Quasi-exact Solvability and Turbiner's Conjecture in Three Dimensions

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Abstract

The results exhibited in this thesis are related to Schrödinger operators in three dimensions and are subdivided in two parts based on two published papers, [15] and [14]. A variant of Turbiner's conjecture is proved in the first paper while a partial classification of quasi-exactly solvable Lie algebras of first order differential operators in dimension three is exhibited in the second paper. This classification is then used to construct new quasi-exactly solvable Schrödinger operators in three dimensions.

Turbiner's conjecture posits that, for a Lie algebraic Schrödinger operator in dimension two, the Schrödinger equation is separable if the underlying metric is locally flat. This conjecture is false in general. However, if the generating Lie algebra is imprimitive and if a certain compactness requirement holds, Rob Milson proved that in two dimensions, the Schrödinger equation separates in a Cartesian or polar coordinate system. In [15], the first paper included in this thesis, a similar theorem is proved in three variables. The imprimitivity and compactness hypotheses are still necessary and another condition, related to the underlying metric, must be imposed. In three dimensions, the separation is only partial and the separation will occur in either a spherical, cylindrical or Cartesian coordinate system.

In the second paper [14], a partial classification of quasi-exactly solvable Lie algebras of first order differential operators is performed in three dimensions. Such a classification was known in one and two dimensions but the three dimensional case was still open before the beginning of this research. These new quasi-exactly solvable Lie algebras are used to construct new quasi-exactly solvable Schrödinger operators with the property that part of their spectrum can be explicitly determined. This classification is based on a classification of Lie algebras of vector fields in three variables due to Lie and Amaldi.

Résumé

Les travaux présentés dans cette thèse portent sur les opérateurs de Schrödinger en dimension trois et se subdivisent en deux parties basées sur deux articles publiés, [15] et [14]. En premier lieu, une variante de la conjecture de Turbiner en dimension trois est démontrée. Dans le second article, une classification partielle des algèbres de Lie quasi-exactement résolubles d'opérateurs différentiels du premier ordre en dimension trois est présentée. Cette classification est par la suite utilisée pour construire de nouveaux opérateurs de Schrödinger quasi-exactement résolubles à trois variables.

La conjecture de Turbiner avance qu'en dimension deux, l'équation de Schrödinger est séparable si l'opérateur est Lie algébrique et si la métrique de la variété est localement plate. Cette conjecture s'avère fausse en général. Cependant, Rob Milson a prouvé que, si l'algèbre de Lie génératrice est imprimitive et si l'on impose un argument de compacité, alors l'équation de Schrödinger en deux dimensions est séparable en coordonnées polaires ou Cartésiennes. Dans le premier article composant cette thèse [15], un théorème similaire est prouvé dans le cas tridimensionnel. Encore une fois les hypothèses d'imprimitivité et de compacité sont nécessaires et une autre condition, liée à la métrique de la variété, doit être imposée. Notons que la séparation est, dans le cas tridimensionnel, seulement partielle et que la séparation se produira en coordonnées sphériques, cylindriques ou Cartésiennes.

Dans le second article [14], une classification partielle des algèbres de Lie quasi-exactement résolubles d'opérateurs différentiels du premier ordre est effectuée en dimension trois. Une telle classification existait en dimensions un et deux mais le cas tridimensionnel demeurait ouvert. Ces nouvelles algèbres permettent de construire de nouveaux opérateurs quasi-exactement résolubles, particulièrement des opérateurs de Schrödinger, dont une partie du spectre peut être déterminée explicitement. Cette classification est basée sur une classification des algèbres de Lie de champs de vecteurs effectuée par Lie et Amaldi.

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Chapter 1

Introduction and Review of the Literature

This thesis is based on two published papers: *Turbiner's Conjecture in Three Dimensions* [15] and *Quasi-exactly Solvable Schrödinger Operators in Three Dimensions* [14]. In the first paper, a modified version of Turbiner's conjecture in three dimensions is proved and a counter-example to the original conjecture is given. The Lie algebraic Schrödinger equations corresponding to flat metrics of a certain restricted type are shown to separate partially in either Cartesian, cylindrical or spherical coordinates. The main contribution of the second paper is to give a partial classification of the quasi-exactly solvable Lie algebras of first order differential operators in three variables, and to show how this can be applied to the construction of new quasi-exactly solvable Schrödinger operators in three variables. We begin by introducing and illustrating the major notions employed in these two papers. To avoid redundancy, some of this material have been deleted from the two papers.

Recall that a Schrödinger operator on a n-dimensional Riemannian manifold

$(\mathbf{M}, g)$ is a second order linear differential operator of the form

$$\mathcal{H}_0 = -\frac{1}{2}\Delta + U,$$

where Δ is the Laplace-Beltrami operator and U is a potential function for the physical system under consideration. A question of fundamental interest in quantum mechanics is to solve the Schrödinger equation $\mathcal{H}_0\psi = E\psi$. In general, this equation can not be solved exactly. However, there exist few potentials for which the equation is exactly solvable, such as the harmonic oscillator and the hydrogen atom, whose point spectrum can be completely determined using algebraic methods. For these examples, the existence of a large group of symmetries and the particular form of the potential are of a significant importance. Furthermore, some systems are known to be only partially solvable in the sense that at least part of the spectrum can be computed exactly by algebraic methods. For instance, the sextic anharmonic oscillator with the Hamiltonian

$$\mathcal{H} = -\frac{\partial^2}{\partial x^2} + \alpha x^2 + \beta x^4 + \gamma x^6,$$

is partially solvable when the parameters α, β and γ satisfy the condition

$$\alpha - \frac{\beta^2}{4\gamma} + \sqrt{\gamma}(4M + 2p + 3) = 0$$

with M an integer and p either 0 or 1. There are a number of quantum mechanical systems, for which the Schrödinger equation is exactly solvable or partially solvable and which don't have evident underlying symmetries, as in the example above. Nevertheless, the notion of Lie group or Lie algebra appears in these examples but in a more subtle way, as we shall see.

One approach to these problems, which is based on the representation theory of Lie algebras, is to consider Schrödinger operators $\mathcal{H}_0$ which are Lie algebraic or quasi-exactly solvable. We start by considering the case of a general linear second-order differential operator $\mathcal{H}$, given in local coordinates by

$$\mathcal{H} = \sum_{i,j=1}^{n} A^{ij}\partial_i\partial_j + \sum_{i=1}^{n} B^i\partial_i + C.$$

The operator $\mathcal{H}$ is said to be *Lie algebraic* if it is an element of the universal enveloping algebra of $\mathfrak{g}$, a finite dimensional Lie algebra of first order differential operators. More explicitly,

$$\mathcal{H} = \sum_{a,b=1}^{m} C_{ab}T^aT^b + \sum_{a=1}^{m} C_aT^a + C_0, \tag{1.1}$$

where

$$T^a = v^a + \eta^a, \qquad 1 \leq a \leq m, \tag{1.2}$$

is a basis of $\mathfrak{g}$ and C_{ab}, C_a and C_0 are constants. In (1.2), the operators $v^1, ..., v^m$ are vector fields, and $\eta^1, ..., \eta^m$ are multiplication operators.

A Lie algebra $\mathfrak{g}$ of first order differential operators is said to be *quasi-exactly solvable* if one can find explicitly a finite dimensional $\mathfrak{g}$-module $\mathcal{N}$ of smooth functions, that is, if there exists $\mathcal{N} = \{h^1, ..., h^r\}$ with $T^a(\mathcal{N}) \subset \mathcal{N}$ for all $1 \leq a \leq m$. A Lie algebraic operator $\mathcal{H}$, is said to be quasi-exactly solvable if it lies in the universal enveloping algebra of a quasi-exactly solvable Lie algebra of first order differential operators. Obviously, $\mathcal{H}(\mathcal{N}) \subset \mathcal{N}$, *i.e.* the module $\mathcal{N}$ will be fixed by the operator $\mathcal{H}$. Moreover, if the functions contained in the module $\mathcal{N}$ are square integrable with respect to the Riemannian measure of the Riemannian manifold, the operator $\mathcal{H}$ is said to be a *normalizable* quasi-exactly solvable operator.

We can see from the above definitions that the formal eigenvalue problem for $\mathcal{H}_0$, a normalizable quasi-exactly solvable Schrödinger operator, can be solved partially by elementary linear algebraic methods. Indeed, the operator $\mathcal{H}_0$ is formally self-adjoint with respect to the inner product associated to the standard measure, so that the restriction of $\mathcal{H}_0$ to the finite dimensional module $\mathcal{N}$ is a Hermitian finite dimensional linear operator, see [1] for details. Thus, one can in principle, compute $r = dim(\mathcal{N})$ eigenvalues of $\mathcal{H}_0$, counting multiplicities, by diagonalizing the $r \times r$ matrix representing $\mathcal{H}_0$ in a basis of $\mathcal{N}$.

Let $g^{(ij)}$ be the contravariant metric of the manifold $\mathbf{M}$ in a local coordinate chart and g its determinant. In that setting, a Schrödinger operator reads locally as

$$\mathcal{H}_0 = -\frac{1}{2}\sum_{i,j=1}^{n}[g^{ij}\partial_{ij} + \partial_i(g^{ij})\partial_j - \frac{g^{ij}\partial_i(g)}{2g}\partial_j] + U.$$

Note that a quasi-exactly solvable second order differential operator is not, in general, a Schrödinger operator. However, this operator might be equivalent to a Schrödinger operator in a way that preserves the formal spectral properties of the operators under consideration. The appropriate notion of equivalence, which will be used throughout our work, is the following. Two differential operators are *locally equivalent* if there is a gauge transformation $\mathcal{H} \rightarrow \mu\mathcal{H}\mu^{-1}$, with gauge factor $\mu = e^{\lambda}$, and a change of variables relating one to the other. In principle, it is possible to verify if a general second order differential operator $\mathcal{H}$ is equivalent to a Schrödinger operator with respect to this notion of equivalence. Indeed, every second order linear differential operator can be written locally as

$$\mathcal{H} = -\frac{1}{2}\sum_{i,j=1}^{n} g^{ij}\partial_{ij} + \sum_{i}^{n} h^i\partial_i + U_0.$$

If the contravariant tensor $g^{(ij)}$ is non-degenerate, that is if g does not vanish, the operator can be expressed as

$$\mathcal{H} = -\frac{1}{2}\Delta + \vec{V} + U_0, \qquad (1.3)$$

where $\vec{V} = b^i\partial_i$ is a vector field. For this operator to be locally equivalent to a Schrödinger operator, the vector field $\vec{V}$ has to be a gradient vector field with respect to the metric $g^{(ij)}$. Locally, this will be the case if and only if $\omega = g_{ij}b^j dx^i$, the one form associated to $\vec{V}$, is closed. For this reason, this condition is named the *closure condition.* Note that if $\vec{V} = \nabla(\lambda)$, the gauge factor is given by $e^{\frac{\lambda}{2}}$. Given an operator of the form (1.1), the closure conditions can be easily verified provided the contravariant metric $g^{(ij)}$ is non-degenerate. Indeed, these conditions can be written as algebraic constraints on the coefficients C_{ab} and C_c, and are the Frobenius compatibility conditions for an overdetermined system that will be described later.

An important point to keep in mind is that the classes of Lie algebraic and quasi-exactly solvable operators are invariant under local equivalence. Indeed, suppose $\mathcal{H}$ is a quasi-exactly solvable operator which is gauge equivalent to another operator $\mathcal{H}_0$ under the rescaling μ. If $\mathcal{H}$ lies in the universal enveloping algebra of $\mathfrak{g}$, whose $\mathfrak{g}$-module is $\mathcal{N}$, then one can easily show that $\mathcal{H}_0$ is quasi-exactly solvable with respect to the finite dimensional Lie algebra

$$\widetilde{\mathfrak{g}} = \mu \cdot \mathfrak{g} \cdot \mu^{-1} = \{\ \mu \cdot T \cdot \mu^{-1} |\ T \in \mathfrak{g}\}$$

which is isomorphic to $\mathfrak{g}$ and posses the finite-dimensional $\widetilde{\mathfrak{g}}$-module

$$\widetilde{\mathcal{N}} = \mu \cdot \mathcal{N} = \{\ \mu \cdot h\ |\ h \in \mathcal{N}\}.$$

Note however that the gauge factor is not necessarily unitary. Thus, a gauge transformation does not necessarily preserve the normalizability property of the functions in $\mathcal{N}$. Therefore, the class of normalizable quasi-exactly solvable operators is not invariant under our notion of local equivalence.

The scheme that we have just outlined originates in the work of several teams of physicists. It appears that Goshen and Lipkin where the first to introduce the concept of a "spectrum generating algebra" in their 1959 paper [26]. Their paper did not elicit much reaction in the physics community at the time. In 1965, two groups of physicists rediscovered independently the spectrum generating algebras. The first group was composed by Barut and Bohm [9] and the second of Dothan, Gell-Mann and Ne'eman [12]. Their work was an impetus for further research in this area as exposed by the conference proceedings [27] and the two volume set of reprints [11]. At the beginning of these two volumes, the review paper [10] of Bohm and Ne'eman, gives a survey of the history and the contribution papers related to the spectrum generating algebras.

Iachello, Levine, Alhassid, Gürsey, Wu and collaborators exhibited the first applications of spectrum generating algebras late 1970s early 1980s. This new concept was used to study models in nuclear physics [7, 8] and was successfully applied to molecular dynamics and spectroscopy in [13, 33]; and to scattering theory in [2, 3, 4, 5]. In these applications, the relevant Hamiltonian is a Lie algebraic operator in the sense described previously. A survey of the theory and applications is given in the book [28].

The concept of quasi-exact solvability appeared in the mid 1980s. To be precise, quasi-exactly solvable classes of operators, as described in this thesis, were introduced by Shifman, Turbiner and Ushveridze in [42, 44, 46, 47]. A survey of the theory and applications of quasi-exactly solvable systems in physics is given in [48]. Independently, Levine [33] posed the problem of classifying the Lie algebraic operators under the equivalence relation defined by smooth changes of independent variables and rescalings of wave functions. This classification problem was then extended to the classification of normalizable quasi-exactly solvable Schrödinger operator.

The general procedure to answer this classification problem is subdivided into four steps. One has first to classify all the classes of Lie algebras of first order differential operators, then one has to determine which of them admit a finite-dimensional module of smooth functions. Then, based on these quasi-exactly solvable Lie algebras, one has to construct second order differential operators and verifies if they are equivalent to Schrödinger operators. Finally one has to check if the eigenfunctions are square integrable in order to determine part of the spectrum of the operator. During the 1990s, a great deal of effort has been put into the classification problem in one and two dimensions, with significant contribution by González-López, Kamran, Olver and Turbiner.

The complete list of one-dimensional quasi-exactly solvable Schrödinger operators is described in a 1988 paper of Turbiner [44], while the solution to Levine's problem is given by Kamran and Olver in [31]. In 1993, González-López, Kamran and Olver gave a complete solution for the normalizability problem for these operators [22]. In spite of some progress, the higher-dimensional case is still open at the moment. A complete list of quasi-exactly solvable Lie algebras of first order differential operators in two complex variables was established in 1991 [20, 21], and the real case was completed in 1996 [25]. The starting point was Lie's complete classification of finite dimensional Lie algebras of vector fields in two complex variables [34]. The two last steps remain to be done in two dimensions. However several new families of normalizable quasi-exactly solvable Schrödinger operators has been exhibited, see for instance [23], [24] and [25].

No attempt had been made to address the three dimensional version of Levine's problem until the work presented in the second paper [14]. Once again, the starting point was a classification of finite dimensional Lie algebras of vector fields in three variables. Part of this classification was performed in 1893 by Lie [34], where all the classes of primitives Lie algebras were determined together with all the classes of two of the three types of imprimitive Lie algebras. The remaining class of imprimitive Lie algebras was worked out by Amaldi in 1901, [6].

Related to the question of quasi-exact solvability, there is recent interest for operators that preserve finite dimensional invariant subspaces and that are not necessarily Lie algebraic [17] or even linear [30]. For instance, if the invariant subspace is generated by polynomials in one or several variables, Gómez-Ullate, Kamran and Milson proposed an explicit basis for the space of such differential operators using the concept of deficiency [18].

In the scope of the exact solvability, the first paper contained in this thesis is related to Turbiner's conjecture. This conjecture was first formulated in 1994 by Turbiner in [45] and states the following. If $\mathcal{H}_0$ is a Lie algebraic Schrödinger operator defined on a locally flat 2-dimensional manifold, then the spectral equation $\mathcal{H}_0\psi = E\psi$ can be solved by a separation of variables. A few years later, in [36, 38], Rob Milson showed that the conjecture is false in general and proved a modified version provided that additional assumptions hold. The question to know if this conjecture holds in three dimensions was still open before the beginning of the research presented in the first paper [15] on which this thesis is based.

We conclude this introduction with two simple examples that will illustrate these concepts. Each of them are related to the two papers on which this thesis is founded. Based on a quasi-exactly solvable Lie algebra of first order differential operators in three variables exhibited in the second paper, the first example displays the construction of a normalizable quasi-exactly solvable Schrödinger operator in three variables.

Example 1.1. *We consider a representation of* $\mathfrak{g} \cong \mathfrak{sl}(2) \times \mathfrak{sl}(2) \times \mathfrak{sl}(2)$ *as a quasi-exactly solvable Lie algebra of first order differential operators. With the standard* *notation* $p = \frac{\partial}{\partial x}$, $q = \frac{\partial}{\partial y}$ *and* $r = \frac{\partial}{\partial z}$, *this Lie algebra representation is given by the following first order differential operators*

$$T^1 = p,\ T^2 = xp,\ T^3 = x^2p - x,\ T^4 = q,\ T^5 = yq,\ T^6 = y^2q - y,$$

$$T^7 = r,\ T^8 = zr,\ T^9 = z^2r - z.$$

Then, the finite dimensional module of smooth functions

$$\mathcal{N}_{111} := \{x^i y^j z^k |\ 0 \leq i \leq 1,\ 0 \leq j \leq 1,\ 0 \leq k \leq 1\}$$

is a $\mathfrak{g}$*-module. With the following choice of coefficients, one constructs a quasi-exactly solvable operator*

$$\begin{aligned} -2\mathcal{H} &= (T^1)^2 + (T^2)^2 + 2[(T^3)^2 + (T^4)^2 + 2(T^5)^2 + 2(T^6)^2] + (T^7)^2 + (T^8)^2 \\ &\quad +2(T^9)^2 + \{T^1, T^3\} + \{T^7, T^9\} - 2T^4 - 2T^5 - 4T^6 - 8, \end{aligned}$$

where $\{T^a, T^b\} = T^a(T^b) + T^b(T^a)$. *The induced contravariant metric associated to this operator is computed to be the following positive definite matrix*

$$g^{(ij)} = \begin{pmatrix} (x^2+1)^2 & 0 & (x^2+1)(z^2+1) \\ 0 & y^4 + 4y^2 + 1 & 0 \\ (x^2-1)(z^2+1) & 0 & 2(z^2+1)^2 \end{pmatrix}, \tag{1.4}$$

whose determinant is $g = (x^2+1)^2(y^4+4y^2+1)(z^2+1)^2$. *Then, with respect to this non-degenerate metric, the operator* $\mathcal{H}$ *can also be written as*

$$-2\mathcal{H} = \Delta + \vec{V} + U_0,$$

where $\vec{V} = -2(x^3 + x + z + zx^2)p - 2(2y + y^3)q - 2(2z^3 + 2z + x + xz^2)r$. *It is not hard to verify, always with respect to the metric (1.4), that the first order term* $\vec{V}$ *is the gradient of the function* $\lambda = -\ln(x^2+1) - 1/2\ln(y^4+4y^2+1) - \ln(z^2+1)$. *Hence, by considering the gauge factor*

$$\mu = e^{\frac{\lambda}{2}} = (x^2+1)^{-1/2}(y^4+4y^2+1)^{-1/4}(z^2+1)^{-1/2},$$

the operator $\mathcal{H}$ is gauge equivalent to a Schrödinger operator

$$-2\mathcal{H}_0 = \Delta + U,$$

were the potential is a rational function of y. Furthermore, it is not hard to show that, after the gauge transformation, the functions in $\widetilde{\mathcal{N}}_{111} = \{\ \mu \cdot x^i y^j z^k \mid 0 \leq i,j,k \leq 1\}$ are square integrable with respect to $\sqrt{g^{-1}}dxdydz$. Recall here that g^{-1} is the determinant of the covariant metric and we have the following equality $\sqrt{g^{-1}} = \mu^2$. Thus, for i,j,k either 0 or 1, one can use Fubini's theorem to decompose the integral

$$\begin{aligned}&\int\int\int_{\mathbb{R}^3} (\mu x^i y^j z^k)^2 \mu^2 dxdydz \\ = &\int\int\int_{\mathbb{R}^3} \frac{x^{2i} y^{2j} z^{2k}}{(x^2+1)^2(y^4+4y^2+1)(z^2+1)^2} dxdydz,\end{aligned}$$

into the product of three finite integrals in one variable. Consequently the operator $\mathcal{H}_0$ is a normalizable quasi-exactly solvable Schrödinger operator and it is possible to compute eight eigenfunctions by diagonalizing the matrix obtained by restricting $\mathcal{H}$ to $\mathcal{N}$. For this operator, one gets two eigenvalues, -3 and 1, both of multiplicity four. The eight eigenfunctions associated to these two eigenvalues are respectively,

$$\psi_{-3,1} = -1 + xz, \quad \psi_{-3,2} = y - xyz, \quad \psi_{-3,3} = xy + yz, \quad \psi_{-3,4} = x + z,$$

$$\psi_{1,1} = y + xyz, \quad \psi_{1,2} = -x + z, \quad \psi_{1,3} = -xy + yz, \quad \psi_{1,4} = 1 + xz.$$

Finally, one gets eight eigenfunctions of the Schrödinger operator $\mathcal{H}_0$ by scaling each of these functions by the gauge factor μ.

The second example illustrates the separation theorem proved in the first paper. It exhibits a flat Lie algebraic Schrödinger operator whose generating Lie algebra is 2-imprimitive and which separates in three different coordinates systems. We mention here that 2-imprimitive Lie algebras admit an invariant foliation by surfaces. If the level sets of the function λ are the leaves of the foliation, the operator T^a is said to acts 2-imprimitively if $T^a(\lambda)$ and λ are functionally dependent. A detailed definition will be given prior to the proof of the theorem.

Example 1.2. *Consider* $\mathbf{M} \cong \mathbb{R}^3$ *and the Lie algebra* $\mathfrak{g} = \mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_1$ *spanned by the first order differential operators*

$$T^1 = \partial_u, \quad T^2 = u\partial_u, \quad T^3 = \partial_v, \quad T^4 = v\partial_v, \quad T^5 = \partial_w, \quad T^6 = w\partial_w.$$

We define a Lie algebraic operator $\mathcal{H}$ *with the following choice of coefficients:*

$$C_{ab} = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad C_a = \begin{pmatrix} 0 \\ 2\alpha \\ 4\beta - 4 \\ 4\alpha \\ 4\beta + 4\gamma - 6 \\ 4\alpha \end{pmatrix},$$

where α, β *and* γ *are real numbers. In terms of the* (u, v, w) *coordinates, the operator reads as follows*

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}\partial_{uu} + 2u\partial_{uv} + 2u\partial_{uw} + 2v\partial_{vv} + 4v\partial_{vw} + 2w\partial_{ww} \\ &\quad + 2\alpha u\partial_u + (4\beta - 3 + 4\alpha v)\partial_v + (4\beta + 4\gamma - 5 + 4\alpha w)\partial_w. \end{aligned}$$

The metric associated to this operator is,

$$g^{(ij)} = -\begin{pmatrix} 1 & 2u & 2u \\ 2u & 4v & 4v \\ 2u & 4v & 4w \end{pmatrix}. \tag{1.5}$$

If we forget the degeneracy issue for a moment, we can rewrite the operator $\mathcal{H}$ *in terms of the Laplace-Beltrami operator associated to the metric (1.5). We obtain*

$$\mathcal{H} = -\frac{1}{2}\Delta + 2\alpha u\partial_u + (4\beta + 4\alpha v)\partial_v + (4\beta + 4\gamma + 4\alpha w)\partial_w.$$

A direct calculation shows that the undesirable first order term can be expressed as the gradient, with respect to (1.5), of the scalar function

$$\lambda = \alpha w + \beta \ln |v - u^2| + \gamma \ln |w - v|.$$

The closure condition is then satisfied and, scaling with the factor $e^{\frac{\lambda}{2}}$, *the operator constructed is gauge equivalent to the following Schrödinger operator:*

$$\mathcal{H}_0 = -\frac{1}{2}\Delta + \alpha(2\beta + 2\gamma + 3) + \alpha^2 w + \frac{\beta(\beta - 1)}{v - u^2} + \frac{\gamma(\gamma - 1)}{w - v}.$$

Observe that the tensor $g^{(ij)}$ *fails to be of full rank when* $g = 16(w-v)(u^2-v) = 0$. *The covariant tensor* $g_{(ij)}$ *is singular on the sets* $\{w = v\}$ *and* $\{v = u^2\}$, *thus the inner product is not defined everywhere. We therefore have to allow degeneracy for the induced metric. To deal with this issue, we will have to introduce a particular structure for the manifold later.*

Let us now illustrate how the 2-imprimitivity of the action affects the operator constructed previously. If we consider only the domain where the metric is positive definite, we can easily check that the Riemannian curvature vanishes identically and the operator reads as follows

$$\mathcal{H} = -\frac{1}{2}\Delta + \nabla(\alpha w + \beta \ln |v - u^2| + \gamma \ln |w - v|).$$

This operator, and its equivalent Schrödinger operator, illustrate clearly that separation arises from invariant foliations by surfaces. Indeed the separation occurs in each of the three possible systems of coordinates. We shall not expect this in general. The three separations reflect the fact that the group action allows not only one but three distinct invariant foliations by surfaces:

$$\{u = const.\}, \qquad \{v = const.\}, \qquad \{w = const.\}.$$

It is now guaranteed that $\mathcal{H}(\lambda) = f(\lambda)$ *for* $\lambda \in \{u, v, w\}$.

In terms of Cartesian coordinates (x, y, z), *the original coordinates are given by*

$$u = x, \qquad v = x^2 + y^2 \qquad w = x^2 + y^2 + z^2.$$

Thus the leaves of the foliations are planes, cylinders or spheres. For each of these foliations we will consider respectively the Cartesian, cylindrical (r, θ, z), *and spherical* (r, θ, ϕ) *coordinates. Hence, in the Cartesian system, the coordinate* x *separates in the Schrödinger equation* $\mathcal{H}_0\psi = E\psi$. *For the two other coordinate systems,* r, *the radial coordinate can be separated. An extra property of this operator is the fact that, for each of these coordinate systems, the operator also separates in the two other coordinates, which shall not be expected in general.*

In these three coordinate systems, the operator is given by

$$\begin{aligned}
\mathcal{H} &= -\frac{1}{2}\Delta + \nabla(\alpha(x^2 + y^2 + z^2) + \beta \ln|y| + \gamma \ln|z|), \\
\mathcal{H} &= -\frac{1}{2}\Delta + \nabla(\alpha(r^2 + z^2) + \beta(\ln|r| + \ln|\sin\theta|) + \gamma \ln|z|), \\
\mathcal{H} &= -\frac{1}{2}\Delta + \nabla(\alpha(r^2) + \beta(\ln|r| + \ln|\sin\phi| + \ln|\sin\theta|) + \gamma(\ln|r| + \ln|\cos\theta|)).
\end{aligned}$$

By applying the operator $\mathcal{H}$ *to* $\psi(x_1, x_2, x_3) = \psi_1(x_1)\psi_2(x_2)\psi_3(x_3)$, *where* x_1, x_2 *and* x_3 *are the respective coordinates, one easily verifies that each of these equations separate into three equations, each of them involving only one variable. After the required gauge transformation, the Schrödinger operator reads as*

$$\begin{aligned}
\mathcal{H}_0 &= -\frac{1}{2}\Delta + \alpha(2\beta + 2\gamma + 3) + \alpha^2(x^2 + y^2 + z^2) + \frac{\beta(\beta-1)}{y^2} + \frac{\gamma(\gamma-1)}{z^2}, \\
\mathcal{H}_0 &= -\frac{1}{2}\Delta + \alpha(2\beta + 2\gamma + 3) + \alpha^2(r^2 + z^2) + \frac{1}{r^2}[\frac{\beta(\beta-1)}{\sin^2\theta}] + \frac{\gamma(\gamma-1)}{z^2}, \\
\mathcal{H}_0 &= -\frac{1}{2}\Delta + \alpha(2\beta + 2\gamma + 3) + \alpha^2(r^2) + \frac{1}{r^2\sin^2\theta}[\frac{\beta(\beta-1)}{\sin^2\phi}] + \frac{1}{r^2}[\frac{\gamma(\gamma-1)}{cos^2\theta}].
\end{aligned}$$

Once again, the three operators separate in their respective coordinate systems since the three potentials satisfy the known separation condition detailed, for instance, in [40].

Chapter 2

Turbiner's Conjecture in Three Dimensions

2.1 Introduction

The aim of this chapter is to extend results related to separation of variables for flat Lie algebraic Schrödinger operators. Originally, in [45], Alexander Turbiner conjectured the following:

Conjecture 1. (Turbiner) " In $\mathbb{R}^2$ there exist no quasi-exactly-solvable or exactly-solvable problems containing the Laplace-Beltrami operator with flat-space metric tensor, which are characterized by non-separable variables."

This conjecture was reformulated in more geometrical terms by Rob Milson [36]. The conjecture, which the work in this paper is based on, now reads as follows.

Conjecture 2. (Turbiner, second version) Let $\mathcal{H}_0$ be a Lie algebraic Schrödinger operator defined on a 2-dimensional manifold. If the symbol of $\mathcal{H}_0$ corresponds to a Euclidean geometry, i.e. if the corresponding Gaussian curvature is zero, then the spectral equation $\mathcal{H}_0\psi = E\psi$ can be solved by a separation of variables.

This conjecture is false in general. A counterexample is given by Rob Milson in [36] and [38], together with a proof of a modified version of the conjecture. By adding two extra assumptions, namely an imprimitive action and a compactness requirement, one can prove that the spectral equation can be solved by separation of variables. Furthermore, the imprimitivity hypothesis implies even more than expected: separation will occur in either a Cartesian or a polar coordinate system.

In this chapter, it is shown that, in three dimensions, the original conjecture is also false and a proof of a modified version of the 3D Turbiner's conjecture is given. Again the compactness requirement and a condition related to the imprimitivity of the action are necessary and and a third condition, related to the contravariant metric, will have to be imposed to prove the theorem. In three dimensions, the invariant foliation of an imprimitive action can be a family of curves or a family of surfaces. In the proof of our result, the leaves of the foliation need to be surfaces and such an imprimitive action will be called 2-imprimitive. Similarly to the two dimensional case, the 2-imprimitivity of the action will ensure that separation, here only partial, will occur in either a Cartesian, cylindrical or a spherical coordinate system.

The proofs of the two modified versions of the conjecture are based on the following ideas. First, the imprimitive action, which is 2-imprimitive in three dimensions, induces an invariant foliation Λ, for which the leaves are surfaces and will be denoted by $\{\lambda = \text{constant}\}$. The Schrödinger operator $\mathcal{H}_0$ is Lie algebraic, thus, it is an element of the enveloping algebra of a Lie algebra of first order differential operators. When applied to λ, the elements of the generating Lie algebra must give back functions of λ. The operator $\mathcal{H}_0$ will enjoy the same property, that is $\mathcal{H}_0(\lambda) = f(\lambda)$. Combining the fact that the operator is Lie algebraic with the imprimitivity of the action, one can prove that the leaves of $\Lambda^{\perp}$, the foliation which is perpendicular to

Λ, are necessarily geodesics. Then, a key result, the Tiling Theorem, gives a global map from a Euclidean space to our manifold. Thus by pulling back the manifold, the leaves of the perpendicular foliation became straight lines.

In this setting, one can show that the invariant leaves can only be prescribed curves or surfaces. In two dimensions the curves need to be straight lines or concentric circles, while in three dimensions the surfaces have to be planes, cylinders or spheres. In this context, λ will be either a Cartesian or a radial coordinate. Finally, using the appropriate coordinate system, one checks that the equation $\mathcal{H}_0\psi = E\psi$ separates with respect to the coordinate λ.

Despite the fact that the path followed to prove the modified 3D Turbiner's conjecture is similar to the one given in [38], there are several important issues to be dealt with, which were absent in the two dimensional case, and which appear in our study. We have first to select the appropriate curvature conditon, while in two dimensions one only has the Gaussian curvature to work with. In order to prove the 3D-Trapping Theorem, which is necessary to prove the 3D-Tiling Theorem, one has to impose that the diagonal terms of the Ricci curvature tensor be zero. For the 3D-Tiling Theorem, it is the Riemannian curvature tensor that needs to vanish. For the proof of the 3D-Trapping and Tiling Theorems, we have to assume that either the metric can be diagonalised, or that M is a transverse, type changing manifold. For the first case, to conclude both theorems, an extra requirement of genericity of the contravariant metric needs to be added. This requirement is related to the non-invertible factors of its components and will be defined later. For the transverse type changing manifold, we will see that such metric can always be diagonalised and is necessarily generic.

The determination of the possible foliations requires a different approach. While in two dimensions one needed only to consider the possible foliations by straight lines to conclude the proof of the result, in three dimensions, one has to keep in mind the entire picture of the two perpendicular foliations in order to determine the three types of leaves. The arguments are not sophisticated but the proof is long enough for us to devote an entire section to it. Another distinction with the two dimensional case is the fact that separation of variables is only partial. Indeed, as in the work of Rob Milson, one can isolate one variable, λ, but we are left, in three dimensions, with two variables for which nothing can be said.

In section 2, we briefly describe the context of Turbiner's conjecture and we define the notions employed in this paper that were not established in the introduction of the thesis. Section 3 fills in the gaps needed to generalize the proof of both 3D-Tiling and 3D-Trapping Theorems. The proofs of these two theorems are omitted since, once this work done, both generalizations are straightforward. In Section 4, we show that, after pulling back the metric to $\mathbb{R}^3$, the only possible leaves of the foliation are planes, cylinders and spheres. This fourth section, involving a succession of simple had hoc arguments, is crucial for its consequences although the proof itself may be skipped at first reading. Using the results exhibited in the two preceding sections, section 5 is devoted to the proof of the tridimensional modified conjecture. Finally a counterexample to the three dimensional general form of Turbiner's conjecture is exhibited in section 6.

We conclude by noting that there are deep connections between separation of variables, exact solvability and superintegrability, see for instance [29] and [41]. However, these lie outside the scope of our thesis.

2.2 General setting

In this section, we complete the introduction of the notions necessary to prove the three dimensional version of the modified Turbiner's conjecture. The two dimensional version, see [38] and [36] for a complete proof, is also discussed.

Recall that, in the construction of Lie algebraic operators detailed in the introduction, the tensor $g^{(ij)}$ can fail to be of full rank. We therefore have to allow degeneracy for the induced metric. However, despite this flexibility, we want the metric to behave reasonably well on the degeneracy locus. For this reason, we introduce a generalization of the pseudo-Riemannian structure.

For $\mathbf{M}$ a real-analytic manifold and $g^{(ij)}$ a type $(2,0)$ tensor field, we denote $\mathbf{D}_g$ the locus of degeneracy of the tensor. The analyticity requirement implies that $\mathbf{D}_g$ is either empty, or a codimension 1 subvariety or $\mathbf{M}$. We set $\mathbf{M}_0 = \mathbf{M}\backslash\mathbf{D}_g$ and we assume that $g^{(ij)}$ is not identically degenerate. Thus $\mathbf{M}_0$ is an open, dense subset of $\mathbf{M}$ and the connected components of $\mathbf{M}_0$ are pseudo-Riemannian manifolds with boundary in $\mathbf{D}_g$.

The pair $(\mathbf{M}, g^{(ij)})$ is called an *almost-Riemannian manifold* if for all pairs u, v of analytic vector fields with non-degenerate plane section $u \wedge v$ on $\mathbf{M}_0$, the sectional curvature function $K(u \wedge v)$ has removable singularities on $\mathbf{D}_g$. Remark that if the sectional curvature is constant on connected components, which is the case if the Riemannian curvature is zero, then $\mathbf{M}$ is an almost-Riemannian manifold. Throughout this paper, we will focus on components of $\mathbf{M}_0$ for which the metric is positive definite and for which the Riemannian curvature tensor is zero.

We will see that separation of variables is, in our context, closely related to foliations by geodesics. Thus we would prefer to work in Euclidean geometry, where the geodesics are straight lines, instead of working on $\mathbf{M}$, a flat analytic manifold. The 3D-Tiling Theorem will help us to achieve this by showing that, under certain conditions, there exists a global real-analytic map from $\mathbb{R}^3$ to $\mathbf{R} \subset \mathbf{M}$ where the contravariant metric of $\mathbf{R}$ is the pushforward of the Euclidean metric. The following definitions and propositions will be necessary to establish this theorem.

To better understand the overall behaviour of the manifold around the degenerate points, we need to quantify the degeneracy of the contravariant metric. The degenerate points can be broken up into two categories. A point $p \in \mathbf{D}_g$ is called *unreachable* if all smooth curves with end points p have infinite length in the metric $g_{(ij)}$. Conversely a degenerate point is called *reachable* if it can be attained by a finite length curve. If $\gamma(t) : (0,1) \to \mathbf{M}_0$ is a geodesic segment, we denote by T the largest number, possibly ∞, such that $\gamma(t)$ can be extended by a geodesic with domain $(0,T)$. For $\mathbf{R}$ an open connected component of $\mathbf{M}_0$, we say that $\mathbf{M}$ is *complete within* $\mathbf{R}$ whenever for all geodesic segment lying within $\mathbf{R}$, either $T = \infty$, or $\lim_{t\to T} \gamma(t)$ is a reachable boundary point of $\mathbf{R}$. This extends the notion of completeness to almost-Riemannian manifold. The following Proposition will be useful.

Proposition 2.1. *Suppose the signature of $g^{(ij)}$ is positive definite within $\mathbf{R}$, and that $\mathbf{R}$ is contained in a compact subset of $\mathbf{M}$. Then $\mathbf{M}$ is complete within $\mathbf{R}$.*

Finally we will say that $\mathbf{M}$ is a transverse, type changing analytic m-dimensional manifold if $\mathbf{M}$ is an analytic manifold with a contravariant metric $g^{(ij)}$ such that at any point x in the degenerate locus $\mathbf{D}_g$, we have:

1. $d(det(g^{(ij)})) \mid_x \neq 0$ for some (and hence any) coordinate system,

2. $Rad_x := \{v_x \in T^*_x\mathbf{M} \mid g^{(ij)}(v, \cdot) = 0\}$ is transverse to $T^*_x\mathbf{D}_g$,

The degenerate points are given by the zero set of the determinant of a 3 by 3 matrix which is, in general, not easy to handle. To circumvent this issue, we will assume that either

(a) $g^{(ij)}$ can be diagonalised,

(b) **M** is a transverse, type changing manifold.

For the case (a), there exists locally a coordinate system for which $g^{(ij)}$ is expressed as

$$g^{(ij)} = \begin{pmatrix} P(x,y,z) & 0 & 0 \\ 0 & Q(x,y,z) & 0 \\ 0 & 0 & R(x,y,z) \end{pmatrix}. \tag{2.1}$$

Thus, the determinant is given by the simple equation $g = PQR$ and we can assume without loss of generality that the metric is degenerate at the origin. We define the *order* of an analytic function to be the smallest total degree of all the monomials with a non-zero coefficient in its Taylor development. Thus, the order of g will be the sum of the orders of the diagonal components of (2.1). Note that the smaller the order of g is, the closer the metric is from being non-degenerate at the origin.

The next requirement will be needed to prove the three dimensional version of the Trapping and Tiling Theorems when the metric is diagonal. This condition does not appear in the two dimensional case and we do not know yet if it is necessary. Recall that the ring of convergent power series with complex coefficients is a unique factorization domain. We say that a contravariant metric tensor $g^{(ij)}$ given as (2.1) is *generic* if the components of the diagonal do not share non-invertible factors. For instance, at the origin, the metric

$$g^{(ij)} = \begin{pmatrix} (1+x)^2 & 0 & 0 \\ 0 & (1+x)y & 0 \\ 0 & 0 & xz \end{pmatrix}$$

is generic while the metric

$$g^{(ij)} = \begin{pmatrix} x^2 & 0 & 0 \\ 0 & xy & 0 \\ 0 & 0 & xz \end{pmatrix}$$

is not.

For the case (b), it can be deduced that the metric can be diagonalized and its diagonal form is generic. Indeed, for a transverse type changing analytic manifold, one can show, see [32] for details, that around any degenerate point, there exists local natural coordinates $\{x^1, ..., x^m\}$ such that

$$g^{(ij)} = \begin{pmatrix} g^{(ab)} & 0 \\ 0 & x^m \end{pmatrix},$$

where $g^{(ab)}$ is non degenerate. In the three dimensional case, $g^{(ab)}$ is a two by two matrix and can be diagonalised into invertible functions. This leads us to a contravariant metric

$$g^{(ij)} = \begin{pmatrix} P(x,y,z) & 0 & 0 \\ 0 & Q(x,y,z) & 0 \\ 0 & 0 & z \end{pmatrix},$$

for which the genericity property is satisfied.

Once the 3D-Tiling Theorem have been established, there will be one last major property needed in our study: the imprimitivity of the action. Since we are dealing with a Lie algebraic operator, we can assume that the domain of the operator is a homogeneous space $\mathbf{M} = \mathbf{G}/\mathbf{H}$ where $\mathfrak{g}_0$ is the Lie algebra corresponding to the tangent space of $\mathbf{G}$ at the identity. Recall that the action of $\mathbf{G}$ on $\mathbf{M}$ is *imprimitive* if there exists a foliation of $\mathbf{M}$ that is invariant under the action of $\mathbf{G}$. In three dimensional Euclidean space, the invariant leaves can be either curves or surfaces, see [34] for a more detailed description of the possible leaves. Throughout this paper we will only consider foliations by surfaces, this type of action will be called a *2-imprimitive action.* Note however that it would be very interesting to study the case of a foliation by curves. In infinitesimal terms, if the leaves of the foliation are given by $\{\lambda = \text{constant}\}$, then $v^a(\lambda) = f(\lambda)$ where v^a is the left invariant vector field associated to a in the lie algebra $\mathfrak{g}_0$. This second criterion can be generalized to extend the notion of 2-imprimitivity to differential operators. If the level sets of the function λ are the leaves of the foliation, the operator T^a is said to acts 2-imprimitively if $T^a(\lambda)$ and λ are functionally dependent. One can easily show, see [38] for details, the following:

Proposition 2.2. *If the operators $\{T^a = v^a + \eta^a : \ a \in \mathfrak{g}_0\}$ act 2-imprimitively , then there is a $\mathbf{G}$-invariant foliation by surfaces on $\mathbf{M}$.*

The central point here is that Lie algebraic operators generated by these 2-imprimitive generators will behave the same way. Indeed the operator $\mathcal{H}$ applied to λ will give back a function of λ. By taking λ as coordinate, the operator will separate in that variable and we will show that the equivalent Schrödinger operator $\mathcal{H}_0$ will also separate partially.

One of the key arguments for the final theorem is that the invariant foliation Λ is perpendicular to a geodesic foliation. This is due to the Lie algebraic construction of the metric $g^{(ij)}$, and, according to the 3D-Tiling Theorem, this perpendicular foliation can be pulled back to the Euclidean space where the geodesics are well known:

straight lines. We will not go into the details, everything being already exhibited in [38] and [36], but we will state the mains results necessary to prove the theorem.

We will denote $\Lambda^{\perp}$ the distribution of tangent vectors that are perpendicular to Λ. For a Lie algebraic metric with invariant foliation Λ, one can prove the following:

Theorem 2.1. *If $\Lambda^{\perp}$ is tangent to a geodesic of M at one point, then the geodesic is an integral manifold of $\Lambda^{\perp}$.*

In the context of the modified three dimensional Turbiner's conjecture, Λ is a rank two **G**-invariant distribution, thus, being a rank 1 distribution, $\Lambda^{\perp}$ is necessarily integrable. We then get:

Corollary 2.1. *If rank ($\Lambda^{\perp}$) $= 1$, then the integral curves of $\Lambda^{\perp}$ are geodesic trajectories.*

After an investigation of the possible foliations of $\mathbb{R}^3$ which are in accordance to our problem, we will be able to show that the partial separation will occur either in Cartesian, cylindrical or spherical coordinates. Note that, as for the two dimensional case, some additional hypothesis, which will be stated below, are necessary. Based on a primitive action, an explicit flat Lie algebraic Schrödinger operator, for which there is no separation of variable, will be exhibited at the end of this paper.

The aim of the next sections is to prove the three dimensional version of the following modified Turbiner's conjecture proved by Rob Milson in [38]:

Theorem 2.2. *Let $\mathcal{H}$ be a second-order Lie algebraic operator generated by the T^a as per (1.1), $g^{(ij)}$ the induced contravariant metric and $\mathbf{R}$ a connected component of $\mathbf{M_0}$ for which $g^{(ij)}$ is positive definite. Suppose the following statements are true:*

1. *$\mathcal{H}$ is gauge equivalent to a Schrödinger operator;*

2. *$(\mathbf{R}, g^{(ij)})$ is isometric to a subset of the Euclidean plane;*

3. *The operators $\{T^a\} \in \mathfrak{g}$ act imprimitively;*

4. *$\mathbf{R}$ is either compact, or can be compactified in such a way that the $\mathbf{G}$-action on $\mathbf{R}$ extends to a real-analytic action on the compactification.*

Then, both the eigenvalue equation $\mathcal{H}\psi = E\psi$, and the corresponding Schrödinger equation separate in either a Cartesian, or a polar coordinate system.

To this end, we will follow a path which is similar to the one followed by Rob Milson. However, as mentioned above, the extra requirement that the metric is diagonalisable and generic, will be required.

2.3 Trapping and Tiling

The main objective of this section is to show that, under some conditions, there is a global map from the Euclidean space to the positive definite region of a flat three dimensional almost-Riemannian compact manifold. As for the planar case, the 3D-Tiling Theorem will follow principally from the 3D-Trapping Theorem. That later assures that the flow of a gradient vector field can never cross the locus of degeneracy. Note that through this section, the contravariant metric will be taken to be diagonal and the genericity property will be needed to prove both theorems.

2.3.1 3D-Trapping theorem

The trapping property is a feature shared by every flat diagonal generic almost-Riemannian metric whose coefficients are analytic functions. All the work involved in the proof is based on an appropriate expression of the diagonal components of the Ricci curvature tensor. We will use local coordinates (x, y, z), that we will sometimes denote (x^1, x^2, x^3) to ease the notation. Thus, we define $H^i = \sum_j g^{ij}\frac{\partial}{\partial x^i} = g^{ii}\frac{\partial}{\partial x^i}$ and evaluate the diagonal components of the Ricci curvature tensor using the frame $\{H^1, H^2, H^3\}$. After some work of simplifications and rearrangements, we obtain the following three expressions:

$$\begin{aligned}
2(R_{11})g^2 &= -3(H^1(g))^2 + 2g(H^1(H^1(g)) \\
&+g^2[P_yQ_y + P_zR_z + 2QP_{yy} + 2RP_{zz} + P_x^2 - 2PP_{xx}] \\
&+g[2P^3Q_xR_x + P^2QP_xR_x + P^2RP_xQ_x - PQ^2P_yR_y \\
&-PR^2P_zQ_z - 3QR^2P_z^2 - 3Q^2RP_y^2],
\end{aligned}$$

$$\begin{aligned}
2(R_{22})g^2 &= -3(H^2(g))^2 + 2g(H^2(H^2(g)) \\
&+g^2[P_xQ_x + Q_zR_z + 2RQ_{zz} + 2PQ_{xx} + Q_y^2 - 2QQ_{yy}] \\
&+g[2Q^3P_yR_y + PQ^2Q_yR_y + Q^2RP_yQ_y - P^2QR_xQ_x \\
&-QR^2P_zQ_z - 3PR^2Q_z^2 - 3P^2RQ_x^2],
\end{aligned}$$

$$\begin{aligned}
2(R_{33})g^2 &= -3(H^3(g))^2 + 2g(H^3(H^3(g)) \\
&+g^2[Q_yR_y + P_xR_x + 2QR_{yy} + 2PR_{xx} + R_z^2 - 2RR_{zz}] \\
&+g[2R^3P_zQ_z + QR^2P_zR_z + PR^2Q_zR_z - Q^2RP_yR_y \\
&-P^2RQ_xR_x - 3PQ^2R_y^2 - 3P^2QR_x^2].
\end{aligned}$$

Proposition 2.3. *Let g^{ij} be a diagonal, generic three dimensional contravariant metric tensor with analytic coefficients. If the diagonal elements of the Ricci curvature tensor are identically zero, then there exists locally defined, analytic functions μ^1, μ^2 and μ^3 such that*

$$H^i(g) = \mu^i \cdot g \quad \textit{for } i = 1, 2, 3.$$

Proof: Obviously, such functions exist around points where the determinant does not vanish. We can assume that g is zero at the origin and we focus on H^1 first. The ring of convergent power series with complex coefficients is a unique factorization domain. Thus, up to multiplication by invertible functions in the ring, in the sense that the degree zero term is non-vanishing, g factors uniquely into a product of irreducible, complex valued, analytic functions that are zero at the origin. Let f be such factor, and let k be its multiplicity, i.e. $g = f^k\sigma$, with f and σ coprime. Since $g^{(ij)}$ is generic, f^k is only a factor of one of the diagonal elements and if k is greater then one, f^{k-1} divides the three partial derivatives of this component. Thus, one easily sees that f^{2k-1} is a factor of the two last summands of $2(R_{11})g^2$:

$$g^2[P_yQ_y + \dots - 2PP_{xx}] + g[2P^3Q_xR_x + \dots - 3Q^2RP_y^2].$$

Since R_{11} is identically zero, the remaining summand, $3(H^1(g))^2 - 2g(H^1(H^1(g))$, must also be divisible by f^{2k-1}. But, the preceding term can be written as

$$k(k+2)\sigma^2(H^1(f))^2 f^{2k-2} + \rho f^{2k-1}$$

where ρ is some analytic function. Thus $k(k+2)\sigma^2(H^1(f))^2$ must be divisible by f. Recall that σ is relatively prime to f and $k(k+2) > 0$, which forces $H^1(f)$ to be divisible by f. The same must be true for all non-invertible irreducible factors of g, (and obviously true for the invertible factors), therefore $H^1(g)$ is divisible by g. The same argument holds for H^2 and H^3. □

Note that, without the genericity requirement, the first term of the last summand of $2(R_{11})g^2$ is only guaranteed to be divisible by f^{2k-2} which does not allows us to establish the claim. However, maybe another rearrangement of the terms could lead to the same conclusion without this extra hypothesis.

From this proposition, the 3D-Trapping Theorem follows immediately. Being identical to the one given in Corollary 6.4.2 of [36], the proof is omitted.

Theorem 2.3. *(The 3D-Trapping Theorem) Let g^{ij} be as in the preceding theorem, and let f be an analytic function. Then the flow of $\nabla(f)$ can never cross the locus of degeneracy. More precisely, the trajectories of the flow of $\nabla(f)$ are either contained in the locus of degeneracy of g^{ij}, or never intersect it.*

2.3.2 3D-Tiling theorem

In what follows, using the 3D-Trapping Theorem, we will prove a three dimensional version of Rob Milson's Tiling Theorem. As before, **M** is a compact, three dimensional, almost-Riemannian manifold endowed with $g^{(ij)}$ a generic and flat metric with diagonal analytic coefficients. **R** is a region where the metric is positive definite.

The key argument for this proof is that either the degenerate points are unreachable, or the metric $g^{(ij)}$ is the push-forward of a non-degenerate metric $\tilde{g}^{(ij)}$. But before proving this proposition, the two following lemmas, deduced from the Proposition 2.3 and the genericity property of the metric, will simplify the subsequent work. Under the same hypothesis, we have the following:

Lemma 2.1. *If f is a non-invertible, irreducible factor of g^{ii}, then, for $i \neq j$, f is a factor of $g^{ii}_{x^j}$ and a factor of f_{x^j}.*

Proof: Suppose that $P = f^k\sigma$, where f is a non-invertible irreducible factor and

$(f, \sigma) = 1$. By the genericity property of the metric, f is also coprime to Q and R, and from Proposition 2.3,

$$H^2(g) = Q(P_yQR + PQ_yR + PQR_y) = \mu^2 \cdot PQR.$$

The function P being a factor of all but one summands of the middle term, P_yQ^2R has to be also divisible by f^k, forcing f^k to divide P_y. Furthermore,

$$P_y = \begin{cases} kf^{k-1}f_y\sigma + f^k\sigma_y & \text{if } k > 1, \\ f_y\sigma + f\sigma_y & \text{if } k = 1, \end{cases}$$

thus f needs to be a factor of f_y. $\square$

Lemma 2.2. *Given g^{ii}, a diagonal component of the contravariant metric g^{ij}, its non-invertible factors are functions of the variable x^i only.*

Proof: Consider f, a non-invertible factor of R. From the analycity requirement, f can be expressed locally by the following convergent power series,

$$f = \sum_{i,j,k=0}^{\infty} f_{ijk}x^iy^jz^k, \quad \text{where} \quad f_{000} = 0.$$

According to Lemma 2.1, $f_x = f \cdot h$, for h an analytic function. The Taylor series of f_x can therefore be given as a product of two series:

$$f_x = \sum_{i,j,k=0}^{\infty} if_{ijk}x^{i-1}y^jz^k = \sum_{i,j,k=0}^{\infty} f_{ijk}x^iy^jz^k \cdot \sum_{a,b,c=0}^{\infty} h_{abc}x^ay^bz^c. \tag{2.2}$$

If we suppose that there exists positive integers i, such that $f_{ijk} \neq 0$, we can fix (α, β, γ), the smallest triple (with respect to the lexicographic order) such that $f_{\alpha\beta\gamma} \neq 0$. The coefficient of the monomial $x^{\alpha-1}y^\beta z^\gamma$ is $\alpha f_{\alpha\beta\gamma}$, and according to (2.2), it can also be given by

$$\sum_{\substack{i+a=\alpha-1 \\ j+b=\beta \\ k+c=\gamma}} f_{ijk} \cdot h_{abc}.$$

But all the coefficients f_{ijk} are zero since $i = \alpha - 1 - a < \alpha$. Consequently $f_{\alpha\beta\gamma} = 0$, a contradiction. So, we have $f_{ijk} = 0$ for all $i \neq 0$, and the same argument is used to show that $f_{ijk} = 0$ for all $j \neq 0$. Therefore

$$f = \sum_{k=0}^{\infty} f_{00k} z^k = f(z).$$

□

We can now prove the following strong criteria for the unreachability of a degenerate point. As for the rest of this paper, the degenerate point will be taken to be the origin.

Proposition 2.4. *If the order of one of the diagonal components g^{ii} is greater than one, then the origin is unreachable.*

Proof: Without loss of generality, $R = z^l(1 + f(x, y, z))$ where $l > 1$. We will compare the metric $g^{(ij)}$ to

$$\tilde{g}^{(ij)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^2 \end{pmatrix},$$

which is a flat metric whose origin is known to be unreachable. We can write the contravariant metric as

$$g^{(ij)} = \begin{pmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & z^2\tilde{R} \end{pmatrix},$$

where $P, Q, \tilde{R}$ are non-singular at the origin. We can find a neighbourhood N and a upper bound $K > 0$ such that $\sup_N\{P, Q, \tilde{R}\} \leq K$. If we consider the region $\mathbf{R} \cap N$, we must have

$$\langle v, v\rangle_g \geq \frac{1}{K}\langle v, v\rangle_{\tilde{g}},$$

for all tangent vectors v. Indeed

$$\langle v,v\rangle_g = \frac{v_1^2}{P}+\frac{v_2^2}{Q}+\frac{v_3^2}{z^2\tilde{R}} \geq \frac{v_1^2}{K}+\frac{v_2^2}{K}+\frac{v_3^2}{z^2K} = \frac{1}{K}\langle v,v\rangle_{\tilde{g}}.$$

The length functional on curves in the metric g is bounded below by $\frac{1}{K}$ time the length functional in the metric $\tilde{g}$. The origin, unreachable with respect to $\tilde{g}$, is therefore unreachable with respect to g as well. □

Corollary 2.2. *If the origin is reachable, then the order of P, Q and R is at most one.*

This leads us, up to relabeling of the variables, to three possibilities:

$$g = PQR = z(1+f(x,y,z)), \tag{2.3}$$

$$g = PQR = yz(1+f(x,y,z)), \tag{2.4}$$

$$g = PQR = xyz(1+f(x,y,z)), \tag{2.5}$$

that enable us to prove the following key lemma.

Proposition 2.5. *A degenerate point is either an unreachable point, or there exists a contravariant, non-degenerate metric tensor $\tilde{g}^{(ij)}$ with analytic coefficients defined on some neighborhood $N \subset \mathbb{R}^3$ and an analytic map $\phi : N \to \mathbf{R}$ such that $\phi_*(\tilde{g}) = g$.*

Proof: If the origin is reachable, we are in one of three previous possibilities, say the case (2.5). Since each diagonal component has order 1, from Lemma 2.2, we can write $P = 4x\tilde{P}$, $Q = 4y\tilde{Q}$ and $R = 4z\tilde{R}$ where $\tilde{P}, \tilde{Q}, \tilde{R}$ are invertible. We consider the analytic map given by

$$\phi_3 := \begin{cases} x &= \xi^2 \\ y &= \eta^2 \\ z &= \mu^2 \end{cases},$$

and we take N, the domain of this map, to be a neighborhood of the origin sufficiently small so that the image of the map is contained in $\mathbf{R}$. One easily verify that, via this map, $g^{(ij)}$ is the pushforward of

$$\tilde{g}^{(ij)} = \begin{pmatrix} \tilde{P} & 0 & 0 \\ 0 & \tilde{Q} & 0 \\ 0 & 0 & \tilde{R} \end{pmatrix},$$

which is non-degenerate at the origin. The cases (2.3) and (2.4) are resolved the same way, by considering respectively the maps

$$\phi_1 := \begin{cases} x &= \xi \\ y &= \eta \\ z &= \mu^2 \end{cases} \text{ and } \phi_2 := \begin{cases} x &= \xi \\ y &= \eta^2 \\ z &= \mu^2 \end{cases}. \tag{2.6}$$

□

The maps ϕ_i will be called *2^ith-fold maps.* The name reflects the fact that the (ξ, η, μ) space generically covers the (x, y, z) space in a 2^i-to-one relationship. The exception being the folding planes, $z = 0$ for ϕ_1, $y = z = 0$ for ϕ_2 and $x = y = z = 0$ for ϕ_3. With this key lemma in hand, we can now assert that the positive-definite region of the almost-Riemannian manifold is isometric to the Euclidean space modulo a discrete group of isometries. The proof is identical to the one given in [36] for the two dimensional case and is based on the fact that reachable degenerate points are, in a way, removable. The three dimensional case is a little simpler since the only analytic maps we need to consider are the 2^ith-fold maps. For these reasons, we will omit the proof.

Theorem 2.4. *(The 3D-Tiling Theorem) Let M be a compact three dimensional flat almost-Riemannian manifold with diagonal generic metric. Then, there exists a globally defined, real-analytic map $\psi : \mathbb{R}^3 \to M$ such that $g^{(ij)}$ is the push forward of* *the Euclidean metric, and such that ψ covers all of* $\mathbf{R}$ *plus the reachable portions of its boundary. Furthermore,* $\mathbf{R}$ *is isometric to the quotient* $\mathbb{R}^3/\Gamma$*, where* Γ *is the group of isometries* γ *such that* $\psi = \psi\gamma$*.*

Note that since ψ is a 2^ith-fold map, the group of isometries is indeed the group of reflections along the folding planes.

2.4 Foliations

In this section, we intend to determine what are the possible rank two foliations of $\mathbb{R}^3$ that are perpendicular to straight lines. This intermediate result, used in conjunction with the 3D-Tiling Theorem, will be used to prove that the function λ, whose level sets are the leaves of the invariant foliation, is a coordinate of either the Cartesian, the cylindrical or the spherical coordinates systems.

The rank two leaves are complete on $\mathbf{M}$ but they may cross the reachable part of the degenerate locus $\mathbf{D}_g$. Is is not clear a priori that the pull back of these leaves is also complete in $\mathbb{R}^3$. Indeed, the rank of these leaves may drop where the Jacobian of $\psi : \mathbb{R}^3 \to \mathbf{R} \subset \mathbf{M}$ is degenerate. To avoid confusion, we denote $\mathbf{S}_g$ the degeneracy locus of the foliation in $\mathbb{R}^3$ and we have the inclusion $\mathbf{S}_g \subseteq \psi^{-1}(\mathbf{D}_g)$.

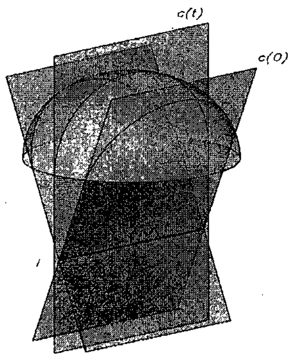
From the 3D-Tiling Theorem, $\mathbf{R}$, the positive definite region of the manifold, is isometric to the quotient $\mathbb{R}^3/\Gamma$ where Γ is a discrete group of reflections. Thus $\mathbb{R}^3$ is tiled into isometric regions where the pull back of each leaf is repeated. If a rank two leaf of $\mathbf{R}$ crosses $\mathbf{D}_g$, its pull back will be reflected on the other side of $\psi^{-1}(\mathbf{D}_g)$. Hence the rank two leaves in $\mathbb{R}^3$ can be extended without restriction but they might fail to be smooth.

However, according to the 3D-Trapping Theorem, the trajectories of the flow of the gradient of λ are either contained in the locus of degeneracy, or never intersect it. This forces the leaves to cross $\psi^{-1}(\mathbf{D}_g)$ perpendicularly, thus, we can conclude that the rank two leaves are also smooth in $\mathbb{R}^3$.

Therefore throughout this section, Λ will denote a foliation of $\mathbb{R}^3$ which is of rank two almost everywhere. By degenerate points we refer to $\mathbf{S}_g$, the points where the rank drops. One easily sees that the rank two leaves never cross the locus of degeneracy. In accordance with Corollary 2.1, the leaves of $\Lambda^{\perp}$, the perpendicular foliation, are straight lines at every non-degenerate point. Our aim is to show that the leaves of Λ can only be planes, infinite cylinders or spheres. Before proving this result we need to establish some notations together with two lemmas.

For any point $x \in \mathbb{R}^3$, we denote $\mathcal{M}_x$ its leaf and, for any curve c contained in a rank two leaf $\mathcal{M}$, we denote S_c the ruled surface generated by the normals of $\mathcal{M}$ along c. Throughout this section the non-degenerate points will be dense and we will use the definitions found in [43] to describe solids.

Lemma 2.3. *Let $c(t)$ be a continuous family of curves parametrized by $t \in (-\delta, \delta)$, contained in $\mathcal{M}$, a rank 2 leaf. Suppose that for every $t_1 \neq t_2 \in (-\delta, \delta)$, the curves $c(t_1)$ and $c(t_2)$ are distinct almost everywhere. If all the surfaces $S_{c(t)}$ intersect each other, then they all intersect at a set $\mathcal{I}$ which is of dimension at most one.*



Proof: Two different curves, $c(t_1)$ and $c(t_2)$, can only intersect at points, hence the two ruled surfaces, $S_{c(t_1)}$ and $S_{c(t_2)}$, can intersect at most in a one dimensional set. For any $\tau \in (-\delta, \delta)$, we define $\mathcal{I}_\tau$ the intersection of $S_{c(\tau)}$ with all the other surfaces.

$$\mathcal{I}_\tau := \bigcup_{\substack{t \in (-\delta,\delta),\\ t \neq \tau}} S_{c(t)} \cap S_{c(\tau)} \neq \varnothing.$$

The set $\mathcal{I}_0$ can not be a surface. Otherwise, by smoothness of the leaf, $\mathcal{I}_\rho$ would also be a surface for $|\rho| < \epsilon$ and ϵ sufficiently small. Thus we would have

$$\mathcal{I} := \bigcup_{|\rho|<\epsilon} \mathcal{I}_\rho,$$

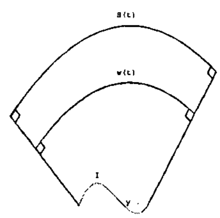
a three dimensional degenerate set. Hence, every surface $S_{c(t)}$ will intersect $S_{c(0)}$ at $\mathcal{I}_0$ which is of dimension is at most one. The same argument holds for each set $\mathcal{I}_t$. Therefore, $\mathcal{I}_0 = \mathcal{I}_t$ for all $t \in (-\delta, \delta)$. $\square$

Lemma 2.4. *Let $c(t)$ be a continuous family of curves parametrized by $t \in (-\delta, \delta)$ and contained in $\mathcal{M}$, a rank 2 leaf. If, for all $t \in (-\delta, \delta)$, each normal of $\mathcal{M}$ along $c(t)$ intersects a degenerate curve $\mathcal{I}$, then $\mathcal{I}$ is parallel to $c(t)$ for all $t \in (-\delta, \delta)$.*

Proof: Let $\mathcal{U} \subset \mathbb{R}^3$ be an open set contained in

$$S := \bigcup_{t\in(-\delta,\delta)} S_{c(t)}.$$

We pick $x \in \mathcal{U}$, a Λ-rank 2 point, and, by completeness, $\mathcal{M}_x$ is crossed perpendicularly by each normal associated to the family $c(t)$. This section of the leaf, given by $\bigcup_{t\in(-\delta,\delta)} \mathcal{M}_x \cap S_{c(t)}$, is therefore parallel to $\mathcal{M}$. We denote $w(t)$ the intersection of $\mathcal{M}_x$ with $S_{c(t)}$ and we note that $c(t)$ is at constant distance from $w(t)$.

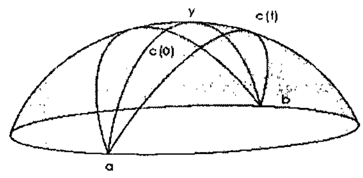


If $\mathcal{I}$ was not parallel to a curve $c(t)$, one could easily choose a rank 2 point, say y, sufficiently closed to $\mathcal{I}$, for which $\mathcal{M}_y$ would intersect the degenerate set $\mathcal{I}$. This is impossible since the leaf $\mathcal{M}_y$ has Λ-rank 2 everywhere. □

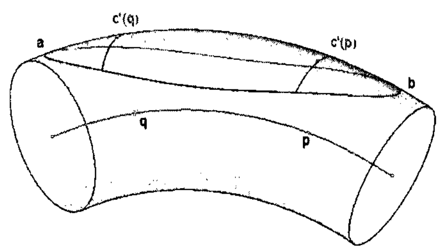
Observe that the intersection curve $\mathcal{I}$, being parallel to the leaves, has to be non-singular. Note also that the curves in the family are necessarily all parallel to each others. We can now prove the following three propositions, which, put together, will enable us to conclude about the three possible foliations.

Proposition 2.6. *Let $\mathcal{M}$ be a rank 2 leaf of the foliation Λ, if there exists an open $\mathcal{U} \subset \mathcal{M}$ for which the Gaussian curvature is positive, then $\mathcal{M}$ is a sphere.*

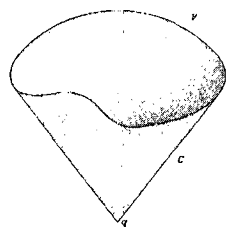
Proof: Let $y \in \mathcal{U}$ and consider $c(0) \subset \mathcal{U}$ the segment of the curve starting at y and following the leaf $\mathcal{M}$ in a given direction $\pm\vec{v}$. If we fix the end points a and b and slide $c(0)$ in the two directions perpendicular to $\vec{v}$, we get a family of curves $c(t)$, $t \in (-\delta, \delta)$, contained in $\mathcal{U}$. We denote $\mathcal{V}$ the subset generated by the curves $c(t)$.



Since the two principal curvatures are non-zero in $\mathcal{U}$, the normal surfaces $S_{c(t)}$ intersect in a connected component and, by Lemma 2.3 the intersection is either a point q, either a connected curve $\mathcal{I}$, parallel to $c(t)$ by Lemma 2.4. Suppose first that the intersection is a curve. Being parallel to $\mathcal{I}$, the surface $\mathcal{V}$ has to be contained in a twisted cylinder centered at $\mathcal{I}$. For each point p on the curve $\mathcal{I}$, we denote $c'(p)$, the intersection of $\mathcal{V}$ with the normal plane of $\mathcal{I}$ at p.

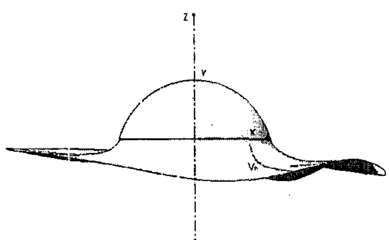


Again, $c'(p)$ is a continuous family of curves, here parametrized by p. Since $\mathcal{I}$ is parallel to the surface $\mathcal{V}$, the surface $S_{c'(p)}$ is contained in the normal plane of p. By the curvature hypothesis, these plane sections intersect, say in $\widetilde{\mathcal{I}}$, and by lemma 2.3, $\widetilde{\mathcal{I}}$ is either a point, either a straight line. The later is impossible since, by lemma 2.4 the line segment $\widetilde{\mathcal{I}}$ would be parallel to every $c'(p)$ whose curvatures are not zero. Therefore, $\widetilde{\mathcal{I}}$ has to be a point q, and all the normals of $\mathcal{V}$ intersect in q. Necessarily, the curve $\mathcal{I}$ needs to restrict to the point q. In that case $\mathcal{V}$ is parallel to q, hence $\mathcal{V}$ is contained in a sphere.



The Gaussian curvature on $\mathcal{V}$ has to be constant, say $\frac{1}{R^2}$, and all the points of $\mathcal{V}$ are at distance R from q. We are left to show that if we extend $\mathcal{V}$ to the entire leaf $\mathcal{M}$, the distance between $\mathcal{M}$ and q will be preserved, i.e. $\mathcal{M}$ is a sphere.

Without loss of generality we consider $\mathcal{V}$ to be the maximal spherical cap with pole y. We denote $\mathcal{C}$ the cone with apex q generated by the normals of $\mathcal{V}$, $\partial\mathcal{V}$ the boundary of $\mathcal{V}$, and $\mathcal{V}_x$ the curve obtained by extending $\mathcal{V}$ through $x \in \partial\mathcal{V}$ perpendicularly to $\partial\mathcal{V}$ along the leaf. We consider as the Z-axis the line containing q and y and we define $\mathcal{P}_w$, the *alignment plane*, containing the Z-axis and the point w.



Remark that if the Gaussian curvature of $\mathcal{M}$ changes along the curve $\mathcal{V}_x$, by smoothness of the leaves, for $y \in \partial\mathcal{V}$ close to x, the curvature will also change along the curves $\mathcal{V}_y$. The key point here is that if there are changes in the curvature, the surfaces $S_{\mathcal{V}x}$ have to eventually leave their alignment planes. Indeed we need to avoid two dimensional intersections with $\mathcal{C}$, otherwise, together with the intersections of the surfaces $S_{\mathcal{V}y}$ with $\mathcal{C}$, we would get a three dimensional degenerate set. We are left with two possibilities. Either $\mathcal{C} \cap S_{\mathcal{V}_x}$ is always q, which is impossible since the curvature changes, either $S_{\mathcal{V}_x}$ eventually leave the cone, which forces the normals of $S_{\mathcal{V}x}$ to leave their alignment plane. Note also that, if a normal stay in the alignment plane, it as to intersect $\mathcal{C}$ at q. The main objective now is to show that if we extend the spherical cap, the normal lines stay in their alignment planes, intersecting the cone at q and preserving the curvature of the spherical cap.

Let γ_ϵ be the closed curve in $\mathcal{M}$ which is at distance ϵ outside $\partial\mathcal{V}$. By completeness, such a curve always exists for ϵ sufficiently small, say $\epsilon < \varepsilon$. We want to show first that along such a path, all the normal lines swing in the same direction with respect to their alignment planes. Let $x \in \gamma_\epsilon$, and assume the curve is traversed in the clockwise direction. Note that, if the dot product between the normal line and the tangent vector of the curve γ_ϵ is positive, then there is an increase of the Z-value of γ_ϵ around x. Suppose we can take two curves, $\mathcal{V}_x$ and $\mathcal{V}_y$, for which the normal lines swing in different directions. Say, without loss of generality, that along each curve γ_ϵ, $\epsilon \in [0, \epsilon_1]$, either the direction changes only once, either the normal is in the alignment plane at $\mathcal{V}_x$ and then rolls in at most one direction. By smoothness of the leaf, for each γ_ϵ, there would be at least a point α_ϵ for which its normal lies in the alignment plane $\mathcal{P}_{\alpha_\epsilon}$, hence intersects $\mathcal{C}$ at q. With an appropriate choice of α_ϵ, for every ϵ in $[0, \epsilon_1]$, we could generate the curve $\alpha(\epsilon)$ parallel to q. Since $\alpha(0) \in \partial\mathcal{V}$ the curve would be at distance R of q.

Necessarily, there should be another change of direction, say between $\mathcal{V}_y$ and $\mathcal{V}_z$, along the curves γ_ϵ for $\epsilon \in [0, \epsilon_2]$. We would then get another curve $\beta(\epsilon)$ at distance R of q. Hence for all $\epsilon \in [0, \tilde{\epsilon}]$, where $\tilde{\epsilon} = \min\{\epsilon_1, \epsilon_2\}$, α_ϵ and β_ϵ would be at distance ϵ from $\partial\mathcal{V}$ and at distance R from q. Thus, for a fixed ϵ, they would necessarily have the same Z-value. But along γ_ϵ, everywhere in between $\alpha(\epsilon)$ and $\beta(\epsilon)$, the normal lines are on the same side of their alignment plane $\mathcal{P}_z$, implying a strict increase (or decrease) of the Z-value between the two points. This is impossible, hence the normal can only swing in one direction.

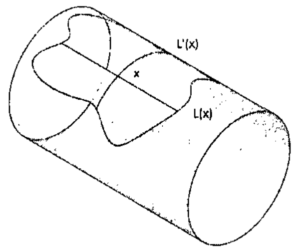
Therefore, the Z-value is monotonic as we follow the closed curve γ_ϵ in a given direction. This is impossible except if the Z-value is constant, that is, if the normals stay in their alignment planes. Thus, for $\epsilon < \varepsilon$ the normal lines of $\mathcal{M}$ along γ_ϵ must intersect the cone at q. Note that the curves $\mathcal{V}_x$ stay parallel to q when they intersect γ_ϵ. So we can increase $\mathcal{V}$ to

$$\widetilde{\mathcal{V}} := \mathcal{V} \bigcup_{\epsilon<\varepsilon} \gamma_\epsilon$$

a bigger spherical cap. This contradicts the maximality of $\mathcal{V}$. Hence $\mathcal{V}$ has to be a sphere and is indeed the entire $\mathcal{M}$. □

Proposition 2.7. *Let $\mathcal{M}$ be a leaf of the foliation Λ, if there exists an open $\mathcal{U} \subset \mathcal{M}$ for which one of the principal curvatures is identically zero and the other is non-vanishing, then $\mathcal{M}$ is an infinite cylinder.*

Proof: Let $\mathcal{L}(x)$ and $\mathcal{L}'(x)$ be the principal curves passing through $x \in \mathcal{U}$ related respectively to the vanishing and the non-vanishing principal curvatures, say $0 \equiv \lambda_1 < \lambda_2$. Note that $\mathcal{L}(x)$ is a line. Since λ_2 is never vanishing on $\mathcal{U}$, the normal surfaces $S_{\mathcal{L}(x)}$ intersect, and from Lemmas 2.3 and 2.4, all the lines $\mathcal{L}(x)$ are parallel to $\mathcal{I}$, which has to be a line also. By following the leaf along $\mathcal{L}'(x)$, the distance between $\mathcal{I}$ and the points in $\mathcal{U}$, say R, has to be preserved. Therefore, $\mathcal{U}$ has to be contained in a radius R cylinder.

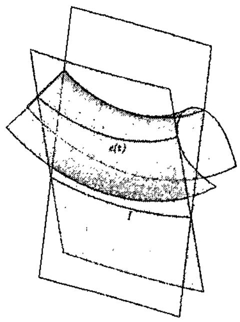


We can extend $\mathcal{L}'(x)$ outside $\mathcal{U}$, and for ϵ sufficiently small, we define the curve γ_ϵ to be the union of the points along these extensions that are at distance ϵ from the boundary of $\mathcal{U}$. The normal surfaces S_{γ_ϵ} have to intersect otherwise it would create a three dimensional degenerate set with the normal lines of $\mathcal{U}$. By Lemma 2.3, these normal surfaces need to intersect in a curve and since the leaf is smooth, this curve has to be $\mathcal{I}$. By Lemma 2.4, these curves are parallel to the line $\mathcal{I}$ so, by extending $\mathcal{L}'(x)$ along $\mathcal{M}$, we get a cylinder $\mathcal{C}$.

If we extend the principal curve $\mathcal{L}(x)$ outside the cylinder $\mathcal{C}$, the curve obtained, say $\mathcal{L}(x)^*$, needs to be a straight line. Otherwise, as for the previous case, it would create a three dimensional degenerate set while intersecting the normal lines of the cylinder. Consequently, $\mathcal{M}$ has to be an infinite cylinder. $\square$

Proposition 2.8. *If $\mathcal{M}$ is a leaf of the foliation Λ, then there is no open $\mathcal{U} \subset \mathcal{M}$ for which $\lambda_1 \cdot \lambda_2 < 0$.*

Proof: Assume the opposite, and pick $y \in \mathcal{U}$. We consider $c(0)$, the intersection of $\mathcal{U}$ with the principal curve through y associated to $\lambda_1 < 0$. We denote $c(t)$, the curves of $\mathcal{U}$ parallel to $c(0)$. By the curvature hypothesis, the normal surfaces $S_{c(t)}$, intersect. Hence by Lemma 2.3 and Lemma 2.4, they intersect at $\mathcal{I}$ a line parallel to the curves $c(t)$.



After crossing the intersection curve, the normal lines of $\mathcal{U}$ will cross leaves for which the two principal curvatures have the same sign, hence by Proposition 2.6, the leaves on the other side of $\mathcal{I}$ will be spheres. The presence of a hyperbolic surface and a spheres is impossible for the foliation Λ. $\square$

Corollary 2.3. *The non-degenerate leaves of the foliation Λ are either planes, cylinders or spheres.*

There seems to be a deep connection between the leaves arising from a 2-imprimitive action and *isoparametric manifolds.* Recall that a hypersurface $\mathbf{M}^n$ of a Riemannian manifold $\mathbf{V}^{n+1}$ is an isoparametric manifold if $\mathbf{M}^n$ is locally a regular level set of a function λ with the property that both $\|\nabla(\lambda)\|$ and $\Delta(\lambda)$ are constant on the level sets of λ. One easily check that the three possible leaves obtained in this section are indeed isoparametric manifolds. The interesting point is that the only complete isoparametric hypersurfaces of $\mathbb{R}^3$ are planes, spheres and round cylinders and this classification holds for any hypersurfaces of $\mathbb{R}^{n+1}$, see [32] for details. Hence, the theory of isoparametric manifolds should provide a good setting to approach Turbiner's conjecture in higher dimensions.

2.5 3D-Turbiner's conjecture

We have now all the tools needed to prove the 3D modified Turbiner's conjecture. This main theorem is a partial affirmation of Turbiner's conjecture in three dimensions. First note that the original conjecture involved complete separation while here we succeed to prove that the equations separate partially. By a partial separation, we mean that the equations separate into two equations, one involving only one variable, the other involving the remaining variables. Also, three assumptions need to be added to the original conjecture: the underlying action has to be 2-imprimitive, the manifold on which the operator is defined has to be compact or can be compactified in such a way that the **G**-action on **R** extends to a real-analytic action on the compactification and the contravariant metric needs to be diagonal and generic. As for the two dimensional case, the recipe is to pull back the invariant foliation to the Euclidean setting where the leaves can only be prescribed surfaces. Then working out the formulas for the operators in the appropriate coordinates, one succeeds in isolating one of the variables. The major differences with the two dimensional case are: the necessity of the genericity requirement and the partial separation obtained. As mentioned previously, at least one of the extra requirements of the modified version of the conjecture is necessary; a counterexample will be given in the next section.

Theorem 2.5. *[3D Modified Turbiner's conjecture] Let $\mathcal{H}$ be a second-order Lie algebraic operator generated by the operators T^a as per* (1.1)*, $g^{(ij)}$ be the induced contravariant metric and* $\mathbf{R}$ *be a connected component of* $\mathbf{M}_0$ *for which $g^{(ij)}$ is positive definite. Suppose that:*

1. *$\mathcal{H}$ is gauge equivalent to a Schrödinger operator;*

2. *$(\mathbf{R}, g^{(ij)})$ is flat;*

3. *The operators $\{T^a\} \in \mathfrak{g}$ act 2-imprimitively ;*

4. **R** *is either compact, or can be compactified in such a way that the* **G***-action on* **R** *extends to a real-analytic action on the compactification;*

5. *The metric* $g^{(ij)}$ *is diagonalizable and generic or* **M** *is a transverse, type changing manifold.*

Then, both the eigenvalue equation $\mathcal{H}\psi = E\psi$*, and the corresponding Schrödinger equation separate partially in either a Cartesian, cylindrical or spherical coordinate system.*

Proof: We denote Λ, the T^a-invariant foliation. The leaves are locally the level sets of a function, say λ and, from Proposition (2.2), this foliation is also **G**-invariant. The almost-Riemannian manifold $(\mathbf{R}, g^{(ij)})$ fulfill the hypothesis of the Tiling theorem, thus there exist a real analytic map $\Phi : \mathbb{R}^3 \to \mathbf{R}$ for which $g^{(ij)}$ is the push forward of the Euclidean metric. It is then possible to pull back the rank 2 foliation Λ to get $\Phi^*(\Lambda)$ which is of rank two almost everywhere. From Corollary 2.1, $\Phi^*(\Lambda)$ is locally orthogonal to a foliation by geodesics that are, in this context, straight lines. The rank two leaves are complete, hence, we can apply Corollary 2.3, to conclude that there exists Cartesian coordinates (x, y, z) such that the leaves are given by the level sets of λ, where λ is either x, $x^2 + y^2$ or $x^2 + y^2 + z^2$.

We will now move the setting to $\mathbb{R}^3$. There is still the local action of the group **G**, but this action is non-degenerate only whenever the Jacobian of Φ is not degenerate. Separation is a local phenomenon, so for the present purpose we can safely ignore the points of degeneracy.

The operator $\mathcal{H}$ is gauge equivalent to a Schrödinger operator $\mathcal{H}_0$, hence it must satisfy the closure condition. That means that there exists a function σ such that

$$\mathcal{H} = \Delta + \nabla(\sigma) + U_0.$$

From the 2-imprimitivity of the action, $\mathcal{H}(\lambda) = f(\lambda)$ and one easily verify that the Laplacian of λ is a function of λ for the three possible coordinate systems. Thus Λ is also invariant with respect to $\nabla(\sigma) + U_0$. But, remark that

$$[\nabla(\sigma) + U_0](\lambda^2) - \lambda[\nabla(\sigma) + U_0](\lambda) = \lambda\nabla(\sigma)(\lambda),$$

which forces both $\nabla(\sigma)$ and U_0 to be functions of λ. Depending of the metric, one easily check that this forces the gauge factor to separate the following way:

$$\begin{aligned}
\sigma(x,y,z) &= \rho(x) + \eta(y,z), \\
\sigma(r,\theta,z) &= \rho(r) + \eta(\theta,z), \\
\sigma(r,\theta,\phi) &= \rho(r) + \eta(\theta,\phi).
\end{aligned}$$

Therefore the equation $\mathcal{H}\psi = E\psi$ separates partially and we are left to show that the Schrödinger equation also separates.

Recall that U, the potential of the Schrödinger operator is given by:

$$U = U_0 + \nabla(\sigma)^2 + \Delta(\sigma), \text{ where } U_0 = U_0(\lambda).$$

After easy computations, the potentials are given respectively by

$$\begin{aligned}
U &= F(x) + G(y,z), \\
U &= F(r) + \frac{1}{r^2}G(\theta,z) + H(\theta,z), \\
U &= F(r) + \frac{1}{r^2}G(\theta,\phi),
\end{aligned}$$

where F depends on the two functions ρ and U_0, while G and H depend on η.

This is sufficient to conclude that the Schrödinger equation

$$(\Delta + U)\Psi = E\Psi$$

separates partially either in Cartesian, cylindrical and spherical coordinates. Indeed, we can perform respectively the following separations:

$$\begin{aligned}
[\partial_{xx} + F(x) - E]\Psi_1(x) &= \alpha\Psi_1(x) \\
[\partial_{yy} + \partial_{zz} + G(y,z)]\Psi_2(y,z) &= -\alpha\Psi_2(y,z),
\end{aligned}$$

$$\begin{aligned}
[\partial_{rr} + \frac{1}{r}\partial_r + F(r) - E]\Psi_1(r) &= (\frac{1}{r^2}\alpha + \beta)\Psi_1(r) \\
[\partial_{\theta\theta} + \partial_{zz} + G(\theta,z) + H(\theta,z)]\Psi_2(\theta,z) &= -(\alpha+\beta)\Psi_2(\theta,z),
\end{aligned}$$

$$\begin{aligned}
[\partial_{rr} + \frac{2}{r}\partial_r + F(r) - E]\Psi_1(r) &= \frac{1}{r^2}\alpha\Psi_1(r) \\
[\frac{1}{\sin^2\phi}\partial_{\theta\theta} + \partial_{\phi\phi} + \cot\phi\partial_\phi + G(\theta,\phi)] &= -\alpha\Psi_2(\theta,\phi),
\end{aligned}$$

where α and β are separation constants. □

2.6 Counterexample

To conclude, we exhibit an example to show that the extra hypotheses can not be omitted. Indeed, we construct a Lie algebraic Schrödinger operator using generating operators that act in a primitive way and we check that the potential can not be separated, even partially. This counterexample is the natural generalization of the one given in [38] for the two dimensional case. It also motivates the notion of almost-Riemannian manifold by realizing the quotient of Euclidean space by an infinite reflection group. The general idea for this type of construction is to find a set of basic invariants and use them as coordinates.

This construction is a bit different from the usual one. Instead of choosing first the coefficients that generates a Lie algebraic operator and then verifying afterwards the closure condition, we proceed in an different order. We first create an almost-Riemannian manifold intimately related to the Lie algebra, then create an operator satisfying the closure condition and finally check if there is a choice of coefficients that generate that operator. We consider in this example the Lie algebra $\mathfrak{sl}_4$, $\mathfrak{h}$ its diagonal Cartan subalgebra equipped with the usual Killing inner product and W, the affine Weyl group associated to the root system. We denote L_1, L_2, L_3 and L_4 the weights associated to the diagonal entries of a trace-free diagonal matrix, where $L_4 = -L_1 - L_2 - L_3$. Taking L_1, L_2 and L_3 as non-orthogonal coordinates, the contravariant form of the metric tensor is given, in an appropriate basis, by

$$\begin{pmatrix} 2 & -2/3 & -2/3 \\ -2/3 & 2 & -2/3 \\ -2/3 & -2/3 & 2 \end{pmatrix}.$$

We define $z_k = e^{2\pi i L_k}$, the generators of the corresponding torus of diagonal unimodular matrices. The algebra of W-invariant elements of the complexified coordinate ring is generated by χ_1, χ_2 and χ_3, the characters of the three fundamental representations of $\mathfrak{sl}_4\mathbb{C}$, see [16] for more details. These three invariants are given by

$$\begin{aligned} \chi_1 &= z_1 + z_2 + z_3 + z_4, \\ \chi_2 &= z_1z_2 + z_1z_3 + z_1z_4 + z_2z_3 + z_2z_4 + z_3z_4, \\ \chi_3 &= z_1z_2z_3 + z_1z_2z_4 + z_1z_3z_4 + z_2z_3z_4, \end{aligned}$$

and one easily computes the contravariant metric associated to this algebra:

$$g^{(ij)} = -8\pi^2 \begin{pmatrix} \chi_1^2 - 8/3\chi_2 & \frac{2}{3}(\chi_1\chi_2 - 6\chi_3) & \frac{1}{3}(\chi_1\chi_3 - 16) \\ \frac{2}{3}(\chi_1\chi_2 - 6\chi_3) & \frac{4}{3}(\chi_2^2 - 2\chi_1\chi_3 - 4) & \frac{2}{3}(\chi_2\chi_3 - 6\chi_1) \\ \frac{1}{3}(\chi_1\chi_3 - 16) & \frac{2}{3}(\chi_2\chi_3 - 6\chi_1) & \chi_3^2 - 8/3\chi_2 \end{pmatrix}.$$

On the real torus, χ_1 and χ_3 are complex conjugate, while χ_2 is real-valued. Thus, fundamental invariants, denoted (x, y, z), are given by the real and imaginary parts of χ_1 and by χ_2. In the real coordinates, the corresponding contravariant metric $g^{(ij)}$, modulo a factor $\frac{-8\pi^2}{3}$, reads as follow:

$$\begin{pmatrix} 2x^2 - z^2 - 4y - 8 & 2(xy - 6y) & 3xz \\ 2(xy - 6y) & 4(y^2 - 2x^2 - 2z^2 - 4) & 2(yz + 6z) \\ 3xz & 2(yz + 6z) & 2z^2 - x^2 + 4y - 8 \end{pmatrix}. \qquad (2.7)$$

For convenience, we will omit this $-8\pi^2/3$ factor and one can verify that the Riemannian curvature tensor is identically zero where the metric is positive definite. The locus of degeneracy of the metric is given by

$$\begin{aligned} \sigma &= -16(x^2 + z^2)^3 + (x^2 + z^2 + 58/39)(320y^2 + 768) \\ &\quad + (x^2 - z^2)(32y^3 - 1152y) + (x^4 + z^4 - 352/39)(-4y^2 + 240) \\ &\quad - 144(x^4 - z^4)y - 8x^2y^2z^2 - 1248x^2z^2 - 64y^4 = 0. \end{aligned}$$

The objective now is to construct a Lie algebraic Schrödinger operator on a space for which the contravariant metric is given by (2.7). The entries of the matrix are degree two polynomials, hence the metric tensor can be generated by $\mathfrak{a}_3$, the Lie algebra of infinitesimal affine transformations of $\mathbb{R}^3$. A set of generators of $\mathfrak{a}_3$ is given by:

$$T^1 = \partial_x, \quad T^2 = \partial_y, \quad T^3 = \partial_z, \quad T^4 = x\partial_x, \quad T^5 = x\partial_y, \quad T^6 = x\partial_z,$$
$$T^7 = y\partial_x, \quad T^8 = y\partial_y, \quad T^9 = y\partial_z, \quad T^{10} = z\partial_x, \quad T^{11} = z\partial_y, \quad T^{12} = z\partial_z,$$

and one easily sees that there is no function λ for which λ and $T^a(\lambda)$ are functionally dependant for all the generators T^a. Thus, these operators do not admit an invariant foliation and the realization of $\mathfrak{a}_3$ is therefore primitive. In terms of these operators, the Laplacian, in the (x, y, z) coordinates, is given by

$$\begin{aligned} \Delta \;=\; & -8(T^1)^2 - 16(T^2)^2 - 8(T^3)^2 + 2(T^4)^2 - 8(T^5)^2 - (T_6)^2 + 4(T^8)^2 - (T^{10})^2 - 8(T^{11})^2 \\ & -2\{T^1, T^7\} + 2\{T^3, T^9\} - 12\{T^2, T^4\} + 12\{T^2, T^{12}\} + 3\{T^4, T^{12}\} \\ & +2\{T^4, T^8\} + 2\{T^8, T^{12}\} - 2T^4 - 4T^8 - 3T^{12}. \end{aligned}$$

For σ the determinant of the contravariant metric (2.7), one easily verifies that

$$\nabla \log \sigma = 12(T^4 + T^{12}) + 16T^8. \tag{2.8}$$

Therefore, the operator

$$\mathcal{H} = -\Delta + \nabla \log \sigma$$

is Lie algebraic and gauge equivalent to $\mathcal{H}_0$, a Schrödinger operator, via the gauge transformation:

$$\mathcal{H}_0 = e^{-\log(\sigma)/2} \circ \mathcal{H} \circ e^{\log(\sigma)/2} = -\Delta + U.$$

The potential U can be computed,

$$\begin{aligned} U \;=\; & 80 - 64[x^6 + 3x^4z^2 + 3x^2z^4 + z^6 - 18x^4y + 2x^2y^3 - 2y^3z^2 + 18yz^4 \\ & +60x^4 + 60x^2y^2 - 312x^2z^2 - 8y^4 + 60y^2z^2 + 60z^4 + \\ & -360x^2y + 360yz^2 + 336x^2 + 192y^2 + 336z^2 - 640]\sigma^{-1}, \end{aligned}$$

and can also be described in terms of the affine coordinates (L_1, L_2, L_3) by

$$U = 80 + \sum_{1 \le j < k \le 4} \frac{1}{\sin^2(\pi(L_j - L_k))}. \tag{2.9}$$

To conclude our counterexample, we need to show that the Schrödinger equation $\mathcal{H}_0$ cannot be solved, even partially, by separation of variables. The potential here is symmetrical in the 3 variables, hence the separation in respect to one variable would imply a separation in the other ones, thus a complete separation of variables. The Schrödinger equation can be solved by separation of variables in only eleven coordinate systems, nine of which (with the exception of paraboloidal coordinates) are particular cases of the ellipsoidal coordinates. According to [40], these coordinates are: rectangular (Cartesian), circular cylinder, elliptic cylinder, parabolic cylinder, spherical, conical, parabolic, prolate spheroidal, oblate spheroidal, paraboloidal, ellipsoidal coordinates.

An appropriate change of coordinates gives the orthonormal system (y_1, y_2, y_3) and one gets the following similar potential:

$$U = 80 + \sum_{1 \le j < k \le 3} \frac{1}{\sin^2(2\sqrt{2/3}\pi(y_j \pm y_k)} \tag{2.10}$$

Since the nine first coordinate systems are particular cases of the last one, we only have to show that there is no separation possible in the last two coordinate systems: ellipsoidal and paraboloidal.

The ellipsoidal system of coordinates (ξ_1, ξ_2, ξ_3) is related to the Cartesian one by

$$\begin{aligned} y_1 &= \sqrt{\frac{(\xi_1^2 - a^2)(\xi_2^2 - a^2)(\xi_3^2 - a^2)}{a^2(a^2 - b^2)}}, \\ y_2 &= \sqrt{\frac{(\xi_1^2 - b^2)(\xi_2^2 - b^2)(\xi_3^2 - b^2)}{b^2(b^2 - a^2)}}, \\ y_3 &= \frac{\xi_1\xi_2\xi_3}{ab}, \quad \text{where } \xi_1^2 \ge a^2 \ge \xi_2^2 \ge b^2 \ge \xi_3^2 \ge 0, \end{aligned}$$

while, for the paraboloidal, we have:

$$\begin{aligned} y_1 &= \sqrt{\frac{(\xi_1^2 - a^2)(\xi_2^2 - a^2)(\xi_3^2 - a^2)}{(a^2 - b^2)}}, \\ y_2 &= \sqrt{\frac{(\xi_1^2 - b^2)(\xi_2^2 - b^2)(\xi_3^2 - b^2)}{(b^2 - a^2)}}, \\ y_3 &= \sqrt{\frac{1}{2}(\xi_1^2 + \xi_2^2 + \xi_3^2 - a^2 - b^2)}, \quad \text{where } \xi_1^2 \geq a^2 \geq \xi_2^2 \geq b^2 \geq \xi_3^2 \geq 0. \end{aligned}$$

For these two systems, a given potential U separates if and only if it is of the form

$$U = \frac{(\xi_2^2 - \xi_3^2)U_1(\xi_1) + (\xi_1^2 - \xi_3^2)U_2(\xi_2) + (\xi_1^2 - \xi_2^2)U_3(\xi_3)}{(\xi_1^2 - \xi_2^2)(\xi_2^2 - \xi_3^2)(\xi_1^2 - \xi_3^2)}.$$

After the suitable substitution, the potential U, in terms of the new coordinates (ξ_1, ξ_2, ξ_3), fails to be of that required form. Therefore, no separation is possible.

Thus, this example emphasis on the necessity of the extra hypotheses we needed to add to the original conjecture. Here, at least one of these hypotheses, the imprimitivity of the action, fails to be satisfied and the Schrödinger equation can not be solved by separation of variables, even partially.

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From Turbiner's Conjecture to Quasi-Exactly Solvable Schrödinger Operators

We now continue our investigation of Schrödinger operators in three dimensions with a different approach. Instead of working out the question of separability we focus on the problem of determining part of the spectrum of the Schrödinger equation. To this effect, a partial classification of quasi-exactly solvable Lie algebras of first order differential operators is performed and these Lie algebras are used to construct new quasi-exactly solvable Schrödinger operators in three variables.

Note that, while the differential operators were imposed to be only Lie algebraic for Turbiner's conjecture, the operators considered in this second paper must be quasi-exactly solvable. Furthermore, the Lie algebras considered for the classification performed in the following pages do not have to be imprimitive. However when it come to chosing which imprimitive Lie algebras we would consider for the partial classification, the 2-imprimitive algebras are chosen due to the separation theorem proved in the above paper. Moreover, the possible application of this separation theorem is discussed for some new tridimensional quasi-exactly solvable Schrödinger operators.

Chapter 3

Quasi-Exactly Solvable Schrödinger Operators in Three Dimensions

3.1 Introduction

It seems that the concept of a "spectrum generating algebra" was first introduced by Goshen and Lipkin in [26] in 1959. However, this paper did not seem to have been be noticed by the community at the time and, ten years later, spectrum generating algebras were independently rediscovered by two groups of physicists, see [9] and [12]. Their work was an impetus for further research in this area as one can see by browsing in the two volume set of reprints [11] and the conference proceedings [27]. A survey of the history and the contribution papers related to the spectrum generating algebras is given in the review paper of Böhm and Ne'eman, which appears at the beginning of [10]. In the early 1980's, Iachello, Levine, Alhassid, Gürsey and collaborators exhibited applications of spectrum generating algebras to molecular spectroscopy; a survey of the theory and applications is given in the book [28]. In these applications, both nuclear and spectroscopic, the relevant Hamiltonian is a Lie algebraic operator in the sense described previously. Finally, the analysis of a new class of Schrödinger operators, the *quasi-exactly solvable* class, was initiated in late 1980's by Shifman, Turbiner and Ushveridze, see [42], [44], [47]. A survey of the theory and applications of quasi-exactly solvable systems in physics is given in [48].

There exists a complete classification of normalizable quasi-exactly solvable Schrödinger operators in one dimension. In two dimensions, this classification is partially complete. Indeed, all the Lie algebraic linear differential operators for which the formal spectral problem is solvable are known, [23]. The main contribution of our paper is to extend these results to three dimensions by giving a partial classification of the quasi-exactly solvable Lie algebras of first order differential operator in three variables, and showing how this can be applied to the construction of new quasi-exactly solvable Schrödinger operators in three dimensions. Our work is based on the classification of finite dimensional Lie algebras of vector fields in three dimensions begun by Lie in [34] and almost completed by Amaldi in [6].

In general, there is no a-priori method for testing whether a given differential operator is Lie algebraic or quasi-exactly solvable. However, one can try to perform a classification of these operators under local equivalence using a general method of classification described by González-López, Kamran and Olver in [23].

The first step toward the classification of normalizable quasi-exactly solvable Schrödinger operators is to classify the finite dimensional Lie algebras of first order differential operators up to local diffeomorphism and rescaling. Then, the task is to determine which of these equivalence classes admit a finite dimensional $\mathfrak{g}$-module $\mathcal{N}$ of smooth functions. Then, from the quasi-exactly solvable Lie algebras found in the second step, one can construct second order differential operator as described in (1.1) from any choice of coefficients C_{ab}, C_c, and C_0. The third step consists in determining which of these operators are equivalent to a Schrödinger operator and this can be performed by verifying the closure condition. Finally, the last step in this classification problem is to check if the functions contained in the $\widetilde{\mathfrak{g}}$-module $\widetilde{\mathcal{N}}$ are square integrable.

As mentioned previously, the entire classification has been established in one dimension. In the scope of the first two steps, every quasi-exactly solvable Lie algebra is locally equivalent to a subalgebra of the Lie algebra

$$\mathfrak{g}_n = \{ \frac{\partial}{\partial x}, x\frac{\partial}{\partial x}, x^2\frac{\partial}{\partial x} - nx \ , 1\},$$

where n is a non negative integer, see [24] for a more detailed explanation. Then, once a second order differential operator is constructed, since all one forms are closed in one dimension, such an operator will always be equivalent to a Schrödinger operator, reducing the third step to a trivial step. Finally González-López, Kamran and Olver determined in [22] necessary and sufficient conditions for the normalizability of the eigenfunctions of the quasi-exacly solvable Schrödinger operators.

In two dimensions, the first two steps of the classification problem were determined by the same authors in [20] and [25]. Based upon Lie's classification of Lie algebras of vector fields, see [34], a complete classification of the quasi-exactly solvable Lie algebras $\mathfrak{g}$ of first order differential operators, together with their finite dimensional $\mathfrak{g}$-modules, was completed. The case of two complex variables is discussed in the first paper while the second paper completed the classification by considering operators on two real variables. However, the last two steps are not yet completed but a wide variety of normalizable quasi-exactly solvable Schrödinger operators has been exhibited, see for instance [23], [24] and [25].

In the next section, a partial classification of quasi-exactly solvable Lie algebras of first order differential operators in three dimensions is given. While the two first steps were successfully completed in one and two dimensions, only part of this work is now done in three dimensions. However, these new quasi-exactly solvable Lie algebras can be used to seek new quasi-exactly solvable Schrödinger operators in three dimensional space. The last section of this paper is devoted to the description of new quasi-exactly solvable Schrödinger operators in three dimensions. Eigenvalues are also computed for two families of Schrödinger operators. These eigenvalues are part of the spectrum of the operators and their eigenfunctions, together with their nodal surfaces, are exhibited. In addition, a connection is made between the separability theorem proved in [15] and the quasi-exactly solvable Schrödinger operators are described. The quasi-exactly solvable models obtained in our paper are new as far as we can tell. In particular they are not part of the list of multi-dimensional quasi-exactly solvable models obtained in [48] by the method of inverse separation of variables.

3.2 Classification of Quasi-Exactly Solvable Lie Algebras of First Order Differential Operators

3.2.1 Lie Algebras of First Order Differential Operators

Our goal in this section is to give a partial classification of quasi-exactly solvable Lie algebras of first order differential operators in three dimensions. A first step toward this goal is to obtain a classification of the finite dimensional Lie algebras $\mathfrak{g}$ of first order differential operators. After this is done, the next step is to impose the existence of an explicit finite dimensional $\mathfrak{g}$-module $\mathcal{N}$ of smooth functions. To this end, we will first summarize the basic theory underlying the classification of Lie algebras of first order differential operators.

For M an n-dimensional manifold, we denote by $\mathcal{F}(\mathrm{M})$ the space of smooth real-valued functions and $\mathcal{V}(\mathrm{M})$ the Lie algebra of vector fields on M. The space $\mathcal{F}(\mathrm{M})$ form a $\mathcal{V}(\mathrm{M})$-module under the usual derivation $\eta \to v(\eta)$, where v is a vector field in $\mathcal{V}(\mathrm{M})$ and η a function in $\mathcal{F}(\mathrm{M})$. The Lie algebra of first order differential operators $\mathcal{D}^1(\mathrm{M})$ can be described as a semidirect product of these two spaces, $\mathcal{D}^1(\mathrm{M}) = \mathcal{V}(\mathrm{M}) \ltimes \mathcal{F}(\mathrm{M})$. Indeed, each element T in $\mathcal{D}^1(\mathrm{M})$ can be written into a sum $T = v+\eta$ and the Lie bracket is given by

$$[T^1, T^2] = [v^1, v^2] + v^1(\eta^2) - v^2(\eta^1), \text{ where } T^i = v^i + \eta^i \in \mathcal{D}^1(\mathrm{M}). \tag{3.1}$$

Note that the space $\mathcal{F}(\mathrm{M})$ is also a $\mathcal{D}^1(\mathrm{M})$-module with $T(\zeta) = v(\zeta) + \eta \cdot \zeta$. Consequently, any finite dimensional Lie algebra of first order differential operators $\mathfrak{g}$ can be written as

$$T^1 = v^1 + \eta^1, ..., T^s = v^s + \eta^s, T^{s+1} = \zeta^1, ..., T^{s+r} = \zeta^r, \tag{3.2}$$

where $v^1, ..., v^s$ are linearly independent vector fields spanning $\mathfrak{h} \subset \mathcal{V}(\mathrm{M})$, a s-dimensional Lie algebra and where the functions $\zeta^1, ..., \zeta^r$ act as multiplication operators and span $\mathcal{M} \subset \mathcal{F}(\mathrm{M})$ a finite dimensional $\mathfrak{h}$-module. Note that restrictions need to be imposed to the functions η^i for $\mathfrak{g}$ to be a Lie algebra. Indeed, without the cohomological conditions that will be described below, the Lie bracket given in (3.1) does not necessarily return an element in the Lie algebra $\mathfrak{g}$.

For $T = v+\eta$, we define a 1-cochain $F : \mathfrak{h} \to \mathcal{F}(\mathrm{M})$ by the linear map $\langle F; v\rangle = \eta$. Since any function $\zeta \in \mathcal{M}$ can be added to T without changing the Lie algebra $\mathfrak{g}$, this map is not well defined. To deal with this issue, we should therefore interpret F as a $\mathcal{F}(\mathrm{M})/\mathcal{M}$-valued 1-cochain. Thus, from the Lie bracket given in (3.1), it is straightforward to see that $\mathfrak{g}$ is a Lie algebra if and only if the 1-cochain F satisfies the bilinear identity

$$v^i\langle F;v^j\rangle - v^j\langle F;v^i\rangle - \langle F;[v^i,v^j]\rangle \in \mathcal{M}, \qquad v^i,v^j\in\mathfrak{h}. \tag{3.3}$$

In terms of Lie algebra cohomology, this condition can be restated as follows, $\langle\delta_1 F;v^i,v^j\rangle \in \mathcal{M}$ for all v^i, v^j in $\mathfrak{h}$, *i.e.* F is a $\mathcal{F}(\mathrm{M})/\mathcal{M}$-valued 1-cocycle on $\mathfrak{h}$. (See [16] for a detailed description of Lie algebra cohomology.)

This classification of Lie algebras of first order differential operators would not be complete without considering the local equivalences between the Lie algebras. Indeed, if a gauge transformation with gauge factor $\mu = e^\lambda$, is performed on an operator $T = v + \eta$ in $\mathfrak{g}$, the resulting differential operator $\widetilde{T} = e^\lambda \cdot T \cdot e^{-\lambda} = v + \eta - v(\lambda)$ will only differ from T by the addition of a multiplication operator $v(\lambda)$. Again, this can be expressed in cohomological terms. Indeed, under the 0-coboundary map $\delta_0 : \mathfrak{h} \to \mathcal{F}(\mathrm{M})/\mathcal{M}$ defined by $\langle\delta_0\lambda;v\rangle = v(\lambda)$, the multiplication factor $v(\lambda)$ can be interpreted as the image, or the 0-coboundary, of the function λ. Hence, combining these two observations, it is possible to conclude that the map F is an element in $H^1(\mathfrak{h},\mathcal{F}(\mathrm{M})/\mathcal{M}) = \ker\delta_1/\mathrm{Im}\delta_0$. Thus, if two differential operators $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ are equivalent with respect to a change of variables φ and a gauge transformation given by $\mu = e^\lambda$, these two operators will correspond to equivalent triples $(\mathfrak{h},\mathcal{M},[F])$, and $(\widetilde{\mathfrak{h}},\widetilde{\mathcal{M}},[\widetilde{F}])$, where $\widetilde{\mathfrak{h}} = \varphi_*(\mathfrak{h})$, $\widetilde{\mathcal{M}} = \varphi_*(\mathcal{M})$, and $\widetilde{F} = \varphi_* \circ F \circ \varphi_*^{-1} + \delta_0\lambda$. This is summarized in the following theorem.

Theorem 3.1. *There is a one to one correspondence between equivalence classes of finite dimensional Lie algebras* $\mathfrak{g}$ *of first order differential operators on* M *and equivalence classes of triples* $(\mathfrak{h},\mathcal{M},[F])$, *where*

1. $\mathfrak{h}$ *is a finite dimensional Lie algebra of vector fields.*

2. $\mathcal{M}$ *is a finite dimensional* $\mathfrak{h}$*-module of functions.*

3. $[F]$ *is a cohomology class in* $H^1(\mathfrak{h}, \mathcal{F}(\mathrm{M})/\mathcal{M})$.

Hence the general classification of finite dimensional Lie algebras of first order differential operators $\mathfrak{g}$ can be reduced to the classification of triples $(\mathfrak{h}, \mathcal{M}, [F])$ under local changes of variables.

In three dimensions, a complete local classification of the finite dimensional Lie algebras of vector fields $\mathfrak{h}$ has been established by Lie in [34] and Amaldi in [6]. Lie's classification distinguishes between the imprimitive Lie algebras, for which their exists an invariant foliation of the manifold, and the primitive Lie algebras, for which no such foliation exists. Lie's work gives a description of the eight different classes of primitive Lie algebras and, based under the possible foliations of the manifold, the imprimitive Lie algebras are subdivided into the following three types :

I The manifold admits locally an invariant foliation by surfaces that does not decompose into a foliation by curves.

II The manifold admits locally an invariant foliation by curves not contained in a foliation by surfaces.

III The manifold admits locally an invariant foliation by surfaces that does decompose into a foliation by curves.

Observe that these three types are not necessarily exclusive. For instance, the Lie algebra $\mathfrak{h} = \{\ p,\ q,\ xq,\ xp - yq,\ yp,\ r\ \}$ belongs to the first two types. The underlying manifold $\mathbb{R}^3$ admits a first indecomposable foliation by planes $\Delta := \{z = \text{constant}\ \}$ and also admits an second invariant foliation by straight lines $\Phi := \{x = \text{constant}\} \cap \{y = \text{constant}\}$ not contained in any invariant surfaces. Lie classified the algebras of type I and II, giving respectively twelve and twenty-one different classes of Lie algebras. Few years latter, the 103 classes of Lie algebras of the third type were exhibited by Amaldi.

The number of finite dimensional Lie algebras of vector fields $\mathfrak{h}$ is large and it did not seem reasonable to consider all the 154 classes. For this first classification attempt, we have chosen to focus on the algebras which seem promising in our aim to construct new quasi-exactly solvable Schrödinger operators. The selection was made upon the following criteria.

We first narrowed our choice based on the results given in [15]; provided the Lie algebra $\mathfrak{g}$ is imprimitive and its invariant foliation consists of surfaces, one can show, adding some other hypothesis on the metric induced, that a Lie algebraic Schrödinger operator generated by $\mathfrak{g}$ separates partially in either Cartesian, cylindrical or spherical coordinates. Since such algebras are good candidates for generating interesting quasi-exactly solvable Schrödinger operators, we restricted our search on the type I and type III imprimitive algebras. In this paper, the classification of the twelve type I Lie algebras is entirely performed while, for the type III Lie algebras, we focused on some of the most general Lie algebras. Since the induced metric $g^{(ij)}$ needs to be non-degenerate, the type III Lie algebras involving only one or two of the three partial derivatives were discarded. Finally we selected our algebras among those that contain other type III algebras as subalgebras.

3.2.2 Classification of Lie Algebras of First Order Differential Operators

Using the equivalence between the Lie algebras of first order differential operators $\mathfrak{g}$ and triples $(\mathfrak{h}, \mathcal{M}, [F])$, it is possible to determine the Lie algebras $\mathfrak{g}$ from the selected Lie algebras of vector fields $\mathfrak{h}$. But first, recall that the second step in the classification of quasi-exactly Schrödinger operators is to determine which of these Lie algebras of first order differential operators $\mathfrak{g}$ are quasi-exactly solvable. It is not hard to see that if $\mathfrak{g}$ is quasi-exactly solvable with non trivial fixed module $\mathcal{N}$, the Lie algebra $\mathfrak{g}$ is finite dimensional if and only if $\mathcal{M}$ is the module of constant functions, see [20] for details. Therefore, instead of working on the general classification of Lie algebras of first order differential operators $\mathfrak{g}$, we will restrict our work to the equivalence classes of triples $(\mathfrak{h}, \{1\}, [F])$. Thus, for each of the selected Lie algebras $\mathfrak{h}$, we first seek for the possible cohomology classes, $[F]$ in $H^1(\mathfrak{h}, \mathcal{F}(\mathrm{M})/\{1\})$. Once this is done, it will be left to find if there exists an explicit finite dimensional $\mathfrak{g}$-module $\mathcal{N}$, where $\mathfrak{g}$ is the Lie algebra equivalent to the triple $(\mathfrak{h}, \{1\}, [F])$. Note that, as in lower dimensions, the existence of a nontrivial module $\mathcal{N}$ will impose a "quantization" condition on $[F]$. Indeed, for each of the Lie algebras worked out in this paper, the possible values for the functions in $[F]$ can only be taken in a discrete set. For detailed results related to the quantization of cohomology, see [19] and [39].

Classification of the Cohomology Classes $[F]$ in $H^1(\mathfrak{h}, \mathcal{F}(\mathrm{M})/\{1\})$

To determine the possible cohomology classes, we first start with $[F]$ as general as possible. For every v in the Lie algebra $\mathfrak{h}$, we denote the value of the 1-cocycle $\langle F; v\rangle$ by η_v, and η_v can be any function in $\mathcal{F}(\mathrm{M})/\{1\}$. Our aim is to find the most general 1-cocycle F, that is the most general functions η_v, satisfying the restrictions imposed by the 1-cocycle conditions (3.3). Then, using the 0-coboundary map, we try to describe the class $[F]$ with representatives η_v as simple as possible. Finally, if $\{v^1, ..., v^r\}$ is a basis for $\mathfrak{h}$, the set $\{v^1 + \eta_{v^1}, ..., v^r + \eta_{v^r}\}$ will be a basis for the Lie algebra $\mathfrak{g}$. Note that in this process, one can alternate the use of the 1-cocycle restrictions with the use of the 0-coboundary cancellations. For instance, if the element p belongs to the algebra $\mathfrak{h}$, the function $\langle F; p\rangle = \eta_p$ can be annihilated by the image of the function $\Psi_p = \int \eta_p dx$ under the 0-coboundary map. Indeed $\langle \delta_0 \Psi_p; p\rangle = p(\int \eta_p dx) = \eta_p$ and $\widetilde{F} = F - \delta_0 \Psi_p$ belong to $[F]$. Thus, we can assume the function η_p to be equivalent to the zero function. Then for another vector field v in $\mathfrak{h}$, using the 1-cocycle restriction for the pair (p, v), that is

$$p\langle F; v\rangle - v\langle F; p\rangle - \langle F; [p, v]\rangle = p\eta_v - 0 - \eta_{[p,v]} \in \{1\},$$

one obtains conditions on the two functions η_v and $\eta_{[p,v]}$. Once again, one might try to absorb part of the function η_v with $\delta_0 \Psi_v$, the image of another function Ψ_v. Note that, in order to maintain $\eta_p \equiv 0$, a restriction is imposed on Ψ_v. Indeed, when the 1-cocycle $F + \delta_0 \Psi_v$ is applied to p, we have to avoid reintroducing a function for η_p. Thus we need to consider only the functions Ψ_v for which $\langle \delta_0 p; \Psi_v\rangle = (\Psi_v)_x$ is a constant function. Then, to complete the determination of $[F]$, the same process is preformed to every vector field of $\mathfrak{h}$, with some care in the choices of the 0-coboundary maps, so as not to undo the simplifications done in the previous steps.

The results of this partial classification of cohomology classes $[F]$, that gives a partial classification of Lie algebras of differential operators $\mathfrak{g}$, are summarized in Tables 1 and 2 at the end of this section. The first table gives a 1-cocycle representative for the twelve type I Lie algebras and Table 2 exhibits the results for some general Lie algebras of vector fields among the type III Lie algebras. For these two tables, the classification numbers, given respectively by Lie and Amaldi, sit in the the first column. The second column gives a basis for the Lie algebra $\mathfrak{h}$ and the third column exhibits the first order differential operators $v+\eta_v$ for which $\langle F; v\rangle = \eta_v$ in not trivial. η_v is taken to be the simplest representative and when the function η_v is trivial, the differential operator is simply the vector field exhibited in the second column.

It would be impractical to present the details of the computations in all cases. Furthermore, the arguments are quite similar for all Lie algebra $\mathfrak{h}$ of vector fields. So, for brevity's sake, we will only give the details for two of the selected Lie algebras. The chosen examples illustrate well the general process and will give to the reader a good idea of how the calculations proceed in general.

Type I, Case 1

This Lie algebra $\mathfrak{h}$ is spanned by the eight vector fields $p, q, xp, yq, xq, yp, x^2p + xyq$ and $xyp + y^2q$. The vector field p belongs to the Lie algebra, hence, as mentioned previously, the function η_p can be assumed to be zero. The 1-cocycle condition for the pair (p, q) imposes the following restriction

$$\langle \delta_1 F; p, q\rangle = (\eta_q)_x - (\eta_p)_y - \langle F; [p, q]\rangle = (\eta_q)_x \in \{1\}.$$

Thus $\eta_q = c_q x + h_q(y, z)$, where c_q is a constant. Hopefully, the function $h_q(y, z)$ can be absorbed by the image, under the 0-coboundary map, of the function $\Psi_q = \int h_q(y, z)dy$. Since $(\Psi_q)_x$ is zero, the 0-coboundary of Ψ_q will not affect η_p.

Similarly, by considering the pair (p, xp), one concludes that $\eta_{xp} = c_{xp}x + h_{xp}(y, z)$, where $c_{xp}x$ can be canceled, without changing the previous functions, by the 0-coboundary of the function $\Psi_{xp} = c_{xp}x$. Then, for the pair (q, xp), the restriction reads as

$$\langle \delta_1 F; q, xp\rangle = (\eta_{xp})_y - x(\eta_q)_x - \langle F; [q, xp]\rangle = (h_{xp}(y, z))_y - x \cdot c_q \in \{1\}.$$

Necessarily, since h_{xp} depends only on y and z, the constant c_q has to be zero and the function $h_{xp}(y,z)$ is forced to be of the form $d_{xp}y + K(z)$, where d_{xp} is a constant. Thus, at this point, $\eta_p = 0$, $\eta_q = 0$ and $\eta_{xp} = d_{xp}y + K(z)$.

Consider now the three vector fields yq, xq and yp. If we pair each of them with p and q, from the six 1-cocycle restrictions, one obtains directly the following

$$\begin{aligned}
\eta_{yq} &= c_{yq}x + d_{yq}y + k_{yq}(z), \\
\eta_{xq} &= c_{xq}x + d_{xq}y + k_{xq}(z), \\
\eta_{yp} &= c_{yp}x + d_{yp}y + k_{yp}(z).
\end{aligned}$$

With the image of the function $\Psi_{yq} = d_{yq}y$, the function η_{yq} can be reduced to $\eta_{yq} = c_{yq}x + k_{yq}(z)$ without undoing the previous work. From the restriction associated to the pair (xp, yp), one easily check that

$$\langle \delta_1 F; xp, yp\rangle = x(\eta_{yp})_x - y(\eta_{xp})_x + \eta_{yp} = x \cdot c_{yp} + c_{yp}x + d_{yp}y + k_{yp}(z) \in \{1\},$$

forcing c_{yp} and d_{yp} to be zero and $k_{yp}(z)$ to be a constant function. Similarly, by considering the pair (xp, yq), one obtains that the function $x \cdot c_{yq} - y \cdot d_{xp}$ must be constant, hence c_{yq} and d_{xp} are zero. To completely determine the functions η_v for these three vector fields, two restrictions, associated to the pairs (xp, xq) and (yp, xq), must be verified. The first imposes that $x(\eta_{xq})_x - x(\eta_{xp})_y - \eta_{xq} = x \cdot c_{xq} - c_{xq}x - d_{xq}y - k_{xq}(z)$ must be constant. Thus it leaves no choice but to take d_{xq} as the constant zero and $k_{xq}(z)$ as a constant function. Finally, the last restriction forces $y(\eta_{xq})_x - x(\eta_{yp})_y - \eta_{yq} + \eta_{xp} = y \cdot c_{xq} - k_{yq}(z) + K(z)$ to be a constant, hence c_{xq} must be zero while $k_{xq}(z)$ must be equal, modulo the constant functions, to the function $K(z)$. Putting together these restrictions, the image of the 1-cocycle F for the first six vector fields of $\mathfrak{h}$ can be described as $\eta_p = \eta_q = \eta_{xq} = \eta_{yp} = 0$ and $\eta_{xp} = \eta_{yq} = K(z)$. One easily checks that the remaining two restrictions are satisfied.

To determine completely the 1-cocycle F, it remains to find its images for the two vector fields $T := x^2p + xyq$ and $Q = xyp + y^2q$. For η_T, three restrictions are needed to reach that $\eta_T = 3xK(z)$. Indeed, from the pair (p, T), the cocycle condition forces the following equality $(\eta_T)_x - 2\eta_{xp} - \eta_{yq} = c_T$, where c_T is a constant. It is not hard to see that η_T must be equal to $3xK(z) + c_Tx + h_T(y, z)$. From the pair (q, T), we get similarly that $\eta_T = 3xK(z) + c_Tx + d_Ty + k_T(z)$. Finally, the restriction for the pair (xp, T) leads to

$$x(\eta_T)_x - T(\eta_{xp}) - \eta_T = x \cdot 3K(z) + x \cdot c_T - (3xK(z) + c_Tx + d_Ty + k_T(z)) \in \{1\}.$$

Hence the constant d_T dies out and $k_T(x)$ has to be a constant function. Note that, with these 3 restrictions, $\eta_T = 3x(K(z) + c_T/3)$, but, by taking $\eta_{xp} = K(z) + c_T/3$, one gets the claimed result. By symmetry on x and y, the exact same arguments lead to $\eta_Q = 3yK(z)$. It is then straightforward to verify that the 1-cocycle F, given by the eight functions

$$\eta_p = \eta_q = \eta_{xq} = \eta_{yp} = 0, \quad \eta_{xp} = \eta_{yq} = K(z), \quad \eta_P = 3xK(z) \text{ and } \eta_Q = 3yK(z),$$

satisfies all the other 1-cocycle conditions. Finally, the Lie algebra $\mathfrak{g}$ associated to this triple $(\mathfrak{h}, \{1\}, [F])$ is the Lie algebra spanned by

$$\{p, q, xp + K(z), yp, xq, yq + K(z), x^2p + xyq + 3xK(z), xyp + y^2q + 3yK(z), 1\},$$

where $K(z)$ can be any function.

Fortunately the calculations performed for a given Lie algebra $\mathfrak{h}$ can be repeated for any other Lie algebra sharing a subset of generators with $\mathfrak{h}$. Note also that some ad hoc lemma's were used trough this work to simplify these calculations. For instance.

Lemma 3.1. *Let* $i : \mathbb{R}^2 \to \mathbb{R}^3$, $(x, y) \mapsto (x, y, z)$ *denote the inclusion map and suppose that* $\mathfrak{h}_0 \subset \Gamma(i_* T\mathbb{R}^2)$, *meaning that the generators of* $\mathfrak{h}_0$ *depend on the variables* x *and* y *only. Let* $\mathfrak{h}$ *be a Lie algebra of vector fields on* $\mathbb{R}^3$ *given by* $\mathfrak{h} = \mathfrak{h}_0 \oplus \{r, zr, z^2 r\}$. *If, for non constant functions* $f(x, y)$ *and* $g(x, y)$, *the vector fields* $f(x, y)p$ *and* $g(x, y)q$ *belong to* $\mathfrak{h}$ *and if their associated images* $\eta_{f(x,y)p}$ *and* $\eta_{g(x,y)q}$ *depend on* x *and* y *only, then*

$$H^1(\mathfrak{h}, \mathcal{F}(\mathbb{R}^3)/\{1\}) = H^1(\mathfrak{h}_0, \mathcal{F}(\mathbb{R}^2)/\{1\}) \oplus H^1(\{r, zr, z^2 r\}, \mathcal{F}(\mathbb{R})/\{1\}).$$

Proof: Denote $A := f(x, y)p$ and $B := g(x, y)q$. From the cocycle restrictions associated to the pairs $(A, z^i r)$, where $i = 0, 1, 2$, we obtain

$$\begin{aligned} \langle \delta_1 F; A, z^i r \rangle &= A(\eta_{z^i r}) - z^i (\eta_A)_z - \langle F; [A, z^i r] \rangle \\ &= f(x, y)(\eta_{z^i r})_x - 0 - \langle F, 0 \rangle \\ &= f(x, y)(\eta_{z^i r})_x \in \{1\}. \end{aligned}$$

Since $f(x, y)$ is not constant, one can show with some extra work that $(\eta_{z^i r})_x$ must vanish, hence the functions $\eta_{z^i r}$ depend on y and z. In a similar way, from the restrictions associated to the pairs $(B, z^i r)$ it is straightforward to conclude that $\eta_{z^i r} = h^i(z)$. Finally, for any element v in $\mathfrak{h}_0$, the function η_v will depend on x and y only. Indeed, since η_{zr} depends on z only,

$$\begin{aligned} \langle \delta_1 F; v, zr \rangle &= v(\eta_{zr}) - z(\eta_v)_z - \langle F; [v, zr] \rangle \\ &= 0 - z(\eta_v)_z - \langle F, 0 \rangle \\ &= -z(\eta_v)_z \in \{1\}. \end{aligned}$$

Therefore $(\eta_v)_z$ must be zero, forcing the function η_v to depend on x and y only. □

Note that $H^1(\{r, zr, z^2 r\}, \mathcal{F}(\mathbb{R})/\{1\})$ is already well known. The 1-cocycle F associated the Lie algebra $\mathfrak{h} = \{r, zr, z^2 r\}$ is determined by three functions and the simplest representative is given by $\eta_r = 0$, $\eta_{zr} = 0$ and $\eta_{z^2r} = dz$, for any constant d. Thus, one can use this lemma to simplify some of the computations required in this classification problem. For instance, given $\mathfrak{h}$ the type I Lie algebra of vector fields given by Case 10 in Table 1, the Lie algebra of differential operators $\mathfrak{g}$ built from $\mathfrak{h}$ is obtained from a direct application of this lemma.

Type I, Case 10

The Lie algebra $\mathfrak{h} = \{p, q, xp, yq, xq, yp, x^2p + xyq, xyp + y^2q, r, zr, z^2r\}$ can be decomposed as $\mathfrak{h}_0 \oplus \{r, zr, z^2r\}$ where $\mathfrak{h}_0$ is the Case 1 Lie algebra from the same table. It was shown in the previous calculations that the functions η_{yp} and η_{xq} are zero, hence functions on x and y only. Thus the Case 10 Lie algebra, along with its two vector fields yp and xq, satisfies the requirements of the Lemma (3.1). Therefore, for the vector fields in the algebra $\mathfrak{h}_0$, the values of the 1-cocycle depend on x and y only, forcing $K(z)$ to be c a constant function. It is then obvious that the 1-cocycle F is defined by eleven functions, were the three non-zero are given by $\eta_T = cx$, $\eta_Q = cy$ and $\eta_{z^2r} = dx$, for c and d any constants. The Lie algebra of first order differential operators $\mathfrak{g}$ corresponding to this triple is then

$$\mathfrak{g} = \{p, q, xp, yq, xq, yp, x^2p + xyq + cx, xyp + y^2q + cy, r, zr, z^2r + dz, 1\}.$$

It should be pointed here that $H^1(\mathfrak{h}, \mathcal{F}(\mathrm{M})/\{1\})$ and $H^1(\mathfrak{h}, \mathcal{F}(\mathrm{M}))$ can also be determined alternatively using isomorphisms given in [37] and [35]. For the case that interests us, that is $H^1(\mathfrak{h}, \mathcal{F}(\mathrm{M})/\{1\})$, we fix a base point e and denote $\mathfrak{i}$ the isotropy subalgebra. Provided the existence of a subalgebra $\mathfrak{a} \subset \mathfrak{h}$ which is complementary to $\mathfrak{i}$, one can show, see [37], that

$$H^1(\mathfrak{h}, \mathcal{F}(\mathrm{M})/\{1\}) \cong H^2(\mathfrak{h}/\mathfrak{i}).$$

This isomorphism leads to an explicit method for constructing 1-cocycle representatives F in $H^1(\mathfrak{h}, \mathcal{F}(\mathrm{M})/\{1\})$. We first choose α, a 2-cocycle representative of a class in $H^2(\mathfrak{h}/\mathfrak{i})$, and, for $\{v^1, ..., v^n\}$ a basis of $\mathfrak{h}$, we denote $\alpha_{ij} = \alpha(v^i, v^j)$. If $\mathfrak{a} = \{v^1, ..., v^m\}$, $\mathfrak{i} = \{v^{m+1}, ..., v^n\}$, and c_{ij}^k are the structure constants of the Lie algebra $\mathfrak{h}$, a 1-cocycle in $H^1(\mathfrak{h}, \mathcal{F}(\mathrm{M})/\{1\})$ will be obtained by solving first the following $m(m-1)$ equations

$$v^i(f_j) - v^j(f_i) - \sum_k c_{ij}^k f_k = \alpha_{ij}, \quad \text{for} \quad 1 \leq i < j \leq m.$$

Once a non unique solution $f_1, ..., f_m$ is obtained, the remaining functions $f_{m+1}, ..., f_n$ are determined as the unique solution to the $m(n-m)$ equations

$$v^i(f_j) - v^j(f_i) - \sum_k c_{ij}^k f_k = \alpha_{ij}, \quad \text{for} \quad 1 \leq i \leq m, \quad m+1 \leq j \leq n,$$

with initial conditions

$$f_i(e) = 0 \quad \text{for} \quad m+1 \leq i \leq n.$$

Note however that this method can not be applied to all the three dimensional Lie algebras since the existence of the complementary Lie subalgebra is not guaranteed. For instance, the type III Case $17A_1$ can not be treated using the isomorphism. Indeed, for the Lie algebra

$$\mathfrak{h} = \{p, q, xp + zr, yq, x^2p + (2x + az)zr, y^2q\},$$

and the base point $e = (0,0,0)$, the isotropy algebra $\mathfrak{i}$ is generated by the last four elements and the algebra $\mathfrak{a} = \{p, q\}$ fails to be complementary, due to the absence of the element r in the Lie algebra. One can easily verify that $Z^2(\mathfrak{h}/\mathfrak{i}) = \{\alpha_1 \wedge \alpha_3, \alpha_1 \wedge \alpha_5, \alpha_2 \wedge \alpha_4, \alpha_2 \wedge \alpha_6\}$ and $B^2(\mathfrak{h}/\mathfrak{i}) = \{\alpha_1 \wedge \alpha_3, \alpha_2 \wedge \alpha_4\}$. One can observe at that point that the theorem does not hold, since the dimension of $H^2(\mathfrak{h}/\mathfrak{i})$ is two while the dimension of $H^1(\mathfrak{h}, \mathcal{F}(\mathrm{M})/\{1\})$ was computed to be three previously. Moreover applying the technique to the 2-cocycle $\alpha = c \cdot \alpha_1 \wedge \alpha_5 + d \cdot \alpha_2 \wedge \alpha_6$, one gets a 1-cocycle that does not satisfies all the conditions that were considered in the technique detailed above.

3.2.3 Classification of Quasi-Exactly Solvable Lie Algebras of First Order Differential Operators and the Quantization Condition

The Lie algebras given in Table 1 and 2 are the candidates for being quasi-exactly solvable Lie algebras, *i.e.* we might expect them to admit $\mathcal{N}$ a finite dimensional module of smooth functions $\mathcal{N}$. In the investigation for these explicit finite dimensional modules, some new restrictions are imposed on the 1-cocycles F. Indeed, as for the quasi-exactly solvable Lie algebras in lower dimensions, it comes out that a finite dimensional module exists only if the values of the functions η_v are taken in a certain discrete set. For this reason, this restriction is named quantization condition. The quasi-exactly solvable Lie algebras and their fixed modules can be found in Tables 3 and 4 for, respectively, the type I and the selected type III Lie algebras. The first column use the same classification numbers as in Tables 1 and 2 and a representative for the non-trivial quantized 1-cocycles is exhibited in the second column. Finally, $\mathcal{N}$, the finite dimensional $\mathfrak{g}$-modules of functions are described in the last column. Once again the detailed calculations are repetitive and the essence of the work can be grasped with one or two examples, together with the following general principles.

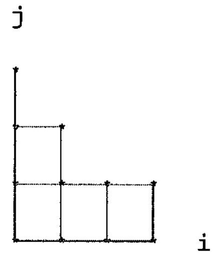
1. A finite dimensional module for the trivial Lie algebra $\mathfrak{g} = \{p\}$ is defined as an *x-translation module.* For instance, any space spanned by a finite set of functions of the form
$$h = \sum_{i=0}^{n} g^i(y,z)x^i,$$
along with all their x derivatives, is an x-translation module. This particular case of x-translation module is referred as a *semi-polynomial x-translation module.* The most general x-translation module is obtained by a direct sum
$$\mathcal{N} = \bigoplus_{\lambda\in\Lambda} \mathcal{N}_\lambda,\ \mathcal{N}_\lambda = \widehat{\mathcal{N}_\lambda} e^{\lambda x},$$

where $\widehat{\mathcal{N}_\lambda}$ are semi-polynomial x-translation modules and the exponents are taken in a finite set Λ, read [21] for more details. Obviously, the y, and the z-translation modules are defined the exact same way.

2. If the Lie algebra $\mathfrak{g}$ under consideration contains the two differential operators p and xp, the module $\mathcal{N}$ will be an x-translation module and the operator xp will impose extra constraints. Firstly, all the exponents λ need be zero. Otherwise, for an non-zero exponent λ, the degree in x of the generating functions in the module $\widehat{\mathcal{N}_\lambda}$ would be unbounded, contradicting the finite dimensionality of $\mathcal{N}$. Moreover, if $h = \sum_{i=0}^{n} g^i(y,z)x^i$ belongs to the module $\mathcal{N}$, the function xh_x also needs to belong to that module. Note that both functions have the same degree in x and are linearly independent if h is not a monomial. Thus, by an appropriate linear combination of these two functions, one can reduce the number of summands in h. By iterating this process, each generating function can be reduced to a monomial in x. Thus, $h = g(y,z)x^i$ where $g(y,z)$ belongs to G^i a finite set of functions in y and z. Since $\mathcal{N}$ is a x-translation module, $h_x = ig(y,z)x^{i-1}$ is also a function in $\mathcal{N}$, hence $g(y,z)$ needs to be also contained in G^{i-1}. Therefore, the module $\mathcal{N}$ decomposes into the following direct sum
$$\mathcal{N} = \bigoplus x^i g^i_k(y,z) \quad i = 0...n, \quad k = 0...l_i,$$
where all the functions $g^i_k(y,z)$ belong to G^i a finite set and where $G^i \subseteq G^{i-1}$.

3. Likewise, if a Lie algebra $\mathfrak{g}$ contains the elements p, q, xp, and yq, a general finite dimensional $\mathfrak{g}$-module for this Lie algebra will be at most
$$\mathcal{N} = \bigoplus x^i y^j g^{i,j}_k(z), \quad i = 0...n, \quad j = 0...m, \quad k = 0...l_{(i,j)},$$
where the functions $g^{i,j}_k(z)$ belong to $G^{(i,j)}$, a finite set of functions of z satisfying $G^{(i,j)} \subseteq G^{(i-1,j)} \cap G^{(i,j-1)}$.

A simple method to describe these modules is to represent each generating function $x^i y^j g^{i,j}(z)$ by a point (i,j), in the Cartesian plane. If a vertex (i,j) belongs to the diagram, since $\mathcal{N}$ is an xy-translation module, the vertices $(i-1,j)$ and $(i,j-1)$ must also sit in the diagram. To complete the description, a finite set $G^{(i,j)}$ is associated to each of these vertices, with the same restriction as above. For instance, such module $\mathcal{N}$ can be represented by



with all the sets $G^{(i,j)}$ being equal to $\{z, e^z\}$, with the exception of $G^{(3,1)}$ that contains only the function z. It is then straightforward to verify that this module is indeed

$$\begin{aligned}\mathcal{N} &= \{z, e^z, xz, xe^z, yz, ye^z, x^2z, x^2e^z, xyz, xye^z, y^2z,\\ &\quad y^2e^z, x^3z, x^3e^z, x^2yz, x^2ye^z, xy^2z, xy^2e^z, y^3z, y^3e^z, x^3yz\},\end{aligned}$$

and that it is a $\mathfrak{g}$-module for the Lie algebra $\mathfrak{g} = \{p, xp, q, yq\}$.

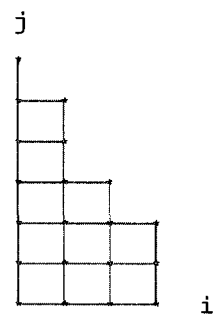
4. If the Lie algebra $\mathfrak{g}$ contains the differential operators p, q, xp, yq and yp, from the three previous principles, the generators for a $\mathfrak{g}$-module are given by $h = x^i y^j g^{i,j}(z)$. After applying the operator yp on h, the resulting function reads as $ix^{i-1}y^{j+1}g^{i,j}(z)$. Thus, iterating this operator, we conclude that all the functions $x^{i-r}y^{j+r}g^{i,j}(z)$ must belong to $\mathcal{N}$, for $r \leq i$. Since p and q also belong to the algebra, all the functions $x^a y^b g^{i,j}(y,z)$ with $a \leq i$, $b \leq j$ and $a+b \leq c$ must belong to $\mathcal{N}$. This condition can be expressed by the following inclusion

$G^{(i,j)} \subseteq G^{(i-1,j)} \cap G^{(i,j-1)} \cap G^{(i-1,j+1)}$, and observe that the first set $G^{(i-1,j)}$ can be omitted without affecting the condition. To summarize, the $\mathfrak{g}$-module will be at most

$$\mathcal{N} = \bigoplus x^i y^j g_k^{i,j}(z), \quad i = 0...n, \quad j...m, \quad k = 0...l_{(i,j)},$$

where the functions $g_k^{i,j}(z)$ belong to $G^{(i,j)}$, a finite set of functions with $G^{(i,j)} \subseteq G^{(i-1,j+1)} \cap G^{(i,j-1)}$.

Once again it is possible to represent such module by a diagram along with a set of functions $G^{(i,j)}$ for each vertex of the diagram. The restrictions for these sets are $G^{(i,j)} \subseteq G^{(i-1,j+1)} \cap G^{(i,j-1)}$ and the conditions on the vertices are slightly different from the one in the previous example. Indeed, if a vertex (i,j), belongs to the diagram, the two vertices $(i-1, j+1)$ and $(i, j-1)$ must also belong to the diagram. Note again that this implies that the vertex $(i-1, j)$ also lies in the diagram. For instance the diagram,



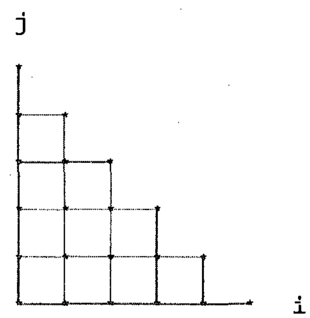
together with twenty appropriate sets of functions $G^{(i,j)}$ for each vertex, would generate a $\mathfrak{g}$-module for the algebra $\mathfrak{g} = \{p, q, xp, yq, yp\}$.

5. Finally, if the elements p, q, xp, yq, yp and xq sit in the Lie algebra under consideration, the module will be at most

$$\mathcal{N} = \bigoplus x^i y^j g_k^{i,j}(z), \quad i + j = 0...n, \quad k = 0...l_{i,j},$$

where the functions $g_k^{i,j}(z)$ belong to $G^{(i+j)}$, a finite set of functions with $G^{(l)} \subseteq G^{(l-1)}$. Indeed, consider $h = x^i y^j g^{i,j}(z)$, a generator of bi-degree $i+j = c$. Since $xq[h]$ and $yp[h]$ must also lie in $\mathcal{N}$, all the functions $x^a y^b g^{i,j}(z)$ with $a + b = c$ will belong to $\mathcal{N}$. Hence $g^{(i,j)} \in G^{(a,b)}$ and, reciprocally, $g^{(a,b)} \in G^{(i,j)}$. Thus for all pairs (a, b) with $a + b = c$, the finite sets $G^{(a,b)}$ are identical and it is therefore well defined to pose $G^{(a,b)} = G^{(a+b)}$. Obviously, since $\mathcal{N}$ is a xy-translation module, the following inclusions hold $G^{(l)} \subseteq G^{(l-1)}$.

For these modules, the possible diagrams are more restricted and have necessarily the shape of a staircase. Also, instead of assigning one set of functions to each vertex, such a set is coupled to all the vertices having same total degree $i + j$. For instance, the module represented by the diagram



will be completely determined after fixing six sets of function in z. Note that this choice must respect the inclusion $G^{(l)}(z) \subseteq G^{(l-1)}(z)$, for $i = 1, ...5$.

This set of principles is of great help in the determination of the possible $\mathfrak{g}$-modules $\mathcal{N}$ for each Lie algebras of first order differential operators $\mathfrak{g}$ described in Tables 1 and 2. Depending on the elements contained in the Lie algebra studied, we started our search of $\mathfrak{g}$-module based on the general module given in this guideline. Once again, the computations are tedious and it would not be relevant to detail each of them. We will concentrate on the same Lie algebras as in the previous step of this classification problem, that is the type I Lie algebras Case 1 and 10.

Type I, Case 1

Since the Lie algebra contains the differential operators p, q, xp, and yq, from the principle (3), the most general module $\mathcal{N}$ will be spanned by functions of the form $h = x^i y^j g^{i,j}(z)$, where $g^{i,j}(z)$ belongs to $G^{(i,j)}$. We now consider the operator $T = x^2p + xyq + 3xK(z)$ in the algebra $\mathfrak{g}$ and its action on $h = x^n y^a g^{n,a}(z)$ a generator of $\mathcal{N}$ with maximal exponent in x. Thus

$$\begin{aligned} T[h] &= nx^{n+1}y^a g^{n,a}(z) + ax^{n+1}y^a g^{n,a}(z) + 3K(z)x^{n+1}y^a g^{n,a}(z) \\ &= (n + a + 3K(z))x^{n+1}y^a g^{n,a}(z). \end{aligned}$$

Since the exponent in x was taken to be maximal, this imposes that $K(z)$ is indeed a constant K equal to $-\frac{n+a}{3}$. Symmetrically, by considering $Q = xyp+y^2q+3yK(z)$ and $h = x^b y^m g^{b,m}(z)$, a function with maximal exponent in y, the following equality holds $K = -\frac{b+m}{3}$. For this to be possible, we necessarily have $n+a = b+m$. Consequently, the differential operators xq and yp belong to the Lie algebra $\mathfrak{g}$ and the module $\mathcal{N}$ is given by the principle (5). Also note that the operators xp and yq force both a and b to be zero. Otherwise $x^{n+1}y^{a-1}g^{n,a}(z)$ and $x^{b-1}y^{m+1}g^{b,m}(z)$ would be in $\mathcal{N}$, contradicting the maximality of n and m. Thus $3K(z) = -n$ and the module

$$\mathcal{N} = \{x^i y^j g^{i,j}(z) \mid i + j \leq n,\ g^{i,j}(z) \in G^{(i+j)}\}, \text{ where } G^{(l)} \subseteq G^{(l-1)} \tag{3.4}$$

is fixed by all the differential operators in $\mathfrak{g}$. Therefore it is possible to conclude that the Lie algebra

$$\mathfrak{g} = \{p, q, xp, yq, xq, yp, x^2p + xyq - nx, xyp + y^2q - ny, 1\},$$

is quasi-exactly solvable with respect to the finite dimensional $\mathfrak{g}$-module $\mathcal{N}$.

Type I, Case 10

Since the Case 10 Lie algebra contains the Case 1 Lie algebra, its module $\mathcal{N}$ will be at best the module given in (3.4). Observe first that the constant c in the Case 10 Lie algebra has to be the negative integer $-n$. Furthermore, the operator r imposes $\mathcal{N}$ to be a z-translation module and the operator zr forces $G^{(l)}$ to be generated by monomials. Then, for z^m a monomial of maximal degree in $G^{(l)}$, the function $h = x^i y^{l-i} z^m$ belongs to $\mathcal{N}$. Since $z^2 r + dz$ belongs to the Lie algebra,

$$\begin{aligned} z^2 r + dz[h] &= m x^i y^{l-i} z^{m+1} + d x^i y^{l-i} z^{m+1} \\ &= [m+d] x^i y^{l-i} z^{m+1}, \end{aligned}$$

should belong to the $\mathfrak{g}$-module $\mathcal{N}$. Thus, from the maximality of the degree in z, the constant d has to be the negative integer $-m$. Since the argument must hold for every set $G^{(l)}$, they will all share the same monomial of maximal degree m. We can therefore conclude that the Lie algebra

$$\mathfrak{g} = \{p, q, xp, yq, xq, yp, x^2p + xyq - nx, xyp + y^2q - ny, r, zr, z^2r - mz, 1\},$$

is quasi-exactly solvable with respect to the module

$$\mathcal{N} = \{x^i y^j z^k \mid i + j \leq n, k \leq m\}.$$

To summarize, a partial classification of quasi-exactly solvable Lie algebras of first order differential operators was accomplished in this section and the description of these Lie algebras $\mathfrak{g}$, along with their $\mathfrak{g}$-modules, can be found in Tables 1-4. In principle, it would be possible to achieve a complete classification using similar arguments. However this gigantic work would require a colossal amount of time. Nevertheless this partial classification is a good starting point for seeking new quasi-exactly solvable Schrödinger operators in three dimensions. In that scope, the next section is devoted to the description of few new quasi-exactly solvable Schrödinger operators.

Table 1. Cohomology for the Type I Lie Algebras of Vector Fields, $m = \{1\}$

	Generators	Cocycles
1	$\{p, q, xp, yq, xq, yp,$ $x^2p + xyq, xyp + y^2q\}$	$xp + K(z), x^2p + xyq + xK(z),$ $yq + K(z), xyp + y^2q + yK(z)$
2	$\{xq, xp - yq, yp,$ $Z^1(z)p, ..., Z^l(z)p, Z^1(z)q, ..., Z^l(z)q\}$	0
3	$\{xq, xp - yq, yp, xp + yq,$ $Z^1(z)p, ..., Z^l(z)p, Z^1(z)q, ..., Z^l(z)q\}$	$xp + yq + K(z)$
4	$\{p, q, xp, yq, xq, yp,$ $x^2p + xyq, xyp + y^2q, r\}$	$x^2p + xyq + cx,$ $xyp + y^2q + cy$
5	$\{xq, xp - yq, yp, z^k e^{\lambda_l z}p, z^k e^{\lambda_l z}q, r\}$ $k \leq n_l,\ l = 0...b$	0
6	$\{xq, xp - yq, yp, xp + yq,$ $z^k e^{\lambda_l z}p, z^k e^{\lambda_l z}q, r\}\ k \leq n_l,\ l = 0...b$	$xp + yq + cz$
7	$\{p, q, xp, yq, xq, yp,$ $x^2p + xy, q, xyp + y^2q, r, zr\}$	$x^2p + xyq + cx,$ $xyp + y^2q + cy$
8	$\{xq, xp - yq, yp, p, zp, ...z^lp,$ $q, zq, ..., z^lq, r, zr + a(xp + yq)\}$	0
9	$\{xq, xp - yq, yp, xp + yq, p, zp, ..., z^lp,$ $q, zq, ..., z^lq, r, zr\}\ k \leq n_l,\ l = 0...b$	0
10	$\{p, q, xp, yq, xq, yp,$ $x^2p + xyq, xyp + y^2q, r, zr, z^2r\}$	$x^2p + xyq + cx,\ xyp + y^2q + cy,$ $z^2r + dz$
11	$\{xq, xp - yq, yp, p, zp, ...z^lp, q, zq, ..., z^lq,$ $r, zr + \frac{a}{2}(xp + yq), z^2r + az(xp + yq)\}$	$z^2r + az(xp + yq) + cz$
12	$\{xq, xp - yq, yp, xp + yq, p, zp, ..., z^lp,$ $q, zq, ..., z^lq, r, zr, z^2r + az(xp + yq)\}$	$z^2r + az(xp + yq) + cz$

Note here that b, l and n_l are positive integers, a, c, d, k and λ_l are arbitrary constants and $Z^1(z), \ldots, Z^l(z)$ and $K(z)$ functions of z.

Table 2. Cohomology for Some Type III Lie Algebras of Vector Fields, $m = \{1\}$

	Generators	Cocycles
4A	$\{p, yq, q, xq, ..., x^t q + r, ...,$ $x^i q + \binom{t}{i} x^{t-i} r, ..., x^s q + \binom{t}{s} x^{s-t} r,$ $xp - tzr, yq + zr\}$ $0 \leq t \leq s$	0
4C	$\{q, xq, ..., x^s q, p, yq, xp,$ $x^l y^{n-b} r, zr\ \}$ $0 \leq b \leq n,\ l \leq l_0 + sb$	0
4D	$\{q, xq, ..., x^s q, p, yq, xp,$ $r, zr, z^2 r\ \}$	$z^2 r + cz$
5A*	$\{q + r, xq + xr, ..., x^s q + x^s r, p,$ $xp, yq + zr, x^2 p + sxyq + sxzr\}$	$x^2 p + sxyq + sr + cx$
5C	$\{q, xq, ..., x^s q, p, yq, xp,$ $x^2 p + sxyq + (l_0 + s\tilde{n}) xzr,$ $x^l y^{\tilde{n}-b} r, zr\ \}$ $0 \leq b \leq \tilde{n},\ l \leq l_0 + sb$	$x^2 p + sxyq + (l_0 + s\tilde{n}) r + cx$
5D	$\{q, xq, ..., x^s q, p, yq, xp$ $x^2 + sxyq, r, zr, z^2 r\}$	$x^2 p + sxyq + cx$ $z^2 r + dz$
7C	$\{p, 2xp + yq, x^2 p + xyq,$ $x^l y^{-n} r, zr\ \}\ 0 \leq l \leq n,$	$x^2 p + xyq + cy^2$
$17A_1$	$\{p, q, xp + zr, yq,$ $x^2 p + (2x + az) zr, y^2 q\ \}$	$x^2 p + (2x + az) zr + bx + cz,$ $y^2 q + dy$
$17A_2$	$\{p, q, xp + azr, yq + zr,$ $x^2 p + 2axzr, y^2 q + 2yzr\ \}$	$x^2 p + 2axzr + cx,$ $y^2 q + 2yzr + dy$
17C	$\{p, q, xp, yq, x^2 p + l_0 xzr,$ $y^2 q + p_0 yzr, x^l y^p r\}$ $l \leq l_0,\ b \leq b_0$	$x^2 p + l_0 xzr + cx,$ $y^2 q + b_0 yzr + dy$
17D	$\{p, xp, x^2 p, q, yp, y^2 q,$ $r, zr, z^2 r\}$	$x^2 p + ax, y^2 q + cy$ $z^2 r + dz$

Note here that b, b_0, l, l_0, $\tilde{n}$, s and t are positive integers, a, c and d are arbitrary constants. Remark that Amaldi's Lie algebra 5A is not a Lie algebra. Indeed, for the space spanned by $\{q, xq, ..., x^t q + r, ..., x^{t+i} q + \binom{t+i}{t} x^i r, ..., x^s q + \binom{s}{t} x^{s-t} r, p, yp, xp - tzr, yq + zr, x^2 p + sxyq + (s-2t)xzr, x^l y^{n-b} r, zr\}$ to be a Lie algebra, the parameter t needs to be zero. We then get the Lie Algebra 5A* given in the table.

Table 3. Type I Quasi-Exactly Solvable Lie Algebras of Differential Operators and their Fixed Modules

	Quantization Condition	Fixed Module
1	$x^2p + xyq - nx$, $xyp + y^2q - ny$	$\{x^iy^jg(z)\mid i+j \le n,\ g(z) \in G^{(i+j)}\}$, for $G^{(l)}$ finite with $G^{(l)} \subseteq G^{(l-1)}$
2	0	$\{x^iy^jg(z)\mid i+j \le n,\ g(z) \in G^{(i+j)}\}$, for $G^{(l)}$ finite with $Z^iG^{(l)} \subseteq G^{(l-1)}$
3	0	$\{x^iy^jg(z)\mid i+j \le n,\ g(z) \in G^{(i+j)}\}$, for $G^{(l)}$ finite, and $Z^iG^{(l)} \subseteq Z^iG^{(l-1)}$
4	$x^2p + xyq - nx$, $xyp + y^2q - ny$	$\{x^iy^jz^ke^{\lambda_l z}\mid i+j \le n,\ k \le m_l,\ l = 0, \ldots p\}$
5	0	$\{x^iy^jz^ke^{\lambda_l z}\mid i+j \le n,\ z^ke^{\lambda_l z} \in G^{(i+j)}\}$, for $G^{(l)}$ a finite z-translation module, and $Z^iG^{(l)} \subseteq G^{(l-1)}$ for $Z^i = z^ke^{\lambda_l z}p$
6	0	$\{x^iy^jz^ke^{\lambda_l z}\mid i+j \le n,\ k \le m_l,\ l = 0, \ldots p\}$, for $G^{(l)}$ a finite z-translation module, and $Z^iG^{(l)} \subseteq G^{(l-1)}$ for $Z^i = z^ke^{\lambda_l z}p$
7	$x^2p + xyq - nx$, $xyp + y^2q - ny$	$\{x^iy^jz^k\mid i+j \le n, k \le m\}$
8	0	$\{x^iy^jz^k\mid i+j \le n,\ l(i+j)+k \le m\}$
9	0	$\{x^iy^jz^k\mid i+j \le n,\ l(i+j)+k \le m\}$
10	$x^2p + xyq - nx$, $xyp + y^2q - ny$, $z^2r - mz$	$\{x^iy^jz^k\mid i+j \le n, k \le m\}$
11	$z^2r + az(xp+yq) - mz$	$\{x^iy^jz^k\mid i+j \le n,\ a(i+j)+k \le m\}$
12	$z^2r + az(xp+yq) - mz$	$\{x^iy^jz^k\mid i+j \le n,\ a(i+j)+k \le m\}$

Note here that m and n are non-negative integers.

Table 4. Some of the Type *III* Quasi-Exactly Solvable Lie Algebras of Differential Operators and their Fixed Modules

	Quantization Condition	Fixed Module
4A	0	$\{x^iy^jz^k \mid i+sj+(s-t)k \leq n,$ $j \leq m_y, k \leq m_z\}$
4C	0	$\{x^iy^jz^k \mid i+sj+(l_0+sn)k \leq n,$ $k \leq b, j \leq b_k\}$ with $b_{k-1} \geq b_k + n$
4D	$z^2r - mz$	$\{x^iy^jz^k \mid i+sj \leq n, k \leq m\}$
5A*	$x^2p + sxyq + sr - nx$	$\{x^iy^jz^k \mid i+s(j+k) \leq n,$ $j \leq m_y, k \leq m_z\}$
5C	$x^2p + sxyq + (l_0+s\tilde{n})xzr - nx$	$\{x^iy^jz^k \mid i+sj+(l_0+s\tilde{n})k \leq n,$ $k \leq b, j \leq b_k\}$ with $b_{k-1} \geq b_k + \tilde{n}$
5D	$x^2p + sxyq - nx,$ $z^2 - mz$	$\{x^iy^jz^k \mid i+sj \leq n, k \leq m\}$
7C	0	0
17A$_1$	$x^2p + (2x+az)zr - m_xx,$ $y^2q - m_yy$	$\{x^iy^j \mid i \leq m_x, j \leq m_y\}$
17A$_2$	$x^2p + 2axzr - m_xx,$ $y^2q + 2yzr - m_yy$	$\{x^iy^jz^k \mid i+2ak \leq m_x,$ $j+2k \leq m_y, k \leq m_z\}$
17C	$x^2p + l_0xzr - m_xx,$ $y^2q + b_0yzr - m_yy$	$\{x^iy^jz^k \mid i+l_0k \leq m_x,$ $j+b_0k \leq m_y, k \leq m_z\}$
17D	$x^2p - m_xx, y^2q - m_yy$ $z^2r - m_zz$	$\{x^iy^jz^k \mid i \leq m_x, j \leq m_y, k \leq m_z\}$

Note here that m, n, m_x, m_y and m_z are non-negative integers.

3.3 New Quasi-Exactly Solvable Schrödinger Operators in Three Dimensions

Recall that in the general classification problem, once the quasi-exactly solvable Lie algebras of differential operators $\mathfrak{g}$ are determined, the next step is to construct second order differential operators $\mathcal{H}$ that are locally equivalent to Schrödinger operators. Given $\mathfrak{g}$, one of the Lie algebras of first order differential operators obtained in the previous section, we obtain a second order differential operator $\mathcal{H}$ by letting

$$\mathcal{H} = \sum_{a,b=1}^{m} C_{ab} T^a T^b + \sum_{a=1}^{m} C_a T^a + C_0, \text{ where } T^a \in \mathfrak{g}, \tag{3.5}$$

as illustrated previously. Then, one has to choose the coefficients C_{ab}, C_a, C_0 in such a way that the closure conditions $d\omega = 0$ are satisfied. Given $T^a = \xi^{ai}\partial_i + \eta^a$, the closure conditions are the Frobenius compatibility conditions for the overdetermined system

$$\sum_{a,b=1}^{m} \xi^{ai} \left[C_{ab} \left(\sum_{j=1}^{n} (\xi^{bj} \frac{\partial \alpha}{\partial x^j} + \frac{\partial \xi^{bj}}{\partial x^j}) - 2\eta^b \right) - C_a \right] = 0,$$

where $\alpha = \lambda + \frac{1}{2}\ln(g)$ and $\mu = e^{\lambda}$ is the gauge factor. Finally, we need to bear in mind that the last step in the classification is to verify that the operators are normalizable, *i.e.* the functions in $\widetilde{\mathcal{N}}$, the module obtained after the gauge transformation, need to be square integrable. These operators will therefore have the property that part of their spectrum can be explicitly computed.

We have now in hand a large variety of generating quasi-exactly solvable Lie algebras $\mathfrak{g}$. The door is therefore wide open to the construction of numerous new quasi-exactly solvable Schrödinger operators in three dimensions. However the Schrödinger operators described in this paper are built only from two of these new quasi-exactly solvable Lie algebras: the type III Cases $17D$ and Case $5A^*$. The reader can therefore see that many more examples can be constructed using this method together with the results of the previous section.

3.3.1 Type III, Case $17D$, $(\mathfrak{sl}(2) \times \mathfrak{sl}(2) \times \mathfrak{sl}(2))$.

The first two families of normalizable quasi-exactly solvable Schrödinger operators displayed in this section are similar to the operator given in the example (1.1). However, these two examples are more general. Indeed, for these two operators, the type III Case $17D$ quasi-exactly solvable Lie algebra $\mathfrak{g}$ is spanned by the first order differential operators

$$p, \quad q, \quad r, \quad xp, \quad yq, \quad zr, \quad x^2p - m_x x, \quad y^2q - m_y y, \quad z^2r - m_z z,$$

where m_x, m_y and m_z are non negative integers and the module $\mathcal{N}_{m_x m_y m_z}$ is generated by the $(m_x+1)(m_y+1)(m_z+1)$ monomials

$$x^i y^j z^k \qquad \text{where} \quad 0 \le i \le m_x, \quad 0 \le j \le m_y \ \text{ and } \ 0 \le k \le m_z.$$

First Example

The first family of operators is constructed with the following choice of coefficients

$$C_{ab} = \begin{pmatrix} A & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & B & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & C & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2A & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2B & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2C & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & A & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & B & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & C \end{pmatrix},$$

$C_a = [0,0,0,-2Am_x,-2Bm_y,-2Cm_z,0,0,0]$ and $C_0 = -Am_x - Bm_y - Cm_z$.

From the order two terms of these second order differential operators, the induced contravariant metric $g^{(ij)}$ is obtained and reads as

$$\begin{pmatrix} A(x^2+1)^2 & 0 & (x^2+1)(z^2+1) \\ 0 & By^4+2(B+1)y^2+B & 0 \\ (x^2+1)(z^2+1) & 0 & C(z^2+1) \end{pmatrix}. \tag{3.6}$$

The determinant of the matrix is $g = (1-AC)(x^2+1)^2(By^4+2(B+1)y^2+B)(z^2+1)^2$ and one easily verifies, for A, B and C positive and $AB > 1$, that the matrix is positive definite on $\mathbb{R}^3$. The operator can therefore be written as

$$-2\mathcal{H} = \Delta + \vec{V} + U_0,$$

where Δ is the Laplace-Beltrami operator related to the metric (3.6) and where

$$\vec{V} = -2(x^2+1)(Am_x x + m_z z)p - 2m_y(By^3 + By + y)q - 2(z^2+1)(Cm_z z + m_x x)r.$$

From a direct computation, the closure conditions are verified and the gauge factor required to gauge transform $\mathcal{H}$ into a Schrödinger operator $\mathcal{H}_0$ is given by

$$\mu = (x^2+1)^{\frac{-m_x}{2}}(By^4 + 2(B+1)y^2 + B)^{\frac{-m_y}{4}}(z^2+1)^{\frac{-m_z}{2}}.$$

Once the transformation is performed, the equivalent operator reads as

$$-2\mathcal{H}_0 = \Delta + U,$$

where the potential of the Schrödinger operator is given by the rational function :

$$\begin{aligned} U \;=\;& 2(m_3BCy^4 - 2m_2Cy^2 - m_2BCy^4 - 2m_2BCy^2 - m_3C^2 \\ & -2m_2By^2 - m_2BC - m_3C^2y^4 - 2m_3C^2y^2 + 2m_3BCy^2 \\ & +2m_2C^2y^2 + 2m_3By^2 + m_3BC + m_2C^2y^4 - 2m_2y^2 - m_2^2y^2 \\ & -2m_2^2Cy^2 + m_2C^2 - 2m_3Cy^2) \setminus (Cy^4 + 2y^2 + 2Cy^2 + C). \end{aligned}$$

The expressions of the potentials for the given examples can all be computed explicitly. However, we only present this potential since the expressions of the two other potentials are very long.

Note that the same three factors arise in both μ and g. This will simplify our computations while testing the square integrability of the functions in $\widetilde{\mathcal{N}}$. Indeed, a function in $\widetilde{\mathcal{N}}$ is given by $h = \mu x^i y^j z^k$ where the exponents i, j and k are smaller or equal to m_x, m_y and m_z respectively. Our aim is to show that the triple integral

$$\int\int\int_{\mathbb{R}^3} (\mu x^i y^j z^k)^2 \sqrt{g^{-1}} dxdydz$$

is finite. Obviously, it is sufficient to show the convergence of this integral for the monomials of maximal exponent. We can therefore focus on

$$\int\int\int_{\mathbb{R}^3} \frac{x^{2m_x} y^{2m_y} z^{2m_z}}{(x^2+1)^{m_x+1}(By^4+2(B+1)y^2+B)^{\frac{m_y}{2}+\frac{1}{2}}(z^2+1)^{m_z+1}} dxdydz.$$

Using Fubini's theorem, this triple integral can be factored into the product of three integrals

$$\int_{-\infty}^{\infty} \frac{x^{2m_x}}{(x^2+1)^{m_x+1}} dx, \int_{-\infty}^{\infty} \frac{y^{2m_y}}{(By^4+2(B+1)y^2+B)^{\frac{m_y}{2}+\frac{1}{2}}} dy, \text{ and } \int_{-\infty}^{\infty} \frac{z^{2m_z}}{(z^2+1)^{m_z+1}} dz,$$

each of which is easily shown to be convergent. We therefore have in hand a normalizable quasi-exactly solvable Schrödinger operator and it is feasible to determine explicitly part of its spectrum.

For instance, if we fix $m_x = 0$, $m_y = 2$ and $m_z = 1$, few manipulations lead to the six eigenfunctions of the operator $\mathcal{H}$ restricted to $\mathcal{N}$. Indeed, with this choice of parameters, the $\mathfrak{g}$-module is

$$\mathcal{N} = \{1, y, y^2, z, yz, y^2z\},$$

and the transformation matrix to be diagonalized reads as

$$\begin{pmatrix} -2 & 0 & 2 & B & 0 & 0 \\ 0 & -4-2B & 0 & 0 & 0 & 0 \\ 2B & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2B \\ 0 & 0 & 0 & 0 & -4-2B & 0 \\ 0 & 0 & 0 & 2B & 0 & -2 \end{pmatrix}.$$

Once the diagonalization is performed, three different eigenvalues $\lambda_1 = -4-2B$, $\lambda_2 = -2-2B$, and $\lambda_3 = -2+2B$ are obtained, each of them having multiplicity two. The six eigenfunctions are respectively

$$\psi_{1,1} = y, \ \psi_{1,2} = yz,$$

$$\psi_{2,1} = 1+y^2, \ \psi_{2,2} = -1+y^2,$$

$$\psi_{3,1} = -z+y^2z, \quad \psi_{3,2} = z+y^2z.$$

Consequently, we obtain three multiplicity two eigenvalues of the Schrödinger operator $\mathcal{H}_0$: $\widetilde{\lambda_1} = 1-B$, $\widetilde{\lambda_2} = 1+B$ and $\widetilde{\lambda_3} = 2+B$, and the six scaled eigenfunctions are

$$\widetilde{\psi_{1,1}} = \mu y, \ \widetilde{\psi_{1,2}} = \mu yz,$$

$$\widetilde{\psi_{2,1}} = \mu(1+y^2), \ \widetilde{\psi_{2,2}} = \mu(-1+y^2),$$

$$\widetilde{\psi_{3,1}} = \mu(-z+y^2z), \quad \widetilde{\psi_{3,2}} = \mu(z+y^2z).$$

As mentioned previously, the metric (3.6) is positive definite on $\mathbb{R}^3$, hence Riemannian, and one can verify that the Riemann curvature tensor is zero everywhere. The change of variables that leads to a Cartesian coordinate system is given by

$$X = \arctan x, Y = \int \frac{1}{\sqrt{By^4 + 2(B+1)y^2 + B}}, Z = \arctan z,$$

where we have some flexibility on B to adjust the roots of the elliptic integral.

Second Example

With the same representation of Lie algebra by first order differential operators but a different choice of coefficients, one constructs another family of second order differential operators $\mathcal{H}$. Indeed, with

$$C_{ab} = \begin{pmatrix} A & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \lambda \\ 0 & D & 0 & 0 & 0 & 0 & 0 & \beta & 0 \\ 1 & 0 & B & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2A & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta D + C & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\lambda B & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & A & 0 & \lambda \\ 0 & \beta & 0 & 0 & 0 & 0 & 0 & \beta C & 0 \\ \lambda & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & \lambda^2 B \end{pmatrix},$$

$$C_a = [0, 0, 0, -2Am_x, -\beta D(1 + 2m_y) + C, -2\lambda B m_z, 0, 0, 0]$$

$$\text{and} \quad C_0 = -Am_x - Bm_y - Cm_z,$$

a family of operators $\mathcal{H}$ is obtained and one easily verifies that all these operators are equivalent to Schrödinger operators $\mathcal{H}_0$. Note that this family of operators is slightly more general than the family obtained in the first example. However, some of the details are lengthy and are omitted for brevity sake. The induced contravariant metric $g^{(ij)}$ is given by:

$$\begin{pmatrix} A(x^2+1)^2 & 0 & (x^2+1)(\lambda z^2+1) \\ 0 & \beta C y^4 + y^2(2\beta + \beta D + C) + D & 0 \\ (x^2+1)(\lambda z^2+1) & 0 & B(\lambda z^2 + z)) \end{pmatrix}, \tag{3.7}$$

and it is positive definite on $\mathbb{R}^3$ provided A, B, C, D and β are positive and $AB > 1$. Its determinant is $g = (AB-1)(x^2+1)^2(\beta C y^4 + 2\beta y^2 + \beta D y^2 + y^2 C + D)(\lambda z^2+1)^2$, and the gauge factor required is the product of three functions in x, y and z respectively. After the gauge transformation the new potential is again a rational function involving only the y variable and the Riemann curvature tensor is null again. The following change of variables leads to Cartesian coordinates

$$X = \arctan x, Y = \int \frac{1}{\sqrt{\beta C y^4 + 2\beta y^2 + 2\beta D y^2 + y^2 c + D}} dy, \, Z = \frac{\arctan \sqrt{\lambda} z}{\sqrt{\lambda}}$$

where we have some flexibility on β, C and D to adjust the roots of the elliptic integral Y.

However, we do not know if the formal eigenfunctions obtained for these operators are all normalizable. But, if we fix $C = \beta D$, the gauge transformation simplifies and becomes, once again, very similar to the determinant of the metric (3.7). Indeed

$$\mu = (x^2+1)^{-\frac{m_x}{2}}(\beta^2 D y^4 + 2\beta y^2(1+D) + D)^{-\frac{m_y}{4}}(\lambda z^2+1)^{-\frac{m_z}{2}}$$

and one verifies, the exact same way as in the previous example, that the functions in $\widetilde{\mathcal{N}}$ are square integrable. Therefore, for any choice of integers m_x, m_y, and m_z, one would obtain $(m_x+1)(m_y+1)(m_z+1)$ eigenfunctions in the spectrum of the Schrödinger operator $\mathcal{H}_0$.

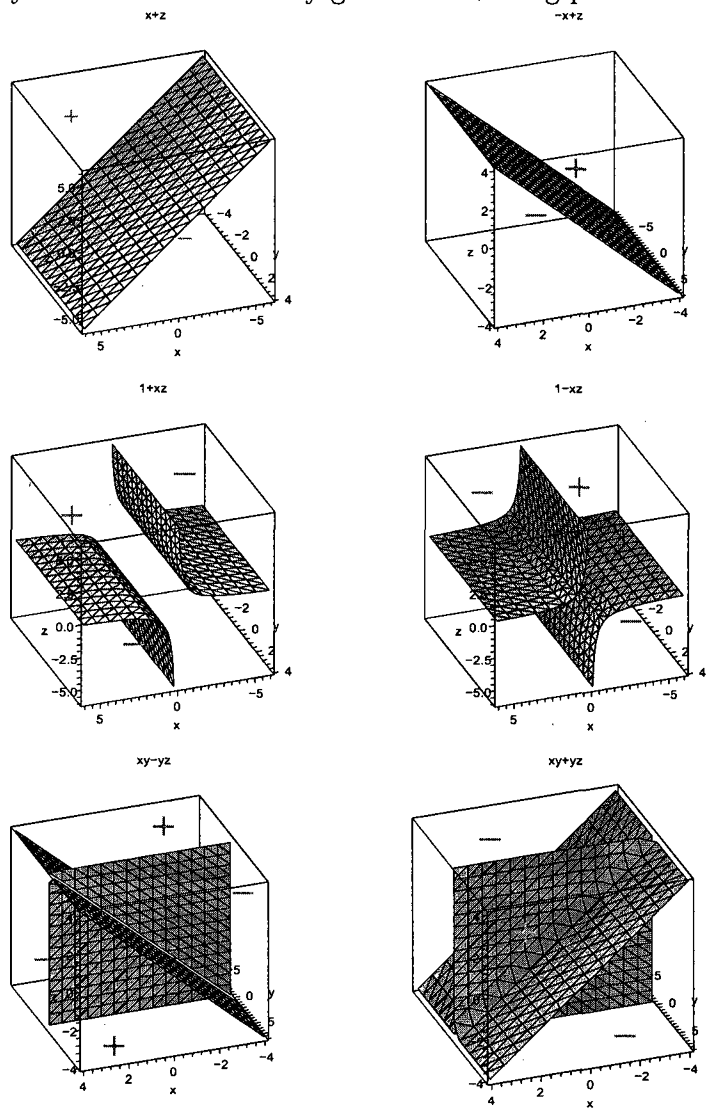
For instance, if we fix $\lambda = 1$, $\beta = 5$ and the three parameters m_x, m_y and m_z to be 1, one gets two eigenvalues, -3 and -7 of multiplicity four, and the following eight eigenfunctions

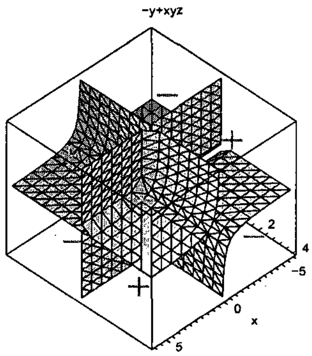
$$\widetilde{\psi_{-7,1}} = \mu(-1+xz), \quad \widetilde{\psi_{-7,2}} = \mu(y-xyz), \quad \widetilde{\psi_{-7,3}} = \mu(xy+yz), \quad \widetilde{\psi_{-7,4}} = \mu(x+z),$$

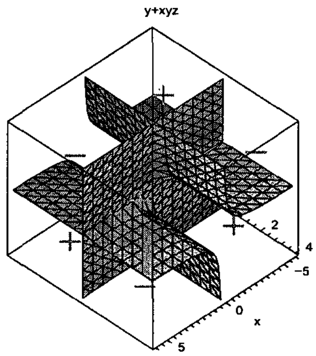
$$\widetilde{\psi_{-3,1}} = \mu(y+xyz), \quad \widetilde{\psi_{-3,2}} = \mu(-x+z), \quad \widetilde{\psi_{-3,3}} = \mu(-xy+yz), \quad \widetilde{\psi_{-3,4}} = \mu(1+xz).$$

Note that the nodal surfaces can described easily in this coordinate system. Indeed, since μ is always positive, the nodal surfaces are simply the zero loci of polynomials.

For these eight eigenfunctions, the surfaces are given by the zeros of degree two factorizable polynomials and one easily gets the following pictures.







3.3.2 Type III, Case $5A^*$, $\mathfrak{sl}(2) \ltimes \mathbb{C}^{s+1}$

For the last example, we consider the type III Case $5A^*$ quasi-exactly solvable Lie algebra and we fix the parameter s to be one. This Lie algebra $\mathfrak{g}$ is therefore spanned by the following six first order differential operators

$$p, \; q+r, \; xp, \; xq+xr, \; yq+zr, \; \text{ and } x^2p+xyq+xzr-mx,$$

and from the Table 4, the $\mathfrak{g}$-module of function is given by

$$\mathcal{N} = \{x^i y^j z^k | \; i+j+k \leq m, j \leq m_y, k \leq m_z\},$$

where n, m_y and m_z are non-negative integers. A family of Schrödinger operators on $\mathbb{R}^3 \backslash \{x=y\}$ is obtained from the following choice of coefficients,

$$C_{ab} = \begin{pmatrix} A & 0 & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 & 0 \\ 0 & 0 & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & D & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C_a = [0, 0, C, 0, -2(1+m)D, 0], \text{ and } C_0 = (1+m)^2 D,$$

where the parameters A, B, C, and D are positive. The induced contravariant metric is given by

$$g^{(ij)} = \begin{pmatrix} Cx^2 + A & 0 & 0 \\ 0 & Dy^2 + B & Dyz + B \\ 0 & Dyz + B & Dz^2 + B \end{pmatrix}, \tag{3.8}$$

its determinant is $g = BD(Cx^2 + A)(y - z)^2$ and the metric is positive definite on $\mathbb{R}^3 \backslash \{x = y\}$. Before performing the gauge transformation the operator reads as

$$\begin{aligned} -2\mathcal{H} &= \Delta + (2Cx - Cmx)p + (-Dy - 2Dmy)q + (-Dz - 2Dmz)r \\ &\quad -1/2Cm + 1/4Cm^2 + (1+m)^2 D, \end{aligned}$$

and one easily verifies that the operator satisfies the closure condition. The gauge factor required to obtain a Schrödinger operator is

$$\mu = (Cx^2 + A)^{\frac{1-m}{4}} (y - z)^{\frac{-2m-3}{2}},$$

and once again, contains the same factors as the determinant of the covariant metric. Finally, after the gauge transformation, the Schrödinger operator reads as,

$$-2\mathcal{H}_0 = \Delta + U,$$

where U depends on the three variables. Although , it is not known if the functions in $\widetilde{\mathcal{N}}$ are square integrable on the domain $\mathbb{R}^3 \backslash \{x = y\}$.

Note that for this example, the scalar curvature is constant and depends on the parameter D while the Riemann curvature tensor is equal to

$$\frac{-1}{B(y-z)^2} dydzdydz.$$

However, the potential does not seem to be separable.

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Conclusion

The motivation for this thesis was to explore some significant aspect of the exact or quasi-exact solubility of Schrödinger operators in three variables. On one hand we successfully proved a modified version of Turbiner's conjecture in three dimensions and on the other hand we made a significant step towards the classification of the quasi-exactly solvable Schrödinger operators for which part of the spectrum can be found algebraically.

Recall that it was proved in the first paper that if a Lie algebraic Schrödinger operator on a locally flat manifold is generated by a 2-imprimitive Lie algebra, then the Schrödinger equation can be solved by partial separation of variables provided that some compactness hypothesis and two constraints on the induced contravariant metric hold: it can be diagonalized and it is generic in a suitable technical sense. Even if many operators involve diagonal metrics, these extra assumptions are quite considerable and it is not known at the moment if they are necessary. The two hypotheses are used in the proof of the $3D$-Trapping Theorem. This theorem is based on a simplified expression of the diagonal components of the Ricci curvature tensor. It might be possible to rearrange this expression in a way that the genericity requirement does not need to hold to conclude the theorem. Moreover, the components of the Ricci tensor are very complicated for a non-diagonal metric and a decent simplification was not achieved. However a simplification may exist and it would worth to spend more efforts on it to try to obtain a more general version on the result. Note that, consequently to these two hypothesis, $3D$-Tiling Theorem was also simplified. Indeed, the possible real-analytic maps $\psi : \mathbb{R}^3 \to M$ restrict to 2^ith-fold maps and one would probably have to deal with a larger variety of maps in a more general setting.

Another improvement to this work would be to simplify the arguments determining the possible foliations. The had hoc arguments for this simple result seems too long and could possibly be simplified. One avenue would be the theory of isoparametric surfaces. Recall that the three possible foliations obtained are exactly the isoparametric hypersurfaces of $\mathbb{R}^3$. It might be interesting to deeply understand the possible link between these two notions in order to simplify the proof of the theorem. Furthermore such a link could be used to generalize the arguments of the proof in higher dimensions. Indeed, the ideas used to prove the separation theorem seem to be extendible, with a lot of work, in dimension n. Provided we succeed to prove the equivalence between the two notions, the geodesic argument might hold for $n-1$-imprimitive Lie algebras since the isoparametric hypersurfaces of $\mathbb{R}^n$ are $n-1$ dimensional hyperplanes, spheres and cylinders. Then one would have to work out the n-dimensional versions of the Trapping and Tiling Theorems. In that scope, it might be prudent to use strong restrictions on the metric for a first attempt.

Related to the second part of this work, number of new quasi-exactly solvable Lie algebras of first order differential operators were described. However, the classification is not complete and it does not seem impossible to perform the entire classification. Note that, in order to complete this classification adequately, one would have first to verify and correct the classification of imprimitive Lie algebras of vector fields made by Amaldi in [6]. Furthermore, we have focused only on the quasi-exactly solvable Lie algebras and a general classification of Lie algebras of first order differential operators in three variables remains an open problem. Another interesting way to continue this work would be to construct new normalizable quasi-exactly solvable Schrödinger operators based on the numerous quasi-exactly solvable Lie algebras obtained in the second paper. The operators presented here being generated by only two among all the new quasi-exactly solvable Lie algebras, this area is certainly a fertile ground.

Appendix A

Overview of the Tiling Theorem

As mentioned in the paper [15], the proof for the three dimensional version of the Tiling theorem is similar to the one given in [36]. The generalization in three dimensions is a straightforward extension of the two-dimensional case except for one argument which uses Proposition (2.5). In order to prove this proposition we need to impose two extra hypotheses: the contravariant metric has to be diagonalizable and generic. It is worth to mention here that for any non-degenerate point on **R**, the positive definite region of the manifold, such a local diagonalization is always possible (Cotton-Darboux Theorem). However, for the proof of the Tiling theorem, the metric needs to be diagonalizable at the reachable part of the degeneracy locus, which is not always guaranteed.

Theorem 2.4. (The 3D-Tiling Theorem) Let M be a compact three dimensional flat almost-Riemannian manifold with diagonal generic metric. Then, there exists a globally defined, real-analytic map $\psi : \mathbb{R}^3 \to M$ such that $g^{(ij)}$ is the push forward of the Euclidean metric, and such that ψ covers all of **R** plus the reachable portions of its boundary. Furthermore, the preimage of the locus of degeneracy $g = 0$, under this map, if it is non empty, consists of surfaces that tile $\mathbb{R}^3$ into isometric cells. These cells are related by the group of isometric symmetries of ψ; indeed $\mathbf{R}$ is isometric to the quotient of $\mathbb{R}^3$ by this group.

Proof: Since the Riemannian curvature tensor vanishes identically, there exists a local isometry from an open neighborhood of $\mathbb{R}^3$ to an open neighborhood of M. The objective is to extend the domain of this isometry to $\mathbb{R}^3$. To do this, we consider the maximal atlas, $\mathcal{A}$, of compatible analytic isometries

$$\psi_\alpha : O_\alpha \to M.$$

Remark here that the range of the maps can include points on M where the metric is degenerate. Thus, in this case, the term "isometry" is used in the sense that the push-forward of the Euclidean metric via ψ_a is equal to the metric $g^{(ij)}$. Assume now that $\mathcal{A}$ does not cover all $\mathbb{R}^3$. Then, there exists a curve

$$\gamma : [0,1] \to \mathbb{R}^3,$$

such that the image of γ lies entirely in some O_α, with the exception of $\gamma(1)$. The manifold M is compact, therefore $\psi_a(\gamma)$ must have a limit point $x \in M$. If x lies in $\mathbf{R}$, the domain of the atlas $\mathcal{A}$ can be extended easily. However, if x is a degenerate point, one need Proposition (2.5) to extend the domain of the atlas.

First, one can compare the length functional on paths in $\mathbf{R}$ with the length functional induced by the metric $dx^2 + dy^2 + dz^2$ and then shows that x is a reachable degenerate point. Therefore, it is possible to apply Proposition (2.5) and obtain an analytic map ϕ from N, an open neighborhood of $\mathbb{R}^3$, to $\mathbf{R}$ such that $\phi_*(\tilde{g}^{(ij)}) = g^{(ij)}$ for $\tilde{g}^{(ij)}$ a non-degenerate metric tensor with analytic coefficients. Since the metric $g^{\tilde{(ij)}}$ is non-degenerate on N, one can extend the domain of the atlas $\mathcal{A}$, a contradiction with its maximality. Therefore the domain of the charts of the atlas is $\mathbb{R}^3$. Since its topology is trivial, one obtains a global map, $\psi : \mathbb{R}^3 \to M$ such that $g^{(ij)}$ is the push-forward of the Euclidean metric.

Hence, the degenerate points of metric $g^{(ij)}$ are the points for which the Jacobian of ψ is degenerate, so that $\mathbb{R}^3$ is tiled into connected cells $\mathcal{C}_i$ that are the preimages of $\mathbf{R}$, and the set of boundary points of these cells is the locus $|J(\psi)| = 0$. If there is more that one cell they must be isometric to each others since they are all isometric to $\mathbf{R}$. If σ is an isometry that relates two of these cells, say $\sigma(\mathcal{C}_1) = \mathcal{C}_2$, then σ and $\psi \circ \sigma$ agree on C_1. Since the germ of ψ completely determines ψ, one gets

$$\psi \circ \sigma = \psi.$$

From Proposition (2.5), the map ψ is a 2^ith-fold map at degenerate points, the group of isometries is therefore the group of reflections along the folding planes. $\square$

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