Some Analytic Properties of Automorphic L-functions

by

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A thesis submitted in partial fulfillment
of the requirements of the degree of
Doctor of Philosophy at McGill University
July 1994

Acknowledgment

I would like to thank my supervisor Ram Murty for his excellent lectures on the introduction of automorphic forms from number theory's view which have led me to both number theory and automorphic forms, for his invaluable encouragement, advice and help on my studying and working, and for his financial support.

I would like to thank professor John Mckay for his advice to pay attention to number theory so that I got a chance to touch this beautiful and fascinating field, and for his financial support.

I would like to thank professors Hershy Kisilevsky, Ian Connell, and Guenther Frei for their lectures and very kind help.

I would like to thank my sister and brother who have definitely played crucial roles in my pursuing mathematics.

Abstract

In this thesis, we study two topics concerning the analytic properties of automorphic L-functions. First, we study mean values of twisted L-functions of unitary automorphic cuspidal forms of GL_r over the rational number field and then, the Langlands' reciprocity law for finite Galois extensions of algebraic number fields. We prove an asymptotic formula, and a bound of the mean value of L-functions twisted by some sort of quadratic characters at critical points and at 1/2 respectively. As a corollary of the asymptotic formula, we prove a nonvanishing result for GL_2. We also establish the Langlands' reciprocity law for certain Frobenius extensions of algebraic number fields.

Resumé

Dans cette thése, nous étudions deux sujets concernant les propriétés analytiques des $L-$fonctions automorphiques. D'abord, nous étudions des valeurs moyennes des $L-$fonctions tordues de formes cuspitales automorphiques unitaires de GL_r sur le corps rationnel de nombre et ensuite la loi de réciprocité de Langlands pour les extensions finies de Galois des corps algébriques de nombre. Nous démontrons une formule asymptotique et une limite de la valeur moyenne des $L-$fonctions tordues par un certain genre de caractères quadratiques aux points critiques et à 1/2 respectivement. Comme un corollaire de la démonstrons un résultat non-trivial pour GL_2. Nous établissons aussi la loi de réciprocité de Langlands pour certaines extensions de Frobenius des corps algébriques de nombre.

Contents

0 Introduction

We study the values of the automorphic cuspidal L-functions of GL_r twisted by characters at critical points and the Langlands' reciprocity law between Galois representations and automorphic representations.

The first one was motivated by the so-called non-vanishing problem. Let F be an algebraic number field, F_A be the ring of adeles of F. Given an irreducible automorphic cuspidal representation of $GL_r(F_A)$, is there a character χ of finite order of the idele class group of F such that

$$L(1/2, \pi \times \chi) \neq 0 \ ?$$

It is well known that such a nonvanishing problem is closely related to some lifting problems. For example, Waldspurger found that for a given irreducible unitary automorphic cuspidal representation π of $PGL_2(F_A)$, the corresponding representation under the Howe correspondence is an automorphic representation of $\overline{SL_2(F_A)}$ if and only if, there is a quadratic character χ such that $L(1/2, \pi \times \chi) \neq 0$ with another hypothesis. For this see [Pia]. In some situations, the non-vanishing result of the L-function twisted by a Hecke character, not necessarily quadratic, already is enough, in, for instance, the recent work of Ginzburg and Ash [G-A].

D. Rohrlich [Roh 1] proved the the following nonvanishing theorem on GL_2. Let π be an irreducible automorphic cuspidal representation of GL_2 over any number field F, and s_0 be a complex number. Then there are infinitely many ray class characters χ of F (of finite order), such that $L(s_0, \pi \times \chi) \neq 0$. It is natural to ask the two questions. One is, can we replace characters of finite order in Rohrlich's theorem by quadratic characters ? The other one is, can we generalize this theorem from GL_2 to GL_r ?

In the preprint paper [B-R] of Barthel and Ramakrishnan, it was proved that for a given irreducible unitary automorphic cuspidal representation π of GL_r over F, $r \geq 2$, then for any complex number s_0 with $Re\ s_0 \notin (\frac{1}{2n-2}, 1 - \frac{1}{2n-2})$, there are infinitely many ray class characters χ of F such that $L(s_0, \pi \times \chi) \neq 0$. After the behavior of the root number and conductor under a twisting by a character is established, the proof is almost a direct generalization of that of Rohrlich's. Now how is the nonvanishing problem of L-functions twisted by quadratic characters ? At the point 1/2, the center of the critical strip of L-functions, it was known that the non-vanishing result does not always hold. In fact, Rohrlich gave some such examples in [Roh 2]. One of them uses Maass forms f of PGL_2 for the full modular group $SL_2(Z)$ satisfying $f(-\bar{z}) = -f(z)$. In this case the L-function twisted by any quadratic character does vanish at 1/2. The proof for this fact is based on a criteiria due to Waldspurger which says that for an irreducible automorphic cuspidal representation of PGL_2 over F, there is a quadratic character χ such that $L(1/2, \pi \times \chi) \neq 0$ if and only if the following holds: if π_v is a principal series representation for every place v of F, then $\epsilon(1/2, \pi) = 1$. However, In [K.Mur], the nonvanishing result of holomorphic modular form twisted by quadratic charcaters was proved. And D. Bump, S. Friedberg and J. Hoffstein [D-F-H] proved a version of a non-vanishing theorem of quadratic twist. Let f be a Maass form of PGL_2 over the imaginary quadratic field $F = Q(i)$ such that $f(\gamma z) = f(z)$ or $f(\gamma z) = \sigma(\gamma) f(z)$, for any $\gamma \in PGL_2(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of F and σ is the trivial character which is trivial on the principal congruence subgroup modulo $1 + i$. They proved that for such an f and a complex number s_0 with $Re\ s_0 \geq 1/2$, there exist infinitely many quadratic characters χ of the idele class group of F such that $L(s_0, \pi \times \chi) \neq 0$. And, recently Hoffstein and Friedberg [F-H] found some sufficient conditions for the nonvanishing theorem of quadratic twist of the GL_2 over any number field. Roughly speaking, they proved that if π satisfies one of the following conditions:

1) $\pi \not\simeq \tilde{\pi}$,

2) $\pi \simeq \tilde{\pi}$ and there is a quadratic character χ_0 such that $\epsilon(1/2, \pi \times \chi_0) = 1$,

then there are infinitely many quadratic characters χ such that $L(1/2, \pi \times \chi) \neq 0$.

For general GL_r, even over the rational number field, we still do not know if there is some similar condition on π which can ensure the non-vanishing of a quadratic twist.

In this thesis we consider only these critical points with their real parts bigger than 1/2, because of the existence of vanishing examples of quadratic twists at 1/2. Our idea is to get an asymptotic formula of the mean value of twisted L-functions by some sort of quadratic characters at point w, $Re\, w > 1/r$. This is our Theorem 2.3. Let $L_f(s, \pi)$ denote the L-function defined originally by the Euler product of the local L-factors over only finite places.

Theorem 2.3 *Let π be an irreducible unitary automorphic cuspidal representation of $GL_r(Q_A)$, with the trivial central character and the conductor A. Assume that*

$$\sum_{m=1}^{\infty} \frac{a_{m^2}(\pi)}{m^s} \text{ converges absolutely in } Re\, s > 1$$

$$\sum_{m=1}^{\infty} \frac{a_{m^2}(\tilde{\pi})}{m^s} \text{ converges absolutely in } Re\, s > 1.$$

Then

i) if we assume GRH for every Dirichlet character, then for every w with $max(1/2, 1-\frac{1}{r}) < Re\, w < 1$ we have

$$\sum_{\substack{0 \le D \le Y \\ D \equiv 1(4A) \\ \text{D square free}}} L_f(w, \pi \times \chi_D) = CY + o(Y);$$

ii) for $r \le 2$, then for $\theta(r) < Re\, w < 1$ the asymptotic formula in i) is also true without GRH, where $\theta(1) = 1/2$, $\theta(2) = 11/16$, where

$$C = \frac{1}{\phi(4A)} \frac{1}{\zeta(2)} \sum_{m_1, m_2} a_{m_1 m_2^2}(\pi)(m_1 m_2^2)^{-w} \prod_{p | m_2} (1 + p^{-1})^{-1}.$$

If the asymptotic constant in the theorem is not zero, then for the complex number w in the theorem, there are infinitely many quadratic characters such that the twisted L-functions are non-vanishing at w. But, unfortunately, we have not yet proved the nonvanishing of the asymptotic constant for $r > 2$. For r=2, it is not zero. Indeed, there is a simple connection between this constant and the symmetric square of π. Gelbart -Jacquet's lifting theorem says that for an irreducible automorphic cuspidal representation π of GL_2, the symmetric square L-function of π is the L-function of an automorphic representation of GL_3. Then by the non-vanishing of automorphic L-functions for $Re\ s \geq 1$, the asymptotic constant in the theorem is nonzero for r=2. Therefore we can prove a nonvanishing result for r=2, i.e, the following corollary.

Corollary 2.3 *Let π be an irreducible unitary automorphic cuspidal representation of $GL_2(Q_A)$ with the trivial central character. Assume GRH for L-functions associated to Dirichlet characters. Then for every w with $1/2 < Re\ w < 1$, there are infinitely many quadratic characters χ_D such that $L(s, \pi \times \chi_D)$ does not vanish at w.*

We should point out that this nonvanishing result for GL_2 is contained in the result of [F-H] already. But our proof is different from [F-H]'s and we give the asymptotic formula here.

The main idea in the proof of Theorem 2.3 is from the paper of [M-M]. That is for given w, $1/2 < w < 1$ we consider the integral

$$\frac{1}{2\pi i}\int_{(\gamma)} L_f(w+s, \pi \times \chi_D) X^s \Gamma(s) ds, \quad \gamma > 1 - w,$$

where $L_f(s, \pi \times \chi_D)$ is the finite part of $L(s, \pi \times \chi_D)$, X is our parameter evaluated with respect to Y at the end of the proof to ensure the error term smaller with respect to the main term. Then we apply the functional equation satisfied by $L(s, \pi \times \chi_D)$ and move the line of the integral to the left of the origin and pick up the residue at the origin. After summing D up to Y we get

$$\sum_{\substack{0 \le D \le Y \\ D \equiv 1(4A) \\ \text{D square free}}} L_f(w, \pi \times \chi_D) = \textit{sum part} + \textit{integral part},$$

where

$$\textit{sum part} = \sum_1^{\infty} \frac{a_m(\pi)}{m^w} exp(-m/x) \sum_{\substack{0 \le D \le Y \\ D \equiv 1(4A) \\ \text{D square free}}} \chi_D(m).$$

The main term in the theorem comes from those m in the sum part which essentially are square. In our estimation on the error terms, two things play the crucial roles. One is Burgess' estimation on the mean value of a character. This estimation says that for a nontrivial Dirichlet character χ mod M and any $\epsilon > 0$ we have

$$|\sum_{X \le n \le X+H} \chi(n)| \ll_\epsilon H^{1/2} M^{3/16+\epsilon}.$$

The other one is the Ramanujan conjecture on average for an irreducible unitary automorphic cuspidal representation of GL_r which is known by a standard argument of applying the analytic property of Rankin-Selberg convolution of two cuspidal representations of GL_r and Landau's theorem in analytic number theory, see [B-R].

In this theorem we work on those automorphic cuspidal representations with their central characters trivial. This restriction is to control the root numbers under the twists which are determined in [B-R]. Among Rohrlich's vanishing examples of quadratic twist, all of them have trivial central characters. Up to now we do not know whether there is a vanishing example of quadratic twists at 1/2, which has a nontrivial central charcter. [F-H]'s results tell us that for GL_2 the central character of π is trivial or quadratic or not, almost determines the possibility of the nonvanishing of quadratic twist. Maybe our restriction on trivial central character puts us in a much more difficult situation for the non-vanishing of quadratic twist at 1/2 even at other critical points, while we can control the twisted root numbers.

At the point 1/2, we prove a bound for the mean value of quadratic twists of L-functions.

Theorem 2.4 *Let π be an irreducible unitary automorphic cuspidal representation of $GL_r(Q_A)$ with the trivial central character and conductor A. Assume the same condition in Theorem 2.3. Then*

$$\sum_{\substack{0 \le D \le Y \\ D \equiv 1(4A) \\ \text{D square free}}} L_f(1/2, \pi \times \chi_D) \ll \begin{cases} Y^{\frac{r+1}{2}} (\log Y)^2 & \text{if } r > 1 \\ & \\ Y^{3/2} \log Y & \text{if } r=1 \end{cases}.$$

This theorem gives an improvement of the corrollary to Theorem 1 of Goldfeld and Viola in [G-V]. The main difference from the proof of Theorem 2.3 is that instead of evaluating X at the end of the proof, we let $X = AD^r$ before we take the summation up to Y, where A is the conductor of π.

The reason we only work on the rational number field is that for arbitray number field some basic estimations such as Polya-Vinogradov inequality, and Burgess' estimation used in our proofs are not yet known to us.

Now let us turn to our second topic, the reciprocity law. We fix a finite Galois extension E/F of number fields. As one particular example of the vast reciprocity conjecture made by Langlands, the reciprocity law says that for every irreducible complex representation ρ of degree n of $Gal(E/F)$ there corresponds a unitary automorphic cuspidal representation $\pi(\rho)$ of $GL_n(F_A)$ such that

$$L(s, \rho,\ E/F) = L_f(s, \pi(\rho)),$$

where the L-function on the left side is the Artin L-function attached to ρ. It is known that an irreducible unitary automorphic cuspidal L-function of GL_n is an entire function except for the case of n=1 and the trivial representation. Therefore, the reciprocity law implies the Artin conjecture which says that the Artin L-function of a nontrivial irreducible complex representation of $Gal(E/F)$ is an entire function. For n=1, the reciprocity law is exactly the statement of abelian class field theory. This is why Artin proved his one dimensional reciprocity law. If ρ is induced from a one dimensional representation, (in this case we call ρ monomial.), then the Artin conjecture is true for ρ. In general we only know that the Artin L-functions at least are meromorphic functions by Brauer's theorem. This theorem expresses every irreducible complex representation of a finite group G as a Z-linear combination of induced representations from one dimensional representations of subgroups of G. If ρ is two dimensional and if the image of $\rho(Gal(E/F))$ in $PGL_2(C)$ is solvable, then the reciprocity law is true for ρ by the work of Langlands and Tunnell, see [Lan], [Tun]. In Langlands' proof there are two important ingredients. The one is the existence of base change map on GL_2 proved in [Lan]. The other one is the lifting result from GL_2 to GL_3 due to Gelbart and Jacquet in [Gel-Jac]. In Tunnell's proof, the existence of the base change map on GL_2 was used, and the base change result on any cubic extension also played an important role. After Langlands and Tunnell's work, Arthur and Clozel obtained the base change map and automorphic induction map on cyclic extentions for GL_r in [A-C]. By this and the analytic property of the Rankin-Selberg convolution of two automorphic cuspidal representations of GL_r, they were able to prove that the reciprocity law is true for nilpotent extensions. Since every irreducible representation ρ of a nilpotent group is monomial, the Artin conjecture is true for ρ. Though this does not say any thing new on Artin's conjecture, it is really new since we have not known how to prove the reciprocity law for monomial representations in general (if we know, we would be able to prove the reciprocity law in general).

In this thesis we prove a new reciprocity law on some Frobenius extensions, basing on the results obtained by Langlands, Tunnell, and Arthur and Clozel. For the definition of a Frobenius group, see section 3. Our result is

Theorem 3.1 *Let G=Gal(E/F) be a finite Frobenius group, K be its Frobenius kernel, H be its Frobenius complement. Let F(H) be the maximal normal nilpotent subgroup of H. If H/F(H) is nilpotent then G is automorphic over F.*

That is, the reciprocity law holds for E/F.

Let us mention Proposition 3.6 which is used in the proof of Theorem 3.1. It is actually a combination of the results of Langlands, Tunnell, Arthur and Clozel. Up to now, as we know, almost all examples for which the reciprocity law holds are not essentially beyond it.

Throughout this thesis we use the language of representation theory on adele groups. In section 1 we introduce some definitions of automorphic representations and their L-functions and Rankin-Selberg convolutions. Meanwhile we record some properties of L-functions and give some explicit formulas of the L-functions, including local, global and twisted L-functions which will be used in section 2. In section 2, Using the explicit formula due to Weil and Mestre, we first prove the two theorems which are on some bounds of the orders of zeros of twisted L-functions at 1/2, under the General Riemann Hypothesis and Ramanujan Conjecture on automorphic L-functions. Then after several lemmas, we prove Theorem 2.3 and Theorem 2.4. In section 3 we prove Theorem 3.1.

1 L-Functions of Automorphic Representation of GL_r

Let F be a global number field or a local field. If F is a global number field and v is a place of F, let F_v be the completion of F corresponding to v. Let R, C denote the field of real numbers and the field of complex numbers respectively, unless we give some specified description.

Denote by GL_r or G_r the reductive group of the r by r invertible matrices defined over F, $GL_r(F)$ the F rational points of GL_r.

Let $P(r_1, r_2, ..., r_n)$ be the standard parabolic subgroup of G_r of type $(r_1, ..., r_n)$ i.e. on its diagonal line are the matrices in G_{r_i} and its left lower part is zeros. We denote the unipotent radical of such a parabolic subgroup by $N(r_1, r_2, ..., r_n)$ which is in $P(r_1, \cdots, r_n)$ and has $I_{r_i}, i = 1, .., n$ on its diagonal. Let $M(r_1, ..., r_n)$ be reductive part of $P(r_1, ..., r_n)$ which has $G_{r_i}, i = 1, ..., n$ on its diagonal, its other entries are 0. If all r_i's are 1, we just write P_r, N_r and M_r for the corresponding parabolic subgroup, its unipotent radical, and its reductive part respectively. For every $P(r_1, ..., r_n)$ we have the decomposition:

$$P(r_1, ..., r_n) = M(r_1, ..., r_n)N(r_1, ..., r_n).$$

When F is a global number field, the ring F_A of adeles of F is the restricted product of $\{F_v\}_v$ with respect to $\{\mathcal{O}_v\}_v$, where $\mathcal{O}_v$ is the integers of F_v. Then $G_r(F_A) = \prod_v G_r(F_v)$ is the restricted product of groups $G_r(F_v)$ with respect to $\{G_r(\mathcal{O}_v)\}_v$. The group $G_r(F_A)$ has its topology inherited from the product topology of the local groups whose topologies come from the local absolute valuations. For this topology $G_r(F_A)$ is a locally compact group. Let $K_v = G_r(\mathcal{O}_v)$ if v is a finite place, $K_v = O(r, R)$ if v is real, $K_v = U(r, C)$ if v is complex. And we define $K = \prod_v K_v$. It is well known that every K_v defined above is a maximal compact subgroup of $G_r(F_v)$, and K is a maximal compact subgroup of the $G_r(F_A)$.

To give a definition of L-functions of automorphic representations of GL_r, we shall adopt the Zeta -Integral approach called the Tate method which was generalized from GL_1 to GL_2 in [J-L] ,to GL_r in [G-J]. Such an approach can give us an explicit form of the L-functions at ramified local places.

1.1 Nonarchimedean Local L-Functions and $\epsilon-$ Factors

Let F be a local nonarchimedean field, q be the cardinality of the residue class field of F, $|.|$ the corresponding absolute value.

A smooth representation (π, V), where V is the complex vector space on which π acts, is a representation of $G_r(F)$ satisfying the condition that for every v in V the stabilizer of v in $G_r(F)$ is an open subgroup. A smooth representation (π, V) is called admissible, if for every open subgroup G of $G_r(\mathcal{O})$ the subspace fixed by G is of finite dimension.

We call an irreducible admissible representation (π, V) unitary, if there is a non-degenerate positive definite Hermitian form on V which is invariant under the action of $G_r(F)$. It is obvious that a character is unitary if and only if it has the absolute value 1. If the restriction of π on $K = G_r(\mathcal{O})$ contains the trivial representation of K, we say that π is unramified.

Let (π, V) be an admissible representation of $G_r(F)$. Then there is an admissible representation $(\tilde{\pi}, \tilde{V})$ characterized by the property that there is a nondegenerate bilinear form on $V \times \tilde{V}$ such that

$$\langle \pi(g)v, \tilde{v} \rangle = \langle v, \tilde{\pi}(g^{-1})\tilde{v} \rangle$$

for all $g \in G_r(F), v \in V, \tilde{v} \in \tilde{V}$. We call this representation $\tilde{\pi}$ the contragredient representation of π.

Remarks. 1) π is irreducible or unitary or unramified if and only if $\tilde{\pi}$ is respectively, and $\tilde{\tilde{\pi}} = \pi$. 2) For an irreducible admissible representation π, the restriction of the π on Z is a scalar, if Z is the center of $G_r(F)$. We call the character of $F^\times$ arising by this scalar the central character of π, and denote it by ω_π.

Let H be a closed subgroup of $G_r(F), (\rho, W)$ be a representation of H, we can define the induced representation from ρ in the usual sense. Let V be the space of all functions from $G_r(F)$ to W satisfying the conditions:
1) $f(hg) = \rho(h)f(g)$ for all $h \in H, g \in G_r$,
2) there is an open compact subgroup G of G_r such that

$$f(gg') = f(g) \text{ for all } g \in G_r,\ \ g' \in G.$$

Then we let $G_r(F)$ act on V by right translations. We denote this representation by

$$Ind_{H(F)}^{G_r(F)} \rho.$$

For a standard parabolic subgroup $P(r_1, ..., r_n)$ of G_r, if σ_i is a representation of $G_{r_i}(F)$ i=1,...,n, then the $\prod_{i=1}^n \sigma_i$ is a representation of $M(r_1, ..., r_n)(F)$. We extend it to a representation σ of the $P(r_1, ..., r_n)(F)$ by letting it trivial on $N(r_1, ..., r_n)$, then induce $\sigma \otimes \delta_P^{1/2}$ to $G_r(F)$, where δ_P is the modulu function of P. We denote such an induced representation π by

$$\pi = I(G_r, P; \sigma_1, ..., \sigma_n).$$

It is known that
1) if every σ_i is admissible, then π is admissible;
2) $\tilde{\pi} = I(G_r, P; \tilde{\sigma}_1, ..., \tilde{\sigma}_n)$.

An irreducible admissible representation (π, V) of $G_r(F)$ is called absolutely cuspidal, if for any maximal parabolic subgroup P of G_r and N the unipotent radical of P, then the integral

$$\int_{N(F)} \pi(n)v\,dn = 0$$

for every v in V.

We call a function f on $G_r(F)$ a coefficient of π, if f is a linear combination of

$$\langle \pi(g)v, \tilde{v} \rangle,$$

for some $v \in V, \tilde{v} \in \tilde{V}$. The function $\check{f}(g) = f(g^{-1})$ is a coefficient of $\tilde{\pi}$.

We call an admissible representation π square integrable, if π admits a unitary central character and all its coefficients are square integrable through the center of $G_r(F)$. If $\pi = \pi_0 \otimes |.|^t$ with π_0 square integrable and t real, we say π is essentially square integrable. Bernstein's result says that if given the data: r=nj and $P(j, ..., j), \sigma_0$ absolutely cuspidal representation of $G_j(F)$, then the representation

$$I(G_r, P; \sigma_1, ..., \sigma_n),$$

where $\sigma_i = \sigma_0 \otimes |.|^{i-1}$, has a unique essentially square integrable component. Conversely every essentially square integrable representation comes in this way.

If we have $P = P(r_1, ..., r_n)$ and $\pi = I(G_r, P; \sigma_1, ..., \sigma_n)$ with every σ_i square integrable, we call π tempered. If $\pi = \pi_0 \otimes |.|^t$, π_0 tempered and t real, we say π is essentially tempered .

Given $P(r_1, ..., r_n), \sigma_i = \sigma_{i,0} \otimes |.|^{t_i}$, with $\sigma_{i,0}$ tempered, t_i real for i=1,...,n, then

$$I(G_r, P; \sigma_1, ..., \sigma_n)$$

has a largest subrepresentation I', and we denote the irreducible representation $\pi = I/I'$ by

$$\pi = J(G_r, P; \sigma_1, ..., \sigma_n).$$

Now we can state the main results on irreducible admissible representations by the following proposition which is the extension of Langlands' theorem for archimedean case to the nonarchimedean case.

Proposition 1.1 *(Silberger and Wallach) Let π be an irreducible admissible representation of $G_r(F)$. Then*

$$\pi = J(G_r, P; \sigma_1, ..., \sigma_n),$$

for some P and σ_i,i=1,...,n.

With the preparation above we can start to define the local L- functions and ϵ-factors.

Define the Zeta function of π by

$$Z(s, \Phi, f) = \int_{G_r(F)} \Phi(g)|det\, g|^s f(g) d^\times g,$$

where the Φ is a Schwartz-Bruhat function on $M_{r\times r}(F)$, the r by r matrices which have their entries in F, f is a coefficient of π and the $d^\times g$ is a Haar measure of $G_r(F)$. It was proved in [G-J] that

i) the $Z(s, \Phi, f)$ converges absolutely in some right half plane;

ii) the integrals are rational functions of q^{-s}, such that they admit a common denominator which does not depend on f and Φ.

iii) for any nontrivial additive character ψ of F, there is a rational function $\gamma(s, \pi, \psi)$ of s such that for all f and Φ the functional equation holds:

$$Z(1 - s + (r - 1)/2, \hat{\Phi}, \check{f}) = \gamma(s, \pi, \psi) Z(s, \Phi, f),$$

where

$$\hat{\Phi}(x) = \int \Phi(y)\psi[tr(yx)]dy,$$

$$\check{f}(g) = f(g^{-1}),$$

the dy is the self dual measure on $M_{r\times r}(F)$.

Let $I(\pi)$ be the subvector space of $C(q^{-s})$ spanned by all integrals $Z(s+(r-1)/2, \Phi, f)$. Then since $X^n I(\pi) = I(\pi)$ for any integer n, $I(\pi)$ actually is a fractional ideal of $C[X, X^{-1}]$, where $X = q^{-s}$. Since $C[X, X^{-1}]$ is PID (note: $C[X]$ is Euclidean, and $C[X, X^{-1}]$ is the localization of $C[X]$ by X.) and $I(\pi)$ contains the constants, $I(\pi)$ is generated by one such an element $P(X)^{-1}, P(X) \in C[X]$ which can be normalized to $P(0) = 1$. Then we define

$$L(s, \pi) = P(q^{-s})^{-1}.$$

Actually we can chose suitable f, Φ, ψ such that $L(s, \pi)$ is the corresponding zeta function. Therefore we can think of our local L-factor as the GCD of all zeta integrals of the π. Furthermore, we define the $\epsilon-$ factor by

$$\epsilon(s, \pi, \psi) = \gamma(s, \pi, \psi) L(s, \pi)/L(1-s, \tilde{\pi}).$$

Remarks. 1) If π is an absolutely cuspidal representation, then $L(s, \pi) = 1$ unless $r = 1$, and $\pi = |.|^t, t$ is a complex number. In the later case $L(s, \pi) = (1 - q^{-(t+s)})^{-1}$. See [G-J] for this. By this, together with the following proposition, we can compute the L-function and ϵ-factor precisely.

2) If r=1, G_1 has no maximal parabolic subgroup. So all characters of $F^\times$ are absolutely cuspidal representations of $F^\times$. Furthermore a character χ of $F^\times$ is unramified if and only if $\chi = |.|^u, u \in C$. So

$$L(s, \chi) = \begin{cases} (1 - q^{-u-s})^{-1} & \text{if } \chi = |.|^u \\ 1 & \text{otherwise} \end{cases}$$

3) $\epsilon(s, \pi, \psi) = 1$, if π is unramified. See [G-J].

Proposition 1.2 *([Jac]) 1) If $\pi = I(G_r, P; \sigma_1, .., \sigma_n)$, then*

$$L(s,\pi) = \prod_{i=1}^{n} L(s,\sigma_i)$$

$$\epsilon(s,\pi,\psi) = \prod_{i=1}^{n} \epsilon(s,\sigma_i,\psi),$$

2) If $\pi = J(G_r, P; \sigma_1, ..., \sigma_n)$, then

$$L(s,\pi) = \prod_{i=1}^{n} L(s,\sigma_i)$$

$$\epsilon(s,\pi,\psi) = \prod_{i=1}^{n} \epsilon(s,\sigma_i,\psi).$$

3) Let r=nj. If π is the unique essential square integrable component of $I(G_r, P; \sigma_1, ..., \sigma_j)$, where the $\sigma_i = \sigma \otimes |.|^{i-1}, \sigma$ is a absolutely cuspidal representation of $G_n(F)$. Then

$$L(s,\pi) = L(s,\sigma).$$

4) If π is an irreducible unramified representation of $G_r(F)$, then it is the unique unramified component of

$$\pi = I(G_r, P_r; \mu_1, ..., \mu_r), \quad \mu_i = |.|^{u_i},$$

and

$$L(s,\pi) = \prod_{i=1}^{r} L(s,\mu_i).$$

Therefore we can deduce the following

Proposition 1.3 *1) for any irreducible admissible representation π of $G_r(F)$,*

$$L(s,\pi) = \prod_{i=1}^{r'} (1 - a_i(\pi) q^{-s})^{-1}$$

with $r' \leq r$ and all the $a_i(\pi)$ are not zero. If π is unramified, $r' = r$.
2) Let $L(s,\pi)$ be as in 1), then

$$L(s,\tilde{\pi}) = \prod_{i=1}^{r'}(1 - a_i^{-1}(\pi)q^{-s})^{-1}$$

3) If π is tempered, then all $a_i(\pi)$ have absolute value 1, hence $L(s,\pi)$ is holomorphic for $Re\, s > 0$.

Proof. The first claim is obvious from the propositions above. 2) comes from the two facts. The one is that the "contragredient" is symmetric for the properties, absolutely cuspidal, unramified. The second one is that if χ is a character then $\tilde{\chi} = \chi^{-1}$. 3) can be proved as follows. By definition,

$$\pi = Ind(G_r, P; \sigma_1, ..., \sigma_n),$$

where the P has the type $(r_1, ..., r_n)$ and the σ_i is a square integrable representation of $G_{r_i}(F)$, i=1,...,n. By proposition 1.2,

$$L(s,\pi) = \prod_i L(s,\sigma_i).$$

Let τ be the square integrable component of

$$I(G, P; \tau_1, ..., \tau_m),$$

with $\tau_j = \tau_0 \otimes |.|^{j-1}, j = 1, ..., m$, and τ_0 is absolutely cuspidal. Then, by proposition 1.2, $L(s,\tau) = L(s,\tau_0)$. The τ is square integrable if and only if the τ_0 is. Since a character is square integrable if and only if it is unitary, we proved 3) by the remarks just before Proposition 1.2. □

1.2 Archimedian Local L-Functions

Let F be a local archimedean field.

The definition of an admissible representation of $G_r(F)$ is not easy to give here, so we only use this terminology to figure out its L-functions. Finally we can introduce the concepts of square integrable, tempered representations analogously as we did in the local nonarchimidean case. In [G-J], the authors also use Zeta-Integals to define the local L-factors and $\epsilon-$ factors. But an amazing thing is that the reciprocity map between the semi-simple representations of degree r of the local Weil group W_F and the irreducible admissible representations of $G_r(F)$ was established in the local archimedean case.

Proposition 1.4 *([Jac]) There is a bijection $\lambda \mapsto \pi$ between the set of all semi-simple representations of degree r of the Weil group W_F to the set of irreducible admissible representations of $G_r(F)$ such that*
1) $\tilde{\lambda} \mapsto \tilde{\pi}$;
2)

$$L(s, \pi) = L(s, \lambda), \quad L(s, \tilde{\pi}) = L(s, \tilde{\lambda}),$$

$$\epsilon(s, \pi, \psi) = \epsilon(s, \lambda, \psi).$$

Remark. 1) such a reciprocity was conjectured to be true for all local places by Langlands.
2) The reciprocity map of degree one is the natural map between the one dimensional representations of W_F and characters of $F^{\times}$.

This gives much information about the local L-functions and ϵ-factors, since the irreducible representations and the L-functions ϵ-factors attached to the representations of W_F are clear to us, see [Tat].

Indeed, the Weil group W_F is $F^\times$ if F is a complex field, while $C^\times \cup jC^\times, j^2 = -1, jcj^{-1} = \bar{c}$, if F is a real field. The $\bar{c}$ is the complex conjugate of the Galois action of C/R. For F complex , all irreducible representations of W_F are abelian. For F real, the only nonabelian irreducible representations of W_F are of the form $Ind_{W_R/W_C}\lambda$, where the λ is not invariant under complex conjugation.

If F is complex, the characters of W_F are

$$\chi(x) = (x\bar{x})^a x^{-N} \ \ or \ \ (x\bar{x})^a \bar{x}^{-N}$$

where a is a complex number, N is a nonnegative integer, and both a and N depend on χ.

If F is real, then the one dimensional representations of W_F are those of $W_F/[W_F, W_F]$, where $[W_F, W_F]$ is the commutator subgroup of W_F. We claim that $W_F/[W_F, W_F]$ is isomorphic to $R^\times$. In fact $[W_F, W_F]$ is the group of the complex numbers which have modulus 1. So the map $r : W_F/[W_F, W_F] \longrightarrow R^\times$, $r(j) = -1, r(x) = x\bar{x}$, for $x \in C^\times$ gives an isomorphism of $W_F/[W_F, W_F]$ with $R^\times$. The characters of $R^\times$ are the form: $\chi(x) = |x|^a x^N, a \in C, N = 0, 1$.

We define the abelian local L- function as $G_R(s+a)$ or $G_C(s+a)$ depending on the F is real or complex, where

$$G_R(s) = \pi^{s/2}\Gamma(s/2), \ \ G_C(s) = (2\pi)^{1-s}\Gamma(s).$$

To ensure our L-function is invariant under induction we define the L-function of $Ind_{W_R/W_C}\lambda$ as $L(s, \lambda)$. This is well defined by the following reasons. Firstly,

$$Ind_{W_R/W_C}\lambda_1 = Ind_{W_R/W_C}\lambda_2,$$

if and only if

$$\lambda_2 = \lambda_1, \ or \ \lambda_1^\tau,$$

where λ^τ is the complex conjugate of λ. Secondly,

$$L(s,\lambda) = L(s,\lambda^\tau).$$

Proposition 1.5 *Let π be an irreducible admissible representation of $G_r(F)$.*

1) If F is real, then

$$L(s,\pi) = \prod_{i=1}^{r_1} G_R(s+a_i) \prod_{j=1}^{r_2} G_C(s+b_j).$$

If F is complex, then

$$L(s,\pi) = \prod_{j=1}^{r_2} G_C(s+b_j).$$

The a_i, b_j are some complex numbers.

2) If π is tempered then

$$a_i = i\alpha_i, 1 + i\alpha_i,$$

$$b_j = m_j/2 + i\beta_j,$$

with α_i, β_j real numbers, m_j's nonnegative integers. And

$$L(s,\tilde{\pi}) = \prod_{i=1}^{r_1} G_R(s+\bar{a}_i) \prod_{i=1}^{r_2} G_C(s+\bar{b}_j).$$

Proof. Suppose that π corresponds to the representation ϕ of W_F. Since ϕ is semisimple we can write

$$\phi = \oplus\lambda_i \oplus \rho_j,$$

with every ϕ_i one dimensional, ρ_j irreducible, two dimensional. The expression of $L(s,\pi)$ is obvious by the proposition above. If π is tempered, then the λ_i, ρ_j are unitary. If $\lambda = |x|^a x^{-N}$ and unitary then $Re\, a - N = 0$, i.e. the real part of a is 0 or 1. If $\rho = Ind_{W_R/W_C}\xi$ and unitary then ξ is unitary also. Suppose $\xi(x) = (x\bar{x})^a x^{-m}, m$ positive integer. Then $2Re\, a - m = 0$, i.e. the real part of a is a positive half integer. Since $\tilde{\pi}$ corresponds to the $\tilde{\phi}$ and $(Ind_{W_F/W_C}\xi)^\sim = Ind_{W_F/W_C}\xi^{-1}$, we have the expression of the $L(s,\tilde{\pi})$. □

From this proposition we see that if π is tempered, its local L-function is holomorphic in $Re\, s > 0$, which means that the Ramanujan conjecture is true at this local place in this case.

1.3 Global L-Function and ϵ-Factors

Let F be a global number field. Let ω be a character of $F^{\times}\backslash F_A^{\times}$, Z the center of G_r.

Let $\mathcal{A}(\omega)$ be the complex vector space of the continuous functions f on $G_r(F)\backslash G_r(F_A)$ satisfying the following conditions:

1) $f(zg) = \omega(z)f(g)$, for every $z \in Z(F_A)$ and $g \in G_r(F_A)$,

2) certain "K-finite" and "slowly increasing" conditions defined in [G-J].

Let $\mathcal{A}_0(\omega)$ be the subspace of $\mathcal{A}(\omega)$ which consists of the functions in $\mathcal{A}(\omega)$ satisfying the cuspidal condition:

$$\int_{N(F)\backslash N(F_A)} f(ng)dn = 0,$$

for the unipotent radical N of every proper parabolic subgroup of G_r.

Now we let $G_r(F_A)$ act on $\mathcal{A}(\omega)$ by right translaton, we call this representation ρ the regular representation of $G_r(F_A)$ with respect to ω, and every irreducible constituent of ρ an automorphic representation of $G_r(F_A)$. Since $\mathcal{A}_0(\omega)$ is invariant under the action of ρ, we call every irreducible constituent of $\mathcal{A}_0(\omega)$ an automorphic cuspidal representation of $G_r(F_A)$.

It was proved in [J-L] that every irreducible automorphic representation of $G_r(F_A)$ can be written as a restricted tensor product of local irreducible admissible representations

$$\otimes_v \pi_v$$

where π_v is unramified, for almost all v, and the local factors are uniquely determined by π.

Remark. If an irrducible automorphic representation π occures in $\mathcal{A}(\omega)$, then the ω is actually the central character of π.

Suppose that $\pi = \otimes\pi_v$ is an irreducible automorphic representation of $G_r(F_A)$. And let ψ be an character of F_A/F, ψ_v its local components. Now we put

$$L(s,\pi) = \prod_v L(s,\pi_v).$$

$$\epsilon(s,\pi) = \prod_v \epsilon(s,\pi_v,\psi_v).$$

Remarks. 1) $\epsilon(s,\pi)$ is independent of the choice of ψ. 2) If d is the absolute norm of the different of F and A_π is the conductor of the π defined locally in [J-P-S 2], then

$$\epsilon(s,\pi) = W_\pi(d^r N(A_\pi))^{1/2-s}.$$

where W_π is a nonzero number independent of s. It is called the root number of π.

Proposition 1.6 *Suppose π is a unitary automorphic representation of $GL_r(F_A)$. Then*

1) the infinite product of local L-functions above converges absolutely in the right half plane $Re\, s > 1$, [J-S 1];

2) it satisfies the following functional equation:

$$L(s,\pi) = \epsilon(s,\pi)L(1-s,\tilde{\pi}), \quad \textit{[G-J];}$$

3) $L(s,\pi)$ can be analytically continued to a meromorphic function of the whole plane, and it has at most finitely many poles, and it is bounded at infinity in every bounded vertical strip. If π is cuspidal then $L(s,\pi)$ can be continued to an entire function on the whole plane, except for the case $r=1$ and π is the trivial representation , [G-J];

4) $L(s,\pi)$ is nonvanishing for $Re\, s \geq 1$, [J-S 1].

Concerning the size of eigenvalues of an automophic cuspidal representation π, there is the

Ramanujan Conjecture: If π is an irreducible unitary automorphic cuspidal representation of $G_r(F_A)$, then π_v is tempered for all places v of F.

1.4 The L-Functions Twisted by Characters

The L-functions twisted by characters are the most simple examples of Rankin-Selberg convolutions. Given π, σ two irreducible automorphic cuspidal representations of $G_r(F_A)$, $G_t(F_A)$ respectively, define

$$L(s, \pi \times \sigma) = \prod_v L(s, \pi_v \times \sigma_v).$$

For every local place v, the local L-function $L(s, \pi_v \times \sigma_v)$ is defined also as the "GCD" of some kind of zeta integrals which are similar with those used to define the local L-functions of an automorphic representation, see [J-P-S] for nonarchimedean case, [J-S 3] for archimedean case.

Firstly we look at the nonarchimedean situation. Let F be a nonarchimedean local field in the following two propositions.

Proposition 1.7 *([J-P-S]) Suppose π is an irreducible absolutely cuspidal representation of $G_r(F)$ and σ is a representation of Whittaker type of $G_t(F)$. Then*
1) if $r > t$, then $L(s, \pi \times \sigma) = 1$,
2) when r=t, $L(s, \pi \times \sigma) = \prod_\chi L(s, \chi^{-1})$,
where the χ are all characters $\chi = |.|^{-u}$ such that $\pi \otimes \chi \simeq \sigma$.

Remark. What we will need is only the fact that any character of $F^\times$ is of Whittaker type.

Proposition 1.8 *([J-P-S]) 1) Let π, σ be irreducible admissible of $G_r(F_A), G_t(F_A)$ respectively. If*

$$\pi = J(G_r, P; \pi_1, ..., \pi_n)$$

$$\sigma = J(G_t, Q; \sigma_1, ..., \sigma_m),$$

where P, Q are some suitable parabolic subgroups of G_r, G_t respectively, $\pi_i = \pi_{i,0} \otimes |.|^{u_i}, u_1 > u_2 > ... > u_n$, $\sigma_i = \sigma_{i,0} \otimes |.|^{v_i}, v_1 > v_2 > ... > v_m$, and $\pi_{i,0}, \sigma_{0,j}$ are tempered. Then

$$L(s, \pi \times \sigma) = \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} L(s, \pi_i \times \sigma_j).$$

2) Let π, σ be two tempered representations

$$: \quad \pi = I(G_r, P; \pi_1, ..., \pi_n)$$

$$\sigma = I(G_t, Q; \sigma_1, ..., \sigma_m),$$

where π_i, σ_j be square integrable. Then

$$L(s, \pi \times \sigma) = \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} L(s, \pi_i \times \sigma_j).$$

3) Let π, σ be square integrable,

$$\pi \subset I(G_r, P; \pi_1, ..., \pi_n)$$

$$\sigma \subset I(G_t, Q; \sigma_1, ..., \sigma_m),$$

where $\pi_i = \pi_0 \otimes |.|^{i-1}, \sigma_j = \sigma_0 \otimes |.|^{j-1}, \pi_0, \sigma_0$ are absolutely cuspidal representations and $t \leq r$.

Then

$$L(s, \pi \times \sigma) = \prod_{1 \leq j \leq m} L(s, \pi_0 \times \sigma_j).$$

4) If π, σ are unramified, and they are the unique unramified components of

$$I(G_r, P_r; \pi_1, \cdots \pi_r)$$

and

$$I(G_t, P_t; \sigma_1, \cdots \sigma_t)$$

respectively, where the π_i, σ_j 's are unramified characters of $F^\times$, then

$$L(s, \pi \times \sigma) = \prod_{\substack{1 \le i \le r \\ 1 \le j \le t}} L(s, \pi_i \times \sigma_j).$$

For the archimedean local case, we have [J-S 3]

$$L(s, \pi \times \sigma) = L(s, \lambda \otimes \phi)$$

$$\epsilon(s, \pi \times \sigma) = \epsilon(s, \lambda \otimes \phi),$$

if the λ, ϕ are the semisimple representations of the local Weil group corresponding to π and σ respectively.

Since the irreducible representation of W_F is one dimensional or induced from a character, it is easy to write down the expression of the $L(s, \pi \times \sigma)$, see [Mor].

Now we state the main theorem on the global Rankin-Selberg convolution of two cuspidal representations.

Proposition 1.9 *Let π, σ be two irreducible unitary cuspidal representations of $G_r(F_A)$, $G_t(F_A)$ respectively.*

1) the Euler product of the local L-functions of Rankin-Selberg convolutions absolutely converges in $Re\, s > 1$, [J-S 1].

2) the L-function satisfies the functional equation

$$L(s, \pi \times \sigma) = \epsilon(s, \pi \times \sigma) L(1 - s, \tilde{\pi} \times \tilde{\sigma})$$

see [Sha].

3) $L(s,\pi\times\sigma)$ can be analytically continued to an entire function unless $\pi\simeq\tilde{\sigma}$, in which case the L-function has the only simple pole at s=1. [J-S 2] .

4) $L(s,\pi\times\sigma)$ is nonvanishing on the line $Re\, s=1$, [J-S 1].

5) $L(s,\pi\times\sigma)$ is bounded at infinity in every bounded vertical strip.

the 3) in this proposition is called the analytic property of Rankin-Selberg convolution of two automorphic cuspidal representations. It has some important consequences. One of them is on the Ramanujan conjecture on average.

Proposition 1.10 *([B-R]) Let π be an irreducible unitary cuspidal representation of $G_r(F_A)$. Write*

$$L_f(s,\pi)=\prod_{v\ finite}L(s,\pi)=\sum_{n}\frac{a_n}{\mathcal{N}n^s},$$

where n runs over the integral ideals of F, the $\mathcal{N}n$ is the absolute norm of n. Then

$$\sum_{\mathcal{N}n\leq x}|a_n|^2\ll x.$$

From now on, we shall concentrate on the $\pi\times\chi$, where the π is an irreducible unitary cuspidal representation of $G_r(Q_A)$, χ is a Dirichlet character mod M.

Lemma 1.1 *Let π be an irreducible unitary cuspidal representation of $G_r(Q_A)$, χ a primitive Dirichlet character mod M. Suppose that the conductors of π and χ are coprime, and*

$$L(s,\pi_p)=\prod_{i=1}^{r_p}(1-\alpha_i(\pi)p^{-s})^{-1}.$$

Then

1) if p does not divide the conductor of χ, then

$$L(s,\pi_p\times\chi_p)=\prod_{i=1}^{r_p}(1-\alpha_i(\pi)\chi(p)p^{-s})^{-1},$$

2) if p divides the conductor of χ, then

$$L(s, \pi_p \times \chi_p) = 1.$$

Proof. By the propositions used to compute local L- functions of π and $\pi \times \chi$, the proof is reduced to the situation of $\sigma \times \chi_p$, where the σ is an absolutely cuspidal representation of $G_1(F_p)$. Now the $L(s, \sigma \times \chi_p) \neq 1$ if and only if $\sigma\chi_p = |.|_p^u$, $u \in C$.

So if p does not divide the conductor of χ,

$$L(s, \pi_p) = \prod_{i=1}^{r_p} (1 - \chi(p)\alpha_i(\pi)p^{-s})^{-1}.$$

If p divides the conductor of χ, π_p is unramified, i.e, π_p is the unique unramified component of

$$I(G_r, P; \mu_1, ..., \mu_r), \quad \mu_i = |.|^{u_i}.$$

So every $\mu_i\chi_p$ is not of the form $|.|^u$, hence $L(s, \pi_p \times \chi_p) = 1$. □

Define

$$L_f(s, \pi) = \prod_{p\ finite} L(s, \pi_p)$$

$$L_f(s, \pi \times \sigma) = \prod_{p\ finite} L(s, \pi_p \times \sigma_p).$$

Corollary 1.1 *Let π, χ be as in the lemma above. If*

$$L_f(s, \pi) = \sum_{n=1}^{\infty} \frac{a_n(\pi)}{n^s},$$

then

$$L_f(s, \pi \times \chi) = \sum_{n=1}^{\infty} \frac{a_n(\pi)\chi(n)}{n^s}.$$

Lemma 1.2 *Let χ be a Dirichlet character mod M. Then if $\chi(-1) = 1$ then χ_∞ is trivial; if $\chi(-1) = -1$ then $\chi_\infty(x) = |x|x^{-1}$.*

Proof. We are working on the rational number field Q. Take the cycle $m = p_\infty M$ of Q. Then the field $K = Q(\xi_M)$, where ξ_M is a Mth primitive root of unity, is the ray class field mod m, $Gal(K/Q) = (Z/MZ)^\times$. Let I_Q be the group of ideles of Q. And let

$$I_Q^m = \{\alpha = (\alpha_p) \in I_Q; \ \ \alpha_p \equiv 1 (mod\ m) \text{ for every } p|m \ \}$$

Under the abelian reciprocity map

$$(\cdot, K/Q) : I_Q/I_Q^m Q^\times \longrightarrow Gal(K/Q),$$

$I_Q/I^m Q^\times$ is isomorphic to Gal(K/Q) . And

$$(\cdot, K/Q) = \prod_p (\cdot, K_p/Q_p),$$

here p runs over all primes of Q including the infinite place, and

$$(\cdot, K_p/Q_p) : Q_p^\times / N_{Q_p}^{K_p} K_p^\times \longrightarrow Gal(K_p/Q_p)$$

is the local reciprocity map.

When $p = \infty$, Q_p is a real field, its nontrivial finite extension is only C/R, the norm $N_R^C C^\times$ is the positive numbers. So if we write $Gal(C/R) = \{1, \sigma\}$, then

$$(x, C/R) = \begin{cases} 1, & \text{if } x > 0 \\ \sigma & \text{if } x < 0 \end{cases}.$$

We know that $Gal(K_p/Q_p)$ is a subgroup of Gal(K/Q), but we have to know how $Gal(K_p/Q_p)$ embeds in $Gal(K/Q)$.

Firstly, when $M = q^n$, q rational prime, then $Gal(K/Q)$ is a cyclic finite group of even order (unless n=1, q=2), so it has only one element of order 2. Hence the σ is this element of order 2.

When $M = q_1^{n_1} q_2^{n_2} \cdots q_t^{n_t}$ then $Gal(K/Q) \simeq Gal(K_1/Q) \times Gal(K_2/Q) \times \cdots \times Gal(K_n/Q)$, where K_i is the ray class field mod $p_\infty q_i^{n_i}$. For every $\alpha_p \in Q_p$, we know that

$$(\alpha_p, K_p/Q_p)|_{K_i} = (\alpha_p, K_{i,p}/Q_p), i = 1, 2, \cdots, t.$$

Since for $p = \infty$ and negative α_p, $(\alpha_p, K_{i,p}/Q_p)$ is the unique element of order 2 in $Gal(K_i/Q)$, the action of this element on $\xi_{q^{n_i}}$ is -1. Since $\xi_M = \xi_{q^{n_1}} \xi_{q^{n_2}} \cdots \xi_{q^{n_t}}$, the action of $(\alpha_p, K_p/Q_p)$ on ξ_M is also -1. Therefore the action of σ on ξ_M is -1. Hence if $\chi(-1) = 1$, χ_∞ is the trivial character of $R^\times$, if $\chi(-1) = -1$, $\chi_\infty(x) = |x|x^{-1}$ on $R^\times$. □

Proposition 1.11 *Let π be a cuspidal representation of $G_r(Q_A)$, χ be a Dirichlet character mod M.*

1) If $\chi(-1) = 1$, then $L(s, \pi_\infty \times \chi_\infty) = L(s, \pi_\infty)$.

2) If $\chi(-1) = -1$ and

$$L(s, \pi_\infty) = \prod G_R(s + a) \prod G_C(s + b)$$

then

$$L(s, \pi_\infty \times \chi_\infty) = \prod G_R(s + a \pm 1) \prod G_C(s + b).$$

Proof. If $\chi(-1) = 1$ then χ_∞ is trivial, by the Lemma above. So $L(s, \pi_\infty \times \chi_\infty) = L(s, \pi_\infty)$.

If $\chi(-1) = -1$ then $\chi_\infty(x) = |x|x^{-1}$, by the Lemma above. Suppose π_∞ corresponds to the representation ϕ of the Weil group, written as

$$\phi = \oplus \phi_i,$$

with each ϕ_i irreducible. So

$$\phi \otimes \chi_\infty = \oplus \phi_i \otimes \chi_\infty.$$

Now ϕ_i is one or two dimensional. If it is one dimensional, then $\phi(x) = |x|^a x^{-N}, a \in C, N = 0, 1$. Hence

$$\phi_i \otimes \chi_\infty(x) = \begin{cases} |x|^{a+1}x^{-1} & \text{if } N = 0 \\ |x|^{a-1} & \text{if } N = 1. \end{cases}$$

So

$$L(s, \phi_i \otimes \chi_\infty) = \begin{cases} G_R(s + a + 1) & \text{if } N = 0 \\ G_R(s + a - 1) & \text{if } N = 1. \end{cases}$$

If ϕ_i is two dimensional, then $\phi_i = Ind_{W_R/W_C}\xi$, hence

$$\phi_i \otimes \chi_\infty = Ind_{W_R/W_C}\xi\chi_\infty|_{W_C}.$$

Under the isomorphism map of $W_F/[W_F, W_F]$ with $R^\times$, the image of $C^\times$ is the positive numbers. Hence the $\chi_\infty|_{C^\times}$ is the trivial representation. Therefore we get

$$\phi_i \otimes \chi_\infty = Ind_{W_R/W_C}\xi.$$

So

$$L(s, \phi_i \otimes \chi_\infty) = L(s, \xi).$$

□

Corollary 1.2 *If π is tempered at infinity and χ is a Dirichlet character, then*

$$L(s, \pi_\infty \times \chi_\infty) = \prod G_R(s + a) \prod G_C(s + b)$$

with the real parts of a's, b's nonnegative numbers, again.

Now we ask how does the ϵ- factor behave when a cuspidal representation is twisted by a character. The following proposition answers this question. Recall first

$$\epsilon(s, \pi) = W_\pi(A_\pi)^{1/2-s}$$

$$\epsilon(s, \chi) = W_\chi(A_\chi)^{1/2-s}$$

$$\epsilon(s, \pi \times \chi) = W_{\pi\times\chi}(A_{\pi\times\chi})^{1/2-s}.$$

Proposition 1.12 *([B-R]) Let π be a irreducible cuspidal representation of $G_r(F_A)$ with the central character ω_π and conductor A_π. Let χ be a character of $F_A^\times/F^\times$, with the conductor A_χ. If their conductors are coprime, then*

1) $A_{\pi\times\chi} = A_\pi A_\chi^r$;

2) $W_{\pi\times\chi} = \omega_\pi(A_\chi)\chi(A_\pi)W_\pi W_\chi^r$.

(B) $\frac{F(x)-F(0)}{x}$ *is a bounded function.*

Define

$$\Phi(\gamma) = \int_{-\infty}^{\infty} F(x)e^{i\gamma x}dx.$$

Then

$$\begin{aligned}\sum_{\gamma}\Phi(\gamma) &= F(0)\log(Cq^r) - \delta_\chi(\Phi(\frac{i}{2}) + \Phi(-\frac{i}{2})) \\ &-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Gamma'}{\Gamma}(1/2+it,\pi\times\chi)\Phi(t)dt \\ &-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Gamma'}{\Gamma}(1/2-it,\tilde{\pi}\times\chi^{-1})\Phi(t)dt \\ &-\sum_{m=1}^{\infty}\frac{\Lambda(m)}{\sqrt{m}}(c_m(\pi)\chi(m)F(\log m) + c_m(\tilde{\pi})\chi(m)^{-1}F(-\log m)),\end{aligned}$$

where γ runs over all the points such that $L(1/2+i\gamma,\pi\times\chi)=0$, $0 \le Re\,(1/2+i\gamma) \le 3/2$, $C = A_\pi A_{\pi,1}A_{\tilde{\pi},2}$, and the δ_χ is 1 or 0 depending on whether $\pi \simeq \chi^{-1}$ or not.

Proof. Since π_∞ is tempered, by Corollary 1.2 we know that $\Gamma(s,\pi_\infty\times\chi_\infty)$ is a product of the gamma fuctions $\Gamma(s+a)$ or $\Gamma(\frac{s+a}{2})$ with the real part of a nonnegative. And by 3) in Proposition 1.9 $L(s,\pi\times\chi)$ is a meromorphic function (having at most one simple pole at s=1) and which is bounded at infinity in every bounded vertical strip . So $L_f(s,\pi\times\chi)$ satisfies all the conditions in the theorem of [Mes]. Hence by the theorem of [Mes] our result follows. □

Theorem 2.1 *Let π be an irreducible unitary automorphic cuspidal representation of $GL_r(Q_A)$ such that π_p is tempered for every place and the GRH (Generalized Riemann Hypothesis) is true for all $L_f(s,\pi\times\chi),\chi$ a Dirichlet character mod a prime q. Then*

$$\sum_{\chi mod q} r_\chi \le \frac{r}{2}\phi(q) + O\left(\frac{\phi(q)}{\log q}\right),$$

where the r_χ is the order of $L_f(s,\pi\times\chi)$ at s=1/2 and $\phi(q)$ is the Euler function.

Proof. We choose the function F in the proposition as the following

$$F(x) = \begin{cases} 2T - |x| & \text{if } |x| \leq 2T \\ 0 & \text{otherwise} \end{cases}$$

We can check that F(x) satifies the conditions in the proposition and that

$$\Phi(\gamma) = \left(\frac{2sin\,\gamma T}{\gamma}\right)^2.$$

Since the Generalized Riemann Hypothesis is true for $L_f(s, \pi \times \chi)$, we know that if $1/2 + i\gamma$ is a nontrivial zero of $L_f(s, \pi \times \chi)$ then γ is real, so the $\left(\frac{2\,sin\,\gamma T}{\gamma}\right)^2$ is a nonnegative number.

Let $T = \frac{1}{2}\log x$. Then by applying the Propsition 2.1 to $L(s, \pi \times \chi)$ and summing over all the characters mod q, we have that

$$\begin{aligned} \sum_{\chi mod q} r_\chi (\log x)^2 \leq & \sum_{\chi mod q} \log x \log(Cq^r) \\ & -\frac{1}{2\pi} \sum_{\chi mod q} \int_{-\infty}^{\infty} \frac{\Gamma'}{\Gamma}(1/2 + it, \pi \times \chi)\Phi(t)dt \\ & -\frac{1}{2\pi} \sum_{\chi mod q} \int_{-\infty}^{\infty} \frac{\Gamma'}{\Gamma}(1/2 - it, \tilde{\pi} \times \chi^{-1})\Phi(t)dt \\ & -\phi(q) \sum_{\substack{m \leq x \\ m \equiv 1\, mod q}} \frac{\Lambda(m)}{\sqrt{m}}(c_m(\pi) + c_m(\tilde{\pi}))\log(x/m). \end{aligned}$$

We first prove the following lemma to deal with the two integrals in the right side above.

Lemma 2.1 *For the function Φ in the proposition and F defined as before,*

$$\int_{-\infty}^{\infty} \frac{\Gamma'}{\Gamma}(1/2 + it, \pi \times \chi)\Phi(t)dt \ll T$$

$$\int_{-\infty}^{\infty} \frac{\Gamma'}{\Gamma}(1/2 - it, \tilde{\pi} \times \chi^{-1})\Phi(t)dt \ll T.$$

Proof. From the description of the local L-function at the infinite place, the gamma factor under our consideration is a product of $\Gamma(s+a)$ or $\Gamma\left(\frac{s+a}{2}\right)$, with $Re\, a \geq 0$.

The Stirling formula says that

$$\frac{\Gamma'}{\Gamma}(s) = \log(s) + O(|s|^{-1})$$

when $|s|$ tends to infinity, and $-\pi + \epsilon \leq arg\, s \leq \pi - \epsilon$. Therefore there is a U such that when $t > U$, $\frac{\Gamma'}{\Gamma}(1/2 + it + a) \leq C \log|t|$, for some constant C. We write the integal as the following

$$\int_0^{\infty} \frac{\Gamma'}{\Gamma}(1/2 + it + a) \left(\frac{2\, sin(tT)}{t}\right)^2 dt = \int_0^{1/T} + \int_{1/T}^{U} + \int_U^{\infty}.$$

Since $Re\, a \geq 0$, our gamma factor is bounded on the segment $0 \leq t \leq U$. Hence the first two integrals are bounded by T. The last one is O(1). So we proved the Lemma. □

By the lemma above, we see that the integral parts are $\ll \phi(q)T$. Now our assumption that every π_p is tempered implies that $|c_m(\pi) + c_m(\tilde{\pi})| \leq 2r$, so the sum part is

$$\leq 2r\phi(q) \sum_{m \leq x, \equiv 1 mod q} \frac{\Lambda(m)}{\sqrt{m}} \log(x/m).$$

It is clear that

$$\sum_{m \leq x, m \equiv 1 mod q} \frac{\Lambda(m)}{\sqrt{m}} = \sum_{p \leq x, p \equiv 1 mod q} \frac{\log p}{p^{1/2}} + \sum_{p^2 \leq x, p^2 \equiv 1 mod q} \frac{\log p}{p} + O(1),$$

and

$$\sum_{p^2 \leq x, p^2 \equiv 1 mod q} \frac{\log p}{p} \ll \frac{2 \log x}{\phi(q)} + O(1).$$

The number of the primes which are less than x and congruent to 1 modq is less than $\frac{3x}{\phi(q)\log(x/q)}$, for $q \leq x$, see [H-R]. So we see

$$\sum_{p \leq x, p \equiv 1 mod q} \frac{\log p}{p^{1/2}} \ll \frac{x^{1/2} \log x}{\phi(q) \log(x/q)},$$

Therefore, by partial summation,

$$2r\phi(q) \sum_{m\leq x, m\equiv 1 \bmod q} \frac{\Lambda(m)}{\sqrt{m}} \ll \frac{(\log x)^2\sqrt{x}}{\log(x/q)}.$$

Hence when we let $x = \phi^2(q)$,

$$\sum_{\chi \bmod q} r_\chi \leq \frac{r}{2}\phi(q) + O\left(\frac{\phi(q)}{\log q}\right).$$

So, we finished the proof of the theorem. □

Corollary 2.1 *Assume the conditions in the theorem above. Then the average order of $L_f(1/2, \pi \times \chi)$ is not bigger than $r/2$.*

Before we state the following theorem we define

$$\chi_D(\cdot) = \left(\frac{D}{\cdot}\right),$$

where the right side of above is the Kronecker symbol. If D is a fundamental discriminant, χ_D is the real character mod $|D|$.

Theorem 2.2 *Let π be an irreducible unitary automorphic cuspidal representation of $G_r(Q_A)$ such that every π_p is tempered. Suppose that GRH is true for all $L(s, \pi \times \chi_p)$ and Dirichlet L-functions, where p is a prime. Then for any $\epsilon > 0$,*

$$\sum_{\substack{p\leq Y \\ p\equiv 1(4A)}} r_p \leq (2 + 3\epsilon + \sum_q \frac{1}{q^2}) r\ \pi(Y; 1, 4A) + O(\frac{Y^{1/2+(1/2+\epsilon)a}}{\log Y})$$

where the r_p is the order of $L(1/2, \pi \times \chi_p), a = \frac{1}{1+3\epsilon}$, $\pi(Y; 1, 4A)$ is the number of primes not bigger than Y congruent to 1 mod 4A, and the sum $\sum_q$ is over all primes q.

Corollary 2.2 *Let π be the same as in the theorem, and all conditions are the same as in the theorem. Then the average order of $L_f(1/2, \pi \times \chi_p)$ is not bigger than* $(2+\epsilon+\sum_q \frac{1}{q^2})r$.

Proof of the Theorem 2.2 . Let F(x), $\Phi(\gamma)$ be the same as in the Theorem 1, and also $T = \frac{1}{2}\log x$. By applying the explicit formula on $L(s, \pi \times \chi_p)$ we get

$$\begin{aligned}\sum_{\substack{p \le Y \\ p \equiv 1(4A)}} r_p (\log x)^2 \le & \sum_{\substack{p \le Y \\ p \equiv 1(4A)}} \log x \log(Cp^r) \\ & - \sum_{\substack{p \le Y \\ p \equiv 1(4A)}} \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Gamma'}{\Gamma}(1/2 + it, \pi_\infty)\phi(it)dt \\ & - \sum_{\substack{p \le Y \\ p \equiv 1(4A)}} \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Gamma'}{\Gamma}(1/2 - it, \tilde{\pi}_\infty)\phi(it)dt \\ & - \sum_{m \le x} \frac{\Lambda(m)}{\sqrt{m}}(c_m(\pi) + c_m(\tilde{\pi}))\log(x/m) \sum_{\substack{p \le Y \\ p \equiv 1(4A)}} \chi_p(m),\end{aligned}$$

where the constant C depends only on π.

From the Lemma in the proof of the previous theorem, the two sums involving integrals are

$$\ll \log x \; \pi(Y; 1, 4A).$$

The first sum in the right side of the inequality above is

$$r \log x \sum_{\substack{p \le Y \\ p \equiv 1(4A)}} \log p \le r \; \log x \; \log Y \; \pi(Y; 1, 4A).$$

For the last sum in the inequality, we have to estimate the sum

$$\sum_{\substack{p \le Y \\ p \equiv 1(4A)}} \chi_p(q^k), \text{q is a prime.}$$

For k even,

$$\sum_{\substack{p \le Y \\ p \equiv 1(4A)}} \chi_p(q^k) = \pi(Y;1,4A).$$

For k odd,

$$\sum_{\substack{p \le Y \\ p \equiv 1(4A)}} \chi_p(q^k) = \sum_{\substack{p \le Y \\ p \equiv 1(4A)}} \chi_p(q) \ll Y^{1/2}(\log(qY))^2.$$

In the last step above we used the fact that the GRH for nonprinciple character χ mod q implies

$$\sum_{p^k \le x} \chi(p^k) \log p \ll x^{1/2}(\log(qx))^2,$$

for this see [Dav].

Now we break the the sum $\sum_{m \le x}$ into two parts according to whether k is even or odd and denote them by S_e, S_o respectively. We see that

$$S_e \le 2r \log x\, \pi(Y;1,4A) \sum_{q^{2m} \le x} \frac{\log q}{q^m}.$$

$$\sum_{q^{2m} \le x} \frac{\log q}{q^m} = \sum_{q \le x^{1/2}} \frac{\log q}{q} + \sum_{q \le x^{1/4}} \frac{\log q}{q^2} + \cdots.$$

Since

$$\sum_{q \le x^{1/4}} \frac{\log q}{q^2} + \cdots \le \sum_q \sum_{2 \le m \le \frac{\log x}{2 \log q}} \frac{\log q}{q^m} \le \frac{1}{2} \log x \sum_q \frac{1}{q^2},$$

we get

$$\begin{aligned} \sum_{q^{2m} \le x} \frac{\log q}{q^m} &\le \sum_{q \le x^{1/2}} \frac{\log q}{q} + \frac{1}{2} \log x \sum_q \frac{1}{q^2} \\ &= \frac{1}{2} \log x \,(1 + \sum_q \frac{1}{q^2}) + O(1). \end{aligned}$$

Therefore

$$S_e \le 2r \log x\, \pi(Y;1,4A) \frac{1}{2} \log x (1 + \sum_q \frac{1}{q^2}) = (\log x)^2 \,(1 + \sum_q \frac{1}{q^2})\, r\, \pi(Y;1,4A).$$

For S_o. By a similar argument for S_e, we have

$$\begin{aligned} S_o &\ll \log x\,(\log(xY))^2 Y^{1/2}(\sum_{q^k \le x} \frac{\log q}{q^{k/2}} \\ &\ll \log x(\log(xY))^2 Y^{1/2} x^{1/2} \log x \\ &= Y^{1/2} x^{1/2} (\log(xY))^2 \log x. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{\substack{p \le Y \\ p \equiv 1(4A)}} r_p &\le \frac{\log Y}{\log x} r\,\pi(Y;1,4A) + (1 + \sum_q \frac{1}{q^2})\, r\,\pi(Y;1,4A) \\ &+ O(Y^{1/2} x^{1/2} \frac{(\log(xY))^2}{\log x}). \end{aligned}$$

Let $x = Y^a$. To get $1/2 + a(1/2) < 1$, we can chose $a = \frac{1}{1+\epsilon}$. Then

$$\sum_{\substack{p \le Y \\ p \equiv 1(4A)}} r_p \le (2 + \epsilon + \sum_q \frac{1}{q^2})\, r\,\pi(Y;1,4A) + O(Y^{1/2+a/2} \log Y),$$

and the order of the error above is lower than the main term. □

2.2 A Mean Value at the Critical Points

Let $\tau(m)$ be the number of positive divisors of m, $\mu(m)$ the Mobius fuction, $\phi(m)$ the Euler function, $\chi_o^{(m)}$ the principal Dirichlet character mod m. We need the following lemmas.

Lemma 2.2

$$\sum_{\substack{0 < D \le Y \\ (D,m) = 1, \\ D squarefree}} 1 = \frac{1}{\zeta(2)} \prod_{p|m} (1 + p^{-1})^{-1} Y + O(Y^{\frac{1}{2}} \tau(m)).$$

Proof.

$$\sum_{\substack{0<D\le Y\\(D,m)=1\\ \text{D square free}}} 1 = \sum_{\substack{0<D\le Y\\(D,m)=1}} \sum_{d^2|D} \mu(d)$$

$$= \sum_{0<d\le Y^{1/2}} \mu(d) \sum_{0<D'\le Y/d^2} \chi_0^{(m)}(d^2D')$$

$$= \sum_{0<d\le Y^{1/2}} \mu(d)\chi_0^{(m)}(d) \sum_{0<D'\le Y/d^2} \chi_0^{(m)}(D')$$

$$= \sum_{0<d\le Y^{1/2}} \mu(d)\chi_0^{(m)}(d) \sum_{0<D'\le Y/d^2} \sum_{r|(D',m)} \mu(r)$$

$$= \sum_{0<d\le Y^{1/2}} \mu(d)\chi_0^{(m)}(d) \sum_{r|m} \mu(r) \sum_{0<rD''\le Y/d^2} 1$$

$$= \sum_{0<d\le Y^{1/2}} \mu(d)\chi_0^{(m)}(d) \sum_{r|m} \mu(r)(\frac{Y}{rd^2} + O(1))$$

$$= \sum_{0<d\le Y^{1/2},(d,m)=1)} \mu(d) \left(\frac{\phi(m)}{m} Y/d^2 + \tau(m)\right).$$

Since

$$\frac{1}{\zeta(2)} = \sum_d \frac{\mu(d)}{d^2}$$

and

$$\frac{\phi(m)}{m} = \prod_{p|m}(1-p^{-1}),$$

the most right side of the equality above is bounded by

$$Y\frac{\phi(m)}{m}\left(\frac{1}{\zeta(2)}\frac{m}{\phi(m)}\prod_{p|m}(1+p^{-1})^{-1} + \sum_{\substack{n>Y^{1/2}\\(n,m)=1}} \frac{1}{n^2}\right) + O(\tau(m)Y^{1/2})$$

which is

$$Y\frac{1}{\zeta(2)}\prod_{p|m}(1+p^{-1})^{-1}+O(Y^{1/2}\tau(m)).$$

□

Lemma 2.3 *For nonprincipal character ψ mod m such that (m, 4A)=1,*

$$\sum_{\substack{0\leq D\leq Y\\ D\equiv 1(4A)\\ \text{D square free}\\ (D,m)=1}}\psi(D)\ll O(Y^{\frac{1}{2}}\log Y\, m^{3/16+\epsilon}).$$

Proof.

$$\sum_{\substack{0\leq D\leq Y\\ D\equiv 1(4A)\\ \text{D square free}}}\psi(D)=\frac{1}{\phi(4A)}\sum_{\chi mod 4A}\sum_{\substack{0<D\leq Y\\ \text{D square free}}}\psi(D)\chi(D).$$

For every χ mod 4A,

$$\begin{aligned}\sum_{\substack{0<D\leq Y\\ D\ square\ free}}\psi(D)\chi(D) &= \sum_{0<D\leq Y}\sum_{d^2|D}\psi(D)\chi(D)\mu(d^2)\\ &= \sum_{0<d\leq Y^{1/2}}\mu(d)\psi(d^2)\chi(d^2)\sum_{0<D'\leq Y/d^2}\psi(D')\chi(D').\end{aligned}$$

Since $(m,4A)=1$,

$$(Z/m4A)^{\times}=(Z/mZ)^{\times}\times(Z/4AZ)^{\times}.$$

So $\psi\cdot\chi$ is a Dirichlet character mod m4A. Since $\psi\cdot\chi$ is the principal character if and only if ψ and χ are the principal characters of $(Z/mZ)^{\times}$ and $(Z/4AZ)^{\times}$respectively, $\psi\cdot\chi$ is not the principal character mod m4A. An estimate due to Burgess [Bur] says that for any nonprincipal character χ mod m, then for any $\epsilon>0$

$$\Big|\sum_{X\leq n\leq X+H}\chi(n)\Big|\ll H^{1/2}m^{3/16+\epsilon}.$$

Applying such an estimate to the character $\psi \cdot \chi$ mod m4A, we see

$$\sum_{\substack{0<D\leq Y \\ D \text{ square free}}} \psi(D)\chi(D) \ll \sum_{0<d\leq Y^{1/2}} \left(\frac{Y}{d^2}\right)^{1/2} m^{3/16+\epsilon} \ll Y^{1/2}(\log Y)\, m^{3/16+\epsilon}.$$

□

Lemma 2.4 *If* $(m, 4A) = 1$, *then*

$$\sum_{\substack{0\leq D\leq Y \\ D\equiv 1(4A) \\ \text{D square free} \\ (D,m)=1}} 1 = \frac{1}{\phi(4A)}\frac{1}{\zeta(2)}\prod_{p|m}(1+p^{-1})^{-1}Y + O(Y^{1/2}\log Y m^{3/16+\epsilon}).$$

Proof.

$$\sum_{\substack{0\leq D\leq Y \\ D\equiv 1(4A) \\ \text{D square free} \\ (D,m)=1}} 1 = \frac{1}{\phi(4A)} \sum_{\chi \bmod 4A} \sum_{\substack{0<D\leq Y \\ (D,m)=1 \\ \text{D square free}}} \chi(D).$$

For the principal character mod 4A, the contribution is the result in Lemma 2.2. For every nonprincipal character χ mod 4A, the contribution is

$$\begin{aligned}
\sum_{0<D\leq Y}\sum_{d^2|D} \chi(D)\chi_0^{(m)}(D)\mu(d) &= \sum_{0<d\leq Y^{1/2}} \mu(d)\chi(d^2)\chi_0^{(m)}(d^2) \sum_{0<D'\leq Y/d^2} \chi(D')\chi_0^{(m)}(D') \\
&\ll \sum_{0<d\leq Y^{1/2}} (Y/d^2)^{1/2} m^{3/16+\epsilon} \\
&\ll Y^{1/2}(\log Y)\, m^{3/16+\epsilon}
\end{aligned}$$

Therefore we proved the lemma. □

Lemma 2.5

$$\sum_{\substack{0 \le D \le Y \\ D \equiv 1(4A) \\ \text{D square free} \\ (D,m)=1}} D^s = \frac{1}{\phi(4A)} \frac{1}{\zeta(2)} \prod_{p|4Am} (1+p^{-1})^{-1} \frac{Y^{1+s}}{1+s} + O(Y^{1/2+s} \log Y m^{3/16+\epsilon}).$$

Proof. We are going to use partial summation.

$$\sum_{\substack{0 \le D \le Y \\ D \equiv 1(4A) \\ \text{D square free} \\ (D,m)=1}} D^s = D^s \sum_{\substack{0 \le D \le Y \\ D \equiv 1(4A) \\ \text{D square free} \\ (D,m)=1}} 1 - \int_1^Y \sum_{\substack{0 \le D \le t \\ D \equiv 1(4A) \\ \text{D square free} \\ (D,m)=1}} 1 \ (s-1) t^{s-1} dt.$$

By Lemma 2.4, we know this is

$$Y^{s+1} \frac{1}{\phi(4A)} \frac{1}{\zeta(2)} \prod_{p|m} (1+p^{-1})^{-1} - \frac{1}{\phi(4A)} \frac{1}{\zeta(2)} \prod_{p|m} (1+p^{-1})^{-1} \int_1^Y (s-1) t^s \, dt$$

$$+ O(Y^{1/2+s} \log Y m^{3/16+\epsilon})$$

$$= Y^{s+1} \frac{1}{\phi(4A)} \frac{1}{\zeta(2)} \prod_{p|m} (1+p^{-1})^{-1} \frac{1}{1+s} + O(Y^{1/2+s} \log Y m^{3/16+\epsilon}).$$

□

Remark. If we assume the GRH on all Dirichlet L-functions, we can replace the 3/16 in the three lemmas by 0.

Theorem 2.3 *Let π be an irreducible unitary automorphic cuspidal representation of $GL_r(Q_A)$, with the trivial central character and the conductor A. Assume that*

$$\sum_{m=1}^{\infty} \frac{a_{m^2}(\pi)}{m^s} \text{ converges absolutely in } Re\, s > 1$$

$$\sum_{m=1}^{\infty} \frac{a_{m^2}(\tilde{\pi})}{m^s} \text{ converges absolutely in } Re\, s > 1.$$

Then

i) if we assume GRH for every Dirichlet character, then for every w with $max(1/2,\ 1-\frac{1}{r}) < Re\ w < 1$ *we have*

$$\sum_{\substack{0 \le D \le Y \\ D \equiv 1(4A) \\ \text{D square free}}} L_f(w, \pi \times \chi_D) = CY + o(Y);$$

ii) for $r \le 2$, *then for* $\theta(r) < Re\ w < 1$ *the asymptotic formula in i) is also true without GRH, where* $\theta(1) = 1/2\ \ \theta(2) = 11/16$. *Here*

$$C = \frac{1}{\phi(4A)} \frac{1}{\zeta(2)} \sum_{m_1, m_2} a_{m_1 m_2^2}(\pi)(m_1 m_2^2)^{-w} \prod_{p | m_2} (1 + p^{-1})^{-1}.$$

Proof. For D satisfying $D \equiv 1\ mod\ 4A$, and D square free, we have $\chi_D(-1) = 1$ and the conductor of χ is D. By Proposition 1.12 and Proposition 1.11,

1) $A_{\pi \times \chi_D} = A_\pi D^r$,

2) $W_{\pi \times \chi_D} = W_\pi$,

3)

$$L(s, \pi_\infty \times \chi_{D,\infty}) = L(s, \pi_\infty),\ \ L(s, \tilde{\pi}_\infty \times \chi_{D,\infty}) = L(s, \tilde{\pi}_\infty).$$

We can assume w is real, since for non-real critical points, the argument in the proof is the same, except replacing w by $Re\ w$ in some estimation.

For given w, $1/2 < w < 1$ we consider the integral

$$\frac{1}{2\pi i} \int_{(\gamma)} L_f(w + s, \pi \times \chi_D) X^s \Gamma(s) ds, \ \ \gamma > 1 - w.$$

On the one hand, since the Euler product of $L(s, \pi \times \chi_D)$ converges absolutely in $Re\, s > 1$, the integral is

$$\sum_{m=1}^{\infty} \frac{a_m(\pi)}{m^w} \chi_D(m) exp(-m/X).$$

We recall that

$$L(s,\pi_\infty) = \prod G_R(s+a)\prod G_C(s+b)$$

$$= A_{\pi,1}A_{\pi,2}^{s}\prod\Gamma\left(\frac{s+a}{2}\right)\prod\Gamma(s+b).$$

Since

$$\Gamma(s)\sim exp(-\pi|t|/2)|t|^{\sigma-1/2}(2\pi)^{1/2},$$

the $\Gamma(-s+a)/\Gamma(s+a')$, a, a' any complex numbers, cancels the exponential factors of $\Gamma(-s+a)$ and $\Gamma(s+a')$, so it is asymptotic with a polynomial in any bounded vertical strip when $|t|$ goes to infinity. Hence

$$\frac{L(1-s,\tilde{\pi}_\infty)}{L(s,\pi_\infty)}$$

is with polynomial growth in every bounded vertical strip. Therefore, $L_f(s,\pi\times\chi_D)$ is, at most, with a polynomial growth in every bounded vertical strip, by the Phragmen-Lindelöf theorem.

Therefore on the other hand, since $L_f(s,\pi\times\chi_D)$ is an entire function and the gamma function is exponentially decreasing in every bounded vertical strip, we can move the line of the integral to $Re\, s=-\eta$, $0<\eta<1$ and pick up the residue at zero. By applying the functional equation we see that the integral under our consideration is

$$L_f(w,\pi\times\chi_D)-\frac{1}{2\pi i}\int_{(-\eta)}(AD^r)^{1/2-w-s}W_\pi\frac{L(1-w-s,\tilde{\pi}_\infty)}{L(w+s,\pi_\infty)}L_f(1-w-s,\tilde{\pi}\times\tilde{\chi}_D)X^s\Gamma(s)ds.$$

Then we let $\eta>w$ so that $1-w-\eta>1$. We have that

$$\sum_{\substack{0\le D\le Y\\ D\equiv 1(4A)\\ \text{D square free}}} L_f(w,\pi\times\chi_D)=\sum_{1}^{\infty}\frac{a_m(\pi)}{m^w}exp(-m/x)\sum_{\substack{0\le D\le Y\\ D\equiv 1(4A)\\ \text{D square free}}}\chi_D(m)$$

$$+\frac{1}{2\pi i}\int_{(-\eta),\eta>w} A^{1/2-w-s} W_\pi \frac{L(1-w-s,\tilde{\pi}_\infty)}{L(w+s,\pi_\infty)}$$
$$\sum_1^\infty \frac{a_m(\tilde{\pi})}{m^{1-w-s}} \sum_{\substack{0\le D\le Y \\ D\equiv 1(4A) \\ D \text{ square free}}} D^{r(1/2-w-s)}\chi_D(m)X^s\Gamma(s)ds.$$

We write $m = m_1 m_2$ such that every prime divisor of m_1 is a prime divisor of 4A and $(m_2, 4A) = 1$. So $\chi_D(m) = \chi_D(m_1 m_2) = \chi_D(m_2)$. We break the sums above into two parts, the square part and the nonsquare part. The square part is the sum over m such that m_2 is a square, the nonsquare part is the sum over the remaining m's.

Consider the nonsquare part of the integral first. We break the sum under the integration at the parameter U. By Proposition 1.10 and Lemma 2.3, the integral part corresponding to the sum over $m \le U$ is

$$\ll \int_{(-\eta)} \frac{L(1-w-s,\tilde{\pi}_\infty)}{L(w+s,\pi_\infty)} \sum_{m\le U} \frac{a_m(\tilde{\pi})}{m^{1-w-s}} Y^{r(1/2-w-s))+1/2} m^{3/16+\epsilon} \log Y\, X^s\Gamma(s)ds$$

$$\ll Y^{r(1/2-w)+1/2}\log Y U^{w+3/16+\epsilon}\int_{(-\eta)} \frac{L(1-w-s,\tilde{\pi}_\infty)}{L(w+s,\pi_\infty)} Y^{-rs}U^s X^s\Gamma(s)ds$$

$$\ll Y^{r(1/2-w)+1/2} Y^{r\eta}\log Y U^{w+3/16+\epsilon} U^{-\eta} X^{-\eta}\int_{(-\eta)} \left|\frac{L(1-w-s,\tilde{\pi}_\infty)}{L(w+s,\pi_\infty)}\right| |\Gamma(s)|\, ds$$

$$\ll Y^{r(1/2-w)+1/2}\log Y U^{w+3/16+\epsilon} Y^{r\eta} U^{-\eta} X^{-\eta}.$$

For the sum with $m > U$, since $\eta > w$, the local L- function $L(1-w-s,\tilde{\pi}_\infty)$ is holomorphic in the left half plane $Re\, s \le -\eta$, we can move the line of the intergal to $Re\, s = -\eta_2$, with $2 > \eta_2 > w + 3/16 + \epsilon$, with $w > 13/16$ (if we assume GRH, we do not need the restriction "$w > 13/16$".) and pick up the residue at $s = -1$ which comes from the pole of $\Gamma(s)$. The residue at $s = -1$ is

$$Res \ll \sum_{m>U} \frac{|a_m(\tilde{\pi})|}{m^{2-w}} Y^{r(1/2-w+1)} Y^{1/2} m^{3/16+\epsilon} X^{-1}\log Y$$

$$\ll Y^{r(1/2-w+1)+1/2}X^{-1}U^{-2+w+3/16+\epsilon+1}\log Y$$

$$= Y^{r(1/2-w+1)+1/2}X^{-1}U^{-1+w+3/16+\epsilon}\log Y.$$

For the integral along the line $Re\, s = -\eta_2$, we also have

$$\sum_{m>U}\frac{a_m(\tilde{\pi})}{m^{1-w-s}}\sum_{\substack{0\le D\le Y\\ D\equiv 1(4A)\\ \text{D square free}}} D^{r(1/2-w-s)}\chi_D(m) \ll Y^{r(1/2-w)+1/2}(\log Y)\, U^{w+3/16+\epsilon}Y^{-rs}U^{s}.$$

Therefore we get the same estimate as that of the part corresponding to $m \le U$.

Let $U = \frac{Y^r}{X}$, we see that the contribution of the integral of nonsquare part is

$$I_n \ll Y^{1/2+r/2+r\,3/16+r\epsilon}X^{-w-3/16-\epsilon}\log Y.$$

Now consider the nonsquare part S_n of the sum part. It is

$$S_n \ll \sum_{m_2\ nonsquare}|a_m(\pi)|m^{-w}exp(-m/X)Y^{1/2}\log Y m_2^{3/16+\epsilon}$$

$$\ll Y^{1/2}\log Y\sum_{m\le X}|a_m(\pi)|m^{-w+3/16+\epsilon}$$

$$\ll Y^{1/2}(\log Y)\, X^{1-w+3/16+\epsilon}.$$

Now we consider the square parts. Firstly we consider the intergal part which we denote it by I_s. We write $I_s = I_{sm} + I_{se}$ where the I_{sm}, I_{se} come from the main term and error term in Lemma 2.5 respectively. Since $L(s,\pi_\infty)$ is holomorphic in $Re\, s > 1/2$, see [B-R], we can move the line of the integral to the line $Re\, s =$

$-\gamma$, $\gamma = w - 1/2 + \epsilon$. Then we see that

$$\begin{aligned} I_{sm} &= \frac{1}{\phi(4A)}\frac{1}{\zeta(2)}Y^{1+r/2-rw}W_\pi\frac{1}{2\pi i}\int_{(-\gamma)}\frac{L(1-w-s,\tilde{\pi}_\infty)}{L(w+s,\pi_\infty)}\sum_m\frac{a_{m_1m_2^2}}{(m_1m_2^2)^{1-w-s}} \\ &\quad \prod_{p|m_2}(1+p^{-1})^{-1}\frac{1}{1+r(1/2-w-s)}\left(\frac{X}{Y^r}\right)^s\Gamma(s)ds \\ &\ll Y^{1+r/2-rw}Y^{r\gamma}X^{-\gamma} \\ &\ll Y^{1+r\epsilon}X^{-(w-1/2)-\epsilon}. \end{aligned}$$

I_{se} has the same bound as I_n has.

So,we have

$$I = I_{sm} + I_{se} + I_n \ll Y^{1+r\epsilon}X^{-(w-1/2)-\epsilon} + Y^{1/2+r/2+r3/16+r\epsilon}X^{-w-3/16-\epsilon}\log Y.$$

Finally we consider the square part of the sum part denoted by S_s. We write $S_s = S_m + S_e$, where the S_m and S_e come from the main term and the error term respectively in Lemma 2.5.

$$\begin{aligned} S_e &\ll Y^{1/2}\log Y\sum_{m\le X}|a_m(\pi)|m^{-w}m^{3/16+\epsilon} \\ &\ll Y^{1/2}(\log Y)\,X^{1-w+3/16+\epsilon}, \end{aligned}$$

where we used the estimate result of Proposition 1.10 in section 1.

Our main term in this theorem will come from the S_m.

$$\begin{aligned} S_m &= \frac{1}{\phi(4A)}\frac{1}{\zeta(2)}Y\sum_{m_1,m_2}a_{m_1m_2^2}(\pi)(m_1m_2^2)^{-w}\prod_{p|m_2}(1+p^{-1})^{-1}exp(-m_1m_2^2/X) \\ &= \frac{1}{\phi(4A)}\frac{1}{\zeta(2)}Y\frac{1}{2\pi i}\int_{(\eta)}\sum_{m_1,m_2}a_{m_1m_2^2}(\pi)(m_1m_2^2)^{-w-s}\prod_{p|m_2}(1+p^{-1})^{-1}X^s\Gamma(s)ds. \end{aligned}$$

for any $\eta > 0$, since $\sum_m \frac{a_{m^2}(\pi)}{m^s}$ converges absolutely in $Re\, s > 1$.

Since the sum under the integral absolutely converges in $Re\, s > -(w - 1/2)$, the sum is a holomorphic function there and $\Gamma(s)$ is exponentially decreasing in the strip $-\eta' \leq Re\, s \leq \eta$, $0 < \eta' < w - 1/2$. So we can move the line of the integral to the line $Re\, s = -\eta'$ and pick up the residue at zero. Therefore

$$\begin{aligned} S_m &= \frac{1}{\phi(4A)}\frac{1}{\zeta(2)}Y\sum_{m_1,m_2} a_{m_1m_2^2}(\pi)(m_1m_2^2)^{-w}\prod_{p|m_2}(1+p^{-1})^{-1} \\ &+\frac{1}{2\pi i}\int_{(-\eta')}\sum_{m_1,m_2} a_{m_1m_2^2}(\pi)(m_1m_2^2)^{-w-s}\prod_{p|m_2}(1+p^{-1})^{-1}X^s\Gamma(s)ds \\ &= CY + O(YX^{-\eta'}). \end{aligned}$$

Therefore

$$\sum_{\substack{0\leq D\leq Y \\ D\equiv 1(4A) \\ \text{D square free}}} L(w,\pi\times\chi_D) = CY + Error$$

where

$$C = \frac{1}{\phi(4A)}\frac{1}{\zeta(2)}\sum_{m_1,m_2} a_{m_1m_2^2}(\pi)(m_1m_2^2)^{-w}\prod_{p|m_2}(1+p^{-1})^{-1}$$

$$Error = Y^{r/2+r3/16+1/2+r\epsilon}X^{-w-3/16-\epsilon} + Y^{1/2}X^{1-w+3/16+\epsilon} + Y^{1+r\epsilon}X^{-(w-1/2)-\epsilon}.$$

Now we analyse the result of the computation above. Our task is to find a positive number α such that if $X = Y^{\alpha}$ then the three terms in Error are lower than the main term Y simultaneously. The last term in the Error has the order lower than Y, since $w > 1/2$. So what we shall be concerned with is only the first two terms in the Error. Firstly we let $X = Y^{\alpha}$; $\alpha > 0$, α to be determined. Then the α must satisfy

$$K_1(\epsilon) < \alpha < K_2(\epsilon),$$

where

$$K_1(\epsilon) = \frac{r/2 + r3/16 - 1/2 + r\epsilon}{w + 3/16 + \epsilon}$$
$$K_2(\epsilon) = \frac{1}{2(1 - w + 3/16 + \epsilon)}.$$

In the above the $K_1(\epsilon)$ comes from the first term in the Error, the $K_2(\epsilon)$ comes from the second term in the Error. We will figure out in which range of w the inequality above has a solution for α.

Consider the inequality

$$K_1(\epsilon) < K_2(\epsilon).$$

Then we get

$$K(\epsilon) < w \quad \text{for small } \epsilon > 0,$$

where

$$K(\epsilon) = \frac{(1 + 3/16 + \epsilon)(r + 2r\,3/16 - 1 + 2r\epsilon) - 3/16 - \epsilon}{r + 2r\,3/16 + 2r\epsilon}.$$

If $\lim_{\epsilon \to 0} K(\epsilon) < 1$ then for all w satisfying $\lim_{\epsilon \to 0} K(\epsilon) < w < 1$, we can find a ϵ_0 depending on w such that $K(\epsilon_0) < 1$, then with this ϵ_0 and w, $K_1(\epsilon_0) < K_2(\epsilon_0)$. Hence we can choose any α satisfying $K_1(\epsilon_0) < \alpha < K_2(\epsilon_0)$. Therefore we get that the Error is o(Y), whenever $w > \lim_{\epsilon \to 0} K(\epsilon)$.

Now

$$\lim_{\epsilon \to 0} K(\epsilon) = \frac{(1 + 3/16)(r + 2r\,3/16 - 1) - 3/16}{r + 2r\,3/16}.$$

$\lim_{\epsilon \to 0} K(\epsilon) < 1$, if and only if $r < \frac{1+3/16}{3/16(1+23/16)} = 4.6060....$

If we assume the GRH for all the Dirichlet characters, we can replace 3/16 by 0, so

$$\lim_{\epsilon \to 0} K(\epsilon) = \frac{r-1}{r} < 1$$

always holds.

Summarize that as the following:

i) If we assume GRH, then whenever $max(1/2,\ 1-\frac{1}{r}) < w$ the Error is o(Y);

ii) For $r \leq 2$, whenever $\theta(r) < w < 13/16$, the Error is o(Y) without GRH, where $\theta(1) = 1/2,\ \ \theta(2) = 11/16$. □

For r=2, since the lifting result of Gelbart and Jacquect [Gel-Jac] says that the symmetric square L-functions of a cuspidal automorphic representation of $GL_2(F_A)$ is the L-function of an automorphic representation of $GL_3(F_A)$, we can prove the asymptotic constant C in the Theorem 2.3 is not zero. Therefore we have the following Corollary.

Corollary 2.3 *Let π be an irreducible unitary automorphic cuspidal representation of $GL_2(Q_A)$ with the trivial central character. Assume GRH for the L-functions associated to Dirichlet characters. Then for every w with $1/2 < Re\ w < 1$, there are infinitely many quadratic characters χ_D such that $L(s, \pi \times \chi_D)$ does not vanish at w.*

Let us firstly prove the following lemma.

Lemma 2.6 *Let π be an irreducible cuspidal representation of $GL_2(Q_A)$, with the central character $\omega = \otimes\omega_p$. If π is unramified at p, then*

$$L(s, Sym^2\pi_p) = g(s, \pi_p)L(2s, \omega_p^2)$$

where

$$g(s, \pi_p) = \sum_{k\geq 0} a_{p^{2k}}(\pi)p^{-ks}.$$

Proof. Suppose that

$$L(s, \pi_p) = (1 - b_1p^{-s})^{-1}(1 - b_2p^{-s})^{-1} = \sum_{m=0}^{\infty} a_{p^k}p^{-ks},$$

then

$$a_{p^k} = \sum_{k_1+k_2=k} b_1^{k_1} b_2^{k_2}.$$

By definition the local symmetric square L-function of π

$$L(s, Sym^2\pi_p) = (1 - b_1^2 p^{-s})^{-1}(1 - b_2^2 p^{-s})^{-1}(1 - b_1 b_2 p^{-s})^{-1}.$$

We can write $L(s, Sym^2\pi_p)$ as the following.

$$\begin{aligned} L(s, Sym^2\pi_p) &= (1 - b_1^2 p^{-s})^{-1}(1 - b_2^2 p^{-s})^{-1}(1 + b_1 b_2 p^{-s})(1 - b_1^2 b_2^2 p^{-2s})^{-1} \\ &= G(s, \pi_p) L(2s, \omega_p^2), \end{aligned}$$

where the $G(s, \pi_p)$ is the product of the first three factors above. Now we need to compute the $G(s, \pi_p)$.

$$\begin{aligned} G(s, \pi_p) &= \left(\sum_{k\geq 0} \sum_{k_1+k_2=k} b_1^{2k_1} b_2^{2k_2} p^{-ks} \right) (1 + b_1 b_2 p^{-s}) \\ &= \sum_{k\geq 0} \sum_{k_1+k_2=k} b_1^{2k_1} b_2^{2k_2} p^{-ks} + \sum_{k\geq 0} \sum_{k_1+k_2=k} b_1^{2k_1+1} b_2^{2k_2+1} p^{-(k+1)s} \\ &= 1 + \sum_{k\geq 1} \left(\sum_{\substack{k_1+k_2=k \\ k_1,k_2\geq 0}} b_1^{2k_1} b_2^{2k_2} + \sum_{\substack{k_1+k_2=k-1 \\ k_1,k_2\geq 0}} b_1^{2k_1+1} b_2^{2k_2+1} \right) p^{-ks} \\ &= 1 + \sum_{k\geq 1} \sum_{\substack{r_1+r_2=2k \\ r_1,r_2\geq 0}} b_1^{r_1} b_2^{r_2}\, p^{-ks} \\ &= \sum_{k\geq 0} \sum_{\substack{r_1+r_2=2k \\ r_1,r_2\geq 0}} b_1^{r_1} b_2^{r_2}\, p^{-ks} = \sum_{k\geq 0} a_{p^{2k}} p^{-ks} \\ &= g(s, \pi_p). \end{aligned}$$

We proved the lemma. $\square$

Proof of the Corollary 2.3. Let

$$L(s)=\sum_{m_1,m_2} a_{m_1m_2^2}(\pi)(m_1m_2^2)^{-s}\prod_{p|m_2}(1+p^{-1})^{-1}.$$

We can write L(s) as an infinite product of the local Euler factors

$$L(s)=\prod_p L_p(s).$$

Firstly we prove that $L_p(s)$ does not vanish for $s \geq 1/2$.

If p does not divide 4A, then by the Lemma above we have

$$\begin{aligned} L_p(s) &= 1+\frac{p}{p+1}\left(\sum_{k\geq 1} a_{p^{2k}}p^{-2ks}+1-1\right) \\ &= 1+\frac{p}{p+1}(g(2s)-1), \end{aligned}$$

where the $g(s)$ is the same as in the lemma. Now $L_p(s)=0$ if and only if

$$g(2s)=-\frac{1}{p}.$$

By the lemma, we write

$$\begin{aligned} g(2s) &= L(2s, Sym^2\pi_p)\ L(4s,\omega_{\pi_p}^2)^{-1} \\ &= (1-b_{p,1}^2p^{-2s})^{-1}(1-b_{p,2}^2p^{-2s})^{-1}(1+p^{-2s}). \end{aligned}$$

In the above, we used the hypothesis that ω_π is the trivial representation.

So $g(2s)=-\frac{1}{p}$ if and only if

$$p^{2s}(\ p(p^{2s}+1)+p^{2s}-(b_{p,1}^2+b_{p,2}^2)\ )=-1,$$

hence

$$|p(p^{2s}+1)+p^{2s}|=|-p^{-2s}+b_{p,1}^2+b_{p,2}^2|.$$

Therefore, by [Sha] $|b_{p,i}| < p^{1/5}$, $i = 1,\ 2$, we have

$$p^{2a}(p+1) < p + p^{-2a} + 2p^{2/5}.$$

where $a = Re\ s$. Let (A) denote the inequality above. We will show (A) is not possible for $Res = a \geq 1/2$. We suppose (A) holds for some $a \geq 1/2$.

Firstly we fix $a = 1/2$, then

$$p^2 < 2p^{2/5} + p^{-1}.$$

When $p = 2$,

$$2^2 < 2 \cdot 2^{2/5} + 1/2 < 2 \cdot 2^{1/2} + 1/2 < 3 + 1/2,$$

which is not possible. If we let

$$f(p) = p^2 - 2p^{2/5} - p^{-1},$$

then $f(2) \geq 0$. Since $f(p)$ is an increasing function, we see that $f(p) \geq 0$, for any p. Therefore, when $a = 1/2$, the inequality (A) is not true for all prime p.

Secondly we fix p and let

$$g(a) = p^{2a}(p+1) - p^{-2a} - 2p^{2/5} - p.$$

By the result above, $g(1/2) \geq 0$. Since $g(a)$ is also an increasing function,

$$g(a) \geq 0, \ \ for \ \ all \ \ a \geq 1/2.$$

Therefore, for every p and $a \geq 1/2$, (A) is not true. Hence we proved that the local factor is not zero, for p not dividing 4A.

If p divides 4A, then $L_p(s) = L(s, \pi_p)$ which are not zero for all s, hence $L_p(s) \neq 0$ for all s.

Secondly we can see that the $L(s)$ converges absolutely in $Re\, s > 1$. We explain this. $L(s)$ is bounded by

$$h(s,\pi) = \sum_{m_1,m_2} a_{m_1 m_2^2}(\pi)(m_1 m_2^2)^{-s},$$

since $\prod_{p|m_2}(1+\frac{1}{p})^{-1} < 1$. While by the Lemma 2.6, the $h(s,\pi)$ differs from

$$L_S(2s, Sym^2\pi)L(4s, \omega_\pi^2)$$

only with the finitely many local factors, where

$$L_S(s, Sym^2\pi) = \prod_{p\notin S} L(s, Sym^2\pi_p)$$

and the S is the set of all primes where π is ramified and the infinite prime. Since $L_S(s, Sym^2\pi)$ converges absolutely for $Re\, s > 1$, $h(s,\pi)$ converges absolutely for $Re\, s > 1/2$.

Therefore when r=2, we do not need the conditions on the series' convergence in the Theorem 2.3, and the asymptotic constant C is not zero. □

2.3 A Bound of the Mean Value at 1/2

Theorem 2.4 *Let π be an irreducible unitary cuspidal representation of $GL_r(Q_A)$ with the trivial central character conductor A. Assume the same conditions in Theorem 2.3. Then*

$$\sum_{\substack{0\le D\le Y \\ D\equiv 1(4A) \\ \text{D square free}}} L_f(1/2, \pi\times\chi_D) \ll \begin{cases} Y^{\frac{r+1}{2}}(\log Y)^2 & \text{if } r>1 \\ Y^{3/2}\log Y & \text{if } r=1 \end{cases}.$$

Proof. Consider the integral

$$\int_{(\gamma)} L_f(1/2+s, \pi\times\chi_D)X^s\Gamma(s)ds \quad ,\gamma > 1/2.$$

For $D \equiv 1 \bmod 4A$, and square free, we have

$$L_f(1/2, \pi \times \chi_D) = \sum_{m=1}^{\infty} \frac{a_m(\pi)}{m^{1/2}} \chi_D(m) exp(-m/X)$$

$$\dot{-}\frac{1}{2\pi i} \int_{(-\eta)} \frac{L(1/2 - s, \tilde{\pi}_\infty)}{L(1/2 + s, \pi_\infty)} L_f(1/2 - s, \tilde{\pi} \times \tilde{\chi}_D) \left(\frac{X}{AD^r}\right)^s \Gamma(s) ds,$$

for any $0 < \eta < 1$.

We let $1/2 < \eta < 1$ and $X = AD^r$. Then we get

$$\sum_{\substack{0 \leq D \leq Y \\ D \equiv 1(4A) \\ \text{D square free}}} L_f(1/2, \pi \times \chi_D) = \sum_{\substack{0 \leq D \leq Y \\ D \equiv 1(4A) \\ \text{D square free}}} \sum_{m=1}^{\infty} \frac{a_m(\pi)}{m^{1/2}} \chi_D(m) \, exp(-m/AD^r)$$

$$- \sum_{\substack{0 \leq D \leq Y \\ D \equiv 1(4A) \\ \text{D square free}}} \frac{1}{2\pi i} \int_{(-\eta)} \frac{L(1/2 - s, \tilde{\pi}_\infty)}{L(1/2 + s, \pi_\infty)} \sum_{m=1}^{\infty} \frac{a_m(\tilde{\pi})}{m^{1/2-s}} \Gamma(s) \, ds = S + I$$

Write

$$S = \sum_{\substack{0 \leq D \leq Y \\ D \equiv 1(4A) \\ \text{D square free}}} \sum_{m \leq AD^r} + \sum_{\substack{0 \leq D \leq Y \\ D \equiv 1(4A) \\ \text{D square free}}} \sum_{m > AD^r} = S_1 + S_2.$$

Also write $m = m_1 m_2$ as we did in the previous theorem . Let S_{1s}, S_{2s}, S_{1n}, S_{2n} be the parts according as m_2 is a square or not.

$$S_{1n} \ll \sum_{\substack{0 \leq D \leq Y \\ D \equiv 1(4A) \\ \text{D square free}}} \sum_{\substack{m \leq AD^r \\ m_2 \text{ not square}}} \frac{a_m(\pi)}{m^{1/2}} \chi_D(m_2)$$

$$= \sum_{\substack{0 < D \leq Y \\ D \equiv 1(4A)}} \sum_{j^2 | D} \mu(j) \sum_{\substack{m \leq AD^r \\ m_2 \text{ not square}}} \frac{a_m(\pi)}{m^{1/2}} \chi_D(m_2)$$

$$= \sum_{\substack{j \leq Y^{1/2} \\ (j, 4A) = 1}} \mu(j) \sum_{\substack{0 < D' \leq Y/j^2 \\ D' \equiv j^2(4A)}} \sum_{\substack{m \leq A(D'j^2)^r \\ m_2 \text{ nonsquare}}} \frac{a_m(\pi)}{m^{1/2}} \chi_{D'}(m_2) \chi_o^{(j)}(m_2)$$

$$= \sum_{\substack{j \le Y^{1/2} \\ (j,4A)=1}} \mu(j) \sum_{\substack{m \le AY^r \\ m_2 \text{ nonsquare}}} \frac{a_m(\pi)}{m^{1/2}} \chi_o^{(j)}(m_2) \times$$

$$\sum_{\substack{(m/A)^{1/r}/j^2 < D \le Y/j^2 \\ D \equiv \bar{j}^2(4A)}} \chi_D(m_2)$$

$$\ll \sum_{\substack{j \le Y^{1/2} \\ (j,4A)=1}} \left(\sum_{\substack{m \le AY^r \\ m_2 \text{ nonsquare}}} \frac{|a_m(\pi)|^2}{m^{1/2}} \right)^{1/2} \times$$

$$\left(\sum_{\substack{m \le AY^r \\ m_2 \text{ nonsquare}}} \frac{1}{m^{1/2}} \left| \sum_{\substack{(m/A)^{1/r}/j^2 < D \le Y/j^2 \\ D \equiv \bar{j}^2(4A)}} \chi_D(m_2) \right|^2 \right)^{1/2}.$$

Now we need the following lemma to estimate the second factor above.

Lemma 2.7

$$\sum_{\substack{m \le AY^r \\ m_2 \text{ nonsquare}}} \left| \sum_{\substack{(m/A)^{1/r}/j^2 < D \le Y/j^2 \\ D \equiv \bar{j}^2(4A)}} \chi_D(m_2) \right|^2 \ll \begin{cases} \frac{Y^{r+1}}{j^2} (\log(AY^r))^2 & \text{if } r > 1 \\ Y^3 (\log Y) \frac{1}{j^5} & \text{if } r=1 . \end{cases}$$

Proof of the lemma. The left side of the inequality in the lemma is

$$\sum_{\substack{m \le AY^r \\ m_2 \text{ nonsquare}}} \sum_{\substack{(m/A)^{1/r}/j^2 < D_1, D_2 \le Y/j^2 \\ D_1, D_2 \equiv \bar{j}^2(4A)}} \chi_{D_1 D_2}(m_2)$$

$$\ll \sum_{\substack{0 < D_1, D_2 \le Y/j^2 \\ D_1, D_2 \equiv \bar{j}^2(4A)}} \sum_{\substack{0 < m \le (D_i j^2)^r A \\ i = 1,2 \\ m_2 \text{ nonsquare}}} \chi_{D_1 D_2}(m_2).$$

Here $\bar{j}$ mod 4A is the inverse of j mod 4A.

We break the outer sum into two parts, the square part and the nonsquare part, according to $D_1 D_2$ is a square or not.

The square part is

$$\sum_{\substack{0<D_1,D_2\le Y/j^2\\ D_1,D_2\equiv j^2(4A)\\ D_1D_2 square}} \sum_{\substack{0<m\le (D_ij^2)^r A\\ i=1,2\\ m_2\ nonsquare}} \chi_{D_1D_2}(m_2).$$

$$\ll \sum_{\substack{0<D^2\le Y^2/j^4\\ D^2\equiv j^4(4A)}} \tau(D^2) \sum_{\substack{0<m\le (D_ij^2)^r A\\ i=1,2\\ m_2\ nonsquare\\ (m_2,D)=1}} 1$$

$$\ll \sum_{0<D\le Y/j^2} \tau(D^2) \sum_{\substack{0<m\le Y^r A\\ (m_2,D)=1\\ m_2\ nonsquare}} 1$$

$$\ll AY^r \sum_{0<D\le Y/j^2} \tau(D^2) \ll AY^r \frac{Y}{j^2}\log^2(Y/j^2) \ll \frac{Y^{r+1}}{j^2}\log^2 Y.$$

In the estimate above we used the fact that

$$\sum_{n\le x} \tau(n^2) \sim \frac{1}{\zeta(2)}\, x\, (\log x)^2.$$

The nonsquare part is, by Polya-Vinogradov inequality [Dav],

$$\sum_{\substack{0<D_1,D_2\le Y/j^2\\ D_1,D_2\equiv j^2(4A)\\ D_1D_2\ nonsquare}} \sum_{\substack{0<m\le D^rA/j^2\\ m_2\ nonsquare}} \chi_{D_1D_2}(m_2).$$

$$\ll \sum_{\substack{0<D_1,D_2\le Y/j^2\\ D_1,D_2\equiv 1j^2(4A)\\ D_1D_2\ nonsquare}} (D_1D_2)^{1/2}\log(D_1D_2)$$

$$\ll (Y^2/j^4)^{1/2}\log(Y^2/j^4)\, Y^2/j^4 \ll Y^3\log Y\,\frac{1}{j^5}.$$

Therefore if $r \geq 2$,

$$\sum_{\substack{m \leq AY^r \\ m_2 \text{ nonsquare}}} \left| \sum_{\substack{(m/A)^{1/r}/j^2 < D \leq Y/j^2 \\ D \equiv j^2(4A)}} \chi_D(m_2) \right|^2 \ll Y^{r+1}(\log Y)^2 \frac{1}{j^2}.$$

We finished the proof of the lemma. □

Hence by the lemma we proved just before and Proposition 1.10, if $r > 1$ then

$$\begin{aligned} S_{1n} &\ll \sum_{\substack{j \leq Y^{1/2} \\ (j,4A)=1}} Y^{r/4} Y^{r/4} Y^{1/2} \log Y \frac{1}{j} \\ &= Y^{\frac{r+1}{2}} \log Y \sum_{j \leq Y^{1/2}} \frac{1}{j} \\ &\ll Y^{\frac{r+1}{2}} (\log Y)^2; \end{aligned}$$

if r=1 then

$$S_{1n} \ll Y^{3/2} \log Y.$$

$$\begin{aligned} S_{2n} &= \sum_{\substack{0 \leq D \leq Y \\ D \equiv 1(4A) \\ \text{D square free}}} \sum_{\substack{m > AD^r \\ m_2 \text{ not square}}} \frac{a_m(\pi)}{m^{1/2}} \chi_D(m_2) exp(\frac{-m}{AD^r}) \\ &\ll \sum_{\substack{0 \leq D \leq Y \\ D \equiv 1(4A) \\ \text{D square free}}} \sum_{\substack{m \leq AD^r \\ m_2 \text{ nonsquare}}} \frac{a_m(\pi)}{m^{1/2}} \chi_D(m_2) \\ &= S_{1n} \ll Y^{\frac{r+1}{2}} (\log Y)^2. \end{aligned}$$

$$\begin{aligned} S_s &= S_{1s} + S_{2s} \\ &= \sum_{m_1, m_2} \frac{a_{m_1 m_2^2}(\pi)}{(m_1 m_2^2)^{1/2}} \sum_{\substack{0 \leq D \leq Y \\ D \equiv 1(4A) \\ \text{D square free}}} \chi_D(m_2^2) exp(\frac{-m_1 m_2^2}{AD^r}) \end{aligned}$$

$$= \frac{1}{2\pi i}\int_{(\gamma)} \sum_{m_1,m_2} \frac{a_{m_1 m_2^2}(\pi)}{(m_1 m_2^2)^{1/2+s}} \sum_{\substack{0\le D\le Y\\ D\equiv 1(4A)\\ \text{D square free}}} \chi_D(m_2^2)(AD^r)^s\Gamma(s)\,ds$$

$$\ll Y^{r\gamma}Y,$$

for any $\gamma > 0$. We want to have this bound lower than that of S_{1n} i.e $1 + r\gamma \le \frac{r+1}{2}$. This is equivalent to the condition $\gamma \le \frac{1}{2}\frac{r-1}{r}$. This condition can be satisfied. So

$$S_s \ll Y^{\frac{r+1}{2}}.$$

Therefore we get

$$S \ll \begin{cases} Y^{\frac{r+1}{2}}(\log Y)^2 & \text{if } r > 1 \\ Y^{3/2}\log Y & \text{if r=1} \end{cases}.$$

Now we deal with the integral part.

$$I = -\frac{1}{2\pi i} \sum_{\substack{0\le D\le Y\\ D\equiv 1(4A)\\ \text{D square free}}} \int_{(-\eta)} \frac{L(1/2-s,\tilde{\pi}_\infty)}{L(1/2+s,\pi_\infty)} L_f(1/2-s,\tilde{\pi}\times\tilde{\chi}_D)\Gamma(s)\,ds$$

$$= -\frac{1}{2\pi i} \sum_{\substack{0\le D\le Y\\ D\equiv 1(4A)\\ \text{D square free}}} \int_{(-\eta)} \frac{L(1/2-s,\tilde{\pi}_\infty)}{L(1/2+s,\pi_\infty)} \sum_{m=1}^{\infty} \frac{a_m(\pi)}{m^{1/2-s}}\chi_D(m)\Gamma(s)\,ds.$$

$$= -\frac{1}{2\pi i}\int_{(-\eta)} \sum_{m=1}^{\infty} \frac{a_{m_1 m_2^2}(\tilde{\pi})}{(m_1 m_2^2)^{1/2-s}} \sum_{\substack{0\le D\le Y\\ D\equiv 1(4A)\\ \text{D square free}}} \chi_D(m)\frac{L(1/2-s,\tilde{\pi}_\infty)}{L(1/2+s,\pi_\infty)}\Gamma(s)\,ds$$

$$\ll Y.$$

Finally we have the result

$$\sum_{\substack{0\le D\le Y\\ D\equiv 1(4A)\\ \text{D square free}}} L_f(1/2,\pi\times\chi_D) \ll \begin{cases} Y^{\frac{r+1}{2}}(\log Y)^2 & \text{if } r > 1 \\ \\ Y^{3/2}\log Y & \text{if r=1} \end{cases}.$$

□

Remark. This improves the corrollary to Theorem 1 of Goldfeld and Viola in [G-V].

3 Some New Reciprocity Laws

3.1 Artin L-Functions

Let E/F be a finite Galois extention of algebraic number fields, $G = Gal(E/F)$, Irr(G) be the set of irreducible representations of G. Let v, w be nonarchimedean place of F and E respectively and $w|v$. Let G_w, T_w be the decomposition group and the inertia group at w respectively, ϕ_w be a generator of G_w/T_w. For $(\rho, V) \in Irr(G)$ the Artin L-function attached to ρ is defined by

$$L(s,\rho,E/F) = \prod_v det(I - \rho(\phi_w)\, q_v^{-s};\ V^{T_w})^{-1}$$

where the infinite product is over all nonarchimedean local places of F, V^{T_w} is the subspace of V fixed by T_w, the $det(I - \rho(\phi_w)\, q_v^{-s};\ V^{T_w})^{-1}$ is the characteristic polynomial of the operator $\rho(\phi_w)$ on the space V^{T_w}, evaluated at q^{-s}.

Remarks. 1) Since G is a finite group, then all eigenvalues of every $\rho(\pi_w)$ have absolute value 1. So the infinite product defining the Artin L-function converges absolutely for $Re\, s > 1$. Every local factor is a polynomial in q_v^{-s} with degree $\leq n$. If v is unramified in E and ρ is unramified at v, then the degree is n.
2) If we let ρ be the trivial representation of G, then the corresponding Artin L-function is the Dedekind Zeta function ζ_F. If we let ρ be the regular representation of G, then the Artin L-function is ζ_E, since in this case $\rho = Ind_1^G 1$ and by the 3) in the following proposition.
3) It is known that the Artin L-functions satisfy nice functional equations, see [Mar].

Artin L-functions have the following basic properties corresponding to operators on representations.

Proposition 3.1 *1) If $\rho = \rho_1 \oplus \rho_2$, then*

$$L(s, \rho, E/F) = L(s, \rho_1, E/F)L(s, \rho_2, E/F).$$

2) If $N \triangleleft G$ and ρ is a representation of G/N, then

$$L(s, \rho, E^N/F) = L(s, \rho, E/F).$$

where E^H is the subfield of E fixed by H.

3) Let H be a subgroup of G, ξ be a representation of H. then

$$L(s, Ind_H^G \xi, E/F) = L(s, \xi, E/E^H).$$

Artin Conjecture: For any representation ρ of G, the $L(s, \rho, E/F)$ can be continued analytically to a meromorphic function on the whole complex plane with only a pole at s=1 of order $< \rho, 1 >$, the multiplicity of the trivial representation occurring in ρ.

If ρ is one dimensional, then Artin conjecture is true by class field theory. Moreover if ρ is monomial, i.e. it is induced from an abelian character of a subgroup of G, then the Artin conjecture is true, by 3) in the proposition above. It is well known that all finite supersolvable groups are M-groups, i.e. every irreducible representation of G is monomial. So the Artin conjecture is true for finite supersolvable extensions. In general, Brauer proved that every irreducible representation of a finite group is a Z-linear combination of induced representations from abelian characters. Hence Artin L-functions are memomorphic functions.

Langlands gave the following conjecture on higher dimensional Galois representations.

Reciprocity Law: For a given irreducible complex representation ρ of dimension n of Gal(E/F), there is an automorphic cuspidal representation $\pi(\rho)$ of $GL_n(F_A)$ such that

$$L_f(s, \pi(\rho)) = L(s, \rho, E/F).$$

We will say that ρ is automorphic over F, if the reciprocity law holds for ρ. We will say that G is automorphic over F, if for any $\rho \in Irr(G)$ the reciprocity law holds for ρ.

Since the L-function of an irreducible unitary cuspidal representation of $GL_n(F_A)$ is an entire function, (except for n=1 and the trivial representation) the reciprocity law and the Hecke's theorem on the analytic properties of L-fuctions associated to Groseecharacters implies the Artin conjecture. For two or three dimensional Galois representations, the Artin conjecture implies the reciprocity law, by the converse theorem for GL_2 and GL_3. By Brauer's result, if we can prove the reciprocity law for monomial representations, we can prove the reciprocity law in general by the method in the proof of the Proposition 3.5. But unfortunately in general we have not found much light on the monomial Galois representations. The non-trivial cases in which the reciprocity law is known true are in the work of Langlands and Tunnell. In fact Serre proved that the image of $Gal(E/F)$ in $PGL_2(C)$ under the composition of an irreducible two dimensional complex representation with the projective map from $GL_2(C)$ to $PGL_2(C)$ is one of the following types:

1) a Dihedral group,

2) the alternative group A_4,

3) the symmetric group S_4,

4) A_5.

For type 1), without loss generality, we can assume that ρ is faithful. Then $G/Z(G) \simeq D_n$, where $Z(G)$ is the center of G and D_n is the dihedral group of order 2n. Since D_n is a semi-direct product of two cyclic subgroups, D_n is supersolvable. Therefore G is a M-group. Hence $L(s, \rho, E/F)$ is entire. So ρ is automorphic over F. Using the base change result on GL_2 which we will discuss later on and the lifting result from

GL_2 to GL_3, Langlands solved the case of A_4. Tunnell solved the S_4 case using the existence of the base change for an arbitrary cubic extension. The case of A_5, the nonsovable case, remains not known. Therefore the following theorem was proved.

Proposition 3.2 *(Langlands, Tunnell) Let E/F be a finite solvable extension of number fields, and ρ be a two dimensional irreducible complex representation of $Gal(E/F)$. Then ρ is automorphic over F.*

3.2 Base Change and Automorphic Induction

To explain the base change and automorphic induction on the automorphic representations of GL_n, we first look at the corresponding restriction and induction operators on Galois representations. Let G=Gal(E/F) be a finite Galois extension of number fields, ρ be an irreducible complex representation of G of degree n. Let H be a subgroup of G. Suppose the reciprocity law holds. Then ρ should correspond to an automorphic representation $\pi(\rho)$ of GL_n over F, the restriction $\rho|_H$ of ρ to H should correspond to an automorphic representation $\pi(\rho|_H)$ of GL_n over E^H. What the "base change" map concerns is for an automorphic representation π of GL_n over F, and a subextension L of F in E, there would exist an automorphic reprensention Π of GL_n over L such that if π is an image $\pi(\rho)$ of Galois representation ρ under the reciprocity law then the Π should be the image $\pi(\rho|_H)$, where H is the subgroup of G corresponding to L. Therefore "base change" map is the automorphic version of the restriction map. Now let σ be a representation of H. Then the reciprocity law implies there is an automorphic representation $\pi(Ind_H^G\sigma)$ of GL_{nl} over F. Now what the "automorphic induction" concerns is for an automorphic representation π_L of GL_n over L, then there would exist an automorphic representation π_F of GL_{nl} over F such that if π_L corresponds to a Galois representation σ then π_F should correspond to $Ind_H^G \sigma$. Therefore we may think as this: if the reciprocity law holds, then it should be compatable with restriction map and induction map of Galois representations and the base change map and the automorphic induction map.

We know that an Artin L-function is invariant under induction map, so it is reasonable to define the automorphic induction map by this property, i.e. let π_E be automorphic representations of GL_n over E, π_F automorphic representation of GL_{nl} over F, we say π_F is an automorphic induction of π if $L(s, \pi_E) = L(s, \pi_F)$.

How to define the base change map ? By looking at the behavoir of Artin L-function under the restriction map, firstly we can define the following weak base change map. Let E/F be a cyclic extension of prime degree l of number fields. Let Π, π be two automorphic representations of $GL_r(E_A)$, $GL_n(F_A)$ respectively. We say Π is a weak base change of π, if f_v is the residual degree of E above an unramified v then for $w|v$:

$$t_{\pi,v}^{f_v} = t_{\Pi,w}$$

holds for almost all v.

By this definition, if Π is a weak base change of π then

$$L_{S_E}(s, \Pi) = \prod_{i=1}^{l} L_S(s, \pi \otimes \eta^i).$$

where S is the set of places of F such that the above relations hold out of S, S_E is the places of F over those of S, the η is the character of the $I_F/N_F^E I_E$ with order l. For the places in S we need a result of [A-C]. Arthur and Clozel [A-C] proved that if Π is a weak base change of π then π is also a strong base change of π, and

$$L(s, \Pi) = \prod_{i=1}^{l} L(s, \pi \otimes \eta^i).$$

Here the strong base change means that every local component of Π is a local base change of the corresponding local component of π which is defined in [A-C] by a equality of the trace of intertwining operator and the trace of the norm of E to F.

Now we summarize

1) if Π is a base change of π, then

$$L(s,\Pi) = \prod_{i=1}^{l} L(s, \pi \otimes \eta^i);$$

1) if π_F is an automorphic induction of π_E, then

$$L(s, \pi_F) = L(s, \pi_E).$$

We say an automorphic representation π of $GL_r(F_A)$ is induced from cuspidal, if there is a cuspidal unitary representation σ of $M(F_A)$, where P=MN is an F-parabolic subgroup of G, such that

$$\pi = Ind_{M(F_A)N(F_A)}^{G(F_A)} \sigma \otimes 1.$$

Proposition 3.3 *([A-C]) Let E/F be a Galois extension of prime degree l.*

a) Every cuspidal representation of $GL_n(F_A)$ has a base change lift to $GL_n(E_A)$;

b) A cuspidal representation of $GL_n(E_A)$ is the base change lift of some π if and only if it is Galois invariant ;

c) If π and π' have the same base change lift to $GL_n(E_A)$ then there is a character η of $I_F/F^\times N_{E/F}(I_E)$ such that $\pi' = \pi \otimes \eta$.

Proposition 3.4 *([A-C]) Let E/F be a Galois extension of prime degree l. Then, if Π is a representation of $GL_r(E_A)$ induced from cuspidal, there exists unique automorphic representation π of $GL_{rl}(F_A)$ automorphically induced from Π. Moreover π is induced from cuspidal.*

Using Proposition 3.4, [A-C] proved the reciprocity law in the nilpotent case.

Proposition 3.5 *([A-C]) Let E/F be a nilpotent extension of number fields. Then the reciprocity law is true.*

Proof. Let $G = Gal(E/F)$ and $\rho \in Irr(G)$. By Brauer's theorem

$$\rho = \sum c_H Ind_H^G \sigma_i,$$

where the sum runs over subgroups H of G, σ_i is an abelian character of H, $c_H \in Z$. Therefore

$$L(s, \rho, E/F) = \prod L_f(s, \sigma_H, E/E^H)^{c_H}.$$

Since G is nilpotent, every subgroup of G is subnormal in G, i.e. there are subgroups $H = N_1, N_2, \cdots, N_t = G$ such that $N_i \lhd N_{i+1}$ and N_{i+1}/N_i has a prime order, $i = 1, \cdots, t$. By applying finitely many steps of automorphic induction, we can prove that there are cuspidal representations $\pi_1, \cdots, \pi_n$ of $GL_{n_i}(F_A)$ and integers $a_1, \cdots, a_n$ such that

$$L(s, \rho) = \prod_{i=1}^{n} L_f(s, \pi_i)^{a_i}.$$

Let

$$\chi(v^m) = Tr\rho(\phi_v^m)$$

$$\pi(v^m) = Trt_{\pi,v}^m.$$

Then

$$\chi(v^m) = \sum_i a_i \pi(v^m),$$

for any nonarchimedean place v of F which is unramified in E and at which ρ and all π_i are unramified. On one hand, by Chebotarev density theorem,

$$\sum_{Nv \leq x} \frac{|\chi(v)|^2}{Nv} = \log\log x + O(1).$$

On the other hand, in our situation the π_i's satisfy the Ramanujan conjecture, hence the analytic property of the Rankin-Selberg convolution of irreducible unitary cuspidal representations of GL_* implies

$$\sum_{i,j} a_i a_j \sum_{Nv \leq x} \frac{\pi_i(v)\overline{\pi_j(v)}}{Nv} = \sum_i a_i^2 \log\log x + O(1).$$

So

$$1 = \sum_i a_i^2,$$

hence $i = 1$, $a_1 = \pm 1$. Since the Artin L-functions has trivial zeros, $a_1 = 1$. □

3.3 Reciprocity Law for Some Frobenius Extensions

Proposition 3.6 *Let $G = Gal(E/F)$ be a solvable finite extension of number fields. If there is a normal subgroup N of G satisfying*

1) G/N is nilpotent,

2) every irreducible representation of N has dimension 1 or 2,

then G is automorphic over F.

Proof. We use induction on the order of G. Since G/N is nilpotent and G is solvable, G is an M-group with respect to N, for this see [Isa]. So for any $\rho \in Irr(G)$, there is a subgroup H of G, $N \leq H \leq G$ such that

$$\rho = Ind_H^G \phi, \quad \phi \in Irr(H), \quad \rho|_H = \phi.$$

If H is not G, we can apply the induction on H, since H satisfies the condition 1) and 2) in the proposition. Hence

$$L(s, \rho) = L(s, \phi, E/E^H) = L(s, \Pi),$$

where Π is an irreducible cuspidal representation of $GL_n((E^H)_A)$. Since G/N is nilpotent, by applying finitely many steps of the automorphic induction, we know that there is an automorphic representation π of $GL_r(F_A)$ induced form cuspidal such that

$$L(s, \Pi) = L(s, \pi).$$

Since ρ is irreducible, the π must be cuspidal by the similar argument in Proposition 3.6. So we are done if H is not G. If H=G, then $\rho|_N \in Irr(N)$. Since every irreducible representation of N has dimension one or two, ρ has dimension one or two. Therefore by the class field theory and Proposition 3.2, ρ is automorphic over F. □

Corollary 3.1 *If G=Gal(E/F) satisfies one of the following conditions*

1) every Sylow subgroup is cyclic,

2) both every proper subgroup and quotient are nilpotent,

3) every proper subgroup of G is abelian,

then G is automorphic over F.

Proof. For 1), G is semi-direct product of two cyclic subgroups, see [M.Hal]. So by the proposition above, G is automorphic over F. For 2) and 3), if G is nilpotent, or abelian respectively, we are done. If G is not in such a case respectively, we know that G is a semi-direct product of two abelian subgroups for both 2) and 3). So the result follows from the proposition above. □

Now we introduce a class of finite groups called Frobenius groups. By definition, a group G is called a Frobenius group, if there is a subgroup $H \neq 1$ of G such that $H^g \cap H = 1$ for all $g \in G - H$.

We record some basic properties of Frobenius groups we will use in the following proposition.

Proposition 3.7 *Let G be a finite Frobenius group.*

1) G=KH, where $K \lhd G$, K is nilpotent, both K and H are Hall subgroups of G and $(|K|, |H|) = 1$.

2) Every Sylow subgroup of G is a cyclic or generalized quaternion group.

3) Let $\rho \in Irr(G)$. If the kernel of ρ, $Ker(\rho)$, does not contain K then $\rho = Ind_K^G \sigma$, $\sigma \in Irr(K)$.

Since for a Frobenius group G, the subgroup K of G stated in the proposition above is uniquely determined by G, we call K the Frobenius kernel of G. The subgroup H of G in the proposition is unique up to conjugation in G, we call H a Frobenius complement of G. For the proof of this, see [Pas].

Theorem 3.1 *Let G=Gal(E/F) be a finite Frobenius group, K be its Frobenius kernel, H be its Frobenius complement. Let F(H) be the maximal normal nilpotent subgroup of H. If H/F(H) is nilpotent then G is automorphic over F.*

Proof. Let $\rho \in Irr(G)$.

If $ker(\rho)$ does not contain K then, by the 3) in the proposition above, $\rho = Ind_K^G \sigma$, $\sigma \in Irr(K)$. So

$$L(s, \rho, E/F) = L(s, \sigma, E/E^K).$$

Since K is nilpotent, ρ is automorphic over E^K. Since K is normal in G, E^K/F is a Galois extension and $Gal(E^K/F) = G/K = H$. Since F(H) and H/F(H) are solvable, H is solvable. So we can apply the automorphic induction finitely many times to get that ρ is automorphic over F.

If $ker(\rho)$ contains K, we can think of $\rho \in Irr(H)$. The result in our theorem follows from the following lemma which deals with a more general situation than our case.

Lemma 3.1 *Let G=Gal(E/F) be a finite solvable group. Suppose that every Sylow p-subgroup of G is abelian for $p > 2$ and every irreducible representation of Sylow 2-subgroups has dimension one or two. If G/F(G) is nilpotent, then G is automorphic over F.*

Proof. We claim that every irreducible representation of F(G) is dimension one or two. In fact, since F(G) is nilpotent, F(G) is the direct product of its Sylow subgroups, hence for $\rho \in Irr(F(G))$, ρ is a tensor product of irreducible representations of the Sylow subgroups of F(G). Since every irreducible representation of the Sylow p-subgroups has dimension one for $p > 2$, and dimension one or two for $p = 2$, every irreducible representation of F(G) has dimension one or two. So we have proved the claim. By this claim and Proposition 3.6, we proved the lemma. □

Now for our H, its Sylow subgroups are cyclic or generalized quaternion group. So the Sylow p-subgroup of F(H) is cyclic for $p > 2$. Since every subgroup of a generalized quaternion group is a cyclic or a generalized quaternion group, the Sylow 2-subroup of F(H) is also cyclic or generalized quaternion. Since a generalized quaternion group Q has a cyclic or abelian normal subgroup of index 2, so every irreducible representation of Q has dimension one or two, to see this, see [Isa] or compute directly. Therefore by the lemma we finish the proof of the theorem. □

Remark. In the theorem above if we assume that H is supersolvable then the commutator subgroup H' of H is nilpotent, hence $H' \leq F(H)$. Therefore H/F(H) is abelian hence is nilpotent. So this situation is included in the theorem above. And note that if the order of H is odd then H is supersolvable, so G is automorphic over F in this case. In particular, if G is a Frobenius group with odd order, then G is automorphic over F.

References

[1] [A-C] J.Arthur and L.Clozel, Simple Algebras, Base Change, and the Advanced Theory of the Trace Formula, *Annals of Mathematics Studies, Princeton University Press*

[2] [A-G] A. Ash and D.Ginzburg, p-adic L-function for $GL(2n)$, *Invent. Math.*, **116**, (1994), 27-73.

[3] [B-R] L. Barthel and D. Ramakrishnan, A nonvanishing Result for Twists of L-Functions of GL(n), *Preprint.*

[4] [Bur] D. Burgess, On characters sums and L-Series II, *Proc. London Math.*, **(3)**, **13**, (1963),524-536.

[5] [B-F-H] D. Bump, S.Friedberg and J.Hoffstein, Eisenstein series on the metaplectic group and nonvanishing theorems for automorphic L-functions and their derivatives, *Annals of Mathematics*, **131** (1990), 53-127.

[6] [Dav] H. Davenport, Multiplicative Number Theory, *Springer-Verlag*, Second Edition, 1980.

[7] [F-H] S.Friedberg and J.Hoffstein, Nonvanishing Theorems For Automorphic L-Functions On GL(2), *Preprint.*

[8] [G-J] R.Godement and H.Jacquet, Zeta Functions of Simple Algeras, Lecture Notes in Mathematics vol. **260**.

[9] [Gel-Jac] S. Gelbart and H. Jacquet, A relation between automorphic representations of GL(2) and GL(3), *Ann. Sci. Ecole Normale Sup.*,4^e serie **11** (1978), 471-552.

[10] [G-V] D. Goldfeld and C.Viola, Mean values of L-functions associated to elliptic Fermat, and other curves at the centre of the critical strip. *J. Number Theory* **11**(1979), 305-320.

[11] [H-R] H. Halberstam and H.Richert, Sieve Methods, *London Mathematical Society Monograghs, No.4*, Academic Press 1975.

[12] [Hal] M. Hall, The Theory of Groups, *The Mcamillan Company* 1981.

[13] [Isa] M. Isaacs Character Theory of Finite Groups, *Academic Press* 1976.

[14] [Jac] H. Jacquet, Principal L-functions of the linear group, in *Proc. of Symp. in Pure Math.*, Vol.**33** (1979), Part 1, 63-86.

[15] [J-L] H. Jacquet and R. Langlands, Automorphic Forms on GL(2), LNM, **114**.

[16] [J-P-S] H. Jacquet, I. Piatetski-Shapiro and J. Shalika, Rankin-Selberg Convolutions, *Amer. J. Math.* **105** (1983), 367-464.

[17] [J-S 1] H. Jacquet, and J. Shalika, On Euler Products and the Classification of Automorphic Representations I, *Amer. J. Math.* **103**, No. 3 (1981), 499-558.

[18] [J-S 2] H. Jacquet, and J. Shalika, On Euler Products and the Classification of Automorphic Representations II, *Amer. J. Math.* **103**, No. 3 (1981), 777-815.

[19] [J-S 3] H. Jacquet, and J. Shalika, Rankin-Selberg Convolutions: Archimedean Theory, in *Piatetski-Shapiro festschrift (Part I)*, IMCP, vol. **2**, 1990.

[20] [Jut] M. Jutila, On the Mean Value of $L(1/2,\chi)$ for Real characters, *Analysis*,**1** (1981), 149-161.

[21] [Lan] R. Langlands, Base Change for GL(2), *Annals of Mathematics Studies, Princeton University Press.*

[22] [Mar] J. Martinet, Character theory and Artin L-functions, in *Algebraic Number fields*, Edited by A. Frohlich, Academic press, 1977.

[23] [Mes] J. Mestre, Formules Explicites et minorations de Conducteurs de Varietes Algebriques, *Comp.Math.*, **58** (1986), 253-281.

[24] [R.Mur] R. Murty, On Simple Zeros of Certain L- Series, *in Number Theory*, Edited by R. Mollin, 427-411.

[25] [M-M] R.Murty and K.Murty, Mean values of derivatives of modular L-series, *Annals of Mathematics*, **133**, (1991), 447-475.

[26] [K.Mur] K. Murty, A non-vanishing theorem for quadratic twists of modular L-functions, *Preprint*.

[27] [Pas] D. Passman, Permutation Groups, *Benjamin,inc.* 1968.

[28] [Pia] I. Piatetski-Shapiro, Work of Waldspurger, *in LNM*, Vol.**1041**, 280-302.

[29] [Roh 1] D. Rohrlich, Non-vanishing of L-functions for GL(2), *Invent. Math.*, **97**(1989), 381-403.

[30] [Roh 2] D. Rohrlich, Nonvanishing of L-functions and Structure of Mordell-Weil groups, *J. reine angew. Math.*, **417**, (1991), 1-26.

[31] [Sha] F. Shahidi, Automorphic L-functions: A Survey, in *Automorphic Forms, Shimura Varieties, and L-functions*, Edited by L. Clozel and J. Milne, vol.I, 415-437.

[32] [Tat] J. Tate, Number theory background, *Proc. of Symp. in Pure Math.*, Vol.**33** (1979), Part 2, 3-26.

[33] [Tun] J. Tunnell, Artin's conjecture for representations of octahedral type, *Bulletin (New Series) of the American Mathematical Society,* Vol.**5**, No.2, 1981.