Study of conformal Minimal Models on non-orientable surfaces and its implications in three-dimensional quantum gravity

Ioannis Tsiares

Master of Science

Physics Department

McGill University
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Abstract

In this thesis we study the consequences of two-dimensional Conformal Field Theories (CFTs) defined on non-orientable surfaces and their three-dimensional gravity duals according to the AdS/CFT correspondence. We focus on a particular class of conformal theories, called Minimal Models, which are conjecturally describing a gravitational theory in a highly quantum regime. The core of this work is summarized in the non-trivial results we provide for the $\mathbb{RP}^2$ one-point function normalization constants of the Ising and the Tricritical Ising model, by numerically solving the CFT consistency conditions on the Real Projective plane, $\mathbb{RP}^2$. These two minimal theories are of particular importance since they constitute *extremal* CFTs, and their gravity dual is already conjectured with success in the literature. We review the construction of CFTs in both orientable and non-orientable surfaces as well as the statement of the AdS/CFT correspondence, and we mainly rely on our results to understand the structure of the path integral of pure gravity in three dimensions. We end by discussing the gravitational consequences of analogous not-yet-obtained results for other Minimal Models, and we sketch a potential utility in the CFT reconstruction program of bulk operators in AdS.

Résumé

Dans cette thèse, nous étudions les conséquences de définir des Théories Conformes des Champs (CFTs) en deux dimensions sur une surface non-orientable ainsi que leurs duals gravitationnels en trois dimensions décrits par la correspondance AdS/CFT. Nous nous concentrons en particulier sur les Modèles Minimaux, des théories conformes qui décrivent de manière conjecturale des théories gravitationnelles dans un régime hautement quantique. La partie la plus importante de ce travail est les résultats non-triviaux que nous fournissons par rapport aux constantes de normalisation des fonctions de corrélation à un point pour les modèles de Ising et Tricritique de Ising sur $\mathbb{RP}^2$, que nous obtenons numériquement en résolvant les conditions de cohérence sur le plan projectif réel, $\mathbb{RP}^2$. Les deux théories minimales que nous étudions sont d'une importance particulière puisquélles constituent des CFTs *extrémales* et leurs duals gravitationnels ont déjà été conjecturés avec succès dans la littérature. Nous résumons ici la construction de CFTs sur des surfaces orientables et non-orientables ainsi que l'énoncé de la correspondance AdS/CFT et nous nous fions principalement sur nos résultats pour comprendre la structure de l'intégrale de chemins pour la gravité pure en trois dimensions. Nous terminons en discutant des conséquences gravitationnelles de futurs résultats similaires pour les autres Modèles Minimaux et nous esquissons une utilité potentielle dans la reconstruction d'opérateurs dans AdS à partir des CFTs.

Contents

1 Conformal Field Theory on Riemann surfaces

1.1 Basics of CFT

We start by setting the general stage of a theory with conformal symmetry, in a suitable d-dimensional conformal compactification[1]of Minkowski spacetime $R^{d-1,1}$. The point of this section is to give a first hint of why conformal symmetry is qualitatively different in $d = 2$ than in $d > 2$, and hence motivate us to study its various aspects in the former case.

A conformal transformation of the coordinates is defined as the invertible map which leaves the metric tensor invariant up to a local function $\Lambda(x)$:

$$g'_{\mu\nu}(x') = \Lambda(x)\eta_{\mu\nu} \tag{1}$$

One first observation is that the special case of $\Lambda(x) = 1$ corresponds to the familiar *Poincare group* consisting of translations and Lorentz transformations.

Therefore, as in the study of the Poincare group (and any Lie group), what we would like to do next is to specify the infinitesimal form of conformal transformations. This will lead us to define the corresponding generators, their algebra and the representations of the group. Our ultimate goal is to build a quantum field theory with conformal symmetry. This will in turn allow us to determine the field content by specifying how various fields transform under the action of the group, and then "promote" appropriately our fields to operators and build a quantum Hilbert space.

Let us consider an infinitesimal coordinate transformation ($\epsilon(x) << 1$) and the induced change in the metric:

$$x^\rho \to x^\rho + \epsilon^\rho(x),\ \eta_{\mu\nu} \to \eta_{\mu\nu} + \partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu \tag{2}$$

Equation (1) then demands that :

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = K(x)\eta_{\mu\nu} \tag{3}$$

where $\Lambda(x) \equiv 1 + K(x) + O(\epsilon^2)$

Eq.(3) in principle determines the form of ϵ. We can however manipulate this relation a bit more to derive some further useful relations. First, by tracing the equation with $\eta^{\mu\nu}$ we can get a specific form of $K(x)$ in terms of ϵ:

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = \frac{2}{d}(\partial\cdot\epsilon)\eta_{\mu\nu} \tag{4}$$

and then, by first taking derivative with respect to ∂^ν and summing over ν:

$$\partial_\mu(\partial\cdot\epsilon) + \Box\epsilon_\mu = \frac{2}{d}\partial_\mu(\partial\cdot\epsilon) \tag{5}$$

[1]by *conformal compactification* here we mean the manifold that also includes the point at infinity, e.g. the Riemann sphere S^2 in 2d. We're going to discuss this in more detail on section 1.2.

and next differentiating (5) with respect to ∂_ν and adding the corresponding expressions with μ and ν interchanged, we can arrive at:

$$(d-1)\Box(\partial\cdot\epsilon) = 0 \tag{6}$$

One last relation that we are going to need is obtained by taking derivatives of (4) with respect to some other index ∂_ρ, and then permuting indices:

$$\begin{aligned}
\partial_\rho\partial_\mu\epsilon_\nu + \partial_\rho\partial_\nu\epsilon_\mu &= \frac{2}{d}\eta_{\mu\nu}\partial_\rho(\partial\cdot\epsilon) \\
\partial_\nu\partial_\rho\epsilon_\mu + \partial_\mu\partial_\rho\epsilon_\nu &= \frac{2}{d}\eta_{\rho\mu}\partial_\nu(\partial\cdot\epsilon) \\
\partial_\mu\partial_\nu\epsilon_\rho + \partial_\nu\partial_\mu\epsilon_\rho &= \frac{2}{d}\eta_{\nu\rho}\partial_\mu(\partial\cdot\epsilon)
\end{aligned} \tag{7}$$

Subtracting the first line from the sum of the last two yields:

$$2\partial_\mu\partial_\nu\epsilon_\rho = \frac{2}{d}(-\eta_{\mu\nu}\partial_\rho + \eta_{\rho\mu}\partial_\nu + \eta_{\nu\rho}\partial_\mu)(\partial\cdot\epsilon) \tag{8}$$

Equations (4),(6) and (8) will be important for our following analysis. For $d = 1$ we can see that the above equations do not impose any constraint on ϵ, and therefore any smooth transformation is conformal in one dimension. In the case of $d = 2$, equation (4) turns out to include some interesting information about the structure of the symmetry but let us postpone the discussion until section 1.2. For the moment we focus on the case $d \geq 3$. Equation (6) suggests that ϵ is at most quadratic in x and so we can make the ansatz:

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho \tag{9}$$

where a,$b_{\mu\nu}$,$c_{\mu\nu\rho}$ are small constants and the latter is symmetric in the last two indices ($c_{\mu\nu\rho} = c_{\mu\rho\nu}$). Their specific form can be studied independently as the constraints for conformal invariance have to be independent of the position x:

$\Rightarrow \epsilon_\mu = a_\mu$. It describes infinitesimal translations for which the generator is the familiar momentum operator $P_\mu = -i\partial_\mu$.

$\Rightarrow \epsilon_\mu = b_{\mu\nu}x^\nu$. We can split b into a symmetric and an antisymmetric tensor and insert this ansatz into (4). What we will find is a constraint on the symmetric part to be proportional to the metric:

$$b_{\mu\nu} = \alpha\eta_{\mu\nu} + m_{\mu\nu} \tag{10}$$

where $m_{\mu\nu} = -m_{\nu\mu}$. We identify the symmetric part with infinitesimal scale transformations $x'^\mu = (1+\alpha)x^\mu$ with generator $D = -ix^\mu\partial_\mu$, and the antisymmetric part with infinitesimal rotations $x'^\mu = (\delta^\mu_\nu + m^\mu_\nu)x^\nu$ with generator being the angular momentum $L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$.

$\Rightarrow \epsilon_\mu = c_{\mu\nu\rho}x^\nu x^\rho$. For this term it's better to use a relation with two derivatives that's why we insert it into equation (8). We obtain:

$$\partial_\nu(\partial\cdot\epsilon) = 2c^\mu_{\mu\nu} \tag{11}$$

from which we can see that:

$$c_{\mu\nu\rho} = \eta_{\mu\rho}B_\nu + \eta_{\mu\nu}B_\rho - \eta_{\nu\rho}B_\mu \tag{12}$$

with $B_\mu \equiv \frac{1}{d}c^\rho_{\rho\mu}$. These kind of transformations go under the name of *Special Conformal Transformations* and their infinitesimal form is $x'^\mu = x^\mu + 2(x\cdot B)x^\mu - (x\cdot x)B^\mu$ with the corresponding generator: $K_\mu = -i(2x_\mu x^\nu\partial_\nu - (x\cdot x)\partial_\mu)$.

It's good to summarize our results in a table:

Transformations	Finite form	Generators
Translations	$x'^\mu = x^\mu + a^\mu$	$P_\mu = -i\partial_\mu$
Dilations	$x'^\mu = \alpha x^\mu$	$D = -ix^\mu\partial_\mu$
Rotations	$x'^\mu = M^\mu_\nu x^\nu$	$L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$
SCT	$x'^\mu = \frac{x^\mu - (x\cdot x)B^\mu}{1-2(B\cdot x)+(B\cdot B)(x\cdot x)}$	$K_\mu = -i(2x_\mu x^\nu\partial_\nu - (x\cdot x)\partial_\mu)$

Table1.1(a): Conformal transformations in $d \geq 3$, their finite form (obtained by exponentiating the infinitesimal form) and the corresponding generators.

The above transformations as derived are *globally defined* and *invertible* in a suitable conformal compactification of $R^{d-1,1}$(i.e. the point at infinity is included), therefore we can go ahead and identify the corresponding group by determining the algebra of the generators. If we perform the identifications:

$$\begin{aligned} J_{\mu,\nu} = L_{\mu\nu},\ J_{-1,\mu} = \frac{1}{2}(P_\mu - K_\mu) \\ J_{-1,0} = D,\ J_{0,\mu} = \frac{1}{2}(P_\mu + K_\mu) \end{aligned} \tag{13}$$

then we can group all the generators in one object $J_{m,n}$, with $m,n = -1,0,1,...,(d-1)$. More than that, the commutation relations can be shown to be:

$$[J_{mn}, J_{rs}] = i(\eta_{ms}J_{nr} + \eta_{nr}J_{ms} - \eta_{mr}J_{ns} - \eta_{ns}J_{mr}) \tag{14}$$

where $\eta_{mn} \equiv diag(-1,-1,...,1)$. These are the commutations relations of the Lie algebra $so(d,2)$. Therefore, the conformal group on $R^{d-1,1}$ for $d \geq 3$ is identified as the Lie group $SO(d,2)$[2]. We fulfilled our goal to identify the conformal group and the algebra of its generators. As a next step, we can go ahead and study representations of $SO(d,2)$. We are not going to do it here though as we will focus mostly in the case of $d = 2$. A good reference can be found in [1].

At this point we should step back and think to which extent the above analysis also carries in the case of $d = 2$. Equation (6) holds when $d = 2$, so one can argue that the result of ϵ being at most quadratic in the coordinates is still valid. However there is much richer structure obtained solely by equation (4) when $d = 2$. We will discover it in the next section.

[2]For a Euclidean d-dimensional space $R^{d,0}$, it can be shown in the same way that the conformal group is $SO(d+1,1)$ which is the Lorentz group in two higher dimensions. In more generality, the conformal group on $R^{p,q}$ is $SO(p+1,q+1)$.

1.2 CFT on S^2

In this section we will sketch the general properties of a two-dimensional conformal field theory defined on the Riemann sphere $S^2 \simeq C \cup \infty$ and then we will focus on a specific class of conformal field theories called *Minimal Models*. The goal of this chapter is to understand the powerful mathematical structure which is hidden in a two-dimensional CFT, which will eventually allow us to highly constrain its spectrum via the so-called *crossing symmetry* of four-point functions. We perform all of our calculations in Euclidean signature unless otherwise specified.

1.2.1 Global and local conformal transformations on S^2

Let's examine (4) in the case of 2d. We obtain two equations:

$$\partial_0\epsilon_0 = \partial_1\epsilon_1,\ \partial_0\epsilon_1 = -\partial_1\epsilon_0 \tag{15}$$

These are the *Cauchy-Riemann* equations that the real and imaginary part of a holomorphic function satisfies in the complex plane.This observation dictates a natural change in the variables:

$$\begin{aligned} \epsilon &= \epsilon^0 + i\epsilon^1, \quad z = x^0 + ix^1, \quad \partial_z = \tfrac{1}{2}(\partial_0 - i\partial_1) \\ \overline{\epsilon} &= \epsilon^0 - i\epsilon^1, \quad \overline{z} = x^0 - ix^1, \quad \partial_{\overline{z}} = \tfrac{1}{2}(\partial_0 + i\partial_1) \end{aligned} \tag{16}$$

and now the infinitesimal form becomes:

$$z' = z + \epsilon(z),\ \overline{z'} = \overline{z} + \overline{\epsilon}(\overline{z}) \tag{17}$$

The conclusion is: for an infinitesimal conformal transformation in two dimensions the function $\epsilon(z)$ has to be *holomorphic* in some open set of the complex plane. This fact immediately implies that we can apply the rich machinery of holomorphic functions in order study the properties of conformal transformations in two dimensions! Furthermore, it's understood that there is actually an *infinite* set of holomorphic mappings one can implement, that eventually constitutes a local conformal transformation.

In general, we can assume that $\epsilon(z)$ is a meromorphic function having isolated poles outside the open set and hence it admits a Laurent expansion around say $z = 0$. A general infinitesimal conformal transformation can be written then as:

$$\begin{aligned} z' &= z + \sum_{n\in Z} \epsilon_n(-z^{n+1}), \\ \overline{z'} &= \overline{z} + \sum_{n\in Z} \overline{\epsilon}_n(-\overline{z}^{n+1}) \end{aligned} \tag{18}$$

for constant ϵ_n and $\overline{\epsilon}_n$. From this expression we can easily read the generators:

$$l_n = -z^{n+1}\partial_z,\ \overline{l_n} = -\overline{z}^{n+1}\partial_{\overline{z}} \tag{19}$$

and in turn the corresponding algebra:

$$\begin{aligned} [l_n, l_m] &= (n-m)l_{n+m} \\ [\bar{l}_n, \bar{l}_m] &= (n-m)\bar{l}_{n+m} \\ [l_n, \bar{l}_m] &= 0 \end{aligned} \tag{20}$$

Thus, the conformal algebra is the direct sum of two isomorphic algebras. The algebra (20) is sometimes called the *Witt algebra*.

The important thing to notice here is that the number of generators is *infinite* (since $n \in Z$), a fact that was expected from the holomorphicity of ϵ. This situation is special in two dimensions and, as we'll see, it has many other interesting consequences.

After the identification of the algebra, the next thing to do is to specify the conformal group of the Riemann sphere S^2. In order to do that though our transformations should be globally defined and invertible.From the expression (19) of the generators, we can see that the points $z = 0$ and $z = \infty$ are actually ambiguous. The generator $l_n = -z^{n+1}\partial_z$ is non-singular at $z = 0$ only for $n \geq -1$, whereas is non-singular at $z = \infty$ only for $n \leq +1$.

This means that *globally defined* conformal transformations on S^2 are generated only by l_{-1}, l_0, l_1. The algebra between these operators closes so this specific subalgebra is to be associated with the global conformal group. From (19) it is manifest that $l_{-1} = -\partial_z$ generates translations on the complex plane, $l_0 = -z\partial_z$ generates scale transformations and rotations, and $l_1 = -z^2\partial_z$ generates special conformal transformations. In general it's easy to show that a finite global conformal transformation in 2d takes the compact form:

$$z \to \frac{az + b}{cz + d}, \quad with\ a, b, c, d \in \mathbb{C} \tag{21}$$

For this transformation to be invertible, we should require that $ad - bc \neq 0$; and we can always scale the constants a, b, c, d such that $ad - bc = 1$. Hence, the total number of real parameters that we have to specify is 6. Furthermore, we notice that (21) is invariant under $(a, b, c, d) \to (-a, -b, -c, -d)$ which means that the corresponding group should be also *projective*. This is exactly the so called *Mobius group* $PSL(2, \mathbb{C})$, which we eventually identify as the *conformal group of the Riemann sphere* S^2.

Let us summarize what we learned so far. In two dimensions there exists an infinite variety of coordinate transformations that, although not everywhere well-defined, are locally conformal: they are holomorphic mappings from the complex plane onto itself. Among this *infinite* set of mappings we should distinguish the 6-parameter *global* conformal group $PSL(2, \mathbb{C})$, made of *one-to-one* mappings of the complex plane into itself.It is actually known that $SL(2, \mathbb{C})$ is isomorphic to the Lorentz group in four dimensions $SO(3, 1)$. Therefore, as far as the conformal group proper is concerned, we have learned nothing new since the previous section. Our previous analysis still holds when considering these transformations only. However, if we want to build a *local* field theory, it better be sensitive to the local symmetries, even if the related transformations are not globally defined. And it's exactly the *local* conformal invariance that enables exact solutions of two-dimensional conformal field theories.

1.2.2 Energy-monetum tensor, primary fields and correlation functions

Having studied to some extent the structure of conformal transformations and conformal group, the next step is to introduce classical fields (i.e. functions of space and time) $\Phi(z, \overline{z})$ on the Riemann sphere. There is a slight difference in terminology between CFTs and more general quantum field theories that we have to point out. Usually in quantum

field theory, one reserves the term *field* for the objects which sit in the action and are integrated over in the path integral. In contrast, in CFT the term *field* refers to any local expression that we can write down. This includes Φ, but also includes derivatives $\partial^n\Phi$ or composite operators such as $e^{i\Phi}$. All of these are thought of as different fields in a CFT.

A distinguished tensor field, on which we're going to focus first, is the *energy-momentum tensor*. The energy-momentum tensor can be in principle deduced from the variation of the action with respect to the metric and so it encodes the behaviour of the theory under infinitesimal transformations $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$. It turns out that in a conformal field theory the constraints of the symmetry are so powerful that we won't need an explicit form of an action. What we will show next is that the energy-momentum tensor of a CFT in d-dimensions (in general) is *traceless*, and that in two dimensions it has two non-vanishing components which are *chiral* and *anti-chiral* fields (to be defined shortly).

In order to study the energy-momentum tensor of a CFT, we should recall Noether's theorem from classical field theory: for every continuous symmetry of a given action, there is a curent j_μ which conserved, i.e. $\partial_\mu j^\mu = 0$, where $\mu = 0, .., d-1$ in a Lorentzian say spacetime (identical analysis holds for Euclidean). For a global conformal transformation $x^\mu \to x^\mu + \epsilon^\mu(x)$, we have a conserved current which can be written as:

$$j_\mu = T_{\mu\nu}\epsilon^\nu \tag{22}$$

where the symmetric tensor $T_{\mu\nu}$ is the *energy-momentum tensor*. For the simple case of $\epsilon^\mu = const.$ the conservation law suggests:

$$\partial^\mu T_{\mu\nu} = 0 \tag{23}$$

For more general conformal transformations $\epsilon^\mu(x)$ now, we can see that:

$$\begin{aligned} 0 = \partial^\mu j_\mu &= (\partial^\mu T_{\mu\nu})\epsilon^\nu + T_{\mu\nu}(\partial^\mu\epsilon^\nu) \\ &= 0 + \frac{1}{2}T_{\mu\nu}(\partial^\mu\epsilon^\nu + \partial^\nu\epsilon^\mu) = \frac{1}{2}T_{\mu\nu}\frac{2}{d}(\partial\cdot\epsilon)\eta_{\mu\nu} = \frac{1}{d}T^\mu_\mu(\partial\cdot\epsilon) \end{aligned} \tag{24}$$

where we have used Eq.(4). The conclusion is that the energy-momentum tensor in a classical conformal field theory is *traceless*.

Let us now investigate what the previous result means in two dimensions in the case of Euclidean signature($\eta_{\mu\nu} = diag(+1, +1)$). Using the change of coordinates (16), we can easily see that $T_{\mu\nu}$ where $\mu, \nu = 0, 1$ is related to T_{ab} where $a, b = z, \overline{z}$ via:

$$\begin{aligned} T_{zz} &= \frac{1}{4}(T_{00} - 2iT_{10} - T_{11}) \\ T_{\overline{z}\overline{z}} &= \frac{1}{4}(T_{00} + 2iT_{10} - T_{11}) \\ T_{z\overline{z}} = T_{\overline{z}z} &= \frac{1}{4}(T_{00} + T_{11}) = \frac{1}{4}T^\mu_\mu = 0 \end{aligned} \tag{25}$$

Employing the traceleness relation we can write: $T_{zz} = \frac{1}{2}(T_{00} - iT_{10})$ and $T_{\overline{z}\overline{z}} = \frac{1}{2}(T_{00} + iT_{10})$, and now the conservation condition in two dimensions:

$$\partial_0 T_{00} + \partial_1 T_{10} = 0, \quad \partial_0 T_{01} + \partial_1 T_{11} = 0 \tag{26}$$

implies:

$$\partial_{\bar{z}} T_{zz} = \frac{1}{4}(\partial_0 + i\partial_1)(T_{00} - iT_{10}) = \frac{1}{4}(\partial_0 T_{00} + \partial_1 T_{10} + i\partial_1 T_{00} - i\partial_0 T_{10}) = 0 \tag{27}$$

where in the last equality we used (26) and $T^{\mu}_{\mu} = 0$. In the same way, one can show that $\partial_z T_{\overline{z}\overline{z}} = 0$.

What we found is that the two non-vanishing components of the energy-momentum tensor are functions seperately of z and $\overline{z}$, namely $T_{zz}(z,\overline{z}) \equiv T(z)$ and $T_{\overline{z}\overline{z}}(z,\overline{z}) \equiv T(\overline{z})$. We call these kind of fields *chiral* and *anti-chiral* respectively. This property will be crucial in the calculations when we proceed with quantization.

Let us now turn to general fields as we advertise them in the beginning of the section. Fields are affected by a conformal transformation as they change according to some representation of the conformal group. In physics, we usually demand that a physical field Φ also belongs to an *irreducible* representation of the symmetry group[3]. The basic picture in words is the following: given an infinitesimal conformal transformation parametrized by ω_g, we seek a matrix representation R_g such that a general multi-component field $\Phi(z,\overline{z})$ change infinitesimally as:

$$\Phi'(z',\overline{z}') = (1 - i\omega_g R_g)\Phi(z,\overline{z}) \tag{28}$$

In order to consider the *actual* change in Φ we should compare Φ' and Φ in the same spacetime point. Hence we also need the infinitesimal expansion of $z, \overline{z}$ in terms of $z', \overline{z}'$ and then, R_g together wth the spacetime part is just a representation of the abstract generators of the conformal group. Therefore, they must also form a matrix representation of the corresponding algebra and this requirement is actually enough to specify them. Then, one can proceed and derive the change in Φ under a finite conformal transformation by exponentiation.

Without delving into the details[4], let us state here the field classification on S^2 under conformal transformations. First, we should introduce the notion of *conformal dimension*: if a field $\Phi(z,\overline{z})$ transforms under finite dilations $z \to \lambda z$ according to:

$$\Phi'(\lambda z, \overline{\lambda}\overline{z}) = \lambda^{-h}\overline{\lambda}^{-\overline{h}}\Phi(z,\overline{z})$$

then it is said to have *conformal dimensions* $(h,\overline{h})$[5].

Now we proceed to the notion of a *primary field*: If a field transforms under finite conformal transformations $z \to f(z)$ according to:

$$\Phi'(f(z),\overline{f}(\overline{z})) = (\frac{\partial f}{\partial z})^{-h}(\frac{\partial \overline{f}}{\partial \overline{z}})^{-\overline{h}}\Phi(z,\overline{z}) \tag{29}$$

it is called *primary field* fo conformal dimension $(h,\overline{h})$. We will soon realize that primary fields are the basic buildings blocks of the representations of the conformal algebra. For later use, let us also figure out how a primary field transforms under infinitesimal conformal transformations. Consider the map $f(z) = z + \epsilon(z)$ where $\epsilon << 1$. Then:

$$\begin{aligned} (\frac{\partial f}{\partial z})^{-h} &= 1 - h\partial_z\epsilon(z) + O(\epsilon^2) \\ \Phi(z+\epsilon(z),\overline{z}) &= \Phi(z) + \epsilon(z)\partial_z\Phi(z,\overline{z}) + O(\epsilon^2) \end{aligned} \tag{30}$$

[3]for example in usual quantum field theory we classify the fields according to irreducible representations of the Poincaré group, labeled by the spin and the mass.

[4]for an extensive discussion of conformal representations see [2].

[5]as one might expect, h and $\overline{h}$ are actually a linear combination of the eigenvalue of the representation matrix of the dilation operator, denoted by Δ, and the planar spin s : $h = \frac{1}{2}(\Delta + s)$ and $\overline{h} = \frac{1}{2}(\Delta - s)$.

Using these two expressions we obtain the infinitesimal *actual* change of a primary field:

$$\delta_{\epsilon,\bar{\epsilon}}\Phi(z,\overline{z}) \equiv \Phi'(z,\overline{z}) - \Phi(z,\overline{z}) = -(h\Phi\partial_z\epsilon + \epsilon\partial_z\Phi) - (\overline{h}\Phi\partial_{\overline{z}}\bar{\epsilon} + \bar{\epsilon}\partial_{\overline{z}}\Phi) \tag{31}$$

Sometimes, it can be the case that the transformation rule (29) holds only for $f \in PSL(2,\mathbb{C})$, i.e for *global* conformal transformations. Then, the field Φ is called *quasi-primary*. As we will see, an important example of a quasi-primary field is the energy-momentum tensor. It is clear that a primary field is always a quasi-primary but the reverse is not true in general. A field which is not a primary (hence neither quasi-primary) is generally called *secondary*. For example, the derivative of a primary field of conformal dimension $h \neq 0$ is a secondary field.

We should emphasize that, up to now, we are dealing with classical fields. When we turn to the quantum aspect of the theory -where fields are promoted to operators- we will actually see that there is a better classification of secondary fields/operators, as *descendants* of a given primary.

Since we identified the conformal group and classified the field content, maybe we can get a first flavor of how quantum correlation functions look like. Correlation functions can in principle be obtained in the path integral formalism where insertions are just 'classical' (commuting or anti-commuting) fields.

Consider a CFT involving a collection of fields Φ with an action $S[\Phi]$ invariant under conformal transformations. A general correlation function has the familiar form:

$$\langle\Phi(x_1)\Phi(x_2)...\Phi(x_n)\rangle = \frac{1}{Z}\int \mathcal{D}\Phi\ \Phi(x_1)\Phi(x_2)...\Phi(x_n)e^{-S[\Phi]} \tag{32}$$

where Z is the vacuum-to-vacuum correlation function (i.e. the partition function), and we denote collectively $z_n, \overline{z_n} \equiv x_n$ for ease.

If we make a conformal transformation $x_1, x_2..x_n \to x_1', x_2'..x_n'$ then, from (32):

$$\begin{aligned}
\langle\Phi(x_1')\Phi(x_2')...\Phi(x_n')\rangle &= \frac{1}{Z}\int \mathcal{D}\Phi\Phi(x_1')\Phi(x_2')...\Phi(x_n')e^{-S[\Phi]} \\
&= \frac{1}{Z}\int \mathcal{D}\Phi'\Phi'(x_1')\Phi'(x_2')...\Phi'(x_n')e^{-S[\Phi']} \\
&= \frac{1}{Z}\int \mathcal{D}\Phi F(\Phi(x_1'))F(\Phi(x_2'))...F(\Phi(x_n'))e^{-S[\Phi]} \\
&= \langle F(\Phi(x_1'))F(\Phi(x_2'))...F(\Phi(x_n'))\rangle
\end{aligned} \tag{33}$$

where $F(\Phi(x_n'))$ describes the functional change of the field. Now for a *primary field*, (29) implies:

$$\langle\Phi(w_1,\overline{w}_1)\Phi(w_2,\overline{w}_2)...\Phi(w_n,\overline{w}_n)\rangle = \prod_{i=1}^{n}\left(\frac{dw}{dz}\right)^{-h_i}_{w=w_i}\left(\frac{d\overline{w}}{d\overline{z}}\right)^{-\overline{h}_i}_{\overline{w}=\overline{w}_i}\langle\Phi(z_1,\overline{z}_1)\Phi(z_2,\overline{z}_2)...\Phi(z_n,\overline{z}_n)\rangle \tag{34}$$

under a conformal map $z \to w(z)$.

Covariance of the correlation function under the *global* transformations, i.e. translations, rotations, scalings and special conformal transformations, is enough to specify exactly the form of the two and three point function, and fix the four point function up to a conformally invariant function.

Using (34) we get:

2-point function:

$$\langle \Phi_1(z_1,\overline{z}_1)\Phi_2(z_2,\overline{z}_2)\rangle = \frac{d_{12}}{(z_1-z_2)^{2h}(\overline{z_1}-\overline{z_2})^{2\overline{h}}}\delta_{h_1h_2}\delta_{\overline{h}_1\overline{h}_2} \tag{35}$$

where d_{12} is just a constant which can always be absorbed in the normalization of the fields, and $h \equiv h_1 = h_2, \overline{h} \equiv \overline{h_1} = \overline{h_2}$. An interesting fact is that two-point functions vanish for fields with different conformal dimensions, indicating how constraining conformal symmetry is.

3-point function:

$$\langle \Phi_1(z_1,\overline{z}_1)\Phi_2(z_2,\overline{z}_2)\Phi_3(z_3,\overline{z}_3)\rangle = C_{123}\frac{1}{(z_1-z_2)^{h_1+h_2-h_3}(z_2-z_3)^{h_2+h_3-h_1}(z_1-z_3)^{h_3+h_1-h_2}}\times \frac{1}{(\overline{z}_1-\overline{z}_2)^{\overline{h}_1+\overline{h}_2-\overline{h}_3}(\overline{z}_2-\overline{z}_3)^{\overline{h}_2+\overline{h}_3-\overline{h}_1}(\overline{z}_1-\overline{z}_3)^{\overline{h}_3+\overline{h}_1-\overline{h}_2}} \tag{36}$$

In contrast to the situation in the two-point function, the coefficients C_{123} are in principle there and cannot be absorbed by some redefinition.These coefficients are actually *dynamical* data of a specific CFT, as we will see next, together with the set of conformal dimensions $h, \overline{h}$ determine completely a CFT on S^2!

As far as the four-point functions (and beyond) conformal invariance does not fix their precise form. With four points on S^2 it is possible to construct conformally invariant expressions, the so called *anharmonic ratios*. In particular:

$$\eta \equiv \frac{z_{12}z_{34}}{z_{13}z_{24}}, \quad 1-\eta = \frac{z_{14}z_{23}}{z_{13}z_{24}}, \quad \frac{\eta}{1-\eta} = \frac{z_{12}z_{34}}{z_{14}z_{23}} \tag{37}$$

The four-point function may then depend on η and $\overline{\eta}$ in an arbitrary way $G(\eta,\overline{\eta})$, provided the result is real.

4-point function:

$$\langle \Phi_1(z_1,\overline{z}_1)\Phi_2(z_2,\overline{z}_2)\Phi_3(z_3,\overline{z}_3)\Phi_4(z_4,\overline{z}_4)\rangle = G(\eta,\overline{\eta})\prod_{i\leq j}^{4} z_{ij}^{h/3-h_i-h_j}\overline{z}_{ij}^{\overline{h}/3-\overline{h}_i-\overline{h}_j} \tag{38}$$

where $h = \sum_{i=1}^4 h_i$ and $\overline{h} = \sum_{i=1}^4 \overline{h}_i$. One important thing to notice here is that, it is always possible to find a *global* confromal transformation that maps three of the four points to three fixed points, e.g. 0,1 and ∞[6]. Hence, a generic four-point function will depend in principle only on one spacetime point $(z,\overline{z})$.

The expression for the four-point amplitude and all the implications that arise from conformal invariance on (38) will be extremely important for our purposes. As we will see later, the function $G(\eta,\overline{\eta})$ can be decomposed into conformally covariant expressions known as *conformal blocks*, which will in turn lead us to the powerful statement of crossing symmetry of four-point functions on S^2. Before we delve into it though we need one really powerful tool: *the operator product expansion*.

[6]indeed, $PSL(2,\mathbb{C})$ involves three independent complex parameters as we pointed out in 2.2.1.

1.2.3 Radial quantization

We now move to the quantum aspects of a conformal field theory. Our fields are going to be promoted to operators in a specific way and a Hilbert space of quantum states will be defined.

The operator formalism of quantum mechanics implies a choice of reference frame, in the sense that one should choose a time axis in order to define quantum states in a fixed time slice. We are always in a Euclidean two-dimensional flat space with coordinates x^0 and x^1. To make the choice of a time slice a little more natural, we can initially define our theory on an infinite cylinder by compacitfying the Euclidean space direction x^1 on a circle of say unit radius. The CFT we obtain in this way will depend on the complex coordinate:

$$w = x^0 + ix^1, \qquad where \;\; w \sim w + 2\pi i \tag{39}$$

where the identification is understood. Constant time slices are obviously circles S^1 surrounding the cylinder. Let us now perform a conformal change of variables by mapping the cylinder (back) to the complex plane (or rather, the Riemann sphere) and identify the slices. This mapping is achieved by:

$$z = e^w = e^{x^0} \cdot e^{ix^1} \tag{40}$$

and is illustrated in Figure1.2(a) below.

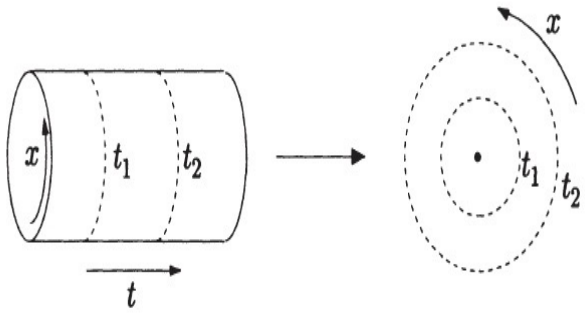

Figure1.2(a) The conformal map of the cylinder to the complex plane

We should emphasize that our theory is covariant after the change, since the coordinate mapping is itself conformal. For this reason, the upcoming consequences of this mapping (state-operator correspondence) are true only for conformal field theories.

On the complex plane the infinite past $t \to -\infty$ is located at the origin $z = 0$, whereas future infinity $t \to \infty$ lies on the point at infinity on the Riemann sphere. Time translations on the cylinder $x^0 \to x^0 + a$ now become complex dilations $z \to e^a z$, and space translations $x^1 \to x^1 + b$ become rotations $z \to e^{ib} z$. In addition to that, surfaces of constant x^0 are mapped to $|z| = const.$, which means that integrals over space $\int dx^1$ get transformed into closed contour integrals, which is a very powerful calculational tool as we will see next.

The next step is to put our fields on the game. The quantization is achieved first by expanding in Laurent modes and then promoting these modes into operators. Let's consider a field $\Phi(z, \overline{z})$ of conformal dimensions $(h, \overline{h})$ for which we can perform a Laurent

expansion around $z_0 = \overline{z}_0 = 0$:

$$\begin{aligned} \Phi(z,\overline{z}) &= \sum_{n,\overline{m}\in\mathcal{Z}} z^{-n-h}\overline{z}^{-\overline{m}-\overline{h}}\phi_{n,\overline{m}} \\ \phi_{n,\overline{m}} &= (\frac{1}{2\pi i})^2 \oint dz \oint d\overline{z}\; z^{n+h-1}\overline{z}^{\overline{m}+\overline{h}-1}\Phi(z,\overline{z}) \end{aligned} \tag{41}$$

We can deduce the action of hermitian conjugation on the Laurent modes from this expression. (41) yields:

$$\Phi(z,\overline{z})^\dagger = \sum_{n,\overline{m}\in\mathcal{Z}} \overline{z}^{-n-h} z^{-\overline{m}-\overline{h}}\phi^\dagger_{n,\overline{m}} \tag{42}$$

But under scalings $\Phi^\dagger$ behaves as:

$$\Phi(z,\overline{z})^\dagger = \overline{z}^{-2h} z^{-2\overline{h}}\Phi(1/\overline{z}, 1/z) \tag{43}$$

The last two equations are compatible provided

$$\phi^\dagger_{n,\overline{m}} = \phi_{-n,-\overline{m}} \tag{44}$$

Having obtained the necessary ingredients, let us now proceed by defining our first quantum states. We first assume the existence of a vacuum $|0\rangle$ upon which a Hilbert space is built by application of (appropriately defined) 'creation' operators, as we know from quantum field theory. We also assume that it's inherently invariant under global conformal transformations, i.e. it's annihilated by l_0, l_{-1}, l_1 (and their anti-holomorphic counterparts). In free theories, the vacuum may be defined as the state annihilated by the 'positive' frequency part of the field. For interacting ones, the basic assumption is that the interaction is turned off as $t \to \pm\infty$ and this helps us to define *in* and *out* states:

$$\begin{aligned} \Phi_{in} &\sim \lim_{t\to-\infty} \Phi(x,t) \\ \Phi_{out} &\sim \lim_{t\to+\infty} \Phi(x,t) \end{aligned} \tag{45}$$

Upon radial quantization, the asymptotic *in field* reduces to a single operator, which, when acting on the vacuum, creates a single asymptotic *in* state:

$$|\Phi_{in}\rangle \equiv \lim_{z,\overline{z}\to 0} \Phi(z,\overline{z})\,|0\rangle \tag{46}$$

The infinite past $t \to -\infty$ is at the point $z, \overline{z} \to 0$ as we pointed out earlier. However, in order for Eq.(46) to be non-singular there, we should require:

$$\phi_{n,\overline{m}}\,|0\rangle = 0 \qquad for \quad n > -h,\ \overline{m} > -\overline{h} \tag{47}$$

from the expansion (41). We can say that this condition *defines* the vacuum at this stage. Using this restriction together with the mode expansion (41), we can simplify (46) by writting:

$$|\Phi_{in}\rangle = \lim_{z,\overline{z}\to 0} \Phi(z,\overline{z})\,|0\rangle = \phi_{-h,-\overline{h}}\,|0\rangle \tag{48}$$

Let's find a similar relation for the *out*-state as well. Naturally, this is achieved by using the hermitian conjugate field which reads:

$$\langle \Phi_{out}| \equiv \lim_{z,\overline{z}\to 0} \Phi^{\dagger}(z,\overline{z}) = \lim_{w,\overline{w}\to\infty} w^{2h}\overline{w}^{2\overline{h}} \langle 0|\, \Phi(w,\overline{w}) \tag{49}$$

where we used (43) together with the change $z \equiv 1/\overline{w}$ and $\overline{z} \equiv 1/w$. However, by the same reasoning as before, in order for the asymptotic *out*-state to be well-defined, we require:

$$\langle 0|\, \phi_{n,\overline{m}} = 0 \qquad for \quad n < h,\ \overline{m} < \overline{h} \tag{50}$$

We can now conclude, as in (48), that the definition of the out state is:

$$\langle \Phi_{out}| = \lim_{\overline{w},w\to\infty} w^{2h}\overline{w}^{2\overline{h}} \langle 0|\, \Phi(w,\overline{w}) = \langle 0|\, \phi_{+h,+\overline{h}} \tag{51}$$

There is actually one particularly important aspect of conformal field theories implied by Eq.(48) and (51): a *map* between states and *local* operators. As we know from Quantum Mechanics in general, states and local operators are quite different objects. Local operators are defined in a specific point in spacetime, whereas states are supported on an entire time slice. Hence it seems quite strange that there might be an isomoprhism between them.

They key point is that the infinite past on the cylinder gets mapped to a *single* point $z = 0$ in the plane. Given an initial state on the cylinder $|\Psi\rangle$ we can make a conformal transformation that 'squashes' it all to the origin on the plane! Thus it actually corresponds to a local disturbance on the plane, which can in turn be interpreted as the action of an operator at the origin(c.f.Eq.(48)). Then we may as well evolve that initial state with the Hamiltonian, and obtain a well-defined final state on the plane. It's understood that the concept of state-operator map is only true in conformal field theories where we can map the cylinder to the plane. It also holds in higher dimensions, where $R \times S^{d-1}$ can be mapped to the plane R^d.

Let us consistently conclude the discussion with the notion of *radial ordering*. As we mentioned above, constant time slices in the complex plane are circles surrounding the origin(c.f.Fig2.2(a)). Since we also have a first notion of operators acting on a Hilbert space, we are soon going to implement correlation functions of these operators. However, from quantum field theory we know that correlation functions are only defined as a time ordered product. Therefore, the change of coordinates (40) dictates that the time ordering now becomes a *radial ordering*. Any product of two operators $A(z,\overline{z})B(w,\overline{w})$ make sense only for $|z| > |w|$. To this end, we define the radial ordering of two operators as:

$$\mathcal{R}(A(z,\overline{z})B(w,\overline{w})) \equiv \begin{cases} A(z,\overline{z})B(w,\overline{w}) & for\ |z| > |w| \\ B(w,\overline{w})A(z,\overline{z}) & for\ |w| > |z| \end{cases} \tag{52}$$

If the two fields are fermions, a minus sign is added in front of the second expression. For what follows, it will be implicit that the product of two operators is already *radially*

ordered, without writing the symbol $\mathcal{R}$ every time. Now, it turns out the the closed contour integral of a radial ordering of two operators is of particular interest in the calculations that will follow. It is actually the integral of the *commutator* of the two fields. Let's examine it here:

Consider two *chiral* fields $A(z)$ and $B(w)$. If we take the integral

$$\oint_{C(w)} A(z)B(w) \tag{53}$$

where the integration contour $C(w)$ circles counterclockwise around w, then we can split the contour into two fixed-time circles (Fig.1.2(b)) going in opposite directions. Our integral is now seen to be just the commutator:

$$\oint_{C(w)} A(z)B(w) = \oint_{|z|>|w|} dzA(z)B(w) - \oint_{|w|>|z|} dzB(w)A(z) = \oint dz[A(z), B(w)] \tag{54}$$

we will see the importance of this calculation in the next section.

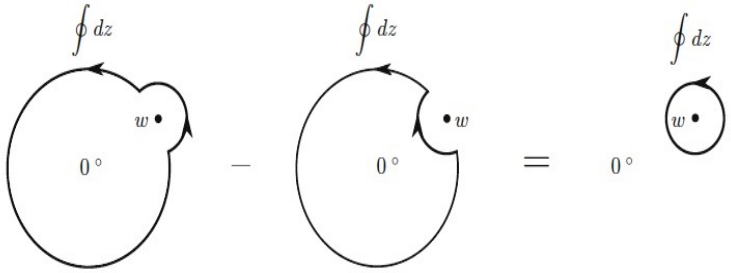

Figure1.2(b) Sum of contour integrals in equation (54).

1.2.4 The Operator Product Expansion and Virasoro algebra

In subsection 1.2.2 we studied the classical aspects of the energy-momentum tensor $T_{\mu\nu}$ on S^2. We saw that T is traceless with its two non-vanishing components (in the $(z, \overline{z})$ coordinate system) being *chiral* fields. We now have the necessary machinery to treat it as quantum operator and see what new knowledge we can gain.

Let us recall that, since the current $j_\mu = T_{\mu\nu}\epsilon^\nu$ is conserved (classically), the time component of that current, when integrated over space, yields a conserved *charge*:

$$Q = \int dx^1 j_0 = \frac{1}{2\pi i}\oint_C (dzT(z)\epsilon(z) + d\overline{z}\overline{T}(\overline{z})\overline{\epsilon}(\overline{z})) \tag{55}$$

In quantum field theory, this charge is an operator and generates symmetry transformations for an operator Φ as:

$$\delta\Phi = [Q, \Phi] \tag{56}$$

where the commutator is evaluated at equal times. Knowing the form of Q,we can evaluate the change of the field Φ under a conformal transformations:

$$\delta\Phi \equiv \delta_{\epsilon,\bar{\epsilon}}\Phi(w, \overline{w}) = \frac{1}{2\pi i}\oint_C dz[T(z)\epsilon(z), \Phi(w, \overline{w})] + \frac{1}{2\pi i}\oint_C d\overline{z}[\overline{T}(\overline{z})\overline{\epsilon}(\overline{z})), \Phi(w, \overline{w})] \tag{57}$$

which leads to:

$$\delta_{\epsilon,\bar{\epsilon}}\Phi(w,\overline{w}) = \frac{1}{2\pi i}\oint_C dz\epsilon(z)\mathcal{R}(T(z),\Phi(w,\overline{w})) + \text{anti-chiral} \tag{58}$$

What happens if the field Φ is a *primary* field of conformal dimension h? We have already computed the infinitesimal change of a primary field under conformal transformations back in subsection 1.2.2. We repeat it here for convenience:

$$\delta_{\epsilon,\bar{\epsilon}}\Phi(w,\overline{w}) = -(h\Phi\partial_w\epsilon + \epsilon\partial_w\Phi) - \text{anti-chiral} \tag{59}$$

Comparing with (58), we can deduce a non-trivial relation for the radial ordering of the energy-momentum tensor and a primary field! In particular, if we use the following identities:

$$\begin{aligned} h\Phi(w,\overline{w})\partial_w\epsilon(w) &= \frac{1}{2\pi i}\oint_{C(w)} dz\ h\ \frac{\epsilon(z)}{(z-w)^2}\Phi(w,\overline{w}) \\ \epsilon(w)\partial_w\Phi(w,\overline{w}) &= \frac{1}{2\pi i}\oint_{C(w)} dz\frac{\epsilon(z)}{z-w}\partial_w\Phi(w,\overline{w}) \end{aligned} \tag{60}$$

we obtain:

$$\mathcal{R}(T(z),\Phi(w,\overline{w})) = -\frac{h}{(z-w)^2}\Phi(w,\overline{w}) - \frac{1}{z-w}\partial_w\Phi(w,\overline{w}) + ... \tag{61}$$

where the ellipsis denotes non-singular terms as $z \to w$ in principle. An expression like (61) is of great significance in a CFT and in quantum field theory in general, and it's called an *Operator Product Expansion*. It basically defines an algebraic product structure on the space of quantum fields. We emphasize once again how important and powerful complex analysis in all these results is.

The idea of Operator Product Expansion(OPE) is known to hold in quantum field theories in general. Any product of two operators, evaluated inside a correlation function at two distinct points, say z and w, can be approximated by a sum of local operators with some c-number coefficient functions (possibly diverging) as $z \to w$:

$$O_i(z,\overline{z})O_j(w,\overline{w}) \sim \sum_k c_{ij}^k(z-w,\overline{z}-\overline{w})O_k(w,\overline{w}) \tag{62}$$

In a CFT, we can proceed more and write it in a conformally covariant fashion. For two primary fields O_i,O_j one can argue that[13]:

$$O_i(z,\overline{z})O_j(w,\overline{w}) \ = \sum_{k\ primary} C_{ij}^k\ \lambda_k(z-w,\overline{z}-\overline{w},\partial,\overline{\partial})O_k(w,\overline{w}) \tag{63}$$

where C_{ij}^k are known as *OPE coefficients*, and λ_k's are functions of derivatives and positions which are completely fixed by conformal symmetry. The expansion in (63) is actually an exact statement in CFT (and not just some representation when $z \to w$) as it was nicely shown in [14]. For the purposes of this thesis, we will not focus too much on the explicit form of λ_k's, but instead on the values of the OPE coefficients. One important fact about these coefficients is that they are basically *equal* to the constant

terms in the three-point function of primary operators (c.f. (36))[7]. One can see this by demanding consistency of (63) with the two and three-point amplitudes we obtained back in 1.2.2. Hence, determining these coefficients for a given CFT amounts to specifying true dynamical data for the theory. We will reutn to the importance of the OPE coefficients in subsections 1.2.6 and 1.2.7 .

Let us next investigate what is the OPE of the energy-momentum tensor with itself. First, one can easily see that the conformal dimensions $(h, \overline{h})$ of the energy momentum tensor $T(z)$ are $(2, 0)$ (from its definition as the variation of the action with respect to the metric,and then performing a scaling transformation). Similarly, $\overline{T}$ has dimensions (0,2). In the spirit of (61) then, the TT OPE should be:

$$T(z)T(w) = ... + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + (\text{non-singular terms}) \tag{64}$$

and similarly for $\overline{T}(\overline{z})\overline{T}(\overline{w})$. The ellipsis on the RHS denotes other possible operators which should in principle be of the form:

$$\frac{O_n}{(z-w)^n} \tag{65}$$

where $h[O_n] = 4 - n$. However, in a unitary CFT there are no operators with $h, \overline{h} < 0$ (we will show it in the next subsection). The only allowed contributions should come from an operator proportional to the identity and an operator of dimension 1. It turns out that the most singular term we can have is coming solely form a term $(z-w)^{-4}$ proportional to the identity[8]:

$$\begin{aligned} T(z)T(w) &= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + (\text{non-singular terms}) \\ \overline{T}(\overline{z})\overline{T}(\overline{w}) &= \frac{\tilde{c}/2}{(\overline{z}-\overline{w})^4} + \frac{2\overline{T}(\overline{w})}{(\overline{z}-\overline{w})^2} + \frac{\partial \overline{T}(\overline{w})}{\overline{z}-\overline{w}} + (\text{non-singular terms}) \end{aligned} \tag{66}$$

The constants $c, \overline{c}$ are called the *central charges* of the theory and they play a very important role in characterizing a tow-dimensional CFT. We will understand this point soon. For now, they constitute just the constant piece in the most singular term of TT OPE[9]. Another conclusion that can be drawn from (66) is that $T_{\mu\nu}$ is *not* a primary operator. Indeed, by comparing the OPE (66) with (61), we see that for non-vanishing central charges, $T(z)$ is generically not a primary. One can actually determine how T transforms under conformal transformations using appropriate Ward identities [1]. We will just state the answer here, which is:

$$T(z) = \big(\frac{\partial f}{\partial z}\big)^2 T(f(z)) + \frac{c}{12} S(f(z), z) \tag{67}$$

where $S(w, z)$ denotes the so-called *Schwarzian derivative* defined as $S(w,z) \equiv \frac{1}{(\partial_z w)^2}\big((\partial_z w)(\partial_z^3 w) - \frac{3}{2}(\partial_z^2 w)^2\big)$. We are going to need this transformation rule in later considerations.

[7] they are actually related with the three-point function constants via the normalization of the two-point functions as: $C_{ijk} = C_{ij}^l d_{lk}$. See for example [12].

[8] one can show that a term of the form $(z-w)^{-3}$ is not compatible with the associativity of the OPE $T(z)T(w) = T(w)T(z)$ inside a correlation function.

[9] the factor of $1/2$ is a convenient normalization.

One important question that we haven't asked so far is how the conformal generators act on the Hilbert space of the quantum theory. In section 1.2.1 we understood the infinite *Witt* algebra of conformal transformations, but this was the algebra acting on the space of functions(classical fields). In order to find the *quantum* algebra we need to understand the commutation relations of the modes of T, which are now promoted to operators. It's at this very point where the OPE (66) will be crucial.

Let us perform a Laurent expansion of T in the following way:

$$T(z) = \sum_{n\in Z} z^{-n-2} L_n \qquad \text{where: } L_n = \frac{1}{2\pi i}\oint dz z^{n+1} T(z) \tag{68}$$

If we use this expansion in the conserved charge (55), along with an conformal transformation $\epsilon(z) = \sum_{n\in Z} \epsilon_n z^{n+1}$, we find that:

$$Q \equiv Q_\epsilon = \sum_{n\in Z} \epsilon_n L_n \tag{69}$$

The mode operators L_n (and $\overline{L}_n$ for the anti-chiral part) are the generators of the local conformal transformations on the Hilbert space. Let's compute their commutation relation and see where it leads us:

$$\begin{aligned}
[L_n, L_m] &= \oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i} z^{n+1} w^{m+1} [T(z), T(w)] \\
&= \frac{1}{(2\pi i)^2}\oint_{C(0)} dw w^{m+1} \oint_{C(w)} dz z^{n+1} \Bigg\{ \frac{c/2}{(z-w)^4} \\
&+ \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \text{non-sing} \Bigg\} \\
&= \frac{1}{2\pi i}\oint_{C(0)} dw w^{m+1} \Bigg\{ \frac{1}{12} c(n+1)n(n-1) w^{n-2} \\
&+ 2(n+1) w^n T(w) + w^{n+1}\partial T(w) + 0 \Bigg\} \\
&= \frac{1}{12} cn(n^2-1)\delta_{n+m,0} + 2(n+1)L_{m+n} \\
&- \frac{1}{2\pi i}\oint dw (n+m+2) w^{n+m+1} T(w) \\
&= (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0}
\end{aligned} \tag{70}$$

This is the celebrated *Virasoro Algebra*. The quantum generators obey almost an identical algebra as the classical generators, except for a central term. If we perform analogous calculations for $\overline{T}$, we end up with the overall algebra:

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0} \\
[\overline{L}_n, \overline{L}_m] &= (n-m)\overline{L}_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0} \\
[L_n, \overline{L}_m] &= 0
\end{aligned} \tag{71}$$

We have to emphasize some important facts about (71) before we go on. First, we notice that the holomorphic and anti-holomorphic components decouple. Therefore, representations of the Virasoro algebra -which is our next goal- can be equivalently constructed for any part (holomorphic or anti-holomorphic) and, in the case of S^2, one can then construct general representations by merely taking the tensor product of the two sectors[10]. Secondly, the commutation relations for L_0, L_{-1}, L_1(and their corresponding anti-holomorhic parts) are the same as the commutation relations of l_0, l_{-1}, l_1. Hence, it is still true that these are exactly the generators of the global $PSL(2,\mathbb{C})$ transformations, i.e. L_{-1} generates translations, L_0 generates dilations and L_{+1} generates special conformal transformations. In the radial quantization framework in particular, the momentum operator is given by $P = i(L_0 - \overline{L}_0)$ and the Hamitonian by $H = L_0 + \overline{L}_0$. In a physical theory with conformal symmetry, we expect the eigenstates of H to fall into representations of the conformal algebra. This will be the topic of the next section.

1.2.5 CFT Hilbert space

In section 1.2.3 we built our first quantum states based on the knowledge of *in* and *out* states from usual quantum field theories. The Φ field we considered was a general field, with the only assumption that admits an expansion in Laurent modes. Let's now examine what happens if our field is a *primary* field, and apply the technology of the OPE.

The commutator of a Virasoro generator with a *primary* field Φ yields:

$$\begin{aligned}[L_n, \Phi(w,\overline{w})] &= \frac{1}{2\pi i}\oint_w dz z^{n+1} T(z)\Phi(w,\overline{w}) \\ &= \frac{1}{2\pi i}\oint_w dz z^{n+1}\left[\frac{h\Phi(w,\overline{w})}{(z-w)^2} + \frac{\partial\Phi(w,\overline{w})}{z-w} + \text{non-sing}\right] \\ &= h(n+1)w^n\Phi(w,\overline{w}) + w^{n+1}\partial\Phi(w,\overline{w}) \quad (\text{for } n \geq -1)\end{aligned} \tag{72}$$

The anti-holomorphic counterpart of this relation is:

$$[\overline{L}_n, \Phi(w,\overline{w})] = \overline{h}(n+1)\overline{w}^n\Phi(w,\overline{w}) + \overline{w}^{n+1}\overline{\partial}\Phi(w,\overline{w}) \quad (\text{for } n \geq -1) \tag{73}$$

After applying the above relations to an asymptotic state $\left|h,\overline{h}\right\rangle \equiv \Phi(0,0)\left|0\right\rangle$ created by the primary field, we get:

$$\begin{aligned} L_0\left|h,\overline{h}\right\rangle &= h\left|h,\overline{h}\right\rangle \quad , \quad \overline{L}_0\left|h,\overline{h}\right\rangle = \overline{h}\left|h,\overline{h}\right\rangle \\ L_n\left|h,\overline{h}\right\rangle &= 0 \qquad\quad , \quad \overline{L}_n\left|h,\overline{h}\right\rangle = 0 \quad \textit{for } n > 0 \end{aligned} \tag{74}$$

where the invariance of the vacuum under $L_0, \overline{L_0}$ has been used. Thus the state $\left|h,\overline{h}\right\rangle$ is an eigenstate of the Hamiltonian and is annihilated by all the modes with $n > 0$. These properties can serve as a definition for a *primary state* in the quantum theory.

Excited states above the the asymptotic state $\left|h,\overline{h}\right\rangle$ can be constructed if we notice that the generators L_{-n} $(m > 0)$ increase the conformal dimension of the state, by virtue of the Virasoro algebra[11]:

$$[L_0, L_{-m}] = mL_{-m} \tag{75}$$

[10] we will examine more carefully this statement when we study conformal symmetry on a torus $\mathbb{T}^2$.

[11] from now on we focus only on the holomorphic sector; analogous results hold for the anti-holomorphic part unless otherwise emphasized.

This means that excited states may be obtained by successive applications of these operators on the asymptotic state $|h\rangle$:

$$L_{-k_1}L_{-k_2}...L_{-k_n}\,|h\rangle \qquad (1 \leq k_1... \leq k_n) \tag{76}$$

The states (76) are eigenstates of the Hamiltonian and called *descendants* of the primary state $|h\rangle$. Sometimes is common to say that they belong to the *conformal family* of $|h\rangle$.Their eigenvalue is:

$$h' = h + k_1 + k_2 + .. + k_n \equiv h + N \tag{77}$$

and the number of distinct,linearly independent states at *level* N is simply the number of partitions $p(N)$ of the integer N.

The subset of the full Hilbert space generated by the asymptotic (or highest-weight) state $|h\rangle$ and its descendants is closed under the action of the Virasoro generators and thus forms a *representation* (or a *module*) of the Virasoro algebra. This subspace is called a *Verma module*, and the representations constructed in the above way are commonly refered to as *highest weight representations*.[12]

We are almost done in specifying a Hilbert space for a 2-dimensional CFT on S^2. We need also an inner product in order to define norms. It turns out we can construct it quite easily from the knowledge we already have. The hermitian conjugate of a Virasoro generator is: $L_m^\dagger = L_{-m}$. Hence, the inner product of two states

$$L_{-k_1}...L_{-k_m}\,|h\rangle \qquad \text{and} \qquad L_{-l_1}...L_{-l_n}\,|h\rangle \tag{78}$$

is simply

$$\langle h|\, L_{k_1}...L_{k_m}L_{-l_1}...L_{-l_n}\,|h\rangle \tag{79}$$

where the dual of a primary state is defined to satisfy: $\langle h|\, L_m = 0$ for $m < 0$. It is worth noticing that the inner product of two states vanishes unless they belong to the same level. One can proceed with the calculations here by passing the L_{k_m}'s over the L_{-l_n}'s using the Virasoro algebra. So two eigenstates of L_0 having different eigenvalues are orthogonal to each other. Hermiticity also renders h real.

Analogously, we can state similar arguments for the Verma modules of the anti-holomorphic sector. Denoting by $V(c,h)$ and $\overline{V}(\overline{h},c)$ the Verma modules generated respectively by the sets L_n and $\overline{L}_n$ for a value c of the central charge and with highest weigths h and $\overline{h}$, we are now in a position to write down the structure of the Hilbert space: it is the direct sum of tensor products $V \otimes \overline{V}$ over all conformal dimensions of the theory:

$$\mathcal{H} = \oplus_{h,\overline{h}}\, V(c,h) \otimes \overline{V}(\overline{h},c) \tag{80}$$

We have to emphasize that the number of terms in the sum can be infinite or finite; we can also have several terms with the same conformal dimension.

We have achieved our goal to construct a Hilbert space of quantum theory with conformal symmetry on S^2. Expression (80) is quite elegant. It tells us that the content

[12]this is reminiscent of the construction of $su(2)$ spin algebra in Quantum Mechanics.

of primary fields and their corresponding conformal dimensions constitute *dynamical* data for a CFT. After knowing that, we can go ahead and construct the descendants of each conformal family, take tensor product of the holomorphic and anti-holomorphic sector and sum over distinct families. Furthermore, energy eigenstates merely fall into representations of $V \otimes \overline{V}$.

One question we want to ask next is, what is the structure of the unitary representations of the Virasoro algebra. This will be the subject of the next section and, as we will see, it will lead us to a particular set of CFTs called *Minimal Models*.

1.2.6 Minimal models: Part I

In order to build a generic unitary theory in quantum mechanics. we need to make sure that a representation of states contains no negative-norm states. The necessary and sufficient conditions for unitarity are found by considering the so-called *Gram matrix* of inner products between all basis states.

For the Verma module, let's denote as $|i\rangle$ a basis state. Then,

$$M_{ij} = \langle i|j\rangle \qquad (M^\dagger = M) \tag{81}$$

is the Gram matrix. In a Verma module, states at different level are orthogonal to each other, hence the Gram matrix is block diagonal, with blocks corresponding to a particular level. We are going to focus only on the matrix elements of particular blocks in what comes next.

Let's now consider a generic state as a linear combination $|a\rangle = \sum_i a_i |i\rangle$. Its norm is:

$$\langle a|a\rangle = a^\dagger M a \tag{82}$$

Since M is hermitian we can diagonalize it with a unitary matrix U: $M = U\tilde{M}U^\dagger$. If also $|b\rangle = U|a\rangle$, then we have:

$$\langle a|a\rangle = \sum_i \lambda_i |b_i|^2 \tag{83}$$

where λ_i are the diagonal elements of $\tilde{M}$, i.e. the eigenvalues of the Gram matrix.

Therefore, it's easy to see that there will be negative-norm states if and only if M has one or more negative eigenvalues. Furthermore, in principle there will be states of zero norm, let us call them *null states* from now on, if one of the eigenvalues λ_i vanishes. The number of linearly independent null states is simply given by the number of roots of the equation $detM = 0$. It turns out that, for the Virasoro representations, null states generate an independent Verma module which is orthogonal to the parent one. The importance of this "orthogonality" will become clear later when we discuss the *fusion rules*.

Let us make a first example and obtain some basic results. If we calculate the norm of the *n-level* state $L_{-n}|h\rangle$ inside the Verma module of some primary state $|h\rangle$ we get:

$$\begin{aligned} \langle h| L_n L_{-n} |h\rangle &= \langle h| L_{-n}L_n + 2nL_0 + \frac{1}{12}cn(n^2-1)\,|h\rangle \\ &= \left[2nh + \frac{1}{12}cn(n^2-1)\right]\langle h|h\rangle \end{aligned} \tag{84}$$

where the Virasoro algebra is used. If we choose $n = 1$ then we have $\langle h| L_1 L_{-1} |h\rangle = 2h$, which means that only only primaries with $h > 0$ constitute a unitary theory. If we

choose n sufficiently large, the term $\frac{1}{12}cn^3$ will dominate in (84), which automatically implies that the central charge is also positive in a unitary theory, $c > 0$. Soon, we are going to find ourselves describing several conclusions about unitarity in an (h, c)-plane; we now know that whatever lies outside the first quadrant of this plane leads to a non-unitary theory, which is relatively a strong constraint.

A second good example includes states at level 2. Here we will see the role of null states determined by the equation $detM = 0$. At level 2 there are two basis states we can choose, namely $L_{-2}|h\rangle$ and $L_{-1}^2|h\rangle$. The Gram matrix reads:

$$M_2(c,h) = \begin{pmatrix} 4h + \frac{c}{2} & 6h \\ 6h & 4h(2h+1) \end{pmatrix} \tag{85}$$

and the determinant:

$$\begin{aligned} detM_2(c,h) &= 32h(h^2 - \frac{5}{8}h + \frac{1}{8}hc + \frac{1}{16}c) \\ &\equiv 32\big(h - h_{1,1}(c)\big)\big(h - h_{1,2}(c)\big)\big(h - h_{2,1}(c)\big) \end{aligned} \tag{86}$$

wherein:

$$\begin{aligned} h_{(1,1)} &= 0 \\ h_{(1,2)} &= \frac{1}{16}(5 - c - \sqrt{(1-c)(25-c)}) \\ h_{(2,1)} &= \frac{1}{16}(5 - c + \sqrt{(1-c)(25-c)}) \end{aligned} \tag{87}$$

The notation will become clear in what follows. In summary, at level 2 we found three *non-trivial* states of vanishing norm. V.Kač in 1978 found a very powerful formula for the determinant of the Gram matrix at arbitrary level N:

$$detM_N(c,h) = \alpha_N \prod_{p,q \leq N, pq \geq 1} \Big(h - h_{p,q}(c)\Big)^{P(N-pq)} \tag{88}$$

with

$$h_{p,q}(c) \equiv \frac{\big((m+1)p - mq\big)^2 - 1}{4m(m+1)}, \qquad m \equiv -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{25-c}{1-c}} \tag{89}$$

α_N a positive constant independent of c and h, and $P(..)$ is the number of partitions of its argument. Also we have to emphasize that m is in general complex for generic values of $c > 0$.

In which sense can the Kac determinant be useful in the classification of the unitary representations? So far, we have seen that the first quadrant of (h, c) plane is allowed. But still, it will be constructive if we had some stronger constraints. Starring at (89), we can see that $h_{p,q}(c)$ defines curves on that plane (Fig1.2(c)). These are the curves indicating the null states. But how one can determine negative norm states -if any- in that first quadrant? It turns out that there is indeed a way to do it systematically. We will not go into the details of this proof here[13], instead we will summarize the interesting results that come up:

[13] we refer the reader to the usual reference [1].

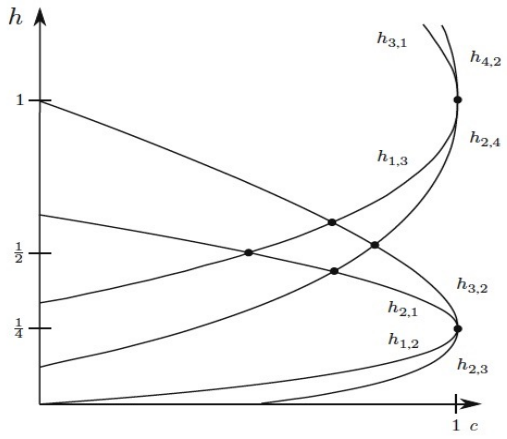

Figure1.2(c) Some of the curves $h_{p,q}(c)$ characterized by vanishing Kač determinant. The various *dots* represent unitary representations.

c>1:
For $h > 0$ and $c > 1$ there are no zeros and all eigenvalues of M_N are positive. Unitary representations can exist for this range.

c=1:
In this case, one can show that the Kac determinant vanishes for $h = \frac{n^2}{4}$ where $n \in \mathbb{Z}$.

c<1:
For this region, it can be shown first that the points which do not lie on a curve $h_{p,q}(c)$ (where $detM_N = 0$) correspond to a non-unitary theory. In fact, only certain *intersection points* of these vanishing curves define a unitary theory with non-negative states. The dots in Fig.1.2(c) indicate this discrete set of theories. We can even right down the expression for the conformal dimensions of the operators of such theories:

$$\begin{aligned} c &= 1 - \frac{6}{m(m+1)} \ , \qquad\qquad m = 3, 4, .. \\ h_{p,q}(m) &= \frac{\big((m+1)p - mq\big)^2 - 1}{4m(m+1)} \ , \quad 1 \leq p \leq m-1 \text{ and } 1 \leq q \leq m \end{aligned} \tag{90}$$

One might notice that these expressions coincides with eq.(89) above. Indeed, the two expressions are the same, except now m is an integer greater than 2, and q, p take values in a different range; to each c there are only $\binom{m}{2}$ allowed values of h.

We should emphasize at this point the power of conformal symmetry. By arguing only with group theory arguments, without writing down any concrete conformal theory model, we saw that one can obtain theories with finite number of primary fields at a specific rational value of the central charge, $c < 1$. Their simplicity in principle allows for a complete solution (i.e. specifying completely the dynamical data and hence, the correlation functions) and strikingly, most of them can be identified with known statistical models at criticality as we will see next. This class of theories deserve a name and called *Minimal Model CFTs*. Determining and examining dynamical data of some of these theories is basically the purpose of this thesis.

Before moving on and investigating further these theories, we should mention at this point that unitarity is not a severe constraint. Indeed, as we will see with the example

of the Lee-Yang model, it seems that the statistical models of so-called hard objects (i.e., of bulky objects that cannot overlap, subject to simple short-ranged interactions) always admit critical continuum descriptions with non-unitary conformal field theories. Without going into much details again, it can be shown that this more general class of minimal CFTs have central charges:

$$\begin{aligned} c &= 1 - \frac{6(p-q)^2}{pq} \ , \qquad && p,q \geq 2 \quad \text{and rel.coprime} \\ h_{r,s}(p,q) &= \frac{(pr-qs)^2-(p-q)^2}{4pq} \ , \qquad && 1 \leq r \leq q-1 \text{ and } 1 \leq s \leq p-1 \end{aligned} \tag{91}$$

Note the obvious symmetry property: $h_{r,s} = h_{q-r,p-s}$. One should compare the general expression (91) with (90). Whenever $|p-q| = 1$ one can parametrize the equations using $p \equiv m+1$ and $q \equiv m$ for some $m > 2$, and in this way we recover the unitary models in (90). In principle though, we can have generic values of p, q (always relative coprime and > 2) with $|p-q| \neq 1$ which correspond to nonunitary minimal CFTs. It is common to denote a minimal model associated with a pair (p,q) by $\mathcal{M}(p,q)$ and we will adopt the convention $p > q$.

Of course, the above discussion is restricted to the holomorphic sector. The full theory requires tensor products of holomorphic and anti-holomorphic sectors, as we mentioned earlier. However, one has to be careful when combining the components of a minimal model into tensor products as we will find out in the next section. For now, a particularly simple solution is to associate to each holomorphic module, let's call it $M(c, h_{r,s})$, the corresponding anti-holomorphic module $\overline{M}(c, \overline{h}_{r,s})$. The Hilbert space of the theory is then,

$$\mathcal{H} = \oplus_{\substack{1 \leq r < q \\ 1 \leq s < p}} M(c, h_{r,s}) \otimes \overline{M}(c, \overline{h}_{r,s}) \tag{92}$$

The resulting theory is termed *diagonal*, since the two factors of each tensor product are identical. We will soon explore what a *non-diagonal* minimal model is.

Let's now try to understand further the impact of the existence of null states in the minimal models. As we will see, there are severe restrictions on the form of OPE between primary fields as well. This is quite remarkable because, not only have we achieved a restriction in the allowed operator dimensions, we can also obtain non-trivial information on the other piece of dynamical data of a CFT, namely the OPE coefficients. We will illustrate this point with the example of a null state at level 2.

A general state at level 2 can be written as a linear combination of $L_{-2}\,|h\rangle$ and $L_{-1}^2\,|h\rangle$. If we assume it is null, then:

$$\big(L_{-2} + a\ L_{-1}^2\big)\,|h\rangle = 0 \tag{93}$$

If we apply L_1 we can fix the constant a:

$$\begin{aligned} 0 &= [L_1, L_{-2}]\,|h\rangle + a\ [L_1, L_{-1}^2]\,|h\rangle \\ &= 3L_{-1}\,|h\rangle + a\big(2L_0L_{-1} + 2L_{-1}L_0\big)\,|h\rangle \\ &= \big(3 + 2a(2h+1)\big)L_{-1}\,|h\rangle \end{aligned} \tag{94}$$

$$\Rightarrow \quad a = -\frac{3}{2(2h+1)}$$

If we apply L_2:

$$\begin{aligned} 0 &= [L_2, L_{-2}]\,|h\rangle + a\,[L_2, L_{-1}^2]\,|h\rangle \\ &= \big(4L_0 + \frac{c}{2}\big)\,|h\rangle + aL_{-1}[L_2, L_{-1}]\,|h\rangle + a[L_2, L_{-1}]L_{-1}\,|h\rangle \\ &= \big(4h + \frac{c}{2} + 6ah\big)\,|h\rangle \end{aligned} \tag{95}$$

$$\Rightarrow \quad h = \frac{1}{16}\big(5 - c \pm \sqrt{(c-1)(c-25)}\big)$$

which is exactly the dimension of the null primary state we obtained previously(c.f.(87)). We reached the conclusion that at level 2 we have a descendant field of $|h\rangle$ which satisfies:

$$\big(L_{-2} - \frac{3}{2(2h+1)}\,L_{-1}^2\big)\,|h\rangle = 0 \tag{96}$$

We would like to somehow use this result in the computation of correlation functions. So let's examine how descendant states define a *descendant field*. Consider, for instance, the descendant $L_{-n}\,|h\rangle$:

$$L_{-n}\,|h\rangle = L_{-n}\Phi(0)\,|0\rangle = \frac{1}{2\pi i}\oint dz z^{1-n}T(z)\Phi(0)\,|0\rangle = (L_{-n}\Phi)(0)\,|0\rangle \tag{97}$$

So the natural definition of the descendant field associated with the state $L_{-n}\,|h\rangle$ is:

$$\widehat{L}_{-n}\Phi(w) \equiv \frac{1}{2\pi i}\oint_w dz \frac{1}{(z-w)^{n-1}}T(z)\Phi(w) \tag{98}$$

From the OPE of $T(z)$ with $\Phi(w)$ (c.f.(61)) we can determine the descendant fields for the first values of n, e.g.

$$\widehat{L}_0\Phi(w) = h\Phi(w)\ , \qquad \widehat{L}_{-1}\Phi(w) = \partial\Phi(w)\ ,\ ... \tag{99}$$

The important take home message from the above expressions is that the physical properties of these fields, i.e. their correlation functions, may be derived from those of the ancestor primary field. Indeed, if we consider the correlator:

$$\langle\ (L_{-n}\Phi)(w)\mathcal{X}\ \rangle \tag{100}$$

where $\mathcal{X} = \Phi_1(w_1)...\Phi_n(w_n)$ a string of primary fields h_i, we can calculate it using (98) and performing the contour integration appropriately. In particular, the contour circles w only, excluding the positions w_i of the other fields. The residue theorem may be applied by *reversing* the contour and hence summing the contributions from the poles at w_i, with the help of the OPE of T with primaries:

$$\begin{aligned} \langle (L_{-n}\Phi)(w)\mathcal{X}\rangle &= \frac{1}{2\pi i}\oint_w dz(z-w)^{1-n}\,\langle T(z)\Phi(w)\mathcal{X}\rangle \\ &= -\frac{1}{2\pi i}\oint_{\{w_i\}} dz(z-w)^{1-n}\sum_i \left\{ \frac{1}{z-w_i}\partial_{w_i}\,\langle\Phi(w)\mathcal{X}\rangle \right\} \\ &\quad + \frac{h_i}{(z-w_i)^2}\,\langle\Phi(w)\mathcal{X}\rangle \\ &\equiv \mathcal{L}_{-n}\,\langle\Phi(w)\mathcal{X}\rangle \qquad (n \geq 1) \end{aligned} \tag{101}$$

wherein the differential operator is:

$$\mathcal{L}_{-n} \equiv \sum_i \left\{ \frac{(n-1)h_i}{(w_i - w)^n} - \frac{1}{(w_i - w)^{n-1}} \partial_{w_i} \right\} \tag{102}$$

The above result is quite important. If our descendant state is also *null*, then we get constraints in the form of correlation functions! Let's study the case of two and three-point functions when our state is (96). The corresponding descendant *field* is:

$$\widehat{L}_{-2}\Phi(z) - \frac{3}{2(2h+1)} \widehat{L}_{-1}^2 \Phi(z) \tag{103}$$

and its correlation with any other primary field should vanish:

$$\begin{aligned} 0 &= \left(\mathcal{L}_{-2} - \frac{3}{2(2h+1)} \mathcal{L}_{-1}^2 \right) \langle \Phi(w) \Phi_1(w_1) .. \Phi_N(w_N) \rangle \\ &= \left(\sum_{i=1}^{N} \left(\frac{h_i}{(w_i - w)^2} - \frac{1}{w_i - w} \partial_{w_i} \right) - \frac{3}{2(2h+1)} \partial_w^2 \right) \langle \Phi(w) \Phi_1(w_1) .. \Phi_N(w_N) \rangle \end{aligned} \tag{104}$$

Therefore, if we consider the example of a *two-point function* we obtain:

$$\begin{aligned} 0 &= \left(\frac{h}{w_1 - w} - \frac{1}{w_1 - w} \partial_{w_1} - \frac{3}{2(2h+1)} \partial_{w_1}^2 \right) \frac{d}{(w - w_1)^{2h}} \\ &= \left(h + 2h - \frac{3}{2(2h+1)} 2h(2h+1) \right) \frac{d}{(w - w_1)^{2h+2}} \end{aligned} \tag{105}$$

which is trivially satisfied. However, for the *three-point function*:

$$\langle \Phi(w) \Phi_1(w_1) \Phi_2(w_2) \rangle = \frac{C_{hh_1h_2}}{(w - w_1)^{h+h_1+h_2} (w_1 - w_2)^{h_1+h_2-h} (w - w_2)^{h-h_1+h_2}} \tag{106}$$

we got a non-trivial constraint between the conformal dimensions of the primary fields involved, which we didn't know before! After using (104) one can obtain the following constraint for the conformal weights $\{h, h_1, h_2\}$:

$$2(2h+1)(h + 2h_2 - h_1) = 3(h - h_1 + h_2)(h - h_1 + h_2 + 1) \tag{107}$$

If we solve this in terms of h_2 for example, then:

$$h_2 = \frac{1}{6} + \frac{h}{3} + h_1 \pm \frac{2}{3} \sqrt{h^2 + 3hh_1 - \frac{1}{2}h + \frac{3}{2}h_1 + \frac{1}{16}} \tag{108}$$

The expression (108) is very crucial. It highlights the fact that the OPE between two primaries in a minimal model includes an explicit and finite number of operators. Any OPE coefficient corresponding to an exchanged field that is not allowed is *zero*.

Let us apply (108) to the primary fields $\Phi_{p,q}$ of the minimal models with central charges given by (90). If we choose $h = h_{2,1}$ and $h_1 = h_{p,q}$, then the two solutions for h_2 are exactly: $\{h_{p-1,q}, h_{p+1,q}\}$. Therefore, at most two of the coefficients $C_{hh_1h_2}$ will be non-zero. It is common notation to write:

$$\left[h_{(2,1)}\right] \times \left[h_{(p,q)}\right] = \left[h_{(p+1,q)}\right] + \left[h_{(p-1,q)}\right] \tag{109}$$

where this equation means that the OPE between a field in the conformal family of $h_{(2,1)}$ and a field in the conformal family of $h_{(p,q)}$ involves only fields belonging to the conformal families of of $h_{(p+1,q)}$ and $h_{(p-1,q)}$. More work is needed if one wants to determine the precise value of the OPE coefficients. We are going to return on this issue when we develop the notion of conformal blocks.

The above manipulations can be generalized to higher level null states. Without going into the detailed derivation[14], it turns out that the conformal families in a *unitary* Minimal Model obey:

$$\left[h_{(p_1,q_1)}\right] \times \left[h_{(p_2,q_2)}\right] = \sum_{\substack{k=1+|p_1-p_2| \\ k+p_1+p_2 \text{ odd}}}^{p_1+p_2-1} \sum_{\substack{l=1+|q_1-q_2| \\ l+q_1+q_2 \text{ odd}}}^{q_1+q_2-1} \left[h_{(k,l)}\right] \tag{110}$$

These are the so-called *fusion rules* in unitary Minimal Models. The closed algebra that is formed between the conformal families is highly non-trivial and basically highlights the power of conformal symmetry in two dimensions with $c < 1$. One can actually obtain explicit formulae for the OPE coefficients of the Minimal Models by using a specific method called the *Coulomb gas formalism*, first developed in [15][16]. We are going to illustrate this formalism in Appendix A as we're going to use the results from there, to numerically specify the OPE coefficients for the Ising and Tricritical Ising model later on.

Concluding this section, we will introduce three physical theories which correspond to two unitary and one non-unitary Minimal Models. The two unitary examples constitute basically the main core of this thesis as later on, we will investigate their dynamical data on a non-orientable surface as well.

The Lee-Yang model $\mathcal{M}(5,2)$

The Lee-Yang model or the Lee-Yang *singularity* occurs in an Ising model above its critical temperature in an non-zero *purely imaginary* magnetic field ih. For h larger than some critical value $h_c(T)$ the partition function acquires zeros, which in the thermodynamic limit (N$\to \infty$) become "dense", in the sense that one can describe the free energy of the system in terms of a well-defined density of that zeros[15]. The density of these zeros near h_c has a power law behaviour $(h - h_c)^\sigma$ (where σ is to be related with the conformal dimension of a primary operator when we identify the model with a rational CFT) . It turns out that the relevant Landau-Ginsburg theory that describes the fluctuations of a scaling field Φ in an imaginary field close to $h = ih_c$ is:

$$\mathcal{L}_{LY} = \frac{1}{2}(\partial_\mu \Phi)^2 + i(h - h_c)\Phi + i\gamma\Phi^3 \tag{111}$$

The model is of course non-unitary, because of the imaginary magnetic field, which translates into an imaginary coupling of the Landau-Ginsburg effective field theory.

As J.Cardy explained in his paper[3], the conformal theory that describes this model is identified as the $\mathcal{M}(5,2)$ non-unitary minimal theory with central charge $c = -22/5$ and operator content consisting merely of two primary fields:

[14] see ch.8 in [1].

[15] Interestingly, people have also speculated about a relationship between the Lee-Yang model and the Riemann hypothesis which involves the non trivial zeros of the *zeta function*. See [Knauf Andreas, *Number theory, dynamical systems and statistical mechanics*, Rev. Math. Phys., 11, 1027 (1999).

Symbol	*Kac* operator	$(h, \overline{h})$
1	$h_{1,1} = h_{1,4}$	$(0, 0)$
Φ	$h_{1,2} = h_{1,3}$	$(-\frac{1}{5}, -\frac{1}{5})$

Table1.2(a): List of the operator content of the $\mathcal{M}(5, 2)$ theory which correspond to the physical Lee-Yang model.

The single fusion rule is:

$$\Phi \times \Phi = 1 + \Phi \tag{112}$$

The Ising model $\mathcal{M}(4, 3)$

The first non-trivial *unitary* minimal model is the $\mathcal{M}(4, 3)$ and, strikingly, it describes the critical Ising model in two-dimensions. The Ising model consists of a square lattice with spin variables at each site taking values ± 1. With a nearest neighbour interaction the theory has a conformally invariant critical point where it's described by three local primary operators, namely the identity, the spin operator, and the energy density:

Symbol	*Kac* operator	$(h, \overline{h})$
1	$h_{1,1} = h_{2,3}$	$(0, 0)$
σ	$h_{1,2} = h_{2,2}$	$(\frac{1}{16}, \frac{1}{16})$
ϵ	$h_{1,3} = h_{2,1}$	$(\frac{1}{2}, \frac{1}{2})$

Table1.2(b): List of the operator content of the $\mathcal{M}(4, 3)$ theory which correspond to the critical Ising model.

The fusion rules are:

$$\begin{aligned} \sigma \times \sigma &= 1 + \epsilon \\ \sigma \times \epsilon &= \sigma \\ \epsilon \times \epsilon &= 1 \end{aligned} \tag{113}$$

The Tricritical Ising model $\mathcal{M}(5, 4)$

This conformal field theory has a statistical interpretation as a simple generalization of the Ising model, where we now allow the spin variable to take values $\{0, \pm 1\}$. This adds to the model a chemical potential associated to the fractional occupation number. This new parameter modifies the structure of the phase diagram, so that there is now a tricritical point where three phases meet: paramagnetic, ferromagnetic and a two-phase region. This tricritical Ising model now has six primary operators, corresponding to the identity, three energy operators and two spin like operators[16]:

Symbol	*Kac* operator	$(h, \overline{h})$
1	$h_{1,1} = h_{3,4}$	$(0, 0)$
ϵ	$h_{1,2} = h_{3,3}$	$(\frac{1}{10}, \frac{1}{10})$
ϵ'	$h_{1,3} = h_{3,2}$	$(\frac{3}{5}, \frac{3}{5})$
ϵ''	$h_{1,4} = h_{3,1}$	$(\frac{3}{2}, \frac{3}{2})$
σ	$h_{2,2} = h_{2,3}$	$(\frac{3}{80}, \frac{3}{80})$
σ'	$h_{2,4} = h_{2,1}$	$(\frac{7}{16}, \frac{7}{16})$

[16]the Tricritical Ising model is also the unique minimal model endowed with supersymmetry, but we are not going need this feature in our analysis.

Table1.2(c): List of the operator content of the $\mathcal{M}(5,4)$ theory which correspond to the Tricritical Ising model.

The fusion rules read:

$$\begin{aligned}
\epsilon \times \epsilon &= 1 + \epsilon' \\
\epsilon \times \epsilon' &= \epsilon + \epsilon'' \\
\epsilon \times \epsilon'' &= \epsilon' \\
\epsilon' \times \epsilon' &= 1 + \epsilon' \\
\epsilon' \times \epsilon'' &= \epsilon \\
\epsilon'' \times \epsilon'' &= 1 \\
\epsilon \times \sigma &= \sigma + \sigma' \\
\epsilon \times \sigma' &= \sigma \\
\epsilon' \times \sigma &= \sigma + \sigma' \\
\epsilon' \times \sigma' &= \sigma \\
\epsilon'' \times \sigma &= \sigma \\
\epsilon'' \times \sigma' &= \sigma' \\
\sigma \times \sigma &= 1 + \epsilon + \epsilon' + \epsilon'' \\
\sigma \times \sigma' &= \epsilon + \epsilon' \\
\sigma' \times \sigma' &= 1 + \epsilon''
\end{aligned} \tag{114}$$

One basic characteristic that Ising and Tricritical Ising model share is that they are the unique CFTs, within the unitary minimal model classification, with non-trivial operator dimensions satisfying $h > c/24$[17]. Theories with that feature are usually termed *Extremal* CFTs [4] and, as we will explore soon, they are really important in the context of three-dimensional gravity.

1.2.7 Crossing symmetry and Conformal Bootstrap

In the previous section we explored the power of conformal symmetry in two dimensions when one wants to impose unitarity for the representations of the Virasoro algebra. In particular, we saw that in some cases ($c \leq 1$) one is able to constraint very strongly the *dynamical data* of a CFT on S^2, namely the spectrum of primary operators and the OPE coefficients (c.f. fusion rules). In the present section we will study further *generic* constraints on these data arising purely from conformal symmetry. This is the basic strategy of the so-called *conformal bootstrap* program -which was initiated in two dimensions by A.Belavin, A.Polyakov and A.Zamolodchikov in [5], and then in higher dimensions by R.Ratazzi, S.Rychkov, E.Tonni and A.Vichi [6]: the approach to solve and classify all CFTs by imposing a set of consistency conditions that constrain the allowed conformal dimensions and OPE coefficients.

In a two-dimensional CFT defined on a closed orientable Riemann surface, the basic statement of the consistency of the theory is summarized elegantly in two sufficient conditions [7],[8]: (a) *crossing symmetry* of the four-point functions on S^2, and (b)

[17] moreover, as we will see, these two theories have a unique *diagonal* partition function, in contrast with the rest minimal models.

modular covariance of the one-point functions on the torus T^2. We will expand on the former here.

Let's start with a generic four-point function on S^2:

$$\langle \Phi_1(z_1,\overline{z}_1)\Phi_2(z_2,\overline{z}_2)\Phi_3(z_3,\overline{z}_3)\Phi_4(z_4,\overline{z}_4)\rangle \tag{115}$$

We have seen (c.f. (38)) that such a function depends continuously on the anharmonic cross ratio $\eta \equiv \frac{z_{12}z_{34}}{z_{13}z_{24}}$. Global conformal symmetry further allows us to put $z_1 = \infty$, $z_2 = 1$, $z_4 = 0$ and $z_3 = z$. The conformally invariant cross ratio becomes $\eta = z$ and the above correlation function may be related to a matrix element between two asymptotic states of a two-field product:

$$\lim_{z_1,\overline{z}_1\to\infty} z_1^{2h_1}\overline{z}_1^{2\overline{h}_1}\,\langle \Phi_1(z_1,\overline{z}_1)\Phi_2(1,1)\Phi_3(z,\overline{z})\Phi_4(0,0)\rangle \equiv G(z,\overline{z}) \tag{116}$$

Now one can actually evaluate (115) in several different ways:

1) We can use the OPE between Φ_1 and Φ_2 and then between Φ_3 and Φ_4. The amplitude, then, can be expressed as:

$$G(z,\overline{z}) = \sum_p C_{12}^p C_{34}^p \mathcal{F}_{12}^{34}(p;z)\overline{\mathcal{F}}_{12}^{34}(p;z) \tag{117}$$

where C_{12}^p, C_{34}^p are the relevant OPE coefficients and the contributions of the primary field Φ_p are factorized into a holomorphic and anti-holomorphic piece. The expressions $\mathcal{F}_{12}^{34}(p;z)$ are called *Virasoro conformal blocks*. They depend only on the conformal dimensions of the primary fields involved, on the central charge of the theory and, in principle, they are completely fixed by conformal symmetry.

A handful closed expression for a generic $\mathcal{F}$ is not known yet, although people have tried to extract some useful expressions [9],[10]. In the case of minimal models, Virasoro conformal blocks obey ordinary differential equations, and in the simplest examples are just hypergeometric functions[5]. There are also some exact interesting results of the blocks at large central charge [5],[9]. In general though, aside from these examples, one simple way we can start doing progress is by expanding the blocks in power series of the cross ratio z, and use conformal symmetry to calculate the different coefficients[11].

2) One can evaluate the four-point function alternatively by first using the OPE between Φ_2 and Φ_3 and between Φ_1 and Φ_4. Effectively, this means exchanging Φ_2 with Φ_4 in the previous result, and which on the level of cross ratios is achieved by $z \to 1-z$. The resulting four-point amplitude is expressed as:

$$G(z,\overline{z}) = \sum_p C_{14}^p C_{23}^p \mathcal{F}_{14}^{23}(p;1-z)\overline{\mathcal{F}}_{14}^{23}(p;1-z) \tag{118}$$

3) Similarly, we can first evaluate the OPE $\Phi_2\Phi_4$ which leads, in turn, to the following form for the amplitude:

$$G(z,\overline{z}) = z^{-2h_2}\overline{z}^{-2\overline{h}_2}\sum_p C_{13}^p C_{24}^p \mathcal{F}_{13}^{24}(p;\frac{1}{z})\overline{\mathcal{F}}_{13}^{24}(p;\frac{1}{z}) \tag{119}$$

The three above channels of the four-point amplitude are commonly called *s-channel*, *t-channel* and *u-channel* respectively. Ordering of fields within correlators does not

change the correlation function (except for signs when dealing with fermions); this means that expressions (117),(118),(119) are equal with each other. These equalities provide the set of consistency conditions that we were looking for. These so-called *crossing symmetry* conditions are depicted below in Fig.1.2(d)

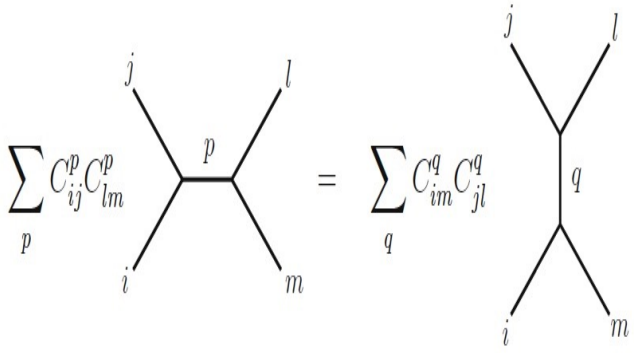

Figure1.2(d): The statement of crossing symmetry between the s and t-channel of the four-point function in diagrammatic language.

Assuming that the conformal blocks are known for arbitrary values of the conformal dimensions, these equalities in principle allow us to determine the coefficients C_{ij}^k and the conformal dimensions h_p. Indeed, if we assume the presence of N conformal families in the theory, the above relation yields, through naive counting, N^4 constraints on the $N^3 + N$ parameters C_{ij}^k and h_p, hence not only we can solve the theory completely but we also can classify the "allowed" CFTs among a vast set of theories that one can write down.

Before concluding the discussion for a CFT on S^2, we will discuss an important aspect of conformal blocks in minimal model CFTs, which will be important for our future calculations. In a minimal model CFT there are only a finite number of conformal families which can "propagate" in the OPE between two primaries. This means that the conformal blocks for the three distinct channels form a finite dimensional vector space. The crossing symmetry then suggests that the different classes of conformal blocks are nothing else than three different choices of basis which must be related by linear transformations. Indeed, the relation between s-channel and t-channel conformal blocks can be written as:

$$\mathcal{F}_{ij}^{kl}(p;z) = \sum_q M \begin{bmatrix} i & l \\ j & k \end{bmatrix}_{p,q} \mathcal{F}_{jk}^{il}(q;1-z) \tag{120}$$

and to clarify the notation is useful to present a graph:

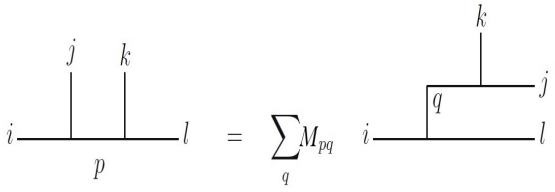

Figure1.2.(e): Diagrammatic illustration of the *fusing* matrices relating s and t-channels of conformal blocks.

The matrices $\mathrm{M}\begin{bmatrix} i & l \\ j & k \end{bmatrix}_{r,s}$ are called the *fusing* or *duality* matrices and depend on the external fusing operators. Notice that (p, q) indices denote the p-primary and q-primary exchanged in the s and t channel respectively. We can also define the so called *braiding matrices* B, which relate the s-channel with u-channel:

$$\mathcal{F}_{ij}^{kl}(p;z) = \sum_q B \begin{bmatrix} j & k \\ i & l \end{bmatrix}_{p,q} \mathcal{F}_{ik}^{jl}(q;\frac{1}{z}) \tag{121}$$

and graphically:

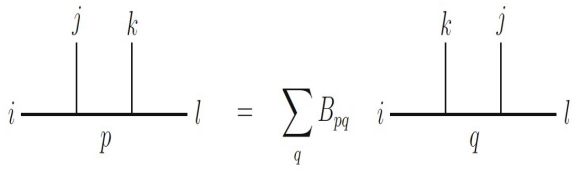

Figure1.2.(f): Diagrammatic illustration of the *braiding* matrices relating s and u-channels of conformal blocks.

Crucial for the purposes of this thesis are going to the *fusing* matrices M for the Minimal Models, as we will see later on. Closed expressions for these matrices were obtained in [15],[16] in the framework of Coulomb gas formalism and we are going to review the related formulas in Appendix A.

Let us finally mention that the braiding and fusing matrices satisfy two non-trivial identities. The fusing matrices satisfy the so called *pentagon identity* and the braiding matrices the so called *hexagon identity*:

$$M \begin{bmatrix} j & k \\ i & s \end{bmatrix}_{r,t} M \begin{bmatrix} t & l \\ i & m \end{bmatrix}_{s,u} = \sum_p M \begin{bmatrix} k & l \\ r & m \end{bmatrix}_{s,p} M \begin{bmatrix} j & p \\ i & m \end{bmatrix}_{r,u} M \begin{bmatrix} j & k \\ u & l \end{bmatrix}_{p,t}$$

$$\sum_p B \begin{bmatrix} j & k \\ i & s \end{bmatrix}_{r,p} B \begin{bmatrix} j & l \\ p & m \end{bmatrix}_{s,t} B \begin{bmatrix} k & l \\ i & t \end{bmatrix}_{p,u} = \sum_q B \begin{bmatrix} k & l \\ r & m \end{bmatrix}_{s,q} B \begin{bmatrix} j & l \\ i & q \end{bmatrix}_{r,u} B \begin{bmatrix} j & k \\ u & m \end{bmatrix}_{q,t} \tag{122}$$

These relations are basically derived by considering a five-point function and successively applying the fusing and braiding operations, respectively [12].

1.3 CFT on $\mathbb{T}^2$

In this section we will sketch the general properties of a two-dimensional conformal field theory defined on a genus one surface, namely the torus $\mathbb{T}^2$. The goal of this section is two-fold. First, we want to illustrate the general properties of a CFT on the next simple example of orientable (Riemann) surface, other than S^2. In fact, the construction we are going to implement in this section is also going to be useful later in defining CFTs on non-orientable surfaces. Secondly, the concept of *modular invariance* (or *covariance* in general), which is introduced specifically for the study of functions on a torus, is the *second* sufficient condition one needs to impose in a 2d CFT, in order to ensure higher point crossing symmetry and higher genus modular invariance[7][8], thus it is certainly very crucial to understand it.

1.3.1 Modular invariance and Partition function

We already know how conformal transformations act on the plane, as holomorphic or antiholomorphic coordinate changes. Now we want to examine conformal symmetry on the torus $\mathbb{T}^2$. All of the usual conformal transformations $z \to z + \epsilon(z)$ are still symmetries, but it turns out that there are also new, *large* conformal transformations, which cannot be continuously connected to the identity as we will find out.

In the previous section we defined a CFT on $S^2 = \mathbb{C} \cup \infty$ from the theory defined on a cylinder via the mapping:

$$z = e^w = e^{x^0} e^{ix^1} \tag{123}$$

where z is the coordinate on $\mathbb{C}$ and $w = x^0 + ix^1$ the coordinate on the cylinder. In this way, we were able to develop the concept of *radial quantization* and we introduced radial ordering for the evaluation of correlation functions as well as for operator product expansion. Our goal now is to define a theory on $\mathbb{T}^2$. A torus is obtained by cutting out a finite piece from the infinite cylinder and identifying the ends so that not only space coordinate but also time coordinate becomes finite.

It is actually useful to formalize this compactification from the point of view of a plane. We notice that a torus can be defined by identifying points w in the complex plane $\mathbb{C}$ as[18]:

$$w \sim w + m\alpha_1 + n\alpha_2, \qquad m, n \in \mathbb{Z} \tag{124}$$

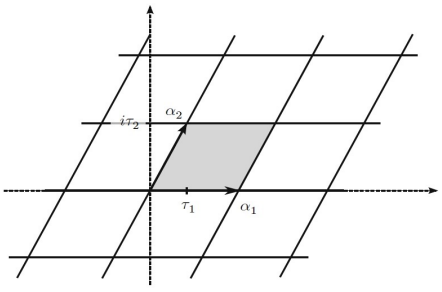

Figure1.3(a): Lattice of a torus generated by the complex vectors (α_1, α_2) as obtained from the plane. The shades region indicates the fundamental domain of the torus.

[18]note that the plane we are considering now is *not* the one defined from an exponential map from a cylinder. It is just a plane helping us to define the periodic identifications. From the plane, one can also start to define the cylinder (though we only identify one direction). This is eventually the reason why we use again the notation w for coordinates on this plane.

where (α_1, α_2) is a pair of complex numbers which spans the lattice whose smallest cell is called the fundamental domain of the torus. One should pay attention to the orientation of these vectors (before gluing together the ends of the cylinder we could have twisted them, for example). The quantity that describes the shape of the torus is actually the ratio of α_1 and α_2 and is called the *modular parameter* of the torus:

$$\tau \equiv \frac{\alpha_1}{\alpha_2} = \tau_1 + i\tau_2 \tag{125}$$

If there is another pair of complex numbers, say (β_1, β_2) which is related with (α_1, α_2) via integer multiples:

$$\begin{aligned} b_1 &= a\ \alpha_1 + b\ \alpha_2 \\ b_2 &= c\ \alpha_1 + d\ \alpha_2 \qquad \text{with} \quad a, b, c, d \in \mathbb{Z} \end{aligned} \tag{126}$$

then this pair is clearly describing the same fundamental domain of the torus, and hence the torus itself. We can compactly write the above statement as:

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \qquad a, b, c, d \in \mathbb{Z} \tag{127}$$

In general, for the inverse of this matrix to also have integer entries, we have to require that $ad - bc = \pm 1$ which just means that the unit cell in each basis should have the same volume (up to a sign, which from now on we choose it to be 1). Furthermore, we notice that the lattice spanned by (α_1, α_2) is equivalent to the one spanned by $(-\alpha_1, -\alpha_2)$, hence we can identify points related by a $\mathbb{Z}_2$ action.

We are led to consider the group of integer, invertible matrices with unit determinant divided by a $\mathbb{Z}_2$ action. This is the so-called *modular group* $SL(2,\mathbb{Z})/\mathbb{Z}_2$ or $PSL(2,\mathbb{Z})$ and is the new infinite set of *large* conformal transformations which basically acts on the modular parameter τ. The important point is that any physical quantity (correlation functions, partition function etc.) defined on the torus should transform covariantly under the action of the modular group. This fact is really crucial and powerful as we will explore soon.

The action of $PSL(2,\mathbb{Z})$ will keep τ in the upper half-plane, so this is where we're going to focus form now on and we call it H_+. This action is in general rather complicated. There is a domain though such that no pair of points within it can be reached through a modular transformation, and any point outside it can be reached from a unique point inside, by some modular transformation. This is the so called *fundamental domain* $H_+/SL(2,\mathbb{Z})$ of the modular group (Fig.1.3(b)):

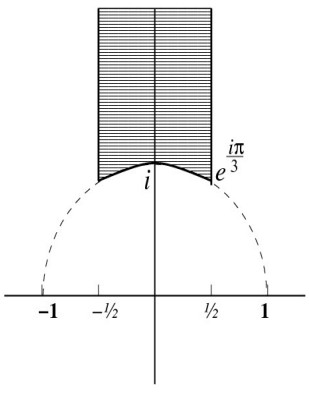

Figure1.3(b):The fundamental domain (*shaded* region) of inequivalent tori in the upper-half τ-plane.

One last interesting thing to know for the modular group is its generators. Consider the particular modular transformations:

$$\begin{aligned} T: &\quad \tau \to \tau + 1, &\qquad T &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\ S: &\quad \tau \to -\frac{1}{\tau}, &\qquad S &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned} \tag{128}$$

it can be shown that these two transformations generate the whole modular group, namely, each modular transformation A may be reduced to successive applications of S and T. In particular:

$$A = T^{n_1} S T^{n_2} S \cdots S T^{n_k} \tag{129}$$

where n_i are integers.

Now let's move to defining our physical quantities. We are going to start by studying the partition function in this subsection. For conformal field theories this is essentially the same object as in statistical mechanics where it is defined as a sum over all possible configurations weighted by the Boltzmann factor $exp(-\beta H)$. This quantity corresponds to the generating functional or the vacuum to vacuum amplitude in a finite temperature QFT where the time is compactified on a circle of radius $R = \beta = 1/T$.

For the case of the torus we can choose Rew to be the space direction and Imw to be the time direction. Refering to Fig.1.3(a), on a torus with non-trivial modular parameter $\tau = \tau_1 + i\tau_2$, we see that a time translation of length τ_2 does not end up at the starting point but is displaced in space by a τ_1. Therefore, a closed time circle on a torus involves also a space translation. This observation motivating the following definition of a CFT partition function[19]:

$$\mathcal{Z}(\tau_1, \tau_2) = \mathrm{Tr}_{\mathcal{H}} \left(e^{-2\pi\tau_2 H} e^{2\pi\tau_1 P} \right) \tag{130}$$

where H is the Hamiltonian and P the momentum operator. The trace is taken over all states in the Hilbert space $\mathcal{H}$. Let us determine these operators from the theory on the cylinder.

The ground state energy of the theory is calculated from the zero-zero component of the energy-momentum tensor T_{00}. But under conformal transformations T changes as in (67), with an extra piece involving a Schwarzian derivative. For the map (123) we obtain:

$$T_{cyl.}(w) = \left(\frac{\partial f}{\partial z}\right)^2 T(f(w)) + \frac{c}{12} S(f(w), w) = z^2 T(z) - \frac{c}{24} \tag{131}$$

It is interesting to see how the modes of T transform:

$$T_{cyl.}(w) = \sum_{n \in \mathbb{Z}} L_n z^{-n} - \frac{c}{24} = \sum_{n \in \mathbb{Z}} \left(L_n - \frac{c}{24} \delta_{n,0} \right) e^{-nw} \tag{132}$$

hence, for the zero mode in particular:

$$\left(L_{cyl.} \right)_0 \equiv L_0 - \frac{c}{24} \tag{133}$$

[19] note that we are always working in Euclidean signature.

Similar results hold for the anti-holomorphic component of T. Now, having in mind relation (25) for T in terms of complex coordinates, we get for the ground state:

$$E_0 = \langle (T_{cyl.})_{00} \rangle = \langle T_{cyl.} \rangle + \langle \overline{T}_{cyl.} \rangle = -\frac{c+\overline{c}}{24} \tag{134}$$

where we have assumed that the energy density $\langle T(z) \rangle$ vanishes on the plane. Since H is the generator of time translations, we can write:

$$H_{cyl.} = -\frac{\partial}{\partial t} + E_0 = -(\partial_w + \partial_{\overline{w}}) - \frac{c+\overline{c}}{24} = \big(L_{cyl.}\big)_0 + \big(\overline{L}_{cyl.}\big)_0 \tag{135}$$

using the fact that $L_0 = -z\partial_z = \partial_w$. Similarly for the momentum operator:

$$P_{cyl.} = i\big(\big(L_{cyl.}\big)_0 - \big(\overline{L}_{cyl.}\big)_0\big) \tag{136}$$

With these observations in hand we can express the partition function (130) as:

$$\begin{aligned} \mathcal{Z}(\tau_1,\tau_2) &= \mathrm{Tr}_{\mathcal{H}} \left(e^{-2\pi\tau_2\left(\left(L_{cyl.}\right)_0 + \left(\overline{L}_{cyl.}\right)_0\right)} e^{2\pi\tau_1\left(i\left(\left(L_{cyl.}\right)_0 - \left(\overline{L}_{cyl.}\right)_0\right)\right)} \right) \\ &= \mathrm{Tr}_{\mathcal{H}} \left(e^{2\pi i\tau (L_{cyl.})_0} e^{-2\pi i\overline{\tau}(\overline{L}_{cyl.})_0} \right) \\ &= \mathrm{Tr}_{\mathcal{H}} \left(q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{\overline{c}}{24}} \right) \qquad\qquad \text{where} \quad q \equiv e^{2\pi i\tau} \end{aligned} \tag{137}$$

We have arrived at the final expression for the partition function of a CFT defined on a torus with modular parameter τ. In general, the modular group is acting in a non-obvious way on $\mathcal{Z}$, but since modular transformations of τ do not change the torus as we saw, the partition function should be *invariant*:

$$\mathcal{Z}(\tau,\overline{\tau}) = \mathcal{Z}\left(\frac{a\tau+b}{c\tau+d}, \frac{a\overline{\tau}+b}{c\overline{\tau}+d}\right) \tag{138}$$

This fact imposes very strong constraints on the dynamical data of a CFT. Probably the most famous result in this context was derived by J.Cardy[17] in his seminal work on the *universal* asymptotic density of states in a two-dimensional CFT. We will not go into the details of this work here, however it's worth mentioning the spirit of the calculation. Modular invariance strikingly allows us to relate the partition function in low and high temperatures; the reason is simply that temperature inversion $\beta \to 4\pi^2/\beta$ is nothing more than the action of a specific modular transformation -and in particular S- in the modular parameter τ.

As we will discover soon, modular invariance allows us also to understand the combination of the holomorphic and anti-holomorphic part of the theory. Remember that so far, in the case of CFT on S^2, we were studying the two sectors as being decoupled from each other.

1.3.2 One-point function

In the light of the results of [7][8], it would be constructive to also study the one-point functions on $\mathbb{T}^2$ to some extent. The analysis of these authors basically states that

in a CFT, the crossing symmetry of four-point functions on the sphere and modular covariance of one-point functions on the torus are the sufficient conditions one needs to impose in order to ensure higher-point crossing symmetry and higher genus modular invariance. So far we have have seen what crossing symmetry means. Let us now move to the covariance of one-point functions on $\mathbb{T}^2$.

The torus one-point function of a primary operator O with conformal dimensions $(h_O, \overline{h}_O)$ is defined as:

$$\begin{aligned} \langle O \rangle_\tau &= \mathrm{Tr}_{\mathcal{H}} \left(O \; q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{\overline{c}}{24}} \right) \\ &= \sum_i \langle i | \, O \, | i \rangle \, q^{h_i - \frac{c}{24}} \overline{q}^{\overline{h}_i - \frac{\overline{c}}{24}} \end{aligned} \tag{139}$$

where in general we have expanded in a basis of states $\{|i\rangle\}$ on the cylinder, with $(h_i, \overline{h}_i)$ the corresponding conformal dimensions on the plane (i.e. $L_0 \, |i\rangle = L_0 O_i \, |0\rangle = h_i \, |i\rangle$ and similarly for the anti-holomorphic part).

One first observation is that, by translation invariance, the one-point function (139) depends only on τ and not on the specific location of the operator O on the torus. This fact indicates that the coefficients $\langle i | \, O \, | i \rangle$ are also constants and basically equal to the *three-point function coefficient* for the correlation function $\langle O_i O O_i \rangle$ on S^2:

$$\langle i | \, O \, | i \rangle = \langle O_i(\infty, \infty) O(1,1) O_i(0,0) \rangle_{S^2} \, , \qquad \text{with} \;\; O_i(\infty, \infty) = \lim_{z \to \infty} z^{2h_i} \overline{z}^{2\overline{h}_i} O_i(z, \overline{z}) \tag{140}$$

where O_i is the corresponding basis of (hermitian) operators which creates the state $|i\rangle$.

There is also another way we can expand the definition of the one-point function in (139). Consider:

$$\begin{aligned} \langle O \rangle_\tau &= \mathrm{Tr}_{\mathcal{H}} \left(O \; q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{\overline{c}}{24}} \right) \\ &= \sum_\alpha \langle \alpha | \, O \, | \alpha \rangle \, q^{h_\alpha - \frac{c}{24}} \overline{q}^{\overline{h}_\alpha - \frac{\overline{c}}{24}} F^{h_O}_{h_\alpha, c}(q) \; \overline{F}^{\overline{h}_O}_{\overline{h}_\alpha, \overline{c}}(\overline{q}) \end{aligned} \tag{141}$$

where $|\alpha\rangle$ are now a set of *primary* states of dimension $(h_\alpha, \overline{h}_\alpha)$. We see that the contributions from the descendants of a given primary $|\alpha\rangle$ factorize in the functions $F^{h_O}_{h_\alpha, c}(q)$ and $\overline{F}^{\overline{h}_O}_{\overline{h}_\alpha, \overline{c}}(\overline{q})$. These objects are called torus one-point function conformal blocks, in analogy with the case of the conformal blocks of the four-point function on the sphere, and they depend only on the dimensions $(h_0, \overline{h}_0)$ and $(h_\alpha, \overline{h}_\alpha)$ and the central charge c. People have studied these blocks and it turns out they can be computed algorithmically using the Virasoro algebra[18]. For example, we can expand $F^{h_O}_{h_\alpha, c}(q)$ in power series of q -exactly as we could expand the conformal blocks in the *cross ratio*- and compute the various coefficients basically by using the null vectors of the Virasoro algebra.

Let us finally study the covarinace of the one-point function under the action of the modular group $PSL(2, \mathbb{Z})$. If we make a modular transformation on τ:

$$\tau \to \gamma\tau \equiv \frac{a\tau + b}{c\tau + d} \tag{142}$$

then using the transformation law of a primary operator, along with the fact that $ad - bc = 1$, we arrive at:

$$\langle O \rangle_{\gamma\tau} = (c\tau + d)^{h_O} (c\overline{\tau} + d)^{\overline{h}_O} \, \langle O \rangle_\tau \tag{143}$$

Mathematically, this transformation law indicates that the one-point function is a *Maass form* of weight $(h_O, \overline{h}_O)$ meaning, roughly, that it falls into a category of objects which have some nice transformation properties under the modular group but need not be necessarily holomorphic (in contrast to the so-called *modular forms*).

It's easy to see that in case where O is the identity operator, the above formula reduces to the modular invariance of the partition function (138). As in the case of the partition function, the covariance of the one-point function is crucial and imposes further constraints on the dynamical data of the CFT. One such constraint was obtained quite recently in [19], where the authors -using covariance under the S modular transformation- were able to derive a universal asymptotic formula for the *average value* of light-heavy-heavy three point function coefficients (c.f. (140)), in analogy with Cardy's result about the universal asymptotic density of states that we mentioned earlier. Results like Cardy's and Kraus-Maloney's formulas are important not only from the CFT point of view, but also for quantum gravity in a three-dimensional spacetime with negative cosmological constant, as we will explore in Chapter 3.

1.3.3 Minimal models: Part II

Let us now turn to the study of modular invariance in the context of minimal models and see what new knowledge we can learn about the structure of these theories. We recall that the Hilbert space of a minimal model with central charge c is a finite collection of holomorphic and anti-holomorphic modules

$$\mathcal{H} = \oplus_{h,\overline{h}}\ M(c,h) \otimes \overline{M}(\overline{h},c) \tag{144}$$

The partition function (137) can be written as:

$$\mathcal{Z}(\tau,\overline{\tau}) = \mathrm{Tr}_{\mathcal{H}}\left(q^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{\overline{c}}{24}}\right) \ = \sum_{h,\overline{h}} N_{h,\overline{h}}\ \chi_h(\tau)\overline{\chi}_{\overline{h}}(\overline{\tau}) \tag{145}$$

where $N_{h,\overline{h}}$ denotes the multiplicity of occurrence of $M(c,h)\otimes\overline{M}(c,\overline{h})$ inside the Hilbert space $\mathcal{H}$. The χ's contain the contribution to the partition function of a given primary with dimensions $(h,\overline{h})$ and called *Virasoro characters* of the corresponding Verma module. Their holomorphic expression is [20]:

$$\begin{aligned}\chi_h(\tau) &\equiv \mathrm{Tr}\, q^{L_0-c/24} \\ &= \sum_{n\geq 0} d(n) q^{(n+h)-c/24} \\ &= q^{h-c/24}\sum_{n\geq 0} p(n) q^n\end{aligned} \tag{146}$$

where in the first line the trace is taken over the corresponding Verma module, and in the second line $d(n)$ is the number of independent vectors at level n which is equal -as we saw back in (77)- to $p(n)$: the number of partitions of the integer n. The Virasoro characters can be viewed, basically, as the analogous conformal blocks of the zero-point correlation function on the torus or as the generating functions for the level degeneracy.

[20] the characters for the anti-holomorphic module are defined in the same manner.

We can actually make a little more progress in the above formula if we notice that[21]:

$$\sum_{n\geq 0} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} \tag{147}$$

then,

$$\chi_h(\tau) = \frac{q^{h+(1-c)/24}}{\eta(\tau)}, \qquad \eta(\tau) \equiv q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \tag{148}$$

The so-called *Dedekind* η function, as defined above, is a special modular function which possesses some nice transformation properties under modular transformations. In particular, under the action of the generators S and T it transforms as:

$$\eta(\tau+1) = e^{\pi i/12}\eta(\tau)$$
$$\eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\eta(\tau)$$

We have to make an important remark at this point. The characters given by (148) describe a *reducible* representation of a Verma module in general. The reason is of course the existence of several null states which are naturally included within the trace in (146). One may construct however an *irreducible* representation of the Virasoro algebra by quotienting out the null states in a systematic way or, in other words, by identifying states that differ by a state of zero norm. These irreducible representations contain relatively "fewer" states than the generic Verma module. For the case of unitary minimal models, the expression of the irreducible characters $\chi_{(r,s)}(\tau)$ in the notation of (91) is given by:

$$\chi_{(r,s)}(\tau) = \frac{q^{(1-c)/24}}{\eta(\tau)}\left[q^{h_{r,s}} + \sum_{k=1}^{\infty}(-1)^k\{q^{h_{r+kq,(-1)^k s+[1-(-1)^k]p/2}} + q^{h_{r,kp+(-1)^k s+[1-(-1)^k]p/2}}\}\right] \tag{149}$$

Although we are not going to delve into the details that lead this calculation[22], this result is going to be crucial for us in Chapter 3, where we discuss the quantum gravity physics behind this construction. From now on, we consider our Hilbert space (144) built out of these irreducible holomorphic/antiholomorphic modules.

We have managed to write the partition function in the form (145), where we know the explicit τ-dependence for the characters χ_h. The next natural question to ask is how these characters transform under the modular group $PSL(2,\mathbb{Z})$, or more accurately, what is their transformation under the generators $S:\tau\to -1/\tau$ and $T:\tau\to\tau+1$. Exactly as the conformal blocks of the four-point function transform into linear combinations of one another according to (120),(121), the Virasoro characters of different modules transform into one another under the action of S and T. Organizing the characters into

[21] expanding the rhs in product of geometric series, it's easy to see that the n-th order coefficient captures exactly the number of ways we can write the integer n.

[22] see ch.8 in [1].

a vector χ_μ, where $\mu = (r, s)$ denotes the Kac indices, the transformation rules are:

$$\begin{aligned} T : \tau \to \tau + 1, \qquad & \chi_\mu(\tau + 1) = \sum_\nu \mathcal{T}_{\mu\nu}\chi_\nu(\tau) \\ S : \tau \to -1/\tau, \qquad & \chi_\mu(-1/\tau) = \sum_\nu \mathcal{S}_{\mu\nu}\chi_\nu(\tau) \end{aligned} \tag{150}$$

where in the case of the Minimal Models the representations $\mathcal{S}_{\mu\nu}$ and $\mathcal{T}_{\mu\nu}$ take a specific form[1]:

$$\begin{aligned} \mathcal{T}_{\mu\nu} &= \mathcal{T}_{rs;\rho\sigma} = \delta_{r,\rho}\delta_{s,\sigma} \exp\left(2\pi i(h_{r,s-c/24})\right) \\ \mathcal{S}_{\mu\nu} &= \mathcal{S}_{rs;\rho\sigma} = 2\sqrt{\frac{2}{pq}}(-1)^{1+s\rho+r\sigma} \sin\left(\pi\frac{p}{q}r\rho\right) \sin\left(\pi\frac{q}{p}s\sigma\right) \end{aligned} \tag{151}$$

and (p, q) are the co-prime integers labeling the specific unitary minimal model as in (91), namely $p > q > 2$ and $1 \leq r \leq q-1, 1 \leq s \leq p-1$. The matrices $\mathcal{T}$ and $\mathcal{S}$ actually form an $\binom{q}{2}$-dimensional representation of the modular group $PSL(2, \mathbb{Z})$, where $\binom{q}{2}$ is the number of primaries of a given unitary minimal model. In particular, we notice that $\mathcal{S}$ is a symmetric,real matrix satisfying $\mathcal{S}^2 = \mathbb{1}$, which implies that is also unitary.

With the above derivations in mind, let's go back and review the modular invariance of the partition function $\mathcal{Z}(\tau, \overline{\tau})$. What we actually want is to construct all partition functions

$$\mathcal{Z}(\tau, \overline{\tau}) = \sum_{h,\overline{h}} N_{h,\overline{h}}\ \chi_h(\tau)\overline{\chi}_{\overline{h}}(\overline{\tau}) \tag{152}$$

which are modular invariant and have unique vacuum. This means, first, that for the identity $N_{1,1} = 1$. Secondly, since we know how the characters transform under the generators of the modular group, the multiplicities N (viewed as a rank-2 tensors with its indices running over the spectrum $(h, \overline{h})$) should *commute* with $\mathcal{T}$ and $\mathcal{S}$, namely:

$$N\mathcal{T} = \mathcal{T}N \qquad \text{and} \qquad N\mathcal{S} = \mathcal{S}N \tag{153}$$

For Minimal Modle CFT's this classification is a well-posed, but difficult, algebraic problem. For minimal theories based on the $sl(2)$ algebra there is a one-to-one correspondence between modular invariants and pairs of the so-called simply laced Lie algebras with Coxeter numbers exactly q and p. This is the *ADE classification* of minimal models developed in [20][21]. One of the main features of this classification of minimal models is that more than one modular-invariant theory can exist at a given value of the central charge $c = 1 - 6(p-q)^2/pq$. This means that one can find different operator algebras, closed under OPE, and built out of the same set of primary fields. The conformal theories discussed so far correspond only to one of this invariants, namely the diagonal invariant, that's why they are termed as *diagonal theories*. In particular the fusion rules discussed in (110) apply only to these theories, and in general we expect different fusion rules for the other theories.

We will not expand into the details of the ADE classification in this thesis. For our purposes, we will just highlight two classes of physical modular invariants, namely the *diagonal* (AA) and *block diagonal* (AD,DA).

Diagonal invariants
Implementing $\mathcal{T}$ invariance yiels a restriction in the relative values of the conformal dimensions, namely:

$$h - \overline{h} = 0 \mod 1 \tag{154}$$

An obvious solution consists of states with $h = \overline{h}$. The corresponding partition function then reads[23]:

$$\mathcal{Z}_{AA} = \sum_{r,s} |\chi_{r,s}|^2 \tag{155}$$

and, because $\mathcal{S}$ is a unitary matrix, modular invariance is ensured. These kind of partition functions are describing the minimal theories, referred to as $\mathcal{M}(p,q)$ so far.

Block diagonal invariants
The other simple solution to (153) ivolves linear combinations of characters whose conformal dimensions differ by integers (so that the whole linear combination is invariant under the $\mathcal{T}$-transformation). In addition, one needs to ensure that the $\mathcal{S}$-transformation relates the relevant linear combinations to one another. Then, one can consider the usual diagonal invariant of this 'extended theory'. For example, for (p,q) with $q = 4m+2$ and $m > 1$, the DA modular invariant is of this type:

$$\mathcal{Z}_{DA} = \frac{1}{2}\sum_{s=1}^{p-1}\Bigg[\sum_{r \ \ odd=1}^{2m-1} |\chi_{r,s} + \chi_{4m+2-r,s}|^2 + 2|\chi_{2m+1,s}|^2\Bigg] \tag{156}$$

The important thing we notice in the above construction is the non-trivial 'interaction' between the holomorphic and anti-holomorphic sector of the theory. Modular invariance dictates which character of the holomorphic module should be combined with the anti-holomorphic one, in order to give a modular invariant expression[24]. This fact was absent in the development of the theory on S^2.

For the end, let us examine three unitary examples:

The Ising model

As we described back in 1.2.6, the Ising model is the $\mathcal{M}(4,3)$ minimal theory with central charge $c = 1/2$. It includes three primary fields with dimensions $h = 0, \frac{1}{16}, \frac{1}{2}$ and the corresponding characters are $\chi_{1,1}, \chi_{1,2}, \chi_{2,1}$. The $\mathcal{T}$ and $\mathcal{S}$ matrices read:

$$\mathcal{T} = \begin{pmatrix} e^{-2\pi i/48} & 0 & 0 \\ 0 & e^{46\pi i/48} & 0 \\ 0 & 0 & e^{2\pi i/24} \end{pmatrix} \qquad \mathcal{S} = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix} \tag{157}$$

[23] the subscript (AA) refers to the A_n algebra of the ADE classification

[24] we should have in mind that the expressions of $\chi_{r,s}$ in (155) and (156) correspond to the *irreducible* representation of the Verma module (c.f. (149)).

Given these matrices, the *unique* modular invariant function is the *diagonal* invariant:

$$\mathcal{Z}_{AA} = \sum_{r,s} |\chi_{r,s}|^2 = |\chi_{1,1}|^2 + |\chi_{1,2}|^2 + |\chi_{2,1}|^2 \tag{158}$$

The Tricritical Ising model

The tri-critical Ising model corresponds to the $\mathcal{M}(5,4)$ minimal theory with central charge $c = 7/10$. There are six primary fields, as we saw back in Table1.2(c), and the characters are $\chi_{1,1}, \chi_{1,2}, \chi_{1,3}, \chi_{1,4}, \chi_{2,2}, \chi_{2,4}$. The $\mathcal{T}$ and $\mathcal{S}$ matrices form a six-dimensional representation of $PSL(2,\mathbb{Z})$:

$$\mathcal{T} = diag\big(e^{2\pi i\frac{233}{240}}, e^{2\pi i\frac{17}{40}}, e^{2\pi i\frac{137}{240}}, e^{2\pi i\frac{113}{240}}, e^{2\pi i\frac{2}{240}}, e^{2\pi i\frac{98}{240}}\big)$$

$$\mathcal{S} = \frac{1}{\sqrt{5}} \begin{pmatrix} s_2 & s_1 & s_1 & s_2 & \sqrt{2}s_1 & \sqrt{2}s_2 \\ s_1 & -s_2 & -s_2 & s_1 & \sqrt{2}s_2 & -\sqrt{2}s_1 \\ s_1 & -s_2 & -s_2 & s_1 & -\sqrt{2}s_2 & \sqrt{2}s_1 \\ s_2 & s_1 & s_1 & s_2 & -\sqrt{2}s_1 & -\sqrt{2}s_2 \\ \sqrt{2}s_1 & \sqrt{2}s_2 & -\sqrt{2}s_2 & -\sqrt{2}s_1 & 0 & 0 \\ \sqrt{2}s_2 & -\sqrt{2}s_1 & +\sqrt{2}s_1 & -\sqrt{2}s_2 & 0 & 0 \end{pmatrix} \tag{159}$$

where $s_1 \equiv sin\frac{2\pi}{5}, s_2 \equiv sin\frac{4\pi}{5}$. The unique modular invariant of this model is again the diagonal one:

$$\mathcal{Z}_{AA} = \sum_{r,s} |\chi_{r,s}|^2 = |\chi_{1,1}|^2 + |\chi_{1,2}|^2 + |\chi_{1,3}|^2 + |\chi_{1,4}|^2 + |\chi_{2,2}|^2 + |\chi_{2,4}|^2 \tag{160}$$

The Three-state Potts model

The next unitary model corresponds to the labelling $\mathcal{M}(6,5)$ and has central charge $c = \frac{4}{5}$. There are 10 primary fields with characters:

$$\chi_{1,1}\ ,\ \chi_{2,1}\ ,\ \chi_{3,1}\ ,\ \chi_{4,1}\ ,\ \chi_{2,2}\ ,\ \chi_{3,2}\ ,\ \chi_{4,2}\ ,\ \chi_{3,3}\ ,\ \chi_{4,3}\ ,\ \chi_{4,4}$$

For this theory we can write two distinct invariants. The diagonal invariant:

$$\mathcal{Z}_{AA} = \sum_{r,s} |\chi_{r,s}|^2 \tag{161}$$

and the block diagonal invariant(c.f.(156)):

$$\mathcal{Z}_{DA} = |\chi_{1,1} + \chi_{4,1}|^2 + |\chi_{2,1} + \chi_{3,1}|^2 + 2|\chi_{3,3}|^2 + 2|\chi_{4,3}|^2 \tag{162}$$

There is a partition function of a physical statistical model at criticality which is identified with (162), and this is the so-called *Three-State Potts model*. It basically describes a model of spin variables σ_i which take 3 different values. The important point is that the operator content of this theory is different than that of the $\mathcal{M}(6,5)$, given by (161). We read that only the operators $h_{1,1}, h_{4,1}, h_{2,1}, h_{3,1}$ are present, together with two copies of the operators $h_{3,3}, h_{4,3}$. This multiplicity actually shows that the Three-State Potts

model is *not* just a subtheory of $\mathcal{M}(6,5)$, as it contains more copies of some of its fields. This is reflected also in the non-trivial structure of the Three-State Potts fusion rules, which are *not* a subset of the $\mathcal{M}(6,5)$ fusion rules, calculated with (110).

In general, the analysis of the $\mathcal{M}(p,q) = \mathcal{M}(m+1,m) = \mathcal{M}(6,5)$ model appears to be representative of all unitary minimal models with $m \geq 5$. Indeed, one can see that all of these models have at least a second (physical) modular invariant[20],[21].

2 Conformal Field Theory on Non-Orientable surfaces

2.1 Boundary CFTs

In this chapter we will start studying conformal field theories on non-orientable surfaces, that is, two dimensional surfaces where an orientation cannot be chosen globally. Examples of these surfaces constitute the *Mobius strip*, the *Real Projective plane* $\mathbb{RP}^2$ or the *Klein bottle* $\mathbb{K}^2$. Our ultimate goal will be to understand the CFT data of minimal models in these surfaces. In order to pursue, though, this kind of construction, it is necessary to introduce first some basic tools from a CFT defined on a surface with boundaries. The study of boundary CFTs, as well as CFTs on non-orientable surfaces, was motivated initially from string theory and in particular the study of open string world-sheets and orientifolds. In this discussion, however, we will describe the basic ideas from the point of view of quantum field theory, without regard to any particular applications in string theory.

So far, we have discussed conformal field theories defined on compact orientable surfaces, such as the sphere and the torus, and obtain several important results regarding the corresponding CFT data, namely the OPE coefficients C_{ijk} and the spectrum of primary (bulk) operators $(h_i, \overline{h}_i)$. We saw that two sufficient consistency conditions - *crossing symmetry* and *modular invariance*- should be respected, in order for correlation functions on these surfaces to be well-defined. When one considers conformal field theories on surfaces with boundaries, we encounter the possibility of operators living solely in the boundary. The spectrum of boundary operators will naturally depend on the boundary conditions considered and need not coincide with those in the bulk. Therefore, it is understood that we can have OPE coefficients arising in the short distance expansion as two boundary operators approach each other, or as a bulk operator approaches the boundary, respectively. These are now *new* piece of data for our theory, and one might wonder whether analogous consistency conditions might apply in order to render the corresponding amplitudes well-defined. The answer is certainly positive and, in rational CFTs, these additional data can actually be given in terms of bulk CFT data. This study was initiated mainly by N.Ishibashi, J.Cardy and D.Lewellen in [23],[24],[25]. These authors termed the boundary CFT consistency conditions as *sewing constraints* and we're going to adopt the same terminology from now on. We will not go into the very details of their construction for general surfaces with boundaries, we will however study the sewing constraints which arise in connection with the framework of

rational CFTs on non-orientable surfaces. To do this, we need to understand first the construction of *boundary* quantum states. Let us start by studying CFT on two simple (orientable) surfaces with boundary: the *upper-half plane* and the *annulus*.

2.1.1 CFT on the Upper-Half Plane

The complex upper-half plane H_+,with coordinates $z : \{Imz \geq 0\}$, is a domain whose boundary is just the real axis. One can actually obtain it from the -more physical- infinite strip, described by the real variables $(\tau, \sigma) : \{\tau \in (-\infty, +\infty) \text{ and } \sigma \in [0, \pi]\}$, via the conformal mapping:

$$z = e^{\tau + i\sigma} \tag{163}$$

Note in particular, as illustrated in Fig.2.1(a), the boundary $\sigma = 0, \pi$ is mapped to the real axis $z = \overline{z}$.

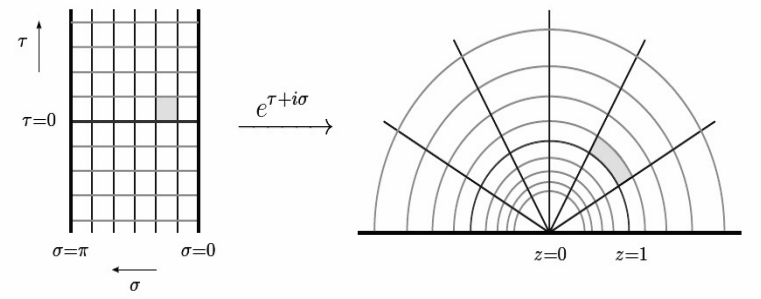

Figure2.1(a):Map from the infinite strip to the complex upper-half plane

For a boundary CFT, we would like to formulate a conformally invariant boundary condition independently of any particular set of fundamental fields and a Lagrangian. The only intrinsic field in our theory is the energy-momentum tensor, which generates the conformal transformations. A natural requirement, then, is that the off-diagonal component $T_{\parallel\perp}$ parallel/perpendicular to the boundary should vanish. This is called the *conformal boundary condition* and is in complete generality. Let's see why it makes sense from the upper-half plane point of view: in the physical picture of the infinite strip our boundaries are at $\sigma = 0, \pi$, which means we don't want any momentum flow across there. This implies that:

$$T_{\tau\sigma} = 0 \tag{164}$$

which is exactly the statement that $T_{\parallel\perp} = 0$. In the coordinates of the upper-half plane we get(c.f.(25)):

$$T(z) = \overline{T}(\overline{z}) \qquad \text{at} \quad z = \overline{z} \tag{165}$$

If we expand T in Laurent modes we obtain a relation:

$$L_n - \overline{L}_n = 0 \tag{166}$$

which indicates that boundaries, in general, introduce relations between the chiral and anti-chiral modes of the theory.

Let's move to quantization now. In radial quantization we mainly want Hilbert spaces defined on different time-slices to be equivalent. Hence, we should choose semi-circles centred on some specific point on the boundary, conventionally the origin. The Hamiltonian is now:

$$H_{uhp} \equiv \frac{1}{2\pi i}\int_S zT(z)dz \; - \; \frac{1}{2\pi i}\int_S \overline{z}\overline{T}(\overline{z})d\overline{z}, \tag{167}$$

where S is a semicircle (Fig.2.1(a)). Splitting the contours into a real and a complex one, and using the conformal boundary condition, this can be written as an integral over a complete circle C around the origin:

$$H_{uhp} = \frac{1}{2\pi i}\int_C zT(z)dz = L_0 \tag{168}$$

It is clear that there is now *one* Virasoro algebra for our boundary CFT. The eigenstates of that L_0 correspond to primary *boundary operators* $\phi_i^{\mathcal{B}}$, which are restricted to lie on the real axis[25] and have conformal dimension $h_i^{\mathcal{B}}$. As in the case of a bulk CFT, conformal invariance fixes the form of the one- and two-point functions of boundary operators and their OPEs:

$$\begin{aligned} \left\langle \phi_i^{\mathcal{B}} \right\rangle &= 0 \\ \left\langle \phi_i^{\mathcal{B}}(0)\phi_j^{\mathcal{B}}(x) \right\rangle &= C^{\mathcal{B}}\frac{\delta_{ij}}{x^{2h_i^{\mathcal{B}}}} \; , \qquad\qquad x>0 \\ \phi_i^{\mathcal{B}}(0)\phi_j^{\mathcal{B}}(x) &\sim \sum_k C_{ijk}^{\mathcal{B}}\; x^{h_k^{\mathcal{B}}-h_i^{\mathcal{B}}-h_j^{\mathcal{B}}}\; \phi_k^{\mathcal{B}}(x) + \cdots \; , \qquad x>0 \end{aligned} \tag{169}$$

The numbers $C^{\mathcal{B}}$ are just some boundary-condition-dependent normalization constants which we are not completely free to choose, having already fixed the bulk normalizations. This connection is due to the fact that bulk and boundary operators are actually related (for more details on that see [24]). In the third expression, the sum runs over boundary primary fields and the omitted terms are descendants under the single Virasoro algebra. It's understood that the OPE coefficients $C_{ijk}^{\mathcal{B}}$ constitute *new* piece of data for our boundary CFT.

Furthermore, as it was first analyzed in [26], it turns out that in a conformal field theory in which the boundary conditions do not break the conformal symmetry (i.e. $T_{\parallel\perp}=0$), a bulk primary operator approaching the boundary can be expanded in terms of boundary primary operators as:

$$\phi_j(z) \sim \sum_i (2Im\; z)^{h_i^{\mathcal{B}}-h_j-\overline{h}_j}\; C_{ji}^{\mathcal{B}}\; \psi_i^{\mathcal{B}}(Re\; z) \tag{170}$$

The new coefficients $C_{ji}^{\mathcal{B}}$ are our second piece of boundary CFT data. Finally, one needs to consider separately the constant amplitude[24]:

$$\langle \mathbb{1} \rangle_{\mathcal{B}} \equiv \langle 0|\mathcal{B}\rangle = \alpha^{\mathcal{B}} \tag{171}$$

[25]the index $\phi^{\mathcal{B}}$ indicates the different behaviour of these these operators under the specific boundary conditions that we choose, e.g. Neumann or Dirichlet. These kind of boundary conditions are unrestricted so far. Our only requirement is the conformally invariant relation (164) for the energy-momentum tensor. In ref.[24], this index is treated more concretely and is specified by two letters $\phi^{\alpha\beta}$, but it's not going to be of great importance for our discussion.

where we can consider a "boundary state" $|\mathcal{B}\rangle$, as a suitable state in the Hilbert space of the *bulk* CFT (without any boundary), and then expressing equivalently all the correlation functions in the geometry of a boundary CFT as $\langle\cdot\rangle_{\mathcal{B}} = \langle 0|\cdot|\mathcal{B}\rangle$. This is always possible when the boundary is at a constant time-slice, and this is actually the reason why we can define a state for it (a good example is the the bulk CFT of the infinite cylinder and the boundary CFT on a semi-infinite cylinder). This observation will be important later for CFT on the annulus. The one-point function of the identity, $\alpha^{\mathcal{B}}$, is therefore a constant that constitutes our last piece of new CFT data that we need to specify.[26]

What about the sewing conditions that constraint these new data? As D.Lewellen described in [24], for CFT on the upper-half plane one should consider *three* type of sewing constraints. First, the crossing symmetry of the boundary four-point function, secondly the sewing involving one bulk operator and two boundary operators, and lastly the sewing between two bulk and one boundary operator. In the subclass of rational CFTs, these conditions are eventually given in terms of data in the bulk, and in particular: the duality matrices M, relating bulk four-point conformal blocks as we saw back in (120), the $\mathcal{S}$ matrix elements, which implements modular transformation on the Virasoro characters, and the bulk OPE coefficients[24],[27].

2.1.2 CFT on the Annulus

Having set the basics with the example of the upper-half plane, let us move to the annulus and see what knowledge we can gain. The annulus can be obtained form the upper-half plane if, starting from the conformally equivalent infinite strip, we identify periodically the time coordinate τ. In this way, we are left with the topology of a finite cylinder as illustrated in Fig.2.1(b) below:

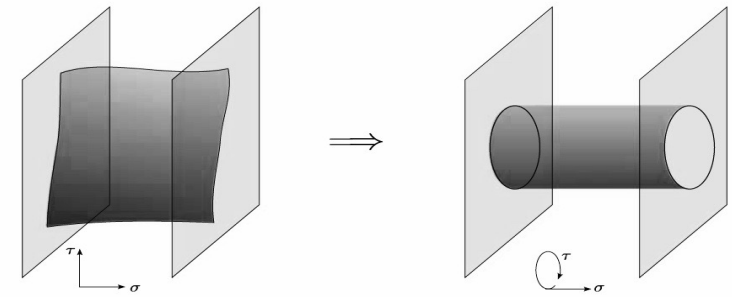

Figure2.1(b):Map from the infinite strip to the finite cylinder or *Annulus*.

There is now a modular parameter t parametrizing different cylinders with coordinates $(\tau, \sigma) : \{\tau \in [0, t) \text{ and } \sigma \in [0, \pi]\}$. This situation is reminiscent of the case of the torus $\mathbb{T}^2$, where there we started with the infinite cylinder, the space coordinate being already periodic, and then identify time. Here σ is not periodically identified, instead we have some boundary conditions on the two ends.

Let's try to calculate the partition function on this surface. We need to determine the Hamiltonian generating translations in time circling the cylinder once along the τ

[26] one might argue that the value of $\langle\mathbb{1}\rangle_{\mathcal{B}}$, being interpreted as a bulk one-point function, is already fixed by the conventions chosen for the bulk theory and hence is not new CFT data. As it shown in [24], however, in the example of the Ising model, one finds that $\alpha^{\mathcal{B}}$ must take non-trivial values in order for all of the four-point boundary operators amplitudes to be crossing-symmetric.

direction, having in mind also the calculation the Hamiltonian in the upper-half plane. Because boundaries lead to the identification (165) between the holomorphic and anti-holomorphic sectors, we see that the Hamiltonian is actually:

$$H_{ann.} \equiv (L_{cyl.})_0 = L_0 - \frac{c}{24} \tag{172}$$

Hence, in analogy to the torus construction, we obtain the cylinder partition function:

$$\mathcal{Z}_{\mathcal{B}}(t) = \mathrm{Tr}\, e^{-2\pi t H_{ann.}} = \mathrm{Tr}_{\mathcal{H}_{\mathcal{B}}}\left(q^{L_0 - c/24}\right) \qquad \text{where} \quad q \equiv e^{-2\pi t} \tag{173}$$

where $\mathcal{H}_{\mathcal{B}}$ denotes in general the Hilbert space of all states satisfying the boundary conditions under consideration, or the space on which $H_{ann.}$ acts. As in the case of the torus, we can decompose the partition function into characters of the $h_{\mathcal{B}}$ representation, which they now come only in one copy:

$$\mathcal{Z}_{\mathcal{B}}(t) = \sum_{h_{\mathcal{B}}} N_{h_{\mathcal{B}}} \chi_{h_{\mathcal{B}}}(t) \tag{174}$$

The non-negative integers $N_{h_{\mathcal{B}}}$ give the boundary primary operators' multiplicity.

In fact, looking at Fig.2.1(b), it seems we can interchange the role of σ and τ and interpret the partition function as the path integral for a CFT on a circle of circumference π:

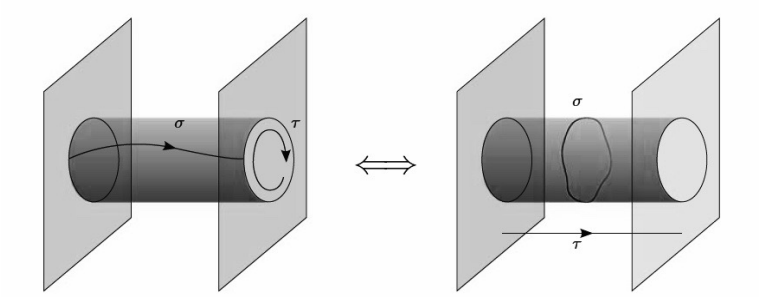

Figure2.1(c):Illustration of the duality between the partition function of the finite cylinder and propagation for time $\tau = \pi/t$ between two boundary states.

Indeed, we can legitimately perform a modular transformation $t \to -1/t$ in the real modular parameter t. From this point of view, the partition function is no longer a trace, but rather the matrix element of the time evolution operator $e^{-\pi H/t}$ for time $\tau = \pi/t$ between two "boundary states" $|a\rangle$ and $|b\rangle$:

$$\mathcal{Z}_{\mathcal{B}}(t) = \langle a|\, e^{-\pi H/t}\, |b\rangle \tag{175}$$

Equation of the two expressions (173) and (175) constitute, together with the three conditions from the upper-half plane, our *fourth* sewing constraint. Again, in rational CFTs, this constraint allows us to calculate, in non-trivial way, the boundary data in terms of the bulk CFT[28].

Let us now try to sketch the analysis that eventually allow us to determine the boundary data in terms of the bulk elements. The remarkable thing to notice in the dual interpretation (174) is that: H is the Hamiltonian of the *infinite cylinder*, that is

$$H = L_0 + \overline{L}_0 - \frac{c + \overline{c}}{24} \tag{176}$$

and the "boundary states" $|a\rangle$ and $|b\rangle$ belong in the Hilbert space

$$\mathcal{H} = \oplus_{h,\overline{h}} M(c,h) \otimes \overline{M}(c,\overline{h})$$

as in (144). Put it differently, we have managed to describe a *boundary* CFT in terms of a *bulk* CFT.

The question that arises now is how are these boundary states at $\tau = 0$ and $\tau = \pi/t$ to be characterized. The conformal boundary condition $T_{\parallel\perp} = 0$ in the dual picture, at $\tau = 0$, yields:

$$T_{cyl.}(w) = \overline{T}_{cyl.}(\overline{w}) \quad \text{at } \tau = 0 \qquad (w = \tau + i\sigma) \tag{177}$$

and if we expand T in modes(c.f. (132)):

$$\begin{aligned} T_{cyl.}(w)\big|_{\tau=0} &= \sum_{n\in\mathbb{Z}} \big(L_n - \frac{c}{24}\delta_{n,0}\big) e^{-in\sigma} \\ \overline{T}_{cyl.}(\overline{w})\big|_{\tau=0} &= \sum_{n\in\mathbb{Z}} \big(\overline{L}_n - \frac{\overline{c}}{24}\delta_{n,0}\big) e^{in\sigma} \end{aligned} \tag{178}$$

Changing variables in the second expression $n \to -n$ we arrive at the result:

$$L_n - \overline{L}_{-n} = 0 \tag{179}$$

Hence, the statement is that any boundary state $|\mathcal{B}\rangle$ should satisfy:

$$\big(L_n - \overline{L}_{-n}\big)\,|\mathcal{B}\rangle = 0 \tag{180}$$

The result (179) deserves some attention (also because we're going to run into a similar expression in the non-orientable case). It says that there is a subspace of special states within the bulk CFT Hilbert space $\mathcal{H}$, that can describe boundary states. If we are able to find a solution to this equation, or even better a nice basis that span these states, then we can characterize any boundary state.

This work was done by Ishibashi in [23]. He showed that to each highest weight representation h_i of the Virasoro algebra one can associate an up to a constant unique state $\big|B_i\big\rangle\big\rangle$ such that the condition (179) is satisfied. If we use $|i;\vec{m}\rangle$ and $\overline{|i;\vec{m}\rangle}$ to denote an orthonormal basis of states for the holomorphic and anti-holomorphic Virasoro descendants of a primary field with conformal dimensions $(h_i, \overline{h}_i)$, then any state in the bulk theory of the form:

$$\big|B_i\big\rangle\big\rangle \equiv \frac{1}{\mathcal{N}} \sum_{\vec{m}} |i;\vec{m}\rangle \otimes \overline{|i;\vec{m}\rangle} \tag{181}$$

satisfies (180). These are called *Ishibashi states*[27]. The notation $\vec{m}$ indicates the descendant states, constructed by acting with each raising operator L_{-j} on the given primary.

[27]We can always have the possibility of an extended symmetry algebra in our theory, e.g. a $\mathcal{W}$-symmetry. Then, there is an analogous condition for the modes of the corresponding current, and Ishibashi states are defined with an insertion of an anti-unitary operator U, acting in a particular way on the extended symmetry generators[23]. We are not going to discuss this situation here, though.

As it turns out, the Ishibashi states are not the boundary states itself but only building blocks guaranteed to satisfy the conformal boundary condition $T_{\parallel\perp} = 0$. A general boundary state can be expressed as a linear combination of Ishibashi states corresponding to different highest weight representations:

$$|\mathcal{B}_a\rangle = \sum_i B_a^i \big|B_i\big\rangle\big\rangle \tag{182}$$

where the complex coefficients B_a^i are usually called *reflection coefficients*; the two indices a, i indicate that for each highest weight representation h_i, there exists not only an Ishibashi state but also a boundary state, that is, the index a also runs from one to the number of highest weight states.

The *bra* vector of $|\mathcal{B}_a\rangle$ is not defined in the usual way, i.e. just as an element of the dual Hilbert space. We have to include the insertion of the CPT operator Θ, and use $\langle\mathcal{B}_a|\,\Theta = \langle\Theta\mathcal{B}_a|$ when calculating matrix elements. The action of Θ is defined formally as:

$$\Theta\,|\mathcal{B}_a\rangle = \sum_i (B_a^i)^* \big|B_{i^+}\big\rangle\big\rangle \tag{183}$$

where $+$ denotes charge conjugation, which in general will not be of interest for our purposes. The reason for considering this operator can roughly be explained by the fact that, the orientation of the boundary where a given process was initiated is opposite to the orientation of the boundary where the process is ended after some time evolution[28].

Having expressed a boundary state in a way that allows us to proceed with calculations (using basically the bulk conformal algebra), our fourth sewing constraint suggests that we can actually express all the boundary data in terms of these reflection coefficients! This fact is indeed manifest in the case of Minimal Model CFTs [24][27], where we have only a finite number of highest weight states and thus, only a finite number of Ishibashi states. The authors of these papers showed that, in diagonal minimal theories, we can write down *explicit* relations -using the various sewing constraints- of the reflection coefficients with the bulk duality matrices M, the modular $\mathcal{S}$ matrix and the bulk OPE coefficients, and in turn relate them with the boundary data $\{C_{ijk}^{\mathcal{B}}, C_{ji}^{\mathcal{B}}, \alpha^{\mathcal{B}}\}$.

2.2 CFT on $\mathbb{RP}^2$

We now turn to the study of CFT on non-orientable surfaces. Although we are going to focus on *closed* non-orientable surfaces, we will shortly realize that the construction of boundary states we did in the previous section resembles the construction of the so-called *crosscap* state, which is going to be relevant for the calculation of correlation functions on these surfaces. Our ultimate goal is to understand the crosscap states in the Minimal Models, so we can implement them in sections 2.4 and 2.5.

The surface of the Real Projective plane $\mathbb{RP}^2$ (or *crosscap surface*) is depicted in the following figure:

[28] an example of a process can be given in the context of string theory, where the amplitude (175) between two boundary states can be interpreted as an emission and absorption of a closed string.

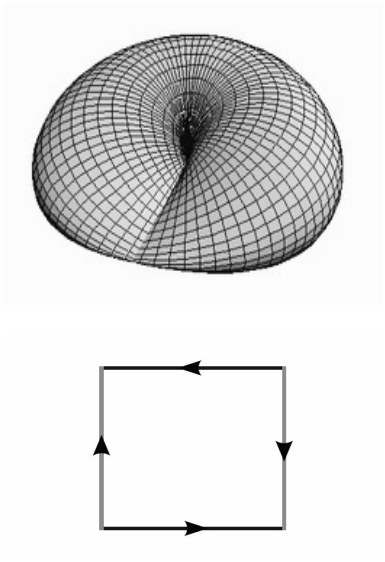

Figure2.2(a):The surface of the crosscap (*above*) and its fundamental domain (*below*) with the arrows indicating how opposite edges are identified leading to the crosscap surface.

It can be defined in several equivalent ways. It can be obtained from a disk by pairwise identifying *opposite* points on its boundary, or it can be obtained from the sphere S^2 by a $\mathbb{Z}_2$ involution that changes the orientation,

$$z \to -\frac{1}{\overline{z}} \tag{184}$$

It's easy to see that (184) has *no* fixed points hence the surface that we end up with has no boundary(closed). We're going to stick to this latter definition, because it actually illustrates a general fact about non-orientable surfaces, namely that any non-orientable surface Σ can be represented as the $\mathbb{Z}_2$ quotient of an orientable manifold $\hat{\Sigma}$, called the *orientation double cover* of Σ. Here, $\mathbb{RP}^2 = S^2/\mathbb{Z}_2$.

If we have any fields in our theory, the involution (184) realizes itself as a specific discrete symmetry operator Ω, say, that acts in a specific way on the fields, relating their values at the two identified points. This situation will of course be true also at the level of the Hilbert space. For conformal field theories on non-orientable surfaces, which are $\mathbb{Z}_2$ quotients of the orientable ones, the Hilbert space will in general be different compared to the one corresponding to the orientable case. In particular, it should contain only states from the orientable-case Hilbert space which are invariant under Ω and for the calculation of the partition function, one should project appropriately onto invariant states (we will see it explicitly for the Klein bottle).

By demanding that the energy momentum tensor is covariantly the same at a pair of points identified via (184), we get for the Laurent modes:

$$L_n - (-1)^n \overline{L}_{-n} = 0 \tag{185}$$

This relation reminds us a lot the analogous identification of the Laurent modes of T in the case of the boundary CFT. Here though we don't have any boundary. In fact, we can define a state $|C\rangle$ in the CFT Hilbert space of its orientation double cover by:

$$\left(L_n - (-1)^n \overline{L}_{-n}\right) |C\rangle = 0 \tag{186}$$

This is the so-called *crosscap state*. We can think of this state, alternatively, as the state prepared by the CFT path integral on a crosscap surface, in the Hilbert space of the CFT on a circle.

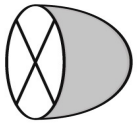

Figure2.2(b):An alternative graphic representation of the crosscap state thought of as the state prepared by a CFT path integral on the crosscap surface

In this sense, we can express any CFT correlator on the Real Projective plane as the amplitude between:

$$\langle \cdot \rangle_{\mathbb{RP}^2} = \langle 0| \cdot |C\rangle \tag{187}$$

exactly as we saw in the case of boundary CFTs with the analogous "boundary states". For example, the partition function can be written as:

$$\mathcal{Z}_{\mathbb{RP}^2} = \langle 0|C\rangle \equiv \Gamma_{\mathbb{1}} \tag{188}$$

where the symbol $\Gamma_{\mathbb{1}}$ will make sense shortly. Similarly, as it was shown in [29], the one point function of primary operators Φ_i is non zero only for $h_i = \overline{h}_i$ (i.e. scalar operators) and has the form:

$$\langle \Phi_i \rangle_{\mathbb{RP}^2} = \langle 0| \Phi_i |C\rangle = \frac{\Gamma_i}{(1+z\overline{z})^{2h}} \tag{189}$$

with Γ_i yet unspecified constants.

Having in mind the analogous constructions for boundary CFT, it is natural to ask whether we can solve (186) explicitly. For a rational CFT one can show[23] that to each highest weight representation of the Virasoro algebra one can associate a *crosscap Ishibashi state* of the form:

$$\big|C_i\big\rangle\big\rangle \equiv \frac{1}{\widetilde{\mathcal{N}}} \sum_{\vec{m}} (-1)^{\sum_j m_j} |i;\vec{m}\rangle \otimes \overline{|i;\vec{m}\rangle} \tag{190}$$

such that the condition (186) is satisfied. The notation $\vec{m}$ denotes, again, all the descendants constructed by acting with L_{-j}, m_j times on the given primary. A general crosscap state is then a linear combination of crosscap Ishibashi states:

$$|C\rangle = \sum_i \Gamma_i \big|C_i\big\rangle\big\rangle \tag{191}$$

where the sum is over all the primary states in the theory. The coefficients Γ_i are often called *one-point function normalizations on* $\mathbb{RP}^2$ and it is now clear why they enter in the zero- and one-point amplitudes. Indeed, using conformal invariance and the orthogonality of different highest weight states, one can obtain[29] the explicit expressions given by (188) and (189).

Taking into consideration the analysis so far, we have seen that in order to specify a CFT on the Real Projective plane an additional piece of data needs to be determined

(along with the OPE coefficients C_{ijk} and the spectrum h_i of primary operators): the Γ_i's, which arise -as we saw- in the one-point functions on $\mathbb{RP}^2$. In fact, the set of data $\{h_i, C_{ijk}, \Gamma_i\}$ is enough to specify a CFT in *any* non-orientable surface. The deeper reason is that, topologically, every non-orientable surface Σ_g can be written as a connected sum of g copies of real projective space

$$\Sigma_g = \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$$

where g denotes the (non-orientable) genus of the surface, and the connected sum $\#$ with $\mathbb{RP}^2$ means a process of removing a disk and gluing in a crosscap surface. The Klein bottle, for example, is a connected sum of two Real Projective spaces $\mathbb{K}^2 = \mathbb{RP}^2 \# \mathbb{RP}^2$, and we will explore it next.

2.3 CFT on $\mathbb{K}^2$

The Klein bottle $\mathbb{K}^2$ is our next example of non-orientable surface (Fig.2.3(a)). It can be constructed by the cylinder $(\tau, \sigma) \sim (\tau, \sigma + 2\pi)$ by modding out by the following action:

$$(\tau, \sigma) \sim (\tau, \sigma + 2\pi) \sim (\tau + 2\pi\tilde{t}, -\sigma) \tag{192}$$

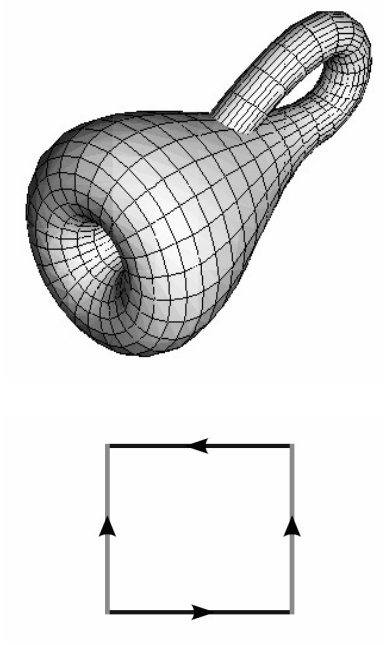

Figure2.3(a):The Klein bottle surface (*above*) and its fundamental domain (*below*) with the arrows indicating how opposite edges are identified leading to the $\mathbb{K}^2$ surface.

It is therefore a cylinder of circumference 2π and length $2\pi\tilde{t}$ with the two ends identified with a parity-reversal transformation. Because of this, there are no boundaries (there are, however, two crosscaps as we will soon find out). The modulus $l \equiv \tilde{t}$ is a positive real number and characterizes topologically different Klein bottles.

An equivalent description of $\mathbb{K}^2$ is by taking the $\mathbb{Z}_2$ quotient of the rectangular torus $(t, \phi) \sim (t + \beta, \phi + 2\pi)$[29] and identify as:

$$(t, \phi) \sim (t + \beta, \phi + 2\pi) \sim (-t, \phi + \pi) \tag{193}$$

Hence, $\mathbb{K}^2 = \mathbb{T}^2/\mathbb{Z}^2$ and the modulus of the Klein bottle in this case is $l = \beta/\pi$ (we will see why that is shortly).

Let us try to compute the partition function, given (193). We start with the usual form of the partition function on $\mathbb{T}^2$:

$$\mathcal{Z}(\tau, \overline{\tau}) = \mathrm{Tr}_{\mathcal{H}} \left(q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{\overline{c}}{24}} \right) \tag{194}$$

where the Hilbert space $\mathcal{H}$ is a tensor product of the holomorphic and anti-holomorphic sector of the theory. Now, as we discussed in the previous section, the identification (193) manifests itself in the Hilbert space of the theory as a discrete operator acting on the spectrum of states. Here, we actually encounter a *parity* operator Ω which changes the orientation. What we want then, is to project the entire Hilbert space $\mathcal{H}$ onto those states which are invariant under the action of Ω. It's easy to construct such a projection operator, namely $\frac{1}{2}(1 + \Omega)$. Therefore, we get:

$$\begin{aligned} \mathcal{Z}_{\Omega}(\tau, \overline{\tau}) &= \mathrm{Tr}_{\mathcal{H}} \left(\frac{1 + \Omega}{2} q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{\overline{c}}{24}} \right) \\ &= \frac{1}{2} \mathcal{Z}(\tau, \overline{\tau}) + \frac{1}{2} \mathrm{Tr}_{\mathcal{H}} \left(\Omega q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{\overline{c}}{24}} \right) \end{aligned} \tag{195}$$

Let us focus on the second term. The insertion of Ω into the trace has the effect of changing the orientation of the t circle of the torus, after performing a loop into itself. Geometrically, this is exactly the construction of the Klein bottle, hence we are led to identify the second term of (195) with the partition function on $\mathbb{K}^2$.

In order to proceed with calculations, we need to know how Ω acts on a basis state in $\mathcal{H}$, namely $\left| i, \overline{j} \right\rangle$ where i denotes a basis state in the holomorphic sector and j a basis state in the anti-holomorphic sector. A logical assumption is that[30]:

$$\Omega \left| i, \overline{j} \right\rangle = \pm \left| \Omega(j), \overline{\Omega(i)} \right\rangle \tag{196}$$

Let us try to explain why that is. The parity operator Ω satisfies naturally $\Omega^2 = \mathbb{1}$. This is the origin of the $\pm$ sign, which indicates whether the state is parity-odd or even. When Ω is acting on the vacuum state, a natural choice (that we're going to adopt from now on) is $\Omega \left| 0 \right\rangle = + \left| 0 \right\rangle$, i.e. the vacuum is invariant under parity. Next, the fact that Ω interchanges the "quantum numbers" of holomorphic and anti-holomorphic sector as $i \to \Omega(j)$ can be understood briefly from the fact that Ω is, by definition, a parity operator which introduces a change in the orientation in the time direction. This change will be manifest as change in the sign of the Laurent modes of any field that we have in the theory, which in turn results to an interchange of holomorphic and anti-holomorphic modes. Finally, the map $\Omega(i)$ can be chosen to be the identity map $\Omega(i) = i$, leading us to:

$$\left\langle i, \overline{j} \right| \Omega \left| i, \overline{j} \right\rangle = \left\langle i, \overline{j} \middle| j, \overline{i} \right\rangle = \delta_{ij} \tag{197}$$

[29] we're always referring to Euclidean time t.

[30] one can explicitly verify this action in several examples where we have a lagrangian description of a CFT, e.g. for the free boson see [12].

Now we are ready to calculate the partition function of the Klein bottle:

$$\begin{aligned}\mathcal{Z}_{\mathbb{K}^2}(\tau,\overline{\tau}) &= \mathrm{Tr}_{\mathcal{H}}\left(\Omega q^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{\overline{c}}{24}}\right)\\ &= \sum_{i,j}\left\langle i,\overline{j}\right|\Omega q^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{\overline{c}}{24}}\left|i,\overline{j}\right\rangle\\ &= \sum_{i,j}\left\langle i,\overline{j}\right|\Omega q^{L_0-\frac{c}{24}}\Omega^{-1}\Omega\overline{q}^{\overline{L}_0-\frac{\overline{c}}{24}}\Omega^{-1}\Omega\left|i,\overline{j}\right\rangle\\ &= \sum_{i}\left\langle i,\overline{i}\right| q^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{\overline{c}}{24}}\left|i,\overline{i}\right\rangle\end{aligned} \tag{198}$$

and since the diagonal subset of state will contribute to the trace we can *effectively* identify L_0 and $\overline{L_0}$ as well as c and $\overline{c}$. Then,

$$\begin{aligned}\mathcal{Z}_{\mathbb{K}^2}(\tau,\overline{\tau}) &= \sum_{i}\left\langle i,\overline{i}\right|(q\overline{q})^{L_0-\frac{c}{24}}\left|i,\overline{i}\right\rangle = \mathrm{Tr}_{\mathcal{H}_{sym.}}\left(e^{-4\pi Im(\tau)(L_0-c/24)}\right)\\ &= \mathrm{Tr}_{\mathcal{H}_{sym.}}\left(e^{-2\beta(L_0-c/24)}\right) = \mathrm{Tr}_{\mathcal{H}_{sym.}}\left(e^{-2\pi l(L_0-c/24)}\right)\end{aligned} \tag{199}$$

where $\mathcal{H}_{sym.}$ denotes the states of the form $\left|i,\overline{i}\right\rangle$, and we have taken into account that the modulus of the torus was defined with $Im(\tau)=\beta/2\pi$ (c.f.(130)). The modulus for the Klein bottle turned out to be $l\equiv\beta/\pi$.

There is a second representation of the Klein bottle partition function, however, where we can represent the Klein bottle as the cylinder between two crosscaps. This is reminiscent of the annulus partition function where, via a modular S-transformation, we could interpret the result as a matrix element between two boundary states. For the Klein bottle, it's simple to illustrate the effect of this S-transformation at the level of the transformation of the fundamental region.

We will describe the large conformal transformations in 5 steps(Fig.2.3(b)). We start *(a)* with the initial fundament domain of $\mathbb{K}^2$ and we divide it *(b)* in half, where the identification of segments and points is indicated explicitly by arrows and symbols. Next, we shift one-half of the fundamental domain *(c)*, and we flipped it to the left *(d)* so that the appropriate edges be identified. Quite remarkably, we end up with a fundamental domain of the form *(e)* which is a cylinder between to crosscaps(c.f. Fig.2.2(a)).

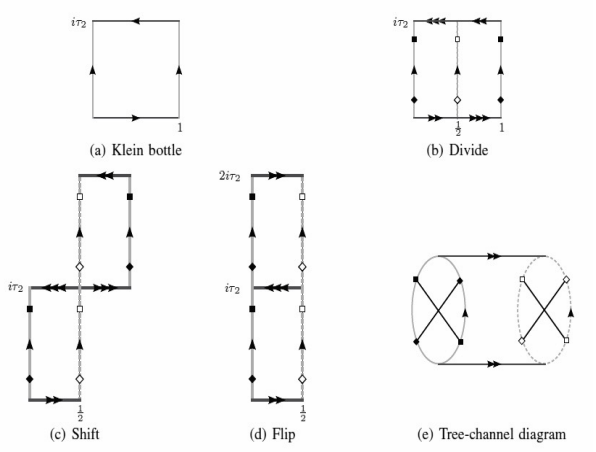

Figure2.3(b):Transformation of the fundamental region of $\mathbb{K}^2$ using *large* conformal transformations, to an equivalent propagation between to crosscap states

Therefore, the Klein bottle partition function can be viewed as matrix element of the time evolution operator propagating between to crosscap states over distance $\frac{\pi}{2Im(\tau)} = \frac{\pi^2}{\beta}$

$$\begin{aligned} \mathcal{Z}_{\mathbb{K}^2}(\beta) &= \langle \Theta C | \, e^{-\frac{\pi^2}{\beta}(L_0+\overline{L_0}-\frac{c+\bar{c}}{24})} \, |C\rangle \\ &= \langle \Theta C | \, e^{-2\pi l(L_0+\overline{L_0}-\frac{c+\bar{c}}{24})} \, |C\rangle \end{aligned} \tag{200}$$

where now $l \equiv \pi/2\beta$. The inclusion of the parity operator Θ is understood (c.f.(183)).

Expression (200) is now easier to manipulate. From the decomposition of the crosscap state into Ishibashi states (191), we can write:

$$\mathcal{Z}_{\mathbb{K}^2}(\beta) = \sum_{i,j} \Gamma_i \Gamma_j \big\langle\big\langle C_i \big| e^{-2\pi l(L_0+\overline{L_0}-\frac{c+\bar{c}}{24})} \big| C_j \big\rangle\big\rangle = \sum_i \Gamma_i^2 \big\langle\big\langle C_i \big| e^{-4\pi l(L_0-\frac{c}{24})} \big| C_i \big\rangle\big\rangle \tag{201}$$

where, again, we can effectively identify L_0 and $\overline{L}_0$, and c and $\bar{c}$, because of the form of the Ishibashi states. For Minimal Model CFTs the sum is running over a finite set of primary fields and expression (201) is more convenient for calculation.

We have managed to write down the partition function of a CFT on the Klein bottle with two expressions, (199) and (201). The important thing to notice about the second expression is that, if we know the Γ_i's of all the primaries, namely their one-point correlation functions on $\mathbb{RP}^2$, we can have all the information we need about the Klein bottle partition function as well. This situation illustrates the general fact we mentioned previously: the CFT partition function on a non-orientable surface $\Sigma_g = \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$ is uniquely determined once the set of data $\{h_i, C_{ijk}, \Gamma_i\}$ is specified. The very goal of this thesis is to examine two examples of minimal model CFTs on non-orientable surfaces, by determining their one-point normalizations on $\mathbb{RP}^2$ via sewing constraints. This will be the topic of the next two sections.

2.4 Sewing constraints on non-orientable surfaces

So far, we have encountered consistency conditions (or *sewing constraints*) for CFTs on closed *orientable* surfaces (e.g. S^2,$\mathbb{T}^2$), and on surfaces with *boundary* (e.g. upper-half plane, annulus). These conditions, as we discussed, restrict the spectrum of the corresponding CFT data. For non-orientable surfaces, the presence of the crosscap imposes a new type of sewing constraint, first discussed in[29], and we're going to study it in this section. As we will see, consistency of CFT on $\mathbb{RP}^2$ requires basically the crossing symmetry of the two-point functions.

Consider the two-point function of two scalar primary operators, Φ_i, Φ_j of dimensions h_i, h_j on $\mathbb{RP}^2$:

$$\langle \Phi_i \Phi_j \rangle_{\mathbb{RP}^2} \equiv \langle 0 | \, \Phi_i(z,\overline{z}) \Phi_j(w,\overline{w}) \, |C\rangle \tag{202}$$

where $(z,\overline{z})$ and $(w,\overline{w})$ are locations on the sphere (not related with a relation similar to the involution $z \to -1/\overline{z}$), and we can choose the crosscap to be at the origin. Topologically, this amplitude corresponds to having two distinct punctures in a crosscap surface. In turn, this situation is equivalent to having 4 punctures on the *sphere* of the

form

$$\left\langle \Phi_i\big(z\big)\Phi_j\big(w\big)I(\Phi_j)\big(-\frac{1}{\overline{w}}\big)I(\Phi_i)\big(-\frac{1}{\overline{z}}\big)\right\rangle_{S^2} \tag{203}$$

and *in addition* imposing the involution $z \to -1/\overline{z}$. The fields $I(\Phi_i), I(\Phi_j)$ are the images of the fields Φ_i, Φ_j under the involution.

The equivalent expression (203) allows us now to use the knowledge about four-point functions on S^2 in order to understand two-point functions on $\mathbb{RP}^2$. The general form of the four-point amplitude on S^2 is:

$$\langle \Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)\Phi_4(z_4))\rangle = G(\eta,\overline{\eta})\prod_{a\le b}^{4} z_{ab}^{h/3-h_a-h_b}\overline{z}_{ab}^{\overline{h}/3-\overline{h}_a-\overline{h}_b} \tag{204}$$

where η is the usual anharmonic ratio (c.f. (37)). For the particular locations of the operators in (203) we get that:

$$P \equiv \prod_{a\le b}^{4} z_{ab}^{h/3-h_a-h_b}\overline{z}_{ab}^{\overline{h}/3-\overline{h}_a-\overline{h}_b} = \big(z-w\big)^{r-h_i-h_j}\big(\overline{z}-\overline{w}\big)^{\overline{r}-\overline{h}_i-\overline{h}_j}\big(1+|z|^2\big)^{r-h_i-\overline{h}_i}\times$$
$$\times\big(1+|w|^2\big)^{r-h_j-\overline{h}_j}\big(1+z\overline{w}\big)^{r-h_i-\overline{h}_j}\big(1+w\overline{z}\big)^{r-h_j-\overline{h}_i}$$

and

$$\eta = \frac{|z-w|^2}{(1+|z|^2)(1+|w|^2)}$$

where $r \equiv \frac{1}{3}(h_i+h_j+\overline{h}_i+\overline{h}_j)$.

For the function G, we know from Section1.2 that it can be decomposed into conformal blocks for distinct OPE channels, and the different results should agree. This was the idea of crossing symmetry. In the present situation, the OPE between Φ,Φ and $\Phi, I(\Phi)$ yield[29]:

$$\Phi_i(z)\Phi_j(w) \sim \sum_k C_{ijk}(z-w)^{h_k-h_i-h_j}(\overline{z}-\overline{w})^{\overline{h}_k-\overline{h}_i-\overline{h}_j}\Phi_k(w) \tag{205}$$

and

$$\Phi_i(z)I(\Phi_j)(-\frac{1}{\overline{w}}) \sim \sum_k C_{ijk}\Gamma_k(1+z\overline{w})^{h_k-h_i-\overline{h}_j}(1+\overline{z}w)^{\overline{h}_k-\overline{h}_i-h_j}\Phi_k(w) \tag{206}$$

where we have to include a factor of Γ_k in the second expression, since the image of the field need not coincide precisely with the field itself, as is dictated by the one-point function (189).

Thus we are now able to compute the G function in two different channels and use crossing symmetry. First, we can fuse Φ_i, Φ_j and $I(\Phi_i)I(\Phi_j)$, i.e. the s-channel with $\eta \to 0$ The holomorphic part of G in this case is:

$$G(\eta) = \sum_k C_{ijk}\Gamma_k\mathcal{F}(k;\eta) \tag{207}$$

If we next fuse in the t-channel($\eta \to 1$), namely $\Phi_i, I(\Phi_j)$ and $\Phi_j, I(\Phi_i)$ we get a similar expression except with a conventional nromalization which is justified from the choice of the location of the image field[29]:

$$G(\eta) = \sum_k (-1)^{h_i - \bar{h}_i + h_j - \bar{h}_j} C_{ijk} \Gamma_k \mathcal{F}(k; 1-\eta) \tag{208}$$

Agreement between the expressions (208) and (209) will generate for us a non-trivial sewing constraint on Γ_i's in terms of bulk CFT data. For Minimal Model CFTs we can further relate the conformal blocks $\mathcal{F}$, evaluated at η and $1-\eta$, with the *duality matrices* M:

$$\mathcal{F}(p;\eta) = \sum_q M \begin{bmatrix} i & l \\ j & k \end{bmatrix}_{p,q} \mathcal{F}_(q; 1-\eta)$$

Hence, the final elegant result of [29] reads[31]:

$$\sum_k C_{ijk} \Gamma_k M \begin{bmatrix} i & \bar{j} \\ j & \bar{i} \end{bmatrix}_{kp} = (-1)^{h_i - \bar{h}_i + h_j - \bar{h}_j} C_{ijp} \Gamma_p \tag{209}$$

2.5 Solving the Ising & Tricitical Ising model on $\mathbb{RP}^2$

We now move to solving the crosscap constraint (209) in terms of Γ_i's for the two diagonal minimal models, namely the Ising and the Tricritical Ising model. The OPE coefficients and the fusing matrices for diagonal minimal theories are obtained via equations (293) and (294), in the Coulomb gas formalism explained in the Appendix A. In addition, the normalization term $(-1)^{h_i - \bar{h}_i + h_j - \bar{h}_j}$ is trivial in our case. Thus, after numerically examining all the different linear equations arising in (209), we summarize our results:

Ising model

$$\boxed{\frac{\Gamma_\epsilon}{\Gamma_{\mathbb{1}}} = \sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}\ , \qquad \Gamma_\sigma = 0} \tag{210}$$

Tricritical Ising model

$$\boxed{\frac{\Gamma_{\epsilon'}}{\Gamma_{\mathbb{1}}} = \sqrt{\sqrt{5}-2}\ , \quad \frac{\Gamma_{\epsilon''}}{\Gamma_\epsilon} = \sqrt{\sqrt{5}+2}\ , \quad \frac{\Gamma_{\epsilon''}}{\Gamma_{\mathbb{1}}} = \sqrt{2}-1\ , \quad \frac{\Gamma_{\epsilon'}}{\Gamma_\epsilon} = \sqrt{2}+1\ , \quad \Gamma_\sigma = \Gamma_{\sigma'} = 0} \tag{211}$$

[31] we have been careful to maintain consistently the notation we have used in 1.2.7 for the entries and indices of the fusing matrix M.

As it was expected from the form of (209), the one-point normalizations Γ_i are fixed up to a normalization constant, that's why we present their ratios above.

This is the main result of this thesis. The values of Γ_i's for the Ising model have been reported previously in the literature[29], however the results for the Tricritical Ising are new as far as we can tell. Given (210) and (211),along with the knowledge of the OPE coefficients and the spectrum of primary operators, the Ising and the Tricritical Ising model are completely specified on $\mathbb{RP}^2$. We would like to proceed and examine these results in the context of gravity next.

3 Quantum gravity on AdS_3

3.1 Basics of the AdS/CFT correspondence

In the final chapter of this thesis we are going to discuss the quantum gravity physics that is hidden behind the study of CFTs. As we will describe, a quantum theory of gravity in a d-dimensional Anti-de Sitter spacetime (to be defined shortly) is conjecturally *equivalent* to a $(d-1)$-dimensional CFT, i.e. a theory *without* gravity in one less spatial dimension. This remarkable observation is commonly attributed to J.Maldacena in his seminal work [32], though some earlier contributions were also made by J.Brown and M.Henneaux in the setting of three-dimensional Anti-de Sitter space[33]. The AdS/CFT correspondence is a manifestation of a general principle that is believed to govern quantum gravity physics, the so-called *holographic principle* [30][31]. For the purposes of this thesis, we are going to focus primarily on AdS_3/CFT_2 correspondence, and analyze the several CFT derivations we obtained in the previous chapters. Our goal, in particular, is to understand the implications of the *crosscap states* from the quantum gravity side point of view. As we will explore soon, the crosscap states will help us -up to some level- in understanding the structure of the quantum mechanical path integral over geometries.

3.1.1 Anti-de Sitter spacetime

We will start by sketching the important general clues that lead to the famous AdS/CFT correspondence, not necessarily following the historical order of the scientific developments. Our basic reference is the lecture notes by J.Kaplan [34]. We are working in Euclidean signature as we did in CFT.

The d-dimensional Euclidean Anti-de Sitter spacetime with length scale l_{AdS} is defined as the set of all points $(X^0, X^1, X^2, \cdots, X^d)$ in a $(d+1)$-dimensional Minkowski spacetime $\mathbb{R}^{d,1}$ that satisfy:

$$-\left(X^0\right)^2 + \left(X^1\right)^2 + \cdots + \left(X^d\right)^2 = -l_{AdS}^2, \qquad (X^0 > 0) \tag{212}$$

It is clear from the above definition that the isometry group of Anti-de Sitter spacetime is $SO(d,1)$, which is exactly the conformal group in $\mathbb{R}^{d-1}$ as we showed in the first chapter. This is our starting naive clue for supporting any correspondence between gravity in AdS

and CFT. The generators are given by:

$$J_{AB} = -i\big(X_A \frac{\partial}{\partial X^B} - X_B \frac{\partial}{\partial X^A}\big) \tag{213}$$

where $A, B = \{0, 1, .., d\}$. The corresponding quadratic Casimir is:

$$\frac{1}{2} J_{AB} J^{BA} = -X^2 \partial_X^2 + X \cdot \partial_X (d + X \cdot \partial_X) \tag{214}$$

where repeated indices are summed over as usual. Using (212), let us define some intrinsic coordinate systems on Euclidean AdS_d. Poincare' coordinates (x^μ, z) are defined by the identification:

$$X^0 \equiv l_{AdS} \frac{1 + x^2 + z^2}{2z} \ , \quad X^\mu \equiv l_{AdS} \frac{x^\mu}{z} \ , \quad X^d \equiv l_{AdS} \frac{1 - x^2 - z^2}{2z} \tag{215}$$

where $x^\mu \in \mathbb{R}^{d-1}$ and $z > 0$. In these coordinates, the metric on AdS_d reads:

$$ds^2 = l_{AdS}^2 \frac{dz^2 + \delta_{\mu\nu} dx^\mu dx^\nu}{z^2} \tag{216}$$

One can easily show that the above metric is a maximally symmetric solution of the Einstein-Hilbert action with negative cosmological constant $\Lambda \equiv -\frac{(d-1)(d-2)}{2 l_{AdS}^2}$ and no matter fields:

$$S_{EH}[g_{\mu\nu}] = \frac{1}{l_{Pl}^{d-2}} \int d^d x \sqrt{g} [\mathcal{R} - 2\Lambda] \tag{217}$$

where $\mathcal{R}$ is the Ricci scalar and $l_{Pl} \equiv G_N^{1/(d-2)}$ is the Planck length, whose relative size with l_{AdS} controls the quantum/classical effects as we will see later. Also, the metric (216) shows that AdS is conformal to $\mathbb{R}^+ \times \mathbb{R}^{d-1}$ whose boundary at $z = 0$ is just $\mathbb{R}^{d-1}$. These coordinates make explicit the subgroup $SO(1,1) \times ISO(d-1)$ of the full isometry group of AdS. These correspond to dilatation and Poincare symmetries inside the $(d-1)$-dimensional conformal group. In particular, the dilatation operator is:

$$D = -i J_{0,d} = -X_0 \frac{\partial}{\partial X^d} + X_d \frac{\partial}{\partial X^0} = -z \frac{\partial}{\partial z} - x^\mu \frac{\partial}{\partial x^\mu} \tag{218}$$

Another useful coordinate system for AdS_d is the one defined by (τ, Ω^μ, ρ) as:

$$X^0 \equiv l_{AdS} \cosh\tau \cosh\rho \ , \quad X^\mu \equiv l_{AdS} \Omega^\mu \sinh\rho \ , \quad X^d \equiv -l_{AdS} \sinh\tau \cosh\rho \tag{219}$$

where Ω^μ $(\mu = 1, .., d-1)$ parametrizes a unit $(d-2)$-dimensional sphere, $\Omega \cdot \Omega = 1$. The metric is given by:

$$ds^2 = l_{AdS}^2 \big[\cosh^2\rho \ d\tau^2 + d\rho^2 + \sinh^2\rho \ d\Omega_{d-2}^2 \big] \tag{220}$$

To understand the global structure of the spacetime it is further convenient to change the radial coordinate ρ via the relation $\tanh\rho \equiv \sin r$, so that $r \in [0, \pi/2]$. Then, the metric becomes

$$ds^2 = \frac{l_{AdS}^2}{cos^2 r} \big[d\tau^2 + dr^2 + \sin^2 r \ d\Omega_{d-2}^2 \big] \tag{221}$$

which is conformal to a *solid* cylinder(Fig.3.1(a)) whose boundary at $r = \pi/2$ is $\mathbb{R} \times S^{d-2}$. In these coordinates, the dilatation operator $D = -i J_{0,d} = -\frac{\partial}{\partial \tau}$ is the Hamiltonian conjugate to global time.

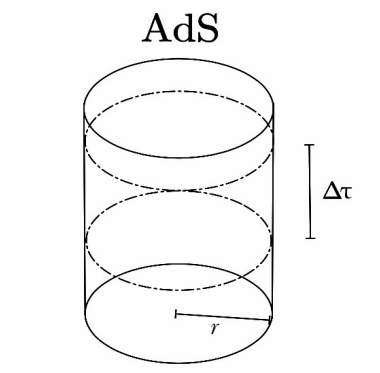

Figure3.1(a): The picture of AdS_d in global coordinates. The boundary is topologically an infinite cylinder $\mathbb{R} \times S^{d-2}$.

3.1.2 Particles and fields in AdS

We would like now to describe real time dynamics of particles or fields in a *fixed* Anti-de Sitter spacetime background. In order to gain the correct intuition it is useful to change the metric back to Lorentzian signature. Lorentzian AdS_d is defined by the universal cover of the manifold

$$-\left(X^0\right)^2 + \left(X^1\right)^2 + \cdots + \left(X^{d-1}\right)^2 - \left(X^d\right)^2 = -l_{AdS}^2 \tag{222}$$

embedded in $\mathbb{R}^{d-1,2}$ with isometry group $SO(d-1,2)$. The analytic continuation of the Euclidean global coordinates $(\tau, \Omega^\mu, \rho) \to (t, \Omega^\mu, \rho)$ via the map $\tau \to it$ (and $X^d \to iX^d$), now reads

$$X^0 \equiv l_{AdS} \cos t \cosh \rho \ , \quad X^\mu \equiv l_{AdS} \Omega^\mu \sinh \rho \ , \quad X^d \equiv -l_{AdS} \sin t \cosh \rho \tag{223}$$

As before, it is convenient to also change the radial coordinate via $\tanh \rho \to \sin r$ so that $r \in [0, \pi/2]$. Then, the metric becomes:

$$ds^2 = \frac{l_{AdS}^2}{cos^2 r}\big[-dt^2 + dr^2 + \sin^2 r \ d\Omega_{d-2}^2\big] \tag{224}$$

which is, again, conformally equivalent to a solid cylinder whose boundary at $r = \pi/2$ is $\mathbb{R} \times S^{d-2}$.

If we want to study the motion of a classical particle in AdS we have to study geodesics of course. Geodesics are basically given by the intersection of AdS with 2-planes through the origin of the embedding space $\mathbb{R}^{d-1,2}$. In global coordinates (224), the simplest timelike geodesic describes a particle sitting at $\rho = 0$. This corresponds to the intersection of $X^\mu = 0$ for $\mu = 1, \cdots, d-1$ with the hyperboloid (222). Performing a boost in the (X^1, X^d)-plane we can obtain an equivalent timelike geodesic $X^1 \cosh \beta = X^d \sinh \beta$ and $X^\mu = 0$ for $\mu = 2, \cdots, d-1$. In global coordinates, this gives an oscillating trajectory

$$\tanh \rho = \tanh \beta \ \sin t \tag{225}$$

with period 2π. This is actually a general fact about timelike geodesics in global time. Therefore, we can say that AdS acts like a *box* that confines massive particles, and it is a very symmetric box since all points are equivalent through the isometry.

Null geodesics in AdS are also null geodesics in the embedding space. For example, the null ray $X^d - X^1 = X^0 - l_{AdS} = X^\mu = 0$ for $\mu = 2, \cdots, d-1$ is a null ray in AdS which in global coordinates is given by $\cosh\rho = \frac{1}{\cos t}$. This equation describes a light ray that passes through the origin at $t = 0$ and reaches the conformal boundary $\rho = \infty$ at $t = \pm\frac{\pi}{2}$. All the light rays in AdS start and end at the conformal boundary travelling for a global time interval equal to π.

Now let's turn to the study of fields in AdS. For simplicity, we can choose a scalar field Φ of mass m obeying the Klein-Gordon equation in global coordinates (224):

$$\nabla^2_{AdS}\Phi = m^2\Phi \tag{226}$$

With a little bit of work we can show that the $SO(d-1,2)$ quadratic Casimir of Lorentzian AdS_d isometry group, acting on a scalar field, is given by the Laplacian as:

$$\frac{1}{2}J_{AB}J^{BA}\Phi = l^2_{AdS}\nabla^2_{AdS}\Phi \tag{227}$$

Therefore, we should identify the mass $l^2_{AdS}m^2$ of the field (measured in AdS units) with the eigenvalues of the quadratic Casimir. But the Casimir of the AdS_d isometry group is equivalent to the quadratic Casimir of the conformal group in $\mathbb{R}^{d-2,1}$ spacetime. In order to establish a connection with CFT, it's necessary to remind ourselves the connection of $J_{A,B}$'s with $D, M_{\mu\nu}, P_\mu, K_\mu$ generators, in Lorentzian signature:

$$\begin{aligned} D &= -J_{0,d} \quad , & P_\mu &= J_{\mu 0} + iJ_{\mu,d} \quad , \\ M_{\mu\nu} &= J_{\mu\nu} \quad , & K_\mu &= J_{\mu 0} - iJ_{\mu,d} \end{aligned} \tag{228}$$

In global coordinates, these generators take the form:

$$\begin{aligned} D &= i\frac{\partial}{\partial t} \quad , & P_\mu &= -ie^{-it}\big[\Omega_\mu(\partial_\rho - i\tanh\rho\,\partial_t) + \frac{1}{\tanh\rho}\nabla_\mu\big] \quad , \\ M_{\mu\nu} &= -i\big(\Omega_\mu\frac{\partial}{\partial\Omega^\nu} - \Omega_\nu\frac{\partial}{\partial\Omega^\mu}\big) \quad , & K_\mu &= ie^{it}\big[\Omega_\mu(-\partial_\rho - i\tanh\rho\,\partial_t) - \frac{1}{\tanh\rho}\nabla_\mu\big] \end{aligned} \tag{229}$$

Now, in analogy with the CFT construction we can look for primary states, which are annihilated by K_μ, $K_\mu\Phi = 0$, and are eigenstates of the Hamiltonian, $D\Phi = \Delta\Phi$. The solution has the form:

$$\Phi \sim \big(\frac{e^{-it}}{\cosh\ \rho}\big)^\Delta = \big(\frac{l_{AdS}}{X^0 - X^d}\big)^\Delta \tag{230}$$

This is the lowest energy state. One can get excited states acting with P_μ. The important thing is that all these states will have the same value under the Casimir which turns out to be

$$\frac{1}{2}J_{AB}J^{BA}\Phi = \Delta\big(\Delta - (d-1)\big)\Phi \tag{231}$$

Therefore, we can extract our first crucial result, that is:

$$l^2_{AdS}m^2 = \Delta(\Delta - d + 1) \tag{232}$$

Let's now return to Euclidean signature and consider *quantum* fields in AdS background. For simplicity, we continue with the example of a free scalar field of mass m with action:

$$S = \int_{AdS} dX \big[\frac{1}{2}(\nabla\Phi)^2 + \frac{1}{2}m^2\Phi^2\big] \tag{233}$$

Our goal is to compute all the correlation functions of this theory. The two-point function is given by the propagator $\Pi(X,Y)$, which obeys:

$$\big[\nabla_X^2 - m^2\big]\Pi(X,Y) = -\delta(X,Y) \tag{234}$$

From the symmetries of ∇_X^2, it is clear that the propagator can depend on the invariant $X\cdot Y$ or equivalently on the so-called *chordal distance* $\zeta \equiv (X-Y)^2/l_{AdS}^2$. From now on we will set $l_{AdS}=1$ and all lengths will be measured with respect to AdS radius. Using the form (231) for the Laplacian we can show that

$$\Pi(X,Y) = \frac{C_\Delta}{\zeta^\Delta}\ {}_2F_1\big(\Delta, \Delta - \frac{d}{2} + 1, 2\Delta - d + 2, -\frac{4}{\zeta}\big) \tag{235}$$

where, here, Δ appears exactly as $\Delta \equiv \frac{d-1}{2} + \frac{\sqrt{(d-1)^2+(2m)^2}}{2}$ (c.f. (232)), and the coefficient C_Δ is:

$$C_\Delta \equiv \frac{\Gamma(\Delta)}{2\pi^{(d-1)/2}\Gamma(\Delta - \frac{d}{2} + \frac{3}{2})} \tag{236}$$

The higher point functions, as in free QFT in flat space, are simply given by Wick contractions. For example a four-point function is

$$\langle\Phi(X_1)\Phi(X_2)\Phi(X_3)\Phi(X_4)\rangle = \Pi(X_1,X_2)\Pi(X_3,X_4) + \Pi(X_1,X_3)\Pi(X_2,X_4) + \Pi(X_1,X_4)\Pi(X_2,X_3) \tag{237}$$

If we want to include weak-coupling interactions in our theory, these can be treated perturbatively as usually in QFT. Suppose, for instance, the action which includes a cubic self-interaction:

$$S = \int_{AdS} dX \big[\frac{1}{2}(\nabla\Phi)^2 + \frac{1}{2}m^2\Phi^2 + \frac{1}{3!}g\Phi^3\big] \tag{238}$$

Then, a three-point function at tree level is given by

$$\langle\Phi(X_1)\Phi(X_2)\Phi(X_3)\rangle = -g\int_{AdS} dY\ \Pi(X_1,Y)\Pi(X_3,Y)\Pi(X_3,Y) + O(g^3) \tag{239}$$

It would be interesting to investigate how two and three-point functions behave if we send all points to the boundary of AdS. More precisely, we can parametrize the point X by introducing a real parameter λ as $X \equiv \lambda P + \cdots$, where P is a future-directed null vector in $\mathbb{R}^{d,1}$ and the $\cdots$ denote terms that do not grow with λ but are there to ensure that the parametrization is compatible with $X^2 = -1$(AdS manifold). Then, we can write

$$O(P) \equiv \frac{1}{\sqrt{C_\Delta}} \lim_{\lambda\to\infty} \lambda^\Delta\ \Phi(X = \lambda P + \cdots) \tag{240}$$

In other words, the operator $O(P)$ is the limit of the field $\Phi(X)$ when X approaches the boundary point P of AdS. We should emphasize that, by definition, the operator $O(P)$ obeys the condition

$$O(\lambda P) = \lambda^{-\Delta} O(P) \tag{241}$$

which reminds us the notion of *conformal dimension* Δ of an operator under scalings. If we ask now how the correlation functions of O behave, the answer is simple. They are naturally defined by the limit of correlation functions of Φ in AdS. For example, the two-point function at tree-level is given by

$$\langle O(P_1)O(P_2)\rangle = \frac{1}{(-2P_1 \cdot P_2)^{\Delta}} + O(g^2) \tag{242}$$

which is exactly the CFT two-point function of a primary operator of dimension Δ. We can show the same for the three-point function at tree-level, namely

$$\langle O(P_1)O(P_2)O(P_3)\rangle = -gC_{\Delta}^{-\frac{3}{2}} \int_{AdS} dX \Pi(X, P_1)\Pi(X, P_2)\Pi(X, P_3) + O(g^3) \tag{243}$$

where

$$\Pi(X, P) = \lim_{\lambda \to \infty} \lambda^{\Delta} \Pi(X, Y = \lambda P + \cdots) = \frac{C_{\Delta}}{(-2P \cdot X)^{\Delta}} \tag{244}$$

is the *bulk-to-boundary propagator*. Using its expression in (243) and manipulating the integral appropriately, we can eventually arrive at the form of a *CFT three-point function* for a primary operator with dimension Δ (c.f. (36)). Same kind of manipulations hold for the four-point amplitude of O at order g^2(tree level), where one confirm the form of a CFT four-point function as well.

Taking into consideration the above, we see that the operators $O(P)$ are of particular importance in our analysis and reinforce a possible duality of QFT in AdS with a quantum conformal field theory[32]. We can go even further actually and prove that the correlators of O obey an *associative OPE*, and that there is also a *state-boundary operator* map that can be derived purely from the AdS symmetries [35]. However, there is still one missing ingredient to obtain a full-fledged CFT: a stress-energy tensor. In the next subsection, we will see that this requires dynamical gravity in AdS. Moreover, one can show in an elegant way[37] that free QFT in AdS_d, on its own right, cannot be dual to a *local* CFT in $(d-1)$-dimensions.

Before we move on, let us mention another equivalent way of obtaining correlation functions in AdS/CFT. This is the so-called GKPW dictionary[35][36] [33]. We can introduce the generating functional

$$Z_{CFT \text{ on } \partial M}\ [\Phi_{\partial}] \equiv \left\langle e^{\int_{\partial M}\ dP\ \Phi_{\partial}(P)O(P)} \right\rangle \tag{245}$$

where the asymptotic value Φ_{∂} of our field Φ is a source for the operator O and, the integral over ∂M denotes formally an integral over a chosen section of the null cone

[32]Sometimes it is commonly said that CFT "lives at the boundary of AdS". This statement, however, is not quite accurate for two basic reasons. First, we do not have both theories at once; we either do CFT or we have an AdS spacetime. Second and more importantly, the CFT is dual to the *entire* gravity theory (i.e. the two Hilbert spaces are equivalent in principle) so in a sense it's not "confined" in some boundary region.

[33]as "opposed" to the BDHM dictionary due to [38], which was the method we used previously.

in $\mathbb{R}^{d,1}$ with its induced measure. We impose $\Phi_\partial(\lambda P) = \lambda^{\Delta-d+1}\Phi_\partial(P)$ and then, the correlation functions are easily obtained with functional derivatives

$$\langle O(P_1)\cdots O(P_n)\rangle = \frac{\delta}{\delta\Phi(P_1)}\cdots\frac{\delta}{\delta\Phi(P_n)}Z[\Phi_\partial]\Big|_{\Phi_\partial=0} \tag{246}$$

If we now set the generating functional to be equal to the gravitational partition function Φ in a *fixed* asymptotically AdS background

$$Z_{CFT\text{ on }\partial M}\ [\Phi_\partial] = Z_{AdS}[\Phi_\partial] \equiv \frac{\int_{\Phi\to\Phi_\partial}\mathcal{D}\Phi\ e^{-S[\Phi]}}{\int_{\Phi\to 0}\mathcal{D}\Phi\ e^{-S[\Phi]}} \tag{247}$$

with the boundary condition that it approaches the source Φ_∂ at the boundary,

$$\lim_{\lambda\to\infty}\lambda^{(d-1-\Delta)}\ \Phi(X=\lambda P+\cdots) = \frac{1}{(2\Delta+1-d)}\frac{1}{\sqrt{C_\Delta}}\Phi_\partial(P) \tag{248}$$

then we recover the correlation functions of O defined above as limits of the correlation functions of Φ.

The statement of the equality of the partition functions is actually more general than expression (247). The UV complete gravitational partition function is a function not only of any matter fields but also of the metric. In the above analysis we were merely describing an effective field theory where the scales of the problem where such that the metric quantum fluctuations are negligible ($l_{AdS}, 1/m, .. >> l_{Pl}$). The generalized statement of the AdS/CFT correspondence is:

$$Z_{CFT\text{ on }\partial M}\ [\Phi_\partial] = Z_{AdS}\ [\Phi_\partial;\partial M] \tag{249}$$

where the symbol ∂M denotes the boundary conditions for the value of the metric (i.e. asymptotically AdS) and the *topology* of the spacetime boundary. This remarkable statement, in a sense, summarizes a non-perturbative definition of quantum gravity in AdS. We will examine it next to some extent.

3.1.3 Dynamical gravity in AdS

Let's consider general relativity in the presence of a negative cosmological constant and a scalar field. The full action reads:

$$S_{EH}[g_{\mu\nu},\Phi] = \frac{1}{l_{Pl}^{d-2}}\int d^dw\sqrt{g}[\mathcal{R}-2\Lambda+\frac{1}{2}(\nabla\Phi)^2+\frac{1}{2}m^2\Phi^2+\cdots] \tag{250}$$

The partition function (omitting the normalization with the zero-point amplitude) includes, at least in principle, a sum over geometries:

$$Z_{AdS}[\Phi_\partial;\partial M] = \int_{\substack{g\to\tilde{g}\\ \Phi\to\Phi_\partial}}\mathcal{D}\Phi\mathcal{D}g\ e^{-S_{EH}[g,\Phi]} \tag{251}$$

Of course we cannot actually compute the path integral (251) by brute force. In fact, the Euclidean action in gravity is not bounded below[39], so the problem is not well-defined on it's own. The best we can do is to approximate it by expanding around

classical saddlepoints, that is, a solution of the classical Einstein field equations

$$Z_{AdS}[\Phi_\partial; \partial M] \sim e^{-S_{EH}[\Phi_{cl.}, g_{cl.}] + \cdots} \tag{252}$$

where the solutions $\Phi_{cl.}, g_{cl.}$ obey the appropriate boundary conditions and the $\cdots$ indicate $O(G_N^0)$ terms, i.e. loop terms.

Whenever our theory is defined on a space with a boundary, we need to add a boundary term to the action (250) in order to have a well posed variational problem for the metric. This is the so-called Gibbons-Hawking-York term (we set $G_N = 1$):

$$S_{EH}[g, \Phi] = \int d^d w \sqrt{g}[\mathcal{R} - 2\Lambda + \ \cdots] + 2\int_{\partial M} \sqrt{h}(K - K_0) \tag{253}$$

Here, h_{ij} is the induced metric on the boundary and the *extrinsic curvature* of ∂M is:

$$K_{ij} \equiv \frac{1}{2}\mathcal{L}_n h_{ij} = \nabla_{(i} n_{j)} \ , \qquad\qquad K = h^{ij} K_{ij} \tag{254}$$

where n is the inward-pointing unit normal to ∂M. The constant term K_0 is just the extrinsic curvature of the boundary embedded in flat spacetime and it serves as a counterterm, as we will see soon.

The Gibbons-Hawking-York term is needed for the action to be stationary around classical solutions. The variation of the Einstein term includes both bulk and boundary terms

$$\delta \int_M \sqrt{g}\mathcal{R} \sim \int_M (EoM)\delta g + \int_{\partial M} [A(g, \partial g)\delta g + B(g, \partial g)\partial\delta g] \tag{255}$$

On a classical solution the bulk term vanishes and if we have fixed the metric at the boundary then $\delta g\big|_{\partial M} = 0$. The term involving $\partial\delta g$, though, is not zero. The Gibbons-Hawking-York term fixes this problem. It is chosen so that the variation of the full action has the form

$$\delta S_{EH}[g] = \int_M (EoM)\delta g - \frac{1}{4\pi}\int_{\partial M} \sqrt{h} T^{ij}\delta h_{ij}. \tag{256}$$

i.e. we eliminated the term $\partial\delta g$. The "quasi-local stress-tensor" associated with the boundary

$$\begin{aligned} T^{ij} &\equiv -\frac{4\pi}{\sqrt{h}}\frac{\delta S^{on-shell}}{\delta h_{ij}} \\ &= -\frac{1}{4}\big(K^{ij} - K h^{ij} - K_0 h_{ij}\big) \end{aligned} \tag{257}$$

is called the *Brown-York stress tensor*. The first two terms in the second expression came from varying the Einstein action plus the Gibbons-Hawking term. The last term comes from the counterterm, with the coefficient set in order to make the answer finite as $r \to \infty$. We would like to compute this stress tensor and see how its expression in terms of the boundary data.We will specialize to the case of AdS_3 to sketch this calculation and study its consequences.

Consider global coordinates in Euclidean AdS_3 ($l_{AdS} = 1$)

$$ds^2 = \cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho \ d\phi^2 \tag{258}$$

The group of symmetries of AdS_3 is $SO(3,1) = SL(2,\mathbb{C})$ or, in Lorentzian signature, $SO(2,2) = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. Let us perform the coordinate change $z, \overline{z} = \tau \pm i\phi$, $\rho = log(2r)$ and send $r \to \infty$. At leading order in $1/r$ we get:

$$ds^2 = \frac{dr^2}{r^2} + r^2 dz d\overline{z} \tag{259}$$

Having this in mind, we define a metric to be *asymptotically AdS* if at large r takes the form:

$$ds^2 = h_{ij}dx^i dx^j = \frac{dr^2}{r^2} + r^2 dz d\overline{z} + h_{zz}dz^2 + h_{\overline{z}\overline{z}}d\overline{z}^2 + 2h_{z\overline{z}}dz d\overline{z} \tag{260}$$

where the h's are arbitrary functions of z and $\overline{z}$ but independent of r, and serve as a continuous perturbation to the AdS metric. A notable example of an asymptotically AdS spacetime is the BTZ black hole[40], which is an "honest"[34] black hole solution in three-dimensional gravity with negative cosmological constant.

Using Einstein equations for this metric imply that the perturbation is traceless and conserved, namely:

$$h_{z\overline{z}} = 0 \ , \qquad \partial h_{\overline{z}\overline{z}} = \overline{\partial} h_{zz} = 0 \tag{261}$$

We can now implement (257) and determine the Brown-York tensor. The appropriate choice for the counterterm in this case is $K_0 \equiv 1/l_{AdS}$, where we restored the AdS units for consistency. Using (261), after a long calculation[35] we arrive at the important result:

$$T_{z\overline{z}} = 0 \ , \qquad T_{zz} = -\frac{1}{4l_{AdS}} h_{zz} \ , \qquad T_{\overline{z}\overline{z}} - \frac{1}{4l_{AdS}} h_{\overline{z}\overline{z}} \tag{262}$$

Thus, the Brown-York tensor is traceless, conserved and therefore holomorphic/anti-holomorphic just like in CFT. We have managed to relate a quantity that is basically related to the boundary metric in the gravity side, with the characteristics of the energy-momentum tensor of a CFT in one less dimension. We can make this relation even stronger actually.

Remaining in our AdS_3 set up, we can ask what vector field ζ preserve the form of the metric (260). The answer is:

$$\begin{aligned} z &\to z + \epsilon(z) - \frac{l_{AdS}^4}{2r^2}\overline{\epsilon}''(\overline{z}) \\ \overline{z} &\to \overline{z} + \overline{\epsilon}(\overline{z}) - \frac{l_{AdS}^4}{2r^2}\epsilon''(z) \\ r &\to r - \frac{r}{2}\epsilon'(z) - \frac{r}{2}\overline{\epsilon}'(\overline{z}) \end{aligned} \tag{263}$$

for arbitrary functions $\epsilon(z)$ and $\overline{\epsilon}(\overline{z})$. Near the boundary, these act on $z, \overline{z}$ just like conformal transformations, whereas the extra ∂_r piece acts as a rescaling. Thus, transformations of AdS_3 that preserve the asymptotics of the metric coincide with 2d conformal

[34] by honest here we mean that it shares all the general features of black holes, like a mass/charge, a horizon of positive area (length) and a corresponding Bekenstein- Hawking entropy in the semi-classical limit[41]

[35] for details of the calculation see[42].

transformations! Hence, the actual group of symmetries of AdS_3 turns out to be the infinite dimensional Virasoro group of orientation-preserving diffeomorphisms of the unit circle, that is $G = diffS^1 \times diffS^1$(which contains $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ as a subgroup).

We can derive a very crucial result from this observation. If we set $\bar{\epsilon} = 0$ and focus only on the holomorphic transformations, under (263) the metric transforms as

$$ds^2 \to ds^2 + \big(-2h_{zz}\epsilon' - \epsilon h'_{zz} + \frac{l^2_{AdS}}{2}\epsilon'''\big)dz^2 \tag{264}$$

Thus the dz^2 piece of the metric, which we interpreted as the gravitational stress tensor up to a factor of $-1/4l_{AdS}$, transforms as

$$\delta_\epsilon T = -\epsilon\partial T - 2T\partial\epsilon - \frac{l_{AdS}}{8}\epsilon''' \tag{265}$$

This is exaclty the infinitesimal transformation law of T in 2d CFT discussed back in (67). Comparing the coefficient coming from the Schwarzian derivative, we see

$$c = \frac{3l_{AdS}}{2G_N} = \frac{3l_{AdS}}{2l_{Pl}} \tag{266}$$

where we restored the factor of G_N from dimensional analysis. This is called the *Brown-Henneaux central charge* after the work of Brown and Henneaux in [33] back in 1987, long before the establishment of AdS/CFT by J.Maldacena. We will come back to the Brown-Henneaux analysis later. For now, the above result, although subject to the case of AdS_3/CFT_2 correspondence, allow us to discuss some interesting conclusions that hold in general in AdS/CFT.

First of all, it seems that the magnitude of the central charge determines how quantum/classical our theory is in the gravity side. A CFT with small central charge (e.g. a minimal model CFT where $c < 1$) should, at least naively, correspond to a highly quantum mechanical regime of gravity in the bulk. In contrast, large central charge should describe a semi-classical theory of gravity as given by the Einstein-Hilbert action (or suitable generalizations), where the AdS radius is sufficiently larger than the Planck length. This last statement should be taken with caution though, as we will explain shortly.

Back in (252), we saw that we can compute the gravitational partition function by saddle point approximation in powers of G_N, which basically organize the loop expansion. If we want to calculate a graviton two-point function at tree-level then we will see that

$$\langle gg \rangle \sim 1/G_N + \cdots \tag{267}$$

From the CFT side, we already know that the two point function of the energy-momentum tensor has the form (c.f. (66))

$$\langle TT \rangle \sim c + \cdots \tag{268}$$

This observation, together with the Brown-Henneaux result suggest that the metric tensor in gravity is dual to the stress-tensor in CFT. Every theory of gravity has a massless spin-2 particle and every CFT has a stress tensor $T_{\mu\nu}$ which generates conformal

trasnformations. In fact, this statement is generic in $\text{AdS}_d/\text{CFT}_{d-1}$. Indeed, one can show that when $l_{AdS} >> l_{Pl}$ the gravitons form an approximate Fock space of states. One can compute the single graviton states and verify that they are in one-to-one correspondence with the CFT stress tensor and its derivatives.

In general, it is believed that the AdS/CFT correspondence, as summarized by (249), holds for any theory of gravity and CFT. That is, given a theory of gravity we can use it to *define* a CFT, and (perhaps) vice-versa. Aside from certain examples however, the correspondence is well defined and useful only in certain regimes, like the semi-classical limit in gravity. And even in that limit, certain requirements should hold for the CFT, in order to reproduce a nice semiclassical theory of gravity given by Einstein-Hilbert action or suitable generalizations thereof (i.e.$f(\mathcal{R})$ gravity)[43].

3.2 Holography on orientable surfaces

In the previous section we were working on coordinates where the boundary, ∂M, of Euclidean AdS_d was topologically a sphere S^{d-1}(or an infinite cylinder $\mathbb{R} \times S^{d-2}$). As we've discussed, the topology of the asymptotic boundary is crucial for the computation of the gravitational path integral. If we don't include any matter fields in our analysis, then the path integral reads:

$$Z_{AdS}\ [\partial M] = \int \mathcal{D}g\ e^{-S[g]} \tag{269}$$

and we say that we deal with *pure* gravity theories. In the present and following section, we're going to study this integral over geometries in its own merit, and use the AdS/CFT correspondence to see what new knowledge we can gain about it. In particular, an interesting question one can ask is: can we have a CFT which is dual to a quantum mechanical theory of just *pure* gravity[36] degrees of freedom? We are going to focus on the case of $\text{AdS}_3/\text{CFT}_2$ and see what answers we can get. The topology of the AdS_3 boundary will "define" for us a dual CFT on surfaces we studied in the previous chapters of this thesis. In this section we discuss the case of an *orientable* boundary, namely the torus $\mathbb{T}^2$, and we follow the analysis of [45],[46]. The authors of [47] were actually the first to interpret a gravitational path integral with torus boundary conditions as a CFT partition function, where they termed the resulting sum over geometries lyrically as *Black Hole Farey Tail*.

The problem we have to solve is the Euclidean path integral (269) over asymptotically AdS manifolds M, with a conformal structure of a torus at the boundary. As it's proposed in [45],[46], to compute this path integral we should sum not just over all the metrics on a fixed topological 3-manifold, but also over all possible topologies. The boundary conditions conditions in (269) fix the topology of spacetime only at the boundary, not in the interior. Thus, the gravitational path integral should take the form:

$$Z_{AdS}\ [\partial M] = \sum_{M} Z(M) \tag{270}$$

[36]the particular theory of gravity in the bulk plays an important role in this question. We are going to focus mostly on three-dimensional Einstein-Hilbert action, but there are other interesting theories of pure 3d-gravity (e.g. Chiral Gravity[44])

where M denotes a specific topological 3-manifold and $Z(M)$ the contribution from the sum over all metrics on M. This is a nice organization of the path integral but we would like to know how to compute the different M. We can get a hint from the saddle point approximation:

$$Z_{AdS}\ [\partial M] = \sum_{g_{cl}} \exp\big(- cS[g_{cl}] + S^{(1)}[g_{cl}] + \frac{1}{c} S^{(2)}[g_{cl}] + \cdots \big) \tag{271}$$

where we have normalized the action S so we can have a factor of $c = 3l_{AdS}/2G_N$ out front, and g_{cl} is a classical saddle, i.e. a solution to the equations of motion. The terms $S^{(k)}$ describe k-loop terms. As we can see, in the semi-classical limit the only topologies which contribute to the path integral are those which admit a classical solution to the equations of motion. In fact, one of the basic assumptions of the authors of [46], is that this picture -that the only topologies which contribute are those which admit a classical solution- continues to hold even in the highly quantum regime where c is of order one. This is a relatively strong assumption but one needs it in order to make progress. In this regime, the seed partition function for a given g_{cl} contains an infinite (in principle) series of quantum corrections and there is no sense in which these corrections are small compared to the classical $O(c)$ term.

A second important assumption in the following analysis is the inclusion only of *smooth* geometries for g_{cl}. Non-smooth geometries, such as those which include conical-type singularities, may solve the equations of motion locally but are usually associated with new degrees of freedom in the bulk and thus do not describe pure theories of gravity. We will come back to question this assumption actually, when we study non-orientable boundaries in AdS.

Given these two assumptions, we can go ahead and write an explicit expression for the sum over topologies, $\sum_M$, and for the seed partition function, $Z(M)$, as we will see. Let's start by thinking of a simple geometry M that contributes to the above sum. The metric of Euclidean AdS_3 in global coordinates reads:

$$ds^2 = l_{AdS}^2 \big(\cosh^2 \rho dt_E^2 + d\rho^2 + \sinh^2 \rho d\phi^2 \big) \tag{272}$$

where the angle ϕ is periodic $\phi \sim \phi + 2\pi$. Defining the complex coordinate $z = i(\phi + i\, t_E)$, we have the identification $z \sim z + 2\pi i n$ for $n \in \mathbb{Z}$. If we further make the identification

$$z \sim z + 2\pi i m\tau\ , \qquad m \in \mathbb{Z} \tag{273}$$

then we get *thermal AdS_3* with a torus boundary of conformal structure τ. Topologically, thermal AdS_3 is a solid torus with one contractible circle -the spatial ϕ circle- and a non-contractible one -the t_E circle. This situation actually makes it clear how one can construct other distinct topologically 3-manifolds. One just needs to consider other solid tori where *other* circles are of the boundary torus are contractible in the bulk. Geometrically, these new solid tori will have the same metric as AdS_3 but will be "glued" in the boundary torus in a different way. Hence, the different 3-manifolds are related to thermal AdS_3 by *large* diffeomorphisms which act non-trivially on the boundary torus. And we actually know this group of large diffeomorphisms from when we studied conformal transformations on $\mathbb{T}^2$, that is, the modular group $PSL(2,\mathbb{Z})$ acting on the conformal structure τ:

$$\tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}\,, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}) \tag{274}$$

The geometries constructed in this way are physically inequivalent and, at first glance, give topologically distinct contributions to the path integral. They also describe *smooth* solutions to the equations of motion as was shown in [45]. A simple but important example is the diffeomorphism that corresponds to the S-transformation

$$z \to z' \equiv i(\phi' + it'_E) = -\frac{1}{\tau} z \tag{275}$$

After this change of variables, the contractible cycle is now a combination of ϕ' and t'_E and not just the spatial cycle alone. The Lorentzian continuation of this geometry will thus have a horizon. Indeed, this geometry is the Euclidean continuation of the BTZ black hole we saw back in 3.1.3.

We have reached the conclusion that different geometries are labelled by the elements γ of the modular group. We will refer to the contribution to the partition function of thermal AdS_3 as $Z_{vac.}(\tau, \overline{\tau})$ or, in other words, the contributions from metrics which are continuously connected to the AdS_3 solution (272). The full path integral therefore seems to take the form of a sum over the modular group $PSL(2,\mathbb{Z})$ of the seed partition function $Z_{vac.}(\gamma\tau, \gamma\overline{\tau})$.

It turns out there is a subtlety, however, which is that not all elements of $PSL(2,\mathbb{Z})$ will give distinct contributions to the path integral. As it's described in [46], there is a subgroup Γ_c of $PSL(2,\mathbb{Z})$ which corresponds to *trivial* diffeomorphisms on the boundary that do not change the topology of the resulting 3-manifold. The important thing in this analysis is that this subgroup depends on the central charge of 2d CFT that "lives" on the boundary torus(via the Brown-Henneaux analysis). If the central charge is $c < 1$, the subgroup Γ_c is a finite index subgroup of $PSL(2,\mathbb{Z})$ which leaves the vacuum character of the conformal theory invariant(we will understand shortly the reason for that). If the central charge is $c >> 1$, the theory is in the semi-classical regime and we can actually determine exactly those trivial diffeomorphisms: Γ_∞ constitutes the infinite subgroup of shifts $\tau \to \tau + n$, generated by n powers of the modular T matrix. These transformations basically add a contractible cycle to a non-contractible cycle but this fact does not affect the topology of the resulting 3-manifold. We will discuss more about the case of $c < 1$ later.

Eventually, the gravitational partition function is not a sum over the elements of the modular group, but rather a sum over the elements of the coset $\Gamma_c / PSL(2,\mathbb{Z})$:

$$Z_{AdS}\ [\mathbb{T}^2] = \sum_{\gamma \in \Gamma_c / PSL(2,\mathbb{Z})} Z_{vac.}(\gamma\tau, \gamma\overline{\tau}) \tag{276}$$

We note that this expression is manifestly modular-invariant and agrees with the modular invariance expected from CFT living on the boundary.

Now let's move to describing the contribution from a fixed topology $Z_{vac.}(\tau, \overline{\tau})$. As we said, $Z_{vac.}(\tau, \overline{\tau})$ describes the contributions from metrics which are continuously connected to the AdS_3 solution (272) with the identifications (273). We would like to determine its non-perturbative expression and not just some semi-classical expansion of the form (271). At least naively, we expect the partition function to be of the form

$$Z_{vac.}(\tau, \overline{\tau}) \sim \mathrm{Tr}_{vac.}\, e^{-\beta H + i\theta J} \tag{277}$$

where the trace is over the Hilbert space of the theory (which we don't know yet and want to determine eventually) and H, J operators which act on this Hilbert space. In the classical limit the theory is described by general relativity. The Hamiltonian H and the angular momentum J are defined via the ADM procedure in terms of the leading behaviour at spatial infinity of the correction to the AdS_3 metric (in our case). Hence, if we analyze the transformations that leave the metric invariant to leading order at spatial infinity, we will be able to know H, J as the conserved charges corresponding to the killing vectors of these transformations. This is exactly what we did back in (263) where we identified the group of transformations as $G = diffS^1 \times diffS^1$, summarizing the seminal work by Brown and Henneaux [33]. Let us actually proceed more concretely now.

We can define a *classical phase space* for our theory as the configuration space of all physically inequivalent excitations of AdS_3 -in the sense that they correspond to diffeomorphisms that do not vanish sufficiently quickly at infinity- which are continuously connected to the to the AdS "ground state" (272). The various states of the theory are obtained by acting on the ground state with one of the phase space charges. This action is dictated by the symplectic structure with which the phase space is equipped and is basically defined by the Dirac bracket[37]. Let us see these quantities explicitly.

In the Lorentzian version ($t_E \to t = -it_E$) of coordinates (272), the standard Brown-Henneaux boundary conditions at large ρ state that the metric is

$$ds^2 = l_{AdS}^2 \left(d\rho^2 + \frac{1}{4} e^{2\rho} (-dt + d\phi) \right) + O(\rho^0)$$

Following [46], we denote by ζ the vector which generates a diffeomorphism that preserves these boundary conditions, and by $H[\zeta]$ the corresponding phase space charge which generates the symmetry. Then, the *Fourier modes* ζ_ν of the killing vectors can be shown to be

$$\zeta_n = e^{inu} \big(\partial_u - \frac{1}{2} n^2 e^{-2\rho} \partial_v - i \frac{n}{2} \partial_\rho \big) + \cdots \ , \quad \overline{\zeta_n} = e^{inv} \big(\partial_v - \frac{1}{2} n^2 e^{-2\rho} \partial_v - i \frac{n}{2} \partial_\rho \big) + \cdots$$

where $u \equiv \frac{1}{2}(t + \phi)$, $v = \frac{1}{2}(t - \phi)$ and the "$\cdots$" denote subleading corrections in ρ that do not affect the charges $H[\zeta]$. Brown and Henneaux calculated the Dirac bracket of the corresponding charges $H[\zeta_n]$ and found the amazing result:

$$i\{H[\zeta_n], H[\zeta_m]\} = (n - m) H[\zeta_{n+m}] + \frac{c}{12} n(n^2 - 1) \delta_{n+m,0} \tag{278}$$

and similarly for $H[\overline{\zeta}_n]$. The Brown-Henneaux central charge is identified, as before, from the value $c = 3l_{AdS}/2G_N$. Hence, the phase space charges $H[\zeta_n]$ correspond to Virasoro generators of 2d conformal transformations. The states of the theory created by $H[\zeta_n]$ are usually called *boundary gravitons* and correspond to genuine physical states(or metrics) in the theory describing excitations above empty AdS.

We have managed to describe the classical phase space of the theory for a fixed topology, namely that of empty AdS. It comprises of the empty AdS solution plus the "boundary gravitons" geometries. Of course the total phase space of the gravitational theory has many more states coming from different topologies in the sum (270). The

[37] a Dirac bracket is the appropriate generalization of the Poisson bracket when we treat classical systems with second class constraints in Hamiltonian mechanics.

topologies different than that of empty AdS describe in general geometries where the spatial cycle is non-contractible and hence, as we mentioned previously, correspond to BTZ-like black hole states[38]. In fact, the phase space for these topologies can be worked out in a similar fashion as that for empty AdS. The only difference is the representations of the killing vectors ζ and charges $H[\zeta]$ that act on the boundary (where the ϕ and t circles are essentially exchanged, at least in the BTZ case). The Virasoro algebra however is still the symmetry algebra of our problem. Therefore, we can summarize saying that the total classical phase space of pure AdS_3 gravity consists of the state of empty AdS, the states of inequivalent black holes, and the states where boundary gravitons are "dressed" on top of the aforementioned states.

Returning to the seed contribution of AdS_3 in (276), one might now hope to quantize its phase space and thus compute $Z_{vac.}$ explicitly. When the central charge is $c > 1$ this can indeed be accomplished with the method of *co-adjoint orbits* [48]. When the central charge is $c < 1$, though, the quantization of the phase space turns out to be a difficult task in general. We will describe an alternative way here, following the arguments of [49].

Having in mind that all we need to compute is the contribution from the ground state of empty AdS, we can make progress by relying on the symmetries of this topology! For every value of c, we can in principle quantize canonically. Let us promote the Dirac bracket in (278) to commutators, and the generators $H[\zeta_n]$ to operators $\hat{L_n} \equiv \hat{H[\zeta_n]}$. We denote also by $|0\rangle$ the empty AdS vacuum state. This state has zero ADM energy and angular momentum, hence will be annihilated by the operators L_0 and $\overline{L}_0$. Moreover, as it is the state with lowest energy it will be annihilated by all of the lowering Virasoro operators L_n, $\overline{L}_n$ with $n > 0$ which decrease L_0 and $\overline{L}_0$. Thus it is a weight-zero primary

$$L_n |0\rangle = 0 \ , \qquad n > 0 \tag{279}$$

Another physical requirement comes from the fact that empty AdS is by definition invariant under global $SO(2,2) = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ transformations. Therefore, we should impose

$$L_{-1} |0\rangle = \overline{L}_{-1} |0\rangle = 0 \tag{280}$$

which indicates that there is one null state in the Verma module of our weight-zero primary. These are all the symmetry requirements that we can impose for the vacuum state. Descendants of this state are obtained by acting with raising operators $L_{-n}, \overline{L}_{-n}$ with $n > 1$ and correspond to boundary gravitons *quantum* states.

$$L_{-n_1} \cdots L_{-n_k} |0\rangle = 0 \ , \qquad n_i > 1 \tag{281}$$

We have managed to describe a Hilbert space for our theory[39]. The partition function (277) is then a trace over the Verma module of $|0\rangle$, and takes the familiar form:

$$Z_{vac.}(\tau, \overline{\tau}) = \mathrm{Tr}_{\mathcal{H}_{vac}} q^{L_0} \overline{q}^{\overline{L}_0} \ , \qquad q = e^{2\pi i \tau} \tag{282}$$

where specifically $\mathcal{H}_{vac}$ should be the Hilbert space of *irreducible* holomorphic and anti-holomorphic Verma modules of $|0\rangle$ (c.f. (144)). As we discussed back in Sect.1.2.3, the irreducible modules are the full Verma modules modded out by the null states. And this

[38] in [45], these states are nicely called $SL(2,\mathbb{Z})$ family of black holes.

[39] one should note that the above analysis is efficient only to the ground state topology. The dynamics of black holes states are in general not known to be fixed by symmetry. We don't know for example where the first black hole states should enter the spectrum.

is exactly what we want in a consistent gravity theory, that is, to count only physical states with positive norm. The result however depends on whether c is greater than or less than one.

When $c > 1$, the norm of any boundary graviton (281) is positive, with the exception of the null state $L_{-1}|0\rangle$. This computation of the norm follows directly from the Virasoro algebra, after we promoted the Dirac bracket to a commutator. The resulting trace is equal to the *character* of the full Verma module of $|0\rangle$ modded out by the $L_{-1}|0\rangle$ null state, which is:

$$Z^{vac.}_{c>1}(\tau,\overline{\tau}) = \left| q^{(1-c)/24}\frac{(1-q)}{\eta(\tau)} \right|^2 = \left| q^{-c/24}(1+q^2+q^3+2q^4+2q^5+4q^6+\cdots) \right|^2 \quad (283)$$

$\eta(\tau)$ is the Dedekind eta function and the factor of $(1-q)$ accounts for the removal of the L_{-1} from the spectrum.

When $c < 1$ the computation is identical to the construction of the minimal models using the Kac determinant. The partition function is simply equal to the irreducible vacuum character of a minimal model CFT as we wrote it back in (149) with the additional factor of $(1-q)$ for the exclusion of the L_{-1} state. Thus we read

$$\begin{aligned} Z^{vac.}_{c<1}(\tau,\overline{\tau}) &= \left|\chi_{1,1}(\tau)\right|^2 \\ &= \left| q^{(1-c)/24}\frac{(1-q)}{\eta(\tau)}\left(1+\sum_{k=1}^{\infty}(-1)^k\{q^{h_{1+k(p-1),(-1)^k s+[1-(-1)^k]p/2}}+q^{h_{1,kp+(-1)^k s+[1-(-1)^k]p/2}}\}\right)\right|^2 \end{aligned} \quad (284)$$

where we follow all the conventions from Section 1.2.6.

We have managed to get a complete picture about the path integral over geometries for pure Einstein-Hilbert gravity. Let us summarize the results:

The partition function of AdS_3 quantum gravity with torus boundary conditions has the general form:

$$Z_{AdS}\ [\mathbb{T}^2] = \sum_{\gamma\in\Gamma_c/PSL(2,\mathbb{Z})} Z_{vac.}(\gamma\tau,\gamma\overline{\tau})$$

The subgroup Γ_c of pure gauge transformations, as well as the seed partition function $Z_{vac.}$ depend on the values of the Brown-Henneaux central charge. For $c > 1$, where the theory admits a semi-classical approximation, the subgroup Γ_c can be determined concretely [45] and the seed partition function reads

$$Z^{vac.}_{c>1}(\tau,\overline{\tau}) = \left| q^{(1-c)/24}\frac{(1-q)}{\eta(\tau)} \right|^2$$

For $c < 1$, we have no perturbative control over the bulk theory in order to determine pure gauge transformations. The form of the seed partition function

$$Z^{vac.}_{c<1}(\tau,\overline{\tau}) = \left|\chi_{1,1}(\tau)\right|^2$$

however dictates the structure of Γ_c. We should take Γ_c to be the subgroup of $PSL(2,\mathbb{Z})$

that leaves the minimal model vacuum character $\left|\chi_{1,1}(\tau)\right|^2$ invariant. In the case where the minimal theory has a unique modular invariant partition function (namely the diagonal), one can explicitly compute the inequivalent contributions to the sum using the transformation of the vacuum character under the modular matrices $\mathcal{S}$ and $\mathcal{T}$ (c.f. Section 1.3.3). In the case, though, where we have more than one modular invariants one should proceed more systematically[46].

All in all, we are in a situation where we can explicitly make computations for the gravitational partition function. In the light of AdS/CFT correspondence, the question we should ask then is: does the gravity result reproduce the partition function of a suitable CFT theory? For the case of pure Einstein-Hilbert gravity the answer is given by the authors of [45],[46] and we summarize it here:

- For $c > 1$ the gravitational partition function *cannot* be written as a well-defined trace over a CFT Hilbert space.
- For $c < 1$ the gravitational partition function *agrees* with a minimal model CFT *only* for the central charge values: $c = 1/2$ and $c = 7/10$.

The (mostly) negative results of these authors for the search of a dual theory of pure Einstein gravity might be suggestive that such a theory doesn't exist quantum mechanically. One can however advocate the argument that some of the assumptions that led to the form (270) were too strict. A basic assumption, for example, was that one should include only smooth geometries in the sum over 3-manifolds. We are going to discuss the validity of this assumption when we examine pure gravity with a non-orientable boundary in the next section.

Finally, the fact that the gravitational partition function in the highly quantum regime ($c < 1$) agrees with *two* minimal theories, namely the $\mathcal{M}(4,3)$ (Ising) and the $\mathcal{M}(5,4)$ (Tricritical Ising) models, deserves some special attention. As we mentioned back in Sections 1.2.6 and 1.3.3, the Ising and Tricritical Ising model are the unique unitary minimal theories with two basic characteristics: (a) their operator spectrum (other than the identity) obeys the bound:

$$h > c/24$$

and, (b) their partition functions is given solely by the diagonal invariant

$$Z_{AA} = \sum_{r,s} |\chi_{r,s}|^2$$

. From the bulk point of view, the fact that their partition function is given by the diagonal invariant was really crucial in the calculations of [46], in order to render the corresponding CFT partition functions equal to the gravitational one. As far as the bound on the spectrum is concerned, in gravity the black hole states are naturally separated by the AdS ground state by a gap. As it was shown in [17],[41], these black holes should correspond to states in the CFT with weight larger than $c/24$ in order to reproduce the semiclassical Bekenstein-Hawking entropy correctly from the CFT. Thus, it seems that $\mathcal{M}(4,3)$ and $\mathcal{M}(5,4)$ theories are the only two diagonal minimal model CFTs with the property that all primary states can be interpreted as black holes in AdS_3 gravity. The rest of the minimal models that failed to reproduce the gravity theory, according to the analysis of [46], include primaries with dimensions (not much) below $c/24$. For example, the diagonal $\mathcal{M}(6,5)$ theory has central charge $c = 4/5$ and includes a single primary operator, namely $h_{2,2}$, with dimension $h = 1/40 < c/24$. Therefore a

possible interpretation is that such primaries correspond to matter fields which do not render the theory dual to pure gravity, but rather to something more complicated.[40]

3.3 Holography on non-orientable surfaces

In this section we will consider the gravitational partition function on *non-orientable* surfaces, focusing on the case of $\mathbb{RP}^2$ and the Klein bottle $\mathbb{K}^2$. Following the recent analysis of [51], we will examine only the semi-classical limit ($c >> 1$) in these cases, and see if our results reproduce the expected CFT results from Chapter 2. One assumption that is maintained from the previous section is that the AdS_3 partition function still admits a semi-classical expansion in saddle-point geometries. In particular,

$$Z_{AdS}\ [\mathbb{RP}^2, \mathbb{K}^2] = \int \mathcal{D}g\ e^{-S[g]} \approx \sum_{g_{cl}} \exp\big(- S[g_{cl.}] + \cdots \big) \tag{285}$$

The problem is to classify all these saddles with the appropriate boundary conditions and study their features and contributions to the partition function.

As we mentioned back in Section 2.2, any non-orientable surface Σ_g can be represented as a $\mathbb{Z}_2$ quotient of an orientable double cover $\hat{\Sigma}_g$. For example, the $\mathbb{RP}^2$ is a $\mathbb{Z}_2$ quotient of the sphere, and the Klein bottle $\mathbb{K}^2$ is the $\mathbb{Z}_2$ quotient of a torus. This observation applies in the bulk as well and allow us to relate the familiar bulk saddles with orientable boundary to bulk saddles with a non-orientable one. Actually, any 3-manifold with a non-orientable boundary must be non-orientable itself and thus, any bulk geometry with Σ_g boundary is the $\mathbb{Z}_2$ quotient of an orientable manifold whose boundary is the orientable double cover $\hat{\Sigma}_g$. The bulk saddles obtained in this way, however, are not guaranteed to be smooth in general.

Let's expand on the previous statement. Imagine we have a manifold M that has discrete symmetry group G. We can consider a new manifold $\tilde{M} = M/G$, which is obtained from the old one by modding out by the symmetry group G. If M has no fixed points under the action of G then M/G is a smooth manifold. On the other hand, if G has fixed points, then M/G is no longer a smooth manifold but has conical singularities at the fixed points. These are known as *orbifold singularities* and are the types of non-smooth saddles we will come across in what follows. A simple example is the case of the real line $\mathbb{R}$. It has a $\mathbb{Z}_2$ symmetry $x \to -x$ which has one fixed point, namely $x = 0$. Thus, the orbifold $\mathbb{R}/\mathbb{Z}_2$ is the manifold of half-line with an orbifold singularity at $x = 0$.

For our purposes, we want to find a locally Euclidean AdS_3 spacetime with isometry group $SL(2, \mathbb{C})$ or, in other words, a locally hyperbolic 3-manifold, whose conformal boundary is a non-orientable surface Σ_g. Let's begin with the case of $\Sigma_g = \mathbb{RP}^2 = S^2/\mathbb{Z}_2$. The bulk saddles with such boundary should be of the form $\mathbb{H}^3/\mathbb{Z}_2$, where $\mathbb{H}^3$ the hyperbolic 3-space. An easy way to visualize these geometries is to imagine starting from the S^2 boundary of $\mathbb{H}^3$ and extending the involution $z \to -1/\overline{z}$ into the interior. We would like to see however if there are any orbifold singularities in such a saddle. In order to specify a point in the bulk of such saddle uniquely, we must choose an element $g \in SL(2, \mathbb{C})$ with $g^2 = 1$. This means, in turn, that every such g has a single fixed point in the bulk; and actually different choices of g correspond to different choices of fixed point. We have come to the conclusion that there are *no* smooth saddles when

[40] see however [50].

the boundary is $\mathbb{RP}^2$. If we stick to the assumption that only smooth saddles should be included in (285), exactly as we did in the orientable case, then we realize that the gravity partition function should vanish. Such a result seems to be in odds with the CFT partition function on $\mathbb{RP}^2$ as we examine it in Chapter 2. In particular, a vanishing partition function would mean that

$$Z_{\mathbb{RP}^2} = \langle 0|C\rangle = \Gamma_{\mathbb{1}} \to 0$$

Let's examine the case of the Klein bottle and return to the above statement again later. As we mentioned, the Klein bottle geometry is obtained as the $\mathbb{Z}_2$ quotient of the rectangular torus: $\mathbb{K}^2 = \mathbb{T}^2/\mathbb{Z}_2$. Restating the identification we wrote on Section 2.3.:

$$(t, \phi) \sim (t + \beta, \phi + 2\pi) \sim (-t, \phi + \pi) \tag{286}$$

The way to identify the bulk saddles with this boundary is to look for saddle points for the torus which are invariant under the $\mathbb{Z}_2$ action. In the previous section, we saw that the bulk saddles with $\mathbb{T}^2$ boundary conditions are solid tori characterized by which non-contractible cycle in the torus becomes contractible in the bulk. Since the $\mathbb{Z}_2$ is a symmetry of the boundary torus, the corresponding bulk saddle will either be invariant under it or will be exchanged with some other saddle. Under the above orientation-reversing, a t-cycle is odd whereas a ϕ-cycle is even. More formally, since the first homotopy group of the torus is $H^1(\mathbb{T}^2) = \mathbb{Z} \oplus \mathbb{Z}$, a natural basis consisting of an α cycle along t and a b cycle along ϕ gives us a general contractible cycle of the form $m\alpha + nb$, where $m, n \in \mathbb{Z}$. Therefore, under orientation-reversing the $m\alpha + nb$ cycle is exchanged with $-m\alpha + nb$. We conclude that there are only two bulk saddles which are invariant under $\mathbb{Z}_2$: the one where the α cycle is contractible, i.e. the non-rotating BTZ black hole, and the one where the b cycle is contractible, i.e. thermal AdS. We distinguish these two particular saddles, and construct the Klein bottle geometries as quotients of these.

It is convenient to start with Poincaré coordinates $(y, z, \overline{z})$ on usual AdS_3

$$ds^2 = \frac{dy^2 + dzd\overline{z}}{y^2}$$

with $y \in (0, \infty)$ and z a complex coordinate on the planes of constant y. The orbifold by $\gamma : (y, z) \to (e^\beta y, e^\beta z)$ turns the geometry in to a solid torus. Furthermore, if we write $z = e^{t+i\phi}$, where t is the Euclidean time on the boundary, this is a torus with $t \sim t + \beta$, $\phi \sim \phi + 2\pi$, and the solid torus is thermal AdS_3. If write instead $z = {}^{\frac{\beta}{2\pi}(\phi+it)}$, the boundary is a torus with $t \sim t + 4\pi^2/\beta$, $\phi \sim \phi + 2\pi$, and the bulk solution is the non-rotating BTZ black hole. To the obtain the saddles for the Klein bottle we just have to take the $\mathbb{Z}_2$ quotient $t \to -t$, $\phi \to \phi + \pi$.

In the case of thermal AdS, where $z = e^{t+i\phi}$, the $\mathbb{Z}_2$ action corresponds to the involution $z \to -1/\overline{z}$, which can also be written as a discrete isometry of $\mathbb{H}^3$ as:

$$\sigma : \qquad z \to -\frac{z}{|z|^2 + y^2}, \quad y \to \frac{y}{|z|^2 + y^2}$$

where $\sigma^2 = \mathbb{1}$. We notice that there two fixed points in this case, namely at $z = 0$ and $y = 1, e^{\beta/2}$. The space we obtain after the orbifold with γ and σ is generally $\mathbb{H}^3/\Gamma$,

where Γ is the subgroup generated by the γ and σ orbifold actions. The boundary of that space is a Klein bottle with fundamental region shown in Fig.3.3(a) below. It is equivalent to the propagation between two crosscaps (corresponding to the two fixed points) as we discussed back in Chapter 2.

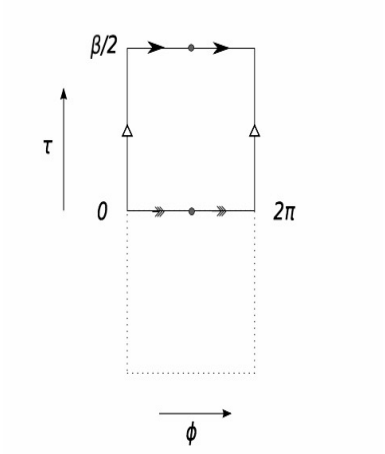

Figure3.3(a): The fundamental domain of the $\mathbb{K}^2$ boundary of the space $\mathbb{H}^3/\Gamma$, with $\Gamma = \{\gamma^n, \sigma\gamma^n | n \in \mathbb{Z}\}$.The propagation between two crosscaps indicate the existence of fixed points for the quotient space.

where the modular parameter reads $l = \beta/4\pi$. If we further want to write the CFT partition function on this boundary, using expression (201) from Chapter 2, we have

$$Z_{\mathbb{K}^2}(\beta) = \langle \Theta C | \, e^{-2\pi l(L_0 + \overline{L_0} - \frac{c}{12})} \, |C\rangle = \sum_i \Gamma_i^2 \big\langle\big\langle C_i \big| e^{-\beta H/2} \big| C_i \big\rangle\big\rangle \tag{287}$$

where $H = L_0 + \overline{L}_0 - c/12$, and $\big|C_i\big\rangle\big\rangle$ the crosscap Ishibashi states.

In the case of the non-rotating BTZ, where $z = e^{\frac{\beta}{2\pi}(\phi + it)}$, the $\mathbb{Z}_2$ action corresponds to identifying $z \to e^{\beta/2}\overline{z}$, which in turn can be written as a discrete transformation

$$\kappa : \qquad z \to e^{\beta/2}\overline{z}, \quad y \to e^{\beta/2} y$$

with no fixed points and $\kappa^2 = \gamma$. The fact that this subgroup is freely acting means that in this case we obtain a smooth manifold $\mathbb{H}^3/\Gamma$, where Γ is the subgroup generated by κ alone. This geometry is actually the Euclidean version of the so-called $\mathbb{RP}^2 geon$[52]. Its boundary is a Klein bottle with modular parameter $l = \pi/\beta$ and fundamental region given below:

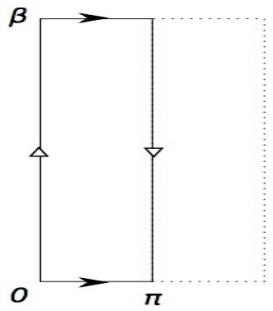

Figure3.3(b):The fundamental domain of the $\mathbb{K}^2$ boundary of the space $\mathbb{H}^3/\Gamma$, with $\Gamma = \{\kappa^n | n \in \mathbb{Z}\}$. This geometry is smooth and corresponds to the Euclidean *geon*.

This is basically the second representation of $\mathbb{K}^2$ as we obtained it back in Section 2.3 using S-duality. It's interesting that we confirm again this construction (this time from the S-duality that basically relates thermal AdS and the non-rotating BTZ) which was otherwise expected from the symmetries of AdS. The partition function in this case is given by

$$Z_{\mathbb{K}^2}(\beta) = \mathrm{Tr}\left(\Omega e^{-2\pi^2 H/\beta}\right) \tag{288}$$

where Ω is the parity operator.

Summarizing so far: in the case of $\mathbb{RP}^2$ boundary we have obtained *no* smooth bulk solution of Einstein gravity, whereas for a $\mathbb{K}^2$ boundary we have one *smooth* solution, namely the Euclidean $\mathbb{RP}^2$ geon, and one *non-smooth* solution from the quotient of thermal AdS_3. We would like now to go back and calculate the gravitational path integral (285) in the large central charge limit, by calculating the contributions of these geometries. We are facing however the question of what kind of saddles -smooth or non-smooth- should be included in the sum over $g_{cl.}$. In the orientable case we didn't have to struggle about this fact because we constructed only smooth saddles and their inclusion in the path integral seemed a natural assumption. Let's use AdS/CFT to get our answers in the present case.

If we assume that only smooth saddles are to be included, we get at first that $Z_{AdS}[\mathbb{RP}^2] \to 0$ which in turns means $Z_{\mathbb{RP}^2} = 0$, as we've discussed already. For the Klein bottle, we do have one smooth saddle so let's investigate the result better. The classical action of the Euclidean $\mathbb{RP}^2$ geon is easy to compute: since the solution is the $\mathbb{Z}_2$ quotient of the BTZ black hole, the action is just half the action of the non-rotating BTZ black hole[41]. This gives a contribution:

$$Z_{AdS}[\mathbb{K}^2] \approx \exp\Big\{\frac{\pi^2 c}{6\beta}\Big\} \tag{289}$$

where $c = 3l_{AdS}/2G_N$ Let us now see if it reproduces the CFT partition function. From (288), at high temperatures (small β) we can approximate the expression by the vacuum state contribution with energy $-c/12$ on the circle. Hence,

$$Z_{\mathbb{K}^2}(\beta \to 0) \approx \exp\Big\{\frac{\pi^2 c}{6\beta}\Big\} \tag{290}$$

where we have assumed that the vacuum is parity symmetric. Therefore, there is an *agreement* between the CFT and gravity in this case. The small temperature(large β) behaviour is a little more subtle though. From the gravity side we see that the action of the geon is zero for $\beta \to \infty$ rendering the partition function finite. However, from the CFT point of view we can use the representation (287) for the $\mathbb{K}^2$ partition function which gives

$$Z_{\mathbb{K}^2}(\beta \to \infty) = \sum_i \Gamma_i^2 \big\langle\big\langle C_i \big| e^{-\beta H/2} \big| C_i \big\rangle\big\rangle \approx Z^2_{\mathbb{RP}^2} \exp\Big\{\frac{\beta c}{24}\Big\} \tag{291}$$

where $\Gamma_{\mathbb{1}} = Z_{\mathbb{RP}^2}$. The only way the gravity and CFT behaviours agree in this case is if $Z_{\mathbb{RP}^2} \to 0$ in order to compensate for the exponential growth. This indeed seems to be a correct statement as we argued independently from the gravitational partition function

[41] for the calculation of the action of the BTZ geometry see for example Note 16 in [53].

on $\mathbb{RP}^2$. So if we take into account also our previous result we again reach an agreement. There is one more non-trivial requirement though, implied by the exact expression of $Z_{\mathbb{K}^2}$. For primary operators ϕ_i with negative energies, i.e. $E_i = h_i + \overline{h}_i - c/12 < 0$, the one-point normalizations Γ_i should *also* vanish in order to render the result finite at large β. This is a non-trivial constraint from the gravity side, especially considering the fact that Γ_i's should also obey the sewing constraints we described back in Section 2.4. Moreover, the vanishing of these Γ_i's should hold even for finite values of the central charge, and not just at some semi-classical limit in $1/c$, or the exponential growth will eventually take over at sufficiently large β. In other words, the general conclusion that we can reach at this point is that: a bulk calculation involving *only smooth* saddles can reprduce the CFT behaviour only if the one-point functions Γ_i vanish for all operators with $h_i + \overline{h}_i < c/12$.

Let's take into consideration our results in Section 2.5 for the one-point normalizations of the Ising and the Tricritical Ising model. As far as the vanishing of $Z_{\mathbb{RP}^2}$ is concerned, it is immediately clear in both cases that, if we set $\Gamma_{\mathbb{1}} = Z_{\mathbb{RP}^2} = 0$, then all the Γ_i vanish. In this case, the Klein bottle partition function should also vanish identically. However, from the gravity side we have already identified a smooth saddle point which contributes to the Klein bottle partition function. We see, therefore, that the assumption of the inclusion of smooth saddles leads to some qualitative contradiction with CFT. Unfortunately, the results we obtained in (210) and (211) are not informative about the vanishing of Γ_i for some non-trivial operator with $h_i + \overline{h}_i < c/12$. As we discussed, the spectrum of these two diagonal theories satisfies the bound $h > c/24$.

The authors of [51] proceeded with considering the inclusion of singular saddles in the above analysis. They examined both the singular semi-classical saddles (as we will also examine here) as well as the one loop determinant term. And they indeed found agreement with the CFT partition function. This fact constitutes a significant modification of the usual rules for calculating the integral over geometries in pure 3d gravity, especially taking into account the (mostly) negative results we discussed in the previous section for the search of a CFT dual. As it's emphasized in [51], one suggestive approach as to what kind of singular geometries should be included in the path integral might be the inclusion of instanton contributions which occur at points in the Euclidean time circle. These saddles do not correspond to some additional "particle-like" degrees of freedom in the bulk and hence, we are still considering a pure theory of gravity.

Before we end this chapter, let us discuss the case of the inclusion of singular saddles at tree level, just so we can enjoy to some extent the agreement of the formulas. In the case of $\mathbb{RP}^2$, the quotient spaces $\mathbb{H}^3/\mathbb{Z}_2$ will have fixed points in the bulk and they will contribute to $Z_{\mathbb{RP}^2}$ rendering it non-zero. For the Klein bottle we have, in addition to the smooth saddle of the geon geometry, we now consider the singular saddle from the Z_2 quotient of thermal AdS. Its action will therefore be half the action of the thermal AdS, plus potentially a contribution from singularities. At small β, the geon geometry dominates and our results agree as in (289),(290). For large β it is the singular saddle that dominates and we get

$$Z_{AdS}[\mathbb{K}^2] \approx \exp\Big\{ -\big(S_{AdS}/2 + S_{sing.}\big)\Big\} = \exp\Big\{\beta\frac{c}{24} + S_{sing.}\Big\} \tag{292}$$

The important thing is that $S_{sing.}$ is *independent* of β since the singularities are localized at points in the t direction. Hence, these contributions add an overall constant to the action and allow us to remarkably match the result with the CFT partition function (291).

4 Discussion

In this thesis we have mostly considered CFTs on non-orientable surfaces and obtain some exact information about the conformal data of two Minimal Model theories, namely the Ising and the Tricritical Ising model, on the surface of Real Projective plane $\mathbb{RP}^2$. This piece of data is specifically the one-point function normalizations constants Γ_i's on $\mathbb{RP}^2$, which together with the OPE coefficients and the operator spectrum, uniquely fix the partition function of a CFT on a general non-orientable surface[29].

According to the AdS/CFT correspondence[32], a quantum gravity theory in a d-dimensional spacetime with asymptotically AdS boundary conditions and specific boundary topology is equivalent to a $(d-1)$-dimensional CFT defined on the boundary of that spacetime. The authors of [46] considered AdS_3 spacetime with torus boundary conditions and examined to which extent the Ising and the Tricritical Ising model on $\mathbb{T}^2$ are equivalent to pure gravity in the bulk by explicitly computing the corresponding partition functions. They found perfect agreement in both sides and thus they were naturally led to conjecture a duality between Ising and Tricritical Ising in the one side, and pure gravity theory at specific order one values of the Brown-Henneaux central charge on the other side. As they explain though, these dualities should remain conjectural since they only provide evidence about the torus partition functions. In principle, one should be able to check the duality in higher genus partition functions, non-orientable surfaces of genus g or even in surfaces with boundaries as well. If agreement persists in all of these cases then the duality should be considered exact. In this spirit, the values of the Γ_i's we calculated in this thesis for the Ising and the Tricritical Ising model, certainly allow us to calculate the Klein bottle partition function(c.f. (201)) for these models. Hence calculating the non-perturbative contributions of the corresponding pure gravity path integral and see if it matches with CFT is a natural next project, although we are not aware of such calculation in the literature.

Furthermore, one can certainly solve the sewing constraints (209) for the rest of the diagonal minimal theories and hence determine their structure on non-orientable surfaces. In connection with the pure gravity program, it would be interesting to find a model with $\Gamma_i = 0$ for the operators with $h + \overline{h} < c/12$, and some $\Gamma_i \neq 0$ for higher dimension operators. This fact would mean that light operators do not contribute in the partition function (e.g. on $\mathbb{K}^2$) and hence the inclusion of only smooth saddles in the orientable-boundary case is probably justified for that reason, but this might not be true in general as an assumption. The Ising and the Tricritical Ising are not good playgrounds for these kind of conclusions because as we've explained they constitute *extremal* CFTs. A good first example to check might be the diagonal $\mathcal{M}(6,5)$ model with central charge $c = 4/5$ which includes an operator $\phi_{2,2}$ of dimension $h_{2,2} = 1/40 < c/24$.

Finally, let us comment on a recent interesting application of CFTs on $\mathbb{RP}^d$ to constructing bulk AdS operators in the context of holography. As we mentioned in 3.1.2, the asymptotic boundary values of any bulk fields in AdS correspond to primary operators of specific conformal dimension (dictated by the mass of the bulk field) in the boundary CFT. However, there is no confirmed correspondence of a purely bulk operator in AdS and a corresponding intrinsic operator in CFT. In the semi-classical limit where the bulk is described by an effective *local* QFT on a fixed background, one can in principle integrate the equations of motion in the bulk and obtain a non-local expression of a bulk operator in terms of integrals of the boundary primary operators. This is the so-called HKLL procedure, after [54]. However, we would like to have an *inherent* non-perturbative operator in CFT that corresponds to such a bulk operator and also one that does not rely on the spacetime geometry as in HKLL. H.Verlinde in

[55] first proposed such an inherent definition from CFT. Working in the framework of AdS_3/CFT_2 he proposed the identification of a bulk operator in AdS with a Virasoro crosscap state in the dual CFT, as we defined it in (191). Recently, the authors of [56] provided non-trivial evidence that this proposition passes a series of non trivial tests. In particular, they rely on the fact that an 'honest' bulk operator should be inherently gravitationally dressed and this fact imposes some strong physical constraints on the form its correlations functions. According to [56], the identification with the crosscap state does seem to obey all these constraints in the case of AdS_3/CFT_2. Another remarkable fact these authors show is that, for holographic CFTs, approximate bulk locality emerges as a dynamical consequence of the crosscap sewing constraint (209). They also point out some interesting connections of this approach with the investigation of the black hole interiors and the regularity of correlation functions at the black hole horizon.

All in all, it seems that the study of CFTs in non-orientable surfaces is a far reaching subject, especially in the context of AdS/CFT. It would be extraordinarily interesting to study all the aforementioned consequences in the case of the minimal theories where we can explicitly compute the crosscap state by determining a finite number of one-point normalizations on $\mathbb{RP}^2$.

Appendix A. Coulomb-gas formalism for Minimal Models

This appendix describes a representation of the conformal Minimal Models in terms of vertex operators built from a free boson with special boundary conditions. We will sketch the analysis as presented in [1] in order to reach to the expressions for the OPE coefficients and the duality matrices.

This representation of the Minimal Models bears the name of *Coulomb gas* from the resemblance of the free boson correlator:

$$\langle \phi(z,\overline{z})\phi(w,\overline{w}))\rangle = -4ln|z-w|$$

with the electric potential energy between two unit charges in two dimensions. The above propagator is obtained by the usual 2d bosonic action:

$$S = \frac{1}{16\pi}\int d^2x \partial_\mu\phi\partial^\mu\phi$$

which corresponds to a conformal field theory with central charge $c = 1$. The energy momentum tensor reads:

$$T(z) = -\frac{1}{4}(\partial\phi)^2(z)$$

The basic idea is to consider a modified stress tensor

$$T(z) = -\frac{1}{4}(\partial\phi)^2(z) + Q\partial^2\phi(z)$$

obtained from the action

$$S = \frac{1}{16\pi}\int d^2x\sqrt{g}[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + 2Q\phi R]$$

The Ricci term R in the action contributes to, among other things, global symmetries and Ward identities as well as the central charge:

$$c = 1 + 24Q^2$$

Conformal primaries of interest are the so-called *vertex operators* of the form

$$V_a(z,\overline{z}) \sim e^{ia\phi(z,\overline{z})}$$

suitable regularized. Since the propagator for ϕ is that of a free boson, arbitrary correlation functions of vertex operators have the form

$$\left\langle \prod_{i=1}^{n} V_{a_i}(z_i) \right\rangle_Q \sim \prod_{i<j} z_{ij}^{2a_ia_j}$$

The subscript Q denotes that these correlation functions are actually only non-zero if a *neutrality condition* is satisfied. This condition arises as following: the Ricci scalar term in the action modifies the nature of the global symmetry $\phi \to \phi + A$, effectively placing a background charge of $-2iQ$ at infinity. More precisely, despite the fact that the Ricci scalar coupling 'breaks' the shift symmetry, a modified Ward identity survives

that forces non-zero correlation functions to have total charge $-2iQ$. Taking $Q \equiv ia_0$ and noting that the vertex operators $V_a(z)$ have charge a under the global symmetry, the neutrality condition is:

$$\sum_i a_i = 2a_0$$

In the case of a two point function, this prescription yields

$$\langle V_a(z) V_{2a_0-a}(0)\rangle \sim z^{2a(2a_0-a)}$$

which implies that one should take $V_a^\dagger = V_{2a_0-a}$ and

$$h_a = a(a-2a_0)$$

In order to generalize the set of correlators that can be non-vanishing consistently with the neutrality condition above, one adds in non-local 'screening charges', which are conformally invariant operators that soak up extra charge:

$$Q_a = \oint_C dz V_a(z)$$

For this to be conformally invariant, the vertex operator must have weight 1 to offset the measure, which requires:

$$a(a-2a_0) = 1 \qquad \rightarrow \qquad a_\pm = a_0 \pm \sqrt{a_0^2+1}$$

One can show that inserting such an operator does not affect the conformal Ward identities. Therefore, this is a constructive method for generating correlation functions that are consistent with crossing symmetry and conformal symmetry, which for minimal models uniquely determines the correlation functions.

The solutions $a_\pm$ satisfy

$$a_+ + a_- = 2a_0 \ , \qquad\qquad a_+ a_- = -1$$

and for later use we define the parameters

$$\rho \equiv a_+^2 \ , \qquad\qquad \rho' = a_-^2 = \frac{1}{\rho}$$

To evaluate, say, the four point function $\langle V_a V_a V_a V_{2a_0-a}\rangle$, one must be able to add in factors of $Q_\pm$ to bring the total charge to $2a_0$. If $2a$ is a linear combination of $a_\pm$, i.e. if

$$2a = (1-r)a_+ + (1-s)a_-$$

then one can consider

$$\left\langle V_a V_a V_a V_{2a_0-a} Q_+^{r-1} Q_-^{s-1}\right\rangle$$

By construction, the operators in this correlation function satisfy the neutrality condition. It is conventional to parametrize these particular charge values of a by:

$$a_{r,s} \equiv \frac{1-r}{2}a_+ + \frac{1-s}{2}a_- = a_0 - \frac{1}{2}(ra_+ + sa_-)$$

corresponding to dimensions of

$$h_{r,s} = a_{r,s}(a_{r,s}-2a_0) = -a_{r,s}a_{-r,-s} = \frac{(ra_+ + sa_-)^2}{4} - a_0^2$$

Similar considerations apply to correlation functions with more than one operator, i.e. $\left\langle V_{a_1} V_{a_2} V_{a_3} V_{2a_0-a_4} Q_+^r Q_-^s \right\rangle$.

The crucial thing with all this formalism is that we can have *integral representations* of the correlators in minimal models. For instance, the correlator $F(z_i) \equiv \langle V_{1,2}(z_1) V_{1,2}(z_2) V_{r,s}(z_3) V_{-r,-s}(z_4) Q_- \rangle$ can be represented as

$$\begin{aligned} F(z_i) &= \oint_C du \, \langle V_{1,2}(z_1) V_{1,2}(z_2) V_{r,s}(z_3) V_{-r,-s}(z_4) V_-(u) \rangle \\ &= z_{12}^{2a_{1,2}^2} (z_{13} z_{23})^{2a_{1,2} a_{r,s}} (z_{14} z_{24})^{2a_{1,2} a_{-r,-s}} z_{34}^{-2\Delta_{r,s}} \times \\ \times \oint du [(z_1)(z_2 - u)]^{2a_{1,2} a_-} &(z_3 - u)^{2a_{r,s} a_-} (z_4 - u)^{2a_{-r,-s} a_-} \end{aligned}$$

Using global conformal invariance to send $z_1 \to \infty$, $z_2 \to 1$, $z_3 \to z$ and $z_4 \to 0$, this reduces to

$$F(z) = (1-z)^{2a_{1,2} a_{r,s}} z^{2a_{1,2}^2} \oint du \; u^{2a_{1,2} a_-} (u-z)^{2a_{1,2} a_-} (u-1)^{2a_{r,s} a_-}$$

up to some phase factors that we will fix independently. This integral depends on the choice of contour. This contour should be single valued, that is the integrand should be single valued upon going around the entire contour, while also enclosing at least one singular point so that it is non-vanishing. A way to do it is to use the so-called *Pochhammer contour*, which encloses two of the singularities twice, once clockwise and once counter clockwise. Since any monodromy obtained by going around a singularity is eventually cancelled by going around in the opposite direction, the integrand is single valued. Furthermore, by collapsing the contour to the line connecting the singularities, the integral reduces to a single integral between the two singular points, though there is a phase that one has keep track of. In any case, there are two independent such contours, which correspond to the two different conformal blocks that are allowed in the OPE of $V_{1,2} \times V_{1,2}$. In the present case, they have simple representations as hypergeometric functions, via the identities

$$\begin{aligned} \int_1^\infty du \; u^a (u-1)^b (u-z)^c &= I_1(a,b,c;z) \\ &= \frac{\Gamma(-a-b-c-1)\Gamma(b+1)}{\Gamma(-a-c)} {}_2F_1(-c, -a-b-c-1; -a-c; z) \; , \\ \int_z^\infty du \; u^a (1-u)^b (z-u)^c &= I_2(a,b,c;z) = z^{1+a+c} \int_0^\infty u^a (1-u)^c (1-zu)^b \\ &= z^{1+a+c} \frac{\Gamma(a+1)\Gamma(c+1)}{\Gamma(a+c+2)} {}_2F_1(a+1, -b; a+c+2; z) \end{aligned}$$

The generalization to higher level degenerate operators is straightforward [1].

So far we have concentrated on the holomorphic correlation functions, but for a physical theory we must construct add in the anti-holomorphic sector. Restricting to the case of scalar primaries, we can construct the physical correlation function as

$$G(z, \overline{z}) = \sum_{k,l=1}^{N} C_{kl} \mathcal{F}_k(z) \overline{\mathcal{F}_l(z)}$$

To specify the matrix C_{kl} we require that the physical correlation function be single valued and hence *monodromy free*. In particular we check the monodromy around $z = 0$ and $z = 1$. The $z = 0$ case is simple and forces C_{kl} to be diagonal $C_{kl} = C_k\delta_{kl}$:

$$G(z,\overline{z}) = \sum_{k,l=1}^{N} C_k|\mathcal{F}_k(z)|^2$$

The $z = 1$ monodromy is much more involved. The approach worked out in [15],[16] is to use the integral expressions to rewrite the $\mathcal{F}_k(z)$ in terms of N new analytic functions $\tilde{\mathcal{F}}_k(z)$ with diagonal monodromy around $z = 1$; physically this procedure is expressing the conformal blocks in the s-channel in terms of the t-channel blocks:

$$\mathcal{F}_k(z) = M\begin{bmatrix} a_2 & a_3 \\ a_1 & a_4 \end{bmatrix}_{k,l} \tilde{\mathcal{F}}_l(z)$$

In terms of the t-channel blocks, the correlation function reads

$$G(z,\overline{z}) = \sum_{k,l,m} C_k M\begin{bmatrix} a_2 & a_3 \\ a_1 & a_4 \end{bmatrix}_{k,l} M\begin{bmatrix} a_2 & a_3 \\ a_1 & a_4 \end{bmatrix}^*_{k,m} \tilde{\mathcal{F}}_l(z)\overline{\tilde{\mathcal{F}}_m(z)} \equiv \sum_{l,m} \tilde{C}_{lm}\tilde{\mathcal{F}}_l(z)\overline{\tilde{\mathcal{F}}_m(z)}$$

Therefore diagonal monodromy around $z = 1$ requires $\tilde{C}_{lm} = 0$ for for $l \neq m$. With this constraint, one can solve for the coefficients C_k up to an overall coefficient

$$\frac{C_k}{C_M} = \frac{M^*_{NN}(M^{-1})_{Nk}}{M^*_{kN}(M^{-1})_{NN}}$$

Provided we normalize the blocks $\mathcal{F}_k(z)$ appropriately, the C_k are just the OPE coefficients! The squared OPE for the *diagonal* Minimal Models read[15],[16]:

$$\left[C^{r_1,s_1}_{(r_2,s_2),(r_3,s_3)}\right]^2 = \frac{a(r_2,s_2)a(r_3,s_3)}{a(r_1,s_1)}\left[D^{(r_1,s_1)}_{(r_2,s_2),(r_3,s_3)}\right]^2 \tag{293}$$

where $a(r,s)$ and $D^{(r_1,s_1)}_{(r_2,s_2),(r_3,s_3)}$ are defined as:

$$\begin{aligned} a(r,s) = &\left[\prod_{i,j=1}^{s-1,r-1}\frac{1+i-\rho(1+j)}{i-j\rho}\right]^2\left[\prod_{i=1}^{s-1}\frac{\Gamma(i\rho')\Gamma(2-\rho'(1+i))}{\Gamma(1-i\rho')\Gamma(\rho'(1+i)-1)}\right] \\ &\times\left[\prod_{j=1}^{r-1}\frac{\Gamma(j\rho)\Gamma(2-\rho(1+j))}{\Gamma(1-j\rho)\Gamma(\rho(1+j)-1)}\right], \\ D^{(r_1,s_1)}_{(r_2,s_2),(r_3,s_3)} = &\,\mu(l,l')\left[\prod_{i,j=0}^{l'-2,l-2}\tilde{\lambda}_{ij}(r_1,s_1)\lambda_{ij}(r_2,s_2)\lambda_{ij}(r_3,s_3)\right] \\ &\times\left[\prod_{j=0}^{l-2}\tilde{\tau}_j(r_1,s_1;\rho)\tau_j(r_2,s_2;\rho)\tau_j(r_3,s_3;\rho)\right] \\ &\times\left[\prod_{i=0}^{l'-2}\tilde{\tau}_i(s_1,r_1;\rho')\tau_i(s_2,r_2;\rho')\tau_i(s_3,r_3;\rho')\right]. \end{aligned}$$

Here $l = \frac{r_2+r_3-r_1+1}{2}$ and $l' = \frac{s_2+s_3-s_1+1}{2}$, while the auxiliary functions μ, λ and τ are defined as

$$\begin{aligned}
\mu(l,l') &= \rho^{4(l-1)(l'-1)} \prod_{i,j=1}^{l'-1,l-1} (i-\rho j)^{-2} \prod_{i=1}^{l'-1} \frac{\Gamma(i\rho')}{\Gamma(1-i\rho')} \prod_{j=1}^{l-1} \frac{\Gamma(j\rho)}{\Gamma(1-j\rho)}\,, \\
\lambda_{ij}(r,s) &= [(s-1-i)-\rho(r-1-j)]^{-2}\,, \\
\tilde{\lambda}_{ij}(r,s) &= [(s+1+i)-\rho(r+1+j)]^{-2}\,, \\
\tau_i(r,s;\rho) &= \frac{\Gamma(s-\rho(r-1-i))}{\Gamma(1-s+\rho(r-1-i))}\,, \\
\tilde{\tau}_i(r,s;\rho) &= \frac{\Gamma(\rho(r+1+i)-s)}{\Gamma(1+s-\rho(r+1+i))}\,.
\end{aligned}$$

We also recall that $\rho = 1/\rho' = a_+^2$, which takes the value $\rho = p/q$ for the minimal model $\mathcal{M}(p,q)$. The unitary series correspond to $p = m+1$, $q = m$.

Finally, we give the closed form expressions for the fusion matrices M:

$$M \begin{bmatrix} \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_4 \end{bmatrix}_{(p_s,p'_s),(q_t,q'_t)} = \frac{N^{(m,n)}_{k_2,k'_2}(b,a,c,d;\rho)}{N^{(m,n)}_{k_1,k'_1}(a,b,c,d;\rho)} \alpha^{(m)}_{k_1,k_2}(a,b,c,d;\rho)\alpha^{(n)}_{k'_1,k'_2}(a',b',c',d';\rho') \quad (294)$$

where the parameters are defined as

$$\begin{aligned}
&a = 2\alpha_+\alpha_1, \quad b = 2\alpha_+\alpha_3, \quad c = 2\alpha_+\alpha_2\,, \quad d = 2\alpha_+\bar{\alpha}_4\,, \quad \rho = \alpha_+^2\,, \\
&a' = 2\alpha_-\alpha_1, \quad b' = 2\alpha_-\alpha_3, \quad c' = 2\alpha_-\alpha_2\,, \quad d' = 2\alpha_-\bar{\alpha}_4\,, \quad \rho' = \alpha_-^2 = 1/\rho\,,
\end{aligned}$$

the indices as (k_i, k'_i, $i = 1,2$ are just convenient parameterizations for the exchanged operators)

$$\begin{aligned}
m &= \frac{r_1+r_2+r_3-r_4}{2}\,, \quad n = \frac{s_1+s_2+s_3-s_4}{2}\,, \\
k_1 &= \frac{r_1+r_2+1-p_s}{2}\,, \quad k'_1 = \frac{s_1+s_2+1-p'_s}{2}\,, \\
k_2 &= \frac{r_2+r_3+1-p_t}{2}\,, \quad k'_2 = \frac{s_2+s_3+1-p'_t}{2}\,,
\end{aligned}$$

the normalization functions as (note that $a' = -a/\rho$, $b' = -b/\rho$, etc.)

$$\begin{aligned}
N^{(m,n)}_{p,p'}(a,b,c,d;\rho) &= \mathcal{J}_{m-p,n-p'}(d,b;\rho)\mathcal{J}_{p-1,p'-1}(a,c;\rho)\,, \\
\mathcal{J}_{p,q}(a,b;\rho) &= \rho^{2pq} \prod_{i,j=1}^{p,q} \frac{1}{i\rho-j} \prod_{i=1}^{p} \frac{\Gamma(i\rho)}{\Gamma(\rho)} \prod_{j=1}^{q} \frac{\Gamma(j\rho')}{\Gamma(\rho')} \\
&\quad \times \prod_{i=0}^{p-1} \frac{\Gamma(1+a+i\rho)\Gamma(1+b+i\rho)}{\Gamma(2-2q+a+b+(p-1+i)\rho)} \\
&\quad \times \prod_{j=0}^{q-1} \frac{\Gamma(1+a'+j\rho')\Gamma(1+b'+j\rho')}{\Gamma(2-2p+a'+b'+(q-1+j)\rho')} \\
&\quad \times \prod_{i,j=0}^{p-1,q-1} \frac{1}{(a+i\rho-j)(b+i\rho-j)[a+b+\rho(p-1+i)-(q-1+j)]}\,,
\end{aligned}$$

and finally,

$$\alpha_{j,k}^{(m)}(a,b,c,d;\rho) = \sum_{p=\max(j,k)}^{\min(m,j+k-1)} \prod_{i=1}^{p-j} \frac{s[(j+k+i-p-1)\rho]}{s(i\rho)} \prod_{i=1}^{m-p} \frac{s[(p-k+i)\rho]}{s(i\rho)}$$
$$\frac{\prod_{i=0}^{m-p-1} s[1+a+(j-1+i)\rho] \prod_{i=0}^{p-k-1} s[1+d+(m-j+i)\rho]}{\prod_{i=0}^{m-k-1} s[a+d+(m-k-1+i)\rho]}$$
$$\frac{\prod_{i=0}^{j+k-p-2} s[1+b+(m-j+i)\rho] \prod_{i=0}^{p-j-1} s[1+c+(j-1+i)\rho]}{\prod_{i=0}^{k-2} s[b+c+(k-2+i)\rho]},$$

$$s(x) = \sin(\pi x)\,.$$

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