RELATIONS IN CATEGORIES

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Ph.D.

Relations are studied in categories which have finite products and (E,M) factorization systems. Relations are the morphisms of a Bénabou's bicategory, provided that pullbacks preserve E-class morphisms. More generally, when there is assigned a full reflexive subcategory Hom'(A,B) to each hom category, Hom(A,B), of a given bicategory, a sufficient condition is obtained for Hom'(A,B) to be the hom category in another bicategory. This is also applied to obtain a bicategory whose morphisms are the pullback spans.

Some properties of a relation R and its converse $\overline{R}$ are investigated. All pullback relations are difunctional (i.e. $R \simeq R \circ \overline{R} \circ R$). The main result in Chapter 2 is concerned with the converse of this statement. Applications are made to Barr's exact categories. Furthermore, some results known in algebraic categories are extended to exact categories.

The last chapter deals with the problem in regular categories when a pair of morphisms $\xrightarrow{g}\xrightarrow{f}$ can be embedded in a pullback square.

RELATIONS IN CATEGORIES

by

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ABSTRACT

Relations are studied in categories which have finite products and ($\underline{E}$,$\underline{M}$) factorization systems. Relations are the morphisms of a Bénabou's bicategory, provided that pullbacks preserve $\underline{E}$-class morphisms. More generally, when there is assigned a full reflexive subcategory Hom'(A,B) to each hom category, Hom(A,B), of a given bicategory, a sufficient condition is obtained for Hom'(A,B) to be the hom category in another bicategory. This is also applied to obtain a bicategory whose morphisms are the pullback spans.

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INTRODUCTION

Relations in special categories have been considered by various authors, for example, in abelian categories by MacLane [8], Puppe [10], and Hilton [3] and in algebraic categories by Lambek [7]. We would like to study relations in general categories, but our approach requires that these must have finite products and factorization systems. Klein [6] has obtained a condition for composition of relations to be strictly associative. Here we consider the possibility that associativity only holds up to a coherent isomorphism, in other words, that the relations are the morphisms of a bicategory in the sense of Bénabou [2].

In Chapter 0, we investigate general properties of factorization systems. Most of the concepts and results in this chapter we learned from a talk by Kelly [4]. However, we believe that propositions(0.9) and (0.11) are new.

In Chapter 1, we look at a bicategory in which, to each hom category, Hom (A,B), there is assigned a full reflexive subcategory Hom' (A,B). We obtain a sufficient condition for Hom' (A,B) to be the hom category in another bicategory. This is applied to the situation in which Hom (A,B) consists of all spans from B to A and Hom' (A,B) consists of all relations from B to A. Klein's result [6] is obtained as a special case.

In Chapter 2, we study the converse $\overline{R}$ of the relation R and the relations $R \circ \overline{R}$ and $\overline{R} \circ R$. The latter are equivalence relations when $R \circ \overline{R} \circ R \simeq R$, in which case R is called difunctional. All pullback relations are difunctional. Our main result (theorem (2.22)) is concerned with the converse of this statement.

In Chapter 3, we apply the above result to the exact categories of Barr. For exact categories we generalise the result known for algebraic categories which asserts that every relation is difunctional if and only if a number of interesting equivalent conditions hold, for example, that every reflexive relation is an equivalence relation.

In Chapter 4, we obtain another application of the sufficient condition in Chapter 1, where now Hom' (A,B) is the category of all pullback spans from B to A. Under certain conditions this bicategory is the same as Rel $\underline{A}$ in Chapter 1.

In Chapter 5, we consider the problem when a pair of maps $\cdot \xrightarrow{g} \cdot \xrightarrow{f} \cdot$ can be embedded in a pullback square.

Originally, this had been planned as an integral part of the thesis, but at present it is unrelated to the other chapters.

Chapter 0

($\underline{E}$, $\underline{M}$) FACTORIZATION OF MORPHISMS

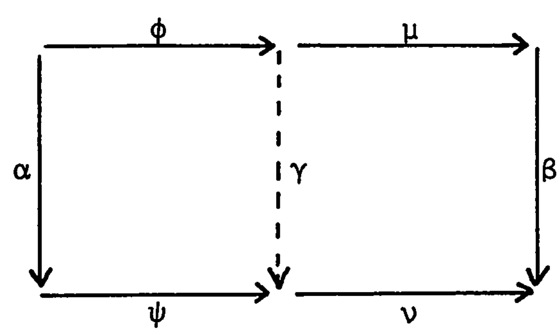
Let $\underline{A}$ be a category. The following definition is due to Kelly [4].

DEFINITION 0.1. An <u>($\underline{E}$,$\underline{M}$) factorization system of morphisms</u> in $\underline{A}$ is defined as follows: there are two classes $\underline{E}$,$\underline{M}$ of morphisms in $\underline{A}$ satisfying the following:

1. Every isomorphism is both in $\underline{E}$ and $\underline{M}$.

2. $\underline{E}$ is closed under composition and $\underline{M}$ is closed under composition.

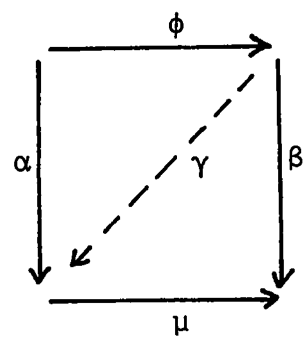
3. In the following commutative diagram, $\phi , \psi \in \underline{E}$ and $\mu , \nu \in \underline{M}$,



there exists a unique morphism γ such that $\nu\gamma = \beta\mu$ and $\gamma\phi = \psi\alpha$.

4. For every morphism α in $\underline{A}$ there exists $\alpha_m \in \underline{M}$ and $\alpha_e \in \underline{E}$ such that $\alpha = \alpha_m\alpha_e$.

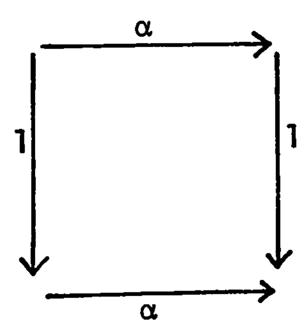
Clearly, (3) is equivalent to the unique diagonal property: for any commutative diagram,



in $\underline{A}$, with $\phi \in \underline{E}$ and $\mu \in \underline{M}$, there exists a unique γ such that $\gamma\phi = \alpha$ and $\mu\gamma = \beta$.

PROPOSITION 0.2. If $\alpha \in \underline{E} \wedge \underline{M}$ then α is an iso.

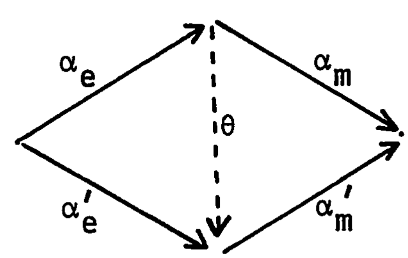
Proof: Applying the unique diagonal property to the commutative square



Definition (0.1) implies that the factorization in (4) is unique; in fact, we have the following result.

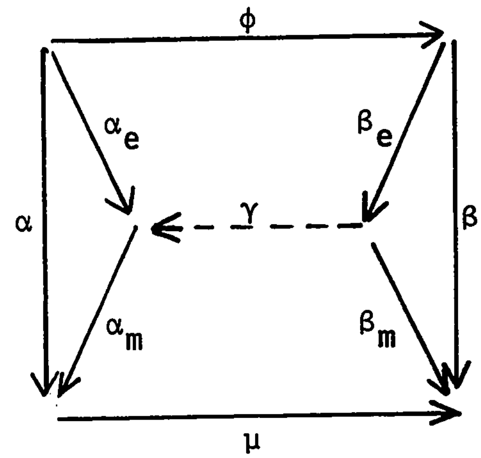
PROPOSITION 0.3. Suppose $\underline{A}$ is a category with two classes $\underline{E}$ and $\underline{M}$ of morphisms such that $\underline{E}$ and $\underline{M}$ are closed under composition and with the property that every morphism α in $\underline{A}$ is $\alpha_m \alpha_e$ where $\alpha_m \in \underline{M}$ and $\alpha_e \in \underline{E}$, then the following are equivalent:

(i) The factorization is unique, i.e. if $\alpha = \alpha_m \alpha_e = \alpha_m' \alpha_e'$ where $\alpha_m' \in \underline{M}$ and $\alpha_e' \in \underline{E}$ then there exists a unique isomorphism θ such that $\theta \alpha_e = \alpha_e'$ and $\alpha_m' \theta = \alpha_m$.



(ii) $\underline{A}$ has the unique $(\underline{E},\underline{M})$ diagonal property.

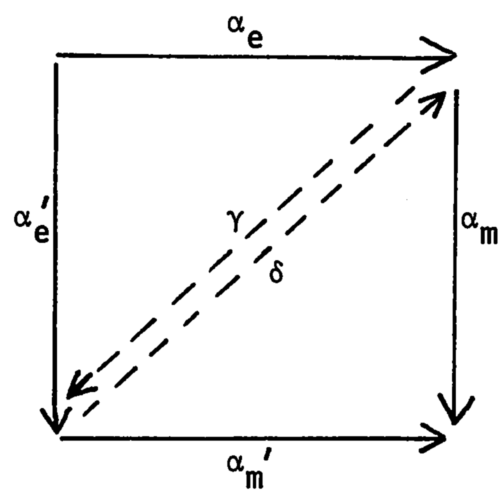
Proof: (i) $\Longrightarrow$ (ii) Let $\beta \phi = \mu \alpha$ with $\phi \in \underline{E}$ and $\mu \in \underline{M}$. Suppose that $\beta_m \beta_e$ and $\alpha_m \alpha_e$ are the $(\underline{E},\underline{M})$ factorizations of β and α, respectively.



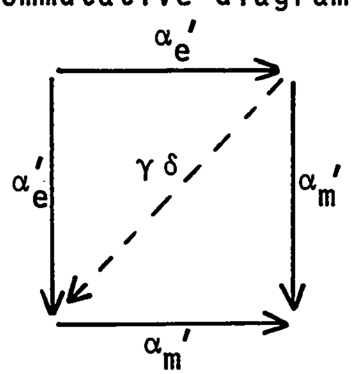
Then $\beta\,\phi = \beta_m\,\beta_e\,\phi = \mu\,\alpha_m\,\alpha_e$ and by the uniqueness of (E,M) factorization of $\beta\,\phi$, there exists a unique iso γ such that $\gamma\,\beta_e\,\phi = \alpha_e$ and $\mu\,\alpha_m\,\gamma = \beta_m$. It follows that $\alpha_m\,\gamma\,\beta_e\,\phi = \alpha_m\,\alpha_e = \alpha$ and $\mu\,\alpha_m\,\gamma\,\beta_e = \beta_m\,\beta_e = \beta$. Thus $\alpha_m\,\gamma\,\beta_e$ is a diagonal morphism. It remains to show its uniqueness.

Suppose χ is another such diagonal map, i.e. $\chi\,\phi = \alpha$ and $\mu\,\chi = \beta$. Let $\chi_m\,\chi_e$ be the (E,M) factorization of χ. Then $\chi_m\,\chi_e\,\phi = \alpha_m\,\alpha_e$ and there exists a unique iso δ such that $\alpha_m\,\delta = \chi_m$ and $\delta\,\chi_e\,\phi = \alpha_e$. Similarly $\mu\,\chi_m\,\chi_e = \beta_m\,\beta_e$ and there is a unique iso σ such that $\mu\,\chi_m\,\sigma = \beta_m$ and $\sigma\,\beta_e = \chi_e$. Hence $\alpha_e = \delta\,\chi_e\,\phi = \delta\,\sigma\,\beta_e\,\phi$ and $\beta_m = \mu\,\chi_m\,\sigma = \mu\,\alpha_m\,\delta\,\sigma$. We recall that γ is a unique iso so that $\gamma\,\beta_e\,\phi = \alpha_e$ and $\mu\,\alpha_m\,\gamma = \beta_m$. Thus $\gamma = \delta\,\sigma$. Therefore $\alpha_m\,\gamma\,\beta_e = \alpha_m\,\delta\,\sigma\,\beta_e = \chi_m\,\chi_e = \chi$.

(ii) $\Longrightarrow$ (i) Let $\alpha = \alpha_m \alpha_e = \alpha_m' \alpha_e'$ with $\alpha_m, \alpha_m' \in \underline{M}$ and $\alpha_e, \alpha_e' \in \underline{E}$. By the unique diagonal property, in the commutative square



there exists unique morphisms γ and δ such that $\gamma \alpha_e = \alpha_e'$, $\alpha_m' \gamma = \alpha_m$, $\delta \alpha_e' = \alpha_e$ and $\alpha_m \delta = \alpha_m'$. We claim that γ is an isomorphism such that $\gamma \delta = 1$ and $\delta \gamma = 1$. It follows from above that $\alpha_m' \gamma \delta = \alpha_m \delta = \alpha_m'$ and $\gamma \delta \alpha_e' = \gamma \alpha_e = \alpha_e'$. In the following commutative diagram



there is a unique diagonal morphism, 1. Hence $\gamma \delta = 1$. Similarly $\delta \gamma = 1$. γ is therefore an isomorphism. ∎

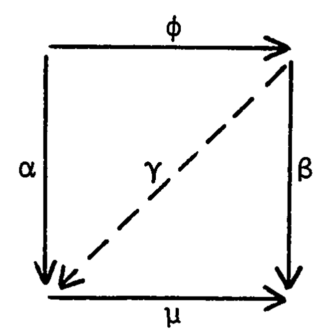
In the following we shall assume that $\underline{A}$ has an $(\underline{E},\underline{M})$ factorization system of morphisms.

PROPOSITION 0.4. $\underline{E}$ and $\underline{M}$ can determine each other as follows:

(i) $\underline{E} = \{\phi \mid \phi \longrightarrow\!\!\!| \ \mu \quad \forall \mu \in \underline{M}\}$,

(ii) $\underline{M} = \{\mu \mid \phi \longrightarrow\!\!\!| \ \mu \quad \forall \phi \in \underline{E}\}$,

where $\phi \longrightarrow\!\!\!| \ \mu$ means that for every commutative square

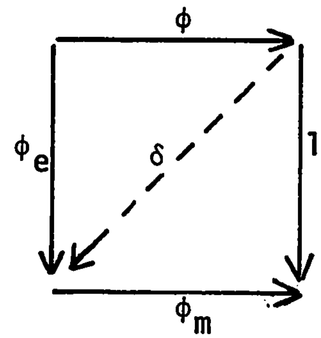


there exists a unique morphism γ such that $\gamma \phi = \alpha$ and $\mu \gamma = \beta$.

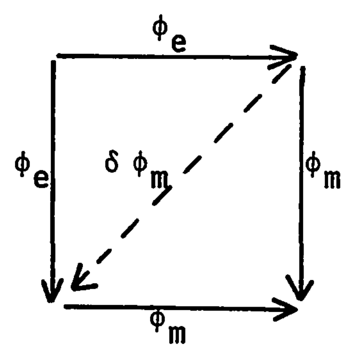
Proof: By duality we need only to prove (i). Suppose $\phi \in \underline{E}$. Then by the unique diagonal property $\forall \mu \in \underline{M}$, $\phi \longrightarrow\!\!\!| \ \mu$.

Now suppose that $\phi \longrightarrow\!\!\!| \ \mu$, $\forall \mu \in \underline{M}$. We want to show that $\phi \in \underline{E}$. Let $\phi_m \phi_e$ be $(\underline{E},\underline{M})$ factorization of ϕ.

Then in the following commutative square

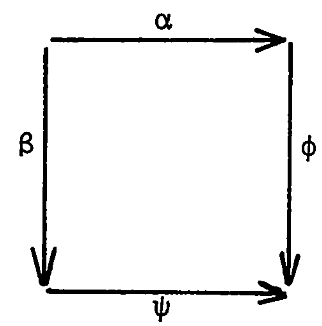


there exists a unique δ such that $\phi_m \delta = 1$ and $\delta\ \phi = \phi_e$. Hence $\phi_m\ \delta\ \phi_m = \phi_m$ and $\delta\ \phi_m\ \phi_e = \phi_e$. Then $\delta\ \phi_m$ is a diagonal morphism in the following square.



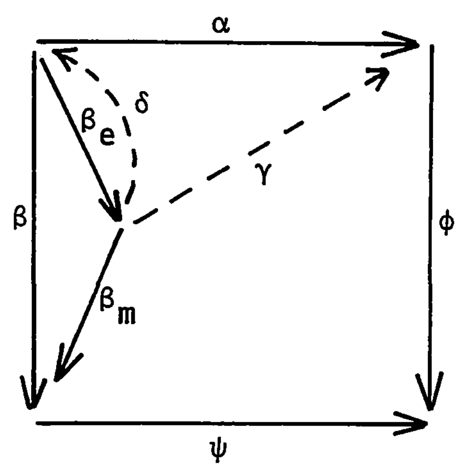
By the unique diagonal property $\delta\ \phi_m = 1$, since 1 is also a diagonal morphism in the above square. Hence ϕ_m is an isomorphism and $\phi = \phi_m\ \phi_e \in \underline{E}$. ∎

PROPOSITION 0.5. Suppose that

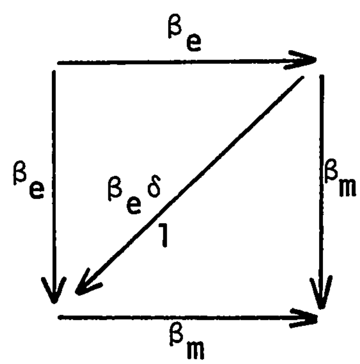


is a pullback and $\phi \in \underline{M}$. Then $\beta \in \underline{M}$.

Proof: Let $\beta_m \, \beta_e$ be the $(\underline{E},\underline{M})$ factorization of β. In the following diagram, since $\phi \in \underline{M}$ there is a unique diagonal morphism γ such that $\gamma \, \beta_e = \alpha$ and $\phi \, \gamma = \psi \, \beta_m$.



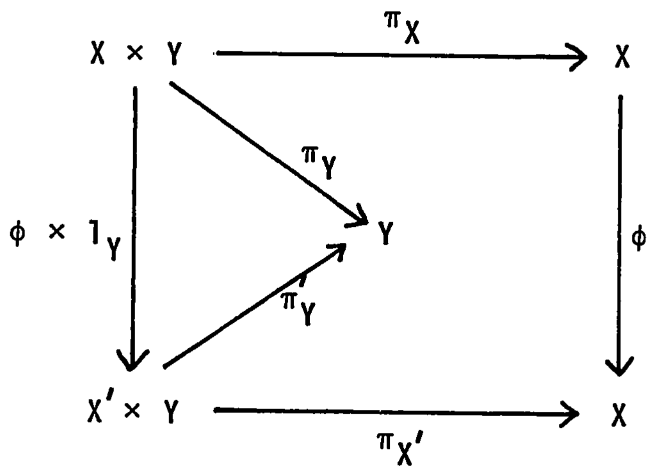
Furthermore, by the pullback square, there exists a unique δ such that $\alpha \, \delta = \gamma$ and $\beta \, \delta = \beta_m$. It follows that $\alpha \, \delta \, \beta_e = \gamma \, \beta_e = \alpha$ and $\beta \, \delta \, \beta_e = \beta_m \, \beta_e = \beta$. By the uniqueness property of the pullback square, $\delta \, \beta_e = 1$. We shall also show that $\beta_e \delta = 1$. By multiplying β_e to the left of $\delta \beta_e = 1$ we have $(\beta_e \delta)\beta_e = \beta_e$. Also we know $\beta_m(\beta_e \delta) = \beta \delta = \beta_m$. Thus in the commutative square



both $\beta_e \delta$ and 1 are diagonal morphisms. Hence $\beta_e \delta = 1$ and β_e is an isomorphism. Therefore $\beta = \beta_m \beta_e \in \underline{M}$. ∎

COROLLARY 0.6. Suppose that $\underline{A}$ has products. Let $\phi : X \to X', \psi : Y \to Y'$ be both $\underline{M}$-class morphisms. Then $\phi \times \psi : X \times Y \to X' \times Y'$ is also in $\underline{M}$.

Proof: We shall show that the following square is a pullback, where π_X, π_Y, $\pi_{X'}$, and π'_Y are projections. Then $\phi \in \underline{M} \Longrightarrow \phi \times 1_Y \in \underline{M}$. We then apply this result twice to obtain $\phi \times \psi \in \underline{M}$.



Let $\phi\,\alpha = \pi_{X'}\,\beta$, where $\alpha : Z \to X$ and $\beta : Z \to X' \times Y$. Then there is a morphism $\pi'_Y\,\beta : Z \to Y$. By the property of the product $X \times Y$, there exists a unique morphism $\gamma : Z \to X \times Y$ such that $\pi_X\,\gamma = \alpha$ and $\pi_Y\,\gamma = \pi'_Y\,\beta = \pi'_Y\,(\phi \times 1_Y)\,\gamma$. But we also have $\pi_{X'}\,\beta = \phi\,\alpha = \phi\,\pi_X\,\gamma = \pi_{X'}(\phi \times 1_Y)\,\gamma$. Hence $\beta = (\phi \times 1_Y)\,\gamma$. So we have a morphism γ, such that $\pi_X\,\gamma = \alpha$ and $(\phi \times 1_Y)\,\gamma = \beta$. We only need to show its uniqueness. Let δ be a morphism such that $\pi_X\,\delta = \alpha$ and $(\phi \times 1_Y)\,\delta = \beta$. Then $\pi_X\,\delta = \pi_X\,\gamma$ and $(\phi \times 1_Y)\,\delta = (\phi \times 1_Y)\,\gamma$. It follows that $\pi_Y\,\delta = \pi'_Y\,(\phi \times 1_Y)\,\delta = \pi'_Y\,(\phi \times 1_Y)\,\gamma = \pi_Y\,\gamma$. By the uniqueness property of the product $X \times Y$, $\delta = \gamma$. ▮

Suppose that $\alpha : A \to X$ and $\beta : A \to Y$. Then we denote the unique morphism $A \to X \times Y$ by $\{\alpha,\beta\}$.

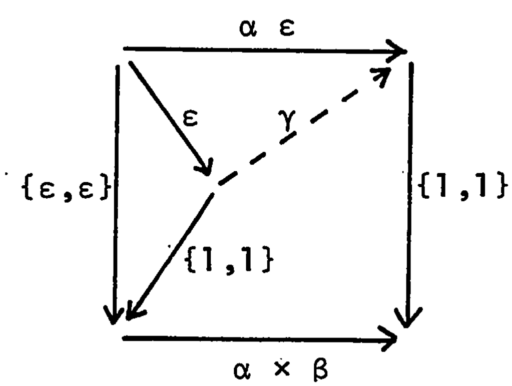
PROPOSITION 0.7. The following are equivalent:

1. For every A in $\underline{A}$, $\{1,1\} : A \longrightarrow A \times A \in \underline{M}$,

2. $\underline{E} \subset$ Epis,

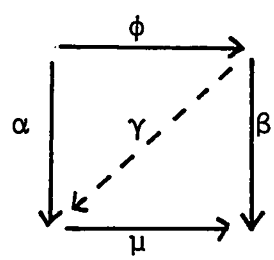
3. Regular monos $\subset \underline{M}$.

Proof: (1) $\Longrightarrow$ (2) Let $\varepsilon \in \underline{E}$ and $\alpha\,\varepsilon = \beta\,\varepsilon$. Then we have the following commutative square.



By definition (0.1), there exists a unique γ such that $\gamma\,\varepsilon = \alpha\,\varepsilon$ and $\{1,1\}\,\gamma = (\alpha \times \beta)\{1,1\}$, i.e. $\{\gamma,\gamma\} = \{\alpha,\beta\}$. Hence $\alpha = \beta$ and ε is an epi.

(2) $\Longrightarrow$ (3) For every commutative square



with $\phi \in \underline{E} \subset$ epis and μ a regular mono $\big($i.e. an equalizer of a pair of morphism, say $(\rho,\tau)\big)$, there exists a unique γ such that $\mu\,\gamma = \beta$, since $\rho\,\beta = \tau\,\beta$. It follows that $\mu\,\gamma\,\phi = \beta\,\phi = \mu\,\alpha$. We have $\gamma\,\phi = \alpha$. Hence, by proposition (0.4), regular monos $\subset$ $\underline{M}$.

(3) $\Longrightarrow$ (1) $\{1,1\}$ is an equalizer of $A \times A \overset{p_1}{\underset{p_2}{\rightrightarrows}} A$. Hence $\{1,1\} \in \underline{M}$. ▮

COROLLARY 0.8. If

$$\begin{array}{ccc} A & \xrightarrow{\ \alpha\ } & X \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\phi} \\ Y & \xrightarrow[\ \psi\ ]{} & Z \end{array}$$

is a pullback then $\{\alpha,\beta\} \in \underline{M}$, provided that $\{1,1\} \in \underline{M}$.

Proof: $\{\alpha,\beta\}$ is a regular mono, since it is the equalizer of

$$X \times Y \overset{\phi\, p_1}{\underset{\psi\, p_2}{\rightrightarrows}} Z$$

where $p_1 : X \times Y \longrightarrow X$ and $p_2 : X \times Y \longrightarrow Y$ are projections. ▮

PROPOSITION 0.9. Assume that pullbacks exist in $\underline{A}$. Then the following are equivalent:

(i) $\underline{M} \subseteq$ monos ,

(ii) $m\,\gamma = m\,\delta \in \underline{M}$ and $m \in \underline{M} \Longrightarrow \gamma = \delta$,

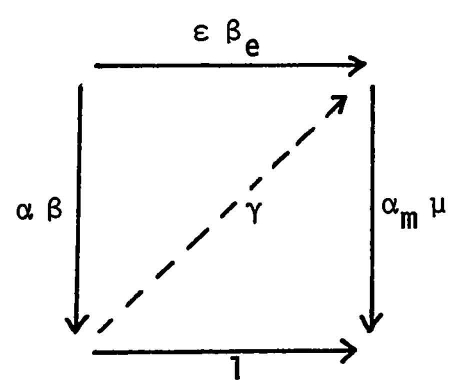
(iii) $\alpha\,\beta \in \underline{E} \Longrightarrow \alpha \in \underline{E}$.

Proof: (i) $\Longrightarrow$ (ii) trivial.

(ii) $\Longrightarrow$ (iii) Let $\alpha_m\,\alpha_e$ and $\beta_m\,\beta_e$ be the $(\underline{E},\underline{M})$ factorizations of α and β, respectively. Then

$$\alpha\,\beta = \alpha_m\,\alpha_e\,\beta_m\,\beta_e = \alpha_m\,\mu\,\varepsilon\,\beta_e \in \underline{E}$$

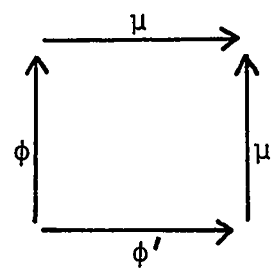
where $\mu\,\varepsilon$ is the $(\underline{E},\underline{M})$ factorization of $\alpha_e\,\beta_m$. Thus



by the unique factorization of $\alpha\,\beta$, there is a unique iso γ such that $\gamma\,\alpha\,\beta = \varepsilon\,\beta_e$ and $\alpha_m\,\mu\,\gamma = 1$. Hence $\alpha_m\,\mu\,\gamma\,\alpha_m = \alpha_m \in \underline{M}$ and, since $\alpha_m \in \underline{M}$, it implies that $\mu\,\gamma\,\alpha_m = 1$. Thus α_m is an iso and has inverse $\mu\,\gamma$.

We have therefore shown that $\alpha \in \underline{E}$.

(iii) $\Longrightarrow$ (i) Let $\mu \in \underline{M}$ and let (ϕ,ϕ') be its kernel pair, i.e.



is a pullback.

By proposition (0.5) $\phi,\phi' \in \underline{M}$. Since (ϕ,ϕ') is a kernel pair, there is a unique χ such that $\phi\,\chi = 1 = \phi'\,\chi$. Hence $\phi \in \underline{E}$. Thus ϕ,ϕ' are both isomorphisms and χ is their inverse. Therefore χ is an iso and (1,1) is the kernel pair of μ. Then it follows that μ is a mono.

A morphism α is said to be a <u>regular epi</u> if it is a coequalizer of some pair of maps. Kelly [5] pointed out that regular epis in general are not closed under composition and if $\alpha\,\beta$ is a regular epi, α is not necessarily a regular epi. However, both Kelly [5] and Barr [1] show that if kernel pairs and coequalizers exist, then in the following, (1) $\Longrightarrow$ (2), (3), (4):

1. A pullback of a regular epi is an epi.

2. Every map has a factorization of the form $\cdot \twoheadrightarrow \cdot \rightarrowtail \cdot$ where $\twoheadrightarrow$ is a regular epi and $\rightarrowtail$ is a mono.

3. If $\alpha\,\beta$ is a regular epi, so is α.

4. Regular epis are closed under composition.

COROLLARY 0.10. Assume that, in $\underline{A}$, a pullback of a regular epi is an epi. Then the following are equivalent:

1. $\underline{E}$ = the class of all regular epis,

2. $\underline{M}$ = the class of all monos.

Proof: (1) $\Longrightarrow$ (2) By proposition (0.9), $\underline{M} \subseteq$ mono.

Let μ be mono and $\mu_m\,\mu_e$ be the $(\underline{E},\underline{M})$ factorization of μ. Then μ_e is a mono and a regular epi. Hence μ_e is an iso and $\mu = \mu_m\,\mu_e$ is in $\underline{M}$.

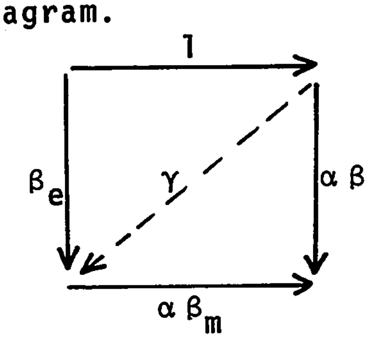
(2) $\Longrightarrow$ (1) Let ε be an $\underline{E}$-class map. Since every map has a factorization of the form $\cdot \twoheadrightarrow \cdot \rightarrowtail \cdot$,

let $\varepsilon = \mu \phi$ where μ is a mono $(\in \underline{M})$ and ϕ is a regular epi. Then $\mu \in \underline{M}$ and by proposition (0.9), $\mu \in \underline{E}$. Therefore μ is an iso and ε is a regular epi.

Now let ϕ be a coequalizer of (γ,δ), i.e., a regular epi, and $\phi_m \phi_e$ be the $(\underline{E},\underline{M})$ factorization of ϕ. Since ϕ_m is mono, $\phi_e \gamma = \phi_e \delta$. Then there exists a unique α such that $\alpha \phi = \phi_e$. Hence $\phi_m \alpha \phi = \phi_m \phi_e = \phi$. Therefore $\phi_m \alpha = 1$. So ϕ_m is an iso and $\phi = \phi_m \phi_e$ is an $\underline{E}$-class morphism. ∎

PROPOSITION 0.11. (i) $\alpha \beta \in \underline{M}$ and $\alpha \in \underline{M} \Longrightarrow \beta \in \underline{M}$. (ii) $\alpha \beta \in \underline{E}$ and $\beta \in \underline{E} \Longrightarrow \alpha \in \underline{E}$.

Proof: By duality we only need to show (i). Let $\beta_m \beta_e$ be the $(\underline{E},\underline{M})$ factorization of β. Then $\alpha \beta_m \in \underline{M}$. By the uniqueness of $(\underline{E},\underline{M})$ factorization of $\alpha \beta$, there is a unique iso γ such that $\gamma = \beta_e$ and $\alpha \beta_m \gamma = \alpha \beta$ as shown in the following diagram.



Hence β_e is an iso and $\beta \in \underline{M}$. ∎

Chapter 1

ON BICATEGORIES OF SPANS AND RELATIONS

Let $\underline{A}$ be a category with finite products, pullbacks and $(\underline{E},\underline{M})$ factorization system. Let A,B,R be objects of $\underline{A}$. According to Benabou [2], a triple (R,α,β) with morphisms $\alpha : R \longrightarrow A$ and $\beta : R \longrightarrow B$ is called a <u>span</u> from B to A in $\underline{A}$. We denote the morphism $R \longrightarrow A \times B$ by $\{\alpha,\beta\}$. The span (R,α,β) is called a <u>relation</u> if $\{\alpha,\beta\} \in \underline{M}$.

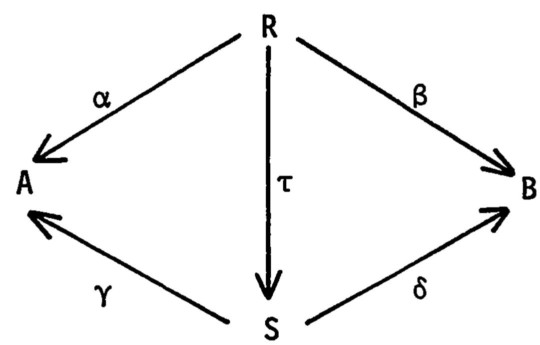
PROPOSITION 1.1. To any span (R,α,β) there corresponds a canonical relation (R^+, α',β') obtained by taking the $(\underline{E},\underline{M})$ factorization of the morphism $\{\alpha,\beta\}$; thus

$$\{\alpha,\beta\} = \{\alpha',\beta'\}\ \varepsilon(R) ,$$

where

$$\{\alpha',\beta'\} \in \underline{M} \quad \text{and} \quad \varepsilon(R) \in \underline{E}. \qquad \blacksquare$$

Suppose (R,α,β) and (S,γ,δ) are both spans or relations from B to A. A <u>map</u> from (R,α,β) to (S,γ,δ) is a commutative diagram in $\underline{A}$,



such that $\gamma\,\tau = \alpha$ and $\delta\,\tau = \beta$.

PROPOSITION 1.2. Let (R,α,β) and (S,γ,δ) be spans from B to A. The following are equivalent:

(i) $\tau : R \longrightarrow S$ such that $\gamma\,\tau = \alpha$ and $\delta\,\tau = \beta$,

(ii) $\tau : R \longrightarrow S$ such that $\{\gamma,\delta\}\,\tau = \{\alpha,\beta\}$. ∎

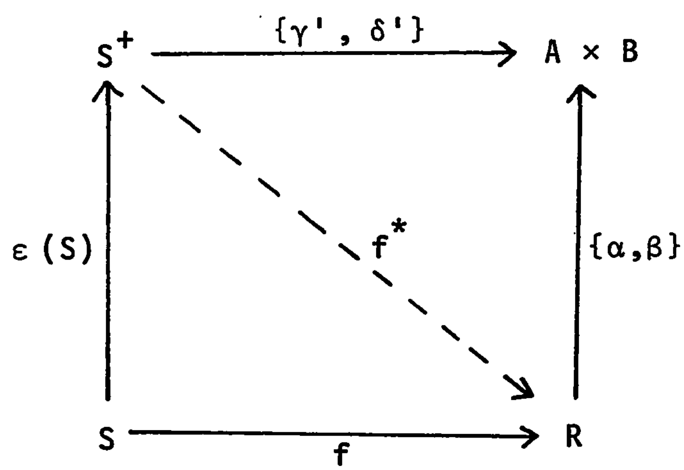
REMARK 1.3. Let (R,α,β) and (S,γ,δ) be relations from B to A. It follows from (ii) in the above and proposition (0.11) that $\tau : (R,\alpha,\beta) \longrightarrow (S,\gamma,\delta)$ is an <u>M</u>-class morphism.

(R,α,β) and (S,γ,δ) are said to be <u>isomorphic</u> if and only if τ is an isomorphism.

In this chapter we shall assume that $\{1,1\} \in \underline{M}$. Then for each object A of $\underline{A}$, (A,1,1) is a relation and is called the <u>identity relation</u> on A.

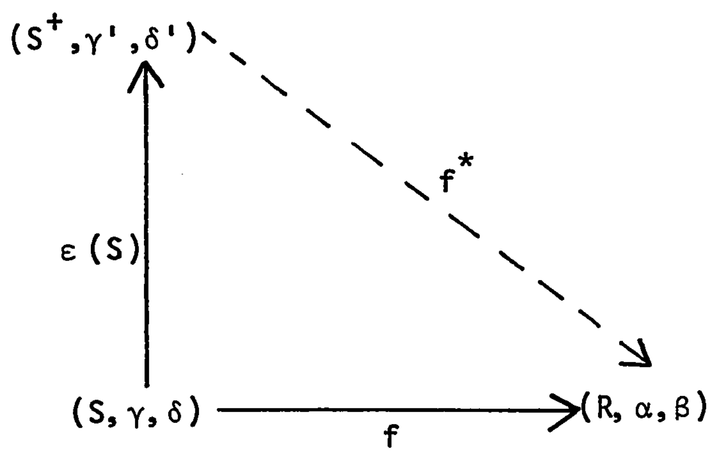
Let Span (A,B) and Rel (A,B) denote the categories of spans and relations from B to A, respectively. Thus Rel (A,B) is a full subcategory of Span (A,B). We will show that the object function $^+$ of proposition (1.1) can be extended to a functor from Span (A,B) to Rel (A,B), in fact the left adjoint of the inclusion functor.

Suppose $(S,\gamma,\delta) \in$ Span (A,B), $(R,\alpha,\beta) \in$ Rel (A,B) and $f : (S,\gamma,\delta) \longrightarrow (R,\alpha,\beta)$. Let f^* be the diagonal morphism in the following square, which commutes by proposition (1.2).

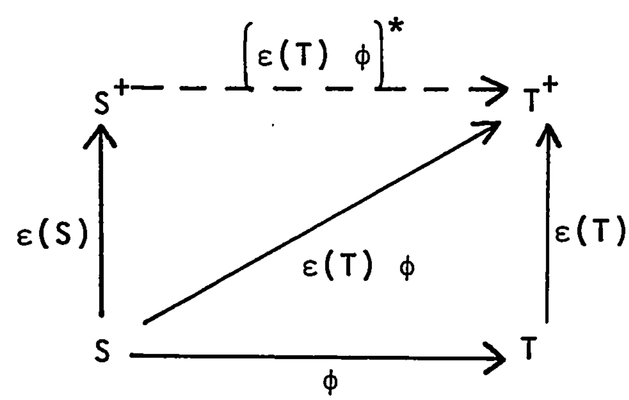


(see definition (0.1)).

By proposition (1.2), f^* is also the unique map $(S^+,\gamma',\delta') \longrightarrow (R,\alpha,\beta)$ such that $f^* \varepsilon(S) = f$.



Given any map $\phi : (S,\gamma,\delta) \longrightarrow (T,\tau,\sigma)$ in Span (A,B) we let ϕ^+ be the unique map $(S^+,\gamma',\delta') \longrightarrow (T^+,\tau',\sigma)$ such that $\phi^+ \varepsilon(S) = \varepsilon(T)\ \phi$, i.e., $\phi^+ = (\varepsilon(T)\ \phi)^*$.



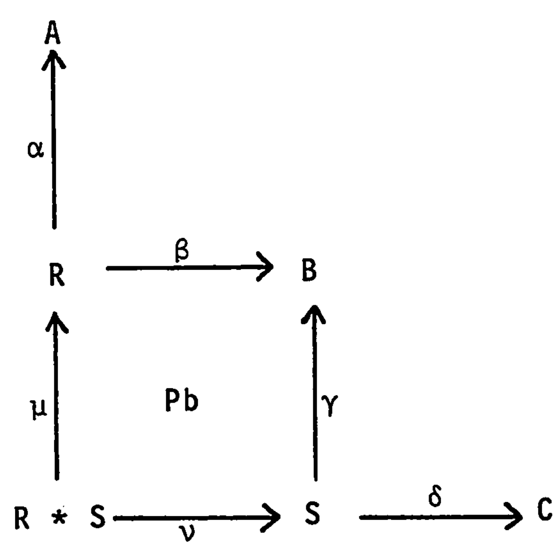
We have thus proved the following.

PROPOSITION 1.3. Rel (A,B) is a full reflexive subcategory of Span (A,B) with reflector $^{+}$. ∎

We recall the notion of a <u>bicategory</u> in the sense of Bénabou [2]. Bénabou himself gave as an example the bicategory Span $\underline{A}$ with objects those of $\underline{A}$, with morphisms $A \longrightarrow B$ the spans from A to B, and with cells the maps between spans. Composition of spans is defined thus:

$$(R,\alpha,\beta) * (S,\gamma,\delta) = (R * S, \alpha\mu, \delta\nu)$$

where $R \xleftarrow{\mu} R * S \xrightarrow{\nu} S$ is the pullback of $R \xrightarrow{\beta} B \xleftarrow{\gamma} S$.



The question now arises whether one can similarly obtain a bicategory Rel $\underline{A}$. We define composition of relations thus:

$$(R,\alpha,\beta) \circ (S,\gamma,\delta) = \left[(R * S)^{+},(\alpha\ \mu)', (\delta\ \nu)'\right]$$

where

$$\{\alpha\ \mu,\delta\ \nu\} = \{(\alpha\ \mu)',(\delta\ \nu)'\}\ \varepsilon(R * S)$$

is the $(\underline{E},\underline{M})$ factorization of $R * S \longrightarrow A \times C$. We may abbreviate $(R * S)^{+}$ as $R \circ S$. Unfortunately this composition is not necessarily associative, as was already observed by Klein [6]. He pointed out that it is associative under a certain condition. We shall generalize his result.

PROPOSITION 1.4. Suppose $\underline{X}$ is a bicategory with composition functor $*$ between morphisms and between cells and to each category $\mathrm{Hom}_{\underline{X}}(A,B)$ with A,B objects in $\underline{X}$, we assign a full reflexive subcategory $\mathrm{Hom}_{\underline{X}'}(A,B)$ with reflector $^{+}$ and adjunction $\eta(S) : S \longrightarrow S^{+}$. Then $\underline{X}'$ is also a bicategory with composition $\circ$ between morphisms and between cells defined by $S \circ T = (S * T)^{+}$, $\phi \circ \psi = (\phi * \psi)^{+}$

provided there exists a natural isomorphism

$$\sigma(S,T) : (S * T)^{+} \xrightarrow{\sim} (S^{+} * T^{+})^{+}$$

whenever $S : B \longrightarrow A$, $T : C \longrightarrow B$, in particular, if $\left[\eta(S) * \eta(T)\right]^{+}$ is an iso.

Proof: $\underline{X}$ is a bicategory with composition $*$ between morphisms and between cells. Hence $*$ is a functor. Since $\phi \circ \psi = (\phi * \psi)^{+}$ where ϕ, ψ are maps in $Hom_{\underline{X}}(A,B)$, $\circ$ is also a functor. Let S,T,U,V be objects of categories $Hom_{\underline{X}}(A,B)$, $Hom_{\underline{X}}(B,C)$, $Hom_{\underline{X}}(C,D)$ and $Hom_{\underline{X}}(D,E)$, respectively. Then there are natural isomorphisms:

$$\begin{aligned} a(S,T,U) &: (S * T) * U \xrightarrow{\sim} S * (T * U) \\ \ell(S) &: I_A * S \xrightarrow{\sim} S \\ r(S) &: S * I_B \xrightarrow{\sim} S \end{aligned}$$

such that they satisfy the following coherence conditions:

(i) $$a(S,T,U*V)\, a(S*T,U,V) = \left[Id_S * a(T,U,V)\right] \cdot a(S,T*U,V) \cdot \left[a(S,T,U) * Id_V\right].$$

(ii) $$r(S) * Id_T = \left[Id_S * \ell(T)\right] \cdot a(S,I_B,T).$$

We need to show the same in $\underline{X}'$. We note that $^+$ preserves natural isomorphisms. Let S,T,U,V be objects of categories $\mathrm{Hom}_{\underline{X}'}(A,B)$, $\mathrm{Hom}_{\underline{X}'}(B,C)$, $\mathrm{Hom}_{\underline{X}'}(C,D)$ and $\mathrm{Hom}_{\underline{X}'}(D,E)$, respectively. Then

$$(SoT)oU = (S*T)^{+}oU = \left[(S*T)^{+}*U\right]^{+} \xrightarrow[\sigma^{-1}(S*T,U)]{\sim}$$

$$\left[(S*T)*U\right]^{+} \xrightarrow[a^{+}(S,T,U)]{} \left[S*(T*U)\right]^{+} \xrightarrow[\sigma(S,T*U)]{} \left[S*(T*U)^{+}\right]^{+} =$$

$$\left[S*(ToU)\right]^{+} = So(ToU) \ .$$

Hence, $\sigma(S,T*U)\ a^{+}(S,T,U)\ \sigma^{-1}(S*T,U) : (SoT)oU \longrightarrow So(ToU)$ is a natural isomorphism.

Similarly, we have natural isomorphisms $\ell^{+}(S) : I_A \circ S \xrightarrow{\sim} S$ and $r^{+}(S) : S \circ I_B \xrightarrow{\sim} S$.

Before proceeding to check the coherence conditions, we introduce abbreviations for the following natural isomorphisms.

$$\tau_1 = \sigma^{-1}\left[(S*T)*U,V\right]\left[\sigma^{-1}(S*T,U)*Id_V\right]^{+}: \left[(SoT)oU\right]oV \longrightarrow \left[\left[(S*T)*U\right]*V\right]^{+}$$

$$\tau_2 = \sigma^{-1}\left[S*(T*U),V\right]\left[\sigma^{-1}(S,T*U)*Id_V\right]^{+}: \left[So(ToU)\right]oV \longrightarrow \left[\left[S*(T*U)\right]*V\right]^{+}$$

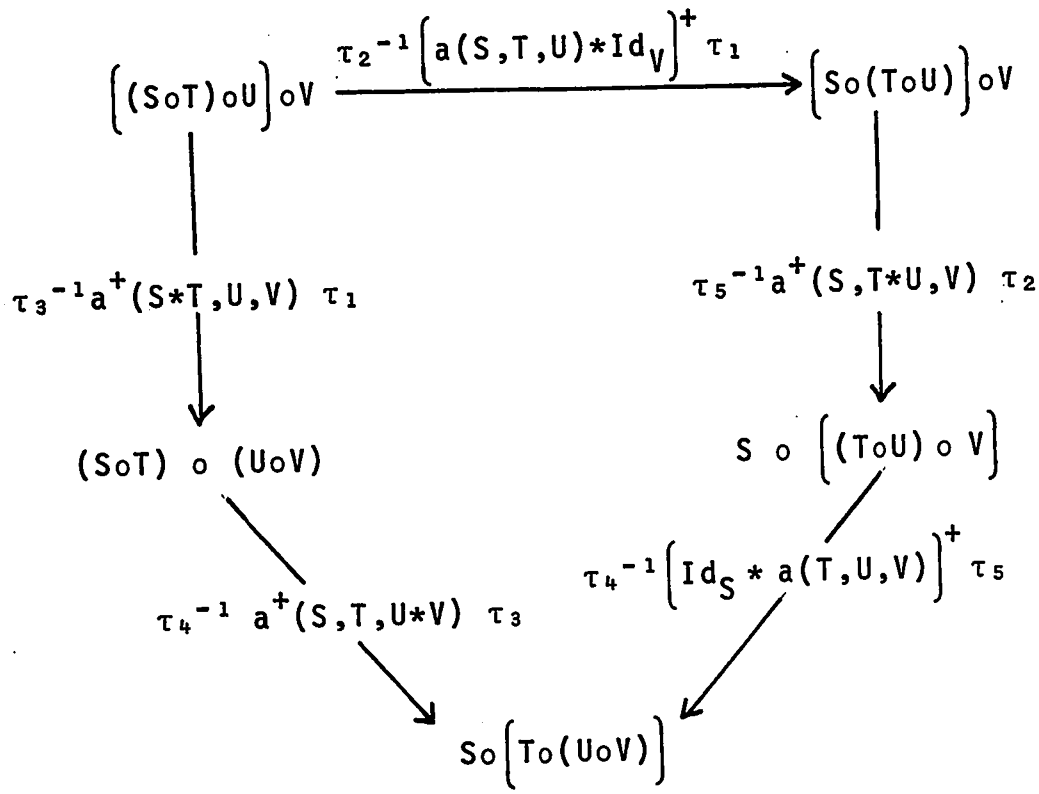
$$\tau_3 = \sigma^{-1}(S*T,\ U*T) : (S\circ T)\circ(U\circ V) \longrightarrow \left[(S*T) * (U*V)\right]^+$$

$$\tau_4 = \left[Id_S*\sigma^{-1}(T,U*V)\right]^+ \sigma^{-1}\left[S,T*(U*V)\right] :\ S\circ\left[T\circ(U\circ V)\right] \longrightarrow \left[S*\left[T*(U*V)\right]\right]^+$$

$$\tau_5 = \left[Id_S*\sigma^{-1}(T*U,V)\right]^+ \sigma^{-1}\left[S,(T*U)*V\right] :\ S\circ\left[(T\circ U)\circ V\right] \longrightarrow \left[S*\left[(T*U)*V\right]\right]^+ .$$

The required coherence conditions are the following:

(i)



This commutes, since by applying coherence condition (i) of the bicategory $\underline{X}$, we obtain

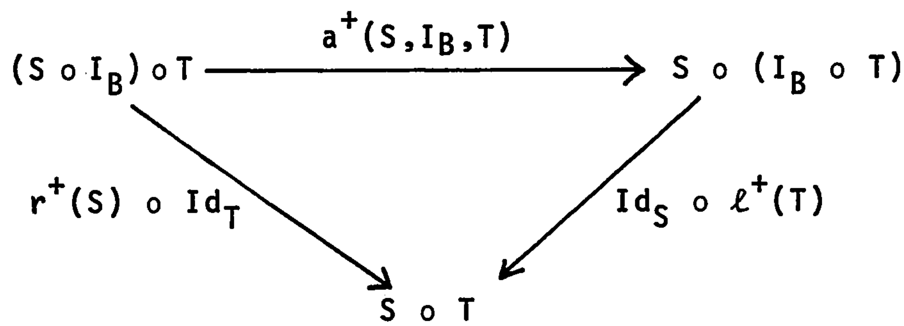
$$\tau_4^{-1}\left[Id_S * a(T,U,V)\right]^+ \tau_5 \ \tau_5^{-1} a^+(S,T*U,V)\tau_2 \ \tau_2^{-1}\left[a(S,T,U)*Id_V\right]^+ \tau_1$$

$$= \tau_4^{-1}\left[Id_S * a(T,U,V)\right]^+ a^+(S,T*U,V) \ \left[a(S,T,U)*Id_V\right]^+ \tau_1$$

$$= \tau_4^{-1} \ a^+(S,T,U*V) \ a^+(S*T,U,V) \ \tau_1$$

$$= \tau_4^{-1} \ a^+(S,T,U*V) \ \tau_3 \ \tau_3^{-1} \ a^+(S*T,U,V) \ \tau_1 \quad .$$

(ii)



This commutes by using coherence condition (ii) of the bicategory X.

Thus X' is also a bicategory. ∎

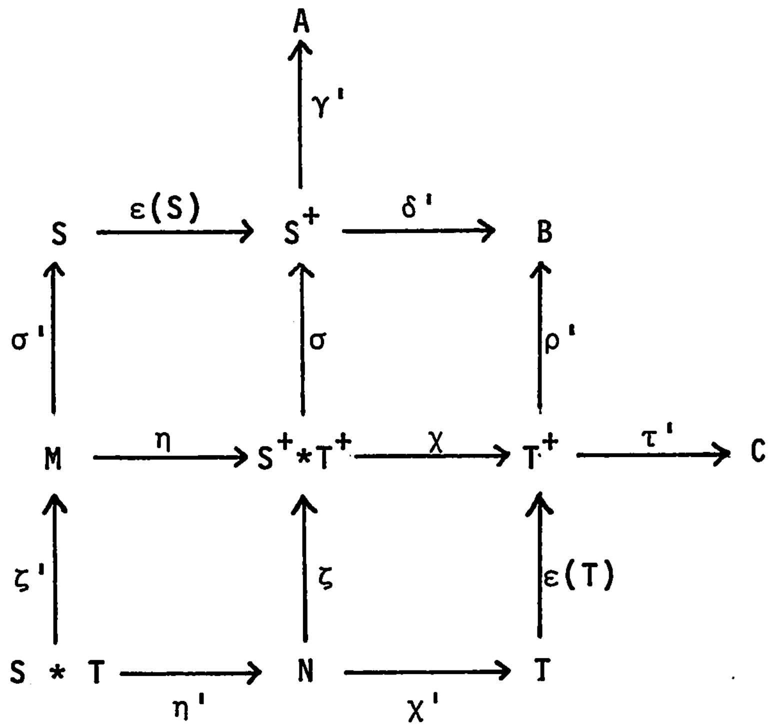
We now return to Rel A. In the following we assume that A is a category with finite products, pullbacks and an (E,M) factorization system. We abbreviate the span (S,γ,δ) from B to A by S whenever there is no ambiguity.

LEMMA 1.5. If pullbacks preserve E-class morphisms, then

$$\sigma(S,T) = \left[\varepsilon(S) * \varepsilon(T)\right]^{+} : (S * T)^{+} \longrightarrow (S^{+} * T^{+})^{+}$$

is an isomorphism, where S and T are spans from B to A and C to B, respectively.

Proof: We recall that $\varepsilon(S) : (S,\gamma,\delta) \longrightarrow (S^{+},\gamma',\delta')$ and $\varepsilon(T) : (T,\rho,\tau) \longrightarrow (T^{+},\rho',\tau')$. $S^{+} * T^{+}$ and $S * T$ are obtained by pullbacks of $S^{+} \xrightarrow{\delta'} B \xleftarrow{\rho'} T^{+}$ and $S \xrightarrow{\delta} B \xleftarrow{\rho} T$, respectively, as shown in the following diagram:

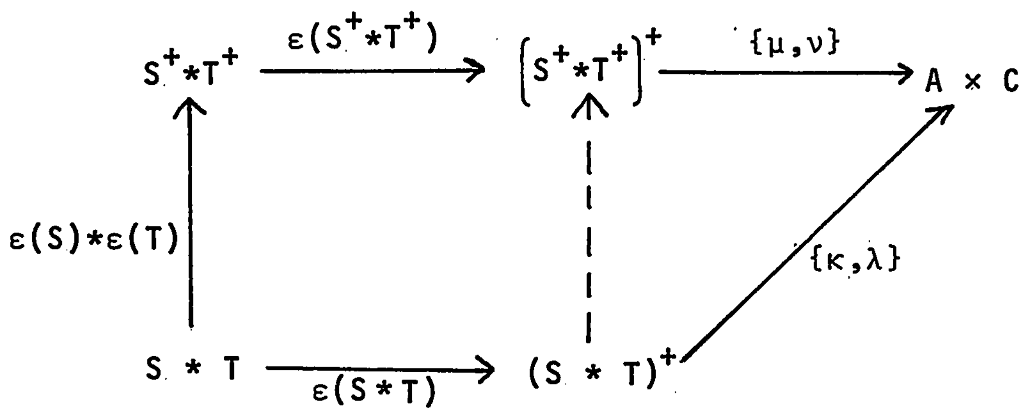


where $\gamma'\varepsilon(S) = \gamma$, $\delta'\varepsilon(S) = \delta$, $\rho'\varepsilon(T) = \rho$, $\tau'\varepsilon(T) = \tau$ and all squares are pullbacks.

Now $\varepsilon(S), \varepsilon(T) \in \underline{E}$. Hence $\eta : M \longrightarrow S^{+}*T^{+}$, $\zeta : N \longrightarrow S^{+}*T^{+} \in \underline{E}$ and $\eta' : S*T \longrightarrow N, \zeta' : S*T \longrightarrow M \in \underline{E}$.

Since $S^{+} \xleftarrow{\sigma} S^{+}*T^{+} \xrightarrow{\chi} T^{+}$ is a pullback, therefore there is a unique map $\varepsilon(S) * \varepsilon(T) : S*T \longrightarrow S^{+}*T^{+}$ such that $\sigma\left[\varepsilon(S) * \varepsilon(T)\right] = \varepsilon(S)\ \sigma'\ \zeta'$ and $\chi\left[\varepsilon(S) * \varepsilon(T)\right] = \varepsilon(T)\ \chi'\ \eta'$. Clearly then $\varepsilon(S) * \varepsilon(T) = \eta\ \zeta'$ and therefore an $\underline{E}$-class morphism.

We consider the following diagram:



where $\{\mu,\nu\}\ \varepsilon(S^+ * T^+)$ and $\{\kappa,\lambda\}\ \varepsilon(S*T)$ are ($\underline{E}$,$\underline{M}$) factorizations of morphisms $S^+ * T^+ \longrightarrow A \times C$ and $S*T \longrightarrow A \times C$, respectively, such that $\{\mu,\nu\}, \{\kappa,\lambda\} \in \underline{M}$ and $\varepsilon(S^+ * T^+)$, $\varepsilon(S*T) \in \underline{E}$.

Since we have shown that $\varepsilon(S) * \varepsilon(T) \in \underline{E}$, by composition $\varepsilon(S^+ * T^+)\left[\varepsilon(S) * \varepsilon(T)\right] \in \underline{E}$. By the unique ($\underline{E}$,$\underline{M}$) factorization of $S * T \longrightarrow A \times C$, there exists an isomorphism $\left[\varepsilon(S) * \varepsilon(T)\right]^+ : (S*T)^+ \longrightarrow (S^+ * T^+)^+$. ∎

From proposition (1.4) and lemma (1.5), we obtain:

THEOREM 1.6. If pullbacks preserve $\underline{E}$-class morphisms, then Rel $\underline{A}$ is a bicategory. ∎

Of special interest is the situation treated by Klein [6], where Rel (A,B) is not a category but a preordered set. We shall regard a preordered set as a special kind of category in which there is at most one morphism between any pair of objects.

PROPOSITION 1.7. The following three statements are equivalent:

(1) Rel (A,B) is a preordered set for each pair (A,B),

(2) Rel $(A,\mathbb{1})$ is a preordered set for each A, where $\mathbb{1}$ is the terminal object of $\underline{A}$,

(e) $\underline{M} \subseteq$ mono.

Proof: Since $\underline{A}$ has finite products, the "empty" product, $\mathbb{1}$, is the terminal object of $\underline{A}$. Clearly (1) $\Longrightarrow$ (2).

(2) $\Longrightarrow$ (3) We shall show that, for any objects A,B and C in $\underline{A}$, if $f,g : C \longrightarrow B$ and $n : B \longrightarrow A$ with $n \in \underline{M}$ and $n\,f = n\,g = m \in \underline{M}$ then $f = g$. Then by proposition (0.9), it follows that $\underline{M}$ is a class of monos.

Let $t(B) : B \longrightarrow \mathbb{1}$. Then $\{n,t(B)\} : B \longrightarrow A \times \mathbb{1} \in \underline{M}$, because $A \times \mathbb{1} \simeq A$ and $n \in \underline{M}$. Hence B can be regarded as a relation $\left[B, n, t(B)\right]$ from $\mathbb{1}$ to A. Similarly we have a relation $\left[C, m, t(C)\right]$ from $\mathbb{1}$ to A, and therefore $f,g : \left[C, m, t(C)\right] \longrightarrow \left[B, n, t(B)\right]$. But Rel $(A,\mathbb{1})$ is a preordered set. Hence $f = g$.

(3) $\Longrightarrow$ (1) Suppose $\underline{M}$ consists of monos only. We need to show that if there exists a map between two relations R,S from B to A, then the map is unique. Let $f,g : R \longrightarrow S$, where $n : R \longrightarrow A \times B$ and $m : S \longrightarrow A \times B$ are elements of $\underline{M}$. By proposition (1.2), $mf = n$ and $mg = n$. Thus $mf = mg$. Since $\underline{M} \subseteq$ monos, m is a mono and therefore $f = g$. ∎

We remark that our result in theorem (1.6) specializes to Bénabou's [2] when $\underline{E}$ = isos , $\underline{M}$ = all maps since $\underline{E}$ is then invariant under pullbacks.

Chapter 2

PROPERTIES OF RELATIONS

Let $\underline{A}$ be a category with finite products, pullbacks, pushouts and $(\underline{E},\underline{M})$ factorization system. Whenever we talk about the bicategory of relations, it will be tacitly assumed that pullbacks preserve $\underline{E}$-class morphisms. We also assume that a pullback of a regular epi is an epi whenever "E = the class of all regular epis" is mentioned. We shall investigate some properties of a relation from B to A in $\underline{A}$.

To any relation (R,α,β) from B to A, there is a converse relation (R,β,α), denoted by $\overline{R}$, from A to B.

PROPOSITION 2.1. (i) $\overline{\overline{R}} \simeq R$. (ii) $\overline{RoS} \simeq \overline{S}o\overline{R}$, where (S,γ,δ) is a relation from C to B.

Proof: (i) Obvious.

(ii) The pullback of $R \xrightarrow{\beta} B \xleftarrow{\gamma} S$ is $R \xleftarrow{\sigma} R{*}S \xrightarrow{\tau} S$. We obtain the span $(R{*}S, \alpha\sigma, \delta\tau)$ from C to A. Its converse, $\overline{R{*}S}$ is $(R{*}S, \delta\tau, \alpha\sigma)$ from A to C.

On the other hand, the composition $\overline{S} * \overline{R}$ is $(\overline{S} * \overline{R}\ , \delta\tau, \alpha\sigma)$, where $\overline{S} \xleftarrow{\tau} \overline{S} * \overline{R} \xrightarrow{\sigma} \overline{R}$ is the pull-back of $\overline{S} \xrightarrow{\gamma} B \xleftarrow{\beta} \overline{R}$.

We deduce $\overline{R * S} \simeq \overline{S} * \overline{R}$. By the uniqueness of $(\underline{E},\underline{M})$ factorization, $\overline{R \circ S} \simeq \overline{S} \circ \overline{R}$. ▮

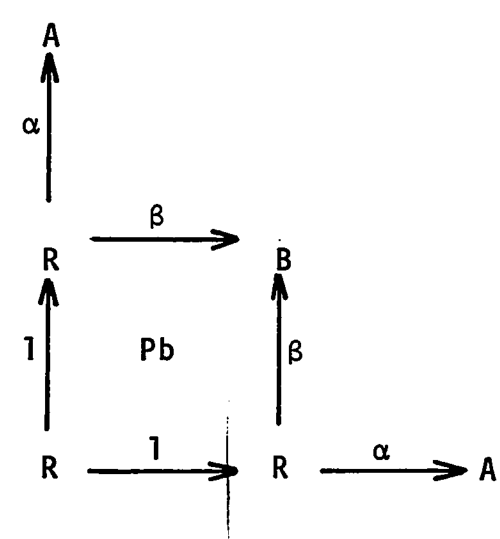
We recall that when $\underline{M} \subseteq$ mono, Rel (A,B) is a pre-ordered set (see proposition (1.7)).

PROPOSITION 2.2. Assume $\underline{M} \subseteq$ mono. Let (R,α,β) be a relation from B to A. Suppose that $\{1,1\} \in \underline{M}$, then

(1) $R \circ \overline{R} \subseteq I_A$ if and only if β is mono.

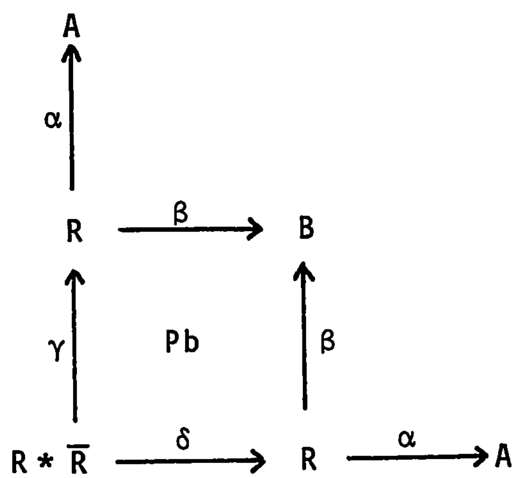
(2) $R \circ \overline{R} \geq I_A$ if and only if $\alpha \in \underline{E}$.

Proof: (1) Let β be mono. Then $(R * \overline{R}, \alpha, \alpha)$ is obtained from the following diagram:



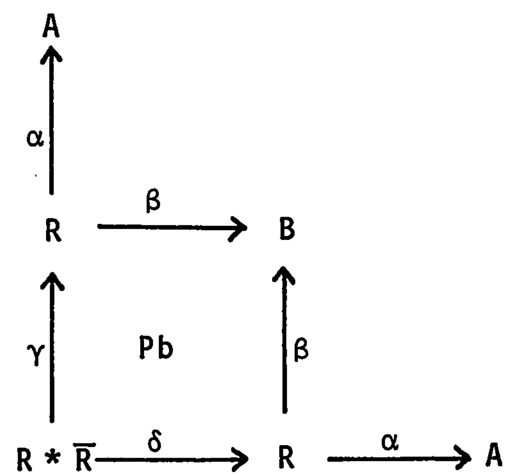
Let $\alpha_m \alpha_e$ be the $(\underline{E},\underline{M})$ factorization of α. Then $\{\alpha_m,\alpha_m\}\,\alpha_e$ is the $(\underline{E},\underline{M})$ factorization of $\{\alpha,\alpha\}$, since $\{\alpha_m,\alpha_m\} = \{1,1\}\,\alpha_m \in \underline{M}$. Thus we obtain $(R \circ \overline{R}, \alpha_m, \alpha_m)$ and there is a map, $\alpha_m : R \circ \overline{R} \longrightarrow I_A$, which belongs to $\underline{M}$.

Conversely, suppose that $R \circ \overline{R} \subseteq I_A$. We note that $(R*\overline{R}, \alpha\gamma, \alpha\delta)$ is obtained as follows:



We have $(R \circ \overline{R}, \mu, \nu)$, where $\{\mu, \nu\}\varepsilon$ is the $(\underline{E},\underline{M})$ factorization of $\{\alpha\gamma, \alpha\delta\}$. Since there is a map $\tau : R\circ\overline{R} \longrightarrow I_A$, $\mu = \tau = \nu$. Hence $\alpha\gamma = \mu\varepsilon = \nu\varepsilon = \alpha\delta$ and since $\beta\gamma = \beta\delta$ we have $\{\alpha,\beta\}\gamma = \{\alpha,\beta\}\delta$. Now since $\{\alpha,\beta\} \in \underline{M} \subseteq$ mono, $\gamma = \delta$. It follows that β is mono because its kernel pair is (γ,δ).

(2) Suppose $\alpha \in \underline{E}$. $(Ro\overline{R}, \mu, \nu)$ is obtained from the $(\underline{E},\underline{M})$ factorization of $\{\alpha\gamma, \alpha\delta\}$ in the following diagram. That is, $\{\alpha\gamma, \alpha\delta\} = \{\mu,\nu\}\varepsilon$ where $\{\mu,\nu\} \in \underline{M}$ and $\varepsilon \in \underline{E}$.



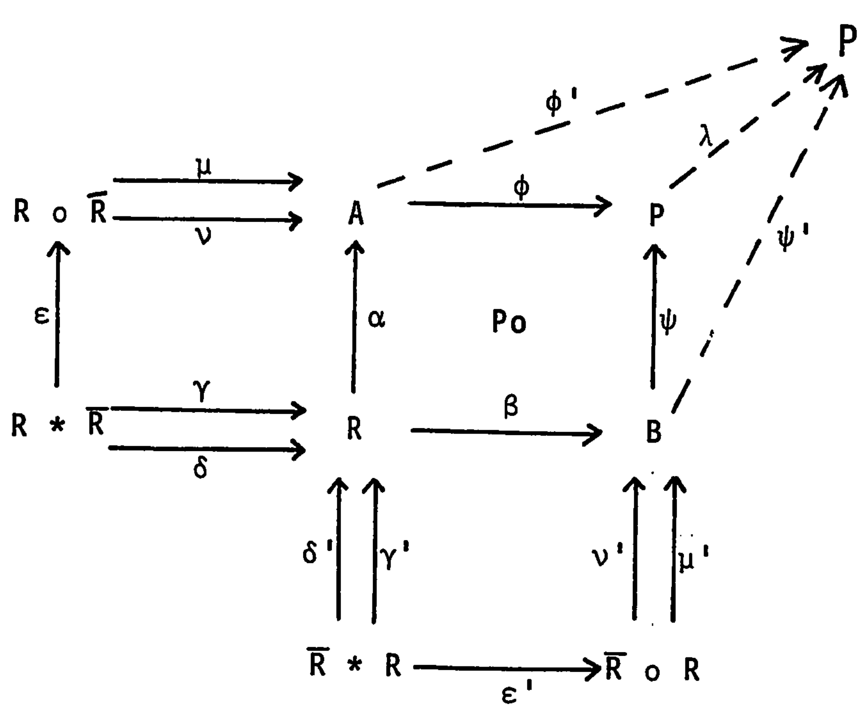
By the pullback square, there exists a λ such that $\gamma\lambda = 1 = \delta\lambda$. It follows that $\mu\varepsilon\lambda = \alpha\gamma\lambda = \alpha = \alpha\delta\lambda = \nu\varepsilon\lambda$. We obtain $\{\alpha,\alpha\} = \{1,1\}\alpha = \{\mu,\nu\}\varepsilon\lambda$, where $\{1,1\}, \{\mu,\nu\} \in \underline{M}$ and $\alpha \in \underline{E}$. Hence by the diagonal property (see definition (0.1)), there exists a unique τ such that $\tau\alpha = \varepsilon\lambda$ and $\{\mu,\nu\}\tau = \{1,1\}$. In other words, there exists a $\tau : I_A \longrightarrow R o \overline{R}$.

Conversely, suppose $\tau : (I_A,1,1) \longrightarrow (R o \overline{R}, \mu,\nu)$. Then $\mu\tau = 1 = \nu\tau$ and therefore, by proposition (0.9), μ

is an $\underline{E}$-class morphism. From the construction of $R \circ \overline{R}$, $\alpha\gamma = \mu\varepsilon \in \underline{E}$. Hence α is an $\underline{E}$-class morphism (see proposition (0.9)). ∎

PROPOSITION 2.3. Assume $\underline{E} \subseteq$ regular epis. Let (R,α,β) be a relation from B to A with both α and β in $\underline{E}$. Then the coequalizer of $R \circ \overline{R} \underset{\nu}{\overset{\mu}{\rightrightarrows}} A$ is isomorphic to that of $\overline{R} \circ R \underset{\nu'}{\overset{\mu'}{\rightrightarrows}} B$. (Explicitly, $(P,\phi) = \text{Coeq.}\ (\mu,\nu)$ and $(P,\psi) = \text{Coeq.}\ (\mu',\nu')$ where $A \xrightarrow{\phi} P \xleftarrow{\psi} B$ is the pushout of $A \xleftarrow{\alpha} R \xrightarrow{\beta} B$.)

Proof: In the following commutative diagram, (γ,δ) and (γ',δ') are the kernel pairs of β and α, respectively. Since α and β both belong to $\underline{E}$, they are coequalizers of (γ',δ') and (γ,δ), respectively. $\{\mu,\nu\}\varepsilon$ and $\{\mu',\nu'\}\varepsilon'$ are $(\underline{E},\underline{M})$ factorizations of $\{\alpha\gamma,\alpha\delta\}$ and $\{\beta\gamma',\beta\delta'\}$, respectively.



Let (P', ϕ') be the coequalizer of (μ,ν). Then $\phi' \alpha \gamma = \phi' \mu \varepsilon = \phi' \nu \varepsilon = \phi' \alpha \delta$. Since β is the coequalizer of (γ,δ), there is a unique $\psi' : B \longrightarrow P'$ such that $\psi' \beta = \phi' \alpha$. By the pushout in the above diagram, there exists a unique $\lambda : P \longrightarrow P'$ such that $\lambda \phi = \phi'$ and $\lambda \psi = \psi'$.

On the other hand, from $\lambda \phi = \phi'$ we obtain $\lambda \psi \beta = \lambda \phi \alpha = \phi' \alpha = \psi' \beta$. Then $\psi' = \lambda \psi$ because $\beta \in \underline{E}$.

λ is therefore the unique map such that $\lambda\,\phi = \phi'$. Hence $(P,\phi) = \text{Coeq.}\ (\mu,\nu)$. Similarly $(P,\psi) = \text{Coeq.}\ (\mu',\nu')$.

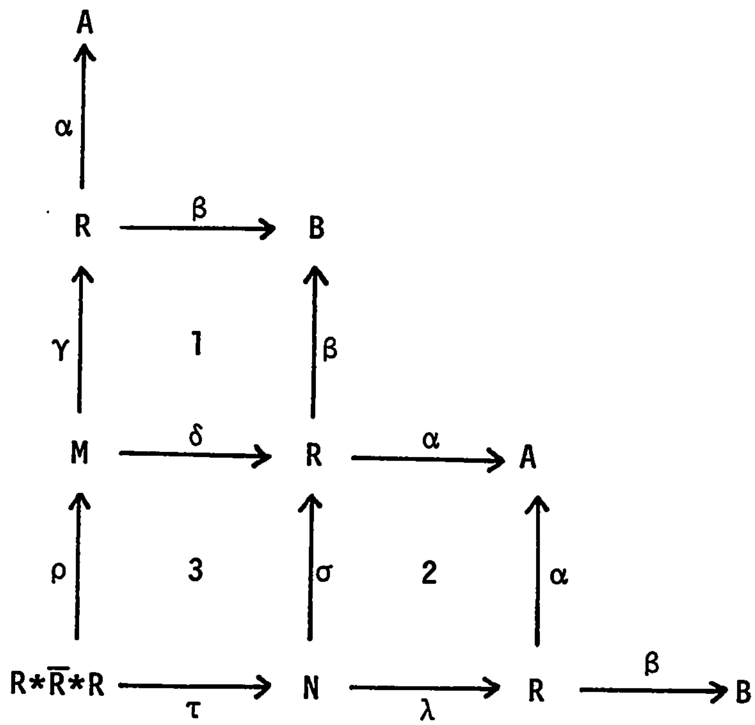
Later when we consider a situation in which $R \circ \overline{R}$ and $\overline{R} \circ R$ are equivalence relations, this result can be written as:

$$A/(R \circ \overline{R}) \simeq B/(\overline{R} \circ R)$$

(compare with [7] proposition 1).

LEMMA 2.4. Let (R,α,β) be a relation from B to A. Then there is a canonical map from R to $R \circ \overline{R} \circ R$ which is an $\underline{M}$-class morphism.

Proof: In the following diagram, we obtain $(R * \overline{R} * R\ ,\alpha\,\gamma\,\rho\ ,\beta\,\lambda\,\tau)$ by successive pullbacks.



By the pullbacks (1) and (2), there exist unique $\eta : R \longrightarrow M$ and $\zeta : R \longrightarrow N$ such that $\gamma\,\eta = 1 = \delta\,\eta$ and $\sigma\,\zeta = 1 = \lambda\,\zeta$. Hence $\delta\,\eta = \sigma\,\zeta = 1$ and, by the pullback square (3), there exists a unique $\kappa : R \longrightarrow R * \overline{R} * R$ such that $\rho\,\kappa = \eta$ and $\tau\,\kappa = \zeta$. We thus obtain $\alpha\,\gamma\,\rho\,\kappa = \alpha\,\gamma\,\eta = \alpha$ and $\beta\,\lambda\,\tau\,\kappa = \beta\,\lambda\,\zeta = \beta$.

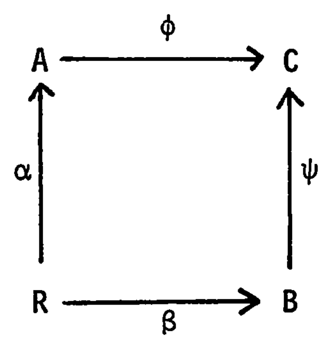
Let $\{\alpha\,\gamma\,\rho\,,\,\beta\,\lambda\,\tau\} = \{\mu,\nu\}\,\varepsilon$ be the ($\underline{E}$,$\underline{M}$) factorization of the morphism $\{\alpha\,\gamma\,\rho\,,\,\beta\,\lambda\,\tau\} : R * \overline{R} * R \longrightarrow A \times B$. Then we have $\varepsilon\,\kappa : R \longrightarrow R \circ \overline{R} \circ R$, which is an $\underline{M}$-class map (see remark (1.3)). ∎

We call a relation R <u>difunctional</u> if it satisfies $R \simeq R \circ \overline{R} \circ R$. In the following proposition, we give a sufficient condition for R to be difunctional. We shall assume $\{1,1\} \in \underline{M}$ in the following and define a <u>pullback relation</u> to be a relation which is a pullback of some pair of maps. We note that if a span (R,α,β) is a pullback of of some pair of maps, then it is a relation (see corollary (0.8)).

THEOREM 2.5. Let $\underline{M} \subseteq$ Monos. Assume that (R,α,β) is a pullback relation from B to A. Then the canonical map from R to $R \circ \overline{R} \circ R$ is an isomorphism.

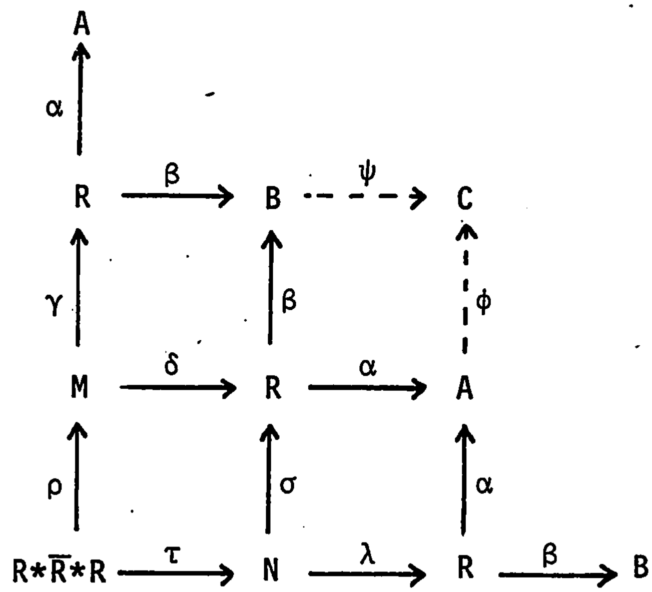
Proof: As is well-known, $A \xleftarrow{\alpha} R \xrightarrow{\beta} B$ is a pullback of its own pushout $A \xrightarrow{\phi} C \xleftarrow{\psi} B$, i.e.

(2.6)



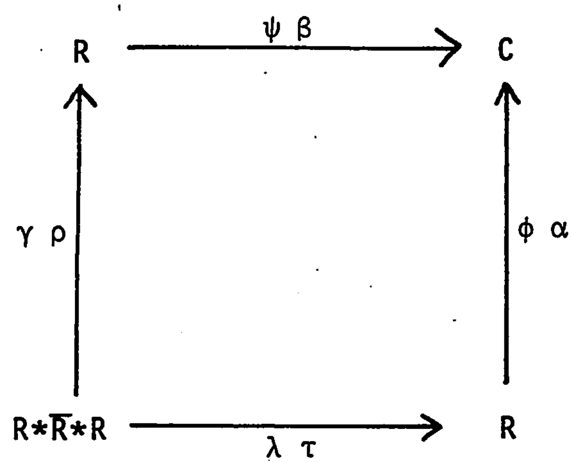
is bicartesian (that is, both a pullback and a pushout square).

In the following diagram, $(R * \overline{R} * R, \alpha\gamma\rho, \beta\lambda\tau)$ is obtained by successive pullbacks.



By composition of pullbacks, we obtain the pullback square :

(2.7)



We claim that the following is almost a pullback square.

$$\begin{array}{ccc} A & \xrightarrow{\quad\varphi\quad} & C \\ {\scriptstyle \alpha\gamma\rho}\Big\uparrow & & \Big\uparrow{\scriptstyle \psi} \\ R*\bar{R}*R & \xrightarrow[\quad\beta\lambda\tau\quad]{} & B \end{array} \tag{2.8}$$

It commutes since, $\varphi\alpha\gamma\rho = \psi\beta\gamma\rho = \varphi\alpha\lambda\tau = \psi\beta\lambda\tau$. Suppose there are $x : X \longrightarrow A$ and $y : X \longrightarrow B$ such that $\varphi x = \psi y$. By the pullback square (2.6), there exists a unique z such that $\alpha z = x$ and $\beta z = y$. Hence $\varphi\alpha z = \psi\beta z$ and, by the pullback (2.7), there exists a unique ω such that $\gamma\rho\omega = z$ and $\lambda\tau\omega = z$. We thus obtain $\alpha\gamma\rho\omega = \alpha z = x$ and $\beta\lambda\tau\omega = \beta z = y$. It follows that the square (2.8) is almost a pullback.

Take $X = R$. Since R is a pullback, there exists θ such that $\alpha\theta = \alpha\gamma\rho$ and $\beta\theta = \beta\lambda\tau$, hence $\alpha\theta\omega = \alpha\gamma\rho\omega$ and $\beta\theta\omega = \beta\lambda\tau\omega$. Since R is a pullback, $\theta\omega = 1$. Therefore $\theta \in \underline{E}$ (see proposition (0.9)). Hence $R \circ \bar{R} \circ R \simeq (R*\bar{R}*R)^{+} \simeq R$.

We shall extend the notion of mono and epi to bicategories : R will be called <u>mono</u> if $R \circ S \simeq R \circ T$ implies that $S \simeq T$ and <u>epi</u> if $S \circ R \simeq T \circ R$ implies that $S \simeq T$. R is called an <u>equivalence</u> if there exists an S such that $R \circ S \simeq 1$ and $S \circ R \simeq 1$.

Clearly, from proposition (2.5) we obtain the following:

COROLLARY 2.9. Assume that (R,α,β) is a pullback relation. Then the following are equivalent:

(i) R is mono,

(ii) $\overline{R}$ is epi,

(iii) $\overline{R} \circ R \simeq 1$. ▮

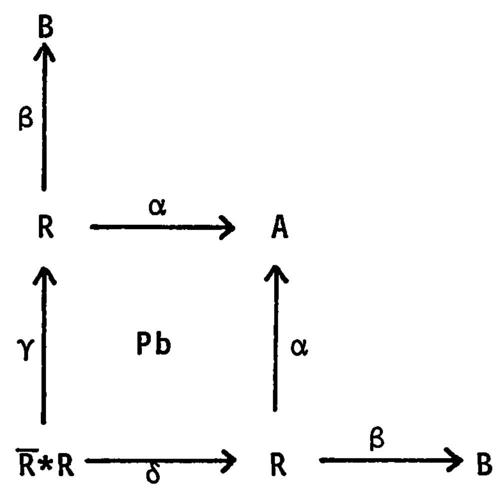
Hence a pullback relation is an equivalence if and only if it is a mono and an epi. We shall describe monos and epis in Rel <u>A</u>.

PROPOSITION 2.10. Let (R,α,β) be a relation. The following are equivalent:

(i) $\overline{R} \circ R \simeq 1$ and $\{\alpha,\beta\}$ is mono ,

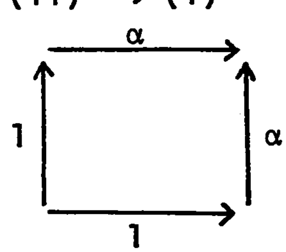
(ii) α is mono and $\beta \in \underline{E}$.

Proof: (i) $\Longrightarrow$ (ii) $(\overline{R} * R , \beta\gamma , \beta\delta)$ is a span obtained as indicated in the following diagram. We note that $\alpha\gamma = \alpha\delta$.



Since $\overline{R} \circ R \simeq 1$, $\beta\gamma = \beta\delta \in \underline{E}$. We thus have $\{\alpha,\beta\}\gamma = \{\alpha,\beta\}\delta$ and hence $\gamma = \delta$. Since (γ,δ) is the kernel pair of α with $\gamma = \delta$, α is mono. Without loss of generality we can take $\gamma = \delta = 1$. Therefore $\beta = \beta\gamma \in \underline{E}$.

(ii) $\Longrightarrow$ (i) Assume that α is mono and $\beta \in \underline{E}$. Since



is a pullback, we have the

span $(\overline{R} * R, \beta, \beta)$. But the $(\underline{E},\underline{M})$ factorization of $\{\beta,\beta\} : \overline{R} * R \longrightarrow B \times B$ is just $\{1,1\}\beta$, since $\beta \in \underline{E}$. We thus obtain the relation $(\overline{R} \circ R, 1, 1)$ and hence $\overline{R} \circ R \simeq I_B$.

Clearly, here $\{\alpha,\beta\}$ is mono because α is mono. ∎

We note that if (R,α,β) is a pullback relation, then $\{\alpha\ \beta\}$ is mono. From corollary (2.9) and proposition (2.10), we have

PROPOSITION 2.11. Assume that (R,α,β) is a pullback relation. Then the following are equivalent:

(i) R is mono,

(ii) $\overline{R}$ is epi,

(iii) $\overline{R} \circ R \simeq 1$,

(iv) α is mono and $\beta \in \underline{E}$. ∎

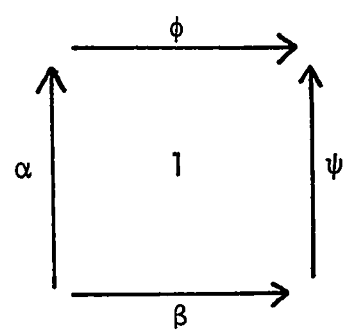
We shall now show that pullback relations can be factored canonically whenever $\underline{M}$ = the class of all monos. We shall first prove the following lemmas.

LEMMA 2.12. Suppose that (R,α,β) is a relation from B to A. Then (R,α_e,β_e) is a relation from B' to A', where $R \xrightarrow{\alpha_e} A' \xrightarrow{\alpha_m} A$ and $R \xrightarrow{\beta_e} B' \xrightarrow{\beta_m} B$ are $(\underline{E},\underline{M})$ factorizations of α and β, respectively.

Proof: We have $\{\alpha,\beta\} = (\alpha_m \times \beta_m)\{\alpha_e,\beta_e\}$. By corollary (0.6), $\alpha_m \times \beta_m \in \underline{M}$ and since $\{\alpha,\beta\} \in \underline{M}$, so does $\{\alpha_e,\beta_e\}$ (see proposition (0.11)). ∎

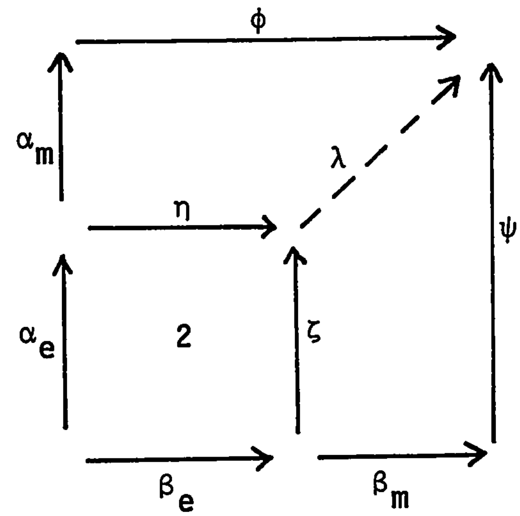
LEMMA 2.13. Assume that $\underline{M} \subseteq$ mono. If (R,α,β) is a pullback relation, so is (R, α_e,β_e), where $\alpha_m\,\alpha_e$ and $\beta_m\,\beta_e$ are $(\underline{E},\underline{M})$ factorizations of α and β, respectively.

Proof: Let $\xrightarrow{\phi}\xleftarrow{\psi}$ be a pushout of $\xleftarrow{\alpha}\;\xrightarrow{\beta}$. Then



is bicartesian. We take $\xrightarrow{\eta}\xleftarrow{\zeta}$ to be a pushout of

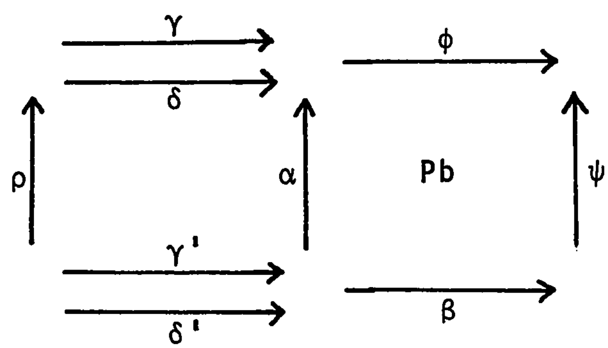
$\xleftarrow{\alpha_e} \xrightarrow{\beta_e}$. Hence there exists a unique λ such that $\lambda\,\eta = \phi\,\alpha_m$ and $\lambda\,\zeta = \psi\,\beta_m$, as follows:



We need to show that the square (2) is a pullback. Suppose that $\eta\,x = \zeta\,y$. We then have $\phi\,\alpha_m\,x = \lambda\,\eta\,x = \lambda\,\zeta\,y = \psi\,\beta_m\,y$. By the pullback square (1), there exists a unique z such that $\alpha\,z = \alpha_m\,x$ and $\beta\,z = \beta_m\,y$, i.e. $\alpha_m\,\alpha_e\,z = \alpha_m\,x$ and $\beta_m\,\beta_e\,z = \beta_m\,y$. Since α_m and β_m are both mono, we obtain $\alpha_e\,z = x$ and $\beta_e\,z = y$. Since $\{\alpha_e,\beta_e\} \in \underline{M} \subseteq$ mono, it follows that there exists a unique z such that $\alpha_e\,z = x$ and $\beta_e\,z = y$. Therefore (R,α_e,β_e) is a pullback relation.∎

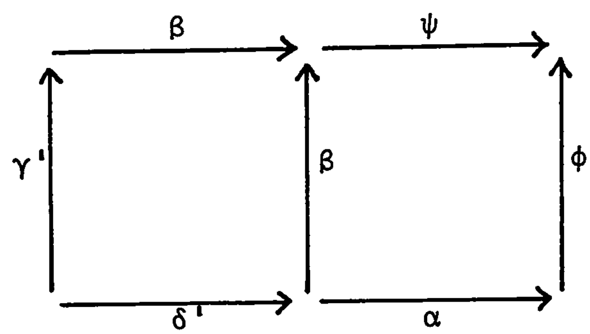
The converse of lemma (2.13) is not always true. We shall give the necessary and sufficient condition in the next proposition. We need the results of the following lemmas.

LEMMA 2.14. Suppose that the right square in the following diagram is a pullback and (γ,δ) , (γ',δ') are kernel pairs of ϕ and β, respectively.

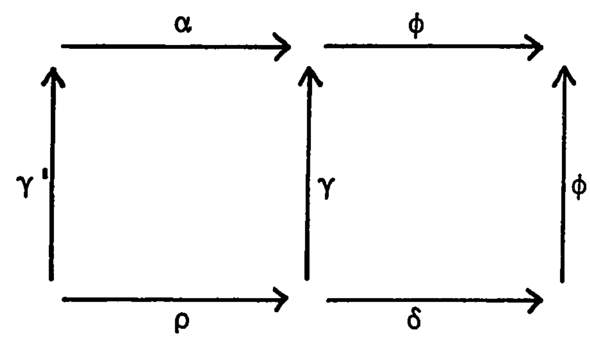


Then the left squares are both pullbacks.

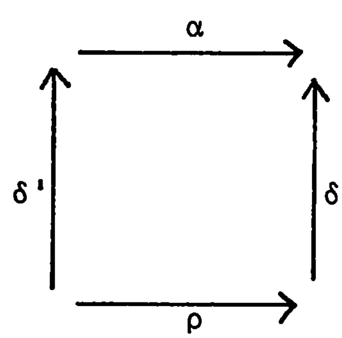
Proof: In the following diagram all squares are pullbacks:



But $\psi\,\beta = \phi\,\alpha$ and $\alpha\,\delta' = \delta\,\rho$. Hence the composite of the following squares is a pullback.

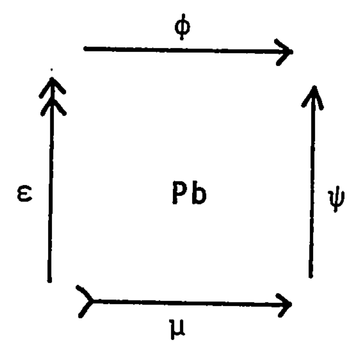


Here the right is a pullback square and therefore so is the left. Similarly,



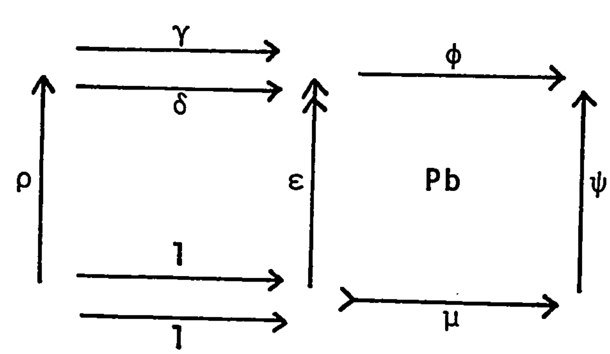
is a pullback. ∎

LEMMA 2.15. Assume that a pullback of a regular epi is an epi. Suppose that



is a pullback with μ mono and ε regular epi. Then ϕ is a mono.

Proof: Since μ is mono, (1,1) is its kernel pair. Let (γ,δ) be the kernel pair of ϕ. Then there exists a unique ρ such that $\gamma\,\rho = \varepsilon$ and $\delta\,\rho = \varepsilon$.



By lemma (2.14), the left squares are pullbacks. Hence ρ is epi. Since $\gamma\rho = \varepsilon = \delta\rho$, we obtain $\gamma = \delta$. Therefore ϕ is a mono. ∎

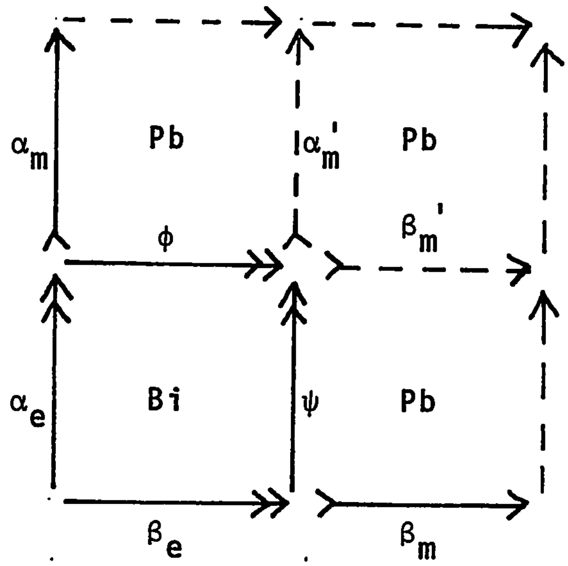
PROPOSITION 2.16. Assume $\underline{E}$ = the class of all regular epis. Let (R,α,β) be a relation from B to A. Then the following are equivalent:

(i) If (R,α_e,β_e) is a pullback relation, so is (R,α,β).

(ii) Any pair of morphisms $\xleftarrow{\mu} \prec \xrightarrow{\beta}$ with μ mono is a pullback of some pair of morphisms.

Proof: (i) $\Longrightarrow$ (ii) Take $\mu = \alpha_m$ and $1 = \alpha_e$. $(R, 1, \beta_e)$ is a pullback relation. Therefore, so is (R,α,β), which is (R,μ,β) in this case.

(ii) $\Longrightarrow$ (i) Let (R,α_e,β_e) be the pullback of its own pushout $\xleftarrow{\phi} \cdot \xrightarrow{\psi}$. By the dual of proposition (0.5), ϕ and ψ both belong to $\underline{E}$. Hence we can complete the pullback squares in the following diagram because by corollary (0.10) and lemma (2.15), α_m' and β_m' are both mono.

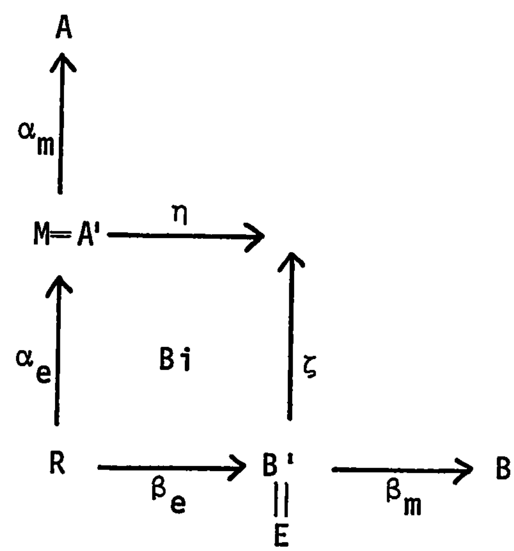


Thus the composite large square is a pullback and (R,α,β) is a pullback relation. ∎

We now return to show that pullback relations can be factored canonically. We remark that $\{\alpha,\beta\}$ is mono whenever one of the α and β is a mono. Thus when $\underline{M}$ = all monos, (R,α,β) is a relation if one of the α and β is mono.

PROPOSITION 2.17. Assume that $\underline{M}$ = the class of all monos and suppose that (R,α,β) is a pullback relation. Then there is a canonical factorization $R \simeq M \circ E$ where (M, α_m, η) and (E, ζ, β_m) are relations with $\alpha_m, \beta_m \in \underline{M}$ and $\eta, \zeta \in \underline{E}$.

Proof: In lemma (2.13) we have shown that the square in the following diagram is bicartesian.



Since $\alpha_e, \beta_e \in \underline{E}$, by the dual of proposition (0.5), we have $\eta, \zeta \in \underline{E}$. Hence $(M, \alpha_m\ \eta)$ and (E, ζ, β_m) are both relations with $\alpha_m, \beta_m \in \underline{M}$ = mono and $\eta, \zeta \in \underline{E}$. We recall the composition of spans and hence $R \simeq M * E$. But (R,α,β) is already a relation, thus $R \simeq M \circ E$. ∎

PROPOSITION 2.18. Suppose that $R \simeq M \circ E$, where (M, μ, η) and (E, ζ, ν) are relations with μ, ν being mono and $\eta, \zeta \in \underline{E}$. Then R is difunctional (i.e. $R \simeq R \circ \overline{R} \circ R$).

Proof: By proposition (2.1), we obtain:

$$R \circ \overline{R} \circ R \simeq M \circ E \circ (\overline{M \circ E}) \circ M \circ E \simeq M \circ E \circ \overline{E} \circ \overline{M} \circ M \circ E .$$

Since $E \circ \overline{E} \simeq 1$ and $\overline{M} \circ M \simeq 1$ (see proposition (2.10)), we have $R \circ \overline{R} \circ R \simeq M \circ E \simeq R$. ∎

We can write a relation (R, α, β) as $M_1 \circ R' \circ \overline{M_2}$ whenever $\underline{M} \subseteq$ mono, where $M_1 = (M_1, \alpha_m, 1)$, $M_2 = (M_2, \beta_m, 1)$ and $R' = (R, \alpha_e, \beta_e)$. Here $\alpha_m \alpha_e$ and $\beta_m \beta_e$ are $(\underline{E}, \underline{M})$ factorizations of α and β, respectively. We therefore have the following lemmas.

LEMMA 2.19. Assume that $\underline{M} \subseteq$ mono. Let (R, α, β) be a relation. Then R is difunctional if and only if R' is, where $R' = (R, \alpha_e, \beta_e)$.

Proof: We write $R = M_1 \circ R' \circ \overline{M}_2$, where $M_1 = (M_1, \alpha_m, 1)$ and $M_2 = (M_2, \beta_m, 1)$. Then

$$\begin{aligned} R \circ \overline{R} \circ R &\simeq (M_1 \circ R' \circ \overline{M}_2) \circ \overline{(M_1 \circ R' \circ \overline{M}_2)} \circ (M_1 \circ R' \circ \overline{M}_2) \\ &\simeq M_1 \circ R' \circ \overline{M}_2 \circ M_2 \circ \overline{R}' \circ \overline{M}_1 \circ M_1 \circ R' \circ \overline{M}_2 \\ &\simeq M_1 \circ R' \circ \overline{R}' \circ R' \circ \overline{M}_2 , \end{aligned}$$

since, by proposition (2.10), $\overline{M}_2 \circ M_2 \simeq 1$ and $\overline{M}_1 \circ M_1 \simeq 1$.

(i) Suppose that $R \circ \overline{R} \circ R \simeq R$. Then, $M_1 \circ R' \circ \overline{R}' \circ R' \circ \overline{M}_2 \simeq R \simeq M_1 \circ R' \circ \overline{M}_2$. Since $\overline{M}_1 \circ M_1 \simeq 1$ and $\overline{M}_2 \circ M_2 \simeq 1$, we obtain $R' \simeq R' \circ \overline{R}' \circ R'$.

(ii) Conversely, suppose that $R' \simeq R' \circ \overline{R'} \circ R'$. Then $R \circ \overline{R} \circ R \simeq M_1 \circ R' \circ \overline{R}' \circ R' \circ \overline{M}_2 \simeq M_1 \circ R' \circ \overline{M}_2 \simeq R$.

∎

We note that, whenever $R' = (R, \alpha_e, \beta_e) \simeq M' \circ E'$ with $M' = (M', \mu, \eta)$ and $E' = (E', \zeta, \nu)$, in which μ, ν are mono and $\eta, \zeta \in \underline{E}$, then μ, ν are isomorphisms. Without loss of generality, we can take $M' = (M', 1, \eta)$ and $E' = (E', \zeta, 1)$

Clearly, $(M, \alpha_m, \eta) = M_1 \circ M'$ and $(E, \zeta, \beta_m) = E' \circ \overline{M}_2$, where $M_1 = (M, \alpha_m, 1)$, $M' = (M, 1, \eta)$, $E' = (E, \zeta, 1)$, and $M_2 = (E, \beta_m, 1)$. Since $R \simeq M_1 \circ R' \circ \overline{M}_2$, we thus obtain:

LEMMA 2.20. $R \simeq M \circ E$ if and only if $R' \simeq M' \circ E'$. ∎

In the following we shall consider mainly the relation (R,α,β) with $\alpha, \beta \in \underline{E}$. We define a relation (R,α,β) as an $\underline{E}$-relation whenever both $\alpha, \beta \in \underline{E}$. We shall first show the converse of proposition (2.17) for $\underline{E}$-relations.

LEMMA 2.21. Let (R,α,β) be an $\underline{E}$-relation and assume that $R \simeq M \circ E$, where $M = (M, 1, \eta)$ and $E = (E, \zeta, 1)$. Then (R,α,β) is a pullback relation.

Proof: $(R,\alpha,\beta) = M * E$ is just a pullback of $\xrightarrow{\eta}\,\xleftarrow{\zeta}$. ∎

PROPOSITION 2.22. Suppose that (R,α,β) is a difunctional $\underline{E}$-relation from B to A. Assume that every equivalence relation is a kernel pair and $\underline{E}$ = the class of all regular epis. Then (R,α,β) can be expressed as $(M, 1, \phi) \circ (E, \psi, 1)$.

Proof: $R \circ \overline{R}$ is an equivalence relation on A, because $I \subseteq R \circ \overline{R}$ (see proposition (2.2)), $\overline{R \circ \overline{R}} \simeq R \circ \overline{R}$ and $(R \circ \overline{R}) \circ (R \circ \overline{R}) \simeq R \circ \overline{R} \circ R \circ \overline{R} \simeq R \circ \overline{R}$. Then $R \circ \overline{R} \underset{\tau}{\overset{\rho}{\rightrightarrows}} A$ is a kernel pair, say, of its own coequalizer ϕ. We note that $\phi \in \underline{E}$ and

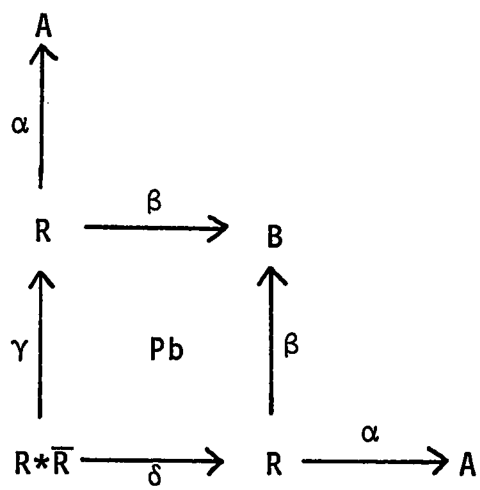
$$\begin{array}{ccc} A & \xrightarrow{\phi} & C \\ \rho \uparrow & & \uparrow \phi \\ R \circ \overline{R} & \xrightarrow[\tau]{} & A \end{array}$$

is a pullback.

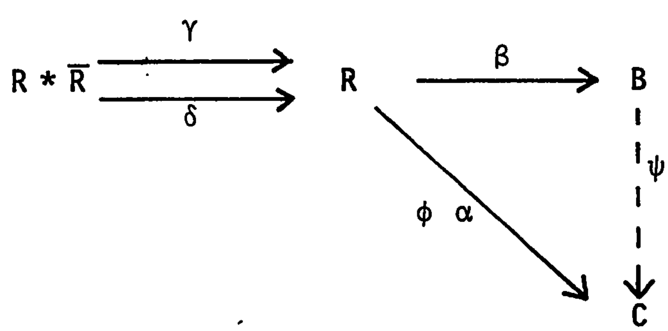
Therefore we can write $R \circ \overline{R} \simeq M \circ \overline{M}$, where $M = (A, 1, \phi)$ is mono in Rel $\underline{A}$. We obtain that $R \circ \overline{R} \circ R \simeq M \circ \overline{M} \circ R$.

We recall the relation $(R \circ \overline{R}, \rho, \tau)$ which is obtained by $(\underline{E},\underline{M})$ factorization of the morphism

$R * \overline{R} \xrightarrow{\{\alpha\gamma, \alpha\delta\}} A \times A$ in the following diagram, i.e. $\{\alpha\gamma, \alpha\delta\} = \{\rho, \tau\}\varepsilon$, where $\{\rho, \tau\} \in \underline{M}$ and $\varepsilon \in \underline{E}$.

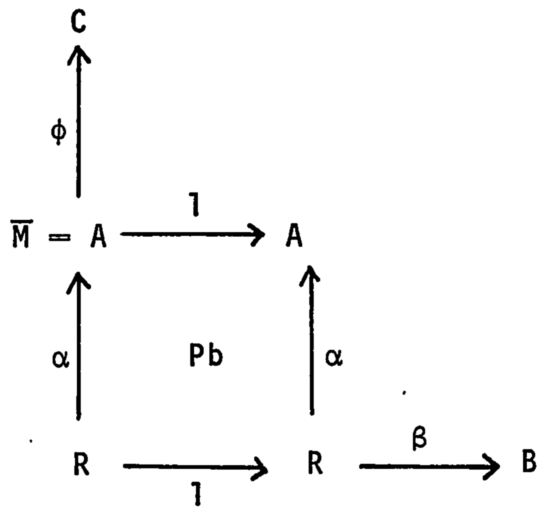


Since β is a regular epi $(\in \underline{E})$, it is a coequalizer of its kernel pair, (γ, δ). But $\phi\alpha\gamma = \phi\rho\varepsilon = \phi\tau\varepsilon = \phi\alpha\delta$.



Hence there exists a unique morphism ψ such that $\psi\beta = \phi\alpha$.

We now return to $R \circ \overline{R} \circ R \simeq M \circ \overline{M} \circ R$ which we obtained earlier, and note that $\overline{M} * R$ is $(R, \phi\,\alpha, \beta)$ as follows:



$\overline{M} \circ R$ is thus obtained by the ($\underline{E}$,$\underline{M}$) factorization of $\{\phi\,\alpha, \beta\}$, which is: $\{\phi\,\alpha, \beta\} = \{\psi\,\beta, \beta\} = \{\psi, 1\}\,\beta$, where $\{\psi, 1\} \in \underline{M}$ and $\beta \in \underline{E}$. We denote $\overline{M} \circ R$ by E and therefore, obtain the relation $(E, \psi, 1)$. Thus, $R \simeq R \circ \overline{R} \circ R \simeq M \circ \overline{M} \circ R$ can be expressed as $(M, 1, \phi) \circ (E, \psi, 1)$. ∎

As a consequence of lemma (2.21) and the above proposition, we obtain the converse of theorem (2.5) for $\underline{E}$-relations.

THEOREM 2.23. Suppose that (R,α,β) is a difunctional $\underline{E}$-relation from B to A. Assume that every equivalence relation is a kernel pair and $\underline{E}$ = the class of all regular epis. Then R is a pullback relation. ▮

Proposition (2.16) gives the necessary and sufficient condition for any difunctional relation to be a pullback relation.

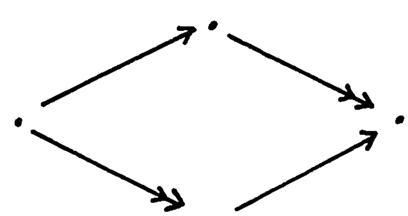
Chapter 3

APPLICATION TO EXACT CATEGORIES

We shall discuss relations in an exact category in the sense of Barr [1], since it includes a wide scope of examples and our assumptions in the previous chapters are satisfied by this category. Barr has given the definition and examples of exact categories in his paper. Here we shall provide the definition for the sake of completeness.

DEFINITION 3.1. Let $\underline{A}$ be a category. We say that $\underline{A}$ is <u>regular</u> if it satisfies (EX 1) below and <u>exact</u> if it satisfies (EX 2) in addition.

(EX 1) The kernel pair of every morphism exists and has a coequalizer; moreover, every pair $\cdot \twoheadrightarrow \cdot \longleftarrow \cdot$ has a pullback of the form



(EX 2) Every equivalence relation is a kernel pair.

Throughout this chapter, A denotes a regular category with finite products, $\rightarrowtail$ a mono and $\twoheadrightarrow$ a regular epi. Barr established that every morphism has a factorization of the form $\cdot \twoheadrightarrow \cdot \rightarrowtail \cdot$. This factorization satisfies our definition (0.1) and is therefore an (E,M) factorization where M = the class of all monos and E = the class of all regular epis.

Since A satisfies the assumption in theorem (1.6), we thus obtain:

PROPOSITION 3.2. Rel A is a bicategory. ∎

We can sum up the results obtained in Chapter 2 as follows:

THEOREM 3.3. Let A be an exact category with finite products and pushouts. Suppose that $R = (R,\alpha,\beta)$ is a relation and $R' = (R,\alpha_e,\beta_e)$ is the canonical E-relation of R. Then the following are equivalent:

(1) R' is a pullback relation,

(2) R is difunctional,

(3) R' is difunctional,

(4) $R = M \circ E$, where $M = (M, \alpha_m, \phi)$ and $E = (E, \psi, \beta_m)$ with $\phi, \psi \in \underline{E}$,

(5) $R' = M' \circ E'$. ∎

Another equivalent statement is

(6) R is a pullback relation,

provided that every pair $\cdot \longleftarrow\!\!\!< \cdot \longrightarrow \cdot$ in $\underline{A}$ is a pullback. This is the case for categories of sets, M-sets but not for the category of groups. The following lemma gives the proof for the category of $\mathbf{S}^M$.

LEMMA 3.4. Let M be a monoid. Then in $\mathbf{S}^M$, $A \xleftarrow{m} B \xrightarrow{f} C$ is a pullback of some pair of maps.

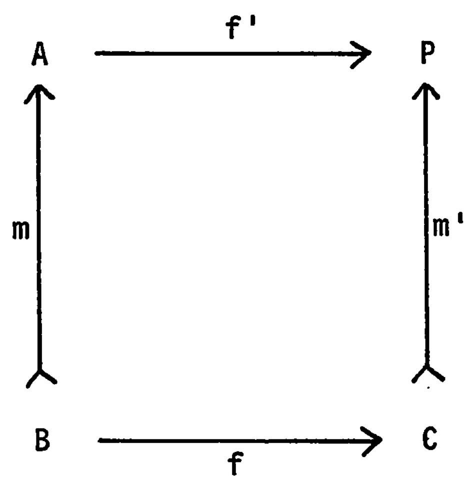
Proof: The pushout of $A \xleftarrow{m} B \xrightarrow{f} C$ is $P = A \dot{\cup} C / \equiv$, where $\equiv$ is the equivalence relation on P such that for $a, a' \in A$ and $c, c' \in C$,

(1) $a \equiv a' \Longleftrightarrow f(a) = f(a')$

(2) $a \equiv c \Longleftrightarrow f(a) = c$

(3) $c \equiv c' \Longleftrightarrow c = c$

Let $m' : C \longrightarrow P$ be defined by $m'(c) = [c]$ modulo $\equiv$. Hence $m' : C \longrightarrow P$ is mono. Let $f' : A \longrightarrow P$ be defined by $f'(a) = [a]$.



Now the pullback of $A \xrightarrow{f'} P \xleftarrow{m'} C$ is

$$\{(a,c) \mid f'(a) = m'(c)\} = \{(a,c) \mid [a] = [c]\}$$

$$= \{(a,c) \mid f(a) = c\} \simeq B \qquad \blacksquare$$

The following theorem generalises some results known for algebraic categories.

THEOREM 3.5. In an exact category, the following are equivalent:

(1) Every relation is difunctional,

(2) Every reflexive relation is an equivalence relation,

(3) The composite of two equivalence relations is an equivalence relation.

(4) $E \circ F \simeq F \circ E$, where E and F are equivalence relations.

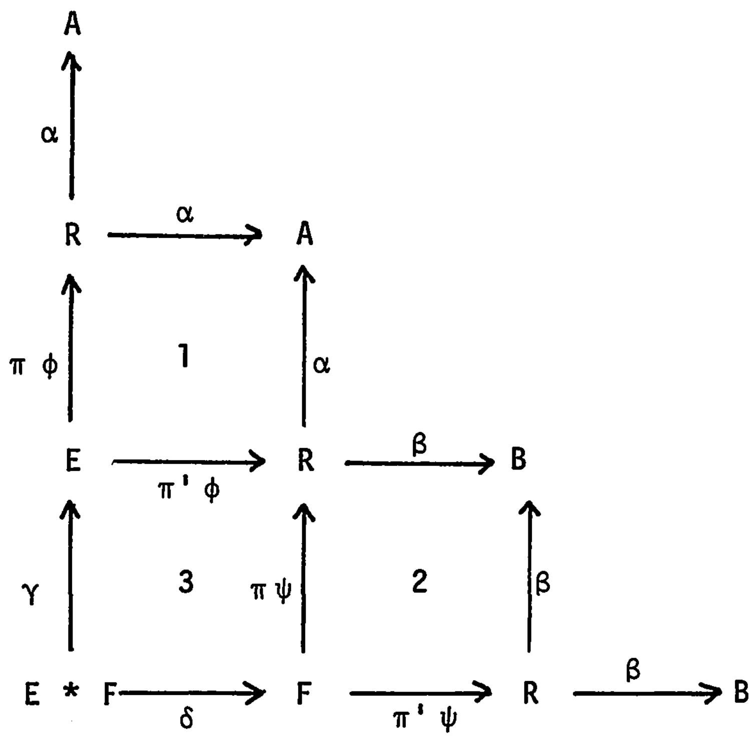
Proof: (1) $\Longrightarrow$ (2) Let (R,α,β) be a reflexive relation on A. Then $I_A \subseteq R$. Let $\gamma : I_A \longrightarrow R$. Then $\alpha\gamma = 1 = \beta\gamma$. Hence α,β are both regular epis and $\in \underline{E}$. R is therefore an $\underline{E}$-relation. Since $R \simeq R \circ \overline{R} \circ R$, by theorem (3.3) R is a pullback relation. Let $A \xrightarrow{\phi} X \xleftarrow{\phi'} B$ be the pushout of $A \xleftarrow{\alpha} R \xrightarrow{\beta} B$. Then we obtain a bicartesian square $\phi \alpha = \phi' \beta$. By multiplying γ on the right, we obtain $\phi = \phi'$, because $\alpha \gamma = 1 = \beta \gamma$. Hence (α,β) is the kernel pair of ϕ and (R,α,β) is an equivalence relation.

(2) $\Longrightarrow$ (3) Let E and F be equivalence relations on A. Then $I_A \subseteq E$ and $I_B \subseteq F$. Hence $I_A \subseteq I_A \circ I_A \subseteq E \circ F$. By (2) $E \circ F$ is an equivalence relation.

(3) $\Longrightarrow$ (4) Let E and F be equivalence relations on A. Then $E \subseteq E$, $I \subseteq F$ and hence $E \simeq E \circ I \subseteq E \circ F$. Similarly $F \subseteq E \circ F$. We claim that $E \circ F$ is the smallest equivalence relation containing E and F. Suppose G is any equivalence relation containing E and F. Then $G \simeq G \circ G \supseteq E \circ F$. We define the smallest equivalence relation containing E and F to be $E \vee F$. Here $E \circ F \simeq E \vee F$. Similarly, $F \circ E \simeq E \vee F$. Thus we obtain $E \circ F \simeq F \circ E$.

(4) $\Longrightarrow$ (1) Let (R,α,β) be a relation from B to A and π , π' be the projection morphisms from $R \times R$ to R. We define (E,ϕ) and (F,ψ) to be the equalizers of $(\alpha \pi , \alpha \pi')$ and $(\beta \pi , \beta \pi')$, respectively. Then $(\pi \phi , \pi' \phi)$ and $(\pi \psi , \pi' \psi)$ are kernel pairs of α and β, respectively.

Hence $(E, \pi\phi, \pi'\phi)$ and $(F, \pi\psi, \pi'\psi)$ are equivalence relations on R. Therefore $E \circ F \simeq F \circ E$. We observe that $F * E \simeq R * \overline{R} * R$ (see the construction of $R * \overline{R} * R$ in lemma (2.4)). It follows that $F \circ E \simeq R \circ \overline{R} \circ R$. We only need to show that $E \circ F \simeq R$.



By the pullback (1) and (2), there exist unique $\sigma : R \longrightarrow E$ and $\rho : R \longrightarrow F$ such that $\pi\phi\sigma = 1 = \pi'\phi\sigma$ and $\pi\psi\rho = 1 = \pi'\psi\rho$. Then by the pullback (3), there exists a unique $\tau : R \longrightarrow E * F$ such that $\gamma\tau = \sigma$ and

$\delta\ \tau = \rho$. Hence $\pi'\ \phi\ \gamma\ \tau = \pi'\ \phi\ \sigma = 1$ and $\pi\ \psi\ \delta\ \tau = 1$. Let $\omega = \pi'\ \phi\ \gamma = \pi\ \psi\ \delta$. Then $\omega\ \tau = 1$. Hence ω is a regular epi. We note that $\alpha\ \omega = \alpha\ \pi\ \phi\ \gamma$ and $\beta\ \omega = \beta\ \pi'\ \psi\ \delta$. Thus ω is a map from $(E * F, \alpha\ \pi\ \phi\ \gamma, \beta\ \pi'\ \psi\ \delta)$ to (R,α,β) . Therefore $E \circ F \simeq R$. We have thus proved that R is difunctional. ∎

From proposition (2.2), we obtain the following proposition.

PROPOSITION 3.6. Let (R,α,β) be a relation in an exact category. Then the following statements are equivalent:

1. $R \circ \overline{R} \subseteq I_A$ and $\overline{R} \circ R \geq I_B$,

2. β is an isomorphism ,

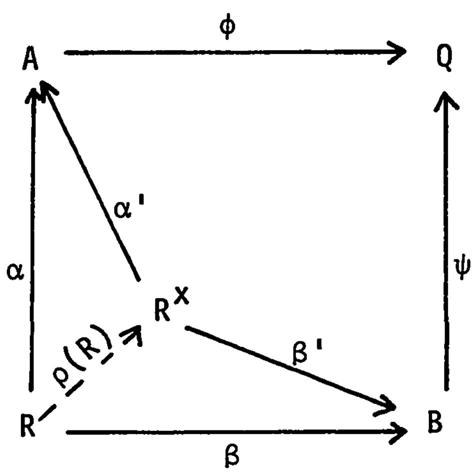
3. $(R,\alpha,\beta) \simeq (B,\gamma,1)$. ∎

Thus (up to isomorphism of morphisms) we can recapture the original category <u>A</u> from Rel <u>A</u>.

Chapter 4

ON THE BICATEGORY OF PULLBACK SPANS

To every span (R,α,β) from B to A, we may associate a pullback span (R^X, α', β') from B to A and a map $\rho(R) : (R,\alpha,\beta) \longrightarrow (R^X, \alpha', \beta')$ as follows:



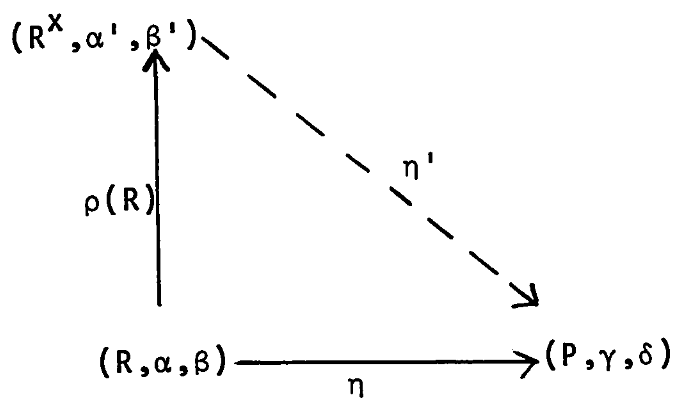
where $A \xrightarrow{\phi} Q \xleftarrow{\psi} B$ is the pushout of $A \xleftarrow{\alpha} R \xrightarrow{\beta} B$ and $A \xleftarrow{\alpha'} R^X \xrightarrow{\beta'} B$ is the pullback of $A \xrightarrow{\phi} Q \xleftarrow{\psi} B$. Hence there is a unique $\rho(R) : R \longrightarrow R^X$ such that $\alpha' \rho(R) = \alpha$ and $\beta' \rho(R) = \beta$.

Let Pull (A,B) denote the category of pullback spans from B to A. We note that it is a preordered set. We will show that the object function ^{X} can be extended to a functor from Span (A,B) to Pull (A,B), in fact the left adjoint of the inclusion functor.

Suppose $\eta : (R, \alpha, \beta) \longrightarrow (P, \gamma, \delta)$ with $(R, \alpha, \beta) \in$ Span (A,B) and $(P, \gamma, \delta) \in$ Pull (A,B), we want to show that there exists a unique map,

$$\eta' : (R^X, \alpha', \beta') \longrightarrow (P, \gamma, \delta)$$

such that $\eta' \rho(R) = \eta$.

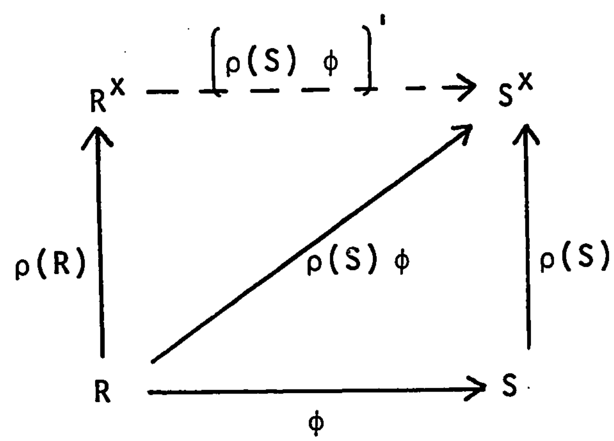


Let $A \xrightarrow{\phi'} Q' \xleftarrow{\psi'} B$ be the pushout of $A \xleftarrow{\gamma} P \xrightarrow{\delta} B$. Then $\phi' \alpha = \phi' \gamma \eta = \psi' \delta \eta = \psi' \beta$. By the pushout $A \xrightarrow{\phi} Q \xleftarrow{\psi} B$ there exists a unique $\kappa : Q \longrightarrow Q'$ such that $\kappa \phi = \phi'$ and $\kappa \psi = \psi'$. It follows that $\phi' \alpha' = \kappa \phi \alpha' = \kappa \psi \beta' = \psi' \beta'$. Hence, by the pullback $A \xleftarrow{\gamma} P \xrightarrow{\delta} B$, there exists a unique $\eta' : R^X \longrightarrow P$ such that $\gamma \eta' = \alpha'$ and $\delta \eta' = \beta'$. Again, by the pullback $A \xleftarrow{\gamma} P \xrightarrow{\delta} B$, this η' is a unique map such that $\eta' \rho(R) = \eta$.

Given any map $\phi : (R, \alpha, \beta) \longrightarrow (S, \tau, \sigma)$ in Span (A,B), we let ϕ^X be the unique map

$$(R^X, \alpha', \beta') \longrightarrow (S^X, \tau', \sigma')$$

such that $\phi^X \rho(R) = \rho(S) \phi$, i.e. $\phi^X = \left[\rho(S)\ \phi\right]'$.



We have thus proved the following proposition. We note that $S^X \simeq (S^+)^X$ since $\varepsilon(S) : S \longrightarrow S^+$ is epi.

PROPOSITION 4.1. Pull (A,B) is a full reflexive subcategory of Span (A,B) (also of Rel (A,B)) with reflector ^{x} . ∎

Then, by proposition (1.4), there is a bicategory Pull $\underline{A}$ whose hom categories are Pull (A,B) with composition $R \otimes S = (R * S)^x$, provided that the map from $(R * S)^x$ to $(R^x * S^x)^x$ is an iso. The following proposition gives the conditions when this map is an iso.

PROPOSITION 4.2. Suppose that $\rho(R) : R \longrightarrow R^x$ is (regular) epi for any span (R,α,β) in $\underline{A}$. Assume that a pullback of a (regular) epi is an epi. Then the map from $(R * S)^x$ to $(R^x * S^x)^x$ is an iso.

Proof: As in the proof of proposition (1.5), $\rho(R) * \rho(S) : R * S \longrightarrow R^x * S^x$ is an epi. Therefore the pushouts of $R * S$ and $R^x * S^x$ are the same. Hence $(R * S)^x$ and $\left(R^x * S^x\right)^x$ are pullbacks of the same pair of maps and thus the map $(R * S)^x \longrightarrow \left(R^x * S^x\right)^x$ is an iso. ∎

COROLLARY 4.3. Under the conditions in proposition (4.2), there is a bicategory Pull $\underline{A}$ whose objects are those of $\underline{A}$ and whose hom categories are the categories Pull (A,B). ∎

PROPOSITION 4.4. Suppose $\underline{E}$ = the class of all (regular) epis. The following are equivalent:

(1) For every span (R,α,β) in $\underline{A}$,
$\rho(R) : R \longrightarrow R^x$ is (regular) epi,

(2) Every relation is a pullback relation.

Proof: (1) $\Longrightarrow$ (2) Let (R,α,β) be a relation in $\underline{A}$. Then, by remark (1.3), $\rho(R) \in \underline{M}$. Therefore $\rho(R)$ is an isomorphism and $R \simeq R^x$ is a pullback relation.

(2) $\Longrightarrow$ (1) Let (R,α,β) be a span from B to A. We factor $\{\alpha,\beta\} : R \longrightarrow A \times B$ into $\{\mu,\nu\} \in \underline{M}$ followed by a (regular) epi ε, i.e.

$$R \xrightarrow{\varepsilon} R^+ \xrightarrow{\{\mu,\nu\}} A \times B$$

where (R^+, μ, ν) is a relation, therefore a pullback relation. The pushouts of $A \xleftarrow{\mu} R^+ \xrightarrow{\nu} B$ and $A \xleftarrow{\alpha} R \xrightarrow{\beta} B$ are the same, because ε is an epi. Hence $R^+ \simeq R^x$ and $\rho(R)$ is then a (regular) epi. ▮

It follows that

$$\text{Pull } \underline{A} = \text{Rel } \underline{A}$$

provided that $\rho(R) : R \longrightarrow R^x \in \underline{E}$ = (regular) epis for every span (R,α,β) in $\underline{A}$ and a pullback of a (regular) epi is an epi. In this case we can apply Hilton's treatment [3] of category of correspondences (corelations).

We shall now compare (1) in proposition (4.4) and (2) in proposition (2.16) which is the condition for any difunctional relation to be a pullback relation.

PROPOSITION 4.5. In the following, (1) implies (2).

(1) $\rho(R) : R \longrightarrow R^x$ is a regular epi for any span (R,α,β) in $\underline{A}$.

(2) Any pair of morphisms $A \xleftarrow{\mu} R \xrightarrow{\beta} B$ in A with μ mono is a pullback of some pair of morphisms.

Proof: Since μ is mono, $\rho(R) : R \longrightarrow R^X$ is a mono. Hence it is an iso and R is a pullback relation. ∎

But the converse of proposition (4.5) is not true in general. We recall lemma (3.4) in which (2) holds in the category of sets but the following example shows that (1) does not hold in the category of sets.

EXAMPLE. Consider $R = \{1,2,3\}$ and a span,

$$\{1, x\} \xleftarrow{f} \{1,2,3\} \xrightarrow{g} \{y, 3\}$$

such that f (1) = 1, f (2) = x, f(3) = x, g (1) = y, g (2) = y and g (3) = 3.

Then $R^X = \{(1,y), (1,3), (x,y), (x,3)\}$ and $\rho(R)$ is not an epi.

Chapter 5

PULLBACKS IN REGULAR CATEGORIES

Given a pair of morphisms

$$A \xrightarrow{g} B \xrightarrow{f} C$$

in a regular category [1], we would like to know whether they form part of a pullback diagram as follows:†

$$\begin{array}{ccc} A & \xrightarrow{f'} & D \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ B & \xrightarrow[f]{} & C \end{array}$$

Throughout this chapter, we use $\rightarrowtail$ to denote a mono and $\twoheadrightarrow$ to denote a regular epi. It will be useful to make the following definition.

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†I am endebted to Basil Rattray for mentioning the solution of this problem for the category of sets.

DEFINITION 5.1. Given three morphisms as follows:

$$A \xrightarrow{g} B \underset{v}{\overset{u}{\leftleftarrows}} K ,$$

we say that they have a <u>common pullback</u>

$$A \underset{v'}{\overset{u'}{\leftleftarrows}} P \xrightarrow{h} K$$

provided both

$$\begin{array}{ccc} P & \xrightarrow{u'} & A \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ K & \xrightarrow[u]{} & B \end{array} \qquad \text{and} \qquad \begin{array}{ccc} P & \xrightarrow{v'} & A \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ K & \xrightarrow[v]{} & B \end{array}$$

are pullback squares.

THEOREM 5.2. Let <u>A</u> be a regular category. A pair of morphisms $A \xrightarrow{g} B \xrightarrow{f} C$ is part of a pullback if and only if the morphisms $A \xrightarrow{g} B \underset{v}{\overset{u}{\leftleftarrows}} K$ with $K \underset{v}{\overset{u}{\rightrightarrows}} B$ being the kernel pair of $B \xrightarrow{f} C$, have a common pullback $A \underset{v'}{\overset{u'}{\leftleftarrows}} P \xrightarrow{h} K$ such that (u', v') is a kernel pair.

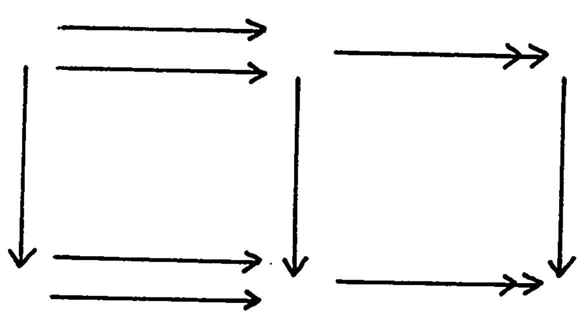
We shall use the following properties of a regular category [1]:

1. Every morphism has a kernel pair.

2. Every pair of morphisms has a coequalizer.

3. Every morphism can be factored into $\cdot \twoheadrightarrow \cdot \rightarrowtail \cdot$.

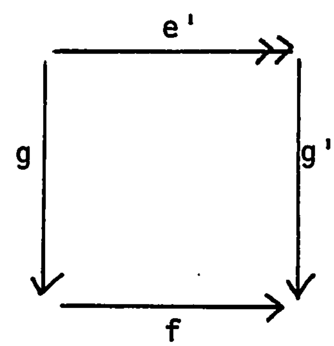
4. In the commutative diagram



let top and bottom rows be exact (that is, at the same time a kernel pair and a coequalizer). Then, if one of the left squares is a pullback, so is the right square.

Proof: (i) ($\Longrightarrow$) See lemma (2.14).

(ii) Let $A \underset{v'}{\overset{u'}{\leftleftarrows}} P \xrightarrow{h} K$ be the common pullback of $A \xrightarrow{g} B \underset{v}{\overset{u}{\leftleftarrows}} K$ and assume that (u', v') is a kernel pair. Let e' be the coequalizer of (u', v'). Since $f\,g\,u' = f\,u\,h = f\,v\,h = f\,g\,v'$, there exists a unique g' such that $g'\,e' = fg$. We claim that



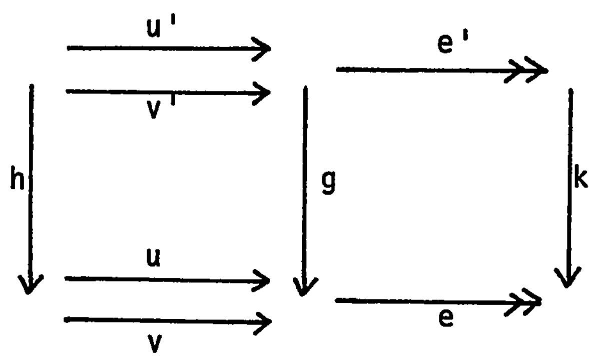
is a pullback.

Let $f = m\, e$ be the factorization of f such that m is a mono and e is a regular epi. Hence e is a coequalizer of (u, v). Since $e\, g\, u' = e\, g\, v'$, there exists a unique morphism k such that $k\, e' = e\, g$. We obtain

$$m\, k\, e' = m\, e\, g = f\, g = g'\, e' ,$$

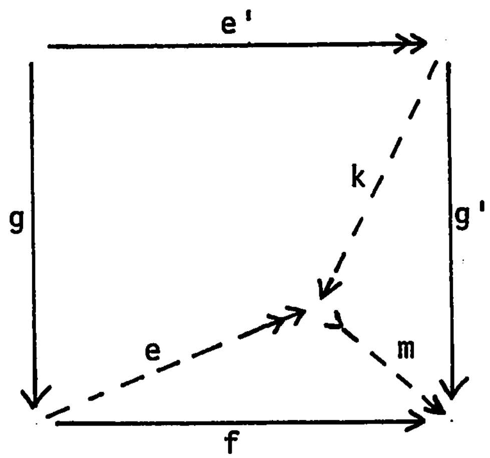
and therefore $m\, k = g'$.

We have exact top and bottom rows in the following diagram:



Also the left squares are pullbacks. Hence by property 4 of a regular category, the right square is also a pullback.

It follows that



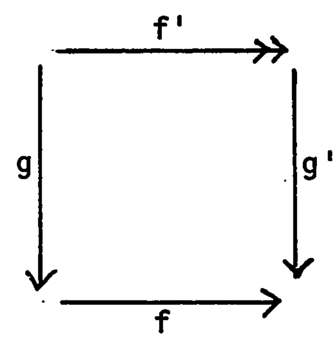
is a pullback, since $g' = m\,k$, $f = m\,e$ and m is mono.

The proof is now complete. ∎

COROLLARY 5.3. A pair of morphisms $A \xrightarrow{g} B \rightarrowtail^{f} C$ with f mono forms part of a pullback.

Proof: The kernel pair of f is $(1, 1)$ and hence the result follows. ∎

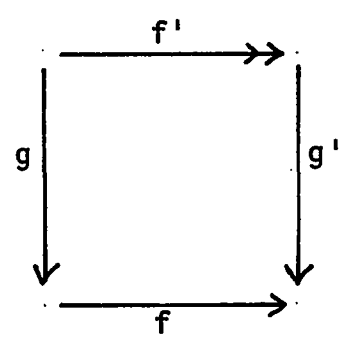
The proof of theorem (5.2) provides the necessary and sufficient condition for a commutative square



with f' regular epi to be a pullback square.

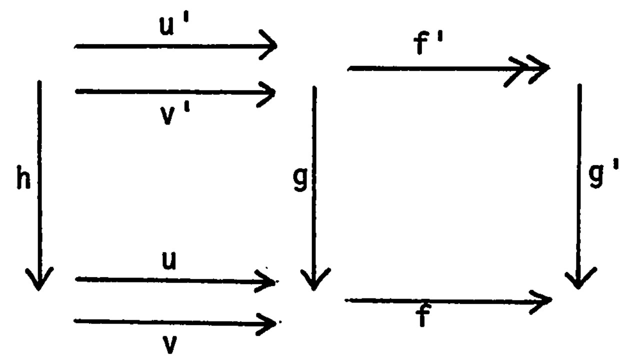
PROPOSITION 5.4. Let

(5.5)



be a commutative square with f' regular epi. Let (u , v) and (u' , v') be kernel pairs of f and f', respectively.

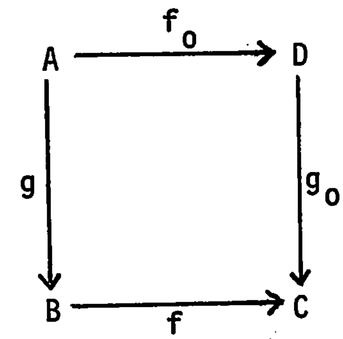
(5.6)



Then (5.5) is a pullback if and only if one of the left squares in (5.6) is a pullback. ▮

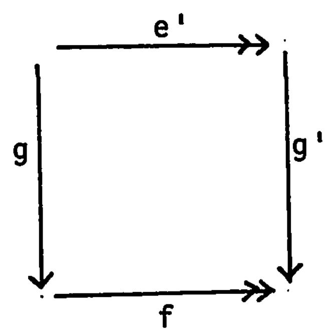
It is of interest to know whether the pullback constructed in theorem (5.2) is essentially unique. This is the case whenever f is a regular epi.

PROPOSITION 5.7. Let $\underline{A}$ be a regular category. Given a pair of morphisms $A \xrightarrow{g} B \xrightarrow{f} C$ in $\underline{A}$, there is an essentially unique pair of morphisms $A \xrightarrow{f_0} D \xrightarrow{g_0} C$ such that

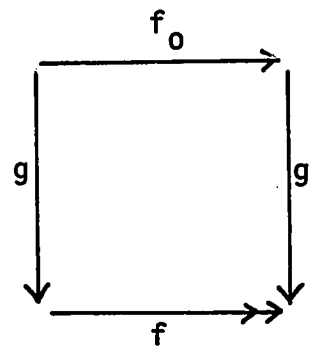


is a pullback, provided that f is a regular epi.

Proof: From theorem (5.2), we constructed the pullback square



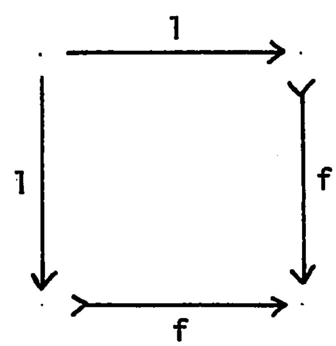
where e' is the coequalizer of its kernel pair (u' , v'). We also showed that if



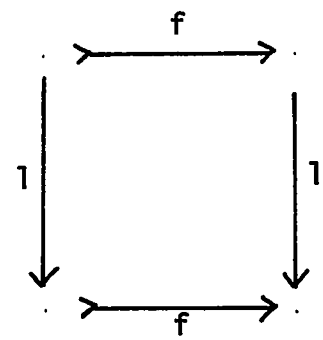
is a pullback, then (u', v') is also a kernel pair of f_o. Since, in a regular category, a pullback of a regular epi is a regular epi, f_o is a regular epi. Then f_o is a coequalizer of its own kernel pair (u', v'). Now both e' and f_o are coequalizers of (u', v'). Hence there exists an iso i such that $i\,e' = f_o$. Since $g'\,e' = f\,g = g_o\,f_o$, we obtain $g'\,e' = g_o\,i\,e'$, and hence $g' = g_o\,i$. We have thus proved proposition (5.7). ∎

The following example shows that we cannot drop the condition that f is a regular epi.

EXAMPLE. Let f be a mono and take $g = 1$. The following squares give two different pullbacks which have $\xrightarrow{1} \overset{f}{\rightarrowtail}$ as part of the squares:

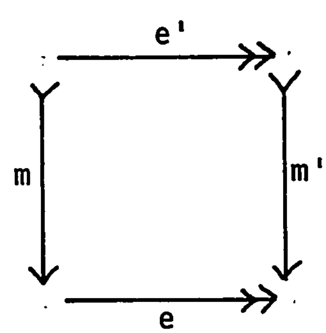


and



By lemma (2.15), we have the following.

COROLLARY 5.8. Let $\overset{m}{\rightarrowtail} \overset{e}{\twoheadrightarrow}$ be part of a pullback. Then the counter part of the pullback is $\overset{e'}{\twoheadrightarrow} \overset{m'}{\rightarrowtail}$, the factorization of $e\,m$, i.e., the pullback is

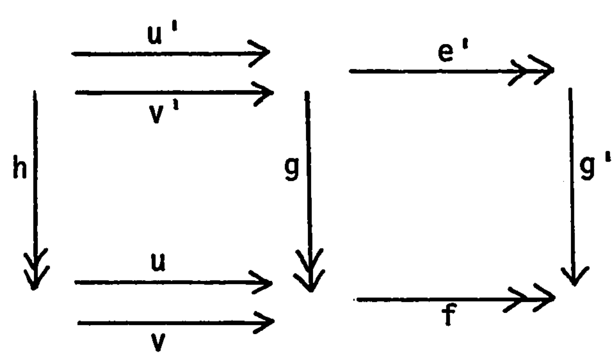


∎

We remark that, in abelian categories, if $\longrightarrow \twoheadrightarrow$ is part of a pullback, then the canonical pullback constructed in theorem (5.2) is also a pushout. However, in regular categories, we have the following.

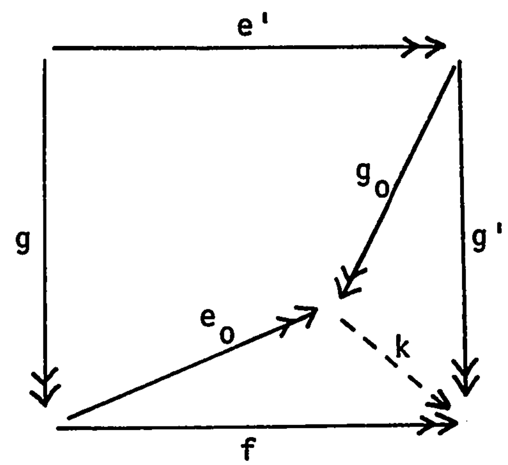
PROPOSITION 5.9. Let $\xrightarrow{g}\!\!\!\!\!\twoheadrightarrow \xrightarrow{f}\!\!\!\!\!\twoheadrightarrow$ be part of a pullback. Then the pullback is also a pushout.

Proof: By proposition (5.7), there is an essentially unique pullback and by theorem (5.2) we have the following pullback squares:



where (u, v) and (u', v') are kernel pairs of f and e', respectively. Since pullbacks preserve regular epis, h is a regular epi.

Let $\xrightarrow{g_0}\!\!\!\!\to \;\; \twoheadleftarrow\!\!\!\xleftarrow{e_0}$ be a pushout of $\xleftarrow{e'}\!\!\!\!\leftarrow \;\; \xrightarrow{g}\!\!\!\!\to$.
Then there exists a unique k such that $k\, g_0 = g'$ and $k\, e_0 = f$.



Then, k is a regular epi and we have

$$e_0\, u\, h = e_0\, g\, u' = g_0\, e'\, u' = g_0\, e'\, v'$$

$$= e_0\, g\, v' = e_0\, v\, h \; .$$

Since h is a regular epi, $e_0\, u = e_0\, v$. Hence k is a mono and it follows that k is an iso. ∎

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