The Culler-Shalen seminorms of pretzel knots

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Abstract

We use the theory of Culler-Shalen seminorms to investigate the $SL_2(\mathbb{C})$-character variety $X(K)$ and $PSL_2(\mathbb{C})$-character variety $\bar{X}(K)$ of a Montesinos knot K. When K has three tangles, $\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3$, the $PSL_2(\mathbb{C})$-character variety of the triangle group $\Delta(\alpha_1, \alpha_2, \alpha_3)$ includes into $\bar{X}(K)$. We apply this observation to make several deductions about (p, q, r) pretzel knots.

For hyperbolic $(-2, 3, n)$ pretzel knots, we calculate the Culler-Shalen seminorms explicitly and deduce that $X(K)$ consists of exactly two (respectively three) algebraic curves when $3 \nmid n$ (respectively $3 \mid n$). This leads to a classification of the finite and cyclic surgeries of these knots. We obtain similar results for some $(-3, 3, n)$ pretzel knots.

We classify cyclic surgeries on $(-2, p, q)$ pretzel knots (p, q odd and positive). The $(-2, 3, 7)$ is the only non-torus knot in this family admitting non-trivial cyclic surgeries. We show that a $(p, q, 2m)$ pretzel knot admits at most one non-trivial finite surgery so long as $\frac{1}{|p|} + \frac{1}{|q|} + \frac{1}{|m|} < 1$. Moreover, for $m < -1$, these knots have no non-trivial finite surgeries.

Combining these results with work of Delman we classify cyclic surgeries on Montesinos knots. If a non-torus Montesinos knot K admits a non-trivial cyclic surgery, then it is the $(-2, 3, 7)$ pretzel knot and the surgery is 18 or 19. Further, if K has a non-trivial finite surgery, then it is either a $(-2, p, q)$ pretzel knot with $5 \le p \le q$ odd, or else it is a $(-2, 3, q)$ pretzel knot with $q = 7$ or 9.

Résumé

On utilise la théorie des seminormes de Culler-Shalen pour examiner $X(K)$ (la $SL_2(\mathbb{C})$-variété de caractères) et $\bar{X}(K)$ (la $PSL_2(\mathbb{C})$-variété de caractères) d'un noeud de Montesinos K. Lorsque K a trois tangles $\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3$, alors la $PSL_2(\mathbb{C})$-variété de caractères du groupe de triangle $\Delta(\alpha_1, \alpha_2, \alpha_3)$ est incluse dans $\bar{X}(K)$. Cette observation permet de tirer plusieurs déductions concernant les (p, q, r) noeuds bretzel .

On calcule explicitement les seminormes de Culler-Shalen pour les $(-2, 3, n)$ noeuds bretzel hyperboliques. On déduit que $X(K)$ est constitué de deux courbes algébriques quand $3 \nmid n$ et de trois courbes algébriques quand $3 \mid n$. Ceci donne la classification des chirurgies cycliques et finies de ce type de noeuds. Des résultats similaires sont obtenus pour certains $(-3, 3, n)$ noeuds bretzel.

On classifie les chirurgies cycliques pour les $(-2, p, q)$ noeuds bretzel (p, q étant impairs et positifs). On prouve que le $(-2, 3, 7)$ noeud bretzel est le seul noeud non-torique dans cette famille admettant des chirurgies cycliques non-triviales. On démontre aussi que le $(p, q, 2m)$ noeud bretzel avec $\frac{1}{|p|} + \frac{1}{|q|} + \frac{1}{|m|} < 1$, admet au plus une chirurgie finie non-triviale. Si de plus, $m < -1$, alors ce noeud n'admet aucune chirurgie finie non-triviale.

En combinant ces résultats avec le travail de Delman, on classifie les chirurgies cycliques des noeuds de Montesinos. Si un noeud non-torique de Montesinos K admet une chirurgie cyclique non-triviale, alors il est le $(-2, 3, 7)$ noeud bretzel et la chirurgie est 18 ou 19. De plus, si K admet une chirurgie finie non-triviale, soit il est un $(-2, p, q)$ noeud bretzel avec $5 \leq p \leq q$, soit il est un $(-2, 3, q)$ noeud bretzel avec $q = 7$ ou 9.

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For me, playing is like unraveling a whole bunch of little knots. Some shows I never get past the knots. I spend the whole night trying to untie them, and my consciousness never breaks through to the first person out there.

–J. Garcia (1942-1995)

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1. Introduction

1.1. Conjectures of Poincaré and Thurston. The Poincaré Conjecture states that any closed, simply-connected 3-manifold is homeomorphic to S^3. That this conjecture has remained open for close to a century points to the subtlety of 3-dimensional topology. Although S^3 can be regarded as the simplest of 3-manifolds, we still do not know that there are no clever imitations or "fake" S^3's out there which share most of its important properties.

The work of Lickorish and Wallace suggests one approach to this conundrum. They showed, independently, that any 3-manifold can be constructed through Dehn surgery on a knot or link in S^3. In particular any fake S^3 counterexample to Poincaré's conjecture could be so constructed. Of course, starting with such a knot or link, one can always recover S^3 through "trivial" surgery. We say that a knot has property P (for "Poincaré") if this is the only Dehn surgery which results in a simply-connected manifold. The property P conjecture states that every non-trivial knot in S^3 has property P. Although property P is known to hold for large classes of knots, this conjecture also remains open.

The work of Culler and Shalen [CS1, CS2, CGLS], and Boyer and Zhang [BZ1, BZ2, BZ4] shows the feasibility of taking a slightly larger perspective. Rather than surgeries leading to a simply-connected manifold they investigate those resulting in a manifold with cyclic or finite fundamental group. Essentially, they show that surgery on a hyperbolic knot in S^3 can produce at most three manifolds with cyclic fundamental group and at most five with finite group (see Theorems 2.2.1 and 2.2.2 for more precise statements). For example, since trivial surgery produces S^3, and as simply-connected manifolds have cyclic fundamental group, a given hyperbolic knot could have at most two surgeries which violate the Poincaré Conjecture. Thus one could imagine a program to prove the Poincaré Conjecture by systematically going through all the hyperbolic knots in S^3, finding the cyclic or finite surgeries (of which there are at most five for a given knot), and checking if any of them is a counterexample to the conjecture. Of course, one would still be left to examine non-hyperbolic knots, not to mention links.

A more realistic appraisal comes in the context of another of the great open problems of 3-dimensional topology, Thurston's hyperbolization conjecture. Roughly speaking, this conjecture contends that "most 3-manifolds are hyperbolic." For example, Thurston has proved that all but a finite number of Dehn surgeries on a hyperbolic knot result in manifolds which are again hyperbolic. Moreover, the conjecture implies that the non-hyperbolic ones either contain an incompressible sphere or torus, have cyclic or finite fundamental group, or else are Seifert fibred manifolds (these outcomes are not disjoint). Along with Gordon and Luecke's studies of incompressible surfaces, the investigation of cyclic and finite surgeries by Culler, Shalen, Boyer, and Zhang forms half of a two-pronged attack on Thurston's conjecture.

Culler-Shalen seminorms have played a central role in these investigations. They were first introduced by Culler and Shalen [CGLS] as part of the proof of the Cyclic Surgery Theorem. Later, Boyer and Zhang [BZ1] extended the idea to the study of finite surgeries eventually leading to a proof of the Finite Filling Conjecture [BZ4].

1.2. **Contributions of this thesis.** By focusing on a specific class of knots, the pretzel knots, we can use the methods of Culler, Shalen, Boyer, and Zhang to obtain stronger results. Rather than just showing that there are very few cyclic or finite surgeries, we can show that in many cases there are none (aside from the trivial surgery) and give a listing of the knots which admit interesting fillings. By further restricting to the $(-2, 3, n)$ pretzel knots, we can explicitly calculate the Culler-Shalen seminorms and use that information to give a complete description of the character variety of these knots. Therefore this thesis makes contributions to the understanding of finite and cyclic surgeries of pretzel knots and illustrates how Culler-Shalen seminorms can be used to describe the character variety of a knot.

Delman [Del] showed that if a hyperbolic Montesinos knot admits a finite surgery, then it is a pretzel knot of the form $(2k+1, 2l+1, -2m)$, k, l, m being positive integers. Our contribution is a thorough understanding of cyclic and finite surgeries on these pretzel knots.

In particular, we prove that there are no non-trivial cyclic or finite surgeries on a $(-2, 3, n)$ pretzel knot K unless one of the following holds.

- K is torus, in which case $n = 1, 3$, or 5.

- $n = 7$, in which case 18 and 19 are cyclic fillings while 17 is a finite, non-cyclic filling.
- $n = 9$, in which case 22 and 23 are finite, non-cyclic fillings.

A consequence is that the $(-2,3,7)$ is the only non-torus $(-2,p,q)$ pretzel knot (p,q odd, positive integers) which admits a non-trivial cyclic surgery. We show that a $(p,q,2m)$ pretzel knot admits at most one non-trivial finite surgery so long as $\frac{1}{|m|} + \frac{1}{|p|} + \frac{1}{|q|} < 1$. For $m < -1$, we show further that $(p,q,2m)$ pretzels admit no non-trivial finite surgeries.

This can be combined with the work of Delman to deduce:

Theorem 4.5.1. *The only non-torus Montesinos knot which admits a non-trivial cyclic surgery is the* $(-2,3,7)$ *pretzel knot. The non-trivial cyclic surgeries on this knot are of slope* 18 *and* 19.

Theorem 4.5.2. *If a non-torus Montesinos knot* K *admits a non-trivial finite surgery, then one of the following holds.*

- K *is a* $(-2,p,q)$ *pretzel knot with* $5 \leq p \leq q$ *odd and the filling is not cyclic.*
- K *is the* $(-2,3,7)$ *pretzel knot and the filling is along slope* 17, 18, *or* 19.
- K *is the* $(-2,3,9)$ *pretzel knot and the filling is along slope* 22 *or* 23.

For certain pretzel knots we are able, not only to classify finite and cyclic surgeries, but also to determine the structure of the character variety. We begin by demonstrating the inclusion of $PSL_2(\mathbb{C})$-character varieties $\bar{X}(\Delta(\alpha_1,\alpha_2,\alpha_3)) \subset \bar{X}(K)$ when $K = K(\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3)$ is a three-tangle Montesinos knot and $\Delta(\alpha_1,\alpha_2,\alpha_3)$ the associated triangle group. The (p,q,r) pretzel knots, for which $K = K(1/p, 1/q, 1/r)$, are important examples. These knots are small, so the $SL_2(\mathbb{C})$-character variety $X(K)$ will consist of algebraic curves. Generally, if K also admits a Seifert surgery, we can proceed to calculate the Culler-Shalen seminorms explicitly and thereby enumerate the curves in $X(K)$. For hyperbolic $(-2,3,n)$ pretzel knots (n an odd integer), we deduce that $X(K)$ consists of exactly two (respectively three) curves when $3 \nmid n$ (respectively $3 \mid n$). We obtain similar results for some $(-3,3,n)$ pretzel knots. (This is an expanded form of the papers [BMZ, Mat1, Mat2].)

It should be emphasized that such a detailed description of the character variety of a knot is rare. It is true that the character varieties of torus knots are well understood and Burde [Bu] has shown that $X(K)$ consists of exactly two curves for certain 2-bridge knots. However, our description of the character varieties and Culler-Shalen seminorms of a family of pretzel knots marks substantial progress in this area.

These seminorm calculations also provide a link with the A-polynomial invariant of [CCGLS]. For example, we determine the Newton polygon for the A-polynomial of the $(-2, 3, n)$ pretzel knots. This is significant, as it remains difficult to compute the A-polynomial. For example, the $(-2, 3, 7)$ is the only one of these knots whose polynomial is included in the table of [CCGLS].

In addition to such concrete contributions, we would like to think of this thesis as a handbook of computational techniques for use in the understanding of Culler-Shalen seminorms. For example, we illustrate how Ohtsuki's [Oht] work on ideal points can be used to deduce the Culler-Shalen seminorms of the twist knots. (This again leads to information about cyclic and finite surgeries and the Newton polygon for the A-polynomial of these knots.) We explain how to show $Z_x(f_\mu) = 1$ when x is the character of a p-representation. We discuss in some detail Hatcher and Oertel's algorithm for calculating the boundary slopes of Montesinos knots using pretzel knots as examples. Finally, in the Appendix, we present calculations of the zeroes of Alexander polynomials for certain pretzel knots.

1.3. **Outline.** Following the introductory first chapter, the second chapter presents a brief overview of the theory of character varieties of knots and the Culler-Shalen seminorm. To illustrate the power of this theory, we explicitly calculate the Culler-Shalen seminorms of the twist knots and show how such a calculation can be used to make deductions about finite surgeries on those knots. We also construct the Newton polygon for the A-polynomial of the twist knots. We next state some preliminary lemmas about commutativity of representations and representations of triangle groups. We complete the second chapter with an argument that $Z_x(f_\mu) = 1$ when x is the character of a p-representation of a twist knot or a $(-2, 3, n)$ pretzel knot.

The third chapter introduces Montesinos knots. After outlining the theory of Seifert fibred spaces, we prove a theorem of Montesinos which illustrates the close connection between Seifert spaces and Montesinos knots. This chapter closes with a proposition showing that representations of the double cover of a knot group can often be extended to the whole knot group. It follows that the character variety of a triangle group includes into the character variety of the corresponding three-tangle Montesinos knot.

This is exploited in the last two chapters where we investigate a specific kind of three-tangle Montesinos knot, the (p, q, r) pretzel knot. In Chapter 4 we look at cyclic and finite surgeries of such knots. We first show that if a $(p, q, 2m)$ pretzel knot admits a finite surgery, that surgery is odd integral and near a non-integral boundary slope. We then present Hatcher and Oertel's algorithm for calculating boundary slopes in order to refine our results. We next classify cyclic surgeries on $(-2, p, q)$ pretzel knots and finite surgeries on $(p, q, -2m)$ pretzel knots $(m > 1)$. Combined with Delman's work, this gives a complete classification of cyclic surgeries on Montesinos knots (there are only two non-trivial cyclic surgeries amongst the non-torus Montesinos knots) and a near complete analysis of finite surgeries.

In Chapter 5, we make explicit calculations of the Culler-Shalen seminorms of several knots. We first determine the minimum of the total seminorm for $(2, p, q)$ pretzel knots. We then calculate the Culler-Shalen seminorms of $(-2, 3, n)$ pretzel knots and thereby deduce the structure of the $SL_2(\mathbb{C})$-character variety of those knots. We also investigate to what extent we can make analogous computations for the $(-3, 3, n)$ pretzel knots.

Finally, in Chapter 6 we suggest some ways in which this work could be extended and discuss questions arising from our research. The thesis concludes with an Appendix concerning zeroes of the Alexander polynomials of $(-2, 3, n)$ pretzel knots.

2. Character Varieties and the Culler-Shalen Seminorm

2.1. **Character varieties.** Character varieties will play a fundamental role in what follows. We give here a brief outline of the important facts for our purposes. The standard references for $SL_2(\mathbb{C})$-character varieties are the two articles of Culler and Shalen [CS1, CS2] as well as their part (Chapter 1) of [CGLS]. Boyer and Zhang's [BZ1, Section 2] is also a good reference and [BZ2] develops the theory of $PSL_2(\mathbb{C})$-character varieties. For a more leisurely introduction, we recommend Tanguay's thesis [Ta].

For a finitely generated group π, let $R = \text{Hom}(\pi, SL_2(\mathbb{C}))$ denote the set of $SL_2(\mathbb{C})$-representations. Then R is an affine algebraic set referred to as the *representation variety*. In particular, if π is generated by $g_1, g_2, \dots g_n$, then the entries of the matrices $\rho(g_i) \in SL_2(\mathbb{C})$ can be taken as coordinates. The algebraic set R is then determined by the equations $\det(\rho(g_i)) = 1$ along with those arising from any relations amongst the g_i in π. Thus $R \subset \mathbb{C}^{4n}$. A different choice of generators will result in a set R' isomorphic to R.

The *character variety* X is the set of characters of representations in R. It is also algebraic and generated by

$$\{\chi_\rho(g) | g \in G\} \text{ where } G = \{g_i\}_{1 \le i \le n} \cup \{g_i g_j\}_{1 \le i < j \le n} \cup \{g_i g_j g_k\}_{1 \le i < j < k \le n}$$

(see [GM, Corollary 4.12] and [V]). If we let $t : R \to \mathbb{C}^{|G|}$ denote the function which takes ρ to $(\chi_\rho(g))_{g \in G}$, then the image $t(R)$ is isomorphic to X. Note that $|G|$, the cardinality of G, is $n(n^2 + 5)/6$.

The focus of this thesis is the case where $\pi = \pi_1(M)$ is the fundamental group of the complement M of a *knot* K in S^3. That is, K is the image of a smooth embedding of S^1 in S^3 and $M = S^3 \setminus N(K)$, $N(K)$ being an open tubular neighbourhood of the knot. We will be especially interested in the case where K is *hyperbolic*. This means $\mathring{M}$, the interior of M, admits a hyperbolic metric. In other words, $\mathring{M} = \mathbb{H}^3/\Gamma$ where $\mathbb{H}^3$ is hyperbolic 3-space and $\Gamma \subset PSL_2(\mathbb{C}) = \text{Isom}_+(\mathbb{H}^3)$, the group of orientation preserving isometries of $\mathbb{H}^3$.

In this context, $\Gamma \cong \pi$ so that Γ provides a natural $\mathrm{PSL}_2(\mathbb{C})$-representation, the *holonomy* representation, $\bar{\rho}_0 : \pi \to \mathrm{PSL}_2(\mathbb{C})$. Thurston has shown (see [CS1, Proposition 3.1.1] for a proof) that $\bar{\rho}_0$ lifts to an irreducible $\mathrm{SL}_2(\mathbb{C})$-representation ρ_0. Moreover, the irreducible component $X_0 \subset X$ containing χ_{ρ_0} is a *curve*, i.e., $\dim_{\mathbb{C}}(X_0) = 1$ [CGLS, Proposition 1.1.1]. We refer to X_0 as the *canonical* curve.

The set of characters of reducible representations also forms a curve which is isomorphic to $\mathbb{C}$. (See for example [Ta, Proposition 2.5.5]. This is true whenever M is the complement of a knot in S^3, hyperbolic or not.) Moreover, when the knot is small, any irreducible component of X containing the character of an irreducible representation is again a curve. (A knot is *small* provided there are no closed essential surfaces in M. A surface is *essential* if it is incompressible, orientable, and properly embedded in M such that no component is parallel into ∂M and no S^2 component bounds a ball.)

For such a curve $X_i \subset X$, let $\widetilde{X}_i$ denote the smooth projective variety birationally equivalent to X_i. The birational equivalence is regular at all but a finite number of points of $\widetilde{X}_i$ called *ideal points*. As in [CGLS, Section 1.5], the complement of the ideal points in $\widetilde{X}_i$ may be identified with X_i^{ν} where $\nu : X_i^{\nu} \to X_i$ is normalization [Shf, Chapter II §5].

We will have occasion to use $\mathrm{PSL}_2(\mathbb{C})$-character varieties, particularly when we investigate r-curves (see Section 2.2.3). We will generally use a bar to denote $\mathrm{PSL}_2(\mathbb{C})$ versions of familiar objects. Thus $\bar{R}$ is a $\mathrm{PSL}_2(\mathbb{C})$-representation variety and $\bar{X}$ a $\mathrm{PSL}_2(\mathbb{C})$-character variety. The theory of these objects is similar to the $\mathrm{SL}_2(\mathbb{C})$ theory and we refer the reader to [BZ2, Sections 3 and 4] for details.

Each ideal point of $\widetilde{X}_i$ can be used to construct a simplicial tree on which π acts non-trivially (see [CGLS, Section 1.2]). This gives a splitting of the group π as the fundamental group of a graph of groups. This in turn yields an incompressible surface in the knot exterior M. In this way ideal points are associated with incompressible surfaces in M and vice versa.

2.2. **Culler-Shalen seminorms.** Culler-Shalen seminorms were introduced in the first chapter of [CGLS]. The authors show that the canonical curve $X_0 \subset X$ (see Section 2.1) can be used to construct a norm on surgery space $V = H_1(\partial M; \mathbb{R})$ and this norm plays a central role in their proof of the Cyclic Surgery Theorem. Boyer and Zhang [BZ1, BZ2] recognized that this construction can be extended to any curve in X containing the character of an irreducible representation. However, in general the result is not a norm, but rather a seminorm. These seminorms, which we will call Culler-Shalen seminorms, again figure prominently in Boyer and Zhang's proof of the Finite Filling Conjecture [BZ4]. Moreover, one of the themes of this thesis is that these seminorms are also of great use in understanding Seifert surgeries.

2.2.1. *Dehn Surgery.* Before coming to the seminorm construction, we briefly introduce Dehn surgery. The classical reference here is Rolfsen's book [Rol]. We also recommend Boyer's survey [B2].

Let $M = S^3 \setminus N(K)$ where $N(K)$ is an open tubular neighbourhood of a knot K in S^3. A Dehn surgery on K (or a Dehn filling of M) is the closed manifold $M(\alpha) = M \cup_\alpha S^1 \times D^2$ obtained by attaching a solid torus, $S^1 \times D^2$, and M along their boundary tori, T^2. This glueing is determined by the image $\alpha \subset \partial M$ of a meridional disc $\{\text{pt}\} \times \partial D^2$.

To be precise, the surgery is determined by the isotopy class of α in ∂M. There is a standard choice of basis $\{\mu, \lambda\}$ of $H_1(\partial M; \mathbb{Z})$ where μ is the class of a *meridian* of the knot (the boundary of a disc transverse to the knot) and λ is the class of a *preferred longitude* (a simple closed curve in ∂M which intersects a meridian once transversally, and whose class in homology $H_1(M)$ is zero) [Rol, Section 2.E]. With respect to this basis, α represents the classes $\pm(a\mu + b\lambda)$ (depending on orientation). We will sometimes (by abuse of notation) write $\alpha = (a, b)$. The ambiguity in sign can be removed by taking the ratio $\frac{a}{b}$. Indeed, since the isotopy class of α can be identified (canonically, see [B2, Proposition 2.4]) with $\frac{a}{b} \in \mathbb{Q} \cup \{\frac{1}{0}\}$, we will frequently confuse α with its isotopy class and $\frac{a}{b}$. Using the basis $\{\mu, \lambda\}$, lines of slope $\frac{a}{b}$ in the universal cover, $\mathbb{R} \times \mathbb{R}$ of $\partial M = T^2 = S^1 \times S^1$, project onto curves of class α in ∂M. For this reason, we will often refer to α (and other primitive classes in $H_1(\partial M; \mathbb{Z})$) as a "slope." Finally, taking advantage of the Hurewicz isomorphism, $H_1(\partial M; \mathbb{Z}) \cong \pi_1(\partial M) \cong \mathbb{Z} \oplus \mathbb{Z}$, we will have occasion to think of α or $\frac{a}{b}$ as a homotopy class although this identification is only valid up to conjugation.

If $\pi_1(M(\alpha))$ is cyclic, we will say α is a *cyclic* surgery. The simplest example is *trivial* surgery: $M(\frac{1}{0}) = M(\mu) = S^3$. On the other hand, the only way to obtain $\pi_1(M(\alpha)) \cong \mathbb{Z}$ is through surgery on a *trivial knot*, i.e., the standard, unknotted, embedding of S^1 in S^3 [Ga1]. So cyclic surgeries will generally be examples of *finite* surgeries, i.e., surgeries for which $\pi_1(M(\alpha))$ is finite. A third important category, from our point of view, is the case where $M(\alpha)$ is a Seifert fibred space. (We will discuss this case in more detail in Section 2.2.4 below.)

2.2.2. *Construction of Culler-Shalen seminorms.* Let us now describe the construction of a Culler-Shalen seminorm given a curve $X_i \subset X$. For $\gamma \in \pi = \pi_1(M)$, define the regular function $I_\gamma : X \to \mathbb{C}$ by $I_\gamma(\chi_\rho) = \chi_\rho(\gamma) = \text{trace}(\rho(\gamma))$. By the Hurewicz isomorphism, a class $\gamma \in L = H_1(\partial M; \mathbb{Z})$ determines an element of $\pi_1(\partial M)$, and therefore an element of π, well-defined up to conjugacy. The function $f_\gamma = I_\gamma^2 - 4$ is again regular and so can be pulled back to $\widetilde{X}_i$, the smooth projective variety birationally equivalent to X_i. For $\gamma \in L$, $\|\gamma\|_i$ is the degree of $f_\gamma : \widetilde{X}_i \to \mathbb{CP}^1$. The seminorm is extended to $V = H_1(\partial M; \mathbb{R})$ by linearity.

The power of Culler-Shalen seminorms is perhaps best illustrated by the following two theorems which relate them to cyclic and finite surgeries. Here $s_i = \min\{\|\gamma\|_i ; \gamma \in L, \|\gamma\|_i > 0\}$ denotes the *minimal norm* and a *boundary slope* $\beta \in H_1(\partial M; \mathbb{Z})$ is the class of the boundary of an essential surface properly embedded in M.

Theorem 2.2.1 (Corollary 1.1.4 [CGLS]). *If α is not a boundary slope and $\pi_1(M(\alpha))$ is cyclic, then $\|\alpha\|_i = s_i$.*

Theorem 2.2.2 (Theorem 2.3 [BZ1]). *If α is not a boundary slope and $\pi_1(M(\alpha))$ is finite, then $\|\alpha\|_i \leq max(2s_i, s_i + 8)$.*

Indeed, boundary slopes play a fundamental role in the theory of Culler-Shalen seminorms. Let $\Delta(\gamma, \beta)$ denote the minimal geometric intersection number of curves representing the classes γ and β so that $\Delta(\frac{a}{b}, \frac{c}{d}) = |ad - bc|$. In the context of a knot in S^3 for which μ is not a boundary slope, Lemma 6.2 of [BZ1] can be rearranged to say:

Lemma 2.2.3 (Lemma 6.2 [BZ1]).

$$\|\gamma\|_i = 2[\sum_j a^i_j \Delta(\gamma, \beta_j)]$$

where the a^i_j *are non-negative integers and the sum is over the set of boundary slopes* β_j.

Roughly speaking the a^i_j count ideal points of $\widetilde{X}_i$ associated to β_j (see Section 2.1). In particular, if there are no such, then $a^i_j = 0$.

2.2.3. *Norm curves and r-curves.* Lemma 2.2.3, provides a practical way to describe the difference between norm curves and r-curves. A *norm* curve X_i is one on which no f_γ is constant, $(1 \neq \gamma \in \pi_1(\partial M))$. In this case $\|\cdot\|_i$ is a norm (rather than just a seminorm). In terms of Lemma 2.2.3, this means at least two of the a^i_j are non-zero. An important example of a norm curve is the canonical curve, X_0, in the case of a hyperbolic knot.

If $X_i \subset X$ is not a norm curve (and X is the character variety of the complement of a small knot in S^3 [BZ2, Section 5]), then there is a boundary slope r such that f_r is constant on X_i and we will refer to X_i as an r-curve. In terms of Lemma 2.2.3, this means that $a^i_j = 0$ if $\beta_j \neq r$ and $\|\gamma\|_i = s_i\Delta(\gamma, r)$. In particular, since $M(\frac{1}{0}) = S^3$, we have $\|\frac{1}{0}\|_i = s_i$ (Theorem 2.2.1. Note that $\frac{1}{0}$ is not a boundary slope by Lemma 2.3.1.) This means r is distance 1 from $\frac{1}{0}$ and is therefore an integer. If K is small, then any r-curve X_i includes into the $\mathrm{PSL}_2(\mathbb{C})$-character variety $\bar{X}(M(r))$ (see [BZ2, Example 5.10]).

As we mentioned in Section 2.1, K small also implies each component X_i of X is a curve. Therefore each component leads to a Culler-Shalen seminorm. In this context, we will often want to look at the *total norm* $\|\cdot\|_T = \sum_i \|\cdot\|_i$ and its corresponding minimal value $S = \sum_i s_i$, the sums being taken over the components X_i containing irreducible characters. If M is hyperbolic, this total norm is a norm (and not just a seminorm) since one of the components is the canonical curve X_0 (for which $\|\cdot\|_0$ is a norm).

2.2.4. *Seifert surgery.* If $M(\alpha)$ is a Seifert fibred space, we will say that α is a Seifert surgery slope. Seifert fibred spaces will be described in more detail in Section 3.1.1.

Briefly, when the base orbifold is S^2, such a space is a surgery on $S^2 \times S^1$ which involves the removal of n solid tori T_i which are then reattached using a homeomorphism of ∂T_i. Just as in the case of Dehn surgery, we can describe the attaching homeomorphisms by fractions $\frac{a_i}{b_i} \in \mathbb{Q} \cup \{\frac{1}{0}\}$. In case $n \leq 3$, we will say that the resulting manifold is a *small* Seifert space.

According to Thurston's hyperbolization conjecture, Dehn surgery on a hyperbolic knot in S^3 will result in a manifold which is either hyperbolic, has finite or cyclic fundamental group, contains an essential torus or sphere, or is small Seifert (these outcomes are not disjoint). Moreover, there are only a finite number of surgeries in $\mathbb{Q} \cup \{\frac{1}{0}\}$ which can result in a non-hyperbolic manifold [Thu]. These are called *exceptional* surgeries.

There are two important approaches to the study of exceptional surgeries. Surgeries producing essential surfaces are amenable to investigation via the theory of intersection graphs developed by Gordon and Luecke (see [Go1] for a survey). On the other hand, our discussion of Culler-Shalen seminorms showed how the techniques of Culler, Shalen, Boyer, and Zhang allow us to understand cyclic and finite surgeries.

An important theme of this thesis is that Seifert surgeries can also be studied using Culler-Shalen seminorms. In particular, when $M(\alpha)$ is a small Seifert surgery we can often show $\|\alpha\|_i \leq s_i + C$ where C is a constant depending only on the surgery coefficients b_1, b_2, b_3 (compare Theorems 2.2.1 and 2.2.2). This idea will be developed more explicitly in Section 5.2 (see also [BB]) where we calculate the Culler-Shalen seminorms of small Seifert fillings of the $(-2, 3, n)$ pretzel knot.

2.2.5. *Fundamental and Newton polygons.* Given a Culler-Shalen seminorm $\|\cdot\|$ arising from a curve in the character variety of a knot K, we call B, the norm-disc of radius s in $V = H_1(\partial M; \mathbb{R})$, a *fundamental polygon* for K. When $\|\cdot\|$ is a norm (rather than just a seminorm), the polygon B is compact, convex, and finite-sided with vertices which are rational multiples of boundary slopes in $L = H_1(\partial M; \mathbb{Z})$. It is symmetric $(-B = B)$ and centred at $(0, 0)$. It also provides a nice way to visualize finite and cyclic surgeries. Any cyclic surgeries will lie on ∂B while finite surgeries must lie within $2B$ (assuming $s \geq 8$ and neglecting cyclic or finite surgeries along boundary slopes, see Theorems 2.2.1 and 2.2.2). Recall that the only $\mathbb{Z}$ surgery of a knot in S^3 is on the trivial knot ([Ga1]), so the cyclic surgeries on ∂B will generally be examples of finite surgeries.

The fundamental polygon also provides a direct connection with the A-polynomial invariant of [CCGLS]. This is a polynomial in $\mathbb{Z}[\mathfrak{l}, \mathfrak{m}]$: $A = \sum_{(i,j)} b_{i,j}\mathfrak{l}^i\mathfrak{m}^j$. The *Newton polygon* of such a polynomial in two variables is the convex hull in $\mathbb{R}^2$ of $\{(i,j)|b_{i,j} \neq 0\}$.

Let N denote the Newton polygon of the A-polynomial for a hyperbolic knot K. Let B_0 be the fundamental polygon of the Culler-Shalen seminorm associated to the canonical curve $X_0 \subset X$ (i.e. X_0 contains the character of the holonomy representation). Boyer and Zhang have shown that these polygons are dual in the following sense.

Theorem 2.2.4 (Theorem 1.4 of [BZ4]). *The line through any pair of antipodal vertices of B_0 is parallel to a side of N. Conversely, the line through any pair of antipodal vertices of N is parallel to a side of B_0.*

Thus given the fundamental polygon B_0, one can deduce N, at least up to scaling and translation. As it remains difficult to calculate the A-polynomial, this is worthwhile and we include diagrams of several Newton polygons in this thesis (Figures 4, 5, 26, 27 and 30). The conventions we use are that N meets the $\mathfrak{l}$ and $\mathfrak{m}$ axes but lies in the first quadrant. The scale is provided by Shanahan's width function:

Definition 2.2.5 (Definition 1.2 of [Shn]). *The p/q width $w(p/q)$ of N is one less than the number of lines of slope p/q which intersect N and contain a point of the integer lattice.*

We require that Shanahan's width correspond to the canonical Culler-Shalen seminorm: $w(p/q) = ||q/p||_0$. Given an expression for $|| \cdot ||_0$ as in Lemma 2.2.3, we can quickly find N as is illustrated by the example of the twist knots which follows.

2.3. **Polygons of the twist knots.** In Chapter 5 we will make a rather detailed calculation of the Culler-Shalen seminorms of some pretzel knots. As a prelude, and to illustrate the power of the seminorm approach, we investigate twist knots. Although these can be treated in much the same fashion as the pretzel knots (see [BMZ]), they have some special features which allow a more direct approach, which we adopt here.

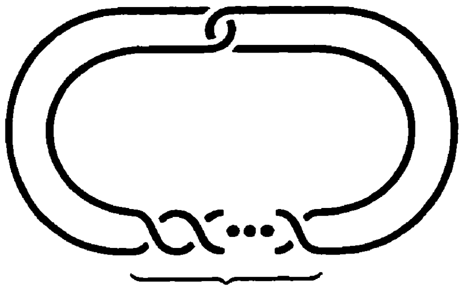


FIGURE 1. The twist knot K_n.

Figure 1 shows the twist knot $K = K_n$. Since the trivial knot K_0 and the trefoil K_1 are not hyperbolic, we will assume $n \neq 0, 1$. Burde [Bu, Section 3] has shown that within the character variety X of a twist knot there is a unique curve X_0 containing irreducible characters (which is therefore the canonical curve).

This allows us to determine s directly for this curve as it is simply related to the number of dihedral representations. One can show (see [Ta, Section 5.3]) that the class $(2, 0) \in H_1(\partial M; \mathbb{Z})$ has norm $s + 2d$ where d is the number of dihedral characters of π. Hence $s = \|(1, 0)\| = \frac{1}{2}\|(2, 0)\| = \frac{1}{2}s + d$ and thus $s = 2d$.

The number d is equal to $(H - 1)/2$ (see for example [Kl, Theorem 10]) where $H = \text{card}(H_1(\Sigma_2))$, Σ_2 being the second branched cyclic cover of the knot ([Rol, Section 10.C]). Since $\text{card}(H_1(\Sigma_2)) = |\Delta(-1)|$ ([Rol, Corollary 8.D.3]), we see that s may be derived from the Alexander polynomial, which is $\Delta(t) = nt^2 + (1 - 2n)t + n$ in the case of the twist knot K_n ([Rol, Exercise 7.B.7]):

$$\begin{aligned} s &= 2d \\ &= |H_1(\Sigma_2)| - 1 \\ &= |\Delta(-1)| - 1 \\ &= \begin{cases} 4|n|, & \text{if } n \leq -1 \\ 4n - 2, & \text{if } n \geq 2 \end{cases} . \end{aligned} \tag{2.1}$$

Twist knots are examples of 2-bridge knots. These knots are sometimes called rational knots as each has an associated rational number. For K_n, the associated number is $\frac{2}{4n-1}$. Hatcher and Thurston [HT] have shown that the incompressible surfaces of such a knot are given by continued fraction expansions of the associated rational number,

$$\frac{p}{q} = r + \frac{1}{b_1 + \frac{1}{b_2 + \ldots + \frac{1}{b_N}}}.$$

We require $|b_j| \geq 2$ and denote such a continued fraction by $[b_1, b_2, \ldots, b_N]$ or simply $[b_j]$.

There are three expansions of $\frac{2}{4n-1}$. The first two we can write immediately:

$$\begin{aligned} \frac{2}{4n-1} &= \frac{1}{2n + \frac{1}{-2}} = [2n, -2] \\ &= \frac{1}{2n - 1 + \frac{1}{2}} = [2n-1, 2]. \end{aligned}$$

The third depends on the sign of n. If n is positive we have

$$[\underbrace{-2, 2, -2, 2, \ldots, -2, 2}_{n-1 \text{ pairs}}, -3]$$

with $n-1$ pairs $-2, 2$, while for n negative,

$$[\underbrace{2, -2, 2, -2, 2, \ldots, 2, -2}_{|n|-1 \text{ pairs}}, 2, -3]$$

with $|n| - 1$ pairs $2, -2$. For each continued fraction, let $b^+ = \text{card}(\{(-1)^{j+1}b_j > 0\})$ and $b^- = \text{card}(\{(-1)^{j+1}b_j < 0\})$.

There is a unique continued fraction whose entries b_j are all even. It corresponds to the longitude of the knot. Let b_0^+ and b_0^- be the b^+, b^- of this surface. For the twist knot, it is the first continued fraction $[2n, -2]$ which has all entries even, so $b_0^+ = 2$, $b_0^- = 0$ when $n > 0$ and $b_0^+ = 1 = b_0^-$ when $n < 0$.

The boundary slope for the surface associated to $[b_j]$ can now be found through comparison with the longitude. It's given as

$$N_{[b_j]} = 2[(b^+ - b^-) - (b_0^+ - b_0^-)].$$

Thus, when $n > 0$, the second continued fraction has $b^+ = 1 = b^-$ and the boundary slope is $2[(1-1) - (2-0)] = -4$. The third boundary slope is $2[(0 - (2n-1)) - 2] = -(4n+2)$. When $n < 0$, we have the boundary slopes -4 and $-4n$.

Ohtsuki [Oht] shows how to determine the Culler-Shalen seminorm given these continued fraction expansions. His strategy is to determine explicitly the number of ideal points for each incompressible surface. He shows that there are $\frac{1}{2}\prod_j(|b_j| - 1)$

such unless all the n_j are even in which case there are $\frac{1}{2}\prod_j(|b_j|-1)-\frac{1}{2}$. This means that, up to a constant k, the Culler-Shalen seminorm is given by

$$||p/q|| = k\sum_{[b_j]}|p - N_{[b_j]}q|[\frac{1}{2}\prod_j(|b_j|-1)] - \frac{1}{2}|p|,$$

the sum being taken over the possible continued fraction expansions of the rational number associated to the knot. (This is almost the formula of [Oht, Proposition 5.2]. Unfortunately, after correctly calculating the number of ideal points on the previous page, Ohtsuki has neglected to subtract $\frac{1}{2}|p|$ in stating the Proposition.)

Remark: Note that there is no ideal point associated to a surface whose continued fraction has all $|b_j| = 2$. These are precisely the surfaces which are the fibres of a fibration of the knot over S^1 (see the remark following the Corollary to [HT, Proposition 1]). In other words, for a fibred 2-bridge knot, there are no ideal points associated to a surface which is a leaf in the fibration. This is true more generally (see for example [CL]). However, such a boundary slope may be represented by other surfaces which are not leaves in a fibration. An example of this is the $(-2, 3, -3)$ pretzel knot, 8_{20}, which admits a fibration with a Seifert surface leaf. In spite of this, as we shall see in Section 5.2, there are ideal points associated to the longitude because there are other surfaces which are not leaves of a fibration, but nonetheless have the longitude as boundary slope.

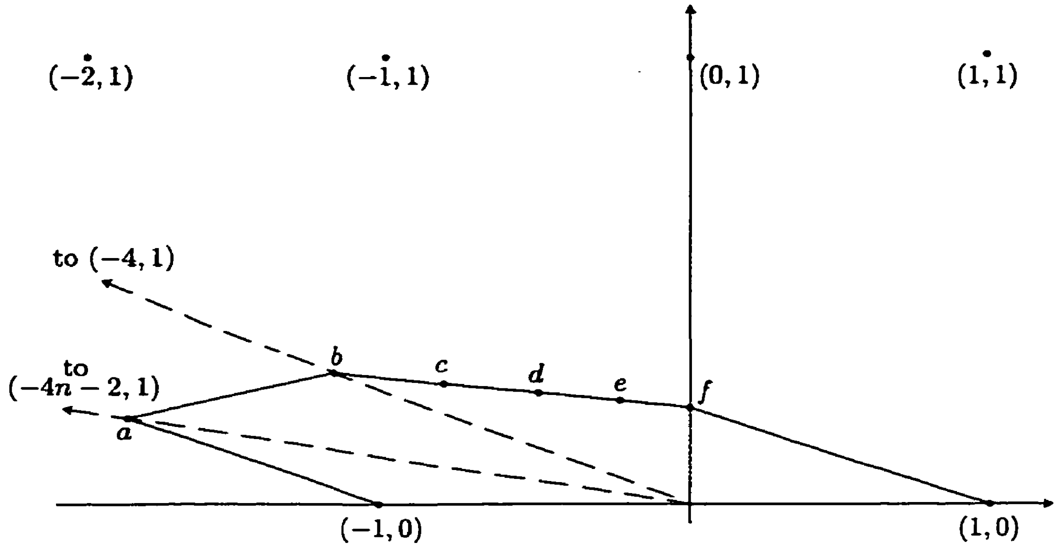
Using Ohtsuki's formula when $n > 0$, we have

$$\begin{aligned} ||p/q|| &= k[(n-1)|p| + (n-1)|p-4q| + |p-(4n+2)q|] \\ &= k[(n-1)\Delta(p/q, 0) + (n-1)\Delta(p/q, -4) + \Delta(p/q, -(4n+2))], \end{aligned}$$

while for $n < 0$,

$$\begin{aligned} ||p/q|| &= k[(|n|-1)|p| - n|p-4q| + |p-4nq|] \\ &= k[(|n|-1)\Delta(p/q, 0) - n\Delta(p/q, -4) + \Delta(p/q, -4n)]. \end{aligned}$$

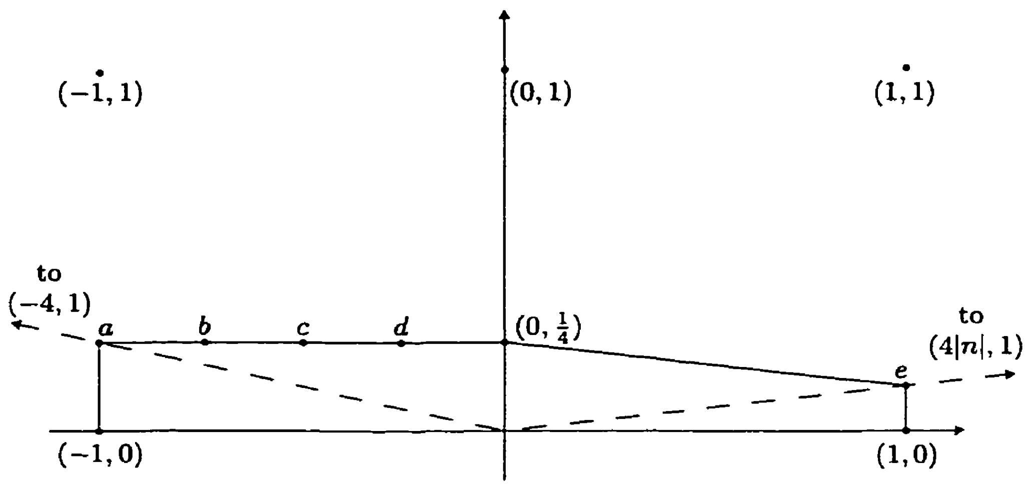
Given that $||1/0|| = s$ (since $M(1/0) = S^3$ has cyclic fundamental group and $1/0$ is not a boundary slope) we can use the calculation of s above (Equation 2.1) to see that $k = 2$.



$a = \frac{s}{2(2sn-s-2)}(-(4n+2), 1)$ $\quad c = \frac{s}{4s-2}(-3, 1)$ $\quad e = \frac{s}{4s+2}(-1, 1)$

$b = \frac{s}{4(s-1)}(-4, 1)$ $\quad d = \frac{1}{4}(-2, 1)$ $\quad f = (0, \frac{s}{4(s+1)})$

FIGURE 2. Fundamental polygon of the twist knot K_n $(n > 1)$.



$a = (-1, \frac{1}{4})$ $\quad c = \frac{1}{4}(-2, 1)$ $\quad e = (1, -\frac{1}{4n})$

$b = \frac{1}{4}(-3, 1)$ $\quad d = \frac{1}{4}(-1, 1)$

FIGURE 3. Fundamental polygon of the twist knot K_n $(n < 0)$.

The fundamental polygon B of the twist knot K_n is illustrated in Figures 2 $(n > 1)$ and 3 $(n < 0)$. In particular, note that the fundamental polygon lies below the line $y = 1/2$. This shows that these knots admit no non-trivial cyclic or finite surgeries as such surgeries would either lie within $2B$ (assuming $n \neq 2$ so that $s \geq 8$, [BZ1, Theorem 2.3]) or else occur at a boundary slope. Since $2B$ includes no integer lattice points (besides $(a, 0)$, $-2 \leq a \leq 2$), the first possibility does not arise. On the other hand, since twist knots are small, the boundary slopes are also not cyclic or finite:

Lemma 2.3.1. *If M is small and α is a boundary slope, then $M(\alpha)$ is not cyclic or finite.*

Proof: By [CGLS, Theorem 2.0.3], $M(\alpha)$ is not finite, and it is cyclic only if $M(\alpha) \cong S^1 \times S^2$. However, Gabai [Ga1] has shown that, amongst knots in S^3, only slope 0 surgery on the trivial knot can produce $S^1 \times S^2$. □

For K_2, $s = 6$ so that $\max(2s, s+8) = s + 8 = 14$ (see Theorem 2.2.2). Here, the highest point of B has $y = s/4(s-1) = 7/26$. So, in this case as well, $\frac{7}{3}B$ includes no integer lattice points other than $(a, 0)$, and therefore K_2 also admits no non-trivial finite surgeries.

Given the Culler-Shalen seminorm calculations, as above, we can also immediately deduce the Newton polygon. For example, for $n > 0$, we see that the edges of slope $\frac{1}{0}$ and $\frac{1}{-4}$ must have "length" $n - 1$, while the edge of slope $\frac{1}{-(4n+2)}$ has only length one. In other words, the vertical segments corresponding to the boundary slope 0 have length $n - 1$ while a segment of slope $-1/4$, like that from $(0, n)$ to $(4(n-1), 1)$ is $(n-1)$ times the vector $(4, -1)$:

$$(0, n) + (n-1)(4, -1) = (0, n) + (4(n-1), 1-n) = (4(n-1), 1).$$

Figures 4 ($n > 1$) and 5 ($n < 0$) illustrate the Newton polygons for these knots.

2.4. **Preliminary lemmas.** Our goal is to develop the theory of $SL_2(\mathbb{C})$-character varieties of Montesinos knots, particularly (p, q, r) pretzel knots. In this section we present several useful lemmas about the structure of $SL_2(\mathbb{C})$- and $PSL_2(\mathbb{C})$-character varieties.

2.4.1. *Commutativity of representations.* The first set of lemmas relate to commutativity. Recall that $A \in SL_2(\mathbb{C})$ is *parabolic* if it is conjugate to a matrix of the form $\begin{pmatrix} \pm 1 & a \\ 0 & \pm 1 \end{pmatrix}$ with $a \neq 0$.

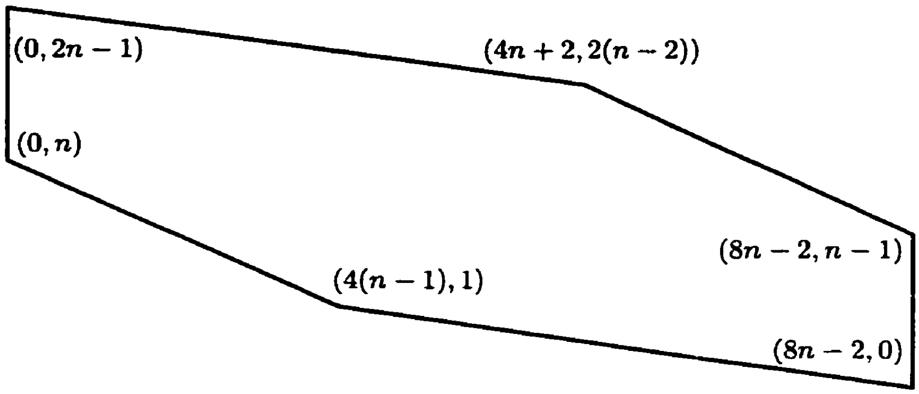


FIGURE 4. Newton polygon of the twist knot K_n $(n > 1)$.

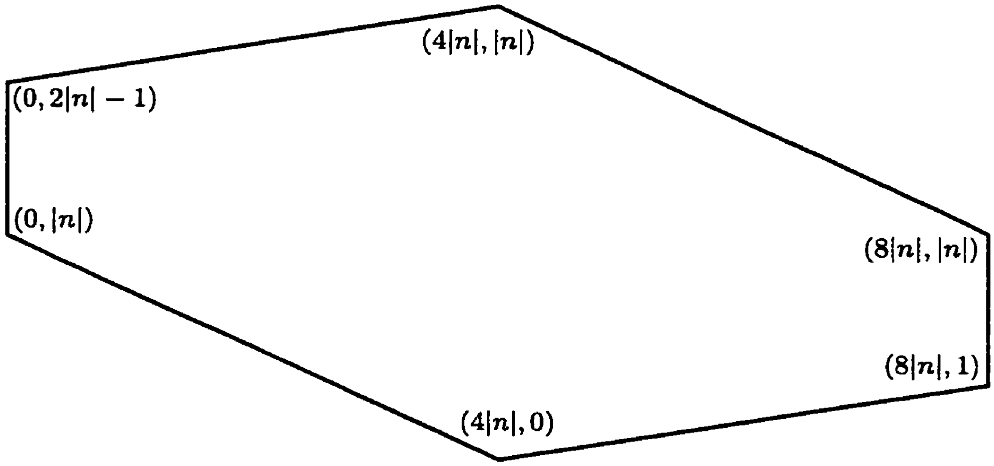


FIGURE 5. Newton polygon of the twist knot K_n $(n < 0)$.

Lemma 2.4.1. *$A, B \in SL_2(\mathbb{C}) \setminus \{\pm I\}$ commute iff they are both diagonal or both parabolic (after an appropriate conjugation).*

Proof: Conjugate so that $A = \begin{pmatrix} \alpha & \beta \\ 0 & 1/\alpha \end{pmatrix}$. Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$AB = \begin{pmatrix} a\alpha + c\beta & b\alpha + d\beta \\ c/\alpha & d/\alpha \end{pmatrix}, \quad \text{and} \quad BA = \begin{pmatrix} a\alpha & a\beta + b/\alpha \\ c\alpha & c\beta + d/\alpha \end{pmatrix}.$$

Suppose A is diagonal. Then $\beta = 0$ and, since $A \neq \pm I$, $\alpha^2 \neq 1$. The lower left entries of AB and BA imply $c = 0$. Similarly, since $\beta = 0$, the upper right entries of AB and BA imply $b = 0$ and so B is diagonal.

Suppose that A is parabolic, i.e., $\alpha = \pm 1$ and $\beta \neq 0$. The lower right entry of AB and BA then implies $c = 0$ while the upper right entry obliges $a = d$. It follows that B is also parabolic. □

Remark: An irreducible $SL_2(\mathbb{C})$-representation is not abelian. However, a reducible $SL_2(\mathbb{C})$-representation could be non-abelian. For example $\rho : \mathbb{Z}/3 * \mathbb{Z}/3 \to SL_2(\mathbb{C})$ defined by taking the generators to $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$ and $\begin{pmatrix} \zeta & 1 \\ 0 & \zeta^{-1} \end{pmatrix}$ where ζ is a primitive third root of unity. On the other hand, the character χ_ρ, of a reducible $SL_2(\mathbb{C})$-representation ρ is always the character χ_{ρ_0} of a diagonal (hence abelian) character ρ_0. Simply replace each matrix $\rho(g) = \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}$ by $\rho_0(g) = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$ and notice that any relations which the $\rho(g)$ satisfy will also hold for $\rho_0(g)$.

Definition 2.4.2. *We will say of a matrix* $A = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{C})$ *that it is diagonal (or parabolic, etc.) if the corresponding matrix* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$ *is diagonal (parabolic etc.). If* $A \in PSL_2(\mathbb{C})$ *is of the form* $\pm \begin{pmatrix} 0 & -a \\ a^{-1} & 0 \end{pmatrix}$, *we will say that* A *is* antidiagonal. *We denote by* N *the set of diagonal and antidiagonal matrices in* $PSL_2(\mathbb{C})$*:*

$$N = \{\pm \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{C}\} \cup \{\pm \begin{pmatrix} 0 & -b \\ b^{-1} & 0 \end{pmatrix} \mid b \in \mathbb{C}\}.$$

E will denote the matrix $\pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in PSL_2(\mathbb{C})$.

Lemma 2.4.3. *Suppose* $A, B \in PSL_2(\mathbb{C}) \setminus \{\pm I\}$ *commute. Then, after an appropriate conjugation, one of the following will obtain.*

1. *A and B are both diagonal.*
2. *A and B are both parabolic.*
3. *$A \in N$ and $B = E$ (or vice versa).*

Proof: Similar to proof of Lemma 2.4.1. □

Definition 2.4.4 ([BZ2]). *A representation $\bar{\rho} : G \to PSL_2(\mathbb{C})$ is called* irreducible *if it is not conjugate to a representation whose image lies in*

$$\{\pm \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \mid a, b \in \mathbb{C}, a \neq 0\}.$$

Otherwise it is called reducible.

Lemma 2.4.5. *Let $\bar{\rho} : G \to PSL_2(\mathbb{C})$ be an irreducible representation. If $\bar{\rho}(G)$ is abelian, then $\bar{\rho}(G) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.*

Proof: Since $\bar{\rho}$ is irreducible, $\exists g \in G \ni \bar{\rho}(g) \neq \pm I$. Conjugate so that $\bar{\rho}(g) = A = \pm \begin{pmatrix} \alpha & \beta \\ 0 & 1/\alpha \end{pmatrix}$. If $\alpha = \pm 1$ and $\beta \neq 0$, the only matrices which commute with A are other parabolics and $\bar{\rho}$ is reducible. If $\alpha \neq \pm 1$, we can conjugate so that $\bar{\rho}(g)$ is diagonal. So without loss of generality, we can assume $\beta = 0$. If $\alpha \neq \pm i$, the only matrices which commute with A are again diagonal and $\bar{\rho}(G)$ is reducible. Therefore $\alpha = \pm i$ and $A = E$. By the previous lemma, the remaining elements of $\bar{\rho}(G)$ are in N. Since $\bar{\rho}$ is irreducible, there is at least one antidiagonal element: $\exists B = \pm \begin{pmatrix} 0 & -b \\ 1/b & 0 \end{pmatrix} \in \bar{\rho}(G)$. Suppose also that $C = \pm \begin{pmatrix} 0 & -c \\ 1/c & 0 \end{pmatrix} \in \bar{\rho}(G)$. Then,

$$BC = \pm \begin{pmatrix} b/c & 0 \\ 0 & c/b \end{pmatrix} \quad \text{and} \quad CB = \pm \begin{pmatrix} c/b & 0 \\ 0 & b/c \end{pmatrix}.$$

So, if B and C commute, then $b^2 = \pm c^2$. Thus, either $B = C$, or else $C = BA = BE$, and these are the only antidiagonal elements of $\bar{\rho}(G)$. Since $\pm I$ and E are the only diagonal elements which commute with B, we see that $\bar{\rho}(G) = \{\pm I, E, B, BE\} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. □

Lemma 2.4.6. *Let $\bar{\rho} : G \to PSL_2(\mathbb{C})$ be an irreducible or non-abelian representation. Suppose A is in the center $Z_{PSL_2(\mathbb{C})}(\bar{\rho}(G)) = \{B \in PSL_2(\mathbb{C}) | BC = CB \ \forall\, C \in \bar{\rho}(G)\}$. Then, either*

1. *$A = \pm I$, or*
2. *A is conjugate to E. Moreover, if $A = D^{-1}ED$, then $D^{-1}\bar{\rho}(G)D \subset N$.*

Proof: Apply Lemma 2.4.3. □

Lemma 2.4.7. *Let $\bar{\rho}: G \to PSL_2(\mathbb{C})$ be an irreducible or non-abelian representation. Let $A \in sl_2(\mathbb{C})$ and suppose $A = Ad\bar{\rho}(g) \cdot A = \bar{\rho}(g)^{-1}A\bar{\rho}(g)$ for each $g \in G$. Then $A = 0$.*

Proof: As $\bar{\rho}$ is irreducible or non-abelian, we can find elements $g_1, g_2 \in G$ so that, after a conjugation, $\bar{\rho}(g_1) \neq \pm I$ is upper triangular while $\bar{\rho}(g_2)$ has non-zero lower left entry. As $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ commutes with $\bar{\rho}(g_1)$, we see that $c = 0$. But, then, in order for A to commute with $\bar{\rho}(g_2)$ as well, a and b must also be zero. □

2.4.2. *The triangle group.* The triangle groups $\Delta(p, q, r)$ are intimately related to the (p, q, r) pretzel knots which are the main focus of Chapters 4 and 5.

Definition 2.4.8. *The triangle group $\Delta(p, q, r)$ has presentation*

$$\begin{aligned} \Delta(p,q,r) &= \langle x, y, z | x^p, y^q, z^r, xyz \rangle \\ &= \langle x, y | x^p, y^q, (xy)^r \rangle. \end{aligned}$$

In particular, we will need to count the characters of $\Delta(p, q, r)$ on several occasions. By [BB, Proposition D], the number of $\mathrm{PSL}_2(\mathbb{C})$-characters of $\Delta(p, q, r)$ is

$$(2.2) \qquad \begin{aligned} &(p - \lfloor \tfrac{p}{2} \rfloor - 1)(q - \lfloor \tfrac{q}{2} \rfloor - 1)(r - \lfloor \tfrac{r}{2} \rfloor - 1) + \lfloor \tfrac{p}{2} \rfloor \lfloor \tfrac{q}{2} \rfloor \lfloor \tfrac{r}{2} \rfloor \\ &+ \lfloor \frac{\gcd(p,q)}{2} \rfloor + \lfloor \frac{\gcd(p,r)}{2} \rfloor + \lfloor \frac{\gcd(q,r)}{2} \rfloor + 1 \end{aligned}$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. This count includes characters of reducible representations. The character of a reducible representation is also the character of a diagonal (hence abelian) representation (see the Remark of Section 2.4.1). So, to count the characters of reducible representations we can look at representations of $H_1(\Delta(p, q, r))$. Let $a = \gcd(p, q, r)$, $b = \gcd(pq, pr, qr)$. Then $H_1(\Delta(p, q, r)) = \mathbb{Z}/a \oplus \mathbb{Z}/(b/a)$ while $|H_1(\Delta(p, q, r))| = b$. Consequently, the number of characters of $H_1(\Delta(p, q, r))$ is

$$(2.3) \qquad \begin{aligned} &\lfloor \tfrac{b}{2} \rfloor + 1, \quad \text{if } a \equiv 1 \pmod 2, \\ &\lfloor \tfrac{b}{2} \rfloor + 2, \quad \text{if } a \equiv 0 \pmod 2. \end{aligned}$$

Lemma 2.4.9. *Let $G = \Delta(p, q, r)$ with p and q odd. If $\bar{\rho}$ is an irreducible $PSL_2(\mathbb{C})$-representation of G and $A \in Z_{PSL_2(\mathbb{C})}(\bar{\rho}(G))$, then $A = \pm I$.*

Proof: Suppose instead that A is conjugate to E (see Lemma 2.4.6). Conjugate so that $\bar{\rho}(G) \subset N$. Since the set of diagonal matrices is closed under multiplication and $\bar{\rho}$ is irreducible, at least one of the generators is taken to an antidiagonal matrix. However, antidiagonal matrices are of order 2 in $\mathrm{PSL}_2(\mathbb{C})$, so we have a contradiction if r is odd. If r is even, the corresponding generator is mapped to an antidiagonal matrix while the other two generators are mapped to diagonal matrices. This is a contradiction since the product of the three generators is the identity. □.

Lemma 2.4.10. *If p and q are odd, then no irreducible $PSL_2(\mathbb{C})$-representation of $\Delta(p,q,r)$ has finite dihedral image.*

Proof: Suppose instead that $\bar{\rho} : \Delta(p,q,r) \to \mathrm{PSL}_2(\mathbb{C})$ has dihedral image D. If $D \cong \mathbb{Z}/2\oplus\mathbb{Z}/2$, then $\bar{\rho}(x^2) = \pm I$ and $\bar{\rho}(y^2) = \pm I$, x and y being the generators of $\Delta(p,q,r)$ (Definition 2.4.8). Since these also have odd order p and q respectively, it follows that $\bar{\rho}(x) = \pm I$ and $\bar{\rho}(y) = \pm I$ and the representations is trivial, a contradiction.

Thus, we can assume the index two cyclic subgroup C of D has order at least 3. It follows that, after an appropriate conjugation, C consists of diagonal matrices (use Lemma 2.4.3 and the infinite order of non-identity parabolics). As $\bar{\rho}$ is irreducible, $\bar{\rho}(x)$, say, is not diagonal. Then $\bar{\rho}(x) \in D \setminus C$ has order two. As above, this implies $\bar{\rho}(x) = \pm I$, a contradiction. □

2.5. **Riley's p-representations.** As we shall see, an effective way of calculating the Culler-Shalen seminorm of a slope α is through comparison with the meridian μ and we will encounter equations of the form

$$\|\alpha\| = s + \sum (Z_x(f_\alpha) - Z_x(f_\mu))$$

where $Z_x(f_\gamma)$ denotes the zero of f_γ at a point x in the character variety (for example, see Equation 5.12). Boyer and Ben Abdelghani [BB, Theorem A] have recently shown that, subject to some mild conditions, the "jump" $Z_x(f_\alpha) - Z_x(f_\mu)$ is 2.

Prior to this work, jump calculations broke down into two cases. Let $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ be a representation with character x. If $\rho(\pi_1(\partial M))$ is not parabolic, arguments like those in [BZ1, Section 4] can often be applied to show $Z_x(f_\mu) = 0$ and $Z_x(f_\alpha) = 2$ resulting in a jump of two. If $\rho(\pi_1(\partial M))$ is parabolic (see Section 2.4.1

for a definition), then following Riley [Ri], we say that ρ is a p-representation. We were able to show $Z_x(f_\mu) = 1$ for the character of a p-representation in two rather disparate situations: the twist knots and the $(-2, 3, n)$ pretzel knots. Given $Z_x(f_\mu) = 1$ one can again apply the techniques of [BZ1, Section 4] to see that $Z_x(f_\alpha) = 3$ and the jump is two.

It may well be that $Z_x(f_\mu)$ is always one at the character x of a p-representation, as we know of no evidence to the contrary. We will therefore include our calculations for these two families of knots as evidence supporting this conjecture.

2.5.1. *The twist knots.* We introduced the twist knots K_n in Section 2.3. Again, we assume $n \neq 0, 1$ so that K_n is hyperbolic. Let $M = S^3 \setminus N(K_n)$ denote the complement of such a knot. Burde [Bu, Section 3] has shown that within the $SL_2(\mathbb{C})$-character variety of a twist knot there is a unique curve X_0 containing the characters of irreducible representations.

Proposition 2.5.1. *The characters of the irreducible p-representations of a hyperbolic twist knot K_n are smooth points of X_0. Furthermore $Z_x(f_\mu) = 1$ at the character x of any such representation.*

Proof: The minimal norm s on X_0 is twice the number of dihedral characters d (Equation 2.1). On the other hand, Riley [Ri] shows that irreducible parabolic characters x with $x(\mu) = 2$ correspond to the roots of a polynomial of degree d. Moreover, he argues ([Ri, Theorem 3]) that the polynomial has no repeated roots, i.e., there are d such points in X_0. Similarly there are d points with $x(\mu) = -2$ and s in all. Let us label them $x_1, \ldots, x_s$ and let $\nu^{-1}(x_i) = \{y_{i1}, \ldots, y_{ik_i}\}$, $i = 1 \ldots s$, where $\nu : X_0^\nu \to X_0$ is normalization [Shf, Chapter II §5]. Then $\|\mu\|$ is equal to the sum of degrees at these points:

$$s = \|\mu\| = \sum_{i=1}^{s} \sum_{j=1}^{k_i} Z_{y_{ij}}(f_\mu).$$

Since $Z_{y_{ij}}(f_\mu) \geq 1$, we must have $k_i = 1$, $(i = 1 \ldots s)$ and, setting $y_i = y_{i1}$, $Z_{y_i}(f_\mu) = 1$. If x_i were singular, then $Z_{y_i}(f_\mu) \geq 2$ ([Ta, Lemme 5.4.2]). We conclude therefore that each x_i is a smooth point of X_0 and $Z_{\nu^{-1}(x_i)}(f_\mu) = 1$. $\square$

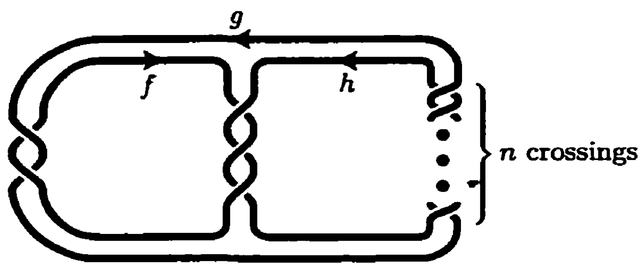


FIGURE 6. The $(-2,3,n)$ pretzel knot.

2.5.2. $(-2,3,n)$ *pretzel knots.* Figure 6 shows the $(-2,3,n)$ pretzel knot K_n. We will look at this knot in detail in Section 5.2. If n is even, K_n is a link. Also, K_1, K_3, and K_5 are torus knots and therefore not hyperbolic. So, we will assume that n is an odd integer, $n \neq 1,3,5$. Let $M = S^3 \setminus N(K_n)$ denote the knot exterior and X its $\mathrm{SL}_2(\mathbb{C})$-character variety.

Proposition 2.5.2. *The characters of the irreducible p-representations of the complement M of a hyperbolic $(-2,3,n)$ pretzel knot are simple points of X. Furthermore $Z_x(f_\mu) = 1$ at the character x of any such representation.*

Proof: Starting from the Wirtinger presentation [Rol, Section 3.D], we can show $\pi = \pi_1(K_n) = \langle f, g, h \mid hfhg = fhgf, gf(hg)^{(n-1)/2} = f(hg)^{(n-1)/2}h \rangle$ where the generators f, g and h are as indicated in Figure 6 (compare [Tr]). Let $x = \chi_\rho$ be the character of an irreducible p-representation with $x(\mu) = 2$. Following Riley, we can assume

$$\rho(f) = \begin{pmatrix} 1-uv & -v^2 \\ u^2 & 1+uv \end{pmatrix}, \ \rho(g) = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}, \text{ and } \rho(h) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Then the relation $hfhg = fhgf$ implies $u - v = \pm 1$. Moreover a representation with $u - v = -1$ can be replaced by one with $u - v = 1$ simply by changing the signs of both u and v. As these two representations will have the same character, we can assume $u - v = 1$. After the substitution $u = v + 1$, the upper left entry of $\rho(hfhg) - \rho(fhgf)$ becomes $v^2(vw - (v+1)(v+2))$. So $v = 0$ or $w = (v+1)(v+2)/v$.

If $v = 0$, the second relation of π implies that the characters of p-representations correspond to the distinct roots of a polynomial of degree $(|n| - 1)/2$ (compare [Ri, Theorem 3]).

If $v \neq 0$ the same relation implies that v is the root of a polynomial $p_n(v)$ of degree $|n-3|-1$. These are the polynomials described in Lemmas 2.5.3 and 2.5.4 below where it is shown that each p_n has distinct roots. Thus, there are $|n-3|-1$ distinct characters with $v \neq 0$.

Moreover, the characters with $v=0$ differ from those with $v \neq 0$ as may be seen by examining $\text{trace}(\rho(fg)) = 2-(v+1)^2$. If $v=0$, the trace is 1. To have the same trace with $v \neq 0$, we must choose $v=-2$. This implies $w=0$ and $\rho(g)$ and $\rho(h)$ commute. The second relation of $\pi_1(K_n)$ then simplifies to $gf = fh$ and it is easy to verify that the choice $v=-2$, $w=0$ and $u=v+1=-1$ is not consistent with $\rho(gf)=\rho(fh)$. Thus there is no character with $v \neq 0$ equal to a character with $v=0$.

So, in all there are $3(|n-2|-1)/2$ irreducible parabolic characters x with $x(\mu)=2$. Similarly there are $3(|n-2|-1)/2$ points with $x(\mu)=-2$ and S in all. (We will see in Section 5.2 that $S=3(|n-2|-1)$, where S is the total minimal norm.) Let us label the points $x_1, \ldots, x_S$ and let $\bigcup_{\{j; x_i \in X_j\}} \nu_j^{-1}(x_i) = \{y_{i1}, \ldots, y_{ik_i}\}$, $i = 1 \ldots S$, where $\nu_j : X_j^\nu \to X_j$ is normalization. Then $\|\mu\|_T$ is equal to the sum of degrees at these points:

$$S = \|\mu\|_T = \sum_{i=1}^{S} \sum_{j=1}^{k_i} Z_{y_{ij}}(f_\mu).$$

Since $Z_{y_{ij}}(f_\mu) \geq 1$, we must have $k_i = 1$, $(i = 1 \ldots S)$ and, setting $y_i = y_{i1}$, $Z_{y_i}(f_\mu) = 1$. If x_i were singular, then $Z_{y_i}(f_\mu) \geq 2$ [Ta, Lemme 5.4.2]. We conclude therefore that each x_i is a simple point of X and $Z_{\nu^{-1}(x_i)}(f_\mu) = 1$. □

Lemma 2.5.3. *Let p_n be the polynomials of degree $2-n$ defined recursively by the equations*

$$p_{-1}(v) = v^3 + 2v^2 + v + 1,\ p_{-3}(v) = -(v^5 + 3v^4 + 4v^3 + 5v^2 + 4v + 2),$$

$$\textit{and } p_{n-2}(v) = -((v^2+v+2)p_n(v) + v^2 p_{n+2}(v))$$

where $n \leq -1$ is odd. Then p_n has distinct roots.

Proof: (I am indebted to Richard Stong [St] for this argument.) First note that p_n has leading term $(-1)^{(n+1)/2} v^{2-n}$ and constant term $p_n(0) = (-2)^{-(n+1)/2}$. Therefore the roots of p_n are all algebraic integers and $v=0$ is never a root. Since $v=0$

is not a root, the roots of p_n are the same as the roots of the Laurent polynomial $L_{(1-n)/2} = v^{(n-1)/2}p_n$.

Notice that for $n \geq 0$,

$$\begin{aligned} L_0(v) &= -v - 1 + 1/v, \\ L_1(v) &= v^2 + 2v + 1 + 1/v, \text{ and} \\ 0 &= L_n + (v + 1 + 2/v)L_{n-1} + L_{n-2}. \end{aligned}$$

Letting $y = -(v + 1 + 2/v)/2$ we see that L_n satisfies the Tchebyshev recursion: $L_n - 2yL_{n-1} + L_{n-2} = 0$.

The standard solutions to this recursion are the Tchebyshev polynomials $T_n(y) = \cos(nt)$ and $P_n(y) = \frac{\sin(nt)}{\sin t}$, where $\cos t = y$. It is well-known that $T_n(y)$ and $P_n(y)$ are polynomials in y, $T_0(y) = 1$, $T_1(y) = y$, $P_0(y) = 0$, $P_1(y) = 1$, and both satisfy the same recursion as L. Therefore,

$$\begin{aligned} L_n(v) &= L_0(v)T_n(y) + (L_1(v) - yL_0(v))P_n(y) \\ &= L_0(v)\cos(nt) + (L_1(v) - yL_0(v))\frac{\sin(nt)}{\sin t} \\ &= A(v)\cos(nt(v) + B(v)), \end{aligned}$$

where $\cos t = y = -(v + 1 + 2/v)/2$,

$$\begin{aligned} A(v) &= (L_0(v)^2 + (L_1(v) - yL_0(v))^2/\sin^2 t)^{1/2} \\ &= \sqrt{\frac{-4}{v(v^2 - v + 2)}}, \end{aligned}$$

and $B(v)$ is given by the formulas

$$\begin{aligned} \cos B(v) &= L_0(v)/A(v) \\ \sin B(v) &= -(L_1(v) - yL_0(v))/(A(v)\sin t). \end{aligned}$$

Note that this formula requires special interpretation if $y = 1$ or -1. In this case $\sin t = 0$ and P_n requires a limit to define. However, in our case, $y = 1$ means $v^2 + 3v + 2 = 0$ so $v = -1$ or -2, and $y = -1$ means $v^2 - v + 2 = 0$ so $v = (1 \pm i\sqrt{7})/2$.

One easily checks that:

$$\begin{aligned} L_n(-1) &= -1, \\ L_n(-2) &= 1/2, \text{ and} \\ L_n((1 \pm i\sqrt{7})/2) &= (-1)^n(2n - 5 \mp (3 + 2n)i\sqrt{7})/4. \end{aligned}$$

Thus none of these four values of v is ever a root of any of the p_n or L_n and we need not worry about this special case.

Returning to the discussion above, for all other v we have $L_n(v) = A(v)\cos(nt(v) + B(v))$ and $A(v)$ is nonzero. Therefore roots of L_n occur exactly for zeroes of the cosine. Now we look for a double root. We have

$$\begin{aligned} L_n'(v) &= A'(v)\cos(nt(v) + B(v)) + A(v)(nt'(v) + B'(v))\sin(nt(v) + B(v)) \\ &= (A'/A)L_n(v) + A(v)(nt'(v) + B'(v))\sin(nt(v) + B(v)). \end{aligned}$$

Since at any root of L_n we have $\sin(nt(v) + B(v)) = \pm 1$, a double root can only occur if we simultaneously have a root of $nt'(v) + B'(v) = 0$.

Since $\cos t = y$,

$$t'(v) = -y'/\sin t = (v^2 - 2)/(2v^2 \sin t),$$

and since $\cos B = L_0/A$,

$$B'(v) = (L_0 A' - L_0' A)/(A^2 \sin B).$$

Now as $A \sin B = (yL_0 - L_1)/\sin t$ we find

$$B'(v) = -\sin y \frac{5v^2 + 4v + 2}{2(v+1)(v+2)(v^2 - v + 2)}.$$

The only zeroes of $t'(v)$ occur at $v = \pm\sqrt{2}$ at which point $B'(v) \neq 0$. Thus the only possible double root left to consider is when

$$n = -B'(v)/t'(v) = -\frac{5v^2 + 4v + 2}{2(v^2 - 2)}.$$

This is equivalent to

$$(2n + 5)v^2 + 4v + (2 - 4n) = 0,$$

or

$$v = \frac{-2 \pm \sqrt{8n^2 + 16n - 6}}{2n + 5}.$$

If this v were a root of L_n, then it would be a double root and these are the only values of v with this property. However the p_n (and therefore the L_n) have roots which are algebraic integers. The quantity whose square root we take in the formula for v is congruent to 2 (mod 4) and hence is not a square. Therefore v is not an algebraic integer unless $2n + 5$ divides 4 (see, for example, Corollary 2 in Chapter 2 of [Mrc]). This only occurs for $n = -2$ or -3. Thus these two bad values of v are never roots of L_n for $n > 0$ (and therefore not roots of the corresponding p_n). Thus p_n has no double root. □

Lemma 2.5.4. *Let p_n be the polynomials of degree $n - 4$ defined recursively by the equations*

$$p_7(v) = -(v^3 + 2v^2 + 8v + 8),\ p_9(v) = v^5 + 4v^4 + 10v^3 + 16v^2 + 24v + 16,$$

$$\text{and } p_{n-2}(v) = -((v^2 + v + 2)p_n(v) + v^2 p_{n+2}(v))$$

where $n \geq 7$ is odd. Then p_n has distinct roots.

Proof: Similar to that of the previous lemma □

3. Montesinos Knots

3.1. Seifert spaces and Montesinos knots. Montesinos knots are knots whose 2-fold branched cyclic cover Σ_2 is a Seifert fibred space having as base orbifold S^2 with a certain number of cone points. Since $\pi_1(\Sigma_2) = \bar{\pi}/\langle\mu^2\rangle$, we can think of Σ_2 as a kind of Seifert surgery of the knot, and we will take advantage of this similarity. To better understand these knots, we begin with a brief introduction to the theory of Seifert spaces.

3.1.1. *Seifert fibred spaces.* This presentation of Seifert spaces is taken from Chapter 4 of Hatcher's [Ha] notes (see also [Sco, Section 3]).

Definition 3.1.1. *A* model Seifert fibring *of $S^1 \times D^2$ is a foliation of $S^1 \times D^2$ by circles, called* fibres *constructed as follows. Starting with $[0,1] \times D^2$ foliated by the segments $[0,1] \times \{pt\}$, identify the disks $\{0\} \times D^2$ and $\{1\} \times D^2$ by a $2\pi\beta/\alpha$ rotation, for $\beta/\alpha \in \mathbb{Q}$.*

Definition 3.1.2. *A* Seifert fibred space *is a three-manifold with a foliation by circles such that each fibre has a neighbourhood diffeomorphic, preserving fibres, to a neighbourhood of a fibre in some model Seifert fibring of $S^1 \times D^2$.*

Definition 3.1.3. *The* multiplicity *of a fibre circle C in a Seifert fibred space is the number of times a small disk transverse to C meets each nearby fibre. The fibres of multiplicity* 1 *are* regular *fibres and the other fibres are* singular.

Definition 3.1.4. *The* base orbifold *of a Seifert fibred space is the two dimensional orbifold obtained by identifying each fibre to a point. The images of the singular fibres are called* cone points, *the* order *of a cone point being the multiplicity of its singular fibre.*

We will be most interested in the case where the base orbifold is S^2 along with some cone points. (For an introduction to the theory of orbifolds, see [Sco, Section 2].) Such a Seifert fibred space is completely determined by a listing of the model fibrings of neighbourhoods of the singular fibres and we will denote it as $V(\beta_1/\alpha_1, \beta_2/\alpha_2, \dots, \beta_r/\alpha_r)$. In this case, the α_i are the cone point indices for the base orbifold which we will denote by $S^2(\alpha_1, \alpha_2, \dots, \alpha_r)$.

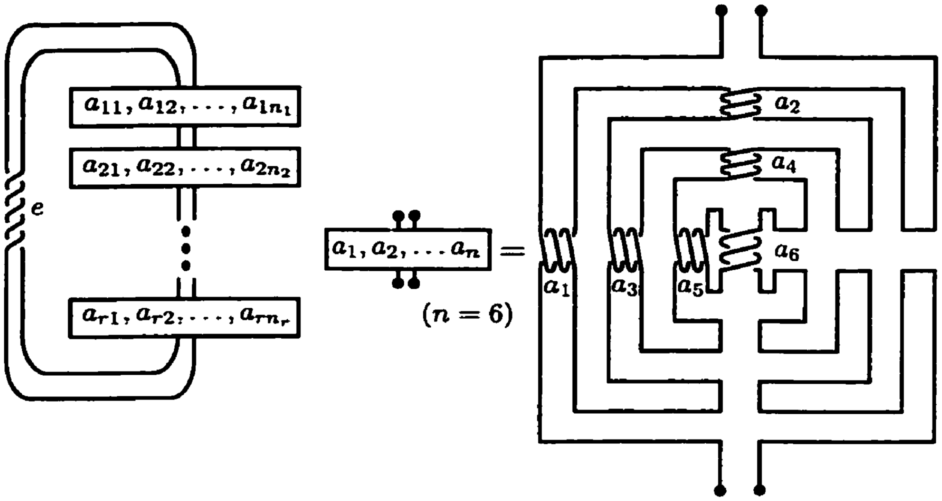


FIGURE 7. The Montesinos link $\mathfrak{m}(e; \beta_1/\alpha_1, \beta_2/\alpha_2, \ldots, \beta_r/\alpha_r)$.

3.1.2. *Montesinos knots.* A link as illustrated in Figure 7 is called a Montesinos link and will be denoted by $\mathfrak{m}(e; \beta_1/\alpha_1, \beta_2/\alpha_2, \ldots, \beta_r/\alpha_r)$. Here β/α describes a rational tangle and we assume that α and β are relatively prime integers and $\alpha > 1$. The a_i give a continued fraction expansion of β/α,

$$\beta/\alpha = \cfrac{1}{a_1 + \cfrac{1}{-a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{\pm a_n}}}}.$$

Generally, we can arrange $e = 0$ by combining those twists with one of the tangles β_i/α_i. In this case, we will write simply $\mathfrak{m}(\beta_1/\alpha_1, \beta_2/\alpha_2, \ldots, \beta_r/\alpha_r)$.

As the rest of the thesis will be concerned exclusively with Montesinos knots, we here introduce some notation for this situation. Let $M = S^3 \setminus N(\mathfrak{m})$ denote the knot complement, $\widetilde{M}$ the 2-fold cyclic cover of M, and Σ_2 the 2-fold branched cyclic cover (see [Rol, Chapter 6] and [Rol, Section 10.C]). Theorem 3.2.1, which we prove in the next section, shows that $\Sigma_2 = V(\beta_0/\alpha_0, \beta_1/\alpha_1, \beta_2/\alpha_2, \ldots \beta_k/\alpha_k)$ with $\beta_0/\alpha_0 = -e$. Much of what follows depends on the relationships amongst the fundamental groups of these manifolds.

First, $\tilde{\pi} = \pi_1(\widetilde{M})$ is an index two subgroup of $\pi = \pi_1(M)$. The group of Σ_2 can be understood in two ways. On the one hand, $\pi_1(\Sigma_2) = \tilde{\pi}/\langle \mu^2 \rangle = \tilde{\pi}/\langle \tilde{\mu} \rangle$ where $\mu \in \pi$ is the class of a meridian of $\mathfrak{m}$ and $\tilde{\mu} \in \tilde{\pi}$ the class of the loop in $\widetilde{M}$ which (double) covers that meridian. On the other hand, due to the Seifert structure, we have the exact sequence (for example, see [Sco, Section 3])

$$0 \to F \to \pi_1(\Sigma_2) \to \pi_1^{\text{orb}}(\mathcal{B}) \to 1 \tag{3.4}$$

where $F \cong \mathbb{Z}$ is the center of $\pi_1(\Sigma_2)$, generated by the class h of a regular fibre, and $\pi_1^{\text{orb}}(\mathcal{B})$ is the fundamental group of the base orbifold $\mathcal{B} = S^2(\alpha_1, \dots \alpha_r)$. In particular, we have the presentations ([Sco, Section 2])

$$\pi_1^{\text{orb}}(\mathcal{B}) = \langle x_1, x_2, \dots, x_r \mid x_1^{\alpha_1}, x_2^{\alpha_2}, \dots, x_r^{\alpha_r}, x_1 x_2 \dots x_r \rangle,$$

and ([BuZ, Equation 12.31])

$$\pi_1(\Sigma_2) = \langle x_1, x_2, \dots, x_r, h \mid x_i^{\alpha_i} h^{\beta_i}, [x_i, h](1 \le i \le r), x_1 x_2 \dots x_r h^{-e} \rangle. \tag{3.5}$$

Note that when $r = 3$, we recover the triangle group (Section 2.4.2), $\pi_1^{\text{orb}}(\mathcal{B}) = \Delta(\alpha_1, \alpha_2, \alpha_3)$. We conclude this section with a lemma pertinent to this case.

Lemma 3.1.5. *Suppose M is Seifert fibred over $S^2(p, q, r)$ with $p, q, r \ge 2$ and that $\bar{\rho} : \pi_1(M) \to PSL_2(\mathbb{C})$ is an irreducible or non-abelian representation. Then $\bar{\rho}$ factors through $\Delta(p, q, r)$.*

Proof: As above, we have the exact sequence

$$0 \to F \to \pi_1(M) \to \Delta(p, q, r) \to 1$$

where $F = \langle h \rangle \cong \mathbb{Z}$. We need to argue that $\bar{\rho}(h) = \pm I$.

Lemma 2.4.6 says that if $\bar{\rho}(h) \ne \pm I$, then we can conjugate so that $\bar{\rho}(h) = E$ and $\bar{\rho}(\pi_1(M)) \subset N$. If p, q, r are all odd, we can argue as in the proof of Lemma 2.4.9. So we can assume p is even and that $\rho(x)$ is antidiagonal where x is the generator of $\pi_1(M)$ satisfying $x^p = h^a$ (see Equation 3.5). As the antidiagonal elements have order two, $\rho(x^2) = \pm I$. Using the relation $x^p = h^a$ we see that $\rho(h)$ has order dividing a where $(a, p) = 1$. However, $\rho(h) = E$ has order two. This contradiction shows that $\rho(h) = \pm I$ in the case of even p as well. □

3.2. **Montesinos' theorem.** In this section we present Théorème 1 of Montesinos' Orsay notes and its proof closely following those notes to which we refer the reader for more details (see also [BuZ, Section 12.D]).

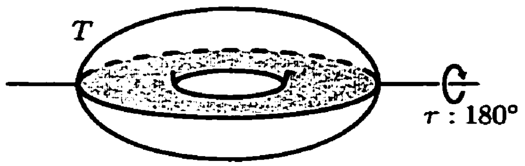


FIGURE 8. An annulus preserved by τ.

Theorem 3.2.1 (Théorème 1 of [Mo]). *Let $V = V(\beta_0/\alpha_0, \beta_1/\alpha_1, \beta_2/\alpha_2, \ldots, \beta_r/\alpha_r)$ be a Seifert fibred space with $(\alpha_0, \beta_0) = (1, -e)$, α_i, β_i relatively prime, and $\alpha_i > 1$ $(1 \leq i \leq r)$. There is an involution τ of V such that the quotient V/τ is homeomorphic to S^3 with the Montesinos link $\mathfrak{m} = \mathfrak{m}(e; \beta_1/\alpha_1, \beta_2/\alpha_2, \ldots \beta_k/\alpha_k)$ as branching set, i.e. $\mathfrak{m}$ is the image of the fixed points under the quotient map $V \to V/\tau$. Moreover, τ is fibre and orientation preserving and acts on the base orbifold S^2 as a reflection in a circle passing through the cone points.*

Remark: It follows that V is Σ_2, the two-fold branched cyclic cover of $\mathfrak{m}$.

The proof requires the following elementary lemma which we state without proof.

Lemma 3.2.2. *Let X be a manifold containing a 3-ball B^3 and let f be an automorphism of $S^2 = \partial B^3$. Suppose F is an extension of f to an automorphism of B^3, the cone of S^2 (for example, $F = cone(f)$). Then there is a homeomorphism $G : (X \setminus \overset{\circ}{B^3}) \cup_f B^3 \xrightarrow{\cong} X$ given by*

$$\begin{aligned} G|_{X\setminus \overset{\circ}{B^3}} &= id|_{X\setminus \overset{\circ}{B^3}} \\ G|_{B^3} &= F. \end{aligned}$$

Proof: (of Montesinos' Théorème 1) We will define the involution τ in several steps.

Consider the solid torus $T = B^2 \times S^1$ equipped with an angle π rotation $\tau : (z_1, z_2) \to (\bar{z}_1, \bar{z}_2)$. In Figure 8 we've shaded an annulus $\{z_1 \in \mathbb{R}\}$ preserved by τ.

The quotient T/τ of T by this rotation is a ball B^3 and the image of the fixed points consists of two unlinked arcs. To the left of Figure 9 we see two choices of fundamental domain of τ with the induced identifications. On the right, we see an equatorial disc of the quotient B^3. The lightly shaded band at right represents the quotient by τ of the annulus of Figure 8.

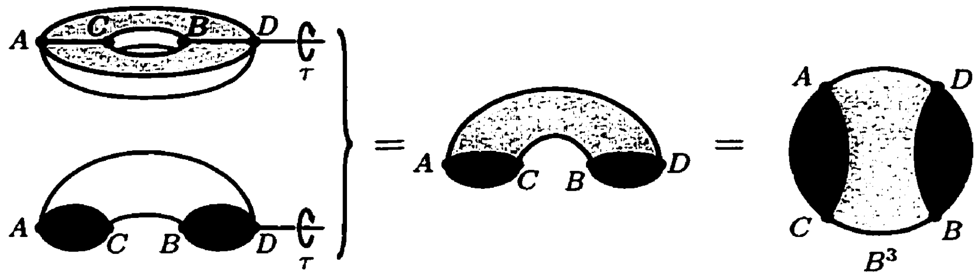


FIGURE 9. The quotient of T by τ.

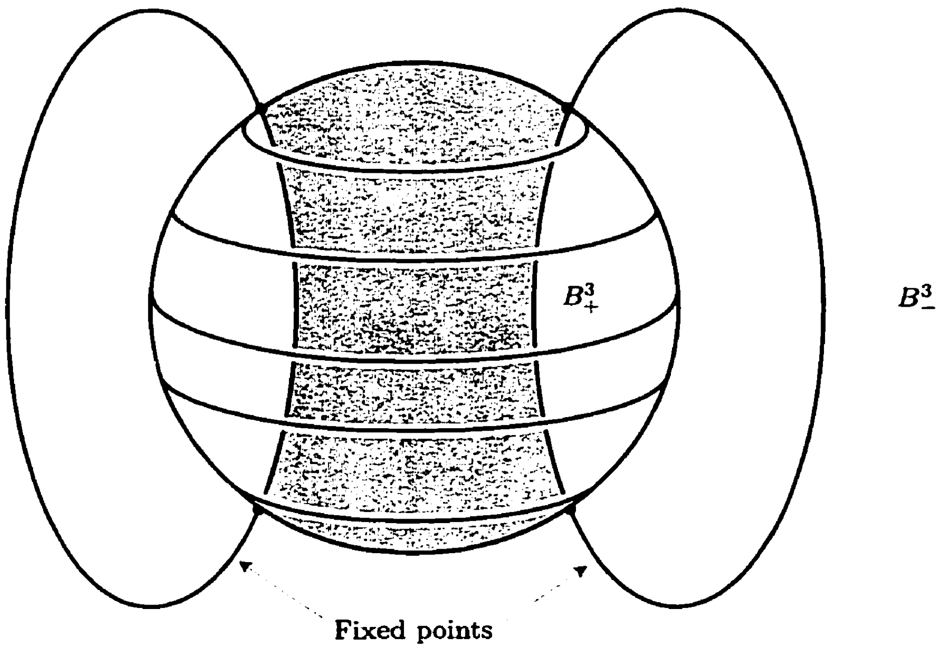


FIGURE 10. The double of the quotient of T.

The rotation τ of T defines an involution τ on the double torus $D(T) = S^2 \times S^1$. The double of the quotient is then $D(B^3) = S^3$ and the image of the fixed points is two disjoint, unknotted circles (Figure 10). On the base $S^2 = D(B^2)$ of the trivial fibration $S^2 \times S^1 = D(T)$, τ acts as reflection in the circle $D(B^1)$.

Now, the Seifert space $V = V(\beta_0/\alpha_0, \beta_1, \alpha_1, \ldots, \beta_r/\alpha_r)$ can be constructed starting with $S^2 \times S^1$ by removing the interiors of $r+1$ disjoint solid tori $T_i = D_i \times S^1$ $(i = 0, \ldots, r)$ and then replacing them using an oriented automorphism ϕ_i of ∂T_i which takes the meridian m_i to the curve $\alpha_i m_i + \beta_i l_i$, l_i being the longitude.

We can assume that the disks D_i used in constructing V are centered on $B^1(\subset B^2 \subset D(B^2))$ as in Figure 11. So we can choose the automorphism ϕ_i of ∂T_i so that

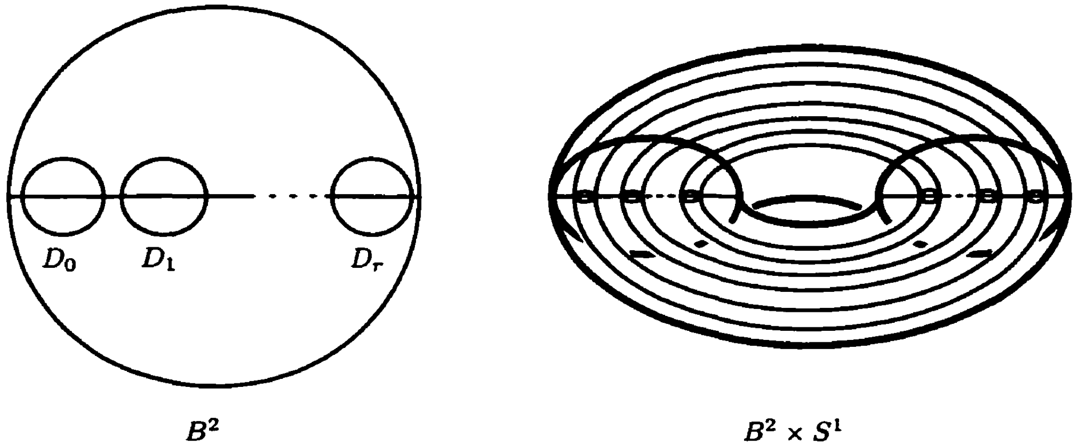


FIGURE 11. Illustration of the disks D_i used to construct V.

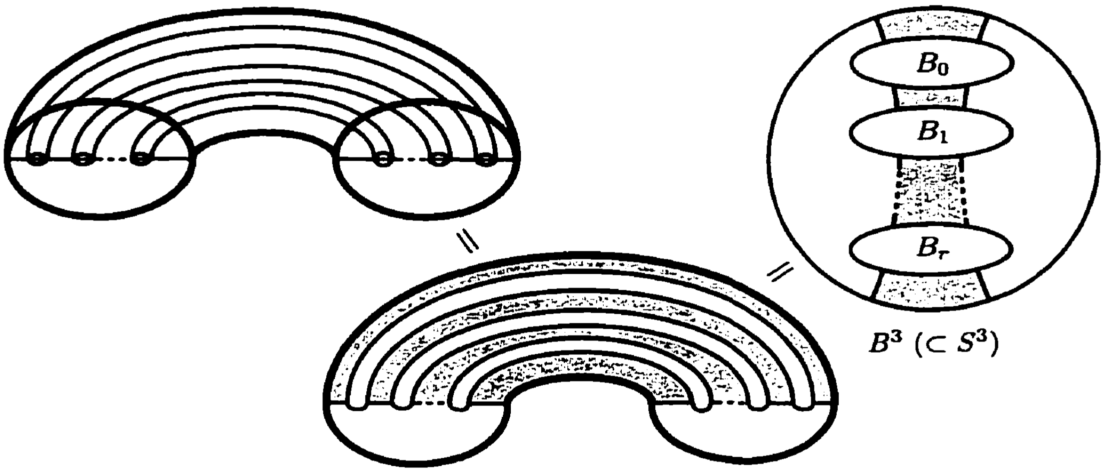


FIGURE 12. Illustration of the balls B_i in S^3.

it commutes with $\tau|_{\partial T_i}$. Thus we can construct an involution on V which we will again call τ.

The quotients T_i/τ are 3-balls B_i. We obtain V/τ by removing the interiors of the $r+1$ balls from $(S^2 \times S^1)/\tau = S^3$ and replacing them using the automorphism $\bar{\phi}_i$ of $\partial B_i = \partial(T_i/\tau)$ induced by ϕ_i. In Figure 12, the balls $B_i = (D_i \times S^1)/\tau$ are lined up along the band $(B^1 \times S^1)/\tau$ just as the D_i were lined up along B^1 in Figure 11.

Applying Lemma 3.2.2 with $X = S^3$ shows that V/τ remains an S^3. Moreover, the lemma shows that the image of the fixed points under the quotient map $V \to V/\tau$, is obtained from the trivial link (of $S^2 \times S^1$, see Figure 10) by replacing the image of this trivial link in each B_i with its image under an extension F_i of $\bar{\phi}_i$ to B_i.

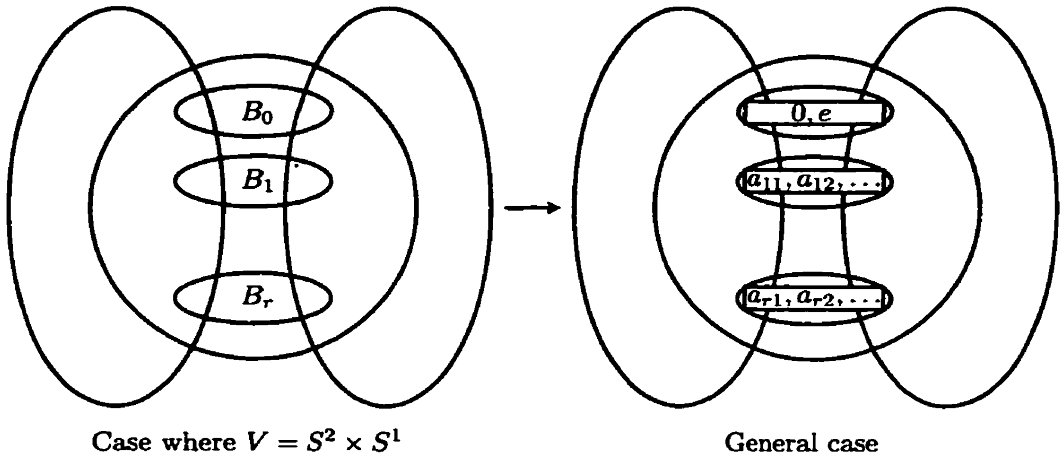


FIGURE 13. The trivial link transforms to the Montesinos link.

Claim 3.2.3. *There is an oriented automorphism ϕ_i of ∂T_i which sends m_i to $\alpha_i m_i + \beta_i \lambda_i$, commutes with $\tau|_{\partial T_i}$ and induces on ∂B_i an automorphism $\bar{\phi}_i$ which extends to an automorphism F_i of B_i such that the image under F_i of the trivial tangle in B^3 is the rational tangle β_i/α_i.*

We refer the reader to [Mo] for a proof of this claim.

The claim completes the proof since it shows that the extensions F_i transform the trivial link to the Montesinos link $\mathfrak{m}(e; \beta_1/\alpha_1, \beta_2/\alpha_2, \ldots \beta_r/\alpha_r)$ (see Figure 13). □

3.3. **Extending representations.** When a Montesinos knot $\mathfrak{m}$ has $r = 3$ tangles, the group of the base orbifold $\pi_1^{\text{orb}}(\mathcal{B})$ is a triangle group, $\Delta(p, q, r)$ (see Section 2.4.2). We can assume that p, q, r are positive and since $\mathfrak{m}$ is a knot, p and q are odd. We show that $\text{PSL}_2(\mathbb{C})$-representations of this triangle group extend to become representations of the knot group $\pi = \pi_1(S^3 \setminus N(\mathfrak{m}))$.

Proposition 3.3.1. *Let $\bar{\rho}_0$ be an irreducible $PSL_2(\mathbb{C})$-representation of $\tilde{\pi}$ which factors through $\Delta(p, q, r)$. Then $\bar{\rho}_0$ has a unique extension to π.*

Proof: Suppose $\bar{\rho}$ and $\bar{\rho}'$ were two extensions. Let $\alpha \in \pi \setminus \tilde{\pi}$. For any $\beta \in \tilde{\pi}$,

$$\begin{aligned}
\bar{\rho}'(\alpha)\bar{\rho}_0(\beta)\bar{\rho}'(\alpha)^{-1} &= \bar{\rho}'(\alpha\beta\alpha^{-1}) \\
&= \bar{\rho}(\alpha\beta\alpha^{-1}) \\
&= \bar{\rho}(\alpha)\bar{\rho}_0(\beta)\bar{\rho}(\alpha)^{-1}
\end{aligned}$$

So $A = \bar{\rho}(\alpha)^{-1}\bar{\rho}'(\alpha)$ commutes with $\bar{\rho}_0(\beta)$ and consequently with each element of $\bar{\rho}_0(\tilde{\pi})$. By Lemma 2.4.9, $A = \pm I$, as $\bar{\rho}$ is irreducible. Thus $\bar{\rho}$ and $\bar{\rho}'$ agree on α, hence on π and there is at most one extension of $\bar{\rho}_0$.

Claim 3.3.2. *There will be an extension $\bar{\rho}$ if and only if there is a matrix $A \in PSL_2(\mathbb{C})$ such that*

1. $A^2 = \pm I$ *and;*
2. $A\bar{\rho}_0(\beta)A^{-1} = \bar{\rho}_0(\mu\beta\mu^{-1})$ *for all* $\beta \in \tilde{\pi}$.

Proof: (of claim) Suppose $\bar{\rho}$ is an extension and let $A = \bar{\rho}(\mu)$. We know that $\bar{\rho}_0$ factors through $\Delta(p, q, r) \subset \pi_1(\Sigma_2) = \tilde{\pi}/\langle\mu^2\rangle = \tilde{\pi}/\langle\tilde{\mu}\rangle$, so $A^2 = \bar{\rho}_0(\mu^2) = \pm I$. Clearly $A\bar{\rho}_0(\beta)A^{-1} = \bar{\rho}_0(\mu\beta\mu^{-1})$ as well.

Conversely, suppose we have such an A. Note that $\pi = \tilde{\pi} \sqcup \mu\tilde{\pi}$. Define $\bar{\rho}$ by

$$\bar{\rho}(\alpha) = \begin{cases} \bar{\rho}_0(\alpha) & \text{if } \alpha \in \tilde{\pi} \\ A\bar{\rho}_0(\mu^{-1}\alpha) & \text{if } \alpha \in \mu\tilde{\pi} \end{cases}$$

Let $\alpha \in \tilde{\pi}$. By hypothesis,

$$\bar{\rho}_0(\mu\alpha\mu^{-1}) = A\bar{\rho}_0(\alpha)A^{-1}$$

Then,

$$\bar{\rho}_0(\alpha) = \bar{\rho}_0(\mu(\mu^{-1}\alpha\mu)\mu^{-1}) = A\bar{\rho}_0(\mu^{-1}\alpha\mu)A^{-1}$$

whence,

$$\bar{\rho}_0(\mu^{-1}\alpha\mu) = A^{-1}\bar{\rho}_0(\alpha)A.$$

So, $\forall\alpha, \beta \in \tilde{\pi}$,

$$\bar{\rho}(\alpha\beta) = \bar{\rho}_0(\alpha\beta) = \bar{\rho}_0(\alpha)\bar{\rho}_0(\beta) = \bar{\rho}(\alpha)\bar{\rho}(\beta);$$

$$\bar{\rho}(\alpha\mu\beta) = \bar{\rho}(\mu(\mu^{-1}\alpha\mu)\beta) = A\bar{\rho}_0(\mu^{-1}\alpha\mu)\bar{\rho}_0(\beta) = A(A^{-1}\bar{\rho}_0(\alpha)A)\bar{\rho}_0(\beta)$$

$$= \bar{\rho}_0(\alpha)(A\bar{\rho}_0(\beta)) = \bar{\rho}(\alpha)\bar{\rho}(\mu\beta);$$

$$\bar{\rho}(\mu\alpha\beta) = A\bar{\rho}_0(\alpha)\bar{\rho}_0(\beta) = \bar{\rho}(\mu\alpha)\bar{\rho}(\beta);$$

and

$$\bar{\rho}((\mu\alpha)(\mu\beta)) = \bar{\rho}(\mu^2(\mu^{-1}\alpha\mu)\beta) = \bar{\rho}_0(\mu^2)(A^{-1}\bar{\rho}_0(\alpha)A)\bar{\rho}_0(\beta) = \pm I(A^{-1}\bar{\rho}_0(\alpha)A)\bar{\rho}_0(\beta)$$

$$= (A\bar{\rho}_0(\alpha))(A\bar{\rho}_0(\beta)) \text{ (since } A^2 = \pm I \text{) } = \bar{\rho}(\mu\alpha)\bar{\rho}(\mu\beta).$$

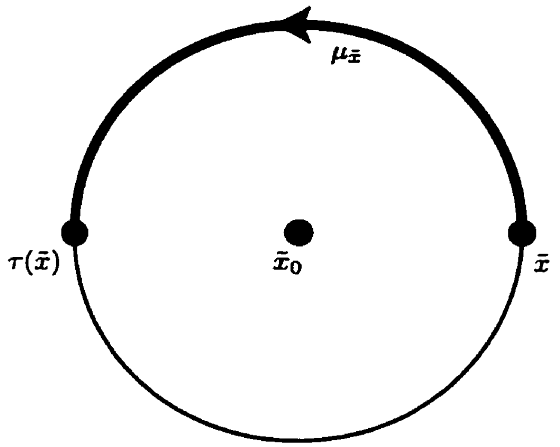


FIGURE 14. A lift of the meridian μ.

Thus $\bar{\rho}$ is a homomorphism which extends $\bar{\rho}_0$. □(claim)

To see that $\bar{\rho}_0$ extends, we must demonstrate the existence of such an A corresponding to conjugation by μ. Let τ (Theorem 3.2.1) be the involution of the 2-fold branched cyclic cover Σ_2 and choose base points $\tilde{x} \in \partial\widetilde{M}$ and $\tilde{x}_0 \in \text{Fix}(\tau) \subset \Sigma_2$. Then conjugation by μ corresponds to

$$\begin{array}{ccc} \pi_1(\widetilde{M}, \tilde{x}) & \xrightarrow{\cong} & \pi_1(\widetilde{M}, \tilde{x}) \\ [\alpha] & \mapsto & [\alpha]^\mu = [\mu_{\tilde{x}} \cdot \tau(\alpha) \cdot (\mu_{\tilde{x}})^{-1}] \end{array}$$

where $\mu_{\tilde{x}}$ is the lift of μ beginning at $\tilde{x}$ (see Figure 14).

So we have the following commutative diagram

$$\begin{array}{ccccc} \pi_1(\widetilde{M}, \tilde{x}) & \longrightarrow & \pi_1(\Sigma_2, \tilde{x}) & \longrightarrow & \pi_1(\Sigma_2, \tilde{x}_0) \\ \Big\downarrow [\alpha]\mapsto[\alpha]^\mu & & \cong \Big\downarrow & & \tau_* \Big\downarrow \\ \pi_1(\widetilde{M}, \tilde{x}) & \longrightarrow & \pi_1(\Sigma_2, \tilde{x}) & \longrightarrow & \pi_1(\Sigma_2, \tilde{x}_0) \end{array}$$

From Theorem 3.2.1, we have the diagram

$$\begin{array}{ccc} \Sigma_2 & \xrightarrow{\tau} & \Sigma_2 \\ \Big\downarrow & & \Big\downarrow \\ S^2(p,q,r) & \xrightarrow{\overline{\tau}} & S^2(p,q,r) \end{array}$$

where $\overline{\tau}$ corresponds to reflection in the equator of S^2 (see Figure 15). If we choose paths as illustrated, then $\overline{\tau}$ has the effect of taking the generators of $\Delta(p,q,r) = \langle a, b \mid a^p, b^q, (ab)^r \rangle$ to their inverses. Combining these ideas with the representation ϕ_0 of $\Delta(p,q,r)$ induced by $\bar{\rho}_0$, we have the diagram:

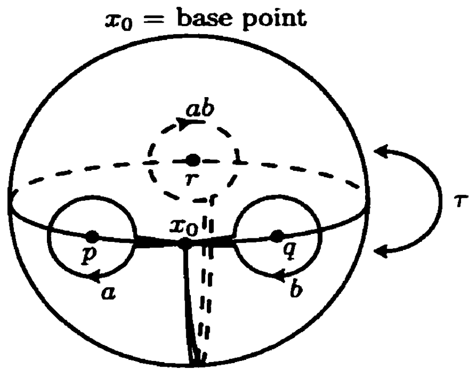


FIGURE 15. The reflection of S^2 induced by τ.

$$\begin{array}{ccccccc}
\tilde{\pi} & \longrightarrow & \pi_1(\Sigma_2) & \longrightarrow & \Delta(p,q,r) & \xrightarrow{\phi_0} & \mathrm{PSL}_2(\mathbb{C}) \\
\downarrow{\scriptstyle [\alpha]\mapsto[\alpha]^{\mu}} & & \cong\downarrow & & \overline{\tau}_\sharp\downarrow & & \\
\tilde{\pi} & \longrightarrow & \pi_1(\Sigma_2) & \longrightarrow & \Delta(p,q,r) & \xrightarrow{\phi_0} & \mathrm{PSL}_2(\mathbb{C})
\end{array}$$

Since $\overline{\tau}_\sharp$ takes a and b of $\Delta(p,q,r)$ to their inverses, and ab to $a^{-1}b^{-1}$ which is conjugate to $(ab)^{-1}$, we see that $\mathrm{tr}(\phi_0 \circ \overline{\tau}_\sharp) = \mathrm{tr}(\phi_0)$. On the other hand, since $\bar{\rho}_0$ is irreducible, ϕ_0 is as well and we deduce (see [CS1, Proposition 1.5.2]) that there is an $A \in \mathrm{PSL}_2(\mathbb{C})$ with $\phi_0 \circ \overline{\tau}_\sharp = A\phi_0 A^{-1}$. In other words, we have found an A such that $A\bar{\rho}_0(\beta)A^{-1} = \bar{\rho}_0(\mu\beta\mu^{-1})$ for all $\beta \in \tilde{\pi}$.

According to the claim, we can complete the proof by showing that $A^2 = \pm I$. Since τ is an involution, $\overline{\tau}_\sharp^2 \equiv 1$ and A^2 commutes with every element of the irreducible representation $\phi_0(\Delta(p,q,r))$. By Lemma 2.4.9, $A^2 = \pm I$. □

Scholium 3.3.3. *Any representation $\bar{\rho}$ which extends $\bar{\rho}_0$ is such that $\bar{\rho}(\mu)$ has order two.*

We have been assuming that $\mathfrak{m} = \mathfrak{m}(\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3)$ is a knot with three tangles. Then Σ_2 is Seifert fibred over S^2 with cone points $p = |\alpha_1|$, $q = |\alpha_2|$ and $r = |\alpha_3|$. This means $\pi_1^{orb}(S^2(p,q,r)) = \Delta(p,q,r)$. In other words, when there are three tangles, we can use the proposition to see that the $\mathrm{PSL}_2(\mathbb{C})$-character variety of $\Delta(p,q,r)$ includes into that of $\mathfrak{m}$: $\bar{X}(\Delta(p,q,r)) \subset \bar{X}(\mathfrak{m})$.

This suggests that for other Montesinos knots as well, $\bar{X}(\pi_1^{\mathrm{orb}}(\mathcal{B})) \subset \bar{X}(\mathfrak{m})$ where $\mathcal{B}$ is the base orbifold of Σ_2, the two-fold branched cyclic cover of $\mathfrak{m}$. Indeed, it is difficult at first glance to see how the proof depends on $\mathfrak{m}$ having three tangles.

Essentially the argument breaks down when $\pi_1^{\mathrm{orb}}(\mathcal{B})$ has more generators because, although $\overline{\tau}_\sharp$ will still take generators to their inverses, it will no longer do so for all products of generators. So there is no guarantee that $\overline{\tau}_\sharp$ will preserve characters. This means we cannot use [CS1, Proposition 1.5.2] to show the existence of A.

On the other hand, it may well be that $\overline{\tau}_\sharp$ preserves the characters of many of the $\pi_1^{\mathrm{orb}}(\mathcal{B})$ representations. For example when $r = 4$, $\pi_1^{\mathrm{orb}}(\mathcal{B})$ has three generators a, b, c so that characters are determined by their value on the words a, b, c, ab, ac, bc and abc (see Section 2.1). Of these, all but one, ac, is taken by $\overline{\tau}_\sharp$ to a word having the same trace. Since the $SL_2(\mathbb{R})$-character variety of $\pi_1^{\mathrm{orb}}(\mathcal{B})$ has (real) dimension two, we know that there are an infinite number of characters. It seems plausible that some of these may in fact be preserved by $\overline{\tau}_\sharp$. It would certainly be interesting to investigate this further. However, in this thesis, we will instead apply the observation $\bar{X}(\Delta(p, q, r)) \subset \bar{X}(\mathfrak{m})$ to deduce some consequences for pretzel knots. This is the subject of the next two chapters.

4. Cyclic and Finite Surgeries on Montesinos Knots

As we have seen, three-tangle Montesinos knots are closely related to the triangle groups. In this and the following chapter, we exploit this connection in a study of the (p, q, r) pretzel knots. These form an important subset of the three-tangle Montesinos knots. The $(-2, 3, n)$ pretzel knots of Section 2.5.2 are examples.

Pretzel knots with cyclic permutations of the indices p, q, r are obviously isotopic and will be considered equivalent. On the other hand, there is also an evident isotopy between the (p, q, r) and the (r, q, p) pretzel knot. (For example, given a diagram of the knot such as that of Figure 6, rotate by 180° about a vertical line to the right of the diagram.) This means that any permutation of the indices p, q, r will result in an isotopic (hence equivalent) knot. Moreover, taking the mirror reflection of a pretzel knot corresponds to changing the signs of all the indices. As this reduces to an isomorphism of the fundamental group π, we will ordinarily consider the knots (p, q, r) and $(-p, -q, -r)$ equivalent. Note however that such a mirror reflection will change the signs of boundary and surgery slopes. For example, it will result in a reflection of the fundamental and Newton polygons in a vertical line.

In this chapter we use the methods of Culler, Shalen, Boyer, and Zhang to come to a thorough understanding of cyclic and finite surgeries on pretzel knots. This complements the work of Delman [Del] who provided a classification of cyclic and finite surgeries on all other Montesinos knots.

4.1. Infinite fillings of $(p, q, 2m)$ pretzel knots.

Let $K = K_{p,q,r}$ be a pretzel knot where $p = 2k + 1$, $q = 2l + 1$ and $r = 2m$. We will be assuming that $1/|p| + 1/|q| + 1/|m| \leq 1$, and, by [Kaw, Theorem III], this ensures that K is hyperbolic.

Lemma 4.1.1. *If $1/|p| + 1/|q| + 1/|m| \leq 1$ then every filling $M(2a/b)$ of the knot complement M is infinite.*

Proof: To simplify the notation, we will assume $p, q, r > 0$. For the general case, one need only take absolute values.

As $\Delta(p, q, m)$ is infinite, our strategy is to construct a representation of $M(2a/b)$ with image $\Delta(p, q, m)$.

Let $\bar{\rho}_0$ be a faithful, non-abelian $PSL_2(\mathbb{C})$-representation of $\Delta(p,q,m)$. Using the obvious homomorphism, $\bar{\rho}_0$ is also a representation of $\Delta(p,q,r)$, the group of the base orbifold of the twofold branched cyclic cover of the knot. Then, as in Proposition 3.3.1, $\bar{\rho}_0$ "extends" to a $PSL_2(\mathbb{C})$-representation $\bar{\rho}$ of the knot group π which in turn lifts to an $SL_2(\mathbb{C})$-representation ρ. (The obstruction to such a lift is in $H^2(\pi;\mathbb{Z}/2)$ [BZ2, Section 3]. For a knot in S^3, the second cohomology is trivial and there is no obstruction.) Moreover, Scholium 3.3.3 shows $Z_x(f_{\mu^2}) > Z_x(f_\mu)$ for $x = \chi_\rho$. It follows that $\bar{\rho}(\mu^2) = \pm I$.

On the other hand, we can determine the image of λ in $\Delta(p,q,r) = \langle f,g,h \mid f^r, g^p, h^q, fgh\rangle$ to be $\tilde{\lambda} = g^k f^m g^{k+1} h^l f^m h^{l+1}$. (This derivation is explained in more detail in Lemma 5.1.2.) Now, as $\bar{\rho}_0$ is a representation of $\Delta(p,q,m)$, we have $\bar{\rho}_0(f^m) = \pm I$ and consequently $\bar{\rho}_0(\tilde{\lambda}) = \pm I$. Then $\bar{\rho}(\lambda) = \pm I$ as well.

So, for any filling of the from $\alpha = 2a/b$, we have $\bar{\rho}(\alpha) = \pm I$ whence $\bar{\rho}$ factors through $\pi_1(M(\alpha))$. Since $\bar{\rho}_0(\Delta(p,q,m))$ is infinite, we see that $\pi_1(M(\alpha))$ must also be infinite.

It remains to construct such a representation $\bar{\rho}_0$. Note that as $\frac{1}{p} + \frac{1}{q} + \frac{1}{m} \leq 1$, $\Delta(p,q,m)$ is infinite. Moreover, either $\{p,q,m\} = \{3,3,3\}$, and $\Delta(p,q,m)$ is a set of isometries of the Euclidean plane $\mathbb{E}^2$, or else $\frac{1}{p} + \frac{1}{q} + \frac{1}{m} < 1$ and $\Delta(p,q,m)$ represents isometries of $\mathbb{H}^2$. Since both $\mathbb{E}^2$ and $\mathbb{H}^2$ imbed isometrically into $\mathbb{H}^3$, in either case, $\Delta(p,q,m)$ is contained in $PSL_2(\mathbb{C})$, the set of orientation preserving isometries of $\mathbb{H}^3$. This provides the required faithful, non-abelian representation. □

So under the hypothesis of the lemma, every $2a/b$ filling of K is infinite. This means that any finite surgeries would have to be of the form $(2a+1)/b$ and therefore would have norm $\|(2a+1)/b\|_T \leq S+8$ [BZ1, Theorem 2.3]. On the other hand, the $2a/b$ fillings will have norm superior to $S+8$. (See Section 2.2.3 for the definition of S and $\|\cdot\|_T$, the total norm.)

Lemma 4.1.2. *If* $1/|p| + 1/|q| + 1/|m| < 1$ *then* $\|2a/b\|_T \geq S + 12$.

Proof: Again, we assume $p,q,r > 0$.

We first observe that there are at least 3 irreducible $PSL_2(\mathbb{C})$ characters of $\Delta(p,q,m)$ using Equations 2.2 and 2.3. This can be verified directly if $\max(p,q,m) \leq 11$. Let us assume then that $\max(p,q,m) > 11$.

As p and q are both odd, we can simplify the two equations somewhat to see that the number of irreducible characters of $\Delta(p, q, m)$ is

$$(p-1)(q-1)(m-1)/4 + [\gcd(p,q) + \gcd(p,m) + \gcd(q,m) - 1]/2 - (b+1)/2$$

where $b = \gcd(pq, pm, qm)$.

Without loss of generality, $p \leq q$, and there are two cases: $m = \max(p, q, m)$, and $q = \max(p, q, m)$. If $m = \max(p, q, m)$, then $b \leq pq$ and the number of irreducible characters is at least

$$\begin{aligned} & (p-1)(q-1)(m-1)/4 + 1 - (pq+1)/2 \\ = \; & (p-1)(q-1)(m-1)/4 + 1 - (p-1)(q-1)/2 - (p+q)/2 \\ = \; & (p-1)(q-1)(m-3)/4 + 1 - (p+q)/2 \end{aligned}$$

If p or q is greater than 3, this is at least $2(m-3) + 1 - m = m - 5$. If $p = q = 3$, we again have the bound $m - 3 + 1 - 3 = m - 5$. So either way there will be at least 7 irreducible characters since we're assuming $m = \max(p, q, m) > 11$.

Suppose next that $q = \max(p, q, m)$. As above we see that there are at least $(p-1)(m-1)(q-3)/4 + 1 - (p+m)/2$ irreducible characters of $\Delta(p, q, m)$. If $p > 5$ or $m > 3$ there are at least $3(q-3)/2 + 1 - q = (q-7)/2 > 2$ characters. On the other hand, the remaining possibilities ($p = 3$ or 5 and $m = 2$ or 3) also yield at least 3 irreducible characters.

Therefore, there are at least 3 irreducible $\mathrm{PSL}_2(\mathbb{C})$-characters of $\Delta(p, q, m)$. As we saw in the previous lemma, these all factor through to become characters of $M(\alpha)$ when α is of the from $2a/b$. None of these characters are dihedral (Lemma 2.4.10), so each is covered twice in $\mathrm{SL}_2(\mathbb{C})$. As they are the characters of irreducible representations of a triangle group, they are smooth points of $X(M)$ (see Lemma 5.1.3). Moreover, they are zeroes of f_α which are not zeroes of f_μ. (As in the previous lemma, these are characters of representations which take μ to an element of order two.) It follows from [BB, Theorem A] that $Z_x(f_\alpha) = Z_x(f_\mu) + 2$, and since we have six such x, we see that $||2a/b||_T = ||\alpha||_T \geq ||\mu|| + 12 = S + 12$. □

Theorem 4.1.3. *If $K = K_{p,q,r}$ is a pretzel knot with p, q odd, r even and $1/|p| + 1/|q| + 2/|r| < 1$, then K admits at most one non-trivial finite surgery. Moreover* *such a surgery slope u is odd integral and there is a non-integral boundary slope in* $(u-1, u+1)$.

Proof: The conditions on p, q, r ensure that K is hyperbolic [Kaw, Theorem III] (see also Lemma 5.2.1).

Let α be a finite surgery of such a knot. We have already observed (Lemma 4.1.1) that $\alpha = (2a+1)/b$. Since meridional surgery is cyclic, we can apply [BZ1, Theorem 1.1] to see that $b \leq 2$.

If $\alpha = (2a+1)/2$ were a finite filling, then, by [BZ1, Theorem 2.3], $\|\alpha\|_T \leq S+8$. At the same time, $\|\mu\|_T = \|-\mu\|_T = S$. The line joining $\alpha = (2a+1, 2)$ and $\mu = (1,0)$ in surgery space $V \cong H_1(\partial M; \mathbb{R}) \cong \mathbb{R}^2$ passes through $(a+1, 1)$ while the line through α and $-\mu$ passes through $(a, 1)$. It follows that $\|a+1\|_T$ and $\|a\|_T$ are both less than $S+4$. Since one of them is even, this contradicts Lemma 4.1.2.

So any non-trivial finite fillings must be odd integral. Suppose there were two such. Each would have norm at most $S+8$. The line joining them would necessarily pass through some even integral surgeries which would therefore also have norm at most $S+8$. This again contradicts Lemma 4.1.2.

Now suppose that $2a+1$ is a non-trivial finite filling. Then $\|2a+1, 1\|_T \leq S+8$ while $\|2a, 1\|_T \geq S+12$ by Lemma 4.1.2. Let $P \subset V$ denote the norm-ball of radius $S+8$. By [CGLS, Proposition 1.1.2], P is a finite-sided convex polygon whose vertices are multiples of boundary slopes. In particular, $(2a+1, 1)$ is not a vertex of P (Lemma 2.3.1).

Since $(2a+1, 1)$ is inside P and $(2a, 1)$ is not, there is a segment of ∂P which intersects the line $y = 1$ between them. Let $k(c, d)$ be the vertex of this segment which lies on or above $y = 1$, i.e., $k \in \mathbb{Q}$ and c/d is a boundary slope. Consider the segment from the origin to $k(c, d)$. As both endpoints are in P, this segment is also. It crosses $y = 1$ at $(c/d, 1)$ which must lie between $(2a, 1)$ and $(2(a+1), 1)$. (Otherwise, the segment joining $(2a+1, 1)$ and $(c/d, 1)$ passes through $(2a, 1)$, say. Since both endpoints are in P, this segment is in P and in particular $(2a, 1)$ is in P, a contradiction.)

Thus $|2a+1-c/d| < 1$, as required. □

Corollary 4.1.4. *If a knot satisfies the conditions of the theorem and has no non-integral boundary slopes, then it admits no non-trivial finite surgeries.*

Corollary 4.1.5. *Alternating knots which satisfy the conditions of the theorem admit no non-trivial finite surgeries.*

Proof: This follows since alternating Montesinos knots have no non-integral boundary slopes (see [HO, p.462]). □

Remark: A (p, q, r) pretzel knot is alternating iff p, q, r are all of the same sign.

More generally, an alternating Montesinos knot $\mathbf{m}$ can be written in the form $\mathbf{m}(e; \beta_1/\alpha_1, \beta_2/\alpha_2, \ldots, \beta_r/\alpha_r)$ with $-e$ and all β_i/α_i having the same sign. Indeed, it is clear that such a knot is alternating. On the other hand, a reduced diagram of an alternating knot will realize the knot's crossing number iff the diagram is alternating (see [Kau, Mu, Thi]). By [LT, Theorem 10], the crossing number of $\mathbf{m}$ is realized by a diagram like Figure 7. If $\mathbf{m}$ is alternating, this diagram will therefore be alternating whence $-e$ and all the β_i/α_i will have the same sign.

Note that the second Corollary also follows from Delman and Robert's [DR] proof that alternating knots satisfy strong property P.

4.2. **Non-integral boundary slopes of pretzel knots.** As we have seen (Theorem 4.1.3), finite surgeries on $(p, q, 2m)$ pretzel knots are intimately related to non-integral boundary slopes. In this section we will use the methods of Hatcher and Oertel [HO] to calculate these slopes. We first illustrate the method using the example of the $(-2, 3, 7)$ pretzel knot. We will assume familiarity with the notation and conventions of [HO].

Boundary slopes are found using "edgepaths" in the Diagram $\mathcal{D}$ (Figure 16). Points of $\mathcal{D}$ are labeled by triples (a, b, c) and have vertical coordinate "slope" $c/(a+b)$ and horizontal coordinate $b/(a+b)$. Thus horizontal lines in $\mathcal{D}$ represent points which all have the same slope. Let $\langle p/q \rangle$ denote the vertex $(1, q-1, p)$ and $\langle p/q, r/s \rangle$ the edge joining $\langle p/q \rangle$ and $\langle r/s \rangle$. Such an edge exists only if the slopes are of distance one: $\Delta(p/q, r/s) = |ps - rq| = 1$. The diagram also includes horizontal segments from $\langle p/q \rangle$ to $(0, q, p)$ on the right-hand edge. We will denote such horizontal segments by $\langle p/q, p/q \rangle$.

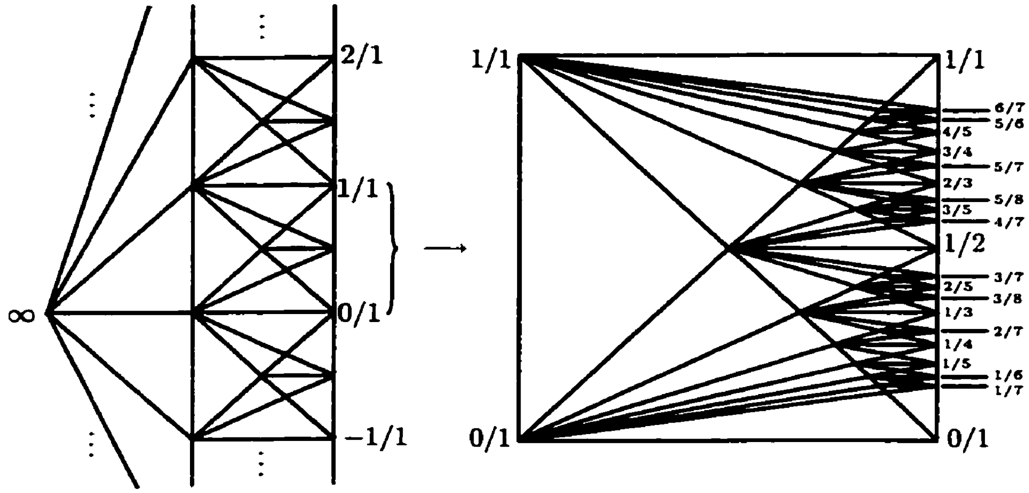


FIGURE 16. The diagram $\mathcal{D}$.

An *edgepath* of $\mathcal{D}$ is a piecewise linear path in the 1-skeleton of $\mathcal{D}$. Boundary slopes of a Montesinos knot $\mathbf{m}(p_1/q_1, p_2/q_2, \ldots p_n/q_n)$ are determined by a sequence of edgepaths γ_i $(i = 1 \ldots n)$ satisfying the following conditions:

E1: The starting point of γ_i lies on the edge $\langle p_i/q_i, p_i/q_i \rangle$. If the starting point is not $\langle p_i/q_i \rangle$, then the edgepath γ_i is constant, i.e., it never leaves $\langle p_i/q_i, p_i/q_i \rangle$.

E2: γ_i is *minimal*, i.e., it never stops and retraces itself, nor does it ever go along two sides of the same triangle of $\mathcal{D}$ in succession.

E3: The ending points of the γ_i's are rational points of $\mathcal{D}$ which all lie on one vertical line and whose vertical coordinates add up to zero.

E4: γ_i proceeds monotonically from right to left, "monotonically" in the weak sense that motion along vertical edges is permitted.

Thus each tangle p_i/q_i gives rise to a tree in $\mathcal{D}$ corresponding to potential edge paths γ_i. For example, the tree for $1/7$ is illustrated in Figure 17.

Non-integral boundary slopes will be given only by edgepaths which have no vertical edges and end at a vertical line $u = u_0 \in \mathbb{Q}$ before reaching the left edge of $\mathcal{D}$ (the "Type I" edgepaths of [HO]). That is, the γ_i end at $(u_0, v_i) = (b/(a+b), c_i/(a+b))$ with $\sum v_i = 0$.

As we have seen (Figure 17), the trees are quite simple in the case of the tangles $1/p_i$ of a pretzel knot. This means finding points where $\sum v_i = 0$ is not so difficult. For example, let's look at the $(-2, 3, 7)$ pretzel knot. The trees corresponding to

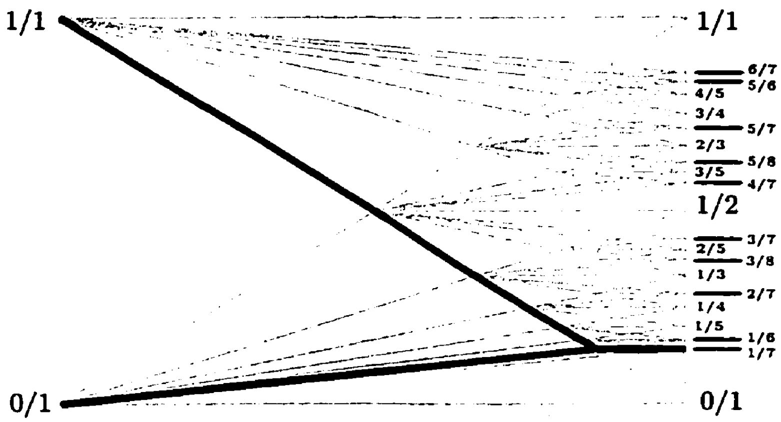


FIGURE 17. The tree of a 1/7 tangle.

the tangles $-1/2$, $1/3$, and $1/7$ are aligned above one another in Figure 18. Let $v_T = \sum v_i$. As v_i is a linear function of u between vertices, it will be helpful to calculate the value of v_i at each u which is a vertex on one of the trees. We have included this data in Figure 18.

Since each tree has two branches, there are at most eight different edgepaths for a given value of u. For example when $u = 0$, the possible values of v_T are 2 $(+++)$, 1 $(++-$, $+-+$, or $-++)$, 0 $(+--$, $-+-$, $--+)$, and -1 $(---)$, where $++-$, for example, is meant to indicate that we've chosen the upper branch of the $-1/2$ tree, the upper branch of the 1/3 tree and the lower branch of the 1/7 tree.

When $u = 1/2$, the possible values of v_T are $1/2$ $(+++$ or $-++)$, $1/4$ $(+-+$ or $--+)$, $1/12$ $(++-$ or $-+-)$, and $-1/6$ $(+--$ or $---)$. Since the value of v_T varies linearly between vertices, we see that it can have no zero for $0 < u \leq 1/2$. For example, for the $+++$ edgepaths, $v_T = 2$ at $u = 0$ and $v_T = 1/2$ at $u = 1/2$. Therefore it is positive throughout the interval $0 \leq u \leq 1/2$.

When $u = 2/3$, there are only two possible values of v_T, namely $1/6$ $(**+)$ and $-1/18$ $(**-)$, where we've used $*$ to mean "+ or $-$," i.e., the two choices lead to the same result here. This shows that v_T has a zero on the $*+-$ paths as $v_T = 1/16$ when $u = 1/2$ and $v_T = -1/18$ when $u = 2/3$.

Finally, when $u = 6/7$, there is only one choice for v_T, namely $v_T = -1/2 + 1/3 + 1/7 = -1/42$. So there is an additional zero of v_T on the $**+$ edgepaths. Indeed, when $u = 5/6$, we have $v_T = -1/2 + 1/3 + 1/6 = 0$ on $**+$. Thus there are exactly

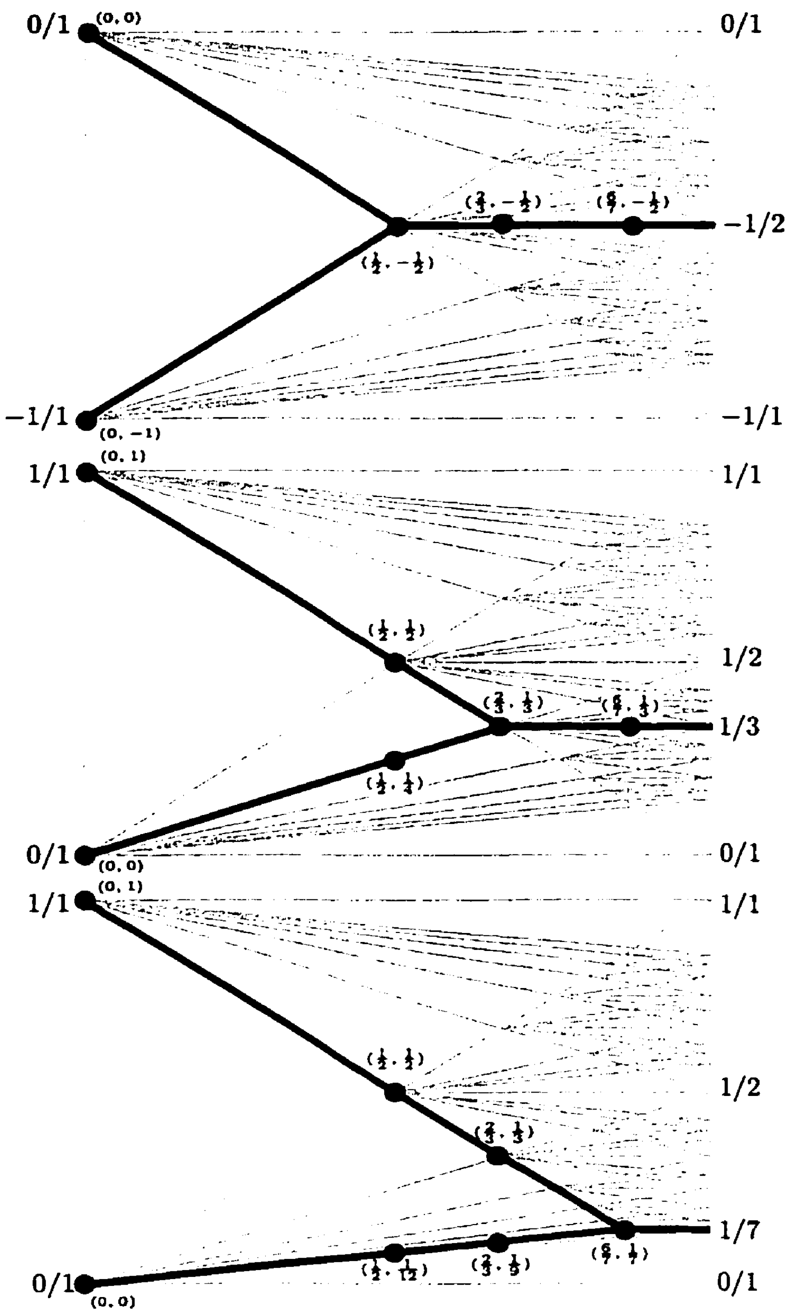


FIGURE 18. The trees for the $(-2, 3, 7)$ pretzel knot.

two Type I edgepaths for the $(-2, 3, 7)$ pretzel knot: $* + -$ with $1/2 < u_0 < 2/3$ and $* * +$ with $u_0 = 5/6$. These are the only candidates which could yield a non-integral boundary slope.

As explained in [HO] an edgepath is a way of describing a surface S in the knot complement. The twist associated to this surface is $\tau(S) = 2(e_- - e_+)$ where e_+ (e_-) is the number of edges in the edge paths which increase (decrease) slope.

For example the twist of the edgepath system which terminates at $u_0 = 5/6$ is -2 since the $-1/2$ and $1/3$ edgepaths are constant and the $1/7$ edgepath increases on one edge from $1/7$ to $1/6$.

The $*+-$ system ending with u_0 between $1/2$ and $2/3$ is more subtle as it involves fractions of edges. The value of v_T for u in this interval is

$$v_T = \frac{1}{12}(1 - \frac{u - \frac{1}{2}}{\frac{2}{3} - \frac{1}{2}}) - \frac{1}{18}(\frac{u - \frac{1}{2}}{\frac{2}{3} - \frac{1}{2}}) = 1/2 - 5u/6.$$

So $v_T = 0$ when $u = u_0 = 3/5$. This is half way along $\langle 1/2, 1/3 \rangle$ in the $1/3$ tangle and $3/4$ of the way along $\langle 0/1, 1/7 \rangle$ in the $1/7$ tangle. Thus the $-1/2$ tangle has no increasing or decreasing edges, the $1/3$ tangle contributes $1/2$ to e_+ and the $1/7$ tangle contributes $3/4$ to e_-. The twist of this surface is then $\tau(S) = 2(e_- - e_+) = 1/2$.

Finally, the boundary slope of a surface is given by $\tau(S) - \tau(S_0)$ where S_0 is a Seifert surface for the knot. For a (p, q, r) pretzel knot, a Seifert surface can be found by taking the $+++$ system of edgepaths and extending to ∞ at the left of $\mathcal{D}$. For the $(-2, 3, 7)$ pretzel knot, the three tangles contribute 1, 2, and 6 upward edges respectively so that $\tau(s_0) = -18$. The two surfaces constructed above therefore have boundary slope 16 and 18 1/2. In particular, this includes the one non-integral boundary slope of this knot: 18 1/2.

We now prove several lemmas about non-integral boundary slopes on pretzel knots which will prove useful in the following sections.

Lemma 4.2.1. *Let K be a $(-2, p, q)$ pretzel knot with p, q odd and $3 \leq p \leq q$. If $p \geq 7$ (respectively $q \geq 7$) then*

$$\frac{p^2 - p - 5}{\frac{p-3}{2}} \qquad (resp.\ \frac{q^2 - q - 5}{\frac{q-3}{2}})$$

is a non-integral boundary slope of K. Moreover, these are the only non-integral boundary slopes of K.

Proof: Let us assume $p \geq 5$. The case $p = 3$ is analogous, but involves small deviations from the general case.

We follow the procedure outlined above and look for Type I edgepaths. As before, at $u = 0$, v_T is $2, 1, 0$, or -1. At $u = \frac{1}{2}$, there are four values for v_T: $\frac{1}{2}$ $(*++)$, $\frac{1}{2(p-1)}$ $(*-+)$, $\frac{1}{2(q-1)}$ $(*+-)$, and $\frac{1}{2}(\frac{1}{p-1} + \frac{1}{q-1} - 1)$ $(*--)$ and so there are no edgepath systems terminating at $u_0 \in (0, \frac{1}{2})$.

At $u = \frac{p-1}{p}$, there are only two values of v_T depending on whether we take the upper or lower branch in the $1/q$ tangle. For $**+$, we have $v_T = -\frac{1}{2} + \frac{1}{p} + \frac{1}{p} = \frac{4-p}{2p}$. Since $p \geq 5$, this is negative and there will be zeroes of v_T on both the $*-+$ and $*++$ edgepaths. At $**-$, $v_T = -\frac{1}{2} + \frac{1}{p} + \frac{p-1}{p(q-1)}$. As $5 \leq p \leq q$, this is again negative and v_T has an additional zero in the $*+-$ system of edgepaths.

Now, $v_T = -\frac{1}{2} + \frac{1}{p} + \frac{1}{q} < 0$ at $u = \frac{q-1}{q}$ and there are no further zeroes of v_T for $u \geq \frac{p-1}{p}$.

Thus there are three candidates surfaces with edgepath systems terminating at $u_0 \in (\frac{1}{2}, \frac{p-1}{p})$. Note that, as each of these systems is constant in the $-1/2$ tangle, these will all be incompressible surfaces (see [HO, Proposition 2.1]).

However, the $*++$ system will not contribute a non-integral boundary slope since it in fact terminates at $u_0 = \frac{3}{4}$ where $v_T = -\frac{1}{2} + \frac{1}{4} + \frac{1}{4}$, and therefore involves no fractional edges.

For the $*-+$ system

$$v_T = \frac{1}{2(p-1)}\left(1 - \frac{u - \frac{1}{2}}{\frac{p-1}{p} - \frac{1}{2}}\right) + \frac{4-p}{2p}\left(\frac{u - \frac{1}{2}}{\frac{p-1}{p} - \frac{1}{2}}\right)$$
$$= \frac{(2-p)2u + p - 1}{2(p-1)}$$

which is zero when $u = u_0 = \frac{p-1}{2(p-2)}$.

In the $1/p$ tangle, u_0 corresponds to a point on the edge $\langle 0/1, 1/p \rangle$:

$$\begin{aligned}
\alpha\langle 0/1\rangle + (1-\alpha)\langle 1/p\rangle &= \alpha(1,0,0) + (1-\alpha)(1, p-1, 1) \\
&= (1, (1-\alpha)(p-1), 1-\alpha) \\
&= \left(\frac{(1-\alpha)(p-1)}{1 + (1-\alpha)(p-1)}, \frac{(1-\alpha)}{1+(1-\alpha)(p-1)}\right).
\end{aligned}$$

Taking the u coordinate from the last line, we have

$$\frac{p-1}{2(p-2)} = u_0 = \frac{(1-\alpha)(p-1)}{1+(1-\alpha)(p-1)}$$

which implies $\alpha = \frac{p-4}{p-3}$. The $1/p$ tangle therefore contributes $\frac{p-4}{p-3}$ to the total of e_-.

Since $p \geq 5$,

$$\frac{1}{2} < u_0 = \frac{p-1}{2(p-2)} \leq \frac{2}{3},$$

and the $1/q$ tangle has $q-3$ increasing edges, plus a certain fraction of the $\langle 1/2, 1/3\rangle$ edge:

$$\begin{aligned}\beta\langle 1/2\rangle + (1-\beta)\langle 1/3\rangle &= \beta(1,1,1) + (1-\beta)(1,2,1) \\ &= (1, 2-\beta, 1) \\ &= (\frac{2-\beta}{3-\beta}, \frac{1}{2-\beta}) \text{ in } (u,v) \text{ coordinates.}\end{aligned}$$

That is,

$$\frac{p-1}{2(p-2)} = u_0 = \frac{2-\beta}{3-\beta},$$

whence $\beta = \frac{p-5}{p-3}$ and the $1/q$ tangle contributes $q-3+\frac{p-5}{p-3}$ to e_+.

Finally, since the edgepath in the $-1/2$ tangle is constant, we find that the twist corresponding to the $* - +$ system is

$$\tau(S) = 2(e_- - e_+) = 2(\frac{p-4}{p-3} - (q-3+\frac{p-5}{p-3})) = 2(3-q+\frac{1}{p-3}).$$

On the other hand, the twist of a Seifert surface ($+++$ extended to ∞) is $\tau(S_0) = 2(0-(1+p-1+q-1))$. Putting it all together, the boundary slope of the surface S given by the $* - +$ system is

$$\begin{aligned}\tau(S) - \tau(S_0) &= 2(3-q+\frac{1}{p-3}) - 2(0-(1+p-1+q-1)) \\ &= 2(2+p+\frac{1}{p-3}) \\ &= \frac{p^2-p-5}{\frac{p-3}{2}}\end{aligned}$$

Analogous arguments show that the $* + -$ surface has boundary slope $\frac{q^2-q-5}{\frac{q-3}{2}}$. □

We turn now to the case of a $(p, q, -r)$ pretzel knot K with $4 \leq r = 2m$, even and $p = 2k+1$ and $q = 2l+1$ both odd. Further, we assume $3 \leq p \leq q$. We will give two lemmas describing the non-integral boundary slopes of K depending on the relative values of p and r.

In each case we will list fractions which are candidates for the non-integral boundary slopes of K with the understanding that for certain choices of p, q, and r, these fractions will turn out to be integers. Moreover, we will only verify that these candidates are truly boundary slopes when we can easily use [HO, Proposition 2.1] to do so, that is, when one of the edgepaths is constant. In general, the fractions are the boundary slope of some surface in M, but when we cannot apply Proposition 2.1, it is tedious to verify whether or not that surface is *essential.* (Also, there is a small error in [HO] when they discuss testing whether or not candidate surfaces are essential. See [Du2].) Since a list of candidates will be adequate for us, we will not endeavour to investigate further.

In short, if K admits a non-integral boundary slope, then that slope will appear in the list of slopes in the appropriate lemma (depending on the values of p and r).

Lemma 4.2.2. *If $p \geq 2r+1$, then*

$$\frac{p(p-1)+1-3r}{\frac{p-1-r}{2}} \quad \textit{and} \quad \frac{q(q-1)+1-3r}{\frac{q-1-r}{2}} \tag{4.6}$$

are the non-integral boundary slopes of K.

Remark: Substituting $r=2$, we see that Lemma 4.2.2 generalizes Lemma 4.2.1.

Proof: As in Lemma 4.2.1, we need to look at edgepath systems *à la* Hatcher and Oertel [HO].

When $u=0$, v_T is $2, 1, 0$, or -1. At $u=\frac{r-1}{r}$, there are four values for v_T: $\frac{1}{r}$ $(*++)$, $\frac{r-1}{r(p-1)}$ $(*-+)$, $\frac{r-1}{r(q-1)}$ $(*+-)$, and $\frac{1}{r}(\frac{r-1}{p-1}+\frac{r-1}{q-1}-1)$ $(*--)$. As the first three values are positive and the last is negative, there are no edgepath systems terminating at $u_0 \in (0, \frac{r-1}{r})$.

When $u=\frac{p-1}{p}$, the two values of v_T are $-\frac{1}{r}+\frac{2}{p}=\frac{2r-p}{rp}$ $(**+)$ and $-\frac{1}{r}+\frac{1}{p}+\frac{p-1}{p(q-1)}$ $(**-)$, so the $*-+$, $*++$ and $*+-$ edgepaths are all Type I each terminating at a $u_0 \in (\frac{r-1}{r}, \frac{p-1}{p})$. As these edgepaths are constant in the $-1/r$ tangle, each results in an incompressible surface [HO, Proposition 2.1]. Since $-\frac{1}{r}+\frac{1}{p}+\frac{1}{q}<0$, there are no other Type I systems for this knot.

Now, the $*++$ edgepath will terminate at $u_0=\frac{2r-1}{2r}$ where $v_T=-\frac{1}{r}+\frac{1}{2r}+\frac{1}{2r}=0$. This system will not lead to a non-integral boundary slope as it involves no fractional edges.

As in Lemma 4.2.1, the $*-+$ and $*+-$ edgepaths will give surfaces having boundary slope

$$\frac{p(p-1)+1-3r}{\frac{p-1-r}{2}} \quad \text{and} \quad \frac{q(q-1)+1-3r}{\frac{q-1-r}{2}}$$

respectively. □

Lemma 4.2.3. *If $p < r$, the only potential non-integral boundary slope of K is*

$$2(p+q+r-1-\frac{(p-1)(q-1)}{p-1+q-1}). \tag{4.7}$$

Proof: As usual, at $u = 0$, the values of v_T are $2, 1, 0$, and -1. When $u = \frac{p-1}{p}$, we have $v_T = \frac{2}{p} - \frac{p-1}{p(r-1)} = \frac{2r-p-1}{p(r-1)}$ $(+*+)$, $v_T = \frac{1}{p}$ $(-*+)$, $v_T = -\frac{p-1}{p(r-1)} + \frac{1}{p} + \frac{p-1}{p(q-1)}$ $(+*-)$, and $v_T = \frac{p-1}{p(q-1)}$ $(-*-)$. Now, these are all positive and, when $u = \frac{q-1}{q}$, $v_T = \frac{1}{p} + \frac{1}{q} - \frac{1}{r}$ is positive as well. So the only Type I system is in the $---$ edgepath and it terminates at a $u_0 \in (0, \frac{p-1}{p})$. The associated boundary slope is as given in Equation 4.7. Note that we cannot use [HO, Proposition 2.1] so it may be that the fraction of Equation 4.7 is not a boundary slope. In that case, K would have no non-integral boundary slopes. □

4.3. **Cyclic surgeries of $(-2, p, q)$ pretzel knots.** In this section, let K be a $(-2, p, q)$ pretzel knot (p, q odd and positive) and $M = S^3 \setminus N(K)$.

Lemma 4.3.1. *$M(2(p+q))$ contains an incompressible torus.*

Proof: The obvious (see Figure 20) spanning surface of such a knot is a once-punctured Klein bottle and it's double cover, T, is a twice-punctured torus which meets ∂M in two parallel curves of slope $2(p+q)$. We can use the methods of Hatcher and Oertel [HO] to see that $2(p+q)$ is in fact a boundary slope of K. Following the proof of [CGLS, Theorem 2.0.3] we see that T is a minimal surface realizing this boundary slope. (Since $2(p+q)$ is not the longitude slope, such a minimal surface is connected, separating and has at least two boundary components.) As T is non-planar, we apply [CGLS, Proposition 2.2.1] (along with the fact that K is a small knot [Oe, Corollary 4]) to see that the filled torus $\hat{T}$ is incompressible in $M(2(p+q))$. □

Proposition 4.3.2. *Suppose K, a $(-2, p, q)$ pretzel knot ($p \leq q$ odd and positive), admits a non-trivial cyclic surgery. Then one of the following holds.*

1. *K is torus and therefore admits an infinite number of cyclic surgeries. In this case either $p = 1$, $q = 1$, or else $\{p, q\} = \{3, 3\}$ or $\{3, 5\}$.*
2. *K is the $(-2, 3, 7)$ pretzel knot and the surgery is 18 or 19.*

Proof: Theorem III of [Kaw] shows that K is torus iff it is as characterized in 1 (see also Lemma 5.2.1).

Since K is small ([Oe, Corollary 4]), we can assume that K is hyperbolic. The cyclic surgeries 18 and 19 of the $(-2, 3, 7)$ pretzel knot were first observed by Fintushel and Stern (see [FS, Section 4]). Our task is to show that there is no other choice for p and q leading to a cyclic surgery.

The case $p = 3$ is the subject of Section 5.2 where we will see that there are no non-trivial cyclic surgeries when $q \geq 9$ and that the cyclic surgeries of the $(-2, 3, 7)$ pretzel knot are as stated.

If $p = 5$, the boundary slopes [HO] are $0, 14, 15, \frac{q^2-q-5}{\frac{q-3}{2}}, 2q + 10$, and $2q + 12$. By [Du1, Theorem 4.1], a non-trivial cyclic surgery could occur only at $2q + 4$ or $2q + 5$. However, as we explain below, a cyclic surgery would have to be within distance 5 of the toroidal surgery $2q + 10$ (see Lemma 4.3.1). So the only candidate is $2q + 5$. Now the $(-2, 5, 5)$ pretzel has no non-integral boundary slopes so ([Du1, Theorem 4.1]) it has no non-trivial cyclic surgeries. As for $(-2, 5, 7)$, SnapPea [Wee] shows that $2q + 5 = 19$ surgery on this knot is hyperbolic. So we can assume $q \geq 9$.

Suppose (for a contradiction) that $2q + 5$ is indeed a cyclic surgery. By [BZ1, Lemma 6.2], the (total) norm can be written

$$\begin{aligned} \|\gamma\| &= 2[a_1\Delta(\gamma, 0) + a_2\Delta(\gamma, 14) + a_3\Delta(\gamma, 15) \\ &\quad + a_4\Delta(\gamma, \frac{q^2 - q - 5}{\frac{q-3}{2}}) + a_5\Delta(\gamma, 2q + 10) + a_6\Delta(\gamma, 2q + 12)]. \end{aligned}$$

If $2q + 5$ is cyclic it has minimal norm s, as does the meridian surgery μ ([CGLS, Corollary 1.1.4] and Lemma 2.3.1). The norm of $2q + 4$ will also be of interest, and it will be bounded by the minimal norm s.

$$\begin{aligned} s &= \|\mu\| = 2[a_1 + a_2 + a_3 + \frac{q-3}{2}a_4 + a_5 + a_6] \\ s &= \|2q + 5\| = 2[(2q + 5)a_1 + (2q - 9)a_2 + (2q - 10)a_3 + \frac{q-5}{2}a_4 + 5a_5 + 7a_6] \\ s &\leq \|2q + 4\| = 2[2q + 4a_1 + (2q - 10)a_2 + (2q - 11)a_3 + a_4 + +6a_5 + 8a_6] \end{aligned}$$

Subtracting the first two equations, we have

$$a_4 = (2q + 4)a_1 + (2q - 10)a_2 + (2q - 11)a_3 + 4a_5 + 6a_6, \tag{4.8}$$

while subtracting the second from the third leaves

$$\begin{aligned} a_5 + a_6 &\geq a_1 + a_2 + a_3 + \frac{q-7}{2}a_4, \\ \Rightarrow \quad \eta(a_5 + a_6 - a_1 - a_2 - a_3) &\geq a_4, \end{aligned}$$

where $\eta = \frac{2}{q-7} \leq 1$.

Combining this with Equation 4.8, we have

$$0 \geq (2q + 4 + \eta)a_1 + (2q - 10 + \eta)a_2 + (2q - 11 + \eta)a_3 + (4 - \eta)a_5 + (6 - \eta)a_6.$$

Since $a_i \geq 0$, this shows $a_1 = a_2 = a_3 = a_5 = a_6 = 0$. On the other hand, for a norm, at least two of the a_i must be non-zero. This contradiction show that there can be no non-trivial cyclic surgery when $p = 5$.

So let us assume $7 \leq p \leq q$. Dunfield[Du1, Theorem 4.1] has shown that any non-trivial cyclic surgery on a knot such as K must lie near a non-integral surgery. Combining this with Lemma 4.2.1, the only candidates for a non-trivial cyclic surgery are $2p+4$, $2p+5$, $2q+4$, and $2q+5$. Suppose that u is one of these candidates slopes and $M(u)$ is a cyclic filling. Since K is strongly invertible, the Orbifold Theorem implies that $M(u)$ admits a geometric decomposition (see [CHK, Corollary 1.7]). Now, as $\Delta(u, 2(p+q)) > 5$, $M(u)$ is irreducible [Oh, Wu] and atoroidal [Go2] and therefore has a geometric structure.

Note that $\pi_1(M(u)) \not\cong \mathbb{Z}$ ([Ga1]), so $\pi_1(M(u))$ is finite. The geometry is therefore S^3, and as $\pi_1(M(u))$ is finite cyclic, we deduce that $M(u)$ is a lens space. However, this contradicts [Go3, Theorem 1.1] which states that the distance between a lens space surgery such as u and a toroidal surgery such as $2(p+q)$ is at most 5. We conclude that there are also no non-trivial cyclic surgeries in this case. □

4.4. **Finite surgeries on pretzel knots.** We turn now to the case of a $(p, q, -r)$ pretzel knot K where $4 \leq r = 2m$ is even and $p = 2k + 1$ and $q = 2l + 1$ are both odd. We will assume $3 \leq p \leq q$.

Lemma 4.4.1. *$M(2(p+q))$ contains an incompressible torus.*

Proof: Identical to that of Lemma 4.3.1. □

Proposition 4.4.2. *If $p > 2r + 1$, then K admits no non-trivial finite surgeries.*

Proof: By Theorem 4.1.3, a non-trivial finite surgery would be close to one of the non-integral boundary slopes of Lemma 4.2.2. However,

$$\begin{aligned}
2(p+q) - \frac{p(p-1)+1-3r}{\frac{p-1-r}{2}} &= 2(p+q) - 2(p+r) - \frac{(r-1)^2}{\frac{p-1-r}{2}} \\
&= 2q - 2r - \frac{(r-1)^2}{\frac{p-1-r}{2}} \\
&\geq 4r + 6 - 2r - \frac{(r-1)^2}{\frac{r}{2}+1} \\
&= \frac{7r+5}{\frac{r}{2}+1} \geq 11
\end{aligned}$$

and similarly for the other slope of Lemma 4.2.2. Therefore, any non-trivial finite surgery would be of distance (in the sense of minimal geometric intersection) greater than 10 from the toroidal surgery $2(p+q)$. However this contradicts work of Agol [Ag] and Lackenby [L] showing that the distance between exceptional surgeries is ≤ 10. □

Proposition 4.4.3. *If $p \leq r - 5$, then K admits no non-trivial finite surgeries.*

Proof: As in the previous proposition, we observe that

$$|2(p+q+r-1-\frac{(p-1)(q-1)}{p-1+q-1}) - 2(p+q)| > 10.$$

Thus the lone non-integral boundary slope of Lemma 4.2.3 is too far from the toroidal boundary slope $2(p+q)$ (by Theorem 4.1.3 a finite filling could only occur at an odd-integral slope, which would therefore have to be within distance 9 of the even number $2(p+q)$). □

Combining the results of the last few sections, we see that we now have a fairly precise description of what a finite filling s on a $(p, q, -r)$ pretzel knot would look like. By Theorem 4.1.3, s would have to be odd-integral and near a non-integral boundary slope and by Propositions 4.4.2 and 4.4.3 we would have to have $p+3 \geq r \geq (p-1)/2$. We now propose to explicitly calculate the fundamental group of such a filling. We will then project onto a smaller group G and observe that G is generically infinite.

The Wirtinger presentation [Rol, Section 3.D] of a $(p, q, -r)$ pretzel knot is (compare [Tr, Equation 1]):

$$\begin{aligned}\pi_1(M) = \langle x, y, z \mid\ & (zx)^{(p-1)/2} z (zx)^{(1-p)/2} = (yx)^{-(q+1)/2} y (yx)^{(q+1)/2},\\ & (yz^{-1})^{-r/2} y (yz^{-1})^{r/2} = (yx)^{(1-q)/2} x (yx)^{(q-1)/2},\\ & (yz^{-1})^{-r/2} z (yz^{-1})^{r/2} = (zx)^{(p+1)/2} x (zx)^{-(p+1)/2}\rangle.\end{aligned}$$

The longitude being

$$l = x^{-2(p+q)} (yx)^{(q-1)/2} (yz^{-1})^{-r/2} (yx)^{(q+1)/2} (zx)^{(p-1)/2} (yz^{-1})^{r/2} (zx)^{(p+1)/2},$$

filling along an odd integral slope s results in

$$\begin{aligned}\pi_1(M(s)) = \langle x, y, z \mid\ & (zx)^{(p-1)/2} z (zx)^{(1-p)/2} = (yx)^{-(q+1)/2} y (yx)^{(q+1)/2},\\ & (yz^{-1})^{-r/2} y (yz^{-1})^{r/2} = (yx)^{(1-q)/2} x (yx)^{(q-1)/2},\\ & (yz^{-1})^{-r/2} z (yz^{-1})^{r/2} = (zx)^{(p+1)/2} x (zx)^{-(p+1)/2}, x^s l\rangle.\end{aligned}$$

We can obtain a more manageable factor group G by adding the relators $(yz^{-1})^{r/2}$, yx^{-1}, and $(zx)^p$:

$$\begin{aligned} G &= \langle y, z \mid (yz^{-1})^{r/2}, (zy)^p, z = (zy)^{(p+1)/2} y (zy)^{-(p+1)/2}, y^{s-2p}\rangle \\ &= \langle w, y \mid (y^2 w^2)^{r/2}, w^p, (wy)^2, y^{s-2p}\rangle,\end{aligned}$$

where $w = (zy)^{(p-1)/2}$. This is an example of a group which Coxeter [C] has called

$$(2, a, b; c) = \langle R, S \mid R^a, S^b, (RS)^2, (R^2S^2)^c\rangle.$$

Thus $G = (2, p, |s - 2p|; r/2)$. Moreover, $\pi_1(M(s))$ will be infinite whenever G is.

And indeed, these groups are usually infinite as Edjvet has shown:

Theorem 4.4.4 (Main Theorem of [E]). *If* $2 \leq a \leq b$, $2 \leq c$ *and* $(2, a, b; c) \neq (2, 3, 13; 4)$, *then the group* $(2, a, b; c)$ *is finite if and only if it is one of the following:*

(i) $(2, 2, b; c)$ $(2 \leq b, 2 \leq c)$;

(ii) $(2, 3, b; c)$ $(3 \leq b \leq 6, 4 \leq c)$;

(iii) $(2, 3, 7; c)$ $(4 \leq c \leq 8)$;

(iv) $(2, 3, b; c)$ $(8 \leq b \leq 9, 4 \leq c \leq 5)$;

(v) $(2, 3, b; 4)$ $(10 \leq b \leq 11)$;

(vi) $(2,4,b;2)$ $(4 \leq b)$;

(vii) $(2,4,4;c)$ $(3 \leq c)$;

(viii) $(2,4,5;c)$ $(3 \leq c \leq 4)$;

(ix) $(2,4,7;3)$;

(x) $(2,5,b;2)$ $(5 \leq b \leq 9)$;

(xi) $(2,6,7;2)$.

Since p is odd, if $p \leq |s-2p|$, then G is infinite unless $p = 3$ or 5.

Similarly, if $|s-2p| \leq p$, we see that G is infinite unless $|s-2p| = 3$ or 5 ($|s-2p|$ is also odd) whence $s \leq 2p+5$. On the other hand, by [Ag, L], the finite filling s and the toroidal filling $2(p+q)$ (Lemma 4.4.1) are separated by at most 10. Since s is odd and $2(p+q)$ is even, they in fact differ by at most 9. Thus, $9 \geq 2(p+q)-s \geq 2(p+q)-(2p+5) = 2q-5$. It follows that $2q \leq 14$, whence $3 \leq p \leq q \leq 7$.

So we can assume that $3 \leq p \leq 7$. That is, if $p \geq 9$, the knot admits no non-trivial finite surgeries.

Now, earlier work shows that there are no non-trivial surgeries unless $\frac{p-1}{2} \leq r \leq p+3$. So, given $r \geq 4$, and assuming $3 \leq p \leq 7$, we see that we are left to consider $4 \leq r \leq 10$. (Since Theorem 4.1.3 does not apply to the knots $(-4,3,3)$, $(-4,3,5)$ and $(-6,3,3)$ we will consider those separately below.)

- **r = 4**
  - $p = 3$. When $q = 7$, there are no non-integral boundary slopes and therefore no non-trivial finite surgeries (Theorem 4.1.3). For $q \geq 9$, the boundary slope of Lemma 4.2.3 is $2q+12-4(q-1)/(q+2) \in (2q+8, 2q+9)$ so that the only candidate for a finite filling is $s = 2q+9$. But then $s-2p = 9+2(q-p) \geq 21$ and G is again infinite (Theorem 4.4.4). Therefore $(-4,3,q)$ pretzel knots admit no finite surgeries $(q \geq 7)$.
  - $p = 5$. Suppose $|s-2p| \leq 3$. Then $s \leq 2p+3$ and since a finite filling must be within distance 10 of the toroidal slope $2(p+q)$ ([Ag, L] and Lemma 4.4.1), $10 \geq 2(p+q)-(2p+3) \geq 2q-3 \Rightarrow q \leq 5$ (q being odd). Since the $(-4,5,5)$ pretzel knot admits no non-integral boundary slopes, we cannot have a non-trivial finite surgery in this case. Therefore we can assume $5 \leq |s-2p|$. In order for G to be finite, we must have $|s-2p| \leq 9$ (Theorem 4.4.4). Since s

would also have to be within distance 10 of $2(p+q)$ we deduce $q \leq 9$. The $(-4,5,7)$ pretzel knot has $126/5$ as its only non-integral boundary slope and therefore $s = 25$ is the only candidate for a finite filling. But then $s - 2p = 15$ contradicting an earlier assumption. Similarly, by examining the non-integral boundary slopes, we find that the only candidate for a finite filling on $(-4,5,9)$ is $s = 29$ with $s - 2p = 19 > 9$. So the $(-4,5,q)$ pretzel knots admit no finite surgeries.

- $p = 7$. Suppose $|s - 2p| \leq 5$. Then $q \leq 7$. However, since the $(-4,7,7)$ admits no non-integral boundary slopes, it cannot lead to a non-trivial finite surgery. So we can assume $|s - 2p| \geq 7$. Since G is then infinite, we see that $(-4,7,q)$ pretzel knots admit no non-trivial finite surgeries.

Except for possibly $(-4,3,3)$ and $(-4,3,5)$, which will be considered separately below, $(-4,p,q)$ pretzel knots admit no non-trivial finite surgeries.

- **r=6**
  - $p = 3$. When $q \geq 9$, the non-integral boundary slope of Lemma 4.2.3 is in $(2q+12, 2q+13)$ and the only candidate for a finite filling is $s = 2q+13$. Then $s - 2p = 13 + 2(q-p) \geq 25$ and G is infinite (Theorem 4.4.4). When $q = 5$, using non-integral boundary slopes, we find that the only candidate for a non-integral boundary slope is $s = 23$, whence $s - 2p = 17$ and G is infinite. When $q = 7$, there are no non-integral boundary slopes. So $(-6,3,q)$ pretzel knots admit no non-trivial finite surgeries ($q \geq 5$).
  - $p = 5$. When $q \geq 15$, the only non-integral boundary slope is in the interval $(2q+12, 2q+14)$ (Lemma 4.2.3) and the only candidate for a finite filling is $s = 2q+13$. Then $s - 2p = 13 + 2(q-p) \geq 33$ and G is infinite. When $q = 5$ or 13, there is no non-integral boundary slope. For $7 \leq q \leq 11$, by non-integral surgeries, the only candidate for finite filling is $s = 2q + 15 \Rightarrow s - 2p = 15 + 2(q-p) \geq 19$ and G is infinite. So $(-6,5,q)$ pretzel knots admit no non-trivial finite surgeries.
  - $p = 7$. We can assume $|s - 2p| \leq 7$ as otherwise G is infinite. Then $q \leq 7$. Since the $(-6,7,7)$ pretzel knot admits no non-integral boundary slopes, we conclude that $(-6,7,q)$ pretzel knots have no non-integral finite surgeries.

Thus, aside from possibly $(-6, 3, 3)$, the $(-6, p, q)$ pretzel knots admit no non-trivial finite surgeries.

- **r=8**
  - $p = 3$. By Proposition 4.4.3 $(-8, 3, q)$ pretzel knots admit no non-trivial finite surgeries.
  - $p = 5$. When $q \geq 15$, by Theorem 4.1.3 and Lemma 4.2.3, the only candidate for a finite filling is $s = 2q + 17$. Then $s - 2p = 17 + 2(q - p) \geq 37$ and G is infinite. When $q = 5$, or 13, there are no non-integral boundary slopes and for $7 \leq q \leq 11$, the only candidate is $s = 2q + 19$ which would again lead to G infinite.
  - $p = 7$. We can assume $|s - 2p| \leq 7$ whence $q \leq 7$. Since $(-8, 7, 7)$ has no non-integral boundary slopes, we are done in this case as well.

  Therefore $(-8, p, q)$ pretzel knots admit no non-trivial finite surgeries.
- **r=10**
  - $p = 7$. Again $|s - 2p| \leq 7 \Rightarrow q \leq 7$. Since $(-10, 7, 7)$ has no non-integral boundary slopes, we're done.

  The $(-10, p, q)$ pretzel knots also have no non-trivial finite surgeries.

It remains to examine the knots $(-4, 3, 3)$, $(-4, 3, 5)$ and $(-6, 3, 3)$ which are not covered by Theorem 4.1.3. We can follow the procedure outlined above to see that the group $\pi_1(M(s/t))$ of a s/t filling on a $(3, q, -r)$ pretzel knot projects onto $G = (2, 3, |s - 2pt|; r/2)$. Since $r/2$ is 2 or 3, this will be an infinite filling unless $|s - 2tp| = |s - 6t| \leq 6$ and $|s - 6t| \neq 2$ (Theorem 4.4.4). By [BZ1, Corollary 1.3], a finite filling s/t has denominator 1 or 2. On the other hand, by the work of Agol [Ag] and Lackenby [L], such a filling must also be close to the toroidal filling $2(p + q)$ (Lemma 4.4.1): $10 \geq \Delta(s/t, 2(p + q)) = |s - 2t(p + q)| = |s - 2t(3 + q)|$. These considerations leave only a few candidates for finite surgery.

- **(-4,3,3)** Since $6 \geq |s - 6t|$, $|s - 6t| \neq 2$, $10 \geq |s - 2t(3 + q)| = |s - 12t|$, and $t = 1$ or 2, the only candidates for a finite surgery on this knot are integral surgeries $2, 3, 5, 6, 7, 9, 10, 11, 12$ and half-integral surgeries $s/2$ with s odd, $15 \leq s \leq 17$. Moreover, Lemma 2.3.1 allows us to eliminate the boundary slope $s = 12$. One can verify directly, using SnapPea [Wee], that none of the remaining candidates are finite surgeries.

- **(-4,3,5)** Here, $6 \geq |s - 6t|$, $|s - 6t| \neq 2$, and $10 \geq |s - 2t(3 + q)|$ imply $s = 6, 7, 9, 11, 12$. We can also eliminate the boundary slope $s = 12$ and again verify directly that these fillings do not yield manifolds with finite fundamental group.
- **(-6,3,3)** Since $q = 3$ and $s = 12$ is again a boundary slope, we are left with the same candidates for a finite filling as we had in the case of the $(-4, 3, 3)$ pretzel knot. We can verify directly that none of these are finite fillings.

In summary then, Theorems 4.1.3 and Theorem 4.4.4 combine to show that a $(p, q, -r)$ admits no non-trivial finite surgery unless $4 \leq r \leq 10$ and $3 \leq p \leq 7$. We then investigated those cases directly to observe that they also admit no non-trivial finite surgeries. We have therefore proved the following.

Theorem 4.4.5. *A $(p, q, -r)$ pretzel knot, with $4 \leq r$ even and $3 \leq p \leq q$ odd admits no non-trivial finite surgeries.*

4.5. **Surgeries on Montesinos knots.** We can combine the results of the last two sections with the work of Delman to classify cyclic surgeries on Montesinos knots.

Theorem 4.5.1. *The only non-torus Montesinos knot which admits a non-trivial cyclic surgery is the $(-2, 3, 7)$ pretzel knot. The non-trivial cyclic surgeries on this knot are of slope 18 and 19.*

Remark: A torus knot admits an infinite number of cyclic fillings.

Proof: Delman [Del] has shown that if such a knot admits a cyclic filling, then it is a pretzel knot of the form $(p, q, -r)$, with $2 \leq r$ even and $3 \leq p \leq q$ odd. As only the trivial knot admits a $\mathbb{Z}$ filling ([Ga1]) Theorem 4.4.5 implies further that r must be 2. Proposition 4.3.2 completes the proof. □

For finite surgeries, we have:

Theorem 4.5.2. *If a non-torus Montesinos knot K admits a non-trivial finite surgery, then one of the following holds.*

- *K is a $(-2, p, q)$ pretzel knot with $5 \leq p \leq q$ odd and the filling is not cyclic.*
- *K is the $(-2, 3, 7)$ pretzel knot and the filling is along slope $17, 18$, or 19.*
- *K is the $(-2, 3, 9)$ pretzel knot and the filling is along slope 22 or 23.*

Proof: Again, Delman [Del] allows us to reduce to the case of a $(p, q, -r)$ pretzel knot and Theorem 4.4.5 further shows that $r = 2$. The finite surgeries on $(-2, 3, n)$ pretzel knots are classified in Section 5.2. That a non-trivial finite filling of $(-2, p, q)$ with $p \geq 5$ is not cyclic follows from the previous theorem. □

Remark: Although the surgeries listed for the $(-2, 3, 7)$ and $(-2, 3, 9)$ knots are finite surgeries, we know of no instances of a finite surgery on a knot $(-2, p, q)$ with $5 \leq p \leq q$. Indeed, we expect that there are none.

5. Character Varieties of Pretzel Knots

We continue our investigation of pretzel knots this time restricting to knots of the form $(-2, 3, n)$ and $(-3, 3, n)$. The Seifert fillings of these knots allow us to make an explicit calculation of their Culler-Shalen seminorms. This in turn leads to a precise description of their $SL_2(\mathbb{C})$-character varieties. Moreover, it allows us to construct the Newton polygon for the A-polynomial of these knots (much as we did for the twist knots in Section 2.3). The study of Seifert fillings of these knots gives a concrete demonstration of the power of Culler-Shalen seminorms, not only in the study of cyclic and finite surgeries, as we saw in the previous chapter, but also in the study of Seifert fillings.

This chapter, which is largely an expanded form of the papers [BMZ, Mat1, Mat2], begins with a calculation of the total minimal seminorm of the $(2, p, q)$ pretzel knots.

5.1. The total norm of (2,p,q) pretzel knots.

Let $K = K(2, p, q)$ be a pretzel knot as illustrated in Figure 19. Since K is a knot, $p = 2k + 1$ and $q = 2l + 1$ are

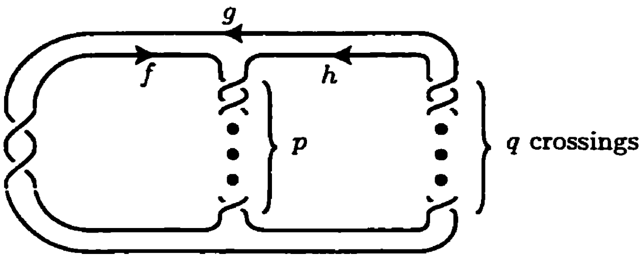


FIGURE 19. The $(2, p, q)$ pretzel knot.

odd. Let $M = S^3 \setminus N(K)$, $\pi = \pi_1(M)$. The two-fold branched cyclic cover Σ_2 is a Seifert manifold with base $\mathcal{B} = S^2(2, |p|, |q|)$ and $\pi_1^{\text{orb}}(\mathcal{B}) = \Delta(2, |p|, |q|)$.

For each irreducible component R_i of the representation variety $R = R(M)$ which contains an irreducible character, let $X_i = t(R_i)$. By [CS1, Proposition 1.4.4], X_i is an affine variety and a closed sub-variety of the character variety $X = X(M)$. We will denote the smooth projective completion of X_i by $\widetilde{X}_i$. Since K is small ([Oe, Corollary 4]), the X_i are curves ([CCGLS, Proposition 2.4]).

Let $x \in \widetilde{X}_i$ with $Z_x(f_{\mu^2}) > Z_x(f_\mu)$. We can find S, the sum of the minima of the Culler-Shalen seminorms over the components X_i of X, by enumerating such "jumping points" and then showing that each contributes two to S. Thus S will simply be twice the number of jumping points.

As a first step, we show that x is not an ideal point using a modification of the argument of [CGLS, Proposition 1.6.1].

Lemma 5.1.1. *For each* $\alpha \in H_1(\partial M; \mathbb{Z})$, $f_{\alpha^2} = (f_\alpha)^2 + 4f_\alpha$.

Proof: Let χ_ρ be a point of X. Let $\rho(\alpha) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$. Then

$$\begin{aligned} f_\alpha(\chi_\rho)(f_\alpha(\chi_\rho)+4) &= [(\mathrm{trace}(\rho(\alpha)))^2 - 4][\mathrm{trace}(\rho(\alpha))]^2 \\ &= (a+d)^4 - 4(a+d)^2 \\ &= (a^2+d^2+2bc)^2 - 4 \text{ (since } ad - bc = 1 \text{)}, \\ &= f_{\alpha^2}(\chi_\rho) \end{aligned}$$

□

Let x be an ideal point with $Z_x(f_{\mu^2}) > Z_x(f_\mu)$. Then $f_{\mu^2}(x) = 0$ so that x is not a pole of f_{μ^2}. By Lemma 5.1.1, x is also not a pole of f_μ and therefore $I_\mu(x) \neq \infty$ as well. It follows that either M admits a closed essential surface, or else μ is a boundary slope (see [CGLS, Proposition 1.3.9]). However, since K is a Montesinos knot with less than four tangles, neither is true (see [Oe, Section 1 and Corollary 4]).

Thus we can assume $x \in X_i^\nu$ and write $\nu(x) = \chi_\rho$ with $\rho \in R_i$. ($\nu : X_i^\nu \to X_i$ is normalization. See Section 2.1 or [Shf, Chapter II, §5].) We will argue that x is the character of an irreducible representation. Suppose instead that ρ were reducible. Then, by conjugating, we can take ρ to be a representation into the upper triangular matrices. Now replace each matrix $\rho(g) = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ by $\rho_0(g) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ to obtain a diagonal representation ρ_0 with the same character $\nu(x)$.

Since

$$\begin{pmatrix} 1/n & 0 \\ 0 & n \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1/n & 0 \\ 0 & n \end{pmatrix}^{-1} = \begin{pmatrix} a & b/n^2 \\ 0 & a^{-1} \end{pmatrix},$$

and R_i is closed under conjugation [CS1, Proposition 1.1.1], we can find representations on R_i arbitrarily close to ρ_0. But, as R_i is closed, $\rho_0 \in R_i$. So without loss of generality, we can assume ρ is diagonal.

The Zariski tangent space at ρ may be identified with a subspace of the space of 1-cocycles $Z^1(\pi; sl_2(\mathbb{C})_{Ad\rho})$ (see [Gl, Section 1.2] or [Wei, Section 3]). We can see that R_i is four dimensional since, as we have mentioned, X_i is one dimensional (the knot being small) and by [CS1, Corollary 1.5.3], $\dim R_i = \dim X_i + 3$. Thus, $\dim(Z^1(\pi; sl_2(\mathbb{C})_{Ad\rho})) \geq 4$.

Now, by Lemma 5.1.1, zeroes of f_μ are also zeroes of f_{μ^2} and moreover, the order of zero agrees at such points. So $Z_x(f_{\mu^2}) > Z_x(f_\mu)$ implies $Z_x(f_\mu) = 0$ and therefore $\text{trace}(\rho(\mu)) \neq \pm 2$. On the other hand, $\text{trace}(\rho(\mu^2))$ is ± 2, so $\rho(\mu^2) = \pm I$ (we're assuming that ρ is a diagonal representation). It follows that $\rho(\mu) = \begin{pmatrix} \pm i & 0 \\ 0 & \mp i \end{pmatrix}$. Since π is normally generated by μ, $\rho(\pi) = \mathbb{Z}/4$ is cyclic.

Given this, we can calculate $\dim(Z^1(\pi; sl_2(\mathbb{C})_{Ad\rho}))$ directly. Using [BN, Theorem 1.1(i)], $\dim(H^1(\pi; sl_2(\mathbb{C})_{Ad\rho})) = b_1(\pi; sl_2(\mathbb{C})_{Ad\rho}) = 1$. (This argument is explained in more detail and in a more general context in Section 5.2.1.) We can also determine $\dim B^1(\pi; sl_2(\mathbb{C})_{Ad\rho})$ as we have the surjection

$$\begin{array}{ccc} sl_2(\mathbb{C}) & \longrightarrow & B^1(\pi; sl_2(\mathbb{C})_{Ad\rho}) \\ A & \mapsto & (u_A : \gamma \mapsto A - Ad\rho(\gamma)(A)). \end{array}$$

Since $\rho(\mu) = \begin{pmatrix} \pm i & 0 \\ 0 & \mp i \end{pmatrix}$, the kernel is the one-dimensional set $\{A \in sl_2(\mathbb{C}) \mid A = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}\}$ while $sl_2(\mathbb{C})$ has dimension 3. Therefore, $\dim(B^1(\pi; sl_2(\mathbb{C})_{Ad\rho})) = 2$ and

$$\begin{aligned} \dim(Z^1(\pi; sl_2(\mathbb{C})_{Ad\rho})) &= \dim(H^1(\pi; sl_2(\mathbb{C})_{Ad\rho})) + \dim(B^1(\pi; sl_2(\mathbb{C})_{Ad\rho})) \\ &= 1 + 2 = 3. \end{aligned}$$

This contradiction with our earlier estimate of the dimension of the cocycles shows that there can be no jump at the character of a reducible representation.

Thus we can assume that ρ is irreducible. Then, by [CGLS, Proposition 1.5.2], $\rho(\mu^2) = \pm I$ and $\bar{\rho}$, the induced $\text{PSL}_2(\mathbb{C})$-representation of $\bar{\pi}$, will factor through $\pi_1(\Sigma_2)$. If $\bar{\rho}$ is non-abelian, Lemma 3.1.5 shows that $\bar{\rho}$ factors through to give an irreducible representation of $\Delta(2, |p|, |q|)$. On the other hand, if $\bar{\rho}$ is abelian, it factors through the finite group $H_1(\Sigma_2)$. In this case $\bar{\rho}(\bar{\pi})$ is cyclic and extending to π and lifting we see that ρ has binary dihedral image in $SL_2(\mathbb{C})$.

Furthermore, any such dihedral representation will result in a jumping point. For let $\nu(x) = \chi_\rho$ be the character of the binary dihedral $SL_2(\mathbb{C})$-representation ρ. The corresponding $PSL_2(\mathbb{C})$ representation $\bar{\rho}$ has as image a dihedral group normally generated by $\bar{\rho}(\mu)$. Therefore $\bar{\rho}(\mu)$ is of order two and consequently $\rho(\mu) \neq \pm I$ while $\rho(\mu^2) = \pm I$. This implies trace$(\rho(\mu)) = 0$ so that $Z_x(f_{\mu^2}) > Z_x(f_\mu)$.

The number of such dihedral characters d can be related to the Alexander polynomial $\Delta_K(t)$. Indeed, d is equal to $(\text{card}(H_1(\Sigma_2)) - 1)/2$ ([Kl, Theorem 10]) while $\text{card}(H_1(\Sigma_2)) = |\Delta_K(-1)|$ ([Rol, Corollary 8.D.3]). So $d = (|\Delta_K(-1)| - 1)/2$.

By Equations 2.2 and 2.3, there are $\lfloor \frac{|p|}{2} \rfloor \lfloor \frac{|q|}{2} \rfloor$ irreducible $PSL_2(\mathbb{C})$-characters of $\Delta(2, |p|, |q|)$. The corresponding representations each extend to an irreducible representation $\bar{\rho}_0$ of $\bar{\pi}$. These in turn can be extended to π (Proposition 3.3.1). Moreover, (Scholium 3.3.3) any representation $\bar{\rho}$ which extends $\bar{\rho}_0$ is such that $\bar{\rho}(\mu)$ has order two. Thus, as was the case for the dihedral representations, the irreducible representations of $\Delta(2, |p|, |q|)$ all lead to jumping points where $Z_x(f_{\mu^2}) > Z_x(f_\mu)$.

There are $\lfloor \frac{|p|}{2} \rfloor \lfloor \frac{|q|}{2} \rfloor$ irreducible $PSL_2(\mathbb{C})$-characters of π which factor through the triangle group $\Delta(2, |p|, |q|)$ and $(|\Delta_K(-1)| - 1)/2 = (|pq + 2(p+q)| - 1)/2$ irreducible dihedral characters. (Note that by [Mru, Proposition 14] or [Hi, Theorem 1.2],

$$\Delta_K(t) \doteq (t-1)(t^{-(p+q)} - 1)/(t+1) + t(t^{-p} + 1)(t^{-q} + 1)/(t+1)^2.)$$

By Lemma 2.4.10, none of the dihedral characters go through $\Delta(2, |p|, |q|)$. Since $PSL_2(\mathbb{C})$-dihedral characters are covered once in $SL_2(\mathbb{C})$ and other characters are covered twice ([BZ1, Lemma 5.5]), we see that there are $(|pq| - (|p| + |q|) + |pq + 2(p+q)|)/2$ jumping points where $Z_x(f_{\mu^2}) > Z_x(f_\mu)$. This allows us to calculate S once we have shown that $Z_x(f_{\mu^2}) - Z_x(f_\mu) = 2$ at each of these points.

The idea is to follow the argument of [BZ1, Section 4] (see also [BB, Theorem A]). The essential requirements are that $\rho(\pi_1(\partial M)) \not\subset \{\pm I\}$ and that $\nu(x)$ is a smooth point of X_i. Suppose first that ρ is a representation which factors through $\Delta(2, |p|, |q|)$. We proceed by showing that the corresponding point $y = \chi_{\rho_0}$ in Y, the character variety of $\bar{\pi}$, also satisfies these requirements.

Lemma 5.1.2. *Let $\bar{\rho}_0$ be the $PSL_2(\mathbb{C})$ representation induced by ρ_0. If $\bar{\rho}_0(\pi_1(\partial\widetilde{M})) \subset \{\pm I\}$ then ρ is an octahedral representation.*

Proof: Actually we will argue that if the preferred longitude λ has trivial image in $\Delta(2, |p|, |q|)$, then ρ is octahedral.

Let $\tilde{\lambda} \in \tilde{\pi}$ denote the class of a lift of a representative of λ. Trotter [Tr] has explained how to find the image of $\tilde{\lambda}$ in $\Delta(p, q, r)$ in the case of a (p, q, r) pretzel knot with p, q, r all odd. We will follow the same procedure for the $(2, p, q)$ pretzel.

Starting with the Wirtinger presentation [Rol, Section 3.D] for π with generators f, g and h as indicated in Figure 19, we look at the index two subgroup consisting of words of even length and quotient out by the relations $f^2 = g^2 = h^2 = 1$. The resulting group is $\pi_1(\Sigma_2) \cong \tilde{\pi}/\langle \mu^2 \rangle$. Quotienting again by the center, brings us to $\Delta(2, |p|, |q|) = \langle a, b, c \mid a^2, b^{|p|}, c^{|q|}, abc \rangle$ where $a = gf$, $b = fh$ and $c = hg$. Beginning with the arc labeled h and tracing out the knot, we find the longitude $\lambda \in \pi$:

$$\lambda = (fh)^k (gf)(fh)^{k+1}(hg)^l (gf)(hg)^{l+1}$$

projects to $\tilde{\lambda} \in \Delta(2, |p|, |q|)$:

$$\tilde{\lambda} = b^k a b^{k+1} c^l a c^{l+1}$$

We can take

$$\bar{\rho}_0(a) = A = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Suppose that $\bar{\rho}_0(\tilde{\lambda}) = \pm I$. Then $P^{-1}AP = \pm A$ where $P = \bar{\rho}_0(b^{k+1}c^{l+1})$. It follows that P is of the form

$$\pm \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \text{ or } \pm \begin{pmatrix} 0 & -\alpha \\ \alpha^{-1} & 0 \end{pmatrix}.$$

Let $\bar{\rho}_0(b) = B$. Note that B^k is not diagonal. If it were, then $B = B^{-2k}$ would be as well and would therefore commute with A. In this case, $\bar{\rho}_0$ would be reducible which is a contradiction. Therefore, after conjugating by diagonal matrices, we can write B^k in one of the following two ways:

$$\pm \begin{pmatrix} u & 1 \\ u(\tau - u) - 1 & \tau - u \end{pmatrix} \text{ or } \pm \begin{pmatrix} u & u(\tau - u) - 1 \\ 1 & \tau - u \end{pmatrix}.$$

where $\pm\tau$ is the trace of B^k and $u \in \mathbb{C}$ (compare [BZ2, Example 3.2]).

Taking $C = \bar{\rho}_0(c)$, we see that $C^{-l} = B^k P$. So the relator abc implies

$$\begin{aligned} \pm I &= ABC \\ &= AB^{-2k}C^{-2l} \\ &= AB^{-k}PB^kP. \end{aligned}$$

In other words, $B^k A = PB^kP$. If

$$P = \pm \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix},$$

then no matter which of the two forms we choose for B^k, the equation $B^k A = PB^kP$ will not be satisfied. Therefore

$$P = \pm \begin{pmatrix} 0 & -\alpha \\ \alpha^{-1} & 0 \end{pmatrix}.$$

For the time being, we will assume

$$B^k = \pm \begin{pmatrix} u & 1 \\ u(\tau - u) - 1 & \tau - u \end{pmatrix}$$

and return to the other case later.

Equating $B^k A$ and PB^kP (in $\mathrm{PSL}_2(\mathbb{C})$) gives rise to the equations

$$\begin{aligned} \alpha^{-2} &= \pm i(u(\tau - u) - 1) \\ \tau &= u(1 \mp i). \end{aligned}$$

Using these, we may simplify

$$\begin{aligned} \mathrm{trace}(C^{-l})^2 &= \mathrm{trace}(B^kP)^2 \\ &= (\pm(\alpha^{-1} - (u(\tau - u) - 1)\alpha))^2 \\ &= \pm i(u(\tau - u) - 1) \pm 2u^2 i + 2 + \mp i(u(\tau - u) - 1) \\ &= 2 \pm 2u^2 i. \end{aligned}$$

On the other hand,

$$\begin{aligned}(\text{trace}(B^{-k}))^2 &= (\pm\tau)^2 \\ &= \tau^2 \\ &= \mp 2u^2 i.\end{aligned}$$

Therefore $(\text{trace}(C^{-l}))^2 + (\text{trace}(B^{-k}))^2 = 2$. Since C and B are of finite order, $\text{trace}(C^{-l}) = \pm(\xi + \xi^{-1})$ and $\text{trace}(B^{-k}) = \pm(\zeta + \zeta^{-1})$ where ξ and ζ are roots of unity and we have

$$\begin{aligned}0 &= \xi^2 + \xi^{-2} + \zeta^2 + \zeta^{-2} + 2 \\ &= a_1\xi^2 + a_2\xi^{-2} + a_3\zeta^2 + a_4\zeta^{-2} + a_5 \end{aligned} \tag{5.9}$$

where $a_i = 1$ $(i = 1 \ldots 4)$ and $a_5 = 2$.

Following Mann [Man, Definition 2], an equation of this form is called irreducible, provided there is no equation of the form

$$b_1\xi^2 + b_2\xi^{-2} + b_3\zeta^2 + b_4\zeta^{-2} + b_5 = 0 \tag{5.10}$$

where $b_i = a_i$ or $b_i = 0$ with at least one but not all the b_i zero. It's clear that at least two of the b_i would have to be zero in Equation 5.10. (Otherwise the complimentary equation would have only one term.) So there are basically two possibilities.

If Equation 5.10 is of the form $\xi^2 + \zeta^2 + 2 = 0$, then we see that $\xi^2 = \zeta^2 = -1$. But then $\xi^{-2} = \zeta^{-2} = -1$ as well in contradiction to Equation 5.9. The other possibility for Equation 5.10 is of the form $\xi^2 + \xi^{-2} + 2 = 0$. Again this implies $\xi^2 = \xi^{-2} = -1$. Now, since $\text{trace}(C^{-l}) = \pm(\xi + \xi^{-1})$, we can diagonalize C^{-l} to put it in the form $\pm\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$. Then $C = C^{-2l}$ diagonalizes to $\pm I$. In other words C is the identity. This contradicts irreducibility of $\bar{\rho}_0$.

We conclude that Equation 5.9 is irreducible. By Mann's Theorem [Man, Theorem 1], the solutions to the equation are 30th roots of unity. Checking amongst the 30th roots of unity, we see that the only solution is $\xi^2+\xi^{-2} = \zeta^2+\zeta^{-2} = -1$ so that ξ^2 and ζ^2 are both third roots of unity. As, before we can argue that B and C diagonalize to

$\pm \begin{pmatrix} \zeta^2 & 0 \\ 0 & \zeta^{-2} \end{pmatrix}$ and $\pm \begin{pmatrix} \xi^2 & 0 \\ 0 & \xi^{-2} \end{pmatrix}$ respectively. We conclude that B and C both have order 3.

If we take

$$B^k = \pm \begin{pmatrix} u & u(\tau - u) - 1 \\ 1 & \tau - u \end{pmatrix},$$

then again we find that B and C have order 3.

We began with the assumption that $\bar{\rho}_0(\tilde{\lambda}) = \pm I$ and deduced that B and C are of order 3. Thus $\bar{\rho}_0(\bar{\pi}) = \Delta(2,3,3)$ with generators A, B and C. In other words $\bar{\rho}_0(\bar{\pi})$ is A_4, the tetrahedral group. As $[\pi : \bar{\pi}] = 2$, the lift to $\bar{\rho}(\pi)$ is either A_4 or S_4. Now, $\bar{\rho}(\mu)$ has order two and normally generates $\bar{\rho}(\pi)$. Since A_4 has no such order two generator, we conclude that $\bar{\rho}(\pi) = S_4$, the octahedral group. Thus $\rho(\pi)$ is a (binary) octahedral representation into $\mathrm{SL}_2(\mathbb{C})$.

This completes the proof of the claim. □

So, as long as ρ is not octahedral, $\bar{\rho}_0(\pi_1(\partial\widetilde{M})) \neq \{\pm I\}$. Then, $\rho_0(\pi_1(\partial\widetilde{M})) \not\subset \{\pm I\}$ and $\rho(\pi_1(\partial M)) \not\subset \{\pm I\}$ as well.

We now turn to the smoothness of $\nu(x)$. Again we will first show y is smooth. As the Zariski tangent space at ρ_0 can be identified with a subspace of the cocycles, we proceed by investigating the group cohomology.

As a first step, we observe that $Z^1(\pi_1(\Sigma_2); sl_2(\mathbb{C})_{Ad\bar{\rho}_0}) \cong Z^1(\Delta(2,|p|,|q|); sl_2(\mathbb{C})_{Ad\bar{\rho}_0})$. Indeed, the Seifert structure of Σ_2 gives the exact sequence (Equation 3.4)

$$0 \longrightarrow F \longrightarrow \pi_1(\Sigma_2) \xrightarrow{\phi} \Delta(2,|p|,|q|) \longrightarrow 1$$

where $F = \langle h \rangle \cong \mathbb{Z}$ is the group of a regular fibre. The projection ϕ induces a homomorphism $\Phi : Z^1(\Delta(2,|p|,|q|); sl_2(\mathbb{C})_{Ad\bar{\rho}_0}) \to Z^1(\pi_1(\Sigma_2); sl_2(\mathbb{C})_{Ad\bar{\rho}_0})$.

To construct the inverse, we show that $u(h) = 0$ for each $u \in Z^1(\pi_1(\Sigma_2); sl_2(\mathbb{C})_{Ad\bar{\rho}_0})$. Indeed, for all $g \in \pi_1(\Sigma_2)$, $u(hg) = u(gh)$. On the other hand, $\bar{\rho}_0(h)$ commutes with $\bar{\rho}_0(\Delta(2,|p|,|q|))$. Since $\bar{\rho}_0$ is irreducible, this implies $\bar{\rho}_0(h) = \pm I$ by Lemma 2.4.9.

Putting it together,

$$\begin{aligned} u(h) + u(g) &= u(h) + Ad\bar{\rho}_0(h) \cdot u(g) \\ &= u(hg) \\ &= u(gh) \\ &= u(g) + Ad\bar{\rho}_0(g) \cdot u(h) \\ &= u(g) + \bar{\rho}_0(g) u(h) \bar{\rho}_0(g)^{-1}. \end{aligned}$$

Thus $u(h) \in sl_2(\mathbb{C})$ commutes with $\bar{\rho}_0(\pi_1(\Sigma_2))$. By Lemma 2.4.7, $u(h) = 0$. Now define $\Psi : Z^1(\pi_1(\Sigma_2); sl_2(\mathbb{C})_{Ad\rho_0}) \to Z^1(\Delta(2,|p|,|q|); sl_2(\mathbb{C})_{Ad\rho_0})$ by $\Psi(u)(\phi(g)) = u(g)$. Since $u(h) = 0$, Ψ is well defined. Moreover, it's an inverse of Φ and we have the required isomorphism. This isomorphism also descends to the level of cohomology: $H^1(\pi_1(\Sigma_2); sl_2(\mathbb{C})_{Ad\bar{\rho}_0}) \cong H^1(\Delta(2,|p|,|q|); sl_2(\mathbb{C})_{Ad\bar{\rho}_0})$. On the other hand, we can argue that the cohomology of the triangle group is trivial directly.

Lemma 5.1.3. *If $\bar{\rho} : \Delta(p,q,r) \to PSL_2(\mathbb{C})$ is an irreducible or non-abelian representation, then $dim_{\mathbb{C}}(H^1(\Delta(p,q,r); sl_2(\mathbb{C})_{Ad\bar{\rho}})) = 0$.*

Proof: We begin with $B^1(\Delta(p,q,r); sl_2(\mathbb{C})_{Ad\bar{\rho}})$. Recall the surjection

$$\begin{array}{ccc} sl_2(\mathbb{C}) & \longrightarrow & B^1(\Delta(p,q,r); sl_2(\mathbb{C})_{Ad\rho}) \\ A & \mapsto & (u_A : \gamma \mapsto A - Ad\bar{\rho}(\gamma)(A). \end{array}$$

By Lemma 2.4.7, the kernel is empty and

$$\dim_{\mathbb{C}}(B^1(\Delta(p,q,r); sl_2(\mathbb{C}))_{Ad\bar{\rho}})) = \dim_{\mathbb{C}}(sl_2(\mathbb{C})) = 3.$$

Thus it suffices to argue that $\dim_{\mathbb{C}}(Z^1(\Delta(p,q,r); sl_2(\mathbb{C})_{Ad\bar{\rho}})) = 3$ and, as the coboundaries are contained in the cocycles, it will in fact be enough to argue that the dimension is at most 3.

Let $\Delta(p,q,r) = \langle a, b \mid a^p = b^q = (ab)^r \rangle$. We can assume

$$\begin{aligned}
\bar{\rho}(a) &= \pm \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix} \text{ where } \eta \neq \pm 1, \eta^p = 1, \\
\bar{\rho}(b) &= \pm A \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} A^{-1} \quad \xi \neq \pm 1, \xi^q = 1, \text{ and} \\
\bar{\rho}(ab) &= \pm B \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} B^{-1} \quad \sigma \neq \pm 1, \sigma^r = 1.
\end{aligned}$$

If $A = \pm \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ then either $a_1 a_2 \neq 0$ or $a_3 a_4 \neq 0$ (otherwise $\bar{\rho}(a)$ and $\bar{\rho}(b)$ commute) and similarly, for $B = \pm \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$, either $b_1 b_2 \neq 0$ or $b_3 b_4 \neq 0$.

Recall that

$$Z^1(\Delta(p,q,r); sl_2(\mathbb{C})_{Ad\bar{\rho}}) = \{u : \Delta(p,q,r) \to sl_2(\mathbb{C}) \mid u(gg') = u(g) + g \cdot u(g')\}$$

where $g \cdot u(g')$ denotes the adjoint action of $\bar{\rho}(g)$ on $u(g')$. Then a cocycle $u \in Z^1(\Delta(p,q,r); sl_2(\mathbb{C})_{Ad\bar{\rho}})$ is determined by its value at a and b:

$$u(a) = \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix}, \qquad u(b) = \begin{pmatrix} y_1 & y_2 \\ y_3 & -y_1 \end{pmatrix}.$$

So $Z^1(\Delta(p,q,r); sl_2(\mathbb{C})_{Ad\bar{\rho}}) \subset sl_2(\mathbb{C}) \oplus sl_2(\mathbb{C}) \cong \mathbb{C}^6$. We can therefore establish the inequality $\dim_{\mathbb{C}}(Z^1(\Delta(p,q,r); sl_2(\mathbb{C})_{Ad\bar{\rho}})) \leq 3$ by finding three independent equations relating the x_i and y_i.

Now,

$$\begin{aligned}
0 &= u(1) \\
&= u(a^p) \\
&= u(a) + a \cdot u(a^{p-1}) \\
&= u(a) + a \cdot u(a) + a^2 \cdot u(a) + \ldots + a^{p-1} \cdot u(a) \\
&= (1 + a + \ldots + a^{p-1}) \cdot u(a),
\end{aligned}$$

so, using

$$a^n \cdot u(a) = \begin{pmatrix} \eta^n & 0 \\ 0 & \eta^{-n} \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix} \begin{pmatrix} \eta^{-n} & 0 \\ 0 & \eta^n \end{pmatrix} = \begin{pmatrix} x_1 & \eta^{2n}x_2 \\ \eta^{-2n}x_3 & -x_1 \end{pmatrix},$$

we have

$$\begin{aligned} 0 &= \begin{pmatrix} px_1 & (1+\eta^2+\ldots+\eta^{2(p-1)})x_2 \\ (1+\eta^{-2}+\ldots+\eta^{-2(p-1)})x_3 & -px_1 \end{pmatrix} \\ &= \begin{pmatrix} px_1 & \frac{1-\eta^{2p}}{1-\eta^2}x_2 \\ \frac{1-\eta^{-2p}}{1-\eta^{-2}}x_3 & -px_1 \end{pmatrix} \\ &= \begin{pmatrix} px_1 & 0 \\ 0 & -px_1 \end{pmatrix}, \end{aligned}$$

whence $x_1 = 0$. This is our first equation.

Similarly, if we let $\begin{pmatrix} w_1 & w_2 \\ w_3 & -w_1 \end{pmatrix} = A^{-1}u(b)A$, then

$$\begin{aligned} 0 &= u(1) \\ &= u(b^q) \\ &= \sum_{i=0}^{q-1} \rho(b^i)u(b)\rho(b^{-i}) \\ &= \sum_{i=0}^{q-1} (A\rho(b)A^{-1})^i \begin{pmatrix} w_1 & w_2 \\ w_3 & -w_1 \end{pmatrix} (A\rho(b)A^{-1})^{-i} \\ &= \begin{pmatrix} qw_1 & 0 \\ 0 & -qw_1 \end{pmatrix} \end{aligned}$$

so that $w_1 = 0$ as well.

Since $w_1 = (a_1a_4 + a_2a_3)y_1 + a_3a_4y_2 - a_1a_2y_3$ and either $a_1a_2 \neq 0$ or $a_3a_4 \neq 0$, this is a second, independent equation in the x_i, y_i coordinates.

Finally, the relation $(ab)^r = 1$ allows us to deduce a third equation

$$b_3b_4x_2 - b_1b_2x_3 + (b_1b_4 + b_2b_3)y_1 + b_3b_4\eta^2y_2 - b_1b_2\eta^{-2}y_3 = 0$$

which is again non-trivial since either b_1b_2 or b_3b_4 is non-zero.

Thus $Z^1(\Delta(p,q,r); sl_2(\mathbb{C})_{Ad\bar{\rho}})$ is an algebraic set in $\mathbb{C}^6$ cut out by at least three independent linear equations and so of dimension at most three. □

Now, using the remarks which precede the lemma, and the observation that the $\mathrm{PSL}_2(\mathbb{C})$ representation $\bar{\rho}_0$ and the $\mathrm{SL}_2(\mathbb{C})$ representation ρ_0 result in exactly the same adjoint action on $sl_2(\mathbb{C})$, we see that $\dim_{\mathbb{C}}(H^1(\pi_1(\Sigma_2); sl_2(\mathbb{C})_{Ad\rho_0})) = 0$. So we can proceed as in [BZ1, Section 4] to show that $\dim_{\mathbb{C}} H^1(\tilde{\pi}; sl_2(\mathbb{C})_{Ad\rho_0}) = 1$ and y is simple in Y. (Note that we will make the distinction between smooth and simple points of a character variety. A *simple* point is a smooth point which lies on a unique irreducible component of the variety. See [Shf, Chapter 2 §2].)

Proposition 5.1.4. *Let ρ be an $SL_2(\mathbb{C})$-representation of a finitely generated group π and ρ_0 the restriction to a normal subgroup of finite index $\tilde{\pi}$. Then*

$$dim_{\mathbb{C}} H^1(\pi; sl_2(\mathbb{C})_{Ad\rho}) \leq dim_{\mathbb{C}} H^1(\tilde{\pi}; sl_2(\mathbb{C})_{Ad\rho_0}).$$

Proof: The Lyndon - Hochschild - Serre spectral sequence gives us the exact sequence (see [Rot, Theorem 11.5])

$$0 \longrightarrow H^1(\pi/\tilde{\pi}; (sl_2(\mathbb{C})_{Ad\rho})^{\tilde{\pi}}) \longrightarrow H^1(\pi; sl_2(\mathbb{C})_{Ad\rho}) \longrightarrow H^1(\tilde{\pi}; sl_2(\mathbb{C})_{Ad\rho_0})^{\pi/\tilde{\pi}},$$

where $A^G = \{a \in A \mid g \cdot a = g, \forall g \in G\}$ denotes the set of fixed points of the module A under the group action G. Now, $H^1(\pi/\tilde{\pi}; (sl_2(\mathbb{C})_{Ad\rho})^{\tilde{\pi}}) = 0$ since $\pi/\tilde{\pi}$ is finite and $(sl_2(\mathbb{C})_{Ad\rho})^{\tilde{\pi}}$ is a complex vector space. So we have

$$H^1(\pi; sl_2(\mathbb{C})_{Ad\rho}) \hookrightarrow H^1(\tilde{\pi}; sl_2(\mathbb{C})_{Ad\rho_0})^{\pi/\tilde{\pi}} \hookrightarrow H^1(\tilde{\pi}; sl_2(\mathbb{C})_{Ad\rho_0}).$$

□

In our case, the proposition shows that $\dim_{\mathbb{C}} H^1(\pi; sl_2(\mathbb{C})_{Ad\rho}) \leq 1$ whence $\nu(x)$ is a smooth point of X_i (and in fact a simple point of X).

Remark: We have been using the ideas of [BZ1, Section 4] whereby, under appropriate conditions, $x = \chi_\rho$ is smooth in $X(\pi)$ exactly when $\dim_{\mathbb{C}} H^1(\pi; sl_2(\mathbb{C})_{Ad\rho}) = 1$. Interpreted in this context, the proposition says "simple points of $X(\tilde{\pi})$ lift to simple points of $X(\pi)$."

Thus if ρ factors through $\Delta(2, |p|, |q|)$ and is not octahedral, then $\nu(x)$ is a smooth point of X_i and $\rho(\pi_1(\partial M)) \not\subset \{\pm I\}$. Following the reasoning of [BZ1, Section 4], we conclude that $Z_x(f_{\mu^2}) - Z_x(f_\mu) = 2$.

We can see that the jump at a dihedral character is also two by adapting Tanguay's [Ta, Propostion 5.3.3] arguments for dihedral characters of two-bridge knots to the present situation. Again, the argument comes down to showing $H^1(\pi; sl_2(\mathbb{C})_{Ad\rho})$ has dimension 1 when ρ is a dihedral representation. Let $\rho(\pi) = D_{4m}$, the binary dihedral group of order $4m$. Then $Ad\rho(\pi) \subset \mathrm{Aut}(\mathrm{SL}_2(\mathbb{C}))$ is isomorphic to D_{2m}, the dihedral group of order $2m$.

Tanguay shows that the Betti number $b_1(\pi; sl_2(\mathbb{C})_{Ad\rho})$ can be related to the Betti numbers of several covers of M:

$$b_1(\pi; sl_2(\mathbb{C})_{Ad\rho}) = b_1(\tilde{\pi}; \mathbb{C}) - b_1(\pi; \mathbb{C}) + \frac{1}{\phi(m)} \sum_{d|m} \mu(\frac{m}{d}) b_1(\pi_d; \mathbb{C}),$$

where the π_d are the kernels of the maps

$$\pi \xrightarrow{Ad\rho} D_{2m} \to D_{2d},$$

and ϕ and μ are the Euler and Möbius Functions respectively. Now, $b_1(\pi; \mathbb{C}) = 1$ [Rol, Exercise 2.E.6] and $b_1(\tilde{\pi}; \mathbb{C}) = 1$ [Rol, Section 8.D]. So showing that $b_1(\pi; sl_2(\mathbb{C})_{Ad\rho}) = 1$ and dihedral characters are smooth reduces to arguing that $b_1(\pi_d; \mathbb{C}) = d$.

Let $\widetilde{M}_d$ be the covering of M corresponding to π_d. Then $\widetilde{M}_d$ also covers $\widetilde{M}$ and this covering may be extended to an orbifold covering $\Sigma_d \to \Sigma_2$. We will argue that $b_1(\Sigma_d) = 0$. Then, since Σ_d is obtained from $\widetilde{M}_d$ by filling along d tori, $0 = b_1(\Sigma_d) \geq b_1(\widetilde{M}_d) - d$ whence $b_1(\widetilde{M}_d) \leq d$. On the other hand, since $\widetilde{M}_d$ has d toral boundary components, Lefschetz duality allows us to argue that $b_1(\pi_d) = \dim H_1(\widetilde{M}_d; \mathbb{C}) \geq d$. Therefore $b_1(\pi_d) = d$, as required.

It remains to show that $b_1(\Sigma_d) = 0$, and this is where we must introduce some new ideas beyond those used by Tanguay. We have the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & \pi_1(\Sigma_d) & \longrightarrow & \pi_1^{\mathrm{orb}}(B_d) & \longrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle P'} & & \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle P''} & & \\ 0 & \longrightarrow & F & \longrightarrow & \pi_1(\Sigma_2) & \longrightarrow & \Delta(2, |p|, |q|) & \longrightarrow & 1 \end{array}$$

where the horizontal rows are the exact sequences arising from the Seifert structure of Σ_d and Σ_2, $E \cong F \cong \mathbb{Z}$ represent regular fibres, and B_d is the base orbifold of Σ_d. Now, $\text{Im}(P)$ is normal since P is a regular cover. This implies $\text{Im}(P'')$ is normal. Since E and F are abelian, $\text{Im}(P')$ is also normal. Thus, the cokernels will be groups and we can use the Snake Lemma to obtain the exact sequence

$$\ker(P'') \xrightarrow{\delta} \text{coker}(P') \xrightarrow{\alpha} \text{coker}(P) \xrightarrow{\beta} \text{coker}(P'') \longrightarrow 1.$$

Now, $\ker(P'') = 0$ since $B_d \to \Delta(2, |p|, |q|)$ is an orbifold covering space. Thus α is injective. Since P comes from the dihedral covering $\widetilde{M_d} \to \widetilde{M} \to M$, we see that $\text{coker}(P) \cong \mathbb{Z}/d$. By the injectivity of α, $\text{coker}(P') \cong \mathbb{Z}/a$ where $a \mid d$. On the other hand, as the degree d of the Seifert cover $\Sigma_d \to \Sigma_2$ is the product of the degree a in the fibres and the degree of the orbifold cover c, we see that $\text{card}(\text{coker}(P'')) = c = d/a$. However, since $\text{Im}(\alpha) = \ker(\beta)$, $\text{Im}(\beta)$ also has cardinality c:

$$\text{coker}(P'') = \text{Im}(\beta) \cong \text{coker}(P)/\ker(\beta) \cong (\mathbb{Z}/d)/(\mathbb{Z}/a) \cong \mathbb{Z}/c.$$

The projection $\Delta(2, |p|, |q|) \to \text{coker}(P'') \cong \mathbb{Z}/c$ is an abelian representation, and as such factors through $H_1(\Delta(2, |p|, |q|)) = \mathbb{Z}/b$, where $b = \gcd(pq, 2p, 2q) = \gcd(p, q)$. So the covering $B_d \to S^2(2, |p|, |q|)$ is either trivial, or else of degree $c > 1$ dividing b. In particular, c is odd. So, B_d is either $S^2(2, |p|, |q|)$ or else $S^2(2, 2, \ldots, 2, |p|/c, |q|/c)$, i.e., c cone-points of order 2. Given B_d, we have an explicit formula (Equation 3.5) for $\pi_1(\Sigma_d)$ involving the orders of the cone points. We can then show that $H_1(\Sigma_d)$ is torsion by examining its order ideal (see [Rol, Section 8.B]). Therefore $b_1(\Sigma_d) = 0$ as required.

Octahedral representations can be treated in a similar fashion. In this case,

$$b_1(\pi; sl_2(\mathbb{C})_{Ad\rho}) = b_1(\hat{\pi}; \mathbb{C}) - b_1(\pi; \mathbb{C})$$

where $\hat{\pi} = \rho^{-1}(D_6)$ with D_6 a dihedral subgroup of index four in the octahedral group S_4. Of course $b_1(\pi; \mathbb{C}) = 1$ as before, so we will want to argue that $b_1(\hat{\pi}; \mathbb{C}) = 2$.

Let $\hat{M}$ be the covering of M corresponding to D_6. Then $\hat{M}$ is an irregular covering of degree 4 which also covers $\widetilde{M}$. As before, we extend the covering $\hat{M} \to \widetilde{M}$ to a degree two map between Seifert spaces: $\hat{\Sigma} \to \Sigma_2$. This leads to a diagram quite similar to that for the dihedral representation:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & \pi_1(\hat{\Sigma}) & \longrightarrow & \pi_1^{\text{orb}}(\hat{B}) & \longrightarrow & 1 \\ \downarrow & & \downarrow P' & & \downarrow P & & \downarrow P'' & & \\ 0 & \longrightarrow & F & \longrightarrow & \pi_1(\Sigma_2) & \longrightarrow & \Delta(2,|p|,|q|) & \longrightarrow & 1 \end{array}$$

In this case, $\hat{B}$, the base orbifold of $\hat{\Sigma}$, is either a $1-1$ or a $2-1$ cover of $S^2(2,|p|,|q|)$. In particular it is a regular cover and $\text{coker}(P'')$ is either trivial or cyclic of order 2. Since $\text{coker}(P'')$ is abelian, it is again a factor of $H_1(\Delta(2,|p|,|q|)) \cong \mathbb{Z}/b$ where $b = \gcd(p,q)$ is an odd number. Thus $\hat{B} = S^2(2,|p|,|q|)$ as well and, as in the dihedral case, we find $b_1(\hat{\Sigma}) = 0$. Since $\hat{M}$ has two boundary components, we may now argue that $b_1(\tilde{\pi};\mathbb{C}) = 2$. Thus $b_1(\pi; sl_2(\mathbb{C})_{Ad\rho}) = 1$ and $\nu(x)$ is again a smooth point of X_i yielding a jump of two.

So each of the jumping points $\nu(x) \in X_i$ is a simple point of X (in particular, $\nu^{-1}(\nu(x)) = x$) and results in a jump of two: $Z_x(f_{\mu^2}) - Z_x(f_\mu) = 2$. This allows us to calculate S once we have observed that $2s_i = 2||\mu||_i = ||\mu^2||_i$ and that $Z_x(f_{\mu^2}) \geq Z_x(f_\mu)$, $\forall x \in X$ ([CGLS, Proposition 1.1.3]).

$$\begin{aligned} S &= \sum_i s_i \\ &= \sum_i ||\mu^2||_i - ||\mu||_i \\ &= \sum_i \sum_{Z_x(f_{\mu^2}) > Z_x(f_\mu), \nu(x)\in X_i} Z_x(f_{\mu^2}) - Z_x(f_\mu) \\ &= \sum_{Z_x(f_{\mu^2}) > Z_x(f_\mu), \nu(x)\in X} Z_x(f_{\mu^2}) - Z_x(f_\mu) \\ &= 2(|pq| - (|p| + |q|) + |pq + 2(p+q)|)/2 \\ &= |pq| - (|p| + |q|) + |pq + 2(p+q)|. \qquad (5.11) \end{aligned}$$

5.2. $(-2,3,n)$ **pretzel knots.** Let K_n denote the $(-2,3,n)$ pretzel knot which we introduced in Section 2.5. If n is even, K_n is a link, so we will take n odd. Since K_n is a Montesinos knot with three tangles, it is small [Oe, Corollary 4] and consequently not a satellite knot. Therefore K_n is either a torus knot or hyperbolic.

Lemma 5.2.1. *K_n is a torus knot iff $n = 1,3,5$.*

Proof: Kawauchi [Kaw, Theorem III] shows more generally that a (p, q, r) pretzel knot with $|p|, |q|, |r| > 1$ is torus only if $\{p, q, r\} = \{-2, 3, 3\}$ or $\{-2, 3, 5\}$. We present here a direct argument specific to the $(-2, 3, n)$ pretzel knots.

We will show that if $n \neq 1, 3, 5$, then the Alexander polynomial $\Delta_{K_n}(t)$ is not the Alexander polynomial of a torus knot. Recall ([Mru, Proposition 14] or [Hi, Theorem1.2]) that

$$\Delta_{K_n}(t) \doteq (t-1)(t^{-(3+n)} - 1)/(t+1) + t(t^{-3}+1)(t^{-n}+1)/(t+1)^2,$$

while the polynomial for the p, q torus knot is [Rol, Section 7.D.8]

$$\Delta_{K_{p,q}}(t) \doteq \frac{(1-t)(1-t^{pq})}{(1-t^p)(1-t^q)}.$$

In the discussion that follows, we will use representatives for these polynomials which have a positive constant term and no negative powers of t.

Suppose first that $n \geq 7$. Then $\Delta_{K_n}(t)$ terminates with the three terms $t^3 - t + 1$ while $\Delta_{K_{p,q}}(t)$ terminates with $t^p - t + 1$ (we will assume $0 < p < q$). Thus, if K_n is a p, q torus knot, then $p = 3$. Since $\deg \Delta_{K_n}(t) = n+3$ and $\deg \Delta_{K_{p,q}}(t) = pq+1-p-q$, we see that $q = (n+5)/2 \geq 6$. Actually $q \geq 7$ since it's also relatively prime to 3. However, although $\Delta_{K_n}(t)$ includes the term t^5 (when $n \geq 7$), $\Delta_{K_{p,q}}(t)$ has no t^5 term (when $p = 3$ and $q \geq 7$.) This contradiction shows that K_n is not a torus knot when $n \geq 7$.

If $n < 0$, $\Delta_{K_n}(t)$ includes the term $-2t$ (or $-3t$ when $n = -1$) while $\Delta_{K_{p,q}}(t)$ always has t coefficient -1. Therefore K_n $(n < 0)$ is not a torus knot either. □

Since we are interested in hyperbolic knots, we will assume that n is an odd integer, $n \neq 1, 3, 5$. In addition, we will generally assume $n \neq -1$ as that case corresponds to the twist knot K_2 treated in Section 2.3. However, we will verify that our conclusions also hold true for this knot.

As in Equation 5.11, $(p = -3,\ q = -n.)$

$$\begin{aligned} S &= |pq| - (|p| + |q|) + |pq + 2(p+q)| \\ &= 3|n| - (|n| + 3) + |3n - 2(n+3)| \\ &= 2|n| + |n-6| - 3 \\ &= 3(|n-2| - 1) \text{ (since } n \notin [0,6] \text{).} \end{aligned}$$

5.2.1. *Bounding the Seifert slopes.* Bleiler and Hodgson [BH, Propositions 16 & 17] have shown that $2n+4$ (respectively $2n+5$) surgery on K_n results in a manifold which is Seifert fibred over $S^2(2,4,|n-6|)$ (respectively $S^2(3,5,|n-5|/2)$). (Note that there is a small error in [BH, Proposition 17] which refers to "$4n+14$ surgery on the $(-2,3,2n+7)$ pretzel knot." It should read "$4n+18$ surgery on the $(-2,3,2n+7)$ pretzel knot.")

We can bound the Culler-Shalen seminorms of these slopes in much the same way we calculated S above. We will count the irreducible characters of the surgered manifold and then show that each such character contributes 2 to the seminorms of the corresponding slope.

2n + 4 By Equations 2.2 and 2.3, there are $|n-6|-1$ irreducible $PSL_2(\mathbb{C})$-characters of $\Delta(2,4,|n-6|)$. Since $d = (|\Delta_K(-1)|-1)/2 = (|n-6|-1)/2$, half of them are dihedral. This gives $\frac{3}{2}(|n-6|-1)$ irreducible $SL_2(\mathbb{C})$-characters.

Recall that

$$||2n+4||_i = \sum_{x\in\tilde{X}_i} Z_x(f_{2n+4})$$

where $Z_x(f)$ denotes the order of zero of f at x. Since the meridian μ of K_n is not a boundary slope, $Z_x(f_\mu) \leq Z_x(f_{2n+4})$ for each x ([CGLS, Proposition 1.1.3]). This suggests that we approach the calculation of the total norm (see Section 2.2.3) $||2n+4||_T$ by comparison with $||\mu||_T = S$:

$$||2n+4||_T = S + \sum_i \sum_{x\in\tilde{X}_i} (Z_x(f_{2n+4}) - Z_x(f_\mu)). \tag{5.12}$$

Since M is small and $2n+4$ is not a strict boundary class, we may apply [CGLS, Proposition 1.6.1] to see that $Z_x(f_{2n+4}) = Z_x(f_\mu)$ at ideal points. Thus the sum of Equation 5.12 may be restricted to $x \in X_i^\nu$.

Reducible characters do not contribute to the sum of Equation 5.12 either. For suppose $\nu(x) = \chi_\rho$ were the character of a reducible representation $\rho \in R_i$ with $Z_x(f_{2n+4}) > Z_x(f_\mu)$. Since R_i is closed and invariant under conjugation, we can assume that ρ is diagonal. Then, as in Section 5.1, we can argue that the dimension of $Z^1(\pi; sl_2(\mathbb{C})_{Ad\rho})$ is at least 4.

On the other hand, $Z_x(f_{2n+4}) > Z_x(f_\mu)$ implies $\rho(2n+4) = \pm I$ ([CGLS, Proposition 1.5.4]). So, if we take $\bar{\rho}$ as the $PSL_2(\mathbb{C})$ representation corresponding to ρ, then $\bar{\rho}$

factors through $H_1(M(2n+4)) \cong \mathbb{Z}/(2n+4)$. Since $\rho(\pi)$ is normally generated by $\rho(\mu)$ and ρ is diagonal, we see that $\rho(\mu) = \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix}$ with $\eta^{2(2n+4)} = 1$.

We can use [BN, Theorem 1.1(i)] to find $b_1(\pi; sl_2(\mathbb{C})_{Ad\rho})$. Indeed,

$$b_1(\pi; sl_2(\mathbb{C})_{Ad\rho}) = b_1(\pi; \mathbb{C}) + 2b_1(\pi; \mathbb{C}_\beta)$$

where $\beta = \eta^2$ is a $(2n+4)$th root of unity. Of course [Rol, Exercise 2.E.6], $b_1(\pi; \mathbb{C}) = 1$. Now $\mathbb{C}_\beta$ is $\mathbb{C}$ with the $\mathbb{Z}$-action induced by $t \cdot c = c\beta$ where t is a generator of $\mathbb{Z}$. Since π surjects onto $H_1(M) \cong \mathbb{Z}$, this gives a π-action on $\mathbb{C}$.

We can also think of $H_1(M)$ as acting on $\hat{M}$, the infinite cyclic cover of M, and define a $\mathbb{C}[t, t^{-1}]$-module structure on $H_1(\hat{M}; \mathbb{C})$ (see [Rol, Section 7.A]). In this context $H^1(\pi; \mathbb{C}_\beta) = \text{coker}(H^1(\hat{M}; \mathbb{C}) \xrightarrow{t-\beta} H^1(\hat{M}; \mathbb{C}))$ where $t - \beta$ represents multiplication by $t - \beta$. Since the Alexander polynomial $\Delta_{K_n}(t)$ is the generator of $H_1(\hat{M}; \mathbb{C})$ as a $\mathbb{C}[t, t^{-1}]$-module, we can argue that $\text{coker}(t - \beta) = 0$ if $\Delta_{K_n}(\beta) \neq 0$.

In Appendix A we show that $\Delta_{K_n}(t)$ admits no roots which are $2n + 4$th roots of unity. Thus $H^1(\pi; \mathbb{C}_\beta) = 0$ and $b_1(\pi; sl_2(\mathbb{C})_{Ad\rho}) = 1$. We can then argue as in Section 5.1 that $\dim B^1(\pi; sl_2(\mathbb{C})_{Ad\rho}) = 2$ and $\dim Z^1(\pi; sl_2(\mathbb{C})_{Ad\rho}) = 3$. This contradicts our earlier estimate for the dimension of the cocycles and we conclude that there can be no jump at the character of a reducible representation ρ.

It remains to examine the $x \in X_i^\nu$ satisfying $Z_x(f_{2n+4}) > Z_x(f_\mu)$ and such that $\nu(x) = \chi_\rho$ is the character of an irreducible representation ρ. As mentioned above, $Z_x(f_{2n+4}) > Z_x(f_\mu)$ implies the corresponding $\text{PSL}_2(\mathbb{C})$ representation $\bar{\rho}$ factors through $\pi_1(M(2n+4))$.

As $M(2n+4)$ is Seifert fibred over $S^2(2, 4, |n-6|)$, $\bar{\rho}$ factors through an irreducible representation $\bar{\rho}' : \Delta(2, 4, |n-6|) \to \text{PSL}_2(\mathbb{C})$ (see Lemma 3.1.5.) Now, as in Section 5.1, $H^1(\pi_1(M(2n+4)); sl_2(\mathbb{C})_{Ad\rho}) \cong H^1(\Delta(2, 4, |n-6|); sl_2(\mathbb{C})_{Ad\bar{\rho}'})$ is trivial. Thus, arguing as in [BZ1, Section 4], we can deduce that $\nu(x)$ is a smooth point of X_i (and in fact a simple point of X) so that $\nu^{-1}(\nu(x)) = x$.

Therefore the jumping points $\nu(x)$ where $Z_x(f_{2n+4}) > Z_x(f_\mu)$ are simple points of X and correspond to irreducible $\text{PSL}_2(\mathbb{C})$ characters $\bar{\rho}$ which factor through $\Delta(2, 4, |n-6|)$. Conversely, any such representation induces a jumping point. This is immediate if the representation is diagonalizable on $\pi_1(\partial M)$ since then $\rho(\mu)$ is of finite order, but not $\pm I$. On the other hand, $\rho(2n+4) = \pm I$. Thus $Z_x(f_{2n+4}) > 0 = Z_x(f_\mu)$. If $\rho(\pi_1(\partial M))$ is parabolic, we can appeal to [BB, Theorem A]. In any case, as the jump $Z_x(f_{2n+4}) - Z_x(f_\mu)$ will be two (see [BB, Theorem A]) at each of the $\frac{3}{2}(|n-6|-1)$ $SL_2(\mathbb{C})$ characters, we can evaluate the sum of Equation 5.12 to find that $||2n+4||_T = S + 3(|n-6|-1)$.

2n + 5 Since $M(2n+5)$ has odd order first homology, $\pi_1(M(2n+5))$ has no dihedral characters and its irreducible characters correspond to those of $\Delta(3,5,|n-5|/2)$.

When $n \not\equiv 5 \pmod{30}$, the Alexander polynomial $\Delta_{K_n}(t)$ admits no zeroes which are $2n+5$th roots of unity (Lemma A.1.1). So, as with $2n+4$, there is no jump at the character of a reducible representation. Equations 2.2 and 2.3 show that there are $|n-4|-1$ irreducible $PSL_2(\mathbb{C})$-characters of $\Delta(3,5,|n-5|/2)$ each of which is double covered in $SL_2(\mathbb{C})$. Thus $\pi_1(M(2n+5))$ has $2(|n-4|-1)$ irreducible $SL_2(\mathbb{C})$-characters. As was the case for $2n+4$, each of these contribute two to the Culler-Shalen seminorms of $2n+5$ so that $||2n+5||_T = S + 4(|n-4|-1)$.

This equation also holds for $n \equiv 5 \pmod{30}$ although by a different argument. In this case, there are $|n-4|-5$ irreducible $PSL_2(\mathbb{C})$-characters of $\Delta(3,5,|n-5|/2)$ and consequently $2(|n-4|-5)$ irreducible $SL_2(\mathbb{C})$-characters of $\pi_1(M(2n+5))$. Four of the reducible characters of $\Delta(3,5,|n-5|/2)$ correspond to reducible representations which project onto $\mathbb{Z}/15$. As shown in Appendix A, the Alexander polynomial admits 15th roots of unity among its zeroes when $15 \mid (n-5)$, so we cannot neglect these reducible characters. Indeed, as we will now show, each one will be covered twice in $SL_2(\mathbb{C})$ by characters contributing two so that we again have $||2n+5||_T = S + 4(|n-4|-1)$.

If $n \equiv 5 \mod 30$, then $|n-5|/2 = 15k$ and $H_1(\Delta(2,3,|n|)) \cong \mathbb{Z}/15$. On the other hand, by Lemma A.1.2, $\Delta_{K_n}(t)$ admits primitive 15th roots of unity and they are simple zeroes of $\Delta_{K_n}(t)$.

Let $\xi = e^{2\pi ji/15}$ be a primitive 15th root of unity and let ρ be the reducible $SL_2(\mathbb{C})$ representation of π induced by

$$\rho(\mu) = \begin{pmatrix} e^{\pi ji/15} & 0 \\ 0 & e^{-\pi ji/15} \end{pmatrix}.$$

Then $\rho(\mu)^{15} = \pm I$ and ρ is a cover of one of the reducible $PSL_2(\mathbb{C})$ representations of $\Delta(2,3,|n|)$ projecting onto $\mathbb{Z}/15$. In other words, we can think of ρ as a representation of π which factors through $M(2n+5)$. Corresponding to the eight primitive 15th roots of unity, we have eight $SL_2(\mathbb{C})$ characters. We will show that the jump at each of these characters is 2.

Frohman and Klassen [FK, Theorem 1.1] show that such a representation ρ is the endpoint of an arc of irreducible representations. So $\rho \in R_i$, a component of the $SL_2(\mathbb{C})$-representation variety containing an irreducible representation. The corresponding character $x = \chi_\rho$ lies on the curve $X_i = t(R_i)$.

Since $\xi = e^{2\pi ji/15}$ is a primitive 15th root of unity, $\chi_\rho(\mu) = 2\cos(\pi j/15) \neq \pm 2$. So $Z_x(f_\mu) = 0$, x is a non-trivial character, and, moreover, $x(\pi_1(\partial M)) \neq \{\pm 2\}$. (A character is *trivial* if $\chi(\pi) \subset \{\pm 2\}$. See [P, Section 3.2].) On the other hand, since ρ factors through $M(2n+5)$, $\rho(2n+5) = I$ and $Z_x(f_{2n+5}) > 0$. So $Z_x(f_{2n+5}) > Z_x(f_\mu)$ and there is a jump at x.

Now Proposition 1.5.2 of [CGLS] shows that there is a non-abelian representation $\rho' \in R_i$ with character x and $\rho'(2n+5) = \pm I$. Since $x(\pi_1(\partial M)) \neq \{\pm 2\}$, we see that $\rho'(\pi_1(\partial M)) \not\subset \{\pm I\}$. Finally, as in Section 5.1, we can argue that $H^1(\pi_1(M(2n+5)); sl_2(\mathbb{C})_{Ad\rho'}) = 0$. This allows us to apply [BB, Theorem A] and conclude that the jump, $Z_x(f_{2n+5}) - Z_x(f_\mu)$, is 2.

5.2.2. *An application of Lemma 6.2.* Lemma 6.2 of [BZ1] relates the Culler-Shalen norm on a norm curve to the boundary slopes. In the case of a knot, such as K_n, for which μ is not a boundary slope we can write

$$\|\gamma\| = 2[a_1\Delta(\gamma,\beta_1) + a_2\Delta(\gamma,\beta_2) + \ldots + a_m\Delta(\gamma,\beta_m)], \tag{5.13}$$

where the a_i are non-negative integers and the β_i are boundary slopes.

The boundary slopes of K_n can be found using the methods of [HO] (see also Section 4.2). We have $\beta_1 = 0$, $\beta_2 = 2n+6$, $\beta_3 = 16$ (respectively 10) and $\beta_4 = \frac{n^2-n-5}{\frac{n-3}{2}}$ (respectively $2(n+1)^2/n$) when $n \geq 7$ (respectively $n \leq -1$).

The calculations of the last two sections imply the following inequalities for any Culler-Shalen seminorm:

$$(5.14) \qquad \|\mu\| = s \leq 3(|n-2|-1);$$

$$(5.15) \qquad s \leq \|2n+4\| \leq s + 3(|n-6|-1) \text{ and}$$

$$(5.16) \qquad s \leq \|2n+5\| \leq s + 4(|n-5|-2).$$

These strongly restrict the possible values of the coefficients a_i.

For example, suppose $n \geq 7$. Then the equations above become

$$(5.17) \qquad 2[a_1 + a_2 + a_3 + \frac{n-3}{2}a_4] = s \leq 3(n-3);$$

$$(5.18) \qquad s \leq 2[(2n+4)a_1 + 2a_2 + (2n-12)a_3 + a_4] \leq s + 3(n-7) \text{ and}$$

$$(5.19) \qquad s \leq 2[(2n+5)a_1 + a_2 + (2n-11)a_3 + \frac{n-5}{2}a_4] \leq s + 4(n-7).$$

It will be useful to subtract s from each of the last two equations:

$$(5.20) \qquad 0 \leq (2n+3)a_1 + a_2 + (2n-13)a_3 - \frac{n-5}{2}a_4 \leq 3(n-7)/2,$$

$$(5.21) \qquad 0 \leq (2n+3)a_1 + (2n-12)a_3 - a_4 \leq 2(n-7)$$

Since $a_i \geq 0$, Equation 5.17 implies $a_4 \leq 3$. In fact, in order to have a norm (rather than a seminorm), we would need at least two of the $a_i > 0$. This condition further restricts $a_4 \leq 2$. (Seminorms which are not norms will be discussed further in Section 5.2.4.)

Given $a_4 \leq 2$, Equation 5.21 implies $(2n+3)a_1 \leq 2(n-6)$ so that $a_1 = 0$. Then, the same equation implies $a_3 \leq 1$. We will argue that, in fact, $a_4 = 2$ and $a_3 = 1$.

Suppose instead that $a_4 \leq 1$. Since $a_1 = 0$, Equation 5.21 becomes $(2n-12)a_3 \leq 2n-13$ so that $a_3 = 0$. This is a contradiction since if a_1 and a_3 are both zero, then Equation 5.21 in fact says $a_4 = 0$ as well, and only a_2 is non-zero. This would mean that $\|\cdot\|$ is not a norm.

Therefore $a_4 = 2$. Since $a_1 = 0$, Equation 5.21 implies that $a_3 > 0$. Thus $a_3 = 1$. Finally, given these values, Equation 5.20 can be rearranged to see that $0 \leq a_2 \leq (n-5)/2$. This implies $s = 2n - 4 + 2a_2$, $\|2n+4\| = s + 2(n-8) + 2a_2$ and $\|2n+5\| = s + 4(n-7)$.

For $n \leq -3$ there are four possible solutions.

1. $a_1 = 0$, $a_3 = a_4 = 1$ and $0 \leq a_2 \leq (1-n)/2$. Then

$$s = 2(1-n) + 2a_2,\ \|2n+4\| = 4(4-n) + 4a_2 = s + 2(7-n) + 2a_2$$

$$\text{and } \|2n+5\| = 2(7-3n) + 2a_2 = s + 4(3-n).$$

2. $a_1 = a_4 = 1$, $a_3 = 0$ and $0 \leq a_2 \leq (1-n)/2$. Then

$$s = 2(1-n) + 2a_2,\ \|2n+4\| = -4(n+1) + 4a_2 = s - 2(n+3) + 2a_2$$

$$\text{and } \|2n+5\| = -6(n+1) + 2a_2 = s - 4(n+2).$$

3. If $n \geq -23$, $a_1 = 1$, $a_3 = a_4 = 0$ and $0 < a_2 \leq (n+25)/2$. Then

$$s = 2 + 2a_2,\ \|2n+4\| = -2(2n+4) + 4a_2 = s - 2(2n+5) + 2a_2$$

$$\text{and } \|2n+5\| = -2(2n+5) + 2a_2 = s - 2(2n+6).$$

4. If $n = -3$, $a_1 = a_4 = 0$ and $a_2 = a_3 = 1$. Then

$$s = 4,\ \|2n+4\| = 28 \text{ and } \|2n+5\| = 24.$$

5.2.3. *Norm Curves.* For $n \geq 7$, we see immediately that there is at most one norm curve in the character variety. Indeed, if there were two, each would have $s \geq 2n-4$ which would force $S > 3(|n-2|-1) = 3(n-3)$. Similarly, for negative n, we see that there can be at most one norm curve of type 1 or 2. In order to see how the type 3 solution interacts with the other two, we use the Seifert slopes.

For a type 1 norm curve, we have $\|2n+5\| = s + 4(3-n)$ which implies all the jumping points for $2n+5$ surgery lie on that curve. We've shown that the jumping points are simple, so they cannot lie on any other curve. Thus any other norm curve would have $\|2n+5\| = s$. However, examining the other solutions we see that $\|2n+5\|$ is greater than s (unless $n = -3$). So for $n \leq -5$, if there is a curve corresponding to a solution of type 1, then there are no other norm curves. Using similar arguments, we can show that for $n \leq -11$, there is at most one norm curve. For $n = -9$, there are either two norm curves each corresponding to a type 3 solution or else there is only one norm curve.

In the following, we will assume that there is only one norm curve X_0 and we will denote its Culler-Shalen norm by $\|\cdot\|_0$. The cases $-9 \leq n \leq -1$ will be treated separately later.

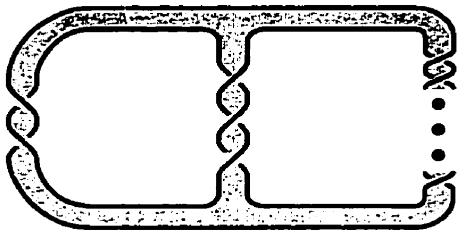


FIGURE 20. A spanning surface of K_n.

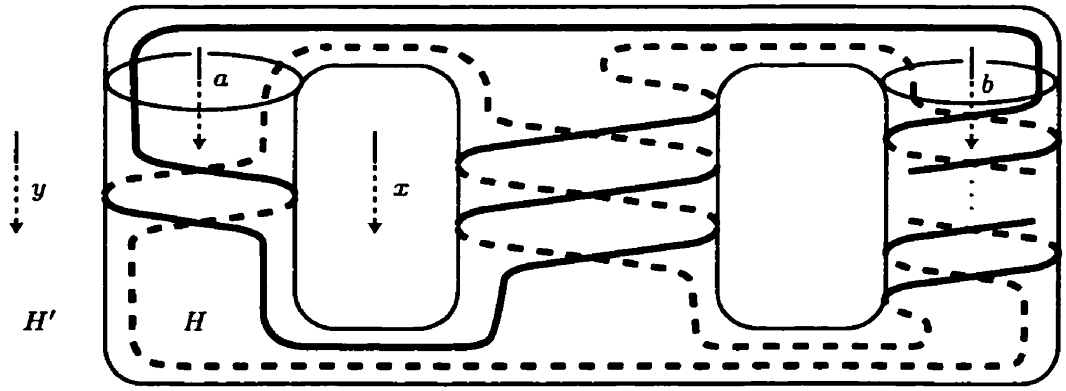


FIGURE 21. The handlebody H.

5.2.4. *r-Curves.* We will argue that K_n admits an r-curve only if $r = 2n+6$. For this, we'll use the graph manifold structure of $M(2n+6)$: $M(2n+6) = M_1 \cup M_2$, is the union of two Seifert fibred manifolds M_1 and M_2 along a torus. We can construct M_1 by thickening the spanning surface of Figure 20. The surface is a punctured Klein bottle so M_1 is a twisted I-bundle over the Klein bottle. We denote the complementary manifold $S^3 \setminus N(M_1)$ by M_2. We can get a better handle on the Seifert structure of the M_i $(i = 1, 2)$ using the ideas of Dean [Dea].

A key observation [Dea, Lemma 2.2.1] is that an irreducible Haken manifold, like M_i, with fundamental group $G_{m,n} = \langle x, y \mid x^m y^n \rangle$ is Seifert fibred, the base orbifold being a disc with cone points of order m and n. Since M_1 is obtained from the obvious genus 2 handlebody H in Figure 21 by adding a 2-handle along the knot, we can compute it's fundamental group. Indeed, with respect to the generators a, b of $\pi_1(H)$, the knot represents the relator $b^{-1}ab^{-1}a^{-1}$. Making the change of basis, $b^{-1}a \to c$, $a \to d^{-1}$, the relator becomes c^2d^2. Thus M_1 is Seifert fibred over $D^2(2,2)$ with $(b^{-1}a)^2$ or a^2 representing a regular fibre and fundamental group $\pi_1(M_1) = \langle c, d \mid c^2d^2 \rangle$.

A similar argument allows us to identify M_2 using the generators x and y of the complementary handlebody H' (see Figure 21). In this context, the knot represents the word $yxy^{(n-1)/2}xy^{(n-1)/2}x$. After the change of basis $y^{(n-1)/2}x \to w$, $y^{-1} \to z$, the word becomes $z^{(n-3)/2}w^3$. Thus M_2 is Seifert fibred over $D^2(3, |n-3|/2)$ and has fundamental group $\pi_1(M_2) = \langle w, z \mid z^{(n-3)/2}w^3 \rangle$. Moreover, a regular fibre corresponds to $(xy^{(n-1)/2})^3$.

We can now argue that the fibres intersect once on the common boundary of M_1 and M_2. Indeed, Figure 22 shows how a fibre of M_1 representing a^2 and a fibre of M_2 representing $(xy^{(n-1)/2})^3$ have intersection number one. Note also that the M_1 fibre a^2 becomes $y^{-2}x^{-1} = y^{(n-5)/2}(xy^{(n-1)/2})^{-1} = z^{(5-n)/2}w^{-1}$ in $\pi_1(M_2)$ whereas the M_2 fibre $(xy^{(n-1)/2})^3$ goes to $b^{-1}ab^{-1}ab^{-1}a = (b^{-1}a)^3a^{-1} = c^3d$.

Proposition 5.2.2. *The* $PSL_2(\mathbb{C})$*-character variety* $\bar{X}(M(2n+6))$ *contains exactly one curve when* $3 \mid n$. *Otherwise* $\dim \bar{X}(M(2n+6)) = 0$.

Proof: We will argue that irreducible $\mathrm{PSL}_2(\mathbb{C})$-characters of $M(2n+6)$ are either isolated or else factor through $\mathbb{Z}/2 * \mathbb{Z}/3$. The result then follows from [BZ2, Example 3.2].

An irreducible representation $\bar{\rho} : M(2n+6) \to \mathrm{PSL}_2(\mathbb{C})$ will be non-abelian or else have image $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ (Lemma 2.4.5). On the other hand, if it's abelian, it also factors through $H_1(M(2n+6)) \cong \mathbb{Z}/(2n+6)$ and there is no cyclic group which contains $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. Therefore if $\bar{\rho}$ is irreducible, it's also non-abelian. Let $\bar{\rho}_i : \pi_1(M_i) \to \mathrm{PSL}_2(\mathbb{C})$ $(i = 1, 2)$ be the induced representations. If one of these is non-abelian, we can show that it kills the corresponding fibre. Let $h_i \in \pi_1(M_i)$ be the class of a regular fibre.

Claim 5.2.3. *If* $\bar{\rho}_i$ *is non-abelian, then* $\bar{\rho}_i(h_i) = \pm I$.

Proof: (of Claim) Suppose $\bar{\rho}_1$ is non-abelian.

Since h_1 generates the center of $\pi_1(M_1)$, by Lemma 2.4.3, if $\bar{\rho}_1(h_1) \neq \pm I$ then we can conjugate so that $\bar{\rho}_1(h_1) = E$ and $\bar{\rho}_1(\pi_1(M_1)) \subset N$ (see Definition 2.4.2 for notation).

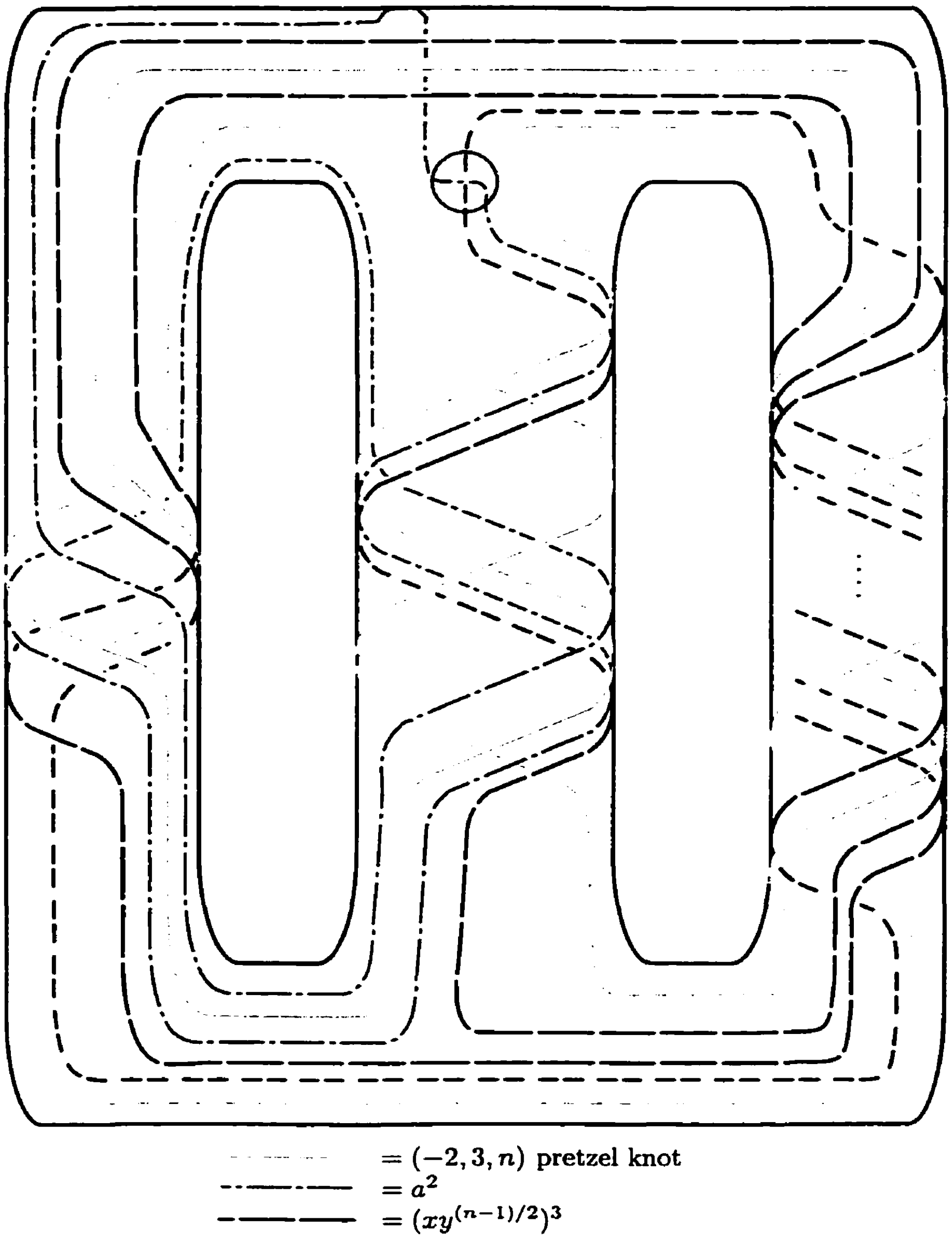


FIGURE 22. The regular fibres a^2 and $(xy^{(n-1)/2})^3$ intersect once inside the circle.

Since $\bar{\rho}_1$ is non-abelian, at least one of the generators, say c, of $\pi_1(M_1) = \langle c, d \mid c^2 d^2 \rangle$ is sent to an antidiagonal element and is therefore of order two. But since c^2 generates the center of $\pi_1(M_1)$, this means that the image of the center is trivial, and, in particular, $\bar{\rho}_1(h_1) = \pm I$, a contradiction.

Similarly, if $\bar{\rho}_2$ is non-abelian, then we can assume (for a contradiction) that $\text{Im}(\bar{\rho}_2) \subset N$ and $\bar{\rho}_2(h_2) = E$. As before, one of the generators of $\pi_1(M_2) = \langle w, z \mid$

$w^3z^{(n-3)/2}\rangle$ must be antidiagonal. If $\bar{\rho}_1(w)$ is antidiagonal, then $\bar{\rho}_1(w^3)$ is as well. This is a contradiction since both w^3 and h_2 generate the center $Z(\pi_1(M_2)) \cong \mathbb{Z}$ and we started out by assuming $\bar{\rho}_2(h_2) = E$. This argument also shows that $\bar{\rho}_2(z)$ cannot be antidiagonal in case $(n-3)/2$ is odd. If $(n-3)/2$ is even, we can repeat the argument we used for $\bar{\rho}_1$. So we also get a contradiction in this case. □(Claim)

Let us assume $\bar{\rho}(\pi_1(T)) \not\subset \{\pm I\}$. We wish to show that $\chi_{\bar{\rho}}$ is then isolated in $\bar{X}(M(2n+6))$. Since regular fibres intersect once on T, their images generate $\bar{\rho}(\pi_1(T))$. The Claim therefore shows that in order to satisfy $\bar{\rho}(\pi_1(T)) \not\subset \{\pm I\}$, at least one $\bar{\rho}_i$ is abelian with $\bar{\rho}_i(h_i) \neq \pm I$.

So, suppose $\bar{\rho}_2$ is abelian and $\bar{\rho}_1$ is not. As above, $\bar{\rho}_1(h_1) = \pm I$. Since the glueing torus T contains regular fibres, we can assume $h_1 \in \pi_1(T)$. As the $\bar{\rho}_i$'s agree on the intersection $\pi_1(T)$, $\bar{\rho}_2(h_1) = \pm I$ as well. However, we've seen earlier that a regular fibre h_1 represents the word $z^{(5-n)/2}w^{-1}$ in $\pi_1(M_2)$. Since this word is killed, $\bar{\rho}_2$ factors through

$$\pi_1(M_2)/\langle h_1\rangle = \langle w, z \mid w^3z^{(n-3)/2}, z^{(5-n)/2}w^{-1}\rangle = \langle z \mid z^{6-n}\rangle$$

which is cyclic of order $|n-6|$.

This means that $\bar{\rho}_2(h_2)$ is of finite odd order (remember that n is odd, so that $n-6 \neq 0$). We can conjugate so that $\bar{\rho}_2(h_2) = \pm \begin{pmatrix} \eta & 0 \\ 0 & 1/\eta \end{pmatrix}$ with $\eta \neq \pm 1, \pm i$. Now, since $\bar{\rho}_1(h_1) = \pm I$, $\bar{\rho}_1$ factors through the orbifold group $\pi_1^{\text{orb}}(\mathcal{B}_1) = \langle c, d \mid c^2, d^2\rangle$ where $\mathcal{B}_1$ is the base orbifold of M_1. Again, $\bar{\rho}_1(h_2) = \bar{\rho}_2(h_2)$ is of finite order dividing $|n-6|$ and represents the word c^3d. Thus $\bar{\rho}_1$ factors through $\langle c, d \mid c^2, d^2, (cd)^{|n-6|}\rangle$ which is dihedral of order $2|n-6|$. Also, $\bar{\rho}_1(h_2) = \bar{\rho}_2(h_2) = \pm \begin{pmatrix} \eta & 0 \\ 0 & 1/\eta \end{pmatrix}$ is in the image of the cyclic subgroup which is therefore diagonal (Lemma 2.4.3).

We have now given a rather specific description of $\bar{\rho}$. Restricted to $\pi_1(M_2)$, it is cyclic of order dividing $|n-6|$ and diagonal. Restricted to $\pi_1(M_1)$ it factors through $D_{2|n-6|}$ with the cyclic subgroup having image in the diagonal matrices. Moreover, $\pm \begin{pmatrix} \eta & 0 \\ 0 & 1/\eta \end{pmatrix}$ with $\eta \neq \pm 1, \pm i$ is common to the images of $\pi_1(M_1)$ and $\pi_1(M_2)$.

There are only a finite number of characters consistent with such a representation. Thus characters of this form are isolated in the sense that they cannot form a curve.

Next, assume $\bar{\rho}_1$ is abelian and $\bar{\rho}_2$ is not. Again $\bar{\rho}_2(h_2) = \pm I$ which implies $\bar{\rho}_1$ goes through $\pi_1(M_1)/\langle h_2 \rangle = \langle c, d \mid c^2 d^2, c^3 d \rangle = \mathbb{Z}/4$. Then $\bar{\rho}(h_1)$ is of order dividing 4 and since it is non-trivial, it has order 2 or 4. On the other hand, as the fibre h_2 is killed, $\bar{\rho}_2$ goes through the orbifold group $\mathbb{Z}/3 * \mathbb{Z}/(|n-3|/2)$. Moreover, since $\bar{\rho}_1(h_1) = \bar{\rho}_2(h_2)$ has order 2 or 4 we deduce that $\bar{\rho}_2$ factors through $\Delta(3, |n-3|/2, 2)$ or $\Delta(3, |n-3|/2, 4)$. So we are again in a rather restrictive situation. The number of characters of such a triangle group is finite (see Equation 2.2) and, since the third generator of the triangle group is of order 2 or 4 and generates $\bar{\rho}_1(M_1)$, there are also only a finite number of characters for such a representation $\bar{\rho}$. So they are isolated.

Finally we turn to the case were both $\bar{\rho}_1$ and $\bar{\rho}_2$ are abelian. This means that $\bar{\rho}(\pi_1(T))$ must commute with everything in the image of the non-abelian representation $\bar{\rho}$. We are assuming $\bar{\rho}(\pi_1(T)) \neq \{\pm I\}$, so, by Lemma 2.4.3 and after an appropriate conjugation, $\bar{\rho}(\pi_1(T)) = \{\pm I, E\}$ and $\bar{\rho}(\pi_1(M(2n+6))) \subset N$.

Now, as $\bar{\rho}_1$ and $\bar{\rho}_2$ are abelian, they will factor through $\mathbb{Z} \oplus \mathbb{Z}/2$ and $\mathbb{Z} \oplus \mathbb{Z}/g$, (where $g = \gcd(3, |n-3|/2)$) respectively. Since $\bar{\rho}$ is non-abelian, there is an antidiagonal matrix in its image. On the other hand, since $E \in \bar{\rho}(\pi_1(T))$, at least one of the $\bar{\rho}_i$'s has image including both diagonal and antidiagonal elements. The only way this could happen, while the $\bar{\rho}_i$ in question remains abelian would be for it to have image $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ (see Lemma 2.4.5). We see that this is feasible only on the M_1 side and conclude that $\bar{\rho}_1(\pi_1(M_1)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

In particular, $\bar{\rho}_1(h_1)$ has order one or two. So $\bar{\rho}_2(\pi_1(M_2))$ either factors through $\mathbb{Z}/|n-6|$ (as above) or else through $\langle w, z \mid w^3 z^{(n-3)/2}, (z^{(5-n)/2} w^{-1})^2, [w, z] \rangle$ which is cyclic of order $2|n-6|$. Moreover, we can assume that $\mathrm{Im}(\bar{\rho}_2)$ consists only of diagonal matrices. Again there are only a finite number of characters corresponding to representations of this form so they are isolated.

In summary then, the only way to construct a curve in $\bar{X}(M(2n+6))$ is by making use of representations $\bar{\rho}$ which kill the glueing torus T and therefore factor through

$$\begin{aligned}\pi_1(M(2n+6))/\pi_1(T) &= (\pi_1(M_1) *_{\pi_1(T)} \pi_1(M_2))/\pi_1(T) \\ &= \pi_1(M_1)/\pi_1(T) * \pi_1(M_2)/\pi_1(T) \\ &= \mathbb{Z}/2 * \mathbb{Z}/g.\end{aligned}$$

where $g = \gcd(3, |n-3|/2)$. If $g = 1$, this is an abelian representation, contradicting an earlier assumption, and there is no curve in $\bar{X}(M(2n+6)$. If $g = 3$, (i.e. if $3 \mid n$), we see that we are looking at representations of $\mathbb{Z}/2 * \mathbb{Z}/3$. Since $\bar{X}(\mathbb{Z}/2 * \mathbb{Z}/3)$ contains exactly one curve (see [BZ2, Example 3.2]), we conclude that this is also the case for $\bar{X}(M(2n+6))$. □

Thus if $3 \mid n$, there is a unique curve in $\bar{X}(M(2n+6))$. Moreover, since the representation $\bar{\rho}_{1/2}$ (see [BZ1, Example 3.2] or Equation 5.22 below) is dihedral, this curve contains the character of a dihedral representation. It follows that the curve is covered by a unique curve in the $SL_2(\mathbb{C})$-character variety $X(M(2n+6))$ (see [BZ1, Lemma 5.5]). Thus there is exactly one r-curve, call it X_1, with $r = 2n+6$ in this case.

Moreover, we can show $s_1 = 2$ for this curve. Recall that $s_1 = \|\mu\|_1$ is the degree of f_μ. So we need to understand the image of μ under the composition $\pi \to \pi_1(M(2n+6)) \to \mathbb{Z}/2 * \mathbb{Z}/3 = \langle c, d \mid c^2, d^3 \rangle$. We can construct μ in terms of a curve γ in the genus two surface which connects points on opposite sides of the knot; see Figure 23. The idea is that we can break up the meridian as the sum of a loop in M_1 and a loop in M_2. We will then show that those project to c and d respectively. The first loop is γ plus a small arc joining the two endpoints of γ in the interior of H, the obvious genus two handlebody. The second loop is γ plus a small arc joining the two endpoints of γ and passing through the complementary genus two handlebody H'.

In $\pi_1(M_1)$, γ represents ab^{-1} which is conjugate to $b^{-1}a$ and therefore projects onto the generator of $\mathbb{Z}/2 = \pi_1(M_1)/\pi_1(T)$. In $\pi_1(M_2)$, γ represents $xy^{(n-1)/2}$ which projects to the generator of $\mathbb{Z}/3$. Thus μ is mapped to cd in $\mathbb{Z}/2 * \mathbb{Z}/3$.

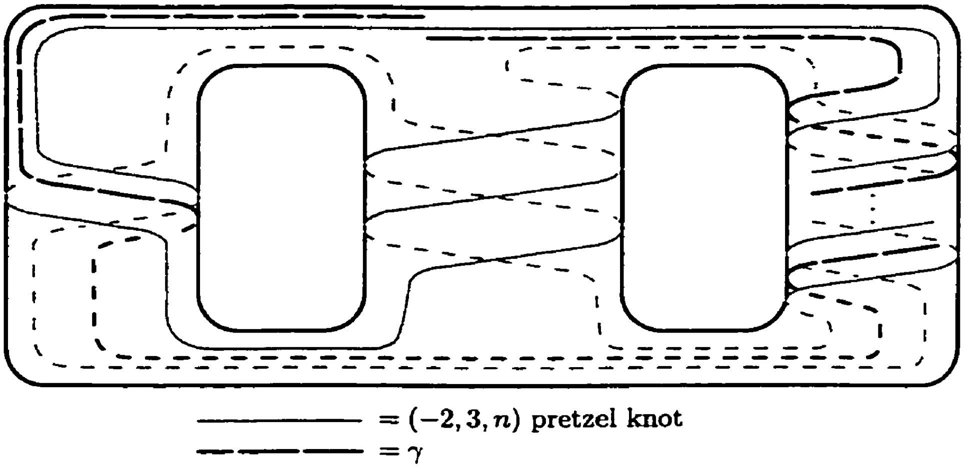


FIGURE 23. The meridian μ represented by a curve γ.

Let $\bar{X}_1$ be the unique curve in $\bar{X}(\mathbb{Z}/2 * \mathbb{Z}/3)$. In [BZ2, Example 3.2], the authors construct a double covering $\mathbb{C} \to \bar{X}_1$ given by mapping $z \in \mathbb{C}$ to the character of $\bar{\rho}_z$:

$$(5.22) \qquad \bar{\rho}_z(c) = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ \bar{\rho}_z(d) = \pm \begin{pmatrix} z & 1 \\ z(1-z)-1 & 1-z \end{pmatrix}.$$

Since $\mathrm{trace}(\bar{\rho}_z(cd)) = i(2z-1)$, we see that $f_{cd}(\chi_{\bar{\rho}_z}) = -(2z-1)^2 - 4$ has degree 2. As this is a double covering of $\bar{X}_1$ by $\mathbb{C}$, the corresponding character on $\bar{X}_1$ has degree 1. Finally, lifting to the curve X_1 in $X(M)$ which double covers $\bar{X}_1 \subset \bar{X}(M(2n+6)) \subset \bar{X}(M)$, we deduce $s_1 = \deg f_\mu = 2$.

For $n \geq 7$, the integral boundary slopes are 0, 16 and $2n+6$. We will show that 0 and 16 do not admit r-curves. Recall that $\|2n+5\|_0 = s_0 + 4(|n-4|-1)$. i.e. all the jumping points for the $2n+5$ surgery are on the norm curve. So we would have $\|2n+5\|_i = s_i$ on any r-curve X_i. However, if $r = 0$ for example, $\|2n+5\|_i = s_i\Delta(2n+5, 0) = (2n+5)s_i$. So there can be no r-curve for $r = 0$. Similarly, there can be no $r = 16$ curve. Analogous arguments show that there is no r curve with $r = 0$ or 10 when $n \leq -11$ so that $r = 2n+6$ is the only candidate in this case as well.

Thus when $n \geq 7$ or $n \leq -11$, there is exactly one norm curve. There will be one r-curve when $3 \mid n$ and otherwise there are no additional curves containing irreducible characters. Since the set of reducible characters forms a complex line ([Ta, Proposition 2.5.5]), we see that $X(K_n)$, the character variety of the knot K_n, consists of two (three) curves when $3 \nmid n$ $(3 \mid n)$ and $n \geq 7$ or $n \leq -11$. This observation also holds true for $-9 \leq n \leq -1$ as we will now verify.

n = −9 Here $3 \mid n$, so there is an r-curve X_1 with $r = 2n + 6 = -12$ and $s_1 = 2$. We can show that there is no r-curve with $r = 0$ or 10 as we did before and we've already mentioned that if there were two norm curves, they would both correspond to a type 3 solution. Let us verify that this cannot happen. Suppose then that there were two norm curves X_0 and X_2. Since $S = 30$ and $s_1 = 2$, we see that

$$28 = s_0 + s_2 = 4 + 2(a_2^0 + a_2^2),$$

where we have given the a_2's (see Equation 5.13) superscripts showing which curve they come from. This implies $a_2^0 + a_2^2 = 12$. But then

$$\|2n+4\|_0 + \|2n+4\|_2 = s_0 + s_2 + -4(2n+5) + 2(a_2^0 + a_2^2) = s_0 + s_2 + 52 + 24$$

which contradicts the equation $\|2n+4\|_T = S + 3(|n-6| - 1) = S + 42$. Thus we see that there is exactly one norm curve and one r-curve when $n = -9$.

n = −7 In this case there is no r-curve for $r = 2n + 6 = -8$. By examining the norm of the $2n + 5 = -9$ slope, we see that if there is a norm curve of type 1, it is the only curve in $X(K_{-7})$ containing an irreducible character. Similarly, if there is a norm curve of type 2, then there is no r-curve with $r = 10$. However, we cannot immediately eliminate the possibility of an r-curve for $r = 0$.

Indeed, we know that for the type 2 norm curve X_0

$$s_0 = 2(1-n) + 2a_2 = 16 + 2a_2, \; \|2n+4\|_0 = \|-10\|_0 = s_0 - 2(n+3) + 2a_2 = s_0 + 8 + 2a_2$$

$$\text{and } \|2n+5\|_0 = \|-9\|_0 = s_0 - 4(n+2) = s_0 + 20.$$

On the other hand, if there were an r-curve X_1 with $r = 0$, then $\|2n+4\|_1 = \|-10\|_1 \leq s_1 + 3(5-n) - 8 = s_1 + 28$ and also $\|2n+4\|_1 = \|-10\|_1 = s_1\Delta(-10, 0) = 10s_1$. This implies

$$\begin{aligned} 10s_1 &\leq s_1 + 28 \\ \Rightarrow 9s_1 &\leq 28 \\ \Rightarrow s_1 &\leq 28/9 \end{aligned}$$

Since s_1 is an even integer, we see that $s_1 = 2$. Similarly, an examination of the -9 slope also leads us to the conclusion that $s_1 = 2$. So we cannot eliminate the possibility of an r-curve with $r = 0$ directly as we did earlier. We need to examine possible combinations of curves.

For example, suppose $X(K_{-7})$, had a type 2 norm curve X_0 and one r-curve X_1 for $r = 0$ and no other norm or r-curves. Then $\|-9\|_0 = s_0 + 20$ and

$$\begin{aligned} \|-9\|_1 &= s_1\Delta(-9,0) \\ &= 9s_1 \\ &= s_1 + 8s_1 \\ &= s_1 + 16. \end{aligned}$$

So

$$\begin{aligned} \|-9\|_T &= \|-9\|_0 + \|-9\|_1 \\ &= s_0 + s_1 + 36 \\ &= S + 36 \\ &< S + 40 \\ &= S + 4(|n-4| - 1). \end{aligned}$$

Thus, if we assume that these are the only two curves, we see that we cannot account for all the jumping points associated with the -9 slope. Therefore this is not a possible configuration for $X(K_{-7})$. By analyzing the possible combinations of norm curves and r-curves in this way, we see that the only possibility is that there is exactly one norm curve of type 1 and no r-curves.

$\mathbf{n = -5}$ A similar analysis shows that $X(K_{-5})$ contains exactly one norm curve and it's of type 1.

$\mathbf{n = -3}$ Since $3 \mid n$, we know that there is an r-curve X_1 for $r = 2n + 6 = 0$:

$$s_1 = 2,\ \|2n+4\|_1 = \|-2\|_1 = s_1 + 2 \text{ and } \|2n+5\|_1 = \|-1\|_1 = s_1.$$

If we follow the same strategy as in the previous cases we find that there are two possible configurations:

I. In addition to the r-curve there is one type 1 norm curve X_0 with

$$s_0 = 10, \; \| -2\|_0 = s_0 + 22 \text{ and } \| -1\|_0 = s_0 + 24.$$

II. Here there is a type 2 norm curve X_0:

$$s_0 = 8, \; \| -2\|_0 = s_0 \text{ and } \| -1\|_0 = s_0 + 4$$

as well as an additional r-curve X_2 with $r = 10$:

$$s_2 = 2, \; \| -2\|_2 = s_2 + 22 \text{ and } \| -1\|_2 = s_2 + 20.$$

Both configurations are consistent with $S = 3(|n-2|-1) = 12$, $\| -2\|_T = S + 3(|n-6|-1) = S + 24$ and $\| -1\|_T = S + 4(|n-4|-1) = S + 24$.

In order to show that the second configuration does not arise, recall that $\bar{X}_2$, the $PSL_2(\mathbb{C})$ analogue of X_2, would include into $\bar{X}(M(10))$ (see Section 2.2.3 or [BZ2, Example 5.10]). Now, $M(-1)$ is Seifert fibred over $S^2(3,4,5)$ and the jumping points for $\| -1\|$ come from the six irreducible $PSL_2(\mathbb{C})$-characters of $\Delta(3,4,5)$. If the second configuration is valid, five of these characters are on $\bar{X}_2$ and therefore come from representations lying in $\bar{R}(M(10))$. We will argue that at least two of them do not.

Indeed two of the characters correspond to representations which factor through $\Delta(2,3,5)$ which has order 60. On the other hand, if such a representation $\bar{\rho}$ is also in $\bar{R}(M(10))$, then it annihilates both the 10 and the -1 slopes. In other words, the kernel of $\bar{\rho}$ contains an index eleven subgroup of $\pi_1(\partial M)$. Therefore $\bar{\rho}(\pi_1(\partial M))$ is either $\mathbb{Z}/11$ or else trivial. On the other hand, $\bar{\rho}(\pi_1(\partial M))$ also factors through $\Delta(2,3,5)$. Thus $\bar{\rho}(\pi_1(\partial M))$ is trivial and since $\pi_1(M)$ is normally generated by the peripheral group, $\bar{\rho}(\pi_1(M)) = \{\pm I\}$ as well. This contradicts the fact that $\bar{\rho}$ is an irreducible representation. Therefore, the irreducible representations which factor through $\Delta(2,3,5)$ are not in $\bar{R}(M(10))$. This shows that the second configuration is not possible.

n $= -1$ This knot was treated using different methods in Section 2.3 (where it is identified as the twist knot K_2). We saw that there is one norm curve of type 1 in the character variety.

Thus for any hyperbolic pretzel knot K_n, the character variety contains one norm curve X_0 and one curve of reducible characters. If $3 \mid n$ there is an additional r-curve X_1 for the slope $r = 2n + 6$ with $s_1 = 2$.

If $3 \nmid n$, then $s_0 = 3(|n-2| - 1)$ and the Culler-Shalen norm is given by

$$\|\gamma\|_0 = 2[\Delta(\gamma, 16) + 2\Delta(\gamma, \frac{n^2 - n - 5}{\frac{n-3}{2}}) + \frac{n-5}{2}\Delta(\gamma, 2n+6)]$$

when $n \geq 7$ and

$$\|\gamma\|_0 = 2[\Delta(\gamma, 10) + \frac{1-n}{2}\Delta(\gamma, 2n+6) + \Delta(\gamma, 2(n+1)^2/n)]$$

when $n \leq -1$.

If $3 \mid n$, then $s_0 = 3|n-2| - 5$ and the Culler-Shalen norm is

$$\|\gamma\|_0 = 2[\Delta(\gamma, 16) + 2\Delta(\gamma, \frac{n^2 - n - 5}{\frac{n-3}{2}}) + \frac{n-7}{2}\Delta(\gamma, 2n+6)]$$

when $n \geq 7$ and

$$\|\gamma\|_0 = 2[\Delta(\gamma, 10) - \frac{n+1}{2}\Delta(\gamma, 2n+6) + \Delta(\gamma, 2(n+1)^2/n)]$$

when $n \leq -1$.

Note that the longitude has an associated ideal point only when $n = -1$ or $n = -3$. That is, for the boundary slope $\beta_i = 0$, the associated a_i is zero unless $n = -1$ or $n = -3$. As we mentioned in the Remark of Section 2.3, this corresponds to the fact that these knots fibre over S^1 with a Seifert surface as fibre unless $n = -1$ (see [Ga2, Section 6]). Although K_{-3} admits such a fibration there is nonetheless at least one ideal point associated to the longitude. This is because there are other essential surfaces which realize the longitude as a boundary slope but are not fibres in a fibration of the knot. Essentially, when $n = -3$, the boundary slope $2n+6$ turns out to be zero, and corresponding to the surfaces which realizes $2n+6$ in all the other knots, there is a surface realizing the longitude slope of 0 for this knot. However, this surface is not the leaf of a fibration of K_{-3}, just as the corresponding $2n+6$ surfaces do not play that role for any other K_n.

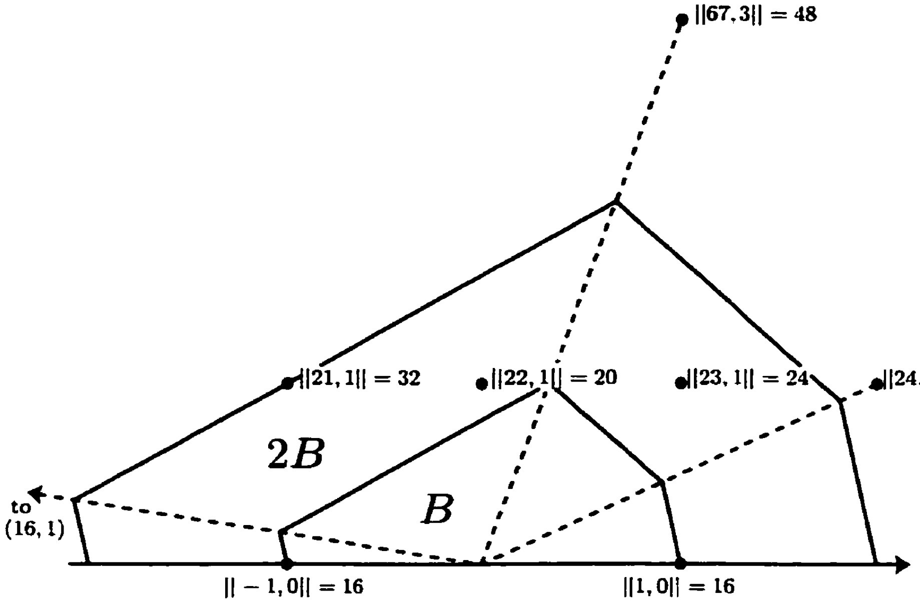


FIGURE 24. The fundamental polygon of K_9.

Fundamental Polygon and Newton Polygon. Figure 24 shows the fundamental polygon of K_9 associated to the norm curve X_0. For K_9, $\max(2s_0, s_0 + 8) = 2s_0$ so any finite surgery slopes must lie in $2B$ (see Section 2.2.5). However, as we see in Figure 24, the only slopes inside $2B$ are 21, 22, 23 and $\mu = 1/0$. According to SnapPea [Wee], $M(21)$ is hyperbolic and so $(-2, 3, 9)$ admits exactly two non-trivial finite surgeries: 22 and 23.

As n increases, the fundamental polygon for the norm curve maintains the same aspect but becomes smaller. For $11 \leq n \leq 19$, the only slopes inside $2B$ are $2n + 4$, $2n+5$ and μ and once $n \geq 21$, only $2n+4$ and μ remain. Now $M(2n+4)$ and $M(2n+5)$ are Seifert fibred over $\Delta(2, 4, |n - 6|)$ and $\Delta(3, 5, |n - 5|/2)$ [BH, Propositions 16 & 17]. Since, for $n \geq 11$, $1/2 + 1/4 + 1/(|n - 6|) < 1$ and $1/3 + 1/5 + 2/(|n - 5|) < 1$, these are hyperbolic orbifolds and consequently $M(2n + 4)$ and $M(2n + 5)$ are not finite surgeries. Thus the $(-2, 3, n)$ pretzel knots admit no non-trivial finite surgeries when $n \geq 11$.

Figure 25 gives the fundamental polygon of the $(-2, 3, -7)$ pretzel knot and illustrates the situation for $n \leq -1$. When $n < 1$, $s_0 < 8$. Since $2B$ lies below the line $y = 1$ there are no non-trivial finite surgeries here. When $n = -1$, we have a twist

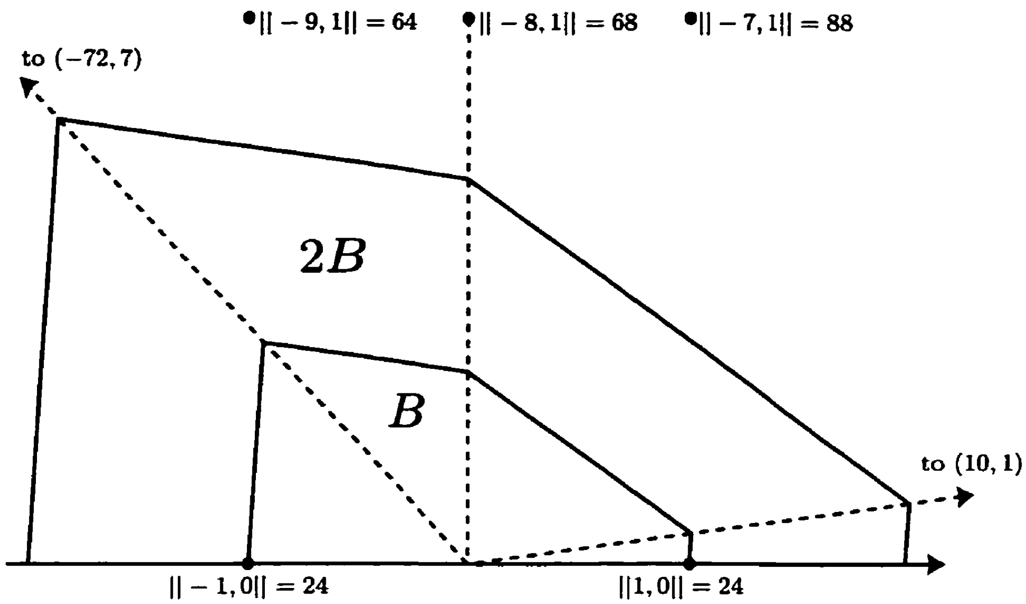


FIGURE 25. The fundamental polygon of K_{-7}.

knot, and we have already observed that they have no non-trivial finite surgeries (Section 2.3).

Thus there are exactly five non-trivial finite surgeries on the $(-2,3,n)$ pretzel knots. As we have mentioned, K_9 admits two. We can use the same methods to see that K_7 has three (see [BMZ, BZ1]), two cyclic fillings 18 and 19 and one non-cyclic finite filling of slope 17. These fillings were already known ([FS, Section 4] and [BH, Propositions 16 and 17]). The content here is that these are the only examples of non-trivial finite surgeries. As none of these fillings are simply connected this constitutes a proof of Property P for these knots. However, Property P was already known as these knots are strongly invertible [BS].

We are also in a position to determine the Newton Polygons for these knots (see Section 2.2.5).

The vertices of the Newton polygon are

$$(0,0), (16,1), (n^2-2n-15, (n-5)/2), (2(n^2-n+3), n-2),$$

$$(3n^2-4n-25, (3n-11)/2), (3n^2-4n-9, 3(n-3)/2)$$

when $n \geq 7$ and $3 \nmid n$ (see Figure 26 which may be compared with [Shn, Figure 5]);

$$(0,0), (16,1), (12(n-7), (n-7)/2), (3(n^2-6n+23), n-2),$$

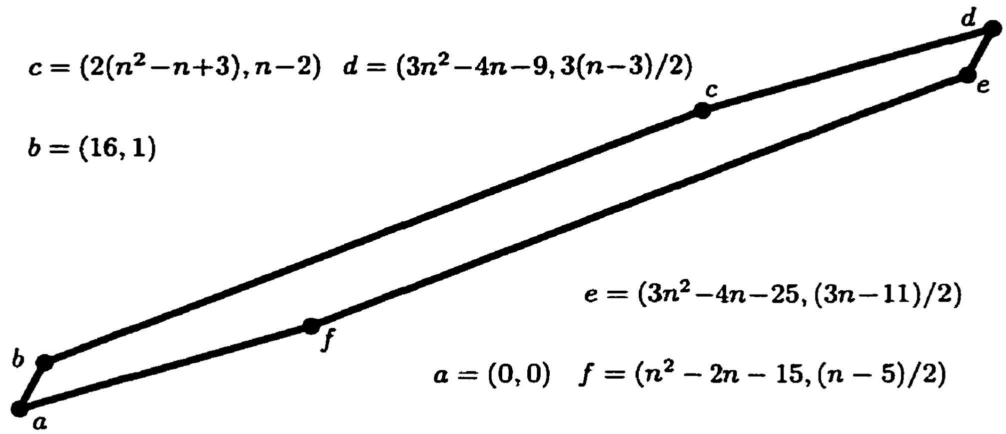


FIGURE 26. The Newton polygon of K_n ($n \geq 7$ and $3 \nmid n$).

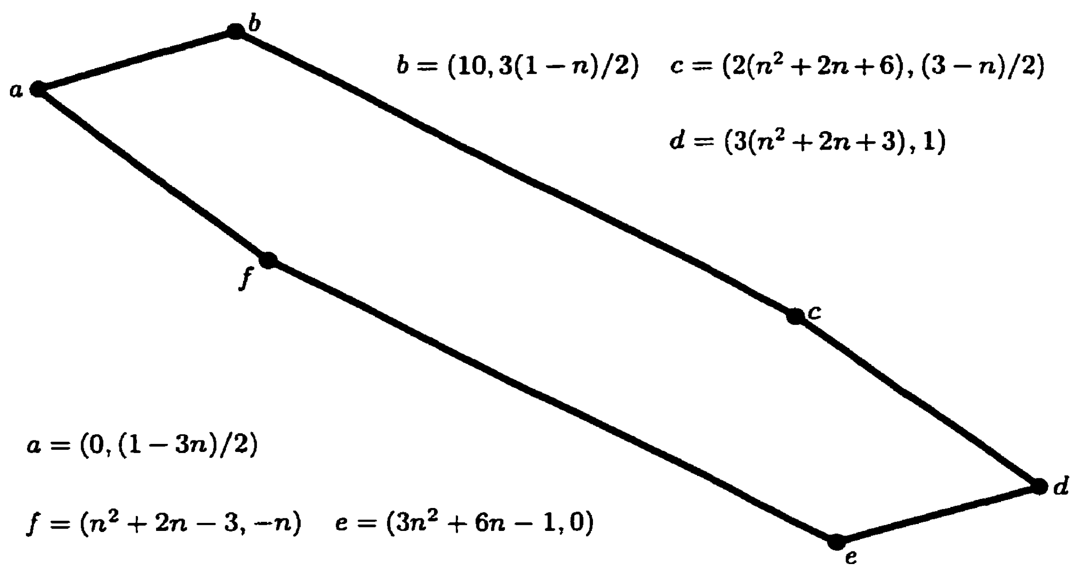


FIGURE 27. The Newton polygon of K_n ($n \leq -5$ and $3 \nmid n$).

$$(3n^2 - 6n - 31, (3n - 13)/2), (3(n^2 - 2n - 5), (3n - 11)/2)$$

when $n = 3k$, $k \geq 3$;

$$(0, (1 - 3n)/2), (10, 3(1 - n)/2), (n^2 + 2n - 3, -n),$$

$$(2(n^2 + 2n + 6), (3 - n)/2), (3n^2 + 6n - 1, 0), (3(n^2 + 2n + 3), 1)$$

when $n \leq -5$ and $3 \nmid n$ (see Figure 27);

$$(0, -(3n + 1)/2), (10, (1 - 3n)/2), (n^2 + 4n + 3, -n),$$

$$(2(n^2 + 2n + 6), (1 - n)/2), (3n^2 + 8n + 5, 0), (3n^2 + 8n + 15, 1)$$

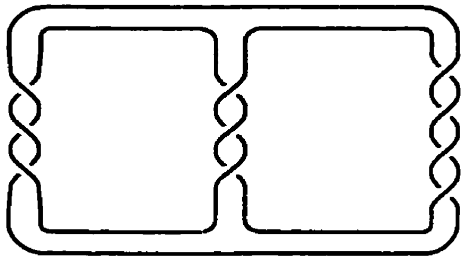


FIGURE 28. The $(-3,3,4)$ pretzel knot.

when $n = 3k$, $k \leq -1$; and

$$(0,0),(0,1),(4,2),(10,1),(14,2),(14,3)$$

when $n = -1$.

5.3. $(-3,3,n)$ **pretzel knots.** We now turn to the $(-3,3,n)$ pretzel knots. We begin by examining the $(-3,3,4)$ pretzel knot K as illustrated in Figure 28. First note that K is hyperbolic. Since it's small [Oe, Corollary 4], it's not a satellite knot. The only other possibility is that it is a torus knot. However, as it has Alexander polynomial ([Mru, Proposition 14] or [Hi, Theorem 1.2]) $\Delta_K(t) \doteq (t^2 - t + 1)^2$ this is not possible (see the argument of Lemma 5.2.1 or [Kaw, Theorem III]).

Note that $|\Delta_K(-1)| = 9$ so that there are 4 dihedral characters contributing to S. On the other hand, Equations 2.2 and 2.3 show that there are three irreducible $\mathrm{PSL}_2(\mathbb{C})$ characters of $\Delta(3,3,4)$. Using the methods of Section 5.1 we see that there are 10 ($= 4+2(3)$) jumping points in the $\mathrm{SL}_2(\mathbb{C})$-character variety, each contributing a jump of 2. Thus $S = 20$.

Using the Montesinos trick (for example, see [BH]), we can see that $M(1)$ is Seifert fibred over $S^2(2,5,7)$. By Equations 2.2 and 2.3, there are 6 irreducible $\mathrm{PSL}_2(\mathbb{C})$-representations of $\Delta(2,5,7)$ and none of these are dihedral representations since we are looking at a surgery $r = 1$ with odd numerator. So each of these is covered by two $\mathrm{SL}_2(\mathbb{C})$-characters each of which in turn contributes two to the seminorms of $r = 1$ surgery. That is, $\|1\|_T = S + 24$ (see Section 5.2.1.)

By [HO], the boundary slopes of K are -14, 0 and 8/5 so that [BZ1, Lemma 6.2] gives us

$$\|\gamma\| = 2[a_1\Delta(\gamma,-14) + a_2\Delta(\gamma,0) + a_3\Delta(\gamma,8/5)].$$

In particular,

$$(5.23) \qquad \|\mu\| = 2(a_1 + a_2 + 5a_3) \;=\; s \leq 20; \text{ and}$$

$$(5.24) \qquad s \leq \|1\| = 2(15a_1 + a_2 + 3a_3) \;\leq\; s + 24.$$

Subtracting, we find

$$(5.25) \qquad 0 \leq 7a_1 - a_3 \leq 6.$$

Now Equation 5.23 shows that $a_3 \leq 1$ on a norm curve. On the other hand, from Equation 5.25 we see that $a_3 = 0$ implies $a_1 = 0$ which is not possible for a norm curve. (Recall that for a norm curve at least two of the a_i's are non-zero.) Therefore $a_3 = 1$. Then Equation 5.25 implies $a_1 = 1$. Finally, Equation 5.23 allows us to bound a_2: $0 \leq a_2 \leq 4$. In particular, on a norm curve, we have $\|1\| = s + 24$. All the jumping points for $r = 1$ surgery lie on the norm curve. As they are simple points, this shows that there is at most one norm curve. Call it X_0.

Since an r curve, X_1, would again have none of the $r = 1$ surgery jumping points, we see that $\|1\|_1 = s_1$. But, as the norm is given by $\|\gamma\|_1 = s_1\Delta(\gamma, r)$, the only candidate for r among the boundary slopes is $r = 0$. Moreover, as was the case for $2n+6$ surgery on the $(-2, 3, n)$ pretzel knot, $M(0) = M_1 \cup M_2$ with M_1 Seifert fibred over $D^2(2,2)$ and M_2 Seifert over $D^2(3,3)$. So as in the previous section, there is a unique r-curve for $r = 0$ and $s_1 = 2$.

In summary then, there is one norm curve, one r-curve with $r = 0$ and one curve of reducible characters in the character variety of K. The minimal norm on the r-curve is $s_1 = 2$. As for the norm curve, $s_0 = 18$ and the norm is given by

$$\|\gamma\|_0 = 2[\Delta(\gamma, -14) + 3\Delta(\gamma, 0) + \Delta(\gamma, 8/5)].$$

Figures 29 and 30 show the fundamental and Newton polygons of this knot. Note that as the fundamental polygon lies below the line $y = 1/2$ (and $2s_0 > s_0 + 8$), K admits no non-trivial cyclic or finite surgeries. (Delman [Del] has already shown that this knot admits a persistent lamination and therefore has no non-trivial finite or cyclic surgeries.)

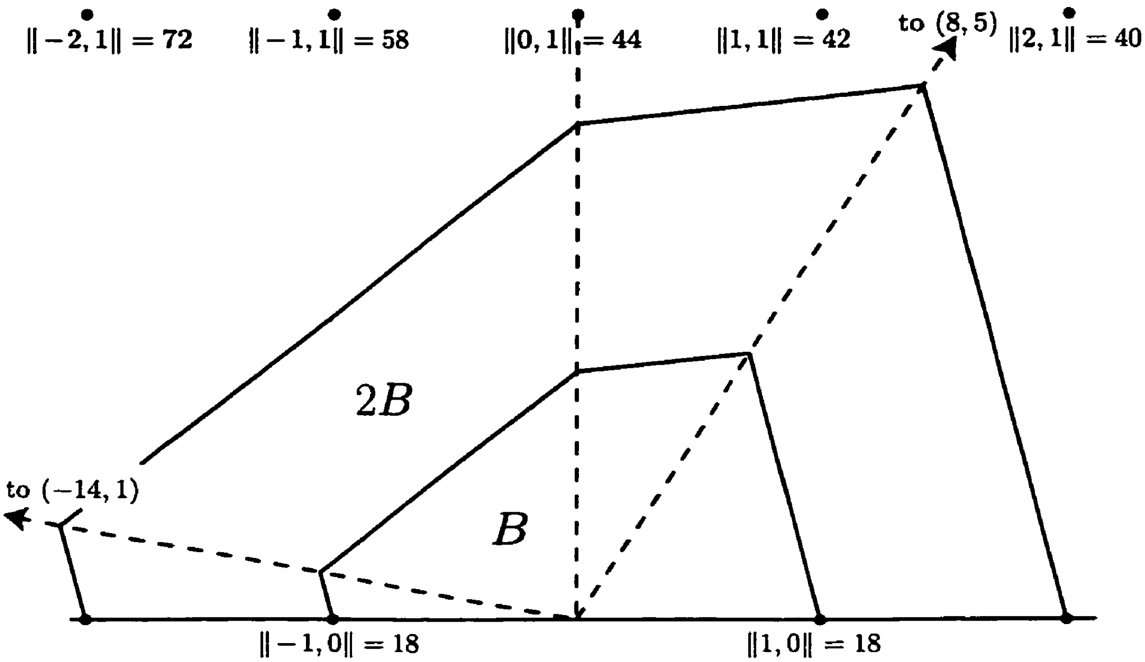


FIGURE 29. The fundamental polygon of the $(-3, 3, 4)$ pretzel knot.

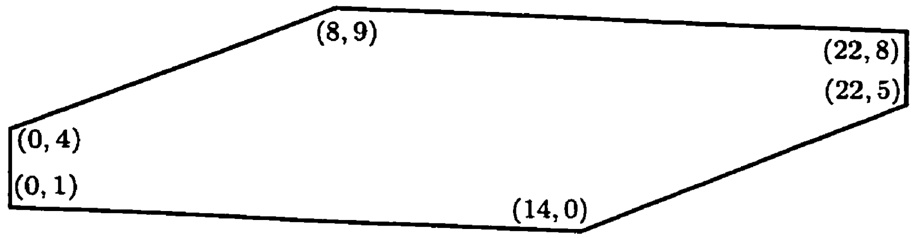


FIGURE 30. The Newton polygon of the $(-3, 3, 4)$ pretzel knot.

5.3.1. *Other n.* We now generalize to the $(-3, 3, n)$ pretzel knot which we will denote by K_n. Note that K_{-n} is the mirror reflection of K_n, so we will assume $n \geq 0$. This family includes some knots we have already looked at: K_1 is a twist knot and K_2 is the reflection of the $(-2, 3, -3)$ pretzel knot. Since K_0 is not prime, it's not hyperbolic and therefore not amenable to the techniques we developed in the last two sections. On the other hand, when $3 \leq n \leq 6$, K_n is small, hyperbolic, and moreover admits a Seifert surgery at slope $r = 1$. (We have verified this for $n = 3, 4, 6$ using the Montesinos trick. For $n = 5$ we have only the evidence of SnapPea [Wee].) This means we can again apply the machinery of the last section to work out the Culler-Shalen seminorms of the knot.

But what of $n \geq 7$? Why stop at $n = 6$? We are obliged to stop since we have no evidence of K_n admitting a Seifert filling for $n \geq 7$. Indeed, the Seifert surgeries occur according to a nice pattern. By [HO], the boundary slopes of K_n are $-(2n+6)$, 0 and $8/(n+1)$. For $1 \leq n \leq 6$, the Seifert surgeries lie between the boundary slopes 0 and $8/(n+1)$ as the following table illustrates.

n	1	2	3	4	5	6	≥ 7
$8/(n+1)$	4	8/3	2	8/5	4/3	8/7	≤ 1
Seifert Surgeries	1, 2, 3	1, 2	1	1	1	1	none

As the boundary slope $8/(n+1)$ moves across the integers toward 0, those integers cease to be available for Seifert surgeries. For example, when $n \geq 7$, the boundary slope is ≤ 1 and there are no more Seifert surgeries. I should emphasize that this is based on experimental evidence. These knots may admit other Seifert surgeries beyond those I've listed in the table. In addition, although I (or others) have shown that all the other surgeries in the table are Seifert, the only evidence I have in the $n = 5$ case comes from SnapPea [Wee]. Nonetheless, it is a curious pattern and it would be nice to understand this phenomenon.

Thus we can only hope to apply our machinery to K_n when $1 \leq n \leq 6$. The first two cases are treated elsewhere and $n = 4$ was discussed in detail above. For K_3 our method breaks down as the equations corresponding to Equations 5.23, 5.24 and 5.25 above don't result in a unique solution for the a_i's. Since K_5 is not strongly invertible, we cannot use the Montesinos trick to work out the indices for the Seifert surgery of slope 1. Without that information, we cannot complete the analysis of that knot.

However, K_6 is tractable. Using the filling $M(1)$, which is Seifert over $S^2(2,3,13)$, we arrive at the same conclusions as for K_4: there's one norm curve with $s_0 = 22$ and

$$||\gamma||_0 = 2[\Delta(\gamma, -18) + 3\Delta(\gamma, 0) + \Delta(\gamma, 8/7)],$$

and one r-curve with $r = 0$ and $s_1 = 2$. This means that K_6 also admits no non-trivial cyclic or finite surgeries. (Again, Delman [Del] had shown this previously using different methods.)

6. Conclusions and Questions

In this thesis we have shown how the techniques developed by Culler, Shalen, Boyer, and Zhang can be fruitfully applied to the study of Montesinos knots and in particular pretzel knots. These techniques allow for a classification of cyclic surgeries as well as a good understanding of the finite surgeries. By taking advantage of the Seifert fillings of the $(-2, 3, n)$ and $(-3, 3, n)$ pretzel knots, we were able to explicitly calculate the Culler-Shalen seminorms for those knots and parlay that information into a precise description of the character varieties. These knots therefore serve as a concrete example of the usefulness of the Culler-Shalen seminorms not only for the study of cyclic and finite surgeries, but also for the study of Seifert fillings.

Along with some progress in understanding pretzel knots, our research raises many questions. We list some of these questions and suggest ways in which our work could be extended.

- In Section 2.3 we used Ohtsuki's [Oht] work to directly compute the Culler-Shalen seminorms of the twist knots. Ohtsuki shows how to explicitly construct the trees associated with ideal points of the character variety of a 2-bridge knot. He can then count these ideal points by enumerating the associated trees. This is a beautiful construction and it is evidently of great use as it allows rapid calculation of the Culler-Shalen seminorms. It would be advantageous to extend his ideas to other classes of knots. Given the good understanding of the Culler-Shalen seminorms of (p, q, r) pretzel knots presented in this thesis it seems natural to experiment with extending Ohtsuki's ideas to pretzel knots as a first step towards eventually looking at larger classes of knots.
- We have presented calculations showing that $Z_x(f_\mu) = 1$ at the character x of a p-representation of a twist knot or $(-2, 3, n)$ knot. These calculations could be extended to other knots. This would be interesting as there is still little known about the zeroes of the f_γ functions. For example it may well be that $Z_x(f_\mu) = 1$ at the character of a p-representation on any knot as we know of no evidence to the contrary. Calculations for other knots might be used to support or disprove this conjecture.

- In Section 3.3 we observed that, when $\mathfrak{m} = \mathfrak{m}(\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3)$ is a three-tangle Montesinos knot, there is an inclusion of the $PSL_2(\mathbb{C})$-character varieties: $\bar{X}(\Delta(\alpha_1, \alpha_2, \alpha_3)) \subset \bar{X}(\mathfrak{m})$. We then went on to use this observation to make many deductions about the pretzel knots $\mathfrak{m}(1/p, 1/q, 1/r)$. An obvious question is, to what extent do our results extend to other three-tangle Montesinos knots?

  On the other hand, as we mentioned in Section 3.3, it seems plausible that there is also an overlap of character varieties when the Montesinos knot has more than three tangles. To see how this might work in a more general context, recall (Section 3.1.2) that Σ_2, the two-fold branched cyclic cover of a Montesinos knot, is a Seifert fibred space with base orbifold $\mathcal{B}$. In the case of a three-tangle Montesinos knot, the triangle group is $\pi_1^{\text{orb}}(\mathcal{B})$.

  Now, the $SL_2(\mathbb{R})$-character variety of $\pi_1^{\text{orb}}(\mathcal{B})$ includes the Teichmüller space of $\mathcal{B}$ which is homeomorphic to $\mathbb{R}^{2(r-3)}$, r being the number of tangles of the Montesinos knot. It follows that the $PSL_2(\mathbb{C})$-character variety $\bar{X}(\pi_1^{\text{orb}}(\mathcal{B}))$ also has dimension at least $2(r-3)$. It seems too much to hope that $\bar{X}(\pi_1^{\text{orb}}(\mathcal{B})) \subset \bar{X}(\mathfrak{m})$ when there are more than $r = 3$ tangles. Nonetheless, given the large dimension of $\bar{X}(\pi_1^{\text{orb}}(\mathcal{B}))$ it seems likely that there is at least some overlap of these character varieties. Moreover, seeing how the dimension of $\bar{X}(\pi_1^{\text{orb}}(\mathcal{B}))$ grows linearly with the number of tangles r, perhaps this is also true of the $\bar{X}(\mathfrak{m})$ character varieties. In other words, Montesinos knots are likely examples of knots having character varieties of arbitrarily large dimension. The solid understanding of three-tangle knots presented in this thesis would be an excellent platform from which to launch an exploration of the rich structure such higher dimensional character varieties likely present.
- The conclusion of the proof of Theorem 4.4.5 relies on Theorem 4.4.4 (the main theorem of [E]), a rather powerful result in group theory. It is clear that this group theoretic result could be of great use in understanding infinite fillings of many other Montesinos knots since fillings of these knots will often admit factors of the type $(2, a, b; c)$ described in Theorem 4.4.4. Indeed, the original theorem of Edjvet's [E] paper actually refers to a more general type of group $(d, a, b; c)$.

On the other hand, it might also be possible to go the other way. Perhaps conclusions about finite fillings derived from Culler-Shalen theory or other topological arguments could be used to make some deductions in group theory. For example, the finiteness of $(2, 3, 13; 4)$ remains an open question. If this group could be recognized as a factor of a finite filling of some knot that would prove the group finite.

- Clearly it would be nice to complete the analysis of finite surgeries on Montesinos knots (Theorem 4.5.2) by understanding finite surgeries on $(-2, p, q)$ pretzel knots with $5 \leq p \leq q$. For example, [Du1, Theorem 4.1] shows that cyclic surgeries are near non-integral boundary slopes while Theorem 4.1.3 shows that the same is true of finite surgeries, at least on $(p, q, -r)$ pretzel knots with $r \geq 4$. It seems plausible that the same is true for $(-2, p, q)$ pretzels and likely for other knots as well. Proving this would be a powerful first step in completing the classification of finite surgeries on Montesinos knots.
- The detailed calculations of Culler-Shalen seminorms in Chapter 5 depend largely on the existence of Seifert fillings of the $(-2, 3, n)$ and $(-3, 3, n)$ pretzel knots. Such calculations could likely be carried out for other Montesinos knots admitting such Seifert fillings. Therefore, one way to extend this research is to use it as motivation for a more thorough investigation of Seifert surgery on Montesinos knots. A particularly provocative point in this regard is the pattern of Seifert surgeries of $(-3, 3, n)$ pretzel knots illustrated by the table in Section 5.3.1.
- Aside from providing information about finite and cyclic surgeries, detailed calculations of the Culler-Shalen seminorm provide a lot of information about the A-polynomial invariant of a knot. Conversely, the A-polynomial can be used to construct Culler-Shalen seminorms. Recently, David Boyd has proposed techniques for efficient calculation of A-polynomials. This is an exciting development since, as difficult as the determination of the A-polynomial is, it is nonetheless even more difficult to get at the Culler-Shalen seminorm, particularly if one wants to consider knots other than the 2-bridge knots (for which Ohtsuki's methods can be employed) or knots admitting Seifert fillings. For example, given A-polynomials for the $(-2, p, q)$ pretzel knots $(5 \leq p \leq q)$ one

could complete the classification of finite surgeries on Montesinos knots begun in Theorem 4.5.2.

Appendix A. Zeroes of Alexander Polynomials

Lemma A.1.1. *Let $\Delta(t)$ be the Alexander polynomial of the $(-2,3,n)$ pretzel knot K. (In particular, n is odd.) Suppose $\Delta(\zeta) = 0$ where ζ is a primitive mth root of unity. Then one of the following is true.*

- $3|n$ *and* $m = 6$.
- $10|(n-1)$ *and* $m = 10$.
- $12|(n-3)$ *and* $m = 12$.
- $15|(n-5)$ *and* $m = 15$.

Proof: The Alexander polynomial never admits zeroes which are prime powers of unity. Indeed, by [BuZ, Theorem 8.21], $H_1(\Sigma_m)$ is finite iff no root of the Alexander polynomial is an mth root of unity. Here Σ_m denotes the m-fold branched cyclic cover of the knot (see [Rol, Section 10.C]). Using the Milnor [Mi] sequence we can show that $b_1(\Sigma_m) = 0$ whenever m is a prime power of unity.

We next show that $\zeta = e^{2\pi i/m}$ is not a root when $m \geq 18$ by looking at a few cases. By [Mru, Proposition 14] or [Hi, Theorem 1.2],

$$\begin{aligned}
\Delta(t) &\doteq t^2 - 2t + 2 + t^{n+2}(t-1) + (t^{n+1}-1)/(t+1) \\
&= \frac{1}{t+1}(t^{n+4} - t^{n+2} + t^{n+1} + t^3 - t^2 + 1) \\
&= \frac{1}{t+1} D_+(t) \\
&= \frac{t^{n+1}}{t+1}(t^{2-n} - t^{1-n} + t^{-n-1} + t^3 - t + 1) \\
&= \frac{t^{n+1}}{t+1} D_-(t).
\end{aligned}$$

Suppose first that $m > 2(2-n) > 0$. Then,

$$\begin{aligned}
\mathrm{Re}(D_-(\zeta)) &= \cos(\frac{(2-n)2\pi}{m}) - \cos(\frac{(1-n)2\pi}{m}) + \cos(\frac{-(n+1)2\pi}{m}) \\
&\quad + \cos(\frac{6\pi}{m}) - \cos(\frac{2\pi}{m}) + 1 \\
&= 2\cos(\frac{(2-n)2\pi}{2m})\cos(\frac{(2-n)2\pi}{2m}) - 2\cos(\frac{(2-n)2\pi}{2m})\cos(-\frac{n2\pi}{2m}) \\
&\quad + 2\cos(\frac{(2-n)2\pi}{2m})\cos(-\frac{(n+4)2\pi}{2m}).
\end{aligned}$$

So, $\text{Re}(D_-(\zeta) = 2\cos(\frac{(2-n)\pi}{m})[\cos(\frac{(2-n)\pi}{m}) - \cos(-\frac{n\pi}{m}) + \cos(\frac{-(n+4)\pi}{m})]$. Since $m > 2(2-n)$, this is positive as long as $n < -4$. In fact, $D_-(\zeta)$ is also positive when $n = -1, -3$, so we see that ζ is not a root of the Alexander polynomial when $n < 0$ and $2(2-n) < m$.

Next suppose $m > 2(n+4)$ and $n > 0$. Then,

$$\text{Re}(D_+(\zeta)) = 2\cos(\frac{(n+4)\pi}{m})[\cos(\frac{(n+4)\pi}{m}) - \cos(\frac{n\pi}{m}) + \cos(\frac{(n-2)\pi}{m})] > 0.$$

and ζ is not a root of the Alexander polynomial.

Since $n \equiv n' \bmod m$ implies $\zeta^n = \zeta^{n'}$, we can assume $-m/2 < n-2 \leq m/2$. We have already looked at the case $m > 2(2-n) \Leftrightarrow -m/2 < n-2$, as well as the case $m > 2(n+4) \Leftrightarrow m/2 > n+4$. So it remains to investigate $n-2 \leq m/2$ and $m/2 \leq n+4$, i.e., $2n-4 \leq m \leq 2n+8$. Let us now assume that $m \geq 18$. Since $m \leq 2n+8$, this implies $n \geq 5$.

If $m = 2n+8$, then in fact $\text{Re}\Delta(\zeta) = 0$, so we work instead with the imaginary part:

$$\begin{aligned}\text{Im}D_+(\zeta) &= 2\sin(\frac{(n+4)\pi}{m})[\cos(\frac{(n+4)\pi}{m}) - \cos(\frac{n\pi}{m}) + \cos(\frac{(n-2)\pi}{m})] \\ &= 2(1)[0 - \cos(\frac{n\pi}{m}) + \cos(\frac{(n-2)\pi}{m})].\end{aligned}$$

Thus ζ is a root of the Alexander polynomial only if $\cos(\frac{n\pi}{m}) = \cos(\frac{(n-2)\pi}{m})$. But since

$$0 < \frac{(n-2)\pi}{m} < \frac{n\pi}{m} < \frac{(n+4)\pi}{m} = \pi/2$$

these two values of cosine are distinct, $\text{Im}D_+(\zeta) \neq 0$, and ζ is not a root of the Alexander polynomial in this case either.

If $m = 2n+j$, with $-4 \leq j \leq 7$, we can use the real value:

$$\begin{aligned}\text{Re}D_+(\zeta) &= 2\cos(\frac{(n+4)\pi}{m})[\cos(\frac{(n+4)\pi}{m}) - \cos(\frac{n\pi}{m}) + \cos(\frac{(n-2)\pi}{m})] \\ &= 2\cos(\frac{(n+4)\pi}{m})[\cos((\frac{2n+j}{2} + \frac{8-j}{2})\frac{\pi}{m}) \\ &\quad - \cos((\frac{2n+j}{2} - \frac{j}{2})\frac{\pi}{m}) + \cos((\frac{2n+j}{2} - \frac{j+4}{2})\frac{\pi}{m})] \\ &= 2\cos(\frac{(n+4)\pi}{m})[-\sin(\frac{(8-j)\pi}{2m}) - \sin(\frac{j\pi}{2m}) + \sin(\frac{(j+4)\pi}{2m})] \\ &= 2\cos(\frac{(n+4)\pi}{m})[2\cos(\frac{(j+2)\pi}{2m})\cos(\frac{\pi}{m}) - \sin(\frac{(8-j)\pi}{2m})].\end{aligned}$$

Since the term in square brackets is never zero (given $m \geq 18$), we see that ζ is not a root of the Alexander polynomial when $m \geq 18$.

Next, let's look at the case where $m = 14$. Again,

$$\text{(A.26)} \qquad \text{Re}D_+(\zeta) = 2\cos(\frac{(n+4)\pi}{m})[\cos(\frac{(n+4)\pi}{m}) - \cos(\frac{n\pi}{m}) + \cos(\frac{(n-2)\pi}{m})],$$

and as we've mentioned this value depends only on $n \bmod m$. By running through the odd integers $1, 3, 5, \ldots, 13$ with $m = 14$ in Equation A.26 we see that the value is never zero. It follows that the Alexander polynomial can never have a root ζ which is a fourteenth root of unity.

Thus far, we have shown that the Alexander polynomial has no zeroes which are mth roots of unity with m a prime power, nor with $m \geq 18$, nor with $m = 14$. The only remaining candidates are $m = 6, 10, 12$, and 15. To find out which values of n yield a root, we again use Equation A.26 and substitute the odd integers mod m to see which gives zero. It turns out that $n \equiv 3 \pmod 6$, $n \equiv 1 \pmod{10}$, $n \equiv 3 \pmod{12}$, and $n \equiv 5 \pmod{15}$ respectively are the only candidates. Verifying that these values also yield zero for $\text{Im}D_+(\zeta)$ completes the proof. □

Kurt Foster [F] has proposed a slightly different proof of this lemma. Here is a sketch of his argument.

Proof: (of Lemma A.1.1) Multiplying $\Delta(t)$ by $t+1$ gives the equation

$$\text{(A.27)} \qquad t^{n+4} - t^{n+2} + t^{n+1} + t^3 - t^2 + 1 = 0.$$

Plugging in $t = e^{2\pi i/m}$, and doing a bit of algebra yields

$$\text{(A.28)} \qquad \cos((n+4)\pi/m) - \cos(n\pi/m) + \cos((n-2)\pi/m) = 0.$$

This may be recast (assuming $(n+1)/m$ isn't a half-integer) as

$$\text{(A.29)} \qquad 2\cos(3\pi/m) = \cos(n\pi/m)/cos((n+1)\pi/m).$$

Direct verification rules out $m = 3, 4$, and 5 as possibilities, and turns up the solution $m = 6$ when $n \equiv 3 \pmod 6$. The left side of Equation A.29 is positive for $m > 6$, and is > 1 for $m > 9$. Inspection rules out any solutions in the case $m = 9$. We assume henceforth that $m > 6$. We clearly may restrict n to $0 \leq n < m$.

Taking logs (assuming m isn't 9) and applying the Mean Value Theorem tells us that

$$\pi/m \tan(x) = \ln(2\cos(3\pi/m)), \qquad n\pi/m < x < (n+1)\pi/m.$$

If m is sufficiently large, we may then write

$$n < m/2 - (m/\pi)\arctan(\pi/m \ln(2\cos(3\pi/m))) < n+1, \tag{A.30}$$

which will, for each m, force the value of n. The limiting value of the expression being subtracted from $m/2$ is $1/\ln(2)$ as m increases without bound, so it will be between 1 and 1.5 for sufficiently large m. It follows that, if m is large enough, the integrality of m forces $n = m/2 - 2$ if m is even, and $n = (m-1)/2 - 1$ if m is odd. Direct substitution of these values then shows the relation of Equation A.29 fails.

All that remains is to mop things up for small values of m. A simple computer routine checked the values of $m = 7$ to 40 (except $m = 9$), to see whether Equation A.29 held to within 2^{-30}. The program flagged the pairs

$$m = 8, n = 6; m = 10, n = 1; m = 12, n = 3; \text{ and } m = 15, n = 5.$$

The first value of n is even, so doesn't fit the original problem - Equation A.27 isn't divisible by $(t+1)$ if n is even - but it does give a solution of Equation A.29, and yields the primitive 8th roots of unity as solutions to Equation A.27 when $n \equiv 6 \pmod 8$. The other pairs do give solutions (direct verification).

The program also indicates that $m \geq 40$ is "sufficiently large" for the "large n" argument to apply. □

Remark: As a consequence of Lemma A.1.1, $\Delta(t)$ has no zero which is a $(2n+4)$th root of unity. For, suppose ζ were such a zero. Then ζ would also be a 6th, 10th, 12th or 15th root of unity. However, if it were a 6th root, then $6 \mid 2n+4$ which precludes $3 \mid n$. Similar arguments apply for 10th, 12th, and 15th roots. Analogous reasoning shows that $\Delta(t)$ admits a $2n+5$th root of unity zero iff $n \equiv 5 \bmod 30$.

Lemma A.1.2. *Let $\Delta_n(t)$ be the Alexander polynomial of the $(-2, 3, n)$ pretzel knot K When $n \equiv 5 \mod 30$, $\Delta_n(t)$ admits primitive 15th roots of unity as zeroes and moreover, they are simple zeroes of Δ_n.*

Proof: $\Delta_n(t) = (t^{n+4} - t^{n+2} + t^{n+1} + t^3 - t^2 + 1)/(t+1)$. Let $\phi_{15}(t) = t^8 - t^7 + t^5 - t^4 + t^3 - t + 1$ be the polynomial whose roots are the primitive 15th roots of unity. We will show that $\phi_{15}(t)$ is a factor of $\Delta_n(t)$ by induction. We illustrate the induction for $n > 0$. The argument for $n < 0$ is similar. When $n = 35$, ϕ_{15} is indeed a factor of $\Delta_n(t)$.

Suppose ϕ_{15} is a factor for some n with $n \equiv 5 \mod 30$.

$$\begin{aligned}\Delta_{n+30}(t) &= (t^{n+34} - t^{n+32} + t^{n+31} + t^3 - t^2 + 1)/(t+1)\\ &= (t^{n+34} - t^{n+4} - t^{n+32} + t^{n+2} + t^{n+31} - t^{n+1})/(t+1) + \Delta_n(t)\\ &= (t^{30} - 1)(t^{n+4} - t^{n+2} + t^{n+1})/(t+1) + \Delta_n(t)\end{aligned}$$

Since ϕ_{15} is a factor of both $t^{30} - 1$ and $\Delta_n(t)$, we see that it's a factor of $\Delta_{n+30}(t)$ as well.

To see that these are simple roots, it suffices to show that they are not also roots of the derivative $\Delta_n'(t)$. Again, we will assume that $n > 0$. The case where $n < 0$ is analogous.

$$\Delta_n'(t) = \frac{(n+3)t^{n+4} + (n+4)t^{n+3} - (n+1)t^{n+2} - 2t^{n+1} + (n+1)t^n + 2t^3 + 2t^2 - 2t - 1}{(t+1)^2}.$$

Suppose ξ is a common root of ϕ_{15} and Δ_n'. Since ξ is a 15th root of unity and $n \equiv 5 \mod 30$, we have

$$(n+3)\xi^9 + (n+4)\xi^8 - (n+1)\xi^7 - 2\xi^6 + (n+1)\xi^5 + 2\xi^3 + 2\xi^2 - 2\xi - 1 = 0.$$

Thus ξ is a root of

$$f(t) = (n+3)t^9 + (n+4)t^8 - (n+1)t^7 - 2t^6 + (n+1)t^5 + 2t^3 + 2t^2 - 2t - 1.$$

This implies that the irreducible polynomial ϕ_{15} divides $f(t)$. However, it's easy to verify that ϕ_{15} does not in fact divide $f(t)$. □

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