Towards a T-dual Cosmology

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Abstract

In this thesis we address the singularity problem within the formalism of Double Field Theory (DFT). This problem arises in the context of General Relativity and is also present in string cosmology. We start by introducing the necessary elements of cosmology, string theory, string cosmology and DFT which allow the understanding of the research presented here. The first result encompasses a generalization of the DFT action in order to consider point particle motion. After deriving the geodesic equations for the point particle, we argue that the geodesic motion can be extended to both past and future infinities once the appropriate physical clock is considered. Then, our second paper entails an argument that upon considering the double geometry of DFT, a singular background cosmology in Einstein gravity corresponds to a universe expanding to infinite size in the dual dimensions. Our third work describes how a hydrodynamical fluid can be coupled to the DFT action, which allows us to recover Friedmann-like equations in the context of DFT when a frame is chosen. We consider two frames, the supergravity and the winding frames, and study their respective cosmology for a gas of closed strings considering the dilaton to be stabilized. The last work included here shows that the solutions found in the supergravity frame have a T-dual correspondent in the winding frame.

Abrégé

Dans cette thèse, nous abordons le problème de la singularité à l'intérieur du formalisme de la théorie double des champs. Ce problème survient dans le contexte de la Relativité Générale et est aussi présent en cosmologie cordiste. Nous commençons par introduire les éléments nécessaires de cosmologie, de la théorie des cordes, de cosmologie cordiste et de la théorie double des champs qui permettent la compréhension de la recherche présentée dans cette thèse. Le premier résultat englobe une généralisation de l'action de la théorie double des champs dans l'objectif de considérer le mouvement d'une particule ponctuelle. Après avoir calculé les équations de géodésique pour la particule ponctuelle, nous argumentons que le mouvement le long d'une géodésique peut être continué jusqu'aux passé et futur infinis quand l'horloge appropriée est choisie. Ensuite, notre deuxième article implique un argument selon lequel, en considérant la géométrie double de la théorie double des champs, un fond cosmologique singulier en gravité einsteinienne correspond à un univers qui prend de l'expansion jusqu'à une taille infinie dans les dimensions duelles. Notre troisième article décrit comment un fluide hydrodynamique peut être couplé à l'action de la théorie double des champs, ce qui nous permet de retrouver des équations semblables aux équations de Friedmann dans le contexte de la théorie double des champs quand un cadre est choisi. Nous considérons deux cadres, le cadre de la supergravité et le cadre de l'enroulement, et nous étudions leur cosmologie respective pour un gaz de cordes fermées, assumant que le dilaton est stabilisé. Le dernier article inclus dans cette thèse démontre que les solutions trouvées dans le cadre de la supergravité ont une correspondance de dualité T dans le cadre d'enroulement.

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Preface

This manuscript thesis contains four peer-reviewed and published articles that are original and consist of distinct contributions to knowledge. These are presented in their original form, due to copyright, in Chapters 6, 7, 8 and 9. It is usual in the field of Cosmology that the authors are listed in alphabetical order. For that reason, below we state the contribution of the author to each of the included works.

Contributions of the Author

Robert Brandenberger, Renato Costa, Guilherme Franzmann, Amanda Weltman, *Point particle motion in double field theory and a singularity-free cosmological solution*, Phys. Rev. D **97** (2018) no. 6, 063530 [1].

This article is presented in Chapter 6. It entails the first paper of a two year collaboration between researchers at McGill University and at University of Cape Town around the main topic of my PhD. The project resulted in three papers, which are also introduced below. For this project, all the authors participated in the discussions, the calculations were performed by Renato and me and the writing was done by Robert, Renato and me.

Robert Brandenberger, Renato Costa, Guilherme Franzmann, Amanda Weltman, *Dual spacetime and nonsingular string cosmology*, Phys. Rev. D**98** (2018) no. 6, 063521 [2].

This paper is the second of the same collaboration introduced above and is presented in Chapter 7. It is a continuation of the previous work, introducing complementary tools and concepts that also led to the third project below. In this work, all the authors participated in the discussions, the calculations were performed by Renato and me and the writing was done by Robert, Renato and me.

Robert Brandenberger, Renato Costa, Guilherme Franzmann, Amanda Weltman, *T-dual cosmological solutions in double field theory* , Phys. Rev. D**99** (2019) no. 2, 023531 [3].

This work is the third and last paper of the previous collaboration and is presented in Chapter 8. Similar to the previous ones, all the authors participated in the discussions, the calculations were performed by Renato and me and the writing was done by Robert, Renato and me.

Heliudson Bernardo, Robert Brandenberger, Guilherme Franzmann, *T-dual cosmological solutions in double field theory. II.*, Phys. Rev. D**99** (2019) no. 6, 063521 [4].

This project is an extension to the latest work introduced above and was envisioned by the visiting PhD student Heliudson Bernardo. The article is presented in Chapter 9. For this paper, my main role was of supervising the work done by Heliudson and participating in all the discussions done between the authors. Most of the calculations were done by Heliudson and me, however I have done them mostly in order to verify the results that were being obtained by Heliudson. The writing was done by Robert and Heliudson, while I have presented my suggestions throughout the manuscript.

Publications not included in this thesis

1. Robert Brandenberger, Renato Costa, Guilherme Franzmann, *Can backreaction prevent eternal inflation?*, Phys. Rev. D**92** (2015) no. 4, 043517 [5].

2. Robert Brandenberger, Guilherme Franzmann, Qiuyue Liang, *Running of the Spectrum of Cosmological Perturbations in String Gas Cosmology*, Phys.Rev. D**96** (2017) no. 12, 123513 [6].

3. Renato Costa, Rodrigo R. Cuzinatoo, Elisa M. G. Ferreira, Guilherme Franzmann, *Covariant c-flation: a variational approach*, International Journal of Modern Physics D**28** (2019) 1950119 [7].

Preprint Publications

- George De Conto, Guilherme Franzmann, *Atomic beings and the discovery of gravity*, arXiv:1511.05431 [astro-ph.CO] [8].

- Guilherme Franzmann, *Varying fundamental constants: a full covariant approach and cosmological applications*, arXiv:1704.07368 [gr-qc] [9].
- Elisa Ferreira, G.M., Guilherme Franzmann, Justin Khoury, Robert Brandenberger, *Unified Superfluid Dark Sector*, arXiv:1810.09474 [astro-ph.CO] [10] *(submitted to JCAP)*.
- Renato Costa, Guilherme Franzmann, Jonas P. Pereira, *A local Lagrangian for MOND as modified inertia*, arXiv:1904.07321 [gr-qc] [11] *(submitted to PRD)*.
- Stephen Angus, Kyoungho Cho, Guilherme Franzmann, Jeong-Hyuck Park and Shinji Mukohyama, *$O(D, D)$ completion of the Friedmann Equations*, arXiv:1905.03620 [hep-th] [12] *(submitted to JCAP)*.
- Guilherme Franzmann and Yigit Yargic, *A democratic Cosmos? (Essay written for the 2019 Essay Prize in Philosophy of Cosmology)*.

Chapter 1

Introduction

The Standard Cosmological Model (SCM) is quite successful. Relying solely on six free parameters, it is able to account for most of the current data, which has become abundant for the last 30 years starting with the COBE (Cosmic Background Explorer) mission. It also provides an exquisite description of the evolution of the Universe that extends from a fraction of a second to its current age, around 13.8 billion years [13].

A recent attachment to the SCM is the inflationary paradigm for the very early universe. Inflation [14, 15, 16, 17, 18] postulates a phase of accelerated expansion in the early universe that explains why the Universe we live in seems to be so spatially flat, so large and nearly homogeneous. It also explains how the small fluctuations in the Cosmic Microwave Background (CMB) are generated and why they are almost scale-invariant, and therefore it also explains how structures such as galaxies and galaxy clusters have been formed in our Universe. Most of the inflationary models rely on a quasi-de Sitter initial phase that smoothly shifts towards a phase of standard radiation decelerated expansion.

Paradoxically, the success of the SMC is also its Achilles' heel. Its potential to provide us with a coherent picture of the Universe relies on mysterious forms of matter and energy, Dark Matter and Dark Energy, which remain elusive to our understanding, regardless of being responsible for about 95% of the Universe's energy budget. Moreover, even after considering the inflationary paradigm, we still lack a satisfying picture of the very early Universe, since it cannot avoid singularity theorems [19, 20, 21, 22]. Currently, it is expected that only a fully-fledged theory of Quantum Gravity (QG) could yield a nonsingular cosmology. This thesis is concerned with the singularity problem.

String Theory (ST) is one of the most promising candidates for a QG theory (see [23, 24, 25] for other alternatives). Among its successes, the theory unifies all the known interactions of nature under the same framework: first (perturbative) [26, 27] and second (nonperturbative) [28, 29, 30] quantization of 1-dimensional strings. All that is observed of low energies and accounted for by the Standard Model of Particle Physics is a result of different degrees of freedom of these strings (however see about the string landscape [31]).

One of the main advantages to consider strings as being fundamental instead of point particles is the fact that the singularity theorems can be avoided. This is easy to understand intuitively, since as the energy scale gets higher, the energy can flow into the additional degrees of freedom present due to the extra dimensionality of the string.

Among their degrees of freedom, one can find three different types: momentum, winding and oscillatory modes. Momentum modes are the same as the ones already encountered in particle physics, corresponding to the centre of mass motion of the string and typically decomposed in an infinite set of Fourier modes. The oscillatory modes are due to transverse oscillations along the string. Finally, winding modes only exist in the presence of closed strings, since it accounts for the amount of times a string winds around a compact dimension in a topologically nontrivial way.

Not only the existence of new degrees of freedom is relevant, but also the existence of new symmetries and dualities [32] which arise in ST. In particular, one can interchange the winding for the momentum modes, and the theory remains the same as long as the radii of the compact dimensions are taken to their reciprocal. This is a particular example of T-duality, which can be roughly understood as the theory being the same for very large and very small radii of the compact dimensions (this translates into the scale-factor duality in cosmological models of supergravity (SUGRA) [33]). Given that the momentum modes are the conjugate of the physical coordinates, it is natural to wonder if there would be coordinates which are also the conjugate of the winding modes. In fact, this idea was already put forward when String Gas Cosmology (SGC) was first proposed [34].

SGC relies fundamentally in the existence of the extra degrees of freedom of the string as well as the symmetries introduced in ST. It considers a thermodynamical gas of closed strings as the relevant substrate in the very early universe, which leads to a maximal temperature, the Hagedorn temperature. Thus, SGC could potentially model a nonsingular cosmology, at least thermodynamically. Unfortunately, the available cosmology provided by SUGRA is bounded to be singular [33], and therefore SGC cannot be embedded into standard SUGRA.

One ingredient that has not been taken into account in formulating string cosmology so far is the potential for the winding modes also to define the physical space.

At the nonperturbative level, string field theory naturally introduces physical coordinates associated with the winding modes [35]. Effectively, the underlying geometry has its number of compact dimensions doubled. In the low-energy limit double coordinates have been introduced in [36, 37, 38, 39], and this led to a new framework called Double Field Theory (DFT) [35, 40].

DFT lifts T-duality to become an underlying geometrical symmetry. Similar to Einstein gravity where the action is invariant under generalized coordinate transformations, the DFT action is invariant under the $O(D, D)$ symmetry group, which accounts for generalized diffeomorphims as well as generalized gauge transformations. Moreover, it also recovers the Buscher rules that are implemented for T-dual transformations in standard SUGRA, now embedded into one of the symmetry transformations of the $O(D, D)$ group.

In DFT, the fundamental degrees of freedom are encoded in the massless modes of the bosonic string: a scalar field (dilaton), a symmetric rank 2 field (metric), and an anti-symmetric rank 2 field (B-field). These fields are responsible for the gravitational sector of the theory, just like in SUGRA. The main difference is that there is an underlying double geometry which allows choosing different coordinate frames, where the frame choice corresponds to which degrees of freedom of the string are excited, which potentially can allow us to avoid the singular behavior as the universe contracts, since this contraction is tied to a frame of choice. Hence, DFT may be able to yield a nonsingular cosmology, and potentially allows SGC to be embedded into it.

In its original formulation, DFT corresponds to the theory in vacuum, since no matter content respecting the underlying symmetries had been considered. Thus, part of the challenge to consider its cosmology is to add matter to the theory in a consistent way. The first paper of this thesis considers this challenge from the point of view of point particle motion [1] (Chapter 6), where we argue that geodesics can be completed arbitrarily to the past and future if the point particle motion is interpreted in terms of physical clocks.

In the next paper [2] (Chapter 7), we have introduced a phenomenological equation of state accounting for the gas of closed strings, allowing us to have an effective treatment of the dynamics of the model in standard SUGRA. In the context of DFT, we argue that a nonsingular cosmology is indeed possible. Moreover, we also formalize the heuristics behind a physical clock, since this extends the dual geometry of DFT to also include a temporal dual coordinate.

The two remaining papers [3, 4] (Chapters 8 and 9) discuss the full cosmology associated with the equation of state of a gas of closed strings in the context of DFT. In order to do so, we first conceive a way to add a hydrodynamical fluid to the DFT action, and thus consider the resulting dynamics in different coordinate frames: the one associated with the momentum coordinates (supergravity frame) and the one associated with the winding coordinates (the winding frame). Each of these frames is kinematically chosen considering the appropriate regime: large and small radii of the compact dimensions, respectively. The final outcome is a T-dual solution between those two different frames, where a contracting solution in the supergravity frame is seen as expanding from the point of view of the winding frame.

Although we still do not have a complete DFT, which would entail a dynamical choice amongst the frames, we believe that the T-dual solution found here is an important development for the DFT cosmology. This solution hints towards two asymptotic regimes of a nonsingular, smooth solution yet to be found in the context of the full theory.

☯

This thesis is organized as follows: Chapter 2 intends to present a review of the cosmological elements necessary for the understanding of the cosmology used in the following sections. Chapter 3 introduces important elements of String Theory used in the remaining chapters. Then, Chapter 4 discusses String Cosmology in the context of a gas of closed strings, and Chapter 5 briefly reviews the Double Field Theory framework, laying down the necessary tools for the final chapters.

The first paper is presented in Chapter 6, where we consider the study of a point particle in the context of Double Field Theory. In Chapter 7, we present the second paper, where we have argued for a dual spacetime and discussed potential generalizations to Double Field Theory. In Chapter 8, the third paper is reproduced where cosmological solutions in the context of Double Field Theory that could be used to connect T-dual regimes in the very early universe are found. Finally, Chapter 9 is the fourth paper, which considers a generalization of these solutions.

Chapter 2

Elements of Cosmology

As we have discussed in the Introduction, the main problem in Cosmology that we would like to make progress on is the initial singularity problem. This challenge emerges in the context of General Relativity when the dynamics of the cosmological background is considered in light of the data available to us. Thus, a good place to start is by introducing the necessary elements to make this issue explicit.

2.1 Theoretical Anchors

Cosmology is concerned with the origin, evolution and structure of the Universe. For most of our scientific history, that entailed the study of the motions of the objects in the sky. The first time that such a line of inquiry was possible came after Newton's theory of gravity, where the universality of that force allowed us to consider the same law ruling objects' motions on Earth to be also defining the mechanics of the Cosmos.

It took us a few centuries to improve on this paradigm. In 1915, Albert Einstein proposed the Theory of General Relativity (GR) [41, 42], in which gravity was promoted from a force to become pure inertia due to the geometry of space and time, now dynamical and represented by their interwoven structure called spacetime. From now on, the study of the Cosmos shifted from the mere analysis of the motions of bodies to the very dynamics of the spacetime seen as a whole. It is here that we start.

The spacetime dynamics is described by Einstein's equations,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} + \Lambda g_{\mu\nu}. \tag{2.1.1}$$

where c is the speed of light, taken to be unity unless otherwise mentioned, G is the Newton's constant and Λ is the cosmological constant, taken to be zero from now on. The left side contains information about the geometry, encoded in the Einstein tensor $G_{\mu\nu}$, which contains information about the curvature of the spacetime through its elements $R_{\mu\nu}$, the Ricci tensor, and R, the Ricci scalar. They are defined as

$$R_{\mu\nu} \equiv \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\nu\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha} \tag{2.1.2}$$

$$R \equiv g_{\mu\nu} R^{\mu\nu}, \tag{2.1.3}$$

where the Christoffel symbols, $\Gamma^\rho_{\mu\nu}$, are defined as,

$$\Gamma^\alpha_{\mu\nu} \equiv \frac{1}{2} g^{\alpha\beta}(\partial_\nu g_{\mu\beta} + \partial_\mu g_{\nu\beta} - \partial_\beta g_{\mu\nu}), \tag{2.1.4}$$

$g_{\mu\nu}$ the metric of the spacetime, the fundamental dynamical field associated to gravitational phenomena. Finally, the right side of (2.1.1) is related to the matter-energy content, described by the energy-momentum tensor, $T_{\mu\nu}$. Thus, the source of the spacetime's dynamics is the energy content within spacetime.

The action associated to Einstein's equations is the Einstein-Hilbert action coupled to the matter action, S_m, given by,

$$S = \frac{c^4}{16\pi G} \int d^4x \sqrt{-g} R + S_m, \tag{2.1.5}$$

that recovers (2.1.1) when varied in relation to the metric, given that the energy-momentum tensor is defined as,

$$T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}. \tag{2.1.6}$$

Einstein's equations are very general and can be applied to any gravitational system. In particular, the first system ever considered was Mercury's orbit around the Sun [43]. His equations described its perihelion motion that could not be properly accounted after using only Newtonian gravity. If we aim to use Einstein's equations to study Cosmology, we need to simplify the problem by making use of its symmetries. In order to do so, we rely on two principles: the Copernican principle and the Cosmological principle. The former states that we do not occupy any particularly special place in the Cosmos, while the latter states that the distribution of matter in the Universe is homogeneous and isotropic[1] given a large enough scale[2]. When those principles were first proposed, we did not have observations for anchoring them. Nowadays, we know that the Universe looks isotropic from our priviledged point of view, which combined with the Coperninan principle results in a Universe that is also homogeneous. In some sense, the Cosmological principle has become obsolete.

Hence, combing Einstein's equations with a universe that is homogeneous and isotropic, we can investigate an ansatz for the metric that respects such symmetries. This is known as the Friedmann-Robertson-Walker (FRW) metric[3] [44], and its line element is written as,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right], \tag{2.1.7}$$

where t is the physical time, (r, θ, φ) are comoving spatial polar coordinates, $a(t)$ is the scale factor which parametrizes the background expansion of the Universe, and k parametrizes the spatial curvature, assuming the values $(-1, 0, +1)$ which represent an hyperbolic, flat and spherical spatial geometry, respectively. We will consider $k = 0$ for the next chapters unless stated otherwise. By imposing the symmetries above, we have reduced all the dynamics of the Universe on large scales to a single function of time.

Making use of the Cosmological principle for the matter-energy content of the Universe, we can also consider its content to be approximately homogeneous and isotropic, and properly described by a perfect fluid,

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}, \tag{2.1.8}$$

where ρ is the energy density of the fluid, p its pressure and u^μ its 4-velocity. Note that ρ and p are defined in the fluid's rest frame and they are functions only of time due to the symmetries we are considering. In comoving coordinates, the 4-velocity is $u_\mu = (-1, 0, 0, 0)$

[1]Homogeneity implies the physical system is invariant under translations while isotropy means it is invariant under rotations.

[2]Usually referred as the cosmological scale, it is currently of the order of hundreds of Megaparsecs, where one parsec corresponds to 3.26 light-years, roughly 3×10^{13} kilometers. This is a dynamical scale and it changes according to the Universe's evolution.

[3]Note that Alexander Friedmann was the first to develop this model, preceding the others by more than a decade.

and the energy-momentum tensor takes the form,

$$T_{\mu\nu} = \text{diag}\ (\rho, p, p, p). \tag{2.1.9}$$

Having (2.1.9) and (2.1.7) at hand, it is just a matter of inserting them into the Einstein's equations in order to have the equations determining the dynamics of the Universe.

2.2 Friedmann Equations

We start realizing that although (2.1.1) entails ten equations coming from the ten independent components (we have two-index symmetric tensors in four dimensions), the symmetries in the FRW metric reduces the problem into only two independent equations. In order to understand how this is so, we can span the spacetime with four normalized 4-vectors, one of them the timelike 4-velocity, u^μ, and the three others being spacelike, s^α. In Cartesian coordinates, they are parametrized as $\{\partial_t, \partial_i;\ i = x, y, z\}$. Imposing isotropy, we have

$$G^\mu{}_\nu u^\nu \propto u^\mu,$$

otherwise we would have a preferred direction in space. Hence, $G^\mu{}_\nu u^\nu s_\mu = 0$. As a result, the time-space components of the Einstein's equations are identically zero. Besides, the diagonal space-space components yield the same equation, as can be inferred by (2.1.8). Therefore, there remains two independent equations,

$$G_{\mu\nu} s^\mu s^\nu = 8\pi T_{\mu\nu} s^\mu s^\nu \to G_{\star\star} = 8\pi p \tag{2.2.10}$$

$$G_{\mu\nu} u^\mu u^\nu = 8\pi T_{\mu\nu} u^\mu u^\nu \to G_{tt} = 8\pi\rho. \tag{2.2.11}$$

The Ricci tensor and scalar can be easily obtained,

$$\begin{aligned} R_{00} &= -3\frac{\ddot{a}}{a} \\ R_{ij} &= \delta_{ij}\left[2\dot{a}^2 + a\ddot{a} + 2\frac{k}{a^2}\right] \\ R &= 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right]. \end{aligned} \tag{2.2.12}$$

Finally, inserting (2.2.12) into (2.2.10) and (2.2.11), we can derive the Friedmann equations

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}. \tag{2.2.13}$$

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{4\pi G}{3}(\rho + 3p) \tag{2.2.14}$$

where we introduced an important quantity, the Hubble parameter H,

$$H \equiv \frac{\dot{a}}{a}, \tag{2.2.15}$$

which is positive for an expanding universe and negative for a collapsing one. It sets the characteristic time- and length-scales of the universe, $t \sim H^{-1}$ and $l \sim H^{-1}$.

The Friedmann equations can be combined, yielding the continuity equation,

$$\dot{\rho} + 3H(\rho + p) = 0, \tag{2.2.16}$$

which corresponds to the local conservation of energy on cosmological scales. Its covariant expression is given by the conservation law $\nabla_\mu T^\mu_\nu = 0$, where ∇_μ is the covariant derivative. In GR, the local conservation law is the corresponding physical statement to the geometrical property that for any (non-)Riemannian manifold the contracted Bianchi identity is zero, which is translated to $\nabla_\mu G^{\mu\nu} = 0$.

The Friedmann equations' set has cardinality two. However, we have three variables: the geometrical scale factor and the two hydrodynamical properties of matter. Therefore, we need one more independent equation in order to have a determined system. The remaining equation to be introduced is the equation of state, which characterizes the fluid that is present in the Universe by relating its energy density and pressure in the following way,

$$p = w\rho, \tag{2.2.17}$$

where w is a function of time. Most components of the Universe can be parametrized by a constant equation of state, which is calculated after thermodynamical considerations in a static background. In particular, in four dimensions we have $w = 0$ for matter (baryonic or dark), $w = 1/3$ for radiation, $w = -1$ for a cosmological constant, and $w = -1/3$ for winding modes (we will discuss them in the next chapter).

For a single fluid with a constant equation of state, we can solve the continuity equation (2.2.16) for the energy density,

$$\rho = \rho_0 a(t)^{-3(1+w)}, \tag{2.2.18}$$

where ρ_0 is the initial energy density. Plugging this solution into the Friedmann equations, we can find the evolution of the scale factor as well (assuming a universe spatially flat),

$$a(t) = a_0 t^{\frac{2}{3(1+w)}}, \quad (w \neq -1) \tag{2.2.19}$$

where a_0 is the initial scale factor. For $w = -1$, the energy density is a constant and so is H, corresponding to maximally symmetric solutions: anti-de Sitter ($H < 0$), de Sitter ($H > 0$) and Minkowski ($H = 0$).

The most relevant aspect of the solution (2.2.19) is that for $w > -1$, the time dependence of the scale factor is a power-law with a positive exponent, meaning that in the limit $t \to 0$, the scale factor vanishes and the spacetime's curvature diverges. Although this is explicit here for a constant equation of state, it has been shown to be a much more general feature of the equations for almost any given matter-energy content of the Universe [19]. This is the initial singularity problem.

2.3 The initial singularity

We have found a limit of GR in which the theory seems to break down. Should we be concerned? Is there any evidence that our Universe has gone through such a phase? After all, our Universe is composed of different kinds of matter-energy contents, and possibly they could balance each other out in such a way that the evolution of the Universe had never reached such extreme conditions.

In fact, when Einstein proposed GR and applied it to the Cosmos, there was a wide belief the Universe was static. He even introduced a cosmological constant to his equations hoping to counterbalance the effect of gravity [45]. However, that did not last.

In 1929, Edwin Hubble observed that distant galaxies were moving away from the Earth [46]. That meant the universe was under expansion instead of being static. If the universe was expanding, it should have been smaller in the past. If that had always been the case arbitrarily to the past, then it meant there was a moment in which the Universe was effectively shrunk to a point. Having the Universe's volume shrinking to zero would imply that quantities as the energy density, temperature, and others would be infinite. That was the physical evidence of the initial singularity that was already presented in the FRW ansatz's dynamics. Nonetheless, this remained controversial for a few decades, since the evidence was indirect.

The controversy ended with the discovery of the Cosmic Microwave Background (CMB), see Fig. 2.1. It had been already theoretically predicted by George Gamow [47], and was finally detected in 1964 by Arno Penzias and Robert Wilson [48]. They observed the existence of a radiation in the microwave spectrum (with temperature around 2.7 K) permeating the Universe, which is understood today as a remnant fossil from a very hot ($T \sim 3 \times 10^3$ K) and dense epoch of the early universe ($\sim 3 \times 10^5$ years). It is mostly because of this discovery that the Big Bang, how it was later called, became part of the current paradigm of Cosmology in the context of Einstein's gravity.

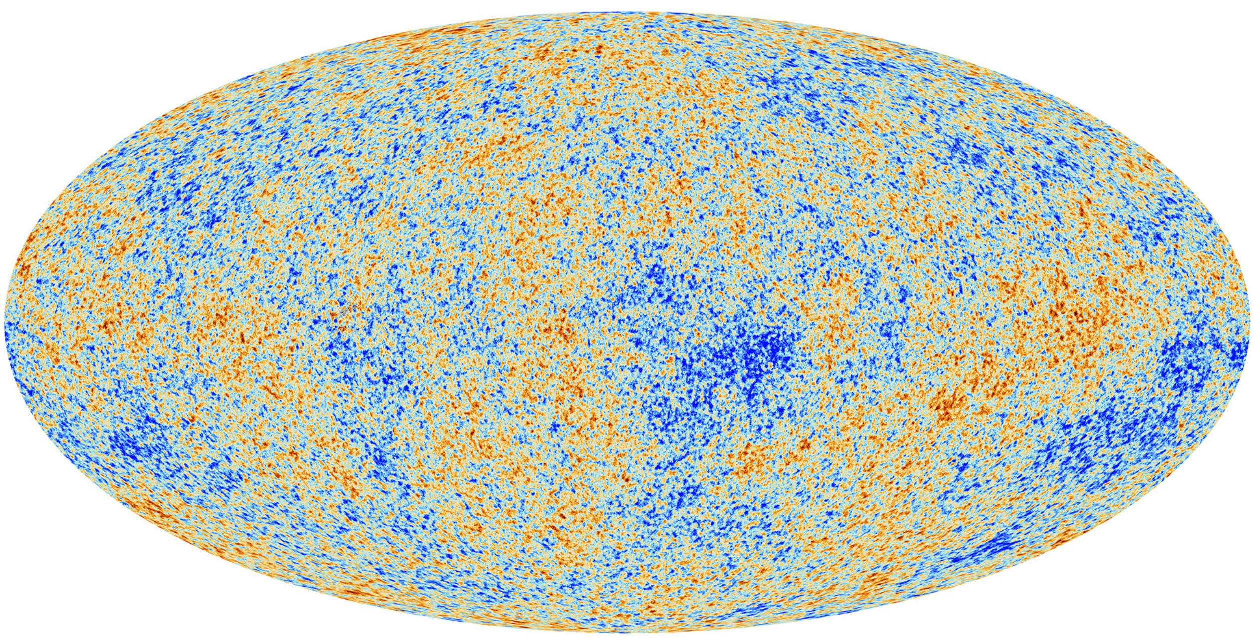

Fig. 2.1 The CMB measured by the Planck Satellite [49]. The average temperature is about 2.7 K and the fluctuations are of order 10^{-5} K. Red and blue spots represent hotter and colder spots in the sky, respectively.

2.4 Can we do better?

As we have already discussed in the Introduction, it is hoped that a fully-fledged theory of Quantum Gravity (QG) should yield a nonsingular cosmology. Among the candidates for a QG theory, String Theory (ST) is one of the most promising candidates (see [23, 50, 24, 25] for other alternatives). Among its successes, the theory unifies all the known interactions of nature under the same framework: first (perturbative) [26, 27] and second (non-perturbative) [28, 29, 30] quantization of 1-dimensional strings. All that is observed at low energies and accounted for in the Standard Model of Particle Physics is a result of different degrees of freedom of these strings (however see about the string landscape [31]).

One of the main advantages to consider extended objects as being fundamental as opposed to point particles is the fact that the singularity theorems [19] can be avoided. Note that these theorems apply either to the Standard Cosmological Model as well to its extensions considering an inflationary period [20, 21] or bouncing cosmologies. The reason strings can avoid those theorems can be intuitively understood after picturing that as the energy scale gets higher, the energy can flow into the additional degrees of freedom provided by the extra dimensionality of the string. Thus, if the universe is best described by the existence of strings as the fundamental objects at very high energies, one should expect a very different picture of the early universe than the one provided by the Standard Cosmological Scenario.

We will take ST as our proxy theory for QG. It is important to say first that ST has been under development for about 40 years and still has not provided a full description of the early universe dynamics in which we do not have a singular universe. Hence, the goal of our research is to develop a description of the early universe which is consistent with the principles of ST and that enables new insights and models attempting to describe the phase where GR breaks down.

Thus, we now attend to review some elements of String Theory.

Chapter 3

Elements of String Theory

We still do not have a full-fledged theory of quantum gravity, but we do have at least a promising candidate: String Theory. Not only does the theory promote gravity to be quantum, but it also provides most of the ingredients we believe to be necessary to make up our Universe: at low-energies it leads naturally to General Relativity, gauge theories, scalar fields and fermions. Thus, it also plays the role of a unifying theory of nature.

For those reasons, regardless of String Theory really being the final description of the fundamental interactions of nature, it is certainly worth taking it as a proxy in our attempts to understand many questions concerning the realm of quantum gravity.

In this chapter, we lay down most of the machinery required to discuss String Cosmology and Double Field Theory. We will follow closely [26, 51, 52, 53].

3.1 String Action

A point particle sweeps out a worldline in Minkowski space and its action corresponds to the worldline's length. Moreover, its action can be made not dependent on the parametrization used to describe its trajectory, thus being *reparametrization invariant*. This is equivalent to say that the worldline is diffeomorphism invariant.

We follow the same prescription for a string. Since the string is one-dimensional, it sweeps out a worldsheet instead. Its action must correspond to the area spanned by the worldsheet and it should also be invariant under reparametrization of the coordinates used to describe it. Such an action exists and it is called the Polyakov action,

$$S_P[X,\gamma] = -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma(-\gamma)^{\frac{1}{2}}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}, \tag{3.1.1}$$

where M denotes the worldsheet, a two-dimensional surface embedded in the target space[1] representing the string *trajectory*; τ and σ are time and space parameters describing the worldsheet, respectively; γ is the two-dimensional dynamical embedded metric; X^μ are the spacetime coordinates of the string, which are vectors in the spacetime but scalars in the worldsheet. Note that their indices are contracted with the D-dimensional flat spacetime metric $\eta_{\mu\nu}$. We have also introduced a coupling, α', that is inversely proportional to the tension of the string,

$$T = 1/2\pi\alpha', \tag{3.1.2}$$

where the tension is equal to the mass per unit length and has dimension $[T] = 2$. Thus, $[\alpha'] = -2$, and we can associate a length scale to it, l_s, by

$$\alpha' = l_s^2, \tag{3.1.3}$$

the *string length* l_s, the natural length that appears in String Theory. Generally, the string length is larger than the Planck length [54].

The action (3.1.1) has the following symmetries:

1. D-dimensional Poincaré invariance:

$$\begin{aligned} X'^\mu(\sigma^a) &= \Lambda^\mu{}_\nu X^\nu(\sigma^a) + a^\mu \\ \gamma'^{ab}(\sigma^a) &= \gamma^{ab}(\sigma^a), \end{aligned} \tag{3.1.4}$$

where Λ is a Lorentz transformation and a^μ is a constant translation. Note that σ^a represent the worldsheet coordinates.

2. Worldsheet diffeomorphism invariance:

$$\begin{aligned} X'^\mu(\sigma^{a\prime}) &= X^\mu(\sigma^a) \\ \frac{\partial\sigma'^c}{\partial\sigma^a}\frac{\partial\sigma'^d}{\partial\sigma^b}\gamma'_{cd}(\sigma^{a\prime}) &= \gamma_{ab}(\sigma^{a\prime}), \end{aligned} \tag{3.1.5}$$

[1] In sub-Section 3.5.1 we discuss more about the relation between target space and spacetime.

for new coordinates $\sigma'^a(\sigma^a)$. This is a gauge symmetry on the worldsheet. The fields X^μ transform as worldsheet scalars, while γ_{ab} transforms as a metric.

3. Two-dimensional Weyl invariance:

$$\begin{aligned} X'^\mu(\sigma^a) &= X^\mu(\sigma^a) \\ \gamma'_{ab}(\sigma^a) &= \exp[2\omega(\sigma^a)]\gamma_{ab}(\sigma^a), \end{aligned}$$

for arbitrary $\omega(\sigma^a)$. This is another gauge symmetry of the string and it means that two metrics related by a Weyl transformation are to be considered as the same physical state.

Note that the Weyl symmetry implies that the theory is invariant under a local change of scale preserving the angles between all lines. The fact that the Polyakov action is invariant under it is particular to two-dimensions, since the scaling factor coming from the determinant $\sqrt{-\gamma}$ cancels the one coming from the inverse metric. Moreover, if we aim to keep this symmetry, then the interactions we can add to the string action are very limited (for instance, neither a potential for the worldsheet scalars nor a worldsheet cosmological constant can be added).

We can define the energy-momentum tensor on the worldsheet by varying the action with respect to γ, i.e.,

$$\begin{aligned} T^{ab}(\sigma^a) :&= -4\pi(-\gamma)^{-1/2}\frac{\delta}{\delta\gamma_{ab}}S_p \\ &= -\frac{1}{\alpha'}\left(\partial^a X^\mu \partial^b X_\mu - \frac{1}{2}\gamma^{ab}\partial_c X^\mu \partial^c X_\mu\right), \end{aligned} \tag{3.1.6}$$

which is conserved, $\nabla_a T^{ab} = 0$, and traceless, $T^a{}_a = 0$, due to diffeomorphism and Weyl invariances, respectively.

The equations of motion can be obtained by varying the action with respect to γ_{ab} and X^μ,

$$T_{ab} = 0 \tag{3.1.7}$$

$$\gamma^{ab}\partial_a\partial_b X^\mu = 0, \tag{3.1.8}$$

respectively. Notice we are ignoring surface terms since we will be exclusively interested in closed strings, for which we consider the following boundary conditions,

$$X^\mu(\tau, 2\pi) = X^\mu(\tau, 0), \quad \partial^\sigma X^\mu(\tau, 2\pi) = \partial^\sigma X^\mu(\tau, 0) \tag{3.1.9}$$

$$\gamma_{ab}(\tau, 2\pi) = \gamma_{ab}(\tau, 0), \tag{3.1.10}$$

where $\tau \in (-\infty, \infty)$ and $\sigma \in [0, 2\pi]$. Thus, the fields are periodic and the string has no ending points, forming closed loops.

3.2 The closed string spectrum

To study the string spectrum we can fix most of the symmetries discussed above and only work with the conformal properties of the Polyakov action. For that purpose, we can consider the *unitary gauge*,

$$\gamma_{ab} = \eta_{ab}. \tag{3.2.11}$$

This gauge fixing leaves a group of local symmetries not fixed, the so called *conformal transformations*, which is a combination of diffeomorphisms and Weyl transformations.

The equations of motion (3.1.8) can be solved after introducing lightcone coordinates on the worldsheet,

$$\sigma^\pm = \tau \pm \sigma,$$

so that the equation of motion (3.1.8) reads

$$\partial_+ \partial_- X^\mu = 0. \tag{3.2.12}$$

The most general solution is,

$$X^\mu(\sigma^a) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-)$$

for arbitrary functions X^μ_L and X^μ_R. These describe left- and right-moving waves, respectively. The most general solution expanded in Fourier modes considering the boundary conditions

(3.1.9) is

$$X_L^\mu(\sigma^+) = \tfrac{1}{2}x^\mu + \tfrac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-in\sigma^+},$$
$$X_R^\mu(\sigma^-) = \tfrac{1}{2}x^\mu + \tfrac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^-}, \quad (3.2.13)$$

where the normalization was chosen for later convenience. The variables x^μ and p^μ are the position and momentum of the center of mass of the string. Moreover, given that X^μ are real, we have

$$\alpha_n^\mu = (\alpha_{-n}^\mu)^\star, \quad \tilde{\alpha}_n^\mu = (\tilde{\alpha}_{-n}^\mu)^\star. \quad (3.2.14)$$

We still need to impose the constraints coming from (3.1.7), which in the worldsheet lightcone coordinates become,

$$(\partial_+ X)^2 = (\partial_- X)^2 = 0. \quad (3.2.15)$$

These equations give constraints on the momenta p^μ and the Fourier coefficients α_n^μ and $\tilde{\alpha}_n^\mu$, which can be elegantly written as,

$$L_n = \tilde{L}_n = 0, \quad n \in \mathbb{Z}, \quad (3.2.16)$$

where we have defined,

$$\alpha_0^\mu = \tilde{\alpha}_0^\mu \equiv \sqrt{\frac{\alpha'}{2}} p^\mu, \quad L_n = \frac{1}{2}\sum_m \alpha_{n-m}\cdot\alpha_m, \quad (3.2.17)$$

and analogously for $\tilde{L}_n$.

In Minkowski space the square of the spacetime momentum is equal to the square of the rest mass of a particle,

$$p_\mu p^\mu = -M^2.$$

Thus, considering the rest frame of the string and (3.2.17), the mass of the string can be expressed as,

$$M^2 = \frac{4}{\alpha'} \sum_{n>0} \alpha_n \cdot \alpha_{-n} = \frac{4}{\alpha'} \sum_{n>0} \tilde{\alpha}_n \cdot \tilde{\alpha}_{-n}, \tag{3.2.18}$$

written in terms of right- or left-moving oscillators. The fact that those two expressions are the same is known as *level matching*, which implies that the number of left and right oscillators must be the same. This has a deeper origin that can be traced to the fact that the string states should be invariant under σ translations, since the origin of the spatial coordinate parametrizing the closed string should not matter. This gauge symmetry is fundamentally what imposes the number of oscillators to be matched in both directions.

3.3 The massless states

Our ultimate goal here is to introduce the necessary elements of string theory so that we can start discussing its resulting cosmology. For that, we need to investigate the massless modes of the string, since they are the ones expected to populate the background and to mediate long-range interactions.

In order to find out the massless modes we need to quantize the string. That can be done in many different ways: lightcone, canonical covariant, path integral and BRST quantization. All of those procedures agree amongst themselves and highlight different aspects of the quantum string. We do not plan to go over any of these though, since the quantization of the string will not play a role for the remaining chapters. Instead, we will highlight the main results below from the canonical quantization point of view.

- The modes are promoted to operators with canonical commutation relations,

$$[x^\mu, p_\nu] = i\delta^\mu_{\ \nu} \quad \text{and} \quad [\alpha^\mu_n, \alpha^\nu_m] = [\tilde{\alpha}^\mu_n, \tilde{\alpha}^\nu_m] = n\,\eta^{\mu\nu}\delta_{n+m,0}\ , \tag{3.3.19}$$

  where the modes α_n can be further normalized to be standard harmonic oscillator creation and annihilation operators; the same follows for $\tilde{\alpha}$. Thus, each scalar field is associated to two infinite tower of creation/annihilation operators corresponding to left- and right-moving modes.

- The vacuum state of a single string, $|0\rangle$, is defined as the state annihilated by all the annihilation operators,

$$\alpha_n^\mu \, |0\rangle = \tilde{\alpha}_n^\mu \, |0\rangle = 0, \qquad \text{for} \;\; n > 0, \tag{3.3.20}$$

while the true vacuum state[2] carries yet another quantum number, p^μ, the eigenvalue of the momentum operator. Thus, the true ground state of the string also obeys,

$$\hat{p}^\mu \, |0; p\rangle = p^\mu |0; p\rangle. \tag{3.3.21}$$

- A generic state of the string is, therefore, written after acting with any number of creation operators on the vacuum,

$$(\alpha_{-1}^{\mu_1})^{n_{\mu_1}} (\alpha_{-2}^{\mu_2})^{n_{\mu_2}} \ldots (\tilde{\alpha}_{-1}^{\nu_1})^{n_{\nu_1}} (\tilde{\alpha}_{-2}^{\nu_2})^{n_{\nu_2}} \ldots |0; p\rangle. \tag{3.3.22}$$

  Each state corresponds to a different excited state of the string, which corresponds to different particles in spacetime. Note that we have an infinite tower of states, and therefore an infinite number of particles in string theory.

- Notice that the covariant canonical quantization scheme above introduces states with a negative norm since the spacetime metric is Lorentzian. These states are called ghosts and cannot appear in any physical process. It turns out that the gauge symmetries of the string gets rid of them, fortunately.

- We still need to impose the classical constraints (3.2.16) at the quantum level. At the end of the day, it suffices to introduce them acting only on states[3]. Thus,

$$L_n |\text{phys}\rangle = \tilde{L}_n |\text{phys}\rangle = 0 \quad \text{for} \;\; n > 0. \tag{3.3.23}$$

  Note that we need to consider the ordering of the mode operators defining L_0 and $\tilde{L}_0$. They can be normal ordered and defined to be,

$$L_0 = \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m + \frac{1}{2} \, \alpha_0^2, \quad \tilde{L}_0 = \sum_{m=1}^{\infty} \tilde{\alpha}_{-m} \cdot \tilde{\alpha}_m + \frac{1}{2} \, \tilde{\alpha}_0^2. \tag{3.3.24}$$

[2]It accounts not only for the oscillations of the string, but as well as for the existence of any string.

[3]This implicitly assumes $L_n^\dagger = L_{-n}$, since the only requirement is to impose the constraints over matrix elements defined by physical states.

However that implies that their constraints on the states are ambiguous,

$$(L_0 - a)|\psi\rangle = (\tilde{L}_0 - a)|\psi\rangle = 0 \tag{3.3.25}$$

where a is a constant and ψ corresponds to any physical state. This implies that the mass spectrum of the string (3.2.18) is given by,

$$M^2 = \frac{4}{\alpha'}\left(-a + \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m\right) = \frac{4}{\alpha'}\left(-a + \sum_{m=1}^{\infty} \tilde{\alpha}_{-m} \cdot \tilde{\alpha}_m\right). \tag{3.3.26}$$

- Getting rid of the ghosts results in two main requirements: a unique choice of a is demanded, $a = 1$, and the number of scalar fields is $D = 26$. The string theory we are considering here only makes sense at the quantum level in a 26-dimensional spacetime. In fact, it also tells us, considering a lightcone quantization of the string, that the remaining degrees of freedom describe the transverse fluctuations of the string, with

$$M^2 = \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n>0} \alpha_n^i \alpha_{-n}^i = \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n>0} \tilde{\alpha}_n^i \tilde{\alpha}_{-n}^i \ . \tag{3.3.27}$$

for $i = 1, \ldots, D-2$. Correspondingly, the Hilbert space of states also becomes restricted,

$$\hat{p}^\mu|0;p\rangle = p^\mu|0;p^\mu\rangle, \qquad \alpha_n^i|0;p\rangle = \tilde{\alpha}_n^i|0;p\rangle = 0 \quad \text{for} \ \ n > 0. \tag{3.3.28}$$

The mass spectrum is obtained after regularizing the zero point energy, being written as,

$$M^2 = \frac{4}{\alpha'}\left(N - \frac{D-2}{24}\right) = \frac{4}{\alpha'}\left(\tilde{N} - \frac{D-2}{24}\right) \ . \tag{3.3.29}$$

where we have introduced the level operators,

$$N = \sum_{i=1}^{D-2} \sum_{n>0} \alpha_{-n}^i \alpha_n^i, \qquad \tilde{N} = \sum_{i=1}^{D-2} \sum_{n>0} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i \ . \tag{3.3.30}$$

differing from the number operators of a harmonic oscillator because of the modes' normalization. Note that the level matching condition can be now written as $N = \tilde{N}$.

We are finally in a position to investigate the string's excitations and their corresponding field content.

3.3.1 The lightest state

The lightest closed string state is

$$|0,0;k\rangle, \quad M^2 = \frac{2-D}{6\alpha'}, \tag{3.3.31}$$

which has negative mass for $D = 26$. Such particles are called tachyons. It is still not clear if bosonic string theory can be made sense having this excitation. However, its presence disappears once one consider superstrings (adding also fermions on the worldsheet).

3.3.2 The first excited states

The first excited states are

$$\alpha^i_{-1}\tilde{\alpha}^j_{-1}|0,0;k\rangle, \quad M^2 = \frac{26-D}{6\alpha'}. \tag{3.3.32}$$

These should be massless states in order for the quantum theory to preserve Lorentz symmetry, and hence $D = 26$. They transform as a 2-tensor under SO$(D-2)$. We can decompose any such tensor e^{ij} as

$$e^{ij} = \frac{1}{2}\left(e^{ij} + e^{ji} - \frac{2}{D-2}\delta^{ij}e^{kk}\right) + \frac{1}{2}(e^{ij} - e^{ji}) + \frac{1}{D-2}\delta^{ij}e^{kk}, \tag{3.3.33}$$

that is, a symmetric traceless tensor, an antisymmetric tensor and a scalar, that do not mix under rotations. These irreducible parts give rise to the following fields: the graviton (symmetric traceless), 2-form gauge boson (antisymmetric) and a dilaton (scalar). These three massless fields are common to all string theories, and that is the main reason we will focus on them quite closely from now on in order to develop string cosmology. To each of these modes, we associate a massless field in spacetime, represented by

$$g_{\mu\nu}(X), \qquad b_{\mu\nu}(X), \qquad \phi(X). \tag{3.3.34}$$

3.4 Strings in a general background

We have just discovered that the massless states of the string correspond to three different fields. Those fields are expected to compose the background in which the string is propagating. The action that accounts for a string moving in such a background is given by

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{\gamma} \left(g_{\mu\nu}(X) \partial_\alpha X^\mu \partial^\alpha X^\nu + i b_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu \epsilon^{\alpha\beta} + \alpha' \phi(X) R^{(2)} \right), \quad (3.4.35)$$

where $R^{(2)}$ is the two-dimensional Ricci scalar of the worldsheet and $\epsilon^{\alpha\beta}$ is the anti-symmetric 2-tensor, normalized such that $\sqrt{\gamma}\epsilon^{12} = +1$. Although this action naturally generalizes the Polyakov action, it defines an interacting theory for the scalar fields. Note that the dilaton's coupling vanishes on a flat worldsheet, $R^{(2)} = 0$. Moreover, as we discussed above, this coupling violates Weyl invariance even classically. How can we make sense of that?

It turns out that the dilaton's coupling is modulated by the presence of α', which comes in purely in terms of dimensional grounds. At the end of the day, that means that this violation of the Weyl invariance can be compensated by a one-loop contribution arising from the couplings to $g_{\mu\nu}$ and $b_{\mu\nu}$ when computing the beta functions.

To make this explicit, we can look for the breakdown of the Weyl symmetry in terms of the trace of the energy-momentum tensor, $\langle T^\alpha{}_\alpha \rangle$. Each field can contribute in a different way to the trace of the energy-momentum tensor, and thus we can define three different beta functions,

$$\langle T^\alpha{}_\alpha \rangle = -\frac{1}{2\alpha'} \beta_{\mu\nu}(g)\, \gamma^{\alpha\beta} \partial_\alpha X^\mu\, \partial_\beta X^\nu - \frac{i}{2\alpha'} \beta_{\mu\nu}(b)\, \epsilon^{\alpha\beta} \partial_\alpha X^\mu\, \partial_\beta X^\nu - \frac{1}{2} \beta(\phi) R^{(2)}. \quad (3.4.36)$$

The one-loop calculation for the beta functions results in [52],

$$\beta_{\mu\nu}(g) = \alpha' R_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \phi - \frac{\alpha'}{4} H_{\mu\lambda\kappa} H_\nu{}^{\lambda\kappa} \quad (3.4.37)$$

$$\beta_{\mu\nu}(b) = -\frac{\alpha'}{2} \nabla^\lambda H_{\lambda\mu\nu} + \alpha' \nabla^\lambda \phi\, H_{\lambda\mu\nu} \quad (3.4.38)$$

$$\beta(\phi) = -\frac{\alpha'}{2} \nabla^2 \phi + \alpha' \nabla_\mu \phi\, \nabla^\mu \phi - \frac{\alpha'}{24} H_{\mu\nu\lambda} H^{\mu\nu\lambda}, \quad (3.4.39)$$

where

$$H_{\mu\nu\rho} = \partial_\mu b_{\nu\rho} + \partial_\nu b_{\rho\mu} + \partial_\rho b_{\mu\nu}. \tag{3.4.40}$$

A consistent background of string theory must preserve Weyl invariance, and therefore the energy-momentum tensor should be traceless, which requires $\beta_{\mu\nu}(g) = \beta_{\mu\nu}(b) = \beta(\phi) = 0$.

3.4.1 The Low-Energy Effective Action

The equations $\beta_{\mu\nu}(g) = \beta_{\mu\nu}(b) = \beta(\phi) = 0$ can be viewed as the equations of motion for the background in which the string propagates. We now change our perspective: we look for a $D = 26$ dimensional spacetime action which reproduces these beta-function equations as its equations of motion. This is the *low-energy effective action* of the bosonic string,

$$S = \frac{1}{2\kappa_0^2} \int d^{26}X \sqrt{-g}\, e^{-2\phi} \left(R - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + 4\partial_\mu \phi\, \partial^\mu \phi \right). \tag{3.4.41}$$

On dimensional grounds alone, the coupling scales as $\kappa_0^2 \sim l_s^{24}$ where $\alpha' = l_s^2$. This action will be called *supergravity action* from now on. The caveat "low-energy" refers to the fact that we only worked with the one-loop beta functions which requires small spacetime curvature.

We can easily see that this action is invariant under spacetime diffeomorphisms,

$$\Phi \to \Phi' = \Phi + L_\lambda \phi, \tag{3.4.42}$$

where Φ is any of the fields above and L_λ is the Lie Derivative parametrized by infinitesimal vectors λ^i. The action is also invariant under gauge transformations of the two form,

$$b_{\mu\nu} \to b_{\mu\nu} + \partial_\mu \tilde{\lambda}_\nu - \partial_\nu \tilde{\lambda}_\mu. \tag{3.4.43}$$

The equations of motion associated to this action are exactly the same as demanding that the β-functions vanish,

$$R_{\mu\nu} - \frac{1}{4}H_{\mu}^{\rho\sigma}H_{\nu\rho\sigma} + 2\nabla_{\mu}\nabla_{\nu}\phi = 0 \tag{3.4.44}$$

$$\frac{1}{2}\nabla^{rho}H_{\rho\mu\nu} - H_{\rho\mu\nu}\nabla^{\rho}\phi = 0 \tag{3.4.45}$$

$$\frac{1}{2}\nabla^2\phi - \nabla_{\mu}\phi\,\nabla^{\mu}\phi + \frac{1}{24}H_{\mu\nu\lambda}H^{\mu\nu\lambda} = 0. \tag{3.4.46}$$

We will refer to these equations as the vacuum supergravity equations.

3.5 Compactification and T-Duality

We have already studied the closed string spectrum and how its excited modes are responsible for the background dynamics through its massless excitations. In doing so, we have seen that the number of spacetime dimensions required for the theory to be consistent is higher than the macroscopically observed number of dimensions. Therefore, it is expected that some of the dimensions in string theory are small and compact. Let us see how that can change the spectrum of the closed strings in the simplest case, when one of the dimensions is a circle.

The first effect is that the spatial momentum in the compact direction becomes quantized[4],

$$p = \frac{n}{R}, \qquad n \in \mathbb{Z}. \tag{3.5.47}$$

This is known from having a free particle in a box, where its wavefunction includes the factor $e^{ip\cdot X}$, which must be single valued. The same happens with a string.

The second effect is that the boundary conditions for the scalar field in that direction becomes more general,

$$X(\sigma + 2\pi) = X(\sigma) + 2\pi m R, \qquad m \in \mathbb{Z}, \tag{3.5.48}$$

in comparison to (3.1.9). The integer m corresponds to how many times the string winds around the circle, and it is called the winding number.

[4] All the spacetime quantities that are index-free refer to the circle.

Those two effects result in a different mass spectrum for an observer living in the non-compact directions. The short story is that the mass spectrum becomes,

$$M^2 = \frac{n^2}{R^2} + \frac{m^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2) \tag{3.5.49}$$

and the level matching no longer tells us that $N = \tilde{N}$, but instead[5]

$$N - \tilde{N} = nm \tag{3.5.50}$$

Therefore, we see that a string with $n > 0$ momentum units gives a contribution to its mass but a similar contribution can be obtained by a string which winds around the compact dimension. What is astonishing about this formula is that those contributions are reciprocal in nature if we consider their dependence on the size of the compact direction. This is the tip of the iceberg of a fundamental duality in string theory: T-duality.

3.5.1 T-duality

The surprising aspect of the presence of the winding modes is that their energy contribution is proportional to the radius of the compact dimension, in contrast to the momentum modes, which is inversely proportional to the radius. Thus, it is reasonable to expect that as the box expands or shrinks, different modes are excited, which can be easily understood by the simple assumption that nature tends to privilege energetically cheap configurations.

More precisely, as $R/\alpha' \to \infty$, the winding modes become very heavy and are irrelevant for the low-energy dynamics while the momentum modes become very light and form a continuum in this limit. In the converse limit, as $R/\alpha' \to 0$, the momentum modes become heavy and can be ignored while the winding modes become light and start to form a continuum.

This dual behavior can be made explicit when we look to the mass spectrum (3.5.49) and

[5] Note that now there are more possibilities for having massless states that do not require $N = \tilde{N} = 1$, since we can have $N = \tilde{N} = 0$ while having either m or n being different than zero, as long as we find the particular radius for which the mass becomes zero.

realize that it remains invariant under this following transformation,

$$n \longleftrightarrow m \tag{3.5.51}$$

$$R \longleftrightarrow \alpha'/R, \tag{3.5.52}$$

which exchanges the number of momentum and winding modes as well as consider the reciprocal of the radius scaled by α'. Although we show this duality here for a very simple example, it is actually a duality of the full theory, including any physical process under consideration. Effectively, the duality exchanges what is meant by winding and what is meant by moving as the strings are not able to distinguish the difference between very large and very small circles.

The most important consequence of such mode-availability is that the notion of position becomes a derived concept. An easy way to understand that is to imagine that in order to measure distances, for instance, we need to build a photon wave-package, which depends on the Fourier modes available. However, as we have just seen, those are dependent on the size of the box in which the observer finds herself. This has been already pointed out in the seminal work on String Gas Cosmology [34], where the notion of two position operators was introduced,

$$|x\rangle = \sum_p e^{ix\cdot p}|p\rangle$$

$$|\tilde{x}\rangle = \sum_w e^{i\tilde{x}\cdot w}|w\rangle,$$

where $p = n/R$ and $w = mR/\alpha'$, while the physical position operator, $|x_p\rangle$, would correspond to a linear combination of them depending on the size of the box. In fact, nowadays this dual coordinate notion has also been explored in cosmological backgrounds for the time coordinate [3, 4]. Thus, string states in general could be seen as point particles propagating in a doubled space,

$$X^M = \left(x^i, \tilde{x}_i\right),$$

where i runs over the compact dimensions.

The progress we make relying on T-duality is in our account of what spacetime really is and how it can be reconstructed. As it turns out, the Fourier transform of momentum space, historically referred to as target space, to which we usually attribute the notion of the physical spacetime is shaken, for now there is an analogous transformation we can consider for the winding modes, defining a *winding space*.

Thus, there are two notions of space: *winding space* and *target space*. The duality between them given the above transformation makes the definition of the physical spacetime subtler [55]. The reason for that is that when an observer probes the World, she will do it using the modes which are available to her: when momentum modes are available, target space will match physical space; while when she uses winding modes for the reconstruction, then winding space will match the physical space.

The key point is to realize that winding and target space are dual to each other by (3.5.51). Hence, even though for small radius, $R < l_s$, momentum modes are not available and the reconstruction is made using winding modes, which results in an effective radius of α'/R, that does correspond to what would be the dynamics of having momentum modes in a space with radius R. Thus, the seemly inaccessible region $l_s > R > l_P$ in target space can be reconstructed by using winding space with an effective radius $l_s < \tilde{R} < \alpha'/l_P$.

One of the physical consequences of this mechanics is that in string theory there is a minimum length scale. As the circle shrinks to smaller and smaller sizes, when it reaches $R = \sqrt{\alpha'}$, the theory acts as if the circle is growing again, with winding modes playing the role of momentum modes. If we can make use of this to tackle the initial singularity, then we can avoid the problem altogether. In order to do so, we need to have a theory that make this duality explicit. The closest we have to it so far is the framework introduced by Double Field Theory, that we review in Chapter 5. Before we consider it, we can study how the tools introduced in this chapter so far can improve our early universe picture over the one coming from general relativity.

Chapter 4

Elements of String Cosmology

Having derived the supergravity action, we now turn to consider its resulting cosmology. We follow similar steps as the ones in Chapter 2 and consider a cosmological background defined by the massless excitations of the string: the metric, the 2-form and the dilaton.

As we have discussed, to compute their low energy action it was assumed that the space-time curvature is small compared to the string scale. We will have to keep this approximation in mind when considering the viable regimes to which our dynamics is applicable.

Moreover, so far we have ignored the presence of all the other massive modes. In fact, the supergravity action is the analog of the Einstein-Hilbert action, corresponding to the gravitational sector in the absence of matter, i.e., vacuum. One of the ways to consider the other massive modes is to consider them all being the matter to which we couple the gravitational sector. We will show how that can be implemented using a hydrodynamical fluid, and then we will focus on a gas of closed strings.

4.1 Equations of motion

We start considering the supergravity action,

$$S_0 = \frac{1}{2\kappa_0^2} \int d^D X \sqrt{-g}\, e^{-2\phi} \left(R - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + 4\partial_\mu \phi\, \partial^\mu \phi \right), \tag{4.1.1}$$

where from now on we consider an arbitrary number of dimensions D and we will be ignoring the 2-form. Therefore, we will be mostly concerned with the dynamics given by the metric and the dilaton field. Note that we are calling this action S_0 since it corresponds to the action defined by gravitational sector, e.g., the massless fields, in the absence of any matter. Later on we will also couple matter to the gravitational sector similarly as it is done in GR.

In order to study the background cosmological solutions, let us start by considering a homogeneous Bianchi-I-type universe,

$$ds^2 = -dt^2 + \sum_{i=1}^{d} a_i^2(t) dx_i^2, \qquad a = e^{\lambda_i(t)} \tag{4.1.2}$$

$$\phi = \phi\left(t\right). \tag{4.1.3}$$

It is useful to introduce the shifted dilaton field d defined to be,

$$2d \equiv 2\phi - \sum_{i=1}^{D-1} \ln a_i, \quad \sqrt{-g} e^{-2\phi} = e^{-2d}. \tag{4.1.4}$$

Thus, for such an ansatz, the action becomes,

$$S_0 = \frac{1}{2\kappa_0^2} \int dt e^{-2d} \sqrt{-g_{00}} \left[-g^{00} \sum_{i=1}^{D-1} H_i^2 + 4g^{00} \dot{d}^2 \right], \tag{4.1.5}$$

where we introduced the Hubble parameter defined as,

$$H_i(t) \equiv \frac{\dot{a}_i}{a_i}, \tag{4.1.6}$$

which measures the expansion rate in the i-th direction. This action is symmetric under

$$\lambda_i \;\rightarrow\; -\lambda_i \tag{4.1.7}$$

$$d \;\rightarrow\; d, \tag{4.1.8}$$

for any i among $1, ..., (D-1)$. This is called scale-factor duality and it is related to the T-duality we have introduced in the last chapter.

In order to introduce matter, we consider the effective string coupling to be small so that we can consider matter to be a gas of (free) strings modes in a thermal equilibrium at temperature β^{-1} [56]. Its action is given by,

$$S_m = \int dt\sqrt{-g_{00}}F\left(\lambda_i, \beta\sqrt{-g_{00}}\right), \tag{4.1.9}$$

where F is the (one loop) free energy and can be represented in terms of the one loop string partition function, Z, on a torus of radius a and periodic Euclidean time of perimeter $\beta\sqrt{-g_{00}}$, such that $Z = -\beta F$.

Considering the full action and varying it in respect to λ_i, ϕ and g_{00} (and then setting $g_{00} = -1$), we find the following equations of motion

$$4\dot{d}^2 - \sum_i^{D-1} H_i^2 = e^{2d}E \tag{4.1.10}$$

$$\dot{H}_i - 2H_i\dot{d} = \frac{1}{2}e^{2d}P_i \tag{4.1.11}$$

$$4\ddot{d} - 4\dot{d}^2 - \sum_i^{D-1} H_i^2 = 0, \tag{4.1.12}$$

where

$$E = -2\frac{\delta S_m}{\delta g_{00}} = F + \beta\frac{\partial F}{\partial \beta} \tag{4.1.13}$$

$$P = -\frac{\partial F}{\partial \ln a}, \tag{4.1.14}$$

E being the total energy of the matter and P_i the pressure in the i-th direction times the volume. These equations can be also combined in a continuity equation (which is equivalent to considering the system to be adiabatic, i.e., having constant entropy),

$$\dot{E} + \sum_{i=1}^{D-1} H_i P_i = 0, \tag{4.1.15}$$

which is equivalent to the conservation of entropy ($S = \beta^2\partial F/\partial\beta$), showing that our matter is indeed evolving adiabatically. The adiabaticity assumption implies that we can replace constant radii and β by functions of time.

Further imposing isotropy, the equations of motion become,

$$4\dot{d}^2 - (D-1)\,H^2 = e^{2d}a^d\rho \tag{4.1.16}$$

$$\dot{H} - 2H\dot{d} = \frac{1}{2}e^{2d}a^d p \tag{4.1.17}$$

$$4\ddot{d} - 4\dot{d}^2 - (D-1)\,H^2 = 0, \tag{4.1.18}$$

where ρ is the energy density and p is the pressure, so that the continuity equation now reads,

$$\dot{\rho} + (D-1)H\,(\rho + p) = 0. \tag{4.1.19}$$

These are our starting point equations for considering the background cosmology given by the supergravity action in the absence of the anti-symmetric field coupled to hydrodynamical perfect fluid.

4.2 String Gas Matter (SGm)

We have just introduced matter to the supergravity equations in the form of a hydrodynamical fluid. The simplest fluid we can consider is a linear barotropic fluid, where the pressure is linearly proportional to the energy density,

$$p = w\rho, \tag{4.2.20}$$

which is the same kind of fluid we have discussed in Chapter 2. When we have this sort of matter sector, all the complexity of the matter interactions is reduced to a single parameter, w, the equation of state parameter. If we aim to characterize the strings' excitations as a linear barotropic fluid, we need to compute w for such a thermodynamical gas.

Our interest here lies in a gas of closed strings given our initial motivation coming from String Gas Cosmology [34]. Since we know the mass spectrum of the string in a compact space is given by (3.5.49), we can write down its energy density as [51],

$$\rho = \frac{1}{a^{D-1}}\sum_s N_s E_s \tag{4.2.21}$$

$$l_s^2 E_s^2 = 2\left(N + \tilde{N} - 2\right) + \frac{n^2}{a^2} + m^2 a^2, \tag{4.2.22}$$

where $s = \left\{n, m, N, \tilde{N}\right\}$ correspond to all possible different states of the string and N_s corresponds to the density of states. Note that, in principle, when we compute the energy spectrum for the strings, it is computed in terms of the radius R, which is the constant size of the compact dimensions. However, this can be lifted to be a time-dependent variable using the adiabatic approximation, where the radius is written as the scale factor times the string length, $R = a(t)l_s$ [51]. The pressure is given by,

$$p = -\frac{\partial\left(\rho V\right)}{\partial V} = -\frac{1}{D-1}a^{1-D}\sum_s \frac{N_s}{l_s^2}\left(-\frac{n^2}{a^2} + a^2 m^2\right). \tag{4.2.23}$$

Both the pressure and energy density rely on the density of states N_s. Unfortunately, describing the above dynamics quantitatively coming from the string spectrum is quite complicated. In particular, the main problem is computing the density of states as a function of the radius of the compact dimensions (though there are results for large radius [57]). What we can do instead is to consider some specific regimes.

As we have discussed in Section 3.5.1, it is clear that when the radius of the box is large, only momentum modes will be existing; while when the radius is small, only winding modes should exist. Finally, for radius around the self-dual radius, $R = l_s$, we expect all the modes to be excitable: momentum, winding and oscillatory modes. These three defining regimes can be characterized by,

small box ($R \ll l_s$)	self-dual ($R \sim l_s$)	large box ($R \gg l_s$)
$\omega = -1/\left(D-1\right)$	$\omega = 0$	$\omega = 1/\left(D-1\right)$

Note that, heuristically, the effective equation of state close enough to the dual radius should vanish, given that the momentum and winding modes contribute equally to the overall pressure but with opposite signs, while the oscillatory modes contribution does not depend on the radius.

Now, given we expect a smooth transition between the energy budget distribution amongst the different modes as the compact dimensions expand, we can use these regimes to model an effective equation of state,

$$\omega\left(a\right) = \frac{2}{\pi(D-1)}\arctan\left(\beta\ln\left(\frac{a}{a_0}\right)\right), \tag{4.2.24}$$

where β tells us the transition rate between the regimes and a_0 is an arbitrary pivot scale in general, here to be taken as unity given the pivot scale for the energy spectrum defined above is the string length. This parameter β has to be related to the cross-section of winding modes annihilating into momentum modes and should depend on the number of dimensions [34]. This function was chosen due to the fact it is symmetric for small/large radius, as is expected from a T-dual perspective. Figure 4.1 shows its behavior, where we can observe the transition between two main regimes: when the dynamics is dominated by the winding modes, small radius, and when the momentum modes dominate, large radius.

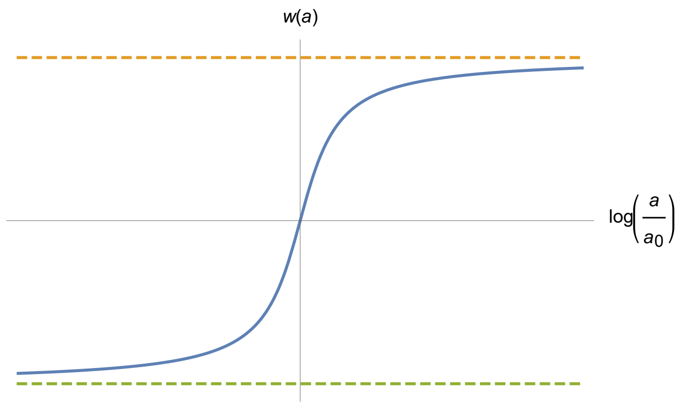

Fig. 4.1 We parametrize the equation of state as $\omega\left(a\right) = \frac{2}{\pi(D-1)}\arctan\left(\beta\ln\left(\frac{a}{a_0}\right)\right)$, where β parametrizes how fast winding modes annihilate themselves as the scale factor evolves towards the self-dual scale factor, $a_0 = 1$. Note that the equation of state transforms as $\omega(a^{-1}) = -\omega\left(a\right)$ under the scale factor duality, as is expected.

Before we attempt solving the background equations for this effective equation of state, we can look at its thermodynamical properties and to the phase space dynamics so that we can develop further heuristics, which tells us the expected behavior of the energy density and pressure.

4.2.1 Thermodynamics of the string excitations

Let us start looking at a linear barotropic equation of state as a function of the temperature,

$$p\left(T\right) = w\left(T\right)\rho\left(T\right), \tag{4.2.25}$$

where T is the temperature. Then, we can use Euler's relation,

$$U = TS - PV, \tag{4.2.26}$$

where U is the internal energy, S is the entropy, P is the pressure and V is the volume, so that we can rewrite the entropy as,

$$S = \frac{U + PV}{T} = \frac{V}{T}\rho(T)\left[1 + \omega(T)\right]. \tag{4.2.27}$$

Using the free energy potential, $F = F(T, V)$, we can write down its differential form as,

$$dF = -SdT - PdV, \tag{4.2.28}$$

which allows us to derive the following Maxwell relation,

$$\left(\frac{\partial P}{\partial T}\right)_V = \left(\frac{\partial S}{\partial V}\right)_T. \tag{4.2.29}$$

Then, for the equation of state (4.2.25), this relation implies

$$\frac{\partial}{\partial T}\left[\omega(T)\rho(T)\right]_V = \frac{\rho(T)}{T}\left[1 + \omega(T)\right]. \tag{4.2.30}$$

We have seen above that the regime of the excited modes of the string can be defined by a constant equation of state. Considering $w =$ const. we end up having,

$$\omega\frac{\partial\rho}{\partial T} = \frac{\rho}{T}(1 + \omega), \tag{4.2.31}$$

which implies $\rho(T) = \rho_0 T^{1+1/\omega}$. Hence,

$$S = (1 + \omega)\rho V T^{1/\omega}. \tag{4.2.32}$$

Since we are considering adiabatic expansion, $dS = 0$, we finally have,

$$a^d T^{1/\omega} = \text{const.}, \tag{4.2.33}$$

where $V = V_0 a^d$, such that,

$$\begin{cases} \omega = 1/d & \to T \propto a^{-1} \\ \omega = -1/d & \to T \propto a \end{cases}. \tag{4.2.34}$$

This shows how the temperature evolves as function of the scale factor for momentum and winding modes, respectively. If $w = 0$, we can show that the temperature is a constant. This result tells us that the singularity present in GR, and observed here with the temperature becoming singular for the momentum modes as the scale factor vanishes, may be resolved if we have winding modes instead when the scale factor becomes much smaller than unity. In fact, if we consider an evolving equation of state, such as the equation of state (7.3.15), we can show that the temperature as a function of the scale factor is given by the one presented in Figure 6.1.

This temperature plot is very compelling: the temperature does not diverge, instead it finds a plateau and then decays as the scale factor becomes smaller and smaller. The peak of this plateau is given by the Hagedorn temperature [58], which is the temperature for which the partition sum diverges in a system with exponential growth in the density of states.

4.2.2 Phase space

We can have a look at the phase space dynamics to see what is expected for the evolution of the energy density and pressure given (7.3.15). The first step is to rewritte the continuity equation in the phase space as,

$$\frac{d\rho}{\rho} + (D-1)\left(1+\omega\left(a\right)\right) d\ln a = 0, \tag{4.2.35}$$

where the time dependence is gone. Given we know the functional form of $\omega\left(a\right)$, we can just integrate the equation, obtaining:

$$\ln\frac{\rho}{\rho_0} = -\left(D-1\right)\ln\frac{a}{a_0} - \frac{2}{\pi}\left\{\ln\left(\frac{a}{a_0}\right)\arctan\left[\beta\ln\left(\frac{a}{a_0}\right)\right] - \frac{1}{2\beta}\ln\left[1+\beta^2\left(\ln\frac{a}{a_0}\right)^2\right]\right\}, \tag{4.2.36}$$

which does reproduce what is expect for large radius, $\rho\left(a \text{ large}\right) \to \rho_0\left(a/a_0\right)^{-D}$, and small radius, $\rho\left(a \text{ small}\right) \to \rho_0\left(a/a_0\right)^{-D+2}$ in supergravity for pure momentum or pure winding modes, respectively. Also, note that ρ_0 is fixed by the energy density at the self-dual radius, parametrized by a_0. Its generic behavior can be seen through the following plot,

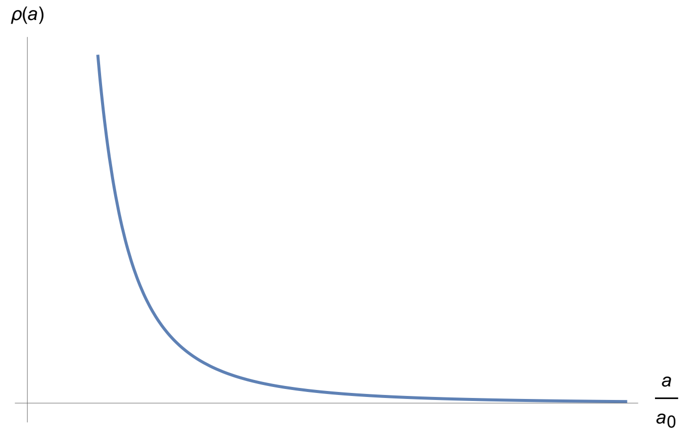

Fig. 4.2 Energy density as a function of the scale factor for SGm in $D = 4$ and $\beta = 1$.

Knowing $\rho(a)$ and $\omega\,(a)$, we also know $p\,(a)$, which can be seen generically represented by,

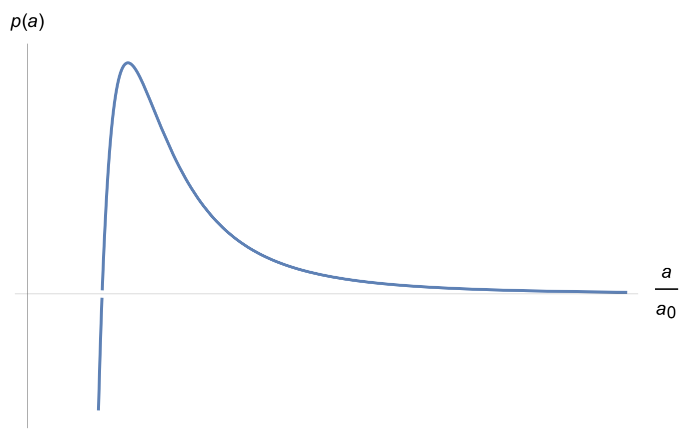

Fig. 4.3 Pressure as a function of the scale factor for $D = 4$ and $\beta = 1$. It vanishes for $a = a_0$.

where we can see a very important feature of SGm: the pressure grows in the early phases before decaying as it naturally happens in standard cosmology. This particular signature will result in interesting effects in our analysis below. Also note that the pressure is negative for radius smaller than the self-dual one, that is due to the abundance of winding modes in relation to the momentum ones.

4.3 Dynamics

We can finally attempt to solve the equations of motion for the scale factor and dilaton. In order to do so, we can consider a piece-wise solution, meaning we solve the equations for each regime (small/self-dual/large radius) and glue the solutions together. The numerical solution after considering the full equation of state can be considered in the future.

We look for a solution that accounts approximately for a constant equation of state for each regime,

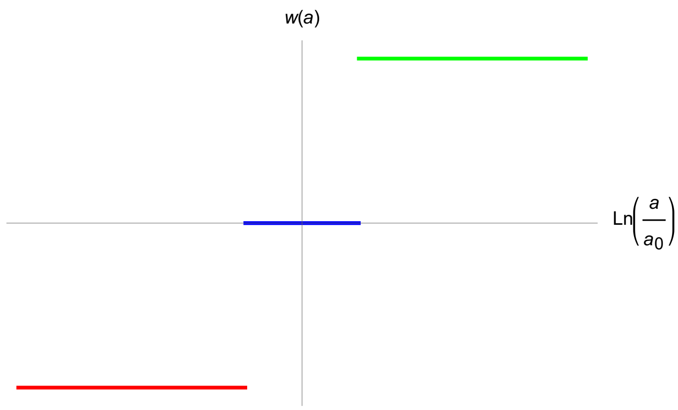

Fig. 4.4 We consider the equation of state to be constant in each regime, represented schemetically in the figure: $w = 1/(D-1)$ for $a \gg 1$, $w = 0$ for a a_0, and $w = -1/(D-1)$ for $a \ll 1$.

We can search for general solutions considering the following power-law ansatz [33],

$$a \sim t^{\alpha}, \quad 2d \sim -\gamma \ln t, \quad w \sim \frac{p}{\rho}, \tag{4.3.37}$$

such that for w =const. the equations of motion imply two constraints,

$$(D-1)\, w\alpha + \gamma = 2 \tag{4.3.38}$$

$$\gamma^2 + (D-1)\,\alpha^2 = 2\gamma. \tag{4.3.39}$$

Thus, we see that the set of parameters $\{\alpha, \gamma\}$ for different regimes are given by,

$w_w = -1/(D-1)$	$w_{osc} = 0$	$w_m = 1/(D-1)$
$\{0,2\}, \left\{-\frac{2}{D}, \frac{2}{D}(D-1)\right\}$	$\{0,2\}$	$\{0,2\}, \left\{\frac{2}{D}, \frac{2}{D}(D-1)\right\}.$

Those solutions are reasonably good away from the singularity [33]. Given the parameters obtained above, a piece-wise solution can be written as,

$$a(t) \sim \begin{cases} a_0 \left(\frac{t}{t_0}\right)^{-2/D}, & t \leq t_0 \\ a_0, & t = t_0 \\ a_0 \left(\frac{t}{t_0}\right)^{2/D}, & t \geq t_0 \end{cases} \tag{4.3.40}$$

$$\bar{\phi}(t) \sim \begin{cases} -\frac{2}{D}(D-1)\ln t, & t \neq t_0 \\ 2, & t = t_0, \end{cases} \tag{4.3.41}$$

where the discontinuity in the shifted dilaton solution is a symptom of the singular behavior of the full solution, and t_0 represents the time for which the system is at the self-dual radius. The scale factor solution is not surprising due to the scale-factor duality, and it corresponds to an early contraction until it reaches the dual radius, for then starting to expand as a radiation-dominated universe.

This heuristic solution tells us that the typical scale-factor dynamics expected in String Gas Cosmology cannot be simply recovered even considering the equation of state of SGm here studied, since it requires having an almost static scale factor during the Hagedorn phase before entering the typical radiation-dominated expansion. It is easy to see from above that the static solution is generally quite short if the equation of state is evolving. It is worth mentioning that if $w = 0$, then the scale-factor goes asymptotically to a constant [56], however there would be no way out of this solution.

We have seen above that even considering the equation of state given by a gas of closed strings where there are momentum, oscillatory and winding modes, the energy density still diverges as the scale factor goes to zero. Then, we considered a piece-wise solution to the background equations in order to investigate if that would happen, and we have seen that even if the scale factor is well-behaved, the dilaton will be singular. Unfortunately, there is a much more general result that tells us that string cosmology is unavoidably singular[1] [33]. Hence, our departure from GR towards ST has not been *stringy* enough to provide us with the heuristic picture expected in ST: a non-singular cosmology. We may wonder if we

[1] By string cosmology we mean supergravity coupled to stringy matter.

can do better with the available tools introduced in Chapter 3. The answer is yes, since we have not yet exhausted our tool kit. We have discussed T-duality in the last chapter and how this duality redefines the notion of position. Unfortunately, supergravity despite being T-dual, it does not have this duality manifest. In fact, all the equations we have written here have taken the target space to be defined by the momentum modes. Recently, a new framework has been proposed in which T-duality is made a manifest symmetry, where double coordinates are introduced in order to have momentum and winding modes on same grounds, very much like (3.5.1). In the next chapter, we review the foundations of such framework, finally introducing the last piece of technology necessary to understand the gist of our work presented in the final chapters of this thesis.

Chapter 5

A short introduction to Double Field Theory

As we have seen in the last chapter, standard string cosmology is not enough to provide us with a nonsingular cosmology. However, we have also seen that the introduction of double coordinates conjugated to the winding modes may be able to improve our situation.

Double Field Theory is the current framework in which one can consistently consider double coordinates. This is implemented by promoting T-duality from a duality in string theory to be the geometrical group underlying a field theory, in the same spirit that $GL(D)$ is the geometrical group in GR. In other words, T-duality becomes a manifest symmetry in DFT. Hence, we intend to explore background cosmological solutions in it.

There are a couple of reviews on DFT [40, 59, 60, 35]. In order to briefly review the formalism, we follow closely [40] that presents a pedagogical introduction to DFT, while we also consider the metric formulation introduced in [61].

5.1 Introduction

Double Field Theory is a proposal to incorporate T-duality as a symmetry of a field theory, reformulating supergravity and going beyond it [62]. The main difference regarding standard Quantum Field Theory is that this theory is also based on winding modes, which are degrees of freedom only present when there are compact dimensions and multidimensional objects, as strings, that can wind around those dimensions. Thus, one can think that Fourier space is naturally doubled for compact dimensions, that is, the Fourier decomposition is made in terms of momenta and winding modes. Therefore, one is tempted to also double the reciprocal space, the configuration space for the compact dimensions, as we have discussed in Section 3.5.1.

The DFT proposal considers that for a $D-$dimensional space with d non-compact space-time dimensions and n compact dimensions, i.e., $D = n+d$, the fields depend on coordinates $X^M = (\tilde{x}_\mu, \tilde{y}_m, x^\mu, y^m)$, where x^μ are the noncompact space-time coordinates, $\tilde{x}_\mu$ are there simply for decoration (we can double the number of coordinates associated with noncompact dimensions as well just to have a simpler notation, even though these coordinates will be sterile), and $\mathbb{Y}^A = (\tilde{y}_m, y^m)$ are $2n$ compact coordinates associate with winding and momentum modes, respectively.

As we have discussed, winding modes correspond to strings winding in one or more of the compact dimensions. When these compact dimensions increase in size (R, the radius of compactification, gets bigger in a toroidal compactification) beyond the string scale, the string tension prevents these modes to be excited, so that they become absent in the large radius limit. From a scattering point of view, as the radius grows larger, we expect that winding and anti-winding modes self-annihilate to produce momentum modes [34]. From this, one can conclude that in the limit of noncompactification all the double coordinates should be dynamically suppressed. Considering now T-duality, in the opposite limit of small radii, all the momentum coordinates should be the ones suppressed in the compact space (the reason for this is that the momentum modes are quantized in compact spaces and their energies go as the inverse of the compactification radius (assuming toroidal compactification from now on), and therefore being absent for small radius. In particular, in the limit of noncompactification we find correspondence with supergravity (SUGRA).

It is important to remember that SUGRA has field content given by the metric, g_{ij}, the two-form field, b_{ij}, and a scalar dilaton, ϕ[1]. Let us assume from now on that all the dimensions are compact, and therefore the framework will be $2D-$dimensional, since we consider the number of compact dimensions to be doubled. The duality group for n-compact dimensions is $O(n,n)$ for string toroidal compactification. We will be seeking representations of the $O(D,D)$ group.

A quick analysis of the degrees of freedom of our field content tells us we have

[1] In this chapter, we will be considering latin indices for all the fields as a reminder that those are the coordinates corresponding to the compact dimensions.

$$\frac{D(D+1)}{2} + \frac{D(D-1)}{2} = D^2,$$

for the metric and 2-form degrees of freedom. In DFT these two objects are considered on the same ground in the gravitational sector and should be responsible for the doubled geometry. This is accomplished considering a generalized $O(D,D)$ symmetric metric, $\mathcal{H}_{MN}$, with $M, N = 1, ..., 2D$, defined in the double space. The dilaton becomes an $O(D,D)$ scalar when combined with the determinant of the metric.

In order to understand the symmetries we will be working with, let us remind ourselves of the SUGRA action already introduced in Section 3.4.1,

$$S = \int d^D x \sqrt{-g} e^{-2\phi} \left(R + 4(\partial\phi)^2 - \frac{1}{12} H_{ijk} H^{ijk} \right), \tag{5.1.1}$$

This is the action that should be recovered in DFT when the dual coordinates are projected out in the large radii limit. This action has two symmetries: *diffeomorphisms* and *gauge symmetry*. In DFT, those symmetries are unified and become a "subgroup" of the group of *generalized diffeomorphisms* of a generalized metric.

Evidently, we cannot simply double the geometry and expect to recover the same theory, since the symmetric $O(D,D)$ metric has $D(2D+1)$ degrees of freedom, and therefore it should be further constrained so that it retains only the original number of degrees of freedom. One particular solution to this problem is given by implementing the so called *strong constraint* (SC), also called section condition.

The SC implies that the fields of the theory and the gauge parameters as well as any product among them only depend on a slice of the double space parametrized by half of the coordinates, such that there always exists a frame in which, locally, the configurations do not depend on the dual coordinates. As we will see, it is not clear if we can consider the theory without implementing the SC. This is still a topic of research and we will discuss it further below.

It is important to mention that, at the nonperturbative level, string field theory naturally introduces physical coordinates associated with the winding modes [35]. Therefore, one could argue that the full closed string theory is a theory for which the physical space is effectively a double geometry. DFT is a consistent way of exploring this double geometry without having to consider second quantization of strings, even though it is still not clear how much beyond

SUGRA the resulting formalism goes.

5.2 T-duality group and fields

The T-duality group is $O(D, D; \mathbb{Z})$. The elements of this group are defined as $2D \times 2D$ matrices h_{MN} that preserve the $O(D, D; \mathbb{Z})$ metric η_{MN},

$$h_M{}^P \eta_{PQ} h_N{}^Q = \eta_{MN}. \tag{5.2.2}$$

The definition of the η matrix is

$$\eta_{MN} = \begin{pmatrix} 0 & \delta^i{}_j \\ \delta_i{}^j & 0 \end{pmatrix}, \qquad \eta^{MN} = \begin{pmatrix} 0 & \delta^j{}_i \\ \delta_j{}^i & 0 \end{pmatrix}, \quad \eta^{MP}\eta_{PN} = \delta^M_N. \tag{5.2.3}$$

Had we compactified more than one dimension, the mass spectrum gets more complicated than the one written in formula (3.5.49). To see that, we can go back at string action (3.4.35) without the dilaton term,

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{\gamma}(\gamma^{ab}g_{mn}\partial_a X^m \partial_b X^n + i\epsilon^{ab}b_{mn}\partial_a X^m \partial_b X^n), \tag{5.2.4}$$

where here the metric and the 2-form are constant background fields. Using the ansatz

$$X^m(\sigma^1, \sigma^2) = x^m(\sigma^2) + w^m R\sigma^1, \tag{5.2.5}$$

which obeys the periodic condition $X^m(\sigma^1 + 2\pi, \sigma^2) = X^m(\sigma^1, \sigma^2) + 2\pi R w^m$, the action becomes

$$S = \frac{1}{2\alpha'}\int d^2\sigma\sqrt{\gamma}\Big[g_{mn}(\dot{x}^m\dot{x}^n + w^m w^n R^2) + 2ib_{mn}\dot{x}^m w^n R\Big]. \tag{5.2.6}$$

The conjugate momenta are,

$$p_m = \frac{\partial\mathcal{L}}{\partial v^m} = \frac{1}{\alpha'}(g_{mn}v^n - b_{mn}w^m R), \tag{5.2.7}$$

where $v^m = i\dot{x}^m$ and $\dot{x}^m = \partial_2 x^m$. Periodicity of the wave-functions implies that the momenta are quantized, i.e., $p_m = n_m/R$, so

$$v_m = \alpha' \frac{n_m}{R} + g_{mn} w^n R.$$

The Hamiltonian is

$$H = p_m v^m + \mathcal{L} = \frac{1}{2\alpha'} g_{mn} \left[v^m v^n + w^m w^n R^2 \right].$$

Looking at the Hamiltonian, we can define the generalized metric, $\mathcal{H}_{MN}$, as

$$\mathcal{H}_{MN} \equiv \begin{pmatrix} g^{ij} & -g^{ik} b_{kj} \\ b_{ik} g^{kj} & g_{ij} - b_{ik} g^{kl} b_{lj} \end{pmatrix}, \tag{5.2.8}$$

which encodes in a unified way the degrees of freedom of the spacetime metric and the 2-form. Then, the mass formula for multiple compact dimensions becomes,

$$\alpha' M^2 = 2(N + \tilde{N} - 2) + \mathcal{P}^P \mathcal{H}_{PQ} \mathcal{P}^Q$$

where

$$\mathcal{P}^M = \begin{bmatrix} \tilde{p}^i \\ p_i \end{bmatrix} \tag{5.2.9}$$

is the generalized momentum, and $\tilde{p}^i$ is defined in terms of w^i, while p_i is defined in terms of n_i. The LMC becomes (3.5.50),

$$N - \tilde{N} = \frac{1}{2} \mathcal{P}^M \mathcal{P}_M.$$

Considering the group $O(D, D)$, we can retrieve the known symmetries that the SUGRA action has through the following transformations:

- Diffeomorphisms, represented by

$$h_M{}^N = \begin{pmatrix} E^i{}_j & 0 \\ 0 & E_i{}^j \end{pmatrix}, \qquad E \in GL(D).$$

- Shifts of the antisymmetric b field, given by

$$h_M{}^N = \begin{pmatrix} \delta^i{}_j & 0 \\ b_{ij} & \delta_i{}^j \end{pmatrix}, \qquad b_{ij} = -b_{ji}.$$

- Factorized T-dualities:

$$h_M^{(k)\ N} = \begin{pmatrix} \delta^i{}_j - t^i{}_j & t^{ij} \\ t_{ij} & \delta_i{}^j - t_i{}^j \end{pmatrix}, \qquad t = \text{diag}(0...010...0).$$

  We see that if we apply successively the above matrix for k going from 1 to D, it becomes the η^M_N matrices, since the diagonal matrices will be filled with zeros and the off-diagonal matrices will transform into identity matrices. We also see that if we apply the $h_M^{(k)N}$ matrix on the $\mathcal{P}_M$ vector, we will change the k-ith p_k momentum to $\tilde{p}^k$; and analogous for X_M.

We can now see how the η-metric and the generalized metric must be elements of the group $O(D, D)$ by considering the transformation,

$$Z' = hZ. \tag{5.2.10}$$

For the LMC to be invariant, we need

$$h\eta h^t = \eta, \tag{5.2.11}$$

such that h preserves the η metric, and therefore $h \in O(D, D)$. Demanding the same for the mass spectrum, we derive that also $\mathcal{H} \in O(D, D)$. Note that here the generalized metric is a constant background, but in DFT it is promoted to be dynamical.

5.3 Double space and generalized fields

After reviewing the string action above, we have learned how to consider the SUGRA degrees of freedom arranged in a T-dual way through the following definitions,

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij} - b_{ik}g^{kl}b_{lj} \end{pmatrix} \tag{5.3.12}$$

$$e^{-2d} = \sqrt{g}e^{-2\phi}, \tag{5.3.13}$$

where now the components of this generalized metric are not constants. The dilaton ϕ is combined with the determinant of the metric g in an $O(D, D)$ scalar d, while the generalized metric is an $O(D, D)$ element. Thus, we have

$$\mathcal{H} \in O(D, D) \qquad \mathcal{H}^{MN} = \eta^{MP}\mathcal{H}_{PQ}\eta^{QN} \qquad \mathcal{H}_{MP}\mathcal{H}^{PN} = \delta_M^N. \tag{5.3.14}$$

Note that the dilaton alone is not a T-dual invariant, which is why it was combined with the determinant of the spacetime metric to form a scalar. The reason for this can be found by considering the low-energy effective action of a string in a $\mathbb{R}^{25} \times S^1$ background, which in the Einstein frame is written as [53],

$$\frac{2\pi R}{2l_s^{D-2}g_s^2} \int d^{D-1}X \sqrt{-\tilde{g}}\; e^{\sigma}\, \mathcal{R} + \dots$$

Since a scientist cannot tell the difference between R and $\tilde{R} = \alpha'/R$, we know that the coupling R/g_s^2 must remain constant under a T-dual transformation, which is only possible if

$$g_s \to \tilde{g}_s = \frac{\sqrt{\alpha'}g_s}{R}, \tag{5.3.15}$$

and we know that the string coupling is the expectation value of the dilaton field. Therefore, the dilaton transforms under T-duality.

Now, since the lowest representation of $O(D, D)$ is the fundamental, which has $2D$ dimensions, we need to introduce a new set of D-coordinates, $\tilde{x}_i$, in order to complete the fundamental representation. Hence, the generalized notion of coordinates is given by

$$X^M = \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix}, \tag{5.3.16}$$

and the generalized fields will depend on this double set of coordinates: $\mathcal{H}_{MN}(X)$ and $d(X)$. Note that the generalized coordinates in a toroidal compactified background will be dual to $\mathcal{P}^M$, and this can be extended to more general backgrounds. Naturally, the generalized coordinates transform under $O(D,D)$ as vectors,

$$X^M \to h^M{}_N X^N, \qquad h \in O(D,D), \tag{5.3.17}$$

while the generalized metric, for instance, transform as a rank 2 tensor,

$$\mathcal{H}_{MN}(X) \;\; \to h_M^P h_N^Q \mathcal{H}_{PQ}(hX)\,, \tag{5.3.18}$$

which can account for the Buscher rules in SUGRA.

Finally, an important observation about the generalized metric is that it is always well defined in the sense that,

$$\det \mathcal{H} = 1. \tag{5.3.19}$$

Thus, it is possible that by considering the generalized metric as the fundamental field defining the true geometrical aspects of spacetime we may avoid singularities by construction.

5.4 The Section Condition

As we have briefly mentioned in the Introduction of the chapter, we cannot simply add a new set of coordinates since that changes the number of degrees of freedom of the theory. In order to recover the initial field content with which we introduced DFT, we need to impose a constraint on all the fields and gauge parameters called the section condition (SC), which is generally written as,

$$\eta^{MN}\partial_M\partial_N\,(...) = 0, \tag{5.4.20}$$

where the ellipsis represent any local operator, which can also be made up of products of fields. A less strict constraint can also be considered, which represents the imposition of the LMC. That is typically called "weak constraint" and can be represented by the same relation above where the ellipsis only represents the fundamental fields of the theory, but not their products.

Note that a solution to the strong constraint can be simply given by considering all the double coordinate dependence to be dropped, such that,

$$\tilde{\partial}(...) = 0. \tag{5.4.21}$$

Each solution to the section condition is said to determine a frame, or polarization, where all the frames are connected by $O(D,D)$ rotations. The frame defined above is called supergravity frame. If we had imposed that the usual coordinates dependence is dropped instead, then we would define what we call the *winding frame*. We will be considering these two frames in the final chapters of this thesis.

5.5 Generalized Lie derivative

In DFT it is possible to define a generalized Lie derivative. This derivative is built such that when acting on the fields it recovers the symmetry transformations that we had in SUGRA as long as we also impose the supergravity frame.

In order to do so, we introduce a generalized gauge parameter

$$\xi^M = (\tilde{\lambda}_i, \lambda^i), \tag{5.5.22}$$

where $\tilde{\lambda}$ is the 1-form associated with the gauge symmetry of the 2-form, while the λ-vector is associated to spacetime diffeomorphisms, as we have seen in Section (3.4.1). It is possible to show that the diffeomorphisms and the gauge transformations for the antisymmetric field are then written in a generalized form as

$$\begin{aligned} \mathcal{L}_\xi e^{-2d} &= \partial_M(\xi^M e^{-2d}), && (5.5.23)\\ \mathcal{L}_\xi \mathcal{H}_{MN} &= L_\xi \mathcal{H}_{MN} + Y^{R}_{M}{}^{P}_{Q}\partial^Q\xi_P\mathcal{H}_{RN} + Y^{R}_{N}{}^{P}_{Q}\partial^Q\xi_P\mathcal{H}_{MR}, && (5.5.24) \end{aligned}$$

where $\mathcal{L}$ is the *generalized Lie derivative.* Here the Lie derivative L_ξ acts on a general vector field V^i as,

$$L_\xi V^i = \xi^j \partial_j V^i - V^j \partial_j \xi^i, \tag{5.5.25}$$

as we typically have in differential geometry.

A generalized Lie derivative with respect to a vector ξ acts on a tensorial density V^M with weight $\omega(V)$ as

$$\mathcal{L}_\xi V^M = \xi^P \partial_P V^M + (\partial^M \xi_P - \partial_P \xi^M) V^P + \omega(V) \partial_P \xi^P V^M. \tag{5.5.26}$$

If $\omega(e^{-2d}) = 1$, then

$$\mathcal{L}_\xi e^{-2d} = \partial_P(\xi^P e^{-2d}), \tag{5.5.27}$$

which recovers the transformation (5.5.24) of the dilaton. For the generalized metric, if $\omega(\mathcal{H}) = 0$, then

$$\mathcal{L}_\xi \mathcal{H}_{MN} = L_\xi \mathcal{H}_{MN} + Y^{R\ P}_{M\ Q} \partial^Q \xi_P \mathcal{H}_{RN} + Y^{R\ P}_{N\ Q} \partial^Q \xi_P \mathcal{H}_{MR}, \tag{5.5.28}$$

where we considered $L_\xi \mathcal{H}_{MN} = \xi^P \partial_P \mathcal{H}_{MN} - \partial^P \xi_M \mathcal{H}_{PN} - \partial^P \xi_N \mathcal{H}_{MP}$ and $Y^{R\ P}_{M\ Q} = \eta^{RP} \eta_{MQ}$. Naturally, the $O(D,D)$ metric η is invariant under these generalized diffeomorphisms,

$$\mathcal{L}_\xi \eta_{MN} = 0. \tag{5.5.29}$$

5.5.1 Consistency constraints

We have defined generalized Lie derivatives. One thing this definition must satisfy is that the action of two successive derivatives is still another Lie derivative, defining a closed group. Under this demand, one can show that we need to impose that,

$$([\mathcal{L}_1, \mathcal{L}_2] - \mathcal{L}_{\xi_{12}}) \xi_3^M = 0. \tag{5.5.30}$$

where ξ_3^M is an arbitrary vector and the indices $1, 2$ represent different generalized diffeomorphism parameters.

5.6 DFT action

After considering all the tools introduced so far, we can write down the DFT action. The requisites to find it are: i) all the terms should be up to second order in derivatives, ii) it must recover the SUGRA action (4.1.1) in the supergravity frame, iii) it must respect the gauge symmetries. Imposing these, the action is

$$S_{DFT} = \int dX e^{-2d}\mathcal{R}, \tag{5.6.31}$$

where

$$\begin{aligned}\mathcal{R} \;=\; & \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_K\mathcal{H}_{NL} \\ & +4\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} - 4\mathcal{H}^{MN}\partial_M d\partial_N d + 4\partial_M\mathcal{H}^{MN}\partial_N d.\end{aligned} \tag{5.6.32}$$

This is the generalized Ricci scalar. Note that this action is similar to the Einstein-Hilbert action of GR, having equation of motion given by

$$\mathcal{R} = 0, \tag{5.6.33}$$

which is a vacuum-like equation of motion. The above action recovers, after imposition of the SC, the SUGRA action, as expected.

5.7 Generalizing the DFT action

We start with the following action,

$$S = S_{DFT} + \int dS e^{-2d} f\left(d, \mathcal{H}, \chi\right), \tag{5.7.34}$$

where f is a general function of the dilaton field, generalized metric $\mathcal{H}$ and a spectator field χ.

Now, we would like to derive the generalized equation of motion for $\mathcal{H}^{MN}$ given the inclusion of the second term in (5.7.34). The variation of S_{DFT} with respect to the generalized metric has been considered in [61] and we reproduce it below. In doing so, we will learn how to deal with the variation of the generalized metric, since this is a constrained object by (5.3.14).

We start off considering,

$$\delta S_1 = \int dX e^{-2d} \delta \mathcal{H}^{MN} K_{MN}, \tag{5.7.35}$$

where

$$\begin{aligned} K_{MN} &= \frac{1}{8}\partial_M \mathcal{H}^{KL}\partial_N \mathcal{H}_{KL} - \frac{1}{4}\left(\partial_L - 2\partial_L d\right)\left(\mathcal{H}^{LK}\partial_K \mathcal{H}_{MN}\right) + 2\partial_M\partial_N d - \frac{1}{2}\partial_{(M}\mathcal{H}^{KL}\partial_L \mathcal{H}_{N)K} \\ &+ \frac{1}{2}\left(\partial_L - 2\partial_L d\right)\left(\mathcal{H}^{KL}\partial_{(M}\mathcal{H}_{N)K} + \mathcal{H}^{K}{}_{(M}\partial_K \mathcal{H}^{L}{}_{N)}\right). \end{aligned} \tag{5.7.36}$$

As $\mathcal{H}$ is constrained to satisfy $\mathcal{H}\eta\mathcal{H} = \eta^{-1}$, the equations of motion are found by considering variations that preserve this constraint. The varied field $\mathcal{H}' = \mathcal{H} + \delta\mathcal{H}$ will also satisfy $\mathcal{H}'\eta\mathcal{H}' = \eta^{-1}$ provided that

$$\delta\mathcal{H}\eta\mathcal{H} + \mathcal{H}\eta\delta\mathcal{H} = 0. \tag{5.7.37}$$

Using $S^M{}_N = \mathcal{H}^M{}_N = \eta^{MP}\mathcal{H}_{PN} = \mathcal{H}^{MP}\eta_{PN}$, with $S^2 = 1$ and $S^t\eta S = \eta$, we have

$$\delta\mathcal{H} = -S\delta\mathcal{H}S^t. \tag{5.7.38}$$

Since $\frac{1}{2}(1 \pm S)$, acting on vectors $V = V^M$ with upper indices can be viewed as projectors onto subspaces with S eigenvalues ± 1, and any matrix $M = M^{MN}$ can be viewed as a bivector and thus written as the sum of four projections onto independent subspaces:

$$\begin{aligned} M &= \frac{1}{4}(1+S)M\left(1+S^t\right) + \frac{1}{4}(1+S)M\left(1-S^t\right) \\ &+ \frac{1}{4}(1-S)M\left(1+S^t\right) + \frac{1}{4}(1-S)M\left(1-S^t\right). \end{aligned} \tag{5.7.39}$$

It then follows that the general solution to (5.7.38) is given by

$$\delta\mathcal{H} = \frac{1}{4}(1+S)\mathcal{M}\left(1-S^t\right) + \frac{1}{4}(1-S)\mathcal{M}\left(1+S^t\right), \tag{5.7.40}$$

since these are the terms that flip sign after the action of S on the left and S^t on the right;

$\mathcal{M}$ is an arbitrary symmetric matrix since $\delta\mathcal{H}$ is symmetric. Thus, we have

$$\begin{aligned}\delta S_1 &= \frac{1}{4}\int dXe^{-2d}\mathrm{Tr}\left\{\left[(1+S)\mathcal{M}\left(1-S^t\right)+(1-S)\mathcal{M}\left(1+S^t\right)\right]\mathcal{K}\right\}\\ &= \int dXe^{-2d}\mathrm{Tr}\left(\mathcal{M}\mathcal{R}\right),\end{aligned} \tag{5.7.41}$$

where

$$\mathcal{R}_{MN} = \frac{1}{4}\left(\delta_M^P - S^P{}_M\right)\mathcal{K}_{PQ}\left(\delta_N^Q + S^Q{}_N\right) + \frac{1}{4}\left(\delta_M^P + S^P{}_M\right)\mathcal{K}_{PQ}\left(\delta_N^Q - S^Q{}_N\right), \tag{5.7.42}$$

and $S^M{}_N = \mathcal{H}^{MP}\eta_{PN}$. Therefore, the equation of motion is

$$\mathcal{R}_{MN} = 0. \tag{5.7.43}$$

Since the second term in the action (5.7.34) is also a function of the generalized metric, we have

$$\delta S_2 = \int dx d\tilde{x} e^{-2d}\frac{\partial f}{\partial \mathcal{H}^{MN}}\delta\mathcal{H}^{MN} \equiv \int dx d\tilde{x} e^{-2d}\mathcal{F}_{MN}\delta\mathcal{H}^{MN}. \tag{5.7.44}$$

Everything follows the same way as above for computing this constrained variation. In the end, we have the general equation of motion given by

$$\mathcal{R}_{MN}\left(\mathcal{K}\right) + \mathcal{R}_{MN}\left(\mathcal{F}\right) = 0, \tag{5.7.45}$$

where the second term just means taking the same form of $\mathcal{R}$ but changing $\mathcal{K}_{MN} \to \delta f/\delta\mathcal{H}^{MN}$ everywhere.

We have now finished introducing all the necessary tools for the understanding of the papers presented in the following chapters of this thesis. It is important to mention that although a recent framework, DFT's literature encompasses many important results and mathematical developments that we are not making explicit here. For those, we recommend any of the reviews cited in the introduction of this chapter.

Chapter 6

Point Particle in DFT and a Singularity-Free Cosmological Solution

6.1 Introduction

If the fundamental building blocks of matter are elementary superstrings instead of point particles, the evolution of the very early universe will likely be very different than in Standard Big Bang cosmology. "String Gas Cosmology" is a scenario for the very early stringy universe which was proposed some time ago [34] (see e.g. [63, 64, 51] for some more recent reviews). String Gas Cosmology is based on making use of the key new degrees of freedom and symmetries which distinguish string theories from point particle theories. The existence of string oscillatory modes leads to a maximal temperature for a gas of strings in thermal equilibrium, the "Hagedorn temperature" T_H [58]. Assuming that all spatial dimensions are toroidal with radius R, the presence of string winding modes leads to a duality,

$$R \to \frac{1}{R}, \tag{6.1.1}$$

(in string units) in the spectrum of string states. This comes about since the energy of winding modes is quantized in units of R, whereas the energy of momentum modes is quantized in units of $1/R$. The symmetry (7.2.7) is realized by interchanging momentum and winding quantum numbers[1].

As was argued in [34], in String Gas Cosmology the temperature singularity of the Big Bang is automatically resolved. If we imagine the radius $R(t)$ decreasing from some initially very large value (large compared to the string length), and matter is taken to be a gas of superstrings, then the temperature T will initially increase, since for large values of R most of the energy of the system is in the light modes, which are the momentum modes, and the energy of these modes increases as R decreases. Before T reaches the maximal temperature T_H, the increase in T levels off since the energy can now go into producing oscillatory modes. For $R < 1$ (in string units) the energy will flow into the winding modes which are now the light modes. Hence,

$$T(R) = T\left(\frac{1}{R}\right) . \tag{6.1.2}$$

A sketch of the temperature evolution as a function of R is shown in Figure 1. As a function of $\ln R$ the curve is symmetric as a reflection of the symmetry (7.2.7). The region of R when the temperature is close to T_H and the curve in Fig. 1 is approximately horizontal is called the "Hagedorn phase". Its extent is determined by the total entropy of the system [34].

In [34] it was furthermore argued that at the quantum level there must be two position operators for every topological direction, one operator X dual to the momentum number (this is the usual position operator for point particle theories) and a dual operator $\tilde{X}$ which is dual to the winding number. The physically measured length $l(R)$ will always be determined by the light modes of the system. Hence, for large R it is determined by X, but for small R it is determined by $\tilde{X}$. Thus,

$$\begin{aligned} l(R) &= R \quad \text{for} \quad R \gg 1 , \\ l(R) &= \frac{1}{R} \quad \text{for} \quad R \ll 1 . \end{aligned} \tag{6.1.3}$$

More recently, a study of cosmological fluctuations in String Gas Cosmology [66] showed that thermal fluctuations in the Hagedorn phase of an expanding stringy universe will evolve into a scale-invariant spectrum of cosmological perturbations on large scales today (see [67] for a review). If the string scale is comparable to the scale of particle physics Grand Unification the predicted amplitude of the fluctuations matches the observations well (see [68] for recent observational results). The scenario also predicts a slight red tilt to the scalar power

[1] See also [65] for an extended discussion of T-duality when branes are added.

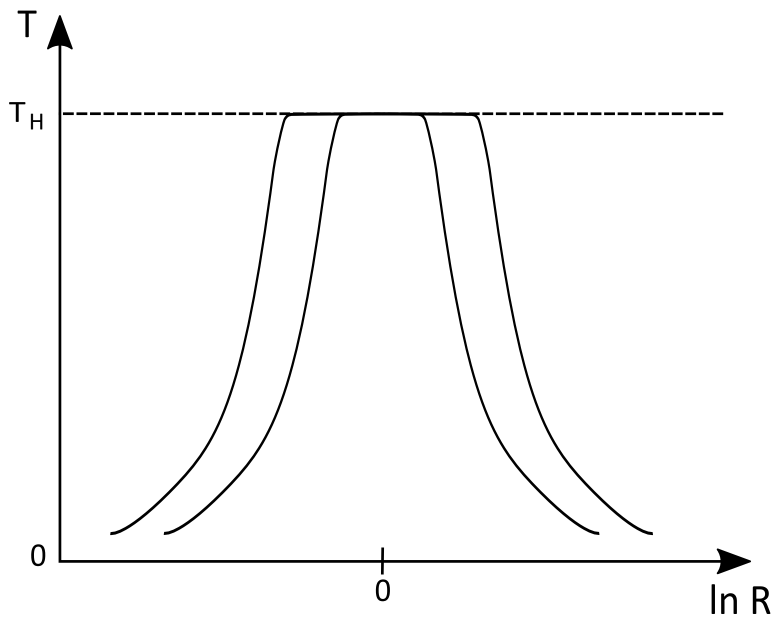

Fig. 6.1 T versus log R for type II superstrings. Different curves are obtained for different entropy values, which is fixed. The larger the entropy the larger the plateau, given by the Hagedorn temperature. For $R = 1$ we have the self-dual point.

spectrum. Hence, String Gas Cosmology provides an alternative to cosmological inflation as a theory for the origin of structure in the Universe. String Gas Cosmology predicts [69, 70] a slight blue tilt in the spectrum of gravitational waves, a prediction by means of which the scenario can be distinguished from standard inflation (meaning inflation in Einstein gravity driven by a matter field obeying the usual energy conditions). A simple modelling of the transition between the Hagedorn phase and the radiation phase leads to a running of the spectrum which is parametrically larger than what is obtained in simple inflationary models [6].

What is missing to date in String Gas Cosmology is an action for the dynamics of space-time during the Hagedorn phase. Einstein gravity is clearly inapplicable since is does not have the key duality symmetry (7.2.7). In “Pre-Big-Bang Cosmology” [71, 33] it was suggested to use dilaton gravity as a dynamical principle since the T-duality symmetry yields a scale factor duality symmetry. However, dilaton gravity does not take into account enough of the stringy nature of the Hagedorn phase.

“Double Field Theory” [72, 39, 73, 35] (see e.g. [40] for a review) has recently been introduced as a field theory which is consistent with the T-duality symmetry of string theory.

Given a topological space, in Double Field Theory (DFT) there are two position variables associated with every direction of the topological space which is compact. Since DFT is based on the same stringy symmetries as String Gas Cosmology, it is reasonable to expect DFT to yield a reasonable prescription for the dynamics of String Gas Cosmology.

On the other hand, even DFT will not yield an ideal dynamical principle for String Gas Cosmology since DFT only contains the massless modes of the bosonic string theory: the metric, the dilaton and the antisymmetric tensor field. We can try to include the other stringy degrees of freedom through a matter action in the same way as done in [34]. Hence, ultimately we would like to study the cosmological equations of motion of DFT in the presence of string gas matter. As a first step towards this goal we will in this paper study point particle motion in DFT. Through this study we can explore time-like and light-like geodesics. We will argue that (taking into account the appropriate definition of time) these geodesics are complete. This yields further evidence that string theory can lead to a nonsingular early universe cosmology. In particular, we also show that vacuum DFT background equations of motion produce a singularity-free cosmological solution when this new definition of time is considered.

6.2 Essentials of Double Field Theory

DFT is a field theory which lives in a "doubled" space in which the number of all dimensions with stable string windings is doubled. From the point of view of string theory this means having one spatial dimension dual to the momentum, and another one dual to the windings. We will here consider a setup in which all spatial dimensions have windings. Thus, our DFT will live in $(2D-1)$-dimensions, where $(D-1)$ is the number of spatial dimensions of the underlying manifold. Note that under toroidal compactifications, the corresponding T-dualily group is $O(n,n)$, where n corresponds to the number of spatial compact dimensions. In DFT, the theory is covariantly formulated in the double space, so that the underlying symmetry group is $O(n,n)$ [40]. Thus, any scalar object should be invariant under this group transformations. This will be relevant for when we build the point particle action in DFT below. We will denote the usual spatial coordinates by x^i and the dual coordinates by $\tilde{x}_i$.

We consider a cosmological space-time in standard General Relativity given by:

$$ds^2 \;=\; -dt^2 + g_{ij}dx^i dx^j\,, \tag{6.2.4}$$

where t is physical time and g_{ij} is the $(D-1)$-dimensional spatial metric. The coordinates i and j run over these original spatial indices. In DFT the metric in doubled space-time (all spatial dimensions doubled, but not time) is written in terms of a generalized metric $\mathcal{H}_{MN}$, where M and N run over all $(2D-1)$ space-time indices:

$$dS^2 \, = \, \mathcal{H}_{MN} dX^M dX^N \, . \tag{6.2.5}$$

The generalized metric depends both on the original metric and on the antisymmetric tensor field b_{ij}. In the case of a cosmological background we will usually separate out the time component and write the line element as:

$$dS^2 \, = \, -dt^2 + \mathcal{H}_{MN} dX^M dX^N \, , \tag{6.2.6}$$

where now M and N run only over spatial indices. In DFT all massless string states are considered. Hence, in addition to the metric there is a dilaton ϕ and an antisymmetric tensor field b_{ij}. The generalized metric is then given by:

$$\mathcal{H}_{MN} \, = \, \begin{bmatrix} g^{ij} & -g^{ik} b_{kj} \\ b_{ik} g^{kj} & g_{ij} - b_{ik} g^{kl} b_{lj} \end{bmatrix} \, , \tag{6.2.7}$$

where the indices are raised with the usual Riemannian metric.

The DFT action is chosen to treat g_{ij} and b_{ij} in a unified way, and to reduce to the supergravity action if there is no dependence on the dual coordinates. It is given by:

$$S \, = \, \int dx d\tilde{x} e^{-2d} \mathcal{R}, \tag{6.2.8}$$

where d contains both the dilaton ϕ and the determinant of the metric,

$$e^{-2d} \, = \, \sqrt{-g} e^{-2\phi} \, , \tag{6.2.9}$$

and where [74],

$$\begin{aligned}\mathcal{R} &= \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_K\mathcal{H}_{NL} \\ &+ 4\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} - 4\mathcal{H}^{MN}\partial_M d\partial_N d \\ &+ 4\partial_M\mathcal{H}^{MN}\partial_N d, \end{aligned} \tag{6.2.10}$$

with the matrix η^{MN} being given by:

$$\eta^{MN} = \begin{bmatrix} 0 & \delta_i{}^j \\ \delta^i{}_j & 0 \end{bmatrix}, \tag{6.2.11}$$

and we are writing the doubled space coordinates as:

$$X^M = (\tilde{x}_i, x^i). \tag{6.2.12}$$

Finally, note that since all the fields now depend on double coordinates, in principle we have doubled the number of degrees of freedom we had started with. In order to eliminate these extra degrees of freedom, one usually considers the section condition [40], which eliminates the dual-coordinate dependence of all the fields.

6.3 Point Particle Motion in Double Field Theory

The action for the massive relativistic point particle with world line coordinates $x^i(t)$ is given by:

$$S = -mc\int ds\,, \tag{6.3.13}$$

where the line element ds is given by (6.2.4). The natural generalization of it which corresponds to the action of a point particle with world line in doubled space given by $X^M(t)$ in a DFT background is written in the following way:

$$S = -mc\int dS, \tag{6.3.14}$$

where the generalized line element dS is given by (6.2.5). This action has also been introduced in [75], although the geodesic equations have not been worked out. Note this action is covariant under $O(n,n)$ transformations, as expected. Moreover, if the section condition is imposed, it recovers (6.3.13).

Before deriving the geodesic equations, a few comments are in order. The generalized metric is a constrained object, which satisfies:

$$\mathcal{H}\eta\mathcal{H} \;=\; \eta^{-1}, \tag{6.3.15}$$

therefore its variation is constrained as well, as showed in [61], and it is given by:

$$\begin{aligned}\frac{\partial\mathcal{H}^{(c)}_{MN}}{\partial X^P} = \frac{1}{4}\bigg[&(\eta_{MQ}+\mathcal{H}_{MQ})\,\frac{\partial\mathcal{H}^{QR}}{\partial X^P}\,(\eta_{RN}-\mathcal{H}_{RN})\,+\\ &+(\eta_{MQ}-\mathcal{H}_{MQ})\,\frac{\partial\mathcal{H}^{QR}}{\partial X^P}\,(\eta_{RN}+\mathcal{H}_{RN})\bigg]\,,\end{aligned} \tag{6.3.16}$$

where the index (c) specifies when we consider constrained objects.

Varying the action with respect to the world sheet coordinates $X^M(t)$ yields the following equations of motion:

$$\mathcal{H}_{MN}\frac{d^2X^N}{dS^2}+\frac{\partial\mathcal{H}^{(c)}_{MN}}{\partial X^P}\frac{dX^P}{dS}\frac{dX^N}{dS}-\frac{1}{2}\frac{\partial\mathcal{H}^{(c)}_{PN}}{\partial X^M}\frac{dX^P}{dS}\frac{dX^N}{dS}=0. \tag{6.3.17}$$

Then, the equation of motion for the dual coordinates $(M=1)$ is:

$$\begin{aligned}g^{ij}\frac{d^2\tilde{x}_j}{dS^2}-g^{ik}b_{kj}\frac{d^2x^j}{dS^2}+(\tilde{\partial}^m g^{in})\frac{d\tilde{x}_n}{dS}\frac{d\tilde{x}_m}{dS}+(\partial_m g^{in})\frac{dx^m}{dS}\frac{d\tilde{x}_n}{dS}+\tilde{\partial}^m(g^{ik}b_{kn})\frac{d\tilde{x}_m}{dS}\frac{dx^n}{dS}\\ +\partial_m(g^{ik}b_{kn})\frac{dx^m}{dS}\frac{dx^n}{dS}-\frac{1}{2}\bigg[(\tilde{\partial}^i g^{mn})\frac{d\tilde{x}_m}{dS}\frac{d\tilde{x}_n}{dS}-\tilde{\partial}^i(g^{mk}b_{kn})\frac{d\tilde{x}_m}{dS}\frac{dx^n}{dS}+\\ \tilde{\partial}^i(b_{mk}g^{kn})\frac{dx^m}{dS}\frac{d\tilde{x}_n}{dS}+\tilde{\partial}^i(g_{mn}-b_{mk}g^{kj}b_{jn})\frac{dx^m}{dS}\frac{dx^n}{dS}\bigg]\;=\;0\,,\end{aligned} \tag{6.3.18}$$

whereas the equation of motion for the regular coordinates ($M = 2$) is:

$$\begin{aligned}
&\left(g_{ij} - b_{ik}g^{kl}b_{lj}\right)\frac{d^2x^j}{dS^2} + \partial_l(g_{in} - b_{ik}g^{kj}b_{jn})\frac{dx^n}{dS}\frac{dx^l}{dS} - \frac{1}{2}\partial_i(g_{mn} - b_{mk}g^{kj}b_{jn})\frac{dx^m}{dS}\frac{dx^n}{dS} \\
&+b_{ik}g^{kj}\frac{d^2\tilde{x}_j}{dS^2} + \tilde{\partial}^l(b_{ik}g^{kn})\frac{d\tilde{x}_l}{dS}\frac{d\tilde{x}_n}{dS} + \partial_j(b_{ik}g^{kn})\frac{d\tilde{x}_n}{dS}\frac{dx^j}{dS} + \tilde{\partial}^l(g_{in} - b_{ik}g^{kj}b_{jn})\frac{d\tilde{x}_l}{dS}\frac{dx^n}{dS} \\
&-\frac{1}{2}\left[\partial_i g^{mn}\frac{d\tilde{x}_m}{dS}\frac{d\tilde{x}_n}{dS} - \partial_i(g^{mk}b_{kn})\frac{d\tilde{x}_m}{dS}\frac{dx^n}{dS} + \partial_i(b_{mk}g^{kn})\frac{d\tilde{x}_n}{dS}\frac{dx^m}{dS}\right] = 0. \qquad (6.3.19)
\end{aligned}$$

These are the most general equations for a point particle in a DFT background with a metric and a 2-form field. From the first line of equation (6.3.19) it is easy to see that after imposing the section condition and setting the two-form to be zero, we are left with the geodesic equation of a relativistic point particle.

6.4 Point Particle Motion in a Cosmological Background

Now we want to specialize the discussion to a homogeneous and isotropic cosmological background with vanishing b_{ij}. We thus consider the cosmological metric:

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j + a^{-2}(t)\delta^{ij}d\tilde{x}_i d\tilde{x}_j \,, \qquad (6.4.20)$$

where $a(t)$ is the scale factor. Setting the antisymmetric tensor field to zero, the general equations of motion of the previous section simplify to:

$$\frac{d}{dS}\left(\frac{d\tilde{x}_a}{dS}\frac{1}{a^2}\right) = 0 \qquad (6.4.21)$$

$$\frac{d}{dS}\left(\frac{dx^a}{dS}a^2\right) = 0. \qquad (6.4.22)$$

These are the geodesic equations of point particle motion of DFT in a cosmological background.

We will now argue that geodesics are complete in the sense that they can be extended to arbitrarily large times both in the future and in the past. This is true for all particle geodesics except for the set of measure zero where either all coordinates x^i or all coordinates $\tilde{x}_i$ vanish. We will consider a given monotonically increasing scale factor $a(t)$, like the scale factor of Standard Big Bang cosmology. Note that in this parametrization, the coordinate t

lies in the interval between $t = 0$ and $t = \infty$.

Consider a trajectory at some initial time t_0 with the property that some x^i and some $\tilde{x}_j$ are non-vanishing. Due to Hubble friction, then the velocity dx^i/dt will decrease. On the other hand, the dual velocity $d\tilde{x}_j/dt$ will approach the speed of light. Hence, the proper distance ΔS in double space between time t_0 and some later time t_2 will be:

$$\Delta S = \int_{t_0}^{t_2} \gamma(t)^{-1}(1+T_2)^{1/2}dt\,, \tag{6.4.23}$$

where T_2 is the contribution from the dual which ceases to increase at late times since the dual velocity goes at the speed of light, and $\gamma(t)$ is the relativistic γ factor of the motion in the x^i direction (for simplicity we consider motion only in one original direction and in one dual direction). Hence, the geodesic can be extended to infinite time in the future.

Now consider evolving this geodesic backwards in time from t_0 to some earlier time t_1. Then it is the motion of the dual coordinates which comes to rest. The proper distance in double space is now:

$$\Delta S = \int_{t_1}^{t_0} \tilde{\gamma}(t)(1+T_1)^{1/2}dt\,, \tag{6.4.24}$$

where $\tilde{\gamma}$ is the relativistic gamma factor for motion in the dual space directions, and T_1 is the contribution to the proper distance which comes from the regular spatial dimensions which is negligible at very early times since the velocity in the regular directions approaches the speed of light.

The expansion of the scale factor in the dual spatial directions as time decreases is analogous to the expansion in the regular directions as time increases. In line with T-duality we propose to view the dynamics of the dual spatial dimensions as t decreases as expansion when the *dual time*,

$$t_d = \frac{1}{t}, \tag{6.4.25}$$

increases. In fact, in analogy to the definition of *physical length* $l(R)$ in (7.2.6) [34], we can define a *physical time* $t_p(t)$ as:

$$\begin{aligned} t_p(t) &= t \quad \text{for} \quad t \gg 1\,, \\ t_p(t) &= \frac{1}{t} \quad \text{for} \quad t \ll 1\,. \end{aligned} \tag{6.4.26}$$

With this definition, the geodesics studied in this paper are geodesically complete in the sense that they can be extended in both directions to infinite time.

We can also justify the above argument by dualizing both space and time, *i.e.*, by also introducing a dual time $\tilde{t}$ which is dual to "temporal winding modes" of the string [76]. This concept can be made rigorous in Euclidean space-time where time is taken to be compact. When the regular Euclidean time domain shrinks in size, the dual time domain increases. This is analogous to the dual space domain increasing as $1/R$ when the regular space domain R is decreasing. From this point of view, the definition (6.4.26) is simply the time component of (7.2.6). We can also write (6.4.26) as:

$$\begin{aligned} t_p(t) &= t \quad \text{for} \quad t \gg 1\,, \\ t_p(t) &= \tilde{t} = t_d \quad \text{for} \quad t \ll 1\,. \end{aligned} \tag{6.4.27}$$

At finite temperatures T, the string partition function $Z(T)$ is periodic in Euclidean time $\beta = 1/T$, and - at least for certain string theory setups - satisfies the temperature duality:

$$Z(\beta) = Z\left(\frac{1}{\beta}\right), \tag{6.4.28}$$

which is a consequence of the T-duality symmetry. Based on this symmetry it was argued [77, 78] that these string theory models correspond to bouncing cosmologies in which the physical temperature is taken to be (always in string units):

$$\begin{aligned} T_p(T) &= T \quad \text{for} \quad T \ll 1\,, \\ T_p(T) &= \frac{1}{T} \quad \text{for} \quad T \gg 1\,. \end{aligned} \tag{6.4.29}$$

Note that two time formalism based on ideas from string theory were also discussed in [79] and [80].

6.5 Singularity-free cosmological background

In this section, we study a geometrical approach to formalize the idea of the physical clock introduced in the last section and conclude that upon considering this definition of time, the vacuum solutions of the DFT background equations are singularity-free. The DFT cosmological background equations of motion in the presence of a hydrodynamical fluid will be discussed in [2].

We start off considering the following ansatz:

$$\begin{aligned} dS^2 &= - dt^2 - d\tilde{t}^2 + a^2(t,\tilde{t}) \sum_{i=1}^{D-1} dx^i dx^i \\ &+ a^{-2}(t,\tilde{t}) \sum_{j=1}^{D-1} d\tilde{x}^j d\tilde{x}^j, \end{aligned} \quad (6.5.30)$$

in the DFT equations of motion, resulting in [81]:

$$\left[4\partial_{\tilde{t}}\partial_{\tilde{t}}d - 4(\partial_{\tilde{t}}d)^2 - (D-1)\tilde{H}^2\right] + \left[4\partial_t\partial_t d - 4(\partial_t d)^2 - (D-1)H^2\right] = 0 \quad (6.5.31)$$

$$\left[-(D-1)H^2 + 2\partial_t\partial_t d\right] - \left[-(D-1)\tilde{H}^2 + 2\partial_{\tilde{t}}\partial_{\tilde{t}}d\right] = 0 \quad (6.5.32)$$

$$\left[\dot{\tilde{H}} - 2\tilde{H}\partial_{\tilde{t}}d\right] + \left[\dot{H} - 2H\partial_t d\right] = 0, \quad (6.5.33)$$

where $H = a^{-1}da/dt$ and $\tilde{H} = a^{-1}da/d\tilde{t}$.

The solution to these equations are given by

$$a_{\pm}(\tilde{t},t) = \left|\frac{t}{\tilde{t}}\right|^{\pm 1/\sqrt{D-1}}, \quad d(t,\tilde{t}) = -\frac{1}{2}\ln|t\tilde{t}| \quad (6.5.34)$$

$$a_{\pm}(\tilde{t},t) = \left|t\tilde{t}\right|^{\pm 1/\sqrt{D-1}}, \quad d(t,\tilde{t}) = -\frac{1}{2}\ln|t\tilde{t}|. \quad (6.5.35)$$

However, so far there was no clear interpretation of these equations and solutions given the presence of the extra time coordinate, $\tilde{t}$.

Within the prescription we have introduced in the last section, we can interpret this extra time coordinate as the geometrical clock associated to the winding modes[2]. In particular, we have provided arguments that this clock, when seen from the momentum perspective, should correspond to:

$$\tilde{t} = \frac{1}{t}. \quad (6.5.36)$$

Thus, the above ansatz can be seen as a way to implement the ideas we have introduced at

[2]We will discuss further about this prescription in [2].

a geometrical level. By doing so, the effective line element becomes:

$$dS^2 = -\left(1+\frac{1}{t^4}\right)dt^2 + a^2(t)\sum_{i=1}^{D-1} dx^i dx^i + a^{-2}(t)\sum_{j=1}^{D-1} d\tilde{x}^j d\tilde{x}^j. \tag{6.5.37}$$

The solutions of the DFT equations of motion become:

$$a_\pm(t) = |t|^{\pm 2\sqrt{D-1}}, \qquad d(t) = \text{const.} \tag{6.5.38}$$

$$a_\pm(t) = \text{const.}, \qquad d(t) = \text{const.} \tag{6.5.39}$$

Now, using instead the physical clock, we can rewrite the line element by considering the following

$$dt_p = \sqrt{1+\frac{1}{t^4}}\,dt, \tag{6.5.40}$$

so that we recover a FRW-like metric in the standard form, meaning $g_{00} = -1$. It is clear that the physical clock reduces to the momentum one for large t. In fact, its functional form in terms of t is very complicated, but its plot is easy to understand:

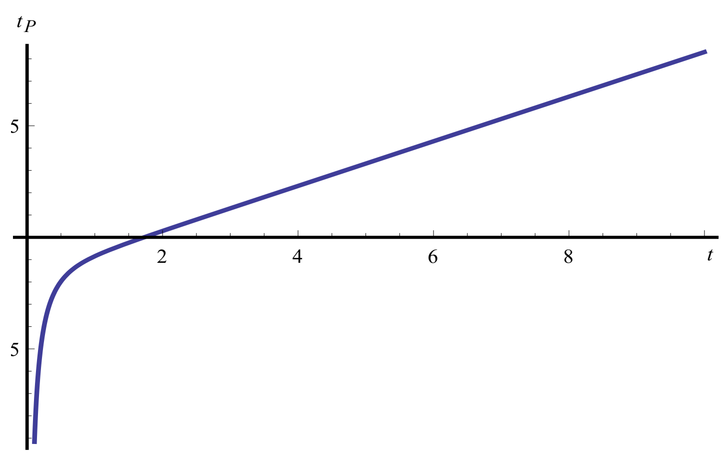

Fig. 6.2 Physical clock as a function of the time coordinate. The physical clock goes from $-\infty$ to ∞ for $t \in (0, \infty)$.

The non-trivial solution for the scale factor (6.5.39) can also be plotted in terms of the physical clock (we consider $D = 4$ for simplicity),

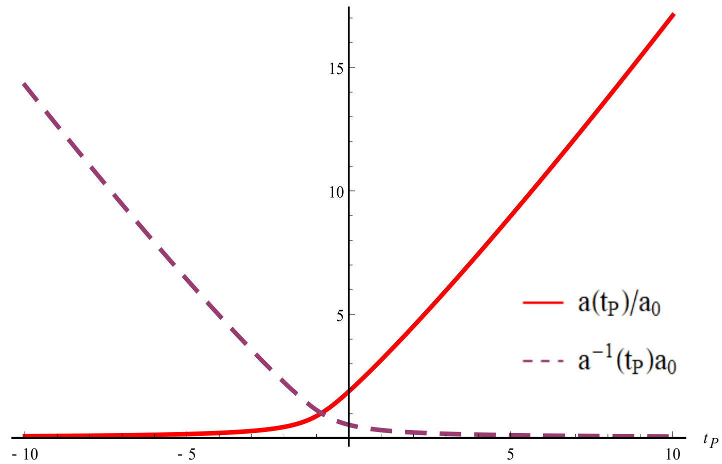

Fig. 6.3 The scale factor goes to zero only at $t_p \to -\infty$. Similarly its inverse goes to zero when $t_p \to \infty$.

Finally, the metric looks like

$$ds^2 = -dt_p^2 + a^2(t_p) \sum_{i=1}^{D-1} dx^i dx^i + a^{-2}(t_p) \sum_{j=1}^{D-1} d\tilde{x}^j d\tilde{x}^j, \tag{6.5.41}$$

as expected. It is important to realize that this effective $(2D-1)$-dimensional geometry reduces effectively to a D-dimensional one for large t_p, the momentum sector, and analogously for large negative t_p, the winding sector. We also observe that there should be a D-dimensional slice that has its volume bounded from below for all t_p. This slice is the physical geometry where we live and which is accessible by physical rulers and clocks (which are always given by the corresponding light modes).

6.6 Conclusions and Discussion

We have studied the geodesics corresponding to point particle motion in Double Field Theory. We derived the general equations of motion, and then considered the special case of a cosmological background with vanishing antisymmetric tensor field. We argued that in this context the geodesics of point particle motion are complete, provided we measure the motion in terms of a physical time which reflects the T-duality symmetry of the setup. Our result provides further support for the expectation that cosmological singularities are resolved in superstring cosmology. Then, we also considered a geometrization of this prescription within the framework of DFT, resulting in a cosmological solution which is singularity-free.

Chapter 7

Dual Space-Time and Nonsingular String Cosmology

7.1 Introduction

The singularities which arise at the beginning of time in both standard and inflationary cosmology indicate that the theories which are being used in cosmology break down as the singularity is approached. If space-time is described by Einstein gravity and matter obeys energy conditions which are natural from the point of view of point particle theories, then singularities in homogeneous and isotropic cosmology are unavoidable [19]. These theorems in fact extend to inflationary cosmology [20, 21, 22].

But we know that Einstein gravity coupled to point particle matter cannot be the correct description of nature. The quantum structure of matter is not consistent with a classical description of space-time. The early universe needs to be described by a theory which can unify space-time and matter at a quantum level. Superstring theory (see e.g. [26, 27], for a detailed overview) is a promising candidate for a quantum theory of all four forces of nature. At least at the string perturbative level, the building blocks of string theory are fundamental strings. Strings have degrees of freedom and new symmetries which point particle theories do not have, and these features may lead to a radically different picture of the very early universe, as discussed many years ago in [34] (see also [82]).

As discussed in [34], string thermodynamic considerations indicate that the the cosmological evolution in the context of string theory should be nonsingular. A key realization is that the temperature of a gas of closed string in thermal equilibrium cannot exceed a limiting value, the *Hagedorn temperature* [58]. In fact, as reviewed in the following section, the temperature of a gas of closed strings in a box of radius R decreases as R becomes much smaller than the string length. If the entropy of the string gas is large, then the range of values of R for which the temperature is close to the Hagedorn temperature T_H is large. This is called the *Hagedorn phase* of string cosmology. The exit from the Hagedorn phase is smooth and is a consequence of the decay of string winding modes into string loops[1]. The transition leads directly to the radiation phase of Strandard Big Bang cosmology (see [63, 64, 51, 86] for reviews of the String Gas Cosmology scenario).

If strings in the Hagedorn phase are in thermal equilibrium, then the thermal fluctuations of the energy-momentum tensor can be computed using the methods of [87]. In particular, it can be shown that in a compact space with stable winding modes the specific heat capacity has holographic scaling as a function of the radius of the volume being considered. As a consequence [66, 67], thermal fluctuations of strings in the Hagedorn phase lead to a scale-invariant spectrum of cosmological perturbations at late times, with a slight red tilt like what is predicted [88] in inflationary cosmology. If the string scale is comparable to the scale of particle physics Grand Unification the predicted amplitude of the fluctuations matches the observations well (see [68] for recent observational results). Hence, String Gas Cosmology provides an alternative to cosmological inflation as a theory for the origin of structure in the Universe. The predicted spectrum of gravitational waves [69, 70] is also scale-invariant, but a slight blue tilt is predicted, in contrast to the prediction in standard inflationary cosmology. This is a prediction by means of which the scenario can be distinguished from standard inflation (meaning inflation in Einstein gravity driven by a matter field obeying the usual energy conditions). A simple modelling of the transition between the Hagedorn phase and the radiation phase leads to a running of the spectrum which is parametrically larger than what is obtained in simple inflationary models [6].

In this paper, we will study the cosmological background dynamics which follow from string theory if the target space has stable winding modes. An example where this is the case is a spatial torus. We will argue that from the point of view of string theory the dynamics is non-singular.

[1]This mechanism suggests that exactly three spatial dimensions can become large [34], the others being confined to the string length by the interaction of the string winding and momentum modes [83, 84, 85].

7.2 Dual Space from T-duality

For simplicity let us assume that space is toroidal with $d = 9$ spatial dimensions, all of radius R. Closed strings then have *momentum modes* whose energies are quantized in units of $1/R$

$$E_n \,=\, \frac{n}{R}\,, \tag{7.2.1}$$

where n is an integer. They also have *winding modes* whose energies are quantized in units of R, i.e.

$$E_m \,=\, mR\,, \tag{7.2.2}$$

where m is an integer and we are working in units where the string length is one. Strings also have a tower of *oscillatory modes* whose energies are independent of R. The number of oscillatory modes increases exponentially with energy.

It follows from (7.2.1) and (7.2.2) that the spectrum of string states is invariant under the T-duality transformation

$$R \,\rightarrow\, \frac{1}{R} \tag{7.2.3}$$

if the momentum and winding numbers are interchanged. The transformation (7.2.3) is also a symmetry of the string interactions, and is assumed to be a symmetry of string theory beyond perturbation theory (see e.g. [27])[2].

As is well known, the position eigenstates $|x\rangle$ are dual to momentum eigenstates $|p\rangle$. In a compact space, the momenta are discrete, labelled by integers n, and hence

$$|x\rangle \,=\, \sum_n e^{inx}|n\rangle\,. \tag{7.2.4}$$

where $|n\rangle$ is the momentum eigenstate with momentum quantum number n. As already discussed in [34], in our string theory setting, windings are T-dual to momenta, and we can

[2] See also [65] for an extended discussion of T-duality when branes are added.

define a T-dual position operator

$$|\tilde{x}\rangle = \sum_m e^{im\tilde{x}}|m\rangle \,, \tag{7.2.5}$$

where $|m\rangle$ are the eigenstates of winding, labelled by an integer m.

As again argued in [34], experimentalists will measure physical length in terms of the position operators which are the lightest. Thus, for $R > 1$ (in string units), it is the regular position operators $|x\rangle$ which determine physical length, whereas for $R < 1$ it is the dual variables $|\tilde{x}\rangle$. Hence, the physical length $l_p(R)$ is given by

$$\begin{aligned} l(R) &= R \quad \text{for} \quad R \gg 1 \,, \\ l(R) &= \frac{1}{R} \quad \text{for} \quad R \ll 1 \,. \end{aligned} \tag{7.2.6}$$

As was argued in [34], in String Gas Cosmology the temperature singularity of the Big Bang is automatically resolved. If we imagine the radius $R(t)$ decreasing from some initially very large value (large compared to the string length), and matter is taken to be a gas of superstrings, then the temperature T will initially increase, since for large values of R most of the energy of the system is in the light modes, which are the momentum modes, and the energy of these modes increases as R decreases. Before T reaches the maximal temperature T_H, the increase in T levels off since the energy can now go into producing oscillatory modes. For $R < 1$ (in string units) the energy will flow into the winding modes which are now the light modes. Hence,

$$T(R) = T\left(\frac{1}{R}\right) \,. \tag{7.2.7}$$

A sketch of the temperature evolution as a function of R is shown in Figure 1. As a function of $\ln R$ the curve is symmetric as a reflection of the symmetry (7.2.7). The region of R when the temperature is close to T_H and the curve in Fig. 1 is approximately horizontal is called the "Hagedorn phase". Its extent is determined by the total entropy of the system [34].

7.3 Cosmological Dynamics and Dual Space-Time

In the following we will couple a gas of strings to a background appropriate to string theory. Since the massless modes of string theory include, in addition to the graviton, the

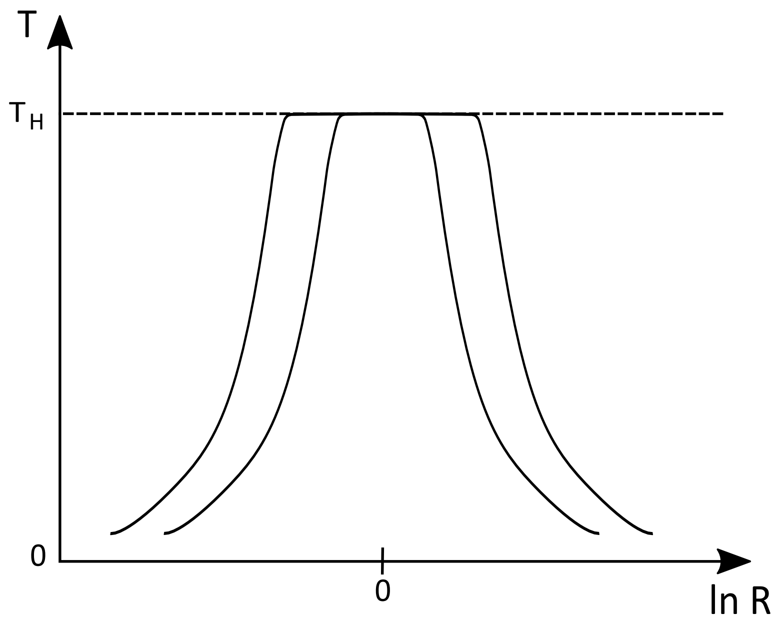

Fig. 7.1 T versus log R for type II superstrings. Different curves are obtained for different entropy values, which is fixed. The larger the entropy the larger the plateau, given by the Hagedorn temperature. For $R = 1$ we have the self-dual point.

dilaton and an antisymmetric tensor field, a cosmological background will contain the metric, the dilaton and the antisymmetric tensor field. For a homogeneous and isotropic cosmology the metric can be written as

$$ds^2 \ = \ -dt^2 + a(t)^2 d\mathbf{x}^2 \,, \tag{7.3.8}$$

where t is physical time, $a(t)$ is the cosmological scale factor and $\mathbf{x}$ are comoving spatial coordinates. We have assumed vanishing spatial curvature for simplicity. We denote the dilaton by $\phi(t)$.

The T-duality symmetry of string theory leads to an important symmetry of the massless background fields, the *scale factor duality* [71, 33]. In the absence of an antisymmetric tensor field these take the form

$$\begin{aligned} a(t) \quad &\rightarrow \quad \frac{1}{a} \\ \bar{\phi}(t) \quad &\rightarrow \quad \bar{\phi}(t) \end{aligned} \tag{7.3.9}$$

where the T-duality invariant combination of the scale factor and the dilaton is

$$\bar{\phi} \equiv \phi - d\ln a\,, \tag{7.3.10}$$

where $d = D - 1$ is the number of spatial dimensions and D the number of space-time dimensions.

The background equations of motion are those of dilaton gravity (we will neglect the antisymmetric tensor field). In the absence of matter, these equations were studied in detailed in the context of Pre-Big-Bang cosmology [71, 33]. In the presence of string matter, they have been analyzed in [56]. The equations in the presence of a gas of matter described by energy density ρ and pressure p are

$$\left(\dot{\phi} - dH\right)^2 - dH^2 = e^{\phi}\rho \tag{7.3.11}$$

$$\dot{H} - H\left(\dot{\phi} - dH\right) = \frac{1}{2}e^{\phi}p \tag{7.3.12}$$

$$2\left(\ddot{\phi} - d\dot{H}\right) - \left(\dot{\phi} - dH\right)^2 - dH^2 = 0\,, \tag{7.3.13}$$

where $H \equiv \dot{a}/a$. These are the equations in the string frame. In particular, we can combine these equations to write a continuity equation,

$$\dot{\rho} + (D-1)H(\rho + p) = 0. \tag{7.3.14}$$

We consider matter to be a gas of strings. For $R \gg 1$ most of the energy is in the momentum modes which act as radiation and hence have an equation of state parameter $w \equiv p/\rho$ given by $w = 1/d$. For $R \ll 1$, however, most of the energy density is in the winding modes whose equation of state parameter is $w = -1/d$. Finally, for $R = 1$ the equation of state is $w = 0$. An interpolating form of the matter equation of state is

$$w\left(a\right) = \frac{2}{\pi d}\arctan\left(\beta\ln\left(\frac{a}{a_0}\right)\right), \tag{7.3.15}$$

where a_0 is the value of the scale factor when $R = 1$, and β is a constant which depends on the total entropy of the gas. The larger the entropy is, the wider the Hagedorn phase as a function of a, and hence the smaller the value of β. For this equation of state, the continuity equation for string gas matter can be integrated and yields

$$\begin{aligned} \ln\frac{\rho}{\rho_0} &= -\ d\ln\frac{a}{a_0} - \frac{2}{\pi}\left\{\ln\left(\frac{a}{a_0}\right)\arctan\left[\beta\ln\left(\frac{a}{a_0}\right)\right]\right\} \\ &- \ \frac{2}{\pi}\left\{\frac{1}{2\beta}\ln\left[1+\beta^2\left(\ln\frac{a}{a_0}\right)^2\right]\right\}, \end{aligned} \tag{7.3.16}$$

where ρ_0 is the energy density at the string length. This result reproduces what is expected for large and small radii,

$$\rho\,(a \text{ large}) \ \rightarrow\ \rho_0\,(a/a_0)^{-(d+1)} \tag{7.3.17}$$

$$\rho\,(a \text{ small}) \ \rightarrow\ \rho_0\,(a/a_0)^{-(d-1)}\,. \tag{7.3.18}$$

for pure momentum or pure winding modes, respectively.

At this point we have a system of background and matter in which both components have the same symmetries. We now turn to an exploration of solutions. Following closely [33], we make the ansatz

$$\begin{aligned} a(t) &\sim \left(\frac{t}{t_0}\right)^{\alpha} \\ \bar{\phi}(t) &\sim -\beta\ln\left(\frac{t}{t_0}\right), \end{aligned} \tag{7.3.19}$$

where α and β are constants, and t_0 is a reference time. Inserting into the dilaton gravity equations gives the following constraints on the constants

$$\begin{aligned} (D-1)\,w\alpha+\beta &= 2 \\ \beta^2+(D-1)\,\alpha^2 &= 2\beta\,. \end{aligned} \tag{7.3.20}$$

Deep in the Hagedorn phase when $w=0$ we get

$$(\alpha,\beta)\,=\,(0,2)\,. \tag{7.3.21}$$

This corresponds to a static scale factor in the string frame. Converting to the Einstein frame in which the scale factor $\tilde{a}(t)$ is given by

$$\tilde{a}(t) \;=\; a(t)e^{-\phi/(d-1)} \tag{7.3.22}$$

we find

$$\tilde{a}(t) \;\sim\; \left(\frac{t}{t_0}\right)^{2/(d-1)} \,. \tag{7.3.23}$$

In the large a phase when $w = 1/d$ we get

$$(\alpha, \beta) \;=\; \left(\frac{2}{D}, \frac{2}{D}(D-1)\right) \,. \tag{7.3.24}$$

In this case, the dilaton is constant and hence the string frame and Einstein frame scale factors are the same. As expected, the scale factor evolves as in a standard radiation dominated universe. There is a second solution of (7.3.20), but that solution is consistent only for $p = 0$.

When $w = -1/d$ we have

$$(\alpha, \beta) \;=\; \left(-\frac{2}{D}, \frac{2}{D}(D-1)\right) \,. \tag{7.3.25}$$

The string frame scale factor is expanding as we go backwards in time. Translating to the Einstein frame we get

$$\tilde{a}(t) \;\sim\; \left(\frac{t}{t_0}\right)^{2/(d-1)} \,. \tag{7.3.26}$$

In the Einstein frame, the scale factor vanishes at $t = 0$ while in the string frame it blows up in this limit.

Let us track the dynamics backwards in time, beginning with a large torus ($R \gg 1$). The energy will hence be in the momentum modes and the equation of state is that of radiation. As we go back in time, the scale factor decreases (it is the same in the two frames), the energy density increases, and eventually the temperature approaches the Hagedorn value at which point oscillatory and winding modes of the string gas get excited, leading to a transition to an equation of state with $p = 0$. We enter a Hagedorn phase during which the string

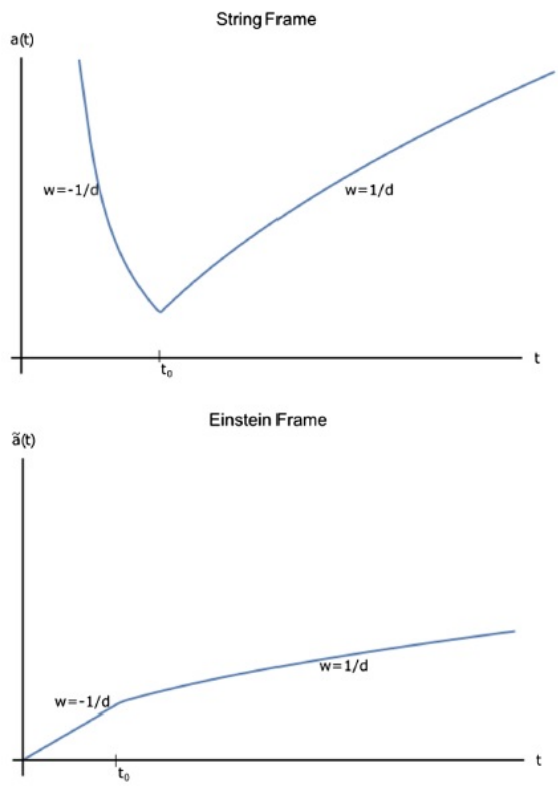

Fig. 7.2 The schematic solution for the scale factor in the String and Einstein frames for $D = 4$. Note that the transition between the winding and momenta equation of state has been smoothed out, as it is expected if (7.3.15) is considered.

frame scale factor is constant while the Einstein frame scale factor is decreasing. This means that the radius of the torus R is decreasing, and it soon becomes energetically preferable for the energy of the string gas to drift to the winding modes, leading to an equation of state $w = -1/d$. In the winding phase the Einstein frame scale factor is still decreasing, which is a self-consistency check on the assumption that the energy of the string gas is mostly in the winding modes[3].

We see that in the string frame, there is no curvature singularity. As the coordinate time t runs from $t = 0$ to $t = \infty$, the scale factor is initially contracting, bounces in the Hagedorn phase and expands afterwards in the radiation phase, as showed schematically in Fig. 2.

Following [1], we argue that in the phase dominated by winding modes we should measure

[3]If we do not allow momentum and winding modes to decay, then, as studied in [56], we obtain solutions where the string frame scale factor oscillates about a_0.

time in terms of the dual time variable

$$t_d \equiv -\frac{t_c^2}{t} \tag{7.3.27}$$

where t_c corresponds to the coordinate time at the center of the Hagedorn phase. In terms of t_d, the solution looks like a contracting universe.

From the point of view of the Einstein frame, the scale factor vanishes at $t = 0$. But from the point of view of a detector made up of winding modes, the measured scale factor is proportional to $a(t)^{-1}$. Hence, the time interval $0 < t < t_c$ corresponds to a contracting universe in terms of the dual position basis.

Heuristically, there are two simple reasons for introducing a dual time coordinate. Let us consider for simplicity a fixed dilaton, so that we have a radiation solution. It is clear that there is an asymmetry between large and small scale factor, since the proper time for the scale factor to go to infinity diverges, while it is finite when the scale factor decreases to zero from some finite value. However, from the point of view of T-duality we should not be able to distinguish between a large and a small universe. This is the first hint towards a more general definition of the physical clock, t_p.

Another qualitative argument follows from special relativity considerations brought together with T-duality. For a large radius, rods are made out of momentum modes, and time measurements for a given physical length, Δx, are given by

$$|\Delta t| = |\Delta x|\,, \tag{7.3.28}$$

where the speed of light has been set to unit. If the universe is composed of closed strings, in principle we could have considered measuring physical length in terms of winding modes as well, and the natural rods built out of these modes are related to the physical length by

$$\Delta \tilde{x} \rightarrow \frac{\alpha'^2}{\Delta x}, \tag{7.3.29}$$

where α' is the string tension. Thus, we can rewrite (7.3.28) as,

$$|\Delta \tilde{x}| \rightarrow \left|\frac{\alpha'^2}{\Delta t}\right|. \tag{7.3.30}$$

Now, if we cannot distinguish large from small, we could have started the argument using winding modes instead, so that we would write the following relation[4],

$$|\Delta\tilde{x}| = \left|\Delta\tilde{t}\right| . \tag{7.3.31}$$

Thus, it is also natural to propose a *winding-clock* that is dual to the momentum-clock by combining the above formulae,

$$\left|\Delta\tilde{t}\right| \to \left|\frac{\alpha'^2}{\Delta t}\right| . \tag{7.3.32}$$

Evidently, physically speaking there is only a single clock. When only winding or momentum modes are light, the existence of a unique time coordinate is already clear. Around the self-dual point, when both modes are energetically favorable, that should also be the case. Therefore, we need a prescription to reduce both time coordinates to a single physical time. We call this prescription *physical clock constraint* and it is given by the identification (7.3.27).

These ideas likely have a very natural interpretation in terms of Double Field Theory [39, 35, 40] (see also [36, 37, 38] for some early work). Double Field Theory is a generalization of supergravity which lives in $2d$ spatial dimensions, with the first d dimensions corresponding to the usual x variables, and the second d dimensions to the dual spatial variables $\tilde{x}$. In Double Field Theory there is a generalized metric which for homogeneous and isotropic cosmology and in the absence of an antisymmetric tensor field is given by

$$ds^2 \;=\; -dt^2 + a^2(t)\delta_{ij}dx^i dx^j + a^{-2}(t)\delta^{ij}d\tilde{x}_i d\tilde{x}_j \,. \tag{7.3.33}$$

The determinant of the generalized metric is one. As space shrinks in the x directions, it opens up in the $\tilde{x}$ directions. This is sketched in Fig. 3. In work in progress [3, 4] we are exploring this connection in more detail, in particular using the $O(D, D)$-formalism for formalizing the introduction of a dual time, and discussing how the physical clock constraint can be seen analogously to the imposition of the section condition in DFT for the dual coordinates [40] [5].

[4]By T-duality one can argue that the dual speed of light is also equal to unit.

[5]See [81] for a study of cosmological vacuum solutions of double field theory including a dilaton potential.

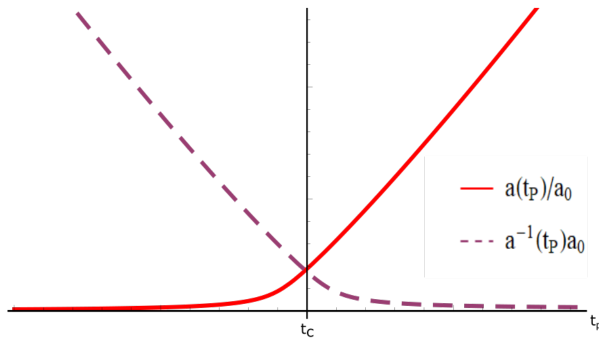

Fig. 7.3 The scale factor goes to zero only at $t_p \to -\infty$. Similarly its inverse goes to zero when $t_p \to \infty$.

7.4 Discussion

We have studied the equations of motion of a cosmological background containing the scale factor $a(t)$ and the dilaton in the presence of string gas matter sources. Both the background action and the matter action are consistent with the T-duality symmetry of string theory. While we do not expect our description to be adequate in the high density phase when truly stringy effects must be considered, our analysis is an improvement over the usual effective field theory of string cosmology where the underlying background geometry is not covariant with the T-duality symmetry.

We find that the solutions are nonsingular, at least when interpreted in the context of double space-time. We conjecture that an improved description could be obtained using the tools of Double Field Theory[6].

[6] For a recent paper exploring the required formalism see [62].

Chapter 8

T-dual cosmological solutions in double field theory

8.1 Introduction

The T-duality symmetry [89] plays an important role in string theory. For example, for strings on a torus of radius R, the symmetry implies that the spectrum of string states is unchanged if $R \to 1/R$ (in string units) and string momentum modes are interchanged with string winding modes. This symmetry is obeyed by string interactions, and it is assumed to be a symmetry of non-perturbative string theory (see e.g. [27, 65]).

T-duality is a key ingredient of the *String Gas Cosmology* (SGC) proposal [34] for early universe cosmology. SGC in the present form (see e.g. [64, 51, 63] for reviews) is based on ideas coming from string thermodynamics. As is well known [58], there is a maximal temperature for a gas of closed strings in thermal equilibrium, the *Hagedorn temperature* T_H. If we consider a box of strings of radius R, then the temperature $T(R)$ of a gas of strings in this box remains close to T_H for a range of values of R about $R = 1$ (in string units), the range increasing as the entropy of the string gas increases. Hence, it was postulated in [34] that the early phase of the universe might be a quasi-static hot string gas phase (see also [82] for similar ideas). As was realized in [66], thermal fluctuations in the quasi-static phase evolve into an approximately scale-invariant spectrum of cosmological fluctuations with a slight red tilt, and [69] a scale-invariant spectrum of gravitational waves with a slight blue tilt. In this framework, the key size and shape moduli of extra dimensions can be stabilized in a natural way [85, 84, 90]. What is missing in string gas cosmology, however, is a dynamical understanding of how the space-time background evolves. Our work is motivated by the aim to make progress on this issue.

As already pointed out in [34], in string theory on a compact space there are two coordinates for each dimension of the topological space, firstly the coordinate x associated with the momentum modes, and secondly the dual coordinate $\tilde{x}$ associated with the winding modes (note that for point particle theories there is only the coordinate x associated with the momentum modes). It is hence to be expected that the dynamics of the background geometry of SGC should live in a doubled space including both x and $\tilde{x}$ coordinates.

Double Field Theory (DFT) [72, 39, 35] is an interesting proposal for a field theory living in doubled space. DFT is given (see e.g. [40] for a review) by an action for a generalized metric in $2D$ space-time dimensions which is constructed from the metric, antisymmetric tensor field and dilaton (the "background") of the massless sector of a D space-time dimensional string theory. In particular, after imposing a *section condition* the dynamical equations for the background reduce to those of supergravity (we will here focusing on bosonic supergravity).

In this paper we will couple a cosmological (i.e. homogeneous and isotropic) background of DFT to matter described by some energy density ρ and some pressure p. We will study various ways of imposing a section condition, assuming that the dilaton is fixed. If we impose the section condition with respect to the regular coordinate (i.e. we assume that the variables do not depend on the dual coordinates), then we find that the equation of state of matter has to be that of regular radiation. On the other hand, if we impose the section condition with respect to the dual coordinates (i.e. we assume that the variables do not depend on the regular coordinates), then we find that the equation of state of matter has to be an equation of state dominated by winding modes. However, even though the equations of state differ, the background dynamics is the same, corresponding to a radiation-like expansion.

We then speculate about a dynamical transition between the two branches of solutions, a correct description of which would have to go beyond the strict framework of DFT and would have to involve more stringy considerations. We argue that this transition might involve a complexification of the scale factor (see also [91] where complexifications of the scale factor have been recently proposed to resolve the cosmological singularity).

The supergravity (SUGRA) equations of motion have been studied extensively for homogeneous and isotropic space-times (see e.g. [56, 33]). If the antisymmetric tensor vanishes they reduce to the ones of dilaton gravity. Recently, we [2] have considered the coupling of dilaton gravity to a perfect fluid matter source whose equation of state is that expected from a gas of fundamental closed strings. This equation of state has the property that it is dominated by momentum modes at large values of the cosmological scale factor (equation of state $w = 1/d$, where d is the dimension of space), by winding modes for small values (equation of state $w = -1/d$), and has zero presssure for intermediate values. The resulting solutions were shown to be nonsingular in both the string and Einstein frames, when interpreted in terms of a dual time variable in the winding mode regime [1]. In this paper we go one step further and consider the cosmological equations in the context of double field theory.

DFT equations of motion for cosmology have been studied in [81], but in the absence of matter sources. Matter sources have been explicitly included in the general DFT equations recently derived in [62], where care was taken to have both background and matter terms considered in the DFT-invariant way. However, in that paper no cosmological solutions were considered. It is such solutions which we consider here.

A word on notation: small Latin letters $i, j, ...$ indicate indices which run over the regular $D = d + 1$ space-time dimensions, capital letters $M, N, ...$ stand for indices which run over both the regular and the dual space-time dimensions. The cosmological scale factor is denoted by $a(t)$, where t is the physical time. The dual time is denoted by $\tilde{t}$. The equation of state parameter w is $w = p/\rho$, where ρ and p are energy density and pressure, respectively.

8.2 Short review of Double Field Theory

The DFT action unifies the metric g_{ij}, the two-form b_{ij} and the dilaton ϕ by rewriting these fields in an $O(D, D, \mathbb{R})$ covariant way, where D is the number of spacetime dimensions, and it reduces to the supergravity action if there is no dependence on the dual coordinates[2]. It is given by,

$$S = \int dx d\tilde{x} e^{-2d} \mathcal{R}, \tag{8.2.1}$$

[1]In a related paper [1] we showed that from this point of view point particle geodesics in DFT can be extended arbitrarily far into the past and future, indicating that DFT cosmology is geodesically complete.

[2]This is usually done by imposing the section condition. See [62] for a derivation of the section condition from a special class of translation invariance allowed by the $O(D, D)$ symmetry group.

where d contains both the dilaton ϕ and the determinant of the metric,

$$e^{-2d} = \sqrt{-g}e^{-2\phi}\,, \tag{8.2.2}$$

and where [74],

$$\begin{aligned}\mathcal{R} &= \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_K\mathcal{H}_{NL} \\ &+ 4\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} - 4\mathcal{H}^{MN}\partial_M d\partial_N d \\ &+ 4\partial_M\mathcal{H}^{MN}\partial_N d,\end{aligned} \tag{8.2.3}$$

where the generalized metric, $\mathcal{H}_{MN}$, is defined as,

$$\mathcal{H}_{MN} = \begin{bmatrix} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij} - b_{ik}g^{kl}b_{lj}\end{bmatrix}\,, \tag{8.2.4}$$

having its $O(D,D)$ index structure being lifted or lowered by the matrix η^{MN}, defined as,

$$\eta^{MN} = \begin{bmatrix} 0 & \delta_i{}^j \\ \delta^i{}_j & 0\end{bmatrix}\,. \tag{8.2.5}$$

Throughout the rest of this paper we will be working with double spacetime coordinates.

8.3 Dual Cosmology

The cosmological background equations of motion coming from DFT are the same as the ones coming from SUGRA for the dynamical fields in terms of the momentum coordinates. Thus, all the solutions found in that context can be automatically brought into the DFT framework. What differs in the latter is that the underlying geometry is (2d+1)-dimensional and given by the following line element,

$$\begin{aligned} dS^2 &= -dt^2 + \mathcal{H}_{MN}dX^M dX^N \\ &= -dt^2 + a^2\left(t\right)d\vec{x}^2 + a^{-2}\left(t\right)d\tilde{x}^2.\end{aligned} \tag{8.3.6}$$

We already considered some physical aspects of this metric in previous work. In [2] we showed that the cosmological solutions of supergravity in the presence of a perfect fluid matter gas with the equation of state appropriate to SGC [3] are nonsingular in the string frame, and can be given a nonsingular interpretation in the Einstein frame making use of a dual time $\tilde{t}$ which replaces the regular time t for small values of the scale factor. Making use of this dual time, it was shown earlier [1] that point particle geodesics can be extended infinitely far both into the future and towards the past.

If we start with two time coordinates, the cosmological ansatz for the line element is

$$\begin{aligned} dS^2 &= \mathcal{H}_{MN} dX^M dX^N \\ &= -dt^2 - d\tilde{t}^2 + a^2(t,\tilde{t})d\vec{x}^2 + a^{-2}(t,\tilde{t})d\tilde{x}^2, \end{aligned} \tag{8.3.7}$$

where now the generalized metric is also defined in terms of the temporal component of the space-time metric.

The vacuum equations of motion for DFT in the presence of a dual-time associated to the winding sector are given by [81],

$$\begin{aligned} 4d'' - 4d'^2 - (D-1)\tilde{H}^2 + 4\ddot{d} - 4\dot{d}^2 - (D-1)\,H^2 = 0 \\ (D-1)\,\tilde{H}^2 - 2d'' - (D-1)H^2 + 2\ddot{d} = 0 \\ \tilde{H}' - 2\tilde{H}d' + \dot{H} - 2H\dot{d} = 0, \end{aligned} \tag{8.3.8}$$

where a prime denotes a derivative with respect to $\tilde{t}$, and a dot a derivative with respect to t. Also, note that $\tilde{H} \equiv a'/a$ and $2d \equiv 2\phi - (D-1)\ln a$ is the shifted dilaton. To derive these equations, we need to vary the DFT action (8.2.1) with respect to d, g_{tt} and g_{ii}, respectively, assuming a cosmological background while taking into account the constraint,

$$\mathcal{H}_{MN}\mathcal{H}^{NP} = \delta_M^P, \tag{8.3.9}$$

which in our case simply implies $g_{\mu\nu}g^{\nu\rho} = \delta_\mu^\rho$[4].

Comparing (8.3.8) to standard string cosmology equations [56] in the presence of matter,

[3]This equation of state corresponds to radiation with $w = 1/d$ for large values of $a(t)$ and to a gas of winding modes with $w = -1/d$ in the small a limit.

[4]Note that associated to $d\tilde{t}$ is $g_{\tilde{t}\tilde{t}} \equiv g_{tt}^{-1}$.

we propose coupling the above equations to matter by the following prescription,

$$\begin{aligned} 4d'' - 4d'^2 - (D-1)\tilde{H}^2 + 4\ddot{d} - 4\dot{d}^2 - (D-1)\,H^2 = 0 \\ (D-1)\,\tilde{H}^2 - 2d'' - (D-1)H^2 + 2\ddot{d} = \frac{1}{2}e^{2d}E(t,\tilde{t}) \\ \tilde{H}' - 2\tilde{H}d' + \dot{H} - 2H\dot{d} = \frac{1}{2}e^{2d}P(t,\tilde{t}), \end{aligned} \tag{8.3.10}$$

where P and E are the pressure and energy associated to the matter sector, respectively[5]. Note that they are also function of $\tilde{t}$. One might wonder if these quantities would have their dual counterparts. That is not the case, since we know that $g_{tt} = g^{-1}_{\tilde{t}\tilde{t}}$ and $g_{ii} = g^{-1}_{\tilde{i}\tilde{i}}$ due to (8.3.9). Therefore, it is easy to see that,

$$\frac{\delta}{\delta g_{\tilde{t}\tilde{t}}} = -g^2_{tt}\frac{\delta}{\delta g_{tt}}, \tag{8.3.11}$$

so that varying (8.2.1) with respect to $g_{\tilde{t}\tilde{t}}$ would result in the same equation of motion derived by varying it with respect to g_{tt}. The same follows for the other components[6][7].

Having the equations of motion in double space-time, we need now to consider the imposition of the section condition. Typically it is imposed in the so called supergravity frame, which means assuming that the dynamical fields do not depend on the dual coordinates. However, one could equally well consider the imposition of any $O\,(D,D)$ rotation of the section condition, in particular assuming that there is no dependence on the regular coordinates rather than no dependence on the dual coordinates. We will consider the resulting dynamics in each frame and draw an interpretation in terms of the overall existence of either

[5]In [56] matter was introduced via the matter action $S = \int dt\sqrt{-g_{tt}}F\left(\log a, \beta\sqrt{-g_{tt}}\right)$, where F is the one loop free energy. In principle, this prescription can be extended in the presence of double time (see note 8.6). It is left for a future work [92] to consider its full covariant formulation.

[6]Regardless, we can still think of the dual counterparts as a matter of definition, even though they are not independent of the normal ones. In particular, it is easy to see that $\tilde{\rho} = -\rho$, after assuming $g_{tt} = -1$, and $\tilde{p} = -a^4\left(t,\tilde{t}\right)p$, given $g_{ii} = a^2\left(t,\tilde{t}\right)$.

[7]Note that fully covariant equations coupling matter to DFT background were proposed in [62]. They arise from varying a generalized action consisting of geometrical action plus matter action with respect to the generalized metric. It reduces to a generalization of supergravity when the section condition is imposed in the supergravity frame, where the gravitational charge can also be acquired through couplings with the dilaton and the antisymmetric tensor field. The cosmology of such framework will be discussed at [92]. In comparison, our equations, derived for homogeneous and isotropic space-times, are not fully covariant, as it is further discussed and explored in [4]. In order to define the energy density and pressure we need to make explicit use of the frame which was being considered. Hence, the equations of [62] have a larger scope of application.

momentum or winding modes in the next section.

Before we proceed, a few comments are in order here. There are two different takes we can consider regarding DFT: as a fundamental theory or as a mathematical framework that connects T-dual solutions of string theory. If one assumes DFT is fundamental, then changing between frames is the same as considering different gauge choices, and even though the solutions look different, this is just due to a gauge choice [93]. On the other hand, one can consider DFT as a theory that connects distinct physical solutions, which in the context of DFT would be accounted for in different frames. For instance, it is known that there are string backgrounds which are non-geometric and do not have a good supergravity description, yet they are captured by alternative frame choices [94, 95, 96]. In fact, an explicit example of such interpretation was considered in [97], where there is a natural choice of frame picked by the string/wave solution as one approaches its core, being the dual frame the natural one. We will consider the latter approach throughout the rest of this paper.

In order to simplify our reasoning, we will be considering the dilaton to be already stabilized. Thus, we have $2\dot{d} = -\left(D-1\right)H$ and $2d' = -\left(D-1\right)\tilde{H}$, and (8.3.10) become,

$$\begin{aligned} 2\left(\tilde{H}' + \dot{H}\right) + D\left(\tilde{H}^2 + H^2\right) &= 0 \\ \left(\tilde{H}^2 - H^2\right) + \left(\tilde{H}' - \dot{H}\right) &= \frac{1}{2\left(D-1\right)} G\rho\left(t, \tilde{t}\right) \\ \left(\tilde{H}' + \dot{H}\right) + \left(D-1\right)\left(\tilde{H}^2 + H^2\right) &= \frac{G}{2} p\left(t, \tilde{t}\right), \end{aligned} \tag{8.3.12}$$

where G depends on $\phi = \phi_0$, the fixed value of the dilaton. The most important feature to be noticed in these equations is the asymmetry between the regular and dual coordinates dependence in the second equation. Now we consider each particular frame.

8.3.1 Supergravity frame: large radius limit

The mass spectrum of a closed string in a one-dimensional space, compactified on a circle, is

$$M^2 = \left(N + \tilde{N} - 2\right) + p^2 \frac{l_s^2}{R^2} + w^2 \frac{l_s^2}{\tilde{R}^2}, \tag{8.3.13}$$

where $N, \tilde{N}$ correspond to oscillatory modes of the string, p corresponds to its momentum modes, associated to the center of mass motion, and w corresponds to winding modes, accounting for the number of times the string has wrap itself around the compact dimension in a topologically non-trivial way.

We expect that as the scale factor becomes larger[8] only momentum modes will be energetically favorable (considering that the radius of the compact dimensions would also become larger), as it can be easily seen from (8.3.13). In this case, we hope that only the t-dependence should be relevant, given the $\tilde{t}$ was introduced exactly to tackle the winding modes dynamics from a T-dual perspective. This is typically called supergravity frame, but here we are putting forward an interpretation associated with this frame. We will do similarly in the next sub-section when considering the winding-frame.

After imposing the section condition on the $\tilde{t}$-coordinates, the equations of motion (8.3.12) reduce to the standard string cosmology equations for a stabilized dilaton [56],

$$\begin{aligned} 2\dot{H} + DH^2 &= 0 \\ -H^2 - \dot{H} &= \frac{G}{2\left(D-1\right)}\rho(t) \\ \dot{H} + \left(D-1\right)H^2 &= \frac{\mathrm{G}}{2}p(t), \end{aligned} \tag{8.3.14}$$

which imply the following equation of state,

$$w = \frac{1}{D-1}, \tag{8.3.15}$$

corresponding to a radiation-like universe. This leads to the scale factor evolving as

$$a\left(t\right) \propto t^{2/D}. \tag{8.3.16}$$

Evidently, the continuity equation is,

$$\dot{\rho} + DH\rho = 0, \tag{8.3.17}$$

and the energy density redshifts as radiation,

$$\rho\left(a\right) \propto a^{-D}(t). \tag{8.3.18}$$

The above result is not surprising since it is well known from studies of dilaton-gravity

[8]Small and large here are always in relation to the string length l_s.

that an expanding universe the dilaton can only be constant if the equation of state of matter is that of radiation.

8.3.2 Winding-frame: small radius limit

Now we consider what we call the winding-frame, in which we impose the section condition on the regular coordinates. We expect this frame to be a good description for the regime in which winding modes dominate, corresponding to the limit of small radius as seen in (8.3.13). In this case the equations of motion become,

$$\begin{aligned} 2\tilde{H}^{'} + D\tilde{H}^2 &= 0 \\ \tilde{H}^2 + \tilde{H}^{'} &= \frac{1}{2\,(D-1)} G\rho(\tilde{t}) \\ \tilde{H}^{'} + (D-1)\,\tilde{H}^2 &= \frac{1}{2} Gp(\tilde{t}), \end{aligned} \tag{8.3.19}$$

implying the following equation of state,

$$w = -\frac{1}{D-1}, \tag{8.3.20}$$

which corresponds to a fluid composed only of winding modes. We thus see that constant dilaton in the winding frame is only consistent if the equation of state of matter is that of a gas of winding modes. This is quite surprising, since this resulted from assuming a regime in which the t-dependence is gone, not an *a priori* assumption about the matter content, reinforcing the interpretation of this frame being associated to a dynamics ruled by only winding modes.

Due to the asymmetry between the frames seen in (8.3.12), and having an equation of state given by winding modes, which is the negative of what we have for radiation, the continuity equation reads,

$$\rho' + D\tilde{H}\rho = 0, \tag{8.3.21}$$

which implies that the energy density will also redshifts as radiation,

$$\rho(a) \propto a^{-D}(\tilde{t}), \tag{8.3.22}$$

despite corresponding to winding modes. This differs from what we find in usual cosmology, where (8.3.20) implies $\rho \propto a^{2-D}$ instead.

Note that our result explicitly shows that the universe is T-dual, *i.e.*, a universe characterized by a winding equation of state in the dual coordinates behaves exactly the same as a universe dominated by momentum modes in the regular coordinates. However, it is important to realize that as we take the limit of small scale factor in the momentum frame, which approaches a singularity, the winding frame expands due to the scale factor duality $a(t) \to a^{-1}(\tilde{t})$ [98, 99]. Imposing the section condition separates both solutions, but if matter in the universe is made of both winding and momentum modes, a smooth transition, not possible in standard DFT, is needed. We investigate this further in a future work [100].

In order to solve for the scale factor, first we notice that unlike the momentum case, the corresponding Friedmann-like equation has a minus sign,

$$\tilde{H}^2 = -\frac{G}{(D-2)(D-1)}\rho. \tag{8.3.23}$$

Thus, we see that either $\tilde{H}$ is complex and $\rho > 0$ (which implies $p < 0$, as usual for winding modes) or $\tilde{H}$ is real but $\rho < 0$ (and $p > 0$).

Considering $\tilde{H}$ to be complex, then we can work with the following ansatz,

$$a\left(\tilde{t}\right) = \tilde{A}\left(\tilde{t}\right)e^{i\theta\left(\tilde{t}\right)}, \tag{8.3.24}$$

so that the Friedmann-like equation becomes,

$$\begin{aligned} \tilde{H}_{\tilde{A}}^2 - \theta'^2 &= -g\rho_0\tilde{A}^{-D}\cos\left(D\theta\right) \\ 2\tilde{H}_{\tilde{A}}\theta' &= g\rho_0\tilde{A}^{-D}\sin\left(D\theta\right), \end{aligned} \tag{8.3.25}$$

where $g \equiv G/\left(D-2\right)\left(D-1\right)$. Note that for $\theta = \pi/D$, the second equation vanishes identically and the first equation becomes,

$$\tilde{H}_{\tilde{A}}^2 = g\rho_0\tilde{A}^{-D}, \tag{8.3.26}$$

so that,

$$a\left(\tilde{t}\right) = \tilde{a}_0\tilde{t}^{2/D}e^{i\pi/D}, \tag{8.3.27}$$

where $\tilde{a}_0$ is a constant in this regime.

Let us take a moment to analyze what we have just derived. Before, for the momentum case, we obtained the solution,

$$a_m(t) = a_0 t^{2/D} e^{i\theta_m}, \tag{8.3.28}$$

where $\theta_m = 0$, since the solution was real. Writing both solutions together, we have,

$$\begin{cases} a_m(t) &= a_0 t^{2/D} \\ a_w(\tilde{t}) &= \tilde{a}_0 \tilde{t}^{2/D} e^{i\pi/D}. \end{cases} \tag{8.3.29}$$

Now, remembering that the scale factor solution associated to the winding modes is the reciprocal of the one associated to the momentum ones (and ignoring those arbitrary constants for the moment),

$$a_m \to a_w^{-1},$$

and we conclude that the solutions are dual given,

$$t \to \tilde{t}^{-1} e^{-i\pi/2}. \tag{8.3.30}$$

Therefore, quite surprisingly, we can also interpret that the winding scale factor solution corresponds to a Wick rotation of the reciprocal of the momentum time-coordinate. Since θ is actually a dynamical variable for us, this rotation happens dynamically as it will be shown below.

For the general case, we still need to solve (8.3.25). Combining the equations, we see that

$$\tilde{H}_A^2 = \frac{g\rho_0 \tilde{A}^{-D}}{2} \left[1 - \cos(D\theta)\right]. \tag{8.3.31}$$

The solution for θ is given by,

$$\theta(\tilde{t}) = \pm \frac{2}{D} \arccos\left[\left(\frac{\tilde{A}}{\tilde{A}_0}\right)^{-D/2}\right], \tag{8.3.32}$$

where $\tilde{A}_0$ is a constant. Therefore,

$$\tilde{H}_A^2 = g\rho_0 \tilde{A}^{-D} \left[1 - \left(\frac{\tilde{A}}{\tilde{A}_0}\right)^{-D}\right], \tag{8.3.33}$$

which implies that,

$$\tilde{A}\left(\tilde{t}\right) = \left[\tilde{A}_0^D - \frac{1}{4}D^2C_1^2 + \frac{D^2}{4}g\rho_0\tilde{t}^2 \pm i\frac{D^2C_1}{2}\sqrt{g\rho_0}\tilde{t}\right]^{1/D}, \tag{8.3.34}$$

for some arbitrary constant C_1. In particular, given that we have chosen $\tilde{A}\left(\tilde{t}\right)$ to be real, this constant should be set to 0 since it would not be there in the first place if we had complied with the assumption that $\tilde{H}_A$ was real. We see that for large $\tilde{t}$ we recover the typical radiation solution as expected with a complex phase.

Having this general solution is quite helpful also to understand the particular case we considered above, the one $\theta = \pi/D$. In principle, we would like to have this happenning dynamically as opposed to just fixing θ by hand. To see that, let us take the large $\tilde{t}$ limit, then

$$\tilde{A}\left(\tilde{t}\right) \to \tilde{t}^{2/D}, \tag{8.3.35}$$

which implies that

$$\theta\left(\tilde{t}\right) \to \pm\frac{2}{D}\arccos\left(\frac{1}{\tilde{t}}\right) \xrightarrow[\tilde{t}\to\infty]{} \pm\frac{\pi}{D}. \tag{8.3.36}$$

Therefore, this shows that it is the case, indeed, that deep in the winding regime this phase is singled out and the oscillations in the scale factor cease to exist. Also, this shows that our temporal duality defined above appears due to the dynamics of our solutions.

Finally, as an illustration, we could have considered $\rho < 0$ in (8.3.25), ending up with,

$$a\left(\tilde{t}\right) = \tilde{a}_0\tilde{t}^{2/D}. \tag{8.3.37}$$

For this case, the winding and radiation solutions would be dual under the following identification,

$$\tilde{t} \to \frac{1}{t}. \tag{8.3.38}$$

This further motivates the heuristic arguments considered in [1, 2]. We see that for positive energy density, the temporal parameter space is complex, while for negative energy density it is $\mathbb{R}^2$.

8.4 Comments

Considering the results from the last section, we can speculate about a different interpretation of these findings. It has been argued before in the context of quantum cosmology and quantum gravity that the ground state of the wave function of the universe should correspond to Euclidean geometry [101], which would have a non-zero probability of tunneling to a de Sitter state of continual expansion.

In fact, such proposals motivated studies about classical change of the metric's signature [102]. In particular, it has been observed that the equation of state for a perfect fluid gets a minus sign when the underlying geometry is Euclidean. Therefore, the radiation equation of state in Euclidean space would mimic the equation of state of winding modes.

If we run this reasoning backwards, we could speculate that winding modes should be understood as radiation when time is Euclidean. The transition we observe from winding frame to momentum frame would correspond simply to a change of the signature of the metric. This is further developed in [100].

8.5 Conclusion

Double Field Theory can be interpreted as a natural generator of T-dual solutions once the section condition is imposed in one or another set of coordinates. We investigated these different solutions after considering a cosmological ansatz for the metric. We have shown that in both frames we have a radiation-like dynamics, but with different equations of state. On one side, the universe is dominated by winding modes while in the other one, by momentum modes. This is exactly what one would expect. The small scale factor limit in one frame approaches a singularity while in the dual frame it expands to an infinite volume. The T-dual mapping between the two frames provides further evidence for the connection between the two time coordinates pointed out previously in [1, 2].

8.6 A note about T-dualizing matter

Typically, matter is introduced in SUGRA by the following action [56],

$$S = \int d^D x \sqrt{-g} e^{-2\phi} f = \int dt \sqrt{-g_{tt}} F \left(\log a, \beta\sqrt{-g_{tt}}, \phi \right), \tag{8.6.39}$$

where F is the free energy,

$$F = \int d^{D-1} x a^{D-1} e^{-2\phi} f.$$

Formally, we can think of a T-dual covariant generalization of it by defining the following action,

$$S = \int d^D x d^D \tilde{x} e^{-2d} \mathcal{F}, \tag{8.6.40}$$

with $\mathcal{F}$ depending on both sets of coordinates. Then, we can write

$$\begin{aligned} S &= \int dx d\tilde{x} dt d\tilde{t} \sqrt{g} e^{-2\phi} \mathcal{F} \\ &= \int dt \sqrt{-g_{tt}} \left(\int d^{D-1} \tilde{x} d\tilde{t} F \right) \end{aligned} \tag{8.6.41}$$

or

$$S = \int d\tilde{t} \sqrt{-\frac{1}{g_{tt}}} \left(\int d^{D-1} \tilde{x} dt F \right). \tag{8.6.42}$$

where F is also function of both sets of coordinates.

Finally, the standard definitions of the energy and pressure of the system follow as usual,

$$E(t, \tilde{t}) = -2 \frac{\delta F}{\delta g_{00}} \tag{8.6.43}$$

$$P_i(t, \tilde{t}) = -\frac{\delta S}{\delta \ln a_i}. \tag{8.6.44}$$

Chapter 9

T-dual cosmological solutions in double field theory II

9.1 Introduction

Target space duality [103, 89, 104, 105, 65] is a key symmetry of superstring theory. Qualitatively speaking, it states that physics on small compact spaces of radius R is equivalent to physics on large compact spaces of radius $1/R$ (in string units). This duality is a symmetry of the mass spectrum of free strings: to each momentum mode of energy n/R (where n is an integer) there is a winding mode of energy mR, where m is an integer. Hence, the spectrum is unchanged under the symmetry transformation $R \to 1/R$ if the winding and momentum quantum numbers m and n are interchanged. The energy of the string oscillatory modes is independent of R. This symmetry is obeyed by string interactions, and it is also supposed to hold at the non-perturbative level (see e.g. [26, 27]).

The exponential tower of string oscillatory modes leads to a maximal temperature for a gas of strings in thermal equilibrium, the Hagedorn temperature [58]. Combining these thermodynamic considerations with the T-duality symmetry lead to the proposal of *String Gas Comology* [34] (see also [82]), a nonsingular cosmological model in which the Universe loiters for a long time in a thermal state of strings just below the Hagedorn temperature, a state in which both momentum and winding modes are excited. This is the *'Hagedorn phase'*. After a phase transition in which the winding modes interact to decay into loops, the T-duality symmetry of the state is spontaneously broken, the equation of state of the matter gas changes to that of radiation, and the radiation phase of Standard Big Bang expansion can begin.

In addition to providing a nonsingular cosmology, String Gas Cosmology leads to an alternative to cosmological inflation for the origin of structure [66]: According to this picture, thermal fluctuations of strings in the Hagedorn phase lead to the observed inhomogeneities in the distribution of matter at late times. Making use of the holographic scaling of matter correlation functions in the Hagedorn phase, one obtains a scale-invariant spectrum of cosmological perturbations with a slight red tilt, like the spectrum which simple models of inflation predict [66]. If the string scale corresponds to that of Grand Unification, then the observed amplitude of the spectrum emerges naturally. String gas cosmology also predicts a roughly scale-invariant spectrum of gravitational waves, but this time with a slight blue tilt [69, 70], a prediction with which the scenario can be distinguished from simple inflationary models (see also [106] and [6] for other distinctive predictions).

The phase transition at the end of the Hagedorn phase allows exactly three spatial dimensions to expand, the others being confined forever at the string scale by the winding and momentum modes about the extra dimension (see [83, 84, 85, 86] for detailed discussions of this point). The dilaton can be stabilized by the addition of a gaugino condensation mechanism [107], without disrupting the stabilization of the radii of the extra dimensions. Gaugino condensation also leads to supersymmetry breaking at a high scale [108]. The reader is referred to [63, 64, 51] for detailed reviews of the String Gas Cosmology scenario.

However, an oustanding issue in String Gas Cosmology is to obtain a consistent description of the background space-time. Einstein gravity is clearly not applicable since it is not consistent with the basic T-duality symmetry of string theory. Dilaton gravity, as studied in *Pre-Big Bang Cosmology* [71, 33] is a promising starting point, but it also does not take into account the fact, discussed in detail in [34], that to each spatial dimension there are two position operators, the first one (x) dual to momentum, the second one ($\tilde{x}$) dual to winding. *Double Field Theory* (DFT) (see [39, 35] for original works and [40] for a detailed review) is a field theory model which is consistent both with the T-duality symmetry of string theory and the resulting doubling of the number of spatial coordinates (see also [36, 37, 38] for some early works). Hence, as a stepping stone towards understanding the dynamics of String Gas Cosmology it is of interest to study cosmological solutions of DFT.

In an initial paper [1], point particle motion in doubled space was studied, and it was argued that, when interpreted in terms of physical clocks, geodesics can be completed arbitrarily far into the past and future. In a next paper [2], the cosmological equations of dilaton gravity were studied with a matter source which has the equation of state of a gas of closed strings. Again, it was shown that the cosmological dynamics is non-singular. The full DFT equations of motion in the case of homogeneous and isotropic cosmology were then studied in [3]. The consistency of DFT with the underlying string theory leads to a constraint. In DFT, in general a stronger version of this constraint is used, namely the assumption that the fields only depend on one subset of the doubled coordinates. There are various possible frames which realize this (see the discussion in the following section). In the *supergravity frame* it is assumed that the fields do not depend on the "doubled" coordinates $\tilde{x}$, while in the *winding frame* it is assumed that the fields only depend on $\tilde{x}$ and not on the x coordinates. It was shown that for solutions with constant dilaton in the supergravity frame, the consistency of the equations demands that the equation of state of matter is that of relativistic radiation, while constant dilaton in the winding frame demands that the equation of state of matter is that of a gas of winding modes. These two solutions, however, are not T-dual. In this paper we will look for solutions which are T-dual. We expand on the analysis of [3] and present improvements in the solutions.

In the following section we discuss different frames which can be used. They can be obtained from each other by T-duality transformations. We also discuss the T-duality transformation of fields. In Section 3 we present the equations of DFT for a homogeneous and isotropic cosmology. In Section 4 we introduce a T-duality preserving ansatz for the solutions, before finding solutions of these equations in Section 5. We conclude with a discussion of our results.

9.2 T-Dual Frames vs. T-Dual Variables

We consider an underlying D-dimensional space-time. The fields of DFT then live in a $2D$ dimensional space with coordinates (t, x) and dual coordinates $(\tilde{t}, \tilde{x})$, where t is time and x denote the $D-1$ spatial coordinates. In general, the *generalized metric* of DFT is made up of the $D-$dimensional space-time metric, the dilaton and an antisymmetric tensor field, all being functions of the $2D$ coordinates.

In this section (like in the rest of this paper) we consider only homogeneous and isotropic space-times and transformations which preserve the symmetries. In this case, the basic fields reduce to the cosmological scale factor $a(t, \tilde{t})$ and the dilaton $\phi(t, \tilde{t})$. It is self-consistent to neglect the antisymmetric tensor field. These are the same fields which also appear in dilaton gravity.

In supergravity, the T-duality transformation of the fields can be defined as

$$a(t) \rightarrow \frac{1}{a(t)} \tag{9.2.1}$$

$$d(t) \rightarrow d(t)\,, \tag{9.2.2}$$

where $d(t)$ is the rescaled dilaton

$$d(t) = \phi(t) - \frac{D-1}{2}\ln a(t) \tag{9.2.3}$$

which is invariant under a T-duality transformation. In DFT this definition can be generalized to be

$$a(t,\tilde{t}) \rightarrow \frac{1}{a(\tilde{t},t)} \tag{9.2.4}$$

$$d(t,\tilde{t}) \rightarrow d(\tilde{t},t)\,. \tag{9.2.5}$$

This implies that dilaton transforms as

$$\phi(t,\tilde{t}) \rightarrow \phi(\tilde{t},t) - (D-1)\ln a(\tilde{t},t)\,. \tag{9.2.6}$$

An important assumption of DFT is the need to impose a *section condition*, a condition which states that the fields only depend on a D-dimensional subset of the space-time variables. The different choices of this section condition are called *frames*, and different frames are related via T-duality transformations. The *supergravity frame* is the frame in which the fields only depend on the (t,x). The second frame which we will consider is the *winding frame* in which the fields only depend on the $(\tilde{t},\tilde{x})$ coordinates.

In this paper we are interested in finding supergravity frame solutions

$$(\phi(t), a(t), d(t)) \tag{9.2.7}$$

and winding frame solutions

$$(\phi(\tilde{t}), a(\tilde{t}), d(\tilde{t})) \tag{9.2.8}$$

which are T-dual to each other, i.e.

$$\begin{aligned} d(\tilde{t}) &= d(t(\tilde{t})) && (9.2.9) \\ a(\tilde{t}) &= \frac{1}{a(t(\tilde{t}))}\,, && (9.2.10) \end{aligned}$$

where $t(\tilde{t}) = \tilde{t}$.

9.3 Equations

Our starting point is the equations for DFT under a cosmological ansatz [81] (Eqs. (8) in [3]):

$$\begin{aligned} 4d'' - 4(d')^2 - (D-1)\tilde{H}^2 + 4\ddot{d} - 4\dot{d}^2 - (D-1)H^2 = 0 \\ (D-1)\tilde{H}^2 - 2d'' - (D-1)H^2 + 2\ddot{d} = 0 \\ \tilde{H}' - 2\tilde{H}d' + \dot{H} - 2H\dot{d} = 0\,, \end{aligned} \tag{9.3.11}$$

where the prime denotes the derivative with respect to $\tilde{t}$, and the overdot the derivative with respect to t. In addition,

$$H = \frac{\dot{a}}{a}\,, \quad \tilde{H} = \frac{a'}{a}\,. \tag{9.3.12}$$

These equations are invariant under T-duality, since $d(t, \tilde{t})$ is a scalar and $H \leftrightarrow -\tilde{H}$ under this transformation. Then, we couple these equations with matter in the following way [3]

$$\begin{aligned} 4d'' - 4(d')^2 - (D-1)\tilde{H}^2 + 4\ddot{d} - 4\dot{d}^2 - (D-1)H^2 = 0 \\ (D-1)\tilde{H}^2 - 2d'' - (D-1)H^2 + 2\ddot{d} = \frac{1}{2}e^{2d}E \\ \tilde{H}' - 2\tilde{H}d' + \dot{H} - 2H\dot{d} = \frac{1}{2}e^{2d}P. \end{aligned} \tag{9.3.13}$$

Now, these new equations are invariant under T-duality provided $E \to -E$ and $P \to -P$. But this is exactly the case since, as explained in [3], the energy and pressure in the winding frame are given by

$$E(t,\tilde{t}) = -2\frac{\delta F}{\delta g_{tt}(t,\tilde{t})} \to -2\left(-g_{tt}^2(\tilde{t},t)\frac{\delta F}{\delta g_{tt}(\tilde{t},t)}\right) = -E(\tilde{t},t),$$
$$P(t,\tilde{t}) = -\frac{2}{D-1}\frac{\delta F}{\delta g_{ij}(t,\tilde{t})}g_{ij}(t,\tilde{t}) = -\frac{\delta F}{\delta \ln a(t,\tilde{t})} \to -\frac{\delta F}{\delta \ln(1/a(\tilde{t},t))} = -P(\tilde{t},t), \quad (9.3.14)$$

where we used $g_{tt} = 1$ for our case and assumed that the matter action in double space F is $O(D,D)$ invariant. The invariance of Eqs. (9.3.13) under T-duality is a strong support for the correctness of the coupling with matter.

Solutions to Eqs. (9.3.13) may be found after imposing the strong condition of DFT. One may impose that all functions are $\tilde{t}$-independent or t-independent, corresponding to the supergravity (SuGra) or winding frames, respectively. In [3], solutions based on either the SuGra or winding frames were found for the case of constant dilaton $\phi(t,\tilde{t}) = \phi_0$. But notice that by (9.2.6) the dilaton transforms non-trivially under T-duality. Hence, the solutions found in [3] in the SuGra and winding frames, respectively, are not T-dual to each other. The fact that two solutions both with constant dilaton in the respective frames are not related by T-duality (or $O(D,D,)$ more generally) can be confirmed by noting that equations (12) in [3] obtained from (9.3.13) after assuming constant dilaton are not T-dual invariant. These equations were obtained by imposing

$$\begin{aligned} 2d(t,\tilde{t}) &= 2\phi_0 - (D-1)\ln a(t,\tilde{t}) \\ &\implies 2\dot{d} = -(D-1)H, \quad 2d' = -(D-1)\tilde{H}\,, \end{aligned} \quad (9.3.15)$$

which is not compatible with T-duality, since $2d'$ does not transform to $2\dot{d}$ as it should.

From the point of view of a field theory with doubled coordinates, there is no problem in considering constant dilaton in the way it was considered in [3]. However, since the SuGra and winding frame solutions are not T-dual to each other, the comparison of these solutions used to motivate the correspondence $\tilde{t} \to t^{-1}$ is tenuous.

In this work, we look for equations and solutions that respect T-duality, and specifically with constant dilaton *only* in the SuGra frame or in the winding frame. We also solve an apparent inconsistency with positive energy density in the winding frame, found in [3].

9.4 T-duality preserving ansatz and equations for each frame

Starting from the supergravity frame, let us look for solutions with constant dilaton. In this case

$$\begin{aligned} 2d(t) \quad &= \quad 2\phi_0 - (D-1)\ln a(t) \\ \Longrightarrow \quad & 2\dot{d} \,=\, -(D-1)H\,. \end{aligned} \tag{9.4.16}$$

We now seek solutions in the winding frame which are T-dual. By the invariance of d, $d(t) = d(\tilde{t}(t))$, we have

$$\begin{aligned} \phi_0 - \frac{D-1}{2}\ln a \quad &= \quad \phi(\tilde{t}) - \frac{D-1}{2}\ln a(\tilde{t}) \\ \Longrightarrow \quad & \phi(\tilde{t}) = \phi_0 - \frac{D-1}{2}\ln\left(\frac{a(t(\tilde{t}))}{a(\tilde{t})}\right)\,. \end{aligned} \tag{9.4.17}$$

Now by the scale-factor duality which comes from the transformation of the generalized metric, $a(t(\tilde{t})) = 1/a(\tilde{t})$, and so

$$\phi(\tilde{t}) \,=\, \phi_0 + (D-1)\ln a(\tilde{t})\,, \tag{9.4.18}$$

and hence

$$\begin{aligned} d(\tilde{t}) \quad &= \quad \phi_0 + \frac{D-1}{2}\ln a(\tilde{t}) \\ \Longrightarrow \quad & 2d'(\tilde{t}) = (D-1)\tilde{H}\,. \end{aligned} \tag{9.4.19}$$

Thus, the ansatz for the rescaled dilaton $d(t,\tilde{t})$ in the winding frame will be such that

$$2\dot{d}(t) \,=\, -(D-1)H, \quad 2d'(\tilde{t}) \,=\, (D-1)\tilde{H}, \tag{9.4.20}$$

which is related to the supergravity frame dilaton by T-duality. Similarly, for a constant dilaton in the winding frame we have

$$2\dot{d}(t) \,=\, (D-1)H, \quad 2d'(\tilde{t}) \,=\, -(D-1)\tilde{H}\,. \tag{9.4.21}$$

Equations (9.4.20) and (9.4.21) are ansaetze compatible with T-duality between the SuGra and winding frames.

To find the equations in each frame under these assumptions, let us consider

$$2\dot{d}(t) \;=\; \alpha(D-1)H, \quad 2d'(\tilde{t}) \;=\; \tilde{\alpha}(D-1)\tilde{H}\,, \tag{9.4.22}$$

which takes both cases into account: for $(\alpha, \tilde{\alpha}) = (-1, 1)$ we have a constant dilaton in the SuGra frame and non-constant dilaton in the winding frame; for $(\alpha, \tilde{\alpha}) = (1, -1)$, we have constant dilaton in the winding frame and non-constant dilaton in the SuGra frame. The case $(\alpha, \tilde{\alpha}) = (-1, -1)$ corresponds to having the dilaton constant in both frames and was considered in [3]. But, as already argued, this breaks the T-duality between the frames. Here,we are looking for solutions in each frame that are T-dual to each other, so we will not consider the case $(\alpha, \tilde{\alpha}) = (1, 1)$.

Applying the section conditions, we get equations for SuGra and winding frame,

$$\begin{aligned} 4\ddot{d} - 4\dot{d}^2 - (D-1)H^2 &= 0 & \qquad 4d'' - 4(d')^2 - (D-1)\tilde{H}^2 &= 0 \\ -(D-1)H^2 + 2\ddot{d} &= \frac{1}{2}e^{2d}E(t) & (D-1)\tilde{H}^2 - 2d'' &= \frac{1}{2}e^{2d}E(\tilde{t}) \\ \dot{H} - 2H\dot{d} &= \frac{1}{2}e^{2d}P(t) & \tilde{H}' - 2\tilde{H}d' &= \frac{1}{2}e^{2d}P(\tilde{t}) \end{aligned} \tag{9.4.23}$$

Before solving them, notice that the energy and pressure in the winding frame are given by

$$\tilde{E}(\tilde{t}) = -2\frac{\delta F}{\delta g_{\tilde{t}\tilde{t}}(\tilde{t})} = -2\left(-g_{tt}^2(\tilde{t})\frac{\delta F}{\delta g_{tt}(\tilde{t})}\right) = -E(\tilde{t}), \tag{9.4.24}$$

$$\begin{aligned} \tilde{P}(\tilde{t}) &= -\frac{2}{D-1}\frac{\delta F}{\delta g_{\tilde{i}\tilde{j}}(\tilde{t})}g_{\tilde{i}\tilde{j}}(\tilde{t}) = -2\frac{\delta F}{\delta(a^{-2}(\tilde{t}))}a^{-2}(\tilde{t}) \\ &= \frac{\delta F}{\delta \ln a(\tilde{t})} = -P(\tilde{t}). \end{aligned} \tag{9.4.25}$$

Thus, under T-duality, $E(t) \to \tilde{E}(\tilde{t})$ and $P(t) \to \tilde{P}(\tilde{t})$. This observation allows to reinterpret the minus sign appearing in the equation for $\tilde{H}^2$ in [3]. In contrast to what happens in the SuGra frame, the energy measured in the winding frame is not simply the function $E(t, \tilde{t})$ projected to $E(\tilde{t})$ upon applying the section condition, but actually the negative of it. The difference appears because the definition of $E(t, \tilde{t})$ selects the SuGra frame as a preferred frame, since $g_{\tilde{t}\tilde{t}}$ does not enter in this definition. As explained in [3], to work only with

$E(t,\tilde{t})$ was a choice since the variations with respect to g_{tt} can be written as $g_{\tilde{t}\tilde{t}}$ variations. But this choice selects t as a preferred variable and so it is natural that the energy in the winding frame is different from $E(\tilde{t})$.

Using (9.4.22) in SuGra frame, we have

$$\begin{aligned} 2\alpha\dot{H} - H^2(\alpha^2(D-1)+1) = 0, \\ \alpha\dot{H} - H^2 = \frac{1}{2(D-1)}e^{2d}E, \\ \dot{H} - \alpha(D-1)H^2 = \frac{1}{2}e^{2d}P, \end{aligned} \tag{9.4.26}$$

which implies

$$\begin{aligned} H^2 &= \frac{e^{2\phi_0}a^{(\alpha+1)(D-1)}}{(D-1)(\alpha^2(D-1)-1)}\rho, \\ w &= -\frac{1}{\alpha}\frac{1}{D-1}, \\ \dot{\rho} &+ (D-1)H(\rho+p) = 0\,. \end{aligned} \tag{9.4.27}$$

Notice that ϕ_0 is the value of the dilaton in the frame where it is constant.

In winding frame we obtain

$$2\tilde{\alpha}\tilde{H}' - \tilde{H}^2(\tilde{\alpha}^2(D-1)+1) = 0, \tag{9.4.28}$$

$$\begin{aligned} -\tilde{H}^2 + \tilde{\alpha}\tilde{H}' = \frac{1}{2(D-1)}e^{2d}\tilde{E}, \\ -\tilde{H}' + \tilde{\alpha}(D-1)\tilde{H}^2 = \frac{1}{2}e^{2d}\tilde{P}, \end{aligned}$$

which are equivalent to

$$\begin{aligned} \tilde{H}^2 &= \frac{e^{2\phi_0}a^{(\tilde{\alpha}+1)(D-1)}}{(D-1)(\tilde{\alpha}^2(D-1)-1)}\tilde{\rho}, \\ w &= \frac{1}{\tilde{\alpha}}\frac{1}{D-1}, \\ \tilde{\rho}' &+ (D-1)\left[\frac{(D-1)-1/\tilde{\alpha}}{(D-1)+1/\tilde{\alpha}}\right]\tilde{H}(\tilde{\rho}+\tilde{p}) = 0\,, \end{aligned} \tag{9.4.29}$$

where w is the equation of state parameter

$$w = \frac{p}{\rho}, \tag{9.4.30}$$

p and ρ being pressure and energy density, respectively.

From these equations, we conclude that the equation of state is the same in both frames regardless in which frame the dilaton is taken to be constant. For constant dilaton in the SuGra frame we obtain the equation of state of radiation, for constant dilaton in the winding frame, on the other hand, the equation of state is that of a gas of winding modes.

9.5 Solutions

Solving the equations of the previous section in the SuGra frame, we obtain

$$\rho(t) \propto a^{-(D-1)+1/\alpha}(t), \tag{9.5.31}$$

$$a(t) \propto \left(\frac{\alpha}{2}(D-1) - \frac{1}{2\alpha}\right)^{\frac{2}{-\alpha(D-1)-1/\alpha}} t^{\frac{2}{-\alpha(D-1)-1/\alpha}}, \tag{9.5.32}$$

while in the winding frame we get

$$\tilde{\rho}(\tilde{t}) \propto a^{-(D-1)+1/\tilde{\alpha}}(\tilde{t}), \tag{9.5.33}$$

$$a(\tilde{t}) \propto \left(\frac{-\tilde{\alpha}}{2}(D-1) - \frac{1}{2\tilde{\alpha}}\right)^{\frac{2}{-\tilde{\alpha}(D-1)-1/\tilde{\alpha}}} \tilde{t}^{\frac{2}{-\tilde{\alpha}(D-1)-1/\tilde{\alpha}}}. \tag{9.5.34}$$

In particular, for constant dilaton in the SuGra frame, we have

$$\rho(t) \propto a^{-D}(t), \qquad \tilde{\rho}(\tilde{t}) \propto a^{-(D-2)}(\tilde{t}), \tag{9.5.35}$$

$$a(t) \propto t^{2/D}, \qquad a(\tilde{t}) \propto \tilde{t}^{-2/D}. \tag{9.5.36}$$

We see that given a radiation equation of state in both frames, the energy density in the winding frame has the same a dependence as a fluid with winding equation of state. The reason for this is that in the winding frame the dilaton is not constant, and hence the relationship between equation of state and scale factor dependence of the energy density which we are used to from Einstein gravity changes.

For constant dilaton in the winding frame, we find

$$\rho(t) \propto a^{-(D-2)}(t), \qquad \tilde{\rho}(\tilde{t}) \propto a^{-D}(\tilde{t}), \tag{9.5.37}$$

$$a(t) \propto t^{-2/D}, \qquad a(\tilde{t}) \propto \tilde{t}^{2/D}, \tag{9.5.38}$$

which shows that a fluid with winding equation of state has time dependence of the scale factor like radiation in the winding frame.

As we can check from the above results, we found solutions in the SuGra and winding frame which are T-dual to each other. Also, the solutions exhibit a symmetry connected with T-duality: if we change t to $\tilde{t}$ in the SuGra frame solution with constant dilaton in that frame, we get the winding frame solution with constant dilaton in the winding frame, and vice-versa.

9.6 Discussion

In this paper we have constructed supergravity and winding frame solutions of the cosmological equations of Double Field Theory which are T-dual to each other. When the correct transformation of the energy and pressure is taken into account, there is no need for complexification of the scale factor.

Since Double Field Theory is based on the same T-duality symmetry which is key to superstring theory, one could hope that Double Field Theory could provide a consistent background for superstring cosmology, and provide a good background for String Gas Cosmology. Let us consider the background space to be toroidal. In this case, as argued in [34], for large values of the radius R of the torus (in string units), the light degrees of freedom correspond to the momenta, and the supergravity frame is hence the one in which observers made up of light degrees of freedom measure physical quantities. In contrast, for small values of R, it is the winding modes which are light, and hence the winding frame is the frame in which observers describe the physics. In the transition region (the Hagedorn phase) the full nature of double space will be important. It is possible that the section condition becomes dynamical [1]. It would be interesting in this context to explore the connection with the recent ideas in [109, 110, 111, 112].

[1] We thank Laurent Freidel for discussions on this point.

Chapter 10

Conclusions

Despite the many successes of the Standard Cosmological Model, it still relies on General Relativity, which has been shown to present singularities for cosmological scenarios ubiquitously. That is also the case for the inflationary paradigm introduced to account the physics of the very early universe, since in generality also relies on Einstein's equations. Therefore, one needs to consider alternative scenarios where this problem can be properly solved.

As we have argued, it is expected that Quantum Gravity may provide a framework in which nonsingular cosmologies can be described. So far, the most prominent Quantum Gravity theory is String Theory, and we have seen that at least thermodynamically it is expected that a gas of closed string in a box should not be singular, in the sense that it reaches a maximum temperature even for ever contracting boxes. This happens because strings have other degrees of freedom which are absent in point particle theories, which include the oscillatory modes of the string and the winding modes.

In order to account for the symmetries introduced by these new degrees of freedom, we introduced Double Field Theory as a framework in which T-duality is made into a manifest symmetry. As a result, dual coordinates are introduced defining an underlying double geometry on which the field content is considered.

This thesis presented works developed with the aim of addressing the singularity problem in the framework of Double Field Theory. We approached this problem gradually by introducing novel extensions to DFT, and then considering the resulting cosmology.

In our first work, we considered the point particle geodesics in DFT. After we derived the equations of motion for a particle in a cosmological background defined in double geometry, we were able to argue that the geodesics can be completed towards the past and future infinities given an extension of DFT in which a physical clock is introduced, reflecting the T-duality symmetry of the setup.

The natural extension of our first paper was to consider the background cosmology in which these particles could propagate in the context of DFT. We have studied the solutions to the standard supergravity equations of motion for a cosmological background in the context of the new dual dimensions. The solutions are found to be nonsingular when interpreted in this context. It remained to consider how those solutions would differ once DFT is coupled to matter sources similarly to General Relavitity.

The third work discussed here attempted to improve the formalism such that a hydrodynamical fluid could be considered as the source of the dual geometry. We have shown that the solutions obtained for a gas of closed strings corresponded to a radiation-like dynamics, even though each frame considered had a different equation of state, namely the momentum and winding ones. Then, it is shown that the scale factor in one of these frames evolves as the reciprocal of the scale factor for the other frame.

In our last paper, it is shown that the solutions obtained in the previous paper for each frame were not T-dual, since we had considered the dilaton to be stabilized for both frames, and it is not possible to consider that in both frames simultaneously while still preserving T-duality. Thus, we considered an ansatz for the dilaton that did not break T-duality symmetry and studied the cosmological solutions for each frame. We found again that the scale factor for one frame was the inverse of the solution for the other frame, and that these solutions were connected by a T-dual transformation between the time coordinates.

DFT remains an area of extensive research. One expected result is the possibility to turn the section condition into a dynamical outcome of the theory, so that each frame is picked out depending on the evolution of the fields being considered, which is driven by the matter content. We expect that this extension of the formalism may provide a smooth transition between the frames we have studied here and could provide a nonsingular cosmology in the context of a gas of closed strings.

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