Some General Estimates for the Heat Kernel on Symmetric Spaces and Related Problems of Integral Geometry

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Abstract

We use a method of analytic continuation introduced by M. Flensted-Jensen to study the asymptotic behaviour of the heat kernel on noncompact symmetric spaces, for values of the time parameter which are arbitrarily small or arbitrarily large The same method is applied to one case of the inversion problem for the Abel transform. The results are illustrated with explicit computations for $SL(3, R)$.

Résumé

Nous utilisons une méthode de prolongement analytique introduite par M. Flensted-Jensen, pour étudier le comportement asymptotique de la solution fondamentale de l'équation de la chaleur sur les espaces symétriques non-compacts, lorsque le paramètre temps prend des valeurs arbitrairement petites ou arbitrairement grandes. Cette même méthode permet d'étudier un cas du problème d'inversion pour la transformation d'Abel. Les résultats sont illustrés par des calculs explicites pour $SL(3, R)$.

Contents

Introduction

Let G be a connected semisimple matrix group of the noncompact type and K a maximal compact subgroup of G. The G-invariant Laplacian on the symmetric space G/K plays a fundamental role for harmonic analysis on G/K. In this respect, the study of the heat equation is of particular interest, since in addition to establishing connections with other areas of analysis, it provides insights into important questions of spectral analysis on Riemannian manifolds in general (see [4] [4bis] for informative surveys). The search for precise information concerning the behaviour of the fundamental solution of the heat equation on G/K has been the subject of many investigations ([1] [8] [17] [4ter] [4quater] [13bis] [15bis] [15ter] to name only a few). The explicitly known solutions of the heat kernel depend almost invariably on the knowledge of an inversion formula for the Abel transform [1] [3]. As a result, explicit solutions are well known for complex semisimple groups, for real rank 1 groups, for $G = SU(p, q)$ Jean-Philippe Anker [1] [4quater] has provided a solution based on Christopher Meaney's inversion formula for the Abel transform [16]. Inversion formulas have also been determined for $SL(3, R)$

[2], $SU^*(6)$ [3, 10bis], $SU^*(8)$ [3], $E_{6(-26)}$ [3], and for $SL(3, R)/SO(3)$, Patrice Sawyer [20bis] gave an explicit expression for the heat kernel. However, for other real groups of higher rank, the problem remains elusive. Another approach to the heat kernel is provided by M. Flensted-Jensen's method of analytic continuation [7]. Let $\mathcal{G}$ be the Lie algebra of G and $\mathcal{G}_C = \mathcal{G} \otimes C$ the complexification of $\mathcal{G}$, we may view G as a subgroup of a complex Lie group G_C with Lie algebra $\mathcal{G}_C$. the method of analytic continuation identifies the spaces of functions which depend on the double cosets $K\backslash G/K$ with a naturally defined family of functions on G_C. Via suitable orbital integrals on G_C it is possible to translate certain questions of spherical analysis on G into equivalent problems on G_C. In particular, the heat kernel for a noncompact reductive matrix group G has an integral representation in term of the heat kernel on the complex group G_C Using this integral representation, we have sought to establish some general estimates for the heat kernel when the time parameter takes arbitrarily small or arbitrarily large values A critical step in these computations is the use of certain geometric estimates which permit control over the behaviour of the heat kernel on the complex group G_C. These computations prove successful in case the time parameter is small.

For large values of the time parameter, we prove the existence of an asymptotic expansion for the heat kernel in case G is a normal real form of G_C. We also venture to conjecture how the estimates for the coefficients of the expansion could be improved in general.

We have also undertaken a discussion of one aspect of the inversion problem for the Abel transform.

Let $G = KAN$ be an Iwasawa decomposition of G. $\mathcal{A}$ the Lie algebra of A, $R = R(\mathcal{G}, \mathcal{A})$ the restricted root system for the pair $(\mathcal{G}, \mathcal{A})$, $W = W(A)$ the Weyl group, and $m : RMod(W) \to N^r$ the multiplicity function which assigns to each root orbit $W \cdot \alpha$ the dimension of the eigenspace $\mathcal{G}_\alpha$. The work of Beerends [3], Opdam [19] and others focuses on a theory of spherical functions which depends only on a root system R with an arbitrarily assigned multiplicity function $m = (m_1, m_2, \ldots, m_r)$. The Abel transform $\mathcal{T}^m : C_c^\infty(K\backslash G/K) \to C_c^\infty(A; W)$ is an isomorphism between the convolution algebra $C_c^\infty(K\backslash G/K)$ of compactly supported, bi-K-invariant, C^∞ functions on G and the Euclidian convolution algebra $C_c^\infty(A; W)$ of compactly supported, C^∞, W-invariant functions on A. Via restriction to A we may simply view $\mathcal{T}^m$ as a bijective correspondence of $C_c^\infty(A; W)$ onto itself which depends only on the root system R and the multiplicity function m. Then, the search for "shift" operators consist of finding all admissible differential operators D on $C_c^\infty(A; W)$ which lower the multiplicity function in the sense that

$$\mathcal{T}^n = \mathcal{T}^m \circ D \quad \text{with } n_i \leq m_i.$$

The elementary "shifts" which have been obtained have even entries (i.e., $(k_1, k_2, \ldots, k_r) = m - n \equiv (0, 0, \ldots, 0)\ Mod(2)$) and explicit inversion formulas for $\mathcal{T}^m$ have been obtained in cases when m is even or when m reduces to a combination of a rank 1 case and a complex case [3] [8] [20] [16] [13quinto] Clearly, a different approach is needed in the case of a normal real form (all roots have multiplicity 1), and we have shown that analytic continuation is particularly well suited to answer this question. An explicit computation is given for $G = SL(3, R)$.

Chapter 1 represents a pertinent collection of basic geometric principles concerning Riemannian and pseudo-Riemannian symmetric spaces, in addition to a brief review of M. Flensted-Jensen's method. In Chapter 2, we apply the principles of the previous chapter to obtain estimates for the fundamental solution of the heat equation on a Riemannian symmetric space. In Chapter 3 we discuss the problems associated with representing the Abel transform via a measure on the group A and prove a version of Aomoto's theorem which is particularly well suited as recursive method for treating $SL(n, R)$. We also state an inversion formula for the Abel transform when G is a normal real form and illustrate the methods developed with explicit computations in the case $G = SL(3, R)$.

Chapter 1

Geometric Preliminaries

1.1 Reductive Groups

By a Euclidian space $(E, \langle,\rangle)$ we will mean a finite dimensional real vector spaced endowed with a postive definitive bilinear symmetric form $\langle,\rangle$: $E \times E \rightarrow R$. $END(E)$ will denote the space of all R-linear transformations, GL(E) the group of invertible ones, and $Pos(E)$ the cone of symmetric positive definite operators, with transposition (with respect to the inner product $\langle,\rangle$) defined in the usual way:

$$\langle Xu, v\rangle = \langle u, X^t v\rangle, \quad u, v \in E \quad \text{and} \quad X \in END(E)$$

Our ultimate interest is the study of certain question of analysis on real semisimple groups. However, we have adopted Harish-Chandra's point of view, by discussing basic principles and associated formulas in the context of real reductive groups.Some technical difficulties are avoided at a latter stage by assuming that our groups are linear. Thus, unless otherwise stated, G will denote a closed connected subgroup of $GL(E)$ which is stable for the action of the fixed Cartan involution:

$$\sigma(g) = (g^{-1})^t, \quad g \in GL(E).$$

Let $\mathcal{G}$ be the Lie algebra of G then the derived involution $(d\sigma)_e(X) = -X^t, \quad X \in END(E)$ (also denoted by σ when no risk of confusion exists) gives rise to a Cartan decomposition $\mathcal{G} = k + p$ into symmetric and skew-symmetric elements of $\mathcal{G}$ (p and k respectively).

The real vector space $\mathcal{G}$ inherits an inner product from $END(E)$:

$$T_\sigma(X,Y) = -Trace(X\sigma(Y)) \quad X,Y \in \mathcal{G}. \tag{1.1}$$

in addition to a positive semidefinite bilinear symmetric form

$$B_\sigma(X,Y) = -B(X,\sigma(Y)) \quad X,Y \in \mathcal{G} \tag{1.2}$$

where $B(X,Y) = Trace(ad(X), ad(Y)), X, Y, \in \mathcal{G}$ is the Killing form of $\mathcal{G}$.

Lemma 1.3: Let G be a connected closed σ-stable subgroup of $GL(E)$ then G is a reductive Lie group and it's Lie algebra $\mathcal{G}$ is the direct sum of two σ stable ideals $\mathcal{Z}$ and $\mathcal{G}'$ with $\mathcal{Z}$ = Centre of $\mathcal{G}$, and $\mathcal{G}'$ the semisimple commutator ideal $[\mathcal{G}, \mathcal{G}]$.

Proof ([13ter]): Clearly G is a Lie group, in fact it is a Lie group without the connectedness assumption ([11], Thm 2.3). With reference to the euclidean space $(\mathcal{G}, T_\sigma)$ we have $ad(X)^t = -ad(\sigma(X)), X \in \mathcal{G}$. Thus, $B_\sigma(X, X) = Trace(ad(X)ad(X)^t) \geq 0$ with equality holding if and only if $ad(X) = 0$, that is, B_σ vanishes precisely on $\mathcal{Z}$. Note that $\mathcal{Z}$ is σ stable since $ad(Z) = 0$ iff $ad(Z)^t = 0$.

Writing $\mathcal{G}$ as a direct sum of T_σ-orthogonal subspaces we have $\mathcal{G} = \mathcal{Z} \dotplus \mathcal{G}'$. σ is an isometry of $T_\sigma(,)$ and so $\mathcal{G}'$ is also σ stable. It is easy to see that $[\mathcal{G}, \mathcal{G}] \subseteq \mathcal{G}'$ and in particular $\mathcal{G}'$ is a subalgebra of $\mathcal{G}$. We have already seen that B_σ (hence B) is a nondegenerate form on $\mathcal{G}'$, but the restriction of B to $\mathcal{G}'$ is simply the Killing form of $\mathcal{G}'$. Consequently $\mathcal{G}'$ is a semisimple subalgebra of $\mathcal{G}$, and the conclusion follows by observing that $\mathcal{G}' = [\mathcal{G}', \mathcal{G}'] \subset [\mathcal{G}, \mathcal{G}] \subset \mathcal{G}'$.
∎

We refer to Helgason's two volumes [11,12] for most of the basic concepts and notations of harmonic analysis on homogenous spaces. Let G be a Lie group with closed subgroups A and B. We will regard $C^\infty(A\backslash G/B)$ interchangeably as the space of the left-A right-B invariant C^∞ functions on G, the space of left-A invariant C^∞ functions on (G/B), or as the space of right-B invariant C^∞ functions on $(A\backslash G)$. Some care must be exercised in treating functions of compact support. $C_c^\infty(G/B)$ will be regarded as the right-B invariant C^∞

functions on G which have compact support modulo B.

Let G be a connected reductive Lie group with Lie algebra $\mathcal{G}$, τ an involutive automorphism of G (not necessarily a Cartan involution), $\mathcal{G}^\tau$ the subalgebra consisting of those elements in $\mathcal{G}$ which are fixed pointwise by τ, $G[\tau]$ the analytic subgroup of G corresponding to τ (i.e the connected component of the fixed point group G^τ), and assume that there exists a trace form $\mathcal{Q}(\cdot,\cdot)$ on $\mathcal{G}$ satisfying:

$$\left.\begin{array}{l} 1.\ \mathcal{Q}(ad(X)Y, Z) + \mathcal{Q}(Y, ad(X)Z) = 0 \quad X, Y, Z \in \mathcal{G}. \\ 2.\ \mathcal{Q} \text{ is nondegenerate bilinear symmetric form on } \mathcal{G}. \\ 3.\ \mathcal{Q} \text{ is } \tau \text{ invariant.} \end{array}\right\} \quad (1.4)$$

(These conditions are automatically satisfied by $\mathcal{Q}$ = Killing form of a semisimple Lie algebra $\mathcal{G}$.)

The quotient space $G/G[\tau]$ inherits a G invariant pseudo-Riemannian metric in the following way:

Let $o \in G/G[\tau]$ denote the coset $\{G[\tau]\}$ (henceforth called the origin of $G/G[\tau]$) and let $t_g : x \mapsto g \cdot x$ designate translation on $G/G[\tau]$ by elements of G. The Lie algebra $\mathcal{G}$ is the direct sum $\mathcal{G}^\tau \oplus q$ of the ± 1 eigenspaces for τ. Using suitable neigborhoods U and V of the origin in q and in $G/G[\tau]$ we have a local diffeomorphism $Exp : q \to G/G[\tau]$ given by $Exp(X) = \exp(X) \cdot o \in G/G[\tau]$ which identifies q with the tangent space at the origin $o \in G/G[\tau]$ in accordance with:

$$X_0 f = \lim_{t \to 0} \left[\frac{f(\exp(tX) \cdot o) - f(o)}{t} \right], \quad f \in C^\infty(G/G[\tau]), \quad X \in q \qquad (1.5)$$

The τ invariance of $\mathcal{Q}(,)$ allows us to conclude that the -1 eigenspace q is contained in the orthogonal complement of $\mathcal{G}^\tau$ (with respect to $\mathcal{Q}(,)$) and a dimension argument shows that $q = (\mathcal{G}^\tau)^\perp$. In particular the restriction of $\mathcal{Q}(,)$ to q and $\mathcal{G}^\tau$ are nondegenerate symmetric bilinear forms. Since $\mathrm{Ad}(G[\tau])$ maps q onto itself ($[\mathcal{G}^\tau, q] \subseteq q$) and $\mathcal{Q}|q$ is $\mathrm{Ad}(G[\tau])$-invariant (see condition 1.4(1)). The pseudo Riemannian metric can be defined (consistently) by group translation in accordance with

$$\langle dt_g(X), dt_g(Y) \rangle_{g\,o} = \langle X_0, Y_0 \rangle_0 = \mathcal{Q}(X, Y), \quad X, Y \in q. \qquad (1.6)$$

Returning to Lemma 1.3 we have:

Lemma 1.7: Let G be a connected closed σ-stable subgroup of $GL(E)$ of the noncompact type ($p \neq (0)$) and let G^σ be the subgroup of G consisting of elements which are fixed pointwise by σ then:

1. the mapping $p \times G^\sigma \to G$ given by $(x, k) \mapsto \exp(x)k$ is a diffeomorphism onto G
2. G^σ is connected and maximal compact in G
3. G/G^σ is the direct product of a Euclidian space (with Euclidian group

of motions) and a symmetric space of the noncompact type with a semisimple group of motions.

Proof ([13ter]): G is connected and hence generated by exponentials of a neighborhood of the origin in $\mathcal{G}$. In particular G is contained in the connected component $GL^+(E)$ of $GL(E)$.

The Cartan decomposition $GL^+(E) = Pos(E) \cdot U$ ($U \approx SO(n)$ the maximal compact subgroup fo $GL^+(E)$) allows us to write every $g \in G$ uniquely as $g = \exp(x)k$, with x a symmetric operator and $k \in U$. We may establish (1) by showing that $x \in \mathcal{G}$ and $k \in G^\sigma$. To do so, we consider the analytic subgroups G_{ss} and Z_0 of G which correspond to the ideals $\mathcal{G}' = [\mathcal{G}, \mathcal{G}]$ and $\mathcal{Z}$. of $\mathcal{G}$. The product map $Z_0 \times G_{ss} \rightarrow G$ defines a covering map ($Z_0 \cap G_{ss} \cap G^\sigma \neq (e)$ in general) onto the connected subgroup $Z_0 G_{ss}$ of G.

Clearly $G = Z_0 G_{ss}$ ($\mathcal{G}$ is also the Lie algebra of $Z_0 G_s$) and it suffices to establish the decompositions $G_{ss} = \exp(p \cap \mathcal{G}') \cdot (G_{ss} \cap G^\sigma)$ and $Z_0 = \exp(p \cap \mathcal{Z}) \cdot (Z_0 \cap G^\sigma)$. The first of these is well known (G_{ss} is semisimple with finite center).

Therefore, let $g = \exp(x)k$ be an element of the analytic subgroup Z_0 and let us show that $x \in \mathcal{Z} \cap p$. Clearly $\exp(2x) = g\sigma(g^{-1}) \in Z_0$. Writing E as a direct sum of irreducible G-submodules $E = \oplus_{i=1}^N E_i$ (G is reductive [13]), we have $\exp(2x) = Diag(e^{\lambda_1} I(k_1), \ldots, e^{\lambda_N} I(k_N))$, where $\lambda_1, \lambda_2, \ldots, \lambda_N \in R$ and $I(k)$ denotes the $k \times k$ identity matrix. The fact that Z_0 is connected allows us to conclude that $\exp(2x)$ lies in the one parameter subgroup $\exp(Diag(\lambda_1 tI(k_1), \ldots, \lambda_N tI(k_N))) = \exp(2tx)$ of Z_0. Hence $x \in \mathcal{Z} \cap p$ and $\exp(x) \in Z_0$.

The remaining statements are easily established.

2) A standard argument shows that G^σ is maximal compact (powers of elements of the form $\exp(x)k$ with $x \neq 0, x \in p$ form unbounded sequences), and the map $G \to G^\sigma$ given by $g \mapsto [g\sigma(g^{-1})]^{-1/2} \cdot g$ is well-defined and continuous so that G^σ is connected.

3) A σ-invariant bilinear symmetric form B_0 may be chosen on $\mathcal{Z}$ so as to satisfy $-B_0(x, \sigma(x)) \rangle 0$ for $x \in \mathcal{Z}, x \neq 0$. If B denotes the Killing form of $\mathcal{G}'$ then the direct sum $\mathcal{Q} = B \oplus B_0$ defines a trace form on $\mathcal{G} = \mathcal{G}' \dotplus \mathcal{Z}$ which satisfies the conditions in 1.4 and is such that its restriction to p is positive definite. We then view G/G^σ as a Riemannian space with a G-invariant metric defined by $\mathcal{Q}$. Every point of G/G^σ has a unique expression of the form $\exp(Z)\exp(x) \cdot o$ where $Z \in \mathcal{Z} \cap p$ and $x \in \mathcal{G}' \cap p$. If $g \in G$ is arbitrary then we may write $g = ag_1 \in Z_0 G_{ss}$ (in a non-unique way) and $a = \exp(A)k$ with $A \in \mathcal{Z} \cap p$ and $k \in Z_0 \cap G^\sigma$. The group action on G/G^σ takes the form

$$t_g(\exp(Z)\exp(x) \cdot o) = \exp(A)kg_1 \cdot \exp(Z)\exp(x) \cdot o,$$

but $Ad(G)$ acts trivially on $\mathcal{Z} \cap p$, so that

$$t_g(\exp(Z)\exp(x) \cdot o) = \exp(Z + A) \cdot g_1 \exp(x) \cdot o,$$

as claimed. ■

1.2 Some Integral Formulas

A decomposition theorem ([18], Theorem 5) for reductive Lie groups in terms of Lie triple subsystems is an essential ingredient in establishing the process described by M. Flensted-Jensen as analytic continuation [7] [7bis] [7ter]. In what follows we derive some differential and integral formulas in connection with particular cases of Mostow's decomposition.

Let G be a closed connected σ-stable subgroup of $GL(E)$, $\mathcal{G}$ its Lie algebra and $\mathcal{G} = k + p$ its Cartan decomposition. As in the proof of Lemma 1.7, we may extend the Killing form of the semisimple ideal $\mathcal{G}' = [\mathcal{G}, \mathcal{G}]$ to a nondegenerate trace form B on $\mathcal{G}$ (see conditions 1.4) such that $B_\sigma(X, Y) = -B(X, \sigma(Y))$ is an inner product on $\mathcal{G}$. Clearly, the restriction of B to p is an $Ad(G^\sigma)$ invariant inner product on p which gives $G/G^\sigma \cong Exp(p)$ the structure of a Riemannian (partly Euclidian) globally symmetric space.

The following observations are immediate consequences of the commutation relations; $[p, p] \subseteq k$, $[k, p] \subseteq p$, $[k, k] \subseteq k$, and of the positivity of B_σ:

Ob 1) If $Z \in p$ then $ad(Z)^2$ is a (B-symmetric) positive semidefinite transformation of p into itself.

Ob 2) For any $Z \in p$, the absolutely convergent power series

$$S(Z) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} ad(Z)^{2k}\Big|_p \tag{1.8}$$

is a (B-symmetric) positive definite transformation of p onto itself with eigenvalues bounded below by 1.

The full statement of Ob2 requires some explanation:

Reductive Lie algebras represent only a slight generalization of semisimple ones, and structure theorems continue to hold with only trivial modifications. Thus, if $\mathcal{A}$ is a maximal abelian subspace of p, we may continue to speak of the root space decomposition for the pair $(\mathcal{G}, \mathcal{A})$. As it turns out, the roots $R(\mathcal{G}, \mathcal{A})$ are simply the roots (in the ordinary sense) for the pair $(\mathcal{G}', \mathcal{A} \cap \mathcal{G}')$ viewed as linear functionals on $\mathcal{A}$ which vanish on $\mathcal{Z} \cap p$ (by necessity $\mathcal{Z} \cap p \subset \mathcal{A}$ for any maximal abelian subspace $\mathcal{A}$ in p).

We formalize our comments in the following:

Lemma 1.9: Let G be a connected closed σ-stable subgroup of $GL(E)$, $\mathcal{G}$, k and p as usual, then:

1. every maximal abelian subspace $\mathcal{A}$ of p decomposes uniquely as $\mathcal{A} = \mathcal{A}' \oplus \tau$, where $\tau = p \cap \mathcal{Z}$ and $\mathcal{A}' = \mathcal{A} \cap [\mathcal{G}, \mathcal{G}]$ is maximal abelian in $p \cap \mathcal{G}'$.

2. any two maximal abelian subspaces of p are $Ad(G^\sigma)$ conjugate.

3. given a maximal abelian subspace $\mathcal{A}$ of p there is a root space decomposition $\mathcal{G} = \mathcal{A} \oplus m \oplus \sum_{\alpha \in R} \mathcal{G}^\alpha$, where R is the set of distinct nonzero

roots for the pair $(\mathcal{G}, \mathcal{A})$ and m is the centralizer of $\mathcal{A}$ in k (necessarily containing $\mathcal{Z} \cap k$).

Proof ([13ter]):

1. Clearly, a maximal abelian subspace $\mathcal{A}$ of p must contain $\tau = p \cap \mathcal{Z}$. If $H \in \mathcal{A}$ then we may write $H = H_1 + H_2$ with $H_1 \in p' = p \cap \mathcal{G}'$ and $H_2 \in \tau$, but then $H_1 = H - H_2 \in \mathcal{A}$, hence $H_1 \in \mathcal{A}'$. If $\mathcal{A}'$ is not maximal abelian in p' then we may linearly adjoin an element $x_0 \in p'$ which commutes with both $\mathcal{A}'$ and τ contradicting the maximality of $\mathcal{A}$.
2. For any pair $\mathcal{A}_1 = \mathcal{A}'_1 \oplus \tau$ and $\mathcal{A}_2 = \mathcal{A}'_2 \oplus \tau$ of maximal abelian subspaces of p, the "semisimple parts" $\mathcal{A}'_1$ and $\mathcal{A}'_2$ are $Ad(G^\sigma \cap G_{ss})$ conjugate, since τ is acted upon trivially by $Ad(G^\sigma)$ the conclusion is immediate.
3. The linear family $\{ad(H) \mid H \in A\}$ is a commuting family of B_σ symmetric transformations of $\mathcal{G}$ (hence simultaneously diagonalizable) and we have a root space decompostion $\mathcal{G} = \mathcal{G}^0 \oplus \sum_{\alpha \in R} \mathcal{G}^\alpha$. If $x \in \mathcal{G}^0$ then $x = x_1 + x_2$ with $x_1 \in k$ and $x_2 \in p$. The commutation relations for k and p show that $x_1 \in \mathcal{G}^0 \cap k$ and $x_2 \in \mathcal{G}^0 \cap p = \mathcal{A}$. Since $\mathcal{Z} \subset \mathcal{A} \oplus m$, it is clear that the root spaces $\mathcal{G}^\alpha$, $\alpha \in R$, are entirely contained in the semisimple ideal $\mathcal{G}'$. Naturally any root $\alpha \in R$ must vanish on τ and we may view R as the root system for the pair $(\mathcal{G}', A')$. ■

Suitable interpretations imposed by Lemma 1.9 lead to the corresponding versions of the polar and Iwasawa decompositions:

$$G = K\overline{A^{+}}K \quad \text{and} \quad G = KAN \text{ respectively.}$$

Note that the Weyl group $W = N_K(A)/C_K(A)$ acts trivially on τ. Thus, a chamber $\mathcal{A}^+$ in $\mathcal{A}$ must be interpreted as $\mathcal{A}^+ = (\mathcal{A}')^+ \oplus \tau$ where $(\mathcal{A}')^+$ is a Weyl chamber in $\mathcal{A}'$.

Returning to Ob2, we see that any $Z \in p$ is contained in some maximal abelian subspace $\mathcal{A}$ of p and the roots of $ad(Z)^2|_p$ are 0 (with multiplicity $\dim(\mathcal{A})$) or $\alpha(Z)^2$ (with multiplicity $m_\alpha = \dim \mathcal{G}^\alpha$). If the "semisimple" part of Z is not regular, then some $\alpha(Z)$ may equal zero. In any event, the roots of $S(Z)$ are of the form $\frac{Sh(\alpha(Z))}{\alpha(Z)}$ if $\alpha(Z) \neq 0$ or 1. (Recall that $\frac{Sh(\lambda)}{\lambda} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}\lambda^{2k}$).

Definition 1.10: Let $\{\gamma_1(t)\}$ and $\{\gamma_2(t)\}$ be two smooth curves in G/G^σ (see Lemma 1.7) defined on an open interval I containing $t = 0$ We will say that the two curves are equivalent at $t = 0$ and write:

$$\gamma_1 \approx \gamma_2 \quad \text{at} \quad t = 0.$$

in case γ_1 and γ_2 have the same tangent vector at $t = 0$ (in particular the two curves must pass through the same point in G/G^σ at $t = 0$).

Remark: Infinitesimal equivalence is a handy notational convention which simplifies the computation of Jacobians in G and G/G^σ. In G, a smooth curve $t \mapsto x(t) \in \mathcal{G}$ defines a smooth curve $\gamma(t) = \exp(x(t))$, and for ϵ small we have $\exp(x(t+\epsilon)) \approx \exp(x(t))\exp(\epsilon V)$ at $\epsilon = 0$, for a uniquely determined $V \in \mathcal{G}$. Other computations in G/G^σ derive from the folowing rules:

Lemma 1.11([11], Ch. II, Theorem 1.7, Lemma 1.8.i): Let $A, B \in \mathcal{G}$, then the following equivalences hold at $\epsilon = 0$:

1. $\exp(A + \epsilon B) \approx \exp(A)\exp(\epsilon L(A)B)$, where
$$L(A) = \left(\frac{I - \exp(-ad(A))}{ad(A)}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k ad(A)^k}{(k+1)!}$$
2. $\exp(\epsilon A + \epsilon B) \approx \exp(\epsilon A)\exp(\epsilon B)$

Proof: The C^∞ curve $C(t) = \exp(-A)\exp(A + tB)$ passes through the identity of the connected group G at $t = 0$. Hence, $C(t)$ is equivalent to $\exp(tV)$ at $t = 0$ for a uniquely determined $V \in \mathcal{G}$. Thus $\exp(A + tB) \approx \exp(A)\exp(tV)$ at $t = 0$. Differentiating at $t = 0$ gives
$$\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k=0}^{n-1} A^{n-1-k} B A^k = \exp(A)V$$

The left-hand side may be rewritten using the identity:
$$BA^k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} A^{k-j} ad(A)^j(B)$$
(see Jacobson [13(p38)]). After rearranging terms using $\sum_{r=j}^{n-1} \binom{r}{j} = \binom{n}{j+1}$ we obtain:
$$\begin{aligned}
\exp(A) \cdot V &= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{j=0}^{n-1} (-1)^j \binom{n}{j+1} A^{n-1-j} ad(A)^j(B) \\
&= \exp(A)L(A)(B),
\end{aligned}$$

which shows that $V = L(A)(B)$. The second equivalence follows from the first (ignoring terms of order ϵ^2 or higher). ■

Lemma 1.12: Let G be a connected closed σ-stable subgroup of $GL(E)$, $\mathcal{G}$, p, $\mathcal{G}^\sigma$ and G/G^σ as before. Then:

1. for $Z, A \in p$ the following equivalence holds at $\epsilon = 0$ in G/G^σ:

$$\exp(Z + \epsilon A) \cdot o \approx \exp(Z) \exp(\epsilon S(Z)(A)) \cdot o \in G/G^\sigma,$$

   where $S(Z) = \left.\frac{Sh(ad(Z))}{ad(Z)}\right|_p$.

2. given $Y, X \in p$, let $Z(t)$ be the uniquely defined C^∞ curve in p satisfying

$$\exp(Z(t)) \cdot o = \exp(tY) \exp(X) \cdot o \in G/G^\sigma, t \in R$$

   then $Z(t)$ satisfies the differential equation

$$S(Z)\dot{Z} = Ch(ad(Z))Y \qquad (1.13)$$

   with initial condition

$$Z(0) = X.$$

These results are due to Mostow [18]. They are easily obtained by applying the infinitesimal rules outlined in Lemma 1.11, and by observing that for $g \in G$ the point $g \cdot o \in G/G^\sigma$ is uniquely expressed as $[g\sigma(g)^{-1}]^{1/2} \cdot o$

Remark: Lemma 1.12 (1) expresses the differential of the exponential map $Exp : p \to G/G^\sigma$ as:

$$d(Exp)_Z(A) = d(t_{\exp(Z)})(S(Z)A), \quad A, Z \in p$$

Using the G invariance of the metric on G/G^σ (see 1.6) one easily obtains the classical integration formula

$$\int_{G/G^\sigma} f(\dot{g})\, d\mu(\dot{g}) = c \cdot \int_{p} f(\exp(Z) \cdot o) J(Z)\, dZ.$$

where

$$f \in C_c^\infty(G/G^\sigma), \quad J(Z) = \det(S(Z)|_p).$$

dZ is the euclidian measure on p. and c is a positive constant determined by the convention that normalizes the invariant measure μ on G/G^σ (see Helgason [12]).

The following "evolution" equation will be used in conjunction with the differential equation 1.13 to establish estimates for the heat kernel on a real semisimple Lie group.

Lemma 1.14: Let G be a closed connected σ-stable subgroup of $GL(E)$, $\mathcal{G}$. k. p. G/G^σ as before and let $t \mapsto Z(t)$ be a smooth curve in p defined for $t \in R$ Then

$$J(Z(t)) = J(Z(t_0)) \exp\left(\int_{t_0}^{t} \kappa(\tau)\, d\tau\right),$$

where

$$\begin{aligned} J(Z) &= \det(S(Z)|_p), \quad \text{and} \\ \kappa(\tau) &= \mathit{Trace}\left[S(Z(\tau))^{-1} \cdot \frac{d}{d\tau} S(Z(\tau))\right] \end{aligned}$$

Proof: More generally, consider a smooth operator valued map $t \mapsto A(t) \in$

$END(V)$, where V is a euclidian space. The logarithmic growth of its determinant is easily described as follows:

Choose a nonzero multilinear alternating n-form $\omega \in \Lambda^n(V^*)$ (ie a volume form on V) and let $\{e_1, e_2, \ldots, e_n\}$ be some basis for V. then

$$\det(A(t))\omega(e_1, \ldots, e_n) = \omega(A(t)e_1, \ldots, A(t)e_n)$$

Differentiating both sides with respect to t gives

$$\begin{aligned}
\frac{d}{dt}\det(A(t))\omega(e_1, \ldots, e_n) &= \sum_{k=1}^{n} \omega(A(t)e_1, \ldots, \dot{A}(t)e_k, \ldots, A(t)e_n) \\
&= \det(A(t)) \sum_{k=1}^{n} \omega(e_1, \ldots, A^{-1}\dot{A}(t)e_k, \ldots, e_n) \\
&= \det(A(t))\, Trace[A^{-1}\dot{A}(t)]\omega(e_1, \ldots, e_n)
\end{aligned}$$

If $A(t)$ is positive definite for each $t \in R$ then the above equation is readily integrated to give

$$\det(A(t)) = \det(A(t_0)) \exp\left(\int_{t_0}^{t} Trace[A(\tau)^{-1}\dot{A}(\tau)]\, d\tau\right)$$

as claimed. ■

Definition 1.15: A Lie triple subsystem of p is a linear subspace $\mathcal{L} \subset p$ satisfying the commutation relation $[\mathcal{L}, [\mathcal{L}, \mathcal{L}]] \subset \mathcal{L}$.

Now consider a second involutive automorphism τ of G and assume that τ commutes with the Cartan involution σ. Let G^τ be the fixed point group for

τ and note that G^τ is reductive with Lie algebra $\mathcal{G}^\tau$ (Lemma 1.3 applies to G^τ without the connectedness assumption). We have an eigenspace decomposition $\mathcal{G} = \mathcal{G}^\tau \oplus q$ for τ. Since τ commutes with σ we also have the direct sum decomposition:

$$\mathcal{G} = \mathcal{G}^\sigma \cap \mathcal{G}^\tau + \mathcal{G}^\sigma \cap q + \mathcal{G}^\tau \cap p + q \cap p \tag{1.16}$$

It is clear that both $\mathcal{G}^\tau \cap p$ and $q \cap p$ are Lie triple subsystems of p. In fact it is worth noting that $\mathcal{G}^\tau \cap p$ and $q \cap p$ have perfectly symmetrical roles in p, since $q \cap p$ may be viewed as $\mathcal{G}^\nu \cap p$, where $\nu = \sigma\tau = \tau\sigma$ is an involutive automorphism of G with Lie algebra $\mathcal{G}^\nu = \mathcal{G}^\tau \cap \mathcal{G}^\sigma + q \cap p$.

Under these conditions we have the following version of Mostow's decomposition theorem ([18], Theorem 5).

Lemma 1.17: Let G be a connected closed σ-stable subgroup of $GL(E)$, τ an involutive automorphism of G which commutes with σ, and $\mathcal{G} = \mathcal{G}^\tau \oplus q$ the eigenspace decomposition of $\mathcal{G}$ relative to τ then:

1. the map $\phi : (\mathcal{G}^\tau \cap p) \times (q \cap p) \to G/G^\sigma$, $\phi(Y, X) = \exp(Y)\exp(X) \cdot o$ is a diffeomorphism onto
2. there exists a constant $C > 0$ such that for any $F \in C_c^\infty(G/G^\sigma)$ we have:

$$\int_{G/G^\sigma} F(\dot{g})\, d\mu(\dot{g}) =$$

$$C \int\limits_{(\mathcal{G}^\tau \cap p) \times (q \cap p)} F(\exp(Y)\exp(X)\cdot o) J_1(Y) K_2(X)\, dY\, dX,$$

where

$$\begin{aligned} J_1(Y) &= \det(S(Y)|_{\mathcal{G}^\tau \cap p}), \quad \text{and} \\ K(X) &= \det(S(X)|_{q\cap p}) \det(Ch(ad(X))|_{\mathcal{G}^\tau \cap p}) \end{aligned}$$

Proof ([7bis] Theorem 2.6, [15] Theorem 1): ϕ is a map between simply connected spaces of the same dimension. To show (1) it suffices to show that ϕ is regular at every point $(Y, X) \in (\mathcal{G}^\tau \cap p) \times (q \cap p)$, and that its image is closed in G/G^σ. The following equivalences hold at $\epsilon = 0$:

$$\begin{aligned} &\exp(Y + \epsilon A)\exp(X + \epsilon B)\cdot o \\ &\quad\approx \exp(Y)\exp(\epsilon L(Y)A)\exp(X)\exp(\epsilon L(X)B)\cdot o \\ &\quad\approx \exp(Y)\exp(X)\exp(\epsilon Q)\cdot o \\ &\quad\approx \exp(Y)\exp(X)\exp(\epsilon T(Y, X)(A \oplus B))\cdot o, \end{aligned}$$

where $T(Y,X)(A \oplus B) = \frac{1}{2}(Q - \sigma Q) \in p$, and $Q = Ad(\exp(-X))L(Y)A + L(X)B$ (see Lemma 1.11(1))

Simplifying $\frac{1}{2}(Q - \sigma Q)$ gives:

$$T(Y,X) = \begin{bmatrix} Ch(ad(X))S(Y) & 0 \\ Sh(ad(X))\left[\dfrac{Ch(ad(Y)) - I}{ad(Y)}\right] & S(X) \end{bmatrix}$$

Hence $\det(T(Y,X)) = J_1(Y)K(X) > 0$, which shows that ϕ is regular. As will be shown later (see 2.12), there is an estimate for the Riemannian distance $d(0, \exp(Y)\exp(X)\cdot o)$ given by

$$\sqrt{|X|^2 + |Y|^2} \leq |\exp(Y)\exp(X)\cdot o|\,.$$

Suppose that $\exp(Z)\cdot o \in Closure\{\phi(Y,X)|Y \in G^\tau \cap p. X \in q\cap p\}$, then there exists a sequence $(Y_n, X_n) \in (\mathcal{G}^\tau \cap p) \times (q \cap p)$ such that $\exp(Y_n)\exp(X_n)\cdot o = \exp(Z_n)\cdot o$ converges to $\exp(Z)\cdot o$.

Since $\{Z_n\}_{n=1}^\infty$ is a Cauchy sequence $\Rightarrow$ $\{Y_n\}_{n=1}^\infty$ and $\{X_n\}_{n=1}^\infty$ are also Cauchy on account of the estimate given above. Hence $Y_n \to Y_0$ and $X_n \to X_0$, which shows that $\exp(Y_0)\exp(X_0)\cdot o = \exp(Z)\cdot o$. Thus the image of ϕ is also closed. ∎

Remark: Let H denote the connected component of G^τ, then $Ad(H \cap G^\sigma)$ (the maximal compact subgroup of H) acts as a group of isometries on $q \cap p$ and $\mathcal{G}^\tau \cap p$. If $F \in C_c^\infty(G)$ is a right-G^σ invariant function. then 1.17(b) can be expressed as:

$$\int_G F(g)\,dg = C \int_H \int_{q\cap p} F(h\exp(X))K(X)\,dX\,dh. \tag{1.18}$$

where dh is a Haar measure on $H = G[\tau]$ ($K(X)$ is $Ad(H \cap G^\sigma)$-invariant!).

Repeating the above argument by induction we obtain:

Lemma 1.19: Let G be a connected closed σ-stable subgroup of $GL(E)$ and $G = G_1 \supset G_2 \supset \cdots \supset G_n$ a descending sequence of closed connected subgroups with corresponding Lie algebras $\mathcal{G} = \mathcal{G}_1 \supset \mathcal{G}_2 \supset \cdots \supset \mathcal{G}_n$, such that $\mathcal{G}_{i+1}$ is the fixed point set in $\mathcal{G}_i$ of an involutive automorphism τ_i of $\mathcal{G}_i$ which commutes with σ. Then, for $F \in C_c^\infty(G/K)$ we have:

$$\begin{aligned} \int_G F(g)\,dg &= C \int_{G_n} \int_{q_{n-1}\cap p_{n-1}} \cdots \int_{q_1 \cap p_1} F(g_n \exp(Y_n) \cdots \exp(Y_1)) \\ & \quad \prod_{i=1}^{n-1} K_i(Y_i)\, dY_1 \ldots dY_{n-1}\, dg_n, \end{aligned}$$

where $\mathcal{G}_i = k_i \oplus p_i$ is the Cartan decomposition of $\mathcal{G}_i$ and $\mathcal{G}_i = \mathcal{G}_{i+1} \oplus q_i$ is its ± 1 eigenspace decomposition with respect to τ_i.

Note: When G is a complex group and τ is a C-linear involution of $\mathcal{G}$ which commutes with σ then

$$\begin{aligned} K(X) &= \det(S(X)|_{q\cap p}) \det(Ch(ad(X))|_{\mathcal{G}^-} \\ &= \det(S(X)Ch(ad(X)|_{q\cap p}) \\ &= \det(S(2X)|_{q\cap p}) \end{aligned}$$

(see Lemma 1.29).

1.3 Analytic Continuation

In this section we review M. Flensted-Jensen's method outlining its main constructions. The explanations are deliberately brief and we refer the reader to M. F. Jensen's original paper [7] [7ter] for a full account as well as for most proofs.

Let L be a closed connected σ-stable subgroup of $GL(E)$, $\mathcal{L}$ its Lie algebra, τ an involution automorphism of L which commutes with σ, $\mathcal{L} = \mathcal{L}^{\sigma} \oplus \mathcal{L}^{-}$ the Cartan decomposition of $\mathcal{L}$, and $\mathcal{L} = \mathcal{L}^{\tau} \oplus q$ the ± 1 eigenspace decomposition with respect to τ. If $L[\sigma]$ stands for the analytic subgroup of L which corresponds to $\mathcal{L}^{\sigma}$ (ie. the maximal compact subgroup of L determined by σ) then we can state Lemma 1.17 by saying that there is a unique decomposition:

$$L = \exp(\mathcal{L}^{\tau} \cap \mathcal{L}^{-}) \exp(q \cap \mathcal{L}^{-}) \cdot L[\sigma] \tag{1.20}$$

Let $L[\tau]$ (respectively $L[\nu], \nu = \sigma\tau = \tau\sigma$) denote the analytic subgroup of L corresponding to $\mathcal{L}^{\tau}$ (resp. $\mathcal{L}^{\nu} = \mathcal{L}^{\sigma} \cap \mathcal{L}^{\tau} \oplus q \cap \mathcal{L}^{-}$), $\mathcal{A}_0$ a maximal abelian subspace of $q \cap \mathcal{L}^{-}$, W the Weyl group associated with the pair $(\mathcal{L}^{\nu}, \mathcal{A}_0)$ and $\mathcal{A}_0^{+}$ a Weyl chamber in $\mathcal{A}_0$. The polar decomposition $L[\nu] = (L[\tau] \cap L[\sigma]) \cdot \overline{A_0^{+}} \cdot (L[\tau] \cap L[\sigma])$ combined with the unique decomposition 1.20 gives a very precise description of the double coset space $L[\tau] \backslash L / L[\sigma]$.

Theorem 1.21 ([7], Theorem 4.1):

1. $L = L[\tau] \overline{A_0^{+}} L[\sigma]$, that is, for every $g \in L$ there exists a *unique* $a \in \overline{A_0^{+}}$ such that $g \in L[\tau] \cdot a \cdot L[\sigma]$.

2. the restriction mapping $C^{\infty}(L[\tau] \backslash L / L[\sigma]) \rightarrow C^{\infty}(A_0)$ onto the space of C^{∞}-W-invariant functions on A_0 is a bijective correspondence (see [12], Ch. II, Theorem 5.8).

The process of analytic continuation depends on two crucial identifications and the use of theorem 1.21. The first of these consists in identifying a connected reductive linear group G with the pseudo-Riemannian symmetric space $G\times G/\Delta(G)$, where $\Delta(G)$ is the diagonal subgroup. The bijective map $\kappa : G\times G/\Delta(G) \to G$ given by $\kappa : (x,y)\Delta(G) \mapsto xy^{-1}$ is a diffeomorphism which allows us to identify the corresponding function spaces $C^\infty(G)$ and $C^\infty(G\times G/\Delta(G))$ via $f \mapsto f^\kappa$ with $f^\kappa(x,y) = f(xy^{-1})$.

Assume that G satisfies the hypothesis of Lemma 1.3 and let K denote its maximal compact subgroup (ie. $K = G[\sigma]$). In the context of 1.21 we have $L = G\times G$. the Cartan involution is $\sigma(x,y) = (\sigma(x),\sigma(y))$. and the involutive automorphism τ is $\tau(x,y) = (y,x)$. $x,y \in G$. Note that σ and τ commute. and that $L[\tau] = \Delta(G)$, $L[\sigma] = K\times K$.

Lemma 1.22 (Helgason [12] Theorem 5.7 and Ch. II, Flensted-Jensen [7] p. 122): The diffeomorphism $\kappa : G\times G/\Delta(G) \to G$ determines:

1. a bijection $f \mapsto f^\kappa$ between $C^\infty(K\backslash G/K)$ (or $C_c^\infty(K\backslash G/K)$) and $C^\infty(K\times K\backslash G\times G/\Delta(G))$ (resp. $C_c^\infty(K\times K\backslash G\times G/\Delta(G))$, where $f^\kappa(x,y) = f(xy^{-1})$, $x,y\in G$).

2. a bijection $D \mapsto D^\kappa$ between the bi-invariant differential operators on G (ie. $Z(G)$) and the left-invariant differential operators on $G\times G/\Delta(G)$ (ie. $D(G\times G/\Delta(G))$).

3. a bijection $D \mapsto D^\kappa$ from $D(K\backslash G)\otimes D(G/K)$ onto $D(K\times K\backslash G\times G)$

The second identification is considerably more subtle. Let us return to the connected reductive linear group L and assume that $\mathcal{L} = \mathcal{L}^\sigma \oplus \mathcal{L}^-$, $\mathcal{L} = \mathcal{L}^\tau \oplus q$ as described earlier. Since L is a linear group, it may be viewed as a subgroup of a connected complex linear group L_C with Lie algebra $\mathcal{L}_C = \mathcal{L} \otimes_R C = \mathcal{L} + j\mathcal{L}$ (here j stands for the complex structure on $\mathcal{L}_C$, $j^2 = -1$). Within $\mathcal{L}_C$, the triple $(\mathcal{L}, \sigma, \tau)$ is closely associated with a dual triple $(\hat{\mathcal{L}}, \hat{\sigma}, \hat{\tau})$ as follows:

Let γ be the conjugation of $\mathcal{L}_C$ with respect to $\mathcal{L}$. We may extend the Cartan involution σ of $\mathcal{L}$ to a Cartan involution (also denoted by σ) of $\mathcal{L}_C$, and the involution τ of $\mathcal{L}$ extends to a C-linear involution τ of $\mathcal{L}_C$.

Note that $\hat{\gamma} = \sigma\tau = \tau\sigma$ (in $\mathcal{L}_C$) is a conjugate linear involution of $\mathcal{L}_C$ whose fixed point set is a certain real form $\hat{\mathcal{L}}$ of $\mathcal{L}_C$. In view of the vector space decomposition 1.16, we have:

$$\mathcal{L} = (\mathcal{L}^\tau \cap \mathcal{L}^\sigma + q \cap \mathcal{L}^\sigma) + (\mathcal{L}^\tau \cap \mathcal{L}^- + q \cap \mathcal{L}^-) \tag{1.23}$$

and

$$\hat{\mathcal{L}} = (\mathcal{L}^\tau \cap \mathcal{L}^\sigma + j\mathcal{L}^\tau \cap \mathcal{L}^-) + (jq \cap \mathcal{L}^\sigma + q \cap \mathcal{L}^-) \tag{1.24}$$

Let $\hat{\sigma}$ be the restriction of $\gamma\sigma = \sigma\gamma$ to $\hat{\mathcal{L}}$ and $\hat{\tau}$ the restriction of τ to $\hat{\mathcal{L}}$ then $(\hat{\mathcal{L}}, \hat{\sigma}, \hat{\tau})$ is the dual triple in question.

We designate the analytic subgroups of L_C which correspond to $\hat{\mathcal{L}}$, $\hat{\mathcal{L}}^{\hat{\sigma}}$, $\hat{\mathcal{L}}^{\hat{\tau}}$, and $\hat{\mathcal{L}}^{\hat{\nu}}$ ($\hat{\nu} = \hat{\sigma}\hat{\tau} = \hat{\tau}\hat{\sigma}$) by $\hat{L}$, $\hat{L}[\hat{\sigma}]$, $\hat{L}[\hat{\tau}]$, and $\hat{L}[\hat{\nu}]$ respectively. Since $\mathcal{L}^\nu =$

$\mathcal{L}^\tau \cap \mathcal{L}^\sigma + q \cap \mathcal{L}^- = \mathcal{L}^{\hat{\nu}} = \mathcal{L} \cap \hat{\mathcal{L}}$, it is clear that the "middle" groups $L[\nu]$ of L and $\hat{L}[\hat{\nu}]$ of $\hat{L}$ are identical, in fact:

$$(L \cap \hat{L})_0 = L[\nu] = \hat{L}[\hat{\nu}], \tag{1.25}$$

where $(L \cap \hat{L})_0$ is the connected component of the identity in $L \cap \hat{L}$. By selecting $\mathcal{A}_0$, a maximal abelian subspace of $q \cap \mathcal{L}^-$, and $\mathcal{A}_0^+$ a Weyl chamber in $\mathcal{A}_0$, we may write the polar decompositions for L and $\hat{L}$ as in Theorem 1.21. $L = L[\tau]\overline{A_0^+}L[\sigma]$ and $\hat{L} = \hat{L}[\hat{\tau}]\overline{A_0^+}\hat{L}[\hat{\sigma}]$, leading to obvious identifications between the corresponding function spaces and algebras of differential operators (see [7] Theorems 4.2 and 4.3).

The second identification is then completed by considering the case when $\mathcal{L}$ is itself a complex Lie algebra (ie. $\mathcal{L}$ is a real Lie algebra with a complex structure denoted by $\imath$).

Specifically, we consider a connected σ-stable linear group G with Lie algebra $\mathcal{G}$ and set $\mathcal{L} = \mathcal{G}_C = \mathcal{G} + i\mathcal{G}$ (ie. $\mathcal{G}$ is a real form of $\mathcal{L}$). The Cartan involution on $\mathcal{G}$ may be extended to a C-linear involution τ of $\mathcal{L}$ and to a conjugate linear involution σ of $\mathcal{L}$. Let us write the Cartan decomposition of $\mathcal{G}$ as $\mathcal{G} = k + p$ and note that $\mathcal{L}^\sigma = k + ip$, $\mathcal{L}^\tau = k + ik = k_C$.

The vector space decompositions in 1.23 and 1.24 are:

$$\mathcal{L} = (k + ip) + (ik + p) \tag{1.26}$$

$$\hat{\mathcal{L}} = (k + jik) + (jip + p) \tag{1.27}$$

The Lie algebras $\mathcal{L} \times \mathcal{L}$ and $\mathcal{L} \otimes_R C$ are isomorphic (over R) via the map $\Theta(X,Y) = \frac{1}{2}(X - jiX) + \frac{1}{2}(\sigma(Y) + ji\sigma(Y)), X, Y \in \mathcal{L}$. Under this isomorphism, the triple $(K \times K, G \times G, \Delta(G))$ corresponds to $(\hat{L}[\hat{\tau}], \hat{L}, \hat{L}[\hat{\sigma}])$ as determined by 1.27. Note that $(L[\sigma], L, L[\tau])$ corresponding to 1.26 is simply (G_C^σ, G_C, K_C), where G_C^σ is the maximal compact subgroup of G_C, and K_C is the analytic subgroup of G_C which corresponds to the involutive (complex) subalgebra k_C of $\mathcal{G}_C$. Combining the above remarks with theorems 1.21 and 1.22 results in the following procedure for lifting functions on G to functions on G_C:

Theorem 1.28 (Flensted-Jensen [7], theorem 5.2)

1. Let $\mathcal{F}$ stand for C, C_c, C_c^∞ or $L^p, 1 \leq p \leq \infty$. There is an isomorphism $f \mapsto f^\eta$ of $\mathcal{F}(K\backslash G/K)$ onto $\mathcal{F}(K_C\backslash G_C/G_C^\sigma)$ such that $f^\eta(g) = f(g\sigma(g)^{-1})$ whenever $g \in G$

2. There is an isomorphism $D \mapsto D^\eta$ of $D(K\backslash G)\otimes D(G/K)$ onto $D(K_C\backslash G_C)$ and of $Z(G)$ onto $D(G_C/G_C^\sigma)$ such that $(Df)^\eta = D^\eta f^\eta$ for all $f \in C^\infty(K\backslash G/K)$.

Note that the middle group G_C^ν $(\nu = \sigma\tau = \tau\sigma)$ which corresponds to the triple (K_C, G_C, G_C^σ) is just G (see 1.25, 1.26 and 1.27).

Let us momentarily digress from our discussion in order to make some general observations concerning the reduced root systems which are associated with a triple $(\mathcal{L}, \sigma, \tau)$, where $\mathcal{L}$ is a linear reductive Lie algebra, σ a Cartan involution and τ an involutive automorphism which commutes with σ. The meanings of $\mathcal{L}^\sigma$, $\mathcal{L}^\tau$, $\mathcal{L}^-$ and q will be those of 1.23.

Consider a maximal abelian subspace $\mathcal{A}_0$ of $\mathcal{L}^- \cap q$ and let $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$ be an extension to a maximal abelian subspace of $\mathcal{L}^-$ ($\mathcal{A}_1 \subset \mathcal{L}^\tau \cap \mathcal{L}^-$). For convenience we will view the real dual $\mathcal{A}_0^*$ (resp. $\mathcal{A}_1^*$) as the subspace of $\mathcal{A}^*$ consisting of all those linear functionals on $\mathcal{A}$ which vanish on $\mathcal{A}_1$ (resp on $\mathcal{A}_0$). The same inclusions will be assumed for the corresponding spaces of complex linear functionals.

Lemma 1.29: Let R and R_0 denote the restricted root systems associated with the pairs $(\mathcal{L}, \mathcal{A})$ and $(\mathcal{L}^\nu, \mathcal{A}_0)$ respectively, and for $\alpha \in R$ let $\tau \cdot \alpha$, $\sigma \cdot \alpha$, and $\nu \cdot \alpha$ denote the compositions $\alpha \circ \tau$, $\alpha \circ \sigma$, and $\alpha \circ \nu$ respectively, then:

1. R is the disjoint union of the following three subsets:

   - R_1 consists of all roots α such that $\nu \cdot \alpha = -\alpha$, i.e., $\alpha|_{\mathcal{A}_0} = 0$
   - S_I consists of all roots α such that $\nu \cdot \alpha = \alpha$, i.e., $\alpha|_{\mathcal{A}_1} = 0$
   - S_M consists of all roots α such that $\nu \cdot \alpha \neq \pm\alpha$, i.e., $\alpha|_{\mathcal{A}_0} \neq 0, \alpha|_{\mathcal{A}_1} \neq 0$

   Furthermore, R_0 is composed of all restrictions to $\mathcal{A}_0$ of elements of S_M in addition to the restrictions to $\mathcal{A}_0$ of all those roots $\alpha \in S_I$ for which $\mathcal{L}_\alpha \cap \mathcal{L}^\nu \neq (0)$ (here $\mathcal{L}_\alpha$ denotes the root space of $\mathcal{L}$ corresponding to $\alpha \in S_I$).

2 Given compatible orderings on $\mathcal{A}_0^*$ and $\mathcal{A}^*$ then

$$\rho - (2\rho_0 + \rho_1) = \frac{1}{2} \sum_{\alpha \in S_I^+} d_\alpha \cdot \alpha. \quad \text{where}$$

$$\rho = \frac{1}{2} \sum_{\alpha \in R^+} m_\alpha \cdot \alpha \quad (m_\alpha = \dim(\mathcal{L}_\alpha)). \quad \rho_1 = \frac{1}{2} \sum_{\alpha \in R_1^+} m_\alpha \cdot \alpha$$

$$(m_\alpha = \dim(\mathcal{L}_\alpha)), \quad \rho_0 = \tfrac{1}{2} \textstyle\sum_{\alpha \in R_0^+} n_\alpha \cdot \alpha \quad (n_\alpha = \dim(\mathcal{L}_\alpha^\nu))$$

and for $\alpha \in S_I^+$, $d_\alpha = \dim(\mathcal{L}_\alpha \cap (\mathcal{L}^\nu)^\perp) - \dim(\mathcal{L}_\alpha \cap \mathcal{L}^\nu)$.

Proof: Note that $\mathcal{A}$ is stable for each of the involutions σ, τ, ν hence R is stable for all three involutions. For a given $\alpha \in R$ exactly one of the conditions $\nu \cdot \alpha = -\alpha$. $\nu \cdot \alpha \neq \alpha, -\alpha$. or $\nu\alpha = \alpha$ holds. Applying σ to the first of these shows that R decomposes as stated in (1) ($\sigma\alpha = -\alpha \; \forall \alpha \in R$). If $\alpha \in S_M$ then ν maps $\mathcal{L}_\alpha$ into $\mathcal{L}_{\nu \cdot \alpha}$ and since the two spaces are distinct there is a nontrivial projection $x \mapsto \frac{1}{2}(x + \nu x)$ onto $\mathcal{L}^\nu$. Each pair $(\alpha, \nu\alpha)$. $\alpha \in S_M$ contributes $\dim(\mathcal{L}_\alpha)$ linearly independent eigenvectors in $\mathcal{L}^\nu$ corresponding to the restricted root $\alpha|_{\mathcal{A}_0} \in R_0$. If $\alpha \in S_I$ then ν maps $\mathcal{L}_\alpha$ onto itself. in which case $\mathcal{L}_\alpha$ may or may not have ν-fixed vectors. If it does then α restricted to $\mathcal{A}_0$ lies in R_0 and $\mathcal{L}_\alpha \cap \mathcal{L}^\nu \neq (0)$. Assuming compatible orderings on $\mathcal{A}_0^*$ and $\mathcal{A}^*$ we write the half sum of the positive roots in R as:

$$\begin{aligned} \rho \;&=\; \rho_1 + \frac{1}{2} \sum_{\alpha \in S_M^+ \mathrm{Mod}(\nu)} m_\alpha(\alpha + \nu\alpha) + \frac{1}{2} \sum_{\alpha \in S_I^+} 2n_\alpha \cdot \alpha \\ &\quad + \frac{1}{2} \sum_{\alpha \in S_I^+} d_\alpha \cdot \alpha, \quad \text{with } n_\alpha = \dim(\mathcal{L}_\alpha \cap \mathcal{L}^\nu). \end{aligned}$$

Taking into account the fact that $\alpha + \nu\alpha = 2\alpha|_{\mathcal{A}_0}$, we obtain the required results as stated in (2). ■

To return to our discussion, let us assume that $\mathcal{A}_0$ is a maximal abelian subspace of p and $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_0 \subset ik + p$ is an extension as in Lemma 1.29. Because ν is a conjugate linear involution it is easy to see that $d_\alpha = 0$ whenever $\alpha \in S_I^+$, thus in this case, $\rho = 2\rho_0 + \rho_1$, where ρ_1 is the half sum of the roots corresponding to the subroot system $R_1 = \{\alpha \in R \mid \alpha(H) = 0\ \forall H \in \mathcal{A}_0\}$ (R_1 is the root system of m_C the centralizer of $\mathcal{A}_0$ in k_C).

The K-spherical functions ([9]) $\phi_\lambda \in C^\infty(K\backslash G/K)$, $\lambda \in \mathcal{A}_0^*$ may be lifted to corresponding left-K_C, right-G_C^σ invariant functions $\phi_\lambda^\eta \in C^\infty(K_C\backslash G_C/G_C^\sigma)$. If the measure on $K_C\backslash G_C$ is suitably normalized so that

$$\int_G f(g)\,dg = \int_{G\times G/\Delta(G)} f^\kappa(x,y)\,d(x,y) = \int_{K_C\backslash G_C} f^\eta(x)\,dx,$$

for $f \in C_c^\infty(K\backslash G/K)$, then we may express Harish-Chandra's spherical Fourier transform as:

$$\tilde{f}(\lambda) = \int_G f(g)\phi_\lambda(g)\,dg = \int_{K_C\backslash G_C} f^\eta(x)\phi_\lambda^\eta(x)\,dx, \tag{1.30}$$

whenever $f \in C_c^\infty(K\backslash G/K)$.

The real advantage of the method of lifting functions on G to functions on G_C is derived from the relations that exist between the function spaces $C_c^\infty(G_C^\sigma\backslash G_C/G_C^\sigma)$ and $C_c^\infty(K_C\backslash G_C/G_C^\sigma)$. These relations are mediated by the two (dual) integral transforms:

$$M_0 : \mathcal{F}(K_C\backslash G_C/G_C^\sigma) \to \mathcal{F}(G_C^\sigma\backslash G_C/G_C^\sigma), \quad \text{where}$$

$$M_0 f(g) = \int_{G_C^\sigma} f(ug)\, du \tag{1.31}$$

and its dual

$$M : C_c^\infty(G_C^\sigma\backslash G_C/G_C^\sigma) \to C_c^\infty(K_C\backslash G_C/G_C^\sigma), \quad \text{where}$$

$$MF(g) = \int_{K_C} F(hg)\, dh \tag{1.32}$$

Suppose that ϕ_λ is an elementary K-spherical function on G then its left-G_C^σ average $\Phi = M_0\phi_\lambda^\eta$ defines a bi-G_C^σ invariant function on G_C. Since every invariant differential operator on G_C/G_C^σ is of the form D^η for some $D \in Z(G)$ (see theorem 1.28 (2)) it follows that $D^\eta\Phi = M_0(D\phi_\lambda)^\eta = \chi(D,\lambda)\Phi$, hence Φ is an elementary spherical function on G_C/G_C^σ. A clever argument ([7] Theorem 5.5) shows that we actually have:

$$\Phi_\Lambda = M_0\phi_\lambda^\eta \quad \text{with} \quad \Lambda = 2\lambda - i\rho_1 \in \mathcal{A}_C^* \tag{1.33}$$

$\lambda \in \mathcal{A}_{0,C}^*$ and ρ_1 is the half sum of the positive roots corresponding to $(m_C, \mathcal{A}_1)$ (see Lemma 1.29 and subsequent remarks).

The above relation is the tool which permits the transfer of the spectral distribution for a certain class of functions on G to a corresponding spectral distribution on G_C.

For instance, if $F \in L^1(G_C^\sigma \backslash G_C / G_C^\sigma)$ and $\phi_\lambda^\eta \in C^\infty(K_C \backslash G_C / G_C^\sigma)$ is a bounded elementary K-sperical function on G then $f\phi_\lambda^\eta \in L^1(G_C)$. Integrating over G_C gives:

$$\int_{G_C} F(g)\phi_\lambda^\eta(g)\,dg = \int_{K_C\backslash G_C} MF(x)\phi_\lambda^\eta(x)\,dx = \int_{G_C} F(g)M_0\phi_\lambda^\eta(g)\,dg.$$

which shows that

$$\widetilde{MF}(\lambda) = F^\sim(\Lambda), \quad \Lambda = 2\lambda - \imath\rho_1, \tag{1.34}$$

where $F^\sim$ is the spherical Fourier transform of F in G_C.

The partial limitations of the method are apparent from 1.34. In general, M (see 1.32) is not onto, which means that certain types of spectral distributions on G may not be lifted to G_C. If $\mathcal{G}$ is a normal real from of $\mathcal{G}_C$ then M is a bijection and in that case the method is a complete success as has been eloquently demonstrated by Jean-Philippe Anker and Noël Lohoué [15] in their study of the multiplier problem for the L^p spaces.

Chapter 2

The Heat Equation

This chapter is devoted to the study of an integral expression for the heat kernel on a connected semisimple Lie group G. We will adhere to the assumptions and notation made in section 1.3.

2.1 The Heat Kernel

The invariant metric on G/K gives rise to the Laplacian Δ, which is a G-invariant second order elliptic differential operator. Given an initial datum

$f \in C_c^\infty(G/K)$ there is a solution $\psi(x,t)$ to the initial value problem:

$$\left.\begin{array}{l}\Delta\psi(x,t) = \dfrac{\partial\psi}{\partial t}(x,t), \quad x \in G/K,\ t \in (0,\infty) \\ \lim_{t\downarrow 0}\psi(x,t) = f(x)\end{array}\right\} \tag{2.1}$$

expressed by a convolution transform:

$$\psi(g,t) = f * H_t(g) = \int_G f(gy^{-1})H_t(y)\,dy,$$

where $H_t \in C^\infty(K\backslash G/K)$ is a fundamental solution of the heat equation [1] [13bis] [6] [8], satisfying:

$$\left.\begin{array}{l}
1.\ \text{for fixed } t > 0\ H_t \text{ is bounded on } G/K \\[2ex]
2.\ \int_G H_t(g)\,dg = 1 \\[2ex]
3.\ H_t * H_s = H_{t+s} \text{ for } s,t \in (0,\infty) \\[2ex]
4.\ \text{for } f \in L^2(G/K),\ \lim_{t\downarrow 0} \| f * H_t - f \|_2 = 0
\end{array}\right\} \tag{2.2}$$

For an arbitrary noncompact symmetric pair (L, L^σ) the Laplacian Δ has the well-known spectral distribution:

$$\Delta\phi_\lambda = -(|\lambda|^2 + |\rho|^2)\phi_\lambda, \lambda \in \mathcal{A}^*, \tag{2.3}$$

where the ϕ_λ's are the elementary spherical functions on L/L^σ, $\mathcal{A}$ is a maximal abelian subspace of $\mathcal{L}^-$ (the Lie algebra $\mathcal{L}$ of L is assumed to have a Cartan decomposition $\mathcal{L} = \mathcal{L}^\sigma \oplus \mathcal{L}^-$), and ρ is the half sum of the positive restricted roots (with multiplicity). This fact leads via spherical Fourier analysis to the construction of the following general expression for the heat kernel on L/L^σ [1] [8]:

$$H_t(g) = \frac{1}{c} \int_{\mathcal{A}^*} \exp\left[-t(|\lambda|^2 + |\rho|^2)\right] \phi_\lambda(g) |C(\lambda)|^{-2}\, d\lambda, \tag{2.4}$$

where $C(\lambda)$ is Harish-Chandra's C-function and c is a constant.

In general, difficulties in dealing with 2.4 result from insufficient information concerning the behaviour of the spherical functions ϕ_λ. Luckily for complex groups $(G_C,\, G_C^\sigma)$ an explicit expression for the heat kernel G_t is known in terms of elementary functions on G_C/G_C^σ [8]:

$$G_t(\exp(Z)) = ct^{-n/2} \exp\left[-|\rho|^2 t\right] \exp\left[-\frac{|Z|^2}{4t}\right] \Phi_0(\exp(Z)) \tag{2.5}$$

where $n = \dim(G_C/G_C^\sigma)$, $Z \in \imath\mathcal{G}_C^\sigma$, $\Phi_0(\exp(Z)) = \det[S(Z)]^{-1/2}$ is the elementary sperical function of index zero on G_C/G_C^σ, and c is a constant determined by 2.2 (b).

Let B_0 and B denote the Killing forms of the real Lie algebras $\mathcal{G}$ and $\mathcal{G}_C$ respectively. If $X \in p$ we let $|X|_0 = \sqrt{B_0(X, X)}$ denote its norm as an element of $\mathcal{G}$ and $|X| = \sqrt{B(X, X)}$ the corresponding norm when viewed as an element of $\mathcal{G}_C$. Comparing root systems in $\mathcal{G}$ and in $\mathcal{G}_C$ gives (see Lemma 1.29):

$$|X|^2 = 2|X|_0^2 \tag{2.6}$$

Let $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_0$ be a maximal abelian subspace of $ik + p$ such that $\mathcal{A}_0$ is maximal abelian in p. For $\lambda \in \mathcal{A}^*$ we define $H_\lambda \in \mathcal{A}$ in the usual way according to $< H_\lambda, H >= \lambda(H)$, $\forall H \in \mathcal{A}$, and $|\lambda|^2 = |H_\lambda|^2$. Similarly, if

$\lambda \in \mathcal{A}_0^*$, then $H_\lambda^0 \in \mathcal{A}_0$ and $|\lambda|_0^2 = |H_\lambda^0|_0^2$ are defined relative to $B_0(.)$ on $\mathcal{A}_0$. Since $\lambda \in \mathcal{A}_0^*$ may also be viewed as a liner functional on $\mathcal{A}$, we have a correspondence between the two norms on $\mathcal{A}_0^*$:

$$|\lambda|_0^2 = \frac{1}{2}|2H_\lambda|^2 = 2|\lambda|^2, \quad \lambda \in \mathcal{A}_0^* \tag{2.7}$$

Lemma 2.8([7], Example, p. 131–132, [1], §2.4, Remark (ii)): Let H_t and G_t be the heat kernels on G/K and G_C/G_C^σ respectively then, for a suitably normalized Haar measure on K_C we have:

$$H_t^\eta(g) = \int_{K_C} G_{t/2}(hg)\,dh, \quad g \in G_C \tag{2.9}$$

Proof: Notice that the right-hand side of 2.9 defines a right G_C^σ-invariant function on G_C which is integrable over $K_C \backslash G_C$.

Indeed, from 2.2(b) we see that:

$$1 = \int_{G_C} G_{t/2}(x)\,dx = \int_{K_C\backslash G_C} \left(\int_{K_C} G_{t/2}(h\dot{g})\,dh \right) d\dot{g},$$

where $d\dot{g}$ stands for the invariant measure on $K_C \backslash G_C$, normalized in such a way that

$$\int_{K_C\backslash G_C} f^\eta(\dot{g})\,d\dot{g} = \int_G f(x)\,dx \quad \text{whenever } f \in C_c^\infty(K\backslash G/K)$$

By theorem 1.28(1), we may identify the right-hand side of 2.9 with a function $\psi_t \in L^1(K\backslash G/K)$ in the following way:

$$\psi_t(\exp(x)) = \psi_t^\eta\left(\exp\left(\frac{x}{2}\right)\right) = \int_{K_C} G_{t/2}\left(h\exp\left(\frac{x}{2}\right)\right)\,dh, \quad \text{if } x \in p.$$

Our objective is to show that $H_t = \psi_t$. If we can show that ψ_t is bounded and continuous for each $t > 0$ then the equality $H_t = \psi_t$ follows from the Plancherel pointwise inversion formula ([9], Theorem 1.6.5) since:

$$\psi_t \in L^1(K\backslash G/K) \cap L^2(K\backslash G/K) \cap C(G)$$

and

$$\psi_t(g) = \frac{1}{c}\int_{\mathcal{A}_0^*} \tilde{\psi}_t(\lambda)\psi_\lambda(g)\,d\lambda = \frac{1}{c}\int_{\mathcal{A}_0^*} G_{t/2}^{\sim}(2\lambda - i\rho_1)\phi_\lambda(g)|C(\lambda)|^{-2}\,d\lambda$$

(see 1.34). But $G_{t/2}^{\sim}(2\lambda - i\rho_1) = \exp\left[-\frac{t}{2}\left(|2\lambda - i\rho_1|^2 + |\rho|^2\right)\right]$. Expanding $|2\lambda - i\rho_1|^2$ and using the fact that $\rho = 2\rho_0 + \rho_1$ (see Lemma 1.29) we obtain:

$$|2\lambda - i\rho_1|^2 + |\rho|^2 = 4|\lambda|^2 - |\rho_1|^2 + 4|\rho_0|^2 + |\rho_1|^2 = 4\left(|\lambda|^2 + |\rho_0|^2\right)$$

Since the norms on p^* and $(ik + p)^*$ are releated as in 2.7, we finally have:

$$\frac{t}{2}\left(|2\lambda - i\rho_1|^2 + |\rho|^2\right) = t\left(|\lambda|_0^2 + |\rho_0|_0^2\right),$$

and hence

$$\psi_t(g) = \frac{1}{c}\int_{\mathcal{A}_0^*} \exp\left[-t\left(|\lambda|_0^2 + |\rho_0|_0^2\right)\right]\phi_\lambda(g)|C(\lambda)|^{-2}\,d\lambda = H_t(g).$$

To show that ψ_t is bounded and continuous, let $x \in p$ and let us rewrite $\psi_t(\exp(x))$ using 2.5:

$$\begin{aligned} \psi_t(\exp(x)) &= Ct^{-n/2}e^{-|\rho|^2 t/2} \int_{ik} \exp\left[-\frac{|\exp(Y)\exp(x/2)|^2}{2t}\right] \\ &\quad \Phi_0\left(\exp(Y)\exp(x/2)\right) J(Y)\, dY \qquad (2.10) \end{aligned}$$

where $|\exp(Y)\exp(x/2)|$ is the Riemannian distance $d(0, \exp(Y)\exp(x/2)\cdot o)$ in G_C/G_C^o, $J(Y) = \det(S(Y)\mid_{ik})$ is the Jacobian of the exponential map on $K\backslash K_C$, and the constant C is determined by the condition 2.2(b).

Later (in Section 2.2) it will be shown that for $X \in p$ and $Y \in ik$ we have.

$$|\exp(Y)\exp(X)| \geq \sqrt{|Y|^2 + |X|^2} \geq |Y|.$$

Since $J(Y) \leq e^{2\rho(Y)} \leq e^{2C\alpha_{MAX}(Y)} \leq e^{C|Y|}$, where $\rho(Y)$ is the half sum of the positive roots of $ad(Y)$, and $\Phi_0(\exp(z)) \leq 1$ for all $z \in ik + p$, it follows that the integrand in 2.10 is bounded uniformly in x by

$$\exp\left(-\frac{|Y|^2}{2t}\right)\exp(c|Y|),$$

for a suitable constant $c \in R^+$. Thus, ψ_t is bounded for each t, and Lebesque's dominated convergence theorem may be applied to show that $\psi_t()$ is continuous.■

2.2 The Behaviour of H_t for small t

Before taking up the question of the estimates for H_t, let us digress for a moment to discuss some elementary estimates of geometric origin, valid for any reductive symmetric pair (L, L^σ).

The Riemannian metric in the exponential coordinate chart, $Exp : \mathcal{L}^- \to L/L^\sigma$ takes the form:

$$g_Z(\partial(A)_Z, \partial(B)_Z) = \langle S(Z)A, S(Z)B\rangle, \quad \text{where } A, B, Z \in \mathcal{L}^-, \tag{2.11}$$

and $\partial(A)$ is the (euclidian) parallel vector field on $\mathcal{L}^-$ determined by $A \in \mathcal{L}^-$ (see 1.6, Lemma 1.12(1) and a subsequent remark).

Consider then a smooth curve $Z_s \in \mathcal{L}^-$, $0 \le s \le 1$, such that $Z_o = -Y \in \mathcal{L}^-$, $Z_1 = X \in \mathcal{L}^-$, and $\exp(Z_s) \cdot o$ is a geodesic in L/L^σ. Clearly, $|\exp(Y)\exp(X)| = distance(\exp(-Y)\cdot o, \exp(X)\cdot o) = \int_0^1 |S(Z_s)\dot{Z}_s|\, ds \ge \int_0^1 |\dot{Z}_s|\, ds \ge |X + Y|$.

On the other hand, the triangle inequality gives:

$$|\exp(Y)\exp(X)| \le d(0, \exp(Y)\cdot o) + d(\exp(Y)\cdot o, \exp(Y)\exp(X)\cdot o) = |X| + |Y|$$

In particular, if X and Y are orthogonal, one obtains the well-known inequalities

$$\sqrt{|X|^2+|Y|^2} \le |\exp(Y)\exp(X)\cdot o| \le |X|+|Y|, \quad X,Y \in \mathcal{L}^-, \quad \langle X,Y\rangle = 0 \tag{2.12}$$

The left-hand estimate in 2.12 may be refined using a Taylor expansion with integral remainder. Consider the smooth curve $Z_s = Exp^{-1}(\exp(sY)\exp(X)$ $o) \in \mathcal{L}^-$, $-\infty < s < \infty$ then 1.13 gives:

$$\begin{aligned}\frac{1}{2}(|Z_s|^2 - |X|^2) &= \int_0^s \langle Z_\tau, \dot{Z}_\tau\rangle\, d\tau = \int_0^s \langle Z_\tau, \Theta(Z_\tau)Y\rangle\, d\tau \\ &= \int_0^s \langle Z_\tau, Y\rangle\, d\tau.\end{aligned}$$

where

$$\Theta(Z) = S(Z)^{-1}Ch(ad(Z)) = \left.\frac{ad(Z)}{\tanh(ad(Z))}\right|_{\mathcal{L}^-}$$

Integrating by parts and assuming that $X, Y \in \mathcal{L}^-$ are orthogonal gives:

$$\frac{1}{2}\left(|Z_s|^2 - |X|^2\right) = \int_0^s \langle \Theta(Z_\tau)Y, Y\rangle\,(s-\tau)\,d\tau. \tag{2.13}$$

Therefore, the n^{th} order Taylor expansion ($n \ge 2$) about $s = 0$ takes the form:

$$\begin{aligned}\frac{1}{2}\left(|Z_s|^2 - |X|^2\right) &= \sum_{k=2}^{n} \frac{1}{k!}\left\langle \Theta(Z_s)^{(k-2)}\Big|_{s=0} Y, Y\right\rangle s^k \\ &\quad + \frac{1}{n!}\int_0^s \left\langle \Theta(Z_\tau)^{(n-1)}Y, Y\right\rangle (s-\tau)^n\, d\tau, \end{aligned} \tag{2.14}$$

where $\Theta(Z_s)^{(k)} = \frac{d^k}{ds^k}\{\Theta(Z_s)\}$.

If $Z \in ik + p$ is a generic element, we let $\alpha_M(Z)$ be the largest positive eigenvalue of $ad(Z)$, $\rho(Z)$ the half sum of the positive eigenvalues (each having multiplicity 2), $|ad(Z)|$ the operator norm of $ad(Z)$ on $\mathcal{G}_C$, $|Z|^2 = 4\sum_{\alpha>0}\alpha(Z)^2$ the squared norm defined by the inner product $B_\sigma(,)$ on $\mathcal{G}_C$, and N the number of positive eigenvalues. The following relations are easily verified:

$$\frac{|Z|}{\sqrt{N}} \leq 2\alpha_M(Z) = 2|ad(Z)| \leq |Z|,$$

and

$$|ad(Z)| = \alpha_M(Z) \leq \rho(Z)$$

Elementary considerations also show that the spherical function Φ_0 of G_C satisfies the inequality:

$$\Phi_0(\exp(Z)) \leq (1 + 2\alpha_M(Z))^N e^{-\rho(Z)} \leq (1 + |Z|)^N e^{-|ad(Z)|}$$

Finally, let us note that if $Y \in ik$ and $X \in p$ then $\Phi_0(\exp(Y)\exp(X))$ is dominated by $\Phi_0(\exp(X))\exp(\frac{n}{2}|Y|)$. To see this, we let $\hat{J}(Z) = \det[S(Z)\mid_{\mathcal{G}_C}]$ and use the results of Lemma 1.14 to write:

$$\begin{aligned}\frac{\partial}{\partial s}\ln \hat{J}(Z_s) &= Trace_{\mathcal{G}_C}\left[S(Z_s)^{-1}S(\dot{Z}_s)\right] = Trace\left[\frac{\Theta(Z_s) - I}{ad(Z_s)}ad(\dot{Z}_s)\right] \\ &= \sum_\alpha \frac{\alpha\mathrm{Coth}(\alpha) - 1}{\alpha}\left\langle ad(\dot{Z}_s)X_\alpha, X_\alpha\right\rangle,\end{aligned}$$

where the α's run through the eigenvalues of $ad(Z_s)$ and $X_\alpha \in \mathcal{G}_{\alpha,C}$ are corresponding normed eigenvectors. The above evaluates to

$$-\sum_\alpha \frac{\alpha \mathrm{Coth}(\alpha)-1}{\alpha}\left\langle \dot{Z}_s, [X_\alpha, \sigma(X_\alpha)]\right\rangle .$$

Since $\dot{Z}_s = \Theta(Z_s)Y$, and $[X_\alpha, \sigma(X_\alpha)]$ is fixed by $\Theta(Z_s)$, we see that

$$\frac{\partial}{\partial s}\ln \hat{J}(Z_s) = Trace\left[\frac{\Theta(Z_s)-I}{ad(Z_s)}ad(Y)\right] \leq \left(Trace\left[\frac{\Theta(Z_s)-I}{ad(Z_s)}\right]^2\right)^{1/2}|Y|.$$

The eigenvalues of $\left[\frac{\Theta(Z_s)-I}{ad(Z_s)}\right]^2$ are bounded above by 1, so that:

$$\left|\frac{\partial}{\partial s}\ln \hat{J}(Z_s))\right| \leq 2n|Y|,$$

where $2n = \dim_R(\mathcal{G}_C)$, i.e., $n = \dim(G_C/G_C^\sigma)$.

The result follows from the fact that $\Phi_0(\exp(Z)) = [\hat{J}(Z)]^{-1/4}$.

Having completed these preliminaries, we may now state the following estimate for H_t:

Theorem 2.15: Let H_t be the heat kernel associated with the symmetric space G/K, and set

$$H_t(\exp(X)) = Ct^{-d/2}e^{-|\rho|^2t/2}e^{-|X|_0^2/4t}U_t(X) \quad \text{for } X \in p,$$

with $d = \dim(G/K)$ and ρ as in 2.9 then:

1. $U_0(X) = \lim_{t\to 0} U_t(X) = \Phi_0(\exp(X/2))\int_{ik} \exp\left[-\frac{1}{2}\left\langle \Theta\left(\frac{X}{2}\right)Y, Y\right\rangle\right] dY$

2. for any $L > 0$ there exists a constant $A(L)$ depending only on L ($A(L) > 0$) such that:

$$|U_t(X) - U_0(X)| \leq A(L) t^{1/2} \Phi_0 \left(\exp\left(\frac{X}{2}\right)\right) \cdot \left(1 + \left|\frac{X}{2}\right|\right),$$

whenever $0 \leq t \leq L^2$.

Proof:

1) We use the integral expression 2.10 for H_t and the fact that $|X|^2 = 2|X|_0^2$ to write:

$$\begin{aligned} U_t(X) &= t^{-k/2} \int_{ik} \exp\left[-\frac{|\exp(Y)\exp(X/2)|^2 - |X/2|^2}{2t}\right] \\ &\quad \Phi_0\left(\exp(Y)\exp\left(\frac{X}{2}\right)\right) J(Y)\, dY \end{aligned}$$

with $k = \dim(ik)$, $X \in p$.

Changing the variable of integration from Y for $\sqrt{t}Y$ results in the expression

$$U_t(X) = \int_{ik} \exp\left[-\frac{1}{t}\int_0^{\sqrt{t}} \langle \Theta(Z_s)Y, Y\rangle (\sqrt{t} - s)\, ds\right] \Phi_0\left(\exp\left(Z_{\sqrt{t}}\right)\right) J(\sqrt{t}Y)\, dY,$$

where

$$Z_s = Exp^{-1}\left(\exp(sY)\exp\left(\frac{X}{2}\right) \cdot 0\right).$$

(see 2.13).

Note that the eigenvalues of $\Theta(Z_s)$ are bounded below by 1, $\Phi_0(\exp(Z)) \leq 1$, and $J(\sqrt{t}Y) \leq \exp(c\sqrt{t}|Y|)$ for a suitably chosen $c > 0$. Thus, the integrand above is dominated by:

$$\exp\left(-\frac{|Y|^2}{2}\right)\exp(c\sqrt{T}|Y|) \quad \text{whenever } 0 < t < T.$$

The stated value for the limit $\lim_{t\to 0^+} U_t(X)$ follows immediately from 2.11 ($n = 2$) and dominated convergence.

2) The error estimate $|U_t(X) - U_0(X)|$ requires a few manipulations.

Recall that:

$$\Phi_0(\exp(Z_\lambda))\, J(\lambda Y) = \Phi_0\left(\exp\left(\frac{X}{2}\right)\right)\exp(-\psi(\lambda)).$$

for an explicitly determined $\psi(\lambda)$ (see Lemma 1.14).

Now, the error term may be written as:

$$\begin{aligned} &U_t(X) - U_0(X) = \\ &\quad \int_{ik}\left[\exp(-\phi(\sqrt{t})) - \exp(-\varphi(0))\right]\Phi_0\left(\exp\left(Z_{\sqrt{t}}\right)\right) J(\sqrt{t}Y)\, dY \\ &\quad + \int_{ik}\exp(-\phi(0))\Phi_0\left(\exp\left(\frac{X}{2}\right)\right)\left[\exp(-\psi(\sqrt{t})) - 1\right] dY. \end{aligned}$$

where

$$\phi(\lambda) = \frac{1}{\lambda^2}\int_0^\lambda \langle\Theta(Z_s)Y, Y\rangle\,(\lambda - s)\, ds$$

and

$$\psi(\lambda) = \frac{1}{4}\int_0^\lambda Tr\left[S(Z_s)^{-1}S(\dot{Z_s})\right]_{\mathcal{G}_C} - \frac{1}{2}\int_0^\lambda \mathrm{Tr}\left[S(sY)^{-1}S(\dot{sY})\right]_{\mathcal{G}_C} ds$$

Applying the mean value property to the integrands which express $U_t - U_0$ gives:

$$|U_t(X) - U_0(X)| \leq \int\limits_{tk} e^{-|Y|^2/2}|\phi(\sqrt{t}) - \phi(0)|\Phi_0(\exp\left(\frac{X}{2}\right))e^{2n\sqrt{t}|Y|}\,dY$$
$$+ \int\limits_{tk} e^{-|Y|^2/2}|\psi(\sqrt{t})|\Phi_0(\exp\left(\frac{X}{2}\right))e^{2n\sqrt{t}|Y|}\,dY$$

In our preliminary discussion we showed that $\psi(\lambda)$ satisfies the inequality:

$$|\psi(\lambda)| \leq \left[\frac{1}{4}(2n)|Y| + \frac{1}{2}(2n)|Y|\right]\lambda \leq 2n|Y|\lambda, \quad \text{for } \lambda > 0.$$

Using 2.14 ($n = 2$) we have:

$$\phi(\lambda) - \phi(0) = \frac{1}{2\lambda^2}\int\limits_0^\lambda \left\langle \dot{\Theta}(Z_s)Y, Y\right\rangle (\lambda - s)^2\,ds,$$

where $\dot{\Theta}(Z_s) = \frac{d}{ds}[\Theta(Z_s)]$.

A lengthy computation allows us to establish an upper bound for the (possibly indefinite) quadratic form:

$$\left\langle \dot{\Theta}(Z_s)U, U\right\rangle, \quad U \in \mathcal{G}_C.$$

An expression for $\langle \ddot{Z}_s, Y\rangle$ obtained from 1.13 in conjunction with the estimate $\left|\frac{x\mathrm{Coth}(x)-1}{x}\right| \leq 1$ for $x \in R\backslash\{0\}$ results in:

$$|\langle \dot{\Theta}(Z_s)Y, Y\rangle| \leq 2n(1 + |Z_s|)|Y|^3$$

A small modification of the argument used in establishing 2.13 shows that for $s_2 > s_1 > 0$ we have $|Z_{s_2}|^2 - |Z_{s_1}|^2 \geq (s_2^2 - s_1^2)|Y| > 0$, hence the function $s \mapsto |Z_s|$ increases, and

$$|\phi(\lambda) - \phi(0)| \leq \frac{n}{3}(1 + |Z_\lambda|)\,|Y|^3\lambda \quad \text{for } \lambda > 0$$

Combining the estimates for $|\psi(\sqrt{t})|$ and $|\phi(\sqrt{t}) - \phi(0)|$ results in:

$$|U_t(X) - U_0(X)| \leq 2nt^{1/2}\Phi_0\left(\exp\left(\frac{X}{2}\right)\right)$$
$$\int_{ik}\left[L|Y|^4 + \left(1 + \left[\frac{X}{2}\right]\right)|Y|^3 + |Y|\right]e^{-|Y|^2/2}e^{2nL|Y|}\,dY$$

whenever $0 \leq t \leq L^2$. ■

2.3 An Asymptotic Expansion for H_t for a Particular Class of Groups

If G is a normal real of of G_C (i.e. p contains a maximal abelian subspace of $ik + p$), then it can be shown a case by case basis that the elementary spherical function $\Phi_0 \in C^\infty(G_C/G_C^\sigma)$ is integrable over K_C (see [7], Theorem 7.1). In fact, there exists $\epsilon > 0$ such that

$$\int_{K_C} e^{\epsilon|h|}\Phi_0(h)\,dh < +\infty$$

For any compact subset Ω of p one may find a constant $C(\Omega) > 0$ such that

$$\Phi_0\left(h\exp\left(\frac{X}{2}\right)\right) \leq C(\Omega)\Phi_0(h)$$

for all $h \in K_C$ and $X \in \Omega \subset p$.

Thus,

$$\phi_0(\exp(X)) = \int_{K_C} \Phi_0\left(h\exp\left(\frac{X}{2}\right)\right)\, dh, \quad X \in p$$

is well-defined and coincides with the elementary spherical function of index zero on G/K (see [7], Corollary 7.4). These facts are exploited to obtain the following asymptotic expansion:

Theorem 2.16: Let G be a normal real form of G_C, H_t the heat kernel associated with the symmetric space G/K. and for $X \in p$ set

$$H_t(\exp(X)) = Ct^{-n/2}\exp\left[-|\rho_0|_0^2 t\right]\exp\left[-\frac{|X|_0^2}{4t}\right] V_t(X)$$

where $n = \dim(G_C/G_C^{\sigma})$, then:

1. $V_t(X) \leq \phi_0(\exp(X))$ and $\lim_{t\to\infty} V_t(X) = \phi_0(\exp(X))$
2. There is an symptotic expansion

$$V_t(X) = \phi_0(\exp(X)) + \sum_{k=1}^{m-1} t^{-k} V_k(X) + E_m(X,t),$$

where

$$V_k(X) = \frac{(-1)^k}{k!}\int_{\imath k}\left\{\frac{|Z(Y,X/2)|^2 - |X/2|^2}{2}\right\}^k \Phi_0(Z(Y,X/2))J(Y)\, dY$$

and for each compact subset Ω of p there exists a constant $A(\Omega) > 0$ such that

$$|E_m(X,t)| \leq \frac{A(\Omega)}{t^m}, \quad \text{whenever } X \in \Omega$$

Proof: Using the integral expression for $H_t(\exp(X))$ we have:

$$V_t(X) = \int\limits_{ik} e^{-\delta(Y,X)/t}\Phi_0\left(\exp(Y)\exp\left(\frac{X}{2}\right)\right) J(Y)\,dY.$$

where $\delta(Y,X) = \frac{1}{2}\left\{\left|Z\left(Y,\frac{X}{2}\right)\right|^2 - \left|\frac{X}{2}\right|^2\right\}$.

As previously indicated, $\delta(Y,X) \geq 0$ for all $X \in p$, $Y \in ik$. In particular

$$V_t(X) \leq \int\limits_{ik} \Phi_0\left(\exp(Y)\exp\left(\frac{X}{2}\right)\right) J(Y)\,dY = \phi_0(\exp(X))$$

for all $t > 0$. Therefore, by dominated convergence,

$$\lim_{t\to\infty} V_t(X) = \phi_0(\exp(X)).$$

The stated asymptotic expansion follows immediately from the Taylor expansion for e^{-x} about $x = 0$. The error term has the form:

$$|E_m(X,t)| = \int\limits_{ik} \frac{\delta(Y,X)^m}{m!t^m}\Phi_0(\exp(Y)\exp\left(\frac{X}{2}\right))J(Y)\,dY$$

But $\left|Z\left(Y,\frac{X}{2}\right)\right| \leq |Y| + \left|\frac{X}{2}\right|$, so that

$$\delta(Y,X)^m \leq 2^{-m}(|X| + |Y|)^m|Y|^m \leq 2^{-m}(1+|X|)^m(1+|Y|)^{2m}$$

We may choose $\epsilon > 0$ such that $e^{\epsilon|h|}\Phi_0(h)$ is integrable over K_C. Given a compact subset Ω of p there is a constant $C(\Omega) > 0$ such that $\Phi_0(Z(Y, \frac{X}{t})) \leq C(\Omega)\Phi_0(\exp(Y))$, $X \in \Omega$, $Y \in ik$. Hence

$$\begin{aligned} |E_m(X,t)| &\leq \frac{C(\Omega)(1+|X|)^m}{2^m m! t^m} \int_{ik} (1+|Y|)^{2m} e^{-\epsilon|Y|} e^{\epsilon|Y|}\Phi_0(\exp(Y))J(Y)\,dY \\ &\leq \frac{C_m C(\Omega)(1+|X|)^m}{t^m}, \quad \text{whenever } X \in \Omega \end{aligned}$$

as claimed. ∎

Remark: It is interesting to compare the bounds on the coefficients of the asymptotic expansion given above with similar results obtained from known solutions of the heat kernel. For instance, if $G = SL(2,R)$ (we could also consider $SL(3,R)$ in view of 3.15) the heat kernel is:

$$H_t(\exp(X)) = ct^{-3/2}e^{-t/8}\int_x^\infty \exp\left(-\frac{h^2}{2t}\right)\frac{h\,dh}{\sqrt{Ch(h) - Ch(x)}}.$$

where $X = diag(x/2, -x/2)$.

Using the notation of Proposition 2.16 we may write

$$V_t(X) = \int_0^\infty e^{-y^2/2t}\frac{y\,dy}{\sqrt{Sh\left(\frac{x+\sqrt{x^2+y^2}}{2}\right)Sh\left(\frac{\sqrt{x^2+y^2}-x}{2}\right)}}$$

The error term $E_m(X,t)$ in the asymptotic expansion for V_t would then take the form

$$|E_m(X,t)| \leq \frac{C_m}{t^m}\int_0^\infty \frac{y^{2m+1}\,dy}{\sqrt{Sh\left(\frac{x+\sqrt{x^2+y^2}}{2}\right)Sh\left(\frac{\sqrt{x^2+y^2}-x}{2}\right)}},$$

where C_m is a constant which does not depend on X or t.

Changing the variable of integration $y \mapsto \frac{\sqrt{x^2+y^2}-x}{2} = \delta$ finally gives

$$
\begin{aligned}
&|E_m(X,t)| \\
&\leq \frac{C'_m(1+x)^{m+1}}{t^m} \int_0^\infty \frac{\delta^m(1+\delta)^{m+1}\,d\delta}{\sqrt{Sh(x+\delta)Sh(\delta)}} \\
&\leq \frac{C'_m}{t_m}(1+x)^{m+1}e^{-x/2} \int_0^\infty \frac{\delta^m(1+\delta)^{m+1}(1+1/\delta)^{1/2}e^{-\delta/2}\,d\delta}{\sqrt{Sh(\delta)}}
\end{aligned}
$$

so that there exists a constant B_m which does not depend on X or t for which

$$|E_m(X,t)| \leq \frac{B_m(1+x)^m}{t^m}\Phi_0\left(\exp\left(\frac{X}{2}\right)\right)$$

holds.

In viewing the nature of the general estimates made in Proposition 2 16, it is reasonable to expect that

$$|E_m(X,t)| \leq \frac{B_m(1+|X|)^m}{t^m}\Phi_0\left(\exp\left(\frac{X}{2}\right)\right)$$

will be satisfied if G is a normal real form G_C.

Equivalently this may be stated by saying that

$$\sup_{X\in p}\int_{ik}(1+|Y|)^m\,|Y|^m \exp\left[-\frac{1}{4}\int_0^1 Trace\left[\frac{\Theta(Z_s)-I}{ad(Z_s)}ad\,Y\right]\right] J(Y)\,dY < \infty$$

Chapter 3

The Abel Transform

In this chapter we indicate how analytic continuation provides a partial solution to the inversion problem for the Abel transform.

3.1 Elementary Subgroups

Let G be a connected semisimple linear Lie group of the noncompact type. We will assign the usual meanings to σ, K, $\mathcal{G}$, k, and p. Let $\mathcal{A}$ be a maximal abelian subspace of p, R the restricted root system associated with the pair $(\mathcal{G}, \mathcal{A})$ and W the corresponding Weyl group. Let us choose a system R^+ of positive restricted roots so that the Iwasawa decompositions $G = KAN$ and

$\mathcal{G} = k + \mathcal{A} + \mathcal{N}$ hold with $\mathcal{N} = \sum_{\alpha \in R^+} \mathcal{G}_\alpha$. The Abel transform of a function $f \in C_c^\infty(K\backslash G/K)$ is defined by

$$\mathcal{T}f(a) = \int_{\mathcal{N}} f(an)\exp[\rho(\log(a))]\,dn, \tag{3.1}$$

where $a \in A$ and $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \dim(\mathcal{G}_\alpha) \cdot \alpha$.

This integral transform establishes an isomorphism between the convolution algebra $C_c^\infty(K\backslash G/K)$ and the (Euclidian) convolution algebra $C_c^\infty(A;W)$ of compactly supported C^∞, W-invariant functions on A. Its significance lies in the fact that it factors Harish-Chandra's spherical Fourier transform in accordance with the following diagram:

$$\begin{tikzcd}
C_c^\infty(K\backslash G/K) \arrow[r, "\textit{Abel Transform}"] \arrow[dr, swap, "\substack{\textit{Haris-Chandra's}\\ \textit{spherical transform}}"] & C_c^\infty(A;W) \arrow[d, "\substack{\textit{Euclidian Fourier}\\ \textit{transform}}"] \\
& \mathcal{H}(\mathcal{A}^*;W)
\end{tikzcd}$$

where $\mathcal{H}(\mathcal{A}^*;W)$ stands for the space of W-invariant functions on $\mathcal{A}^*$ which are of exponential type (see Helgason [12]). Thus, certain problems of Fourier analysis on G/K may be translated to equivalent questions of classical Fourier analysis on A. The establishment on an inversion formula for the Abel transform which is "sufficiently explicit" and preserves the support of functions is of central importance in the use of this dictionary.

By a representing measure for the Abel transform we mean an assignment which to each $a \in A$ associates a measure ν_a on A such that

$$\mathcal{T}f(a) = \int_A f(x)\, d\nu_a(x), \quad a \in A$$

Its existence and properties may be easily established by first considering the well-known representing measure for the dual of the Abel transform.

Definition 3.2: The dual Abel transform is the map $\mathcal{T}^* : C^\infty(A: W) \to C^\infty(K\backslash G/K)$ defined by:

$$\mathcal{T}^* F(g) = \int_K F(\exp(H(gk))) \exp[-\rho(H(gk))]\, dk, \tag{3.3}$$

where the map $g \mapsto H(g)$ is the Iwasawa projection of G onto A. Using the expression $d(kan) = \exp(2\rho(\log(a))) \cdot dk\, da\, dn$ for the Haar measure on G in the Iwasawa decomposition, one easily establishes the following duality relation:

$$\int_A \mathcal{T}f(a)F(a)\, da = \int_G f(g)\mathcal{T}^* F(g)\, dg. \tag{3.4}$$

where $f \in C_c^\infty(K\backslash G/K)$ and $F \in C^\infty(A: W)$.

If $a \in A$ we let $C - Hull(a) = \exp[\text{Convex hull of } W \cdot \log(a)]$. It is well known (see Helgason [12], Ch. IV, Theorem 10.5 and Corollary 10.12) that for each regular $a \in A$ there exists a positive function $K(a, \cdot) \in L^1(A)$ such that:

$$\left.\begin{array}{l} 1.\ \mathcal{T}^{*}F(a) = \int\limits_{A} F(x)K(a,x)\,dx,\ \ F \in C^{\infty}(A;W),\ a \in A \\ 2.\ \text{The support of the measure } d\mu_a(x) = K(a,x)\,dx \text{ coincides} \\ \quad \text{with the set } C - Hull(a) \text{ whenever } a \text{ is a regular element} \\ \quad \text{of } A. \end{array}\right\} \quad (3.5)$$

Lemma 3.6: Let $f \in C_c^{\infty}(K\backslash G/K)$, then its Abel transform $\mathcal{T}f$ may be expressed as:

$$\mathcal{T}f(a) = \frac{1}{|W|}\int\limits_{A} f(x)K(x,a)\,|\delta(x)|\,dx, \quad \forall a \in A.$$

where $\delta(\exp(H)) = \prod_{\alpha\in R^+}(e^{\alpha(H)} - e^{-\alpha(H)})^{\dim(\mathcal{G}_\alpha)}_{\alpha t(H)}$, and $K(.)$ is as in 3.5

Proof: The polar decomposition $G = K\overline{A^+}K$ leads to the integration formula:

$$\int\limits_{G}\phi(g)\,dg = \int\limits_{A^+}\phi(a)\delta(a)\,da = \frac{1}{|W|}\int\limits_{A}\phi(a)\,|\delta(a)|\,da$$

where $\delta(a) = \prod_{\alpha\in R^+}(e^{\alpha(H)} - e^{-\alpha(H)})^{m}_{\log(a)}$. The stated result follows immediately from the duality relation 3.4. ■

Consider an involutive automorphism τ of G which commutes with σ. As usual we let $G[\tau]$ stand for the analytic subgroup of G which corresponds to the subalgebra $\mathcal{G}^{\tau}$ of $\mathcal{G}$ and we write

$$\mathcal{G} = k\cap\mathcal{G}^{\tau} + k\cap q + p\cap\mathcal{G}^{\tau} + p\cap q$$

as in 1.16.

Definition 3.7: An involutive subgroup $G[\tau]$ will be called elementary in case the following conditions are satisfied:

1. $p \cap \mathcal{G}^\tau$ contains a maximal abelian subspace $\mathcal{A}$ of p.
2. for each root $\alpha \in R(\mathcal{G}, \mathcal{A})$, either $\mathcal{G}_\alpha \subseteq \mathcal{G}^\tau$ or $\mathcal{G}_\alpha \subseteq q$.
3. if $\beta \in R(\mathcal{G}, \mathcal{A})$ is such that $\mathcal{G}_\beta \subseteq q$ then β does not vanish on $Centre(\mathcal{G}^\tau) \cap \mathcal{A}$.

Remark: Under the above conditions, the root system R associated with the pair $(\mathcal{G}, \mathcal{A})$ decomposes into two disjoint subsets, $R = R_{(\tau)} \dot{\cup} S$, where $R_{(\tau)}$ is the root system of $(\mathcal{G}^\tau, \mathcal{A})$ and S is the set of all those roots in R which fail to vanish on $Centre(\mathcal{G}^\tau) \cap \mathcal{A}$. If we choose compatible orderings on R and $R_{(\tau)}$ we then write:

$$\mathcal{N} = \sum_{\alpha \in R^+} \mathcal{G}_\alpha, \quad \mathcal{N}_{(\tau)} = \sum_{\alpha \in R^+_{(\tau)}} \mathcal{G}_\alpha, \quad \text{and} \quad \hat{\mathcal{N}} = \sum_{\alpha \in S^+} \mathcal{G}_\alpha$$

so that $\mathcal{N} = \mathcal{N}_{(\tau)} \dot{+} \hat{\mathcal{N}}$.

Lemma 3.8: Let $G[\tau]$ be an elementary subgroup of G. Given compatible orderings on $R(\mathcal{G}, \mathcal{A})$ and $R(\mathcal{G}^\tau, \mathcal{A})$ then:

1. $\hat{\mathcal{N}}$ is a commutative ideal of $\mathcal{N}$ and $G[\tau]$ normalizes $\hat{\mathcal{N}}$.
2. if N is the analytic subgroup of G corresponding to $\mathcal{N}$ then the map $\phi : \mathcal{N}_{(\tau)} \times \hat{\mathcal{N}} \to N$, $\phi(X, Y) = \exp(X)\exp(Y)$ is a diffeomorphism onto.

Proof:

1. Recall that $\mathcal{N}_{(\tau)} \subseteq \mathcal{G}^{\tau}$ and $\hat{\mathcal{N}} \subseteq q$, so that $[\mathcal{N}_{(\tau)}, \hat{\mathcal{N}}] \subseteq \hat{\mathcal{N}}$. Also, if $\alpha, \beta \in S^+$ then for $X_\alpha, X_\beta \in \hat{\mathcal{N}}$ we must have $[X_\alpha, X_\beta] \in \mathcal{N}_{(\tau)}$; however, the linear functional $\alpha + \beta$ fails to vanish on $Centre(\mathcal{G}^{\tau}) \cap \mathcal{A}$, which means that $\alpha + \beta$ is not a root in $R_{(\tau)}$. Therefore $[X_\alpha, X_\beta] = 0$ and consequently $\hat{\mathcal{N}}$ is abelian. Clearly $[\mathcal{N}, \hat{\mathcal{N}}] \subseteq [\mathcal{N}_{(\tau)}, \hat{\mathcal{N}}] + [\hat{\mathcal{N}}, \hat{\mathcal{N}}] \subseteq \hat{\mathcal{N}}$. We can see from the eigenspace decomposition for $\mathcal{G}^{\tau}$, $\mathcal{G}^{\tau} = m \cap \mathcal{G}^{\tau} + \mathcal{A} + \sum_{\alpha \in R_\tau} \mathcal{G}_\alpha$, with $m =$ centralizer of $\mathcal{A}$ in k, that $\hat{\mathcal{N}}$ is normalized by $G[\tau]$ if and only if $[\mathcal{G}_{-\alpha}, \hat{\mathcal{N}}] \subset \hat{\mathcal{N}}$ whenever $\alpha \in R^+_{(\tau)}$. But, for any root $\beta \in S^+$, $-\alpha + \beta$ is either not a root or $-\alpha + \beta \in S^+$, since its restriction to $Centre(\mathcal{G}^{\tau}) \cap \overline{\mathcal{A}^+}$ coincides with β and hence must be positive.

2. If N_τ, $\hat{N}$ and N are the analytic subgroups which correspond to $\mathcal{N}_{(\tau)}$, $\hat{\mathcal{N}}$ and $\mathcal{N}$ respectively, then it is clear that N_τ, $\hat{N}$, N are all simply connected and $N = N_{(\tau)} \cdot \hat{N}$ with $N_1 \cap \hat{N} = (e)$. The rest of the argument is standard (see Helgason [12], Ch. IV, Lemma 6.8) ∎

One often needs to express a subdeterminant of a given matrix. For convenience we introduce the following notation:

Let $(E, \langle , \rangle)$ be a Euclidian space, F a subspace with basis $\{f_1, f_2, \ldots, f_l\}$ and $g \in Aut(E)$. Let

$$j(g; F) = \left[\frac{\det(\langle g f_i, g \cdot f_j \rangle)}{\det(\langle f_i, f_j \rangle)} \right]^{1/2} \tag{3.9}$$

$j(g; F)$ is a numerical invariant for the pair (g, F) which does not depend on the particular choice of basis $\{f_1, \ldots, f_l\}$ for F. In addition, it satisfies the following properties:

1. $j(kg; F) = j(g; F)$ if k is an isometry of E.
2. $j(gx; F) = j(g; xF)j(x; F)$ if $x \in Aut(E)$.

On the Euclidian space $(\mathcal{G}, B_\sigma)$ we see that

$$j(Ad(g); \mathcal{N})^{1/2} = \exp(\rho(H(g))) \quad \forall g \in G,$$

where $\mathcal{N} = \sum_{\alpha \in R^+} \mathcal{G}_\alpha$.

For technical reasons, we prefer to express the Abel transform as a map $\mathcal{T} : C_c^\infty(K\backslash G/K) \to C_c^\infty(K\backslash G/MN)$ according to:

$$\mathcal{T}F(g) = \int_N f(gn) j(Ad(g); \mathcal{N})^{1/2}\, dn, \tag{3.10}$$

where $f \in C_c^\infty(K\backslash G/K)$, $g \in G$.

In connection with Lemma 3.6, we may now state the following version of Aomoto's theorem (see Aomoto [2], Theorem 1).

Proposition 3.11([3bis] [10ter]): Let $G[\tau]$ be an elementary subgroup of G satisfying the assumptions of Lemma 3.8, and for every $f \in C_c^\infty(K\backslash G/K)$

let

$$\hat{\mathcal{T}} f(g) = \int\limits_{\hat{N}} f(g\hat{n}) j(Ad(g); \hat{\mathcal{N}})^{1/2}\, d\hat{n}, \quad g \in G$$

for a suitably normalized Haar measure $d\hat{n}$ on $\hat{N}$, then:

1. $\hat{\mathcal{T}} f \in C_c^\infty(K\backslash G/K \cap G^\tau)$

2. via restriction to $G[\tau]$, the Abel transform $\mathcal{T}$ on G factors as $\mathcal{T} = \mathcal{T}_{(\tau)} \circ \hat{\mathcal{T}}$, where $\mathcal{T}_{(\tau)}$ is the Abel transform associated with the subgroup $G[\tau]$.

Proof: We normalize the measures dn and $dn_{(\tau)}$ on N and $N_{(\tau)}$ respectively in accordance with standard convention, then we fix $d\hat{n}$ so that $dn = dn_{(\tau)} d\hat{n}$.

1. Now, let $g \in G$ and $k \in K \cap G[\tau]$ be arbitrary then

$$\hat{\mathcal{T}} f(gk) = \int\limits_{\hat{N}} f(gk\hat{n}) j(Ad(g)Ad(k); \hat{\mathcal{N}})^{1/2}\, d\hat{n}$$

   $K \cap G[\tau]$ normalizes $\hat{N}$ and acts as a group of isometries on its Lie algebra $\hat{\mathcal{N}}$, thus preserving the measure. Using the properties of 3.9 we easily obtain

$$\hat{\mathcal{T}} f(gk) = \int\limits_{\hat{N}} f(g\hat{n}) j(Ad(g); \hat{\mathcal{N}})^{1/2}\, d\hat{n} = \hat{\mathcal{T}} f(g)$$

2. Since $\hat{\mathcal{T}} f \in C_c^\infty(K_{(\tau)}\backslash G[\tau]/K_C)$, where $K_{(\tau)} = K \cap G[\tau]$, it follows that $\mathcal{T}_{(\tau)} \circ \hat{\mathcal{T}} f$ is well-defined for elements $g \in G[\tau]$. the result is an immediate consequence of Lemma 3.8(2) and the properties of 3.9 ∎

An Example $G = SL(3, R)$

As an example, let us find an explicit expression for the Abel transform on the group of 3×3 unimodular real matrices. Here, K is the subgroup of orthogonal matrices, A the diagonal matrices of determinant 1 and N the subgroup of upper triangular matrices with 1's in the diagonal.

Let τ be the involutive automorphism of G, $\tau(g) = \mu g \mu$, $\forall g \in G$, where

$$\mu = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

It is clear that τ commutes with σ and $G[\tau] \approx GL(2, R)$ is an elementary subgroup. It can be easily verified that:

$$\begin{aligned} N_{(\tau)} &= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix} \middle| \; u \in R \right\}, \quad \text{and} \\ \hat{N} &= \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \; x, y \in R \right\} \end{aligned}$$

The Lie algebra $\mathcal{A}$ of A consists of those diagonal matrices with entries $\lambda_1, \lambda_2, \lambda_3$ for which $\sum \lambda_i = 0$, and we have:

$$\mathcal{N}_{(\tau)} = \mathcal{G}_{\lambda_2 - \lambda_3} \text{ and } \hat{\mathcal{N}} = \mathcal{G}_{\lambda_1 - \lambda_2} + \mathcal{G}_{\lambda_1 - \lambda_3}$$

For convenience, we will use the K-spherical functions $Z_1(g) = Trace(g^t g)$ and $Z_2(g) = Trace((g^t g)^{-1})$, $\forall g \in G$, as W-invariant coordinates on A. The jacobian of the map $a \mapsto (Z_1(a), Z_2(a))$ is given by

$$(dZ_1 \wedge dZ_2)_a = c\,|\delta(a)|\; da, \quad \forall a \in \overline{A^+}$$

with c a positive constant.

Proposition 3.12: If $G = SL(3,R)$, the Abel transform may be expressed as

$$\mathcal{T} f(a) = \int_A f(b) K(b,a)\,|\delta(b)|\; db, \quad a \in A$$

where $K(b,a)$ is a positive multiple of the real period associated with the elliptic curve

$$y^2 = (w-e_1)(w-e_2)(w-e_3)(w-e_4), \quad \text{where}$$

$$\begin{aligned} e_1 &= e^{a_2-a_3} + e^{-a_2+a_3}, & e_2 &= e^{a_1+2b_3} + e^{-a_1-2b_3}, \\ e_3 &= e^{a_1+2b_2} + e^{-a_1-2b_2}, & e_4 &= e^{a_1+2b_1} + e^{-a_1-2b_1}, \\ a &= diag(a_1,a_2,a_3), & b &= diag(b_1,b_2,b_3) \end{aligned}$$

Proof: We first consider writing $\hat{\mathcal{T}} f(d) = \int_{\hat{N}} f(d\hat{n}) j(Ad(d):\hat{N})^{1/2}\, d\hat{n}$ as $\int_A \Psi(b,d) f(b)\,|\delta(b)|\; db$, where $d \in A$. To do so, we compute for a generic $\hat{n} \in \hat{N}$ the values of $Z_1(d\hat{n})$ and $Z_2(d\hat{n})$:

$$\begin{aligned} Z_1(d\hat{n}) &= Z_1(d) + e^{2d_1}x^2 + e^{2d_1}y^2 \\ Z_2(d\hat{n}) &= Z_2(d) + e^{-2d_2}x^2 + e^{-2d_3}y^2, \quad x,y \in R \end{aligned}$$

where $d = diag(d_1, d_2, d_3) \in A$. Let $b \in \overline{A^+}$ be the unique representative of $d\hat{n}$ for which $d\hat{n} \in K \cdot b \cdot K$, and let b_1, b_2, b_3 be its diagonal entries. The jacobian of the map $\hat{n} \mapsto (Z_1(d\hat{n}), Z_2(d\hat{n}))$ gives

$$\mathcal{T}f(d) = \int_{\hat{N}} f(d\hat{n}) j(Ad(d); \hat{N})^{1/2} \, d\hat{n} = c \int_{A} \Psi(b,d) f(b) \, |\delta(b)| \, db,$$

with

$$\begin{aligned} \Psi(b,d) \;\; = \;\; & \Big[-e^{-d_1}(Z_1(b) - Z_1(d))^2 - e^{d_1}(Z_2(b) - Z_2(d))^2 \\ & + \left(e^{d_2-d_3} + e^{d_3-d_2}\right)(Z_1(b) - Z_1(d))(Z_2(b) - Z_2(d)) \Big]^{-1/2}. \end{aligned}$$

Now let us fix an element $a \in diag(a_1, a_2, a_3) \in A$, and for each $n_1 \in N_{(\tau)}$ we write $d(an_1) = diag(d_1, d_2, d_3)$ the diagonal matrix which corresponds to an_1 via the polar decomposition $K\overline{A^+}K = G$. Then by Proposition 3.11(2) we may write:

$$K(b,a) = \int_{N_{(\tau)}} \Psi(b, d(an_1)) j(Ad(a); N_\tau)^{1/2} \, dn_1$$

Changing the variable of integration from n_1 to

$$w(an_1) = e^{d_2-d_3} + e^{d_3-d_2} = w(a) + u^2 e^{a_2-a_3}, \quad u \in R$$

yields, after a few manipulations, the required expression for $K(b,a)$. ∎

3.2 An Inversion Formula

For a complex semisimple group G_C, it is known that the Abel transform is inverted by the differential operator $f \mapsto \bigcirc f$, where $f \in C^{\infty}(A; W)$ and

$$\bigcirc f(\exp(H)) = \frac{1}{\delta(\exp(H))} \cdot \prod_{\alpha > 0} \partial(H_\alpha) f(\exp(H))$$

(see [3] [8]). Analytic continuation may then be used to invert the Abel transform for normal real forms.

Proposition 3.13: Let G be a normal real form of G_C, $F \in C_c^{\infty}(A; W)$, and $\check{F} \in C_c^{\infty}(A; W)$ the uniquely defined function satisfying $\check{F}(\exp(H)) = F(\exp(2H))\ \forall H \in \mathcal{A}$. Then the inverse Abel transform for G may be expressed as:

$$(\mathcal{T}^{-1}F)(\exp(X)) = c \int_{K_C} (\bigcirc \check{F}) \left(h \exp\left(\frac{X}{2}\right)\right) dh, \quad x \in p. \qquad (3.14)$$

where $\bigcirc_{\exp(H)} = \frac{1}{\delta(\exp(H))} \left[\prod_{\alpha>0} \partial(H_\alpha)\right]_{\exp(H)}$, and c is a positive constant.

Proof: Recall that the Abel transform may be viewed as the spherical transform followed by the inverse (Euclidian) Fourier transform. Let $\psi(\exp x)$ denote the right-hand side of 3.14 and let us apply 1.34 to both sides, then

$$\tilde{\psi}(\lambda) = (\bigcirc \check{F})^{\sim}(2\lambda) \quad \forall \lambda \in \mathcal{A}^*.$$

Consequently,

$$\mathcal{T}\psi(\exp(H)) \;=\; \int_{\mathcal{A}^*} \tilde{\psi}(\lambda) e^{i\lambda(H)}\, d\lambda = c \cdot \int_{\mathcal{A}^*} (\bigcirc \check{F})^{\sim}(\Lambda) e^{i\Lambda(H/2)}\, d\Lambda$$

$$= \check{F}\left(\exp\left(\frac{H}{2}\right)\right) = F(\exp(H))$$

as required. ∎

As an example, we give an explicit expression for the inverse Abel transform in the case $G = SL(3, R)$.

Let us consider a regular element $x \in p$ of the form

$$x = \begin{pmatrix} x_1 & 0 & x_2 \\ 0 & -2x_1 & 0 \\ x_2 & 0 & x_1 \end{pmatrix}$$

and fix and Iwasawa decomposition for K_C such that every element $h \in K_C$ has a unique expression

$$h = k \cdot \exp\left[U\left(\frac{w}{\sqrt{2}}\right)\right] \exp[A(t)],$$

with $k \in K$, $w \in C$, $t \in R$.

$$U(z) = \begin{pmatrix} 0 & z & 0 \\ -z & 0 & iz \\ 0 & -iz & 0 \end{pmatrix} \quad \text{and} \quad A(t) = \begin{pmatrix} 0 & 0 & it \\ 0 & 0 & 0 \\ -it & 0 & 0 \end{pmatrix}$$

One easily verifies that the unitary transformation

$$u = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & i \\ 0 & \sqrt{2} & 0 \\ i & 0 & 1 \end{pmatrix}$$

commutes with x and transforms by conjugation $A(t)$ into the diagonal matrix $H(t) = diag(t, 0, -t)$, resp. $U(w/\sqrt{2})$ into the upper triangular matrix

$$V(w) = \begin{pmatrix} 0 & w & 0 \\ 0 & 0 & iw \\ 0 & 0 & 0 \end{pmatrix}.$$

Writing $w = Re^{-i\theta}$ and applying the unitary transformation $g_\theta = diag(e^{i\theta}, 1, e^{-i\theta})$ finally gives:

$$\mathcal{T}^{-1}F(\exp(x)) = c \cdot \int_{-\infty}^{+\infty} \int_0^{\infty} \int_0^{2\pi} \bigcirc \check{F}\left(\exp[V(R)]\exp[H_t]\exp\left[Ad(g_\theta)\frac{x}{2}\right]\right) \, d\theta \, R dR \, dt$$

Again, we may use G_C^x spherical coordinates on G_C in order to express $\mathcal{T}^{-1}F(\exp(x))$ in the format of Proposition 3.12. If we let $Z_1(g) = Trace(g^*g)$ and $Z_2(g) = Trace((g^*g)^{-1})$ then for $g = \exp[V(R)]\exp[H_t]\exp[Ad(g_\theta)x/2]$ we have:

$$\begin{aligned} Z_1(g) &= e^{x_1} Ch(x_2)\left[e^{2t} + \left(1 + \frac{R^2}{2}\right)^2 e^{-2t}\right] + (1 + R^2)e^{-2x_1} \\ &\quad + e^{x_1} Sh(x_2) R^2 \sin(2\theta) \\ Z_2(g) &= e^{-x_1} Ch(x_2)\left[e^{2t} + \left(1 + \frac{R^2}{2}\right)^2 e^{-2t}\right] + (1 + R^2)e^{2x_1} \\ &\quad - e^{x_1} Sh(x_2) R^2 \sin(2\theta) \end{aligned}$$

By performing computations similar to those that appear in Proposition 3.12 we finally obtain:

Theorem 3.15: Let $G = SL(3, R)$ and $F \in C_c^{\infty}(A; W)$ then the inverse Abel transform $\mathcal{T}^{-1}F$ of F may be expressed as:

$$\mathcal{T}^{-1}F(\exp(x)) = \int_A \bigcirc \check{F}(b) K(b, \exp(x)) \, |\delta(b)| \, db$$

where $K(b, \exp(x))$ is a positive (constant) multiple of the real period which corresponds to the elliptic curve

$$y^2 = (B^2 - D^2)(1 - w^2)\left(w - \frac{A - C}{B - D}\right)\left(w - \frac{A + C}{B + D}\right),$$

where

$$\begin{aligned}
A &= e^{2x_1} Z_1(b) - e^{-2x_1} Z_2(b) \\
B &= Sh(x_2)\left[e^{-x_1} Z_1(b) + e^{x_1} Z_2(b) - 2Ch(3x_1)\right] \\
C &= Ch(x_2)\left[-e^{-x_1} Z_1(b) + e^{x_1} Z_2(b) + 2Sh(3x_1)\right] \\
D &= 4Ch(x_2)Sh(x_2), \\
X &= diag(x_1 + x_2, -2x_1, x_1 - x_2), \text{ and} \\
\bigcirc f(\exp(H)) &= \frac{1}{\delta(\exp(H))} \prod_{\alpha > 0} \partial(H_\alpha) f(\exp(H)).
\end{aligned}$$

Remark: The Abel transform $U_t = \mathcal{T}H_t$ of the heat kernel on a symmetric space G/K is the fundamental solution of the following Euclidian diffusion problem:

$$L_A U_t - |\rho_0|^2 U_t = \frac{\partial U_t}{\partial t}$$

(see Helgason [11], Corollary 5.20).

Let H_t be the heat kernel for $SL(3, R)$ then we have the simple expression

$$\mathcal{T}H_t(\exp(H)) = ct^{-1} e^{-|\rho_0|_0^2 t} \exp\left[-\frac{|H|_0^2}{4t}\right] \quad \text{for } H \in \mathcal{A}.$$

We may then apply Theorem 3.15 to solve for H_t:

$$H_t(exp(X)) = ct^{-4}e^{-|\delta_0|_0^2 t} \int_{\mathcal{D}^+(X)} \exp\left[-\frac{|H|_0^2}{2t}\right] K(\exp(H), \exp(X)) \left|\prod_{\alpha>0} \alpha(H)\right| dH, \tag{3.16}$$

where $X = diag(x_1 + x_2, -2x_1, x_1 - x_2)$, $K(\cdot,\cdot)$ is the kernel described in Theorem 3.15, and the integration is to be carried out over the domain

$$\mathcal{D}^+(X) = \{H \in \mathcal{A} \mid \exp(X) \in C - Hull(\exp(H))\}$$

Notes

Chapter 1

The elementary discussion in the first two parts of Chapter 1 constitutes what could be described as standard knowledge in Lie Group Theory. Whenever possible, a specific reference is given (sources are mainly [11], [12], [22], and [13ter]). In some cases, no direct reference could be found. Thus, Lemma 1.29 generalizes a discussion on root systems in [7]§2 and Lemma 1.14 is completely elementary and probably well-known.

Chapter 2

The proof of Proposition 2.8 differs slightly from the argument given by Flensted-Jensen ([7], Example, p 131–132). It reflects my own understanding at the time of writing. However, as has been pointed out by Carl Herz, the relation between the Casimir operators for G and G_C ([7], 4.12 and 4.14) leads to a much shorter proof.

Proposition 2.15 represents one of my contributions to the subject. Jean Philippe Anker had obtained, in [1]§2.4, the upper bound

$$H_t(\exp(H)) \leq C(T)t^{-d/2} \prod_{\alpha \in R_0^+} (1 + \alpha(H)^{\dim \mathcal{G}_\alpha} e^{-\rho_0(H) - |H|_0^2/4t}.$$

with $0 < t \leq T$, $H \in \overline{\mathcal{A}_0^+}$, and he conjectured that $\dim(\mathcal{G}_\alpha)$ could be replaced by $1/2 \dim \mathcal{G}_\alpha$. Our estimate is qualitatively different, since it is expressed as a first order expansion in $t^{1/2}$. Our estimate does not prove the conjecture, but appears to support it, since

$$U_0(H) \approx \prod_{\alpha \in R_0^+} (1 + \alpha(H))^{1/2 \dim(\mathcal{G}_\alpha)} e^{-\rho_0(H)}$$

As far as the error term is concerned (see Proposition 2.15(2)), it has been pointed out by both Carl Herz and Jean-Philippe Anker that an improved version is possible on account of the parity of the function $s \mapsto |Z_s|^2$. Indeed this leads to an asymptotic expansion for H_t (t small) in powers of t rather than $t^{1/2}$. However, an estimate for

$$\frac{\left|Z_{\sqrt{t}}\right|^2 - |Z_o|^2}{2t} - \frac{1}{2}\left\langle \Theta\left(\frac{x}{2}\right) \dot{Y}, \dot{Y} \right\rangle = \frac{1}{6t} \int_0^{\sqrt{t}} \left\langle \Theta(\ddot{Z}_s) \dot{Y}, \dot{Y} \right\rangle (\lambda - s)^3 ds$$

must then be provided, and this appears to give some difficulties

As Professor Anker has indicated, there is a completely different approach to the study of the small timee asymptotic behaviour of the Heat Kernel based on general principles (see Minakshisundaram-Pleijel asymptotics [4] Ch III §3-4). It is not clear if such an approach may provide an existence proof for the small time asymptotic expansion as a convergent power series.

Proposition 2.16 represents our second partial contribution to the subject. However, it must be noted that the cases $SL(2, R)$ and $SL(3, R)$ had already been established by Patrice Sawyer [20bis] using different methods. An upper bound in the case of normal real forms has also been given by Jean-Philippe Anker in [1]§2.4, namely

$$H_t(\exp(H)) \leq C(T)t^{-n/2} \prod_{\alpha>0}(1+\alpha(H))e^{-|\rho_0|_0^2-\rho_0(H)-|H_0|_0^2/4t},$$

with $t > 0$, $H \in \overline{\mathcal{A}^-}$.

Chapter 3

The slightly more elaborate version of Aomoto's theorem (Proposition 3.11) has been known in perhaps slightly different form by Hba A. [10ter]. Furthermore, the generalization of Aomoto's theorem was also studied by Beerends R. J. in [3bis].

For $SL(3, R)$, Proposition 3.12 expresses the Abel transform in a slightly more elegant form than in [3]§6 (also [3bis] Chap. III). However, the result itself was known to Patrice Sawyer [20bis].

The inversion formula (Proposition 3.13) appears to have been known to the experts but was apparently never used in explicit form. It must be noted that in the case $SL(3, R)$ (Proposition 3.15) the result is implied in the work of Patrice Sawyer [20bis], who obtained an explicit expression for the Heat Kernel using different methods.

We wish to acknowledge the assistance of Professor Carl Herz in removing some obscurities and inadequacies in the original version of the text. As a result, Lemma 1.17 now has a complete proof, and various misprints or awkward notational inconveniences have been lifted. We are also indebted to Professor Jean-Philippe Anker for his thorough review of the original text. His various comments were incorporated alongside those of Carl Herz to form the greater part of these notes. The extended bibliography (appearing with additional entries [...bis], [...ter], etc.,) also reflects Professor Anker's contribution and has allowed us to focus on the relevance of the results in relation to previous work in the area.

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