On the p-adic variation of Heegner points

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To my parents and sisters

Abstract

In this thesis we study the so-called *big Heegner points* introduced and first studied by Ben Howard [**How07b**]. By construction these are global cohomology classes, with values in the Galois representation associated to a twisted Hida family, interpolating in weight 2 the twisted Kummer images of CM points.

In the first part, we relate the higher weight specializations of the big Heegner point of conductor one to the p-adic étale Abel–Jacobi images of Heegner cycles. This is based on a new p-adic limit formula of Gross–Zagier type obtained in the recent work [**BDP13**] of Bertolini–Darmon–Prasanna, a formula that we extend to a setting allowing arbitrary ramification at p. As a first consequence of the aforementioned relation, we deduce an interpolation of the p-adic Gross–Zagier formula of Nekovář over a Hida family.

In the second part, we extend some of these formulas in the anticyclotomic direction, and find that the p-adic L-function introduced in [**BDP13**] can be obtained as the image of a compatible sequence of big Heegner points of p-power conductor via a generalization of the Coleman power series map. By an application of Kolyvagin's method of Euler systems, we then exploit this alternate construction of the p-adic L-function to establish certain new cases of the Bloch–Kato conjecture for the Rankin–Selberg convolution of a cusp form with a theta series of higher weight, and deduce one divisibility in the associated anticyclotomic Iwasawa main conjecture.

Abrégé

Cette thèse est consacrée aux “points de Heegner en famille” introduits par Ben Howard dans [**How07b**]. Par définition, ce sont des classes de cohomologie globales à valeurs dans la représentation Galoisienne associée à une famille de Hida, interpolant en poids 2 les images de points CM par l'application de Kummer.

La première partie de cette thèse relie les spécialisations de la classe de Howard en poids $k \geq 2$ aux images de certains cycles de Heegner par l'application d'Abel–Jacobi p-adique. Notre démonstration de cette relation repose sur une formule de Gross–Zagier p-adique obtenue dans les travaux récents [**BDP13**] de Bertolini–Darmon–Prasanna, et que nous étendons ici à un cadre permettant de travailler avec des formes modulaires de niveau divisible par p. On déduit de nos résultats une interpolation de la formule de Gross–Zagier p-adique de Nekovář sur une famille de Hida.

La deuxième partie étend la définition de la classe de Howard “le long de la droite anticyclotomique”, pour obtenir une classe de cohohomologie à deux variables. On montre que la fonction L p-adique de Hida–Rankin, telle que décrite dans [**BDP13**], est l'image de cette classe par une généralisation de l'isomorphisme de Coleman. La méthode des systèmes d'Euler de Kolyvagin, telle que réinventée par Kato et Perrin-Riou, permet d'en déduire certains nouveaux cas de la conjecture de Bloch–Kato pour la convolution de Rankin–Selberg d'une forme parabolique avec une série thêta de poids supérieur, et une divisibilité dans la conjecture principale de la théorie d'Iwasawa associée à cette famille de motifs.

Contents

Preface

It is a somewhat vexing fact that, to the embarrassment of many mathematicians, the most convincing theoretical evidence in support of the Birch and Swinnerton-Dyer conjecture still rests largely on the foundational works of Gross–Zagier [**GZ86**] and Kolyvagin [**Kol88**], where the classical Heegner point construction attached to the auxiliary choice of an imaginary quadratic field was stunningly exploited to establish the conjecture in the case of analytic rank at most 1 for a class of elliptic curves that now, after Wiles's breakthrough [**Wil95**] culminating in [**BCDT01**], is known to be rich enough to include all rational elliptic curves.

In this thesis we aim to further scrutinize the wealth of information accounted for by Heegner points and their p-adic variation, examining a two-variable construction by Howard [**How07b**] that extends over a Hida family and over the anticyclotomic tower.

Our new results in these directions are contained in Chapters 1 and 2, which are slightly modified versions of the papers [**Cas13a**] (to appear in Mathematische Annalen) and [**Cas13b**] (submitted for publication), and are ultimately based on the study of an anticyclotomic p-adic L-function introduced in [**BDP13**] for which the characters relevant for the Birch and Swinnerton-Dyer conjecture lie *outside* the range of classical interpolation. Because of this feature, the p-adic Gross–Zagier formulae of [**BDP13**] are certainly a less natural analogue of the result of Gross–Zagier than the p-adic formulae proven by Perrin-Riou [**PR87**] and Nekovář [**Nek95**], but *a posteriori* they have been found to be useful for arithmetic applications.

Starting with Leopoldt's formula, similar formulae for the values of p-adic L-functions outside their range of classical interpolation have been discovered and exploited in most situations where interesting Euler systems can be shown to exist. This point of view, which is sometimes not completely apparent in the classical literature, is stressed in [**BCD$^+$13**], where the reader can see most clearly how our results fit within a broader perspective.

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CHAPTER 1

Higher weight specializations of big Heegner points

Summary

Let $\mathbf{f}$ be a p-ordinary Hida family of tame level N, and let K be an imaginary quadratic field satisfying the Heegner hypothesis relative to N. By taking a compatible sequence of twisted Kummer images of CM points over the tower of modular curves of level $\Gamma_0(N) \cap \Gamma_1(p^s)$, Howard [**How07b**] has constructed a canonical class $\mathfrak{Z}$ in the cohomology of a self-dual twist of the big Galois representation associated to $\mathbf{f}$. If a p-ordinary eigenform f on $\Gamma_0(N)$ of weight $k > 2$ is the specialization of $\mathbf{f}$ at ν, one thus obtains from $\mathfrak{Z}_\nu$ a higher weight generalization of the Kummer images of Heegner points. In this chapter we relate the classes $\mathfrak{Z}_\nu$ to the étale Abel–Jacobi images of Heegner cycles when p splits in K.

Introduction

Fix a prime $p > 3$ and an integer $N > 4$ such that $p \nmid N\phi(N)$. Let

$$f_o = \sum_{n>0} a_n q^n \in S_k(X_0(N))$$

be a p-ordinary newform of even weight $k = 2r \geq 2$ and trivial nebentypus. Thus f_o is an eigenvector for all the Hecke operators T_n with associated eigenvalues a_n, and a_p is a p-adic unit for a choice of embeddings $\iota_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$ that will remain fixed throughout this paper. Also let $\mathcal{O}$ denote the ring of integers of a (sufficiently large) finite extension $L/\mathbf{Q}_p$ containing all the a_n.

For $s > 0$, let X_s be the compactified modular curve of level

$$\Gamma_s := \Gamma_0(N) \cap \Gamma_1(p^s),$$

and consider the tower

$$\cdots \longrightarrow X_s \xrightarrow{\alpha} X_{s-1} \longrightarrow \cdots$$

with respect to the degeneracy maps described on the non-cuspidal moduli by

$$(E, \alpha_E, \pi_E) \longmapsto (E, \alpha_E, p \cdot \pi_E),$$

where α_E denotes a cyclic N-isogeny on the elliptic curve E, and π_E a point of E of exact order p^s. The group $(\mathbf{Z}/p^s\mathbf{Z})^\times$ acts on X_s via the diamond operators

$$\langle d \rangle : (E, \alpha_E, \pi_E) \longmapsto (E, \alpha_E, d \cdot \pi_E)$$

compatibly with α under the reduction $(\mathbf{Z}/p^s\mathbf{Z})^\times \longrightarrow (\mathbf{Z}/p^{s-1}\mathbf{Z})^\times$. Set $\Gamma := 1 + p\mathbf{Z}_p$. Letting J_s be the Jacobian variety of X_s, the inverse limit of the system induced by Albanese functoriality,

$$\cdots \longrightarrow \mathrm{Ta}_p(J_s) \otimes_{\mathbf{Z}_p} \mathcal{O} \longrightarrow \mathrm{Ta}_p(J_{s-1}) \otimes_{\mathbf{Z}_p} \mathcal{O} \longrightarrow \cdots, \tag{1.0.1}$$

is equipped with an action of the Iwasawa algebras $\widetilde{\Lambda}_{\mathcal{O}} := \mathcal{O}[[\mathbf{Z}_p^\times]]$ and

$$\Lambda_{\mathcal{O}} := \mathcal{O}[[\Gamma]].$$

Let $\mathfrak{h}_s$ be the $\mathcal{O}$-algebra generated by the Hecke operators T_ℓ $(\ell \nmid Np)$, $U_\ell := T_\ell$ $(\ell | Np)$, and the diamond operators $\langle d \rangle$ $(d \in (\mathbf{Z}/p^s\mathbf{Z})^\times)$ acting on the space $S_k(X_s)$ of cusp forms of weight k and level Γ_s. Hida's ordinary projector

$$e^{\mathrm{ord}} := \lim_{n\to\infty} U_p^{n!}$$

defines an idempotent of $\mathfrak{h}_s$, projecting to the maximal subspace of $\mathfrak{h}_s$ where U_p acts invertibly. We make each $\mathfrak{h}_s$ into a $\widetilde{\Lambda}_{\mathcal{O}}$-algebra by letting the group-like element attached to $z \in \mathbf{Z}_p^\times$ act as $z^{k-2}\langle z\rangle$.

Taking the projective limit with respect to the restriction maps induced by the natural inclusion $S_k(X_{s-1}) \hookrightarrow S_k(X_s)$ we obtain a $\widetilde{\Lambda}_{\mathcal{O}}$-algebra

$$\mathfrak{h}^{\mathrm{ord}} := \varprojlim_s e^{\mathrm{ord}}\mathfrak{h}_s \tag{1.0.2}$$

which can be seen to be *independent* of the weight $k \geq 2$ used in its construction.

After a highly influential work [**Hid86b**] of Hida, one can associate with f_o a certain local domain $\mathbb{I}$ quotient of $\mathfrak{h}^{\mathrm{ord}}$, finite flat over $\Lambda_{\mathcal{O}}$, with the following properties. For each n, let $\mathbf{a}_n \in \mathbb{I}$ be the image of T_n under the projection $\mathfrak{h}^{\mathrm{ord}} \longrightarrow \mathbb{I}$, and consider the formal q-expansion

$$\mathbf{f} = \sum_{n>0} \mathbf{a}_n q^n \in \mathbb{I}[[q]].$$

We say that a continuous $\mathcal{O}$-algebra homomorphism $\nu : \mathbb{I} \longrightarrow \overline{\mathbf{Q}}_p$ is an *arithmetic prime* if there is an integer $k_\nu \geq 2$, called the *weight* of ν, such that the composition

$$\Gamma \longrightarrow \mathbb{I}^\times \longrightarrow \overline{\mathbf{Q}}_p^\times$$

agrees with $\gamma \longmapsto \gamma^{k_\nu - 2}$ on an open subgroup of Γ of index $p^{s_\nu - 1} \geq 1$. Denote by $\mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ the set of arithmetic primes of $\mathbb{I}$, which will often be seen as sitting inside $\mathrm{Spf}(\mathbb{I})(\overline{\mathbf{Q}}_p)$. If $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$, F_ν will denote its residue field. Then:

- for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$, there exists an ordinary p-stabilized newform[1]
$$\mathbf{f}_\nu \in S_{k_\nu}(X_{s_\nu})$$
such that $\nu(\mathbf{f}) \in F_\nu[[q]]$ gives the q-expansion of $\mathbf{f}_\nu$;
- if $s_\nu = 1$ and $k_\nu \equiv k \pmod{2(p-1)}$, there exists a normalized newform $\mathbf{f}_\nu^\sharp \in S_{k_\nu}(X_0(N))$ such that
$$\mathbf{f}_\nu(q) = \mathbf{f}_\nu^\sharp(q) - \frac{p^{k_\nu - 1}}{\nu(\mathbf{a}_p)}\mathbf{f}_\nu^\sharp(q^p); \tag{1.0.3}$$
- there exists a unique $\nu_o \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ such that $f_o = \mathbf{f}_{\nu_o}^\sharp$.

In particular, after "p-stabilization" (1.0.3), the form f_o fits in the p-adic family $\mathbf{f}$.

[1]As defined in [**NP00**, (1.3.7)].

Similarly for the associated Galois representation V_{f_o}: the continuous $\mathfrak{h}^{\mathrm{ord}}$-linear action of the absolute Galois group $G_{\mathbf{Q}}$ on the module

$$\mathbb{T} := \mathbb{T}^{\mathrm{ord}} \otimes_{\mathfrak{h}^{\mathrm{ord}}} \mathbb{I}, \qquad \text{where} \qquad \mathbb{T}^{\mathrm{ord}} := \varprojlim_{s} e^{\mathrm{ord}}(\mathrm{Ta}_p(J_s) \otimes_{\mathbf{Z}_p} \mathcal{O}), \tag{1.0.4}$$

gives rise to a "big" Galois representation $\rho_{\mathbf{f}} : G_{\mathbf{Q}} \longrightarrow \mathrm{Aut}_{\mathbb{I}}(\mathbb{T})$ such that

$$\nu(\rho_{\mathbf{f}}) \cong \rho^*_{\mathbf{f}_\nu}$$

for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$, where $\rho^*_{\mathbf{f}_\nu}$ is the contragredient of the (cohomological) p-adic Galois representation $\rho_{\mathbf{f}_\nu} : G_{\mathbf{Q}} \longrightarrow \mathrm{Aut}(V_{\mathbf{f}_\nu})$ attached to $\mathbf{f}_\nu$ by Deligne; in particular, one recovers $\rho^*_{f_o}$ from $\rho_{\mathbf{f}}$ by specialization at ν_o.

Assume from now on that the residual representation $\bar{\rho}_{f_o}$ is irreducible; then $\mathbb{T}$ can be shown to be free of rank 2 over $\mathbb{I}$. (See [**MT90**, Théorème 7] for example.) Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$ containing an ideal $\mathfrak{N} \subset \mathcal{O}_K$ with

$$\mathcal{O}_K/\mathfrak{N} \cong \mathbf{Z}/N\mathbf{Z}, \tag{1.0.5}$$

and denote by H the Hilbert class field of K. Under this *Heegner hypothesis* relative to N (but with no extra assumptions on the prime p), the work [**How07b**] of Howard produces a compatible sequence $U_p^{-s} \cdot \mathfrak{X}_s$ of cohomology classes with values in a certain twist of the ordinary part of (1.0.1), giving rise to a canonical "big" cohomology class $\mathfrak{X}$, the *big Heegner point* (of conductor 1), in the cohomology of a self-dual twist $\mathbb{T}^\dagger$ of $\mathbb{T}$. Moreover, if every prime factor of N splits in K, it follows from his results that the class

$$\mathfrak{Z} := \mathrm{Cor}_{H/K}(\mathfrak{X})$$

lies in Nekovář's extended Selmer group $\widetilde{H}^1_f(K, \mathbb{T}^\dagger)$. In particular, for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ with $s_\nu = 1$ and $k_\nu \equiv k \pmod{2(p-1)}$ as above, the specialization $\mathfrak{Z}_\nu$ belongs to the Bloch–Kato Selmer group $H^1_f(K, V_{\mathbf{f}^\sharp_\nu}(k_\nu/2))$ of the self-dual representation $\mathbb{T}^\dagger \otimes_{\mathbb{I}} F_\nu \cong V_{\mathbf{f}^\sharp_\nu}(k_\nu/2)$. The classes $\mathfrak{Z}_\nu$ may thus be regarded as a natural higher weight analogue of the Kummer images of Heegner points, on modular Abelian varieties (associated with weight 2 eigenforms).

But for any of the above $\mathbf{f}^\sharp_\nu$, one has an alternate (and completely different!) method of producing such a higher weight analogue. Briefly, if $k_\nu = 2r_\nu > 2$, associated to any elliptic curve A with CM by $\mathcal{O}_K$, there is a null-homologous cycle $\Delta^{\mathrm{heeg}}_{A,r_\nu}$, a so-called *Heegner cycle*, on the $(2r_\nu - 1)$-dimensional Kuga–Sato variety W_{r_ν} giving rise to an H-rational class in the Chow group $\mathrm{CH}^{r_\nu+1}(W_{r_\nu})_0$ with $\mathbf{Q}$-coefficients. Since the representation $V_{\mathbf{f}^\sharp_\nu}(r_\nu)$ appears in the étale cohomology of W_{r_ν}:

$$H^{2r_\nu-1}_{\text{ét}}(\overline{W}_{r_\nu}, \mathbf{Q}_p)(r_\nu) \xrightarrow{\pi_{\mathbf{f}^\sharp_\nu}} V_{\mathbf{f}^\sharp_\nu}(r_\nu),$$

by taking the images of the cycles $\Delta^{\text{heeg}}_{A,r_\nu}$ under the p-adic étale Abel-Jacobi map

$$\Phi^{\text{ét}}_H : \text{CH}^{r_\nu+1}(W_{r_\nu})_0(H) \longrightarrow H^1(H, H^{2r_\nu-1}_{\text{ét}}(\overline{W}_{r_\nu}, \mathbf{Q}_p)(r_\nu))$$

and composing with the map induced by $\pi_{\mathbf{f}^\sharp_\nu}$ on H^1's, we may consider the classes

$$\Phi^{\text{ét}}_{\mathbf{f}^\sharp_\nu,K}(\Delta^{\text{heeg}}_{r_\nu}) := \text{Cor}_{H/K}(\pi_{\mathbf{f}^\sharp_\nu}\Phi^{\text{ét}}_H(\Delta^{\text{heeg}}_{A,r_\nu})).$$

By the work [**Nek00**] of Nekovář, these classes are known to lie in the same Selmer group as $\mathfrak{Z}_\nu$, and the question of their comparison thus naturally arises.

Main Theorem (Thm. 1.4.12). *Assume that p splits in $K = \mathbf{Q}(\sqrt{-D})$ and that the class $\mathfrak{Z}$ is not $\mathbb{I}$-torsion. Then for all but finitely many $\nu \in \mathcal{X}_{\text{arith}}(\mathbb{I})$ of weight $k_\nu = 2r_\nu > 2$ with $k_\nu \equiv k \pmod{2(p-1)}$ and trivial character, we have*

$$\langle \mathfrak{Z}_\nu, \mathfrak{Z}_\nu \rangle_K = \left(1 - \frac{p^{r_\nu-1}}{\nu(\mathbf{a}_p)}\right)^4 \frac{\langle \Phi^{\text{ét}}_{\mathbf{f}^\sharp_\nu,K}(\Delta^{\text{heeg}}_{r_\nu}), \Phi^{\text{ét}}_{\mathbf{f}^\sharp_\nu,K}(\Delta^{\text{heeg}}_{r_\nu})\rangle_K}{u^2(4D)^{r_\nu-1}},$$

where $\langle,\rangle_K$ is the cyclotomic p-adic height pairing on $H^1_f(K, V_{\mathbf{f}^\sharp_\nu}(r_\nu))$, and $u := |\mathcal{O}_K^\times|/2$.

Thus assuming the non-degeneracy of the p-adic height pairing, it follows that the étale Abel–Jacobi images of Heegner cycles are p-adically interpolated by $\mathfrak{Z}$. We also note that $\mathfrak{Z}$ is conjecturally *always* not $\mathbb{I}$-torsion ([**How07b**, Conj. 3.4.1]), and that by [**How07a**, Cor. 5] this conjecture can be verified in any given case by exhibiting the non-vanishing of an appropriate L-value (a derivative, in fact). But arguably the main interest of the above result is to be found in connection with p-adic L-functions, as we indicate below.

Let $\mathscr{G}_\infty$ be the Galois group of the unique $\mathbf{Z}_p^2$-extension of K. In their recent proof [**SU13**] one divisibility in the Iwasawa Main Conjecture for $\mathbf{GL}_2$, Skinner and Urban construct an element $\mathcal{L}_p(\mathbf{f} \otimes K) \in \mathbb{I}[[\mathscr{G}_\infty]]$ which interpolates a certain two-variable p-adic L-function $\mathcal{L}_p(\mathbf{f}_\nu \otimes K) \in \mathcal{O}_\nu[[\mathscr{G}_\infty]]$ attached to the specializations $\mathbf{f}_\nu$. For any ν as in the above Main Theorem, the work [**Nek95**] of Nekovář proves a p-adic analogue of the Gross–Zagier formula for $\mathcal{L}_{\mathbf{f}_\nu,K}$. Combined with the existence of an $\mathbb{I}$-valued "height pairing" $\langle,\rangle_{K,\mathbb{T}^\dagger}$ on $\widetilde{H}^1_f(K, \mathbb{T}^\dagger)$, we can easily deduce the following.

Corollary (Thm. 3.1.3). *Let $\mathcal{L}'_{\mathbf{f},K}$ be the linear term in the expansion of $\mathcal{L}_{\mathbf{f},K}$ restricted to the cyclotomic line. Under the assumptions of the Main Theorem, we have*

$$\mathcal{L}'_{\mathbf{f},K}(\mathbb{1}_K) = \langle \mathfrak{Z}, \mathfrak{Z} \rangle_{K,\mathbb{T}^\dagger} \pmod{\mathbb{I}^\times}.$$

This paper is organized as follows. Section 1.1 is aimed at proving an expression for the formal group logarithms of ordinary CM points on X_s using Coleman's theory of p-adic integration. Our methods here are drawn from [**BDP13**, §3], which we extend in weight 2 to the case of level divisible by an arbitrary power of p, but with ramification restricted to a *potentially crystalline* setting. Not quite surprisingly, this restriction turns out to make our computations essentially the same as theirs, and will suffice for our purposes.

In Section 1.2 we recall the generalised Heegner cycles and the formula for their p-adic Abel-Jacobi images from *loc.cit.*, and discuss the relation between these and the more classical Heegner cycles.

In Section 1.3 we deduce from the work [**Och03**] of Ochiai the construction of a "big" logarithm map that will allow as to move between different weights in the Hida family.

Finally, in Section 1.4 we prove our results on the arithmetic specializations of the big Heegner point $\mathfrak{Z}$. The key observation is that, when p splits in K, the combination of CM points on X_s taken in Howard's construction appears naturally in the evaluation of the critical twist of a p-adic modular form at a canonical trivialized elliptic curve. The expression from Section 1 thus yields, for infinitely many ν of weight 2, a formula for the p-adic logarithm of the localization of $\mathfrak{Z}_\nu$ in terms of certain values of a p-adic modular form of weight 0 associated with $\mathbf{f}_\nu$ (Theorem 1.4.9). When extended by p-adic continuity to an arithmetic prime ν of higher even weight, this expression is seen to agree with the formula from Section 1.2, and by the interpolation properties of the big logarithm map it corresponds to the p-adic logarithm of the localization of $\mathfrak{Z}_\nu$. The above Main Theorem then follows easily from this.

We also note that an extension "in the anticyclotomic direction" of some of the results in this paper leads to a number of arithmetic applications arising from the connection between Howard's big Heegner points and a certain p-adic L-function introduced in [**BDP13**]. This connection appears implicitly here and is developed in [**Cas13b**].

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1.1. Preliminaries

1.1.1. p-adic modular forms. To avoid some issues related to the representability of certain moduli problems, in this section we change notations from the Introduction, letting X_s be the compactified modular curve of level $\Gamma_s := \Gamma_1(Np^s)$, viewed as a scheme over $\mathrm{Spec}(\mathbf{Q}_p)$. Let $\pi : \mathcal{E}_s \longrightarrow \tilde{X}_s$ be the universal elliptic curve over the complement $\tilde{X}_s \subset X_s$ of the cuspidal subscheme $Z_s \subset X_s$, and let $\underline{\omega}_{X_s}$ be the invertible sheaf X_s given by the extension of $\pi_* \Omega_{\mathcal{E}_s/\tilde{X}_s}$ to the cusps Z_s as described in [**Gro90**, §1], for example.

Algebraically, $H^0(X_s, \underline{\omega}_{X_s}^{\otimes 2})$ gives the space of modular forms of weight 2 and level Γ_s (defined over $\mathbf{Q}_p$). Consider the complex

$$\Omega^\bullet_{X_s/\mathbf{Q}_p}(\log Z_s) : 0 \longrightarrow \mathcal{O}_{X_s} \xrightarrow{d} \Omega^1_{X_s/\mathbf{Q}_p}(\log Z_s) \longrightarrow 0 \tag{1.1.1}$$

of sheaves on X_s. The algebraic *de Rham cohomology* of X_s

$$H^1_{\mathrm{dR}}(X_s/\mathbf{Q}_p) := \mathbb{H}^1(X_s, \Omega^\bullet_{X_s/\mathbf{Q}_p}(\log Z_s))$$

is a finite-dimensional $\mathbf{Q}_p$-vector space equipped with a *Hodge filtration*

$$0 \subset H^0(X_s, \Omega^1_{X_s/\mathbf{Q}_p}(\log Z_s)) \subset H^1_{\mathrm{dR}}(X_s/\mathbf{Q}_p),$$

and by the Kodaira-Spencer isomorphism $\underline{\omega}_{X_s}^{\otimes 2} \cong \Omega^1_{X_s/\mathbf{Q}_p}(\log Z_s)$, every cusp form $f \in S_2(X_s)$ (in particular) defines a cohomology class $\omega_f \in H^1_{\mathrm{dR}}(X_s/\mathbf{Q}_p)$.

Let X be the complete modular curve of level $\Gamma_1(N)$, also viewed over $\mathrm{Spec}(\mathbf{Q}_p)$, and consider the subspaces of the associated rigid analytic space X^{an}:

$$X^{\mathrm{ord}} \subset X_{<1/(p+1)} \subset X_{<p/(p+1)} \subset X^{\mathrm{an}}.$$

To define these, let $\mathcal{X}_{/\mathbf{Z}_p}$ be the canonical integral model of X over $\mathrm{Spec}(\mathbf{Z}_p)$, and let $X_{\mathbf{F}_p} := \mathcal{X} \times_{\mathbf{Z}_p} \mathbf{F}_p$ denote its special fiber. The *supersingular points* $SS \subset X_{\mathbf{F}_p}(\overline{\mathbf{F}}_p)$ is the finite set of points corresponding to the moduli of supersingular elliptic curves (with $\Gamma_1(N)$-level structure) in characteristic p.

Let E_{p-1} be the Eisenstein series of weight $p-1$ and level 1, seen as a global section of the sheaf $\underline{\omega}_X^{\otimes(p-1)}$. (Recall that we are assuming $p \geq 5$.) The reduction of E_{p-1} to $X_{\mathbf{F}_p}$ is the *Hasse invariant*, which defines a section of the reduction of $\underline{\omega}_X^{\otimes(p-1)}$ with SS as its locus of (simple) zeroes. If $x \in X(\overline{\mathbf{Q}}_p)$, let $\bar{x} \in X_{\mathbf{F}_p}(\overline{\mathbf{F}}_p)$ denote its reduction. Each point

$\bar{x} \in SS$ is smooth in $X_{\mathbf{F}_p}$, and the *ordinary locus* of X

$$X^{\mathrm{ord}} := X^{\mathrm{an}} \smallsetminus \bigcup_{\bar{x} \in SS} D_{\bar{x}}$$

is defined to be the complement of their residue discs $D_{\bar{x}} \subset X^{\mathrm{an}}$. The function $|E_{p-1}(x)|_p$ defines a local parameter on $D_{\bar{x}}$, and with the normalization $|p|_p = p^{-1}$, $X_{<1/(p+1)}$ (resp. $X_{<p/(p+1)}$) is defined to be complement in X^{an} of the subdiscs of $D_{\bar{x}}$ where $|E_{p-1}(x)|_p \leq p^{-1/(p+1)}$ (resp. $|E_{p-1}(x)|_p \leq p^{-p/(p+1)}$), for all $\bar{x} \in SS$.

Using the *canonical subgroup* H_E (of order p) attached to every elliptic curve E corresponding to a closed point in $X_{<p/(p+1)}$, the *Deligne-Tate map*

$$\phi_0 : X_{<1/(p+1)} \longrightarrow X_{<p/(p+1)}$$

is defined by sending $E \longmapsto E/H_E$ (with the induced action on the level structure) under the moduli interpretation. This map is a finite morphism which by definition lifts to characteristic zero the absolute Frobenius on $X_{\mathbf{F}_p}$. (See [**Kat73**, Thm. 3.1].)

For every $s > 0$, the Deline-Tate map ϕ_0 can be iterated $s-1$ times on the open rigid subspace $X_{<p^{2-s}/(p+1)}$ of X^{an} where $|E_{p-1}(x)|_p > p^{-p^{2-s}/(p+1)}$. Letting $\alpha_s : X_s \longrightarrow X$ be the map forgetting the "$\Gamma_1(p^s)$-part" of the level structure, define

$$\mathcal{W}_1(p^s) \subset X_s^{\mathrm{an}}$$

to be the open rigid subspace of X_s whose closed points correspond to triples (E, α_E, π_E) whose image under α_s lands inside $X_{<p^{2-s}(p+1)}$ and are such that π_E generates the canonical subgroup of E *of order* p^s (as in [**Buz03**, Defn. 3.4]).

Define $\mathcal{W}_2(p^s) \subset X_s^{\mathrm{an}}$ is the same manner, replacing $p^{2-s}/(p+1)$ by $p^{1-s}/(p+1)$ in the definition of $\mathcal{W}_1(p^s)$. Then we obtain a lifting of Frobenius $\phi = \phi_s$ on X_s making the diagram

$$\begin{array}{ccc} \mathcal{W}_2(p^s) & \xrightarrow{\phi} & \mathcal{W}_1(p^s) \\ \downarrow{\scriptstyle \alpha_s} & & \downarrow{\scriptstyle \alpha_s} \\ X_{<p^{1-s}(p+1)} & \xrightarrow{\phi_0} & X_{<p^{2-s}(p+1)}. \end{array}$$

commutative by sending a point $x = (E, \alpha_E, \imath_E) \in \mathcal{W}_2(p^s)$, where $\imath_E : \boldsymbol{\mu}_{p^s} \hookrightarrow E[p^s]$ is an embedding giving the $\Gamma_1(p^s)$-level structure on E, to $x' = (\phi_0 E, \phi_0 \alpha_E, \imath'_E)$, where $\imath'_E$ is determined by requiring that $\alpha_s(x')$ lands in $X_{<p^{2-s}/(p+1)}$ and for each $\zeta \in \boldsymbol{\mu}_{p^s} - \{1\}$, $\imath'_E(\zeta) = \phi_0 Q$ if $\imath_E(\zeta) = pQ$. (Cf. [**Col97b**, §B.2].)

Let $I_s := \{v \in \mathbf{Q} \ : \ 0 \leq v < p^{2-s}/(p+1)\}$, and for $v \in I_s$ define the affinoid subdomain $X_s(v)$ of X_s^{an} inside $\mathcal{W}_1(p^s)$ whose closed points x satisfy $|E_{p-1}(x)|_p \geq p^{-v}$. Then $X_s(0)$ is the connected component of the ordinary locus of X_s containing the cusp ∞.

Denote by $\underline{\omega}_{X_s^{\mathrm{an}}}$ the rigid analytic sheaf on X_s^{an} deduced from $\underline{\omega}_{X_s}$ and fix $k \in \mathbf{Z}$. The space of *p-adic modular forms* of weight k and level Γ_s (defined over $\mathbf{Q}_p$) is the p-adic Banach space

$$M_k^{\mathrm{ord}}(X_s) := H^0(X_s(0), \underline{\omega}_{X_s^{\mathrm{an}}}^{\otimes k}),$$

and the space of *overconvergent p-adic modular forms* of weight k and level Γ_s is the p-adic Fréchet space

$$M_k^{\mathrm{rig}}(X_s) := \varinjlim_{v} H^0(X_s(v), \underline{\omega}_{X_s^{\mathrm{an}}}^{\otimes k}),$$

where the limit is with respect to the natural restriction maps as $v \in I_s$ increasingly approaches $p^{2-s}/(p+1)$. By restriction, a classical modular form in $H^0(X_s, \underline{\omega}_{X_s}^{\otimes k})$ defines an (obviously) overconvergent p-adic modular form of the same weight an level. Moreover, the action of the diamond operators on X_s gives rise to an action of $(\mathbf{Z}/p^s\mathbf{Z})^\times$ on the spaces of p-adic modular forms which agrees with the action on $H^0(X_s, \underline{\omega}_{X_s}^{\otimes k})$ under restriction.

We say that a ring R is a *p-adic ring* if the natural map $R \longrightarrow \varprojlim R/p^nR$ is an isomorphism. For varying s, the data of a compatible sequence of embeddings $\boldsymbol{\mu}_{p^s} \hookrightarrow E$ as R-group schemes, amounts to the data of an embedding $\boldsymbol{\mu}_{p^\infty} \hookrightarrow E[p^\infty]$ of p-divisible groups, and also to the given of a *trivialization* of E over R, i.e. an isomorphism

$$\imath_E : \hat{E} \longrightarrow \hat{\mathbf{G}}_m$$

of the associated formal groups. The space $\mathbf{M}(N)$ of *Katz p-adic modular functions* of tame level N (over $\mathbf{Z}_p$) is the space of functions f on trivialized elliptic curves with $\Gamma_1(N)$-level structure over arbitrary p-adic rings, assigning to the isomorphism class of a triple $(E, \alpha_E, \imath_E)$ over R a value $f(E, \alpha_E, \imath_E) \in R$ whose formation is compatible under base change. If R is a fixed p-adic ring, by only considering p-adic rings which are R-algebras, we obtain the notion of Katz p-adic modular functions defined over R, forming the space $\mathbf{M}(N)\widehat{\otimes}_{\mathbf{Z}_p}R$ which will also be denoted by $\mathbf{M}(N)$ by an abuse of notation.

The action of $z \in \mathbf{Z}_p^\times$ on a trivialization gives rise to an action of $\mathbf{Z}_p^\times$ on $\mathbf{M}(N)$:

$$\langle z \rangle f(E, \alpha_E, \imath_E) := f(E, \alpha_E, z^{-1} \cdot \imath_E),$$

and given a character $\chi \in \mathrm{Hom}_{\mathrm{cont}}(\mathbf{Z}_p^\times, R^\times)$, we say that $f \in \mathbf{M}(N)$ has *weight-nebentypus* χ if $\langle z \rangle f = \chi(z) f$ for all $z \in \mathbf{Z}_p^\times$. If k is an integer, denoting by z^k the k-th power character on $\mathbf{Z}_p^\times$, the subspace $M_k^{\mathrm{ord}}(Np^s, \varepsilon)$ of $M_k^{\mathrm{ord}}(X_s)$ consisting of p-adic modular forms with nebentypus $\varepsilon : (\mathbf{Z}/p^s\mathbf{Z})^\times \longrightarrow R^\times$ can be recovered as

$$M_k^{\mathrm{ord}}(Np^s, \varepsilon) \cong \{f \in \mathbf{M}(N) : \langle z \rangle f = z^k \varepsilon(z) f, \quad \text{for all } z \in \mathbf{Z}_p^\times\}. \tag{1.1.2}$$

Since it will play an important role later, we next recall from [**Gou88**, §III.6.2] the definition in terms of moduli of the twist of p-adic modular forms by characters of not necessarily finite order. Let R be a p-adic ring, and let $(E, \alpha_E, \imath_E)$ be a trivialized elliptic curve with $\Gamma_1(N)$-level structure over R. For each s, consider the quotient $E_0 := E/\imath_E^{-1}(\boldsymbol{\mu}_{p^s})$, and let $\varphi_0 : E \twoheadrightarrow E_0$ denote the projection. Since $p \nmid N$, φ_0 induces a $\Gamma_1(N)$-level structure α_{E_0} on E_0, and since $\ker(\varphi_0) \cong \boldsymbol{\mu}_{p^s}$, the dual $\check{\varphi}_0 : E_0 \longrightarrow E$ is étale, inducing an isomorphism of the associated formal groups. Thus (with a slight abuse of notation) $\imath_{E_0} := \imath_E \circ \check{\varphi}_0 : \hat{E}_0 \xrightarrow{\sim} \hat{\mathbf{G}}_m$ is a trivialization of E_0, and since we have an embedding $\mathbf{Z}/p^s\mathbf{Z} \cong \ker(\check{\varphi}_0) \hookrightarrow E_0[p^s]$, we deduce an isomorphism

$$E_0[p^s] \cong \boldsymbol{\mu}_{p^s} \oplus \mathbf{Z}/p^s\mathbf{Z}$$

which we use to bijectively attach a p^s-th root of unity ζ_C to every étale subgroup $C \subset E_0[p^s]$ of order p^s, in such a way that 1 is attached to $\ker(\check{\varphi}_0)$.

Now for $f \in \mathbf{M}(N)$ and $a \in \mathbf{Z}_p$, define $f \otimes \mathbb{1}_{a+p^s\mathbf{Z}_p}$ to be the rule on trivialized elliptic curves given by

$$f \otimes \mathbb{1}_{a+p^s\mathbf{Z}_p}(E, \alpha_E, \imath_E) = \frac{1}{p^s} \sum_C \zeta_C^{-a} \cdot f(E_0/C, \alpha_C, \imath_C) \tag{1.1.3}$$

where the sum is over the étale subgroups $C \subset E_0[p^s]$ of order p^s, and where α_C (resp. $\imath_C$) denotes the $\Gamma_1(N)$-level structure (resp. trivialization) on the quotient E_0/C naturally induced by α_{E_0} (resp. $\imath_{E_0}$).

Lemma 1.1.1. *The assignment*

$$a + p^s\mathbf{Z}_p \quad \rightsquigarrow \quad \big(f \longmapsto f \otimes \mathbb{1}_{a+p^s\mathbf{Z}_p}\big)$$

gives rise to an $\mathrm{End}_R\mathbf{M}(N)$*-valued measure* μ_{Gou} *on* $\mathbf{Z}_p$*.*

Proof. Let $\sum_n a_n q^n$ be the q-expansion of f, i.e. the value that it takes at the triple $(\mathrm{Tate}(q), \alpha_{\mathrm{can}}, \imath_{\mathrm{can}}) = (\mathbf{G}_m/q^{\mathbf{Z}}, \zeta_N, \boldsymbol{\mu}_{p^\infty} \hookrightarrow \mathbf{G}_m/q^{\mathbf{Z}})$ over the p-adic completion of $R((q))$. By the q-expansion principle, the claim follows immediately from the equality

$$f \otimes \mathbb{1}_{a+p^s\mathbf{Z}_p}(q) = \sum_{n \equiv a \bmod p^s} a_n q^n,$$

which is shown by adapting the arguments in [**Gou88**, p. 102]. □

DEFINITION 1.1.2 (Gouvêa). Let $f \in \mathbf{M}(N)$ and $\chi : \mathbf{Z}_p \longrightarrow R$ be any continuous multiplicative function. The *twist* of f by χ is

$$f \otimes \chi := \left(\int_{\mathbf{Z}_p} \chi(x) d\mu_{\mathrm{Gou}}(x) \right)(f) \in \mathbf{M}(N).$$

This operation is compatible with the usual character twist of Hecke eigenforms:

LEMMA 1.1.3. *Let $\chi : \mathbf{Z}_p^\times \longrightarrow R^\times$ be a continuous character extended by zero on $p\mathbf{Z}_p$. If $f \in \mathbf{M}(N)$ has q-expansion $\sum_n a_n q^n$, then $f \otimes \chi$ has q-expansion $\sum_n \chi(n) a_n q^n$, and if f has weight-nebentypus $\kappa \in \mathrm{Hom}_{\mathrm{cts}}(\mathbf{Z}_p^\times, R^\times)$, then $f \otimes \chi$ has weight-nebentypus $\chi^2\kappa$.*

PROOF. See [**Gou88**, Cor. III.6.8.*i*] and [**Gou88**, Cor. III.6.9]). □

In particular, twisting by the identity function of $\mathbf{Z}_p$ we obtain an operator

$$d : \mathbf{M}(N) \longrightarrow \mathbf{M}(N)$$

whose effect on q-expansions is $q\frac{d}{dq}$. For every $k \in \mathbf{Z}$, we see from (1.1.2) and Lemma 1.1.3, that this restricts to a map

$$d : M_k^{\mathrm{ord}}(X_s) \longrightarrow M_{k+2}^{\mathrm{ord}}(X_s)$$

which increases the weight by 2 and preserves the nebentypus. Moreover, for $k = 0$, the arguments in [**Col96**, Prop. 4.3] can be adapted to show that d gives rise to a linear map $M_0^{\mathrm{rig}}(X_s) \longrightarrow M_2^{\mathrm{rig}}(X_s)$, viewing $M_k^{\mathrm{rig}}(X_s)$ as a subspace of $M_k^{\mathrm{ord}}(X_s)$ via the natural restriction map.

1.1.2. Comparison isomorphisms. Let ζ_s be a primitive p^s-th root of unity, and let F be a finite extension of $\mathbf{Q}_p(\zeta_s)$ over which X_s acquires stable reduction, i.e. such that the base extension $X_s \times_{\mathbf{Q}_p} F$ admits a stable model over the ring of integers $\mathcal{O}_F$ of F. For the ease of notation, from now on we will denote $X_s \times_{\mathbf{Q}_p} F$ (as well as the associated rigid analytic space) simply by X_s.

Let $\mathscr{X}_s$ be the minimal regular model of X_s over $\mathcal{O}_F$, and denote by F_0 the maximal unramified subfield of F. The work [**HK94**] of Hyodo-Kato endows the F-vector space $H^1_{\mathrm{dR}}(X_s/F)$ with a canonical F_0-structure

$$H^1_{\mathrm{log-cris}}(\mathscr{X}_s) \hookrightarrow H^1_{\mathrm{dR}}(X_s/F) \tag{1.1.4}$$

equipped with a semi-linear Frobenius operator φ.

After the proof [**Tsu99**] of the so-called semistable conjecture of Fontaine–Jannsen, these structures are known to agree with those attached by Fontaine's theory to the p-adic

G_F-representation

$$V_s := H^1_{\text{ét}}(\overline{X}_s, \mathbf{Q}_p). \tag{1.1.5}$$

More precisely, since X_s has semistable reduction, V_s is semistable in the sense of Fontaine, and there is a canonical isomorphism $D_{\text{st}}(V_s) \xrightarrow{\sim} H^1_{\log-\text{cris}}(\mathscr{X}_s)$, inducing an isomorphism

$$D_{\text{dR}}(V_s) \xrightarrow{\sim} H^1_{\text{dR}}(X_s/F) \tag{1.1.6}$$

as filtered φ-modules after extending scalars to F.

Consider the étale Abel–Jacobi map $\text{CH}^1(X_s)_0(F) \longrightarrow H^1(F, V_s(1))$ constructed in [**Nek00**], which in this case agrees with the usual Kummer map

$$\delta_F : J_s(F) \longrightarrow H^1(F, \mathbf{Q}_p \otimes \text{Ta}_p(J_s)),$$

where $J_s = \text{Pic}^0(X_s)$ is the connected Picard variety of X_s. (See [*loc.cit.*, Example(2.3)].)

Let $g \in S_2(X_s)$ be a newform with primitive nebentypus, denote by V_g the p-adic Galois representation associated to g, which is equipped with a Galois-equivariant projection $V_s \longrightarrow V_g$, and let V_g^* be the representation contragredient to V_g, so that $V_g(1)$ and V_g^* are in Kummer duality. Also, let L_g be a finite extension of $\mathbf{Q}_p$ over which the Hecke eigenvalues of g are defined. By [**BK90**, Example 3.11], the image of the induced composite map

$$\delta_{g,F} : J_s(F) \xrightarrow{\delta_F} H^1(F, V_s(1)) \longrightarrow H^1(F, V_g(1)) \tag{1.1.7}$$

lies in the Bloch–Kato "finite" subspace $H^1_f(F, V_g(1))$, and by our assumption on g, the Bloch-Kato exponential map gives an isomorphism

$$\exp_{F,V_g(1)} : \frac{D_{\text{dR}}(V_g(1))}{\text{Fil}^0 D_{\text{dR}}(V_g(1))} \longrightarrow H^1_f(F, V_g(1)) \tag{1.1.8}$$

whose inverse will be denoted by $\log_{F,V_g(1)}$.

Our aim in this section is to compute the images of certain degree 0 divisors on X_s under the p-adic Abel–Jacobi map $\delta^{(p)}_{g,F}$, defined as the composition

$$J_s(F) \xrightarrow{\delta_{g,F}} H^1_f(F, V_g(1)) \xrightarrow{\log_{F,V_g(1)}} \frac{D_{\text{dR}}(V_g(1))}{\text{Fil}^0 D_{\text{dR}}(V_g(1))} \xrightarrow{\sim} (\text{Fil}^0 D_{\text{dR}}(V_g^*))^\vee, \tag{1.1.9}$$

where the last identification arises from the de Rham pairing

$$\langle\ ,\ \rangle_{\text{dR}} : D_{\text{dR}}(V_g(1)) \times D_{\text{dR}}(V_g^*) \longrightarrow D_{\text{dR}}(\mathbf{Q}_p(1)) \otimes_{\mathbf{Q}_p} L_g \cong F \otimes_{\mathbf{Q}_p} L_g \tag{1.1.10}$$

with respect to which $\mathrm{Fil}^0 D_{\mathrm{dR}}(V_g(1))$ and $\mathrm{Fil}^0 D_{\mathrm{dR}}(V_g^*)$ are exact annihilators of each other. A basic ingredient for this computation will be the following alternate description of the logarithm map $\log_{F,V_g(1)}$.

Recall the interpretation of $H^1(F, V_g(1))$ as the space $\mathrm{Ext}^1_{\underline{\mathrm{Rep}}(G_F)}(L_g, V_g(1))$ of extensions of $V_g(1)$ by L_g in the category of p-adic G_F-representations. Since F contains $\mathbf{Q}_p(\zeta_s)$, V_g is a crystalline G_F-representation in the sense of Fontaine, and under that interpretation the Bloch–Kato "finite" subspace corresponds to those extensions which are crystalline (see [**Nek93**, Prop. 1.26], for example):

$$H^1_f(F, V_g(1)) \cong \mathrm{Ext}^1_{\underline{\mathrm{Rep}}_{\mathrm{cris}}(G_F)}(L_g, V_g(1)). \tag{1.1.11}$$

Now consider a crystalline extension

$$0 \longrightarrow V_g(1) \longrightarrow W \longrightarrow L_g \longrightarrow 0. \tag{1.1.12}$$

Since $D_{\mathrm{cris}}(V_g(1))^{\varphi=1} = 0$ by our assumptions, the resulting extension of φ-modules

$$0 \longrightarrow D_{\mathrm{cris}}(V_g(1)) \longrightarrow D_{\mathrm{cris}}(W) \longrightarrow F_0 \otimes_{\mathbf{Q}_p} L_g \longrightarrow 0 \tag{1.1.13}$$

admits a unique section $s_W^{\mathrm{frob}} : F_0 \otimes_{\mathbf{Q}_p} L_g \longrightarrow D_{\mathrm{cris}}(W)$ with $s_W^{\mathrm{frob}}(1)$ landing in the φ-invariant subspace $D_{\mathrm{cris}}(W)^{\varphi=1}$. Extending scalars from F_0 to F in (1.1.13) and taking Fil^0-parts, we take an arbitrary section $s_W^{\mathrm{fil}} : F \otimes_{\mathbf{Q}_p} L_g \longrightarrow \mathrm{Fil}^0 D_{\mathrm{dR}}(W)$ of the resulting exact sequence of F-vector spaces

$$0 \longrightarrow \mathrm{Fil}^0 D_{\mathrm{dR}}(V_g(1)) \longrightarrow \mathrm{Fil}^0 D_{\mathrm{dR}}(W) \longrightarrow F \otimes_{\mathbf{Q}_p} L_g \longrightarrow 0 \tag{1.1.14}$$

and form the difference

$$t_W := s_W^{\mathrm{fil}}(1) - s_W^{\mathrm{frob}}(1),$$

which can be seen in $D_{\mathrm{dR}}(V_g(1))$, and whose image modulo $\mathrm{Fil}^0 D_{\mathrm{dR}}(V_g(1))$ is well-defined.

LEMMA 1.1.4. *Under the identification* (1.1.11)*, the above assignment*

$$0 \to V_g(1) \to W \to L_g \to 0 \quad \rightsquigarrow \quad t_W \bmod \mathrm{Fil}^0 D_{\mathrm{dR}}(V_g(1))$$

defines an isomorphism which agrees with the Bloch–Kato logarithm map

$$\log_{F,V_g(1)} : H^1_f(F, V_g(1)) \xrightarrow{\sim} \frac{D_{\mathrm{dR}}(V_g(1))}{\mathrm{Fil}^0 D_{\mathrm{dR}}(V_g(1))}.$$

PROOF. See [**Nek93**, Lemma 2.7], for example. □

Let $\Delta \in J_s(F)$ be the class of a degree 0 divisor on X_s with support contained in the finite set of points $S \subset X_s(F)$. The extension class $W = W_\Delta$ (1.1.12) corresponding to

$\delta_{g,F}(\Delta)$ can then be constructed from the étale cohomology of the open curve $Y_s := X_s \smallsetminus S$, as explained in [**BDP13**, §3.1]. We describe the associated $s^{\mathrm{fil}}_{W_\Delta}$ and $s^{\mathrm{frob}}_{W_\Delta}$.

By [**Tsu99**] (or also [**Fal02**]), denoting g-isotypical components by the superscript g, there is a canonical isomorphism of $F_0 \otimes_{\mathbf{Q}_p} L_g$-modules

$$D_{\mathrm{cris}}(V_g) \cong H^1_{\log-\mathrm{cris}}(\mathscr{X}_s)^g \tag{1.1.15}$$

compatible with φ-actions and inducing an $F \otimes_{\mathbf{Q}_p} L_g$-module isomorphism

$$D_{\mathrm{dR}}(V_g) \cong H^1_{\mathrm{dR}}(X_s/F)^g \tag{1.1.16}$$

after extension of scalars.

Writing $\Delta = \sum_{Q\in S} n_Q.Q$ for some $n_Q \in \mathbf{Z}$, we assume from now on that S contains the cusps, and that the reductions of the points $Q \in S$ are smooth and pair-wise distinct. We also assume that the reduction of S in the special fiber is stable under the absolute Frobenius. Like $H^1_{\mathrm{dR}}(X_s/F)$, the F-vector space $H^1_{\mathrm{dR}}(Y_s/F)$ is equipped with a canonical F_0-structure

$$H^1_{\log-\mathrm{cris}}(\mathscr{Y}_s) \hookrightarrow H^1_{\mathrm{dR}}(Y_s/F), \tag{1.1.17}$$

a Frobenius operator still denoted by φ, and a Hecke action compatible with that in (1.1.4). Thus for $W = W_\Delta$ the exact sequence (1.1.13) is obtained as the pullback

$$\begin{array}{ccccc} D_{\mathrm{cris}}(V_g(1)) & \hookrightarrow & D_{\mathrm{cris}}(W_\Delta) & \xrightarrow{\rho} & F_0 \otimes_{\mathbf{Q}_p} L_g \\ \| & & \downarrow & & \downarrow \Delta \\ H^1_{\log-\mathrm{cris}}(\mathscr{X}_s)^g(1) & \hookrightarrow & H^1_{\log-\mathrm{cris}}(\mathscr{Y}_s)^g(1) & \xrightarrow{\oplus \mathrm{res}_Q} & (F_0 \otimes_{\mathbf{Q}_p} L_g)^{\oplus S}_0 \end{array} \tag{1.1.18}$$

of the bottom extension of φ-modules with respect to the $F_0 \otimes_{\mathbf{Q}_p} L_g$-linear map sending $1 \mapsto (n_Q)_{Q\in S}$, where the subscript 0 indicates taking the degree 0 subspace.

On the other hand, after extending scalars from F_0 to F and taking Fil^0-parts, (1.1.14) is given by the pullback[2]

$$\begin{array}{ccccc} \mathrm{Fil}^0 D_{\mathrm{dR}}(V_g(1)) & \hookrightarrow & \mathrm{Fil}^0 D_{\mathrm{dR}}(W_\Delta) & \xrightarrow{\rho} & F \otimes_{\mathbf{Q}_p} L_g \\ \| & & \downarrow & & \downarrow \Delta \\ \mathrm{Fil}^1 H^1_{\mathrm{dR}}(X_s/F)^g & \hookrightarrow & \mathrm{Fil}^1 H^1_{\mathrm{dR}}(Y_s/F)^g & \xrightarrow{\oplus \mathrm{res}_Q} & (F \otimes_{\mathbf{Q}_p} L_g)^{\oplus S}_0 \end{array} \tag{1.1.19}$$

of the bottom exact sequence of free $F \otimes_{\mathbf{Q}_p} L_g$-modules with respect to the $F \otimes_{\mathbf{Q}_p} L_g$-linear map sending $1 \mapsto (n_Q)_{Q\in S}$.

[2]Notice the effect of the Tate twist on the filtrations.

Let $\varepsilon_g = \varepsilon_{g,p} \cdot \varepsilon_g^{(p)}$ be the nebentypus of g, decomposed as the product of its "wild" component $\varepsilon_{g,p}$ on $(\mathbf{Z}/p^s\mathbf{Z})^\times$ and its "tame" component $\varepsilon_g^{(p)}$ on $(\mathbf{Z}/N\mathbf{Z})^\times$. Let $g^* \in S_2(X_s)$ be the form *dual* to g, defined as the newform associated with the twist $g \otimes \varepsilon_{g,p}^{-1}$, and let $\omega_{g^*} \in H^0(X_s, \Omega^1_{X_s/F})$ be its associated differential, so that

$$\mathrm{Fil}^0 D_{\mathrm{dR}}(V_g^*) = (F \otimes_{\mathbf{Q}_p} L_g).\omega_{g^*}.$$

The image of the functional $\delta_{g,F}^{(p)}(\Delta)$ is thus determined by the value

$$\delta_{g,F}^{(p)}(\Delta)(\omega_{g^*}) = \langle t_{W_\Delta}, \omega_{g^*} \rangle_{\mathrm{dR}} \tag{1.1.20}$$

of the pairing (1.1.10), which corresponds to the Poincaré pairing on $H^1_{\mathrm{dR}}(X_s/F)$ under the identification (1.1.16). Using rigid analysis, we will now give an expression for the latter pairing that will make (1.1.20) amenable to computations.

Let $\mathcal{X}_s$ be the canonical balanced model of X_s over $\mathbf{Z}_p[\zeta_s]$ constructed by Katz and Mazur. The special fiber $\mathcal{X}_s \times_{\mathbf{Z}_p[\zeta_s]} \mathbf{F}_p$ is a reduced disjoint union of Igusa curves over $\mathbf{F}_p$ intersecting at the supersingular points, with exactly two of them isomorphic to the Igusa curve $\mathrm{Ig}(\Gamma_s)$ representing the moduli problem $([\Gamma_1(N)], [\Gamma_1(p^s)])$ over $\mathbf{F}_p$ (see [**KM85**, §13]); we let I_∞ be the one that contains the reduction of $\mathcal{W}_1(p^s) \times_{\mathbf{Q}_p} \mathbf{Q}_p(\zeta_s)$, and let I_0 be the other. (We note that these components are the two "good" components in the terminology of [**MW86**].)

By the universal property of the regular minimal model, there exists a morphism

$$\mathscr{X}_s \longrightarrow \mathcal{X}_s \times_{\mathbf{Z}_p[\zeta_s]} \mathcal{O}_F \tag{1.1.21}$$

which reduces to a sequence of blow-ups on the special fiber. Letting κ be the residue field of F, define $\mathcal{W}_\infty \subset X_s$ (resp. $\mathcal{W}_0 \subset X_s$) to be the inverse image under the reduction map via $\mathscr{X}_s$ of the unique irreducible component of $\mathscr{X}_s \times_{\mathcal{O}_F} \kappa$ mapping bijectively onto $I_\infty \times_{\mathbf{F}_p} \kappa$ (resp. $I_0 \times_{\mathbf{F}_p} \kappa$) in $\mathcal{X}_s \times_{\mathbf{Z}_p[\zeta_s]} \kappa$ via the reduction of (1.1.21). Similarly, define $\mathcal{U} \subset X_s$ by considering the irreducible components of $\mathcal{X}_s \times_{\mathbf{Z}_p[\zeta_s]} \kappa$ different from $I_\infty \times_{\mathbf{F}_p} \kappa$ and $I_0 \times_{\mathbf{F}_p} \kappa$. Letting SS denote (the degree of) the supersingular divisor of $\mathrm{Ig}(\Gamma_s)$, one can show that $\mathcal{U}$ intersects $\mathcal{W}_\infty$ (resp. $\mathcal{W}_0$) in a union of SS supersingular annuli.

Since they reduce to smooth points, the residue class D_Q of each $Q \in S$ is conformal to the open unit disc $D \subset \mathbf{C}_p$. Fix an isomorphism $h_Q : D_Q \xrightarrow{\sim} D$ that sends Q to 0, and for a real number $r_Q < 1$ in $p^{\mathbf{Q}}$, denote by $\mathcal{V}_Q \subset D_Q$ the annulus consisting of the points $x \in D_Q$ with $r_Q < |h_Q(x)|_p < 1$.

Attached to any (oriented) annulus $\mathcal{V}$, there is a *p-adic annular residue map*

$$\mathrm{Res}_{\mathcal{V}} : \Omega^1_{\mathcal{V}} \longrightarrow \mathbf{C}_p$$

defined by expanding $\omega \in \Omega^1_{\mathcal{V}}$ as $\omega = \sum_{n\in\mathbf{Z}} a_n T^n \frac{dT}{T}$ for a fixed uniformizing parameter T on $\mathcal{V}$ (compatible with the orientation), and setting $\mathrm{Res}_{\mathcal{V}}(\omega) = a_0$. This descends to a linear functional on $\Omega^1_{\mathcal{V}}/d\mathcal{O}_{\mathcal{V}}$. (See [**Col89**, Lemma 2.1].)

For any *basic wide-open* $\mathcal{W}$ (as in [**Buz03**, p. 34]), define

$$H^1_{\mathrm{rig}}(\mathcal{W}) := \mathbb{H}^1(\mathcal{W}, \Omega^\bullet(\log Z)) \cong \Omega^1_{\mathcal{W}}/d\mathcal{O}_{\mathcal{W}}, \tag{1.1.22}$$

where $\Omega^\bullet(\log Z)$ denotes the complex of rigid analytic sheaves on $\mathcal{W}$ deduced from (1.1.1) by analytification and pullback, and consider the basic wide-opens

$$\widetilde{\mathcal{W}}_\infty := \mathcal{W}_\infty \smallsetminus \bigcup_{Q\in S}(D_Q \smallsetminus \mathcal{V}_Q) \qquad \widetilde{\mathcal{W}}_0 := \mathcal{W}_0 \smallsetminus \bigcup_{Q\in S}(D_Q \smallsetminus \mathcal{V}_Q).$$

The space $H^1_{\mathrm{dR}}(X_s/F)$ is equipped with a natural action of the diamond operators, and following [**Col97a**, §2] we define $H^1_{\mathrm{dR}}(X_s/F)^{\mathrm{prim}}$ to be the subspace of $H^1_{\mathrm{dR}}(X_s/F)$ spanned by (the pullbacks of) the classes in $H^1_{\mathrm{dR}}(X_r/F)$, for $0 \le r \le s$, with primitive nebentypus at p. Also, we define $H^1_{\mathrm{dR}}(Y_s/F)^{\mathrm{prim}}$ to be the image of $H^1_{\mathrm{dR}}(X_s/F)^{\mathrm{prim}}$ under the natural restriction map $H^1_{\mathrm{dR}}(X_s/F) \longrightarrow H^1_{\mathrm{dR}}(Y_s/F)$. Finally, let $H^1_{\mathrm{rig}}(\widetilde{\mathcal{W}}_\infty)^*$ be the subspace of $H^1_{\mathrm{rig}}(\widetilde{\mathcal{W}}_\infty)$ consisting of classes ω with $\mathrm{res}_{\mathcal{V}_x}(\omega) = 0$ for all supersingular annuli $\mathcal{V}_x$ and $\mathrm{res}_{\mathcal{V}_Q}(\omega) = 0$ for all $Q \in S$, and define $H^1_{\mathrm{rig}}(\widetilde{\mathcal{W}}_0)^*$ in the analogous manner.

LEMMA 1.1.5 (Coleman). *The natural restriction maps induce an isomorphism*

$$H^1_{\mathrm{dR}}(Y_s/F)^{\mathrm{prim}} \xrightarrow{\sim} H^1_{\mathrm{rig}}(\widetilde{\mathcal{W}}_\infty)^* \oplus H^1_{\mathrm{rig}}(\widetilde{\mathcal{W}}_0)^*,$$

and if η and ω are any two classes in $H^1_{\mathrm{dR}}(X_s/F)^{\mathrm{prim}}$, their Poincaré pairing is given by

$$\langle \eta, \omega\rangle_{\mathrm{dR}} = \sum_{x\in S\cup SS} \mathrm{Res}_{\mathcal{V}_x}(F_{\omega_\infty|_{\mathcal{V}_x}} \cdot \eta_\infty|_{\mathcal{V}_x}) + \sum_{x\in S\cup SS} \mathrm{Res}_{\mathcal{V}_x}(F_{\omega_0|_{\mathcal{V}_x}} \cdot \eta_0|_{\mathcal{V}_x}), \tag{1.1.23}$$

where for each annulus $\mathcal{V}_x$, $F_{\omega_{\mathcal{V}_x}}$ denotes any solution to $dF_{\omega_{\mathcal{V}_x}} = \omega_{\mathcal{V}_x}$ on $\mathcal{V}_x$.

PROOF. By an excision argument, the first assertion follows from [**Col97a**, Thm. 2.1], and the second is shown by adapting the arguments in [**Col96**, §5] for each of the two components, as done in [**Col94a**, Prop. 1.3] for $s = 1$. (See also [**Col97a**, §3].) □

1.1.3. Coleman p-adic integration.

Coleman's theory of p-adic integration provides a coherent choice of local primitives that will allow us to we compute (1.1.20) using the formula (1.1.23). The key idea is to exploit the action of Frobenius.

Recall the lift of Frobenius $\phi : \mathcal{W}_2(p^s) \longrightarrow \mathcal{W}_1(p^s)$ described in Section 1.1.1, where $\mathcal{W}_i(p^s)$ are the strict neighborhoods of the connected component $X_s(0)$ of the ordinary locus of X_s containing the cusp ∞ described there. Recall also the wide open space $\mathcal{W}_\infty$ described in the preceding section, which also contains $X_s(0)$ by construction.

PROPOSITION 1.1.6 (Coleman). *Let $g = \sum_{n>0} b_n q^n \in S_2(X_s)$ be a normalized newform with primitive nebentypus of p-power conductor, so that b_p is such that $U_p g = b_p g$. There exists a unique locally analytic function F_{ω_g} on $\mathcal{W}_\infty$ with the following three properties:*

- *$dF_{\omega_g} = \omega_g$ on $\mathcal{W}_\infty$,*
- *$F_{\omega_g} - \frac{b_p}{p}\phi^* F_{\omega_g} \in M_0^{\mathrm{rig}}(X_s)$, and*
- *F_{ω_g} vanishes at ∞.*

PROOF. This follows from the general result of Coleman [**Col94b**, Thm. 10.1]. Indeed, a computation on q-expansions shows that the action of the Frobenius lift ϕ on differentials agrees with that of pV, with V the map acting as $q \mapsto q^p$ on q-expansions, in the sense that $\phi^*\omega_g = p\omega_{Vg}$ on $\mathcal{W}'_\infty := \phi^{-1}(\mathcal{W}_\infty \cap \mathcal{W}_1(p^s))$. Since the differential $\omega_{g^{[p]}} = \omega_g - b_p\omega_{Vg}$ attached to

$$g^{[p]} = \sum_{(n,p)=1} b_n q^n$$

becomes exact upon restriction to $\mathcal{W}'_\infty$, this shows that the polynomial $L(T) = 1 - \frac{b_p}{p}T$ is such that $L(\phi^*)\omega_g = 0$. Finally, since g has primitive nebentypus, b_p has complex absolute value $p^{1/2}$, and hence [**Col94b**, Thm. 10.1] can be applied with $L(T)$ as above. □

Attached to a primitive p^s-th root of unity ζ, there is an automorphism w_ζ of X_s which interchanges the components $\mathcal{W}_\infty$ and $\mathcal{W}_0$. (See [**BE10**, Lemma 4.4.3].)

COROLLARY 1.1.7. *Set $\phi' := w_\zeta \circ \phi \circ w_\zeta$. With hypotheses as in Proposition 1.1.6, there exists a unique locally analytic function F'_{ω_g} on $\mathcal{W}_0$ with the following three properties:*

- *$dF'_{\omega_g} = \omega_g$ on $\mathcal{W}_0$,*
- *$F'_{\omega_g} - \frac{b_p}{p}(\phi')^* F'_{\omega_g}$ is rigid analytic on a wide-open neighborhood $\mathcal{W}'_0$ of $w_\zeta X_s(0)$ in $\mathcal{W}_0$, and*
- *F'_{ω_g} vanishes at 0.*

PROOF. Proposition 1.1.6 applied to the differential $\omega'_g := w_\zeta^*\omega_g$ gives the existence of a locally analytic function $F_{\omega'_g}$ with $F'_{\omega_g} := w_\zeta^* F_{\omega'_g}$ having the desired properties. The uniqueness of F'_{ω_g} follows immediately from that of $F_{\omega'_g}$. □

We refer to the locally analytic function F_{ω_g} (resp. F'_{ω_g}) appearing in Proposition 1.1.6 as the *Coleman primitive* of g on $\mathcal{W}_\infty$ (resp. $\mathcal{W}_0$).

Let $g = \sum_{n>0} b_n q^n$ be as in Proposition 1.1.6. The q-expansion $\sum_{(n,p)=1} \frac{b_n}{n} q^n$ corresponds to a p-adic modular form g' vanishing at ∞ and satisfying $dg' = g^{[p]}$, where d is the operator described at the end of Section 2.1, which here corresponds to the differential operator $\mathcal{O}_{\mathcal{W}} \longrightarrow \Omega^1_{\mathcal{W}}$ for any subspace $\mathcal{W} \subset X_s$. Set $d^{-1}g^{[p]} := g'$.

Corollary 1.1.8. *If F_{ω_g} is the Coleman primitive of g on $\mathcal{W}_\infty$, then*

$$F_{\omega_g} - \frac{b_p}{p}\phi^* F_{\omega_g} = d^{-1}g^{[p]}.$$

Proof. Since $d^{-1}g^{[p]}$ is an overconvergent rigid analytic primitive of $\omega_{g^{[p]}}$, and the operator $L(\phi^*) = 1 - \frac{b_p}{p}\phi^*$ acting on the space of locally analytic functions on $\mathcal{W}_\infty$ is invertible, we see that $L(\phi^*)^{-1}(d^{-1}g^{[p]})$ satisfies the defining properties of F_{ω_g}. Since $d^{-1}g^{[p]}$ vanishes at ∞, the result follows. □

We can now give an explicit formula for the p-adic Abel–Jacobi images of certain degree 0 divisors on X_s. Note that this formula it is key in all what follows.

Proposition 1.1.9. *Assume that $s > 1$. Let $g \in S_2(X_s)$ be a normalized newform with primitive nebentypus of p-power conductor, let P be an F-rational point of X_s factoring through $X_s(0) \subset X_s$, and let $\Delta \in J_s(F)$ be the divisor class of $(P) - (\infty)$. Then*

$$\delta^{(p)}_{g,F}(\Delta)(\omega_{g^*}) = F_{\omega_{g^*}}(P), \tag{1.1.24}$$

where $F_{\omega_{g^*}}$ is the Coleman primitive of ω_{g^*} on $\mathcal{W}_\infty$.

Proof. By (1.1.20), we must evaluate $\langle t_{W_\Delta}, \omega_{g^*}\rangle_{\mathrm{dR}}$, where (with a slight abuse of notation) $t_{W_\Delta} = s^{\mathrm{fil}}_{W_\Delta} - s^{\mathrm{frob}}_{W_\Delta}$ with

- $s^{\mathrm{fil}}_{W_\Delta} \in \mathrm{Fil}^1 D_{\mathrm{dR}}(W_\Delta)$ is such that $\rho(s^{\mathrm{fil}}_{W_\Delta}) = 1$ in (1.1.19), and
- $s^{\mathrm{frob}}_{W_\Delta} \in D_{\mathrm{cris}}(W_\Delta)^{\varphi=1}$ is such that $\rho(s^{\mathrm{frob}}_{W_\Delta}) = 1$ in (1.1.18).

By Lemma 1.1.5, we see that these can be represented, respectively, by

- $\eta^{\mathrm{fil}}_\Delta$ a section of $\Omega^1_{X_s/F}$ over Y_s with simple poles at P and ∞ and with
  - $\mathrm{Res}_P(\eta^{\mathrm{fil}}_\Delta) = 1$,
  - $\mathrm{Res}_\infty(\eta^{\mathrm{fil}}_\Delta) = -1$,
  - $\mathrm{Res}_Q(\eta^{\mathrm{fil}}_\Delta) = 0$ for all $Q \in S - \{P, \infty\}$;
- $\eta^{\mathrm{frob}}_\Delta = (\eta^{\mathrm{frob}}_\infty, \eta^{\mathrm{frob}}_0) \in \Omega^1_{\widetilde{\mathcal{W}}_\infty} \times \Omega^1_{\widetilde{\mathcal{W}}_0}$ with
  - $(\phi^*\eta^{\mathrm{frob}}_\infty, (\phi')^*\eta^{\mathrm{frob}}_\Delta) = (p \cdot \eta^{\mathrm{frob}}_\infty + dG_\infty, p \cdot \eta^{\mathrm{frob}}_0 + dG_0)$ with G_∞ and G_0 rigid analytic on $\phi^{-1}\widetilde{\mathcal{W}}_\infty$ and $(\phi')^{-1}\widetilde{\mathcal{W}}_0$, respectively,
  - $\mathrm{Res}_{\mathcal{V}_x}(\eta^{\mathrm{frob}}_\Delta) = 0$ for all supersingular annuli $\mathcal{V}_x$, and
  - $\mathrm{Res}_{\mathcal{V}_Q}(\eta^{\mathrm{frob}}_\Delta) = \mathrm{Res}_Q(\eta^{\mathrm{fil}}_\Delta)$ for all $Q \in S$.

The arguments in [**BDP13**, Prop. 3.21] can now be straightforwardly adapted to deduce the result. Indeed, using the defining properties of the Coleman primitives $F_{\omega_{g^*}}$ and $F'_{\omega_{g^*}}$ of ω_{g^*} on $\mathcal{W}_\infty$ and $\mathcal{W}_0$, respectively, one first shows that

$$\sum_{x \in S \cup SS} \mathrm{Res}_{\mathcal{V}_x}(F_{\omega_{g^*}} \cdot \eta_\infty^{\mathrm{frob}}) = 0 \quad \text{and} \quad \sum_{x \in S \cup SS} \mathrm{Res}_{\mathcal{V}_x}(F'_{\omega_{g^*}} \cdot \eta_0^{\mathrm{frob}}) = 0 \tag{1.1.25}$$

as in [*loc.cit.*, Lemma 3.20]. On the other hand, using the same primitives, one shows as in [*loc.cit.*, Lemma 3.19] that

$$\sum_{x \in S \cup SS} \mathrm{Res}_{\mathcal{V}}(F_{\omega_{g^*}} \cdot \eta_\Delta^{\mathrm{fil}}) = F_{\omega_{g^*}}(P) \quad \text{and} \quad \sum_{x \in S \cup SS} \mathrm{Res}_{\mathcal{V}}(F'_{\omega_{g^*}} \cdot \eta_\Delta^{\mathrm{fil}}) = 0. \tag{1.1.26}$$

Substituting (1.1.26) and (1.1.25) into the formula (1.1.23) for the Poincaré pairing (and using that $s > 1$, so that there is no overlap between the supersingular annuli in $\widetilde{\mathcal{W}}_\infty$ and the supersingular annuli in $\widetilde{\mathcal{W}}_0$), the result follows. □

1.2. Generalised Heegner cycles

Let $X_1(N)$ be the compactified modular curve of level $\Gamma_1(N)$ defined over $\mathbf{Q}$, and let $\mathcal{E}$ be the universal generalized elliptic curve over $X_1(N)$. (Recall that $N > 4$.) For $r > 1$, denote by W_r the $(2r-1)$-dimensional Kuga–Sato variety[3], defined as the canonical desingularization of the $(2r-2)$-nd fiber product of $\mathcal{E}$ with itself over $X_1(N)$. By construction, the variety W_r is equipped with a proper morphism

$$\pi_r : W_r \longrightarrow X_1(N)$$

whose fibers over a noncuspidal closed point of $X_1(N)$ corresponding to an elliptic curve E with $\Gamma_1(N)$-level structure is identified with $2r-2$ copies of E. (For a more detailed description, see [**BDP13**, §2.1].)

Let K be an imaginary quadratic field of odd discriminant $-D < 0$. It will be assumed throughout that K satisfies the following hypothesis:

Assumptions 1.2.1. *All the prime factors of N split in K.*

Denote by $\mathcal{O}_K$ the ring of integers of K, and note that by this assumption we may choose an ideal $\mathfrak{N} \subset \mathcal{O}_K$ with

$$\mathcal{O}_K/\mathfrak{N} \cong \mathbf{Z}/N\mathbf{Z}$$

that we now fix once and for all.

[3]Perhaps most commonly denoted by W_{2r-2}; cf. [**Zha97**] and [**Nek95**], for example.

Let A be a fixed elliptic curve with CM by $\mathcal{O}_K$. The pair $(A, A[\mathfrak{N}])$ defines a point P_A on $X_0(N)$ rational over H, the Hilbert class field of K. Choose one of the square-roots $\sqrt{-D} \in \mathcal{O}_K$, let $\Gamma_{\sqrt{-D}} \subset A \times A$ be the graph of $\sqrt{-D}$, and define

$$\Upsilon^{\mathrm{heeg}}_{A,r} := \Gamma_{\sqrt{-D}} \times \overset{(r-1)}{\cdots} \times \Gamma_{\sqrt{-D}}$$

viewed inside W_r by the natural inclusion $(A \times A)^{r-1} \longrightarrow W_r$ as the fiber of π_r over a point on $X_1(N)$ lifting P_A. Let ϵ_W be the projector from [**BDP13**, (2.1.2)], and set

$$\Delta^{\mathrm{heeg}}_{A,r} := \epsilon_W \Upsilon^{\mathrm{heeg}}_{A,r}, \tag{1.2.1}$$

which is an $(r-1)$-dimensional null-homologous cycle on W_r defining an H-rational class in the Chow group $\mathrm{CH}^r(W_r)_0$ (taken with $\mathbf{Q}$-coefficients, as always here) which is independent of the chosen lift of P_A.

These cycles (1.2.1) are the so-called *Heegner cycles* (of conductor one, weight $2r$), and they share with classical Heegner points many of their arithmetic properties (see [**Nek92, Nek95, Zha97**]).

We next recall a variation of the previous construction introduced in the recent work [**BDP13**] of Bertolini, Darmon, and Prasanna. Let A be the CM elliptic curve fixed above, and consider the variety[4]

$$X_r := W_r \times A^{2r-2}.$$

For each class $[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)$, represented by an ideal $\mathfrak{a} \subset \mathcal{O}_K$ prime to N, let $A_{\mathfrak{a}} := A/A[\mathfrak{a}]$ and denote by $\varphi_{\mathfrak{a}}$ the degree $\mathrm{N}\mathfrak{a}$-isogeny

$$\varphi_{\mathfrak{a}} : A \longrightarrow A_{\mathfrak{a}}.$$

The pair $\mathfrak{a} * (A, A[\mathfrak{N}]) := (A_{\mathfrak{a}}, A_{\mathfrak{a}}[\mathfrak{N}])$ defines a point $P_{A_{\mathfrak{a}}}$ in $X_0(N)$ rational over H. Let $\Gamma^t_{\varphi_{\mathfrak{a}}} \subset A_{\mathfrak{a}} \times A$ be the transpose of the graph of $\varphi_{\mathfrak{a}}$, and set

$$\Upsilon^{\mathrm{bdp}}_{\varphi_{\mathfrak{a}},r} := \Gamma^t_{\varphi_{\mathfrak{a}}} \times \overset{(2r-2)}{\cdots} \times \Gamma^t_{\varphi_{\mathfrak{a}}} \;\subset\; (A_{\mathfrak{a}} \times A)^{2r-2} = A_{\mathfrak{a}}^{2r-2} \times A^{2r-2} \xrightarrow{(\imath_{\mathfrak{a}}, \mathrm{id}_A)} X_r,$$

where $\imath_{\mathfrak{a}}$ is the natural inclusion $A_{\mathfrak{a}}^{2r-2} \longrightarrow W_r$ as the fiber of π_r over a point on $X_1(N)$ lifting $P_{A_{\mathfrak{a}}}$. Letting ϵ_A be the projector from [**BDP13**, (1.4.4)], the cycles

$$\Delta^{\mathrm{bdp}}_{\varphi_{\mathfrak{a}},r} := \epsilon_A \epsilon_W \Upsilon^{\mathrm{bdp}}_{\varphi_{\mathfrak{a}},r} \tag{1.2.2}$$

define classes in $\mathrm{CH}^{2r-1}(X_r)_0(H)$ and are referred to as *generalised Heegner cycles*.

We will assume for the rest of this paper that K also satisfies the following:

[4]Notice that our indices differ from those in [**BDP13**].

Assumptions 1.2.2. *The prime p splits in K.*

Let $g \in S_{2r}(X_0(N))$ be a normalized newform, and let V_g be the p-adic Galois representation associated to g by Deligne. By the Künneth formula, there is a map

$$H^{4r-3}_{\text{ét}}(\overline{X}_r, \mathbf{Q}_p(2r-1)) \longrightarrow H^{2r-1}_{\text{ét}}(\overline{W}_r, \mathbf{Q}_p(1)) \otimes \mathrm{Sym}^{2r-2} H^1_{\text{ét}}(\overline{A}, \mathbf{Q}_p(1)),$$

which composed with the natural Galois-equivariant projection

$$H^{2r-1}_{\text{ét}}(\overline{W}_r, \mathbf{Q}_p(1)) \otimes \mathrm{Sym}^{2r-2} H^1_{\text{ét}}(\overline{A}, \mathbf{Q}_p(1)) \xrightarrow{\pi_g \otimes \pi_{\mathrm{N}^{r-1}}} V_g(r)$$

induces a map

$$\pi_{g,\mathrm{N}^{r-1}} : H^1(F, H^{4r-3}_{\text{ét}}(\overline{X}_r, \mathbf{Q}_p(2r-1))) \longrightarrow H^1(F, V_g(r))$$

over any number field F. In the following we fix a number field F containing H.

Now consider the étale Abel–Jacobi map

$$\Phi^{\text{ét}}_F : \mathrm{CH}^{2r-1}(X_r)_0(F) \longrightarrow H^1(F, H^{4r-3}_{\text{ét}}(\overline{X}_r, \mathbf{Q}_p)(2r-1))$$

constructed in [**Nek00**]. Let F_p be the completion of $\imath_p(F)$, and denote by loc_p the induced localization map from G_F to $\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_p)$. Then we may define the *p-adic Abel–Jacobi map* AJ_{F_p} by the commutativity of the diagram

$$(1.2.3)\qquad \begin{array}{ccccc} \mathrm{CH}^{2r-1}(X_r)_0(F) & \xrightarrow{\pi_{g,\mathrm{N}^{r-1}} \circ \Phi^{\text{ét}}_F} & H^1(F, V_g(r)) & \xrightarrow{\mathrm{loc}_p} & H^1(F_p, V_g(r)) \\ & \searrow^{\cdots} & & & \cup \\ & & & & H^1_f(F_p, V_g(r)) \\ & & \searrow^{\mathrm{AJ}_{F_p}} & & \downarrow {\scriptstyle \log_{F_p, V_g(r)}} \\ & & & & \mathrm{Fil}^1 D_{\mathrm{dR}}(V_g(r-1))^{\vee}, \end{array}$$

where the existence of the dotted arrow follows from [**Nek00**, Thm.(3.1)(i)], and the vertical map is given by the logarithm map of Bloch–Kato of $V_g(r)$ as G_{F_p}-representation, similarly as it appeared in (1.1.9) for $r = 1$. Using the comparison isomorphism of Faltings [**Fal89**], the map AJ_{F_p} may be evaluated at the class $\omega_g \otimes e_\zeta^{\otimes r-1}$, with e_ζ an F_p-basis of $D_{\mathrm{dR}}(\mathbf{Q}_p(1)) \cong F_p$.

The main result of [**BDP13**] yields the following formula for the p-adic Abel–Jacobi images of the generalised Heegner cycles (1.2.2) which we will need.

THEOREM 1.2.3 (Bertolini–Darmon–Prasanna). *Let $g = \sum_n b_n q^n \in S_{2r}(X_0(N))$ be a normalized newform of weight $2r \geq 2$ and level N prime to p. Then*

$$(1 - b_p p^{-r} + p^{-1}) \sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)} \mathrm{N}\mathfrak{a}^{1-r} \cdot \mathrm{AJ}_{F_p}(\Delta^{\mathrm{bdp}}_{\varphi_{\mathfrak{a}},r})(\omega_g \otimes e_\zeta^{\otimes r-1})$$
$$= (-1)^{r-1}(r-1)! \sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)} d^{-r} g^{[p]}(\mathfrak{a} * (A, A[\mathfrak{N}])),$$

where $g^{[p]} = \sum_{(n,p)=1} b_n q^n$ is the p-depletion of g.

PROOF. See the proof of [**BDP13**, Thm. 5.13]. □

We end this section by relating the images of Heegner cycles and of generalised Heegner cycles under the p-adic height pairing. (Cf. [**BDP13**, §2.4].) Consider $\Pi_r := W_r \times A^{r-1}$ seen as a subvariety of $W_r \times X_r = W_r \times W_r \times (A^2)^{r-1}$ via the map

$$(\mathrm{id}_{W_r}, \mathrm{id}_{W_r}, (\mathrm{id}_A, \sqrt{-D})^{r-1}).$$

Denoting by π_W and π_X the projections onto the first and second factors of $W_r \times X_r$, the rational equivalence class of the cycle Π_r gives rise to a map on Chow groups

$$\Pi_r : \mathrm{CH}^{2r-1}(X_r) \longrightarrow \mathrm{CH}^{r+1}(W_r)$$

induced by $\Pi_r(\Delta) = \pi_{W,*}(\Pi_r \cdot \pi_X^* \Delta)$.

LEMMA 1.2.4. *We have*

$$\langle \Delta^{\mathrm{heeg}}_{A,r}, \Delta^{\mathrm{heeg}}_{A,r} \rangle_{W_r} = (4D)^{r-1} \cdot \langle \Delta^{\mathrm{bdp}}_{\mathrm{id}_A,r}, \Delta^{\mathrm{bdp}}_{\mathrm{id}_A,r} \rangle_{X_r},$$

where $\langle , \rangle_{W_r}$ and $\langle , \rangle_{X_r}$ are the p-adic height pairings of [**Nek93**] *on $\mathrm{CH}^{r+1}(W_r)_0$ and $\mathrm{CH}^{2r-1}(X_r)_0$, respectively.*

PROOF. The image $\Phi^{\text{ét}}_F(\Delta^{\mathrm{heeg}}_{A,r})$ remains unchanged if we replace the cycle $\Gamma_{\sqrt{-D}}$ by the modification

$$Z_A := \Gamma_{\sqrt{-D}} - (A \times \{0\}) - D(\{0\} \times A)$$

(see [**Nek95**, §II(3.6)]). Since clearly $Z_A \cdot Z_A = -2D$, we thus see from the construction of Π_r that

$$\Phi^{\text{ét}}_F(\Delta^{\mathrm{heeg}}_{A,r}) = (-2D)^{r-1} \cdot \Phi^{\text{ét}}_F(\Pi_r(\Delta^{\mathrm{bdp}}_{\mathrm{id}_A,r})). \tag{1.2.4}$$

On the other hand, if $\langle , \rangle_A$ denotes the Poincaré pairing on $H^1_{\mathrm{dR}}(A/F)$, we have

$$\langle (\sqrt{-D})^* \omega, (\sqrt{-D})^* \omega' \rangle_A = D \cdot \langle \omega, \omega' \rangle_A,$$

for all $\omega, \omega' \in H^1_{\rm dR}(A/F)$. By the definition of the p-adic height pairings $\langle , \rangle_{W_r}$ and $\langle , \rangle_{X_r}$ (factoring through $\Phi^{\text{ét}}_F$), we thus see that

$$\langle \Delta^{\rm bdp}_{{\rm id}_A,r}, \Delta^{\rm bdp}_{{\rm id}_A,r} \rangle_{X_r} = D^{r-1} \cdot \langle \Pi_r(\Delta^{\rm bdp}_{{\rm id}_A,r}), \Pi_r(\Delta^{\rm bdp}_{{\rm id}_A,r}) \rangle_{W_r}. \tag{1.2.5}$$

Combining (1.2.4) and (1.2.5), the result follows. □

1.3. The big logarithm map

Let

$$\mathbf{f} = \sum_{n>0} \mathbf{a}_n q^n \in \mathbb{I}[[q]]$$

be a Hida family passing through (the ordinary p-stabilization of) a p-ordinary newform $f_o \in S_k(X_0(N))$ as described in the Introduction. We begin this section by recalling the definition of a certain twist of $\mathbf{f}$ such that all of its specializations at arithmetic primes of *even weight* correspond to p-adic modular forms with trivial weight-nebentypus.

Decompose the p-adic cyclotomic character $\varepsilon_{\rm cyc}$ as the product

$$\varepsilon_{\rm cyc} = \omega \cdot \epsilon : G_{\mathbf{Q}} \longrightarrow \mathbf{Z}_p^\times = \boldsymbol{\mu}_{p-1} \times \Gamma.$$

Since k is even, the character ω^{k-2} admits a square root $\omega^{\frac{k-2}{2}} : G_{\mathbf{Q}} \longrightarrow \boldsymbol{\mu}_{p-1}$, and in fact two different square roots, corresponding to the two different lifts of $k-2 \in \mathbf{Z}/(p-1)\mathbf{Z}$ to $\mathbf{Z}/2(p-1)\mathbf{Z}$. Fix for now a choice of $\omega^{\frac{k-2}{2}}$, and define the *critical character* to be

$$\Theta := \omega^{\frac{k-2}{2}} \cdot [\epsilon^{1/2}] : G_{\mathbf{Q}} \longrightarrow \Lambda_{\mathcal{O}}^\times, \tag{1.3.1}$$

where $\epsilon^{1/2} : G_{\mathbf{Q}} \longrightarrow \Gamma$ denotes the unique square root of ϵ taking values in Γ.

Remark 1.3.1. As noted in [**How07b**, Rem. 2.1.4], the above choice of Θ is for most purposes largely indistinguishable from the other choice, namely $\omega^{\frac{p-1}{2}}\Theta$, where

$$\omega^{\frac{p-1}{2}} : \mathrm{Gal}(\mathbf{Q}(\sqrt{p^*})/\mathbf{Q}) \xrightarrow{\sim} \{\pm 1\} \qquad (p^* = (-1)^{\frac{p-1}{2}} p).$$

Nonetheless, for a given f_o as above, our main result (Theorem 1.4.12) will specifically apply *to only one* of the two possible choices for the critical character.

The *critical twist* of $\mathbb{T}$ is then defined to be the module

$$\mathbb{T}^\dagger := \mathbb{T} \otimes_{\mathbb{I}} \mathbb{I}^\dagger \tag{1.3.2}$$

equipped with the diagonal $G_{\mathbf{Q}}$-action, where $\mathbb{I}^\dagger = \mathbb{I}(\Theta^{-1})$ is the free $\mathbb{I}$-module of rank one equipped with the $G_{\mathbf{Q}}$-action via the character $G_{\mathbf{Q}} \xrightarrow{\Theta^{-1}} \Lambda_{\mathcal{O}}^\times \longrightarrow \mathbb{I}^\times$.

LEMMA 1.3.2. *Let $\rho_{\mathbb{T}^\dagger} : G_{\mathbf{Q}} \longrightarrow \mathrm{Aut}_{\mathbb{I}}(\mathbb{T}^\dagger)$ be the Galois representation carried by $\mathbb{T}^\dagger$. Then for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of even weight $k_\nu = 2r_\nu \geq 2$ we have*

$$\nu(\rho_{\mathbb{T}^\dagger}) \cong \rho_{\mathbf{f}'_\nu} \otimes \varepsilon_{\mathrm{cyc}}^{r_\nu},$$

where $\mathbf{f}'_\nu$ is a character twist of $\mathbf{f}_\nu$ of weight k_ν and with trivial nebentypus. In other words, defining $\mathbb{V}^\dagger_\nu := \mathbb{T}^\dagger \otimes_{\mathbb{I}} F_\nu$ and letting $V_{\mathbf{f}'_\nu}$ be the representation space of $\rho_{\mathbf{f}'_\nu}$, we have

$$\mathbb{V}^\dagger_\nu \cong V_{\mathbf{f}'_\nu}(r_\nu), \tag{1.3.3}$$

and in particular $\mathbb{V}^\dagger_\nu$ is isomorphic to its Kummer dual.

PROOF. This follows from a straightforward computation explained in [**NP00**, (3.5.2)] for example (where $\mathbb{T}^\dagger$ is denoted by T). □

Let $\theta : \mathbf{Z}_p^\times \longrightarrow \Lambda_{\mathcal{O}}^\times$ be such that $\Theta = \theta \circ \varepsilon_{\mathrm{cyc}}$. It follows from the preceding lemma that the formal q-expansion

$$\mathbf{f}^\dagger = \mathbf{f} \otimes \theta^{-1} := \sum_{n>0} \theta^{-1}(n)\mathbf{a}_n q^n \in \mathbb{I}[[q]]$$

(where we put $\theta^{-1}(n) = 0$ whenever p divides n) is such that, for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of even weight, $\mathbb{V}^\dagger_\nu$ is the Galois representation attached to the specialization $\mathbf{f}_\nu \otimes \theta_\nu^{-1}$ of $\mathbf{f}^\dagger$, which by Lemma 1.1.3 is a p-adic modular form of weight 0 and trivial nebentypus.

We next recall some of the local properties of the big Galois representation $\mathbb{T}$. Let $I_w \subset D_w \subset G_{\mathbf{Q}}$ be the inertia and decompositon groups at the place w of $\overline{\mathbf{Q}}$ above p induced by our fixed embedding $\imath_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. In the following we will identify D_w with the absolute Galois group $G_{\mathbf{Q}_p}$. Then by a result of Mazur and Wiles (see [**Wil88**, Thm. 2.2.2]) there exists a filtration of $\mathbb{I}[D_w]$-modules

$$0 \longrightarrow \mathscr{F}_w^+\mathbb{T} \longrightarrow \mathbb{T} \longrightarrow \mathscr{F}_w^-\mathbb{T} \longrightarrow 0 \tag{1.3.4}$$

with $\mathscr{F}_w^\pm\mathbb{T}$ free of rank one over $\mathbb{I}$ and with the Galois action on $\mathscr{F}_w^-\mathbb{T}$ unramified, given by the character $\alpha : D_w/I_w \longrightarrow \mathbb{I}^\times$ sending an arithmetic Frobenius σ_p to $\mathbf{a}_p$. Twisting (2.2.1) by Θ^{-1} we define $\mathscr{F}_w^\pm\mathbb{T}^\dagger$ in the natural manner.

Let $\mathbb{T}^* := \mathrm{Hom}_{\mathbb{I}}(\mathbb{T}, \mathbb{I})$ be the contragredient[5] of $\mathbb{T}$, and consider the $\mathbb{I}$-module

$$\mathbb{D} := (\mathscr{F}_w^+\mathbb{T}^* \widehat{\otimes}_{\mathbf{Z}_p} \widehat{\mathbf{Z}}_p^{\mathrm{nr}})^{G_{\mathbf{Q}_p}}, \tag{1.3.5}$$

where $\mathscr{F}_w^+\mathbb{T}^* := \mathrm{Hom}_{\mathbb{I}}(\mathscr{F}_w^-\mathbb{T}, \mathbb{I}) \subset \mathbb{T}^*$, and $\widehat{\mathbf{Z}}_p^{\mathrm{nr}}$ is the completion of the ring of integers of the maximal unramified extension of $\mathbf{Q}_p$ in $\overline{\mathbf{Q}}_p$.

[5] So that $\mathbb{T}^* \otimes_{\mathbb{I}} F_\nu \cong V_{\mathbf{f}_\nu}$ for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$.

Fix a compatible system $\zeta = (\zeta_s)_{s\geq 0}$ of primitive p^s-th roots of unity $\zeta_s \in \overline{\mathbf{Q}}_p$, and let e_ζ be the basis of $D_{\mathrm{dR}}(\mathbf{Q}_p(1))$ corresponding to $1 \in \mathbf{Q}_p$ under the resulting identification $D_{\mathrm{dR}}(\mathbf{Q}_p(1)) = \mathbf{Q}_p$.

LEMMA 1.3.3. *The module $\mathbb{D}$ is free of rank one over $\mathbb{I}$, and for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of even weight $k_\nu = 2r_\nu \geq 2$ there is a canonical isomorphism*

$$\mathbb{D}_\nu \otimes D_{\mathrm{dR}}(\mathbf{Q}_p(r_\nu)) \xrightarrow{\sim} \frac{D_{\mathrm{dR}}(V_{\mathbf{f}_\nu}(r_\nu))}{\mathrm{Fil}^0 D_{\mathrm{dR}}(V_{\mathbf{f}_\nu}(r_\nu))}. \tag{1.3.6}$$

PROOF. Since the action on $\mathscr{F}_w^+\mathbb{T}^*$ is unramified, the first claim follows from [**Och03**, Lemma 3.3] in light of the definition (1.3.5) of $\mathbb{D}$. The second can be deduced from [**Och03**, Lemma 3.2] as in the proof of [**Och03**, Lemma 3.6]. □

With the same notations as in Lemma 1.3.3, we denote by $\langle\ ,\ \rangle_{\mathrm{dR}}$ the pairing

$$\langle\ ,\ \rangle_{\mathrm{dR}} : \mathbb{D}_\nu \otimes D_{\mathrm{dR}}(\mathbf{Q}_p(r_\nu)) \times \mathrm{Fil}^1 D_{\mathrm{dR}}(V_{\mathbf{f}_\nu^*}(r_\nu - 1)) \longrightarrow F_\nu \tag{1.3.7}$$

deduced from the usual de Rham pairing

$$\frac{D_{\mathrm{dR}}(V_{\mathbf{f}_\nu}(r_\nu))}{\mathrm{Fil}^0 D_{\mathrm{dR}}(V_{\mathbf{f}_\nu}(r_\nu))} \times \mathrm{Fil}^0 D_{\mathrm{dR}}(V_{\mathbf{f}_\nu}^*(1 - r_\nu)) \longrightarrow F_\nu$$

via the identification (1.3.6) and the isomorphism $V_{\mathbf{f}_\nu}^* \cong V_{\mathbf{f}_\nu^*}(k_\nu - 1)$.

THEOREM 1.3.4 (Ochiai). *Assume that the residual representation $\bar{\rho}_{f_o}$ is irreducible, fix an $\mathbb{I}$-basis η of $\mathbb{D}$, and set $\lambda := \mathbf{a}_p - 1$. There exists an $\mathbb{I}$-linear map*

$$\mathrm{Log}^\eta_{\mathbb{T}^\dagger} : H^1(\mathbf{Q}_p, \mathscr{F}_w^+\mathbb{T}^\dagger) \longrightarrow \mathbb{I}[\lambda^{-1}]$$

such that for every $\mathfrak{Y} \in H^1(\mathbf{Q}_p, \mathscr{F}_w^+\mathbb{T}^\dagger)$ and every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of even weight $k_\nu = 2r_\nu \geq 2$ with $k_\nu \equiv k \pmod{2(p-1)}$, we have

(1.3.8)

$$\nu(\mathrm{Log}^\eta_{\mathbb{T}^\dagger}(\mathfrak{Y})) = \frac{(-1)^{r_\nu - 1}}{(r_\nu - 1)!} \times \begin{cases} \left(1 - \frac{p^{r_\nu - 1}}{\nu(\mathbf{a}_p)}\right)^{-1} \left(1 - \frac{\nu(\mathbf{a}_p)}{p^{r_\nu}}\right) \langle \log_{V^\dagger_{\mathbf{f}_\nu}}(\mathfrak{Y}_\nu), \eta'_\nu\rangle_{\mathrm{dR}} & \text{if } \vartheta_\nu = \mathbb{1}; \\ G(\vartheta_\nu^{-1})^{-1} \left(\frac{\nu(\mathbf{a}_p)}{p^{r_\nu - 1}}\right)^{s_\nu} \langle \log_{V^\dagger_{\mathbf{f}_\nu}}(\mathfrak{Y}_\nu), \eta'_\nu\rangle_{\mathrm{dR}} & \text{if } \vartheta_\nu \neq \mathbb{1}, \end{cases}$$

where

- $\log_{V^\dagger_{\mathbf{f}_\nu}}$ *is the Bloch–Kato logarithm map for the representation $V^\dagger_{\mathbf{f}_\nu}$ over $\mathbf{Q}_p$,*
- $\eta'_\nu \in \mathrm{Fil}^1 D_{\mathrm{dR}}(V_{\mathbf{f}_\nu^*}(r_\nu - 1))$ *is such that $\langle \eta_\nu \otimes e_\zeta^{\otimes r_\nu}, \eta'_\nu\rangle_{\mathrm{dR}} = 1$ under (1.3.7),*
- $\vartheta_\nu : \mathbf{Z}_p^\times \longrightarrow F_\nu^\times$ *is the finite order character $z \longmapsto \theta_\nu(z) z^{1 - r_\nu}$,*
- $s_\nu > 0$ *is such that the conductor of ϑ_ν is p^{s_ν}, and*

- $G(\vartheta_\nu^{-1})$ *is the Gauss sum* $\sum_{x \bmod p^{s_\nu}} \vartheta_\nu^{-1}(x)\zeta_{s_\nu}^x$.

PROOF. Let $\Lambda(C_\infty) := \mathbf{Z}_p[[C_\infty]]$, where C_∞ is the Galois group of the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}_p$, and let Λ_{cyc} be the module $\Lambda(C_\infty)$ equipped with the natural action of $G_{\mathbf{Q}_p}$ on group-like elements. Also, let γ_o be a topological generator of C_∞ and define

$$\mathcal{I} := (\lambda, \gamma_o - 1),$$

seen as an ideal of height 2 inside $\mathbb{I}\widehat{\otimes}_{\mathbf{Z}_p}\Lambda(C_\infty) \cong \mathbb{I}[[C_\infty]]$.

Consider the $\mathbb{I}\widehat{\otimes}_{\mathbf{Z}_p}\Lambda(C_\infty)$-modules

$$\mathcal{D} := \mathbb{D}\widehat{\otimes}_{\mathbf{Z}_p}\Lambda(C_\infty), \qquad \mathscr{F}_w^+\mathcal{T}^* := \mathscr{F}_w^+\mathbb{T}^*\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\mathrm{cyc}} \otimes \omega^{\frac{k-2}{2}},$$

the latter being equipped with the diagonal action of $G_{\mathbf{Q}_p}$. By [**Och03**, Prop. 5.3] there exists an *injective* $\mathbb{I}\widehat{\otimes}_{\mathbf{Z}_p}\Lambda(C_\infty)$-linear map

$$\mathrm{Exp}_{\mathscr{F}_w^+\mathcal{T}^*} : \mathcal{I}\mathcal{D} \longrightarrow H^1(\mathbf{Q}_p, \mathscr{F}_w^+\mathcal{T}^*),$$

with cokernel killed by λ, which interpolates the Bloch–Kato exponential map over the arithmetic primes of $\mathbb{I}$ and of $\Lambda(C_\infty)$.

As in (1.3.1), let $\epsilon^{1/2} : C_\infty \longrightarrow \Gamma \subset \mathbf{Z}_p^\times$ be the unique square-root of the wild component of the p-adic cyclotomic character $\varepsilon_{\mathrm{cyc}}$, and let

$$\mathrm{Tw}^\dagger : \mathbb{I}[[C_\infty]] \longrightarrow \mathbb{I}[[C_\infty]]$$

be the $\mathbb{I}$-algebra isomorphism given by $\mathrm{Tw}^\dagger([\gamma]) = \epsilon_{\mathrm{w}}^{1/2}(\gamma)[\gamma]$ for all $\gamma \in C_\infty$. Then letting $\mathscr{F}_w^+\mathcal{T}^\dagger$ be the $\mathbb{I}[[C_\infty]]$-module $\mathscr{F}_w^+\mathcal{T}^*$ with the C_∞-action twisted by $\epsilon^{1/2}$, there is a natural projection $\mathrm{Cor} : \mathscr{F}_w^+\mathcal{T}^\dagger \longrightarrow \mathscr{F}_w^+\mathbb{T}^\dagger$ induced by the augmentation map $\mathbb{I}[[C_\infty]] \longrightarrow \mathbb{I}$.

Setting $\mathbb{D}^\dagger := \mathcal{I}\mathcal{D}\widehat{\otimes}_{\mathbf{Z}_p}\mathbb{I}[[C_\infty]]/(\epsilon^{1/2}(\gamma_o)[\gamma_o] - 1)$, the composition

$$\mathcal{I}\mathcal{D} \xrightarrow{(\mathrm{Tw}^\dagger)^{-1}} \mathcal{I}\mathcal{D} \xrightarrow{\mathrm{Exp}_{\mathscr{F}_w^+\mathcal{T}^*}} H^1(\mathbf{Q}_p, \mathscr{F}_w^+\mathcal{T}^*) \xrightarrow{\otimes\epsilon^{1/2}} H^1(\mathbf{Q}_p, \mathscr{F}_w^+\mathcal{T}^\dagger) \xrightarrow{\mathrm{Cor}} H^1(\mathbf{Q}_p, \mathscr{F}_w^+\mathbb{T}^\dagger),$$

factors through an injective $\mathbb{I}$-linear map

$$\mathrm{Exp}_{\mathscr{F}_w^+\mathbb{T}^\dagger} : \mathbb{D}^\dagger \longrightarrow H^1(\mathbf{Q}_p, \mathscr{F}_w^+\mathbb{T}^\dagger) \tag{1.3.9}$$

making, for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ as in the statement, the diagram

$$\begin{array}{ccc} \mathbb{D}^\dagger & \xrightarrow{\mathrm{Exp}_{\mathscr{F}_w^+\mathbb{T}^\dagger}} & H^1(\mathbf{Q}_p, \mathscr{F}_w^+\mathbb{T}^\dagger) \\ \big\downarrow \mathrm{Sp}_\nu & & \big\downarrow \mathrm{Sp}_\nu \\ D_{\mathrm{dR}}(\mathscr{F}_w^+V_{\mathbf{f}_\nu}^\dagger) & \longrightarrow & H^1(\mathbf{Q}_p, \mathscr{F}_w^+V_{\mathbf{f}_\nu}^\dagger) \end{array}$$

commutative, where the bottom horizontal arrow is given by

$$(-1)^{r_\nu-1}(r_\nu-1)!\times\begin{cases}\left(1-\frac{p^{r_\nu-1}}{\nu(\mathbf{a}_p)}\right)\left(1-\frac{\nu(\mathbf{a}_p)}{p^{r_\nu}}\right)^{-1}\exp_{V^\dagger_{\mathbf{f}_\nu}} & \text{if } \vartheta_\nu=\mathbb{1};\\ \\ \left(\frac{p^{r_\nu-1}}{\nu(\mathbf{a}_p)}\right)^{s_\nu}\exp_{V^\dagger_{\mathbf{f}_\nu}} & \text{if } \vartheta_\nu\neq\mathbb{1},\end{cases}$$

with $\exp_{V^\dagger_{\mathbf{f}_\nu}}$ the Bloch–Kato exponential map for $V^\dagger_{\mathbf{f}_\nu}$, which factors as

$$\frac{D_{\mathrm{dR}}(V^\dagger_{\mathbf{f}_\nu})}{\mathrm{Fil}^0 D_{\mathrm{dR}}(V^\dagger_{\mathbf{f}_\nu})}\xleftarrow{\sim}D_{\mathrm{dR}}(\mathscr{F}^+_w V^\dagger_{\mathbf{f}_\nu})\xrightarrow{\exp_{V^\dagger_{\mathbf{f}_\nu}}}H^1(\mathbf{Q}_p,\mathscr{F}^+_w V^\dagger_{\mathbf{f}_\nu})\longrightarrow H^1(\mathbf{Q}_p,V^\dagger_{\mathbf{f}_\nu}).$$

Now if $\mathfrak{Y}$ is an arbitrary class in $H^1(\mathbf{Q}_p,\mathscr{F}^+_w\mathbb{T}^\dagger)$, then $\lambda\cdot\mathfrak{Y}$ lands in the image of the map $\mathrm{Exp}_{\mathscr{F}^+_w\mathbb{T}^\dagger}$ and so

$$\mathrm{Log}_{\mathbb{T}^\dagger}(\mathfrak{Y}):=\lambda^{-1}\cdot\mathrm{Exp}^{-1}_{\mathscr{F}^+_w\mathbb{T}^\dagger}(\lambda\cdot\mathfrak{Y})$$

is a well-defined element in $\mathbb{I}[\lambda^{-1}]\otimes_{\mathbb{I}}\mathbb{D}^\dagger$. Thus defining $\mathrm{Log}^\eta_{\mathbb{T}^\dagger}(\mathfrak{Y})\in\mathbb{I}[\lambda^{-1}]$ by the relation

$$\mathrm{Log}_{\mathbb{T}^\dagger}(\mathfrak{Y})=\mathrm{Log}^\eta_{\mathbb{T}^\dagger}(\mathfrak{Y})\cdot\eta\otimes 1,$$

the result follows. □

1.4. The big Heegner point

In this section we prove the main results of this paper, relating the étale Abel–Jacobi images of Heegner cycles to the specializations at higher even weights of the big Heegner point $\mathfrak{Z}$ (whose definition is recalled below). Their proof is based on two key ingredients: the properties of the big logarithm map deduced from the work of Ochiai as explained in the preceding section, and the local study of (almost all) the weight 2 specializations of $\mathfrak{Z}$ taken up in the following.

1.4.1. Weight two specializations. As in Section 1.2, let K be a fixed imaginary quadratic field in which all prime factors of N split, equipped with an ideal $\mathfrak{N}\subset\mathcal{O}_K$ such that $\mathcal{O}_K/\mathfrak{N}\cong\mathbf{Z}/N\mathbf{Z}$. We also assume that p splits in K, and let $\mathfrak{p}$ be the prime of K above p induced by our fixed embedding $\imath_p$, and by $\bar{\mathfrak{p}}$ the other. Finally, A is a fixed elliptic curve with CM by $\mathcal{O}_K$ defined over the Hilbert class field H of K, and recall that in Section 1.3 we fixed a compatible system $\zeta=(\zeta_s)_{s\geq 0}$ of primitive p^s-th roots of unity $\zeta_s\in\overline{\mathbf{Q}}_p$.

Let $R_0=\widehat{\mathbf{Z}}_p^{\mathrm{nr}}$ be the completion of the ring of integers of the maximal unramified extension of $\mathbf{Q}_p$, which we view as an overfield of H via $\imath_p$. Since p splits in K, A admits a trivialization

$$\imath_A:\hat{A}\longrightarrow\hat{\mathbf{G}}_m$$

over R_0 with $\imath_A^{-1}(\boldsymbol{\mu}_{p^s}) = A[\mathfrak{p}^s]$ for every $s > 0$. Letting α_A be the cyclic N-isogeny on A with kernel $A[\mathfrak{N}]$, the triple $(A, \alpha_A, \imath_A)$ thus defines a trivialized elliptic curve with $\Gamma_0(N)$-level structure defined over R_0.

Set $A_0 := A/A[\mathfrak{p}^s]$ and let $(A_0, \alpha_{A_0}, \imath_{A_0})$ be the trivialized elliptic curve deduced from $(A, \alpha_A, \imath_A)$ via the projection $A \longrightarrow A_0$. Let $C \subset A_0[p^s]$ be any étale subgroup of order p^s, and set $A_s := A_0/C$. Finally, let $(A_s, \alpha_{A_s}, \imath_{A_s})$ be the trivialized elliptic curve with $\Gamma_0(N)$-level structure deduced from $(A_0, \alpha_{A_0}, \imath_{A_0})$ via the projection $A_0 \longrightarrow A_s$, and consider

$$h_s = (A_s, \alpha_{A_s}, \imath_{A_s}(\zeta_s)), \tag{1.4.1}$$

which defines an algebraic point on the modular curve X_s.

Write $p^* = (-1)^{\frac{p-1}{2}}p$, and let ϑ be the unique continuous character

$$\vartheta : G_{\mathbf{Q}(\sqrt{p^*})} \longrightarrow \mathbf{Z}_p^\times/\{\pm 1\} \tag{1.4.2}$$

such that $\vartheta^2 = \varepsilon_{\rm cyc}$. Notice the inclusion $G_{H_{p^s}} \subset G_{\mathbf{Q}(\sqrt{p^*})}$ for any $s > 0$, where H_{p^s} denotes the ring class field of K of conductor p^s.

Lemma 1.4.1. *The curve A_s has CM by the order $\mathcal{O}_{p^s}$ of K of conductor p^s, and the point h_s is rational over $L_{p^s} := H_{p^s}(\boldsymbol{\mu}_{p^s})$. In fact we have*

$$h_s^\sigma = \langle \vartheta(\sigma) \rangle \cdot h_s \tag{1.4.3}$$

for all $\sigma \in \mathrm{Gal}(L_{p^s}/H_{p^s})$.

Proof. The first assertion is clear, and immediately from the construction we also see that α_{A_s} is the cyclic N-isogeny on A_s with kernel $A_s[\mathfrak{N} \cap \mathcal{O}_{p^s}]$. It follows that the point (2.3.5) gives rise to precisely the point $h_s \in X_s(\mathbf{C})$ in [**How07b**, Eq. (4)]. The result thus follows from [*loc.cit.*, Cor. 2.2.2]. □

If ν is an arithmetic prime of $\mathbb{I}$, we let ψ_ν denote its *wild character*, defined as the composition of $\nu : \mathbb{I} \longrightarrow \overline{\mathbf{Q}}_p$ with the structure map $\Gamma = 1 + p\mathbf{Z}_p \longrightarrow \mathbb{I}^\times$, which we view as a Dirichlet character of p-power conductor in the obvious manner. The nebentypus of $\mathbf{f}_\nu$ is then given by

$$\varepsilon_{\mathbf{f}_\nu} = \psi_\nu \omega^{k-k_\nu},$$

where $\omega : (\mathbf{Z}/p\mathbf{Z})^\times \longrightarrow \boldsymbol{\mu}_{p-1} \subset \mathbf{Z}_p^\times$ is the Teichmüller character.

Recall the critical characters Θ and θ from Section 1.3, and for every $\nu \in \mathcal{X}_{\rm arith}(\mathbb{I})$ of weight 2, consider the $F_\nu^\times$-valued Hecke character of K given by

$$\chi_\nu(x) = \Theta_\nu(\mathrm{art}_{\mathbf{Q}}(\mathrm{N}_{K/\mathbf{Q}}(x))) \tag{1.4.4}$$

for all $x \in \mathbb{A}_K^\times$. Notice that since χ_ν has finite order, it may alternately be seen as character on G_K via the Artin reciprocity map $\mathrm{art}_K : \mathbb{A}_K^\times \longrightarrow G_K^{\mathrm{ab}}$.

For every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$, after fixing an embedding $F_\nu \longrightarrow \overline{\mathbf{Q}}_p$, the form $\mathbf{f}_\nu \in S_{k_\nu}(X_{s_\nu})$ defines a p-adic modular form $\mathbf{f}_\nu \in \mathbf{M}(N)$. Finally, recall the dual form $\mathbf{f}_\nu^*$ defined as in the paragraph before (1.1.20).

LEMMA 1.4.2. *Let $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ have weight 2 and non-trivial wild character, and let $s > 1$ be the p-power of the conductor of ψ_ν. Then*

$$d^{-1}\mathbf{f}_\nu^{*[p]} \otimes \theta_\nu(A, \alpha_A, \imath_A) = \frac{u}{G(\theta_\nu^{-1})} \sum_{\sigma \in \mathrm{Gal}(H_{p^s}/H)} \chi_\nu^{-1}(\tilde{\sigma}) \cdot d^{-1}\mathbf{f}_\nu^{*[p]}(h_s^{\tilde{\sigma}}), \tag{1.4.5}$$

where $u = |\mathcal{O}_K^\times|/2$, $G(\theta_\nu^{-1}) = \sum_{x \bmod p^s} \theta_\nu^{-1}(x)\zeta_s^x$ is a usual Gauss sum, and for every $\sigma \in \mathrm{Gal}(H_{p^s}/H)$, $\tilde{\sigma}$ is any lift of σ to $\mathrm{Gal}(L_{p^s}/H)$.

REMARK 1.4.3. Since ν has weight $k_\nu = 2$, we have $\theta_\nu = \vartheta_\nu$, where ϑ_ν is the finite order character in the statement of Theorem 1.3.4.

PROOF. We begin by noting that the expression in the right hand side of (1.4.5) does not depend on the choice of lifts $\tilde{\sigma}$. Indeed, as explained in [**How07a**, p. 808] the character $\chi_{0,\nu} := \chi_\nu|_{\mathbb{A}_{\mathbf{Q}}^\times}$, seen as a Dirichlet character in the usual manner, is such that $\chi_{0,\nu}^{-1} = \theta_\nu^2$. But since the weight of ν is 2, we have

$$\theta_\nu^2 = \varepsilon_{\mathbf{f}_\nu} = \varepsilon_{\mathbf{f}_\nu^*}^{-1}$$

(see [**How07a**, p. 806]), and our claim thus follows immediately from (1.4.3).

To compute the above value of the twist $d^{-1}\mathbf{f}_\nu^{*[p]} \otimes \theta_\nu$ we follow Definition 1.1.2. The integer $s > 1$ in the statement is such that θ_ν factors through $(\mathbf{Z}/p^s\mathbf{Z})^\times$, therefore

$$\begin{aligned} d^{-1}\mathbf{f}_\nu^{*[p]} \otimes \theta_\nu(A, \alpha_A, \imath_A) &= \sum_{a \bmod p^s} \theta_\nu(a) \left(\int_{a+p^s\mathbf{Z}_p} d\mu_{\mathrm{Gou}}(x) \right) (d^{-1}\mathbf{f}_\nu^{*[p]})(A, \alpha_A, \imath_A) \\ &= \frac{1}{p^s} \sum_{a \bmod p^s} \theta_\nu(a) \sum_C \zeta_C^{-a} \cdot d^{-1}\mathbf{f}_\nu^{*[p]}(A_0/C, \alpha_C, \imath_C), \end{aligned} \tag{1.4.6}$$

where as before $A_0 := A/\imath_A^{-1}(\boldsymbol{\mu}_{p^s}) = A/A[\mathfrak{p}^s]$ and the sum is over the étale subgroups $C \subset A_0[p^s]$ of order p^s. Letting γ_s be a generator of $\mathbf{Z}/p^s\mathbf{Z}$, these subgroups correspond bijectively with the cyclic subgroups $C_u = \langle \zeta_s^u . \gamma_s \rangle \subset \boldsymbol{\mu}_{p^s} \times \mathbf{Z}/p^s\mathbf{Z}$, with u running over the integers modulo p^s, and we set $\zeta_{C_u} = \zeta_s^u$.

Since θ_ν does not factor through $(\mathbf{Z}/p^{s-1}\mathbf{Z})^\times$, we have $\sum_{a \bmod p^s} \theta_\nu(a)\zeta_s^{-ua} = 0$ whenever $u \notin (\mathbf{Z}/p^s\mathbf{Z})^\times$. Continuing from (1.4.6), we thus obtain

$$\begin{aligned} d^{-1}\mathbf{f}_\nu^{*[p]} \otimes \theta_\nu(A, \alpha_A, \imath_A) &= \frac{1}{p^s} \sum_{a \bmod p^s} \theta_\nu(a) \sum_{u \bmod p^s} \zeta_s^{-ua} \cdot d^{-1}\mathbf{f}_\nu^{*[p]}(A_{C_u}, \alpha_{C_u}, \imath_{C_u}) \\ &= \frac{1}{p^s} \sum_{u \in (\mathbf{Z}/p^s\mathbf{Z})^\times} d^{-1}\mathbf{f}_\nu^{*[p]}(A_{C_u}, \alpha_{C_u}, \imath_{C_u}) \sum_{a \bmod p^s} \theta_\nu(a)\zeta_s^{-ua} \\ &= \frac{1}{G(\theta_\nu^{-1})} \sum_{u \in (\mathbf{Z}/p^s\mathbf{Z})^\times} \theta_\nu^{-1}(u) \cdot d^{-1}\mathbf{f}_\nu^{*[p]}(A_{C_u}, \alpha_{C_u}, \imath_{C_u}), \end{aligned}$$

with the last equality obtained by a change of variables. The result thus follows from the relation

$$\sum_{u \in (\mathbf{Z}/p^s\mathbf{Z})^\times} \theta_\nu^{-1}(u) \cdot d^{-1}\mathbf{f}_\nu^{*[p]}(A_{C_u}, \alpha_{C_u}, \imath_{C_u}) = u \sum_{\sigma \in \mathrm{Gal}(H_{p^s}/H)} \chi_\nu^{-1}(\tilde{\sigma}) \cdot d^{-1}\mathbf{f}_\nu^{*[p]}(h_s^{\tilde{\sigma}}),$$

where $u = |\mathcal{O}_K^\times|/2$, and for each $\sigma \in \mathrm{Gal}(H_{p^s}/H)$, $\tilde{\sigma} \in \mathrm{Gal}(L_{p^s}/H)$ lifts σ. □

Keeping the above notations, let $\Delta_s \in J_s(L_{p^s})$ be the divisor class of $(h_s) - (\infty)$, and consider the element in $J_s(L_{p^s}) \otimes_{\mathbf{Z}} F_\nu$ given by

$$\widetilde{Q}_{\chi_\nu} := \sum_{\sigma \in \mathrm{Gal}(H_{p^s}/H)} \Delta_s^{\tilde{\sigma}} \otimes \chi_\nu^{-1}(\tilde{\sigma}), \tag{1.4.7}$$

where for every $\sigma \in \mathrm{Gal}(H_{p^s}/H)$, $\tilde{\sigma}$ is any lift to $\mathrm{Gal}(L_{p^s}/H)$.

Let F_s be the completion of $\imath_p(L_{p^s})$, and consider the p-adic Abel–Jacobi map $\delta^{(p)}_{\mathbf{f}_\nu, F_s}$ defined in (1.1.9) which we extend by F_ν-linearity to a map

$$\delta^{(p)}_{\mathbf{f}_\nu, F_s} : J_s(L_{p^s}) \otimes_{\mathbf{Z}} F_\nu \longrightarrow (\mathrm{Fil}^1 D_{\mathrm{dR}}(V_{\mathbf{f}_\nu^*}))^\vee.$$

Proposition 1.4.4. *Let $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ and $s > 1$ be as in Lemma 1.4.2. Then*

$$\sum_{\sigma \in \mathrm{Gal}(H_{p^s}/H)} \chi_\nu^{-1}(\tilde{\sigma}) \cdot d^{-1}\mathbf{f}_\nu^{*[p]}(h_s^{\tilde{\sigma}}) = \delta^{(p)}_{\mathbf{f}_\nu, F_s}(\widetilde{Q}_{\chi_\nu})(\omega_{\mathbf{f}_\nu^*}). \tag{1.4.8}$$

Proof. The integer $s > 1$ in the statement is so that the nebentypus $\varepsilon_{\mathbf{f}_\nu}$ of $\mathbf{f}_\nu$ is primitive modulo p^s. Moreover, since p splits in K, we see from the construction that the point h_s lies in the connected component $X_s(0)$ of the ordinary locus of X_s containing the cusp ∞. Thus Proposition 1.1.9 applies, giving

$$\delta^{(p)}_{\mathbf{f}_\nu, F_s}(\Delta_s)(\omega_{\mathbf{f}_\nu^*}) = F_{\omega_{\mathbf{f}_\nu^*}}(h_s),$$

where $F_{\omega_{\mathbf{f}_\nu^*}}$ is the Coleman primitive of $\omega_{\mathbf{f}_\nu^*}$ from Proposition 1.1.6, and by linearity

$$\sum_{\sigma\in\mathrm{Gal}(H_{p^s}/H)} \chi_\nu^{-1}(\tilde{\sigma})\cdot F_{\omega_{\mathbf{f}_\nu}}(h_s^{\tilde{\sigma}}) = \delta^{(p)}_{\mathbf{f}_\nu,F_s}(\widetilde{Q}_{\chi_\nu})(\omega_{\mathbf{f}_\nu^*}). \tag{1.4.9}$$

Since ϕ lifts the Deligne–Tate map to X_s, we see that ϕh_s is defined over the subfield $H_{p^{s-1}}(\zeta_s)\subset L_{p^s}$. If b_p denotes the U_p-eigenvalue of $\mathbf{f}_\nu^*$, by Corollary 1.1.8 we obtain

$$\begin{aligned}\sum_\sigma \chi_\nu^{-1}(\tilde{\sigma})\cdot d^{-1}\mathbf{f}_\nu^{*[p]}(h_s^{\tilde{\sigma}}) &= \sum_\sigma \chi_\nu^{-1}(\tilde{\sigma})\cdot F_{\omega_{\mathbf{f}_\nu^*}}(h_s^{\tilde{\sigma}}) - \frac{b_p}{p}\sum_\sigma \chi_\nu^{-1}(\tilde{\sigma})\cdot F_{\omega_{\mathbf{f}_\nu^*}}(\phi h_s^{\tilde{\sigma}})\\ &= \sum_\sigma \chi_\nu^{-1}(\tilde{\sigma})\cdot F_{\omega_{\mathbf{f}_\nu^*}}(h_s^{\tilde{\sigma}}),\end{aligned}$$

where all the sums are over $\sigma\in\mathrm{Gal}(H_{p^s}/H)$, and the second equality follows immediately from the fact θ_ν is primitive modulo p^s. The result thus follows from (1.4.9). □

Still with the same notations, recall Hida's ordinary projector (1.0.2) and set $y_s := e^{\mathrm{ord}}h_s$, which naturally lies in $e^{\mathrm{ord}}J_s(L_{p^s})$ (see [**How07b**, p.100]). Equation (1.4.3) then amounts to the fact that

$$y_s^\sigma = \Theta(\sigma)\cdot y_s \tag{1.4.10}$$

for all $\sigma\in\mathrm{Gal}(L_{p^s}/H_{p^s})$, where Θ is the critical character (1.3.1). Denoting by $J_s^{\mathrm{ord}}(L_{p^s})^\dagger$ the module $e^{\mathrm{ord}}J_s(L_{p^s})$ with the Galois action twisted by Θ^{-1}, and by $y_s^\dagger$ the point y_s seen in this new module, (1.4.10) translates into the statement that

$$y_s^\dagger\in H^0(H_{p^s}, J_s^{\mathrm{ord}}(L_{p^s})^\dagger).$$

Lemma 1.4.5 (Howard). *The classes*

$$x_s := \mathrm{Cor}_{H_{p^s}/H}(y_s^\dagger)\in H^0(H, J_s^{\mathrm{ord}}(L_{p^s})^\dagger) \tag{1.4.11}$$

are such that

$$\alpha_* x_{s+1} = U_p\cdot x_s, \qquad \textit{for all } s>0$$

under the Albanese maps induced from the degeneracy maps $\alpha: X_{s+1}\longrightarrow X_s$.

Proof. This is shown in the course of the proof of [**How07b**, Lemma 2.2.4]. □

Abbreviate by $\mathrm{Ta}_p^{\mathrm{ord}}(J_s)$ the module $e^{\mathrm{ord}}(\mathrm{Ta}_p(J_s)\otimes_{\mathbf{Z}_p}\mathcal{O})$ from the Introduction, and denote by $\mathrm{Ta}_p^{\mathrm{ord}}(J_s)^\dagger$ this same module with the Galois action twisted by Θ^{-1}. By the Galois and Hecke-equivariance of the twisted Kummer map

$$\mathrm{Kum}_s : H^0(H, J_s^{\mathrm{ord}}(L_{p^s})^\dagger)\longrightarrow H^1(H, \mathrm{Ta}_p^{\mathrm{ord}}(J_s)^\dagger)$$

constructed in [**How07b**, p. 101], Lemma 1.4.5 implies that the cohomology classes $\mathfrak{X}_s := \mathrm{Kum}_s(x_s)$ are such that $\alpha_* \mathfrak{X}_{s+1} = U_p \cdot \mathfrak{X}_s$, for all $s > 0$.

Definition 1.4.6 (Howard). The *big Heegner point* of conductor one is the cohomology class $\mathfrak{X}$ given by the image of

$$\varprojlim_s U_p^{-s} \cdot \mathfrak{X}_s$$

under the natural map induced by the $\mathfrak{h}^{\mathrm{ord}}[G_{\mathbf{Q}}]$-linear projection $\varprojlim_s \mathrm{Ta}_p^{\mathrm{ord}}(J_s)^\dagger \longrightarrow \mathbb{T}^\dagger$.

Our object of study is in fact the big cohomology class

$$\mathfrak{Z} := \mathrm{Cor}_{H/K}(\mathfrak{X}), \tag{1.4.12}$$

which is predicted to be non-trivial by [**How07b**, Conj. 3.4.1].

Conjecture 1.4.7 (Howard). *The class $\mathfrak{Z}$ is not $\mathbb{I}$-torsion.*

Recall from [**How07b**, §2.4] that the *strict Greenberg Selmer group* $\mathrm{Sel}_{\mathrm{Gr}}(K, \mathbb{T}^\dagger)$ is defined to be the subspace of $H^1(K, \mathbb{T}^\dagger)$ consisting of classes c which are unramified outside the places above p and such that $\mathrm{loc}_v(c)$ lies in the kernel of the natural map

$$H^1(K_v, \mathbb{T}^\dagger) \longrightarrow H^1(K_v, \mathscr{F}_w^- \mathbb{T}^\dagger)$$

for all $v|p$.

For every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2, let $L(s, \mathbf{f}_\nu, \chi_\nu)$ be the Rankin–Selberg convolution L-function of [**How09**, §1]. In the spirit of the classical Gross–Zagier theorem, one has the following criterion for the non-triviality of (the specializations of) $\mathfrak{Z}$.

Theorem 1.4.8 (Howard). *The class $\mathfrak{Z}$ belongs to $\mathrm{Sel}_{\mathrm{Gr}}(K, \mathbb{T}^\dagger)$, and if there is some $\nu' \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2 and non-trivial nebentypus such that $L'(1, \mathbf{f}_{\nu'}, \chi_{\nu'}) \neq 0$, then Conjecture 1.4.7 holds.*

Proof. The first assertion is shown in [**How07b**, Prop. 2.4.5]. The second is shown in [**How07a**, Prop. 3] and follows from the equivalence

$$\mathfrak{Z}_{\nu'} \neq 0 \quad \Longleftrightarrow \quad L'(1, \mathbf{f}_{\nu'}, \chi_{\nu'}) \neq 0 \tag{1.4.13}$$

combined with the freeness of $\mathrm{Sel}_{\mathrm{Gr}}(K, \mathbb{T}^\dagger) \otimes_{\mathbb{I}} \mathcal{O}_\nu$ ([**Nek06**, Prop. 12.7.13.4(iii)]) for any $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$. ☐

The following result shows that Proposition 1.4.4 may be reformulated as giving an explicit formula, for all but finitely many $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2, for the image of the classes $\mathfrak{Z}_\nu$ under the inverse of the Bloch–Kato exponential map.

For any class $[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)$, taking a representative $\mathfrak{a} \subset \mathcal{O}_K$ prime to Np, define

$$\mathfrak{a} * (A, \alpha_A, \imath_A) := (A_{\mathfrak{a}}, \alpha_{A_{\mathfrak{a}}}, \imath_{A_{\mathfrak{a}}}),$$

where $A_{\mathfrak{a}} = A/A[\mathfrak{N}]$, $\alpha_{A_{\mathfrak{a}}} = A_{\mathfrak{a}}[\mathfrak{N}]$, and $\imath_{A_{\mathfrak{a}}}$ is the trivialization $\hat{A}_{\mathfrak{a}} \xrightarrow{\hat{\varphi}_{\mathfrak{a}}^{-1}} \hat{A} \xrightarrow{\imath_A} \hat{\mathbf{G}}_m$ induced by the projection $\varphi_{\mathfrak{a}} : A \longrightarrow A_{\mathfrak{a}}$.

THEOREM 1.4.9. *Let $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ have weight 2 and non-trivial wild character ψ_ν, and let $s > 1$ be the p-power of the conductor of ψ_ν. Then*

$$\sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)} d^{-1}\mathbf{f}_\nu^{[p]} \otimes \theta_\nu^{-1}(\mathfrak{a} * (A, \alpha_A, \imath_A)) = u\frac{\nu(\mathbf{a}_p)^s}{G(\theta_\nu^{-1})} \log_{s, V_{\mathbf{f}_\nu}(1)}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}_\nu))(\omega_{\mathbf{f}_\nu^*}), \tag{1.4.14}$$

where $u = |\mathcal{O}_K^\times|/2$, and $G(\theta_\nu^{-1})$ is the Gauss sum $\sum_{x \bmod p^s} \theta_\nu^{-1}(x)\zeta_s^x$.

PROOF. Since clearly

$$d^{-1}\mathbf{f}_\nu^{[p]} \otimes \theta_\nu^{-1} = d^{-1}\mathbf{f}_\nu^{*[p]} \otimes \theta_\nu,$$

letting F_s be the completion of $\imath_p(L_{p^s})$ it suffices to establish the equality

$$d^{-1}\mathbf{f}_\nu^{*[p]} \otimes \theta_\nu(A, \alpha_A, \imath_A) = u\frac{\nu(\mathbf{a}_p)^s}{G(\theta_\nu^{-1})} \log_{F_s, V_{\mathbf{f}_\nu}(1)}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{X}_\nu))(\omega_{\mathbf{f}_\nu^*}). \tag{1.4.15}$$

Combining the formulas from Lemma 1.4.2 and Proposition 1.4.4, we have

$$d^{-1}\mathbf{f}_\nu^{*[p]} \otimes \theta_\nu(A, \alpha_A, \imath_A) = \frac{u}{G(\theta_\nu^{-1})} \delta^{(p)}_{\mathbf{f}_\nu, F_s}(\widetilde{Q}_{\chi_\nu}). \tag{1.4.16}$$

Now the integer $s > 1$ is such that the natural map $\mathbb{T} \longrightarrow \mathbb{V}_\nu$ can be factored as

$$\mathbb{T} \longrightarrow \mathrm{Ta}_p^{\mathrm{ord}}(J_s) \longrightarrow \mathbb{V}_\nu, \tag{1.4.17}$$

and we have $\mathbb{V}_\nu^\dagger \cong \mathbb{V}_\nu$ as $G_{L_{p^s}}$-modules. Tracing through the construction of $\mathfrak{X}$, we see that the image of $U_p^s \cdot \mathfrak{X}_\nu$ in $H^1(L_{p^s}, \mathbb{V}_\nu^\dagger)$ agrees with the image of $\widetilde{Q}_{\chi_\nu}$ under the composite map (where the unlabelled arrow is induced by (1.4.17))

$$\begin{aligned} J_s(L_{p^s}) \otimes_{\mathbf{Z}} F_\nu \xrightarrow{\mathrm{Kum}_s} H^1(L_{p^s}, \mathrm{Ta}_p(J_s) \otimes_{\mathbf{Z}} F_\nu) \xrightarrow{e^{\mathrm{ord}}} H^1(L_{p^s}, \mathrm{Ta}_p^{\mathrm{ord}}(J_s) \otimes_{\mathbf{Z}} F_\nu) \longrightarrow \\ \longrightarrow H^1(L_{p^s}, \mathbb{V}_\nu) \cong H^1(L_{p^s}, \mathbb{V}_\nu^\dagger). \end{aligned} \tag{1.4.18}$$

Since U_p acts on $\mathbb{V}_\nu^\dagger$ as multiplication by $\nu(\mathbf{a}_p)$, we thus arrive at the equality

$$\mathrm{Kum}_s(e^{\mathrm{ord}}\widetilde{Q}_{\chi_\nu}) = \nu(\mathbf{a}_p)^s \cdot \mathrm{res}_{L_{p^s}/H}(\mathfrak{X}_\nu) \in H^1(L_{p^s}, \mathbb{V}_\nu). \tag{1.4.19}$$

By [**Rub00**, Prop. 1.6.8], this shows that the restriction to $\mathrm{loc}_{\mathfrak{p}}(\mathfrak{X}_\nu)$ to G_{F_s} is contained in the Bloch–Kato finite subspace $H^1_f(F_s, \mathbb{V}_\nu) \cong H^1_f(F_s, \mathbb{V}_\nu^\dagger)$. Since the map $\delta^{(p)}_{\mathbf{f}_\nu, F_s}$ is defined by the commutativity of the diagram

$$\begin{array}{ccccc} J_s(L_{p^s}) \otimes_{\mathbf{Z}} F_\nu & \xrightarrow{(1.4.18)} & H^1(L_{p^s}, \mathbb{V}_\nu) & \xrightarrow{\mathrm{loc}_{\mathfrak{p}}} & H^1(F_s, \mathbb{V}_\nu) \\ & & & & \cup \\ & & & \overset{\cdots\cdots}{\searrow} & H^1_f(F_s, \mathbb{V}_\nu) \\ & & \delta^{(p)}_{\mathbf{f}_\nu, F_s} \searrow & & \downarrow \log_{F_s, \mathbb{V}_\nu} \\ & & & & (\mathrm{Fil}^0 D_{\mathrm{dR}}(\mathbb{V}_\nu))^\vee, \end{array}$$

we thus see that (1.4.15) follows from (1.4.16) and (1.4.19). □

Corollary 1.4.10. *Assume that there is some $\nu' \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2 and non-trivial nebentypus such that $L'(1, \mathbf{f}_{\nu'}, \chi_{\nu'}) \neq 0$. Then, for all but finitely many $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$, the localization map*

$$\mathrm{loc}_{\mathfrak{p}} : \mathrm{Sel}_{\mathrm{Gr}}(K, \mathbb{V}^\dagger_\nu) \longrightarrow H^1(\mathbf{Q}_p, \mathbb{V}^\dagger_\nu) \tag{1.4.20}$$

is injective.

Proof. By Howard's Theorem 1.4.8, our nonvanishing assumption implies that $\mathfrak{Z}$ is not $\mathbb{I}$-torsion, and hence by the exact sequence

$$0 \longrightarrow \frac{\widetilde{H}^1_f(K, \mathbb{T}^\dagger)_\nu}{\nu \cdot \widetilde{H}^1_f(K, \mathbb{T}^\dagger)_\nu} \longrightarrow \widetilde{H}^1_f(K, \mathbb{V}^\dagger_\nu) \longrightarrow \widetilde{H}^2_f(K, \mathbb{T}^\dagger)_\nu[\nu] \longrightarrow 0$$

(see [**How07b**, Cor. 3.4.3]), combined with the finite generation over $\mathbb{I}$ of $\widetilde{H}^1_f(K, \mathbb{T}^\dagger)$ and [**How07b**, Lemma 2.1.6], it implies that the image of $\mathfrak{Z}$ in $\widetilde{H}_f(K, \mathbb{V}^\dagger_\nu)$ is nonzero for almost all $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$; in particular, for all but finitely many ν of weight 2 and non-trivial nebentypus, $\mathfrak{Z}_\nu \neq 0$ in $\widetilde{H}_f(K, \mathbb{V}^\dagger_\nu) \cong \mathrm{Sel}_{\mathrm{Gr}}(K, \mathbb{V}^\dagger_\nu)$ (see [**How07b**, Eq. (22)] for the comparison).

Now, by [**How07b**, Cor. 3.4.3] it follows that $\mathrm{Sel}_{\mathrm{Gr}}(K, \mathbb{T}^\dagger)$ has rank one, and hence

$$\mathrm{Sel}_{\mathrm{Gr}}(K, \mathbb{V}^\dagger_\nu) = \mathfrak{Z}_\nu . F_\nu \tag{1.4.21}$$

for almost all $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$. Thus to prove the result it suffices to show that the implication

$$\mathfrak{Z}_\nu \neq 0 \quad \Longrightarrow \quad \mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}_\nu) \neq 0$$

holds for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2 and non-trivial nebentypus. (Indeed, by (1.4.21) this will show that the localization map (1.4.20) is injective at infinitely many ν, and by [**How07b**, Lemma 2.1.7] it will follow that the kernel of the localization map

$$\mathrm{loc}_{\mathfrak{p}} : \widetilde{H}^1_f(K, \mathbb{T}^\dagger) \longrightarrow H^1(\mathbf{Q}_p, \mathbb{T}^\dagger)$$

must be $\mathbb{I}$-torsion, hence supported only at a finite number of arithmetic primes.)

Let $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ and $s > 1$ be as in Theorem 1.4.9, and assume that $\mathfrak{Z}_\nu \neq 0$. Since the restriction map

$$H^1(K, \mathbb{V}_\nu^\dagger) \xrightarrow{\mathrm{res}_{L_{p^s}}} H^1(L_{p^s}, \mathbb{V}_\nu^\dagger) \cong H^1(L_{p^s}, \mathbb{V}_\nu)$$

is injective, the class $\mathrm{res}_{L_{p^s}}(\mathfrak{Z}_\nu)$ is non-zero, and it arises as the image of a necessarily non-torsion point in $J_s(L_{p^s})$ under the composite map (cf. (1.4.18))

$$J_s(L_{p^s}) \longrightarrow H^1(L_{p^s}, \mathrm{Ta}_p^{\mathrm{ord}}(J_s)) \longrightarrow H^1(L_{p^s}, \mathbb{V}_\nu), \tag{1.4.22}$$

where the first arrow is the Kummer map composed with the ordinary projector e^{ord}. Let $\mathcal{L}_{p^s}$ be the completion of $\imath_p(L_{p^s}) \subset \overline{\mathbf{Q}}_p$. Then both the natural map

$$J_s(L_{p^s}) \otimes \mathbf{Q} \longrightarrow J_s(\mathcal{L}_{p^s}) \otimes \mathbf{Q}_p$$

and the local Kummer map

$$J_s(\mathcal{L}_{p^s}) \otimes \mathbf{Q}_p \longrightarrow H^1(\mathcal{L}_{p^s}, \mathrm{Ta}_p(J_s) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p)$$

are injective, and hence by the commutativity of the resulting diagram

$$\begin{array}{ccc} J_s(L_{p^s}) \otimes \mathbf{Q} & \xrightarrow{(1.4.22)\otimes\mathbf{Q}} & H^1(L_{p^s}, \mathbb{V}_\nu) \\ \downarrow & & \downarrow \mathrm{loc}_{\mathfrak{p}} \\ J_s(\mathcal{L}_{p^s}) \otimes \mathbf{Q}_p & \longrightarrow & H^1(\mathcal{L}_{p^s}, \mathbb{V}_\nu), \end{array}$$

it follows that $\mathrm{loc}_{\mathfrak{p}}(\mathrm{res}_{L_{p^s}}(\mathfrak{Z}_\nu)) = \mathrm{res}_{\mathcal{L}_{p^s}}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}_\nu)) \neq 0$, whence $\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}_\nu) \neq 0$ as desired. □

1.4.2. Higher weight specializations. Now we can prove our main result. Recall from the introduction that f_o is a p-ordinary newform of level N prime to p, even weight $k \geq 2$ and trivial nebentypus, and that $\mathbf{f} = \sum_{n>0} \mathbf{a}_n q^n \in \mathbb{I}[[q]]$ is the Hida family passing through the ordinary p-stabilization of f_o. Let ν_o be the arithmetic prime of $\mathbb{I}$ such that $\mathbf{f}_{\nu_o}$ is the ordinary p-stabilization of f_o, and let $\mathbb{T}^\dagger = \mathbb{T} \otimes \Theta^{-1}$ be the critical twist of $\mathbb{T}$ such that ϑ_{ν_o} is the trivial character (as opposed to $\omega^{\frac{p-1}{2}}$.)

If $\mathbf{f}_\nu$ is the ordinary p-stabilization of a p-ordinary newform $\mathbf{f}_\nu^\sharp$ of even weight $2r_\nu > 2$ and trivial nebentypus, the Heegner cycle $\Delta_{A,r_\nu}^{\mathrm{heeg}}$ has been defined in Section 1.2 (attached to a suitable choice of an imaginary quadratic field K), and by [**Nek00**, Thm. (3.1)(i)] the class

$$\Phi^{\text{ét}}_{\mathbf{f}_\nu^\sharp, K}(\Delta_{r_\nu}^{\mathrm{heeg}}) := \mathrm{Cor}_{H/K}(\Phi^{\text{ét}}_{\mathbf{f}_\nu^\sharp, H}(\Delta_{A,r_\nu}^{\mathrm{heeg}})) \tag{1.4.23}$$

lies in the Bloch–Kato Selmer group $H^1_f(K, V_{\mathbf{f}_\nu^\sharp}(r_\nu))$.

On the other hand, by [**How07b**, Prop. 2.4.5], the big Heegner point $\mathfrak{X}$ lies in the strict Greenberg Selmer group $\mathrm{Sel}_{\mathrm{Gr}}(H, \mathbb{T}^\dagger)$ (defined in [*loc.cit.*, Def. 2.4.2]), and since $\mathrm{Sel}_{\mathrm{Gr}}(K, \mathbb{V}_\nu^\dagger) \cong H^1_f(K, \mathbb{V}_\nu^\dagger)$ as explained in [**How07b**, p. 114]) and $\mathbb{V}_\nu^\dagger \cong V_{\mathbf{f}_\nu^\sharp}(r_\nu)$ by Lemma 1.3.2, the class

$$\mathfrak{Z}_\nu = \mathrm{Cor}_{H/K}(\mathfrak{X}_\nu)$$

naturally lies in $H^1_f(K, V_{\mathbf{f}_\nu^\sharp}(r_\nu))$ as well. Our main result relates these two classes.

ASSUMPTIONS 1.4.11. (1) *The residual representation $\bar\rho_{f_o}$ is absolutely irreducible,*
(2) *$\bar\rho_{f_o}|_{G_{\mathbf{Q}_p}}$ has non-scalar semi-simplication,*
(3) *The prime p splits in K,*
(4) *Every prime divisor of N splits in K.*

THEOREM 1.4.12. *Together with Assumptions 1.4.11, suppose that there exists some $\nu' \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2 and non-trivial nebentypus such that*

$$L'(1, \mathbf{f}_{\nu'}, \chi_{\nu'}) \neq 0. \tag{1.4.24}$$

Then for all but finitely many $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight $2r_\nu > 2$ with $2r_\nu \equiv k \pmod{2(p-1)}$, we have

$$\langle \mathfrak{Z}_\nu, \mathfrak{Z}_\nu \rangle_K = \left(1 - \frac{p^{r_\nu - 1}}{\nu(\mathbf{a}_p)}\right)^4 \cdot \frac{\langle \Phi^{\text{ét}}_{\mathbf{f}_\nu^\sharp, K}(\Delta^{\mathrm{heeg}}_{r_\nu}), \Phi^{\text{ét}}_{\mathbf{f}_\nu^\sharp, K}(\Delta^{\mathrm{heeg}}_{r_\nu})\rangle_K}{u^2 (4D)^{r_\nu - 1}}, \tag{1.4.25}$$

where $\langle , \rangle_K$ is the cyclotomic p-adic height pairing on $H^1_f(K, V_{\mathbf{f}_\nu^\sharp}(r_\nu))$, $u = |\mathcal{O}_K^\times|/2$, and $-D < 0$ is the discriminant of K.

PROOF. By [**How07b**, Prop. 2.4.5] the class $\mathfrak{Z}$ lies in the strict Greenberg Selmer group $\mathrm{Sel}_{\mathrm{Gr}}(K, \mathbb{T}^\dagger)$ (note that under our assumptions we may take $\lambda = 1$ in Howard's result), and hence its restriction $\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z})$ at $\mathfrak{p}$ lies in the kernel of the natural map

$$H^1(\mathbf{Q}_p, \mathbb{T}^\dagger) \longrightarrow H^1(\mathbf{Q}_p, \mathscr{F}_w^- \mathbb{T}^\dagger)$$

induced by (2.2.1) (twisted by Θ^{-1}). Since $H^0(\mathbf{Q}_p, \mathscr{F}_w^- \mathbb{T}^\dagger) = 0$ by [**How07b**, Lemma 2.4.4], the class $\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z})$ can therefore be seen as sitting inside $H^1(\mathbf{Q}_p, \mathscr{F}_w^+ \mathbb{T}^\dagger)$. Thus upon taking an $\mathbb{I}$-basis η of $\mathbb{D}$, we can form

$$\mathcal{L}_{\mathfrak{p}}^{\mathrm{arith}}(\mathbf{f}^\dagger) := u \cdot \mathrm{Log}^{\eta}_{\mathbb{T}^\dagger}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z})) \in \mathbb{I}[\lambda^{-1}] \qquad (\lambda := \mathbf{a}_p - 1).$$

On the other hand, consider the continuous function on $\mathrm{Spf}(\mathbb{I})(\overline{\mathbf{Q}}_p)$ given by

$$\mathcal{L}_{\mathfrak{p}}^{\mathrm{analy}}(\mathbf{f}^\dagger) : \nu \longmapsto \sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)} d^{-1} \mathbf{f}_\nu^{[p]} \otimes \theta_\nu^{-1}(\mathfrak{a} * (A, \alpha_A, \imath_A)).$$

(Its continuity can be checked by staring at the q-expansion of $d^{-1}\mathbf{f}_\nu^{[p]} \otimes \theta_\nu^{-1}$ and appealing to the results in [**Gou88**, §I.3.5], for example.)

By the specialization property (1.3.8) of the map $\mathrm{Log}^{\eta}_{\mathbb{T}^\dagger}$, we immediately see that Theorem 1.4.9 can be reformulated as follows: For every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2 and non-trivial wild character, there exists a unit $\Omega_\nu^\eta \in \mathcal{O}_\nu^\times$ such that

$$\nu\big(\mathcal{L}_{\mathfrak{p}}^{\mathrm{analy}}(\mathbf{f}^\dagger)\big) = \Omega_\nu^\eta \cdot \nu\big(\mathcal{L}_{\mathfrak{p}}^{\mathrm{arith}}(\mathbf{f}^\dagger)\big). \tag{1.4.26}$$

In fact,

$$\Omega_\nu^\eta = \langle \eta_\nu \otimes e_\zeta^{\otimes r_\nu}, \omega_{\mathbf{f}_\nu^*}\rangle_{\mathrm{dR}} \tag{1.4.27}$$

under the pairing (1.3.7), so that $\omega_{\mathbf{f}_\nu^*} = \Omega_\nu^\eta \cdot \eta'_\nu$ with η'_ν as defined in Theorem 1.3.4. (That Ω_ν^η, which a priori just lies in F_ν, is indeed a unit is shown in [**Och06**, Prop. 6.4].) Since both $\mathfrak{L}_{\mathfrak{p}}^{\mathrm{arith}}(\mathbf{f}^\dagger)$ and $\mathfrak{L}_{\mathfrak{p}}^{\mathrm{analy}}(\mathbf{f}^\dagger)$ are continuous functions of ν, (1.4.26) shows that the map $\nu \longmapsto \Omega_\nu^\eta$ is continuous, and hence the relation (1.4.27) is valid for all $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$.

Now let $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ be as in the statement. Then $\theta_\nu(z) = z^{r_\nu - 1}\vartheta_\nu(z) = z^{r_\nu-1}$ as characters on $\mathbf{Z}_p^\times$, from where if follows that

$$\begin{aligned}\nu\big(\mathcal{L}_{\mathfrak{p}}^{\mathrm{analy}}(\mathbf{f}^\dagger)\big) &= \sum_{[\mathfrak{a}]\in \mathrm{Pic}(\mathcal{O}_K)} d^{-1}\mathbf{f}_\nu^{[p]} \otimes \theta_\nu^{-1}(\mathfrak{a} * (A, \alpha_A, \imath_A)) \\ &= \sum_{[\mathfrak{a}]\in \mathrm{Pic}(\mathcal{O}_K)} d^{-r_\nu}\mathbf{f}_\nu^{[p]}(\mathfrak{a} * (A, A[\mathfrak{N}])).\end{aligned} \tag{1.4.28}$$

By Theorem 1.2.3, setting

$$\Delta_{r_\nu}^{\mathrm{bdp}} := \sum_{[\mathfrak{a}]\in \mathrm{Pic}(\mathcal{O}_K)} \mathrm{N}\mathfrak{a}^{1-r} \cdot \Delta^{\mathrm{bdp}}_{\varphi_{\mathfrak{a}}, r_\nu} \in \mathrm{CH}^{2r_\nu - 1}(X_{r_\nu})_0(K), \tag{1.4.29}$$

the equation (1.4.28) can be rewritten as

$$\begin{aligned}\nu\big(\mathcal{L}_{\mathfrak{p}}^{\mathrm{analy}}(\mathbf{f}^\dagger)\big) &= \mathcal{E}_\nu(r_\nu)\mathcal{E}^*_\nu(r_\nu)\frac{(-1)^{r_\nu-1}}{(r_\nu-1)!}\cdot \mathrm{AJ}_{\mathbf{Q}_p}(\Delta_{r_\nu}^{\mathrm{bdp}})(\omega_{\mathbf{f}_\nu^\sharp} \otimes e_\zeta^{\otimes r_\nu - 1}) \\ &= \mathcal{E}_\nu(r_\nu)\mathcal{E}^*_\nu(r_\nu)\frac{(-1)^{r_\nu-1}}{(r_\nu-1)!}\cdot \log_{\mathbb{V}_\nu^\dagger}(\mathrm{loc}_{\mathfrak{p}}(\Phi^{\text{ét}}_{\mathbf{f}_\nu^\sharp, K}(\Delta_{r_\nu}^{\mathrm{bdp}})))(\omega_{\mathbf{f}_\nu^\sharp} \otimes e_\zeta^{\otimes r_\nu - 1}),\end{aligned} \tag{1.4.30}$$

where

$$\mathcal{E}_\nu(r_\nu) := \left(1 - \frac{p^{r_\nu-1}}{\nu(\mathbf{a}_p)}\right), \qquad \mathcal{E}^*_\nu(r_\nu) := \left(1 - \frac{\nu(\mathbf{a}_p)}{p^{r_\nu}}\right),$$

and $\Phi^{\text{ét}}_{\mathbf{f}_\nu^\sharp, K} := \pi_{\mathbf{f}_\nu^\sharp, \mathrm{N}^{r_\nu - 1}} \circ \Phi^{\text{ét}}_K$ with notations as in the diagram (1.2.3) defining $\mathrm{AJ}_{\mathbf{Q}_p}$.

On the other hand, by the specialization property of the map $\mathrm{Log}^{\eta}_{\mathbb{T}^\dagger}$ we have

$$\nu\big(\mathcal{L}_{\mathfrak{p}}^{\mathrm{arith}}(\mathbf{f}^\dagger)\big) = u\frac{(-1)^{r_\nu-1}}{(r_\nu-1)!}\mathcal{E}_\nu(r_\nu)^{-1}\mathcal{E}_\nu^*(r_\nu)\cdot\log_{\mathbb{V}_\nu^\dagger}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}_\nu))(\eta_\nu'). \tag{1.4.31}$$

Comparing (1.4.31) and (1.4.30), we thus conclude form (1.4.26) that

$$\log_{\mathbb{V}_\nu^\dagger}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}_\nu))(\omega_{\mathbf{f}_\nu^\sharp}\otimes e_\zeta^{\otimes r_\nu-1}) = \frac{1}{u}\mathcal{E}_\nu(r_\nu)^2\cdot\log_{\mathbb{V}_\nu^\dagger}(\mathrm{loc}_{\mathfrak{p}}(\Phi^{\text{ét}}_{\mathbf{f}_\nu^\sharp,K}(\Delta_{r_\nu}^{\mathrm{bdp}})))(\omega_{\mathbf{f}_\nu^\sharp}\otimes e_\zeta^{\otimes r_\nu-1}).$$

Since $\mathrm{Fil}^1 D_{\mathrm{dR}}(V_{\mathbf{f}_\nu^\sharp}(r_\nu-1))$ is spanned by $\omega_{\mathbf{f}_\nu^\sharp}\otimes e_\zeta^{\otimes r_\nu-1}$, it follows that

$$\log_{\mathbb{V}_\nu^\dagger}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}_\nu)) = \frac{1}{u}\mathcal{E}_\nu(r_\nu)^2\cdot\log_{\mathbb{V}_\nu^\dagger}(\mathrm{loc}_{\mathfrak{p}}(\Phi^{\text{ét}}_{\mathbf{f}_\nu^\sharp,K}(\Delta_{r_\nu}^{\mathrm{bdp}}))),$$

and since $\log_{\mathbb{V}_\nu^\dagger}$ is an isomorphism, that

$$\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}_\nu) = \frac{1}{u}\mathcal{E}_\nu(r_\nu)^2\cdot\mathrm{loc}_{\mathfrak{p}}(\Phi^{\text{ét}}_{\mathbf{f}_\nu^\sharp,K}(\Delta_{r_\nu}^{\mathrm{bdp}})). \tag{1.4.32}$$

Our nonvanishing assumption (1.4.24) implies by Corollary 1.4.10 that the localization map $\mathrm{loc}_{\mathfrak{p}}$ on $\mathrm{Sel}_{\mathrm{Gr}}(K,\mathbb{V}_\nu^\dagger)\cong\mathrm{Sel}_{\mathrm{Gr}}(K,V_{\mathbf{f}_\nu^\sharp}(r_\nu))$ is injective for almost all ν, and hence

$$\mathfrak{Z}_\nu = \frac{1}{u}\mathcal{E}_\nu(r_\nu)^2\cdot\Phi^{\text{ét}}_{\mathbf{f}_\nu^\sharp,K}(\Delta_{r_\nu}^{\mathrm{bdp}}) \tag{1.4.33}$$

for all but finitely many $\nu\in\mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight $k_\nu=2r_\nu$ as in the statement. In particular, we each such ν we have

$$\begin{aligned}\langle\mathfrak{Z}_\nu,\mathfrak{Z}_\nu\rangle_K &= \frac{1}{u^2}\mathcal{E}_\nu(r_\nu)^4\cdot\langle\Phi^{\text{ét}}_{\mathbf{f}_\nu^\sharp,K}(\Delta_{r_\nu}^{\mathrm{bdp}}),\Phi^{\text{ét}}_{\mathbf{f}_\nu^\sharp,K}(\Delta_{r_\nu}^{\mathrm{bdp}})\rangle_K\\ &= \frac{1}{u^2}\mathcal{E}_\nu(r_\nu)^4\cdot\frac{\langle\Phi^{\text{ét}}_{\mathbf{f}_\nu^\sharp,K}(\Delta_{r_\nu}^{\mathrm{heeg}}),\Phi^{\text{ét}}_{\mathbf{f}_\nu^\sharp,K}(\Delta_{r_\nu}^{\mathrm{heeg}})\rangle_K}{(4D)^{r_\nu-1}},\end{aligned}$$

where the last equality follows from Lemma 1.2.4 in light of (1.4.23) and (1.4.29). The result follows. □

Remark 1.4.13. As shown in the course of the proof of Theorem 1.4.12, we deduce in fact the equality (1.4.33) of global cohomology classes for almost all ν as in the statement.

1.5. $\mathbb{I}$-adic Gross–Zagier formula

Let $\mathscr{G}_\infty$ be the Galois group of the unique $\mathbf{Z}_p^2$-extension of K, and denote by

$$\mathcal{L}_p(\mathbf{f}\otimes K)\in\mathbb{I}[[\mathscr{G}_\infty]]$$

the "three-variable" p-adic L-function attached to $\mathbf{f}$ over K constructed in [**SU13**, §12.3]. Letting D_∞ (resp. C_∞) denote the Galois group of the anticyclotomic (resp. cyclotomic)

$\mathbf{Z}_p$-extension of K, we identify $\mathbb{I}[[\mathscr{G}_\infty]]$ with $\mathbb{I}_\infty[[C_\infty]]$ where $\mathbb{I}_\infty := \mathbb{I}[[D_\infty]]$, and choosing a generator γ_o of C_∞, we may thus expand

$$\mathcal{L}_p(\mathbf{f}\otimes K) = \mathcal{L}_{\mathbf{f},K} + \mathcal{L}'_{\mathbf{f},K}(\gamma_o - 1) + \mathcal{L}''_{\mathbf{f},K}(\gamma_o-1)^2 + \cdots \tag{1.5.1}$$

with coefficients $\mathcal{L}^{(i)}_{\mathbf{f},K} \in \mathbb{I}_\infty$.

Recall that the big Heegner point $\mathfrak{Z}$ lies in strict Greenberg Selmer group $\mathrm{Sel}_{\mathrm{Gr}}(K, \mathbb{T}^\dagger)$, and that (as explined in [**How07b**, p. 113] for example) this group identified with Nekovář's extended Selmer group $\widetilde{H}^1_f(K, \mathbb{T}^\dagger)$.

By [**Nek06**, §11], there exists an $\mathbb{I}$-bilinear "height pairing"

$$\langle\ ,\ \rangle_{K,\mathbb{T}^\dagger} : \widetilde{H}^1_f(K,\mathbb{T}^\dagger) \times \widetilde{H}^1_f(K,\mathbb{T}^\dagger) \longrightarrow \mathbb{I}$$

such that

$$\nu\left(\langle \mathfrak{Y}, \mathfrak{Y}'\rangle_{K,\mathbb{T}^\dagger}\right) = \langle \mathfrak{Y}_\nu, \mathfrak{Y}'_\nu\rangle_K \tag{1.5.2}$$

for all $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ and $\mathfrak{Y}, \mathfrak{Y}' \in \widetilde{H}^1_f(K, \mathbb{T}^\dagger)$.

THEOREM 1.5.1. *With notations and assumptions as in Theorem 1.4.12, if* $\mathcal{L}'_{\mathbf{f},K} \in \mathbb{I}_\infty$ *is the linear term in the expansion* (1.5.1), *then*

$$\mathcal{L}'_{\mathbf{f},K}(\mathbb{1}_K) = \langle \mathfrak{Z}, \mathfrak{Z}\rangle_{K,\mathbb{T}^\dagger}$$

up to a unit in $\mathbb{I}^\times$.

PROOF. For every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of even weight as Theorem 1.4.12, the work [**Nek95**] of Nekovář produces a two-variable p-adic L-function $\mathcal{L}_p(\mathbf{f}_\nu \otimes K) \in F_\nu[[\mathscr{G}_\infty]]$. After expanding

$$\mathcal{L}_p(\mathbf{f}_\nu\otimes K) = \mathcal{L}_{\mathbf{f}_\nu,K} + \mathcal{L}'_{\mathbf{f}_\nu,K}(\gamma_o - 1) + \mathcal{L}''_{\mathbf{f}_\nu,K}(\gamma_o-1)^2 + \cdots$$

similarly as in (1.5.1), [**SU13**, Thm. 12.3.2(ii)] implies on the one hand that

$$\nu(\mathcal{L}'_{\mathbf{f},K}(\mathbb{1}_K)) = \mathcal{L}'_{\mathbf{f}_\nu,K}(\mathbb{1}_K) \tag{1.5.3}$$

up to a unit in $\mathcal{O}_L^\times$, and on the other the main result of [**Nek95**] can be rewritten as the p-adic Gross–Zagier formula

$$\mathcal{L}'_{\mathbf{f}_\nu,K}(\mathbb{1}_K) = \left(1 - \frac{p^{r_\nu - 1}}{\nu(\mathbf{a}_p)}\right)^4 \frac{\langle \Phi^{\text{ét}}_{\mathbf{f}^\sharp_\nu,K}(\Delta^{\text{heeg}}_{r_\nu}), \Phi^{\text{ét}}_{\mathbf{f}^\sharp_\nu,K}(\Delta^{\text{heeg}}_{r_\nu})\rangle_K}{u^2(4D)^{r_\nu - 1}}. \tag{1.5.4}$$

Combining (1.5.2) applied to $\mathfrak{Z}$ with (1.5.3) and (1.5.4), the result follows immediately from Theorem 1.4.12. $\square$

CHAPTER 2

p-adic L-functions and the p-adic variation of Heegner points

Summary

A construction due to Ben Howard [**How07b**], obtained by patching twisted Kummer images of CM points over a tower of modular curves, produces a two-variable cohomology class $\mathfrak{Z}_\infty$ extending in the anticyclotomic direction and over a Hida family. In this paper we find the relation, via a generalization of the Coleman power series map, between $\mathfrak{Z}_\infty$ and a two-variable anticyclotomic p-adic L-function interpolating central critical Rankin–Selberg L-values. Based on this relation, and in the extension of Kolyvagin's methods developed by Mazur–Rubin [**MR04**] and Howard [**How04a**], we then deduce new results on the vanishing of Selmer groups for the Rankin–Selberg convolution of a cusp form with a theta series of higher weight in cases predicted by the Bloch–Kato conjecture, as well as a weak divisibility in an anticyclotomic Iwasawa main conjecture.

Introduction

In this paper we study the Bloch–Kato conjecture for the Rankin–Selberg convolution of a cusp form f with a theta series of higher weight, as well as an associated anticyclotomic Iwasawa main conjecture. Perhaps motivated by the influencial work [**GZ86**] of Gross–Zagier, this setting appears to have received less attention than the case where the weight of f is the largest, when in fact most often only the theta series associated to finite order characters are considered (see [**PR87**], [**Nek95**] and [**Zha97**] for example).

As is well-known, the aforementioned conjectures relate the sizes of certain Selmer groups to special values of L-functions and to p-adic L-functions, respectively, and after the methods introduced by Kolyvagin, the Euler system of Heegner points and their higher weight generalizations are, in the setting of [**GZ86**], the fundamental input in the proof of one of the two inequalities predicted by those conjectures. Here we show that the same anticyclotomic Euler systems can be exploited to prove results towards those conjectures in the setting in which we have placed ourselves, and in fact by methods that appear to be more parallel to Kato's [**Kat04**] than in previous studies of the Euler systems derived from the classical Heegner points and cycles.

Let p be an odd prime and let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$. Let $f \in S_k(\Gamma_0(N))$ be a normalized newform of even weight $k \geq 2$ and level N prime to p, and assume that f is ordinary at p and that every prime divisor of N splits in K. Let $\mathbf{f}$ be the Hida family passing through (the ordinary p-stabilization of) f and let $\mathbb{T}$ be the big Galois representation associated with $\mathbf{f}$.

A construction due to Ben Howard, obtained by patching twisted Kummer images of CM points over a tower of modular curves, produces a two-variable cohomology class

$$\mathfrak{Z}_\infty \in \widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty, \mathbb{T}^\dagger),$$

where $\mathbb{T}^\dagger$ is a so-called critical twist of $\mathbb{T}$, and $\widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty, \mathbb{T}^\dagger) := \varprojlim_t \widetilde{H}^1_f(K_t, \mathbb{T}^\dagger)$ is Nekovář's extended Selmer group for the self-dual representation $\mathbb{T}^\dagger$ over the anticyclotomic $\mathbf{Z}_p$-extension of K. By construction, letting $\rho_f : G_{\mathbf{Q}} \longrightarrow \mathrm{Aut}_L(V_f)$ be the self-dual Tate twist of the p-adic Galois representation associated to f, with coefficients in the finite extension $L/\mathbf{Q}_p$, there is a Galois equivariant specialization map $\nu_f : \mathbb{T}^\dagger \longrightarrow V_f$ which induces a map on cohomology also denoted by ν_f in the following. Similarly, attached to every continuous character $\phi : \mathrm{Gal}(K_\infty/K) \longrightarrow L^\times$ one may define specialization maps

$$\nu_\phi : \widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty, V_f) \longrightarrow H^1(K, V_f \otimes \phi),$$

where $V_f \otimes \phi$ denotes the twist of V_f by ϕ as a G_K-representation.

Assume that ϕ has finite order, and let $t > 0$ be the smallest positive integer such that ϕ factors through $\mathrm{Gal}(K_t/K)$. If t is large enough, then the class $\nu_\phi(\nu_f(\mathfrak{Z}_\infty)) =: \nu_f(\mathfrak{Z}_\infty)^\phi$ is known, as a consequence of [**How07b**, Cor. 3.1.2], to generate a rank 1 subspace of the Bloch–Kato Selmer group $\mathrm{Sel}(K, V_f \otimes \phi)$. The resulting systematic growth of the rank of the Selmer groups $\mathrm{Sel}(K_t, V_f)$ as one goes up the anticyclotomic tower $K_\infty = \bigcup_t K_t$ is in agreement with the equality

$$\dim_L \mathrm{Sel}(K, V_f \otimes \phi) = \mathrm{ord}_{s=k/2} L(f, \phi^{-1}, s) \tag{2.0.1}$$

predicted by the Bloch–Kato conjecture [**BK90**], since for all finite order ϕ as above the sign in the functional equation for the Rankin–Selberg L-function $L(f, \phi^{-1}, s)$ is -1, forcing the vanishing of $L(f, \phi^{-1}, s)$ at $s = k/2$.

Now let χ be an anticyclotomic Hecke character of K of infinity type $(k/2, -k/2)$ and conductor dividing a fixed integral ideal $\mathfrak{N}$ of K with

$$\mathcal{O}_K/\mathfrak{N} \cong \mathbf{Z}/N\mathbf{Z}.$$

Then, in contrast to the above, if ϕ is a any continuous character of $\mathrm{Gal}(K_\infty/K)$ of finite order, the Rankin–Selberg L-function $L(f, \chi^{-1}\phi^{-1}, s)$ has a functional equation with sign $+1$, and so the values $L(f, \chi^{-1}\phi^{-1}, k/2)$ are not necessarily zero. Moreover, general conjectures (as in [**Gre94a**]) lead to the expectation that the self-dual L-functions $L(f, \chi^{-1}\phi^{-1}, s)$, as ϕ varies, should generically have the least order of vanishing at the central point $s = k/2$ allowed by the sign in the functional equation, and hence one expects the nonvanishing of $L(f, \chi^{-1}\phi^{-1}, k/2)$ to occur generically, in the sense that $L(f, \chi^{-1}\phi^{-1}, k/2)$ should not vanish for all but finitely many ϕ.

Let $\mathcal{O}$ be the ring of integers of L, let $T_f \subset V_f$ be a fixed $G_{\mathbf{Q}}$-stable $\mathcal{O}$-lattice, and denote by $\bar{\rho}_f$ the (semi-simple) mod p representation obtained by reducing the resulting $\rho_f : G_{\mathbf{Q}} \longrightarrow \mathrm{Aut}_{\mathcal{O}}(T_f)$ modulo the maximal ideal of $\mathcal{O}$. Our first main result gives evidence for the Bloch–Kato conjecture (2.0.1) for the representation $V_f \otimes \chi\phi$ over K in the above generic case (cf. Theorem 2.4.2 and Corollary 2.5.3).

Theorem A. *Assume that p splits in K and that $\bar{\rho}_f|_{G_K}$ is absolutely irreducible. Then*

$$L(f, \chi^{-1}\phi^{-1}, k/2) \neq 0 \quad \Longrightarrow \quad \mathrm{Sel}(K, V_f \otimes \chi\phi) = 0.$$

In other words, Theorem A shows that under the above hypotheses there are no nonzero classes in $H^1(K, V_f \otimes \chi\phi)$ which are unramified outside the places in K above p and "finite" (in the sense of Bloch–Kato) at the places above p.

Note that under the above assumption on $\mathfrak{N}$ it follows from [**How07b**, Prop. 2.4.5] that the classes $\nu_f(\mathfrak{z}_\infty)^\phi$ are in fact "finite" at the places above p, and that by [**How07b**, Cor. 3.1.2] they are known nonzero for all but finitely many ϕ. However, due to the twist by the character χ (of higher weight), the classes $\nu_f(\mathfrak{z}_\infty)^{\chi\phi}$ do not satisfy in general the right local conditions at the places above p in order to lie in the Bloch–Kato Selmer group of $V_f \otimes \chi$, and in fact as a key step towards the proof of Theorem A we exhibit the value $L(f, \chi^{-1}\phi^{-1}, k/2)$ as the *obstruction* for the class $\nu_f(\mathfrak{z}_\infty)^{\chi\phi}$ to be "finite" (or equivalently in this case, trivial) at a place above p. We refer the reader to Theorem 2.5.1 for the precise statement, while in the next paragraph we just indicate the two key ingredients that enter into the proof.

Based on an explicit form of Waldspurger's special value formula, a number of recent works (starting with [**BDP13**], and followed by [**Bra11**], [**Hsi12**]) construct an anticyclotomic p-adic L-function $\mathscr{L}(f, \chi)$ interpolating the square root of the central critical values $L(f, \chi^{-1}\phi^{-1}, k/2)$ for varying ϕ. The first key ingredient in the proof of Theorem 2.5.1 is the relation that we establish between $\mathscr{L}(f, \chi)$ and the class $\nu_f(\mathfrak{z}_\infty)$ via a generalization of the Coleman power series map (see Theorem 2.3.1); the second is the "explicit reciprocity law" of Perrin-Riou proven by Colmez [**Col98**], to which we can appeal[1] to easily deduce the aforementioned interpretation of the L-value $L(f, \chi^{-1}\phi^{-1}, k/2)$.

The square of the p-adic L-function $\mathscr{L}(f, \chi)$ conjecturally governs the arithmetic of the G_K-representation $V_f \otimes \chi$ over the anticyclotomic $\mathbf{Z}_p$-extension K_∞/K, which as hinted at above differs markedly from that of the representation V_f itself.

Set $A_f := V_f/T_f \cong \mathrm{Hom}_{\mathcal{O}}(T_f, \boldsymbol{\mu}_{p^\infty})$. Various generalizations of Iwasawa's theory (especially as developed in [**Gre94b**]) lead to the expectation that the Greenberg Selmer group

$$\mathrm{Sel}(K_\infty, A_f \otimes \chi^{-1}) := \varinjlim_t \mathrm{Sel}(K_t, A_f \otimes \chi^{-1}),$$

is *cotorsion* over the Iwasawa algebra $\Lambda_{\mathcal{O}}(K_\infty) := \mathcal{O}[[\mathrm{Gal}(K_\infty/K)]]$, and that the characteristic ideal of its Pontryagin dual is generated by $\mathscr{L}(f, \chi)^2$. In the direction of this conjecture, we can show the following result (cf. Thm. 2.5.5.)

THEOREM B. *Assume that p splits in K and that $\bar{\rho}_f|_{G_K}$ is absolutely irreducible, and let $X(f, \chi)$ be the Pontryagin dual of* $\mathrm{Sel}(K_\infty, A_f \otimes \chi^{-1})$. *Then*

$$\mathrm{char}_{\Lambda_{\mathcal{O}}(K_\infty)}(X(f, \chi)) \quad \textit{divides} \quad (p^n \cdot \mathscr{L}(f, \chi)^2)$$

for some $n \geq 0$. In particular, $X(f, \chi)$ is $\Lambda_{\mathcal{O}}(K_\infty)$-torsion.

[1] In fact, we use a variant due to Shaowei Zhang [**Zha04**] for height 1 Lubin–Tate formal groups over $\mathbf{Z}_p$.

Similarly as Theorem A, the proof of Theorem B follows after Theorem 2.3.1 from an application of Kolyvagin's methods in the form axiomatized by Mazur–Rubin [**MR04**] and adapted by Howard [**How04a**] to anticyclotomic settings germane to ours.

This paper is organized as follows. In the next section we briefly recall the construction of the class $\mathfrak{Z}_\infty$ and of the p-adic L-function $\mathscr{L}(f,\chi)$. In Section 2.2, by combining the work of Ochiai [**Och03**] with some ideas in the recent work [**LZ11**] of Loeffler–Zerbes, we deduce a two-variable generalization of Perrin-Riou's regulator map adapted to the local situation at p that arises in our setting. (As well as in other places, the assumption that p splits in K is crucial at this point.) In Section 2.3 we extend the computations in [**Cas13a**, §4.1] to show that a two-variable p-adic L-function specializing to $\mathscr{L}(f,\chi)$ is obtained as the image of $\mathfrak{Z}_\infty$ via the two-variable regulator map from Section 2.2, and with this relation at hand, the proof of our arithmetic applications is given in Section 2.5.

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2.1. Preliminaries

As implicit in the above introduction, we fix throughout a choice of embeddings $\imath_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and $\imath_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$.

We begin by describing our basic set-up. Let $p \geq 5$ be a prime, and let

$$\mathbf{f} = \sum_{n=1}^{\infty} \mathbf{a}_n q^n \in \mathbb{I}[[q]] \tag{2.1.1}$$

be a Hida family of tame level N prime to p, where $\mathbb{I}$ is an integral complete noetherian local ring finite flat over $\Lambda_{\mathcal{O}} := \mathcal{O}[[1+p\mathbf{Z}_p]]$. Let $K = \mathbf{Q}(\sqrt{-D})$ be an imaginary quadratic field, let $\mathfrak{p}$ be the prime in K above p induced by the embedding $\imath_p$, and denote by $h_K = \#\mathrm{Pic}(\mathcal{O}_K)$ the class number of K. It will be assumed throughout that K satisfies the following hypotheses:

Assumptions 2.1.1. (1) *$p \nmid h_K$;*

(2) *D is odd and $\neq 3$, so that $\mathcal{O}_K^\times = \pm 1$;*

(3) *Every prime divisor of N splits in K;*

(4) *p splits in K;*

REMARK 2.1.2. For the purposes of this paper, the first two assumptions are made mostly for simplicity and could be relaxed with some more work; the third one is more serious, but could also be relaxed by working with nonsplit quaternion algebras. In contrast, the assumption that p splits in K is a crucial one for us, and new fundamentally new ideas would seem necessary in order to treat the case where p is inert or ramifies in K.

For each $t \geq 0$, let $\mathcal{O}_{p^{t+1}} = \mathbf{Z} + p^{t+1}\mathcal{O}_K$ be the order of K of conductor p^{t+1}, and let $H_{p^{t+1}}$ be the ring class field of K of conductor p^{t+1}. Class field theory gives a canonical isomorphism

$$\mathrm{Pic}(\mathcal{O}_{p^{t+1}}) \xrightarrow{\sim} \mathrm{Gal}(H_{p^{t+1}}/K)$$

sending the class associated with a proper $\mathcal{O}_{p^{t+1}}$-ideal $\mathfrak{b}$ of K to the Artin symbol $\sigma_{\mathfrak{b}} \in \mathrm{Gal}(H_{p^{t+1}}/K)$. In the same manner $\mathrm{Pic}(\mathcal{O}_K)$ is identified with the Galois group $\mathrm{Gal}(H/K)$ of the Hilbert class field of K.

Let $H_{p^\infty} = \bigcup_{t\geq 0} H_{p^{t+1}}$ and set $\mathcal{G} = \mathrm{Gal}(H_{p^\infty}/K)$. The field H_{p^∞} contains K_∞, the maximal anticyclotomic $\mathbf{Z}_p$-extension of K, and we have a decomposition

$$\mathcal{G} = \mathcal{G}_{\mathrm{tors}} \times D_\infty, \tag{2.1.2}$$

with $\mathcal{G}_{\mathrm{tors}} = \mathrm{Gal}(H_{p^\infty}/K_\infty)$ finite, of order prime to p after our assumption on h_K, and with $D_\infty = \mathrm{Gal}(K_\infty/K)$ topologically isomorphic to $\mathbf{Z}_p$.

We say that a continuous $\mathcal{O}$-algebra homomorphism $\nu : \mathbb{I} \longrightarrow \overline{\mathbf{Q}}_p^\times$ is an *arithmetic prime* if there is an integer $k_\nu \geq 2$, called the weight of ν, such that the composition $\Gamma \longrightarrow \mathbb{I}^\times \xrightarrow{\nu} \overline{\mathbf{Q}}_p^\times$ agrees with the map $\gamma \longmapsto \gamma^{k_\nu - 2}$ on some open subgroup of Γ. The set of arithmetic primes of $\mathbb{I}$ is denoted by $\mathcal{X}_{\mathrm{arith}}(\mathbb{I})$.

Let $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ have even weight $k_\nu = 2r_\nu \geq 2$, and denote by $\mathbf{f}_\nu$ the ordinary p-stabilized newform of weight k_ν whose q-expansion is given by $\nu(\mathbf{f})$. (See [**Hid86a**, Cor. 1.3].)

Let $\theta : \mathbf{Z}_p^\times \longrightarrow \mathbb{I}^\times$ be a fixed choice of a *critical character* as defined in [**How07a**, §2], and consider the p-adic modular form

$$\mathbf{f}_\nu^\dagger := \mathbf{f}_\nu \otimes \theta_\nu^{-1} \tag{2.1.3}$$

obtained by twisting $\mathbf{f}_\nu$ by the character $\theta_\nu : \mathbf{Z}_p^\times \xrightarrow{\theta} \mathbb{I}^\times \xrightarrow{\nu} \overline{\mathbf{Q}}_p^\times$. (See [**Gou88**, §III.6.2] for the definition of the twist of p-adic modular forms by characters not necessarily of finite order, or [**Cas13a**, §2.1] for a brief review of the facts that we use here.)

REMARK 2.1.3. By the construction of θ, we have $\theta_\nu(z) = z^{r_\nu - 1}\vartheta_\nu(z)$ for all $z \in \mathbf{Z}_p^\times$, with ϑ_ν a finite order character, and hence $\mathbf{f}_\nu^\dagger = d^{1-r_\nu}(\mathbf{f}_\nu \otimes \vartheta_\nu^{-1})$, where d is the operator on

p-adic modular forms acting as $q\frac{d}{dq}$ on q-expansions. Moreover, as explained in [**How07a**, p. 808], the nebentypus of $\mathbf{f}_\nu$ agrees with ϑ_ν^2, and hence the nebentypus of $\mathbf{f}_\nu \otimes \vartheta_\nu^{-1}$ is trivial.

2.1.1. p-adic Rankin L-series. We define a canonical measure on D_∞ attached to the p-adic modular form $\mathbf{f}_\nu^\dagger$. Using the decomposition (2.1.2), this will be obtained from a measure $d\mu_{\mathbf{f}_\nu^\dagger}$ on $\mathcal{G}$ by restricting to characters

$$\chi = \chi_t \cdot \chi_w \in \mathrm{Hom}_{\mathrm{cont}}(\mathcal{G}, \overline{\mathbf{Q}}_p^\times)$$

with a fixed tame part $\chi_t : \mathcal{G}_{\mathrm{tors}} \longrightarrow \overline{\mathbf{Q}}_p^\times$.

Let A/H be a fixed elliptic curve with CM by the maximal order $\mathcal{O}_K$ of K, and denote by $\widehat{\mathbf{Z}}_p^{\mathrm{nr}}$ the completion of the ring of integers of the maximal unramified extension of $\mathbf{Q}_p$. Since p splits in K, the curve A has ordinary reduction at the prime of H above $\mathfrak{p}$ induced by $\imath_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$, and we may fix a trivialization of A over $\widehat{\mathbf{Z}}_p^{\mathrm{nr}}$, i.e. an isomorphism

$$\imath_A : \hat{A} \xrightarrow{\sim} \hat{\mathbf{G}}_m$$

as formal groups. By the third of our Assumptions 2.1.1, we may choose an $\mathcal{O}_K$-ideal with $\mathcal{O}_K/\mathfrak{N} \cong \mathbf{Z}/N\mathbf{Z}$ that we also fix from now on.

The group $\mathrm{Pic}(\mathcal{O}_K)$ acts naturally on the set of triples $(A', \alpha_{A'}, \imath_{A'})$ consisting of an elliptic curve A' with CM by $\mathcal{O}_K$ equipped with an arithmetic $\Gamma_1(N)$-level structure $\alpha_{A'} : \boldsymbol{\mu}_N \hookrightarrow A'[\mathfrak{N}]$ and a trivialization $\imath_{A'} : \hat{A}' \xrightarrow{\sim} \hat{\mathbf{G}}_m$ over $\widehat{\mathbf{Z}}_p^{\mathrm{nr}}$. After representing each class $[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)$ by an $\mathcal{O}_K$-ideal $\mathfrak{a}$ prime to $\mathfrak{N}p$, and writing

$$\varphi_{\mathfrak{a}} : A' \longrightarrow A'_{\mathfrak{a}} := A'/A'[\mathfrak{a}]$$

for the natural projection and $\hat{\varphi}_{\mathfrak{a}}$ for the induced isomorphism on the associated formal groups, this action is defined by setting

$$\mathfrak{a} * (A', \alpha_{A'}, \imath_{A'}) = (A'_{\mathfrak{a}}, \alpha_{A'_{\mathfrak{a}}}, \imath_{A'_{\mathfrak{a}}}),$$

where $\alpha_{A'_{\mathfrak{a}}} := \varphi_{\mathfrak{a}} \circ \alpha_{A'}$ and $\imath_{A'_{\mathfrak{a}}} := \imath_{A'} \circ \hat{\varphi}_{\mathfrak{a}}^{-1}$.

For each $[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)$, define $\mathrm{d}\mu_{\mathbf{f}_\nu^\dagger,\mathfrak{a}}$ to be measure on $\mathbf{Z}_p^\times$ such that

$$\int_{\mathbf{Z}_p^\times} \phi(z) \mathrm{d}\mu_{\mathbf{f}_\nu^\dagger,\mathfrak{a}}(z) = d^{-1}\mathbf{f}_\nu^\dagger \otimes \phi(\mathfrak{a} * (A, \alpha_A, \imath_A)),$$

for all $\phi \in \mathrm{Hom}_{\mathrm{cont}}(\mathbf{Z}_p^\times, \overline{\mathbf{Q}}_p^\times)$, and using the (non-canonical) decomposition

$$\mathcal{G} \cong \bigsqcup_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)} \mathfrak{a}^{-1}\mathbf{Z}_p^\times,$$

define $\mathrm{d}\mu_{\mathbf{f}_\nu^\dagger}$ by setting

$$(2.1.4)\qquad \int_{\mathcal{G}} \phi(x)\mathrm{d}\mu_{\mathbf{f}_\nu^\dagger}(x) = \sum_{[\mathfrak{a}]\in\mathrm{Pic}(\mathcal{O}_K)} \mathrm{N}\mathfrak{a}^{-m}\int_{\mathbf{Z}_p^\times}\phi(\mathfrak{a}^{-1}z)\mathrm{d}\mu_{\mathbf{f}_\nu^\dagger,\mathfrak{a}}(z),$$

for all $\phi\in\mathrm{Hom}_{\mathrm{cont}}(\mathcal{G},\overline{\mathbf{Q}}_p^\times)$, where $m\in\mathbf{Z}$ is such that

$$(2.1.5)\qquad \phi(\mathfrak{a}^{-1}z) = \phi^{-1}(\mathfrak{a})z^m\phi_0(z),\quad \text{for all } z\in\mathbf{Z}_p^\times,$$

with ϕ_0 a finite order character.

In particular, if a character $\phi\in\mathrm{Hom}_{\mathrm{cont}}(D_\infty,\overline{\mathbf{Q}}_p^\times)\subset\mathrm{Hom}_{\mathrm{cont}}(\mathcal{G},\overline{\mathbf{Q}}_p^\times)$ is the p-adic avatar of a Hecke character of K of infinity type $(m,-m)$ with $m\in\mathbf{Z}$, it follows immediately from the definition (2.1.4) and Remark 2.1.3 that

$$(2.1.6)\quad \int_{D_\infty}\phi(x)\mathrm{d}\mu_{\mathbf{f}_\nu^\dagger}(x) = \sum_{[\mathfrak{a}]\in\mathrm{Pic}(\mathcal{O}_K)}\phi^{-1}(\mathfrak{a})\mathrm{N}\mathfrak{a}^{-m}\cdot d^{m-r_\nu}(\mathbf{f}_\nu\otimes\vartheta_\nu^{-1})\otimes\phi_0(\mathfrak{a}*(A,\alpha_A,\imath_A)).$$

We will use $\mathscr{L}_\mathfrak{p}(\mathbf{f}_\nu^\dagger)$ to denote the function (2.1.6) of the character ϕ^{-1}, and refer to it as the (square-root) *anticyclotomic p-adic L-function* attached to $\mathbf{f}_\nu$, K, and $\mathfrak{p}$. This choice of terminology is justified by the next Theorem 2.1.4.

Denote by Θ the Galois character given by the composition $G_\mathbf{Q}\xrightarrow{\varepsilon_{\mathrm{cyc}}}\mathbf{Z}_p^\times\xrightarrow{\theta}\mathbb{I}^\times$, where $\varepsilon_{\mathrm{cyc}}$ is the p-adic cyclotomic character, and let χ_ν be the Hecke character of K defined by

$$\chi_\nu(x) = \Theta_\nu(\mathrm{Art}_\mathbf{Q}(\mathrm{N}_{K/\mathbf{Q}}(x)))\qquad \text{for all } x\in\mathbb{A}_K^\times,$$

where $\Theta_\nu=\theta_\nu\circ\varepsilon_{\mathrm{cyc}}$. As shown in [**How07a**, §3] for arithmetic primes of weight 2, the central character $\chi_\nu|_{\mathbb{A}_\mathbf{Q}^\times}$ of χ_ν is the inverse of the nebentypus of $\mathbf{f}_\nu$.

THEOREM 2.1.4 (Bertolini–Darmon–Prasanna, Brakočević, Hsieh). *The rule $\mathscr{L}_\mathfrak{p}(\mathbf{f}_\nu^\dagger)$ defines a measure on D_∞ characterized by the following interpolation property. For every anticyclotomic Hecke character ϕ of K of infinity type $(m,-m)$ with $m\geq k_\nu/2$:*

$$\left(\frac{\mathscr{L}_\mathfrak{p}(\mathbf{f}_\nu^\dagger)(\phi)}{\Omega_p^{k_\nu+2m}}\right)^2 = C(\mathbf{f}_\nu,\chi_\nu\phi^{-1})\left(1-\frac{\nu(\mathbf{a}_p)}{\chi_\nu^{-1}\phi(\bar{\mathfrak{p}})}\right)^2\frac{L^{\{DN\}}(\mathbf{f}_\nu,\chi_\nu\phi^{-1},1)}{\Omega_\infty^{2(k_\nu+2m)}},$$

where $(\Omega_\infty,\Omega_p)\in\mathbf{C}^\times\times\mathbf{C}_p^\times$ are periods associated with the CM elliptic curve A, $C(\mathbf{f}_\nu,\chi_\nu\phi^{-1})$ is an explicit nonzero constant, and $L^{\{DN\}}(\mathbf{f}_\nu,\chi_\nu\phi^{-1},1)$ is the central critical value of the DN-imprimitive Rankin–Selberg L-function attached to $\mathbf{f}_\nu$ and the theta series of $\chi_\nu\phi^{-1}$.

PROOF. After noting [**Hid10**, Remark 4.2(a)], this follows from the main result of [**Bra11**, §9]. □

REMARK 2.1.5. Let $\widehat{\mathbf{Z}}_p$ be the completion of the ring of integers of $\overline{\mathbf{Q}}_p$, and denote by $\mathrm{Meas}(D_\infty, \widehat{\mathbf{Z}}_p)$ the space of $\widehat{\mathbf{Z}}_p$-valued measures on D_∞. It is easy to define a "two-variable" p-adic L-function interpolating $\mathscr{L}_{\mathfrak{p}}(\mathbf{f}_\nu^\dagger)$ for different ν.

Indeed, letting ν vary over the Zariski dense subset $\mathcal{X}_{\mathrm{arith}}(\mathbb{I}) \subset \mathrm{Spf}(\mathbb{I})(\overline{\mathbf{Q}}_p)$, we obtain from (2.1.6) a continuous map $\mathrm{Spf}(\mathbb{I})(\overline{\mathbf{Q}}_p) \longrightarrow \mathrm{Meas}(D_\infty, \widehat{\mathbf{Z}}_p)$. In the terminology of [**Hid88**, §3], we thus have a *generalized measure* on $D_\infty \times \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ corresponding to an element $\mathscr{L}_{\mathfrak{p}}(\mathbf{f}^\dagger) \in \mathrm{Meas}(D_\infty, \widehat{\mathbf{Z}}_p)\widehat{\otimes}_{\mathbf{Z}_p}\mathbb{I} \cong \mathbb{I}[[D_\infty]]$, which can alternatively be seen as an $\mathbb{I}$-valued measure on D_∞. (Cf. [**Bra12**].)

One knows the nontriviality of $\mathscr{L}_{\mathfrak{p}}(\mathbf{f}_\nu^\dagger)$ in most cases. For the precise statement, let

$$\rho_{\mathbf{f}} : G_{\mathbf{Q}} \longrightarrow \mathrm{Aut}_{\mathbb{I}}(\mathbb{T}) \tag{2.1.7}$$

be the Galois representation associated with $\mathbf{f}$ by Hida, and denote by $\bar{\rho}_{\mathbf{f}}$ the mod p representation obtained by reducing $\rho_{\mathbf{f}}$ modulo the maximal ideal of $\mathbb{I}$.

THEOREM 2.1.6 (Hsieh). *Assume that $\bar{\rho}_{\mathbf{f}}$ restricted to G_K is absolutely irreducible. Then for all $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$, the p-adic L-function $\mathscr{L}_{\mathfrak{p}}(\mathbf{f}_\nu^\dagger)$ is not identically zero.*

PROOF. This follows *a fortiori* from the vanishing of the μ-invariant established in Theorem 6.2 of [**Hsi12**]. (See also [*loc.cit.*, Remark. 6.4].) □

REMARK 2.1.7. The irreducibility assumption in Theorem 2.1.6 is clearly not satisfied when $\mathbf{f}$ is the Hida family associated to an $\mathbb{I}$-adic character $\varphi : G_K \longrightarrow \mathbb{I}^\times$, but in that case one also knows the nontriviality of $\mathscr{L}_{\mathfrak{p}}(\mathbf{f}_\nu^\dagger)$.

Indeed, in the CM case for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ the p-adic Rankin L-series $\mathscr{L}_{\mathfrak{p}}(\mathbf{f}_\nu^\dagger)$ factors as the product of two conjugate Katz p-adic L-functions (as it is easily seen by comparing their values at the characters within the range of p-adic interpolation; see [**BDP12**, Thm. 3.16] for the calculations in the unramified case), from where the result follows from the nontriviality of the latter.

2.1.2. Big Heegner points. Define the critical twist of $\mathbb{T}$ to be $\mathbb{T}^\dagger := \mathbb{T} \otimes_{\mathbb{I}} \mathbb{I}(\Theta^{-1})$, where $\Theta = \theta \circ \varepsilon_{\mathrm{cyc}}$ with $\theta : \mathbf{Z}_p^\times \longrightarrow \mathbb{I}^\times$ the critical character chosen in Section 2.1.1.

We briefly recall from [**How07b**, §3.3] the definition of an element of a canonical class $\mathfrak{Z}_\infty = \varprojlim_t \mathfrak{Z}_t \in \widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty, \mathbb{T}^\dagger)$ in Nekovář's extended Selmer group for the self-dual representation $\mathbb{T}^\dagger$ over the anticyclotomic $\mathbf{Z}_p$-extension K_∞/K.

For each $t \geq 0$, K_t is contained in $H_{p^{t+1}}$, and the class $\mathfrak{Z}_t$ is defined to be the image of $U_p^{-t} \cdot \mathfrak{X}_{p^{t+1}}$ under the corestriction map

$$\widetilde{H}^1_f(H_{p^{t+1}}, \mathbb{T}^\dagger) \longrightarrow \widetilde{H}^1_f(K_t, \mathbb{T}^\dagger).$$

By an extension of the arguments of Cornut and Vatsal in their proof of Mazur's conjecture on higher Heegner points, the nontriviality of the classes $\nu(\mathfrak{Z}_\infty)$, for all $\nu \in \mathcal{X}_{\text{arith}}(\mathbb{I})$, was proven by Howard. (See [**How07b**, Thm. 3.1.1].)

2.2. p-adic logarithm maps

In [**Och03**], Ochiai constructs a map interpolating over the arithmetic primes in a Hida family the cyclotomic p-adic regulator of Perrin-Riou. In this section, using ideas from the recent work of Loeffler–Zerbes [**LZ11**], we show how to assemble Ochiai's construction at each of the finite layers in the unramified $\mathbf{Z}_p$-extension of $\mathbf{Q}_p$ into a two-variable p-adic regulator map.

2.2.1. Nearly-ordinary deformation. Let $\mathbf{f}$ be a Hida family and $\mathbb{T}$ be its associated Galois representation as introduced in (2.1.1) and (2.1.7), respectively.

Let K be an imaginary quadratic field for which the Assumptions 2.1.1 are satisfied (in particular, p splits in K), denote by C_∞ be the Galois group of the *cyclotomic* $\mathbf{Z}_p$-extension of K, and let Λ_{cyc} be the free $\Lambda(C_\infty) := \mathbf{Z}_p[[C_\infty]]$-module of rank one equipped with the natural G_K-action on group-like elements. Finally, set $\mathcal{I} := \mathbb{I}\widehat{\otimes}_{\mathbf{Z}_p}\Lambda(C_\infty)$.

Definition 2.2.1. The *nearly-ordinary deformation* of $\mathbb{T}$ is the $\mathcal{I}$-module

$$\mathcal{T} := \mathbb{T}\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\text{cyc}}$$

equipped with the diagonal Galois action.

By a result of Mazur–Wiles (see [**Wil88**, Thm. 2.2.2] for example), the big G_K-representation $\mathbb{T}$ is *ordinary* at p in the sense of Greenberg. That is, for every prime ideal $\mathfrak{q}|p$ in K, if w is any place in $\overline{\mathbf{Q}}$ above $\mathfrak{q}$ and $D_w \subset G_K$ denotes the corresponding decomposition group at $\mathfrak{q}$, there exists an exact sequence of $\mathbb{I}[[D_w]]$-modules

$$0 \longrightarrow \mathscr{F}_w^+\mathbb{T} \longrightarrow \mathbb{T} \longrightarrow \mathscr{F}_w^-\mathbb{T} \longrightarrow 0 \tag{2.2.1}$$

with each $\mathscr{F}_w^\pm\mathbb{T}$ free of rank one over $\mathbb{I}$. Moreover, the action of D_w on $\mathscr{F}_w^-\mathbb{T}$ is unramified, given by the character $\alpha : D_w \longrightarrow \mathbb{I}^\times$ sending an arithmetic Frobenius σ_p to $\mathbf{a}_p \in \mathbb{I}^\times$, the p-th coefficient of the Hida family $\mathbf{f}$.

Tensoring (2.2.1) with Λ_{cyc} we obtain a similar exact sequence of $\mathcal{I}[[D_w]]$-modules

$$0 \longrightarrow \mathscr{F}_w^+\mathcal{T} \longrightarrow \mathcal{T} \longrightarrow \mathscr{F}_w^-\mathcal{T} \longrightarrow 0, \tag{2.2.2}$$

where of course the D_w-action on neither of $\mathscr{F}_w^\pm\mathcal{T}$ is unramified.

Let $\mathbb{T}^* := \mathrm{Hom}_{\mathbb{I}}(\mathbb{T}, \mathbb{I})$ be the contragredient of $\mathbb{T}$ equipped with the dual Galois action, and $\mathscr{F}_w^+\mathbb{T}^* := \mathrm{Hom}_{\mathbb{I}}(\mathscr{F}_w^-\mathbb{T}, \mathbb{I})$ which is thus unramified as a D_w-module, and consider the modules (over $\mathbb{I}$ and $\mathcal{I}$, respectively)

$$\mathbb{D} := (\mathscr{F}_w^+\mathbb{T}^*\widehat{\otimes}_{\mathbf{Z}_p}\widehat{\mathbf{Z}}_p^{\mathrm{nr}})^{G_{\mathbf{Q}_p}}, \qquad \mathcal{D} := \mathbb{D}\widehat{\otimes}_{\mathbf{Z}_p}\Lambda(C_\infty).$$

Since p splits in K, the completion of $K_{\mathfrak{q}}$ of K at every prime $\mathfrak{q}$ above p can be identified with $\mathbf{Q}_p$. In the following, we will take $\mathfrak{q} = \mathfrak{p}$ and w corresponding to $\imath_p$, and make the identifications $D_w = G_{K_{\mathfrak{p}}} = G_{\mathbf{Q}_p}$, so that $\mathbb{T}$ may be regarded as $G_{\mathbf{Q}_p}$-representation by restriction

$$\mathrm{loc}_{\mathfrak{p}} : G_K \longrightarrow G_{\mathbf{Q}_p}. \tag{2.2.3}$$

The map (2.2.3) identifies C_∞ with $\mathrm{Gal}(\mathbf{Q}_{p,\infty}/\mathbf{Q}_p)$, the Galois group of the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}_p$. We will also repeatedly use this identification below.

Let $\Gamma = 1 + p\mathbf{Z}_p$ be the group of 1-units in $\mathbf{Z}_p^\times$, and denote by ϵ the isomorphism

$$\epsilon : C_\infty \xrightarrow{\sim} \Gamma \tag{2.2.4}$$

induced by the p-adic cyclotomic character $\varepsilon_{\mathrm{cyc}}$.

Define $\mathcal{X}_{\mathrm{arith}}(\Lambda(C_\infty))$ to be the set of all continuous characters $\sigma : C_\infty \longrightarrow \overline{\mathbf{Q}}_p^\times$ of the form $\sigma = \epsilon^{m_\sigma}\sigma_0$ for some integer $m_\sigma \geq 0$ and with σ_0 of finite order. Similarly as for $\mathcal{X}_{\mathrm{arith}}(\mathbb{I})$, we then say that m_σ is the weight of σ.

For every pair $(\nu, \sigma) \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I}) \times \mathcal{X}_{\mathrm{arith}}(\Lambda(C_\infty))$, let $\mathcal{O}_\nu$ be the ring of integers of the residue field F_ν of ν, and define the specialization $\mathrm{Sp}_{\nu,\sigma}(\mathcal{T}^*)$ to be

$$\mathrm{Sp}_{\nu,\sigma}(\mathcal{T}^*) := (\mathbb{T}^* \otimes_{\mathbb{I}} \mathcal{O}_\nu) \otimes_{\mathbf{Z}_p} (\Lambda(C_\infty)/\ker(\sigma)) \cong T_{\mathbf{f}_\nu}(\sigma), \tag{2.2.5}$$

where $T_{\mathbf{f}_\nu} \subset V_{\mathbf{f}_\nu}$ denotes a $G_{\mathbf{Q}}$-stable $\mathcal{O}_\nu$-lattice inside the Galois representation associated with $\mathbf{f}_\nu$ by Deligne.

Let $F_\infty/\mathbf{Q}_p$ be the unramified $\mathbf{Z}_p$-extension of $\mathbf{Q}_p$, and denote by $\mathcal{O}_\infty$ its ring of integers. Recall that the Galois group $\mathrm{Gal}(F_\infty/\mathbf{Q}_p)$ is identified with the additive group $\mathbf{Z}_p$ by sending the arithmetic Frobenius σ_p to 1. For each $n \geq 0$ we let F_n be the unique subfield of F_∞ with $\mathrm{Gal}(F_n/\mathbf{Q}_p) \cong \mathbf{Z}/p^n\mathbf{Z}$, and let $\mathcal{O}_n$ denote the ring of integers of F_n.

Fix a compatible system $\zeta = (\zeta_s)_{s\geq 0}$ of primitive p^s-th roots of unity $\zeta_s \in \overline{\mathbf{Q}}_p$. From (2.2.5), for every $n \geq 0$ we have induced maps

$$\mathrm{Sp}_{\nu,\sigma}^{(n)} : H^1(F_n, \mathscr{F}_w^+\mathcal{T}^*) \longrightarrow H^1(F_n, \mathscr{F}_w^+V_{\mathbf{f}_\nu}(\sigma))$$

on cohomology. In parallel to these, for every pair $(\nu, \sigma) \in \mathcal{X}_{\rm arith}(\mathbb{I}) \times \mathcal{X}_{\rm arith}(\Lambda(C_\infty))$ we have specialization maps

$$\mathrm{Sp}_{\nu,\sigma}^{(n)} : \mathcal{D} \otimes_{\mathbf{Z}_p} \mathcal{O}_n \longrightarrow D_{\rm cris}^{(F_n)}(\mathscr{F}_w^+ V_{\mathbf{f}_\nu}(\sigma)) \tag{2.2.6}$$

defined as follows (assuming for simplicity that the values of σ_0 are contained in $\mathcal{O}_\nu$; see [**Och03**, Def. 3.12] for the general case.) Assume that $\sigma \neq \mathbb{1}$, and let $s > 1$ be the smallest positive integer such that $\sigma_0 \circ \epsilon^{-1}$ is trivial on $1 + p^s\mathbf{Z}_p \subset \Gamma$. Upon identifying $\mathrm{Gal}(\mathbf{Q}_p(\zeta_s)/\mathbf{Q}_p)$ with $(\mathbf{Z}/p^s\mathbf{Z})^\times$ via $\varepsilon_{\rm cyc}$, we have

$$D_{\rm dR}(F_\nu(m_\sigma)) \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(\zeta_s) = D_{\rm dR}(F_\nu(m_\sigma) \otimes_{\mathbf{Q}_p} \mathbf{Q}_p[(\mathbf{Z}/p^s\mathbf{Z})^\times])$$

by Shapiro's lemma, and applying $\sigma_0 : (\mathbf{Z}/p^s\mathbf{Z})^\times \longrightarrow \mathcal{O}_\nu^\times$ we deduce a map

$$D_{\rm dR}(F_\nu(m_\sigma)) \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(\zeta_s) \longrightarrow D_{\rm dR}(F_\nu(m_\sigma) \otimes_{\mathbf{Q}_p} F_\nu(\sigma_0)) = D_{\rm dR}(F_\nu(\sigma)). \tag{2.2.7}$$

Then setting $\Lambda_{\mathcal{O}_\nu}(C_\infty) := \mathcal{O}_\nu \otimes_{\mathbf{Z}_p} \Lambda(C_\infty)$, the specialization map

$$\mathrm{Sp}_{\nu,\sigma} : \mathcal{D} \longrightarrow D_{\rm cris}(\mathscr{F}_w^+ V_{\mathbf{f}_\nu}(\sigma)) \tag{2.2.8}$$

is defined to be the composite map

$$\begin{aligned}
(\mathscr{F}_w^+ \mathbb{T}^* \widehat{\otimes}_{\mathbf{Z}_p} \widehat{\mathbf{Z}}_p^{\rm nr})^{G_{\mathbf{Q}_p}} \widehat{\otimes}_{\mathbf{Z}_p} \Lambda(C_\infty) &\xrightarrow{\mathrm{Sp}_\nu \otimes 1} (\mathscr{F}_w^+ T_{\mathbf{f}_\nu} \otimes_{\mathbf{Z}_p} \widehat{\mathbf{Z}}_p^{\rm nr})^{G_{\mathbf{Q}_p}} \otimes_{\mathcal{O}_\nu} \Lambda_{\mathcal{O}_\nu}(C_\infty) \\
&\xrightarrow{1 \otimes \mathrm{Sp}_\sigma} (\mathscr{F}_w^+ T_{\mathbf{f}_\nu} \otimes_{\mathbf{Z}_p} \widehat{\mathbf{Z}}_p^{\rm nr})^{G_{\mathbf{Q}_p}} \otimes_{\mathcal{O}_\nu} D_{\rm dR}(F_\nu(\sigma)) \\
&\hookrightarrow D_{\rm dR}(\mathscr{F}_w^+ V_{\mathbf{f}_\nu}) \otimes_{F_\nu} D_{\rm dR}(F_\nu(\sigma)) \\
&\cong D_{\rm dR}(\mathscr{F}_w^+ V_{\mathbf{f}_\nu}(\sigma)),
\end{aligned}$$

where Sp_σ denotes the composite of (2.2.7) with the map

$$\Lambda_{\mathcal{O}_\nu}(C_\infty) = \mathcal{O}_\nu[[C_\infty]] \longrightarrow D_{\rm dR}(F_n(m_\sigma)) \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(\zeta_s)$$

sending $g \in C_\infty$ to $e^{\otimes m_\sigma} \otimes \zeta_s^g$, with e the basis of $D_{\rm dR}(\mathbf{Q}_p(1))$ corresponding to 1 under the identification $D_{\rm dR}(\mathbf{Q}_p(1)) \cong \mathbf{Q}_p$ induced by $(\zeta_s)_{s\geq 0}$. Finally, the map (2.2.6) on $\mathcal{D} \otimes_{\mathbf{Z}_p} \mathcal{O}_n$ is defined as $\mathrm{Sp}_{\nu,\sigma}^{(n)} := \mathrm{Sp}_{\nu,\sigma} \otimes \mathrm{id}_{\mathcal{O}_n}$.

Remark 2.2.2. By construction, $\mathrm{Sp}_{\nu,\sigma}(\mathcal{D})$ is a $\mathbf{Z}_p$-lattice inside $D_{\rm cris}(\mathscr{F}_w^+ V_{\mathbf{f}_\nu}(\sigma))$. Hence $\mathrm{Sp}_{\nu,\sigma}^{(n)}(\mathcal{D} \otimes_{\mathbf{Z}_p} \mathcal{O}_n)$ is an $\mathcal{O}_n$-lattice inside

$$D_{\rm cris}^{(F_n)}(\mathscr{F}_w^+ V_{\mathbf{f}_\nu}(\sigma)) = D_{\rm cris}(\mathscr{F}_w^+ V_{\mathbf{f}_\nu}(\sigma)) \otimes_{\mathbf{Q}_p} F_n.$$

Let γ_o be a topological generator of C_∞, and for each $n \geq 0$ consider

$$\mathcal{J}_n := (\mathbf{a}_p^{p^n} - 1, \gamma_o - 1),$$

seen as an ideal of height two inside $\mathcal{I}$.

THEOREM 2.2.3 (Ochiai). *For every $n \geq 0$ there exists an injective $\mathcal{I}$-linear map*

$$\mathrm{Exp}^{(n)}_{\mathscr{F}_w^+\mathcal{T}^*} : \mathcal{J}_n(\mathcal{D} \otimes_{\mathbf{Z}_p} \mathcal{O}_n) \longrightarrow H^1(F_n, \mathscr{F}_w^+\mathcal{T}^*)$$

with pseudo-null cokernel such that for every pair $(\nu, \sigma) \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I}) \times \mathcal{X}_{\mathrm{arith}}(\Lambda(C_\infty))$ with $1 \leq m_\sigma \leq k_\nu - 1$ the diagram

$$\begin{array}{ccc} \mathcal{J}_n(\mathcal{D} \otimes_{\mathbf{Z}_p} \mathcal{O}_n) & \xrightarrow{\mathrm{Exp}^{(n)}_{\mathscr{F}_w^+\mathcal{T}^*}} & H^1(F_n, \mathscr{F}_w^+\mathcal{T}^*) \\ \downarrow{\scriptstyle \mathrm{Sp}^{(n)}_{\nu,\sigma}} & & \downarrow{\scriptstyle \mathrm{Sp}^{(n)}_{\nu,\sigma}} \\ D^{(F_n)}_{\mathrm{cris}}(\mathscr{F}_w^+ V_{\mathbf{f}_\nu}(\sigma)) & \longrightarrow & H^1(F_n, \mathscr{F}_w^+ V_{\mathbf{f}_\nu}(\sigma)) \end{array} \tag{2.2.9}$$

commutes, where the bottom horizontal arrow is given by the map

$$(-1)^{m_\sigma - 1}(m_\sigma - 1)! \times \begin{cases} \left(1 - \frac{p^{m_\sigma - 1}}{\nu(\mathbf{a}_p)}\right)\left(1 - \frac{\nu(\mathbf{a}_p)}{p^{m_\sigma}}\right)^{-1} \exp^{(n)}_{V_{\mathbf{f}_\nu}(\sigma)} & \text{if } \sigma_0 = \mathbb{1}; \\ \left(\frac{\nu(\mathbf{a}_p)}{p^{m_\sigma - 1}}\right)^{-s} \exp^{(n)}_{V_{\mathbf{f}_\nu}(\sigma)} & \text{if } \sigma_0 \neq \mathbb{1}, \end{cases} \tag{2.2.10}$$

with s the smallest positive integer such that $\sigma_0 \circ \epsilon^{-1}$ is trivial on $1 + p^s\mathbf{Z}_p$, and $\exp^{(n)}_{V_{\mathbf{f}_\nu}(\sigma)}$ the Bloch–Kato exponential map for $V_{\mathbf{f}_\nu}(\sigma)$ over F_n.

PROOF. First note that by the arguments in the proof of [**Och03**, Lemma 3.2] there exists a canonical isomorphism

$$D^{(F_n)}_{\mathrm{cris}}(\mathscr{F}_w^+ V_{\mathbf{f}_\nu}(\sigma)) \xrightarrow{\sim} \frac{D^{(F_n)}_{\mathrm{dR}}(V_{\mathbf{f}_\nu}(\sigma))}{\mathrm{Fil}^0 D^{(F_n)}_{\mathrm{dR}}(V_{\mathbf{f}_\nu}(\sigma))},$$

and the exponential map

$$\exp^{(n)}_{V_{\mathbf{f}_\nu}(\sigma)} : \frac{D^{(F_n)}_{\mathrm{dR}}(V_{\mathbf{f}_\nu}(\sigma))}{\mathrm{Fil}^0 D^{(F_n)}_{\mathrm{dR}}(V_{\mathbf{f}_\nu}(\sigma))} \longrightarrow H^1(F_n, V_{\mathbf{f}_\nu}(\sigma))$$

factors through the map on cohomology

$$H^1(F_n, \mathscr{F}_w^+ V_{\mathbf{f}_\nu}(\sigma)) \longrightarrow H^1(F_n, V_{\mathbf{f}_\nu}(\sigma))$$

induced by the exact sequence (2.2.2) specialized at (ν, σ).

The result follows as in [**Och03**, Prop. 5.3], noting that all the arguments in the proof of Ochiai's result still go through after replacing $\mathbf{Q}_p$ by the finite unramified extension $F_n/\mathbf{Q}_p$ (and accordingly the arithmetic Frobenius σ_p by $\sigma_p^{p^n}$). □

2.2.2. Going up the unramified $\mathbf{Z}_p$-extension. We now recall a construction from [**LZ11**, §3.2] that we will need in what follows.

Set $U = \mathrm{Gal}(F_\infty/\mathbf{Q}_p)$ and $U_n = \mathrm{Gal}(F_\infty/F_n)$, so that U/U_n is the Galois group of the unramified extension F_n of $\mathbf{Q}_p$ of degree p^n. The group ring $\mathcal{O}_n[U/U_n]$ is equipped with two natural commuting actions of U, and we let $\mathcal{S}_n \subset \mathcal{O}_n[U/U_n]$ be the $\mathbf{Z}_p$-submodule where these two different actions agree, i.e.,

$$\mathcal{S}_n := \left\{ \sum_{\sigma\in U/U_n} a_\sigma.\sigma \in \mathcal{O}_n[U/U_n] \;:\; \tau a_\sigma = a_{\tau^{-1}\sigma} \quad \text{for all } \tau \in U \right\}.$$

If $x_n \in \mathcal{O}_n$, then the element $y_n(x_n) = \sum_{\sigma\in U/U_n} x_n^{\sigma^{-1}}.\sigma$ lies in $\mathcal{S}_n$, and in fact the resulting map

$$y_n : \mathcal{O}_n \longrightarrow \mathcal{S}_n$$

is easily seen to be an isomorphism of $\mathbf{Z}_p[U/U_n]$-modules. For varying n these maps fit into a commutative diagram

$$\begin{array}{ccccc} \mathcal{O}_{n+1} & \xrightarrow{y_{n+1}} & \mathcal{S}_{n+1} & \subset & \mathcal{O}_{n+1}[U/U_{n+1}] \\ \downarrow{\scriptstyle \mathrm{Tr}_{F_{n+1}/F_n}} & & \downarrow & & \\ \mathcal{O}_n & \xrightarrow{y_n} & \mathcal{S}_n & \subset & \mathcal{O}_n[U/U_n], \end{array}$$

where the right vertical map is induced by the natural projection $U/U_{n+1} \twoheadrightarrow U/U_n$.

Lemma 2.2.4 (Loeffler–Zerbes). *The* Yager module

$$\mathcal{S}_\infty := \varprojlim_n \mathcal{S}_n$$

is free of rank 1 *over* $\mathbf{Z}_p[[U]]$ *and the maps* y_n *induce an isomorphism of* $\mathbf{Z}_p[[U]]$*-modules*

$$\varprojlim_n \mathcal{O}_n \xrightarrow{\sim} \mathcal{S}_\infty.$$

Proof. See [**LZ11**, Prop. 3.5]. □

Corollary 2.2.5. *The module* $\mathcal{D}_\infty := \mathcal{D}\widehat{\otimes}_{\mathbf{Z}_p}\mathcal{S}_\infty$ *is free of rank one over* $\mathcal{I}_\infty := \mathcal{I}[[U]]$.

Proof. By Lemma 2.2.4, we have

$$\mathcal{D}_\infty \cong \varprojlim_n(\mathcal{D} \otimes_{\mathbf{Z}_p} \mathcal{O}_n),$$

where the limit is taken with respect to the maps $1 \otimes \mathrm{Tr}_{F_{n+1}/F_n}$. Since $\mathcal{D} \cong \mathcal{I}$ by [**Och03**, Lemma 3.3], the result follows. □

For a Galois extension of fields E'/E and a module M equipped with a continuous action of the absolute Galois group of E, we define the Iwasawa cohomology group $H^1_{\rm Iw}(E'/E, M)$ to be

$$H^1_{\rm Iw}(E'/E, M) := \varprojlim_{E''} H^1(E'', M),$$

where the limit is with respect to corestriction maps over the finite extensions E'' of E contained in E'. Often if the base field is clear from the context, it will be omitted from the notation.

In particular, let $L_\infty := F_\infty(\boldsymbol{\mu}_{p^\infty})$ and $G_\infty := {\rm Gal}(L_\infty/\mathbf{Q}_p)$, and note that

$$H^1_{\rm Iw}(F_\infty, \mathscr{F}^+_w\mathcal{T}^*) \;\cong\; H^1_{\rm Iw}(L_\infty, \mathscr{F}^+_w\mathbb{T}^*)$$

by Shapiro's Lemma. Also note that as it follows from local Class Field Theory, the extension L_∞ contains many distinguished $\mathbf{Z}_p^\times$-subextensions obtained from adjoining to $\mathbf{Q}_p$ the torsion points on various one-dimensional Lubin–Tate formal groups $\mathcal{F}$ over $\mathbf{Z}_p$.

2.2.3. Two-variable p-adic regulator map. We keep the notations as introduced in Sections 2.2.1 and 2.2.2. Let $\Lambda_{\rm anti}$ be the free $\Lambda(D_\infty)$-module of rank one endowed with the natural action of G_K on group-like elements, and set

$$\mathbb{T}^*_\infty := \mathbb{T}^*\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\rm anti}$$

equipped with the diagonal G_K-action and $\mathbb{D}_\infty := \mathbb{D}\widehat{\otimes}_{\mathbf{Z}_p}\Lambda(D_\infty)$. We explain a convenient way to view the characters of D_∞ locally at the places of K above p.

Let $K_{\infty,\mathfrak{p}}$ be the completion of K_∞ at the prime above p induced by $\imath_p$. Since $p \nmid h_K$ by one of our Assumptions 2.1.1, the field $K_{\infty,\mathfrak{p}}$ is a totally ramified $\mathbf{Z}_p$-extension of $K_\mathfrak{p}$, and if $\pi \in \mathcal{O}_K$ is a generator of the principal ideal $\mathfrak{p}^{h_K}$, by Class Field Theory we know that

$$\varpi := \pi/\bar{\pi}$$

is a universal norm of $K_{\infty,\mathfrak{p}}/K_\mathfrak{p}$. Moreover, setting $k_\infty := K_{\infty,\mathfrak{p}}(\boldsymbol{\mu}_p)$, it follows that the $\mathbf{Z}_p^\times$-extension $k_\infty/K_\mathfrak{p}$ can be obtained from the adjunction of the ϖ-power torsion points of a one-dimensional height one Lubin–Tate formal group $\mathcal{F}$ associated with ϖ. More precisely, letting $K_{t,\mathfrak{p}}$ denote the completion of the t-th intermediate subfield $K \subset K_t \subset K_\infty$ at the unique prime above $\mathfrak{p}$, so that ${\rm Gal}(K_{t,\mathfrak{p}}/K_\mathfrak{p}) \cong \mathbf{Z}/p^t\mathbf{Z}$, and setting $k_{t+1} := K_{t,\mathfrak{p}}(\boldsymbol{\mu}_p)$, we have

$$k_t = K_\mathfrak{p}(\mathcal{F}[\varpi^t]),$$

where we may take $\mathcal{F}$ to be the one-dimensional formal group over $\mathcal{O}_\mathfrak{p}$ associated to a "good lift of Frobenius" corresponding to ϖ in the sense of [**IP06**, §4] for example.

The action of $G_{K_{\mathfrak{p}}}$ on $\varprojlim_t \mathcal{F}[\varpi^t]$ gives rise to a canonical isomorphism

$$\varepsilon_{\varpi} : \mathrm{Gal}(k_\infty/K_{\mathfrak{p}}) \xrightarrow{\sim} \mathbf{Z}_p^{\times}. \tag{2.2.11}$$

Together with the decomposition $\mathrm{Gal}(k_\infty/K_{\mathfrak{p}}) \cong \Gamma_{\varpi} \times \Delta$, where

$$\Gamma_{\varpi} := \mathrm{Gal}(k_\infty/K_{\mathfrak{p}}(\boldsymbol{\mu}_p)) \cong \mathrm{Gal}(K_{\infty,\mathfrak{p}}/K_{\mathfrak{p}}) \qquad \Delta := \mathrm{Gal}(K_{\mathfrak{p}}(\boldsymbol{\mu}_p)/K_{\mathfrak{p}}),$$

we easily see that (2.2.11) composed with the restriction map (2.2.3) associated with the embedding $\imath_p$ induces an identification

$$\epsilon_{\varpi} : D_\infty \xrightarrow{\sim} \Gamma_{\varpi} \xrightarrow{\sim} \Gamma, \tag{2.2.12}$$

where $\Gamma = 1 + p\mathbf{Z}_p$ analogous to (2.2.4), which we will use to identify characters of D_∞ with certain character of $\mathrm{Gal}(k_\infty/K_{\mathfrak{p}})$.

Letting $\xi = (\xi_t)_{t\geq 0}$ be a compatible system of primitive elements $\xi_t \in \mathcal{F}[\varpi^t]$, and replacing $\epsilon : C_\infty \xrightarrow{\sim} \Gamma$ by the identification (2.2.12) and ζ by ξ in the definition of the specialization maps $\mathrm{Sp}_{\nu,\sigma}^{(n)}$ given in (2.2.8), we may similarly define specialization maps

$$\mathrm{Sp}_{\nu,\phi}^{(n)} : \mathbb{D}_\infty \otimes \mathcal{O}_n \longrightarrow D_{\mathrm{cris}}^{(n)}(\mathscr{F}_w^+ V_{\mathbf{f}_\nu}(\phi))$$
$$\mathrm{Sp}_{\nu,\phi}^{(n)} : H^1(F_n, \mathscr{F}_w^+ \mathbb{T}_\infty^*) \longrightarrow H^1(F_n, \mathscr{F}_w^+ V_{\mathbf{f}_\nu}(\phi))$$

for every $n \geq 0$ and every pair $(\nu,\phi) \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I}) \times \mathcal{X}_{\mathrm{arith}}(\Lambda(D_\infty))$, where similarly as before $\mathcal{X}_{\mathrm{arith}}(\Lambda(D_\infty))$ denotes the set of all continuous characters $\phi : D_\infty \longrightarrow \overline{\mathbf{Q}}_p^{\times}$ such that, for some integer $m_\phi \geq 0$ called the weight of ϕ, the character $\phi_0 := \phi\epsilon_{\varpi}^{-m_\phi}$ has finite order.

We now return to the setting of Theorem 2.2.3. The corestriction maps $\mathrm{Tr}_{F_{n+1}/F_n}$ send $\mathcal{J}_{n+1}$ onto $\mathcal{J}_n$, and we let

$$\mathcal{J}_\infty := \varprojlim_n \mathcal{J}_n$$

be the corresponding inverse limit.

Proposition 2.2.6. *There exists an injective $\mathbb{I}[[G_\infty]]$-linear pseudo-isomorphism*

$$\mathrm{Exp}_{\mathscr{F}_w^+\mathbb{T}^*}^{G_\infty} : \mathcal{J}_\infty \mathcal{D}_\infty \longrightarrow H^1_{\mathrm{Iw}}(L_\infty, \mathscr{F}_w^+ \mathbb{T}^*)$$

such that for every $(\nu,\phi) \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I}) \times \mathcal{X}_{\mathrm{arith}}(\Lambda(D_\infty))$, with $0 \leq m_\phi \leq k_\nu - 2$, the diagram

$$\begin{array}{ccc} \mathcal{J}_\infty \mathcal{D}_\infty & \xrightarrow{\mathrm{Exp}_{\mathscr{F}_w^+\mathbb{T}^*}^{G_\infty}} & H^1_{\mathrm{Iw}}(L_\infty, \mathscr{F}_w^+ \mathbb{T}^*) \\ \downarrow{\scriptstyle \mathrm{Sp}_{\nu,\phi}\circ\mathrm{pr}_{D_\infty}} & & \downarrow{\scriptstyle \mathrm{Sp}_{\nu,\phi}\circ\mathrm{pr}_{D_\infty}} \\ D_{\mathrm{cris}}(\mathscr{F}_w^+ V_{\mathbf{f}_\nu}(\phi)) & \longrightarrow & H^1(\mathbf{Q}_p, \mathscr{F}_w^+ V_{\mathbf{f}_\nu}(\phi)) \end{array} \tag{2.2.13}$$

commutes, where pr_{D_∞} *denotes the map induced by the projection* $G_\infty \twoheadrightarrow D_\infty$ *and the bottom horizontal arrow is given by the map*

$$(2.2.14)\qquad (-1)^{m_\phi-1}(m_\phi-1)! \times \begin{cases} \left(1-\frac{\varpi^{m_\phi}}{\nu(\mathbf{a}_p)p}\right)\left(1-\frac{\nu(\mathbf{a}_p)}{\varpi^{m_\phi}}\right)^{-1}\exp_{V_{\mathbf{f}_\nu}(\phi)} & \text{if } \phi_0 = \mathbb{1}; \\ \left(\frac{\nu(\mathbf{a}_p)p}{\varpi^{m_\phi}}\right)^{-t}\exp_{V_{\mathbf{f}_\nu}(\phi)} & \text{if } \phi_0 \neq \mathbb{1}, \end{cases}$$

with $t > 0$ *is the power of* p *in the conductor of* ϕ_0.

Proof. Setting

$$\mathrm{Exp}^{G_\infty}_{\mathscr{F}_w^+\mathbb{T}^*} := \varprojlim_n \mathrm{Exp}^{(n)}_{\mathscr{F}_w^+\mathcal{T}^*},$$

where $\mathrm{Exp}^{(n)}_{\mathscr{F}_w^+\mathcal{T}^*}$ is the big exponential map in Theorem 2.2.3, we obtain a map that clearly has the desired properties with respect to the projection to C_∞ in place D_∞. Hence to conclude it suffices to see that $\mathrm{Exp}^{G}_{\mathscr{F}_w^+\mathbb{T}^*}$ also interpolates over F_∞ the extended logarithm map associated with any one-dimensional Lubin–Tate formal group over $\mathbf{Z}_p$. But since $\mathscr{F}_w^+\mathbb{T}^* \cong \mathbb{I}$, this follows from the properties of the Yager map as in [**LZ11**, §7.4.3]. □

Definition 2.2.7. We say that an arithmetic prime $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ is *exceptional* if it has weight $k_\nu = 2$, $\nu(\mathbf{a}_p) = \pm 1$, and the composite map

$$(2.2.15)\qquad \Gamma \longrightarrow \mathbb{I}^\times \xrightarrow{\nu} \overline{\mathbf{Q}}_p^\times$$

is the trivial character on Γ.

Set $\Lambda_{\mathbb{I}}(D_\infty) := \mathbb{I}[[D_\infty]] \cong \mathbb{I}\widehat{\otimes}_{\mathbf{Z}_p}\Lambda(D_\infty)$ and $\lambda := \mathbf{a}_p - 1$.

Corollary 2.2.8. *Fix an* $\mathbb{I}$*-basis* η *of* $\mathbb{D}$. *There exists a* $\Lambda_{\mathbb{I}}(D_\infty)$*-linear map*

$$\mathrm{Log}^\eta_{\mathscr{F}_w^+\mathbb{T}^\dagger_\infty} : H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}}, \mathscr{F}_w^+\mathbb{T}^\dagger) \longrightarrow \Lambda_{\mathbb{I}}(D_\infty) \otimes_{\mathbb{I}} \mathbb{I}[\lambda^{-1}]$$

such that for every pair $(\nu,\phi) \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I}) \times \mathcal{X}_{\mathrm{arith}}(\Lambda(D_\infty))$ *of weights* $k_\nu \geq 2$ *and* $m_\phi \geq 0$, *with* $k_\nu = 2r_\nu$ *and* $m_\phi < r_\nu$, *if* $\mathfrak{Y}_\infty \in H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}}, \mathscr{F}_w^+\mathbb{T}^\dagger)$ *then*

$$\nu(\mathrm{Log}^\eta_{\mathscr{F}_w^+\mathbb{T}^\dagger_\infty}(\mathfrak{Y}_\infty))(\phi) = (-1)^{r_\nu-m_\phi-1}(r_\nu-m_\phi-1)!^{-1}$$
$$\times \begin{cases} \left(1-\frac{p^{r_\nu-1}}{\nu(\mathbf{a}_p)\varpi^{m_\phi}}\right)^{-1}\left(1-\frac{\nu(\mathbf{a}_p)\varpi^{m_\phi}}{p^{r_\nu}}\right)\langle \log_{V^\dagger_{\mathbf{f}_\nu}(\phi)}(\nu(\mathfrak{Y}_\infty)^\phi), \eta'_\nu\rangle_{\mathrm{dR}} & \text{if } \vartheta_\nu\phi_0 = \mathbb{1}; \\ G(\phi_0^{-1})^{-1}\left(\frac{\nu(\mathbf{a}_p)\varpi^{m_\phi}}{p^{r_\nu-1}}\right)^t\langle \log_{V^\dagger_{\mathbf{f}_\nu}(\phi)}(\nu(\mathfrak{Y}_\infty)^\phi), \eta'_\nu\rangle_{\mathrm{dR}} & \text{if } \vartheta_\nu\phi_0 \neq \mathbb{1}, \end{cases}$$

where $t > 0$ is the power of p in the conductor of ϕ_0, and $\eta'_\nu \in \mathrm{Fil}^0 D_{\mathrm{dR}}(V^\dagger_{\mathbf{f}_\nu}(\varepsilon_\varpi^{-m_\phi}))$ is the dual to $\eta^\dagger_\nu \otimes e_\xi^{\otimes m_\phi}$ under the de Rham pairing

$$\frac{D_{\mathrm{dR}}(V^\dagger_{\mathbf{f}_\nu}(\varepsilon_\varpi^{m_\phi}))}{\mathrm{Fil}^0 D_{\mathrm{dR}}(V^\dagger_{\mathbf{f}_\nu}(\varepsilon_\varpi^{m_\phi}))} \times \mathrm{Fil}^0 D_{\mathrm{dR}}(V^\dagger_{\mathbf{f}_\nu}(\varepsilon_\varpi^{-m_\phi})) \longrightarrow F_\nu. \tag{2.2.16}$$

Proof. This follows from an argument completely analogous to that used in the proof of [**Cas13a**, Thm. 3.4]. Indeed consider the $\mathbb{I}$-algebra isomorphism

$$\mathrm{Tw}^\dagger : \Lambda_{\mathbb{I}}(C_\infty) \longrightarrow \Lambda_{\mathbb{I}}(C_\infty) \tag{2.2.17}$$

given by

$$\mathrm{Tw}^\dagger(\gamma) = \epsilon^{1/2}(\gamma)[\gamma]$$

for all $\gamma \in C_\infty$, where $\epsilon^{1/2}$ is the unique square-root of the wild component ϵ of $\varepsilon_{\mathrm{cyc}}$. Letting $\Lambda^\dagger_{\mathrm{cyc}}$ be the module Λ_{cyc} with the Galois action twisted by $\epsilon^{1/2}$, and setting

$$\mathscr{F}^+_w \mathcal{T}^\dagger_\infty := \mathscr{F}^+_w \mathbb{T}_\infty \widehat{\otimes}_{\mathbf{Z}_p} \Lambda^\dagger_{\mathrm{cyc}}$$

equipped with the diagonal G_K-action, there is natural projection

$$\mathrm{Cor} : \mathscr{F}^+_w \mathcal{T}^\dagger_\infty \longrightarrow \mathscr{F}^+_w \mathbb{T}^\dagger_\infty$$

induced by the augmentation map on $\Lambda(C_\infty)$. Setting

$$\mathbb{D}^\dagger_\infty := \mathcal{J}_\infty \mathcal{D}_\infty \widehat{\otimes}_{\mathbf{Z}_p} \Lambda_{\mathbb{I}}(C_\infty)/(\epsilon^{1/2}(\gamma_o)[\gamma_o] - 1),$$

the composition

$$\begin{aligned} \mathcal{J}_\infty \mathcal{D}_\infty \xrightarrow{(\mathrm{Tw}^\dagger)^{-1}} \mathcal{J}_\infty \mathcal{D}_\infty &\xrightarrow{\mathrm{Exp}^{G_\infty}_{\mathscr{F}^+_w \mathbb{T}^*}} H^1_{\mathrm{Iw}}(L_\infty, \mathscr{F}^+_w \mathbb{T}^*) \cong H^1(\mathbf{Q}_p, \mathscr{F}^+_w \mathcal{T}^*_\infty) \\ &\xrightarrow{\otimes \epsilon^{1/2}} H^1(\mathbf{Q}_p, \mathscr{F}^+_w \mathcal{T}^\dagger_\infty) \xrightarrow{\mathrm{Cor}} H^1(\mathbf{Q}_p, \mathscr{F}^+_w \mathbb{T}^\dagger_\infty) \end{aligned}$$

factors through an *injective* $\Lambda_{\mathbb{I}}(D_\infty)$-linear map

$$\mathrm{Exp}_{\mathscr{F}^+_w \mathbb{T}^\dagger_\infty} : \mathbb{D}^\dagger_\infty \longrightarrow H^1(\mathbf{Q}_p, \mathscr{F}^+_w \mathbb{T}^\dagger_\infty) \tag{2.2.18}$$

making, for every pair (ν, ϕ) as in the statement, the diagram

$$\begin{array}{ccc} \mathbb{D}^\dagger_\infty & \xrightarrow{\mathrm{Exp}_{\mathscr{F}^+_w \mathbb{T}^\dagger_\infty}} & H^1(\mathbf{Q}_p, \mathscr{F}^+_w \mathbb{T}^\dagger_\infty) \\ \downarrow{\scriptstyle \mathrm{Sp}_{\nu,\phi}} & & \downarrow{\scriptstyle \mathrm{Sp}_{\nu,\phi}} \\ D_{\mathrm{cris}}(V^\dagger_{\mathbf{f}_\nu}(\phi)) & \longrightarrow & H^1(\mathbf{Q}_p, \mathscr{F}^+_w V^\dagger_{\mathbf{f}_\nu}(\phi)) \end{array}$$

commutative, where the bottom horizontal arrow is given by (cf. (2.2.14))

$$\pm(r_\nu - m_\phi - 1)! \begin{cases} \left(1 - \frac{p^{r_\nu - 1}}{\nu(\mathbf{a}_p)\varpi^{m_\phi}}\right)\left(1 - \frac{\nu(\mathbf{a}_p)\varpi^{m_\phi}}{p^{r_\nu}}\right)^{-1} \exp_{V^\dagger_{\mathbf{f}_\nu}(\phi)} & \text{if } \vartheta_\nu\phi_0 = \mathbb{1}; \\ G(\phi_0^{-1})\left(\frac{p^{r_\nu - 1}}{\nu(\mathbf{a}_p)\varpi^{m_\phi}}\right)^t \exp_{V^\dagger_{\mathbf{f}_\nu}(\phi)} & \text{if } \vartheta_\nu\phi_0 \neq \mathbb{1}, \end{cases} \tag{2.2.19}$$

where $\pm = (-1)^{r_\nu - m_\phi - 1}$.

Now if $\mathfrak{Y}_\infty \in H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}}, \mathscr{F}^+_w\mathbb{T}^\dagger) \cong H^1(\mathbf{Q}_p, \mathscr{F}^+_w\mathbb{T}^\dagger_\infty)$, then $\lambda \cdot \mathfrak{Y}_\infty$ lands in the image $\mathrm{Exp}_{\mathscr{F}^+_w\mathbb{T}^\dagger_\infty}$ and so

$$\mathrm{Log}_{\mathscr{F}^+_w\mathbb{T}^\dagger}(\mathfrak{Y}) := \lambda^{-1} \cdot \mathrm{Exp}^{-1}_{\mathscr{F}^+_w\mathbb{T}^\dagger_\infty}(\lambda \cdot \mathfrak{Y}_\infty)$$

is a well-defined element in $\mathbb{I}[\lambda^{-1}] \otimes_{\mathbb{I}} \mathbb{D}^\dagger_\infty$. Finally, the chosen $\mathbb{I}$-basis η of $\mathbb{D}$ induces a $\Lambda_{\mathbb{I}}(D_\infty)$-basis $\widetilde{\eta}$ of $\mathbb{D}^\dagger_\infty$, and defining $\mathrm{Log}^\eta_{\mathscr{F}^+_w\mathbb{T}^\dagger_\infty}(\mathfrak{Y}_\infty)$ to be the element in $\Lambda_{\mathbb{I}}(D_\infty) \otimes_{\mathbb{I}} \mathbb{I}[\lambda^{-1}]$ determined by the relation

$$\mathrm{Log}_{\mathscr{F}^+_w\mathbb{T}^\dagger_\infty}(\mathfrak{Y}_\infty) = \mathrm{Log}^\eta_{\mathscr{F}^+_w\mathbb{T}^\dagger_\infty}(\mathfrak{Y}_\infty) \cdot \widetilde{\eta} \otimes 1,$$

the result follows. ☐

2.2.4. Explicit reciprocity law. Recall from Section 2.2.3 the height one Lubin–Tate formal group $\mathcal{F}$ over $\mathcal{O}_{\mathfrak{p}} \cong \mathbf{Z}_p$ associated with the uniformizer $\varpi = \pi/\bar{\pi}$, and the extension $k_\infty/K_{\mathfrak{p}}$ obtained by adjoining the ϖ-power torsion points of $\mathcal{F}$. We will use the decomposition

$$\mathrm{Gal}(k_\infty/K_{\mathfrak{p}}) \cong \mathrm{Gal}(K_{\infty,\mathfrak{p}}/K_{\mathfrak{p}}) \times \mathrm{Gal}(K_{\mathfrak{p}}(\boldsymbol{\mu}_p)/K_{\mathfrak{p}})$$

to identify the local restriction at $\mathfrak{p}$ of characters of the Galois group $D_\infty = \mathrm{Gal}(K_\infty/K)$ of the anticyclotomic $\mathbf{Z}_p$-extension of K with characters of $\mathrm{Gal}(k_\infty/K_{\mathfrak{p}})$ with a fixed tame component.

The following result due to Shaowei Zhang [**Zha04**] will be of great importance for us here. It generalizes, from the formal multiplicative group to more general one-dimensional height one Lubin–Tate formal groups over $\mathbf{Z}_p$, Colmez's work [**Col98**] on the cyclotomic p-adic regulator map of Perrin-Riou.

THEOREM 2.2.9 (Colmez, Zhang). *Let $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ have weight $k_\nu = 2r_\nu \geq 2$. Then for each $\eta_o \in D_{\mathrm{dR}}(V^\dagger_{\mathbf{f}_\nu})$, there exists a unique map*

$$\mathfrak{L}^{\eta_o}_{V^\dagger_{\mathbf{f}_\nu}} : H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}}, V^\dagger_{\mathbf{f}_\nu}) \longrightarrow \Lambda_{\mathcal{O}_\nu}(D_\infty)$$

characterized by either of the properties (i) or (ii): for every $\nu(\mathfrak{Y}_\infty) \in H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}}, V^\dagger_{\mathbf{f}_\nu})$ and $\phi \in \mathcal{X}_{\mathrm{arith}}(\Lambda(D_\infty))$ of weight $m_\phi \equiv 0 \pmod{p-1}$, (*i*) *If* $m_\phi \geq r_\nu$, *then*

$$\mathfrak{L}^{\eta_o}_{V^\dagger_{\mathbf{f}_\nu}}(\mathfrak{Y}_\infty)(\phi) = (m_\phi - r_\nu)! \times \begin{cases} \left(1 - \frac{p^{r_\nu - 1}}{\nu(\mathbf{a}_p)\varpi^{m_\phi}}\right)^{-1}\left(1 - \frac{\nu(\mathbf{a}_p)\varpi^{m_\phi}}{p^{r_\nu}}\right)\langle \exp^*_{V^\dagger_{\mathbf{f}_\nu}(\phi)}(\nu(\mathfrak{Y}_\infty)^\phi), \eta_o\rangle_{\mathrm{dR}} & \text{if } \vartheta_\nu\phi_0 = \mathbb{1}; \\ G(\phi_0^{-1})^{-1}\left(\frac{\nu(\mathbf{a}_p)\varpi^{m_\phi}}{p^{r_\nu - 1}}\right)^t \langle \exp^*_{V^\dagger_{\mathbf{f}_\nu}(\phi)}(\nu(\mathfrak{Y}_\infty)^\phi), \eta_o\rangle_{\mathrm{dR}} & \text{if } \vartheta_\nu\phi_0 \neq \mathbb{1}, \end{cases}$$

where $t > 0$ *is the power of* p *in the conductor of* ϕ_0.

(*ii*) *If* $m_\phi < r_\nu$, *then*

$$\mathfrak{L}^{\eta_o}_{V^\dagger_{\mathbf{f}_\nu}}(\mathfrak{Y}_\infty)(\phi) = \frac{(-1)^{r_\nu - m_\phi - 1}}{(r_\nu - m_\phi - 1)!} \times \begin{cases} \left(1 - \frac{p^{r_\nu - 1}}{\nu(\mathbf{a}_p)\varpi^{m_\phi}}\right)^{-1}\left(1 - \frac{\nu(\mathbf{a}_p)\varpi^{m_\phi}}{p^{r_\nu}}\right)\langle \log_{V^\dagger_{\mathbf{f}_\nu}(\phi)}(\nu(\mathfrak{Y}_\infty)^\phi), \eta_o'\rangle_{\mathrm{dR}} & \text{if } \vartheta_\nu\phi_0 = \mathbb{1}; \\ G(\phi_0^{-1})^{-1}\left(\frac{\nu(\mathbf{a}_p)\varpi^{m_\phi}}{p^{r_\nu - 1}}\right)^t \langle \log_{V^\dagger_{\mathbf{f}_\nu}(\phi)}(\nu(\mathfrak{Y}_\infty)^\phi), \eta_o'\rangle_{\mathrm{dR}} & \text{if } \vartheta_\nu\phi_0 \neq \mathbb{1}, \end{cases}$$

where $t > 0$ *is the power of* p *in the conductor of* ϕ_0, *and* η_o' *is the dual to* η_o *under the de Rham pairing* (2.2.16)*:*

$$\frac{D_{\mathrm{dR}}(V^\dagger_{\mathbf{f}_\nu}(\varepsilon_\varpi^{m_\phi}))}{\mathrm{Fil}^0 D_{\mathrm{dR}}(V^\dagger_{\mathbf{f}_\nu}(\varepsilon_\varpi^{m_\phi}))} \times \mathrm{Fil}^0 D_{\mathrm{dR}}(V^\dagger_{\mathbf{f}_\nu}(\varepsilon_\varpi^{-m_\phi})) \longrightarrow F_\nu.$$

PROOF. Since $\mathrm{Fil}^0 D_{\mathrm{dR}}(V^\dagger_{\mathbf{f}_\nu}(\phi)) = D_{\mathrm{dR}}(V^\dagger_{\mathbf{f}_\nu}(\phi))$ whenever $m_\phi \geq r_\nu$, the existence of a map $\mathfrak{L}^{\eta_o}_{V^\dagger_{\mathbf{f}_\nu}}$ with property (*i*) is shown in [**Zha04**, Thm. 3.6]. The property (*ii*) of this map is then a reformulation of the explicit reciprocity law of [**Zha04**, Thm. 6.4]. Finally, the implicit growth properties in the above statement, i.e. the claim that the image of the map $\mathfrak{L}^{\eta_o}_{V^\dagger_{\mathbf{f}_\nu}}$ lands in $\Lambda_{\mathcal{O}_\nu}(D_\infty)$, follows from the fact that $\mathbf{f}_\nu$ has slope zero. □

Using Theorem 2.2.9 we can now deduce the extended interpolation property of the map $\mathrm{Log}^\eta_{\mathscr{F}^+_w\mathbb{T}^\dagger_\infty}$ from Corollary 2.2.8:

COROLLARY 2.2.10. *Fix an* $\mathbb{I}$-*basis* η *of* $\mathbb{D}$, *and let*

$$\mathrm{Log}^\eta_{\mathscr{F}^+_w\mathbb{T}^\dagger_\infty} : H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}}, \mathscr{F}^+_w\mathbb{T}^\dagger) \longrightarrow \Lambda_{\mathbb{I}}(D_\infty) \otimes_{\mathbb{I}} \mathbb{I}[\lambda^{-1}]$$

be the map from Corollary 2.2.8. Then for every pair $(\nu,\phi)\in\mathcal{X}_{\mathrm{arith}}(\mathbb{I})\times\mathcal{X}_{\mathrm{arith}}(\Lambda(D_\infty))$ *of weights* $k_\nu=2r_\nu\geq 2$ *and* $m_\phi\geq r_\nu$ *respectively, if* $\mathfrak{Y}_\infty\in H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}},\mathscr{F}_w^+\mathbb{T}^\dagger)$ *then*

$$\nu(\mathrm{Log}^\eta_{\mathscr{F}_w^+\mathbb{T}^\dagger_\infty}(\mathfrak{Y}_\infty))(\phi)=(m_\phi-r_\nu)!$$
$$\times\begin{cases}\left(1-\frac{p^{r_\nu-1}}{\nu(\mathbf{a}_p)\varpi^{m_\phi}}\right)^{-1}\left(1-\frac{\nu(\mathbf{a}_p)\varpi^{m_\phi}}{p^{r_\nu}}\right)\langle\exp^*_{V^\dagger_{\mathbf{f}_\nu}(\phi)}(\nu(\mathfrak{Y}_\infty)^\phi),\eta'_\nu\rangle_{\mathrm{dR}} & \text{if } \vartheta_\nu\phi_0=\mathbb{1};\\ \\ G(\phi_0^{-1})^{-1}\left(\frac{\nu(\mathbf{a}_p)\varpi^{m_\phi}}{p^{r_\nu-1}}\right)^t\langle\exp^*_{V^\dagger_{\mathbf{f}_\nu}(\phi)}(\nu(\mathfrak{Y}_\infty)^\phi),\eta'_\nu\rangle_{\mathrm{dR}} & \text{if } \vartheta_\nu\phi_0\neq\mathbb{1},\end{cases}$$

where $t>0$ *is the power of* p *in the conductor of* ϕ_0, *and* $\eta'_\nu\in\mathrm{Fil}^0D_{\mathrm{dR}}(V^\dagger_{\mathbf{f}_\nu}(\varepsilon_\varpi^{-m_\phi}))$ *is the dual to* $\eta^\dagger_\nu\otimes e_\xi^{\otimes m_\phi}$ *under the de Rham pairing* (2.2.16)*:*

$$\frac{D_{\mathrm{dR}}(V^\dagger_{\mathbf{f}_\nu}(\varepsilon_\varpi^{m_\phi}))}{\mathrm{Fil}^0D_{\mathrm{dR}}(V^\dagger_{\mathbf{f}_\nu}(\varepsilon_\varpi^{m_\phi}))}\times\mathrm{Fil}^0D_{\mathrm{dR}}(V^\dagger_{\mathbf{f}_\nu}(\varepsilon_\varpi^{-m_\phi}))\longrightarrow F_\nu.$$

Proof. Comparing the formulas in Corollary 2.2.8 and Theorem 2.2.9, we see that

$$\nu(\mathrm{Log}^\eta_{\mathscr{F}_w^+\mathbb{T}^\dagger_\infty})=\mathfrak{L}^{\eta_\nu}_{V^\dagger_{\mathbf{f}_\nu}},\tag{2.2.20}$$

since both sides have the same interpolation properties at all characters ϕ of finite order, and these are enough to uniquely determine the map $\mathfrak{L}^{\eta_\nu}_{V^\dagger_{\mathbf{f}_\nu}}$. The result thus follows after the explicit reciprocity law of Theorem 2.2.9(i). □

2.3. Comparison of p-adic L-functions

Recall from Section 2.1.1 the anticyclotomic p-adic L-function $\mathscr{L}_\mathfrak{p}(\mathbf{f}^\dagger)\in\Lambda_\mathbb{I}(D_\infty)$ attached to the critical twist $\mathbf{f}^\dagger$ of a Hida family $\mathbf{f}$ and the imaginary quadratic field K (see Remark 2.1.5), and from Section 2.1.2 Howard's construction of the norm-compatible system $\mathfrak{Z}_\infty\in\widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty,\mathbb{T}^\dagger)$ of big Heegner points.

2.3.1. Weight two specializations. The purpose of this section is to explain the proof of the following result, which will be completed in Section 2.3.2.

Theorem 2.3.1. *Assume that* $\bar\rho_\mathbf{f}|_{G_K}$ *is absolutely irreducible, and fix an* $\mathbb{I}$*-basis* η *of* $\mathbb{D}$*. Then there exists a class* $\mathfrak{Z}^\eta_\infty\in\widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty,\mathbb{T}^\dagger)$ *such that the composite map*

$$\widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty,\mathbb{T}^\dagger)\xrightarrow{\mathrm{loc}_\mathfrak{p}}H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}},\mathscr{F}_w^+\mathbb{T}^\dagger)\xrightarrow{\mathrm{Log}^\eta_{\mathscr{F}_w^+\mathbb{T}^\dagger_\infty}}\Lambda_\mathbb{I}(D_\infty)\otimes_\mathbb{I}\mathbb{I}[\lambda^{-1}]$$

sends $\mathfrak{Z}^\eta_\infty$ *to* $\mathscr{L}_\mathfrak{p}(\mathbf{f}^\dagger)$*.*

PROOF. We begin with some rather standard reductions. By [**How07b**, Lemma 2.1.7] and the commutativity of the diagram

$$\begin{array}{ccccc}
\widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty,\mathbb{T}^\dagger) & \xrightarrow{\mathrm{loc}_\mathfrak{p}} & H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}},\mathscr{F}^+_w\mathbb{T}^\dagger) & \xrightarrow{\mathrm{Log}^\eta_{\mathscr{F}^+_w\mathbb{T}^\dagger_\infty}} & \Lambda_{\mathbb{I}}(D_\infty)\otimes_{\mathbb{I}}\mathbb{I}[\lambda^{-1}] \\
\downarrow \nu & & \downarrow \nu & & \downarrow \nu \\
\widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty,V^\dagger_{\mathbf{f}_\nu}) & \xrightarrow{\mathrm{loc}_\mathfrak{p}} & H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}},V^\dagger_{\mathbf{f}_\nu}) & \xrightarrow{\mathfrak{L}^{\eta\nu}_{V^\dagger_{\mathbf{f}_\nu}}} & \Lambda_{\mathcal{O}_\nu}(D_\infty)\otimes_{\mathcal{O}_\nu}\mathcal{O}_\nu[\lambda_\nu^{-1}]
\end{array}$$

at every $\nu\in\mathcal{X}_{\mathrm{arith}}(\mathbb{I})$, it suffices to show that for infinitely many arithmetic primes ν there exists a class $\mathfrak{Z}^\eta_{\infty,\nu}\in\widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty,V^\dagger_{\mathbf{f}_\nu})$ (depending on ν) which is sent to $\mathscr{L}_\mathfrak{p}(\mathbf{f}^\dagger_\nu)$ via the bottom sequence of maps; we will consider all ν of weight 2 and non-trivial wild character. Letting ν be such an arithmetic prime, we must thus show that for some $\mathfrak{Z}^\eta_{\infty,\nu}$ as above the equality

$$\mathscr{L}_\mathfrak{p}(\mathbf{f}^\dagger_\nu)(\phi)=\nu(\mathrm{Log}^\eta_{\mathscr{F}^+_w\mathbb{T}^\dagger_\infty}(\mathrm{loc}_\mathfrak{p}(\mathfrak{Z}^\eta_{\infty,\nu})))(\phi) \tag{2.3.1}$$

holds for every $\phi\in\mathrm{Hom}_{\mathrm{cont}}(D_\infty,\overline{\mathbf{Q}}_p^\times)$, and in fact if $s>0$ is the power of p in the conductor of the critical character ϑ_ν, it suffices to prove that (2.3.1) holds for all such ϕ factoring through D_t for some integer $t\geq s$.

The proof of (2.3.1) and hence of Theorem 2.3.1 will be based on the following.

PROPOSITION 2.3.2. *Let $\nu\in\mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ have weight 2 and non-trivial wild character (2.2.15), with ϑ_ν of conductor p^s for some $s>0$. Let $\phi_0: D_\infty\longrightarrow\overline{\mathbf{Q}}_p^\times$ be a continuous character of finite order, let $t>0$ be the smallest positive integer such that ϕ_0 factors through D_t, and assume that $t\geq s$. Then*

$$d^{-1}\mathbf{f}^\dagger_\nu\otimes\phi_0(A,\alpha_A,\imath_A)=\frac{\nu(\mathbf{a}_p)^t}{G(\phi_0^{-1})}\langle\log_{V^\dagger_{\mathbf{f}_\nu}(\phi_0)}(\mathrm{loc}_\mathfrak{p}(\nu(\mathfrak{Z}_t))^{\phi_0}),\omega^\dagger_{\mathbf{f}^*_\nu}\rangle_{\mathrm{dR}},$$

where $G(\phi_0^{-1})=\sum_{v \bmod p^t}\phi_0^{-1}(v)\zeta_t^v$ is the Gauss sum of ϕ_0^{-1}, and $\omega^\dagger_{\mathbf{f}^*_\nu}\in\mathrm{Fil}^0D_{\mathrm{dR}}(V^\dagger_{\mathbf{f}_\nu})$ is the class associated with the twist $\mathbf{f}^*_\nu\otimes\vartheta_\nu$.

PROOF. This can be shown by the same methods as in the proof of the analogous [**Cas13a**, Lemma 4.2, Prop. 4.3]. Indeed, using the "modular" description of the character twist (as recalled in [*loc.cit.*, Def. 1.2] for example), we easily see that

$$d^{-1}\mathbf{f}^\dagger_\nu\otimes\phi_0(A,\alpha_A,\imath_A)=G(\phi_0^{-1})^{-1}\sum_{v \bmod p^t}\phi_0(v)\cdot d^{-1}\mathbf{f}^\dagger_\nu(A^{(v)},\alpha_A^{(v)},\imath_A^{(v)}), \tag{2.3.2}$$

where the triples $(A^{(v)}, \alpha_A^{(v)}, \imath_A^{(v)})$ are obtained from $(A, \alpha_A, \imath_A)$ by the procedure described in [**Cas13a**, Lemma 4.2], and ϕ_0 is seen as a Dirichlet character by composing with $(\mathbf{Z}/p^{t+1}\mathbf{Z})^\times \twoheadrightarrow \mathbf{Z}/p^t\mathbf{Z} \cong D_t$.

For a fixed $v \in \mathbf{Z}/p^t\mathbf{Z}$, set

$$(A_{p^{t+1}}, \alpha_{p^{t+1}}, \imath_{p^{t+1}}) = (A^{(v)}, \alpha_A^{(v)}, \imath_A^{(v)}).$$

Then $A_{p^{t+1}}$ has CM by $\mathcal{O}_{p^{t+1}}$ and it follows easily that (2.3.2) can be rewritten as

$$d^{-1}\mathbf{f}_\nu^\dagger \otimes \phi_0(A, \alpha_A, \imath_A) = G(\phi_0^{-1})^{-1} \sum_{[\mathfrak{b}]} \phi_0^{-1}(\mathfrak{b}) \cdot d^{-1}\mathbf{f}_\nu^\dagger(\mathfrak{b} * (A_{p^{t+1}}, \alpha_{p^{t+1}}, \imath_{p^{t+1}})), \tag{2.3.3}$$

where the sum is over the classes $[\mathfrak{b}]$ in the kernel of the map $\mathrm{Pic}(\mathcal{O}_{p^{t+1}}) \longrightarrow \mathrm{Pic}(\mathcal{O}_K)$ induced by $\mathfrak{b} \longmapsto \mathfrak{b}\mathcal{O}_K$. For each such $[\mathfrak{b}]$, the same calculation from *loc.cit.* as above, but this time applied to the critical twist $d^{-1}\mathbf{f}_\nu^\dagger = d^{-1}\mathbf{f}_\nu^{*[p]} \otimes \vartheta_\nu$, yields

$$\begin{aligned} &d^{-1}\mathbf{f}_\nu^\dagger(\mathfrak{b} * (A_{p^{t+1}}, \alpha_{p^{t+1}}, \imath_{p^{t+1}})) \\ &\qquad = G(\vartheta_\nu^{-1})^{-1} \sum_{u \bmod p^s} \vartheta_\nu^{-1}(u) \cdot d^{-1}\mathbf{f}_\nu^{*[p]}(\mathfrak{b} * (A_{p^{t+1}}^{(u)}, \alpha_{p^{t+1}}^{(u)}, \imath_{p^{t+1}}^{(u)})), \end{aligned} \tag{2.3.4}$$

where the triples $(A_{p^{t+1}}^{(u)}, \alpha_{p^{t+1}}^{(u)}, \imath_{p^{t+1}}^{(u)})$ are deduced from $(A_{p^{t+1}}, \alpha_{p^{t+1}}, \imath_{p^{t+1}})$ in the same manner as before.

Now for a fixed $u \in \mathbf{Z}/p^s\mathbf{Z}$, set

$$h_{p^{t+1-s},s} = (A_{p^{t+1}}^{(u)}, \alpha_{p^{t+1}}^{(u)}(\boldsymbol{\mu}_N), (\imath_{p^{t+1}}^{(u)})^{-1}(\zeta_s)), \tag{2.3.5}$$

which defines a point on the modular curve X_s of level $\Gamma_0(N) \cap \Gamma_1(p^s)$ rational over

$$L_{p^{t+1-s},s} := H_{p^{t+1}}(\zeta_s);$$

indeed, [**How07b**, Lemma 2.2.1] gives

$$h_{p^{t+1-s},s}^\tau = \langle \vartheta(\tau) \rangle \cdot h_{p^{t+1-s},s} \qquad \text{for all } \tau \in G_{H_{p^{t+1}}},$$

where $\vartheta : G_{H_{p^{t+1}}} \to \mathbf{Z}_p^\times/\{\pm 1\}$ is such that $\zeta_s^\tau = \zeta_s^{\vartheta(\tau)^2}$.

Since by construction both collections

$$\left\{ A_{p^{t+1}}^{(u)} \ : \ u \bmod p^s \right\}$$

and

$$\{\mathfrak{c} * A_{p^{t+1}} \ : \ [\mathfrak{c}] \in \ker(\mathrm{Pic}(\mathcal{O}_{p^{t+1}}) \longrightarrow \mathrm{Pic}(\mathcal{O}_{p^{t+1-s}}))\}$$

consist of p^s pairwise non-isomorphic elliptic curves p^s-isogenous to $A_{p^{t+1}}$, we see that (2.3.4) can be rewritten as

$$\begin{aligned} (2.3.6)\qquad & d^{-1}\mathbf{f}_\nu^\dagger(\mathfrak{b} * (A_{p^{t+1}}, \alpha_{p^{t+1}}, \imath_{p^{t+1}})) \\ &= G(\vartheta_\nu^{-1})^{-1} \sum_{\sigma\in \mathrm{Gal}(H_{p^{t+1}}/H_{p^{t+1-s}})} \chi_\nu^{-1}(\tilde{\sigma}) \cdot d^{-1}\mathbf{f}_\nu^{*[p]}(\mathfrak{b} * h_{p^{t+1-s},s}^{\tilde{\sigma}}), \end{aligned}$$

where for each $\sigma \in \mathrm{Gal}(H_{p^{t+1}}/H_{p^{t+1-s}})$, $\tilde{\sigma}$ is any lift of σ to $\mathrm{Gal}(L_{p^{t+1-s},s}/H_{p^{t+1-s}})$.

Now, for each $[\mathfrak{b}]$ as above, consider

$$_{\mathfrak{b}}\Delta_{p^{t+1-s},s} := (\mathfrak{b} * h_{p^{t+1-s},s}) - (\infty) \in \mathrm{Div}^0(X_s)(L_{p^{t+1-s},s})$$

and let

$$(2.3.7)\qquad {}_{\mathfrak{b}}\widetilde{Q}^{\chi_\nu}_{p^{t+1-s},s} := \sum_{\sigma\in \mathrm{Gal}(H_{p^{t+1}}/H_{p^{t+1-s}})} {}_{\mathfrak{b}}\Delta^{\tilde{\sigma}}_{p^{t+1-s},s} \otimes \chi_\nu^{-1}(\tilde{\sigma}) \in J_s(L_{p^{t+1-s},s}) \otimes_{\mathbf{Z}} F_\nu,$$

where J_s denotes the Jacobian variety of X_s, and for each $\sigma \in \mathrm{Gal}(H_{p^{t+1+s}}/H_{p^{t+1}})$, $\tilde{\sigma}$ is any lift of σ to $\mathrm{Gal}(L_{p^{t+1},s}/H_{p^{t+1}})$.

Let $F_{p^{t+1-s},s}$ be the composite of the completion of $L_{p^{t+1-s},s}$ at the prime above p induced by our fixed embedding $\imath_p : \overline{\mathbf{Q}} \longrightarrow \overline{\mathbf{Q}}_p$ and the finite extension $F_s/\mathbf{Q}_p(\zeta_s)$ over which the base change $X_s \times_{\mathbf{Q}_p} F_s$ admits a stable model.

By the explicit calculation in [**Cas13a**, Prop. 1.9], which applies to the pair $\mathbf{f}_\nu^*$ and ${}_{\mathfrak{b}}\Delta_{p^{t+1-s},s} \in J_s(F_{p^{t+1-s},s})$, we have

$$\begin{aligned} \log_{\omega_{\mathbf{f}_\nu^*}}({}_{\mathfrak{b}}\Delta_{p^{t+1-s},s}) &= \mathrm{AJ}_{F_{p^{t+1-s},s}}({}_{\mathfrak{b}}\Delta_{p^{t+1-s},s})(\omega_{\mathbf{f}_\nu^*}) \\ (2.3.8)\qquad &= F_{\omega_{\mathbf{f}_\nu^*}}(\mathfrak{b} * h_{p^{t+1-s},s}), \end{aligned}$$

where $F_{\omega_{\mathbf{f}_\nu^*}}$ is the Coleman primitive of $\omega_{\mathbf{f}_\nu^*}$ which vanishes at ∞. From the defining properties of $F_{\omega_{\mathbf{f}_\nu^*}}$ one can show that

$$(2.3.9)\qquad F_{\omega_{\mathbf{f}_\nu^*}} - \frac{\nu(\mathbf{a}_p)}{p}\mathrm{Frob}\ F_{\omega_{\mathbf{f}_\nu^*}} = d^{-1}\mathbf{f}_\nu^{*[p]}$$

as in [**Cas13a**, Cor. 1.8], and from the definition (2.3.5) we see that the Frobenius operator Frob appearing in (2.3.9) is such that

$$(2.3.10)\qquad \mathrm{Frob}\ h_{p^{t+1-s},s} = h_{p^{t-s},s}.$$

Combining (2.3.3) and (2.3.6), we thus arrive at (with sums taken over the same $[\mathfrak{b}]$ and σ as in those two equations)

$$\begin{aligned} d^{-1}\mathbf{f}_\nu^\dagger \otimes \phi_0(A, \alpha_A, \imath_A) &= G(\phi_0^{-1})^{-1} G(\vartheta_\nu^{-1})^{-1} \sum_\sigma \chi_\nu^{-1}(\tilde{\sigma}) \sum_{[\mathfrak{b}]} \phi_0^{-1}(\mathfrak{b}) \cdot d^{-1}\mathbf{f}_\nu^{*[p]}(\mathfrak{b} * h^{\tilde{\sigma}}_{p^{t+1-s},s}) \\ &= G(\phi_0^{-1})^{-1} G(\vartheta_\nu^{-1})^{-1} \sum_\sigma \chi_\nu^{-1}(\tilde{\sigma}) \sum_{[\mathfrak{b}]} \phi_0^{-1}(\mathfrak{b}) \cdot F_{\omega_{\mathbf{f}_\nu^*}}(\mathfrak{b} * h^{\tilde{\sigma}}_{p^{t+1-s},s}) \\ &= G(\phi_0^{-1})^{-1} G(\vartheta_\nu^{-1})^{-1} \sum_{[\mathfrak{b}]} \phi_0^{-1}(\mathfrak{b}) \cdot \log_{\omega_{\mathbf{f}_\nu^*}}({}_{\mathfrak{b}}\widetilde{Q}^{\chi_\nu}_{p^{t+1-s},s}), \end{aligned} \tag{2.3.11}$$

where the second equality follows from the combination of (2.3.9) and (2.3.10), and the last one from (2.3.8).

The integer $s > 0$ is such that the map $\mathbb{T} \longrightarrow V^*_{\mathbf{f}_\nu}$ can be factored as

$$\mathbb{T} \longrightarrow \mathrm{Ta}_p^{\mathrm{ord}}(J_s) \longrightarrow V^*_{\mathbf{f}_\nu},$$

and tracing through the definitions we see that the image of ${}_{\mathfrak{b}}\widetilde{Q}^{\chi_\nu}_{p^{t+1-s},s}$ under the induced (unlabelled) map

$$\begin{aligned} J_s(L_{p^{t+1-s},s}) \otimes_{\mathbf{Z}} F_\nu &\xrightarrow{\mathrm{Kum}_s} H^1(L_{p^{t+1-s},s}, \mathrm{Ta}_p(J_s) \otimes_{\mathbf{Z}} F_\nu) \\ &\xrightarrow{e^{\mathrm{ord}}} H^1(L_{p^{t+1-s},s}, \mathrm{Ta}_p^{\mathrm{ord}}(J_s) \otimes_{\mathbf{Z}} F_\nu) \\ &\longrightarrow H^1(L_{p^{t+1-s},s}, V^*_{\mathbf{f}_\nu}) \cong H^1(L_{p^{t+1-s},s}, V^\dagger_{\mathbf{f}_\nu}), \end{aligned}$$

agrees with the image of $\nu(\mathfrak{X}^{\sigma_{\mathfrak{b}}}_{p^{t+1}})$ in $H^1(L_{p^{t+1-s},s}, V^\dagger_{\mathbf{f}_\nu})$ under restriction, from where

$$\begin{aligned} \log_{\omega_{\mathbf{f}_\nu^*}}({}_{\mathfrak{b}}\widetilde{Q}^{\chi_\nu}_{p^{t+1-s},s}) &= \langle \log_{V^\dagger_{\mathbf{f}_\nu}}(\mathrm{Kum}_s(e^{\mathrm{ord}}({}_{\mathfrak{b}}\widetilde{Q}^{\chi_\nu}_{p^{t+1-s},s}))), \omega_{\mathbf{f}_\nu^*}\rangle_{\mathrm{dR}} \\ &= \langle \log_{V^\dagger_{\mathbf{f}_\nu}}(\mathrm{loc}_{\mathfrak{p}}(\nu(\mathfrak{X}^{\sigma_{\mathfrak{b}}}_{p^{t+1}}))), \omega_{\mathbf{f}_\nu^*}\rangle_{\mathrm{dR}}. \end{aligned} \tag{2.3.12}$$

Substituting (2.3.12) into (2.3.11) we thus arrive at

$$\begin{aligned} d^{-1}\mathbf{f}_\nu^\dagger \otimes \phi_0(A, \alpha_A, \imath_A) &= G(\phi_0^{-1})^{-1} G(\vartheta_\nu^{-1})^{-1} \sum_{[\mathfrak{b}]} \phi_0^{-1}(\mathfrak{b}) \langle \log_{V^\dagger_{\mathbf{f}_\nu}}(\mathrm{loc}_{\mathfrak{p}}(\nu(\mathfrak{X}^{\sigma_{\mathfrak{b}}}_{p^{t+1}}))), \omega_{\mathbf{f}_\nu^*}\rangle_{\mathrm{dR}} \\ &= \frac{\nu(\mathbf{a}_p)^t}{G(\phi_0^{-1})} \langle \log_{V^\dagger_{\mathbf{f}_\nu}(\phi_0)}(U_p^{-t} \cdot \mathrm{loc}_{\mathfrak{p}}(\nu(\mathfrak{X}_{p^{t+1}}))^{\phi_0}, \omega^\dagger_{\mathbf{f}_\nu^*}\rangle_{\mathrm{dR}} \\ &= \frac{\nu(\mathbf{a}_p)^t}{G(\phi_0^{-1})} \langle \log_{V^\dagger_{\mathbf{f}_\nu}(\phi_0)}(\mathrm{loc}_{\mathfrak{p}}(\nu(\mathfrak{Z}_t))^{\phi_0}, \omega^\dagger_{\mathbf{f}_\nu^*}\rangle_{\mathrm{dR}}, \end{aligned}$$

where the last equality follows from the construction of $\mathfrak{Z}_t$ from $U_p^{-t} \cdot \mathfrak{X}_{p^{t+1}}$ as outlined in Section 2.1.2. The proof of Proposition 2.3.2 is thus concluded. ☐

If $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ has even weight $k_\nu = 2r_\nu \geq 2$ and if e_ζ is the $\mathbf{Q}_p$-basis of $D_{\mathrm{dR}}(\mathbf{Q}_p(1))$ corresponding to a fixed choice of a compatible system $(\zeta_s)_{s\geq 0}$ of primitive p^s-th roots of unity $\zeta_s \in \overline{\mathbf{Q}}_p$, we define the p-adic period $\Omega_\nu^\eta \in \mathbf{C}_p^\times$ associated to the $\mathbb{I}$-basis $\eta \in \mathbb{D}$ by the rule

$$\Omega_\nu^\eta = \langle \eta_\nu^\dagger \otimes e_\zeta^{\otimes r_\nu}, \omega_{\mathbf{f}_\nu^*}^\dagger \rangle_{\mathrm{dR}}. \tag{2.3.13}$$

REMARK 2.3.3. It follows from the proof of [**Och05**, Prop. 6.4] that Ω_ν^η is in fact a p-adic unit for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2 and trivial nebentypus.

COROLLARY 2.3.4. *Fix an $\mathbb{I}$-basis η of $\mathbb{D}$, and let ν be as in Proposition 2.3.2. Then*

$$\nu(\mathrm{Log}^\eta_{\mathscr{F}_w^+\mathbb{T}_\infty^\dagger}(\mathrm{loc}_\mathfrak{p}(\mathfrak{Z}_\infty))) = \frac{\mathscr{L}_\mathfrak{p}(\mathbf{f}_\nu^\dagger)}{\Omega_\nu^\eta} \tag{2.3.14}$$

as elements in $\Lambda_{\mathcal{O}_\nu}(D_\infty)$.

PROOF. By the definition of the p-adic L-function $\mathscr{L}_\mathfrak{p}(\mathbf{f}^\dagger)$ (see especially (2.1.6)) and the specialization properties of the map $\mathrm{Log}^\eta_{\mathscr{F}_w^+\mathbb{T}_\infty^\dagger}$ of Corollary 2.2.8, we see that Proposition 2.3.2 may be reformulated as the statement that the two sides of (2.3.14) agree when evaluated at ϕ, where ϕ is any finite order character of D_∞ of sufficiently large conductor. Since an element in $\Lambda_{\mathcal{O}_\nu}(D_\infty)$ is uniquely determined by these values, the result follows. $\square$

2.3.2. A patching argument. In this section we complete the proof of Theorem 2.3.1. Notice that Corollary 2.3.4 shows that (2.3.1) is satisfied, *for each* $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2 and non-trivial wild character, by taking $\mathfrak{Z}_{\infty,\nu}^\eta$ to be $\mathfrak{Z}_{\infty,\nu}^\eta := \Omega_\nu^\eta \cdot \nu(\mathfrak{Z}_\infty)$ for example, where Ω_ν^η is the p-adic period (2.3.13). Thus to conclude the proof of Theorem 2.3.1 it remains to show that the classes $\mathfrak{Z}_{\infty,\nu}^\eta$ can be patched together for different ν, i.e. that they arise as specializations of a *single* class $\mathfrak{Z}_\infty^\eta \in \widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty, \mathbb{T}^\dagger)$. This is the content of the next result, which is an analogue of [**Och06**, Thm. 6.1] in our context, and in which we closely follow Ochiai's arguments.

In the following, we will adopt the usual abuse of terminology and identify arithmetic primes $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ with the corresponding height one prime ideals $\ker(\nu) \subset \mathbb{I}$.

PROPOSITION 2.3.5. *Assume that $\bar{\rho}_\mathbf{f}|_{G_K}$ is absolutely irreducible, and fix an $\mathbb{I}$-basis η of $\mathbb{D}$. Let $\mathfrak{n}$ be the intersection of a finite number of $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2 and non-trivial wild character, and set $\mathbb{T}_\mathfrak{n}^\dagger := \mathbb{T}^\dagger \otimes_\mathbb{I} \mathbb{I}/\mathfrak{n}$. Then there exists a class $\mathfrak{Z}_{\infty,\mathfrak{n}}^\eta \in \widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty, \mathbb{T}_\mathfrak{n}^\dagger)$ such that the equality*

$$\nu(\mathrm{Log}^\eta_{\mathscr{F}_w^+\mathbb{T}_\infty^\dagger}(\mathrm{loc}_\mathfrak{p}(\mathfrak{Z}_{\infty,\mathfrak{n}}^\eta))) = \mathscr{L}_\mathfrak{p}(\mathbf{f}_\nu^\dagger) \tag{2.3.15}$$

holds for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ with $\mathfrak{n} \subset \ker(\nu)$.

PROOF. We argue by induction on the number of $\nu \in \mathcal{X}_{\rm arith}(\mathbb{I})$ with $\mathfrak{n} \subset \ker(\nu)$. As noted above, the case where there is a unique such ν follows from Corollary 2.3.4. So assume that the proposition holds for an ideal $\mathfrak{n}$ as in the statement. Then if ν is an arbitrary arithmetic prime of $\mathbb{I}$ of weight 2 and non-trivial wild character with $\mathfrak{n} \not\subseteq \ker(\nu)$, we will show that the result also holds for $\mathfrak{n}'' := \mathfrak{n} \cap \mathfrak{n}'$, where $\mathfrak{n}' := \ker(\nu)$.

There is an exact sequence

$$(2.3.16) \qquad 0 \longrightarrow \widetilde{H}^1_{f,\rm Iw}(K_\infty, \mathbb{T}^\dagger_{\mathfrak{n}''}) \xrightarrow{\ \alpha\ } \widetilde{H}^1_{f,\rm Iw}(K_\infty, \mathbb{T}^\dagger_{\mathfrak{n}}) \oplus \widetilde{H}^1_{f,\rm Iw}(K_\infty, T^\dagger_{\mathbf{f}_\nu}) \xrightarrow{\ \beta\ } \widetilde{H}^1_{f,\rm Iw}(K_\infty, \mathbb{T}^\dagger_{\mathfrak{n}\oplus\mathfrak{n}'}),$$

where α is the natural diagonal map, and β sends a class $(\mathfrak{X}_{\mathfrak{n}}, \mathfrak{Y}_\nu)$ to the difference $\mathfrak{X}_{\mathfrak{n}} - \mathfrak{Y}_\nu \bmod \mathfrak{n} \oplus \mathfrak{n}'$.

As above, the class $\mathfrak{Z}^\eta_{\infty,\mathfrak{n}'} := \Omega^\eta_\nu \cdot \mathfrak{Z}_\infty$ is such that

$$(2.3.17) \qquad \nu(\mathrm{Log}^\eta_{\mathscr{F}^+_w\mathbb{T}^\dagger_\infty}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}^\eta_{\infty,\mathfrak{n}'}))) = \mathscr{L}_{\mathfrak{p}}(\mathbf{f}^\dagger_\nu)$$

as elements in $\Lambda_{\mathcal{O}_\nu}(D_\infty)$. Evaluating (2.3.17) at a finite order character $\phi \in \mathcal{X}_{\rm arith}(\Lambda(D_\infty))$ of sufficiently large conductor, we obtain

$$(2.3.18) \qquad \nu(\mathrm{Log}^\eta_{\mathscr{F}^+_w\mathbb{T}^\dagger_\infty}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}^\eta_{\infty,\mathfrak{n}'})))(\phi) = \sum_{[\mathfrak{a}]\in \mathrm{Pic}(\mathcal{O}_K)} \phi^{-1}(\mathfrak{a}) \cdot d^{-1}\mathbf{f}^\dagger_\nu \otimes \phi_0(\mathfrak{a} * (A, \alpha_A, \imath_A)),$$

and on the other hand if $\nu' \in \mathcal{X}_{\rm arith}(\mathbb{I})$ is any arithmetic prime as in the statement such that $\mathfrak{n} \subset \ker(\nu')$, then by our induction hypothesis there exists a class $\mathfrak{Z}^\eta_{\infty,\mathfrak{n}} \in \widetilde{H}^1_{f,\rm Iw}(K_\infty, \mathbb{T}^\dagger_{\mathfrak{n}})$ such that

$$(2.3.19) \qquad \nu'(\mathrm{Log}^\eta_{\mathscr{F}^+_w\mathbb{T}^\dagger_\infty}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}^\eta_{\infty,\mathfrak{n}})))(\phi) = \sum_{[\mathfrak{a}]\in \mathrm{Pic}(\mathcal{O}_K)} \phi^{-1}(\mathfrak{a}) \cdot d^{-1}\mathbf{f}^\dagger_{\nu'} \otimes \phi_0(\mathfrak{a} * (A, \alpha_A, \imath_A)),$$

where in both equations (2.3.18) and (2.3.19) we decompose ϕ as in (2.1.5).

Since the q-expansions of the twists $d^{-1}\mathbf{f}^\dagger_\nu \otimes \phi_0$ and $d^{-1}\mathbf{f}^\dagger_{\nu'} \otimes \phi_0$ are congruent modulo $\mathfrak{n}' + \ker(\nu')$, the same is true for their values at the CM points appearing in (2.3.18) and (2.3.19), and hence the class $(\phi \circ \beta)(\mathfrak{Z}^\eta_{\infty,\mathfrak{n}}, \mathfrak{Z}^\eta_{\infty,\mathfrak{n}'})$ lies in the kernel of the composite map

$$(2.3.20) \qquad \widetilde{H}^1_{f,\rm Iw}(K_\infty, T^\dagger_{\mathbf{f}_\nu}) \xrightarrow{\mathrm{Sp}_\phi} \mathrm{Sel}_{\rm Gr}(K, T^\dagger_{\mathbf{f}_\nu}(\phi)) \xrightarrow{\mathrm{loc}_{\mathfrak{p}}} H^1(K_{\mathfrak{p}}, (\mathscr{F}^+_w T^\dagger_{\mathbf{f}_\nu})(\phi)) \xrightarrow{\log^{\eta_\nu}_{V^\dagger_{\mathbf{f}_\nu}}} \mathcal{O}_{\nu,\phi},$$

where $\mathcal{O}_{\nu,\phi}$ is the ring extension of $\mathcal{O}_\nu$ generated by the values of ϕ.

But from [**How07b**, Cor. 3.1.2] if follows that the map (2.3.20) is injective for all characters ϕ of finite order and sufficiently large conductor (cf. [**Cas13a**, Cor. 4.10]), and hence

$$\beta(\mathfrak{Z}^\eta_{\infty,\mathfrak{n}}, \mathfrak{Z}^\eta_{\infty,\mathfrak{n}'}) = 0.$$

By the exactness of the sequence (2.3.16), it follows that $(\mathfrak{Z}^\eta_{\infty,\mathfrak{n}}, \mathfrak{Z}^\eta_{\infty,\mathfrak{n}'})$ arises as the image under α of a class $\mathfrak{Z}^\eta_{\infty,\mathfrak{n}''} \in \widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty, \mathbb{T}_{\mathfrak{n}''})$ which satisfies (2.3.15) for all $\nu'' \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ with $\mathfrak{n}'' \subset \ker(\nu'')$. By induction, the proof of Proposition 2.3.5 is concluded. □

Now we can complete the proof of Theorem 2.3.1. Indeed, setting

$$\mathfrak{Z}^\eta_\infty := \varprojlim_{\mathfrak{n}} \mathfrak{Z}^\eta_{\infty,\mathfrak{n}}, \tag{2.3.21}$$

for $\mathfrak{n}$ running over a directed set of ideals as in Lemma 2.3.5 with $\bigcap_{\mathfrak{n}} \mathfrak{n} = 0$, we see that $\mathfrak{Z}^\eta_\infty$ satisfies (2.3.1) for infinitely many ϕ and ν. The result follows. □

Corollary 2.3.6. *Assume that $\bar{\rho}_{\mathbf{f}}|_{G_K}$ is absolutely irreducible, and fix an $\mathbb{I}$-basis η of $\mathbb{D}$. Then for each non-exceptional $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$, we have*

$$\alpha^\eta_\nu \cdot \nu(\mathrm{Log}^\eta_{\mathscr{F}^+_w \mathbb{T}^\dagger_\infty}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}_\infty))) = \mathscr{L}_{\mathfrak{p}}(\mathbf{f}^\dagger_\nu)$$

for some unit $\alpha^\eta_\nu \in \mathcal{O}^\times_\nu$.

Proof. By [**Fou13**, Thm. A(iii)] the group $\widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty, \mathbb{T}^\dagger)$ is torsion-free of rank one over $\Lambda_{\mathbb{I}}(D_\infty)$, and hence the class $\mathfrak{Z}^\eta_\infty$ from Theorem 2.3.1 is such that

$$\mathfrak{Z}^\eta_\infty = \alpha_\eta \cdot \mathfrak{Z}_\infty \tag{2.3.22}$$

for some element $\alpha_\eta \in \Lambda_{\mathbb{I}}(D_\infty)$. Since the construction (2.3.21) of the class $\mathfrak{Z}^\eta_\infty$ shows that $\nu(\alpha_\eta)$ is a unit in $\mathcal{O}^\times_\nu$ for some $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ (in fact for infinitely many ν; see Remark 2.3.3 and Corollary 2.3.4), it follows that in fact α_η lies in $\mathbb{I}^\times$. The result thus follows from the combination of (2.3.22) and Theorem 2.3.1 by setting $\alpha^\eta_\nu := \nu(\alpha_\eta)$. □

Remark 2.3.7. As follows immediately from the preceding proof of Corollary 2.3.6, the unit α^η_ν can be taken so that

$$\langle -, \omega^\dagger_{\mathbf{f}^*_\nu}\rangle_{\mathrm{dR}} = \alpha^\eta_\nu \cdot \langle -, \eta'_\nu\rangle_{\mathrm{dR}},$$

where $\eta'_\nu \in \mathrm{Fil}^0 D_{\mathrm{dR}}(V^\dagger_{\mathbf{f}_\nu})$ is such that $\langle \eta^\dagger_\nu \otimes e_\zeta^{\otimes r_\nu}, \eta'_\nu\rangle_{\mathrm{dR}} = 1$.

2.4. The method of Euler systems

In this section we use Kolyvagin's machinery of Euler systems [**Kol90**], in the form extended by Mazur–Rubin and Howard, to deduce the vanishing of certain Selmer groups.

2.4.1. Greenberg's Selmer groups. Throughout this section we fix an imaginary quadratic field K for which the Assumptions 2.1.1 hold.

Let $\nu \in \mathcal{X}_{\rm arith}(\mathbb{I})$ have even weight $k_\nu \geq 2$, let $\mathbf{f}_\nu$ be the ordinary p-stabilized newform with q-expansion $\nu(\mathbf{f})$, and denote by $V_{\mathbf{f}_\nu}$ the Galois representation associated with $\mathbf{f}_\nu$ by Deligne, regarded as a representation of G_K. Let $\chi \in \mathcal{X}_{\rm arith}(\Lambda(D_\infty))$ correspond to an anticyclotomic Hecke character of K of infinity type $(m, -m)$ with $m \geq k_\nu/2$.

Definition 2.4.1. If L/K is a finite extension, define

$$\mathrm{Sel}_{\mathfrak{p}}(L, V_{\mathbf{f}_\nu}^\dagger(\chi)) = \ker\left(H^1(L, V_{\mathbf{f}_\nu}^\dagger(\chi)) \longrightarrow \bigoplus_v \frac{H^1(L_v, V_{\mathbf{f}_\nu}^\dagger(\chi))}{H^1_f(L_v, V_{\mathbf{f}_\nu}^\dagger(\chi))}\right), \tag{2.4.1}$$

where v runs over all the places of L, and for $v \nmid p$ we put

$$H^1_f(L_v, V_{\mathbf{f}_\nu}^\dagger(\chi)) = \ker\left(H^1(L_v, V_{\mathbf{f}_\nu}^\dagger(\chi)) \longrightarrow H^1(L_v^{\rm nr}, V_{\mathbf{f}_\nu}^\dagger(\chi))\right),$$

whereas for $v|p$,

$$H^1_f(L_v, V_{\mathbf{f}_\nu}^\dagger(\chi)) = \begin{cases} H^1(L_v, V_{\mathbf{f}_\nu}^\dagger(\chi)) & \text{if } v|\bar{\mathfrak{p}}; \\ \\ 0 & \text{if } v|\mathfrak{p}. \end{cases} \tag{2.4.2}$$

In particular, the classes in $\mathrm{Sel}_{\mathfrak{p}}(K, V_{\mathbf{f}_\nu}^\dagger(\chi))$ are unramified outside p, satisfy no specific local condition at $\bar{\mathfrak{p}}$, and they have trivial restriction at $\mathfrak{p}$.

For $v|p$, it is easy to see that the subspaces (2.4.2) agree with the "finite" subspaces

$$H^1_f(L_v, V_{\mathbf{f}_\nu}^\dagger(\chi)) := \ker\left(H^1(L_v, V_{\mathbf{f}_\nu}^\dagger(\chi)) \longrightarrow H^1(L_v, V_{\mathbf{f}_\nu}^\dagger(\chi) \otimes B_{\rm cris})\right),$$

of Bloch–Kato, and hence $\mathrm{Sel}_{\mathfrak{p}}(L, V_{\mathbf{f}_\nu}^\dagger(\chi))$ is the same as the Bloch–Kato Selmer group.

Let $T_{\mathbf{f}_\nu}^\dagger(\chi) \subset V_{\mathbf{f}_\nu}^\dagger(\chi)$ be a G_K-stable lattice, and define $A_{\mathbf{f}_\nu}^\dagger(\chi)$ by the exactness of the sequence

$$0 \longrightarrow T_{\mathbf{f}_\nu}^\dagger(\chi) \longrightarrow V_{\mathbf{f}_\nu}^\dagger(\chi) \longrightarrow A_{\mathbf{f}_\nu}^\dagger(\chi) \longrightarrow 0. \tag{2.4.3}$$

Then if χ is as in Definition 2.4.1, by replacing the local subspaces $H^1_f(L_v, V_{\mathbf{f}_\nu}^\dagger(\chi))$ by their natural images in $H^1(L_v, A_{\mathbf{f}_\nu}^\dagger(\chi))$ (resp. preimages in $H^1(L_v, T_{\mathbf{f}_\nu}^\dagger(\chi))$ under the map induced by (2.4.3) on cohomology, we use the same formula (2.4.1) to define the Selmer groups $\mathrm{Sel}_{\mathfrak{p}}(L, A_{\mathbf{f}_\nu}^\dagger(\chi)) \subset H^1(L, A_{\mathbf{f}_\nu}^\dagger(\chi))$ (resp. $\mathrm{Sel}_{\mathfrak{p}}(L, T_{\mathbf{f}_\nu}^\dagger(\chi)) \subset H^1(L, T_{\mathbf{f}_\nu}^\dagger(\chi))$).

2.4.2. Anticyclotomic Kolyvagin systems. The following key result follows from an application of Kolyvagin's method of Euler systems as systematized by Mazur–Rubin [**MR04**] and adapted by Howard [**How04a**] to anticyclotomic settings.

THEOREM 2.4.2. *Let $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ be a non-exceptional arithmetic prime of even weight. If $\phi \in \mathcal{X}_{\mathrm{arith}}(\Lambda(D_\infty))$ is such that $\mathrm{loc}_{\mathfrak{p}}(\nu(\mathfrak{Z}_\infty)^{\chi\phi}) \neq 0$, then $\mathrm{Sel}_{\mathfrak{p}}(K, T^\dagger_{\mathbf{f}_\nu}(\chi\phi)) = 0$.*

PROOF. Our assumptions on ν imply that

$$\widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty, T^\dagger_{\mathbf{f}_\nu}) = \mathrm{Sel}_{\mathrm{Gr}}(K, T^\dagger_{\mathbf{f}_\nu}\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\mathrm{anti}})$$

by the discussion in [**How07b**, §2.4] preceding Prop. 2.4.5 in *loc.cit.*. Arguing as in [**Büy**, Thm. 4.28], we thus see that there exists a Kolyvagin system

$$\{\nu(\kappa_n)^\chi\}_n \in \mathbf{KS}(T^\dagger_{\mathbf{f}_\nu}(\chi)\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\mathrm{anti}}, \mathcal{F}_{\mathrm{Gr}})$$

for the strict Greenberg Selmer structure $\mathcal{F}_{\mathrm{Gr}}$ on $T^\dagger_{\mathbf{f}_\nu}(\chi)\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\mathrm{anti}}$ such that

$$\nu(\kappa_1)^\chi = \nu(\mathfrak{Z}_\infty)^\chi \tag{2.4.4}$$

in $\mathrm{Sel}_{\mathrm{Gr}}(K, T^\dagger_{\mathbf{f}_\nu}(\chi)\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\mathrm{anti}})$.

The discussion in [**How04a**, §2.2] preceding Thm. 2.2.10 in *loc.cit.* shows that there exists a map

$$\mathrm{Sp}_\phi : \mathbf{KS}(T^\dagger_{\mathbf{f}_\nu}(\chi)\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\mathrm{anti}}, \mathcal{F}_{\mathrm{Gr}}) \longrightarrow \mathbf{KS}(T^\dagger_{\mathbf{f}_\nu}(\chi\phi), \widetilde{\mathcal{F}}_{\mathrm{Gr}})$$

induced by localization at the height one prime of $\Lambda(D_\infty)$ corresponding to ϕ sending the class (2.4.4) to $\mathrm{Sp}_\phi(\nu(\kappa_1)^\chi) = \nu(\mathfrak{Z}_\infty)^{\chi\phi}$, where $\widetilde{\mathcal{F}}_{\mathrm{Gr}}$ denotes the Selmer structure on $T^\dagger_{\mathbf{f}_\nu}(\chi\phi)$ induced from $\mathcal{F}_{\mathrm{Gr}}$, so that in particular for each prime $\mathfrak{q}|p$ in K,

$$\begin{aligned} H^1_{\widetilde{\mathcal{F}}_{\mathrm{Gr}}}(K_{\mathfrak{q}}, T^\dagger_{\mathbf{f}_\nu}(\chi\phi)) &= \ker\Big(H^1(K_{\mathfrak{q}}, T^\dagger_{\mathbf{f}_\nu}(\chi\phi)) \longrightarrow H^1(K_{\mathfrak{q}}, (\mathscr{F}^-_w T^\dagger_{\mathbf{f}_\nu})(\chi\phi))\Big) \\ &\cong H^1(K_{\mathfrak{q}}, (\mathscr{F}^+_w T^\dagger_{\mathbf{f}_\nu})(\chi\phi)), \end{aligned}$$

where the isomorphism follows from the assumption that ν is nonexceptional, which implies that $H^0(K_{\mathfrak{q}}, (\mathscr{F}^-_w T^\dagger_{\mathbf{f}_\nu})(\chi\phi)) = 0$ (see [**How07b**, Lemma 2.4.4]) and so the map

$$H^1(K_{\mathfrak{q}}, (\mathscr{F}^+_w T^\dagger_{\mathbf{f}_\nu})(\chi\phi)) \longrightarrow H^1(K_{\mathfrak{q}}, T^\dagger_{\mathbf{f}_\nu}(\chi\phi))$$

induced by (2.2.1) is injective.

Our nonvanishing assumption obviously ensures that $\nu(\mathfrak{Z}_\infty)^{\chi\phi} \neq 0$, and by [**How04a**, Thm. 1.6.1] it follows that

$$H^1_{\widetilde{\mathcal{F}}_{\mathrm{Gr}}}(K, T^\dagger_{\mathbf{f}_\nu}(\chi\phi)) \cong \mathcal{O} \qquad \text{and} \qquad H^1_{\widetilde{\mathcal{F}}_{\mathrm{Gr}}}(K, A^\dagger_{\mathbf{f}_\nu}(\chi\phi)) \cong F/\mathcal{O} \oplus M^{\oplus 2} \tag{2.4.5}$$

for some finite $\mathcal{O}$-module M. It remains to compare the Selmer modules (2.4.5) with $\mathrm{Sel}_{\mathfrak{p}}(K, T^\dagger_{\mathbf{f}_\nu}(\chi\phi))$ and $\mathrm{Sel}_{\mathfrak{p}}(K, A^\dagger_{\mathbf{f}_\nu}(\chi\phi))$, respectively.

For any continuous character $\psi : D_\infty \longrightarrow \mathcal{O}^\times$ consider the auxiliary Selmer groups

$$\mathrm{Sel}_{\mathrm{Gr},\emptyset}(K, A^\dagger_{\mathbf{f}_\nu}(\psi)) := \{c \in \mathrm{Sel}^{\{\bar{\mathfrak{p}}\}}(K, A^\dagger_{\mathbf{f}_\nu}(\psi)) \mid \mathrm{loc}_\mathfrak{p}(c) \in H^1_{\widetilde{\mathcal{F}}_{\mathrm{Gr}}}(K_\mathfrak{p}, A^\dagger_{\mathbf{f}_\nu}(\psi))\};$$
$$\mathrm{Sel}_{\mathrm{Gr},0}(K, A^\dagger_{\mathbf{f}_\nu}(\psi)) := \mathrm{Sel}_{\mathrm{Gr},\emptyset}(K, A^\dagger_{\mathbf{f}_\nu}(\psi)) \cap \mathrm{Sel}_{\bar{\mathfrak{p}}}(K, A^\dagger_{\mathbf{f}_\nu}(\psi)).$$

Directly from the definitions, we have the chain of inclusions

$$\mathrm{Sel}_{\mathrm{Gr},0}(K, A^\dagger_{\mathbf{f}_\nu}(\psi)) \subset \mathrm{Sel}_\mathfrak{p}(K, A^\dagger_{\mathbf{f}_\nu}(\psi)) \subset \mathrm{Sel}_{\mathrm{Gr},\emptyset}(K, A^\dagger_{\mathbf{f}_\nu}(\psi)). \tag{2.4.6}$$

LEMMA 2.4.3. *There is a noncanonical isomorphism of $\mathcal{O}$-modules*

$$\mathrm{Sel}_{\mathrm{Gr},\emptyset}(K, A^\dagger_{\mathbf{f}_\nu}(\chi\phi)) \cong F/\mathcal{O} \oplus \mathrm{Sel}_{\mathrm{Gr},0}(K, A^\dagger_{\mathbf{f}_\nu}(\chi^{-1}\phi^{-1}))$$

PROOF. As in [**AH06**, Prop. 1.2.3], we have

$$\mathrm{Sel}_{\mathrm{Gr},\emptyset}(K, A^\dagger_{\mathbf{f}_\nu}(\chi\phi))[p^i] \cong (F/\mathcal{O})^r[p^i] \oplus \mathrm{Sel}_{\mathrm{Gr},0}(K, A^\dagger_{\mathbf{f}_\nu}(\chi^{-1}\phi^{-1}))[p^i]$$

for every i, where the integer r is given by

$$\mathrm{corank}_\mathcal{O} H^1(K_\mathfrak{p}, (\mathscr{F}^+_w A^\dagger_{\mathbf{f}_\nu})(\chi\phi)) + \mathrm{corank}_\mathcal{O} H^1(K_{\bar{\mathfrak{p}}}, A^\dagger_{\mathbf{f}_\nu}(\chi\phi)) - \mathrm{corank}_\mathcal{O} H^0(K_v, A^\dagger_{\mathbf{f}_\nu}(\chi\phi))$$

in light of the Poitou–Tate duality as formulated in [**Wil95**] (see also [**DDT94**, §2.3]), and where v denotes the unique archimedean place of K. Hence $r = 1$, and letting $i \longrightarrow \infty$ the result follows. ☐

Now since $\mathrm{loc}_\mathfrak{p}(\nu(\mathfrak{Z}_\infty)^{\chi\phi}) \neq 0$ by assumption, we see from (2.4.5) and (2.4.6) together with Lemma 2.4.3, that $\mathrm{Sel}_{\mathrm{Gr},0}(K, A^\dagger_{\mathbf{f}_\nu}(\chi^{-1}\phi^{-1}))$ is necessarily finite and that

$$\mathrm{Sel}_{\mathrm{Gr},\emptyset}(K, A^\dagger_{\mathbf{f}_\nu}(\chi\phi)) \cong F/\mathcal{O} \oplus N \tag{2.4.7}$$

for some finite $\mathcal{O}$-module N, and hence the module $\mathrm{Sel}_{\mathrm{Gr},\emptyset}(K, T^\dagger_{\mathbf{f}_\nu}(\chi\phi))$, which is just the Tate module of (2.4.7), is a free of rank one over $\mathcal{O}$, generated by $\nu(\mathfrak{Z}_\infty)^{\chi\phi}$.

We thus see that under our nonvanishing assumption the localization map

$$\mathrm{loc}_\mathfrak{p} : \mathrm{Sel}_{\mathrm{Gr},\emptyset}(K, T^\dagger_{\mathbf{f}_\nu}(\chi\phi)) \longrightarrow H^1(K_\mathfrak{p}, (\mathscr{F}^+_w T^\dagger_{\mathbf{f}_\nu})(\chi\phi)) \tag{2.4.8}$$

is injective, and hence the Selmer group $\mathrm{Sel}_\mathfrak{p}(K, T^\dagger_{\mathbf{f}_\nu}(\chi\phi))$, which is just the kernel of the map (2.4.8), vanishes. ☐

2.5. Arithmetic applications

In this section we deduce the main arithmetic applications of this paper. These will be deduced from the general results of Section 2.4, using Theorem 2.3.1 to relate the nontriviality of various specializations of $\mathfrak{Z}_\infty$ to the nonvanishing of certain L-values.

2.5.1. Bounding Selmer groups. Here we deduce the vanishing of certain Selmer groups attached to the Rankin–Selberg convolution of a cusp form with a theta series of higher weight in cases predicted by the Bloch–Kato conjecture.

THEOREM 2.5.1. *Let $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ be a non-exceptional arithmetic prime of even weight $k_\nu = 2r_\nu \geq 2$ and trivial nebentypus, and let $\phi \in \mathcal{X}_{\mathrm{arith}}(\Lambda(D_\infty))$ have weight $m_\phi \geq r_\nu$. Then*

$$\begin{aligned} &\exp^*(\mathrm{loc}_{\mathfrak{p}}(\nu(\mathfrak{Z}_\infty)^\phi)) = (m_\phi - r_\nu)!^{-1} \\ &\qquad \times \begin{cases} \left(1 - \frac{p^{r_\nu - 1}}{\nu(\mathbf{a}_p)\varpi^{m_\phi}}\right)\left(1 - \frac{\nu(\mathbf{a}_p)\varpi^{m_\phi}}{p^{r_\nu}}\right)^{-1} \cdot \mathscr{L}_{\mathfrak{p}}(\mathbf{f}_\nu^\dagger)(\phi) \cdot \omega^\dagger_{\mathbf{f}_\nu^*, -m_\phi} & \text{if } \phi_0 = \mathbb{1}; \\ \\ G(\phi_0^{-1})\left(\frac{p^{r_\nu - 1}}{\nu(\mathbf{a}_p)\varpi^{m_\phi}}\right)^t \cdot \mathscr{L}_{\mathfrak{p}}(\mathbf{f}_\nu^\dagger)(\phi) \cdot \omega^\dagger_{\mathbf{f}_\nu^*, -m_\phi} & \text{if } \phi_0 \neq \mathbb{1}, \end{cases} \end{aligned}$$

where $\exp^* : H^1(K_{\mathfrak{p}}, V^\dagger_{\mathbf{f}_\nu}(\phi)) \longrightarrow \mathrm{Fil}^0 D_{\mathrm{dR}}(V^\dagger_{\mathbf{f}_\nu}(\phi))$ is the dual exponential map, $\omega^\dagger_{\mathbf{f}_\nu^*}$ is the class in $D_{\mathrm{dR}}(V^\dagger_{\mathbf{f}_\nu})$ associated with $\mathbf{f}_\nu^* \otimes \varepsilon_{\mathrm{cyc}}^{r_\nu - 1}$, $\omega^\dagger_{\mathbf{f}_\nu^*, -m_\phi}$ its image in $D_{\mathrm{dR}}(V^\dagger_{\mathbf{f}_\nu}(\varepsilon_\varpi^{-m_\phi}))$, and $t > 0$ is the power of p in the conductor of ϕ_0. In particular,

$$\exp^*(\mathrm{loc}_{\mathfrak{p}}(\nu(\mathfrak{Z}_\infty)^\phi)) \neq 0 \iff \mathscr{L}_{\mathfrak{p}}(\mathbf{f}_\nu^\dagger)(\phi) \neq 0. \tag{2.5.1}$$

PROOF. Let $\phi \in \mathcal{X}_{\mathrm{arith}}(\Lambda(D_\infty))$ be as in the statement. Then upon choosing an $\mathbb{I}$-basis η of $\mathbb{D}$, and combining Corollary 2.2.10 with Corollary 2.3.6, there exists a unit $\alpha_\nu^\eta \in \mathcal{O}_\nu^\times$ such that

$$\begin{aligned} \mathscr{L}_{\mathfrak{p}}(\mathbf{f}_\nu^\dagger)(\phi) &= \alpha_\nu^\eta \cdot \nu(\mathrm{Log}^\eta_{\mathscr{F}_w^+ \mathbb{T}_\infty^\dagger}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}_\infty)))(\phi) \\ &= \alpha_\nu^\eta \cdot \mathfrak{L}^{\eta_\nu}_{V^\dagger_{\mathbf{f}_\nu}}(\mathrm{loc}_{\mathfrak{p}}(\nu(\mathfrak{Z}_\infty)))(\phi) \\ &= \alpha_\nu^\eta \cdot (m_\phi - r_\nu)! \\ &\quad \times \begin{cases} \left(1 - \frac{p^{r_\nu - 1}}{\nu(\mathbf{a}_p)\varpi^{m_\phi}}\right)^{-1}\left(1 - \frac{\nu(\mathbf{a}_p)\varpi^{m_\phi}}{p^{r_\nu}}\right) \langle \exp^*(\mathrm{loc}_{\mathfrak{p}}(\nu(\mathfrak{Z}_\infty)^\phi)), \eta_\nu' \rangle_{\mathrm{dR}}; \\ \\ G(\phi_0^{-1})^{-1}\left(\frac{\nu(\mathbf{a}_p)\varpi^{m_\phi}}{p^{r_\nu - 1}}\right)^t \langle \exp^*(\mathrm{loc}_{\mathfrak{p}}(\nu(\mathfrak{Z}_\infty)^\phi)), \eta_\nu' \rangle_{\mathrm{dR}}, \end{cases} \end{aligned} \tag{2.5.2}$$

depending on whether ϕ_0 is trivial or not.

Since neither of the factors $\left(1 - \frac{p^{r_\nu - 1}}{\nu(\mathbf{a}_p)\varpi^{m_\phi}}\right)$ or $\left(1 - \frac{\nu(\mathbf{a}_p)\varpi^{m_\phi}}{p^{r_\nu}}\right)$ vanishes, the result follows from (2.5.2) in light of Remark 2.3.7. ◻

COROLLARY 2.5.2. *If $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ and $\phi \in \mathcal{X}_{\mathrm{arith}}(\Lambda(D_\infty))$ are as in Theorem 2.5.1, then*

$$\mathrm{loc}_{\mathfrak{p}}(\nu(\mathfrak{Z}_\infty)^\phi) = 0 \iff L(\mathbf{f}_\nu, \chi_\nu \phi^{-1}, 1) = 0. \tag{2.5.3}$$

PROOF. Since $m_\phi \geq k_\nu/2$, the character ϕ lies within the range of classical interpolation of $\mathscr{L}_\mathfrak{p}(\mathbf{f}_\nu^\dagger)$, and hence the square of the value $\mathscr{L}_\mathfrak{p}(\mathbf{f}_\nu^\dagger)(\phi)$ is given by Theorem 2.1.4 to be a nonzero multiple of $L(\mathbf{f}_\nu, \chi_\nu\phi^{-1}, 1)$. Also for $m_\phi \geq k_\nu/2$, the map

$$\exp^* : H^1(K_\mathfrak{p}, V_{\mathbf{f}_\nu}^\dagger(\epsilon_\varpi^{-m_\phi})) \longrightarrow D_{\mathrm{dR}}(V_{\mathbf{f}_\nu}^\dagger(\epsilon_\varpi^{-m_\phi}))$$

is an isomorphism, and hence the equivalence (2.5.3) follows from (2.5.1). □

As an immediate corollary, we can now complete the proof of the first main arithmetic application of this paper.

THEOREM 2.5.3. *Let $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ be a non-exceptional arithmetic prime of even weight $k_\nu = 2r_\nu \geq 2$ and trivial nebentypus, and let $\phi \in \mathcal{X}_{\mathrm{arith}}(\Lambda(D_\infty))$ have weight $m_\phi \geq r_\nu$. Then the implication*

$$L(\mathbf{f}_\nu, \chi_\nu\phi^{-1}, 1) \neq 0 \quad \Longrightarrow \quad \mathrm{Sel}_\mathfrak{p}(K, T_{\mathbf{f}_\nu}^\dagger(\phi)) = 0$$

holds.

PROOF. Combine Theorem 2.4.2 and Corollary 2.5.2. □

2.5.2. Iwasawa theory. In this section we give an interpretation of the p-adic L-function $\mathscr{L}_\mathfrak{p}(\mathbf{f}_\nu^\dagger)$ from Section 2.1.1 in terms of the Iwasawa theory of the representation $V_{\mathbf{f}_\nu}^\dagger$ over the anticyclotomic $\mathbf{Z}_p$-extension K_∞/K.

Recall that $D_\infty := \mathrm{Gal}(K_\infty/K)$, and consider the $\Lambda_{\mathcal{O}_\nu}(D_\infty)$-module

$$\mathcal{S}_\mathfrak{p}(\mathbf{f}_\nu^\dagger) := \varinjlim_t \mathrm{Sel}_{\bar{\mathfrak{p}}}(K_t, A_{\mathbf{f}_\nu}^\dagger(\chi^{-1})),$$

where χ is any character in $\mathcal{X}_{\mathrm{arith}}(\Lambda(D_\infty))$ of weight $m_\phi \geq k_\nu/2$. This can be seen to be independent of the chosen χ arguing similarly as in [**Kat04**, §17.10], and its Pontryagin dual

$$X_\mathfrak{p}(\mathbf{f}_\nu^\dagger) := \mathrm{Hom}_{\mathbf{Z}_p}(\mathcal{S}_\mathfrak{p}(\mathbf{f}_\nu^\dagger), \mathbf{Q}_p/\mathbf{Z}_p)$$

can be seen to be finitely generated over $\Lambda_{\mathcal{O}_\nu}(D_\infty)$ adapting the arguments in the proof of the Proposition in [**Gre94b**, §4].

The following anticyclotomic Iwasawa main conjecture appears in the literature to be first formulated in the work [**Hsi12**] of Hsieh, and it is suggested by the point of view in Iwasawa theory developed in [**Gre94b**].

CONJECTURE 2.5.4. *The module $X_\mathfrak{p}(\mathbf{f}_\nu^\dagger)$ is $\Lambda_{\mathcal{O}_\nu}(D_\infty)$-torsion, and*

$$\mathrm{char}_{\Lambda_{\mathcal{O}_\nu}(D_\infty)}(X_\mathfrak{p}(\mathbf{f}_\nu^\dagger)) = (\mathscr{L}_\mathfrak{p}(\mathbf{f}_\nu^\dagger)^2)$$

as ideals in $\Lambda_{\mathcal{O}_\nu}(D_\infty)$.

As a result towards this conjecture, we can prove the following.

THEOREM 2.5.5. *Assume that $\overline{\rho}_{\mathbf{f}}|_{G_K}$ is absolutely irreducible, and let $\nu \in \mathcal{X}_{\rm arith}(\mathbb{I})$ be a non-exceptional arithmetic prime of even weight $k_\nu \geq 2$ and trivial nebentypus. Then*

$$\mathrm{char}_{\Lambda_{\mathcal{O}_\nu}(D_\infty)}(X_{\mathfrak{p}}(\mathbf{f}_\nu^\dagger)) \quad \textit{divides} \quad (p^n \cdot \mathscr{L}_{\mathfrak{p}}(\mathbf{f}_\nu^\dagger)^2) \tag{2.5.4}$$

for some $n \geq 0$. In particular, $X_{\mathfrak{p}}(\mathbf{f}_\nu^\dagger)$ is $\Lambda_{\mathcal{O}_\nu}(D_\infty)$-torsion.

PROOF. For the ease of notation, set $\Lambda_{\mathcal{O}_\nu} := \Lambda_{\mathcal{O}_\nu}(D_\infty)$ in what follows. By [**How07b**, Cor. 3.1.2], the class $\nu(\mathfrak{Z}_\infty)$ is not $\Lambda_{\mathcal{O}_\nu}$-torsion, and therefore the twist $\nu(\mathfrak{Z}_\infty)^\chi$ is not $\Lambda_{\mathcal{O}_\nu}$-torsion (see [**Rub00**, Thm. 6.4.1] for example). Arguing as in [**How04a**, Thm. 2.2.10], we thus deduce that

$$H^1_{\mathcal{F}_{\rm Gr}}(K, T^\dagger_{\mathbf{f}_\nu}(\chi)\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\rm anti}) \cong \Lambda_{\mathcal{O}_\nu}$$

and that there is a pseudo-isomorphism

$$H^1_{\mathcal{F}_{\rm Gr}}(K, A^\dagger_{\mathbf{f}_\nu}(\chi^{-1})\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\rm anti})^\vee \sim \Lambda_{\mathcal{O}_\nu} \oplus M^{\oplus 2} \tag{2.5.5}$$

for some torsion $\Lambda_{\mathcal{O}_\nu}$-module such that

$$\mathrm{char}_{\Lambda_{\mathcal{O}_\nu}}(M) \quad \text{divides} \quad \mathrm{char}_{\Lambda_{\mathcal{O}_\nu}}\left(\frac{H^1_{\mathcal{F}_{\rm Gr}}(K, T^\dagger_{\mathbf{f}_\nu}(\chi)\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\rm anti})}{\Lambda_{\mathcal{O}_\nu}\cdot\nu(\mathfrak{Z}_\infty)^\chi}\right). \tag{2.5.6}$$

Now we consider the Selmer modules

$$X_{\mathrm{Gr},\emptyset}(\mathbf{f}_\nu^\dagger) := \mathcal{S}_{\mathrm{Gr},\emptyset}(\mathbf{f}_\nu^\dagger)^\vee, \quad \text{where} \quad \mathcal{S}_{\mathrm{Gr},\emptyset}(\mathbf{f}_\nu^\dagger) := \varinjlim_t \mathrm{Sel}_{\emptyset,\mathrm{Gr}}(K_t, A^\dagger_{\mathbf{f}_\nu}(\chi^{-1}));$$

$$X_{\mathrm{Gr},0}(\mathbf{f}_\nu^\dagger) := \mathcal{S}_{\mathrm{Gr},0}(\mathbf{f}_\nu^\dagger)^\vee, \quad \text{where} \quad \mathcal{S}_{\mathrm{Gr},0}(\mathbf{f}_\nu^\dagger) := \varinjlim_t \mathrm{Sel}_{0,\mathrm{Gr}}(K_t, A^\dagger_{\mathbf{f}_\nu}(\chi^{-1})),$$

where the groups appearing in the right-hand sides are defined in the same way as the auxiliary Selmer groups introduced in the proof of Theorem 2.4.2, but replacing K by K_t. Similarly as in (2.4.6), we have the chain of inclusions

$$\mathcal{S}_{\mathrm{Gr},0}(\mathbf{f}_\nu^\dagger) \subset \mathcal{S}_{\mathfrak{p}}(\mathbf{f}_\nu^\dagger) \subset \mathcal{S}_{\mathrm{Gr},\emptyset}(\mathbf{f}_\nu^\dagger), \tag{2.5.7}$$

and arguing as in the proof of [**AH06**, Thm. 1.2.2] (cf. also Lemma 2.4.3) we find that

$$\begin{aligned}\mathrm{rank}_{\Lambda_{\mathcal{O}_\nu}}(\mathcal{S}_{\mathrm{Gr},\emptyset}(\mathbf{f}_\nu^\dagger)) &= \mathrm{rank}_{\Lambda_{\mathcal{O}_\nu}}(X_{\mathrm{Gr},\emptyset}(\mathbf{f}_\nu^\dagger)) \\ &= 1 + \mathrm{rank}_{\Lambda_{\mathcal{O}_\nu}}(X_{\mathrm{Gr},0}(\mathbf{f}_\nu^\dagger)) = 1 + \mathrm{rank}_{\Lambda_{\mathcal{O}_\nu}}(\mathcal{S}_{\mathrm{Gr},0}(\mathbf{f}_\nu^\dagger)),\end{aligned} \tag{2.5.8}$$

and that

$$\mathrm{char}_{\Lambda_{\mathcal{O}_\nu}}(X_{\mathrm{Gr},\emptyset}(\mathbf{f}_\nu^\dagger)_{\rm tors})[1/p] = \mathrm{char}_{\Lambda_{\mathcal{O}_\nu}}(X_{\mathrm{Gr},0}(\mathbf{f}_\nu^\dagger)_{\rm tors})[1/p], \tag{2.5.9}$$

where the subscript tors indicates $\Lambda_{\mathcal{O}_\nu}$-torsion.

Since $\nu(\mathfrak{Z}_\infty)^\chi$ is not $\Lambda_{\mathcal{O}_\nu}$-torsion, we easily see that $\mathrm{loc}_\mathfrak{p}(\nu(\mathfrak{Z}_\infty)^\chi)$ is also not $\Lambda_{\mathcal{O}_\nu}$-torsion, and since $H^1_{\mathcal{F}_{\mathrm{Gr}}}(K, T^\dagger_{\mathbf{f}_\nu}(\chi)\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\mathrm{anti}}) \cong \Lambda_{\mathcal{O}_\nu}$, we see from (2.5.7) and (2.5.8) that both $X_{\mathrm{Gr},0}(\mathbf{f}^\dagger_\nu)$ and the quotient $\mathrm{Sel}_{\mathrm{Gr},\emptyset}(K, T^\dagger_{\mathbf{f}_\nu}(\chi)\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\mathrm{anti}})/\Lambda_{\mathcal{O}_\nu}\cdot\nu(\mathfrak{Z}_\infty)^\chi$ are $\Lambda_{\mathcal{O}_\nu}$-torsion. Moreover, combining (2.5.5), (2.5.6) and (2.5.9) it follows that

$$\mathrm{char}_{\Lambda_{\mathcal{O}_\nu}}(X_{\mathrm{Gr},0}(\mathbf{f}^\dagger_\nu)) \quad \text{divides} \quad p^n\cdot\mathrm{char}_{\Lambda_{\mathcal{O}_\nu}}\left(\frac{\mathrm{Sel}_{\mathrm{Gr},\emptyset}(K, T^\dagger_{\mathbf{f}_\nu}(\chi)\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\mathrm{anti}})}{\Lambda_{\mathcal{O}_\nu}\cdot\nu(\mathfrak{Z}_\infty)^\chi}\right)^2 \tag{2.5.10}$$

for some $n \geq 0$. By Poitou–Tate duality we have the exact sequence

$$\begin{aligned} 0 \longrightarrow \frac{\mathrm{Sel}_{\mathrm{Gr},\emptyset}(K, T^\dagger_{\mathbf{f}_\nu}(\chi)\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\mathrm{anti}})}{\Lambda_{\mathcal{O}_\nu}\cdot\nu(\mathfrak{Z}_\infty)^\chi} &\longrightarrow \frac{H^1_{\mathcal{F}_{\mathrm{Gr}}}(K_\mathfrak{p}, T^\dagger_{\mathbf{f}_\nu}(\chi)\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\mathrm{anti}})}{\Lambda_{\mathcal{O}_\nu}\cdot\mathrm{loc}_\mathfrak{p}(\nu(\mathfrak{Z}_\infty)^\chi)} \\ &\longrightarrow X_\mathfrak{p}(\mathbf{f}^\dagger_\nu) \longrightarrow X_{\mathrm{Gr},0}(\mathbf{f}^\dagger_\nu) \longrightarrow 0, \end{aligned}$$

which together with (2.5.10) immediately implies that

$$\mathrm{char}_{\Lambda_{\mathcal{O}_\nu}}(X_\mathfrak{p}(\mathbf{f}^\dagger_\nu)) \quad \text{divides} \quad p^n\cdot\mathrm{char}_{\Lambda_{\mathcal{O}_\nu}}\left(\frac{H^1_{\mathcal{F}_{\mathrm{Gr}}}(K_\mathfrak{p}, T^\dagger_{\mathbf{f}_\nu}(\chi)\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\mathrm{anti}})}{\Lambda_{\mathcal{O}_\nu}\cdot\mathrm{loc}_\mathfrak{p}(\nu(\mathfrak{Z}_\infty)^\chi)}\right)^2 \tag{2.5.11}$$

for some $n \geq 0$.

By Theorem 2.5.1 the p-adic regulator map $\mathfrak{L}^{\eta_\nu}_{V^\dagger_{\mathbf{f}_\nu}}$ sends $\mathrm{loc}_\mathfrak{p}(\nu(\mathfrak{Z}_\infty))$ to $\mathscr{L}_\mathfrak{p}(\mathbf{f}^\dagger_\nu)$, inducing a $\Lambda_{\mathcal{O}_\nu}$-module isomorphism

$$\frac{H^1_{\mathcal{F}_{\mathrm{Gr}}}(K_\mathfrak{p}, T^\dagger_{\mathbf{f}_\nu}(\chi)\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\mathrm{anti}})}{\Lambda_{\mathcal{O}_\nu}\cdot\mathrm{loc}_\mathfrak{p}(\nu(\mathfrak{Z}_\infty)^\chi)} \cong \frac{H^1(K_\mathfrak{p}, (\mathscr{F}^+_w T^\dagger_{\mathbf{f}_\nu})\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\mathrm{anti}})}{\Lambda_{\mathcal{O}_\nu}\cdot\mathrm{loc}_\mathfrak{p}(\nu(\mathfrak{Z}_\infty))} \xrightarrow{\sim} \frac{\Lambda_{\mathcal{O}_\nu}}{\Lambda_{\mathcal{O}_\nu}\cdot\mathscr{L}_\mathfrak{p}(\mathbf{f}^\dagger_\nu)}. \tag{2.5.12}$$

Finally, combining (2.5.11) and (2.5.12), the divisibility (2.5.4) follows, and since $\mathscr{L}_\mathfrak{p}(\mathbf{f}^\dagger_\nu)$ is not identically zero by Hsieh's Theorem 2.1.6, the proof of Theorem 2.5.5 is concluded. ☐

CHAPTER 3

Conclusion

Summary

In this last chapter we propose a few lines of investigation suggested by the problems and ideas explored in this thesis, and rise a number of questions and conjectures. For some of these we believe to have a good sense of how a proof would go, but for others we are admittedly being more speculative.

Future directions

Thanks to the combined efforts of many mathematicians, many results are known on the arithmetic of Heegner points, both in the classical setting of Gross–Zagier and in its subsequent generalizations, where the "Heegner hypothesis" (in which we have placed ourselves throughtout this thesis) is relaxed by working on Shimura curves attached to an appropriate non-split quaternion algebra over $\mathbf{Q}$.

After Howard's construction of big Heegner points, and its extension by Longo–Vigni to a general quaternionic setting, a natural line of enquiry is the study of the extent to which the results that are know about Heegner points might be extended to their "big" counterparts over Hida families. Some key steps in this direction were already undertaken by Howard in [**How07b**] and [**How07a**], and this thesis might be seen as a further development of this study in which p-adic L-functions are introduced in the form of two different $\mathbb{I}$-adic Gross–Zagier formulae for big Heegner points, namely Corollary 3.1.3, and Theorem 2.3.1.

In the following paragraphs we indicate some natural extensions of these results and their potential arithmetic applications, as we would like to pursue in our future work.

3.1. Specializations at exceptional primes

The study of the specializations of the big Heegner point at exceptional primes of the Hida family has been completely avoided throughout this thesis, but we expect that such study will have applications to *an anticyclotomic analogue* of the p-adic Birch and Swinnerton-Dyer conjecture of Mazur–Tate–Teitelbaum [**MTT86**] in the rank one case for primes p of split multiplicative reduction. As we outline below, our approach is reminiscent of the strategy taken in [**GS93**] in their proof of the rank zero case of the original (cyclotomic) conjecture, with a twisted form of the $\mathbb{I}$-adic Gross–Zagier formula of Theorem 3.1.3 playing the role of the "improved" p-adic L-function of Greenberg–Stevens.

Suppose for simplicity that the imaginary quadratic field K has class number $h_K = 1$. Let $\pi \in \mathcal{O}_K$ be a generator of the prime ideal $\mathfrak{p}$ of K above p, and denote by ψ and ϕ the Hecke characters of K defined by

$$\psi(\mathfrak{a}) = \alpha, \qquad \phi(\mathfrak{a}) = \alpha/\bar{\alpha}, \qquad \text{if } \mathfrak{a} = \alpha\mathcal{O}_K,$$

respectively. Denote by $\Phi : G_K \longrightarrow \mathbb{I}[[D_\infty]]^\times$ the "universal" anticyclotomic character sending each $g \in G_K$ to the group-like element in $\mathbb{I}[[D_\infty]]^\times$ associated with $(g|_{K_\infty})^{1/2} \in D_\infty$, and note that if $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ has even weight $2r_\nu \geq 2$, then $\Phi_\nu = \phi^{r_\nu - 1}$.

For $r \geq 1$, recall from Section 1.2 the generalised Heegner cycle $\Delta_r^{\mathrm{bdp}} \in \mathrm{CH}^{2r-1}(X_r)_0(K)$ on the Kuga–Sato variety $X_r = W_r \times A^{2r-2}$. For any $0 \leq j < r$, the cycle $W_r \times A^{r-1}$, seen as a subvariety of $W_r \times X_r = W_r \times W_r \times (A^2)^{r-1}$ via the map

$$(\mathrm{id}_{W_r}, \mathrm{id}_{W_r}, (\mathrm{id}_A, \mathrm{id}_A)^j, (\mathrm{id}_A, \sqrt{-D})^{r-1-j}),$$

induces a correspondence

$$\Pi_r^j : \mathrm{CH}^{2r-1}(X_r)_0(K) \longrightarrow \mathrm{CH}^r(W_r)_0(K)$$

sending $[\Delta] \longmapsto [\pi_{W*}\pi_X^*\Delta]$.

On the other hand, if $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ has even weight $k_\nu = 2r_\nu \geq 2$, the non-vanishing of $\nu(\mathfrak{Z}) = \nu(\mathfrak{Z}_\infty)_0$ predicted by [**How07b**, Conj. 3.4.1] should hold if and only if

$$\nu(\mathfrak{Z}_\infty)_0^{\phi^j} \neq 0, \qquad \text{for any } -r_\nu < j < r_\nu.$$

In that case, the results and methods exploited in this thesis would lead to the following "twisted" version of Theorem 1.4.12.

Proposed Theorem 3.1.1. *Together with Assumptions 1.4.11, suppose that there exists some $\nu' \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2 and non-trivial nebentypus such that*

$$L'(1, \mathbf{f}_{\nu'}, \chi_{\nu'}) \neq 0.$$

Then for all but finitely many $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight $2r_\nu > 2$ with $2r_\nu \equiv k \pmod{2(p-1)}$, we have

$$(3.1.1) \quad \langle \nu(\mathfrak{Z}_\infty^\Phi)_0, \nu(\mathfrak{Z}_\infty^\Phi)_0 \rangle_K = \left(1 - \frac{\bar{\pi}^{2r_\nu - 2}}{\nu(\mathbf{a}_p)}\right)^4 \cdot \langle \Phi^{\text{ét}}_{\mathbf{f}_\nu^\sharp, K}(\Pi_{r_\nu}^{r_\nu - 1}\Delta_{r_\nu}^{\mathrm{bdp}}), \Phi^{\text{ét}}_{\mathbf{f}_\nu^\sharp, K}(\Pi_{r_\nu}^{r_\nu - 1}\Delta_{r_\nu}^{\mathrm{bdp}}) \rangle_K,$$

where $\langle\ ,\ \rangle_K$ is the cyclotomic p-adic height pairing on $H^1_f(K, V_{\mathbf{f}_\nu^\sharp}(\psi^{2r_\nu-2}))$.

Notice that, as opposed to the p-adic multiplier appearing in (1.4.25), the factor

$$\mathcal{E}_\nu(\mathbf{f}^\natural \otimes K) := \left(1 - \frac{\bar{\pi}^{2r_\nu - 2}}{\nu(\mathbf{a}_p)}\right)$$

appearing in (3.1.1) depends p-adically analytically on $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I}) \subset \mathrm{Spf}(\mathbb{I})(\overline{\mathbf{Q}}_p)$.

As expressed in [**BDP13**, §2.4], one expects that the étale Abel–Jacobi images of the generalised Heegner cycles Δ_r^{bdp} bear a relation with the p-adic L-function $\mathcal{L}_p(f_{2r} \otimes K)$ of [**Nek95**] similar to that of the classical Heegner cycles in Nekovář's p-adic Gross–Zagier formula. In fact, under the simplifying assumptions of this section we propose the following.

Conjecture 3.1.2. *Let $f \in S_{2r}(\Gamma_0(N))$ be a p-ordinary eigenform of weight $2r \geq 2$, and let $\alpha_p(f)$ be the root of $X^2 - a_p(f)X + p^{2r-1}$ which is a p-adic unit. Then*

$$\frac{d}{ds}\mathcal{L}_p(f \otimes K)(\phi^{r-1}\langle\rho_{\mathrm{cyc}}\rangle^s)|_{s=0} = \left(1 - \frac{\bar{\pi}^{2r-2}}{\alpha_p(f)}\right)^4 \langle\Phi^{\text{ét}}_{f,K}(\Pi_r^{r-1}\Delta_r^{\mathrm{bdp}}), \Phi^{\text{ét}}_{f,K}(\Pi_r^{r-1}\Delta_r^{\mathrm{bdp}})\rangle_K,$$

where $\langle\ ,\ \rangle_K$ is the cyclotomic p-adic height pairing on $H^1_f(K, V_{\mathbf{f}^\sharp_\nu}(\psi^{2r_\nu - 2}))$.

Assume now that $\nu_o \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ is exceptional with $\nu_o(\mathbf{a}_p) = 1$, so that in particular $k_{\nu_o} = 2$, and set $f_o := \nu_o(\mathbf{f})$. Then

$$\mathcal{E}_{\nu_o}(\mathbf{f}^\natural \otimes K) = 0,$$

i.e. the left-hand side of (3.1.1) has an *exceptional zero*, and one then hopes to recover the right-hand side of (3.1.1) from the *second* cyclotomic derivative of $\mathcal{L}_p(f_o \otimes K)$ at $\mathbb{1}_K$.

We believe that a proof of Conjecture 3.1.2 would follow without major difficulties from an adaptation of the methods of [**Nek95**] to generalised Heegner cycles. As a consequence, we could then show the following result.

Proposed Corollary 3.1.3. *With notations and assumptions as in Theorem 1.4.12, there is a factorization*

$$\mathcal{L}'_p(\mathbf{f}^\natural \otimes K) = \mathcal{E}(\mathbf{f}^\natural \otimes K)^4 \cdot \widetilde{\mathcal{L}}'_p(\mathbf{f}^\natural \otimes K) \pmod{\mathbb{I}^\times}.$$

Moreover, the function $\widetilde{\mathcal{L}}'_p(\mathbf{f}^\natural \otimes K)$ is such that

$$\nu_o(\widetilde{\mathcal{L}}'_p(\mathbf{f}^\natural \otimes K)) = \langle\Phi^{\text{ét}}_{f_o,K}(\Delta_1^{\mathrm{heeg}}), \Phi^{\text{ét}}_{f_o,K}(\Delta_1^{\mathrm{heeg}})\rangle_K.$$

In other words, $\widetilde{\mathcal{L}}'_p(\mathbf{f}^\sharp \otimes K)$ is an "improved" derivative p-adic L-function, which one would hope to exploit, in a similar fashion as in [**GS93**], to obtain progress towards an anticyclotomic analogue of the following conjecture, deduced from the combination of the classical Birch and Swinnerton-Dyer conjecture and its p-adic variant by [**MTT86**] in the exceptional rank one case.

Conjecture 3.1.4. *Let $E/\mathbf{Q}$ be an elliptic curve with split multiplicative reduction at p, and assume that $\mathrm{ord}_{s=1}L(E,s) = 1$. Then there exists a nontorsion point $P_E \in E(\mathbf{Q})\otimes\mathbf{Q}$ such that*

$$\frac{d^2}{ds^2}L_p^{\mathrm{MTT}}(f_E, s)|_{s=1} = \mathscr{L}(f_E)\frac{\langle P_E, P_E\rangle_p}{\langle P_E, P_E\rangle_\infty}L'(E,1),$$

*where $L_p^{\mathrm{MTT}}(f_E, s)$ is the cyclotomic p-adic L-function constructed in [**MTT86**], $\langle\ ,\ \rangle_p$ and $\langle\ ,\ \rangle_\infty$ are the cyclotomic and Neron-Tate height pairings on $E(\mathbf{Q}) \otimes \mathbf{Q}$ respectively, and $\mathscr{L}(f_E)$ is the L-invariant of $E/\mathbf{Q}_p$.*

3.2. Big Heegner points and Kato elements

Let K be an imaginary quadratic field for which the Assumptions 2.1.1 are satisfied, and denote by K_∞^{cyc} the unique $\mathbf{Z}_p^2$-extension of K, which can be obtained as the compositum of the anticyclotomic $\mathbf{Z}_p$-extension K_∞/K and the cyclotomic $\mathbf{Z}_p$-extension K^{cyc}/K. Set

$$G_\infty := \mathrm{Gal}(K_\infty^{\mathrm{cyc}}/K).$$

We let f, V_f, $\mathbf{f}$, and $\mathbb{T}$ be as in the introduction to Chapter 2, but we restrict now to the case where the weight of f is $k = 2$. We start by recalling the following.

CONJECTURE 3.2.1 (Loeffler–Zerbes). *There is a special class* $\mathfrak{c}_{f,\infty} \in \bigwedge^2 H^1_{\mathrm{Iw},S}(K_\infty^{\mathrm{cyc}}, V_f)$ *such that*

$$(\mathcal{L}^{G_\infty}_{\mathfrak{p},V_f} \wedge \mathcal{L}^{G}_{\bar{\mathfrak{p}},V_f})(\mathfrak{c}_{f,\infty}) = \mathcal{L}_p(f \otimes K) \pmod{\mathcal{O}_L^\times},$$

where, for each prime ideal $\mathfrak{q}|p$ *in* K, $\mathcal{L}^{G_\infty}_{\mathfrak{q},V_f}$ *is the two-variable* p*-adic regulator map deduced from* [**LZ11**, Thm. 4.7]*, and* $\mathcal{L}_p(f \otimes K) \in \mathcal{O}_L[[G_\infty]]$ *is the two-variable* p*-adic* L*-function constructed in* [**PR87**].

The ongoing work [**LLZ13**] of Lei–Loeffler–Zerbes on the construction of a cyclotomic Euler system for the Rankin–Selberg convolution of two modular forms of weight 2 is expected to yield very substantial progress towards an eventual proof of Conjecture 3.2.1.

Inspired by a conjecture of Perrin-Riou [**PR93**] relating the Beilinson–Kato elements to rational points on an elliptic curve, one expects a relation between the conjectural class $\mathfrak{c}_{f,\infty}$ and the Kummer images of Heegner points.

CONJECTURE 3.2.2. *The class* $\mathfrak{c}_{f,\infty}$ *predicted by Conjecture 3.2.1 satisfies*

$$\mathrm{Cor}_{K_\infty^{\mathrm{cyc}}/K_\infty}(\mathfrak{c}_{f,\infty}) = \nu_f(\mathfrak{Z}_\infty)$$

up to an explicit element in $L^\times$.

It is natural to upgrade the preceding two conjectures over the entire Hida family:

CONJECTURE 3.2.3. *There exists a big special class* $\mathfrak{C}_\infty \in \bigwedge^2 H^1_{\mathrm{Iw,S}}(K_\infty^{\mathrm{cyc}}, \mathbb{T})$ *such that*

$$(\mathrm{Log}^{G_\infty}_{\mathscr{F}_\mathfrak{p}\mathbb{T}^*} \wedge \mathrm{Log}^{G_\infty}_{\mathscr{F}_{\bar{\mathfrak{p}}}\mathbb{T}^*})(\mathfrak{C}_\infty) = \mathcal{L}_p(\mathbf{f} \otimes K) \pmod{\mathbb{I}^\times},$$

where, for each $\mathfrak{q}|p$ *in* K, $\mathrm{Log}^{G_\infty}_{\mathscr{F}_\mathfrak{q}\mathbb{T}^*}$ *is a three-variable regulator map deduced from Prop. 2.2.6, and* $\mathcal{L}_p(\mathbf{f} \otimes K) \in \mathbb{I}[[G_\infty]]$ *is the three-variable* p*-adic* L*-function from* [**SU13**, §12.3]*. Moreover, we have*

$$\mathrm{Cor}_{K_\infty^{\mathrm{cyc}}/K_\infty}(\mathrm{Tw}^\dagger(\mathfrak{C}_\infty)) = \mathfrak{Z}_\infty$$

up to an explicit element in $\mathbb{I}^\times$*, where* $\mathrm{Tw}^\dagger$ *is defined as in* (2.2.17).

3.3. Quaternionic settings and others

Howard's construction of big Heegner points has been generalized by Longo–Vigni [**LV11**] to arbitrary quaternion algebras over $\mathbf{Q}$. A remarkable feature of their work is the ability to give constructions treating the definite and the indefinite cases on an equal footing; as expressed by the authors themselves in *loc.cit.*, this holds the promise of being a first step towards an eventual development in a Hida-theoretic context of the program carried out by Bertolini–Darmon in a series of papers having [**BD05**] perhaps as its greatest landmarks, and where the interplay between the definite and indefinite settings plays a crucial role in the arguments (cf. [**How06**]).

In a different line of investigation, Fouquet [**Fou13**] has obtained a construction of big Heegner points attached to *indefinite* quaternion algebras over a totally real field F. To briefly describe his construction, recall that the (2-dimensional) Galois representation ρ_f associated to Hilbert modular eigenforms f over F is not found in the étale cohomology of a Hilbert modular variety, but rather on the cohomology of a Shimura curve attached to an appropriate indefinite quaternion algebra over F, at least when either of the following conditions is satisfied:

- $[F : \mathbf{Q}]$ is odd, or
- there exists a finite place v of F such that $\pi(f)_v$ special or supercuspidal,

where $\pi(f)$ is the automorphic representation of $\mathbf{GL}_2(\mathbb{A}_F)$ associated with f.

Either of these conditions guarantees that $\pi(f)$ arises a the Jacquet–Langlands lift of an automorphic form on an indefinite quaternion algebra over F. One can then construct ρ_f from the étale cohomology of the associated Shimura curves, and Fouquet's construction (as well as that of Longo–Vigni in the indefinite case) is obtained by taking the appropriate twisted Kummer images of certain CM points over a tower of these Shimura curves with growing $\Gamma_1(p^s)$-level structure, in complete analogy with Howard's.

Let E be a CM field extension of F, and fix a CM type Σ for E/F, i.e. a set Σ of embeddings $F \hookrightarrow \overline{\mathbf{Q}}$ with the property that

$$\Sigma \cup \overline{\Sigma} = \mathrm{Hom}(F, \overline{\mathbf{Q}}) \qquad \text{and} \qquad \Sigma \cap \overline{\Sigma} = \emptyset.$$

Under the assumption that Σ is *ordinary* at $\imath_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$, meaning that $\imath_p \circ \sigma \neq \imath_p \circ \overline{\tau}$ for all $\sigma \in \Sigma$, $\overline{\tau} \in \overline{\Sigma}$, the work of Hsieh [**Hsi12**] constructs in this level of generality an anticylotomic p-adic L-function $\mathscr{L}_\Sigma(f, \lambda)$ as recalled in Theorem 2.1.4, with λ a suitable Hecke character of E of infinity type $(k/2, -k/2)$. Here k is the parallel even weight of f. On the other hand, Howard has extended in [**How04b**] his anticyclotomic Kolyvagin system arguments to prove an analogue of Perrin-Riou's Main Conjecture for Heegner points on Hilbert modular varieties.

Thus, if we further assume that the rational prime p *splits completely* in F, it seems that many of the constructions from Section 2.2 and the arguments from Sections 2.4 and 2.5 would apply almost verbatim to prove analogues of Theorems 2.4.2 and Theorem 2.5.5 for ordinary Hilbert modular forms, with Fouquet's big cohomology classes (or a slight modification thereof) playing the role of Howard's big Heegner points in this thesis.

In light of the well-known absence of an analogue of modular units, and therefore of Beilinson–Kato elements, for Hilbert modular varieties when $F \neq \mathbf{Q}$, a proof of these arithmetic applications would represent (at least to our knowledge) the first unconditional realization of the approach to p-adic L-functions first envisioned by Perrin-Riou [**PR95**] (cf. [**Rub00**, §7]) via p-adic Euler systems in a context where the base field is not $\mathbf{Q}$ or an imaginary quadratic field.

Of course, motivated by Stark's conjectures, there are further conjectural realizations of this approach to p-adic L-functions over a general totally real base field F. In particular, and not quite irrelevantly to the theme of this thesis, Darmon's p-adic construction [**Dar01**] of the so-called *Stark–Heegner points* attached to real quadratic fields where p is stays prime (and generalized by Matt Greenberg in [**Gre09**] to totally real fields with $[F : \mathbf{Q}] > 2$), and their higher dimensional analogue by Rotger–Seveso [**RS12**], the so-called *Darmon cycles*, are expected to have a similar connection to p-adic L-functions as we have exhibited in this thesis for classical Heegner points and Heegner cycles.

We thus feel naturally led to consider the following problem:

Propose a construction of "big" Stark–Heegner points attached to Hida families, and relate their arithmetic specializations to "classical" Stark–Heegner points and Darmon cycles.

Perhaps it is not unreasonable to believe that a study of this question and related ones might lead to some insights into the elusive properties that these p-adic constructions are conjectured to share with the objects of study in this thesis.

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