THE APPLICATION OF CONFORMAL MAPPING TO THE SOLUTION OF ELECTROSTATIC PROBLEMS.

Martin D. Burrow

Submitted to the Faculty of Graduate Studies and Research in partial fulfilment of the requirements for the degree of Master of Science.

McGill University

May 1946.

CON TENTS

CHAPTER I.

CHAPTER II.

CHAPTER III.

CHAPTER IV.

CHAPTER V.

CHAPTER VI.

CHAPTER VII.

I am indebted to Dr. E. Rosenthall

for his assistance and criticism.

PREFACE

Maxwell used conjugate functions to solve problems in electrostatics [1]. His method depended on a guessing of the suitable complex transformation for any given problem. In 1867 Christoffel, and in 1869 Schwarz discussed a general theorem in Transformations which became finally known as the Schwarz-Christoffel Theorem. The use of this Theorem provides a direct method for deducing the correct transformation for the electrostatic problems in two dimensions when the boundaries of the conductors involved, or some of the equipotentials and lines of force are straight. The conductors considered are cylinders with parallel generators and of infinite length; however they and the entire field have the same cross-section in all perpendicular planes and so the terms of plane geometry are used and the problem is spoken of as two dimensional. The picture is of course an idealized one but it is approximated very nearly by long cylinders in regions remote from their ends. Attempts to extend the theorem to deal with cylinders which contain curved as well as straight lines have been made by a number of writers, notably W.M.Page [2] and W.H.Richmond [3].

In this paper the essential elements of the function theory and the required electrostatics background are briefly developed or discussed and it is shown why and how the one may be applied to the other. Proofs of the invariance of charge, equipotentials, and lines of force under complex transformation, which are not usually found in the texts, have been supplied. A number of problems illustrating the use of the complex potential and the complex transformation are solved. The Schwarz-Christoffel Theorem is stated and a fairly detailed discussion of its application made, after which problems illustrating its use are worked out. Lastly methods of extending the solution to sylinders of curved cross-section as occur in papers by the authors quoted above are considered. The whole subject has been well known for many years and no attempt to add anything new has been undertaken. However, the solutions and treatment of a number of the problems are the writers own.

CHAPTER I

INTRODUCTION

1. The most general and valid system of equations governing Electromagnetic phenomena are those of Maxwell. These constitute a system of partial differential equations for the four vectors E, B, D, H. The vectors which are functions of position and time are assumed to be finite and continuous and to possess continuous first derivatives at all regular points. Discontinuities in the vectors or their first derivatives may occur at points where there is an abrupt change in the physical properties of the medium; for example on the boundary separating different media. Hence we may define an ordinary point as one in whose neighbourhood the properties of the medium are continuous. The domain of the vectors E,B,D,H are known as the electromagnetic field. The source of an electromagnetic field is a distribution of charge and current.

2. In this account we shall not be dealing with the general electromagnetic field but with the more special electrostatic field. The source of an electrostatic field is a stationary distribution of charge.

For this particular field Maxwell's equations take the form:

$$\nabla \times E = 0 \qquad \text{-------} \quad (1 . 1)$$

$$\nabla . D = 4\pi \rho \qquad \text{-------} \quad (1 . 2)$$

where E is the electric intensity at any point, defined as the force that would be exerted on a unit positive charge placed at the point if the original distribution were not disturbed by the presence of the charge. D is the electric displacement defined as the product of the dielectric constant K by the electric intensity

i.e. $D = K E$

In free space $K = 1$; in air it has very nearly the same value. We shall assume $K = 1$ in air. ρ is the volume density at a point. If Δq is the charge contained within an element of volume $\Delta \tau$ then

$$\rho = \frac{\Delta q}{\Delta \tau}$$

When the charge is confined to a surface we conceive of a surface density σ defined as

$$\sigma = \frac{\Delta q}{\Delta s}$$

where Δs is an element of area of the surface and Δq is the charge on that element. All charges are in electrostatic units, which we shall employ throughout.

Equations (1·1) and (1·2) are the fundamental equations for the electrostatic field and we shall consider them more closely.

Equation (1·1) is the condition that:

$$E_x dx + E_y dy + E_z dz \quad \text{is a perfect differential}$$

and that a scalar function V(x, y, z), say, exists such that:

$$E = - \text{grad } V \qquad ------ (1\cdot3)$$

E_x, E_y, E_z are the components of E in a system of rectangular Cartesian coordinates.

The minus sign in (1·3) is chosen from convention to make E be directed outward from positive charge.

The function V(x, y, z) may be identified with the potential, for

$$E_x dx + E_y dy + E_z dz = - dV$$

is an expression for the work done by the field in a displacement of unit charge from x,y,z to x + dx, y + dy, z + dz. Therefore -dV is a measure of the expenditure of work done by the field and expresses the loss in potential energy, or the potential difference between the points. Since $E_x dx + E_y dy + E_z dz$ is an exact differential the potential function is single valued and the work done in taking unit charge from a point A to another B is independent of the path.

This is the principle of the conservation of energy.
Using (1·3), (1·1) may be written:

$$\nabla \times (-\operatorname{grad} V) = 0$$

but also:
$$\nabla \times \left[-\operatorname{grad}(V+V_0)\right] = 0$$

where V_0 is an arbitrary constant. Thus V is not unique within an arbitrary additive constant.

It is usual to select the zero of the potential function at some convenient point. Then the potential at x,y,z is

$$V_{(x,y,z)} = -\int_A^{x,y,z} E.dr$$

when $V = 0$ at A.
This condition fixes the value of V .

3. The surfaces represented by $V(x,y,z,) =$ const are equi-potential surfaces. Any displacement along such a surface leaves $dV = 0$

that is: $$\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = 0$$

Hence the quantities $\dfrac{\partial V}{\partial x} : \dfrac{\partial V}{\partial y} : \dfrac{\partial V}{\partial z}$ at any point are in proportion to the direction cosines of the normal to the surface through the point.

Thus at any point the field intensity is normal to the equipotential surface through the point. A line in the electrostatic field with the property that at any point on it the vector E is tangent to it is called a line of force. Thus lines of force are the orthogonal trajectories of equipotential surfaces. We may conceive an electrostatic field as mapped out by a set of equipotential surfaces and lines of force everywhere orthogonal so that through every point in the field passes a line of force and an equipotential surface. If:

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = \frac{\partial V}{\partial z} = 0$$

we have the condition for a double point of V(x,y,z) and so the equipotentials cross at a null point.

4. It was remarked earlier that discontinuities in the field vectors or their first derivatives would occur on the boundaries separating different media. In general there were two categories of material media: dielectric, and conducting. In a dielectric medium charges do not move freely under the action of electric forces, but permit a condition of strain.

In conducting material charges move freely and are incapable of resisting any electric force however small. Any closed domain in which charges move freely is considered to be a conductor. The potential throughout the conductor must be constant if it is in electrostatic equilibrium, otherwise a flow of current would take place. Thus the surface of a conductor is an equipotential, and lines of force leave it normally and at interior points the electric intensity vanishes.

Since E is normal to the surface of the conductor, at a point just outside the conductor the intensity is:

$$E = -\frac{\partial V}{\partial \nu} \qquad \text{---- (1.4)}$$

where the differentiation is along the outward normal i.e. away from the conductor into the medium.

We know from Coloumb's law that if σ is the surface density of electrification on a conductor, then R the outward intensity at a point just outside the conductor is given by:

$$R = 4\pi\sigma$$

Using (1·4) we have:

$$\sigma = \frac{1}{4\pi} \cdot \frac{\partial V}{\partial \nu} \qquad \text{-------(1·5)}$$

The flux across an area is defined as the surface integral of the normal component of D taken over the surface or $\int_S D \cdot n ds$, where n represents the unit outward normal vector to the surface. Using Green's Theorem:

$$\int D.n\,ds = \int \nabla\cdot D\,dV$$
$$= 4\pi \int \rho\,dV \qquad by\ (1\cdot 2)$$
$$= 4\pi q \qquad \text{-----}(1.6)$$

where q is the total charge inside the region.

5. In all the problems we shall discuss only conductors in air will be considered. Outside the conductors $\rho = 0$ and $D = E$. Then $1\cdot 2$ becomes:

$$\nabla \cdot E = 0$$
$$\therefore \quad \nabla \cdot (grad\ V) = 0$$
$$\therefore \quad \nabla^2 V = 0 \qquad \text{-----}(1.7)$$

This is Laplaces equation which must be satisfied by the scalar potential function at all points outside conductors. Moreover we have seen that on any conductor V must be a constant.

The fundamental problem of electrostatics is the determination of the scalar potential function V. Once this function is known the field intensity at any point and the distribution of charge may be obtained from the relations $(1\cdot 3)$ to $(1\cdot 5)$. The magnitude of any point charge may be obtained by considering the flux leaving a region surrounding the charge and using equation $(1\cdot 6)$

For the determination of V we have the fundamental differential equation $\nabla^2 V = 0$ in free space and the boundary conditions of which we encounter the following types:

(i) For each separate conductor we are given its potential.

(ii) The total charge on each conductor is known. Either of these conditions enable V to be determined uniquely. This statement may be deduced from Green's Theorem:

Thus:

$$\int_s \phi \frac{\partial \phi}{\partial \nu} ds = \int_V (\nabla \phi)^2 dV \qquad \text{(Green's Theorem)}$$

where ϕ is the potential function and the integral on the left is taken over the surfaces of all conductors. The integral on the right is essentially positive. Suppose two possible solutions ϕ_1, ϕ_2 exist and let $\phi = \phi_1 - \phi_2$, then on every surface $\phi = 0$, or $\int \frac{\partial \phi}{\partial \nu} ds = 0$ therefore $\nabla \phi = 0$ everywhere. This would mean however that ϕ_1 and ϕ_2 differ at most by a constant and then only in the case where the charge on each surface was the same. If the potential ϕ is prescribed for even one of the surfaces the value of ϕ is then definite everywhere. If any point charges or other singular sources are involved their magnitude and distribution must also be known.

The fundamental differential equation is then:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

In particular if V is independent of one coordinate, say Z, then:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

This means that the electric field is the same in all planes parallel to the x,y plane and so the problem is virtually a two dimensional one, and we may employ the terms of plane geometry. This case arises when we are dealing with very long cylinders whose ends are sufficiently remote not to affect the problem in regions under discussion. As an ideal case we consider infinite cylinders. In our discussion we shall deal exclusively with this case.

Chapter II

1. Holomorphic Functions

Let $W = F(Z) = U(x,y) + i\,V(x,y)$

where $Z = x + i\,y$

and U, V are real functions of x, y.

If f(z) fulfills the following conditions

(i) It is finite and single valued for all values of Z in a certain region of the Z plane

(ii) it possesses a finite single valued derivative $f'(Z)$ within this region.

Then it is said to be holomorphic or analytic within the region.

Condition (ii) leads at once to the well known results - the Cauchy-Riemann conditions.

$$\text{(a)} \quad \frac{\partial u}{\partial x} = \frac{\partial V}{\partial y} \qquad\qquad \text{(b)} \quad \frac{\partial u}{\partial y} = -\frac{\partial V}{\partial x} \qquad (2\cdot1)$$

These conditions are necessary. To be sufficient it is further required that all the partial derivatives be continuous. In our discussions these derivatives represent components of force which are in general continuous.

Deriving (a) and (b) we have:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 V}{\partial x \partial y} \quad ; \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 V}{\partial y \partial x}$$

Now assuming the derivatives continuous

$$\frac{\partial^2 V}{\partial x\, \partial y} = \frac{\partial^2 V}{\partial y\, \partial x}$$

By addition: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Similarly: $$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

2. <u>The Complex Potential.</u>

From above we see that the real and imaginary parts of a complex holomorphic function are solutions of Laplace's equation. This we saw was precisely the important property of a potential function. We may therefore appropriately adopt either of these as a possible potential function. The function W is then called the complex potential.

Now if we take $(U(x,y)=C_1$ and $V(x,y)=C_2$ we get two families of curves in the x,y plane. At any point x,y the slopes of the tangents of the curves are

$$\left(\frac{dy}{dx}\right)_1 = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \; ; \; \left(\frac{dy}{dx}\right)_2 = -\frac{\frac{\partial V}{\partial x}}{\frac{\partial V}{\partial y}}$$ respectively.

By virtue of equation (2.1) a and b we see:

$$\left(\frac{dy}{dx}\right)_1 \times \left(\frac{dy}{dx}\right)_2 = -1$$

Hence the curves are orthogonal.

Thus if $U(x,y) = C_1$ represents a family of equipotentials, then $V(x,y) = C_2$ represents a family of lines of force everywhere orthogonal. $V(x,y)$ is called the stream function from its application in Hydrodynamics.

3. Suppose $W = U(x,y) + i\,V(x,y)$ is holomorphic and let $U(x,y) = C$ become $f(x,y) = 0$

It is then clear that the general function $U(x,y)$ will be a solution of Laplace's equation subject to the condition of having $U(x,y) = C$ over the boundary $f(x,y) = 0$. It will therefore be the appropriate potential function in an electrostatic field in which the curve $f(x,y) = 0$ is a conductor at potential C. Hence given any complex holomorphic function it is obviously possible to deduce the solution of a number of dependent electrostatic problems. We are thus provided with an indirect method for the investigation of a great variety of special problems; it is merely necessary to try out a number of holomorphic functions and see to what kinds of problem they provide a solution. Before indicating some of the possibilities of this approach we proceed to develop a few properties of the complex potential.

4. Let $W = U(x,y) + i\,V(x,y)$ be a holomorphic function for some region of the complex variable Z.

Then as we have seen:

$$U(x,y) = \text{constant}$$

may represent a family of equipotentials.

Now

$$\frac{\partial W}{\partial z} = \frac{\partial u}{\partial x} + i\frac{\partial V}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

by the C.R. conditions (2.1)

We saw from equation (1.3) that:

$$-\frac{\partial u}{\partial x} = E_x \quad \text{AND} \quad -\frac{\partial u}{\partial y} = E_y$$

where E_x, E_y are the components of the intensity E.

$$\left.\begin{aligned} \therefore \text{ Real part of } \frac{\partial W}{\partial z} &= -E_x \\ \text{and Imaginary part } &= E_y \end{aligned}\right\} \quad \text{----(2.2)}$$

Also

$$\left|\frac{\partial W}{\partial z}\right| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2} = |E| \quad \text{----(2.3)}$$

The resultant intensity is necessarily at right angles to the equipotential. In particular if $U(x,y) = C$ represents the surface of a conductor the field intensity at a point just outside the conductor $= 4\pi\sigma$ where σ is surface density of charge at the point.

Thus $$4\pi\sigma = \left|\frac{\partial W}{\partial z}\right|$$

$$\therefore \quad \sigma = \frac{1}{4\pi}\cdot\left|\frac{\partial W}{\partial z}\right| \quad \text{-------(2.4)}$$

The Complex Potential and Charge.

FLUX:

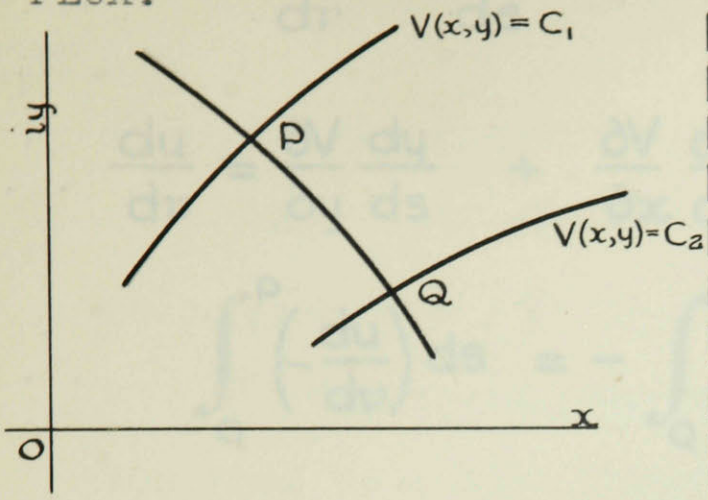


Fig. 14.1

Let P,Q be points on any two stream lines $V = C_1$ and $V = C_2$ respectively. (Fig. 14.1)

Consider the flux across any line joining P and Q.

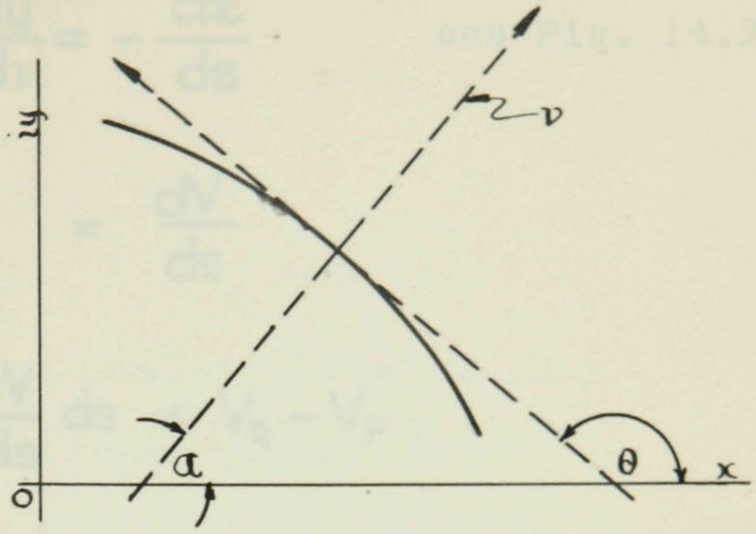


Fig. 14.2

Let $d\nu$ be an element along the normal to a curve and ds one along the curve itself.

$$\frac{dx}{d\nu} = \cos\alpha = \cos(\theta - 90^\circ)$$

$$= \sin\theta = \frac{dy}{ds}$$

$$\frac{dy}{d\nu} = \sin\alpha = \sin(\theta - 90^\circ)$$

$$= -\cos\theta$$

$$= \frac{-dx}{ds}$$

ds is an element along the curve

Flux $\displaystyle\int_Q^P E_n dS = \int\left(-\frac{du}{d\nu}\right)dS$ --

where dS is element of length along QP. If the flux passes to the right across PQ we consider the positive direction along QP to be from Q to P.

Now $$\frac{du}{d\nu} = \frac{\partial u}{\partial x}\frac{dx}{d\nu} + \frac{\partial u}{\partial y}\frac{dy}{d\nu}$$

$$= \frac{\partial V}{\partial y}\frac{dx}{d\nu} - \frac{\partial V}{\partial x}\frac{dy}{d\nu}$$ from the C.R. conditions

but: $\dfrac{dx}{d\nu} = \dfrac{dy}{ds}$; $\dfrac{dy}{d\nu} = -\dfrac{dx}{ds}$ see Fig. 14.2

$$\therefore \frac{du}{d\nu} = \frac{\partial V}{\partial y}\frac{dy}{ds} + \frac{\partial V}{\partial x}\frac{dx}{ds} = \frac{dV}{ds}$$

$$\therefore \int_Q^P \left(-\frac{du}{d\nu}\right) ds = -\int_Q^P \frac{dV}{ds}\, ds = V_Q - V_P$$

Hence the flux passing across PQ = change in the stream function in going from Q to P. If PQ is a portion of a conductor then

$$V_Q - V_P = 4\pi \times \text{charge on PQ.} \quad \text{------(2.5)}$$

for: $\dfrac{dU}{d\nu} = 4\pi\sigma = \dfrac{dV}{ds}$

$$\therefore \int_Q^P \frac{dV}{ds}\, ds = 4\pi \int_Q^P \sigma\, ds = 4\pi \times \text{charge on PQ.}$$

Chapter III

APPLICATIONS

Example I

Consider the transformation:

$$z = a \sin W \qquad (3.1)$$

where $W = \phi + i\psi$ is the complex potential.

Let ϕ represent the stream function and ψ the real potential function.

Now $Z = a \operatorname{Sin}(\phi + i\psi) = a(\operatorname{Sin}\phi \operatorname{Cosh}\psi + i\cos\phi \operatorname{Sinh}\psi)$

$\therefore$ $x = a \operatorname{Sin}\phi \operatorname{Cosh}\psi$

and $y = a \operatorname{Cos}\phi \operatorname{Sinh}\psi$

Then $$\frac{x^2}{a^2\operatorname{Cosh}^2\psi} + \frac{y^2}{a^2\operatorname{Sinh}^2\psi} = 1 \quad \text{---(i)}$$

and $$\frac{x^2}{a^2\operatorname{Sin}^2\phi} - \frac{y^2}{a^2\operatorname{Cos}^2\phi} = 1 \quad \text{(ii)}$$

From (i) we see that the surfaces along which $\psi = \text{const.}$ are elliptic cylinders, for $\psi = 0$, $y = 0$ and $x = a \sin\phi$

We then have the limiting case of an infinite strip with edges at $x = \pm a$. Equation (3.1) then gives the field about such a strip at zero potential.

Suppose now we require the capacitance between a flat earthed strip and an elliptic cylinder with foci at the ends of the strip and with semi major axis A.

On the strip $\psi = 0$

$\therefore$ $x = a \operatorname{Sin}\phi$: at $x = -a$ $\phi = \frac{-\pi}{2}$

and " $x = a$ $\phi = \frac{\pi}{2}$

$\therefore$ change in stream func. $\phi = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$.

$\therefore$ charge on upper portion $= \frac{1}{4\pi} \times \pi = \frac{1}{4}$

$\therefore$ Total charge $= \frac{1}{2}$

Also $A = a \operatorname{Cosh}\psi_1$ where ψ_1 is the potential on the elliptic cylinder

$\therefore \psi_1 = \operatorname{Cosh}^{-1} \frac{A}{a}$

Hence required capacitance $= \frac{\text{Charge}}{\text{Potential}} = \left[2 \operatorname{Cosh}^{-1} \frac{A}{a}\right]^{-1}$

Example II

The above by no means exhausts the possibilities of the transformation $Z = a \operatorname{Sin} W$.

Thus: as before $x = a \operatorname{Sin} \phi \operatorname{Cosh}\psi$; $y = a \operatorname{Cos} \phi \operatorname{Sinh} \psi$

If we take $-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$ and $0 \leq \psi \leq \infty$ we cover the z plane. Now let $\phi = \pm \frac{\pi}{2}$ and ψ range from $0 \longrightarrow \infty$

Then $y = 0$, and x goes from $+a \longrightarrow +\infty$, for $\phi = +\frac{\pi}{2}$

and x goes from $-a \longrightarrow -\infty$, for $\phi = -\frac{\pi}{2}$

So if ϕ is the potential function equation (3.1) provides the solution for two semi-infinite co-planar planes at potentials $\frac{\pi}{2}$ & $-\frac{\pi}{2}$ with a gap 2a wide between them. Applying equation (2.4) the density per unit area at any point x is given by

$$\sigma = \frac{1}{4\pi} \cdot \left|\frac{\partial \psi}{\partial x}\right|_{y=0} = \frac{1}{4\pi\, a \sinh\psi} = \frac{1}{4\pi}\sqrt{x^2 - a^2}$$

This expression shows that the density becomes infinite at the edges and we may also deduce that the total charge on the planes per unit depth is infinite.

Example III

Transformations involving Jacobian elliptic functions lead to a number of interesting results. For a list of the properties of these functions used here, see the table in the Appendix.

Consider $z = b\,\mathrm{Sn}(\phi + i\psi)$ ------ (3.2)

where ϕ is the potential and ψ the stream function. Let k be the modulus and $2K$, $2iK^1$ the real and imaginary periods.

Consider the equipotential $\phi = K$

Then:

$$x + iy = \mathrm{Sn}(K + i\psi) = \frac{b\,\mathrm{Cn}\,i\psi}{\mathrm{dn}\,i\psi} = \frac{b}{\mathrm{Cn}(\psi,k')} \cdot \frac{\mathrm{Cn}(\psi,k')}{\mathrm{dn}(\psi,k')} \text{ (see table)}$$

$$= \frac{b}{\mathrm{dn}(\psi,k')}$$ where k^1 shows we are dealing with an elliptic function modulus $k^1 = \sqrt{1 - k^2}$

Equating real and imaginary parts we have:

$$y = 0 \quad \text{and} \quad x = \frac{b}{\mathrm{dn}\,(\psi,k')}$$

Now dn is always positive and its max. value = 1 when

$\psi = 0$, or $2\,m\,K^1$ $(m = 1,2,3)$ and its minimum value = k when $\psi = K^1$ or $(2m + 1)K^1$

$\therefore z = \dfrac{b}{\mathrm{dn}(\psi,k')}$ represents the portion of the x axis lying between $x = b$ and $x = \dfrac{b}{k}$

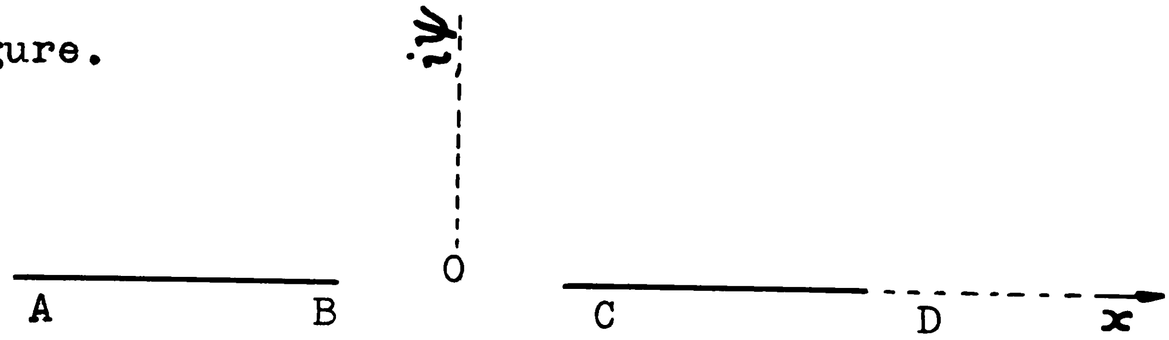
Similarly when $\phi = -K$ we have:

$$z = -\frac{b}{\mathrm{dn}\,(\psi,k')}$$

In this case we get that portion of the x axis lying between

$$x = -\frac{b}{k} \quad \text{and} \quad x = -b$$

Thus the transformation solves the problem of two infinite strips AB, CD and of equal width $\left(\frac{b}{k} - b\right)$ lying in the same plane with a gap $2\,b$ between them and at potentials K and -K. See figure.



Making use of the result (2.5) we see that the quantity of electricity on top of CD

$$\frac{[\psi]_c^0}{4\pi} = \frac{K'-0}{4\pi} = \frac{K'}{4\pi}$$

$\therefore$ total charge on CD $\frac{K'}{2\pi}$ AB has an equal negative charge and since their potential difference = 2K

The capacity = $\frac{1}{4\pi} \cdot \frac{K'}{K}$

In a practical problem k may be calculated from the widths and positions of the strips, in fact k = $\frac{BC}{AD}$ and the ratio $\frac{K^1}{K}$ for any k may be obtained from tables.

We may obtain the surface density at any point from the relationship $\sigma = \frac{1}{4\pi} \cdot \left.\frac{\partial \psi}{\partial x}\right|_{y=0}$ in the same manner as in Example II.

The above may be adapted to provide the solution of a number of different problems.

A. Thus: if $k \rightarrow 0$ we have the breadth of the planes becoming infinite: also from the properties of elliptic functions:

$$K \rightarrow \frac{\pi}{2} \quad \text{and } \operatorname{Sn} W = \operatorname{Sin} W.$$

Hence we have two semi-infinite planes at potentials $\pm \frac{\pi}{2}$ and with a gap 2b between and with $Z = b \operatorname{Sin} W$ as the appropriate transformation, as in Example II.

B. If an infinite conducting plane at zero potential is placed along the y axis it is clear that the field will be undisturbed. If AB is then removed and the infinite plane still kept at zero potential, the field to the right, on the side of CD still remains unchanged. Thus $Z = b \operatorname{Sn} W$ also provides the solution for a charged infinite strip of finite width placed perpendicularly before an infinite plane at zero potential.

C. Let $k \rightarrow 1$. For this limiting value of the modulus the elliptic function has the property that:

$$\operatorname{Sn} W \rightarrow \tanh W; \; K^1 \rightarrow \frac{\pi}{2}$$

Equation (3.2) then takes the form:

$$Z = b \tanh W$$

$$\therefore \; W = \tanh^{-1} \frac{Z}{b}$$

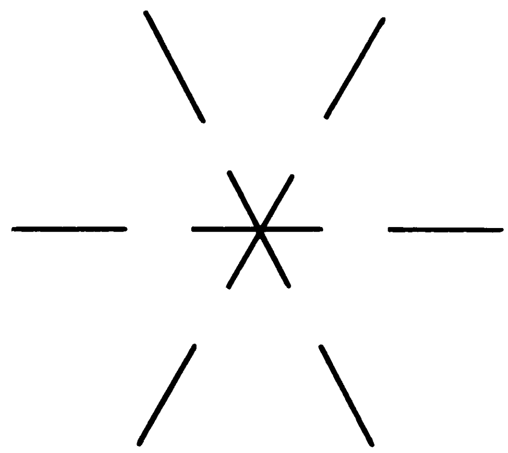
$$W = \tfrac{1}{2} \log \frac{z+b}{z-b} \qquad \text{- - -(3.3)}$$

But as $k \rightarrow 1$, the stripes shrink to two line charges 2b apart with charges of $\frac{K'}{2\pi} = \pm \frac{1}{4}$ per unit length respectively. Therefore equation (3.3) gives the appropriate complex potential for such a case. The argument of case B clearly remains valid here so that this equation also provides the solution for a line charge placed parallel before an infinite conducting plane.

In a similar way other simple relationships with elliptic functions provide solutions for problems with strips arranged in different ways.

Thus: $$z^n = b^n \,\mathrm{dn}\,(\phi + i\psi)$$

with ϕ taken as the potential function provide a solution for the case illustrated in the following diagram.



$\phi = 0$ on outer plates; these extend to ∞

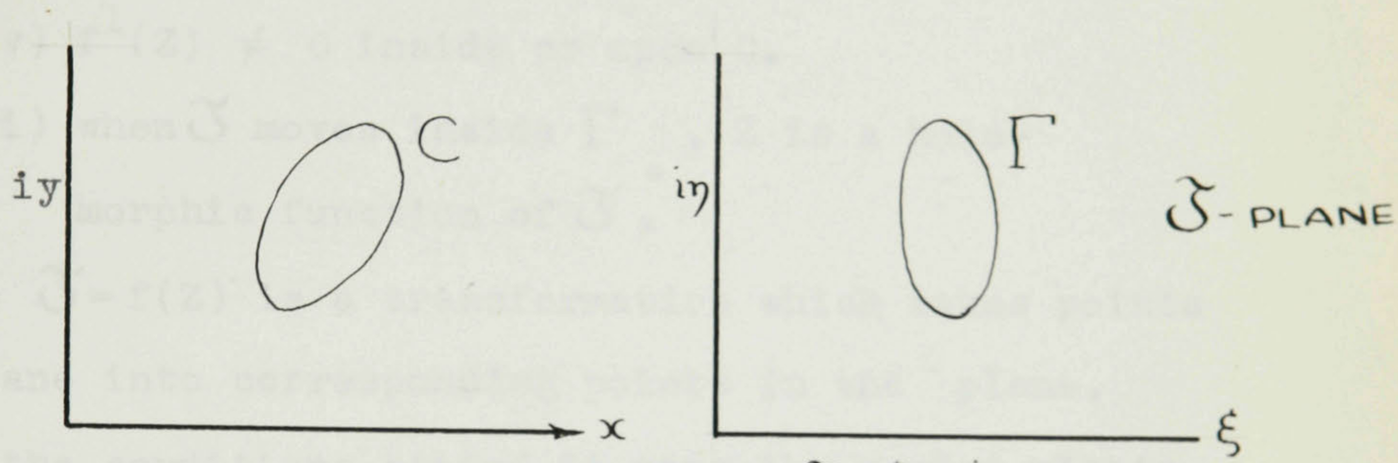
$\phi = K$ on inner plates.

Chapter IV

1. Notes on Mapping. Transformations

Consider a complex function which is holomorphic inside and upon a simple closed curve C in the Z plane.

Z plane



Further consider another complex variable $\zeta = \xi + i\eta$ and form a second Argand diagram in the ζ plane.

Now consider the relation:

$$\zeta = f(z)$$

Since f(Z) is holomorphic (single valued), to each point on the Z plane corresponds only one point on the ζ plane. Thus the points on C and its interior are mapped upon certain points in the ζ plane.

If we make the following assumptions

(a) $f(Z_1) \neq f(Z_2)$ $\quad Z_1, Z_2$ on C.

(b) $f^1(Z) \neq 0$ for any Z on C,

it is easy to prove the following results (See Milne-Thomson (VI) etc.) which we state without proof.

(i) When Z describes C once: ζ describes a closed curve Γ in the ζ plane and the curve has no double points

(ii) Given Z inside C the corresponding

point ζ_0 is inside Γ

(iii) If Z describes C in a positive sense i.e. the circuit taken to have the inside on the left always, ζ describes Γ in the positive sense

(iv) Given ζ_0 inside Γ there is exactly one point Z_0 inside C such that $\zeta_0 = f(Z_0)$

(v) $f^1(Z) \neq 0$ inside or upon C.

(vi) when ζ moves inside Γ , Z is a holomorphic function of ζ.

We see that $\zeta = f(Z)$ is a transformation which takes points in the Z plane into corresponding points in the plane. Subject to the conditions stated it maps the region within C uniquely and reversibly point by point on the region within Γ. If condition (b) holds the boundary is included. In practical cases of mapping, it will in general be possible to select contours in such a way as to omit singular points and ensure conditions (a) and (b). We shall frequently be engaged in mapping infinite regions where the question of interiors becomes more involved. From result (iii) it is apparent that if we know the sense of description along corresponding boundaries the corresponding regions may be inferred. Also the question may in general be decided from the limiting form of the finite case.

2. Invariants under Analytic Transformation.

A. Suppose $\xi + i\eta = f(Z)$ is a holomorphic function, Then $$\frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial y} \quad ; \quad -\frac{\partial \eta}{\partial x} = \frac{\partial \xi}{\partial y}$$

$$\nabla^2 \xi = 0 \quad ; \quad \nabla^2 \eta = 0 \quad \text{-----}(4\cdot1)$$

We show that if V satisfies:

$$\frac{\partial^2 V(x,y)}{\partial x^2} + \frac{\partial^2 V(x,y)}{\partial y^2} = 0$$

then

$$\frac{\partial^2 V(\xi\eta)}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2} = 0$$

or any solution of Laplaces Equation in the ξ, η plane is a solution in the xy plane under an analytic transformation.

Proof: $$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial V}{\partial \eta}\frac{\partial \eta}{\partial x}$$

AND $$\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + 2\frac{\partial^2 V}{\partial \xi \partial \eta}\cdot\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial^2 V}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2$$

$$+ \frac{\partial V}{\partial \xi}\left(\frac{\partial^2 \xi}{\partial x^2}\right) + \frac{\partial V}{\partial \eta}\left(\frac{\partial^2 \eta}{\partial x^2}\right)$$

and we have a similar expression for $\dfrac{\partial^2 V}{\partial y^2}$

Adding we obtain after using conditions (4·1)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial \xi^2}\left[\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial x}\right)^2\right] + \frac{\partial^2 V}{\partial \eta^2}\left[\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial x}\right)^2\right]$$

$$= \left[\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2}\right]\cdot\left[\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial x}\right)^2\right] \quad \text{-----}(4\cdot2)$$

Now the 2nd bracket on the right cannot vanish for an analytic transformation $\xi + i\eta - f(z)$

Hence if:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

then:

$$\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2} = 0 \quad \text{also.}$$

B. A conformal transformation leaves every equipotential family, equipotentials.

Proof

The condition that $\phi(x,y) = C$ represent a family of equipotentials is (ii)

$$\frac{\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2}}{\left(\dfrac{\partial \phi}{\partial x}\right)^2 + \left(\dfrac{\partial \phi}{\partial y}\right)^2} = f(C) \text{ only}$$

See Smythe

Let $\phi(x,y) = C$ be a family of equipotentials.

Under the transformation let this become

$$\phi\left\{x(\xi,\eta),\ y(\xi,\eta)\right\} = \phi(\xi,\eta) = C$$

Now as in previous case

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \left\{\frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2}\right\} \cdot \left\{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial x}\right)^2\right\}$$

and:

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \phi}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\left(\frac{\partial \phi}{\partial x}\right)^2 = \left(\frac{\partial \phi}{\partial \xi}\right)^2\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial \eta}\right)^2\left(\frac{\partial \eta}{\partial x}\right)^2 + 2\,\frac{\partial \phi}{\partial \xi}\cdot\frac{\partial \phi}{\partial \eta}\cdot\frac{\partial \xi}{\partial x}\cdot\frac{\partial \eta}{\partial x}$$

Similarly:

$$\left(\frac{\partial \phi}{\partial y}\right)^2 = \left(\frac{\partial \phi}{\partial \xi}\right)^2\left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial \eta}\right)^2\left(\frac{\partial \eta}{\partial y}\right)^2 + 2\left\{\frac{\partial \phi}{\partial \xi}\cdot\frac{\partial \phi}{\partial \eta}\ \frac{\partial \xi}{\partial y}\cdot\frac{\partial \eta}{\partial y}\right\}$$

$$= \left(\frac{\partial\phi}{\partial\xi}\right)^2 \cdot \left(\frac{\partial\eta}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial\eta}\right)^2 \cdot \left(\frac{\partial\xi}{\partial x}\right)^2 - 2 \, \frac{\partial\phi}{\partial\xi} \cdot \frac{\partial\phi}{\partial\eta} \, \frac{\partial\xi}{\partial x} \cdot \frac{\partial\eta}{\partial x} .$$

using the CR conditions

$$\therefore \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 = \left[\left(\frac{\partial\phi}{\partial\xi}\right)^2 + \left(\frac{\partial\phi}{\partial\eta}\right)^2\right] \cdot \left[\left(\frac{\partial\xi}{\partial x}\right)^2 + \left(\frac{\partial\eta}{\partial x}\right)^2\right]$$

$$\therefore \frac{\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2}}{\left(\dfrac{\partial\phi}{\partial x}\right)^2 + \left(\dfrac{\partial\phi}{\partial y}\right)^2} = \frac{\dfrac{\partial^2\phi}{\partial\xi^2} + \dfrac{\partial^2\phi}{\partial\eta^2}}{\left(\dfrac{\partial\phi}{\partial\xi}\right)^2 + \left(\dfrac{\partial\phi}{\partial\eta}\right)^2} = f(z) \text{ only}$$

Thus $\phi(\xi, \eta) = C$ is a family of equipotentials.

C. <u>Charge is preserved.</u>

Under change of variable from x,y to (ξ, η) through the relation $\xi + i\eta = f(c)$

$$d\xi \, d\eta = \frac{\partial(\xi, \eta)}{\partial(x, y)} \cdot dx \, dy$$

where $\dfrac{\partial(\xi \eta)}{\partial(x,y)}$ represents the Jacobian $\left\{\dfrac{\partial\xi}{\partial x} \cdot \dfrac{\partial\eta}{\partial y} - \dfrac{\partial\xi}{\partial y} \cdot \dfrac{\partial\eta}{\partial x}\right\}$

which by the C.R. conditions becomes $\left\{\left(\dfrac{\delta\xi}{\partial x}\right)^2 + \left(\dfrac{\partial\eta}{\partial x}\right)^2\right\}$

$$\therefore d\xi \, d\eta = \left\{\left(\frac{\partial\xi}{\partial x}\right)^2 + \left(\frac{\partial\eta}{\partial x}\right)^2\right\} dx \, dy = K \, dx \, dy .$$

Now we know from equation (4.2) that:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = K\left[\frac{\partial^2 V'}{\partial \xi^2} + \frac{\partial^2 V'}{\partial \eta^2}\right]$$

$$\therefore \iint\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) dx\,dy = \iint\left(\frac{\partial^2 V'}{\partial \xi^2} + \frac{\partial^2 V'}{\partial \eta^2}\right) K\,dx\,dy$$

$$= \iint\left(\frac{\partial^2 V'}{\partial^2 \xi^2} + \frac{\partial^2 V'}{\partial \xi^2}\right) d\xi\,d\eta$$

but
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -4\pi\rho$$

using the fundamental equation (1.2)

$$\therefore \iint \rho\,dx\,dy = \iint \rho'\,d\xi\,d\eta$$

i.e. total charge = total charge in corresponding region.

We illustrate the application of the foregoing with the discussion of a specific problem.

Chapter V

1. We have an earthed conducting wedge bounded by $\theta = 0$, $\theta = \alpha$ in the presence of a line charge at $\theta = \beta$ Fig.29.1. To find an expression for the potential anywhere between the planes

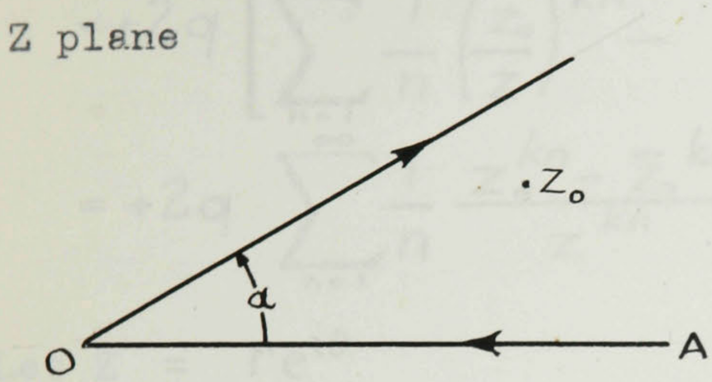


Fig 29·1

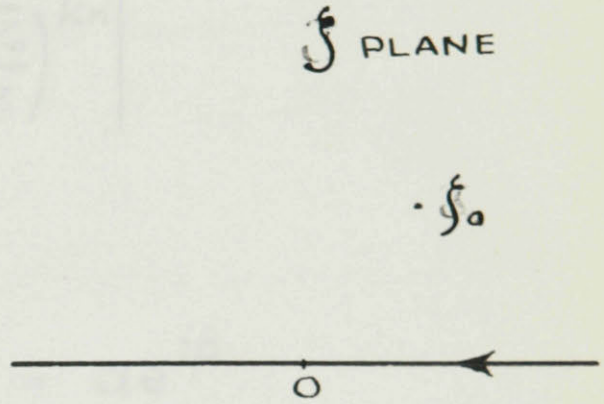


Fig 29·2

Consider the transformation $\zeta = z^{\frac{\pi}{\alpha}}$

$$\therefore \quad \zeta = r^{\frac{\pi}{\alpha}} \cdot e^{i\frac{\pi}{\alpha}\theta}$$

when $\quad \theta = 0 \quad \zeta = r^{\frac{\pi}{\alpha}}$

and the plane $\theta = 0$ of the Z plane becomes the real positive axis of the ζ plane, Fig. 29.2

Similarly for $\theta = \alpha \quad \zeta = -r^{\frac{\pi}{\alpha}}$ and so $\theta = \alpha$ becomes the negative real axis of the ζ plane. Also if the figure in the Z plane is described in the manner indicated by the arrow then that in the ζ plane is described as shown. Then from our notes on mapping the interior of the wedge is mapped on the upper half of the ζ plane.

Let the point Z_0 correspond to ζ_0. Now a charge of q at Z_0 corresponds to one of q at ζ_0. Hence by virtue of this transformation we have transformed our problem into that of a line charge placed before and parallel to an infinite plane. The appropriate complex potential for this case

is: $W = -2q \log\frac{\zeta-\zeta_0}{\zeta-\bar{\zeta}_0}$ as was indicated on Page 20
(here real part represents the potential function.

$$\therefore W = -2q\left[\log(\zeta-\zeta_0) - \log(\zeta-\bar{\zeta}_0)\right]$$

$$= -2q\left[\log(z^k - z_0{}^k) - \log(z^k - \bar{z}_0{}^k)\right] \quad \text{where } k = \frac{\pi}{\alpha}$$

$$= -2q\left\{\log\left[1-\left(\frac{z_0}{z}\right)^k\right] - \log\left[1-\left(\frac{\bar{z}_0}{z}\right)^k\right]\right\} \text{ if } |z| > |z_0|$$

$$= +2q\left[\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{z_0}{z}\right)^{kn} - \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\bar{z}_0}{z}\right)^{kn}\right]$$

$$= +2q\sum_{n=1}^{\infty}\frac{1}{n}\,\frac{z_0{}^{kn} - \bar{z}_0{}^{kn}}{z^{kn}}.$$

Let $Z = re^{i\theta}$ then $Z_0 = ae^{i\beta}$

$\bar{Z}_0 = ae^{-i\beta}$

$$\therefore W = +2q\sum_{n=1}^{\infty}\frac{a^{kn}e^{ikn\beta} - a^{kn}e^{-ikn\beta}}{r^{kn}\,e^{ink\theta}}$$

$$= +2q\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^{kn}\cdot\left[2i\sin kn\beta\right]\left[\cos kn\theta - i\sin kn\theta\right]$$

thus:

$$W = +2q\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^{kn}\cdot\left\{2i\sin kn\beta\right\}\left\{\cos kn\theta - i\sin kn\theta\right\}$$

Now $W = \phi + i\psi$

where ϕ the real part, represents the potential function

$$\phi = \text{REAL PART OF} +2q\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{a}{r}\right)^{kn}\cdot\left[2i\sin kn\beta\right]\left[\cos kn\theta - i\sin kn\theta\right]$$

$$= 4q\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{a}{r}\right)^{\frac{n\pi}{\alpha}}\cdot\sin\left(\frac{\pi}{\alpha}n\beta\right)\cdot\sin\left(\frac{\pi}{\alpha}n\theta\right)$$

This is potential function required when

$|z| > |z_0|$ i.e. for $r > a$

Similarly it may be shown that for $r < a$

$$\phi = 4q\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{r}{a}\right)^{\frac{n\pi}{\alpha}} \sin\left(\frac{\pi}{\alpha} n\beta\right)\left(\sin\frac{\pi}{\alpha}n\theta\right).$$

2. The power of the method of analytic transformation lies in the fact that with it a given electrostatic problem may be transformed into another which may be more easily solved. By a reverse transformation the solution of the original problem may be obtained. For example the solution of the problem of a charged line parallel to an infinite thin strip with an equal opposite charge and symmetrically placed with respect to it is fairly readily obtained. The figure in a Z plane would be as shown:

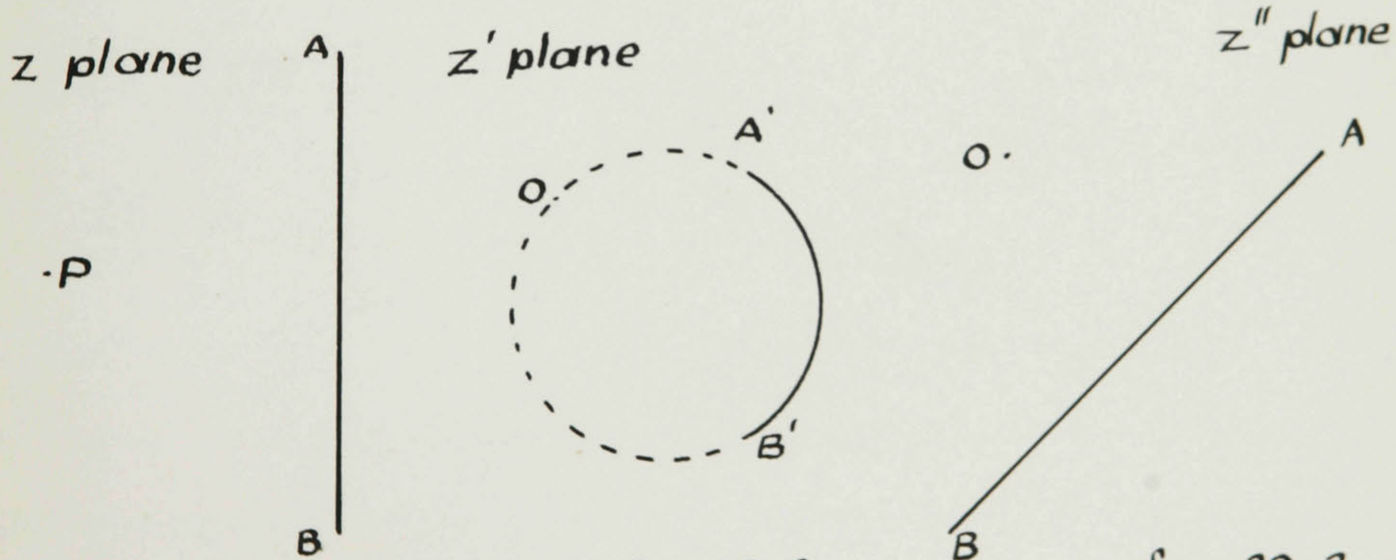


fig 32.1 fig 32.2 fig 32.3.

P is the trace of the line charge, AB that of the strip both going perpendicularly into the plane of the paper. If we transform this figure by a complex inversion with P as centre, AB becomes the arc of a circle and the charged point recedes to infinity. Then we have the solution of the problem of <u>free</u> distribution of charge on a cylindrical sheet with parallel edges (fig. 32.2). Further if in Fig. 32.2 we make a second complex inversion about a point O on the unoccupied arc, A' B' will be transformed into a straight line segment and we shall have the solution for the problem of a strip and parallel line charge in a general position, Fig. 32.3 In his paper[3] "Some electrostatic distributions in two dimensions" A.E.H. Love deals with these cases.

Chapter VI

1. The Schwarz-Christoffel Theorem

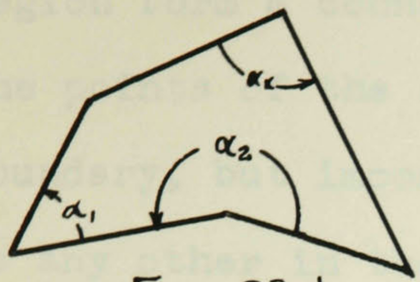


Fig 33.1

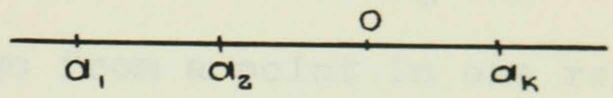


Fig 33 · 2

Let a_1, a_2 --- a_n be n points on the real axis of a $\zeta = \xi + i\eta$ plane, fig. 33.2 such that: $a_{K-1} < a_K$

Further let $\alpha_1, \alpha_2 \cdots \alpha_n$ be the interior angles of the "n" vertices of a closed polygon in the Z plane.

Then

$$\sum_1^n \alpha_K = (n-2)\pi$$

The transformation defined by:

$$\frac{dz}{d\zeta} = K\left(\zeta - a_1\right)^{\frac{\alpha_1}{\pi}-1} \cdot \left(\zeta - a_2\right)^{\frac{\alpha_2}{\pi}-1} \cdots\cdots \left(\zeta - a_n\right)^{\frac{\alpha_n}{\pi}-1} \qquad (6.1)$$

transforms the real ζ axis into the boundary of a closed polygon in the Z plane in such a way that the vertices of the polygon correspond to the points a_k and the interior angles of the polygon are α_K. Moreover when the polygon is simple the interior is mapped upon the upper half of the ζ plane. K is a constant real or complex.

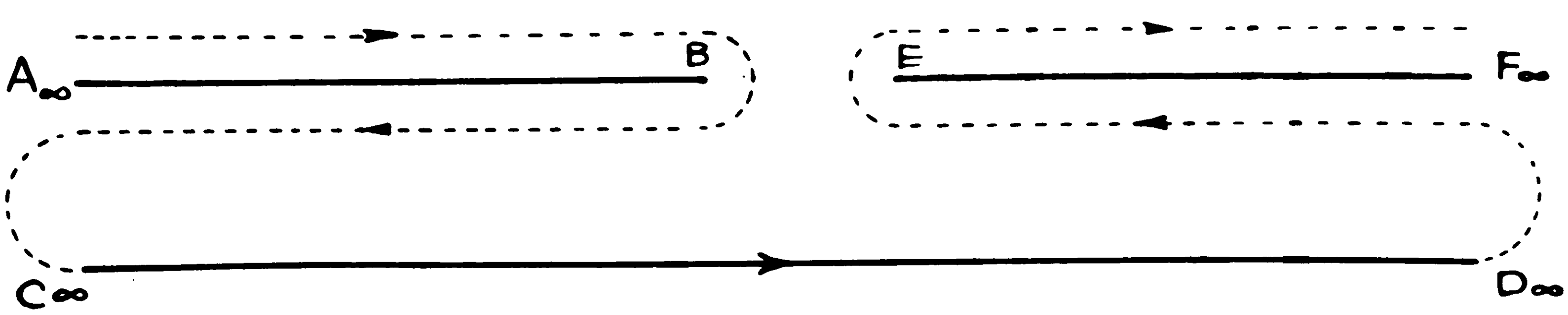
By a simple closed polygon we mean any configuration of straight lines in a plane possessing the following properties.

a) They form a connected boundary i.e. it is possible to go from one assigned point on the boundary to another, also on the boundary, without leaving the

b) The boundary divides the points of the plane into two regions such that the points of either region form a connected system. It is possible to join the points of the same region without crossing the boundary, but impossible to go from a point in one region to any other in the other without doing so.

The result stated above is the Schwarz-Christoffel Theorem. A rigorous proof of this theorem would require a good deal of space and we are chiefly concerned with its application so we shall confine ourselves to this statement of the theorem and to a number of needed comments:

(i) The simple polygons can and usually do extend to infinity. For example a pair of parallel straight lines may be regarded as a polygon with two vertices at infinity. The magnitude of the interior angle may be obtained by noting the change in direction after a vertex is passed in a positive description of the contour and substracting this value from π. A good example is the following: consider the polygon formed by an infinite line parallel to two semi-infinite ones in the same plane. The boundaries here are: the upper side of the infinite line and the upper and under sides of the two semi-infinite one.



The vertices of this polygon are shown in continuous order. They may be listed thus: A_∞ B C_∞ D_∞ E F

A_∞, F_∞ may be considered to coincide at the point at infinity. The interior of this polygon will be the space between the lines and the region above A B E F. Consider the interior angle at B. Going from A to B, after B is passed our direction has been rotated clockwise through π radius. Hence the exterior angle = $-\pi$

$\therefore$ the interior angle at B $= \pi - (-\pi) = 2\pi$

Passing through the vertex C_∞ our direction is rotated counter clockwise through π

$\therefore$ the interior angle at $C = \pi - \pi = 0$

(ii) In transforming a given polygon three of the numbers a_k may be chosen arbitrarily to correspond to three of the vertices of the polygon. The others will then be fixed by the shape of the polygon. The proper choice of K will then fix the scale and orientation.

(iii) When a vertex of the polygon corresponds to a point at infinity on the ζ axis, the factor corresponding to $a_k = \infty$ is omitted from the equation of the transformation and α_K does not appear either.

We may see this thus:

$$\frac{dz}{d\zeta} = K(\zeta - a_1)^{\frac{\alpha_1}{\pi}-1} \cdot (\zeta - a_2)^{\frac{\alpha_2}{\pi}-1} \cdots\cdots (\zeta - a_n)^{\frac{\alpha_n}{\pi}-1}$$

$$= K_1 \left(\frac{\zeta}{?} - a_1\right)^{-\frac{\alpha_1}{\pi}+1} \cdot (\zeta - a_1)^{\frac{\alpha_1}{\pi}-1} (\zeta - a_2)^{\frac{\alpha_2}{\pi}-1} \cdots (\zeta - a_n)^{\frac{\alpha_n}{\pi}-1}$$

by choice of K_1

$$\therefore \frac{dz}{d\zeta} = K_1 \left(\frac{\zeta - a_1}{-a_1}\right)^{\frac{\alpha_1}{\pi}-1} \cdot \left(\zeta - a_2\right)^{\frac{\alpha_2}{\pi}-1} \cdots\cdots \left(\zeta - a_n\right)^{\frac{\alpha_n}{\pi}-1}$$

and when $a \longrightarrow \infty$

$$\mathop{\mathrm{L}} \left(\frac{\zeta - a_1}{-a_1}\right)^{\frac{\alpha_1}{\pi}-1} \longrightarrow 1$$

Hence this factor disappears from the expression.

(iv) When the transformation leads to a simple polygon it is holomorphic as the conditions noted in Section II p 10 hold, if we avoid the points a_k in the description of our contour. This may always be done by small indentations which may be made infinitismal.

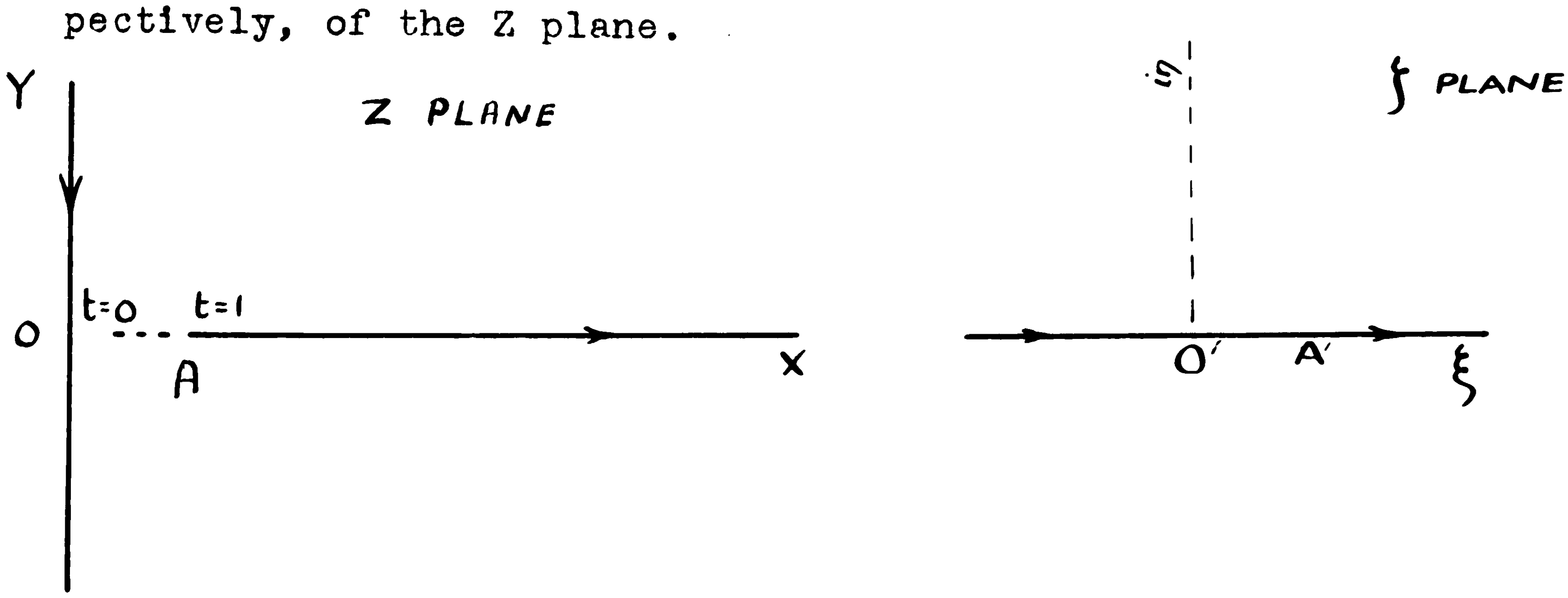
2. The Schwarz-Christoffel Theorem provides a general direct method for finding the proper complex transformations for electrostatic and hydrodynamic problems in two dimensions when the lines over which the potential is given are straight. It can also be adapted in various ways to the solution of other special problems involving curved boundaries. It is our aim to give some account of these methods.

It may be noted that in general the application of the Schwarz-Christoffel Theorem leads to elliptic integrals.

Chapter VII

1. Consider the Following Problem:

A horizontal plane at potential zero has its edge parallel to and at a distance "C" from an infinite vertical plane at potential $\frac{\pi}{2}$. Required an expression for the charge density at any point of each plane.
As the planes are infinite it is clear we are dealing with a two dimensional problem. e take a transverse plane at right angles to both planes as the plane of the complex veriable Z. In this plane let the trace of the vertical and horizontal planes lie along the y and x axis, respectively, of the Z plane.



From symmetry it is clear that the fields on either side of OX are identical and we need only deal with the field within YOX.

First, let us apply the Schwarz-Christoffel Theorem to obtain a transformation which will take the polygon YOX in the Z plane, into a straight boundary - the real axis is an auxiliary plane, the t plane so that O coincides with O', the point $Z = C$ in the X plane with the point $t = 1$ in the t plane.

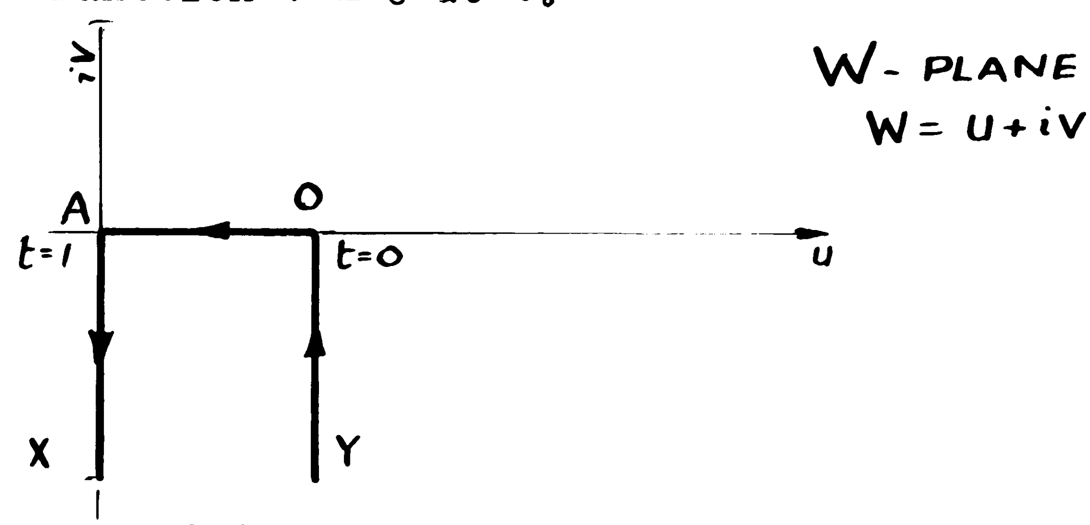
Then $\frac{dz}{dt} = Kt^{-\frac{1}{2}}$

$$\therefore\ z = 2Kt^{\frac{1}{2}} + \text{const} = 2Kt^{\frac{1}{2}} \quad \because t=0 \text{ when } z=0$$

Now $C = 2K$, for $Z = C$, when $t = 1$

$\therefore\ Z = Ct^{\frac{1}{2}}$

Next let us draw the potential diagram on a third auxiliary plane, the W plane. Regarding the real part as representing the potential, suppose the stream function $V = 0$ at O.



As we proceed from Y, along YO, u is constant and V increases to zero at O. Along OA we are on a stream-line $V = 0$. From A to X, $u = 0$ and V decreases. Our choice of the origin of V is arbitrary: the manner of its increase along YOAX we infer from the fact that along YO the flux must be positive and along AX negative. The sense of description ocrresponds to t increasing so that the area on the left of the contour YOAX as described is to be mapped on the upper half of the t plane.

For the W plane the interior angles at O

Along X = 0, Z is pure imaginary

$$\therefore z = ct^{\frac{1}{2}}$$

$$iy = ct^{\frac{1}{2}} \quad \text{or} \quad t = -\frac{y^2}{c^2}$$

$$\therefore \left.\frac{dW}{dz}\right|_{x=0} = \frac{1}{\sqrt{y^2 + c^2}}$$

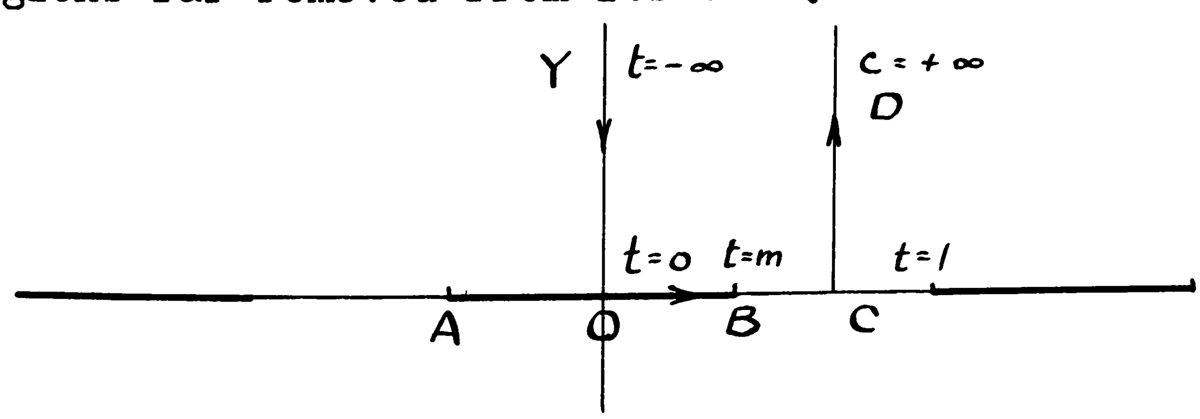
$$\text{and } \sigma_{oy} = (4\pi)^{-1}(y^2+c^2)^{-\frac{1}{2}}$$

These are the expressions required by the problem as stated by Smythe (ii) P 102 No 14

2. Infinite Charged Grating

Consider an infinite number of similar charged strips lying uniformly spaced and parallel to each other in the same plane. We suppose the strips to be equally wide, of infinite length and no thickness.

The solution of this problem will constitute the approximate one for a flat plane grating in the regions far removed from its ends.



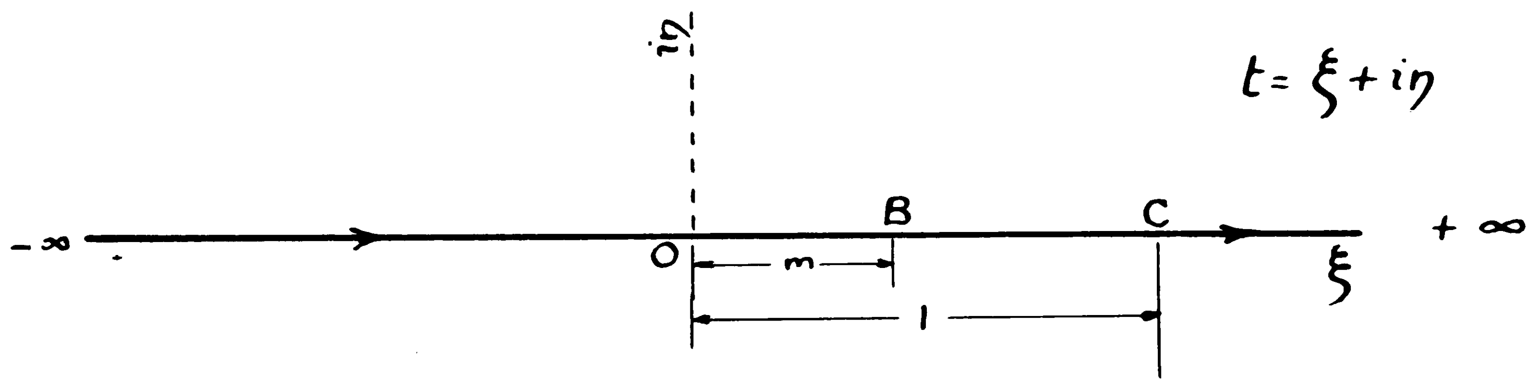
Let AB be the trace of any one of these strips in a plane at right angles to the grating.

It is clear from symmetry that YO, BC, CD will be lines of force and that the field within the rectangle YOBCD will be typical.

Let 2a be the width of each strip and 2b be the distance apart of the centres of any two adjacent strips.

Take O as the origin of the Z plane.

Now we make a Schwarz transformation which will take this rectangle into the real axis of an auxiliary t plane so that the points correspond as indicated.



From the Schwarz theorem the desired relation is

$$\frac{dz}{dt} = \frac{K}{\sqrt{t(t-1)}}$$

$$\therefore \quad z = 2K\cosh^{-1}\sqrt{t} + c$$

$$\text{Now } z = b, \text{ when } t = 1$$

$$\therefore \quad z = 2K\cosh^{-1}\sqrt{t} + b$$

Also when $t = 0$ $Z = 0$

$$\therefore \quad K = \frac{bi}{\pi}.$$

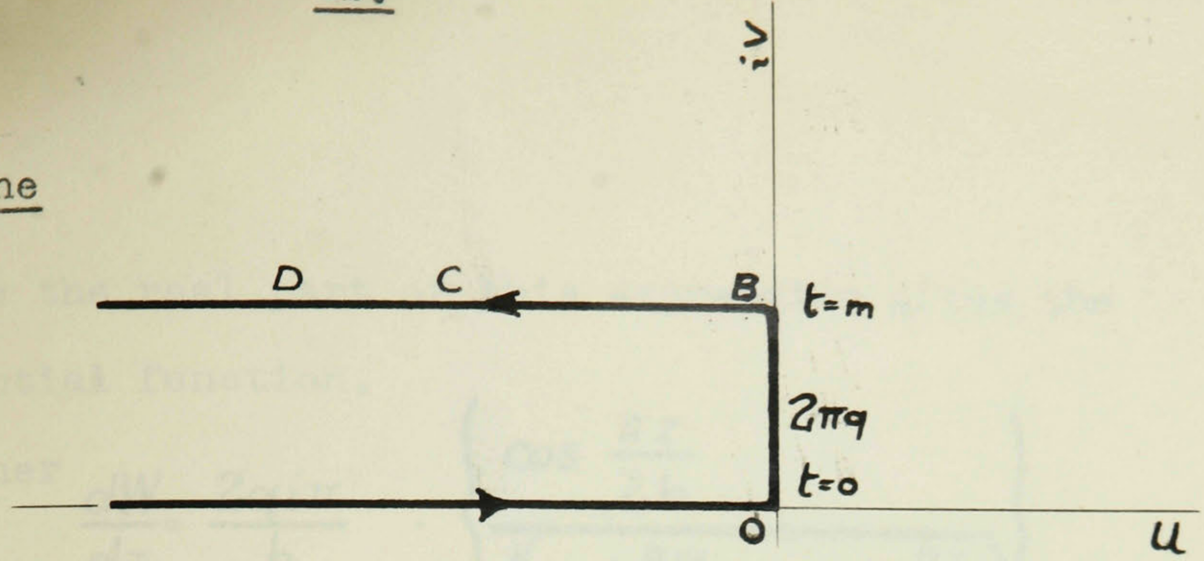
$$\therefore \quad z - b = \frac{2bi}{\pi}\cosh^{-1}\sqrt{t}$$

$$\therefore \cosh \frac{(z-b)\pi}{2ib} = \sin\frac{\pi z}{2b} = \sqrt{t}$$

Now when t = m: Z = a: $\quad \therefore \sqrt{m} = \sin\frac{\pi a}{2b}$

Let us take the potential on the strip to be zero. YO is a line of force and as we approach O along it, the value of the stream function remains constant but the potential increases steadily to the value zero at O. From O to B the potential remains constant but the stream function increases. If we suppose each strip to carry a charge q, then from symmetry the portion OB has a charge $q/2$ and the flux out of this region = $2\pi q$ This is equal to the change in the stream function from O to B. BCD is again a line of force and the stream function is now constant as we describe it. The potential function then decreases steadily. Let the stream function = 0 along YO, then it = $2\pi q$ along BCD. Hence the potential diagram on the W plane is as shown in the figure below, where W = U + i V and U represents the stream function and V the potential.

W plane



We now map this on the real axis of t so that the points correspond, as shown.

The geometrical lines of the problem are now matched with their appropriate stream and potential values and tied together in the real t axis. The proper transformation is:

$$\frac{dW}{dt} = \frac{K}{\sqrt{t(t-m)}}$$

$$\therefore W = 2K \cosh^{-1}\sqrt{\tfrac{t}{m}} + c$$

when $W = 2\pi iq, \quad t = m$

when $\therefore c = z\, 2\pi iq$

$W = 0 \quad t = 0$

$$\therefore -2\pi iq = 2K\pi\, i/2$$

$$\therefore K = -2q$$

$$\therefore W = -4q\cosh^{-1}\sqrt{t/m} + 2\pi iq$$

$$\therefore \cosh\left(\frac{W - 2\pi iq}{-4q}\right) = \sqrt{\frac{t}{m}} = \frac{\sin\frac{\pi z}{2b}}{\sin\frac{\pi a}{2b}} =$$

$$\therefore -\sin\frac{iW}{4q} = \text{''}$$

$$\therefore W = i\,4q\sin^{-1}\left\{\frac{\sin\frac{\pi z}{2b}}{\sin\frac{\pi a}{2b}}\right\}$$

where the real part of this expression gives the potential function.

Further $$\frac{dW}{dz} = \frac{2qi\pi}{b} \cdot \left\{ \frac{\cos\frac{\pi z}{2b}}{\sqrt{\left(\sin^2\frac{\pi a}{2b} - \sin^2\frac{\pi z}{2b}\right)}} \right\}$$

Now if Z is real a we are on the strip between OB; the expression above is pure imaginary and we thus have:

$$\frac{dW}{dz} = \left\{\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right\} = \frac{2\pi i q}{b}\left\{ \frac{\cos\frac{\pi x}{2b}}{\sqrt{\left(\sin^2\frac{\pi a}{2b} - \sin^2\frac{\pi z}{2b}\right)}} \right\}$$

$$\therefore \frac{\partial u}{\partial x} = 0$$

AND $$\frac{\partial u}{\partial y} = \frac{2\pi q}{b}\left\{ \frac{\cos\frac{\pi x}{2b}}{\sqrt{\left(\sin^2\frac{\pi a}{2b} - \sin^2\frac{\pi x}{2b}\right)}} \right\}$$

Hence the intensity is at right angles to the strip, as we should expect.

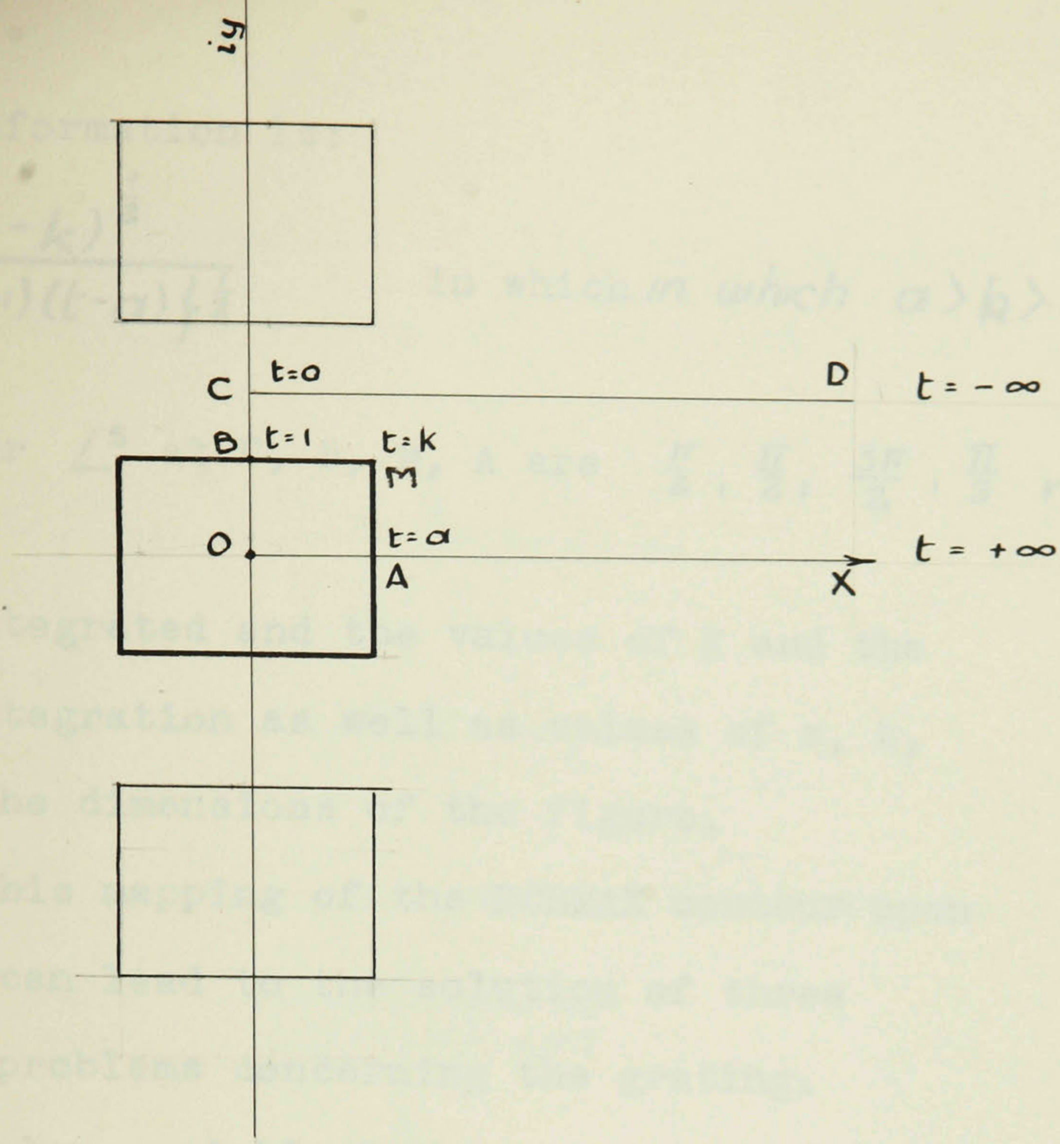
The surface density at any point is:

$$\sigma = \frac{q}{2b}\left\{ \frac{\cos\frac{\pi x}{2b}}{\sqrt{\left(\sin^2\frac{\pi a}{2b} - \sin^2\frac{\pi x}{2b}\right)}} \right\}$$

Thus at x = a, the edge, the density becomes infinite and at the centre

$$\sigma = \frac{q}{2b} \cdot \operatorname{cosec} \frac{\pi a}{2b}$$

3. If the grating discussed, in the last example, consists of similar rectangular bars equally spaced and of finite dimensions, the solution is not more difficult, at any rate in principle.



In the figure we show the section of such a grating by a plane at right angles to the length of the bars. Take this as the plane of a complex variable Z and let the origin O be at the centre of one of the bars. Adopt the axis as shown, where OX and OY are normal to the faces.

It is clear that the grating is symmetrical about the lines OX, OY and CD, where C is midway between the centres of the two adjacent bars.

By Schwarz's theorem we may obtain a relation which will map the contour DCBMAX on the real axis of an auxiliary t plane ($t = \xi + i\eta$) We may arbitrarily assign corresponding values of t for three angular points.
We choose $t = \infty, 0, 1$ to correspond as shown.
Suppose the points A, M correspond to $t = a$; $t = k$ respectively.

Then the transformation is:

$$\frac{dz}{dt} = \frac{K(t-k)^{\frac{1}{2}}}{\{t(t-1)(t-a)\}^{\frac{1}{2}}}$$ in which *in which* $a > k > 1$ -- $\underline{I}$

as the interior $\angle^s$ at C, B, M, A are $\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{2}$ *resp* respectively.

This may be integrated and the values of K and the constant of integration as well as values of a, k, deduced from the dimensions of the figure.

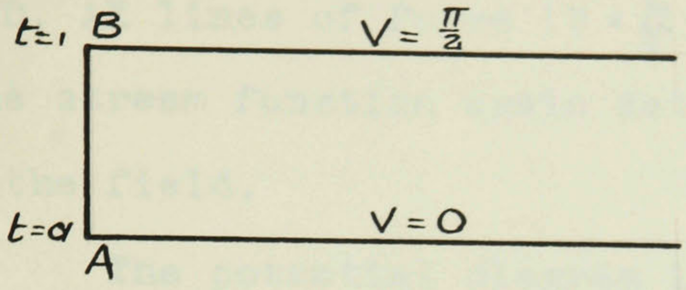
This mapping of the DCBMAX contour upon a real t axis can lead to the solution of three electrostatic problems concerning the grating.

Consider a complex variable W = U + i V, where U = constant, and V = constant are the potential and stream functions as before. The three problems are as follows:

I. <u>The bars carry equal charges.</u>

Then BMA, is an equipotential (U = 0 suppose) and DCB, Ax are lines of force ($V = \frac{\pi}{2}$, V = 0). This choice of the stream values determines the charge on the bar which cannot therefore be arbitrary. However this does not entail a loss of generality in the problem because the field intensity at any point will simply be proportional to the charge.

The diagram on the W plane is as shown, with the corresponding values of t indicated on the figure.



This rectangle is mapped upon the real t axis by the equation

$$\frac{dW}{dt} = \frac{K_1}{\{t(t-1)(t-a)\}^{\frac{1}{2}}}$$

$$= \frac{K_1}{\{(t-\frac{a+1}{2})^2-(\frac{a-1}{2})^2\}}$$

Hence: $$W = 2K_1 \cosh^{-1}\frac{2t-a-1}{a-1} + \text{const}$$

Now when $$t=a\,,\quad W=0\,,\quad \therefore\ \text{const}=0$$

$$t=1\,,\quad W=\frac{\pi i}{2}\,,\quad \therefore\ K_1=\pm\frac{1}{4}$$

$$\therefore W = \frac{1}{2}\cosh^{-1}\frac{2t-a+1}{a-1}$$

$$\therefore \cosh 2W = \frac{2t-a+1}{a-1}$$

$$\therefore a[1+\cosh 2W]+[1-\cosh 2W] = 2t$$

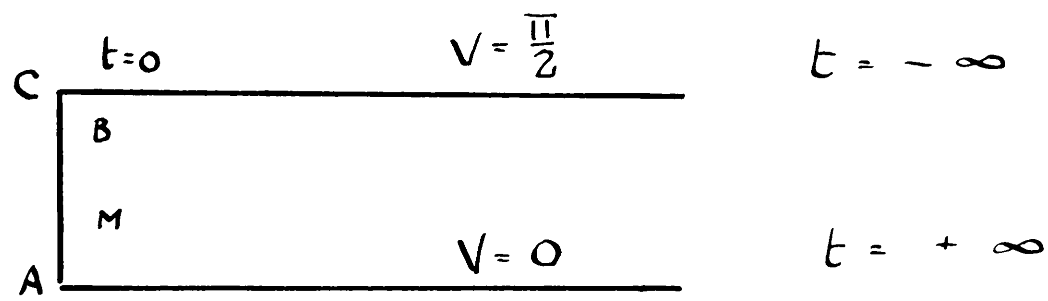
$$\therefore a\cosh^2 W + \sinh^2 W = t \qquad \ldots\ldots \underline{II}\,a$$

This equation taken with No. I provides the solution by giving the relation between W and Z.

II. The grating is uncharged and in a uniform field parallel to OX.

In this case CBMA is an equipotential ($u = 0$) and CD, AX lines of force ($V = \frac{\pi}{2}$, $V = 0$). This choice of the stream function again determines the strenghth of the field.

The potential diagram is seen below



The transformation is:

$$\frac{dW}{dt} = \frac{K_2}{\{t(t-a)\}^{\frac{1}{2}}}$$

$$\therefore W = 2K_2 \cosh^{-1}\sqrt{\frac{t}{a}} + C \quad \text{Now } w = 0 \quad t = a$$

$$\therefore C = 0$$

when $t = 0$, $W = \frac{\pi i}{2}$

$$\therefore \frac{\pi i}{2} = 2K_2 \frac{\pi i}{2}$$

$$\therefore K_2 = \frac{1}{2}$$

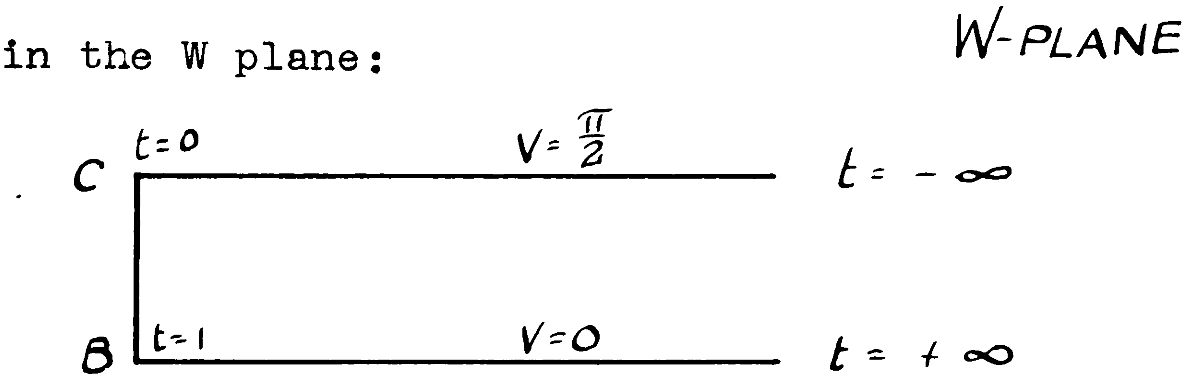
$$\therefore W = \cosh^{-1}\sqrt{\frac{t}{a}} \quad \text{----} \quad \underline{II\,b}.$$

In conjunction with (I), IIb gives the relation between W and Z for this case.

III **The uniform field is parallel to OY.**

Now XAMB and CD are equipotentials ($V = 0$; $V = \frac{\pi}{2}$) BC is a line of force, $U = 0$. We interchange the use of U and V as this simplifies the constants of integration. Choice of the potential differences once more determines the strength of the uniform field.

The diagram in the W plane:



$$\text{Here} \qquad \frac{dW}{dt} = \frac{K_3}{\{t(t-1)\}^{\frac{1}{2}}}$$

$$\text{Thus } W = \cosh^{-1}\sqrt{t} \qquad - \; - \; - \quad \underline{IIc}$$

and as in the other cases IIc with I provides an expression for the complex potential.

<u>4.</u> The developments outlined above are straightforward. W. H. Richmond, who notes these cases in his paper (1) proceeds to indicate how the method may be modified so that the rectangles may be rounded off into ovals of various shapes without invalidating the results. In brief his method is as follows: In Equation I replace $K : (t-k)^{\frac{1}{2}}$ by $P(t-a)^{\frac{1}{2}} + Q(t-1)^{\frac{1}{2}}$ where P, Q are positive constants.

Then: $$\frac{dz}{dt} = \frac{P(t-a)^{\frac{1}{2}} + Q(t-1)^{\frac{1}{2}}}{\{(t-a)(t-1)t\}^{\frac{1}{2}}}$$

$$= \frac{P}{\{t(t-1)\}^{\frac{1}{2}}} + \frac{Q}{\{t(t-a)\}^{\frac{1}{2}}}$$

where a $>$ 1.

Now when t increases from $-\infty$ to $+\infty$ as long as t does not lie between 1 and a, this expression yeidls the same outline DCB, AX, on the Z plane, as the following shows:

ex. when a $< t < \infty$: $\frac{dz}{dt}$ = a real number

$\therefore \frac{dy}{dt} = 0 \quad \therefore y = const$ (zero for our figure)

also x decreases with diminishing t,

when $0 < t < 1$;

$\frac{dz}{dt}$ = is pure imaginary

$\therefore \frac{dx}{dt} = 0$ and $x = const$ (" " " ")

Now $\frac{dy}{dt}$ = a real number as $i\frac{dy}{dt} = \frac{Q}{i\{\ ]}$

$\therefore y$ increases with diminishing t.

Now when a $> t > 1$ $\frac{P}{\{t(t-1)\}^{\frac{1}{2}}}$ is real $\frac{Q}{\{t(t-a)\}^{\frac{1}{2}}}$ is imaginary

$\frac{dx}{dt}$ is plus ; $\frac{dy}{dt}$ is minus

$\therefore$ x decreases and y increases with diminishing t.

also $\frac{dy}{dt} = -\frac{Q}{P}\left(\frac{t-1}{a-t}\right)^{\frac{1}{2}}$ and diminishes in absolute value from ∞ to 0. $\therefore$ the point Z describes some sort of oval cutting OX, OY at right angles.

It is clear from symmetry that the rectangular bars have been replaced by rounded oval bars of unknown shape but defined by the values of P, Q and a. Results II(a), (b), (c) are still valid and so these with:

$$\frac{dz}{dt} = \frac{P(t-a)^{\frac{1}{2}} + Q(t-1)^{\frac{1}{2}}}{\{t(t-1)(t-a)\}^{\frac{1}{2}}}$$

provide solutions for the three cases with oval cylinders

For the particular case where P = Q = 1:

$$Z = 2\left\{\cosh^{-1}\sqrt{t} + \cosh^{-1}\sqrt{t/a}\right\} + C$$

an integration. Let us consider such a case further. Let us require the dimensions OA, OB (using figure as for rectangular bars Page 44) of the oval to be such that C = 0, a = 2 in equation above.

Now the points t = 1, t = a correspond to B, A respectively on the Z plane.

For t = 1, $Z = 2\cosh^{-1}\frac{1}{2} = \frac{\pi i}{2}$

$\therefore\ y = +\frac{\pi}{2}$

when t = a = 1, $Z = 2\cosh^{-1}\sqrt{2}$ = a real number

$\therefore\ x = 2\cosh^{-1}\sqrt{2}$

Hence for the special case under consideration the dimensions of the oval are as follows:

$$OB = \frac{\pi}{2} \qquad OA = 2\cosh^{-1}\sqrt{2}$$

Now for $2 > t > 1$, $\cosh^{-1}\sqrt{t}$ is real

and $\cosh^{-1}\sqrt{t/2}$ is imaginary.

$\therefore$ for this range of t,

$$x = 2\cosh^{-1}\sqrt{t}$$

$$iy = 2\cosh^{-1}\sqrt{t/2}$$

or finally :

$$t = \operatorname{Cosh}^2 \frac{x}{2} \qquad \text{for } 1 < t < 2.$$
$$t = 2\cos^2 \frac{y}{2}$$

These are the parametric equations of this oval in the first quadrant.

Let us now consider an infinite grating formed of these special ovals in uniform a field of force. Let the field be parallel to OY. Then we saw from Case III (Equation IIc) $t = \operatorname{Cosh} W$, and we have as before:

$$z = 2\left\{\cosh^{-1}\sqrt{t} + \cosh^{-1}\sqrt{t/2}\right\}$$
$$\therefore \frac{dW}{dz} = \frac{(t-2)^{\frac{1}{2}}}{(t-1)^{\frac{1}{2}} + (t-2)^{\frac{1}{2}}}$$

Real values of t give the value of $\frac{dW}{dz}$ on the contour DCBMAX, where BMA is now a portion of an oval. For $1 < t < 2$ we are on the surface BMA of the oval conductor.

When $t = 2$: $\frac{dW}{dz} = 0$ and hence the intensity vanishes at the point A.

For $t > 2$ $\quad \frac{dW}{dz}$ is real plus

$\therefore \frac{\partial u}{\partial x} + i\frac{\partial V}{\partial x}$ " " "

$\therefore \frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x}$ " " " from the C.R. conditions

thus : $\frac{\partial V}{\partial x} = 0$ & $\frac{\partial V}{\partial y} = \alpha$ + real no.

i.e. the intensity is in the Y direction.

This is natural as we are now on Ax which is an equipotential.

The strength of the field is determined by:

$$\left(-\frac{\partial V}{\partial y}\right)_{t\to\infty} = \underset{t\to\infty}{\text{L}} \frac{(1-2/t)^{\frac{1}{2}}}{(1-1/t)^{\frac{1}{2}}+(1-2/t)^{\frac{1}{2}}} = -\frac{1}{2}$$

At $t=1$ i.e. at B on the oval:

$$\frac{dW}{dz} = \frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x} = \frac{i}{i} = 1$$

$$\therefore \frac{\partial V}{\partial y} = 1 \quad ; \quad \frac{\partial V}{\partial x} = 0$$

$$\therefore \left(-\frac{\partial V}{\partial y}\right) = -1$$

and the line of force enters the conductor at right angles at B.

Hence we see the uniform field turns out to be in the direction YO. This is because of our initial choice of CD as being at higher potential ($V = \frac{\pi}{2}$) than BMAX ($V = 0$) in Case III.

At $t = 1$ $W = U + iV = 0$ from --- IIc.

$\therefore$ $U = 0$

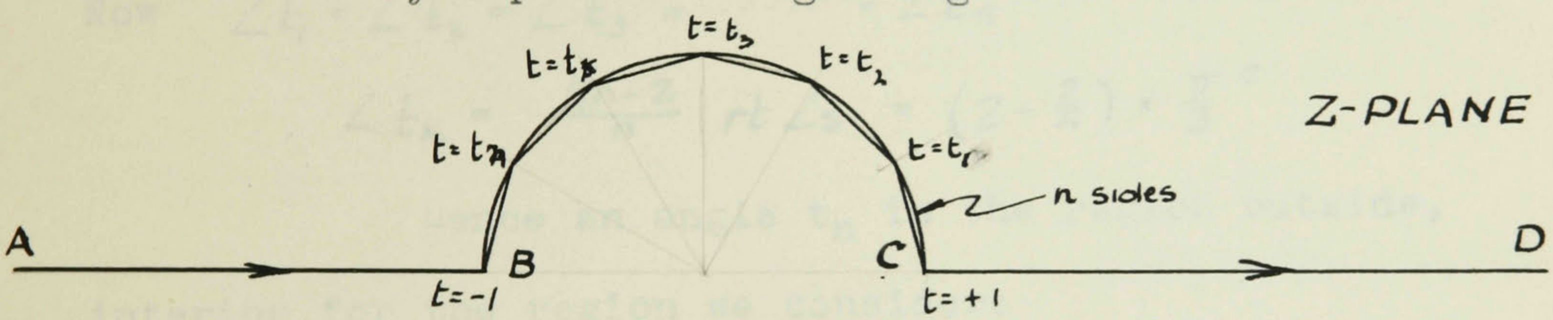
At $t = 2$ $W = U + iV = \text{Cosh}^{-1}\sqrt{2}$

$\therefore$ $U = \text{Cosh}^{-1}\sqrt{2}$

the charge on the surface BMA $= \frac{1}{4\pi}\left[U\right]_1^2 = \frac{1}{4\pi}\cosh^{-1}\sqrt{2}$

5. W. M. Page[2] has discussed a similar method for dealing with cylinders whose cross-sections contain curved as well as straight lines (2). We will discuss his method briefly.

A curved surface may be regarded as the limit of an equiangular polygon. Hence in a problem involving curved contours we may attempt to form a Schwarzian transformation for a polygonal contour of straight edges and proceed to a limit. For example consider the case of an infinite plane having a curved cylindrical boss. The cross-section by a plane at right angles is shown



In the boss consider a portion of an equiangular polygon as in figure. We attempt to transform this contour into the real axis of an auxiliary t plane in the usual manner. Assign the values $t \pm 1$ to the points C, B. respectively, and let the angular points correspond to t_1 t_2 t_3 $\cdots$ t_n where these values are not at our disposal but are fixed after our choice of the points B, C, AD if the size of the polygon is fixed.

Alternatively we may give values to $t_1, t_2 \cdots t_n$ then the polygon in the Z plane has the length of its sides determined and may, in theory be calculated. We shall adopt this procedure. Let us take $C t_1$ and $B t_n$ as $\perp$ to A D. We require the field above this figure hence for the interior of our polygon we must have the region above A, B, t_n D. The interior angles at B, C are $\frac{\pi}{2}$ each. Consider the polygon B C $T_1 t_2 \cdots t_n$ it has $n+2$ sides and $\therefore$ 2n rt. angles $\therefore$ angles $t_1 + t_2 + t_3 + \cdots t_n$ $= (2n - 2)$ rt. angles.

Now $\angle t_1 = \angle t_2 = \angle t_3 = \cdots = \angle t_n$

$$\therefore \angle t_K = \frac{2n-2}{n} \; rt \angle s = \left(2 - \frac{2}{n}\right) \times \frac{\pi}{2}^c$$

Hence an angle t_n in the region outside, interior for the region we consider:

$$2\pi - \left(2 - \frac{2}{n}\right) \times \frac{\pi}{2}^c = \pi + \frac{\pi}{n}$$

The transformation: $\frac{dz}{dt} = K(t-t_1)^{\frac{\alpha_1}{\pi}-1}(t-t_2)^{\frac{\alpha_2}{\pi}-1} \cdots (t-t_n)^{\frac{\alpha_n}{\pi}-1}$

$\therefore$ becomes
$$\frac{dz}{dt} = \frac{K[(t-t_1)(t-t_2)\cdots(t-t_n)]^{\frac{1}{n}}}{(t^2-1)^{\frac{1}{2}}}$$

Suppose the following values are assigned:

$t_1 = \cos\alpha \qquad t_2 = \cos 3\alpha \qquad t_n = \cos(2n-1)\alpha.$

and require that $2n\alpha = \pi$

with this choice, because $t_n = -t_1$, etc. we might expect to get a semi-circle in the Z plane on proceeding to the limit.

Using these values:

$$(t-t_1)(t-t_2)\cdots(t-t_n) = 2^{\frac{1}{n}}\left[(t+\sqrt{t^2-1})^n + (t-\sqrt{t^2-1})^n\right]$$

as may easily be proved.

Hence we have to determine:

$$\lim_{n\to\infty}\left\{\frac{(t-t_1)(t-t_2)----(t-t_n)}{2^n}\right\}^{\frac{1}{n}}$$

or

$$\lim_{n\to\infty}\frac{1}{2}\left\{(t+\sqrt{t^2-1})+(t-\sqrt{t^2-1})\right\}^{\frac{1}{n}}$$

If $t>1$ by putting $t = \operatorname{Cosh} U$, we see

$$\lim_{n\to\infty}\frac{1}{2}\left[e^{nu}+e^{-nu}\right]^{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{2}e^{u}\left[1+e^{-2nu}\right]^{\frac{1}{n}}$$

$$= \frac{1}{2}e^{u} = \frac{1}{2}\left(t \pm \sqrt{t^2-1}\right)$$

When $t < 1$ the expression has no definite limit.

Now for a boss whose cross-section is a circular cylinder of radius R, the transformation is known to be

$$\frac{dz}{dt} = \frac{R(t+\sqrt{t^2-1})}{(t^2-1)^{1/2}}$$

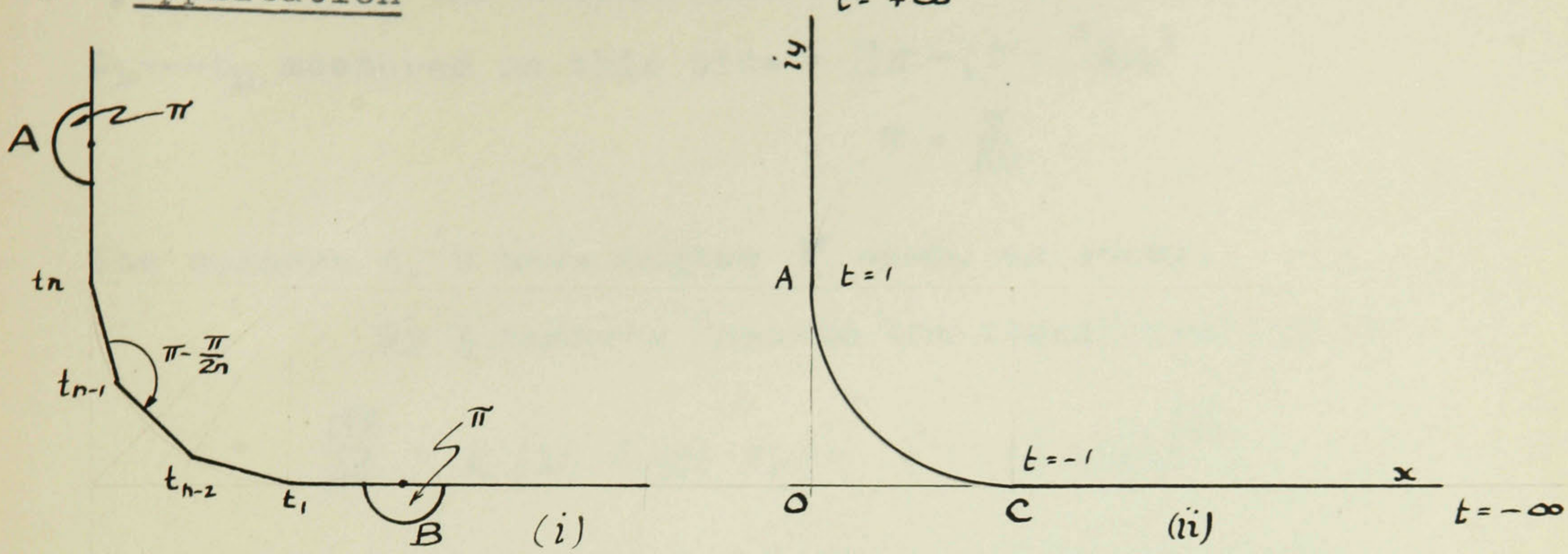
which is exactly the value obtained above as a limiting value from the polygon when $t>1$.

Following a suggestion of W. H. Richmond,[1] Mr. Page[2] simply replaces the indeterminate limiting value when $t<1$ by the expression $(t+\sqrt{t^2-1})$ and then adopts the transformation

$$\frac{dz}{dt} = \frac{K(t+\sqrt{t^2-1})}{2(t^2-1)^{1/2}}$$

and proceeds to investigate the consequences that follow from this transformation.

6. Application



Consider an infinite wedge formed by two semi-infinite planes at right angles and with the edge rounded off. (Fig.ii) The figure shows a cross-section by a plane at right angles. Choose the axis as indicated. We suppose the system carries a charge and will determine the field outside (i.e. to the left of X C A Y.) As usual we seek to map this contour X C A Y on a real t axis. The values of t chosen to correspond to the points C, A are shown. The values for 3 points may always be assigned aribtrarily. X, Y is regarded as one point:- the point at ∞

Suppose the curve AC is taken as the limit of n sides of a equiangular polygon of 4n sides. (fig. ii). The points t_1 ---- t_n (see figure (i)) will clearly have

angles $$= \frac{8n-4}{4n} \times \frac{\pi}{2}$$

$$= \pi - \frac{\pi}{2n}$$

each.

Now the area on the left of XCAY is to map into the upper half of the t plane and the interior angles at t_1---t_n measured on this side $= 2\pi - (\pi - \pi/2n)$

$$= \pi + \frac{\pi}{2n}$$

The corners A, B have angles π each, as shown.

By Schwarz's Theorem the transformation is:

$$\frac{dz}{dt} = K\left[(t-t_1)(t-t_2) - - - - (t-t_n)\right]^{\frac{1}{2n}}$$

According to the method outlined previously we now replace $\left[(t-t_1)(t-t_2)\cdots(t-t_n)\right]^{\frac{1}{n}}$ by $\left\{t+\sqrt{t^2-1}\right\}$

$$\therefore \frac{dz}{dt} = K\left[\frac{1}{2}\left(t+\sqrt{t^2-1}\right)\right]^{\frac{1}{2}} = K\left[\frac{\sqrt{t+1}+\sqrt{t-1}}{2}\right]$$

$$\therefore Z = \frac{K}{3}\left[(t+1)^{3/2} + (t-1)^{3/2}\right] + C$$

when Z = OA, OC, $\therefore$ t = ±1 respectively.

Choose OA, OC so that C = 0.

So: $Z = \frac{K}{3}\left[(t+1)^{3/2} + (t-1)^{3/2}\right]$

when t = -1 we require Z = X; Y = 0; when t = +1: we require Z = iY ; X = o (see fig. ii)

$$\therefore x_c = \frac{K}{3} \times -i2^{3/2} \quad \& \quad iy = \frac{K}{3}2^{3/2}$$

Take K = $\frac{3i}{2^{3/2}}$ then $x_c = 1$ & $y_A = 1$

In our problem then the rounding begins at unit distance from the edge and

$$Z = i2^{-3/2}\left[(t+1)^{3/2} + (t-1)^{3/2}\right] \quad \ldots . (i)$$

For real values t we are on the contour XCAY.

Values $-1 < t < 1$ correspond to points on AC.

Then $(t+1)^{3/2}$ is real and $(t-1)^{3/2} = -i(1-t)^{3/2}$

$$\therefore\ x+iy = i\,2^{-3/2}(t+1)^{3/2} + 2^{-3/2}(1-t)^{3/2} \quad \text{from } (i)$$

$$\therefore\ x = 2^{-3/2}(1-t)^{3/2}$$

$$y = 2^{-3/2}(1+t)^{3/2}$$

$$\therefore\ x^{2/3} + y^{2/3} = 1$$

and thus the rounding curve is a four cusped hypocycloid.

Let $W = U + iV$ be the complex potential function. We adopt V as the potential and suppose $V = 0$ along the conductor. Then the potential diagram is a straight line $V = 0$, and we may have:

$$W = t \quad \cdots\cdots\cdots (ii)$$

This will map the straight line polygon on the real t axis.

By virtue of (i) and (ii) we have: $\quad\cdots\cdots\cdots (iii)$

$$z = i\,2^{-3/2}\left[(W+1)^{3/2} + (W-1)^{3/2}\right]$$

which completely determines the field in the region considered. From (i) and (ii)

$$\frac{dz}{dt} = i\,.\,3\,.\,2^{-5/2}\left[(t+1)^{1/2} + i\,(1-t)^{\frac{1}{2}}\right]$$

$$\therefore\ \frac{dW}{dt} = 1$$

$$\therefore \frac{dW}{dz} = \frac{2^{5/2}}{3i} \cdot \frac{1}{[(1+t)^{1/2} + i(1-t)^{1/2}]}$$

$$\therefore \left|\frac{dW}{dz}\right| = \frac{2^{5/2}}{3} \times \left|\frac{1}{(1+t)^{1/2} + i(1-t)^{1/2}}\right|$$

for $-1 < t < 1$ and real t we are on AC

also $$\left|\frac{dW}{dz}\right| = \frac{2^{5/2}}{3} \cdot \frac{1}{\sqrt{2}} = \frac{4}{3}$$

$\therefore$ Surface density on AC $= \sigma = \frac{1}{4\pi} \times \frac{4}{3} = \frac{1}{3\pi}$

and is therefore constant on the curve.

Also the charge on Ac $= \frac{1}{4\pi}$ (difference in stream value at A and C), now $W = U + iV = t$

$\therefore$ $U = \pm 1$ at A and C respectively.

$\therefore$ Charge on AC $= \frac{1}{4\pi} \times 2 = \frac{1}{2\pi}$

So far we have only considered the case when the limit of $\{(t-t_1)(t-t)\cdots(t-t_n)\}^{1/n}$ is replaced by $\frac{1}{2}(t+\sqrt{t^2-1})$

We decided on this substitution out of analogy to the known case of the circular cylinder.

Let us return to the case of the infinite plane with a cylindrical boss. As before the transformation is

$$\frac{dz}{dt} = \frac{K\{(t-t_1)(t-t_2)\cdots(t-t_n)\}^{1/n}}{(t^2-1)^{1/2}}$$

If the boss is elliptical in shape the solution is known to be

$$\frac{dz}{dt} = \frac{At + B\sqrt{t^2-1}}{(t^2-1)^{\frac{1}{2}}}$$

$\therefore$ for some way of assigning values of t_1, t_2----t_n

$\lim\limits_{n\to\infty} \left[(t-t_1)(t-t_2)\cdots(t-t_n)\right]^{\frac{1}{n}}$ must take the form $At + B\,(t^2-1)^{\frac{1}{2}}$

By using this substitution we may obtain curves of a different shape to round off the ends of conductors.

When the substitution $\frac{1}{2}\left(t+\sqrt{t^2-1}\right)$ is used the resulting curve will usually have its surface density constant when freely charged. This is so because for the proper range of t, $\left|t+\sqrt{t^2-1}\right|$ is independent of t and hence $\left|\frac{dW}{dz}\right|$ may be constant. In the substitution $At + B\,(t^2 - 1)^{1/2}$ this is not the case.

APPENDIX

Jacobian Elliptic Functions.

Suppose:

$$u = \int_0^x \frac{dx}{(1-x^2)\ (1-k^2x^2)}$$

This expression defines u as a function of x.

The inverse, expressing x as a function of u, is defined as the Jacobian sn function, modulus k.

i.e. $x = \mathrm{sn}\, u$.

cn and dn may be defined via the relations:

$$\mathrm{cn}^2 u = 1-\mathrm{sn}^2 u$$

$$\mathrm{dn}^2 u = 1-k^2\mathrm{sn}^2 u.$$

sn, cn, dn are doubly periodic functions having the following periods:

sn u	: 4K, 2iK'
cn u	: 4K, 2K 2iK'
dn u	: 2K, 4iK'

where $K = \displaystyle\int_0^1 \frac{dx}{(1-x^2)\ (1-k^2x^2)}$; $K' = \displaystyle\int_0^1 \frac{dx}{(1-x^2)\ (1-k'^2x^2)}$

$k' = \sqrt{1-k^2}$; k' is termed the co-modulus.

We make use of the following properties of these Jacobian functions:

$$\mathrm{sn}\,(K+u) = \frac{\mathrm{cn}\,u}{\mathrm{dn}\,u} \quad --- (i)$$

$$\mathrm{cn}\,(iu) = \frac{1}{\mathrm{cn}\,(u,k')} \quad --- (ii)$$

$$\mathrm{dn}\,(iu) = \frac{\mathrm{dn}\,(u,k')}{\mathrm{cn}\,(u,k')} \quad --- (iii)$$

where the k' in (ii) and (iii) denote that these resulting elliptic functions have for their modulus k', the co-modulus of the original ones.

For the limiting case where $k = o$

$$\mathrm{sn}\,u = \sin u$$

$$\mathrm{cn}\,u = \cos u$$

$$\mathrm{dn}\,u = 1$$

$$K = \pi/2$$

$$K' \rightarrow \infty$$

When $k = 1$

$$\mathrm{sn}\,u = \tanh u$$

$$\mathrm{cn}\,u = \mathrm{sech}\,u$$

$$\mathrm{dn}\,u = \mathrm{sech}\,u$$

$$K \rightarrow \infty$$

$$K' = \pi/2$$

BIBLIOGRAPHY

(i) THOMSON, J.J. Recent Researches in Electricity and Magnetism.

(ii) SMYTHE, W.R. Static and Dynamic Electricity.

(iii) STRATTON, J.A. Electromagnetic Theory.

(iv) ABRAHAM, M. & BECKER, R. The Classical Theory of Electricity and Magnetism.

(v) ROTHE, R., OLLENDORFF, F., and POHLHAUSEN, K. Theory of Functions as applied to Engineering Problems.

(vi) MILNE-THOMSON, L.M. Theoretical Hydrodynamics.

(1) RICHMOND, H.W. Proc. of the Londn.Math. Soc., Series 2, Vol. 22, 483-494 (1924)

(2) PAGE, W.M. Proc. of the Londn. Math. Soc., Series 2, Vol. 11, 313-328 (1913)

(3) LOVE, A.E.H. Proc. of the Londn. Math. Soc., Series 2, Vol. 22, 337-369 (1924)