrm{N}}^{(\mathrm{N})}\mathbf{H}_{\mathrm{N}}(k_0)}{-it} - \frac{2\cdot \mathrm{N}!\mathbb{C}_{\mathrm{N}}^{(\mathrm{N})}\mathbf{H}_{\mathrm{N}-1}(k_0)}{(\mathrm{N}-1)!(-it)^2} + \frac{2^2\cdot \mathrm{N}!\mathbb{C}_{\mathrm{N}}^{(\mathrm{N})}\mathbf{H}_{\mathrm{N}-2}(k_0)}{(\mathrm{N}-2)!(-it)^3} + \ldots. + \frac{(-1)^{\mathrm{N}}2^{\mathrm{N}}\cdot \mathrm{N}!\mathbb{C}_{\mathrm{N}}^{(\mathrm{N})}\mathbf{H}_0(k_0)}{(-it)^{\mathrm{N}+1}}\right]e^{-ik_0t} \\ &\sim \frac{\mathbb{C}_{\mathrm{N}}^{(\mathrm{N})}}{[\mathrm{N}!]^{\alpha}}\,[\mathrm{N}!]^{\alpha}\,\mathbf{H}_{\mathrm{N}}(k_0)\,\frac{\sin\mu t}{t}. \qquad (3.49)\end{aligned}$$

The dominant contribution gathered from all the terms of the infinite sum is given by

$$\mathcal{S}_{\mathrm{dom}}^{(\mathrm{N})} = g^{\mathrm{N}}\left[\sum_{r=1}^{\mathrm{N}-1}\frac{\mathbb{C}_{\mathrm{N}-r}^{(\mathrm{N})}}{[(\mathrm{N}-r)!]^{\alpha}}\,[(\mathrm{N}-r)!]^{\alpha}\,\mathbf{H}_{\mathrm{N}-r}(k_0) + \frac{\mathbb{C}_{\mathrm{N}}^{(\mathrm{N})}}{[\mathrm{N}!]^{\alpha}}\,[\mathrm{N}!]^{\alpha}\,\mathbf{H}_{\mathrm{N}}(k_0) + \sum_{s=1}^{\infty}\frac{\mathbb{C}_{\mathrm{N}+s}^{(\mathrm{N})}}{[(\mathrm{N}+s)!]^{\alpha}}\,[(\mathrm{N}+s)!]^{\alpha}\,\mathbf{H}_{\mathrm{N}+s}(k_0)\right]_{k_{\mathrm{IR}}}^{\mu},$$

, where only the coefficient for the $\frac{\sin\mu t}{t}$ is given. Now we can see that a generalized case of the Gevrey-α series captures the asymptotic growth. To apply the Borel resummation procedure, we use the Stirling approximation

$$(\mathrm{N}!)^{\alpha} = \frac{(\alpha\mathrm{N})!}{\alpha^{\alpha\mathrm{N}}} \approx \ (\alpha\mathrm{N})!, \tag{3.50}$$

which works for the case $\alpha << \mathrm{N}$. Now our dominant coefficient has the following form, which can be resumed

$$\mathcal{S}^{(\mathrm{N})}_{\mathrm{dom}} = g^{\mathrm{N}}\left[\sum_{r=1}^{\mathrm{N}-1}\frac{\mathbb{C}^{(\mathrm{N})}_{\mathrm{N}-r}}{(\alpha(\mathrm{N}-r))!}(\alpha(\mathrm{N}-r))!\,\mathbf{H}_{\mathrm{N}-r}(k_0) + \frac{\mathbb{C}^{(\mathrm{N})}_{\mathrm{N}}}{(\alpha\mathrm{N})!}(\alpha\mathrm{N})!\,\mathbf{H}_{\mathrm{N}}(k_0) + \sum_{s=1}^{\infty}\frac{\mathbb{C}^{(\mathrm{N})}_{\mathrm{N}+s}}{(\alpha(\mathrm{N}+s))!}(\alpha(\mathrm{N}+s))!\,\mathbf{H}_{\mathrm{N}+s}(k_0)\right]^{\mu}_{k_{\mathrm{IR}}}.$$

Now it is time to recall that denominator of our path integral formulation of the Glauber-Sudarshan state

$$\langle\varphi_1\rangle_{\overline{\sigma}} \equiv \frac{\int\mathcal{D}\varphi_1\mathcal{D}\varphi_2\mathcal{D}\varphi_3\ e^{i\mathbf{S}_{\mathrm{tot}}}\ \mathbb{D}^{\dagger}(\overline{\alpha},\overline{\beta},\overline{\gamma})\varphi_1(x,y,z)\mathbb{D}(\overline{\alpha},\overline{\beta},\overline{\gamma})}{\int\mathcal{D}\varphi_1\mathcal{D}\varphi_2\mathcal{D}\varphi_3\ e^{i\mathbf{S}_{\mathrm{tot}}}\ \mathbb{D}^{\dagger}(\overline{\alpha},\overline{\beta},\overline{\gamma})\mathbb{D}(\overline{\alpha},\overline{\beta},\overline{\gamma})}, \tag{3.51}$$

is independent of the source and thus should be convergent. This fact gives us an additional very useful piece of information, i.e. a constrain condition on the coefficients $\mathbb{C}^{(\mathrm{N})}_{\mathrm{N}}$:

$$\sum_{\mathrm{N}=0}^{\infty} g^{\mathrm{N}}\sum_{m=0}^{\infty}\frac{(-1)^p 2^p\ m!\ \mathbb{C}^{(\mathrm{N})}_m\ \mathbf{H}_{m-p}(k_0)}{(m-p)!}\Bigg|^{\mu}_{k_{\mathrm{IR}}} = f(p), \tag{3.52}$$

for all $p\in\mathbb{Z}_+$ starting with $p=0$. To make this condition useful, we find the momentum representation of $f(p)$ using the Fourier cosine transform and the Riemann-Lebesgue lemma. The cosine Fourier transform is more appropriate here since we need to take into account our simplification condition that all the functions are real and the Fourier transform range should contain only positive momentum modes. We also note that the left-hand side of the last equation is the expectation value of the graviton field on a coherent state. Thus, we need to find the cosine Fourier transform of $\mathsf{g}_{\mu\nu} \equiv \left(\frac{g_s}{HH_o}\right)^{-\frac{8}{3}} = \left(\sqrt{\Lambda}t\right)^{-\frac{8}{3}}$.

We start by introducing a generic cosine Fourier transform

$$\mathbb{F}_{\cos}(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} dt\ f(t)\cos\omega t. \tag{3.53}$$

In our case $\omega = k_0$ and $f(t) = \left(\frac{1}{\sqrt{\Lambda}t}\right)^{\mu-1}$, where $\mu - 1 = -8/3$. We also need the value of the definite integral

$$\int_0^{\infty} dt\ t^{\mu-1}\cos\omega t = \frac{1}{\omega^{\mu}}\Gamma(\mu)\cos\frac{\mu\pi}{2} \tag{3.54}$$

for $\omega > 0$ and $0 < \mathbf{Re}\mu < 1$. In our particular case

$$\mathbb{F}_{\cos}(k_0) = \sqrt{\frac{2}{\pi}}\Lambda^{\frac{\mu-1}{2}} \int_0^\infty dt\ t^{\mu-1} \cos k_0 t = \sqrt{\frac{2}{\pi}}\Lambda^{\frac{\mu-1}{2}} \frac{1}{k_0^\mu}\Gamma(\mu)\cos\frac{\mu\pi}{2}, \tag{3.55}$$

where $\mu = -5/3$. We find

$$\mathbb{F}_{\cos}(k_0) = \sqrt{\frac{2}{\pi}}\Lambda^{-\frac{4}{3}}\Gamma\left(-\frac{5}{3}\right)\cos\left(-\frac{5\pi}{6}\right) k_0^{\frac{5}{3}} = -\sqrt{\frac{3}{2\pi}}\frac{1}{\Lambda^{4/3}}\Gamma\left(-\frac{5}{3}\right)\ k_0^{\frac{5}{3}}. \tag{3.56}$$

In the next step, we take to use the Riemann-Lebesgue lemma to write this result back in the finite range $\kappa_{IR} \leq k_0 \leq \mu$. We need to add the complex conjugate part to extract the sin and cos parts of the series.

$$\int_{\kappa_{IR}}^{\mu} dk_0\ e^{ik_0 t} k_0^{5/3} + \int_{\kappa_{IR}}^{\mu} dk_0\ e^{-ik_0 t} k_0^{5/3} \tag{3.57}$$

$$= \sum_{p=0}^{N} (-1)^p \frac{5}{3}\left(\frac{5}{3}-1\right)...\left(\frac{5}{3}-p\right) k_0^{5/3-p}\left(\frac{e^{ik_0 t}}{(it)^{p+1}} + \frac{e^{-ik_0 t}}{(-it)^{p+1}}\right). \tag{3.58}$$

The expression in brackets generates a series in $\frac{\sin \mu t}{t^{2p+1}}$ and $\frac{\cos \mu t}{t^{2p}}$:

$$\sum_{p=0}^{N}\left(\frac{e^{ik_0 t}}{(it)^{p+1}} + \frac{e^{-ik_0 t}}{(-it)^{p+1}}\right) = \left(e^{ik_0 t} - e^{-ik_0 t}\right)\frac{1}{it} - \left(e^{ik_0 t} + e^{-ik_0 t}\right)\frac{1}{t^2}$$
$$- \left(e^{ik_0 t} - e^{-ik_0 t}\right)\frac{1}{it^3} + \left(e^{ik_0 t} + e^{-ik_0 t}\right)\frac{1}{t^4} + ...$$
$$= 2\frac{\sin k_0 t}{t} - 2\frac{\cos k_0 t}{t^2} - 2\frac{\sin k_0 t}{t^3} + 2\frac{\cos k_0 t}{t^4} + ... \tag{3.59}$$

Now we are ready to write down a generic coefficient of the $f(p)$ series

$$f(p) = (-1)^{p+1} 2\sqrt{\frac{3}{2\pi}}\frac{1}{\Lambda^{4/3}}\Gamma\left(-\frac{5}{3}\right)\frac{5}{3}\left(\frac{5}{3}-1\right)...\left(\frac{5}{3}-p\right) k_0^{5/3-p}\Big|_{\kappa_{IR}}^{\mu}. \tag{3.60}$$

From our previous discussion, we know that $\mathbb{C}_m^{(\mathrm{N})}$ combinatoric factors have at least $(\mathrm{N}!)^2$ growth. We generalize this case to include all possible cases:

$$\mathbb{C}_m^{(\mathrm{N})} \ = \ \mathcal{A}^{\otimes \mathrm{N}} \sum_{n=1}^{\infty}\left(\int d\Omega c_{2n,m}\ \mathbf{H}_{2n}(\overline{\mathbf{k}}_{\mathrm{IR}})\right)(\mathrm{N}!)^\alpha\,, \tag{3.61}$$

with $\alpha \geq 2$ and where coefficients $\mathcal{A}$ are given by

$$\mathcal{A} \ \equiv \ \mathrm{A}(\{m_i\},\{n_i\},\{f_i\}) \tag{3.62}$$

$$= \mathbb{B}_1(m_1,..,m_q)\mathbb{B}_2(n_1,..,n_r)\mathbb{B}_3(f_1,..,f_{s-1})\prod_{i=1}^{q}\int_{k_{\rm IR}}^{\mu} d^{11}k_i\frac{\overline{\alpha}(k_i)k_i^{m_i}}{a(k_i)}\prod_{j=1}^{r}\int_{k_{\rm IR}}^{\mu} d^{11}l_j\frac{\overline{\beta}(l_j)l_j^{n_j}}{b(l_j)}\prod_{t=1}^{s-1}\int_{k_{\rm IR}}^{\mu} d^{11}f_t\frac{\overline{\gamma}(f_t)f_t^{p_t}}{c(f_t)},$$

To streamline the following calculations, we will consider the simpler case $\mathcal{A}^{\otimes \rm N} \approx \mathcal{A}^{\rm N}$. In this case, the combinatoric factors $\mathbb{C}_m^{(\rm N)}$ grows as

$$\mathbb{C}_m^{(\rm N)} \;=\; \mathcal{A}^{\rm N}\,({\rm N}!)^{\alpha}\,\mathbf{a}_m, \tag{3.63}$$

with $\mathbf{a}_m = \sum\limits_{n=1}^{\infty}\left(\int d\Omega c_{2n,m}\ \mathbf{H}_{2n}(\overline{\mathbf{k}}_{\rm IR})\right)$. The Borel resummation of this asymptotic series gives us the following result

$$\begin{aligned}\sum_{\rm N=0}^{\infty} g^{\rm N}\left[\sum_{m=0}^{\infty}\mathbb{C}_m^{(\rm N)}\mathbf{H}_m(k_0)\right]_{k_{\rm IR}}^{\mu} &= \frac{1}{g^{1/\alpha}}\int_0^{\infty} d{\rm S}\,\exp\left(-\frac{\rm S}{g^{1/\alpha}}\right)\sum_{\rm N=0}^{\infty}\left[\sum_{m=0}^{\infty}\frac{\mathbb{C}_m^{(\rm N)}}{(\alpha{\rm N}!)}\Big(\mathbf{H}_m(\mu)-\mathbf{H}_m(k_{\rm IR})\Big){\rm S}^{\alpha{\rm N}}\right]\\ &= \frac{1}{g^{1/\alpha}}\int_0^{\infty} d{\rm S}\,\exp\left(-\frac{\rm S}{g^{1/\alpha}}\right)\left[\frac{\sum\limits_{m=0}^{\infty}\mathbf{a}_m\Big(\mathbf{H}_m(\mu)-\mathbf{H}_m(k_{\rm IR})\Big)}{1-\mathcal{A}{\rm S}^{\alpha}}\right],\end{aligned} \tag{3.64}$$

To make the last result more clear, we will go over the calculations step by step and will concentrate on the case $\alpha = 1$. First, we take outside the N dependence from the sum over m:

$$\left[\sum_{m=0}^{\infty}\mathbb{C}_m^{(\rm N)}\mathbf{H}_m(k_0)\right]_{k_{\rm IR}}^{\mu} = \mathcal{A}^N(N!)^{\alpha}\sum_{m=0}^{\infty}\mathbf{a}_m\left[\mathbf{H}_m(\mu)-\mathbf{H}_m(k_{\rm IR})\right] \simeq c\mathcal{A}^N(\alpha N!). \tag{3.65}$$

where $c = \sum_{m=0}^{\infty}\mathbf{a}_m\left[\mathbf{H}_m(\mu)-\mathbf{H}_m(k_{\rm IR})\right]$ is a constant independent of N and we take $\alpha = 1$. We have a formal divergent asymptotic series with $N!$ growth

$$\phi(g) = \sum_{N}^{\infty} g^N c\mathcal{A}^N N!. \tag{3.66}$$

To remove this asymptotic growth, we make a Borel transform on the series by redefinition $g^N \to \frac{S^N}{N!}$ and obtain a new infinite series

$$\hat{\phi}(S) = \sum_{N}^{\infty} S^N c\mathcal{A}^N = \frac{c}{1-S\mathcal{A}}. \tag{3.67}$$

The last series is convergent only for the case $|S\mathcal{A}| < 1$. In the next step, we analytically continue the result for all values of $S\mathcal{A}$ and resume the new series using the Borel resummation procedure

$$s(\phi)(g) = \int_0^{\infty} dS\ \hat{\phi}(gS)\ e^{-S} = \frac{1}{g}\int_0^{\infty} dS\hat{\phi}(S)e^{-\frac{S}{g}} = \frac{1}{g}\int_0^{\infty} dSe^{-\frac{S}{g}}\frac{c}{1-S\mathcal{A}}. \tag{3.68}$$

There is one additional subtle point about the off-shell contributions, coming from the $k_0 = 0$ modes. We introduce them by making small modifications to the source wave function:

$$\psi_k(\mathrm{X})e^{-ik_0t} \equiv \frac{k^2}{\pi|\omega|\mathbf{k}^9}\exp\left(-\frac{(\mathbf{k}-\overline{\mathbf{k}}_{\mathrm{IR}})^2}{\omega^2} - ik_0t\right)\left[1 - \frac{1}{\pi|\omega|}\exp\left(-\frac{k_0^2}{\omega^2}\right)\right]\ \mathcal{F}\left(\mathbf{k}, \mathrm{X}\right), \tag{3.69}$$

where $\mathcal{F}\left(\mathbf{k}, \mathrm{X}\right)$ solves the Schrödinger equation constructed over the solitonic configuration. In the limit $\omega \to 0$ it satisfies $\mathcal{F}\left(\mathbf{k} = \overline{\mathbf{k}}_{\mathrm{IR}}, \mathrm{X}\right) = 1$.

Since the presence of the delta function $\delta(k_0)$ removes the necessity of the Riemann-Lebesgue integral, we obtain the following result after moving to the Fourier space, which contains only one equation:

$$\sum_{\mathrm{N}=0}^{\infty} g^{\mathrm{N}} \sum_{m=0}^{\infty} \mathbb{C}_m^{(\mathrm{N})}\mathbf{H}_{2m}(0) = \frac{1}{\Lambda^{4/3}}. \tag{3.70}$$

We also note that the coefficients $\mathbb{C}_m^{(\mathrm{N})}$ undergo some modifications:

$$\begin{aligned}\mathbb{C}_m^{(\mathrm{N})} \quad &\equiv \quad \prod_{t=1}^{\mathrm{N}q+1}\left(\frac{\pi_t}{b_t}\right)^{1/2}\sum_{\{m_i\}}\mathbb{D}\left(m_1, m_2, m_3, ..., m_{\mathrm{N}q-1}, m_{\mathrm{N}q}\right)\mathbb{D}_{\varphi_2}(\mathrm{N}r)\mathbb{D}_{\varphi_3}(\mathrm{N}(s-1)) \\ &\times \quad \int_{k_{\mathrm{IR}}}^{\mu} d^{11}k_1\frac{\overline{\alpha}(k_1)k_1^{m_1}}{a(k_1)}\int_{k_{\mathrm{IR}}}^{\mu} d^{11}k_{\mathrm{N}q}\ \frac{\overline{\alpha}(k_{\mathrm{N}q})k_{\mathrm{N}q}^{m_{\mathrm{N}q}}}{a(k_{\mathrm{N}q})}\sum_{n=1}^{\infty}\left(\int d\Omega\ c_{2n,2m}\ \mathbf{H}_{2n}(\overline{\mathbf{k}}_{\mathrm{IR}})\right),\end{aligned} \tag{3.71}$$

We observe, that we still have the same factorial growth for these coefficients, which gives us, after the Borel resummation procedure of the left-hand side, the following equation

$$\frac{1}{g^{1/\alpha}}\int_0^{\infty} d\mathrm{S}\ \exp\left(-\frac{\mathrm{S}}{g^{1/\alpha}}\right)\sum_{m=0}^{\infty}\left(\frac{\mathbf{a}_m\mathbf{H}_{2m}(0)}{1-\mathcal{A}\mathrm{S}^{\alpha}}\right) = \frac{1}{\Lambda^{4/3}}, \tag{3.72}$$

where $\mathbf{a}_m = \sum\limits_{n=1}^{\infty}\left(\int d\Omega c_{2n,2m}\ \mathbf{H}_{2n}(\overline{\mathbf{k}}_{\mathrm{IR}})\right)$.

3.3 The positivity of the cosmological constant

The analysis from the previous subsection suggests that under some conditions, we can obtain the positive value for the cosmological constant. By absorbing all the constants and redefining cosmological constant, we find that cosmological constant may be expressed in terms of a non-perturbative series in the coupling constant g in the following nice way:

$$\int_0^{\infty} d\mathrm{S}\ \exp\left(-\frac{\mathrm{S}}{g^{1/\alpha}}\right)\frac{1}{1-\mathcal{A}\mathrm{S}^{\alpha}} = \frac{1}{\Lambda^{4/3}}. \tag{3.73}$$

We require the left-hand side of the equation to be a positive definite quantity for all positive integer values of α, and all values of $\mathcal{A}$. Under such conditions, the cosmological constant remains positive. We also note that the last expression is based on specific conditions, which demand a positive cosmological constant background. Thus, although the right-hand side remains positive independently of the sign if Λ, the obtained result is only possible for the case of $\Lambda > 0$.

Another very satisfying outcome of the last expression is that there are no perturbative limits to the value of the Λ, suggesting that perturbative corrections are never enough to change the sign of the cosmological constant. It seems, that the cosmological constant can only be realized non-perturbatively.

The analysis of the left-hand side suggests that there is a trivial case with $\mathcal{A} < 0$, which makes the whole integral positive independently of the values of α. In case when $\mathcal{A} > 0$ the integral develops poles on the Borel axis, which makes the analysis less trivial but also more interesting. The general solution of the integral, in this case, is given by some expression in terms of the Exponential integral $\mathtt{Ei}(x) \equiv -\int_{-x}^{\infty} \frac{dt}{t}\, e^{-t}$ in the following way

$$\int d\mathrm{S} \exp\left(-\frac{\mathrm{S}}{g^{1/\alpha}}\right) \frac{1}{1-\mathcal{A}\mathrm{S}^{\alpha}} = -\sum_{j=1}^{\alpha} \frac{1}{\alpha \mathcal{A} a_j^{\alpha-1}} \exp\left(-\frac{a_j}{g^{1/\alpha}}\right) \mathbf{Ei}\left(\frac{a_j - \mathrm{S}}{g^{1/\alpha}}\right) + \text{constant}, \tag{3.74}$$

where a_j are the roots of the polynomial

$$\mathrm{S}^{\alpha} - \frac{1}{\mathcal{A}} \equiv \prod_{i=1}^{\alpha} (\mathrm{S} - a_i)\,, \tag{3.75}$$

implying that the value of the integral may be expressed in terms of the sum over the roots of this polynomial. We note that only one positive real root contributes since S lies in the range $S \in [0, \infty]$. The problem comes from the fact that we need to compute the principal value of the integral to extract a real answer. It can be rigorously proved using complex analysis that the integral

$$\text{P.V.} \int_0^{\infty} d\mathrm{S} \frac{e^{-\mathrm{S}/g^{1/\alpha}}}{1-\mathcal{A}\mathrm{S}^{\alpha}} \tag{3.76}$$

is positive definite.

To make things more visually attractive, we will analyze in some detail the tree-level contribution to the equation determining the cosmological constant and introduce some figures showing the possibility of the Λ being positive and small. In this tree-level approximation, we can write down the expectation value of the scalar field ϕ_1 in the following suggestive way

$$\langle \varphi_1 \rangle_{\overline{\sigma}} = \mathbb{T}_1 + \delta\mathbb{T}, \tag{3.77}$$

where the value of the small contribution due to the presence of all the remaining diagrams can be deduced from our previous analysis of the nodal diagrams

$$\delta\mathbb{T} = \int d^{11}k \left[\frac{\overline{\alpha}(k)}{a(k)} - \frac{k^{m^*}}{\mathrm{V}} \left(\frac{\overline{\alpha}^2(k)}{a^2(k)} + \frac{1}{2a(k)} \right) + \mathcal{O}\left(\frac{1}{\mathrm{V}^2} \right) \right] \psi_k(\mathrm{X}) e^{-i(k_0 - \overline{\kappa}_{\mathrm{IR}})t}, \tag{3.78}$$

where $\mathcal{O}\left(\frac{1}{\mathrm{V}^2}\right)$ indicates the contribution of the higher order nodal diagrams. We are interested in the values of $\overline{\alpha}(k)$ for which the integrand vanishes,

$$\overline{\alpha}(k) = \frac{a(k)}{2k^{m^*}} \left(\mathrm{V} \pm \mathrm{V}\sqrt{1 - \frac{2k^{2m^*}}{a(k)\mathrm{V}^2}} \right), \tag{3.79}$$

without the higher-level contributions. The potential solution which would make $\delta\mathbb{T}$ very small is

$$\overline{\alpha}(k) = \frac{1}{2\mathrm{V}} \left(k^{11/3} + \chi(k) \right) + \mathcal{O}\left(\frac{1}{\mathrm{V}^2} \right), \tag{3.80}$$

with the suggested scaling $m^* = \frac{11}{3}$. We will use the last result slightly later.

Right now, we move back to the equation that determines the possible values of the cosmological constant and make some further analysis.

$$\mathrm{P.V.} \int_0^\infty d\mathrm{S} \, \exp\left(-\frac{\mathrm{S}}{g^{1/\alpha}} \right) \frac{1}{1 - \mathcal{A}\mathrm{S}^\alpha} + \mathcal{O}\left(\frac{1}{\mathrm{V}} \right) = \frac{1}{\Lambda^{4/3}} \left(\frac{2\mathrm{V}g^{1/\alpha}}{b_a} \right), \tag{3.81}$$

where $b_a = \overline{k}_{\mathrm{IR}}^{\nu(k_{\mathrm{IR}})-2}$. The volume V appears because the denominator of the path-integral eliminates all nodal diagrams in which the source does not couple with the interactions. From here, we find the expression for the cosmological constant

$$\Lambda = \left(\frac{2\mathrm{V}g^{1/\alpha}}{b_a} \right)^{3/4} \frac{\Lambda_0}{\left(\mathrm{P.V.} \int\limits_0^\infty d\mathrm{S} \, \exp\left(-\frac{\mathrm{S}}{g^{1/\alpha}} \right) \frac{1}{1 - \mathcal{A}\mathrm{S}^\alpha} \right)^{3/4}}, \tag{3.82}$$

where $\Lambda_0 \equiv a_0^{-2}$ with a_0 takes care of the dimensions. We can rewrite (3.82) in a slightly simpler and suggestive way by using the parameter $c \equiv (\mathcal{A}g)^{1/\alpha}$ as:

$$\Lambda = \left(\frac{2\mathrm{V}\Lambda_0^{4/3}}{b_a}\right)^{3/4} \frac{c^{3/4}}{\left(\mathrm{P.V.}\int\limits_0^\infty du\ \frac{e^{-u/c}}{1-u^\alpha}\right)^{3/4}} = \left(\frac{2\mathrm{V}\Lambda_0^{4/3}}{b_a}\right)^{3/4}\left(\frac{c}{\mathbb{I}_{c,\alpha}}\right)^{3/4}, \tag{3.83}$$

where $\mathbb{I}_{c,\alpha}$ is the principal value integral from the LHS. Looking at Figure 3.1 and Figure 3.2, where we plot the behaviour of $\frac{c}{\mathbb{I}_{c,\alpha}}$ from (3.83) for $\alpha = 2, 3, 4$ respectively, we see that in the limit $c \to 0$, $\frac{c}{\mathbb{I}_{c,\alpha}} \to 1$ irrespective of the choice of α. This suggests that Λ takes a definite value of:

$$\Lambda = \left(\frac{2\mathrm{V}\Lambda_0^{4/3}}{b_a}\right)^{3/4} + \mathcal{O}\left(e^{-1/c}\right), \tag{3.84}$$

which may be made small. There is also a possibility that the cosmological constant Λ may be determined for a given choice of α by the minima of the curves. These numerical plots suggest that the minima approach *smaller* values as we increase the values for α.

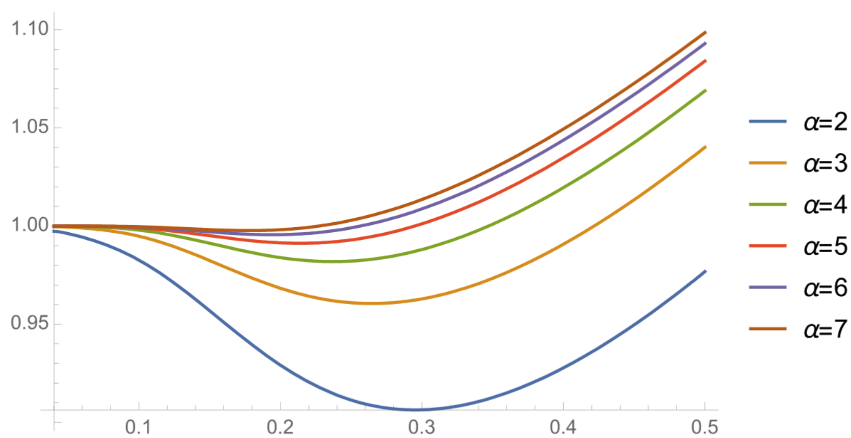


Figure 3.1: Plot of $\left(\frac{c}{\mathbb{I}_{c,3}}\right)^{3/4}$ versus $c \equiv (g\mathcal{A})^{1/3}$ where $\mathbb{I}_{c,3}$ is the principal value integral given in (3.83) for $\alpha = 3$. Again, observe that for $c \to 0$, $\frac{c}{\mathbb{I}_{c,3}} \to 1$ and the cosmological constant Λ from (3.83) takes the specific value of (3.84).

The vanishing of the perturbative contributions led us to impose the vanishing of the integrand in (3.78). This gave us the following constraint:

$$\frac{\overline{\alpha}(k)}{a(k)} - \frac{k^{\nu(k)}}{\mathrm{V}}\left(\frac{\overline{\alpha}^2(k)}{a^2(k)} + \frac{1}{2a(k)}\right) + \mathcal{O}\left(\frac{1}{\mathrm{V}^2}\right) = 0, \tag{3.85}$$

Such a constraint immediately fixes $\overline{\alpha}(k)$ as in (3.80), which we may re-express in a slightly more suggestive way as a scaling relation of the form:

$$\overline{\alpha}(k) = \frac{k^{\nu(k)}}{2\mathrm{V}} + \mathcal{O}\left(\frac{1}{\mathrm{V}^2}\right) \tag{3.86}$$

where the Glauber-Sudarshan states associated with $\overline{\alpha}(k) = \frac{k^{11/3}}{2\mathrm{V}}$ are shown in Figure 3.2.

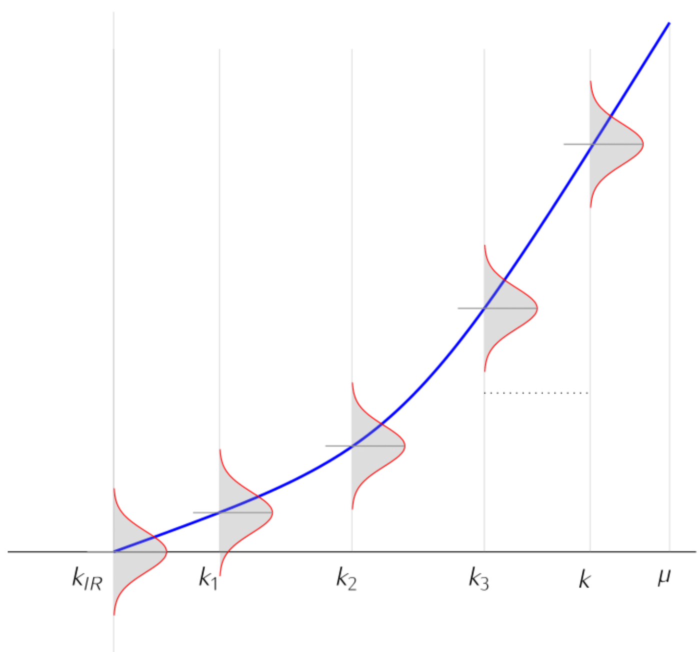


Figure 3.2: The Glauber-Sudarshan states associated with $y \equiv 2\mathrm{V}\overline{\alpha}(k) = k^{11/3}$ with $k_{\mathrm{IR}} \leq k \leq \mu$ in the configuration space. This is part of the full form for $\overline{\alpha}(k)$ given in (3.86) with $\nu(k) \approx \frac{11}{3}$

Chapter 4

Heterotic strings, fermions, and M-theory uplift

This chapter includes some computations only tangentially related to the main theme of the work but is included as an example of possible extensions of the basic toy model presented in the thesis. In this sense, this part is very incomplete and unsatisfactory. At the same time, it may provide some insights into the future developments.

4.1 Conformally related metric

The question that we want to ask here is that whether a generic background of the form:

$$ds^2 = \frac{a^2(t)}{\mathrm{H}^2(y)}\left[-dt^2 + g_{ij}dx^i dx^j + g_{33}(dx^3)^2\right] + \mathrm{H}^2(y)\left[\mathrm{F}_2(t)g_{mn}dy^m dy^n + \mathrm{F}_1(t)g_{\alpha\beta}dt^\alpha dy^\beta\right], \tag{4.1}$$

can also be realized as a Glauber-Sudarshan state. Here $\mathrm{F}_i(t)$ captures the dominant temporal scalings, and in what sense they do will be elaborated when we lift this configuration to M-theory. Note that $a^2(t)$, with t being the dimensionful conformal time, is kept arbitrary, with the only condition being that it becomes large at a late time. This means the background (4.1) naturally *expands* at late time. The other factors, $g_{ij}(\mathbf{x}), g_{33}(\mathbf{x}), g_{mn}(y), g_{\alpha\beta}(y)$ and $\mathrm{H}^2(y)$ are the unwarped spatial metric components and the warp-factor respectively. The coordinate $y \equiv (y^m, y^\alpha) \in \mathcal{M}_4 \times \mathcal{M}_2$ and $x = (t, \mathbf{x}) \in \mathbf{R}^{2,1}$, so that nothing depends on the third spatial direction parametrized by x^3 here. We will soon make a further restriction by converting $g_{\alpha\beta} = \delta_{\alpha\beta}$, so that $\mathcal{M}_2 = \frac{\mathbb{T}^2}{\mathbb{Z}_2}$ where $\mathbb{Z}_2$ will be an orientifolding operation. Such a choice will give us a way to reach the heterotic background by making a series of duality transformations. In that case $y \equiv y^m \in \mathcal{M}_4$.

For various choices of $(a^2(t), \mathrm{F}_i(t))$ and the internal sub-manifolds, we can study the possibilities of realizing de Sitter state in various string theories (including also in M-theory). As was discussed previously, it is possible to realize a four-dimensional type IIB superstring background containing dS isometries as a coherent Glauber-Sudarshan state. We want to study a generic background with some temporal dynamics incorporated into the $F_i(t)$ scaling functions:

$$\begin{aligned} ds^2 =& a^2(t)/H^2(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) \\ &+ H^2(F_1(t) g_{\alpha\beta} dy^\alpha dy^\beta + F_2(t) g_{mn} dy^m dy^n) \end{aligned} \tag{4.2}$$

where $H^2 = H^2(y)$ is a warp-factor , and (y^m, y^α) are coordinates of internal six-manifold $\mathcal{M}^6 = \mathcal{M}_4 \times \mathcal{M}_2$. Here y^m parameterize M_4 and y^α parameterize M_2 such that $\mathcal{M}^6 = M_2 \times M_4$. $F_i(t)$ for $i = 1, 2$ are some time-dependent functions that satisfy $F_1 F_2^2 = 1$. The time variable is, as usual, the dimensionful conformal time. The expansion of the background is generated by an arbitrary parameter $a^2(t)$, which grows in time. The factors $g_{\alpha\beta}(y)$ and $g_{mn}(y)$ represent the unwarped spatial metric components. The restriction of the two-manifold $\mathcal{M}_2$ to a special case $\mathcal{M}_2 = \frac{\mathbb{T}^2}{\mathbb{Z}_2}$ with $\mathcal{Z}_2$ being responsible for an orientiefolding operation, and a respective conversion $g_{\alpha\beta} = \delta_{\alpha\beta}$ will help to change to the heterotic background by means of duality transformations.

The ultimate task of this section is to compute Einstein's tensors $G_{\mu\nu}, G_{mn}, G_{\alpha\beta}$ etc., for this generic metric on the Type IIB side.

Following some suggestions, I can change the original metric (4.2) to a more simple conformally related metric

$$ds^2 = \frac{1}{H^2}\left(g_{\mu\nu} dx^\mu\, dx^\nu + H^4\ F_1(t) g_{\alpha\beta} dy^\alpha dy^\beta + H^4\ F_2(t) g_{mn} dy^m dy^n\right), \tag{4.3}$$

where $g_{\mu\nu}$ the Friedmann isotropic and homogeneous universe

$$g_{\mu\nu}dx^\mu\, dx^\nu = -d\eta^2 + a^2(t)\,\delta_{ij}dx^i\, dx^j \tag{4.4}$$

and η is the conformal time

$$\eta = \int^t \frac{dt}{a(t)} \tag{4.5}$$

The relation between the Ricci tensors of the conformally related metrics is relatively easy to find. We define two metrics as

$$ds^2(typeIIB) = \frac{1}{H^2}\, d\tilde{s}^2 \tag{4.6}$$

where $d\tilde{s}^2$ is given by

$$d\tilde{s}^2 = g_{\mu\nu}dx^\mu\, dx^\nu + H^4\, F_1(t)g_{\alpha\beta}dy^\alpha dy^\beta + H^4\, F_2(t)g_{mn}dy^m dy^n \tag{4.7}$$

Then the Christoffel connection for $d\tilde{s}^2$ is calculated to be

$$\Gamma^d_{ab} = \tilde{\Gamma}^d_{ab} - \frac{\partial_b H}{H}\,\delta^d_a - \frac{\partial_a H}{H}\,\delta^d_b + \frac{\partial_c H}{H}\,g^{cd}\,g_{ab}. \tag{4.8}$$

After some algebra, we find the conformally related Ricci tensor to be

$$\tilde{R}_{ab} = R_{ab} + 8\,\frac{\nabla_a\,\nabla_b H}{H} + \left(\frac{\nabla^2 H}{H} - 9\,\frac{(\nabla H)^2}{H^2}\right) g_{ab}, \tag{4.9}$$

where ∇ is a covariant derivative, such that

$$\begin{aligned} \nabla_a H &= \partial_a H, \\ \nabla_a \nabla_b H &= \nabla_a(\partial_b H) = \partial_a\,\partial_b H - \Gamma^c_{ab}\,\partial_c H. \end{aligned} \tag{4.10}$$

We first need to find Christoffel connection on $d\tilde{s}^2$. To reduce the clutter, it is beneficial to incorporate all the details of the inner metric on $\mathcal{M}^6$ into a single metric tensor h_{ij}:

$$h_{ij}(y,t)dy^i\, dy^j = H^4\, F_1(t)g_{\alpha\beta}dy^\alpha dy^\beta + H^4\, F_2(t)g_{mn}dy^m dy^n. \tag{4.11}$$

The non-vanishing components of the Christoffel connection for the total metric

$$d\tilde{s}^2 = g_{\mu\nu}dx^\mu\, dx^\nu + h_{ij}dy^i\, dy^j \tag{4.12}$$

are

$$\begin{aligned}
\tilde{\Gamma}^{\mu}_{ij} &= -\frac{1}{2} g^{\mu\nu} \partial_\nu h_{ij}, \\
\tilde{\Gamma}^{\sigma}_{\mu\nu} &= \Gamma^{\sigma}_{\mu\nu}(g_{\mu\nu}), \\
\tilde{\Gamma}^{i}_{\mu j} &= \frac{1}{2} h^{ij} \partial_\mu h_{kj}, \\
\tilde{\Gamma}^{i}_{jk} &= \Gamma^{i}_{jk}(h_{ij}).
\end{aligned} \tag{4.13}$$

Now the components of the Riemann tensor can be calculated

$$\begin{aligned}
\tilde{R}^{\sigma}_{\tau\mu\nu} &= R^{\sigma}_{\tau\mu\nu}, \\
\tilde{R}^{\sigma}_{kij} &= \frac{1}{2} \left(\nabla_i \, \partial^\sigma h_{jk} - \nabla_j \, \partial^\sigma h_{ik} \right), \\
\tilde{R}^{\sigma}_{\tau ij} &= \frac{1}{4} \left(h^{kl} \partial_\tau h_{lj} \, \partial^\sigma h_{ik} - h^{kl} \partial_\tau h_{li} \, \partial^\sigma h_{jk} \right), \\
\tilde{R}^{l}_{kij} &= R^{l}_{kij}(h_{ij} + \frac{1}{4} h_{nj} \partial^\rho h_{ik} \, \partial_\rho h^{ln} - \frac{1}{4} h_{ni} \partial^\rho h_{jk} \, \partial_\rho h^{ln}, \\
\tilde{R}^{\sigma}_{ji\mu} &= \frac{1}{4} g^{\sigma\rho} h^{kl} \partial_\mu h_{jl} \, \partial_\rho h_{ik} - \frac{1}{2} g^{\sigma\rho} \partial_\mu \partial_\rho h_{ij}.
\end{aligned} \tag{4.14}$$

We first compute the Ricci tensors of the conformally related metric by contracting the relevant indices in the components of the Riemann tensors (4.14):

$$\begin{aligned}
\tilde{R}_{\mu\nu} &= R_{\mu\nu}(g_{\mu\nu} - \frac{1}{4} \partial_\mu h^{ij} \, \partial_\nu h_{ij} - \frac{1}{2} h^{ij} \partial_\mu \, \partial_\nu h_{ij}, \\
\tilde{R}_{ij} &= R_{ij}(h_{ij}) - \frac{1}{2} g^{\rho\sigma} \partial_\rho \, \partial_\sigma h_{ij} - \frac{1}{2} h_{ik} \partial^\rho h_{jl} \, \partial_\rho h^{kl} + \frac{1}{4} h_{kl} \partial^\rho h_{ij} \, \partial_\rho h^{kl}, \\
\tilde{R}_{i\mu} &= \frac{1}{2} \left(h_{kl} \nabla_i \partial_\mu h^{kl} - h_{ki} \nabla_l \partial_\mu h^{kl} \right).
\end{aligned} \tag{4.15}$$

To obtain the Ricci tensors for our original metric (4.27) we need to move back to the hidden variables of the h_{ij} metric tensor and recalculate all derivatives with respect to them as well as add a term coming from the conformal factor. After some computation, we have the following results for Ricci tensors for the metric in type IIB string theory

$$\begin{aligned}
\tilde{R}^{10}_{\mu\nu} &= R_{\mu\nu}(g_{\mu\nu}) + \frac{1}{2}\frac{\partial_\mu F_1\, \partial_\nu F_1}{F_1^2} + \frac{\partial_\mu F_2\, \partial_\nu F_2}{F_2^2} - \frac{\partial_\mu\, \partial_\nu F_1}{F_1} - 2\frac{\partial_\mu\, \partial_\nu F_2}{F_2} \\
&+ g_{\mu\nu}\left(\frac{\nabla^2 H}{H} - 9\,\frac{(\nabla H)^2}{H^2}\right), \\
\tilde{R}^{10}_{\alpha\beta} &= R_{\alpha\beta}(g_{\alpha\beta}) - \left(\frac{1}{2}\frac{\ddot{F}_1}{F_1} + \frac{\dot{F}_1\,\dot{F}_2}{F_1\,F_2}\right) H^4 F_1 g_{\alpha\beta} + 8\,\frac{\nabla_\alpha\,\nabla_\beta H}{H} \\
&+ \left(\frac{\nabla^2 H}{H} - 9\,\frac{(\nabla H)^2}{H^2}\right) g_{\alpha\beta}, \\
\tilde{R}^{10}_{mn} &= R_{mn}(g_{mn}) - \left(\frac{1}{2}\frac{\ddot{F}_2}{F_2} + \frac{1}{2}\frac{\dot{F}_2^2}{F_2^2} + \frac{\dot{F}_1\,\dot{F}_2}{F_1\,F_2}\right) H^4 F_2 g_{mn} + 8\,\frac{\nabla_m\,\nabla_n H}{H} \\
&+ \left(\frac{\nabla^2 H}{H} - 9\,\frac{(\nabla H)^2}{H^2}\right) g_{mn}, \\
\tilde{R}^{10}_{\alpha\mu} &= -4\frac{\partial_\mu F_1}{F_1}\,\frac{\partial_\alpha H}{H}, \\
\tilde{R}^{10}_{m\mu} &= -4\frac{\partial_\mu F_2}{F_2}\,\frac{\partial_m H}{H}, \\
\tilde{R}^{10}_{\alpha m} &= R_{\alpha m}(h) + 8\frac{\nabla_\alpha \nabla_m H}{H}.
\end{aligned} \tag{4.16}$$

From the know values of the Ricci tensors of our metric in 10 dimensions, we can compute the Ricci curvature scalar

$$R^{10} = R_{MN} g^{MN} = R^{10}_{\mu\nu}\, g^{\mu\nu} + R^{10}_{\alpha\beta}\, g^{\alpha\beta} H^{-4} F_1^{-1} + R^{10}_{mn}\, g^{mn} H^{-4} F_2^{-1}. \tag{4.17}$$

Some algebraic manipulations provide the following result

$$R^{10} = R + \frac{1}{2}\frac{\dot{F}_1^2}{F_1^2} - 2\frac{\ddot{F}_1}{F_1} - \frac{\dot{F}_2^2}{F_2^2} - 4\frac{\ddot{F}_2}{F_2} - 4\frac{\dot{F}_1\,\dot{F}_2}{F_1\,F_2} + 18\frac{\nabla^2 H}{H} - 90\frac{(\nabla H)^2}{H^2}. \tag{4.18}$$

Now we have all the ingredients to evaluate the Einstein tensors

$$G_{MN} = R_{MN} - \frac{1}{2} g_{MN}\, R. \tag{4.19}$$

First we write down the different components of G_{MN} in terms of both F_1 and F_2:

$$
\begin{aligned}
G^{10}_{\mu\nu} = \tilde{G}_{\mu\nu} &- \frac{\partial_\mu\,\partial_\nu\,F_1}{F_1} + \frac{1}{2}\,\frac{\partial_\mu\,F_1\,\partial_\nu\,F_1}{F_1^2} - 2\,\frac{\partial_\mu\,\partial_\nu\,F_2}{F_2} + \frac{\partial_\mu\,F_2\,\partial_\nu\,F_2}{F_2^2} \\
&+ \left(-\frac{1}{4}\,\frac{\dot{F}_1^2}{F_1^2} + \frac{\ddot{F}_1}{F_1} + \frac{1}{2}\,\frac{\dot{F}_2^2}{F_2^2} + 2\,\frac{\ddot{F}_2}{F_2} + 2\,\frac{\dot{F}_1}{F_1}\frac{\dot{F}_2}{F_2}\right)\,g_{\mu\nu} \\
&+ \left(36\,\frac{(\nabla H)^2}{H^2} - 8\,\frac{\nabla^2 H}{H}\right)\,g_{\mu\nu},
\end{aligned}
\tag{4.20}
$$

$$
\begin{aligned}
G^{10}_{\alpha\beta} = \tilde{G}_{\alpha\beta}+\hat{g}_{\alpha\beta}\, &\left(\frac{1}{2}\,\frac{\ddot{F}_1}{F_1} - \frac{1}{4}\,\frac{\dot{F}_1^2}{F_2^1} + \frac{1}{2}\,\frac{\dot{F}_2^2}{F_2^2} + 2\,\frac{\ddot{F}_2}{F_2} + \frac{\dot{F}_1}{F_1}\frac{\dot{F}_2}{F_2} + 36\,\frac{(\nabla H)^2}{H^2} - 8\,\frac{\nabla^2 H}{H}\right) \\
&+ 8\,\frac{\nabla_\alpha\,\nabla_\beta H}{H},
\end{aligned}
\tag{4.21}
$$

$$
\begin{aligned}
G^{10}_{mn} = \tilde{G}_{mn}+\hat{g}_{mn}\, &\left(\frac{\ddot{F}_1}{F_1} - \frac{1}{4}\,\frac{\dot{F}_1^2}{F_1^2} + \frac{3}{2}\,\frac{\ddot{F}_2}{F_2} + \frac{3}{2}\,\frac{\dot{F}_1}{F_1}\frac{\dot{F}_2}{F_2} + 36\,\frac{(\nabla H)^2}{H^2} - 8\,\frac{\nabla^2 H}{H}\right) \\
&+ 8\,\frac{\nabla_m\,\nabla_n H}{H}.
\end{aligned}
\tag{4.22}
$$

Next, we can introduce the condition on F_1, F_2

$$
F_1 F_2^2 = 1, \tag{4.23}
$$

which generates the additional two relations between the derivatives of F_1 and F_2:

$$
\frac{\dot{F}_1}{F_1} + 2\,\frac{\dot{F}_2}{F_2} = 0, \tag{4.24}
$$

$$
\frac{\ddot{F}_1}{F_1} - 6\,\frac{\dot{F}_2^2}{F_2^2} + 2\,\frac{\ddot{F}_2}{F_2} = 0 \tag{4.25}
$$

These constraints on the derivatives of the time-dependent F_i allow substantial simplifications of the 10-dimensional Einstein tensors:

$$
\begin{aligned}
G^{10}_{\mu\nu} &= \tilde{G}_{\mu\nu} + g_{\mu\nu}\left(\frac{9}{2}\,\frac{\dot{F}_2^2}{F_2^2} + 36\frac{(\nabla H)^2}{H^2} - 8\,\frac{\nabla^2 H}{H}\right),\\
G^{10}_{\alpha\beta} &= \tilde{G}_{\alpha\beta} + \left(\frac{1}{2}\,\frac{\dot{F}_2^2}{F_2^2} + \frac{\ddot{F}_2}{F_2} + 36\,\frac{(\nabla H)^2}{H^2} - 8\,\frac{\nabla^2 H}{H}\right)\,H^4 F_1 g_{\alpha\beta}\\
&\quad + 8\,\frac{\nabla_\alpha\,\nabla_\beta H}{H},\\
G^{10}_{mn} &= \tilde{G}_{mn} + \left(2\,\frac{\dot{F}_2^2}{F_2^2} - \frac{1}{2}\,\frac{\ddot{F}_2}{F_2} + 36\,\frac{(\nabla H)^2}{H^2} - 8\,\frac{\nabla^2 H}{H}\right) H^4 F_1 g_{mn}\\
&\quad + 8\,\frac{\nabla_m\,\nabla_n H}{H}.
\end{aligned}
\tag{4.26}
$$

4.2 From type IIB to heterotic SO(32) superstrings

The question about the possible realizations of dS coherent state in other string theories besides Type IIB heavily relies on our ability to move between different string theories using duality transformations. Thus, an appropriate T-duality transformation with a subsequent orientifolding operation would help us to investigate the prospects of dS states in Type I theory. An additional S-duality transformation would bring us to the $SO(32)$ heterotic theory.

T-duality acts on the background fields present in a string theory.

For the type IIB superstring theory, the massless bosonic sector consists of the graviton $g_{\mu\nu}$, the dilaton ϕ, and the Kalb-Ramond field $B_{\mu\nu}$, for the $NS-NS$ sector and the antisymmetric gauge fields $C^{(0)}, C^{(2)}_{\mu\nu}, C^{(4)}_{\kappa\lambda\mu\nu}$, for the $R-R$ sector.

On the other hand, the massless bosonic sector of the type I superstring theory in 10 dimensions consists of the graviton $g^I_{\mu\nu}$, the dilaton ϕ^I, the $R-R$ two-form C^I_2 and the $SO(32)$ Yang-Mills gauge field A_μ.

In our particular case, the metric on the type IIB side has the form

$$
ds^2 = \frac{a^2(t)}{H^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + H^2(F_1(t)\delta_{\alpha\beta}dy^\alpha dy^\beta + F_2(t) g_{mn} dy^m dy^n) \tag{4.27}
$$

where $H = H(y)$, and (y^m, y^α) = coordinates of internal six-manifold $\mathcal{M}^6$. y^m parametrizes

M_4 and y^α parametrizes T^2 such that $\mathcal{M}^6 = T^2 \times M_4$. $F_i(t)$ for $i = 1, 2$ are some time-dependent functions.

The non-zero gauge fields are $NS - NS$ $B_{\alpha n}$ gauge field and $R - R$ $C_{\beta m}$ 2-form. The dilaton has the value of zero: $\phi_B = 0$.

To obtain a type I solution, we perform a T-duality transformation along both sides of the torus T^2. Let's introduce the torus coordinates x, y

$$z = x + iy.$$

Then we use Buscher's duality rules adapted to our situation. The metric terms of the form g_{xm}, g_{yn} are zero and can be ignored in the duality mapping. We also remember that we are doing a double transformation which generates additional terms. The type I metric changes to

$$\begin{aligned}
g^I_{mn} &= g_{mn} + \frac{1}{g_{xx}} B_{xn} B_{xm} + \frac{1}{g_{yy}} B_{yn} B_{ym} = H^2 F_2 g_{mn} + \frac{1}{H^2 F_1} B_{xn} B_{xm} + \frac{1}{H^2 F_1} B_{yn} B_{ym}, \\
g^I_{xn} &= \frac{1}{g_{xx}} B_{xn} = \frac{1}{H^2 F_1} B_{xn}, \\
g^I_{yn} &= \frac{1}{g_{yy}} B_{yn} = \frac{1}{H^2 F_1} B_{yn}, \\
g^I_{xx} &= \frac{1}{g_{xx}} = \frac{1}{H^2 F_1}, \qquad g^I_{yy} = \frac{1}{g_{yy}} = \frac{1}{H^2 F_1}.
\end{aligned} \tag{4.28}$$

Then the T-dualized part of the metric can be written as

$$\begin{aligned}
& g^I_{mn}\, dy^m dy^n + g^I_{xn}\, dx^I dy^n + g^I_{yn}\, dy^I dy^n + g^I_{xx}\, (dx^I)^2 + g^I_{yy}\, (dy^I)^2 \\
&= H^2 F_2\, g^I_{mn} dy^m dy^n + \frac{1}{H^2 F_1} \left(dx^I + B_{xn} dy^n\right)^2 + \left(dy^I + B_{yn} dy^n\right)^2 .
\end{aligned} \tag{4.29}$$

The type I dilaton is given by

$$e^{\phi^I} = \frac{e^{\phi^B}}{\sqrt{g_{xx}\, g_{yy}}} = \frac{1}{H^2 F_1}. \tag{4.30}$$

S-duality has the effect of sending the dilaton to minus itself

$$\phi^H = -\phi^I. \tag{4.31}$$

The dilaton in heterotic theory is given by

$$e^{\phi^H} = e^{-\phi^I} = e^{-\phi^B} \sqrt{g_{xx}\, g_{yy}} = H^2 F_1. \tag{4.32}$$

Finally, our original type IIB solutions become a new, heterotic superstring solution which in a string frame has the form

$$\begin{aligned} ds^2_{Het} &= e^{\phi_H}\ ds^2_I \\ &= F_1 a^2(t)\left(-dt^2 + g_{ij}dx^i dx^j\right) + H^4 F_1 F_2\ g_{mn}dx^m dx^n + \left(dx^I + B_{xn}dy^n\right)^2 + \left(dy^I + B_{yn}dy^n\right)^2. \end{aligned} \tag{4.33}$$

Using similar Buscher's duality rules for the gauge fields, we can calculate the 2-form fields on the Heterotic side from the original $NS - NS\ B_{\alpha n}$ gauge field and $R - R\ C_{\beta m}$ 2-form field.

4.3 Fermionic bilinear series in terms of the generalized metric

A possible extension of the G-flux components, by incorporating eleven-dimensional gravitino, by generalizing the picture further by first introducing a the matrix-valued operator of the form:

$$\mathcal{D}_{\rm M} \equiv \Gamma_{\rm M} + i\mathbb{I}\ {\rm M}_p^{-1}\partial_{\rm M}, \tag{4.34}$$

with the imaginary part scaling as the vielbein (and thus would have the corresponding $\frac{g_s}{{\rm HH}_o}$ scaling), where the real part would simply be the derivative (but not the covariant derivative as the latter would be automatically generated from the curvature and the flux terms in the series (**??**)). This distinction between the real and the imaginary parts comes from the purely imaginary representations of the eleven-dimensional Gamma matrices because of the presence of Majorana gravitino.

Let us start by defining the following Majorana Rarita-Schwinger field $\lambda_{\rm M}$ as:

$$\lambda_{\rm M} = \mathcal{O}^{(1)}_{\rm MA}\Psi^{\rm A} \equiv \sum_{\{k_i\}}\left(a_{k_1}{\rm Re}\ \mathcal{D}^{2k_1} + b_{k_2}{\rm Im}\ \mathcal{D}^{2k_2} + d_{k_3}{\rm Re}\ |\mathcal{D}|^{2k_3} + e_{k_4}{\rm Im}\ |\mathcal{D}|^{2k_4}\right)_{\rm MA}\Psi^{\rm A}, \tag{4.35}$$

with the Gamma-matrices in the Majorana representation and $\Psi_{\rm M}$ being the eleven-dimensional gravitino.

By noting that Γ^{A_k} is a curved space Gamma-matrix and therefore $\Gamma^{\mathrm{A}_k} \equiv \Gamma^a e_a^{\mathrm{A}_k}$ by using eleven-dimensional vielbeins $e_a^{\mathrm{A}_k}$. Using this, one could replace the metric $\mathbf{g}_{\mathrm{CD}}$, which is symmetric in C and D, by the following:

$$\mathbf{g}_{(\mathrm{CD})} \ \rightarrow \ \hat{\mathbf{g}}_{\mathrm{CD}} \equiv e_{\mathrm{C}}^a e_{\mathrm{D}}^b \Big[\eta_{ab}\mathbb{I} + c_1\Gamma_{(ab)} + c_2\Gamma_{[ab]}\Big], \tag{4.36}$$

for some constants (c_1, c_2) and $\mathbb{I}$ is the identity matrix in the same representations as the Gamma-matrices ((a, b) are the *internal* coordinates and should not be confused with the toroidal coordinates (w^a, w^b). Choosing the coefficients in (4.35) appropriately, we can modify the metric (4.36) by adding fermionic contributions as:

$$\hat{\mathbf{g}}_{\mathrm{MN}} \equiv e_{\mathrm{M}}^a e_{\mathrm{N}}^b \Big[\eta_{ab}\mathbb{I} + c_2\Gamma_{[ab]}\Big] + c_3 \left(\bar{\lambda}\mathcal{O}\lambda\right)_{\mathrm{MN}}, \tag{4.37}$$

Where the Lorentz indices over the Rarita-Schwinger fermion as well as the operator $\mathcal{O}$ will have to be inserted in, and c_3 is a dimensionless parameter.

The fermionic bilinear series

$$\mathbb{Q}_{met} = \sum_n c_{(n)} \ \mathtt{tr}(\bar{\lambda}\mathcal{O}\lambda)^n \tag{4.38}$$

can also be represented using the generalized metric $\hat{\mathbf{g}}_{CD}$.

First, we use the notation $\mathbb{X} = \bar{\lambda}\mathcal{O}\lambda$ to make the calculations more tractable. Then we recall the definition of the generalized metric $\hat{\mathbf{g}}_{CD}$:

$$\hat{\mathbf{g}}_{CD} = e_C^a \ e_D^b \ \left[\eta_{ab}\mathbb{1} + c_2\Gamma_{[ab]}\right] + c_3(\bar{\lambda}\mathcal{O}\lambda)_{CD} = \mathbf{g}_{CD} + c_3\mathbb{X}_{CD}. \tag{4.39}$$

In the following calculations, we set $c_3 = 1$ to reduce the clutter. It will not affect the final results.

Next, we note that

$$\begin{aligned}
&\mathtt{tr}\mathbb{X} == \mathbf{g}^{CD}(\bar{\lambda}\mathcal{O}\lambda)_{CD} = \mathbf{g}^{CD}\left(\hat{\mathbf{g}}_{CD} - \mathbf{g}_{CD}\right), \\
&\mathtt{tr}\mathbb{X}^2 = \mathbf{g}^{CD}\mathbf{g}^{MN} \ (\bar{\lambda}\mathcal{O}\lambda)_{CM} \ (\bar{\lambda}\mathcal{O}\lambda)_{DN} = \mathbf{g}^{CD}\mathbf{g}^{MN} \ \left(\hat{\mathbf{g}}_{CM} - \mathbf{g}_{CM}\right) \ \left(\hat{\mathbf{g}}_{DN} - \mathbf{g}_{DN}\right),
\end{aligned}$$

and so on. It allows us to write down $\mathbb{Q}_{met}$ using only the metric:

$$\mathbb{Q}_{met} = \sum_n c_{(n)} \ \mathtt{tr}\mathbb{X}^m \ \mathtt{tr}\mathbb{X}^n \cdots \mathtt{tr}\mathbb{X}^s = c_{(1)} \ \mathbf{g}^{CD}\left(\hat{\mathbf{g}}_{CD} - \mathbf{g}_{CD}\right) \tag{4.40}$$

$$+ c_{(2)}\, \mathbf{g}^{CD}\mathbf{g}^{MN}\ \left(\hat{\mathbf{g}}_{CM} - \mathbf{g}_{CM}\right)\ \left(\hat{\mathbf{g}}_{DN} - \mathbf{g}_{DN}\right) + ... \tag{4.41}$$

Since this quantum series does not contain all possible elements, such as $(\mathtt{tr}\mathbb{X})^p$, we introduce another variation of the quantum series, which includes the previous one but also generates an additional variety of the quantum terms:

$$\mathbb{Q}^{(2)}_{met} = \sum_{\{p_i\}} c_{p_1 p_2 \ldots p_\infty} \prod_{i=1}^{\infty} \mathtt{tr}\mathbb{X}^{p_i} = \sum_{m,n,..,s} \mathtt{tr}\mathbb{X}^m\ \mathtt{tr}\mathbb{X}^n \cdots \mathtt{tr}\mathbb{X}^s. \tag{4.42}$$

This expression is not very easy to handle, so we rewrite it in a more wieldy form as a series in powers of $\mathtt{tr}\mathbb{X}^n$:

$$\mathbb{Q}^{(2)}_{met} = \sum_{k_p} \left(\sum_p c^{(p)}_{k_p} \left[\mathtt{tr}(1+\mathbb{X})^p\right]^{k_p} \right). \tag{4.43}$$

To make sure that this new quantum series contains all the elements of the previous variation of it, we explicitly calculate several first terms. For $p = 1$ we have the following

$$\sum_{k_1} c^{(1)}_{k_1} \left[\mathtt{tr}(1+\mathbb{X})\right]^{k_1} = \sum_{k_1} c^{(1)}_{k_1} \left[\mathtt{tr}\mathbb{1} + \mathtt{tr}\mathbb{X}\right]^{k_1} = \sum_{k_1} c^{(1)}_{k_1} \left[D + \mathtt{tr}\mathbb{X}\right]^{k_1}, \tag{4.44}$$

where we have used the fact that $\mathtt{tr}\mathbb{1} = D$, the dimension of the space-time.

In the case of $p = 2$ we have a slightly more complicated term coming from the trace part:

$$\mathtt{tr}(1+\mathbb{X})^2 = \mathtt{tr}(1 + 2\mathbb{X} + \mathbb{X}^2) = D + 2\mathtt{tr}\mathbb{X} + \mathtt{tr}\mathbb{X}^2,$$

and thus the sum term is

$$\sum_{k_2} c^{2)}_{k_2} \left(D + 2\mathtt{tr}\mathbb{X} + \mathtt{tr}\mathbb{X}^2\right)^{k_2}.$$

In a very similar way, we generate the third term of the series

$$\mathtt{tr}(1+\mathbb{X})^3 = D + 3\mathtt{tr}\mathbb{X} + 3\mathtt{tr}\mathbb{X}^2 + \mathtt{tr}\mathbb{X}^3,$$

and

$$\sum_{k_2} c^{2)}_{k_2} \left(D + 3\mathtt{tr}\mathbb{X} + 3\mathtt{tr}\mathbb{X}^2 + \mathtt{tr}\mathbb{X}^3\right)^{k_3}.$$

We can see that all the powers of the $\mathtt{tr}\mathbb{X}$ appear in this version of the series and much more.

An additional benefit of using this particular form of the quantum series is that we can easily rewrite it in terms of the powers of the generalized metric. We note that

$$\mathtt{tr}\mathbb{X} = \mathbf{g}^{CD}\,\mathbb{X}_{CD} = \mathbf{g}^{CD}(\hat{\mathbf{g}}_{CD} - \mathbf{g}_{CD}) = \mathbf{g}^{CD}\hat{\mathbf{g}}_{CD} - D. \tag{4.45}$$

Then

$$D + \mathtt{tr}\mathbb{X} = D + \mathbf{g}^{CD}\hat{\mathbf{g}}_{CD} - D = \mathbf{g}^{CD}\hat{\mathbf{g}}_{CD},$$

and

$$\begin{aligned} D + 2\mathtt{tr}\mathbb{X} + \mathtt{tr}\mathbb{X}^2 &= D + 2\mathbf{g}^{CD}\hat{\mathbf{g}}_{CD} - 2D + \mathbf{g}^{CD}\mathbf{g}^{MN}\,\hat{\mathbf{g}}_{CM}\hat{\mathbf{g}}_{DN} - 2\mathbf{g}^{CD}\hat{\mathbf{g}}_{CD} + D \\ &= \mathbf{g}^{CD}\mathbf{g}^{MN}\,\hat{\mathbf{g}}_{CM}\hat{\mathbf{g}}_{DN}, \end{aligned}$$

and so on. Thus, we observe that a generic fermionic interaction of written as a quantum series $\mathbb{Q}^{(2)}_{met}$ cab be completely expressed in terms of the generalized metric.

To bring the quantum series $\mathbb{Q}^{(2)}_{met}$ into a trans-series form, we rewrite the coefficients $c^{(p)}_{k_p}$ in the following form

$$c^{(p)}_{k_p} = \sum_{l\geq 1} \frac{d^{(p)}_l\,(-l)^{k_p}}{k_p!}. \tag{4.46}$$

Next, write down the first term of the $\mathbb{Q}^{(2)}_{met}$ quantum series in a suggestive way

$$\sum_{k_1} c^{(1)}_{k_1}\,[D + \mathtt{tr}\mathbb{X})]^{k_1} = c^{(1)}_1(D + \mathtt{tr}\mathbb{X}) + c^{(1)}_2(D + \mathtt{tr}\mathbb{X})^2 + c^{(1)}_3(D + \mathtt{tr}\mathbb{X})^3 + ..., \tag{4.47}$$

where the coefficients $c^{(1)}_{k_1}$, in their turn, are given by the following series

$$c^{(1)}_{k_1} = \sum_{l\geq 1} \frac{d^{(1)}_l\,(-l)^{k_1}}{k_1!} = \frac{d^{(1)}_1(-1)^{k_1}}{k_1!} + \frac{d^{(1)}_2(-2)^{k_1}}{k_1!} + \frac{d^{(1)}_3(-3)^{k_1}}{k_1!} + ... \tag{4.48}$$

In the next step, we combine the two series to find

$$c^{(1)}_1(D + \mathtt{tr}\mathbb{X}) = \frac{d^{(1)}_1(-1)}{1!}(D + \mathtt{tr}\mathbb{X}) + \frac{d^{(1)}_2(-2)}{1!}(D + \mathtt{tr}\mathbb{X}) + \frac{d^{(1)}_3(-3)}{1!}(D + \mathtt{tr}\mathbb{X}) + ...$$

$$c_2^{(1)}(D+\mathtt{tr}\mathbb{X})^2 = \frac{d_1^{(1)}(-1)^2}{2!}(D+\mathtt{tr}\mathbb{X})^2 + \frac{d_2^{(1)}(-2)^2}{2!}(D+\mathtt{tr}\mathbb{X})^2 + \frac{d_3^{(1)}(-3)^2}{2!}(D+\mathtt{tr}\mathbb{X})^2 + ...$$

Summing all these infinite series terms by term, we obtain the following interesting result

$$\sum_{k_1} c_{k_1}^{(1)}\,[D+\mathtt{tr}\mathbb{X})]^{k_1} = d_1^{(1)}\,[\,\exp(-D-\mathtt{tr}\mathbb{X})-1] + d_2^{(1)}\,[\,\exp(-2D-2\mathtt{tr}\mathbb{X})-1]$$
$$+d_3^{(1)}\,[\,\exp(-3D-3\mathtt{tr}\mathbb{X})-1] + ... = \sum_{l\geq 1} d_l^{(1)}\,[\,\exp(-lD-l\mathtt{tr}\mathbb{X})-1]\,.$$

The rest of the terms of the original $\mathbb{Q}_{met}^{(2)}$ quantum series can be rewritten similarly. At the same time, we can write all the above computations in a very condensed formal way, ignoring some mathematical subtleties but giving the written result anyway. For instance, the second term can be formally manipulated as

$$\sum_{k_2} c_{k_2}^{2)}\left(D+2\mathtt{tr}\mathbb{X}+\mathtt{tr}\mathbb{X}^2\right) = \sum_{k_2}\sum_{l\geq 1}\frac{d_l^{(2)}\,(-l)^{k_2}}{k_2!}\left(D+2\mathtt{tr}\mathbb{X}+\mathtt{tr}\mathbb{X}^2\right)^{k_2}$$
$$= \sum_{l\geq 1} d_l^{(2)}\sum_{k_2=0}^{\infty}\frac{(-l)^{k_2}}{k_2!}\left(D+2\mathtt{tr}\mathbb{X}+\mathtt{tr}\mathbb{X}^2\right)^{k_2} = \sum_{l\geq 1} d_l^{(2)}\,\left[\,\exp\left(-lD-2l\mathtt{tr}\mathbb{X}-l\mathtt{tr}\mathbb{X}^2\right)-1\right].$$

The beauty of these series comes from the fact that they are all convergent due to the presence of the e^{-lD} and $e^{-l\mathtt{tr}\mathbb{X}^{2p}}$ suppression.

4.4 M-theory uplift and Riemann tensor scalings.

Interestingly, the *form* of the M-theory metric always remains the same for any type IIB cosmology expressed using conformal coordinates as in (4.1). The only change is the value of the dual IIA string coupling: $\frac{g_s}{\mathrm{HH}_o} = \frac{1}{a(t)}$ which is sensitive to the functional form of $a(t)$. Clearly, and as mentioned earlier, for expanding cosmologies, the IIA coupling can be made small. Demanding $\frac{g_s}{\mathrm{HH}_o} < 1$ provides the temporal domain in which controlled quantum computations may be performed in M-theory. For the usual de Sitter case, irrespective of the choice of the de Sitter slicings, this temporal domain remains perfectly consistent with the so-called Trans-Planckian Cosmic Censorship (TCC) [11]. The question is, what happens now? We must find the functional form of $a(t)$ that provides a de Sitter metric on the *dual* side to answer this. We will return to the issue soon, but not before we elucidate the consistency of the IIB background (4.1) from M-theory. In M-theory, the uplifted metric takes the following standard form:

$$ds^2 = \left(\frac{g_s}{\mathrm{HH}_o}\right)^{-8/3}\left(-\tilde{g}_{00}dt^2 + \tilde{g}_{ij}dx^i dx^j\right) + \left(\frac{g_s}{\mathrm{HH}_o}\right)^{-2/3}\left[\mathrm{F}_1\left(\frac{g_s}{\mathrm{H}_1}\right)\tilde{g}_{\alpha\beta}dy^\alpha dy^\beta + \mathrm{F}_2\left(\frac{g_s}{\mathrm{H}_1}\right)\tilde{g}_{mn}dy^m dy^n\right] + \left(\frac{g_s}{\mathrm{HH}_o}\right)^{4/3}\tilde{g}_{ab}dw^a dw^b, \tag{4.49}$$

where $\mathrm{H}_1(\mathbf{x}, y) \equiv \mathrm{H}(y)\mathrm{H}_o(\mathbf{x})$, which means $\mathrm{F}_i(g_s/\mathrm{H}_1)$ depends on the temporal factor $a(t)$, and we shall discuss their functional form soon. The other metric components may be related to the metric components in (4.1) in the following way:

$$\begin{aligned}\tilde{g}_{ab}(\mathbf{x}, y) &\equiv \left[\mathrm{H}(y)\mathrm{H}_o(\mathbf{x})\right]^{4/3} g_{ab}(\mathbf{x}, y)\\ \tilde{g}_{\mu\nu}(\mathbf{x}, y) &\equiv \frac{g_{\mu\nu}(\mathbf{x})}{\left[\mathrm{H}^4(y)\mathrm{H}_o(\mathbf{x})\right]^{2/3}}, \quad \tilde{g}_{\mathrm{MN}}(\mathbf{x}, y) \equiv \left[\frac{\mathrm{H}^2(y)}{\mathrm{H}_o(\mathbf{x})}\right]^{2/3} g_{\mathrm{MN}}(y)\end{aligned} \tag{4.50}$$

where we have taken $(\mathrm{M}, \mathrm{N}) \in \mathcal{M}_4 \times \mathcal{M}_2$. We have taken the un-warped metric components along the toroidal direction depending on both (x^i, y^{M}). In fact, for the computations of the curvature scaling, we will take both the un-warped and the warped metric components to depend on all the coordinates (except of course, the toroidal direction). Once we go to the heterotic side, we will see that the dependence on the coordinates of $\mathcal{M}_2$ has to be removed.

Let us now come to the functional form for the temporal factors $\mathrm{F}_i(g_s/\mathrm{H}_1)$. These factors did not change the dominant scalings of the metric components as they were constrained by $\mathrm{F}_i(g_s/\mathrm{H}_1) \to 1, g_s \to 0$ and $\mathrm{F}_1\mathrm{F}_2^2 = 1$ to preserve the Newton's constant and to avoid late-time singularities. Both of these conditions are not essential now if we want to dualize to any of the other string and M-theories because only in the *dual* landscape do we want a time-independent Newton's constant with no late time singularities. This means the dominant scalings of the internal metric could, in principle *change*; we can then propose the following dominant scalings:

$$\mathrm{F}_1 \equiv \sum_{k=0}^{\infty}\mathrm{A}_k\left(\frac{g_s}{\mathrm{HH}_o}\right)^{\beta_o + 2k/3}, \quad \mathrm{F}_2 \equiv \sum_{k=0}^{\infty}\mathrm{B}_k\left(\frac{g_s}{\mathrm{HH}_o}\right)^{\alpha_o + 2k/3}, \quad \frac{\partial}{\partial t}\left(\frac{g_s}{\mathrm{HH}_o}\right) \equiv \sum_{k=0}^{\infty}\mathrm{C}_k\left(\frac{g_s}{\mathrm{HH}_o}\right)^{\gamma_o + 2k/3}, \tag{4.51}$$

where $(\mathrm{A}_k, \mathrm{B}_k, \mathrm{C}_k)$ are all integers, positive or negative with $(\alpha_o, \beta_o, \gamma_o)$ being the dominant scalings. Note that, as we demonstrated rigorously, when $\gamma_o < 0$, EFT breaks down along with a violation of the four-dimensional NEC. Here, in the generic setting, we will see whether this continues to hold. On the other hand, (α_o, β_o) are not *a-priori* required to be positive definite. An interesting question would be to find whether there is a connection between the three dominant scalings. If there is one, then it would lead to an even deeper connection between the three disparate facts: the existence of EFT from M-theory, preserving four-dimensional NEC from IIB, and temporal dependence of the internal six-dimensional manifold.

We analyze the perturbative series of quantum effects that include both local and non-local terms embedded in the eleven-dimensional action. We write down only the quantum term of the action

$$\mathtt{S}_1 = \sum_{l_i, n_i} \int d^{11}x \sqrt{-g_{11}} \frac{\mathbb{Q}(l_i, n_i}{M_p^{\sigma(l_i}}, \tag{4.52}$$

where the quantum term $\mathbb{Q}_T(l_i, n_i)$ may be expressed as

$$\hat{\mathbb{Q}}_{\mathrm{T}}^{(\{l_i\}, n_i)} = [\hat{\mathbf{g}}^{-1}] \prod_{i=0}^{3} [\partial]^{n_i} \left(\hat{\mathbf{g}}^{\mathrm{C}_1}_{\mathrm{C}_2} \hat{\mathbf{g}}^{\mathrm{C}_2}_{\mathrm{C}_3} .....\hat{\mathbf{g}}^{\mathrm{C}_{n_4}}_{\mathrm{C}_1}\right)^{n_5} \prod_{\mathrm{k}=1}^{41} \left(\hat{\mathbf{R}}_{\mathrm{A_k B_k C_k D_k}}\right)^{l_{\mathrm{k}}} \prod_{\mathrm{r}=42}^{81} \left(\hat{\mathbf{G}}_{\mathrm{A_r B_r C_r D_r}}\right)^{l_{\mathrm{r}}}, \tag{4.53}$$

which includes the generalized metric $\hat{\mathbf{g}}_{\mathrm{CD}}$ as well as generalized curvature $\hat{\mathbf{R}}_{\mathrm{ABCD}}$ and generalized G-flux components $\hat{\mathbf{G}}_{\mathrm{ABCD}}$ which include the fermionic contribution discussed previously. Our aim is to understand how the quantum series influences the dynamics in the configuration spaces of both the gravitons and the G-flux components by analyzing the $\frac{g_s}{\mathrm{HH}_o}$ scaling of the quantum series. Since the scaling calculations are relatively simple but somewhat tedious and take a good deal of space, they are relegated to the Appendix. Here we only provide the final answer in the form of Table 1.

All the fermionic contributions, from both the curvature and the G-flux components, are sub-dominant. Despite this, and as mentioned above, the $\frac{g_s}{\mathrm{HH}_o}$ scaling is not simple. Additionally, looking at the form of the $\frac{g_s}{\mathrm{HH}_o}$ scaling, we see that there are too many relative minus signs now. This is not good, as uncontrolled relative signs would signify a breakdown of the EFT description. The scalings of the curvature and the derivative terms put the following bounds on the values of α_o, β_o and γ_o:

$$\alpha_o < +\frac{2}{3}, \quad \beta_o < +\frac{2}{3}, \quad \gamma_o > -\frac{1}{3} \tag{4.54}$$

implying that the dominant scalings of $\mathtt{F}_1$ and $\mathtt{F}_2$ cannot exceed the aforementioned bounds, and $\gamma_o \geq 0$ because the scalings jump as $\pm\frac{\mathbb{Z}}{3}$. The latter is consistent with NEC violation. Assuming $(\alpha_o, \beta_o) \geq (0,0)$, this means that two and the four-manifolds, $\mathcal{M}_2$ and $\mathcal{M}_4$ respectively, cannot shrink to zero sizes at a late time when $g_s \to 0$. While this is good for the four-manifold $\mathcal{M}_4$, the non-shrinking of $\mathcal{M}_2$ would mean that the system cannot dynamically go to either the Type I or the heterotic side. In fact, at the late time (when $g_s \to 0$), the two-manifold $\mathcal{M}_2$ would blow up, leading to a heterotic manifold with extremely high curvature. This leads to our first trouble realizing a de Sitter Glauber-Sudarshan state in the heterotic landscape.

There are, however, a couple of ways out of this. One, by demanding the metric components to be *independent* of the coordinates of $\mathcal{M}_2$, , *i.e.* independent of the (α, β) directions. One may easily see that imposing the derivative constraint removes all the scalings with $-\beta_o$ in the curvature and the derivative terms. The $-\beta_o$ terms survive in the G-flux scalings but do not pose any immediate problems. The above considerations lead us to conclude that EFT on the heterotic side is only valid if the metric components are *independent* of the coordinates of $\mathcal{M}_2$. Any non-trivial dependence on the coordinates of $\mathcal{M}_2$ will rule out an EFT description in the dual heterotic side.

Riemann tensors	g_s scalings
$\mathbf{R}_{mnpq}$	$\text{dom}\left(-\frac{2}{3}+\alpha_o, -\frac{2}{3}+2\alpha_o+2\gamma_o, \frac{4}{3}+2\alpha_o, -\frac{2}{3}+2\alpha_o-\beta_o\right)$
$\mathbf{R}_{mn\alpha\beta}$	$\text{dom}\left(-\frac{2}{3}+\alpha_o, -\frac{2}{3}+\beta_o, -\frac{2}{3}+\alpha_o+\beta_o+2\gamma_o, \frac{4}{3}+\alpha_o+\beta_o\right)$
$\mathbf{R}_{\alpha\beta i0}, \mathbf{R}_{m\alpha\beta 0}, *\mathbf{R}_{0\alpha\alpha\beta}$	$-\frac{5}{3}+\beta_o+\gamma_o$
$\mathbf{R}_{\alpha\beta\alpha\beta}$	$\text{dom}\left(-\frac{2}{3}+\beta_o, -\frac{2}{3}+2\beta_o+2\gamma_o, -\frac{2}{3}+2\beta_o-\alpha_o, \frac{4}{3}+2\beta_o\right)$
$\mathbf{R}_{mni0}, \mathbf{R}_{mnp0}, *\mathbf{R}_{mn\alpha 0}$	$-\frac{5}{3}+\alpha_o+\gamma_o$
$\mathbf{R}_{mnab}$	$\text{dom}\left(\frac{4}{3}+2\gamma_o+\alpha_o, \frac{4}{3}+\alpha_o-\beta_o, \frac{10}{3}+\alpha_o, \frac{4}{3}\right)$
$\mathbf{R}_{m\alpha\beta i}, *\mathbf{R}_{i\alpha\alpha\beta}, *\mathbf{R}_{m\alpha\alpha\beta}$	$-\frac{2}{3}+\beta_o$
$\mathbf{R}_{mnpi}, *\mathbf{R}_{mnp\alpha}, *\mathbf{R}_{mn\alpha i}$	$-\frac{2}{3}+\alpha_o$
$\mathbf{R}_{mabi}, *\mathbf{R}_{\alpha abi}, *\mathbf{R}_{m\alpha ab}$	$\frac{4}{3}$
$\mathbf{R}_{\alpha\beta ab}$	$\text{dom}\left(\frac{4}{3}, \frac{4}{3}+\beta_o+2\gamma_o, \frac{4}{3}+\beta_o-\alpha_o, \frac{10}{3}+\beta_o\right)$
$\mathbf{R}_{abab}$	$\text{dom}\left(\frac{10}{3}+2\gamma_o, \frac{10}{3}-\alpha_o, \frac{10}{3}-\beta_o, \frac{16}{3}\right)$
$\mathbf{R}_{0mab}, \mathbf{R}_{abi0}, *\mathbf{R}_{0\alpha ab}$	$\frac{1}{3}+\gamma_o$
$\mathbf{R}_{m0ij}, \mathbf{R}_{ijk0}, *\mathbf{R}_{\alpha 0ij}$	$-\frac{11}{3}+\gamma_o$
$\mathbf{R}_{m0i0}, \mathbf{R}_{mijk}, *\mathbf{R}_{0\alpha i0}$	$-\frac{8}{3}$
$*\mathbf{R}_{\alpha ijk}, *\mathbf{R}_{m\alpha ij}, *\mathbf{R}_{0m0\alpha}$	$-\frac{8}{3}$
$\mathbf{R}_{ijij}, \mathbf{R}_{0i0j}$	$\text{dom}\left(-\frac{8}{3}, -\frac{14}{3}+2\gamma_o, -\frac{14}{3}-\alpha_o, -\frac{14}{3}-\beta_o\right)$
$\mathbf{R}_{abij}, \mathbf{R}_{0a0b}$	$\text{dom}\left(\frac{4}{3}, -\frac{2}{3}+2\gamma_o, -\frac{2}{3}-\alpha_o, -\frac{2}{3}-\beta_o\right)$
$\mathbf{R}_{mnij}, \mathbf{R}_{0m0n}$	$\text{dom}\left(-\frac{8}{3}, -\frac{2}{3}+\alpha_o, -\frac{8}{3}+\alpha_o+2\gamma_o, -\frac{8}{3}+\alpha_o-\beta_o\right)$
$\mathbf{R}_{\alpha\beta ij}, \mathbf{R}_{0\alpha 0\beta}$	$\text{dom}\left(-\frac{8}{3}, -\frac{2}{3}+\beta_o, -\frac{8}{3}+\beta_o+2\gamma_o, -\frac{8}{3}+\beta_o-\alpha_o\right)$

Table 4.1: The $\frac{g_s}{\mathrm{HH}_o}$ expansions of the components of the curvature tensors associated with the M-theory metric. The warp factor $\mathrm{H}(y)$ is the universal warp factor, whereas $\mathrm{H}_o \equiv \mathrm{H}_o(x,y)$ depends on the choice of the de Sitter slicings. The components of the Riemann tensors are defined in the usual way: $(m,n) \in \mathcal{M}_4$, $(\alpha,\beta) \in \mathcal{M}_2$, $(a,b) \in \frac{\mathbb{T}^2}{\mathcal{G}}$ and $(\mu,\nu) \in \mathbf{R}^{2,1}$; with $x \equiv (x^i, x^j)$ and $y^m \in \mathcal{M}_4 \times \mathcal{M}_2$. The modes of the curvature tensor are defined as $\mathrm{R}^{(k)}_{\mathrm{A}_1\mathrm{A}_2\mathrm{A}_3\mathrm{A}_4} = \mathrm{R}^{(k)}_{\mathrm{A}_1\mathrm{A}_2\mathrm{A}_3\mathrm{A}_4}(x,y)$ where $\mathrm{A}_i \in \mathbf{R}^{2,1} \times \mathcal{M}_4 \times \mathcal{M}_2 \times \frac{\mathbb{T}^2}{\mathcal{G}}$ and $k \in \frac{\mathbb{Z}}{2}$.

Chapter 5

Conclusion

It is very a very hard and non-trivial problem to construct the de Sitter space cosmological solution in string theory. It is still an open question whether it is possible to realize de Sitter spacetime in string theory. Classical de Sitter vacuum fails to be obtained due to various no-go theorems. The effective field theory version of the de Sitter vacuum is ruled out by the ill-defined Wilsonian action over an accelerating background. And yet there is still some hope.

This work investigates some aspects of the string theory realization of de Sitter space as a coherent state. This almost classical solution demands a fully quantum analysis, making it more involved and, simultaneously, more satisfactory. There was made effort to simplify the model by working with the limited set of scalar fields instead of going to the whole theory containing the set of 44 gravitons, 84 fluxes and 128 Rarita-Schwinger fermions. This interpretation allowed us to reveal the main features of the construct and avoid unnecessary complexity. We started with the general definition of a coherent state in a straightforward set-up for a non-interacting vacuum with a brief discussion on how to transition to a more elaborate case of a coherent state on an interacting vacuum.

Next, we introduced a path-integral formalism for doing quantum field theory for a shifted interacting vacuum, known as the Glauber-Sudarshan state. A toy model of free massive scalar field theory in 3+1 dimensions was introduced and solved using the abovementioned formalism. At the next level of complexity, we attempted to include the full interacting action with all the quantum corrections in our analysis of the Glauber-Sudarshan state. As was mentioned previously, to make our computations tractable, we included only three scalar fields representing different components of the graviton, flux, and fermionic fields. A geometric tool named nodal diagrams was introduced, as a slightly modified version of Feynman diagrams, to keep track of the path-integral calculations. As a consequence of the analysis, it was shown that the path-integral structure for the three fields over the shifted vacuum breaks down as a collection of nodal diagrams. The nodal diagrams representation helps significantly deal with the integrating diagrams' momenta and momentum conservation.

In the second part of our work, we introduce another mathematical tool known as the Borel resummation technique and the resurgent trans-series. The necessity of such an approach was dictated by the fact that the path-integral calculations of the expectation value of the metric lead to the answer in the form of asymptotic series. And although it is common knowledge that perturbative analyses in quantum field theory generate asymptotic solutions, the non-perturbative part of such investigation is usually lost. We tried to recover some additional knowledge by resuming the divergent series and thus making statements about the non-perturbative nature of the calculations.

The expectation value of the metric generates the divergent asymptotic series because of the factorial growth coming from the nodal diagrams for higher orders in the coupling constant. The metric computation only makes sense if we can summarise all the information contained in the asymptotic series in some meaningful way. The Borel resummation technique is the very tool which allows us to boost our results with the knowledge of the non-perturbative part of the series. It is very natural to suggest that we need to include non-perturbative physics in our solution to construct de Sitter space in string theory.

We show that for the simplified case of 3 scalar fields, the expectation value of the metric produces the factorial growth of the generalized Gevrey-α series. In particular, we found at least the result $\alpha = 2$, which is expected to be much higher for the complete solution in M-theory—the correct Borel resummation of the Gevrey-α series results in a closed-form expression for the cosmological constant. One of our analysis's conclusions is that the cosmological constant's value in our model can be small. It is based on the fact that the displacement generated by the Glauber-Sudarshan state from the vacuum in the configuration space is inversely proportional to the spacetime volume and, thus, minimal. This means that coherent de Sitter space is close to its vacuum configuration. Another important outcome of our analysis is a positive value of the cosmological constant.

We also included some results from our investigation of the possibility of realizing de Sitter space as a Glauber-Sudarshan state for the case of a generic background. This generalization allows us to construct the de Sitter state in various string theories through duality transformations such as T-dualities and S-dualities. We explicitly show one such transition from type IIB background to heterotic $SO(32)$ theory.

The results of this work add some new perspectives to the quest of finding a valid de Sitter space construction from the string theory point of view. Although the fundamental analysis is based on a toy model and does not consider the complexity of the full realization, at least it shows some potential original direction that has never been considered before. Besides, the asymptotic series resummation technique proved helpful in this set-up and thus, arguably, might find its application in other related physical problems.

Appendix A

Riemann tensor scalings

In M-theory, the uplifted metric takes the following standard form:

$$ds^2 = \left(\frac{g_s}{HH_0}\right)^{-8/3}\left(-\tilde{g}_{00}dt^2 + \tilde{g}_{ij}dx^i dx^j\right) + \left(\frac{g_s}{HH_0}\right)^{4/3}\tilde{g}_{ab}dw^a dw^b$$
$$+ \left(\frac{g_s}{HH_0}\right)^{-2/3}\left[\mathrm{F}_1\left(\frac{g_s}{H_1}\right)\tilde{g}_{\alpha\beta}dy^\alpha dy^\beta + \mathrm{F}_2\left(\frac{g_s}{H_1}\right)\tilde{g}_{mn}dy^m dy^n\right]$$

We introduce the following dominant scalings:

$$\begin{aligned} F_1 &= \sum_{k=0}^{\infty} A_k\left(\frac{g_s}{HH_0}\right)^{\beta_0+2k/3}, \qquad F_2 = \sum_{k=0}^{\infty} B_k\left(\frac{g_s}{HH_0}\right)^{\alpha_0+2k/3} \\ \frac{\partial}{\partial t}\left(\frac{g_s}{HH_0}\right) &= \sum_{k=0}^{\infty} C_k\left(\frac{g_s}{HH_0}\right)^{\gamma_0+2k/3}. \end{aligned} \tag{A.1}$$

Then we have the following scalings for the metrics:

$$g_{\mu\nu} \sim g_s^{-8/3}, \qquad g_{mn} \sim g_s^{-2/3+\alpha}, \qquad g_{\alpha\beta} \sim g_s^{-2/3+\beta}, \qquad g_{ab} \sim g_s^{4/3}. \tag{A.2}$$

The time derivative of any of the metric components generates the scaling

$$\begin{aligned} &\frac{\partial}{\partial t} g_s \sim g_s^{\gamma}, \\ &\frac{\partial}{\partial t} g_{MN} \sim \frac{\partial}{\partial t} g_s^{A} \sim g_s^{\gamma} g_s^{A-1}. \end{aligned} \tag{A.3}$$

Now we are ready to find the scalings of the Riemann tensors.

We need to find all the possible permutations of the tensor indices:

$$\boxed{\mathbf{R_{mnpq}}}: \qquad m \quad n \quad p \quad q \qquad + \qquad m \quad m \quad p \quad q \qquad + \qquad m \quad m \quad p \quad q \tag{A.4}$$

Each particular index permutation generates the unique contractions of the first and second derivatives of the metric tensor:

$$m \quad n \quad p \quad q: \qquad g_{mn,pq} \qquad\qquad g_{pq,mn}$$

$$g_{mn,R}\ g_{pq,S}\ g^{RS}$$

$$g_{Rm,n}\ g_{Sp,q}\ g^{RS}$$

Now we can easily compute the scalings for each combination of the metric tensors and their derivatives:

$$g_{mn,pq} \sim g_s \left\{ -\frac{2}{3} + \alpha_0 \right\},$$

$$g_{mn,R}\ g_{pq,S}\ g^{RS} = g_{mn,0}\ g_{pq,0}\ g^{00} + g_{mn,i}\ g_{pq,j}\ g^{ij} + g_{mn,\alpha}\ g_{pq,\beta}\ g^{\alpha\beta} + g_{mn,r}\ g_{pq,s}\ g^{rs}$$
$$\sim g_s \left\{ -\frac{2}{3} + 2\alpha_0 + 2\gamma_0 \right\} + g_s \left\{ \frac{4}{3} + 2\alpha_0 \right\} + g_s \left\{ -\frac{2}{3} + 2\alpha_0 - \beta_0 \right\} + g_s \left\{ -\frac{2}{3} + \alpha_0 \right\},$$

$$g_{Rm,n}\ g_{Sp,q}\ g^{RS} = g_{rm,n}\ g_{sp,q}\ g^{rs} \sim g_s \left\{ -\frac{2}{3} + \alpha_0 \right\},$$

Both the second and the third permutation of indices generates the same g_s scalings:

$$g_s \left\{ m \quad m \quad p \quad q \right\} \qquad \sim \qquad g_s \left\{ m \quad n \quad p \quad q \right\}$$
$$g_s \left\{ m \quad m \quad p \quad q \right\} \qquad \sim \qquad g_s \left\{ m \quad n \quad p \quad q \right\}$$

Finally, we find the scalings of the R_{nmpq} Riemann tensor to be

$$R_{nmpq} \sim g_s \ \left\{ -\frac{2}{3} + \alpha_0, \frac{4}{3} + 2\alpha_0, -\frac{2}{3} + 2\alpha_0 + 2\gamma_0, -\frac{2}{3} + 2\alpha_0 - \beta_0 \right\}.$$

$$\boxed{\mathbf{R_{mn\alpha\beta}}}: \qquad m \quad n \quad \alpha \quad \beta \qquad + \qquad m \quad m \quad \alpha \quad \beta \qquad + \qquad m \quad m \quad \alpha \quad \beta \tag{A.5}$$

The first permutation of indices generates

$$m \quad n \quad \alpha \quad \beta: \qquad g_{mn,\alpha\beta} \qquad\qquad g_{\alpha\beta,mn}$$

$$g_{mn,R}\ g_{\alpha\beta,S}\ g^{RS}$$

$$g_{Rm,n}\ g_{S\alpha,\beta}\ g^{RS}$$

The scalings of the metric tensor combinations are

$$g_{mn,\alpha\beta} \sim g_s\left\{-\frac{2}{3}+\alpha_0\right\}, \qquad g_{\alpha\beta,mn} \sim g_s\left\{-\frac{2}{3}+\beta_0\right\}$$

$$g_{mn,R}\ g_{\alpha\beta,S}\ g^{RS} = g_{mn,0}\ g_{\alpha\beta,0}\ g^{00} + g_{mn,i}\ g_{\alpha\beta,j}\ g^{ij} + g_{mn,\alpha}\ g_{\alpha\beta,\beta}\ g^{\alpha\beta} + g_{mn,r}\ g_{\alpha\beta,s}\ g^{rs}$$
$$\sim g_s\left\{-\frac{2}{3}+\alpha_0+\beta_0+2\gamma_0\right\} + g_s\left\{\frac{4}{3}+\alpha_0+\beta_0\right\} + g_s\left\{-\frac{2}{3}+\beta_0\right\} + g_s\left\{-\frac{2}{3}+\alpha_0\right\},$$

$$g_{Rm,n}\ g_{Sp,q}\ g^{RS} \sim 0.$$

The second permutation of indices gives similar scalings:

$$m \quad m \quad \alpha \quad \beta : \qquad g_{m\alpha,n\beta} \qquad g_{n\beta,m\alpha}$$
$$g_{m\alpha,R}\ g_{n\beta,S}\ g^{RS}$$
$$g_{Rm,\alpha}\ g_{Sn,\beta}\ g^{RS}$$

with the following scalings

$$g_{m\alpha,n\beta} \sim 0, \qquad g_{n\beta,m\alpha} \sim 0,$$

$$g_{m\alpha,R}\ g_{n\beta,S}\ g^{RS} \sim 0,$$

$$g_{Rm,\alpha}\ g_{Sn,\beta}\ g^{RS} = g_{pm,\alpha}\ g_{qn,\beta}\ g^{pq} \sim g_s\left\{-\frac{2}{3}+\beta_0\right\}$$

And the last index permutation is the same as the second one:

$$g_s\left\{m \quad m \quad \alpha \quad \beta\right\} \qquad \sim \qquad g_s\left\{m \quad m \quad \alpha \quad \beta\right\}.$$

The full set of scaling is

$$R_{nm\alpha\beta} \sim g_s\left\{-\frac{2}{3}+\alpha_0+\beta_0+2\gamma_0, \frac{4}{3}+\alpha_0+\beta_0, -\frac{2}{3}+\beta_0, -\frac{2}{3}+\alpha_0\right\}.$$

$$\boxed{\mathbf{R}_{\alpha\beta\mathbf{i0}}} : \quad i \quad 0 \quad \alpha \quad \beta \quad + \quad i \quad 0 \quad \alpha \quad \beta \quad + \quad i \quad 0 \quad \alpha \quad \beta \qquad \text{(A.6)}$$

The first permutation of indices generates

$$
\begin{array}{lll}
i \quad 0 \quad \alpha \quad \beta : & g_{i0,\alpha\beta} & g_{\alpha\beta,i0} \\
 & g_{i0,R}\ g_{\alpha\beta,S}\ g^{RS} & \\
 & g_{Ri,0}\ g_{S\alpha,\beta}\ g^{RS} & \\
 & g_{R0,i}\ g_{S\alpha,\beta}\ g^{RS} &
\end{array}
$$

The scalings of the metric tensor combinations are

$$g_{\alpha\beta,i0} \sim g_s \left\{ -\frac{5}{3} + \beta_0 + \gamma \right\}, \qquad g_{i0,\alpha\beta} \sim 0$$

$$g_{i0,R}\ g_{\alpha\beta,S}\ g^{RS} \sim 0,$$

$$g_{R0,i}\ g_{S\alpha,\beta}\ g^{RS} \sim 0,$$

$$g_{Ri,0}\ g_{S\alpha,\beta}\ g^{RS} \sim 0.$$

The second permutation of indices:

$$
\begin{array}{lll}
i \quad 0 \quad \alpha \quad \beta : & g_{i\alpha,0\beta} & g_{0\beta,i\alpha} \\
 & g_{i\alpha,R}\ g_{0\beta,S}\ g^{RS} & \\
 & g_{Ri,\alpha}\ g_{S0,\beta}\ g^{RS} &
\end{array}
$$

with the following scalings

$$g_{i\alpha,0\beta} \sim 0, \qquad g_{0\beta,i\alpha} \sim 0,$$

$$g_{i\alpha,R}\ g_{0\beta,S}\ g^{RS} \sim 0,$$

$$g_{Ri,\alpha}\ g_{S0,\beta}\ g^{RS} \sim 0.$$

And the last index permutation is the same as the second one:

$$g_s \left\{ i \quad 0 \quad \alpha \quad \beta \right\} \qquad \sim \qquad g_s \left\{ i \quad 0 \quad \alpha \quad \beta \right\} \sim 0.$$

The full set of scalings is

$$R_{\alpha\beta i0} \sim g_s \left\{ -\frac{5}{3} + \beta_0 + \gamma \right\}.$$

$$\boxed{\mathbf{R_{m\alpha\beta0}}}: \quad m \quad \alpha \quad \beta \quad 0 \quad + \quad m \quad \alpha \quad \beta \quad 0 \quad + \quad m \quad \alpha \quad \beta \quad 0 \tag{A.7}$$

The first permutation of indices generates

$$m \quad \alpha \quad \beta \quad 0: \qquad g_{m\alpha,\beta0} \qquad\qquad g_{\beta0,m\alpha}$$

$$g_{m\alpha,R}\ g_{\beta0,S}\ g^{RS}$$

$$g_{Rm,\alpha}\ g_{S\beta,0}\ g^{RS}$$

$$g_{R\alpha,m}\ g_{S\beta,0}\ g^{RS}$$

The scalings of the metric tensor combinations are

$$g_{m\alpha,\beta0} \sim 0, \qquad g_{\beta0,m\alpha} \sim 0,$$

$$g_{m\alpha,R}\ g_{\beta0,S}\ g^{RS} \sim 0,$$

$$g_{Rm,\alpha}\ g_{S\beta,0}\ g^{RS} \sim 0,$$

$$g_{R\alpha,m}\ g_{S\beta,0}\ g^{RS} = g_{\beta\alpha,m}\ g_{\alpha\beta,0}\ g^{\alpha\beta} \sim g_s \left\{-\frac{5}{3} + \beta_0 + \gamma\right\}$$

The second index permutation is the same as the third one:

$$g_s \left\{ m \quad \alpha \quad \beta \quad 0 \right\} \qquad \sim \qquad g_s \left\{ m \quad \alpha \quad \beta \quad 0 \right\}.$$

The last permutation of indices generates the scaling

$$m \quad \alpha \quad \beta \quad 0: \qquad g_{m0,\alpha\beta} \qquad\qquad g_{\alpha\beta,m0}$$

$$g_{m0,R}\ g_{\alpha\beta,S}\ g^{RS}$$

$$g_{Rm,0}\ g_{S\alpha,\beta}\ g^{RS}$$

with the following scalings

$$g_{m0,\alpha\beta} \sim 0, \qquad g_{\alpha\beta,m0} \sim g_s \left\{-\frac{5}{3} + \beta_0 + \gamma\right\},$$

$$g_{m0,R}\ g_{\alpha\beta,S}\ g^{RS} \sim 0,$$

$$g_{Rm,0}\; g_{S\alpha,\beta}\; g^{RS} \sim 0.$$

The full set of scalings is

$$R_{m\alpha\beta 0} \sim g_s \left\{ -\frac{5}{3} + \beta_0 + \gamma \right\}.$$

$$\boxed{\mathbf{R_{0\alpha\alpha\beta}}} : \quad 0 \;\; \alpha \;\; \alpha \;\; \beta \quad + \quad 0 \;\; \alpha \;\; \alpha \;\; \beta \quad + \quad 0 \;\; \alpha \;\; \alpha \;\; \beta \tag{A.8}$$

The first permutation of indices generates

$$0 \;\; \alpha \;\; \alpha \;\; \beta : \qquad g_{0\alpha,\alpha\beta} \qquad g_{\alpha\beta,0\alpha}$$

$$g_{0\alpha,R}\; g_{\alpha\beta,S}\; g^{RS}$$

$$g_{R\alpha,0}\; g_{S\alpha,\beta}\; g^{RS}$$

The scalings of the metric tensor combinations are

$$g_{0\alpha,\alpha\beta} \sim 0, \qquad g_{\alpha\beta,0\alpha} \sim g_s \left\{ -\frac{5}{3} + \beta_0 + \gamma \right\},$$

$$g_{0\alpha,R}\; g_{\alpha\beta,S}\; g^{RS} \sim 0,$$

$$g_{R\alpha,0}\; g_{S\alpha,\beta}\; g^{RS} = g_{\beta\alpha,0}\; g_{\beta\alpha,\beta}\; g^{\beta\beta} \sim g_s \left\{ -\frac{5}{3} + \beta_0 + \gamma \right\},$$

The second index permutation is the same as the first one:

$$g_s \left\{ 0 \;\; \alpha \;\; \alpha \;\; \beta \right\} \quad \sim \quad g_s \left\{ 0 \;\; \alpha \;\; \alpha \;\; \beta \right\} \sim g_s \left\{ -\frac{5}{3} + \beta_0 + \gamma \right\}.$$

The last permutation of indices generates the scaling

$$0 \;\; \alpha \;\; \alpha \;\; \beta : \qquad g_{0\beta,\alpha\alpha} \qquad g_{\alpha\alpha,0\beta}$$

$$g_{m\beta,R}\; g_{\alpha\alpha,S}\; g^{RS}$$

$$g_{R\beta,0}\; g_{S\alpha,\alpha}\; g^{RS}$$

with the following scalings

$$g_{0\beta,\alpha\alpha} \sim 0, \qquad g_{\alpha\alpha,0\beta} \sim 0,$$

$$g_{m\beta,R}\ g_{\alpha\alpha,S}\ g^{RS} \sim 0,$$

$$g_{R\beta,0}\ g_{S\alpha,\alpha}\ g^{RS} = g_{\alpha\beta,0}\ g_{\beta\alpha,\alpha}\ g^{\alpha\beta} \sim g_s\left\{-\frac{5}{3}+\beta_0+\gamma\right\},$$

The full set of scalings is

$$R_{0\alpha\alpha\beta} \sim g_s\left\{-\frac{5}{3}+\beta_0+\gamma\right\}.$$

$$\boxed{\mathbf{R}_{\alpha\beta\alpha\beta}}: \quad \alpha\ \ \beta\ \ \alpha\ \ \beta \quad + \quad \alpha\ \ \beta\ \ \alpha\ \ \beta \quad + \quad \alpha\ \ \beta\ \ \alpha\ \ \beta \tag{A.9}$$

The first permutation of indices generates

$$\alpha\ \ \beta\ \ \alpha\ \ \beta: \quad g_{0\alpha\beta,\alpha\beta} \qquad g_{\alpha\beta,\alpha\alpha}$$
$$g_{\alpha\beta,R}\ g_{\alpha\beta,S}\ g^{RS}$$
$$g_{R\alpha,\beta}\ g_{S\alpha,\beta}\ g^{RS}$$

The scalings of the metric tensor combinations are

$$g_{0\alpha\beta,\alpha\beta} \sim \left\{-\frac{2}{3}+\beta_0\right\},$$

$$g_{\alpha\beta,R}\ g_{\alpha\beta,S}\ g^{RS} = g_{\alpha\beta,0}\ g_{\alpha\beta,}\ g^{00} + g_{\alpha\beta,i}\ g_{\alpha\beta,j}\ g^{ij} + g_{\alpha\beta,\alpha}\ g_{\alpha\beta,\beta}\ g^{\alpha\beta} + g_{\alpha\beta,m}\ g_{\alpha\beta,n}\ g^{mn}$$
$$\sim g_s\left\{-\frac{2}{3}+2\beta_0+2\gamma\right\} + g_s\left\{\frac{4}{3}+2\beta_0\right\} + g_s\left\{-\frac{2}{3}+\beta_0\right\} + g_s\left\{-\frac{2}{3}+2\beta_0-\alpha\right\}$$
$$g_{R\alpha,\beta}\ g_{S\alpha,\beta}\ g^{RS} = g_{\beta\alpha,\beta}\ g_{\beta\alpha,\beta}\ g^{\alpha\beta} \sim g_s\left\{-\frac{2}{3}+\beta_0\right\},$$

The second and the third index permutations are the same as the first one:

$$g_s\left\{\alpha\ \ \beta\ \ \alpha\ \ \beta\right\} \quad \sim \quad g_s\left\{\alpha\ \ \beta\ \ \alpha\ \ \beta\right\}$$
$$g_s\left\{\alpha\ \ \beta\ \ \alpha\ \ \beta\right\} \quad \sim \quad g_s\left\{\alpha\ \ \beta\ \ \alpha\ \ \beta\right\}.$$

The full set of scalings is

$$R_{\alpha\beta\alpha\beta} \sim g_s\left\{-\frac{2}{3}+2\beta_0+2\gamma,\ \frac{4}{3}+2\beta_0,\ -\frac{2}{3}+\beta_0,\ -\frac{2}{3}+2\beta_0-\alpha\right\}.$$

$$\boxed{\mathbf{R_{mnp0}}}: \quad m \quad n \quad p \quad 0 \quad + \quad m \quad m \quad p \quad 0 \quad + \quad m \quad m \quad p \quad 0 \tag{A.10}$$

Each particular index permutation generates the unique contractions of the first and second derivatives of the metric tensor:

$$m \quad n \quad p \quad 0: \qquad g_{mn,p0} \qquad\qquad g_{p0,mn}$$

$$g_{mn,R}\; g_{p0,S}\; g^{RS}$$

$$g_{Rm,n}\; g_{Sp,0}\; g^{RS}$$

Now we can easily compute the scalings for each combination of the metric tensors and their derivatives:

$$g_{mn,p0} \sim g_s \left\{-\frac{5}{3} + \alpha_0 + \gamma\right\},$$

$$g_{p0,mn} \sim 0,$$

$$g_{mn,R}\; g_{p0,S}\; g^{RS} \sim 0$$

$$g_{Rm,n}\; g_{Sp,0}\; g^{RS} = g_{pm,n}\; g_{qp,0}\; g^{pq} \sim g_s \left\{-\frac{5}{3} + \alpha_0 + \gamma\right\},$$

Both the second and the third permutation of indices generates the same g_s scalings:

$$g_s \left\{ m \quad m \quad p \quad 0 \right\} \qquad \sim \qquad g_s \left\{ m \quad n \quad p \quad 0 \right\}$$

$$g_s \left\{ m \quad m \quad p \quad 0 \right\} \qquad \sim \qquad g_s \left\{ m \quad n \quad p \quad 0 \right\}$$

Finally, we find the scalings of the R_{nmpq} Riemann tensor to be

$$R_{nmp0} \sim g_s \left\{-\frac{5}{3} + \alpha_0 + \gamma\right\}.$$

$\boxed{\mathbf{R_{mni0}}}$ is equivalent to $R_{\alpha\beta i0}$ and has the same g_s scaling:

$$R_{mnp0} \sim R_{\alpha\beta i0} \sim g_s \left\{-\frac{5}{3} + \alpha_0 + \gamma\right\}. \tag{A.11}$$

$\boxed{\mathbf{R_{mn\alpha 0}}}$ is equivalent to $R_{m\alpha\beta 0}$ and has the same g_s scaling:

$$R_{mn\alpha 0} \sim R_{m\alpha\beta 0} \sim g_s \left\{-\frac{5}{3} + \alpha_0 + \gamma\right\}. \tag{A.12}$$

$$\boxed{\mathbf{R_{mnab}}} : \quad m \quad n \quad a \quad b \quad + \quad m \quad n \quad a \quad b \quad + \quad m \quad n \quad a \quad b \tag{A.13}$$

Each particular index permutation generates the unique contractions of the first and second derivatives of the metric tensor:

$$m \quad n \quad a \quad b : \qquad g_{mn,ab} \qquad\qquad g_{ab,mn}$$
$$g_{mn,R}\ g_{ab,S}\ g^{RS}$$
$$g_{Rm,n}\ g_{Sa,b}\ g^{RS}$$

Now we can easily compute the scalings for each combination of the metric tensors and their derivatives:

$$g_{mn,ab} \sim 0,$$
$$g_{ab,mn} \sim g_s \left\{ \frac{4}{3} \right\},$$
$$g_{mn,R}\ g_{ab,S}\ g^{RS} = g_{mn,0}\ g_{ab,0}\ g^{00} + g_{mn,i}\ g_{ab,j}\ g^{ij} + g_{mn,\alpha}\ g_{ab,\beta}\ g^{\alpha\beta} + g_{mn,r}\ g_{ab,s}\ g^{rs}$$
$$\sim g_s \left\{ \frac{4}{3} + \alpha_0 + 2\gamma_0 \right\} + g_s \left\{ \frac{10}{3} + \alpha_0 \right\} + g_s \left\{ \frac{4}{3} + \alpha_0 - \beta_0 \right\} + g_s \left\{ \frac{4}{3} \right\},$$
$$g_{Rm,n}\ g_{Sa,b}\ g^{RS} \sim 0.$$

The second index permutation gives

$$m \quad n \quad a \quad b : \qquad g_{ma,nb} \qquad\qquad g_{nb,ma}$$
$$g_{ma,R}\ g_{nb,S}\ g^{RS}$$
$$g_{Rm,a}\ g_{Sn,b}\ g^{RS}$$
$$g_{Ra,m}\ g_{Sb,n}\ g^{RS}$$

Then

$$g_{ma,nb} \sim g_{nb,ma} \sim 0,$$
$$g_{ma,R}\ g_{nb,S}\ g^{RS} \sim 0,$$
$$g_{Rm,a}\ g_{Sn,b}\ g^{RS} \sim 0,$$

$$g_{Ra,m}\, g_{Sb,n}\, g^{RS} = g_{ba,m}\, g_{ab,n}\, g^{ab} \sim g_s \left\{ \frac{4}{3} \right\}.$$

The third permutation is equivalent to the second one:

$$g_s \left\{ m \quad n \quad a \quad b \right\} \qquad \sim \qquad g_s \left\{ m \quad n \quad a \quad b \right\} \sim g_s \left\{ \frac{4}{3} \right\}.$$

Finally, we find the scalings of the R_{nmpq} Riemann tensor to be

$$R_{nmab} \sim g_s \left\{ \frac{4}{3} + \alpha_0 + 2\gamma_0, \frac{10}{3} + \alpha_0, \frac{4}{3} + \alpha_0 - \beta_0, \frac{4}{3} \right\}.$$

$$\boxed{\mathbf{R_{m\alpha\beta i}}} : \qquad m \quad \alpha \quad \beta \quad i \qquad + \qquad m \quad \alpha \quad \beta \quad i \qquad + \qquad m \quad \alpha \quad \beta \quad i \tag{A.14}$$

The first permutation of indices generates

$$m \quad \alpha \quad \beta \quad i : \qquad g_{m\alpha,\beta i} \qquad\qquad g_{\beta 0,m\alpha}$$

$$g_{m\alpha,R}\, g_{\beta i,S}\, g^{RS}$$

$$g_{Rm,\alpha}\, g_{S\beta,i}\, g^{RS}$$

$$g_{R\alpha,m}\, g_{S\beta,i}\, g^{RS}$$

The scalings of the metric tensor combinations are

$$g_{m\alpha,\beta i} \sim 0, \qquad g_{\beta i,m\alpha} \sim 0,$$

$$g_{m\alpha,R}\, g_{\beta i,S}\, g^{RS} \sim 0,$$

$$g_{Rm,\alpha}\, g_{S\beta,i}\, g^{RS} \sim 0,$$

$$g_{R\alpha,m}\, g_{S\beta,i}\, g^{RS} = g_{\beta\alpha,m}\, g_{\alpha\beta,i}\, g^{\alpha\beta} \sim g_s \left\{ -\frac{2}{3} + \beta_0 \right\}$$

The second index permutation is the same as the first one:

$$g_s \left\{ m \quad \alpha \quad \beta \quad i \right\} \qquad \sim \qquad g_s \left\{ m \quad \alpha \quad \beta \quad i \right\}.$$

The last permutation of indices generates the scaling

$$
\begin{array}{lll}
m \quad \alpha \quad \beta \quad i: & g_{mi,\alpha\beta} & g_{\alpha\beta,mi} \\
 & g_{mi,R}\; g_{\alpha\beta,S}\; g^{RS} & \\
 & g_{Rm,i}\; g_{S\alpha,\beta}\; g^{RS} &
\end{array}
$$

with the following scalings

$$g_{mi,\alpha\beta} \sim 0, \qquad g_{\alpha\beta,mi} \sim g_s \left\{ -\frac{2}{3} + \beta_0 \right\},$$

$$g_{mi,R}\; g_{\alpha\beta,S}\; g^{RS} \sim 0,$$

$$g_{Rm,i}\; g_{S\alpha,\beta}\; g^{RS} \sim 0.$$

The full set of scalings is

$$R_{m\alpha\beta i} \sim g_s \left\{ -\frac{2}{3} + \beta_0 \right\}.$$

The $\boxed{\mathbf{R}_{\mathbf{i}\alpha\beta}}$ is very similar to $R_{0\alpha\alpha\beta}$ with the difference that since there is no time differentiation, we don't have a γ scale dependence and thus

$$R_{i\alpha\alpha\beta} \sim R_{0\alpha\alpha\beta} \sim g_s \left\{ -\frac{2}{3} + \beta_0 \right\}.$$

Very similarly, the $\boxed{\mathbf{R}_{\mathbf{m}\alpha\beta}}$ gives similar scaling as $R_{0\alpha\alpha\beta}$ without time derivatives and changes in the main metric scaling:

$$R_{m\alpha\alpha\beta} \sim R_{0\alpha\alpha\beta} \sim g_s \left\{ -\frac{2}{3} + \beta_0 \right\}.$$

The scaling of $\boxed{\mathbf{R}_{\mathbf{mnpi}}}$ we can find by comparing to the previously computed scaling of R_{mnp0} again taking into account that the absence of the time derivative will remove the γ dependence and lift up the total scaling by $+1$:

$$R_{mnpi} \sim R_{mnp0} \sim g_s \left\{ -\frac{2}{3} + \beta_0 \right\}.$$

The scaling of $\boxed{\mathbf{R_{mnp\alpha}}}$ is absolutely the same since in this case i and α are equivalent:

$$R_{mnp\alpha} \sim R_{mnpi} \sim g_s \left\{ -\frac{2}{3} + \beta_0 \right\}.$$

The scaling of $\boxed{\mathbf{R_{mn\alpha i}}}$ can be deduced in a similar manner from the $R_{mn\alpha 0}$ scaling:

$$R_{mn\alpha i} \sim R_{mn\alpha 0} \sim g_s \left\{ -\frac{2}{3} + \beta_0 \right\}.$$

$$\boxed{\mathbf{R_{mabi}}} : \quad m \quad a \quad b \quad i \quad + \quad m \quad a \quad b \quad i \quad + \quad m \quad a \quad b \quad i \tag{A.15}$$

Each particular index permutation generates the unique contractions of the first and second derivatives of the metric tensor:

$$m \quad a \quad b \quad i : \qquad g_{ma,bi} \qquad\qquad g_{bi,ma}$$
$$g_{ma,R}\ g_{bi,S}\ g^{RS}$$
$$g_{Rm,a}\ g_{Sb,i}\ g^{RS}$$
$$g_{Ra,m}\ g_{Sb,i}\ g^{RS}$$

Now we can easily compute the scalings for each combination of the metric tensors and their derivatives:

$$g_{ma,bi} \sim 0, \qquad g_{bi,ma} \sim 0,$$

$$g_{ma,R}\ g_{bi,S}\ g^{RS} \sim 0,$$

$$g_{Rm,a}\ g_{Sb,i}\ g^{RS} \sim 0,$$
$$g_{Ra,m}\ g_{Sb,i}\ g^{RS} = g_{ba,m}\ g_{ab,i}\ g^{ab} \sim g_s \left\{ \frac{4}{3} \right\}.$$

The second permutation of indices generates the same g_s scaling as the first one:

$$g_s \left\{ m \quad a \quad b \quad i \right\} \qquad \sim \qquad g_s \left\{ m \quad a \quad b \quad i \right\}$$

The third permutation again gives the same scaling:

$$m \quad a \quad b \quad i: \qquad g_{ab,mi} \qquad\qquad g_{mi,ab}$$

$$g_{ab,R}\; g_{mi,S}\; g^{RS}$$

$$g_{Ra,b}\; g_{Sm,i}\; g^{RS}$$

The only non-zero scaling comes from

$$g_{ab,mi} \sim g_s \left\{ \frac{4}{3} \right\}.$$

Finally, we find the scaling of the R_{mabi} Riemann tensor to be

$$R_{mabi} \sim g_s \left\{ \frac{4}{3} \right\}.$$

The scaling of the $\boxed{\mathbf{R_{\alpha abi}}}$, in this case, is the same as the scaling of the above tensor R_{mabi} since α and m are interchangeable in the previous calculations:

$$R_{\alpha abi} \sim R_{mabi} \sim g_s \left\{ \frac{4}{3} \right\}.$$

After some thinking, we agree that the scaling of the $\boxed{\mathbf{R_{m\alpha ab}}}$, in this case, is also the same as the scaling of the two above tensors since only the indices a and b play the crucial role in the calculations when the other two indices are different and not 0:

$$R_{m\alpha ab} \sim R_{\alpha abi} \sim R_{mabi} \sim g_s \left\{ \frac{4}{3} \right\}.$$

$$\boxed{\mathbf{R_{\alpha\beta ab}}}: \qquad \alpha \quad \beta \quad a \quad b \qquad + \qquad \alpha \quad \beta \quad a \quad b \qquad + \qquad \alpha \quad \beta \quad a \quad b \tag{A.16}$$

Each particular index permutation generates the unique contractions of the first and second derivatives of the metric tensor:

$$\alpha \quad \beta \quad a \quad b: \qquad g_{\alpha\beta,ab} \qquad\qquad g_{ab,\alpha\beta}$$

$$g_{\alpha\beta,R}\; g_{ab,S}\; g^{RS}$$

$$g_{R\alpha,\beta}\; g_{Sa,b}\; g^{RS}$$

Now we can easily compute the scalings for each combination of the metric tensors and their derivatives:

$$g_{\alpha\beta,ab} \sim 0, \qquad g_{ab,\alpha\beta} \sim g_s \left\{ \frac{4}{3} \right\}.$$

$$g_{\alpha\beta,R}\; g_{ab,S}\; g^{RS} = g_{\alpha\beta,0}\; g_{ab,0}\; g^{00} + g_{\alpha\beta,i}\; g_{ab,j}\; g^{ij} + g_{\alpha\beta,\alpha}\; g_{ab,\beta}\; g^{\alpha\beta} + g_{\alpha\beta,m}\; g_{ab,n}\; g^{mn}$$

$$g_s \left\{ \frac{4}{3} + \beta_0 + 2\gamma \right\} + g_s \left\{ \frac{10}{3} + \beta_0 \right\} + g_s \left\{ \frac{4}{3} \right\} + g_s \left\{ \frac{4}{3} + \beta_0 - \alpha_0 \right\},$$

$$g_{R\alpha,\beta}\; g_{Sa,b}\; g^{RS} \sim 0.$$

The second permutation of indices generates only one non-zero scaling:

$$\alpha \quad \beta \quad a \quad b :$$

$$g_{Ra,\alpha}\; g_{Sb,\beta}\; g^{RS} = g_{ba,\alpha}\; g_{ab,\beta}\; g^{ab} \sim g_s \left\{ \frac{4}{3} \right\}.$$

The third permutation again gives the same scaling as the second:

$$g_s \left\{ \alpha \quad \beta \quad a \quad b \right\} \qquad \sim \qquad g_s \left\{ \alpha \quad \beta \quad a \quad b \right\} \sim g_s \left\{ \frac{4}{3} \right\}.$$

Finally, we find the scaling of the $R_{\alpha\beta ab}$ Riemann tensor to be

$$R_{\alpha\beta ab} \sim g_s \left\{ \frac{4}{3} + \beta_0 + 2\gamma, \frac{10}{3} + \beta_0, \frac{4}{3}, \frac{4}{3} + \beta_0 - \alpha_0 \right\}.$$

$$\boxed{\mathbf{R_{abab}}} : \qquad a \quad b \quad a \quad b \qquad + \qquad a \quad b \quad a \quad b \qquad + \qquad a \quad b \quad a \quad b \qquad \text{(A.17)}$$

Each particular index permutation generates the unique contractions of the first and second derivatives of the metric tensor:

$$a \quad b \quad a \quad b : \qquad g_{ab,ab} \qquad\qquad g_{ab,ab}$$

$$g_{ab,R}\; g_{ab,S}\; g^{RS}$$

$$g_{Ra,b}\; g_{Sa,b}\; g^{RS}$$

Now we can easily compute the scalings for each combination of the metric tensors and their derivatives:

$$g_{ab,ab} \sim 0,$$

$$g_{ab,R}\; g_{ab,S}\; g^{RS} = g_{ab,0}\; g_{ab,0}\; g^{00} + g_{ab,i}\; g_{ab,j}\; g^{ij} + g_{ab,\alpha}\; g_{ab,\beta}\; g^{\alpha\beta} + g_{ab,m}\; g_{ab,n}\; g^{mn}$$
$$g_s\left\{\frac{10}{3}+2\gamma\right\} + g_s\left\{\frac{16}{3}\right\} + g_s\left\{\frac{10}{3}-\beta_0\right\} + g_s\left\{\frac{10}{3}-\alpha_0\right\},$$

$$g_{R\alpha,\beta}\; g_{Sa,b}\; g^{RS} \sim 0.$$

The rest of the permutations generate the same scaling dependence. Thus, we find the scaling of the R_{abab} Riemann tensor to be

$$R_{abab} \sim g_s\left\{\frac{10}{3}+2\gamma, \frac{16}{3}, \frac{10}{3}-\beta_0, \frac{10}{3}-\alpha_0\right\}.$$

$$\boxed{\mathbf{R_{0mab}}}: \quad 0 \;\; m \;\; a \;\; b \quad + \quad 0 \;\; m \;\; a \;\; b \quad + \quad 0 \;\; m \;\; a \;\; b \tag{A.18}$$

Each particular index permutation generates the unique contractions of the first and second derivatives of the metric tensor:

$$0 \;\; m \;\; a \;\; b: \quad g_{0m,ab} \qquad g_{ab,0m}$$
$$g_{0m,R}\; g_{ab,S}\; g^{RS}$$
$$g_{R0,m}\; g_{Sa,b}\; g^{RS}$$

The only meaningful combination is

$$g_{ab,0m} \sim g_s\left\{\frac{4}{3}-1+\gamma\right\} \sim g_s\left\{\frac{1}{3}+\gamma\right\}.$$

The second permutation generates the same scaling dependence:

$$0 \;\; m \;\; a \;\; b: \quad g_{0a,mb} \qquad g_{mb,0a}$$
$$g_{0a,R}\; g_{mb,S}\; g^{RS}$$
$$g_{Ra,0}\; g_{Sb,m}\; g^{RS}$$

with the only fruitful combination

$$g_{Ra,0}\; g_{Sb,m}\; g^{RS} = g_{ba,0}\; g_{ab,m}\; g^{ab} \sim g_s\left\{\frac{1}{3}+\gamma\right\}.$$

The last possible permutation is the same as the second one.

Thus, we find the scaling of the R_{0mab} Riemann tensor to be

$$R_{0mab} \sim g_s \left\{ \frac{1}{3} + \gamma \right\}.$$

The scaling of the $\boxed{\mathbf{R_{abi0}}}$ can be easily obtained from the scaling of the above calculated R_{0mab} tensor by the simple replacement i into m:

$$R_{abi0} \sim R_{0mab} \sim g_s \left\{ \frac{1}{3} + \gamma \right\}.$$

The next scaling of the $\boxed{\mathbf{R_{0\alpha ab}}}$ is obtained from the scaling of the above calculated R_{0mab} tensor by the replacement α into m:

$$R_{0\alpha ab} \sim R_{0mab} \sim g_s \left\{ \frac{1}{3} + \gamma \right\}.$$

$$\boxed{\mathbf{R_{m0ij}}} : \quad m \quad 0 \quad i \quad j \quad + \quad m \quad 0 \quad i \quad j \quad + \quad m \quad 0 \quad i \quad j \tag{A.19}$$

The first permutation of indices generates

$$m \quad 0 \quad i \quad j : \qquad g_{m0,ij} \qquad\qquad g_{ij,m0}$$

$$g_{m0,R}\; g_{ij,S}\; g^{RS}$$

$$g_{Rm,0}\; g_{Si,j}\; g^{RS}$$

The non-zero scaling of the metric tensor combinations is

$$g_{ij,m0} \sim g_s \left\{ -\frac{11}{3} + \gamma \right\}$$

The second index permutation gives the same scaling as the first one:

$$m \quad 0 \quad i \quad j : \qquad g_{mi,0j} \qquad\qquad g_{0j,mi}$$

$$g_{mi,R}\; g_{0j,S}\; g^{RS}$$

$$g_{Rm,i}\; g_{S0,j}\; g^{RS}$$

The only survived scaling is

$$g_{Ri,m}\; g_{Sj,0}\; g^{RS} = g_{ki,m}\; g_{ij,0}\; g^{ik} \sim g_s \left\{ -\frac{11}{3} + \gamma \right\}$$

The last permutation of indices is the same as the second.

The full set of scalings is simply

$$R_{m0ij} \sim g_s \left\{ -\frac{11}{3} + \gamma \right\}.$$

As long as we keep the pair ij and 0 among the indices and the last index different from the above, we obtain the same scaling. Thus, we have the same scaling dependence for $\boxed{\mathbf{R_{ijk0}}}$:

$$R_{ijk0} \sim R_{m0ij} \sim g_s \left\{ -\frac{11}{3} + \gamma \right\}.$$

As well as for the $\boxed{\mathbf{R_{\alpha 0ij}}}$:

$$R_{\alpha 0ij} \sim R_{m0ij} \sim g_s \left\{ -\frac{11}{3} + \gamma \right\}.$$

$$\boxed{\mathbf{R_{m0i0}}} : \quad m \;\; 0 \;\; i \;\; 0 \quad + \quad m \;\; 0 \;\; i \;\; 0 \quad + \quad m \;\; 0 \;\; i \;\; 0 \tag{A.20}$$

The first permutation of indices generates

$$\begin{array}{lll} m \;\; 0 \;\; i \;\; 0 : & g_{m0,i0} & g_{i0,m0} \\ & g_{m0,R}\; g_{i0,S}\; g^{RS} & \\ & g_{Rm,0}\; g_{Si,0}\; g^{RS} & \end{array}$$

The non-zero scaling of the metric tensor combinations is

$$g_{R0,m}\; g_{S0,i}\; g^{RS} = g_{00,m}\; g_{00,i}\; g^{RS} \sim g_s \left\{ -\frac{8}{3} \right\}.$$

The second index permutation gives the same scaling as the first one:

$$m \quad 0 \quad i \quad 0: \qquad g_{mi,00} \qquad\qquad g_{00,mi}$$
$$g_{mi,R}\; g_{00,S}\; g^{RS}$$
$$g_{Rm,i}\; g_{S0,0}\; g^{RS}$$

The only surviving scaling is

$$g_{00,mi} \sim g_s \left\{-\frac{8}{3}\right\}$$

The last permutation of indices is the same as the first one.

The scaling of the Riemann tensor is

$$R_{m0i0} \sim g_s \left\{-\frac{8}{3}\right\}.$$

The variation of the last tensor is $\boxed{\mathbf{R_{0\alpha i0}}}$ which has the same scaling:

$$R_{0\alpha i0} \sim R_{m0i0} \sim g_s \left\{-\frac{8}{3}\right\}.$$

The scaling of R_{mijk} coincides with the two previous ones:

$$\boxed{\mathbf{R_{mijk}}}: \qquad m \quad i \quad j \quad k \qquad + \qquad m \quad i \quad j \quad k \qquad + \qquad m \quad i \quad j \quad k \tag{A.21}$$

The first permutation of indices generates

$$m \quad i \quad j \quad k: \qquad g_{mi,jk} \qquad\qquad g_{jk,mi}$$
$$g_{mi,R}\; g_{jk,S}\; g^{RS}$$
$$g_{Rm,i}\; g_{Sj,k}\; g^{RS}$$

The non-zero scaling of the metric tensor combinations are

$$g_{jk,mi} \sim g_s \left\{-\frac{8}{3}\right\}. \qquad g_{Ri,m}\; g_{Sj,k}\; g^{RS} = g_{ki,m}\; g_{ij,k}\; g^{ik} \sim g_s \left\{-\frac{8}{3}\right\}.$$

Both the second and the third index permutations give the same scaling as the first one.

The scaling of the Riemann tensor is

$$R_{mijk} \sim g_s \left\{-\frac{8}{3}\right\}.$$

The next Riemann tensor, $\boxed{\mathbf{R_{\alpha ijk}}}$, has the same scaling dependence as the R_{mijk}, as can be easily figured out by interchanging α and m:

$$R_{\alpha ijk} \sim R_{mijk} \sim g_s \left\{ -\frac{8}{3} \right\}.$$

If we look closely, we find that the next Riemann tensor, $\boxed{\mathbf{R_{m\alpha ij}}}$, has the same scaling dependence as the R_{mijk}, since the contribution to the scaling comes from the ij couple and the $m\alpha$ index contribution does not influence the final result:

$$R_{m\alpha ij} \sim R_{mijk} \sim g_s \left\{ -\frac{8}{3} \right\}.$$

The final Riemann tensor in this family, $\boxed{\mathbf{R_{0m0\alpha}}}$, is actually secretly the R_{m0i0} Riemann tensor (at least it has the same scaling) which can be seen from their index content:

$$R_{0m0\alpha} \sim R_{m0i0} \sim g_s \left\{ -\frac{8}{3} \right\}.$$

$$\boxed{\mathbf{R_{ijij}}} : \quad i \;\; j \;\; i \;\; j \quad + \quad i \;\; j \;\; i \;\; j \quad + \quad i \;\; j \;\; i \;\; j \tag{A.22}$$

The first permutation of indices generates

$$\begin{aligned} i \;\; j \;\; i \;\; j : \quad & g_{ij,ij} \qquad\qquad g_{ij,ij} \\ & g_{ij,R}\; g_{ij,S}\; g^{RS} \\ & g_{Ri,j}\; g_{Si,j}\; g^{RS} \end{aligned}$$

The non-zero scaling of the metric tensor combinations are

$$g_{ij,ij} \sim g_s \left\{ -\frac{8}{3} \right\}.$$

$$\begin{gathered} g_{ij,R}\; g_{ij,S}\; g^{RS} = g_{ij,0}\; g_{ij,0}\; g^{00} + g_{ij,k}\; g_{ij,j}\; g^{kj} + g_{ij,\alpha}\; g_{ij,\beta}\; g^{\alpha\beta} + g_{ij,m}\; g_{ij,n}\; g^{mn} \\ \sim g_s \left\{ -\frac{14}{3} + 2\gamma \right\} + g_s \left\{ -\frac{8}{3} \right\} + g_s \left\{ -\frac{14}{3} - \beta_0 \right\} + g_s \left\{ -\frac{14}{3} - \alpha_0 \right\}, \\ g_{Ri,j}\; g_{Si,j}\; g^{RS} = g_{ki,j}\; g_{ji,j}\; g^{kj} \sim g_s \left\{ -\frac{8}{3} \right\}. \end{gathered}$$

Both the second and the third index permutations give the same scaling as the first one.

The scaling of the Riemann tensor is

$$R_{ijij} \sim g_s \left\{ -\frac{14}{3} + 2\gamma, -\frac{8}{3}, -\frac{14}{3} - \beta_0, -\frac{14}{3} - \alpha_0 \right\}.$$

$$\boxed{\mathbf{R_{0i0j}}} : \quad 0 \;\; i \;\; 0 \;\; j \quad + \quad 0 \;\; i \;\; 0 \;\; j \quad + \quad 0 \;\; i \;\; 0 \;\; j \tag{A.23}$$

The first permutation of indices generates

$$0 \;\; i \;\; 0 \;\; j : \qquad g_{0i,0j} \qquad\qquad g_{0j,0i}$$

$$g_{0i,R}\; g_{0j,S}\; g^{RS}$$

$$g_{Ri,0}\; g_{Sj,0}\; g^{RS}$$

$$g_{R0,i}\; g_{S0,j}\; g^{RS}$$

The non-zero scaling of the metric tensor combinations are

$$g_{Ri,0}\; g_{Sj,0}\; g^{RS} = g_{ki,0}\; g_{ij,0}\; g^{ki} \sim g_s \left\{ -\frac{14}{3} + 2\gamma \right\},$$
$$g_{R0,i}\; g_{S0,j}\; g^{RS} = g_{00,i}\; g_{00,j}\; g^{00} \sim g_s \left\{ -\frac{8}{3} \right\}.$$

The second index permutation is the most prolific here:

$$0 \;\; i \;\; 0 \;\; j : \qquad g_{00,ij} \qquad\qquad g_{ij,00}$$

$$g_{00,R}\; g_{ij,S}\; g^{RS}$$

$$g_{R0,0}\; g_{Si,j}\; g^{RS}$$

The non-zero scaling of the metric tensor combinations are

$$g_{00,ij} \sim g_s \left\{ -\frac{8}{3} \right\}, \qquad g_{ij,00} \sim g_s \left\{ -\frac{14}{3} + 2\gamma \right\},$$

$$g_{00,R}\; g_{ij,S}\; g^{RS}$$
$$\sim g_s \left\{ -\frac{14}{3} + 2\gamma \right\} + g_s \left\{ -\frac{8}{3} \right\} + g_s \left\{ -\frac{14}{3} - \beta_0 \right\} + g_s \left\{ -\frac{14}{3} - \alpha_0 \right\},$$

The last permutation coincides with the first one.

The scaling of the Riemann tensor is

$$R_{0i0j} \sim g_s \left\{ -\frac{14}{3} + 2\gamma, -\frac{8}{3}, -\frac{14}{3} - \beta_0, -\frac{14}{3} - \alpha_0 \right\}.$$

$$\boxed{\mathbf{R_{abij}}} : \quad a \quad b \quad i \quad j \quad + \quad a \quad b \quad i \quad j \quad + \quad a \quad b \quad i \quad j \tag{A.24}$$

The first permutation of indices generates

$$\begin{array}{llll} a \quad b \quad i \quad j : & g_{ab,ij} & & g_{ij,ab} \\ & g_{ab,R}\ g_{ij,S}\ g^{RS} & & \\ & g_{Ra,b}\ g_{Si,j}\ g^{RS}. & & \end{array}$$

The non-zero scaling of the metric tensor combinations are

$$g_{ab,ij} \sim g_s \left\{ \frac{4}{3} \right\},$$

$$g_{ab,R}\ g_{ij,S}\ g^{RS} = g_{ab,0}\ g_{ij,0}\ g^{00} + g_{ab,i}\ g_{ij,k}\ g^{ik} + g_{ab,\alpha}\ g_{ij,\beta}\ g^{\alpha\beta} + g_{ab,m}\ g_{ij,n}\ g^{mn}$$
$$\sim g_s \left\{ -\frac{2}{3} + 2\gamma \right\} + g_s \left\{ \frac{4}{3} \right\} + g_s \left\{ -\frac{2}{3} - \beta_0 \right\} + g_s \left\{ -\frac{2}{3} - \alpha_{)} \right\}.$$

The second and the third index permutations are the same and give the non-zero scaling:

$$g_s \left\{ a \quad b \quad i \quad j \right\} \sim g_s \left\{ a \quad b \quad i \quad j \right\} \sim g_s \left\{ \frac{4}{3} \right\}.$$

The scaling of the Riemann tensor is

$$R_{abij} \sim g_s \left\{ -\frac{2}{3} + 2\gamma, \frac{4}{3}, -\frac{2}{3} - \beta_0, -\frac{2}{3} - \alpha_{)} \right\}.$$

The $\boxed{\mathbf{R_{0a0b}}}$ Riemann tensor has almost the same index structure as the R_{abij} tensor. The only elements which can potentially generate unexpected scaling are the following :

$$g_{ab,00} \sim g_s \left\{ -\frac{2}{3} + 2\gamma \right\}, \qquad g_{ab,R}\ g_{ij,S}\ g^{RS} \sim g_s \left\{ -\frac{2}{3} + 2\gamma \right\},$$

which are included in the main dependence sequence. Thus we have the same scaling structure:

$$R_{0a0b} \sim g_s \left\{ -\frac{2}{3} + 2\gamma, \frac{4}{3}, -\frac{2}{3} - \beta_0, -\frac{2}{3} - \alpha_{)} \right\}.$$

$$\boxed{\mathbf{R_{mnij}}}: \quad m \quad n \quad i \quad j \quad + \quad m \quad n \quad i \quad j \quad + \quad m \quad n \quad i \quad j \qquad \text{(A.25)}$$

The first permutation of indices generates

$$\begin{aligned} m \quad n \quad i \quad j: \quad & g_{mn,ij} \qquad\qquad g_{ij,mn} \\ & g_{mn,R}\ g_{ij,S}\ g^{RS} \\ & g_{Rm,n}\ g_{Si,j}\ g^{RS}. \end{aligned}$$

The non-zero scaling of the metric tensor combinations are

$$g_{mn,ij} \sim g_s \left\{ -\frac{2}{3} + \alpha_0 \right\}, \qquad g_{ij,mn} \sim g_s \left\{ \frac{4}{3} \right\},$$

$$g_{mn,R}\ g_{ij,S}\ g^{RS} = g_{mn,0}\ g_{ij,0}\ g^{00} + g_{mn,i}\ g_{ij,k}\ g^{ik} + g_{mn,\alpha}\ g_{ij,\beta}\ g^{\alpha\beta} + g_{mn,p}\ g_{ij,q}\ g^{pq}$$
$$\sim g_s \left\{ -\frac{8}{3} + \alpha_0 + 2\gamma \right\} + g_s \left\{ -\frac{2}{3} + \alpha_0 \right\} + g_s \left\{ -\frac{8}{3} + \alpha_0 - \beta_0 \right\} + g_s \left\{ -\frac{8}{3} \right\}.$$

The second index permutations give the non-zero scalings:

$$\begin{aligned} g_s \left\{ m \quad n \quad i \quad j \right\}: \quad & g_{mi,nj} \qquad\qquad g_{nj,mi} \\ & g_{mi,R}\ g_{nj,S}\ g^{RS}, \\ & g_{Rm,i}\ g_{Sn,j}\ g^{RS}. \end{aligned}$$

The non-zero scaling of the metric tensor combinations are

$$g_{Rm,i}\ g_{Sn,j}\ g^{RS} = g_{pm,i}\ g_{qn,j}\ g^{pq} \sim g_s \left\{ -\frac{2}{3} + \alpha_0 \right\}, \qquad g_{ik,m}\ g_{ij,n}\ g^{kj} \sim g_s \left\{ -\frac{8}{3} \right\}.$$

The third possible permutation is the same as the second and has the scale dependence.

The scaling of the Riemann tensor is

$$R_{mnij} \sim g_s \left\{ -\frac{8}{3} + \alpha_0 + 2\gamma, -\frac{2}{3} + \alpha_0, -\frac{8}{3} + \alpha_0 - \beta_0, -\frac{8}{3} \right\}.$$

Using the analogy with the previous tensors, we conclude that the Riemann tensor $\boxed{\mathbf{R_{0m0n}}}$ has the same scalings as the R_{mnij} tensor: The scaling of the Riemann tensor is

$$R_{0m0n} \sim R_{mnij} \sim g_s \left\{ -\frac{8}{3} + \alpha_0 + 2\gamma, -\frac{2}{3} + \alpha_0, -\frac{8}{3} + \alpha_0 - \beta_0, -\frac{8}{3} \right\}.$$

The scalings for the last couple of Riemann tensors can be guessed from the following considerations. The scale dependence of the $\boxed{\mathbf{R_{\alpha\beta ij}}}$ should not be very different from the R_{mnij} tensor. The only difference would come from the g_{mn} versus $g_{\alpha\beta}$ scalings. Thus, we need to interchange the α_0 and β_0 to obtain the right g_s scale dependence:

$$R_{\alpha\beta ij} \sim g_s \left\{ -\frac{8}{3} + \beta_0 + 2\gamma, -\frac{2}{3} + \beta_0, -\frac{8}{3} + \beta_0 - \alpha_0, -\frac{8}{3} \right\}.$$

The scale dependence of the $\boxed{\mathbf{R_{0\alpha 0\beta}}}$ can be found following our reasoning from a penultimate couple of Riemann tensors. It the same as the scalings of the $R_{alpha\beta ij}$ tensor:

$$R_{0\alpha 0\beta} \sim R_{\alpha\beta ij} \sim g_s \left\{ -\frac{8}{3} + \beta_0 + 2\gamma, -\frac{2}{3} + \beta_0, -\frac{8}{3} + \beta_0 - \alpha_0, -\frac{8}{3} \right\}.$$

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