Kadomtsev-Petviashvili type differential systems; their symmetries and an application to solitary wave propagation in nonuniform channels.

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June 1987

A thesis submitted to the *Faculty of Graduate Studies and Research* in partial fulfillment of the requirements for the degree of *Philosophiae Doctor* in physics.

Acknowledgements

The author would like to express his gratitude towards many people without whom the elaboration of the present thesis would not have been possible. First of all, I am deeply indebted in the person of my thesis director, Dr. Pavel Winternitz from the *Centre de Recherches Mathématiques* (CRM) at "l'Université de Montréal", who directed my research work in the past several years and from whom I learned more than mathematical physics. Indeed, in working with Dr. Winternitz, I had the rare opportunity to acquire a great deal of experience in practical scientific research, something which I value more than anything else. I also wish to thank him for introducing me directly, or indirectly by subsidizing my participation to many international meetings, to a great many deal of world reknowned experts in the fields of nonlinear partial differential equations and applications of Lie groups to partial differential equations at large. I am also indebted towards my *official* director, Dr. R.T. Sharp from the department of physics at McGill University, who, although he was not directly involved with my research work, nevertheless provided many encouragements and showed a deep interest in the evolution of my work. Many thanks are due to my two other co-authors whom it has been a pleasure to work with: Dr. Niki Kamran, now at University of Waterloo, and especially Dr. Decio Levi from Università di Roma - *La Sapienza*. Dr. Levi has visited the CRM for over a year and we learned quite a lot from him about the theory of solitary waves propagating in shallow water, as well as on Bäcklund transformations and the inverse spectral transform framework. I should also thank him for the several discussions I had with him on subjects other than those presented in this thesis, in particular on the feasibility of extending the method of Zakharov and Shabat by allowing the dressing matrix to have a more involved singular form. We have benefitted from several very useful conversations with many people. In particular, thanks are due to Drs. M.J. Ablowitz, B. Fuchssteiner, E. Levi, R. Moody, F. Neuman, P.J. Olver, J. Patera, H. Sandstrom, H. Segur, and Y. Saint-Aubin. I would like to thank Dr. Rand (CRM) for his useful advice on computer programming and also for helping me to harness the diabolical mainframe computers of the *Centre de Calcul*. I would finally like to acknowledge support from the *Fonds F.C.A.R. pour l'Aide et le Soutien à la Recherche du Gouvernement du Québec*, as well as the *National Science and Engineering Research Council of Canada*.

Abstract

The present work is devoted to two distinct aspects of Kadomtsev-Petviashvili (KP) like differential systems. In the first part of this work we analyze the symmetry structure of the classical KP equation. Starting from its infinite-dimensional Lie algebra of invariance, L_{KP}, we derive several results. First we show that L_{KP} exhibits a hidden loop structure; inded, its "analytic" component can be embedded into a Kac-Moody loop algebra constructed from sl(5,R). This remarkable property which we discovered for the KP equation was afterwards also observed for other integrable 2-dimensional (in the space coordinates) nonlinear evolution equations. Second, we proceed to completely classify its 1, 2, and 3 dimensional subalgebras into equivalence classes under the adjoint action of the corresponding Lie group G_{KP}. Third, we use the one-dimensional ones in order to perform *symmetry reduction* on the KP equation; we show that this equation is reducible to one of three simpler equations in one space dimension, namely the Boussinesq equation, the KdV equation, and a linear second order equation. Fromn these reductions we get special invariant solutions of the KP equation. Fourth, we consider the coupled system formed of the KP equation in its potential form together with the associated Bäcklund transformation and we apply the symmetry reduction method to this system. This *novel* way of applying this technique yields an interesting group theoretical interpretation for the spectral parameter appearing in the Lax pair associated with the KP equation. The method also yields several explicit nontrivial solutions of our original equation, among them a special *forklike* solution related to so-called *soliton resonances*. The second part of this thesis is devoted to the presentation of a model for describing the propagation of solitary waves through nonuniform channels or straits. More specifically, we attack the problem of the existence of integrable equations for describing this propagation. Using a standard multiple scaling technique we first derive a quite complicated equation which we call the *generalized KP* (GKP) equation, whose coefficients are functions depending on the geometry of the propagation medium; this equation, in its generic form, is not integrable. We then look for special transformations of the dependent and independent variables which, under certain restrictions on the geometry of the strait and the vorticity, reduce the GKP equation to simpler equations that are known to be integrable, for instance the pure KP, KdV, or cylindrical KdV equations. Having

specified these transformations, we then proceed to construct some *exact* solutions which represent *curved* (with respect to the transverse direction) solitary waves moving over special geophysical configurations. Finally we examine some of the conservation laws associated to the GKP equation.

Résumé

La présente thèse est consacrée à l'étude de deux aspects de systèmes différentials du type Kadomtsev-Petviashvili (KP). Dans la première partie nous analysons la structure de symétrie de l'équation KP classique. A partir de la donnée de son algèbre infini-dimensionnelle de symétrie L_{KP}, nous dérivons plusieurs résultats. Nous montrons d'abord que sa composante "analytique" peut être plongée dans une algèbre de lacet construite sur l'algèbre $sl(5,\mathbb{R})$. Ceci constitue une propriété remarquable qui s'avéra ultérieurement être également partagée par d'autres équations nonlinéaires de type évolution à deux dimensions spatiales. Nous classifions ensuite les sous-algèbres de L_{KP} à 1, 2, et 3 dimensions en classes de conjugaison sous l'action adjointe du groupe de Lie G_{KP} correspondant. Nous utilisons celles à une dimension et appliquons la méthode de *réduction par symétrie* sur l'équation KP; il est montré que cette équation peut être réduite à l'une de trois équations plus simples à une dimension d'espace: les équations de Boussinesq et de Korteweg-de Vries, ainsi qu'une équation linéaire du second ordre. Nous considérons enfin le système formé de l'équation KP sous sa forme potentielle et de la transformation de Bäcklund qui lui y est associée, et nous appliquons de nouveau la méthode de réduction par symétrie. Cette nouvelle façon d'appliquer cette technique amène une interprétation intéressante, en termes de groupe, du paramètre spectral qui apparaît dans la paire de Lax associée à l'équation KP. La méthode nous permet aussi de construire plusieurs solutions explicites *exactes* de notre équation originale; en particulier, une solution apparentée aux solutions solitoniques de type résonnant est trouvée. La seconde partie de cette thèse est consacrée à un modèle décrivant la propagation d'ondes solitaires dans des canaux à géométrie non-uniforme. Nous considérons la possibilité d'utiliser des équations intégrables pour décrire cette propagation. Utilisant une méthode standard de changements d'échelles multiples, nous dérivons une équation compliquée que nous appellons l'équation *généralisée de KP* (GKP) dont les coefficients dépendent de la géométrie du milieu de propagation; sous sa forme générale, cette équation n'est pas intégrable. Nous essayons donc de construire des transformations des coordonnées qui, sous certaines hypothèses sur la géométrie du canal et sur la vorticité, réduisent l'équation GKP à des équations que nous savons être intégrables, telles les équations KP, KdV et KdV cylindrique. Enfin, nous construisons quelques exemples de solutions *exactes*

qui représentent des ondes solitaires *courbées* (par rapport à la direction transverse à la direction principale de propagation), Nous terminons en jetant un bref regard sur les lois de conservation qui sont associées à l'équation GKP.

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INTRODUCTION

Systems of partial differential equations (PDE's) have always been of prime importance in theoretical physics. Indeed they form the very adequate mathematical formulation needed in order to describe the time evolution or space distribution of continuously varying functions representing the physical reality. Accordingly they appear in every branch of physics, from classical mechanics to quantum field theories and relativistic mechanics, electrodynamics, hydrodynamics, etc... In fact, the foundations of all these fields essentially lay upon some particular differential system (Hamilton equations, Schrödinger and Dirac equations, Navier-Stokes equations, Maxwell equations, Yang-Mills equations, etc...) and most of our physical knowledge has been gained following a systematic study and analysis of such systems. Further, it is now becoming very obvious from the current research that *real-life* physics, as opposed to *academic* physics, is, for better or worse, genuinely *nonlinear*. It is therefore an urgent task for mathematicians and physicists to understand better the structure of nonlinear systems of partial differential equations, and to devise ways of solving such systems. Important progress has been achieved during the last decade or so (e.g. in nonlinear optics, nonlinear field theories, nonlinear hydrodynamics, etc...), but clearly much remains to be done.

From a historical point of view one had to wait until our century to acknowledge the birth of a formal theory of differential equations because the mathematical concepts underlying this theory were not all seen correctly, or set upon a proper basis, before this time. A typical example example reflecting this situation is our understanding of the structure of the set **R** of real numbers for which the important step had been undertaken by Dedekind when introducing the concept of *cuts* which was indeed necessary in order to consistently understand limits, derivatives, and continuity for functions. In spite of this, it is interesting to look back at the efforts made by mathematicians back in the 19th century. More specifically, it is worth noticing that differential equations were then considered in some way as representations of *geometrici loci*, just as algebraic equations were seen as defining intrinsic algebraic curves and surfaces. In this geometrical perception were hidden the concepts of differentiable and algebraic manifolds which are now considered to be fundamental as to our ability to lay out the formal foundations for the modern study of equations. Manifolds, in their modern meaning, were first introduced by Cartan (see CAR1). As an example, any student undertaking an introductory course on differential equations learns that the ordinary differential equation $x + yy' = 0$ has circles as its integral curves; indeed the integral (or solution) manifold associated to this equation is a one real positive parameter family of circles $x^2 + y^2 - a^2 = 0$ with $a^2 \in \mathbb{R}_+$; i.e. a $S^1 \times \mathbb{R}_+$ structure coordinatized by $\{t, a\}$, with $x = a\cos(t)$ and $y = a\sin(t)$. This correspondence between (systems of) differential equations and geometric sets is indeed *universal* and its importance is to be emphasized when studying nonlinear differential equations. The recent developments in the theory of such equations acknowledge this point very clearly.

The culminant points of the *old* geometric theory of differential equations were reached with the works of Sophus Lie, A.V. Bäcklund (circa 1875), and later on (1918) of Emmy Noether; the key results actually bear Lie's signature. In fact Lie's first achievements, namely the role played by the groups afterwards named after him, were so fundamental that he spent his whole remaining mathematical life pursuing their study and applications. Lie groups, or groups of continuous transformations, are now well known to physicists who use them, for instance in trying to comprehend the nature of the fundamental forces, as starting points for elaborating *unified* field theories; e.g. consider the so-called *electro-weak* theory based on the gauge group $SU(2) \times U(1)$. However physicists mostly use the abstract classical Lie groups taken from Cartan's classification which have, as it is well known, all kinds of very nice properties but often forget about the origins of these groups, namely that they were first introduced in the context of differential equations as groups of *geometrical transformations*. Although Lie mainly preoccupied himself with ordinary differential systems (i.e. depending on a *single* independent variable) of first and second order, most of the group theoretic features of *partial* differential systems still take their origin in one or another of his papers.

The implicit initial goal of Lie was to obtain for differential systems the equivalent of what is known as Galois theory for algebraic equations, but we had to wait for Vessiot and Picard to see a first consistent exposition of such a theory. His principal construction was that of groups of surface transformations, namely *contact* (or first order tangent) transformations, preserving the form of a differential system of order 2 (see LIE1); in modern language these were local groups of diffeomorphic mappings acting on the integral (or solution) manifolds (actually their prolongations to some jet bundles) of these given differential systems. He naturally raised the question about the possible existence of higher-order invertible tangent transformation groups. This was answered by Bäcklund who proved that groups of *finite* higher-order tangent transformations are trivial, in the sense that they are just differential prolongations of groups of first-order tangent transformations (see BAC1). However he also proved that there exist groups for which tangency of *infinite* order is indeed an invariant condition; the transformations belonging to such groups are now called Lie-Bäcklund tangent transformations. Lie and Bäcklund later showed that the generalization of first-order tangent transformation groups is attained by considering *many-valued* surface transformations (see LIE2 and BAC2). This yielded what is now called Bäcklund (or Schlesinger) transformations which have gained much popularity in the past ten years or so. At the level of differential systems, these transformations, although they should rigorously be termed *correspondences,* map solutions of a given system to solutions of another system, possibly the same one. In the latter case, these transformations provide a very powerful mean for constructing infinite families of non-trivial solutions to *a priori* somewhat difficult nonlinear systems, for instance *(multi-)soliton* solutions, thus explaining the great resurgence of interest towards these transformations.

From an applied point of view, the knowledge of the symmetry group of a given differential system is very useful, and Lie perceived that also. In fact one can notice that he laid down the basis for the general procedure of finding the invariants of a symmetry group, as well as for the method of finding solutions by solving auxiliary *reduced* equations (see LIE3); this method is now better known as the *symmetry reduction* technique and will be one of the principal tools used in the present work. As modern texbooks on Lie groups and differential equations, let us mention the works of Ovsiannikov in the late fifties which gave the impetus to this method (see e.g. OVS1), the classic book by Bluman and Cole (see BLU1), and the paper by Olver which recasts the method in its differential-geometric setting in terms of jet bundles (see OLV1).

The other important milestone was reached with the works of Emmy Noether and two results are especially worth mentioning (see NOE1). The first is her extension of Lie's and

Bäcklund's search for groups connected with differential systems. She laid down the basic structure of what are now known as *generalized symmetries*. These symmetries are often confused with Lie-Bäcklund symmetries but are really technically different (see OLV2, pp. 365-366; for a comprehensive treatment of the so-called Bäcklund problem and its possible generalizations see GOU1). The second result accounts for what Noether is most recognized, namely for establishing the connection between symmetries and conservation laws for differential systems derivable from some variational principles. In fact she provided a procedure for calculating the conserved quantities associated to both Lie point symmetries and generalized symmetries. An example of a conserved quantity associated to a non-trivial generalized symmetry that is known since a long time, but was at first not connected to such a kind of symmetry (or to *any* symmetry at all), is the so-called *Laplace-Runge-Lenz* vector encountered in the problem of planetary motion as well as the Schrödinger problem for hydrogenoid atoms.

The second half of the sixties saw the oncoming of a new era concerning the study of *nonlinear* systems of partial differential equations which, it goes without saying, are much more difficult to analyze for various reasons. An important paper to mention is that of Gardner et al. (see GAR1) where the Cauchy problem for the now famous Korteweg-de Vries (KdV) equation was solved by considering a linear problem. The point was that a KdV solution can be related to a potential for the linear Schrödinger equation and that the inverse problem for this equation, namely the reconstruction of the potential from a given set of *scattering data* by solving a Gel'fand-Levitan-Marchenko type integral equation, does yield solutions to the KdV equation. In fact the remarkable observations made in this paper are the following ones. First, the KdV equation can be rewritten in the form of a Lax equation $L' = [L,A]$, where L is a Schrödinger operator $L = u(t,x) - \partial_{xx}$, and $A = 4\partial_{xxx} - 6u\partial_x - 3u_x$, "u" being a solution of the KdV equation. Second, the non-constant part of the scattering data, namely the reflection coefficients for the Schrödinger scattering problem, obey simple linear differential equations with constant coefficients. This procedure later proved to be applicable to other equations, for instance the nonlinear cubic Schrödinger equation. Thereafter the now famous Zakharov-Shabat-AKNS (Ablowitz, Kaup, Newell & Segur) method of the "Inverse Scattering (or Spectral) Transform" (IST) was devised and proved itself to be very powerful as a means of solving the Cauchy problem for a large class of nonlinear equations of evolution type having important applications in theoretical physics (see ABL1; CAL1; ECK1; ~~NOV1, chapter~~ 1). Let us mention the Sine-Gordon equation, the Kadomtsev-Petviashvili (KP) equation which we study in the present work, the modified KdV equation, etc...

It was later realized that the IST method had a deep connection with the theory of complex functions on the Riemann sphere. In fact, the very heart of the method is the classical Riemann-Hilbert-Privalov (RHP) problem of decomposing a given function F, defined on some contour Γ, as a product of two functions F_1 and F_2, respectively holomorphic inside and outside of Γ. V.E. Zakharov and A.V. Shabat noticed that a particular case of the matrix RHP problem, called the problem with zeroes, makes it possible to recast the IST method into a *completely* algebraic procedure which is known as the *Dressing Method* (see ZAK2; NOV1, chapter 3). According to this scheme, it is possible to integrate nonlinear differential systems which are expressible as compatibility conditions for a linear system $\Psi_x = U\Psi$, $\Psi_y = V\Psi$, where U and V are some matrix fields; the procedure then consists in the construction of a Bäcklund transformation having the form of a gauge transformation $\Psi' = \chi\Psi$, $U' = (\chi U + \chi_x)\chi^{-1}$, $V' = (\chi V + \chi_y)\chi^{-1}$, where it is required that the matrices U' and V' be characterized by the same meromorphic structure as that of U and V. This method, in its original formulation but also in a more recent and powerful scheme, along with the so-called *reduction method* (not to be identified with the symmetry reduction mentioned in a previous paragraph; see MIK1), makes it very easy to find all the soliton type solutions of a given equation and has been applied to numerous differential systems such as classical relativistically invariant spinor field theories, the Thirring model and the class of σ-models defined over Riemannian symmetric spaces (see ZAK3,4; DAV1; HAR2,3).

Many other tools for studying various aspects of nonlinear differential systems also appeared during the last decade. Let us mention the *prolongation structures* which are useful for constructing Lax pairs (these are essential for pursuing with the IST method; see WAH1, DAV2), algebraico-geometric methods which are needed to solve the Cauchy problem for periodic or quasi-periodic initial data (see NOV1, chapter 2), and the bilinear formalism of Hirota. The latter is at the origin of the now very popular τ-functions which are quite important in the recent works by the Kyoto School on infinite hierarchies of nonlinear partial differential equations based on infinite-dimensional Kac-Moody groups (we shall come back on that subject later on). For a good compendium of all the recent findings and connections between the many tools see NEW1.

The main topic of the present piece of work is the Kadomtsev-Petviashvili equation, or KP for short. As its name indicates, it originates from a 1970 publication by B.B. Kadomtsev and V.I. Petviashvili (see KAD1). In this paper the authors recall that the Korteweg-de Vries equation (hereafter denoted KdV) gives a good description of stable nonlinear waves in weakly dispersive media, e.g. waves in shallow water, iono-acoustic and magneto-acoustic plasma waves, etc... In the hydrodynamical case, one considers mainly velocity and pressure waves. They ask whether stability can be preserved for a solitary wave with a weak bending distortion in the transverse direction and indeed show that the correction to the KdV equation must then be small and be of the form

$$u_t + uu_x + u_{xxx} = \varphi_y, \qquad \varphi_x \pm cu_y = 0, \qquad c>0,$$

where the signs "+" and "-" stand for negative and positive dispersion, respectively. This is the KP system, which is obviously equivalent to the single equation

$$(u_t + uu_x + u_{xxx})_x \pm cu_{yy} = 0.$$

It is this last equation which is known as the KP equation, up to some rescaling of the constant coefficients. The name was coined later, apparently by Manakov et al. (see MANA1). Throughout this work we shall use a subscript notation for derivatives of functions; thus $u_x = \partial_x u = \partial u/\partial x$. The remaining part of KAD1 is devoted to stability analysis for the case characterized by phase amplitudes that are much smaller than the solitary waves' widths, and by very large wavelengths. The result stated is that solitons of the KP system show instability under transverse perturbations, but are stable otherwise. Further stability analysis, this time for periodic waves obeying the KP system, is done by Petviashvili, using Fourier transform methods (see PETV1). In the case of negative dispersion, he shows that the results of the previous paper still hold when a periodic wave tends to a soliton. However, for positive dispersion, it is shown that stability does not generally occur; for example, solitons may be unstable under local perturbations near them. More recently a "modified" KP equation was analyzed for weak negative dispersion; this equation is obtained by replacing the uu_x term of the above equations by a u^2u_x term (its relationship to the KP equation is similar to that relating the "modified" KdV equation to the KdV equation itself). Some conservation laws were found and it was shown that the local energy asymptotically vanishes with time (see LIN1).

The KP equation reappears about three years after this last paper in a publication by Oikawa et al. (see OIK1) where the argumentation made in KAD1 is extended to a more general situation. They actually look at the KP system in the context of studying the effect of a slightly undulated bottom on shallow water surface waves. In fact KP is a very natural equation for describing such waves, as we shall see later. Next comes a paper by Dryuma who announces the existence of a Lax pair for this equation and actually gives its expression. KP is obtained as the Lax equation $L_t = i[L,A]$, where L and A are operators given by

$$L = 6\partial_{xx} - \sqrt{6}\,\partial_y + u(t,x,y)\,, \qquad A = -4i\partial_{xxx} - iu\partial_x - \frac{i}{2}u_x - \frac{i}{2\sqrt{6}}\int^x u_y\,dx$$

for the negative dispersion case; a similar pair is found for the case of positive dispersion (see DRY1). The author thus shows that the Cauchy problem can be solved, in principle, by using the inverse scattering method. He concludes by giving a Lagrange function from which KP can be obtained by a variational principle. The explicit solution of the Cauchy problem by the IST method is in fact the object of a paper by Zakharov and Shabat (see ZAK1). Naturally, the soliton solution is found and, in addition, the N-solitons are also given. This paper is also important for two additional reasons. First, it presents for the first time the inverse scattering formalism applied to an equation in 2+1 dimensions; previously, only 1+1 dimensional equations had been solved this way. The second point of interest is that this paper already contains the germs yielding the dressing method. A review of the IST problem for the KP equation is also the subject of an article by Manakov in 1981 (see MANA3). Soon after, a Bäcklund transformation was found for the KP equation (see CHE1) in the form

$$(w'-w)^2 + 2(w'+w)_x - \frac{2\varepsilon}{\sqrt{3}}\int^x (w'-w)_y\,dx = 0,$$

$$4(w'-w)_x + (w'-w)^3 + 3(w'+w)_x(w'-w) + 3(w-w')_{xx}$$
$$+ \varepsilon\sqrt{3}(w'+w)_y + (w'-w) + \int^x (w'-w)_t\,dx = 0.$$

where $\varepsilon = \pm 1$ (according to the sign of the dispersivity) and $q = w_x$ is a solution of the KP equation $q_{xt} + q_{yy} + q_{xx} + (3q^2)_{xx} + q_{xxxx} = 0$. A superposition formula is also found:

$$w_3 + w_0 = w_1 + w_2 + 2\frac{(w_1 - w_2)_x}{(w_1 - w_2)}.$$

When $w_0 = 0$ and w_1, w_2 are 1-solitons; w_3 is then the 2-soliton solution of the KP equation. The existence of such superposition formulas, in general, is related to whether or not the set of Bäcklund transformations associated to an equation form an Abelian group (see BOI3). N-soliton solutions were also found, later on, by Satsuma (see SATS1) through the bilinear formalism of Hirota (see HIR1; more on this later). The bilinear form of the KP equation is

$$ff_{xt} - f_x f_t + ff_{xxxx} - 4f_x f_{xxx} + 3(f_{xx})^2 = \pm 12(ff_{yy} - f_y^2),$$

where $u = 2(\ln f)_{xx}$ solves KP. Although Hirota's method was considered to be somewhat heuristic at the time, it yielded N-soliton solutions that could be nicely expressed in closed form. In this paper, Satsuma gives the formulas

$$\xi_i = k_i x + l_i y - \omega_i, \qquad k_i\omega_i - 4k_i^4 \pm 12 l_i^2 = 0, \qquad p_i = l_i/k_i.$$

$$f_N = \sum_{\varepsilon=\pm 1} \exp\{\sum_{i=1}^{N} \varepsilon_i \xi_i + \sum_{i<j}^{(N)} \Phi_{ij}\}, \quad \Phi_{ij} = \tfrac{1}{2}\log|(\varepsilon_i k_i - \varepsilon_j k_j) \pm (p_i - p_j)^2|.$$

It was later shown that this formula, as well as other N-soliton formulas in general, can be written as determinants of some matrices. Satsuma finishes by justifying the term "N-soliton" by showing that such a solution asymptotically breaks down into N distinct 1-soliton solutions. The more compact determinantal form for the N-soliton is proposed by Manakov et al. in 1977 (see MANA1) as $u = 2(\ln \det(A))_{xx}$, where A is some matrix expressed in terms of N^2 characteristic *directions* $at + bx + cy$. The principal result brought out in this paper is an important property, namely that KP solitons interact *trivially*. Indeed, the authors consider the 2-soliton solution, find its asymptotic expressions, and show that the changes of phase, which are seen to be non-zero for soliton solutions of one-dimensional equations, are identically zero. This means that the KP solitons do not interact at all, i.e. they pass through each others and re-emerge just as if each soliton would be alone. In fact this property was later seen to be characteristic of two-dimensional soliton equations. Solving the KP equation through the inverse scattering method was considered again in (JOH3) where the authors solve the inverse step, i.e. the Gel'fand-Levitan integral equation, by using the method of separation of variables. They are thus able to reconstruct already known solutions such as exponential, rational, N-terms exponential, N-rational solutions, and also new ones of the exponential-rational type. As for the pure rational solutions (of decreasing type), they were found in generality by Krichever in 1978 (see KRI2, and also VES1). N-solitons for the KP equation were more recently looked at again by Freeman and Nimmo. Inspired by their determinantal form mentioned above, they recast them as Wronskian determinants and showed that their derivatives then take rather simple forms. It is simple to verify that the N-soliton obeys the KP equation and that a pair made of a N-soliton and a (N+1)-soliton does satisfy the associated Bäcklund transformation. The authors also proceed to perform IST on KP, using this formalism together with the bilinear formalism (see FRE1).

The Bäcklund problem for the KP equation and other equations was considered again in a paper by Hirota and Satsuma (see HIR2) where it is demonstrated that the defining relations for the Bäcklund transformation, as well as the superposition formulas, have a common structure for all the equations that they consider, when Hirota's bilinear formalism is used. For KP written in the bilinear form

$$(D_xD_t + D_x^4 + \alpha D_y^2)f \cdot f = 0,$$

where, for instance, D_x is the bilinear operator acting as $D_x f \cdot g \equiv [\partial_\varepsilon f(x+\varepsilon)g(x-\varepsilon)]\,|_{\varepsilon=0}$. They proceed to show that the Bäcklund transformation is defined through the formulas

$$(D_x^2 - bD_y)g \cdot f = \lambda g \cdot f,$$

$$(D_t + 3\lambda D_x + D_x^3 + 3bD_xD_y)g \cdot f = 0,$$

where b is related to α, and that the superposition formula is given by $f_0f_3 = \text{const.}D_xf_1f_2$; for example, f_0 could be the trivial solution, f_1 and f_2 be 1-soliton solutions: f_3 is then the 2-soliton solution [recall that the KP solution is $u = 2\partial_{xx}\ln(f)$]. The Bäcklund problem is also investigated in LEV1 where the general form of the Bäcklund transformation for (2+1)-dimensional nonlinear evolution equations solvable through the Zakharov-Shabat method is found; as a particular example, that of the KP equation is derived.

In a series of two papers, Satsuma and Ablowitz study the *lump* solutions for nonlinear evolution equations. The first one is devoted to the KdV equation and the nonlinear cubic Schrödinger equation. In the second paper (see SATS2) they consider both KP and a two-dimensional version of the nonlinear Schrödinger equation (the Davey-Stewartson system). They obtain these lumps (or rational solutions) and show that they decay to uniform states along all space directions. These solutions are non-singular and constructed by taking a *long wave* limit of the usual N-soliton solutions, and thus N-lumps that actually describe the physical mutual interaction of a set of several single finite amplitude lumps are defined. As for N-lumps, it is shown that the asymptotic behaviour of a lump is the same for both limits, therefore implying that there are no changes of phase, so that lumps also act trivially. The asymptotic forms of KP solutions for the limits $|t|\to\infty$ is also investigated in a paper by Manakov and his collaborators (see MANA2) where they are shown to be essentially different, corresponding to distinct solutions of the linear (i.e. diffusion free) KP equation. It is remarked that in contrast to one-dimensional equations, the classical scattering matrix for a two-dimensional differential system is *not* diagonal; thus it is not surprising that the direct scattering problem associated to KP is genuinely non-trivial. The authors also investigate the converse problem, i.e. to reconstruct the solutions from the knowledge of the asymptotic forms, and give the procedure for doing that. Similarity-type decay-mode solutions (termed "ripplons") are derived in the 1981 papers by Nakamura (see NAK1 and NAK2). As will be seen in the present work, these are just a single example of solutions that may be obtained by the *symmetry reduction procedure* that we shall later define and use on the KP equation. Nakamura considers the bilinear form of this equation for obtaining these solutions and makes no use of group theoretical arguments other than the defining scale invariance property of such solutions. He shows that such solutions may be superposed non-trivially; indeed these similarity solutions originate from a Bäcklund transformation and thus the superposition formula given above may be used. The "1-ripplon" solution is expressed in terms of a solution Ai(z) of the Airy equation $w_{zz} = zw$:

$$f = 1 + \frac{\rho^2}{(12t)^{2/3}}\int^{x} \mathrm{Ai}(z)dz, \qquad z = \frac{(x+a)}{(12t)^{1/3}} + \frac{(y+b)^2}{c^2(12t)^{4/3}}.$$

It is also known that the KdV limit of these solutions is not defined; in that sense they are genuine KP solutions. A further aspect of the KP equation was found in 1979 by Kaliappan and Lakshmanan (see KAL1), namely that a particular reduction, in the group theoretical sense, exists to the Painlevé transcendental equation of the first kind: $Z_{xx} = 6x^2 + 6\sqrt{6}Z$. Painlevé type equations, i.e. equations whose solutions have no moving singularities other than poles, appear to have an important connection with the IST formalism. In fact, we have many reasons to believe that nonlinear differential systems solvable through the IST formalism with asymptotically vanishing solutions (actually at least of class L^2), are of Painlevé type, in the sense that all their reductions to ordinary differential equations yield Painlevé type equations; the converse of this statement is conjectured but has not yet been proved. In a different paper, Redekopp proceeds to show that the KP equation also admits a reduction to the Painlevé transcendental equation of the second kind: $Z_{xx} - xZ = \pm 2Z^3$. He discusses some solutions of the reduced equations and, in particular, he obtains exact dispersive solutions (see RED1). Similarity solutions were also studied later by Tajiri et al. (see TAJ1). The authors show that there exists a *chain* of reduction for KP, using similarity variables. In fact they first reduce KP to either the KdV or Boussinesq equations; a further reduction (in Lie's sense) brings these equations to the first or second Painlevé transcendent equations. They also consider reductions to ordinary differential equations and discuss solitons moving in a non-steady and non-uniform background. In a second publication (see TAJ2), two of the above authors look at similarity solutions for the modified KP (mKP) equation. It is shown that this equation admits a reduction chain more complicated than that found for KP itself: in a first step, mKP can be reduced to mKdV, a modified Boussinesq equation, and many other equations. In a second step, only the mKdV equation yields an interesting reduction, namely the Painlevé II equation.

Similarity solutions, and many other special types of solutions, whenever they exist, are related to some subgroups of the symmetry (or invariance) group of a given equation. In this perspective, the corresponding related invariants have some importance. As we shall see, the KP equation has quite a rich set of symmetries and invariants. We shall restrict ourselves to Lie point symmetries, but (infinitely many) other types of symmetries do exist, local as well as non local. In INF1, some special conservation laws are derived using an observation relating pairs of conserved quantities. Thus if M is a conserved quantity and if N is integrable and such that $\int N_t dxdy = \int Mdxdy$, then $\int (tM - N)dxdy$ is a conserved quantity as well. The authors of LIN2 find an infinite set of conserved quantities from the singular form of the dispersion relation associated to the KP equation. Similarity solutions and some related special solutions, the so-called soliton resonances, are studied in OHK1 and NIS1. The first of these papers uses a method called the *trace method* which the authors show to be related to the IST formalism, and resonances are shown to occur when virtual solitons turn into exact solitons; these solutions take the form of multi-branched solitary objects. KP also possesses generalized symmetries (see CHE2 and FOK4); these permit to define a hierarchy of equations which can be built by recursively acting on an equation with a so-called recursion (or hereditary) operator. Such operators have been found for many one-dimensional soliton equation hierarchies, but it seemed, until very recently, that there were no such objects for (2+1)-dimensional ones. Ref. FOK4 is in fact devoted to the explicit construction of this operator for the KP equation.

The KP equation played a role, between 1980 and 1982, when the IST formalism, in its original formulation, was seen to be inadequate for certain equations. Consider the KP equation in the form $(u_t + 6uu_x + u_{xxx})_x = -3\sigma u_{yy}$, and distinguish between the cases $\sigma = -1$ (KPI) and $\sigma = 1$ (KPII). It was assumed, up to then, that the IST formalism was intimately connected with a specific type of boundary value problem on the complex plane, namely the *local* Riemann-Hilbert (RH) problem. All equations solved through this formalism indeed involved such a problem. For KPI and some other equations, it was then observed that the RH problem was *non-local:* this fact explains why the KP solitons, viewed as generalizations of the KdV solitons, do not decay at infinity; algebraic lumps, however, do present such a behaviour. The inverse problem for KPI is analyzed in detail in FOK2. KPII is even more dismaying in that there was no RH problem associated with it, so the usual IST formalism was simply inadequate for studying this equation. It was then noticed by Beals and Coifman that the RH problem was in fact a special case of a more general problem encountered in the theory of functions of complex variables, called the DBAR ($\bar{\partial}$) problem (see FOK1 for an exposition of this problem and its distinctions from the classical RH problem). The modification of the IST method for equations associated with this new type of boundary value problem was developed by Ablowitz, Fokas, and Bar Yaacov, and has been proved to be very effective for solving the Cauchy problem for the KPII equation (see FOK3, ABL2).

A very interesting feature of the Kadomtsev-Petviashvili equation is its connection with infinite-dimensional Lie groups which resulted in the sequel of a series of papers written mainly by M. Sato, M. Jimbo, and T. Miwa, on holonomic quantum fields. The now famous τ-functions are a by-product of their studies; in fact they were introduced as expectation values of certain field operators belonging to the Clifford group of free fermions. It was further observed that the linear Lax-Zakharov-Shabat equations and the bilinear equations of Hirota come out in a unified manner when using the language of free fermions. Fairly important results were obtained by Sato (see for instance SATO1). Considering infinite dimensional Lie algebras defined over some functional space, he noticed that the group orbit of the highest weight vector is an infinite dimensional Grassmann manifold, the defining equations of which, when put into the form of differential equations, turn out to be soliton equations. This picture was first established when studying the KP equation, in fact the KP infinite *hierarchy* of soliton equations, for which the infinite dimensional complex Lie algebra is $gl(\infty)$. In his paper, Sato shows that generic points belonging to the above Grassmann manifold GM give generic solutions of the KP equation, whereas points defined in some particular submanifolds of GM yield special solutions, such as rational, quasi-periodic, multi-soliton, and similarity solutions; other submanifolds give rise to generic solutions of other important nonlinear partial differential equations: KdV, mKdV, Boussinesq, nonlinear Schrödinger, Benjamin-Ono equations, as well as the Toda lattice (semi-discrete) differential system; for instance the affine infinite dimensional subalgebra $A_1^{(1)} \subset gl(\infty)$ yields the KdV hierarchy, and $B_1^{(1)}$ yields the so-called BKP hierarchy. He also interprets the automorphism group $GL(\infty) \subset GM$ as the group of *hidden symmetries* of the KP equation. Various specific aspects of the connection between some infinite dimensional Lie groups and the KP hierarchy, or reductions thereof, are studied in DAT1, 2, 3, 4, 5 and KAS1, and an excellent review is given by Jimbo and Miwa (see JIM1) who exhaustively construct the τ-functions for the KP hierarchy; they also study the various reductions of $GL(\infty)$ in order to define many other types of hierarchies based on interesting nonlinear evolution equations. An interesting particular reduction is studied in SATS3. Another excellent review paper is that of Segal and Wilson in which the emphasis is put on the geometric picture of this formalism (see SEGA1), in particular the authors try to give a geometrical meaning to the τ-functions, the Baker-Akhiezer function, and to the Dubrovin construction.

Some recent results have been obtained on periodic and quasi-periodic solutions of the KP equation, by making use of tools of algebraic geometry. Such types of solutions for nonlinear equations are not new. In fact periodic solutions to the KdV equation were discovered by Korteweg and de Vries themselves in the form $u = \sigma^2 k^2 \mathrm{cn}[\sigma(x - x_0 + ct); k]$, where cn(z;k) is the usual Jacobi elliptic cosine function. These solutions were thus called *cnoidal waves*. It was shown by Krichever that the KP equation does admit quasi-periodic solutions (see KRI1,3) of the form $u = 2\partial_x^2[\ln \vartheta(\varphi_1, \ldots, \varphi_n; Z)]$. ϑ is a Riemann theta function of order "n", φ_i is a linear function of t, x, y, and Z is a so-called Riemann matrix, i.e. a symmetric n by n matrix with negative definite real part. Krichever showed that any genus n Riemann surface naturally induces a particular Riemann matrix that then serves to generate a KP solution. He actually gave the form for the genus 1 type. Solutions of higher genera are more difficult to find, because one must determine the class of Riemann matrices that yield KP solutions. It must be noted that not every Riemann matrix does. The characterization problem of Riemann matrices for genera 2 and 3 was solved by Dubrovin (see DUB1). Genus 2 type solutions were worked out by Segur and Finkel (see SEGU1, FIN1). These are bi-periodic solutions that exhibit a dependence on 8 different parameters and they are shown to degenerate into solitons when some particular limit is taken; they may thus be considered as the genuine generalization, in two dimensions, of the one-dimensional cnoidal waves.

Surface and internal waves in channels provide us with remarkable examples of solitary waves; in fact, solitons were first observed as lumps of water propagating through such physical settings, in 1834, as reported by J.S. Russell who indeed followed one of these lumps over a distance of several kilometers (see RUS1). A consistent mathematical approach to these waves was however not to be achieved immediately, and it even was the opinion of Airy, who himself built up a shallow-water theory, that the whole thing was a hoax! The first serious attempts to derive the dynamics of solitary water waves from the basic facts of hydrodynamics are due to Lord Rayleigh. Boussinesq then introduced the famous equation which was later called after him. This equation was however not entirely satisfying and one had to wait untill 1895 when Korteweg and de Vries introduced the now celebrated KdV equation (see KOR1):

$$u_t + c[u_x + \tfrac{3}{2}d^{-1}uu_x + \tfrac{1}{6}u_{xxx}] = 0,$$

where $c=[g(d+a)]^{1/2}$, g being the gravity constant. This very equation was the correct one for describing solitons of amplitude "a" moving with characteristic speed "c" in a straight uniform rectangular channel of depth "d". More specifically, in the case when waves are long waves propagating in shallow water and in the no-dispersion limit, the structure of the KdV equation, as it is now well known and understood, is such that the nonlinear term counterbalances the effects of the dispersion term. Thus a *localized* solution of this equation remains as such through its entire time evolution: it does not flatten out as a linear wave would do. These soliton solutions are typically of the form

$$u = a\,\mathrm{sech}^2[(x-ct)/l],$$

"l" being a characteristic measure of the length of the wave. In the paper KOR1, the authors derived the soliton solution of the KdV equation as the limit case of a family of periodic waves, the so-called *cnoidal* (they coined the term) waves, when the period becomes infinite. The multi-soliton solutions of the KdV equation, i.e. solutions describing the interaction between many distinct solitons, were characterized only much later, in 1965, in a paper by Zabusky and Kruskal who introduced the word *soliton,* and observed, through numerical experiments, that two or more solitons, after colliding together, asymptotically recover their respective identity, up to an apparent change in their phases; this is indeed a very remarkable property (see ZAB1). The stability of soliton solutions was established only relatively recently, in 1972, by Benjamin (see BEN2). The KdV equation is appropriate for describing waves in a rectangular channel but fails to be applicable to the description of waves propagating through otherwise shaped channels. A more realistic situation requires that the equation be modified by allowing its coefficients to become variable. If solutions are allowed to have some genuine two-dimensional character, it may be necessary to replace the KdV equation by a Kadomtsev-Petviashvili equation, possibly with variable coefficients also. We point out that the KP equation *does* have solutions that exhibit a non-trivial two-dimensional behaviour and they may be obtained, as we shall see later, through the use of the symmetry group of the equation; it is apparent that this was not observed by hydrodynamicists studying water waves. Boussinesq was the first to notice, through considerations about energy conservation, that the amplitude of a solitary wave in a non-uniform channel would locally vary inversely as the depth of the channel (see BOU1). The characterization of solitary waves within gradually varying channels is therefore of interest and much literature has been devoted to this subject. For a detailed historical account the reader is directed to review papers such as MIL3 (see also LEB1 and OSB1). Here we shall browse through this literature, mentioning only typical papers about the principal steps which were undertaken in the theoretical developments and works which are of direct pertinence with respect to the subject of this thesis.

The method of multiple scaling or *stretching* method, introduced in 1948 by Friedrichs (see FRI1), applied to the Navier-Stokes system, is a standard procedure for generating water wave equations; a variant of this method was indeed used by Peters in 1966 in order to obtain three-dimensional (i.e. time and two space dimensions) solitary waves on the surface of a horizontal channel of infinite length with arbitrary cross-section, but *constant* along the main direction of propagation (see PETE1). The equations governing three-dimensional solitary waves propagating through a *rectangular* channel were first systematically derived by Ursell (see URS1). After using the stretching method to set up an appropriate dimensionless hydrodynamical differential system, he proceeds to show that there exist rotational solitary waves obeying a time-dependent KdV-like equation, $m_0\eta_1'''(x) = m_1\eta_1\eta_1'_x + m_2\eta_1'$, where m_0, m_1, and m_2 are functions depending on the velocity components of the stream which are determined from some specific boundary conditions requirements. In fact he shows that the above equation has the solution $\eta_1 = -3(m_2/m_1)\mathrm{sech}^2[\frac{1}{2}x\sqrt{m_2/m_0}]$. The case of irrotational solitary waves is much simplified as compared with rotational ones; these are also obtained through similar considerations. Peregrine, in a 1968 paper (see PER1), ventures along a path parallel to that of Peters. Although he considers only irrotational flows, he derives the equations of motion for long gravity waves and shows how these equations may be transformed into those for two-dimensional motion in a rectangular channel (i.e. the KdV theory); his results are thus applicable to more general channels. It must be pointed out, however, that he considers channels of arbitrary cross-section as long as they do *not* have gently sloping banks and are constrained not to be very wide compared with their characteristic breadth; i.e. he considers *deep thin* channels. The above authors only studied surface waves. Shen, in 1968, actually extended the results obtained by Peters, by considering unsteady long waves propagating at the interface of a two-layer stratified fluid (see SHE1). He finds that the dynamics is again ruled by a KdV-type equation with varying coefficients. After characterizing the wave speed, he derives the solitary waves and, in addition, cnoidal waves, breaking waves and Airy waves. We mention that continuously stratified fluids have also been considered in the literature; for instance see BEN1. We also point out that the various KdV-like equations which have been introduced to model solitary wave propagation through gradually varying channels admit very few conservation laws (see MIL2); as we shall see in chapter three, this will also be true of our generalized KP equation.

It was observed in 1969 by Madsen and Mei that when a solitary wave, moving over a region of unchanging depth, enters a distinct region where the shape of the bottom abruptly changes, several solitons form (see MAD1). More recent observations and measurements from space (e.g. radar measurements and photographs from space shuttles) over oceanic regions at the border of continental shelves confirm this fact. A mathematical modelization of this phenomenon is therefore of interest for oceanography. This situation was considered by Johnson in 1973 (see JOH1,2). The relevant equation is again a KdV-type equation, namely $H_X + \frac{3}{2}d^{-7/4}HH_\xi + \frac{1}{6}\kappa d^{1/2}H_{\xi\xi\xi} = 0$, where $d(\varepsilon X)$ is the depth function; X and ξ are some appropriate coordinates. Johnson assumes the solution to be a perturbation of a pure solitary wave and studies the "Cauchy problem", with the following interpretation (such an initial value problem will also be considered here in chapter three). The point is that the coordinate X, although it is a pure space variable, plays the role of time in the above dynamical equation. At $X = 0$ (and physically for $X < 0$), we are given the profile of a solitary wave moving over a region of depth $d(0) = 1$. Assuming that the depth function changes smoothly in the semi-infinite interval $X \geq 0$ and that the solution H vanishes for asymptotic values of the coordinate ξ, the author seeks asymptotic solutions for $\varepsilon \to 0$ of the form $H = H_0 + \varepsilon H_1 + O(\varepsilon)$, where H_0 is a solitary wave of the form $b(\varepsilon X) + a(\varepsilon X)\text{sech}^2\{\alpha(\varepsilon X)[\xi - c(\varepsilon X)X]\}$. The solution is shown to be non-uniform both ahead of and behind the solitary wave; the behaviour ahead is rectified by matching to an appropriate exponential form. He discusses the nature of the solution behind the solitary wave which yields a solution with an oscillatory tail. It is found that as depth increases, the amplitude of the soliton decreases and that for large enough depth, the soliton becomes in fact indistinguishable from the oscillatory tail. The author also considers various limits, showing how his results agree with those obtained in other works. This description, it is to be pointed out, treats the case of channels characterized by very slowly varying depths only. Miles extended the above results to the case of channels with slowly varying breadth as well (see MIL1).

A 1974 paper by Oikawa et al. is of special interest as it is at last considering the use of the Kadomtsev-Petviashvili equation for studying water waves (see OIK1). The KP equation, in pure form or with variable coefficients, is of considerable importance for it permits to study waves propagating over *genuine* two-dimensional bodies of water. In this paper the authors study the effect of an undulated bottom on shallow water waves, in the case when the depth is a slowly varying function and that the characteristic length is much larger than the wave length. In particular, they investigate the conditions for the existence of trapped mode solutions through a linearization of the basic system of equations for gravity waves; they thus obtain a KP-like system and show that it has soliton solutions that are stable under small two-dimensional perturbations. A variable coefficient Kadomtsev-Petviashvili equation was derived in 1978 by Djordjevic and Redekopp (see DJO1), motivated by experimental evidence about internal waves generated near the edge of a shelf and propagating shoreward. This equation is dependent on the local fluid depth. The authors discuss how its solutions can describe waves propagating into shallower water (e.g. waves crossing a continental shelf) for a two-layer (specifically a two-density stratification) fluid body. They also look at the disintegration, under special conditions, of a solitary wave into a dispersive packet.

Waves in a two-layer fluid were also studied by Grimshaw, but expressed within a KdV framework. In GRI1 he develops the equations describing long internal waves in a channel of arbitrary cross-section. Further assumptions are made; namely he restricts to the shallow water approximation and considers a cross-section which is allowed to vary slowly in the direction of propagation of the waves. It is also implicitly assumed that the horizontal dimension of the channel is of the same order of magnitude as its vertical dimension, but, as he points out, this is not as severe a restriction as it may appear. The crucial point is that the wave amplitude is taken to have a quite greater variation in the direction of wave propagation than in the transverse direction; he thus considers an essentially one-dimensional motion. After reviewing the theory for a uniform cross-section, he proceeds to look at a channel with varying cross-section and derives a variable coefficient KdV equation which is the counterpart, for internal waves, of those obtained in PER1. Grimshaw discusses special situations such as nonlinear steepening processes which are experimentally known to follow from an internal surge, the formation of solitons when such a surge approaches breaking distances, and slowly varying solitary waves. These latter are actually considered in GRI2 where the author derives them as asymptotic solutions of a variable coefficient KdV equation obtained through the usual multiple scale perturbation procedure: $q_t + N(t)qq_x + L(x)q_{xxx} + G(x)q = 0$. He shows that when the coefficients N, L, and G satisfy a certain constraint, the above equation reduces to a pure constant coefficient KdV equation and hence he obtains exact solutions for the original equation.

A 1981 paper by Santini looks at the Kadomtsev-Petviashvili equation as a possible equation for the description of the evolution of two-dimensional wave packets over an uneven bottom (see SAN1). The author specializes to nearly one-dimensional long waves of small amplitude, these characteristics being balancing in some convenient manner. Considering a depth function which varies slowly along the longitudinal and transverse directions, he derives a variable coefficient KP equation which, under a certain limit, he shows to yield the results of JOH2. A second special limit yields a nonlinear equation of the form $q_T + \frac{3}{2}\sigma\gamma d^{-7/4} qq_X = 0$, where "d" is the depth function. The Cauchy problem for this equation, with initial datum $q(X,Y,0) = q_0(X,Y)$, is completely solved and therefore yields exact solutions for the above KP equation. The conditions for breaking waves are also examined.

In a more geophysical perspective, let us mention the work of Artale et al. in which the problem of the generation of internal solitary waves in marine straits is examined. In ART1 the authors study the effect of an air-sea surface on the dynamics of internal solitary waves. Extending Whitham's treatment of a homogeneous fluid (see WHI1) to a two-fluid system, they show that the dynamics is described by means of two sets of coupled equations of Boussinesq type which, in turn, yield an inhomogeneous KdV equation whose forcing term depends on the air-sea elevation as well as on the bottom depth. In ART2, they again consider a two-layer fluid and a system of two uncoupled KdV equations is obtained; they are however related by the fact that the amplitudes of the surface wave and of the internal wave are proportional to each other. Their results are discussed in relation to various experimental measurements taken in the Strait of Gibraltar where such waves occur as tide-generated solitons.

Eckhaus, in a very recent paper, brings out a perspective which is worth mentioning (see ECK2). He states that all nonlinear partial differential equations of evolution type that are obtained through the formalism of multiple scaling can also be obtained in the following manner. He points out that the cumbersome machinery of formal series expansion appears to be irrelevant. His procedure is to supply transformations of the variable which produce, from a given perturbed equation, another perturbed equation; the model equations are thus formal limits of the various perturbed equations. Using such considerations, the author proceeds to derive several equations. In particular, he obtains a KP equation describing slightly curved waves, and he examines such waves propagating through channels of slowly varying depth; the author connects his results with our own work (i.e. chapter three of this thesis).

A large amount of literature has been written on experimental measurements of marine solitary waves, or direct observations of these. Measurements on solitary waves passing through the Strait of Gibraltar were done as early as 1964 (see FRA1). More recent observations of internal waves (see CAV1 and LAC1) show that when a tidal flow occurs, solitary wave trains are generated; these waves may have huge amplitudes: Lacombé and Richez measured amplitudes as large as 100 meters at some locations (see LAC1). Needless to say, such a magnitude is a blatant proof that a linear theory is just out of question to modelize these localized waves. Internal waves of that size can also explain the transport of lumps of cold water from the Atlantic Ocean into the Mediterranean Sea (see LAV1,3). Radar measurements are considered in LAV2 for the Strait of Gibraltar, and also in FU1 and FU2 for other areas, for instance the Gulf of California, by the SEASAT satellite. As a matter of fact, interesting phenomena occur in almost any strait or channel-like marine region; let us mention the Georgia Strait in British Columbia (see HUG1), the Archipelago of la Maddalena (see MANZ1), and Scylla & Charybdis (see ALP1).

The present thesis is divided into three chapters. The original contribution is presented in chapters two and three, and is based on the content of our recent publications (see DAV3, 4, 5, 6); part III of chapter three is unpublished as of now. Chapter one is used to introduce the mathematical background pertinent to the subsequent chapters. We review some of the basics about the theory of manifolds and the theory of Lie groups and algebras, actually the definitions, concepts, theorems, and constructions that are necessary for us to define and study what is the symmetry structure of a given system of partial differential equations. The important section for us is unquestionably that about the symmetry reduction technique which will be the principal tool to be eventually used in chapter two to get special KP solutions. Bäcklund transformations will also be needed at some point so we also review what they are.

In chapter two we deal with means of constructing special solutions of the KP equation. This chapter is subdivided into two parts. In the first one we get the Lie symmetry algebra of the equation; it is an infinite-dimensional algebra. We derive some special subalgebras of it. In particular we mention one from which the usual special solutions of the KP equation, such as the similarity solutions, are obtained through symmetry reduction. We also observe that this algebra hides a Kac-Moody loop algebra structure. This *novel* feature, which we first observed for the KP equation, seems to be a property shared by many, if not all, (2+1)-dimensional integrable soliton equations. We then proceed to classify the low dimensional subalgebras of dimensions 1, 2, and 3; these are classified in classes under group conjugation, and also under isomorphism of abstract algebras when the case occurs. This classification then permits us to perform symmetry reduction on the KP equation. We observe that this equation is actually reducible to special equations involving fewer dependent variables. The main result of this technique is that solutions of the reduced equations can be mapped back to solutions of the original KP equation; these solutions are generically nontrivial ones. The second part of chapter two consists in applying the symmetry reduction technique again, this time to the simultaneous system consisting of the KP equation, in its potential form, and of its associated Bäcklund transformation. It is, to our knowledge, the first time that such a procedure has been applied and the results are definitely of interest. The symmetry structure of the "potential" KP (PKP) equation is similar to that for the KP equation. The important fact here is that the symmetry group of the PKP equation acts nontrivially on the Bäcklund transformation and actually induces some functional dependence in this transformation. We then find the symmetry structure of the PKP equation together with this generalized Bäcklund transformation. We follow by applying the symmetry reduction technique. Again this yields a list of special KP solutions; among these we mention a special class which resembles the so-called resonance solitons mentioned earlier.

Chapter three is devoted to a model, based on a generalized KP (GKP) equation with variable coefficients, describing the dynamics of solitary waves in fluids with variable depths, and channels with variable geometry, in fact channels whose boundaries are small deformations of those for a rectangular channel. This is of interest with regards to actual observed, or measured, soliton phenomena in various emplacements in open seas and marine straits. In the first part of this chapter we begin by exposing the physical problem and specify the appropriate geophysical setting. The pertinent equations are the Euler system with boundary conditions, and we introduce dimensionless variables which we use in order to renormalize the system. We then use a conventional scaled perturbation expansion which we solve at lowest order to define new *wave frame* coordinates which are more appropriate for pursuing the problem. These are substituted back in the original system and we make a new perturbation expansion which results in a wave amplitude equation and some constraints which take the boundaries into account. This constitutes the GKP system which we then discuss. Although we derived the wave amplitude equation in order to describe surface waves, we mention that a system of two coupled similar equations is also a good description for describing internal waves as well. In the second part of the chapter, we proceed to reduce the GKP system (not in the same meaning as in chapter two). Indeed the GKP system is not integrable as such. However, for special geometries of the depth and of the walls, i.e. special geophysical constraints, it is then possible to construct appropriate algebraic differential transformations of the variables under which the GKP system can be transformed to a completely integrable system. As we show, some possible reductions are the pure KP equation, the KdV equation, and the cylindrical KdV (cKdV) equation; these equations may have variable coeficients. In the third part we use some of these reductions and thus obtain solutions of the GKP system from solutions of the reduced equations. In fact, we see that some specific geometries, such as a parabolic or hyperbolic tangent shaped depth, do yield *exact* solutions of the GKP system. Their most interesting feature is that they represent *curved* solitary waves, unlike the *straight* solitons which satisfy, say, the pure KP equation. These waves agree, in a certain measure, with those which are indeed observed in the oceans. Finally, in the fourth part, we derive and discuss some conservation laws and conserved quantities associated with the GKP equation.

CHAPTER ONE

Group theoretic aspects of symmetries of differential systems

As mentioned in the Introduction, group theory provides a very useful tool for the analysis of differential systems. The fact is that groups, in particular continuous local Lie groups, appear everywhere, explicitly or not, in the theory of differential equations. Solution spaces have generically a differentiable manifold structure having a Lie group substructure. Differential systems, seen as differential operators acting on some functional space, commute under specific groups of differential operators (vector fields): this is most important in practice for constructing natural invariant quantities, conservation laws, and is the basis at the origin of the symmetry reduction technique which we shall use later on. Less known is the fact that many of the integration techniques for special kinds of ordinary differential equations have direct group theoretic interpretations. Needless to say, group theoretic tools have a definitive importance for studying *nonlinear* differential systems, especially since these are often difficult to study with conventional analytic tools, these being usually linear ones and thus often inappropriate. In this chapter we review the basic concepts about Lie groups and Lie algebras, Bäcklund transformations, etc..., that we shall need in this work. We also spend some time to explain in some detail the technique of symmetry reduction which is the heart of chapter two. Most, if not all, of what is given in the present chapter may be found in several specialized books and articles (see IBR1; OLV1,2; OVS2; AND1; PIR1) and therefore all theorems will be stated without giving proofs. Concerning notations, we shall use a compact notation for derivations and derivatives; thus $\partial_x \equiv \partial/\partial x$, and $u_x \equiv \partial u/\partial x$. We do not use any special symbols, except perhaps boldface characters, to denote quantities other than scalar ones (e.g. vectors): it will usually be clear from the context what types of objects are being considered. Also, implicit summation over repeated indices is everywhere understood. Any other special notations or terminology will be introduced as we go along.

Let M be a connected Hausdorff topological space and $\varphi: U \subset M \to V \subset \mathbf{R}^m$ a homeomorphism, where U is open in M. The pair (U,φ) is called a local coordinate chart on M. The components φ^i of φ, $1 \leq i \leq m$, defined as $\varphi^i = \pi^i \circ \varphi$, where π^i is the canonical i-th projection in $\mathbf{R}^m$, are called the local coordinate maps, and $\forall p \in U$, the $x^i = \varphi^i(p)$ are called the coordinates of the point p. M is said to be an m-dimensional manifold if there exists a set of local charts $\{(U_\alpha,\varphi_\alpha)\}$ such that the U_α's form a covering of M. The concept of charts is fundamental since it permits to do calculus on manifolds by simply *projecting* down the objects at the manifold level onto appropriate objects in some convenient Euclidean spaces. Two charts (U_1,φ_1) and (U_2,φ_2) are said to be C^r-compatible if either $U_1 \cap U_2 = \{\}$, or else if $\varphi_1 \circ \varphi_2^{-1}$ and $\varphi_2 \circ \varphi_1^{-1}$ are both maps of class C^r on $U_1 \cap U_2$. A C^r-atlas is a family of compatible charts on M. Two atlases A_1 and A_2 are equivalent if $A_1 \cup A_2$ is also an atlas. We define a C^r-differentiable structure on M as a maximal equivalence class of C^r-atlases on M and say that M is an m-dimensional C^r-manifold whenever it is an m-dimensional manifold equiped with a C^r-differentiable structure. We shall deal with smooth (C^∞) and analytic manifolds which are defined similarly (replacing the above infinite differentiability condition by the condition of analyticity).

Let M and N be smooth manifolds. A map $F: M \to N$ is called smooth if for any pair (U_1,φ_1) and (U_2,φ_2) of local charts, its local representative $\varphi_2 \circ F \circ \varphi_1^{-1}: R^m \to R^n$ is smooth wherever defined. F is said to be of maximal rank if the matrix with entries $\partial F^i/\partial x^j$ has maximal rank itself; when such is the case there exist local coordinate maps such that F takes the local form $y = (x^1, \ldots, x^m, 0, \ldots, 0)$ if $n > m$, or $y = (x^1, \ldots, x^n)$ if $m > n$. A subset $N \subset M$ together with a smooth 1-1 map $\phi: P \to N$ of maximal rank is called a submanifold of M; P is called the parameter space. Let $0 \in I \subset R$, I being an open interval. A smooth map $\gamma: I \to M$, $\gamma(0) = x$, is called a curve on M through x. This curve γ is locally given by m smooth functions $f(\varepsilon) = [f_1(\varepsilon), \ldots, f_m(\varepsilon)]$ of the variable ε. Two curves γ_1 and γ_2 are called tangent at $x \in M$ if their local derivatives coincide at $\varepsilon = 0$: $\gamma_1'(0) = \gamma_2'(0)$. Tangency at $x \in M$ is actually an equivalence relation; in fact the class $\gamma'(0) = [\gamma]_x = \{\delta \text{ tangent to } \gamma \text{ at } x\}$ is called the vector tangent to M at x. This vector has the local form $f'(\varepsilon) = df/d\varepsilon = [f_1'(\varepsilon), \ldots, f_m'(\varepsilon)]$ and we shall adopt the notation $f'(\varepsilon) \equiv f_1'(\varepsilon)\partial/\partial x^1 + f_2'(\varepsilon)\partial/\partial x^2 + \cdots + f_m'(\varepsilon)\partial/\partial x^m$, which has definite advantages when calculating. The set $TM_x = \{[\gamma]_x\}$ is the tangent space to M at x; it can be identified with the set of all possible curves passing through x and has the structure of a vector space. The collection $TM = \{TM_x \mid x \in M\}$ is the tangent bundle of M.

A smooth manifold G with group operation $\cdot: G \times G \to G$ is a Lie group if the mapping $(a,b) \to a \cdot b^{-1}$ is smooth. A local group is a space G with a distinguished element "e" called the identity element, neighbourhoods U and V of the identity, with $V \subset U$, and a local group operation $\cdot : U \times U \to U$ such that

$$
\begin{aligned}
& V \cdot V \subset U, \\
& \forall a \in U,\ e \cdot a = a \cdot e = a, \\
& \forall a,b,c \in U,\ (a \cdot b) \cdot c = a \cdot (b \cdot c), \\
& \forall a \in U, \exists a^{-1} \in U,\ a \cdot a^{-1} = a^{-1} \cdot a = e, \\
& a \cdot b^{-1} \text{ is continuous on } U \times V.
\end{aligned}
\tag{1.1}
$$

$H \subset G$ is a local subgroup of G if $e \in H$ and if H, equipped with the restriction $\cdot|_H$, is a local group by itself. $H \subset G$ is called invariant (or normal) subgroup of G if there exists a neighbourhood $U \subset G$ of the identity such that $\forall a \in U, \forall b \in U \cap H$, $a^{-1} \cdot b\, a \in H$; G/H denotes the set of cosets (mod H) of elements living in some neighbourhood of the identity. Two groups G and G' are locally isomorphic is there exists a homeomorphism $\phi: U \to U'$, $\phi(V) \subset V'$, such that $\forall a,b \in V$, $\phi(a \cdot b) = \phi(a) \cdot \phi(b)$. A local r-dimensional Lie group is a local group G_r together with a local chart (U,φ), $e \in U$, and a local group operation $\cdot : U \times U \to U$ satisfying to (1.1) for open subsets $V \subset U$ and being smooth in $V \times V$. G_r is called solvable if there exists a descending tower $G_r \supset G_{r-1} \supset \cdots \supset G_1$, where G_i is a normal subgroup of G_{i+1}. G_r is called simple if it has no proper invariant subgroups other than $\{e\}$.

A Lie algebra is a vector space L, together with a bilinear map $L \times L \to L$: $(\xi,\eta) \to [\xi,\eta]$ called the Lie bracket (or its multiplication), such that $\forall \xi,\eta,\rho \in L$, $[\xi,\eta] + [\eta,\xi] = 0$, and $[\xi,[\eta,\rho]] + [\eta,[\rho,\xi]] + [\rho,[\xi,\eta]] = 0$; this last formula is called the Jacobi identity. Let $\{\xi_i\}$ be a basis for L. Then $\forall \xi_i,\xi_j \in L$, $[\xi_i,\xi_j] = C_{ij}{}^k \xi_k$. The $C_{ij}{}^k$ are called the structure constants of L and determine it uniquely. A mapping $f: L \to K$ is a Lie algebra homomorphism if $\forall \xi,\eta \in L$, $f([\xi,\eta]) = [f(\xi),f(\eta)]$. We define $\mathrm{Ker}(f) = f^{-1}(0) = \{\xi \in L \mid f(\xi) = 0\}$; f is an isomorphism if it is onto and if $\mathrm{Ker}(f) = 0$. Let K and N be subspaces of L. $K+N = \{\xi + \eta \mid \xi \in K, \eta \in N\}$ is called the sum of K and N. $[K,N]$ is their product. If $K \cap N = \{0\}$ then K+N is called a direct sum. $K \subset L$ is a subalgebra if $[K,K] \subset K$, and an ideal if $[K,L] \subset K$. If K, N are ideals in L then so are $K \cap N$, $[K,N]$, and K+N; moreover, if $K \cap N = \{0\}$, then $[K,N] = \{0\}$ and we write $K+N \equiv K \oplus N$. If K is an ideal and N a subalgebra with $K \cap N = \{0\}$, then K+N is called a semi-direct sum. Let $K \subset L$ be an ideal. The set $L/K = \{\xi + K \mid \xi \in L\}$ is called the quotient algebra of L by K and it also possesses the structure of a Lie algebra by itself. The mapping $\phi_K: L \to L/K: \xi \to \xi + K$ is the canonical homomorphism, with $\mathrm{Ker}(\phi_K) = K$; we then say that L is the extension of L/K by K. The sequence of ideals recursively defined by $L^{(1)} = [L,L]$, $L^{(i+1)} = [L^{(i)},L^{(i)}]$, are called the derived subalgebras of L. L is said to be solvable if there is an integer "n" such that $L^{(n)} = \{0\}$. The maximum solvable ideal R in L is the radical of L. L is called simple if it has no proper ideal (i.e. if L is not a commutative algebra), and semi-simple if $R = \{0\}$ (i.e. if it has no abelian ideal other than $\{0\}$).

To each Lie group G_r there corresponds a Lie algebra L_r, and every finite dimensional Lie algebra L_r is isomorphic to the Lie algebra corresponding to some Lie group of the same dimension. If G corresponds to an algebra L, and $H \subset G$ is an invariant subgroup corresponding to the algebra K, then $K \subset L$ is an ideal. (Semi-) simplicity, (semi-) direct product, and solvability are carried through by this natural correspondence. An important result is that a local Lie group may be reconstructed from its Lie algebra by means of the Lie equation and the exponential map. Consider a curve $\gamma(\varepsilon) \subset G$, $\varepsilon \in I \subset \mathbf{R}$, passing through the identity e. This curve is said to be a 1-parameter subgroup of G if $\gamma: I \to G$ is a local homomorphism, i.e. if $\gamma(0) = e$ and $\forall \varepsilon,\zeta \in I$, $\gamma(\varepsilon)\cdot\gamma(\zeta) = \gamma(\varepsilon+\zeta)$. Now $\forall \xi \in L$, there exists a 1-parameter subgroup $\gamma(\varepsilon)$ such that $[\gamma]_e = \gamma'(0) = \xi$: this is Lie's equation. The collection of all the curves $\gamma(\varepsilon)$ for all $\xi \in L$ is a local Lie group isomorphic to G. From Lie's equation it then follows that $\gamma(\varepsilon) = e^{\varepsilon\xi}$. All that reflects the fact that $L = TG_e$.

Let G be a Lie group with corresponding Lie algebra L. Each $a \in G$ induces an automorphism $f_a: x \to a \cdot x \cdot a^{-1}$ and the set $\mathrm{Aut}(G) = \{f_a \mid a \in G\}$ has the property to be a local Lie group. Furthermore Aut(G) induces, in its turn, a local Lie group of automorphisms acting on L; this group, G^A, is called the adjoint group of G. Its corresponding Lie algebra, L^A, is called the adjoint algebra of L and is constructed as follows. Let $\xi \in L$. We then define the adjoint map to ξ as $\mathbf{ad}\xi: \eta \to [\xi,\eta]$. Then $L^A = \{\mathbf{ad}\xi \mid \xi \in L\}$ and is equipped with the Lie bracket $[\mathbf{ad}\xi, \mathbf{ad}\eta] = \mathbf{ad}[\xi,\eta]$. Two subgroups H and H' of G are called conjugate if there exists $f \in \mathrm{Aut}(G)$ such that $H = f(H')$. Similarly, two subalgebras K and K' of L are conjugate if there exists an automorphism $\phi = e^{\mathbf{ad}\xi} \in G^A$ such that $K = \phi(K')$; the set θ_r of all conjugacy classes is called the optimal system and plays an important role in the symmetry reduction technique. Indeed this method, for instance when classifying invariant solutions, and in general group analysis of differential systems, require us to describe all non-conjugate subgroups of the invariance group of the system. One usually obtains θ_1 by choosing appropriate automorphisms in G^A. When dealing with *solvable* algebras, one may find θ_{s+1}, $s > 1$, from θ_s first by extending θ_s to an (s+1)-dimensional subalgebra and then by discarding the conjugate subalgebras using the fact that any (s+1)-dimensional *solvable* subalgebra contains an s-dimensional subalgebra. The case of *non-solvable* subalgebras of dimension higher than two is a little harder to deal with and one uses the

Levi Theorem: Let L be a finite-dimensional Lie algebra. Then L admits a decomposition as a semi-direct sum R+N. R is the radical and N is semi-simple. This decomposition is unique, up to a conjugation.

The construction of θ_{s+2}, $s > 1$, then consists in listing all the non-conjugate subalgebras of L and the non-conjugate subalgebras of N (considered independently of L). For instance θ_3 is obtained by assembling a family of non-conjugate subalgebras of L as well as an optimal (or representative) system of 3-dimensional subalgebras of N.

Let $B \subset R^r$ be an open ball containing the origin and coordinatized by collections $a \in B$ of "r" real numbers. Consider a smooth map $f: R^n \times B \to R^n$ which induces another mapping $T_a: R^n \to R^n : x \to T_a(x) = f(x,a)$ called a transformation. The set G_r of all such induced mappings is called a continuous r-parameter local group of transformations (on R^n) if it also has the structure of an r-dimensional local Lie group equipped with the group product $(T_a \cdot T_b)(x) = (T_a(T_b(x)) = f(f(x,a),b)$. For a given $x \in R^n$, the set $O_x = \{T_a(x) \mid T_a \in G_r\}$ is a local manifold called the G_r-orbit through x. More generally, transformation groups are defined on manifolds, but the above definition will be sufficient for our purposes. Transformation groups prove to be of prime importance when studying differential systems. Indeed the symmetry group of such a system has the property to be a local transformation group on the integral (or solution) manifold of this system. Let G be a 1-parameter transformation group characterized by $T_a(x) = f(x,a)$, $a \in I$, I being some open interval in R containing 0. The orbit G(x) is just the curve $a \to f(x,a)$ passing through x. The tangent vector of G at x is thus given by the map $\xi: R^n \to R^n : x \to \partial f(x,a)/\partial a \,|_{a=0}$ or, equivalently, by the tangent vector field $X = \xi^i(x)\partial/\partial x^i$, i.e. the infinitesimal generator nearby the identity.

The following theorem is of prime importance:

Lie's Theorem: $G(x)$ is an integral curve of Lie's equation $df/da = \xi(f)$, with initial condition $f(x,a=0) = x$. Conversely, $\forall \xi: \mathbf{R}^n \to \mathbf{R}^n$, $\forall x \in \mathbf{R}^n$, Lie's equation has a unique solution.

Let G_r be an r-parameter transformation group. For each 1-parameter subgroup of G_r we construct the corresponding tangent vector fields. These generate an r-dimensional vector space L_r which is a Lie algebra with respect to the Lie product $[\xi,\eta] = \eta'\xi - \xi'\eta$, or equivalently, $[X,Y] = XY - YX = \{X(\eta^i) - Y(\xi^i)\}\partial/\partial x^i$. In fact, the set $\{\xi_\nu = \partial f(x,a)/\partial a\,|_{a=0}\}$ is a basis for L_r. Lie's equation reads:

$$\frac{\partial f}{\partial a^\nu} = V^\mu{}_\nu(a)\xi_\mu(f), \qquad V^\mu{}_\nu(a) = \frac{\partial \varphi^\mu(a,b)}{\partial a^\nu}\Bigg|_{b=a^{-1}},$$

where $\varphi: G_r \times G_r \to G_r$ is the group multiplication. Thus to reconstruct the group, we first choose a basis $\{\xi_\nu\}$, solve the system $df/da^\nu = \xi^\nu$ with $f(x,a^\nu=0) = x$; G_r is then produced by composing the transformations of the resulting 1-parameter subgroups. $a = (a^1, ..., a^r)$ provides coordinates in the local group G_r.

A function $F(x)$ is called an invariant of a transformation group G_r in $\mathbf{R}^n$ if F is constant along orbits $G_r(x)$: $F(f(x,a)) = F(x)$. For a 1-parameter transformation group G_1 with infinitesimal generator X, this condition writes simply as $XF = \xi^i(x)\partial F(x)/\partial x^i = 0$, by virtue of Lie's equation. Invariants are at the heart of several group theoretic considerations about differential equations, in particular of the symmetry reduction technique. Any given collection $\{I_1(x), ..., I_{n-1}(x)\}$ of functionally independent solutions of $XF = 0$ forms a basis of invariants, and every invariant F can be written in the form $F(x) = \Phi[I_1(x), ..., I_{n-1}(x)]$. More generally, for an r-parameter transformation group G_r, a function F is an invariant if it is a solution of the following set of r equations: $X_\nu F \equiv \xi^i{}_\nu(x)\partial F(x)/\partial x^i = 0$, $1 \leq \nu \leq r$. Consider the matrix with elements $\xi^i{}_\nu$ and define the quantity $r_*(\xi) = \mathrm{rank}[\xi^i{}_\nu(x)]$. Then the number of solutions to the above system of equations is $n-r_*$; when $r_* = n$, the group G_r is called transitive. Now let $M \subset \mathbf{R}^n$ be a local manifold, with local coordinates $(x^1, ..., x^m; 0, ..., 0)$, parametrized by some monomorphic map $h: U \to \mathbf{R}^n$, $U \subset \mathbf{R}^m$ open. The tangent space to M at x is $TM_x = \{dx \in TR^n{}_x \mid dx = h'(y)dy,\ dy \in TR^n{}_y\}$. M is said to be invariant under a transformation group G_r if $\forall x \in M$, $G_r(x) \subset M$. The infinitesimal invariance criterion is that M is invariant if, and only if, $\forall \xi \in L_r, \forall x \in M$, $\xi(x) \in TM_x$. Let $F: R^n \rightarrow R^{n-m}$ be a differentiable map, with $\text{rank}[F'(x)] = n-m$, and M be the solution set of $F(x) = 0$. Then the condition $\xi(x) \in TM_x$ is equivalent to $\xi^i{}_v(x)\partial F(x)/\partial x^i \,|_M = 0$, $1 \leq v \leq r$, where $\{\xi_v\}$ is a basis for L_r. The equation $F(x) = 0$ can be recast as $\Phi^k[I_1(x), ..., I_{n-\sigma}(x)] = 0$, $1 \leq k \leq n-m$, $\sigma = r_*$, and $\{I_k\}$ is a basis of invariants of G_r. This last condition provides a representation of the m-dimensional invariant manifold M through a manifold of dimension $\rho = m - r_*$ in the space of invariants; ρ is called the rank of M. Let $M \subset R^n$ be an arbitrary manifold. The orbit $G_r(M)$ is the minimal manifold which is invariant under G_r and, in addition, containing M as a submanifold of codimension $\delta = \dim G_r(M) - \dim M$. δ is called the defect of M relative to G_r and can also be written as $\delta = \text{rank}[\,\xi^i{}_v(x)\partial F^k(x)/\partial x^i\,]\,|_M$, and then $\rho = m - r_* + \delta$.

Given a certain differential system, a symmetry group for it can basically be viewed as a transformation group whose elements map solutions into solutions. In this thesis we shall primarily deal with so-called Lie *point* groups. They are not the most general kind of symmetry groups that can be associated with a differential system; for instance they are contained in what are called groups of generalized symmetries, but they are well understood and most of the group theoretic methods applicable to differential systems that we know of are based on point groups. Furthermore these groups admit a very natural interpretation since their elements are geometric transformations, hence their physical meaning is rather direct, in contrast with the transformations corresponding to generalized symmetries which are defined to act on some functional spaces whose physical significance is usually far from being obvious, although their importance is not to be neglected. Indeed, following Noether's work, it is known that there is a 1-1 correspondence between the generalized symmetries of a given set of differential equations and the underlying conservation laws. Since we shall work with geometric transformations it would be natural to provide a geometric setting for differential systems. In fact, it is *the natural* way to look at them: clearly we can think of such a system as some manifold embedded in some Euclidean space of usually high dimension.

At this point it is useful to introduce the notion of jet spaces. In what follows, all subsets are presumed to be open and all functions are supposed to be smooth. Let M and N be two manifolds of dimensions m and n, respectively. N has local coordinates $\{x^i\}$, called the independent coordinates, and those of M are the dependent coordinates $\{u^\alpha\}$. Let $\Gamma(N,M)$ be the set of all $U \subset N \to M$ mappings. Consider the set $S = N \times \Gamma(N,M)$. On this set S we define an equivalence relation $\equiv$ by $(x,f) \equiv (x',f') \Leftrightarrow x = x'$ and f, f' have the same Taylor expansion at x, up to order k inclusively. The quotient set $S/\equiv$ is called the k-jet bundle and noted $J^k(N,M)$. By convention, we make the identification $J^0(N,M) = N \times M$. An equivalence class $j^k{}_x f$ is called the k-jet of f at x. $J^k(N,M)$ is itself a manifold, with local coordinate functions $[x; u, u^{(1)}, \ldots, u^{(k)}]$, where $u^{(J)}$ is a collection of symbols $\{u^\alpha{}_J\}$ with "J" a multi-index; these symbols will later on represent the set of J-th order derivatives of the u^α with respect to the x^i. We define two special kinds of maps, the source maps A_k and the target maps B_k, as well as projection maps $\pi^k{}_l$ $(k > l)$ as

$$A_k : J^k(N,M) \longrightarrow N : j^k{}_x f \longmapsto x\,,$$

$$B_k : J^k(N,M) \longrightarrow M : j^k{}_x f \longmapsto f(x)\,,$$

$$\pi^k{}_l : J^k(N,M) \longrightarrow J^l(N,M) : j^k{}_x f \longmapsto j^l{}_x f.$$

These maps are useful for performing calculus in the jet bundles. Let P be a manifold with local coordinates $\{v^\alpha\}$ and consider a mapping $\phi : J^k(N,M) \to P$. We then define the l-th prolongation of ϕ as the unique mapping $p^l\phi$ such that the diagram

$$\begin{tikzcd}
N \arrow[r, "j^{k+1}f"] \arrow[dr, "j^l(\phi \circ j^k f)"'] & J^{k+1}(N,M) \arrow[d, "p^l\phi"] \\
 & J^l(N,M)
\end{tikzcd}$$

commutes. Locally, $p^l\phi$ has the representation

$$x = x\,,$$

$$v = \phi(x; u, u^{(1)}, \ldots, u^{(k)})\,,$$

$$v^{(l)} = [D_J{}^l\phi](x; u, u^{(1)}, \ldots, u^{(k)})\,,$$

where $D_J = \prod D_i$, D_i being the total differentiation operator with respect to x^i. Let us also define the contact modulus on $J^k(N,M)$:

$$\Omega^k(N,M) = \{\omega \in \Lambda[J^k(N,M)] \mid (j^k f)^* \omega = 0\}$$
$$= \mathrm{Span}\,\{\omega^\alpha{}_J \equiv du^\alpha{}_J - u^\alpha{}_{J,i} dx^i \mid 0 < J < k\}.$$

A differential system Z of order k with domain N and range M is then interpreted as a differential subset $Z \subset J^k(N,M)$, i.e. the zero-set of a finite ideal of functions on $J^k(N,M)$. The local meaning of this definition is that a mapping $F = (F^1, \ldots, F^m)\colon J^k(N,M) \to \mathbf{R}^m$ (or $\mathbf{C}^m$) is given and that a solution of Z is any map $f \in \Gamma(N,M)$ such that $F \circ j^k f = 0$. In what follows, note that we use $N = \mathbf{R}^n$ and $M = \mathbf{R}^m$. For instance, the heat equation can be viewed as the submanifold $Z \subset J^2(\mathbf{R}^2,\mathbf{R})$ with the map $F(x; u, u^{(1)}, u^{(2)}) = u_t - u_{xx}$. A solution to the heat equation is then any particular function $u = f(t,x)$, having a second prolongation $j^2 f$ of local representation $[f(t,x), \partial f/\partial t, \partial f/\partial x, \partial^2 f/\partial t^2, \partial^2 f/\partial t\partial x, \partial^2 f/\partial x^2]$ which satisfies the condition $F \circ j^2 f \equiv f_t - f_{xx} = 0$.

A crucial point when defining the symmetry group of a differential system is the following one. The group transformations are basically characterized by how they act in the space $J^0(\mathbf{R}^n,\mathbf{R}^m)$, even though a differential system is essentially defined as living in some non-trivial jet bundle $J^k(\mathbf{R}^n,\mathbf{R}^m)$. The point is that the natural projections $\pi^j{}_{j-1}$ completely specify how the transformations will act in the $J^k \backslash J^0$ part of $J^k(\mathbf{R}^n,\mathbf{R}^m)$. In fact if G is the group acting in $J^0(\mathbf{R}^n,\mathbf{R}^m)$, then its k-th prolongation $G^{(k)}$ is uniquely defined. Let G be a 1-parameter transformation group acting in the space $J^0(\mathbf{R}^n,\mathbf{R}^m)$ coordinatized by (n+m)-tuples $(x,u) \equiv (x^1, \ldots, x^n; u^1, \ldots, u^m)$, with transformations given by maps f and φ defined through the formulas

$$\begin{aligned} x' &= f(x, u, a), & f(x, u, a{=}0) &= x, \\ u' &= \varphi(x, u, a), & \varphi(x, u, a{=}0) &= u. \end{aligned} \tag{1.2}$$

We introduce new variables $u^{(1)} \equiv \{u^\alpha{}_i \mid \alpha = 1, \ldots, m;\ i = 1, \ldots, n\}$ subject to the following transformations:

$$u'^{(1)} = \psi(x, u, u^{(1)}, a), \qquad \psi(x, u, u^{(1)}, a{=}0) = u^{(1)}. \tag{1.3}$$

We require that (1.3) and the transformations of the $\partial\varphi^\alpha/\partial x^i$ under (1.2) be compatible with the equalities $u^\alpha_i = \partial u^\alpha/\partial x^i$, since we would identify the symbols u^α_i with the derivatives of the u^α, as we stated before. This compatibility requirement actually defines the maps ψ^α_i of (1.3) in a unique manner and therefore the first prolongation $G^{(1)}$ of G as a geometric transformation group acting on $J^1(R^n,R^m)$ with the point transformations (1.2), as well as the new transformations (1,3). Let X belong to the Lie algebra L corresponding to G; X has the form

$$X = \xi^i(x,u)\partial/\partial x^i + \eta^\alpha(x,u)\partial/\partial u^\alpha, \quad \xi = \partial f/\partial a\,|_{a=0}, \quad \eta = \partial\varphi/\partial a\,|_{a=0}. \tag{1.4}$$

Then the first prolongation $X^{(1)} \equiv p^1X \in L^{(1)}$ of X must take the form

$$X^{(1)} = X + \zeta^\alpha_i\partial/\partial u^\alpha_i, \quad \zeta^\alpha_i = \partial\psi^\alpha_i/\partial a\,|_{a=0}.$$

By construction, $L^{(1)}$ is the Lie algebra corresponding to $G^{(1)}$, and the quantities ζ^α_i are to be determined by our above compatibility requirement on the prolongation. Introduce now a collection of 1-forms $\omega^\alpha \equiv du^\alpha - u^\alpha_j dx^j \in \Omega^1(R^n,R^m)$; then this requirement writes as $\omega = 0$ and defines a manifold M invariant under a group G^* with action specified by (1.2,3) and

$$dx' = (\partial f/\partial x^i)dx^i + (\partial f/\partial u^\alpha)du^\alpha, \quad du' = (\partial\varphi/\partial x^i)dx^i + (\partial\varphi/\partial u^\alpha)du^\alpha,$$

acting in the space coordinatized by $(x, u, u^{(1)}, dx, du)$. Introduce the vector field X^* defined as

$$X^* = X^{(1)} + \xi^{i*}\partial/\partial(dx^i) + \eta^{\alpha*}\partial/\partial(du^\alpha),$$
$$\xi^* \equiv \partial(dx')/\partial a\,|_{a=0} = (\partial\xi/\partial x^i)dx^i + (\partial\xi/\partial u^\alpha)du^\alpha,$$
$$\eta^* \equiv \partial(du')/\partial a\,|_{a=0} = (\partial\eta/\partial x^i)dx^i + (\partial\eta/\partial u^\alpha)du^\alpha,$$

The necessary and sufficient condition for M to be invariant under G^* is that

$$X^*\omega^\alpha\,|_{\omega=0} \equiv (\eta^{\alpha*} - u^\alpha_i\xi^{i*} - \zeta^\alpha_i dx^i)\,|_{\omega=0} = 0.$$

Substituting the above expressions for ξ^* and η^* yields the identification

$$\zeta^\alpha_i = D_i(\eta^\alpha) - u^\alpha_j D_i(\xi^j),$$

where $D_i \equiv \partial/\partial x^i + u^\alpha_i\partial/\partial u^\alpha$ is the total differentiation operator with respect to the variable x^i. Thus $X^{(1)}$ is completely specified:

$$X^{(1)} = \xi^i(x,u)\partial/\partial x^i + \eta^\alpha(x,u)\partial/\partial u^\alpha + [D_i(\eta^\alpha) - u^\alpha_j D_i(\xi^j)]\partial/\partial u^\alpha_i .$$

The prolongation of G to $G^{(k)}$, $k>1$, is obtained in an analogous manner by defining the action of G on the variables $\{u^\alpha_J\}$, where $J>1$ is a multi-index, representing the derivatives of the u^α's of all orders up to order k, and by also taking into account the compatibility conditions $\omega^{(j)}=0$, $0\le j\le k-1$, with $\omega^{(0)}\equiv\omega$ as defined above, and $\omega^{(1)\alpha}_i = du^\alpha_i - u^\alpha_{ij}dx^j$, $\omega^{(2)\alpha}_{ij} = du^\alpha_{ij} - u^\alpha_{ijk}dx^k$, etc.; i.e. we require that $\Omega^k(\mathbf{R}^n,\mathbf{R}^m)$ be trivial. This thus yields the k-th prolongation $G^{(k)}$ with the following infinitesimal operators:

$$X^{(k)} = X + \zeta^\alpha_i\partial/\partial u^\alpha_i + \zeta^\alpha_{ij}\partial/\partial u^\alpha_{ij} + \dots,$$

$$\zeta^\alpha_i = D_i(\eta^\alpha) - u^\alpha_j D_i(\xi^j),$$

$$\zeta^\alpha_{ij} = D_j(\zeta^\alpha_i) - u^\alpha_{ij} D_k(\xi^k),$$

$$D_i \equiv \partial/\partial x^i + u^\alpha_i\partial/\partial u^\alpha + u^\alpha_{ij}\partial/\partial u^\alpha_j + \dots$$

In terms of multi-indices we have the rather simple formulas

$$p^kX = X^{(k)} = X + \zeta^\alpha_J\partial/\partial u^\alpha_J ,$$
$$\zeta^\alpha_J = D_J(\eta^\alpha - \xi^i u^\alpha_i) + \xi^i u^\alpha_{J,i} . \tag{1.5}$$

The quantities $Q^\alpha \equiv \eta^\alpha - \xi^i u^\alpha_i$ are called the characteristics of the vector field X. Let us define a new vector field $X_Q = Q^\alpha\partial/\partial u^\alpha$. Then (1.5) writes simply as

$$p^kX = p^kX_Q + \xi^i D_i , \qquad p^kX_Q = D_J Q^\alpha\partial/\partial u^\alpha_J , \tag{1.6}$$

where the sum over the multi-index J is performed for $0\le J\le k$. From a practical point of view, the form (1.6) for the k-th prolongation of X permits to perform the calculations in a more efficient manner. r-parameters groups and algebras are prolonged in a very similar way and it may be shown that they have exactly the *same* structures as the initial groups and algebras from which they are constructed. The invariants of $G^{(k)}$ are called the k-th order differential invariants of G.

Consider now a differential system of order k, specified by Ω: $F(x, u, \dots, u^{(k)}) = 0$, where $F\equiv(F^1, \dots, F^p)$. This system can be viewed as defining some manifold in the space $J^k(\mathbf{R}^n,\mathbf{R}^m)$. If this manifold is invariant under $G^{(k)}$ then we say that the differential system Ω

admits the group G as a symmetry group. Let L be the algebra corresponding to G. Then the above criterion has the following infinitesimal version. Ω admits G as a symmetry group if, and only if,

$$\forall X \in L,\ (p^kX)\Omega = 0 \pmod{\Omega}. \tag{1.7}$$

The "(mod Ω)" means that the object "$(p^kX)\Omega$" must be calculated taking into account the differential system Ω and all its differential consequences, i.e. its k-th prolongation $p^k\Omega$; the reason for this is that the above condition (1.7) is required to hold *only* on solutions of $F=0$. This equation, which is termed the *defining equation*, is in fact equivalent to a system of linear homogeneous differential equations for the components $\xi(x,u)$ and $\eta(x,u)$ of the vector field X. It is a simple matter to check that the set L of those vector fields that satisfy the defining equation (1.7) is indeed a vector space and that $\forall X_1, X_2 \in L$, $[X_1, X_2] \in L$; hence L is indeed a Lie algebra. Its corresponding local Lie group is the maximal point transformation group admitted by the system Ω. It is important to note that the defining equation is of polynomial type in the space $J^k(R^n,R^m)$ and this means that the quantities $x, u, u^{(1)}, \ldots, u^{(k)}$ are to be considered as *independent variables* that satisfy $F=0$, while the defining equation, by itself, must be identically satisfied with respect to the *free variables* x and u. It therefore follows that (1.7) is actually an *overdetermined system*. As a typical example of how to find the symmetry group of a given partial differential equation, consider the heat equation, given by Ω: $u_t - u_{xx} = 0$. The infinitesimal generator X of G is assumed to take the form of a vector field $X = f(t,x,u)\partial/\partial t + g(t,x,u)\partial/\partial x + \eta(t,x,u)\partial/\partial u$. Calculating the second prolongation $X^{(2)}$ of X, it is clear that (1.7) reads as

$$u^t - u^{xx} = 0 \pmod{\Omega}, \tag{1.8a}$$

$$\begin{aligned} u^t &= D_t(\eta - fu_t - gu_x) + fu_{tt} - gu_{tx} \\ &= \eta_t + \eta_u u_t - f_t u_t - f_u u_t^2 - g_t u_x - g_u u_x u_t, \\ u^{xx} &= D_x^2(\eta - fu_t - gu_x) + fu_{txx} - gu_{xxx} \\ &= \eta_{xx} + 2\eta_{xu}u_x + \eta_{uu}u_x^2 + \eta_u u_{xx} - f_{xx}u_t - 2f_{xu}u_x u_t \\ &\quad - 2f_x u_{tx} - f_{uu}u_t u_x^2 - 2f_u u_{tx}u_x - f_u u_t u_{xx} - g_{xx}u_x \\ &\quad - 2g_{xu}u_x^2 - 2g_x u_{xx} - g_{uu}u_x^3 - 3g_u u_x u_{xx}. \end{aligned} \tag{1.8b}$$

Substituting (1.8b) into (1.8a) and replacing all occurences of u_{xx} by u_t (since these two quantities are equal on the solution manifold of the heat equation; this is our taking into account of "mod Ω"), we obtain

$$\begin{aligned}(\eta_t - \eta_{xx}) - (f_t - f_{xx} - 2g_x)u_t - (2\eta_{xu} + g_t - g_{xx})u_x \\ + 2(f_{xu} + g_u)u_t u_x + (\eta_{uu} - 2g_{xu})u_x^2 - 2(f_x)u_{tx} \\ - 2(f_u)u_x u_{tx} - (f_{uu})u_t u_x^2 - (g_{uu})u_x^3 = 0 .\end{aligned}$$

This last equation is satisfied if, and only if, all the coefficients, i.e. the quantities enclosed within parentheses, vanish identically. This provides a set of differential equations for the functions f(t,x,u), g(t,x,u), and η(t,x,u) which is very easy to solve. One finds

$$\begin{aligned}&f = a_0 + 2a_1 t + 4a_2 t^2 , \\ &g = a_3 + 2a_4 t + a_1 x + 4a_2 tx , \\ &\eta = \varphi(t,x) + (a_5 - a_4 x - 2a_2 t - a_2 x^2)u , \qquad \text{with} \quad \varphi_t = \varphi_{xx} .\end{aligned}$$

We note that these functions involve 6 real constants a_0 to a_5 and one function φ(t,x) which is constrained to obey the heat equation, but is otherwise arbitrary. The generators of the Lie algebra corresponding to the symmetry group G of the heat equation are thus obtained (we set $a_5 = b_5 - \frac{1}{2}a_1$ for reasons of convenience):

$$\begin{aligned}&X_0 = \partial_t , \\ &X_1 = 2t\partial_t + x\partial_x - \tfrac{1}{2}u\partial_u , \\ &X_2 = 4t^2\partial_t + 4tx\partial_x - (2t + x^2)u\partial_u , \\ &X_3 = \partial_x , \\ &X_4 = 2t\partial_x - xu\partial_u , \\ &X_5 = u\partial_u , \\ &X_6(\varphi) = \varphi(t,x)\partial_u .\end{aligned}$$

Let K = Span $\{X_0, X_1, X_2\}$, H = Span $\{X_3, X_4, X_5\}$, and N = Span $\{X_6(\varphi)\}$, where φ is defined as above. Calculating the commutation relations for the generators of L, it is easily seen that K generates a sl(2,R) subalgebra which is an ideal in L and that H generates an

Heisenberg type algebra; N is an infinite dimensional Abelian subalgebra. Furthermore, L admits the Levi decomposition $K \oplus (H+N)$. Each of the vector fields X_i yields a particular 1-parameter transformation subgroup of the symmetry group G. For example the transformations corresponding to X_4 are determined by (1.2) and (1.4):

$$dt'/d\alpha = 0, \qquad dx'/d\alpha = 2t', \qquad du'/d\alpha = -x'u',$$

$$x'(t, x, u, \alpha=0) = x, \qquad t'(t, x, u, \alpha=0) = t, \qquad u'(t, x, u, \alpha=0) = u,$$

This yields

$$t' = t, \qquad x' = x + 2\alpha t, \qquad u' = u(t, x)\exp(-\alpha x - \alpha^2 t).$$

Finally, the concrete meaning of this 1-parameter invariance subgroup is that if $u(t,x)$ solves the heat equation, then so does $u'(t', x') = u(t', x' - 2\alpha t')\exp(-\alpha x' + \alpha^2 t')$ in the new variables t' and x'. Therefore $u(t, x - 2\alpha t)\exp(-\alpha x + \alpha^2 t)$ solves our original equation.

By construction, the elements of G induce inner mappings on the solution manifold S of Ω, i.e. if $F(u) = 0$ then $F(e^X u) = 0$. Introduce an equivalence relation $\equiv$ on S by $u_1 \equiv u_2$ if there exists some $g \in G$, with corresponding infinitesimal generator X, such that $u_2 = e^X u_1$. The quotient set $S/\equiv$ is called the resolvent system of Ω. This system may be completed by an automorphic (relatively to the group G) system which yields all the solutions of Ω. In fact, starting from some representative member u in the class $[u] \in S/\equiv$, this automorphic system explicitates all the members of the class.

The symmetry reduction technique is a clever way of obtaining special solutions of given differential systems, and proves itself to be particularly worth of using when one is confronted with complicated differential equations, i.e. equations for which no solutions are obvious, except perhaps trivial ones (e.g. $u = 0$). Let G be the symmetry (or invariance) group admitted by a differential system $\Omega: F = 0$, and consider a subgroup $H \subset G$. Its solution space $\{u = f(x)\}$ is some n-dimensional manifold $U \subset J^0(R^n, R^m)$; in fact U is specified as the set of all pairs (x,u) such that $u - f(x) = 0$. Assume that U is non-singular with respect to H, i.e. that $X|_H F = 0 \pmod{\Omega}$ is of maximal rank; the defect of U is then given by $\delta = \dim H(U) - n$. If $\delta = 0$, which amounts to the situation when U is invariant under H, then we say that the solution $u = f(x)$ is invariant relative to the subgroup H. From a geometrical point of view, this means that these invariant solutions are actually associated with solutions of a differential system, called the reduced system, which defines the manifold $\Omega_H = \Omega/H$. If $0 < \delta \leq m$, then u is termed partially invariant and rank(U) is called the rank of the particular solution f(x). The procedure of reduction by symmetry can be described as follows. Let H be an s-parameter group; usually H is chosen to be an s-parameter subgroup of the symmetry group G of the differential system Ω of order r. Let $\{(\xi_v,\eta_v)\}$ be a given basis of the Lie algebra L_H corresponding to H and assume that $r_*(\xi_v,\eta_v) < n$. We first choose a basis of invariants $\{I_k(x,u)\}$ of H, $1 \leq k \leq m+n-r_*$. We then look for non-singular invariant solutions in the implicit form

$$\Phi^{\alpha}(I_1, ..., I_{\sigma}) = 0, \tag{1.9}$$

$\sigma = m+n-r_*$, $1 \leq \alpha \leq m$. The manifold described by (1.9) can be recast as $U: u - f(x) = 0$ only if the invariants I_k are independent of u, i.e. only if $\mathrm{rank}(\partial I_k/\partial u^{\alpha}) = m$. Suppose that this is indeed the case for the first m invariants I_1 to I_m and introduce a new set of variables $v^{\alpha} = I_{\alpha}(x,u)$, $1 \leq \alpha \leq m$, and $y^i = I_{m+i}(x,u)$, $1 \leq i \leq n-r_*$. We can rewrite (1.9) as $v^{\alpha} = v^{\alpha}(y)$. Then $u, u^{(1)}, ..., u^{(s)}$ can be written completely in terms of x, y, v, and the y-derivatives of v. Substituting into Ω: $F = 0$, we obtain a system for v: it is this system that we shall call the *reduced system* and that defines Ω_H. The rank of an invariant solution, $\rho = n-r_*$, is precisely the number of independent variables in this new system. Obviously, the interesting feature is that $\rho < n$: the reduced system involves a fewer amount of independent variables; in fact the reduced system has r_* fewer independent variables than the initial system has. For partially invariant solutions, we proceed as follows. We first give δ, which must be such that $\max\{r_*-n, 0\} \leq \delta \leq \min\{r_*-1, m-1\}$. We then find the invariants $I_1(x,u)$ to $I_{m-\delta}(x,u)$ of H, with $\mathrm{rank}[\partial I_k/\partial u^{\alpha}] = m-\delta$. We set $v^j = I_j(x,u)$, $1 \leq j \leq m-\delta$, and $y^i = I_{m-\delta+i}(x,u)$, $1 \leq i \leq \rho$, where $\rho = n+\delta-r_*$ is the rank of the solution. The equation of the minimal invariant manifold, containing the unknown partially invariant solution as a manifold of codimension δ, then writes as $v^j = v^j(y)$, $1 \leq j \leq m-\delta$. We reexpress $u^i, ..., u^{n-\delta}$ in terms of x, y, v, and $w^{\sigma} = u^{m-\delta+\sigma}$, $1 \leq \sigma \leq \delta$. This yields a system for the $m-\delta$ functions v^j depending on the ρ variables $y^1, ..., y^{\rho}$, as well as a system of δ functions w^{σ} depending on the n variables $x^1, ..., x^n$. The resulting equations then describe all partially invariant solutions of defect δ and rank ρ.

In practice, one does not seek all the (partially) invariant solutions for all the subgroups of the symmetry group G of Ω. Instead, we classify these solutions. The point is that if $H \subset G$ and $H' \subset G$ are conjugate subgroups, then the (partially) invariant solutions under H and H' can be taken one to another by a transformation by the same $g \in G$ which relates H and H' through conjugation. It is therefore enough to investigate the question for an optimal system. Note that sometimes the symmetry group can be enlarged to contain some additional discrete symmetries that may possibly reduce an optimal system a bit further. One should also realize that this classification will generally imply that some interesting invariant solutions will not be obtained directly: they can however be recovered by applying some transformation induced by an appropriate group conjugation. As an example consider the heat equation again, with the symmetry algebra generated by the vector fields X_0 to $X_6(\varphi)$. It can be shown that an optimal system of generators consists of $X_0 + aX_5$, $X_1 + aX_5$, $X_0 - X_4$, $X_0 + X_4$, $X_0 + X_2 + aX_5$, X_3, X_5, $a \in R$, as well as other classes involving vector fields of the type X_6 alone but we do not bother about these since they are known not to yield any invariant solutions. Note further that space inversion, $x \to -x$, is a discrete symmetry which permits us to eliminate, say, $X_0 - X_4$, since this symmetry maps it into $X_0 + X_4$. As a specific example of symmetry reduction, consider first the generator

$$X_0 + X_2 + aX_5 = (4t^2 + 1)\partial_t + 4tx\partial_x - (2t + x^2 - 2a)\partial_u .$$

The invariants associated with this vector field are

$$v = (4t^2 + 1)^{1/4} u \exp[(4t^2 + 1)^{-1} tx^2 + a \tan^{-1}(2t)],$$

$$y = (4t^2 + 1)^{-1/2} x.$$

Setting $v = v(y)$ and substituting for u in the heat equation yields the reduced equation

$$v_{yy} + (2a + y^2)v = 0.$$

The solution of this equation of Weber type is expressed as a linear combination of two parabolic cylinder functions and we find, upon substituting back in the above equation relating v with u, that this gives us the following invariant solutions

$$u(t,x) = (4t^2 + 1)^{-1/4} \{k_1 W(-a, x/\sqrt{8t^2+2}) + k_2 W(-a, -x/\sqrt{8t^2+2})\} \exp[-tx^2/(4t^2 + 1) + a \tan^{-1}(2t)].$$

As a second example, consider the generator $X_0 - X_4 = \partial_t - 2t\partial_x + xu\partial_u$. The invariants are

$$v = u\exp(-xt - \tfrac{2}{3}t^3), \qquad y = x + t^2.$$

Proceeding as above, we find that the reduced equation is an Airy equation

$$v_{yy} - yv = 0,$$

and consequently that the invariant solutions are expressed in terms of linear combinations of Airy functions:

$$u(t,x) = [k_1 Ai(x + t^2) + k_2 Bi(x + t^2)]\exp(xt + \tfrac{2}{3}t^3).$$

We shall use this technique of symmetry reduction extensively in parts of the next chapter in order to find special solutions of the Kadomtsev-Petviashvili equation, some of them being not obvious at all.

An important application of the knowledge of the symmetry group of a differential system is that of finding conservation laws when the system can be derived from some variational principle. In fact, Noether has given the algorithm for constructing the conservation laws for the Euler-Lagrange equations early in our century (see NOE1). Consider the variational problem associated with the functional

$$\Lambda[u] = \int_{\Phi} L(x, u, ..., u^{(k)})dx, \tag{1.10}$$

whose solution is prescribed by requiring that the variational derivative of $\Lambda[u]$ vanishes identically; this yields the Euler-Lagrange equation

$$\Omega: \delta\Lambda[u] \equiv E(L) = 0,$$

where E is the Euler operator with components given by $E^{\alpha} = (-D_J)\partial/\partial u^{\alpha}{}_J$. For example, if L has the dependence $L(x,u,u_x)$, then the Euler-Lagrange equation is

$$\begin{aligned} E(L) &= \partial L/\partial u - D_x[\partial L/\partial u_x] \\ &= \partial L/\partial u - \partial^2 L/\partial x\partial u_x - u_x\partial^2 L/\partial u\partial u_x - u_{xx}\partial^2 L/\partial u_x{}^2 = 0. \end{aligned}$$

A local transformation group G is called a variational symmetry group admitted by $\Lambda[u]$ if this functional is invariant under $G^{(k)}$. In infinitesimal form, this criterion is that G is admitted by $\Lambda[u]$ if, and only if, $\forall(x, u, ..., u^{(k)}) \in J^k(R^n,R^m)$, and $\forall X = \xi^i(x,u)\partial/\partial x^i +$

$\eta^\alpha(x,u)\partial/\partial u^\alpha$ belonging to the Lie algebra of G, $(p^k X)L + L\mathbf{Div}\xi = 0$. **Div** is the total divergence operator with components $\mathbf{Div}^i = D_i$; thus $\mathbf{Div}\xi = D_i\xi^i$. An important theorem is that if G is a variational symmetry group admitted by $\Lambda[u]$ then it also is a symmetry group admitted by Ω: $E(L) = 0$. It is to be emphasized that the converse is *not* true. Consider a differential system Ω: $F(x, u, ..., u^{(k)}) = 0$. By a conservation law we mean a divergence expression

$$\mathbf{Div}P = 0 \pmod{\Omega}, \tag{1.11}$$

where $P(x, u, ..., u^{(k)}) = (P^1, ..., P^n)$. When considering dynamical systems, as many differential systems of physics are, one of the independent variables, usually denoted t, is distinguished from the other ones and (1.11) takes the form $D_t T + \nabla \cdot \mathbf{D} = 0$, where ∇ is the space component of **Div**. T is called the conserved density and **D** is called the flux associated to T. The physical significance of the locution *conservation law* is that the integral

$$\int_\Phi T(x, u, ..., u^{(k)})dx,$$

where Φ is bounded, is a constant of the motion, provided that $\mathbf{D} \pmod{\Omega}\,|_{\partial\Phi} = 0$. As mentioned in the Introduction, Noether established the connection between groups of generalized symmetries admitted by an appropriate differential system and the corresponding conservation laws. Here we shall restrict to the special case when the symmetries are pure geometric symmetries associated to a 1-parameter group. The important result from the applied point of view is the following theorem

Noether's Theorem: Let G be a local 1-parameter variational symmetry group admitted by a functional $\Lambda[u]$ of the form (1.10). Let $X = \xi^i(x,u)\partial/\partial x^i + \eta^\alpha(x,u)\partial/\partial u^\alpha$ be an infinitesimal operator of G, with characteristics $Q^\alpha = \eta^\alpha - \xi^i u^\alpha_i$. Then there exists a vector $(P^1, ..., P^n)$ such that $\mathbf{Div}P = Q\cdot E(L) = Q^\alpha E^\alpha(L)$ is a conservation law for $E(L) = 0$.

A corollary to this theorem is that if L has the special form $L(x,u,u^{(1)})$, then P is explicitly given by

$$P^i = Q^\alpha \partial L/\partial u^\alpha_i + \xi^i L = (\eta^\alpha - \xi^j u^\alpha_j)\partial L/\partial u^\alpha_i + \xi^i L. \tag{1.12}$$

We point out again that it may very well be the case that not all the 1-parameter subgroups of the symmetry group admitted by a differential system Ω in the form of some Euler-Lagrange equation $L(E) = 0$ should be variational symmetry groups admitted by the corresponding functional $\Lambda[u]$. Therefore one must be careful to check the admissibility of a given symmetry before actually constructing the would be conserved densities. It is however known that Noether's theorem is a little bit too restrictive as far as construction of conservation laws is concerned. In fact, a theorem by Ibragimov (see AND1) gives the necessary and sufficient condition for the existence of conservation laws; it states that if $\Lambda[u]$ admits G as a variational symmetry group, then the vector P given by (1.11) provides a conservation law if, and only if, the extremal values of L are invariant under G. A slight generalization is permissible. Consider some functional $\Lambda[u] = \int L dx$. A vector field X acting on $J^0(\mathbf{R}^n,\mathbf{R}^m)$ is defined to be an infinitesimal divergence symmetry of $\Lambda[u]$ if there exists a vector $\Pi(x, u, u^{(1)}, \ldots) = (\Pi^1, \ldots, \Pi^n)$ such that $\forall (x,u) \in J^0(\mathbf{R}^n,\mathbf{R}^m)$, $(p^k X)L + \mathbf{Div}\xi = \mathbf{Div}\Pi$. If X is an infinitesimal divergence symmetry then it also generates a symmetry group admitted by E(L) for the functional $\Lambda[u] = \int L(x, u, u^{(1)})dx$. The associated conservation law to such a symmetry is not specified by the condition (1.11) but rather by

$$\mathbf{Div}(P - \Pi) = 0 \ (\mathrm{mod}\ \Omega),$$

where P is defined as in (1.12), and Π is such that it satisfies the equation

$$\begin{aligned} D_i\Pi^i &= \xi^i \partial L/\partial x^i \\ &= \eta^\alpha \partial L/\partial u^\alpha + (D_i\eta^\alpha - u^\alpha{}_j D_i\xi^j)\partial L/\partial u^\alpha{}_i + LD_i\xi^i. \end{aligned}$$

Conservation laws, as we now realize, clearly have a group theoretical nature. We should mention that there are other means to construct them, although these are indirect ways. For instance, infinite families of local conserved densities can sometimes be found for special nonlinear completely integrable equations derivable from some variational principle, e.g. the celebrated Sine-Gordon equation equation, if their Bäcklund transformations are known.

Bäcklund transformations played for an important part in our understanding of many nonlinear differential equations, especially in the case of soliton equations. In practice, they are a powerful means of constructing nontrivial solutions from given trivial ones: typically, solitons are constructed from the zero solution. Backlund transformations are often viewed as rules which relate solutions of a pair of differential systems. A classical example of such a transformation is given by the Cauchy-Riemann relations which indeed give us such a rule, as it relates pairs of solutions of the Laplace equation. In their historical context, these transformations arised from the study of surfaces with constant negative curvature by Lie, Bianchi, Bäcklund, and Darboux; such surfaces are characterized by the well known Sine-Gordon equation $u_{tx} = \sin(u)$. Bäcklund's result was that two surfaces of constant negative curvature $-1/a^2$, associated to functions u and v, are related by the following pair of equations:

$$
B(a): \quad \begin{aligned} v_t - u_t &= 2a \sin[(v+u)/2], \\ v_x + u_x &= 2a^{-1} \sin[(v-u)/2]. \end{aligned} \tag{1.13}
$$

Due to the invariance of the Sine-Gordon equation under the group of dilatations, t and x can be rescaled in order to absorb the constant "a" and (1.13) then reduces to Bianchi-Lie's result. Bäcklund transformations are also useful for obtaining infinite lattices of solutions by means of a superposition formula. Thus for the Sine-Gordon equation, consider a given solution u_0 as well as two solutions u_1 and u_2 obtained through (1.13) with $B(a_1)$ and $B(a_2)$, respectively. Then there exists a fourth solution u which is related to the three above solutions according to the following *Bianchi diagram:*

$$
\begin{tikzcd}
 & u_1 \arrow[dr, "B(a_2)"] & \\
u_0 \arrow[ur, "B(a_1)"] \arrow[dr, "B(a_2)"'] & & u \\
 & u_2 \arrow[ur, "B(a_1)"'] &
\end{tikzcd} \tag{1.14}
$$

One may easily show, using (1.13), that u possesses the following implicit expression in terms of the three other solutions

$$
(a_1 - a_2)\tan[(u - u_0)/4] = (a_1 + a_2)\tan[(u_1 - u_2)/4].
$$

This type of rule is known as a *permutability theorem.* It is to be pointed out that such a property does *not* characterize all nonlinear equations: it is rather exceptional.

We now formally define what are Bäcklund transformations. Let N, M, P be smooth manifolds. Practically, M and P will stand for the spaces of the *old* and *new* dependent variables u and v, respectively. Consider then a smooth mapping $\psi: J^1(N,M) \times P \rightarrow J^1(N,P)$ leaving unchanged the local coordinates on N and M; thus ψ is completely determined, locally, if it can be assigned a representation $v^{(1)} = \psi^{(1)}(x,u,u^{(1)},v)$. ψ is called a Bäcklund map if its integrability conditions contain a differential system Z^* on $J^2(N,M) \times P$. If there exists a differential system Z on $J^2(N,M)$ such that $Z \times P \subset Z^*$, then ψ is called an ordinary Bäcklund map. In addition, if the image of $p^1\psi$, such that the diagram

$$\begin{tikzcd}
J^2(N,M) \times P \arrow[r, "p^1\psi"] \arrow[d, "\pi^2{}_1 \times id"'] & J^2(N,P) \arrow[d, "\pi^2{}_1"] \\
J^1(N,M) \times P \arrow[r, "\psi"'] & J^1(N,P)
\end{tikzcd}$$

commutes, is the zero-set of a system Z' on $J^2(N,P)$, then the correspondence between Z and Z' is called the Bäcklund transformation determined by the ordinary Bäcklund map ψ. Furthermore, if Z and Z' are describing a same manifold, one then speaks of a self (or auto) Bäcklund transformation.

There is no best algorithm for finding a Bäcklund transformation. The oldest known method is due to Bäcklund and Clairin, and is also very tedious. For instance consider two systems Z and Z' defined on $J^2(R^2,R)$. Introduce the quantities $p = u_x$, $q = u_y$, $r = u_{xx}$, $s = u_{xy}$, $t = u_{yy}$, and also $p' = u'_x$, $q' = u'_y$, $r' = u'_{xx}$, $s' = u'_{xy}$, $t' = u'_{yy}$. u and u' stand for solutions of Z and Z', respectively. The Bäcklund map can be locally written in the form

$$p = f(x,y,u,u',p',q'), \qquad q = g(x,y,u,u',p',q').$$

Its integrability condition is

$$p_y - q_x = \partial f/\partial y + g\partial f/\partial u + q'\partial f/\partial u' + s'\partial f/\partial p' + t'\partial f/\partial q'$$
$$- \partial g/\partial x - f\partial g/\partial u - p'\partial g/\partial u' - r'\partial g/\partial p' - s'\partial g/\partial q' \equiv 0.$$

The technique then consists in differentiating this expression up to the point where u' no longer explicitly appears. This will yield functional expressions for some higher derivatives of f and g. These may then be integrated back to give the explicit dependence of f an g. A second procedure, which has been very popular a few years ago, is due to Wahlquist and Estabrook (see WAH1) and is called the prolongation method. It works on Pfaff systems of differential forms equivalent to the given differential system. It proved to be very useful for equations as various as the Sine-Gordon equation, the KdV and modified KdV equation, the nonlinear cubic Schrödinger equation, etc... Other techniques do exist (see ROGT), for instance adapted to Hirota's bilinear formalism.

CHAPTER TWO

Symmetry structure and invariant solutions of the Kadomtsev-Petviashvili equation

As mentioned in the introductory chapter, many special solutions of the Kadomtsev-Petviashvili, or KP for short, equation have been found. The most famous ones are the soliton and multi-soliton solutions, rational, quasi-periodic, and similarity solutions; ripplons and soliton resonances are other examples (see several references mentioned earlier). In the present chapter we show that there exist exactly three non-equivalent classes of reductions to differential equations in two independent variables, and therefore three corresponding non-equivalent classes of invariant solutions of the KP equation. There also exists a large number of reductions to ordinary differential equations, or to algebraic ones. We shall give one representative of each class. As we explained in the preceding chapter, any invariant solution, e.g. the special solutions mentioned above, can be mapped to the ones we give through some particular conjugation, and maybe also through an additional transformation due to the discrete symmetries of the equation.

The present chapter consists of two parts. Part I deals with the symmetries and the invariant solutions of the KP equation itself and is divided into three sections. In the first we give and study the symmetry structure of the equation. We state what the Lie algebra of symmetry is and then reconstruct the corresponding local Lie group of transformations leaving the KP equation invariant; from now on we shall speak of the KP symmetry group and algebra. We explicitly specify the form of the transformations induced by this group on the space $\mathbf{R}^3 \times \mathbf{R}$ coordinatized by the independent and dependent variables $\{t,x,y;u\}$; the action of a generic element of the group is seen to be quite complicated. We then proceed to study the structure of the KP symmetry algebra in more details. We establish a Levi decomposition and give a *physical* finite-dimensional subalgebra which has been used in other works to generate similarity solutions of the KP equation. An important fact is that the KP symmetry algebra is infinite-dimensional and we show, as the main result of this section, that a certain subalgebra of it, also infinite-dimensional, can actually be embedded into an affine loop algebra; this feature is a remarkable property which is shared by only a few equations that we know of. The second section is devoted to a classification of low-dimensional subalgebras of the KP symmetry algebra, namely those of dimensions one, two, and three. As explained in the preceding chapter, we are interested in finding the conjugacy classes of subalgebras under the adjoint action of the KP symmetry group, i.e. the group of inner automorphisms of the KP symmetry algebra. This is undertaken because these classes will provide us with the optimal systems that we shall need afterwards in order to perform symmetry reduction. As a secondary goal, we do this as an example to demonstrate that the tools developed for classifying subalgebras of finite-dimensional Lie algebras can also be applied to infinite-dimensional ones. We limit ourselves to subalgebras of dimensions at most equal to three since, in view of looking for invariant solutions, subalgebras of higher dimensions are not needed.

Symmetry reduction is finally performed in the third section. We first use the one-dimensional subalgebras and show, as we stated before, that three distinct reductions are possible. The reduced equations involve two independent variables and are the Boussinesq equation, a once-differentiated Korteweg-de Vries (KdV for short) equation, and a linear equation. Each solution of the Boussinesq equation will provide a family of KP solutions that depend on three arbitrary functions of the time variable. Each solution of the KdV equation yields a family of KP solutions depending on two arbitrary functions. As for the linear equation, it can be solved explicitly. Its solution involves two arbitrary functions of time and it provides another family of KP solutions, involving altogether three arbitrary functions. We also briefly discuss the physical meaning of those solutions. We end by discussing symmetry reduction using two and three-dimensional subalgebras; for example we give the particular reduction to the Painlevé equation of the first kind.

In part II we present a new way of applying the symmetry reduction technique, the novelty consisting in considering *both* the KP equation and its Bäcklund transformation. We actually consider a potential form of the KP equation, referred to as the PKP equation, since the Bäcklund transformation is best defined for this equation. We begin by giving the symmetry algebra of the PKP equation which, and not surprisingly, resembles that of the KP equation. In fact the PKP symmetry algebra has a very similar structure, the essential difference being a pair of additional vector fields that reflect that the KP equation is a differentiated form of the PKP equation. It is again an infinite-dimensional algebra and we point out that, again, there exists an embedding into a particular affine loop algebra. The corresponding symmetry group has a more complicated action than that of the KP symmetry group and we do not reproduce the general transformations, since we shall not use them. In fact we only use the group to generalize the Bäcklund transformation by introducing two arbitrary functions of the time variable; these functions are the analogues of the parameters that appear in the Bäcklund transformations for some soliton equations like the sine-Gordon equation. Our next step is to give the *joint* symmetry algebra of the PKP equation *with* its associated Bäcklund transformation. This algebra is four-dimensional and has four distinct classes of non-conjugate one-dimensional subalgebras. We then use representatives of these classes and perform symmetry reduction. In each case the resulting reduced system consists of a reduced PKP equation *and* an associated reduced Bäcklund transformation. Solutions of the KP equation can then be constructed according to the following scheme. We start from a given solution of the reduced PKP equation and get a new one through the use of the reduced Bäcklund transformation. This new solution can then be mapped to a PKP solution which, finally, can be integrated once to give a KP solution. One of the four possible reductions is of particular interest as it yields a new kind of solution to the KP equation. This solution, which we have termed *splitton*, is related, in some way, to soliton resonances.

I. Symmetry structure of the Kadomtsev-Petviashvili equation

1. The infinite-dimensional symmetry structure

We consider the Kadomtsev-Petviashvili equations given in the standard form

$$\Omega^{\sigma}: [4u_t + 6uu_x + u_{xxx}]_x + 3\sigma u_{yy} = 0 \tag{2.1}$$

where $\sigma = \pm 1$ distinguishes between the so-called KPII and KPI equations, respectively (see Introduction). The KP symmetry algebra L_{KP} has first been derived by Schwarz, using a symbolic package written in REDUCE (SCH1); we shall use his result, correcting a slight misprint, which we have checked with our own MACSYMA written package (see CHA1). A general element V of L_{KP} has the form

$$\begin{aligned}
&V = X(f) + Y(g) + Z(h);\\
&X(f) = f\partial_t + [\tfrac{1}{3}xf' - \tfrac{2}{9}\sigma y^2 f'']\partial_x + \tfrac{2}{3}yf'\partial_y - [\tfrac{4}{27}\sigma y^2 f''' - \tfrac{2}{9}xf'' + \tfrac{2}{3}uf']\partial_u,\\
&Y(g) = g\partial_y - \tfrac{2}{3}\sigma yg'\partial_x - \tfrac{4}{9}\sigma yg''\partial_u,\\
&Z(h) = h\partial_x + \tfrac{2}{3}h'\partial_u,
\end{aligned} \tag{2.2}$$

where $f \equiv f(t)$, $g \equiv g(t)$, and $h \equiv h(t)$ are arbitrary real smooth functions in some open interval $U \subset \mathbf{R}$; primes indicate derivatives with respect to t. The commutation relations for this Lie algebra are easy to compute. We obtain:

$$\begin{aligned}
&[X(f_1), X(f_2)] = X(f_1 f_2' - f_1' f_2),\\
&[X(f), Y(g)] = Y(fg' - \tfrac{2}{3}f'g),\\
&[X(f), Z(h)] = Z(fh' - \tfrac{1}{3}f'h),\\
&[Y(g_1), Y(g_2)] = \tfrac{2}{3}\sigma Z(g_1' g_2 - g_1 g_2'),\\
&[Y(g), Z(h)] = 0,\\
&[Z(h_1), Z(h_2)] = 0.
\end{aligned} \tag{2.3}$$

We emphasize that it is because of these relations that (2.2) form a Lie algebra only if f(t), g(t), and h(t) are real smooth functions. From (2.3) it is also clear that the algebra $L_{KP} = \{X(f), Y(g), Z(h)\}$ admits a Levi decomposition (see Chapter one)

$$L_{KP} = S \oplus N, \tag{2.4}$$

where the symbol $\oplus$ denotes a semi-direct sum. $N \doteq \{Y(g), Z(h)\}$ is a solvable nilpotent ideal in L_{KP}, namely the nilradical (or maximal nilpotent ideal). $S = \{X(f)\}$ is a simple Lie algebra. In order to see that, observe that S is isomorphic to $\mathbf{J(R)} = \{f(\xi)\partial_\xi \mid f \in C^\infty(\mathbf{R})\}$, the algebra of derivations of the real line; this algebra is simple according to a theorem of Cartan (see CAR2, Theorem XI, p.893). The isomorphism is given by the mapping

$$\Psi: \ \mathbf{J(R)} \longrightarrow S$$
$$f(\xi)\partial_\xi \longrightarrow X[f(t)] .$$

Given L_{KP} we can reconstruct the KP symmetry group G_{KP}; integrating the infinitesimal operators $V = X(f) + Y(g) + Z(h)$ given by (2.2) yields a general 1-parameter subgroup of the identity component of the group. We begin with the simplest case, namely when $f(t) = g(t) = 0$, and h(t) arbitrary. Following (1.4) we have to solve the first-order system of ordinary differential equations

$$dt^*/d\lambda = 0, \quad dx^*/d\lambda = h(t^*), \quad dy^*/d\lambda = 0, \quad du^*/d\lambda = \tfrac{2}{3}h'(t^*).$$

Integrating and requiring that we get the identity transformation for $\lambda = 0$, we find

$$\begin{aligned} &t^* = t, \quad x^* = x + \lambda h(t), \quad y^* = y, \\ &u^*(t^*, x^*, y^*) = u[t^*, x^* - \lambda h(t^*), y^*] + \tfrac{2}{3}\lambda h'(t^*). \end{aligned} \tag{2.5}$$

Thus if u(t,x,y) is a solution of the KP equation (2.1) then $u^*(t^*, x^*, y^*)$ also solves (2.1) in the variables t^*, x^*, y^*. The second case is that for which $f(t) = 0$, $g(t) \neq 0$, h(t) arbitrary. The system to solve is

$$dt^*/d\lambda = 0, \quad dx^*/d\lambda = h(t^*) - \tfrac{2}{3}\sigma y^* g'(t^*),$$

$$dy^*/d\lambda = g(t^*), \quad du^*/d\lambda = \tfrac{2}{3}h'(t^*) - \tfrac{4}{9}\sigma y^* g'(t^*).$$

Integrating and imposing appropriate initial conditions for $\lambda = 0$, we obtain the following transformation formulas:

$$t^* = t, \quad x^* = x + [h(t) - \tfrac{2}{3}\sigma y g'(t)]\lambda - \tfrac{1}{3}\sigma g(t)g'(t)\lambda^2, \quad y^* = y + \lambda g(t),$$
$$u^*(t^*, x^*, y^*) = u[t^*, x^* - \lambda h(t^*) + \tfrac{2}{3}\lambda\sigma g'(t^*)y^*, y^* - \lambda g(t^*)] \tag{2.6}$$
$$+ \{\tfrac{2}{3}h'(t^*) - \tfrac{4}{9}\sigma g''(t^*)[y^* - \lambda g(t^*)]\}\lambda - \tfrac{2}{9}\sigma g(t^*)g''(t^*)\lambda^2.$$

Finally, consider the generic case when $f(t) \neq 0$, with the functions $g(t)$ and $h(t)$ being arbitrary. In order to obtain the group transformations we must integrate the equations

$$\frac{dt^*}{f(t^*)} = \frac{9dx^*}{3x^*f'(t^*) - 2\sigma y^{*2}f''(t^*) + 9h(t^*) - 6\sigma y^* g'(t^*)} = \frac{3dy^*}{2y^*f'(t^*) + 3g(t^*)}$$
$$= \frac{-27du^*}{4\sigma y^{*2}f'''(t^*) - 6x^*f''(t^*) + 18u^*f'(t^*) - 18h'(t^*) + 12\sigma y^* g''(t^*)} = d\lambda .$$

Introducing the notations

$$\Phi(t) = \int_{t_0}^{t} \frac{ds}{f(s)}, \qquad G(t^*,t) = [f(t)]^{2/3}\int_{t}^{t^*} g(s)f^{-5/3}(s)ds,$$
$$H(t^*,t) = [f(t)]^{1/3}\int_{t}^{t^*} [\tfrac{2}{3}\sigma f^{-7/3}g^2(s) + h(s)f^{-4/3}(s)]ds, \tag{2.7a}$$

we obtain the transformations

$$t^*(t) = \Phi^{-1}[\lambda + \Phi(t)],$$
$$y^*(t,y) = [y + G(t^*(t),t)]\left(\frac{f(t^*(t))}{f(t)}\right)^{2/3},$$
$$x^*(t,x,y) = \left(\frac{f(t^*(t))}{f(t)}\right)^{1/3}\Bigg\{x + H(t^*(t),t) - \frac{2\sigma y^2[f'(t^*(t)) - f'(t)]}{9f(t)} \tag{2.7b}$$
$$- \frac{2\sigma y}{9f(t)}\left[2G(t^*(t),t)f'(t^*(t)) - 3g(t) + 3g(t^*(t))\left(\frac{f(t^*(t))}{f(t)}\right)^{-2/3}\right]$$
$$- \frac{2\sigma}{9f(t)}\left[G(t^*(t),t)^2 f'(t^*(t)) + 3G(t^*(t),t)g'(t^*(t))\left(\frac{f(t^*(t))}{f(t)}\right)^{-2/3}\right]\Bigg\},$$

$$u^*(t^*,x^*,y^*) = \left(\frac{f(t(t^*))}{f(t^*)}\right)^{2/3}\Big\{ u[t(t^*), x(t^*,x^*,y^*), y(t^*,y^*)] - \frac{2x^*[f'(t(t^*)) - f'(t^*)]}{9f(t(t^*))^{2/3} f(t^*)^{1/3}}$$

$$- \frac{4\sigma y^*}{81f(t(t^*))^{4/3} f(t^*)^{2/3}} \{2G(t^*,t)[3f(t(t^*))f''(t(t^*)) - f'(t(t^*)^2]$$

$$- 9f(t(t^*))g'(t(t^*)) + 3g(t(t^*))f'(t(t^*)) + 3g(t^*)f'(t(t^*)) + 9f(t^*)g'(t^*) - 6g(t^*)f'(t^*)\}$$

$$- \frac{4\sigma {y^*}^2}{81f(t(t^*))^{2/3}f(t^*)^{4/3}}\{3f(t^*)f''(t^*) - 2f'(t^*)^2 + f'(t^*)f'(t(t^*)) + 2f'(t(t^*))^2 - 3f(t(t^*))f''(t(t^*))\} \tag{2.7b}$$

$$+ \frac{2f'(t(t^*))}{9f'(t(t^*))} H(t(t^*),t^*) + \frac{4\sigma G^2(t(t^*),t^*)}{81f(t^*)^2}[3f(t(t^*))f''(t(t^*)) - f'(t(t^*)^2]$$

$$- \frac{4\sigma G(t(t^*),t^*)}{27f(t^*)^2}[3f(t(t^*))g'(t(t^*)) - g(t(t^*)f'(t(t^*))]$$

$$+ \frac{2}{3}\left[\frac{h(t^*)}{f(t^*)^{1/3} f(t(t^*)^{2/3}} - \frac{h(t^*)}{f(t(t^*)}\right] + \frac{2\sigma}{9}\left[\frac{f'(t(t^*)^2}{f(t^*)^{1/3}f(t(t^*)^{2/3}} - \left(\frac{g(t(t^*))}{f(t(t^*)}\right)^2\right]\Big\},$$

where Φ^{-1} denotes the local inverse of the function Φ and where

$$t(t^*) = \Phi^{-1}(-\lambda + \Phi(t^*)),$$

$$y(t^*,y^*) = [y^* + G(t(t^*),t^*)]\left(\frac{g(t(t^*)}{f(t^*)}\right)^{-2/3}$$

$$x(t^*,x^*,y^*) = \left(\frac{f(t(t^*))}{f(t^*)}\right)^{1/3}\Big\{x^* + H(t(t^*),t^*) - \frac{2\sigma {y^*}^2[f'(t(t^*)) - f'(t^*)]}{9f(t^*)} \tag{2.7c}$$

$$- \frac{2\sigma y^*}{9f(t^*)}\left[2G(t(t^*),t^*)\, f'(t(t^*)) - 3g(t^*) + 3g(t(t^*))\left(\frac{f(t(t^*))}{f(t^*)}\right)^{-2/3}\right]$$

$$- \frac{2\sigma}{9f(t^*)}\left[G(t(t^*),t^*)^2\, f'(t(t^*)) + 3G(t(t^*),t^*)g'(t(t^*))\left(\frac{f(t(t^*))}{f(t^*)}\right)^{-2/3}\right]\Big\}.$$

Formulas (2.5), (2.6), and (2.7) thus give us new solutions (u^*) from known ones (u). In particular, if we start from the trivial solution $u = 0$, then (2.7) provides a family of solutions depending on the arbitrary functions f(t), g(t), and h(t). We should also point out that formulas (2.7) represent the action of the most general *1-parameter* subgroup of G_{KP}; in order to get the action of the whole group, we would need to label the functions f(t), g(t), and h(t) with *three* parameters λ_1, λ_2, λ_3 instead of the single group parameter λ.

Besides the Lie symmetries that we have just looked at, the KP equation is also invariant under certain specific discrete transformations that cannot be obtained by integrating the infinitesimal transformations (2.2). These discrete transformations are the reflections R_y and R_{tx} defined as

$$\begin{aligned} R_y:\ & t \longrightarrow t, \quad x \longrightarrow x, \quad y \longrightarrow -y, \quad u \longrightarrow u, \\ R_{tx}:\ & t \longrightarrow -t, \quad x \longrightarrow -x, \quad y \longrightarrow y, \quad u \longrightarrow u. \end{aligned} \tag{2.8}$$

We shall discuss the systematic classification of low-dimensional subalgebras of L_{KP} in the next section. For the moment we may point out that all the *obvious* physical symmetries of the KP equation are actually obtained by restricting the arbitrary functions f(t), g(t), and h(t) to be first-order polynomials in the variable t. Indeed, in obvious notations, we then get a subalgebra generated by the following vector fields:

$$\begin{aligned} T &\equiv X(1) = \partial_t\,, \\ Y &\equiv Y(1) = \partial_y\,, \\ X &\equiv Z(1) = \partial_x\,, \\ D &\equiv X(t) = t\partial_t + \tfrac{1}{3}x\partial_x + \tfrac{2}{3}y\partial_y - \tfrac{2}{3}u\partial_u\,, \\ R &\equiv Y(t) = t\partial_y - \tfrac{2}{3}\sigma y\partial_x\,, \\ B &\equiv Z(t) = t\partial_x + \tfrac{2}{3}\partial_u\,. \end{aligned} \tag{2.9}$$

Thus T, Y, X generate translations, D generates dilations, R has some properties of a rotation, and B yields a Galilei transformation in the x-direction. Integrating, for instance, the vector field R, we obtain the transformation

$$t^* = t, \quad x^* = x - \tfrac{2}{3}\sigma(\lambda y + \tfrac{1}{2}\lambda^2 t), \quad y^* = y + \lambda t, \quad u^* = u\,.$$

This transformation has been extensively used, e.g., by Segur and Finkel (SEGU1) to "rotate" solutions of the Korteweg-de Vries equation into solutions of the KP equation. The dilatation symmetry $D = X(t)$ has been used to generate similarity solutions of the KP equation (for instance see RED1, NAK2). The infinitesimal operators (2.9) form a basis of a six-dimensional Lie algebra $L_P = \{D, R, B, X, Y, T\}$. This algebra has a five-dimensional nilpotent ideal $N = \{R, B, X, Y, T\}$ (the nilradical). The commutation relations for L_P are given in the following table.

	D	R	B	X	Y	T
D	0	$\frac{1}{3}R$	$\frac{2}{3}B$	$-\frac{1}{3}X$	$-\frac{2}{3}Y$	-T
R	$-\frac{1}{3}R$	0	0	0	$\frac{2}{3}\sigma X$	-X
B	$-\frac{2}{3}B$	0	0	0	0	-Y
X	$\frac{1}{3}X$	0	0	0	0	0
Y	$\frac{2}{3}Y$	$-\frac{2}{3}\sigma X$	0	0	0	0
T	T	X	Y	0	0	0

It is a relatively simple matter to classify and construct all the subalgebras of L_P through the use of known classification methods; we shall not present these results here since we think a classification of all the low-dimensional subalgebras of the infinite-dimensional symmetry algebra of the KP equation is both more interesting and more useful for performing symmetry reduction (see the next two sections below).

Another finite-dimensional algebra, not contained in L_P, can be obtained by restricting L_{KP} to be generated by the vector fields of the form $X(a + bt + ct^2)$, $a,b,c \in \mathbf{R}$. We thus obtain the algebra $sl(2,\mathbf{R})$ with generators $C = X(t^2) = t^2\partial_t + \frac{2}{3}(tx - \frac{2}{3}\sigma y^2)\partial_x + \frac{4}{3}ty\partial_y + \frac{4}{9}(x - 3tu)\partial_u$, and D, T defined as above. The commutation rules are

$$[D,T] = -T, \quad [D,C] = C, \quad [T,C] = 2D.$$

C generates a type of conformal transformation:

$$t^* = t(1 - \lambda t)^{-1},$$

$$y^* = y(1 - \lambda t)^{-4/3},$$

$$x^* = \left[x - \frac{4\sigma\lambda y^2}{3(1 - \lambda t)}\right](1 - \lambda t)^{-2/3},$$

$$u^* = \left[u + \frac{4\lambda x}{1 - \lambda t} + \frac{8\lambda y^2 [2(1 - \sigma) - (2 - \sigma)\lambda t]}{81 t(1 - \lambda t)^2}\right](1 - \lambda t)^{4/3}.$$

A most interesting feature of the KP symmetry algebra (2.2) is that it contains a subalgebra that can be embedded into an affine loop algebra. In fact, let us consider the subalgebra $L_\pi \subset L_{KP}$ obtained by restricting the functions f(t), g(t), and h(t) to be Laurent *polynomials* in the variable t. A basis for this subalgebra is provided by

$$X(t^n) = t^n \partial_t + \left[\frac{nx}{3} t^{n-1} - \frac{2\sigma}{9} n(n - 1)y^2 t^{n-2}\right]\partial_x + \frac{2n}{3} y t^{n-1} \partial_y$$
$$- \left[\frac{4\sigma}{27} n(n - 1)(n - 2)y^2 t^{n-3} - \frac{2n}{9}(n - 1)x t^{n-2} + \frac{2n}{3} u t^{n-1}\right]\partial_u ,$$

$$Y(t^n) = t^n \partial_y - \frac{2\sigma}{3} n y t^{n-1}\partial_x - \frac{4\sigma}{9} n(n - 1) y t^{n-2} \partial_u ,$$

$$Z(t^n) = t^n \partial_x + \frac{2n}{3} t^{n-1}\partial_u ,$$

where $n \in Z$. The commutation relations of this subalgebra L_π are (see (2.3))

$$
\begin{aligned}
[X(t^n), X(t^m)] &= (m - n)X(t^{n+m-1}) , \\
[X(t^n), Y(t^m)] &= (m - \tfrac{2}{3}n)Y(t^{n+m-1}) , \\
[X(t^n), Z(t^m)] &= (m - \tfrac{1}{3}n)Z(t^{n+m-1}) , \qquad (2.10) \\
[Y(t^n), Y(t^m)] &= \tfrac{2}{3}\sigma(n - m)Z(t^{n+m-1}) , \\
[Y(t^n), Z(t^m)] &= [Z(t^n), Z(t^m)] = 0 .
\end{aligned}
$$

Let us now consider the following set of vector fields:

$$\begin{aligned} &\Delta = x\partial_x + 2y\partial_y - 2u\partial_u\,, && Q = y\partial_x\,, && Y = \partial_y\,, && X = \partial_x\,, \\ &A = -\sigma y^2\partial_x + x\partial_u\,, && S = y\partial_u\,, && P = y^2\partial_u\,, && U = \partial_u\,. \end{aligned} \tag{2.11}$$

These form an eight-dimensional Lie algebra L_0 with the following nonvanishing commutators:

$$\begin{aligned} &[\Delta,A] = 3A\,, && [\Delta,Y] = -2Y\,, && [\Delta,P] = 6P\,, && [\Delta,Q] = Q\,, && [\Delta,S] = 4S\,, \\ &[\Delta,X] = -X\,, && [\Delta,U] = 2U\,, && [A,Y] = 2\sigma Q\,, && [A,Q] = -S, && [A,X] = -U\,, \\ &[Y,P] = 2S\,, && [Y,Q] = X\,, && [Y,S] = U\,. \end{aligned}$$

This Lie algebra is solvable, its nilradical is spanned by $\{Y, A, P, Q, X, S, U\}$, and it contains a five-dimensional Abelian ideal spanned by $\{P, Q, X, S, U\}$. It should be noted that L_0 is *not* a subalgebra of the KP symmetry algebra. L_0 can be embedded into a simple Lie algebra. The simple Lie algebra of lowest dimension that contains a five-dimensional Abelian subalgebra is A_4 (in Cartan's classification), in particular sl(5,R) in our case. Indeed, it is easy to verify that the traceless matrices

$$\Xi \equiv \begin{bmatrix} 3\delta & -a & \sigma p & s & u \\ 0 & 0 & -a & q & x \\ 0 & 0 & -3\delta & -2\sigma y & 0 \\ 0 & 0 & 0 & -\delta & -y \\ 0 & 0 & 0 & 0 & \delta \end{bmatrix}$$

provide a representation of the Lie algebra L_0 with the prescription that the matrix representing Δ is obtained by setting $\delta = 1$ and all other entries equal to zero in Ξ, similarly for Y, etc. We observe that the Abelian subalgebra spanned by $\{P, Q, X, S, U\}$ is contained in a maximal Abelian subalgebra of sl(5,R) with Kravchuk signature (2,0,3) (see SUP1, WIN1, PAT3). Let us now establish a natural grading on the algebra L_π by attributing the degree "n" to a monomial t^n and the degree μ $(0 \le \mu \le 4)$, equal to the distance from the diagonal in Ξ to elements of L_0. Thus Δ has degree 0, A and Y degree 1, P and Q degree 2, S and X degree 3, and U has degree 4 (the usual grading in the weight space of sl(5,R)). We now construct a loop algebra out of (2.11) following the procedure usually applied to simple Lie algebras (KAC1). Thus, setting

$$X(t^n) = t^n\partial_t + \tfrac{1}{3}nt^{n-1}\Delta + \tfrac{2}{9}n(n-1)t^{n-2}A - \tfrac{4}{27}\sigma n(n-1)(n-2)t^{n-3}P\,,$$
$$Y(t^n) = t^nY - \tfrac{2}{3}\sigma nt^{n-1}Q - \tfrac{4}{9}\sigma n(n-1)t^{n-2}S\,, \qquad (2.12)$$
$$Z(t^n) = t^nX + \tfrac{2}{3}nt^{n-1}U\,,$$

we see that the vector fields $X(t^n)$, $Y(t^n)$, and $Z(t^n)$ form a Lie algebra isomorphic to the subalgebra L_π of the KP symmetry algebra whose commutation relations are given by (2.10). Each element has a well-defined degree in the grading, namely n-1, n+1, and n+3 for $X(t^n)$, $Y(t^n)$, and $Z(t^n)$, respectively. From the embedding constructed above for L_0 into sl(5,R) and from the representation given by (2.12) for the Lie algebra L_π, we see that L_π is a subalgebra of the affine loop algebra $A_4^{(1)}$ without its centre:

$$L_\pi \subset \{R[t,t^{-1}] \otimes sl(5,R)\} \oplus R[t,t^{-1}]\frac{d}{dt}\,.$$

The Levi decomposition (2.4) also holds for L_π. Indeed, from the commutation rules (2.10) we see that $N = \{Y(t^n), Z(t^n)\}$ forms a nilpotent ideal. The elements $X(t^n)$ form a Lie algebra isomorphic to the Z-graded algebra $\delta = R[t,t^{-1}]d/dt$. A basis for δ is given by the collection of derivations defined by $d_k = t^k\, d/dt$ with commutation relations

$$[d_j\,, d_k] = (k-j)d_{j+k-1}\,.$$

As we restrict ourselves to Laurent polynomials, i.e. functions with only a *finite* number of non-vanishing coefficients in their Laurent expansion, it follows directly from the above commutation relations that δ is simple, i.e. it admits no nontrivial ideal. Let us finally remark that the relationship between δ and the algebra of regular vector fields on S^1 has recently been investigated by Goodman and Wallach (GOO1), who also study the Virasoro algebra which is the universal central extension of δ. It is finally worth mentioning that, following our study of the KP equation, it has been found that other integrable nonlinear equations of physical interest in 2+1 dimensions also have infinite-dimensional symmetry groups and, moreover, have some specific loop structure, for instance the Davey-Stewartson equation and the so-called modified KP equation (see CHA2). However no equations in 3+1 dimensions have been yet shown to present such a feature; for instance the equation following the KP equation in the KP hierarchy of the Kyoto School (see, e.g., JIM1) does have an infinite-dimensional Lie algebra as its symmetry algebra, but this algebra hides no loop structure (see DOR1). In fact, it is conjectured that the KP equation is the only equation of the whole hierarchy presenting this feature.

2. The low-dimensional subalgebras of the KP symmetry algebra

In order to perform symmetry reduction to obtain invariant solutions of the KP equation we first need to know the low-dimensional subalgebras of the KP symmetry algebra L_{KP}. More specifically we need subalgebras that correspond to Lie subgroups having orbits of codimension 3, 2, 1 in the 4-dimensional manifold $J^0(R^3,R)$ coordinatized by (t, x, y; u) (see Chapter one). We obtain all the required subalgebras and also derive a better understanding of L_{KP} by classifying all its one-, two-, and three-dimensional subalgebras into conjugacy classes under the adjoint action of the corresponding KP symmetry group G_{KP}. In other words we shall construct optimal systems of subalgebras from which we shall be able to calculate appropriate subgroup invariants.

2.1 - Classification of the 1-dimensional subalgebras of L_{KP} under the adjoint action of G_{KP}

Here we show that there exists exactly three conjugacy classes of one-dimensional subalgebras of L_{KP} under the adjoint action of G_{KP}, with representatives respectively spanned by $X(1)=T$, $Y(1)=Y$, and $Z(1)=X$. The approach we undertake is in all respects similar to the one followed in the classification of the subalgebras of finite-dimensional Lie algebras; for more details on this approach see PAT1 and references therein. The difference between the finite- and infinite-dimensional cases arises in that in the latter case one obtains equations for the arbitrary functions labeling the group elements (whose adjoint action is used to cast the generators of the subalgebras into normal forms). These replace the pure algebraic conditions on the parameters labeling the elements of the finite-dimensional group. We shall use the explicit form of the finite transformations of the variables (t, x, y, u) associated with each of the infinitesimal generators $X(F)$, $Y(G)$, $Z(H)$ (i.e. $\exp\lambda X(F)$, $\exp\lambda Y(G)$, $\exp\lambda Z(H)$, respectively). They are respectively obtained by setting $f = F$, $g = h = 0$ in (2.7); $g = G$, $h = 0$ in (2.6); and $h = H$ in (2.5). We point out that the adjoint action under some given group element e^A can be calculated through the well known Baker-Campbell Haussdorff (BCH) formula,

$$e^A{}_*V = e^A V e^{-A} = \sum_{k=0}^{\infty} \Phi^{(k)}/k!\,, \quad \Phi^{(0)} = V, \quad \Phi^{(i+1)} = [A,\Phi^{(i)}]\,,$$

except when A contains $X(F)$ as one of its terms. In fact, using the BCH formula in the particular case when A contains a term $F\partial_t$ will give us an infinite series of terms, and therefore an infinite family of differential conditions on the function F, whereas the explicit calculation of the adjoint action through the transformations on the variables will yield a single manageable functional equation as a condition on F. As an alternative to the BCH formula, we can also calculate the adjoint action as follows. Consider an arbitrary function $f(x^*)$ of the transformed coordinates $x^* \in M$ under the action of the group element $e^A \in G$, and let $V = w(x)\cdot\nabla$ belong to its Lie algebra L, ∇ being the usual gradient operator in the variables "x"; V is also a vector field on the manifold M. Then

$$\begin{aligned}[e^A V e^{-A} f](x^*) &= e^A V f(x) = e^A[w(x)\cdot\nabla f(x)] \\ &= w(x(x^*)\cdot J(x;x^*)\nabla^* f(x^*) \equiv [e^A{}_* V](x^*)\,,\end{aligned} \tag{2.13}$$

where $J(x;x^*)$ is the Jacobian determinant associated to the transformation $x \longrightarrow x^*$. There are three cases to be considered when classifying the one-dimensional subalgebras, generated by typical elements of the form $V = X(f) + Y(g) + Z(h)$, into conjugacy classes.

<u>Case 1</u>. $f = 0$, $g = 0$, $h \neq 0$. We claim that $V = Z(h)$, with $h(t) \neq 0$, can always be transformed into $X = Z(1)$ by an element of G_{KP}. Actually, the nature of the commutation relations (2.3) for elements of L_{KP} is indicative of how we should proceed. Clearly, acting with $\exp[\lambda Y(G)]$ or $\exp[\lambda Z(H)]$ serves no purpose. However

$\exp[\lambda X(F)]$ induces a new vector field of the type Z and that is what we want. In fact, we can choose F(t) so as to normalize h(t) to "1", identically, in Z(h) by acting on it with $\{\exp[\lambda X(F)]\}_*$ (the adjoint action). Indeed we get

$$\{\exp[\lambda X(F)]\}_* Z(h(t)) = h(t(t^*))\left[\frac{F(t^*)}{F(t(t^*))}\right]^{1/3}\frac{\partial}{\partial x^*}$$
$$+\frac{2}{3}\left[\frac{F(t(t^*))}{F(t^*)}\right]^{2/3}\left[\frac{h(t(t^*))[F'(t^*) - F'(t(t^*))]}{3F(t(t^*))} + h'(t(t^*))\right]\frac{\partial}{\partial u^*}. \qquad (2.14a)$$

Now, it has been shown by Neuman (see NEU1) that there always exists a function F satisfying the following functional equation:

$$F(t(t^*))^{1/3} = h(t(t^*))F(t^*)^{1/3}. \qquad (2.14b)$$

It is then easily verified that (2.14b) together with its differential consequences, when substituted into (2.14a), yields (dropping the stars)

$$\{\exp[\lambda X(F)]\}_* Z(h(t)) = Z(1). \qquad (2.14c)$$

The existence of a solution to (2.14b) may be argued *a posteriori* as follows. Suppose that, for some function $h(t) \neq 0$, (2.14b) has no solution F. Then it should follow that symmetry reduction by the corresponding Z(h) would yield a reduced equation that could not be equivalent, under the action of any element of G_{KP}, to the equation obtained by reducing the KP equation by Z(1), namely $u_{yy} = 0$. But we shall see below, when we shall construct all the solutions of the KP equation that are invariant under the action of a one-dimensional subgroup of G_{KP} having orbits of codimension 3 in the space coordinatized by (t, x, y, u), that symmetry reduction under the subgroup corresponding to any infinitesimal generator $V = Z(h)$, $h \neq 0$, always gives rise to a reduced equation that is equivalent, under the action of an element of G_{KP}, to the linear equation $u_{yy} = 0$. Thus we arrive at a contradiction, and therefore a solution to the functional equation (2.14b) must exist.

<u>Case 2</u>. $f=0$, $g\neq 0$. We claim that $V = Y(g) + Z(h)$, where $g(t)\neq 0$, can always be transformed into $Y(1)$ by some element of the KP symmetry group. Indeed, the form of the commutation rules (2.3) for L_{KP} tells us how to proceed. First, we act on V through a conjugation by a group element $e^{\lambda Y(G)}$, aiming at getting rid of the function $h(t)$. Specifically, using the BCH formula (the series stops at order one: $\forall k > 1$, $\Phi^{(k)} \equiv 0$), we obtain

$$\{\exp[\lambda Y(G)]\}_* V = Y(g) + Z(h) + \tfrac{2}{3}\sigma\lambda Z(G'g - Gg') . \tag{2.15a}$$

It is then straightforward matter to show that if we make the choice

$$G(t) = cg(t) - \tfrac{3}{2}\sigma\lambda^{-1} g(t)\int_0^t h(s)g(s)^{-2}ds , \tag{2.15b}$$

where c is an arbitrary constant, as the function labeling the element $Y(G)$ of the KP symmetry algebra, then we obtain that (2.15a) identically reduces to

$$\{\exp[\lambda Y(G)]\}_* V = Y(g) .$$

Now, as it is again suggested by the commutation rules (2.3) for the algebra L_{KP}, we can attempt to find out a function $F(t)$ such as to allow us to normalize $g(t)$ to "1" in $Y(g)$ by acting on it with $e^{\lambda X(F)}{}_*$. As a matter of fact, using formula (2.13), we actually obtain

$$\begin{aligned} e^{\lambda X(F)}{}_* Y(g) \;=\; & g(t(t^*))\frac{F(t(t^*))^{-2/3}}{F(t^*)^{-2/3}}\frac{\partial}{\partial y^*} - \tfrac{2}{3}\sigma y^* \frac{F(t(t^*))^{1/3}}{F(t^*)^{1/3}} \\ & \Big[g'(t(t^*)) + \tfrac{2}{3} g(t(t^*))\frac{F'(t^*) - F'(t(t^*))}{F(t(t^*))}\Big]\Big[\frac{\partial}{\partial x^*} - \tfrac{2}{9}\frac{F'(t^*) - F'(t(t^*))}{F(t^*)}\frac{\partial}{\partial u^*}\Big] \\ & - \tfrac{4}{9}\sigma y^* \frac{F(t(t^*))^{4/3}}{F(t^*)^{4/3}}\Big[g'(t(t^*)) + \tfrac{2}{3} g(t(t^*))\frac{F'(t^*) - F'(t(t^*))}{F(t(t^*))}\Big]'\frac{\partial}{\partial u^*} \end{aligned} \tag{2.15c}$$

By Neuman's result (NEU1) we know that, given $g(t)\neq 0$, there will always exist a function F satisfying the functional equation

$$F(t(t^*))^{2/3} = g(t(t^*))F(t^*)^{2/3} . \tag{2.15d}$$

As in preceding case, it is easily verified that this equation and its differential consequences, when substituted into (2.15c), yields (dropping the stars)

$$e^{\lambda X(F)}{}_*Y(g) = Y(1). \tag{2.15e}$$

Thus we have found a pair of functions G(t) (2.15b) and F(t) (2.15d) such that the generator $V = Y(g) + Z(h)$ is shown to be equivalent to Y(1) through the composed adjoint actions of two one-parameter subgroups of the KP symmetry group as follows:

$$e^{X(F)}{}_*[e^{Y(G)}{}_*V] = Y(1).$$

Case 3. $f \neq 0$. We now claim that $V = X(f) + Y(g) + Z(h)$, with $f(t) \neq 0$, can always be transformed into X(1) by some element of the KP symmetry group. The main steps are as follows. As suggested by the commutation rules (2.3) for L_{KP}, we first act on V with $e^{\lambda Y(G)}{}_*$ in order to transform the function g away from V. Indeed, it may easily be seen that the BCH formula yields

$$\begin{aligned} e^{\lambda Y(G)}{}_*V = X(f) + Y(g) + Z(h) - \lambda Y(fG' - \tfrac{2}{3}f'G) + \tfrac{2}{3}\sigma\lambda Z(G'g - Gg') \\ - \tfrac{2}{3}\sigma\lambda^2 Z[G'(fG' - \tfrac{2}{3}f'G) - G(fG' - \tfrac{2}{3}f'G)'] . \end{aligned} \tag{2.16a}$$

If we make the choice

$$G(t) = c_1 f(t)^{2/3} + \lambda^{-1} f(t)^{2/3} \int_0^t g(s) f(s)^{-5/3} ds, \tag{2.16b}$$

where c_1 is an arbitrary real constant, as the function labeling the element Y(G) of the algebra L_{KP}, then (2.16a) identically reduces to $e^{\lambda Y(G)}{}_*V = X(f) + Z(h)$. The second step consists now in acting on $X(f) + Z(h)$ with a group element of the type $e^{\lambda Z(H)}$; we calculate

$$e^{\lambda Z(H)}{}_*[X(f) + Z(h)] = X(f) + Z(h) - \lambda Z(fH' - \tfrac{1}{3}f'H). \tag{2.16c}$$

Here we choose the function H(t) as

$$H(t) = c_2 f(t)^{1/3} + \lambda^{-1} f(t)^{1/3} \int_0^t h(s) f(s)^{-4/3} ds, \tag{2.16d}$$

where c_2 is an arbitrary real constant, as the function labeling $Z(H)$. The result (2.16c) then identically reduces to $e^{\lambda Z(H)}{}_*[X(f)+Z(h)] = X(f)$. As our final step we act on $X(f)$ with $e^{\lambda X(F)}{}_*$; after a tedious but otherwise straightforward calculation, we explicitly obtain:

$$
\begin{aligned}
e^{\lambda X(F)}{}_*X(f) = {} & f(t(t^*))\frac{F(t^*)}{F(t(t^*))}\frac{\partial}{\partial t^*} \\
& + \left[f'(t(t^*)) + f(t(t^*))\frac{F(t^*)-F(t(t^*))}{F(t(t^*))}\right]\left[\tfrac{2}{3}y^*\frac{\partial}{\partial y^*} + \tfrac{1}{3}x^*\frac{\partial}{\partial x^*} - \tfrac{2}{3}u^*\frac{\partial}{\partial u^*}\right] \\
& - \frac{F(t(t^*))}{F(t^*)}\left[f'(t(t^*)) + f(t(t^*))\frac{F(t^*)-F(t(t^*))}{F(t(t^*))}\right]'\left[\tfrac{2}{9}\sigma y^{*2}\frac{\partial}{\partial x^*} - \tfrac{2}{9}x^*\frac{\partial}{\partial u^*}\right] \\
& - \tfrac{4}{27}\frac{F(t(t^*))^2}{F(t^*)^2}\left[f'(t(t^*)) + f(t(t^*))\frac{F(t^*)-F(t(t^*))}{F(t(t^*))}\right]''\sigma y^{*2}\frac{\partial}{\partial u^*}\,.
\end{aligned}
\tag{2.16e}
$$

We choose the function $F(t)$ such that it satisfies the functional equation

$$F(t(t^*)) = f(t(t^*))F(t^*)\,. \tag{2.16f}$$

As in the preceding cases, the existence of a solution to (2.16f) is guaranteed from Neuman's result, and it is then easily verified that (2.16f) and its differential consequences, when substituted into (2.16e), yields (dropping the stars)

$$e^{\lambda X(F)}{}_*X(f) = X(1)\,. \tag{2.16g}$$

Thus we have found a pair of functions $G(t)$ (2.16b) and $H(t)$ (2.16d), as well as a third function $X(F)$ satisfying equation (2.16f), such that the generator $V = X(f) + Y(g) + Z(h)$ is shown to be equivalent to $X(1)$ through the composed adjoint actions of three one-parameter subgroups of the KP symmetry group as follows:

$$e^{X(F)}{}_*\{e^{Z(H)}{}_*[e^{Y(G)}{}_*V]\} = X(1).$$

Our proof of the existence of three conjugacy classes of one-dimensional subalgebras of L_{KP} under the adjoint action of G_{KP} with representatives spanned by $X(1) = T$, $Y(1) = Y$, and $Z(1) = X$ is thus complete. To summarize, an arbitrary one-dimensional subalgebra of

L_{KP} is conjugate, under the KP symmetry group G_{KP}, to precisely one of the following ones:

$$L_{1,1} = \{X(1)\}, \quad L_{1,2} = \{Y(1)\}, \quad L_{1,3} = \{Z(1)\}. \tag{2.17}$$

2.2 - Classification of the two-dimensional subalgebras of L_{KP}

It is a well known result (see, e.g., JAC1) that precisely two types of two-dimensional Lie algebras $\{R_1, R_2\}$ exist, both over the fields **R** and **C**, namely Abelian algebras and solvable non-Abelian algebras, satisfying, in an appropriate basis, the commutator equation $[R_1, R_2] = R_1$. We shall take R_1 in one of the three possible forms established above in the preceding section 2.1, and let R_2 be a general element of the KP symmetry algebra. The procedure to find the subalgebras will consist in first imposing the above commutator equation and then to simplify R_2 by using the isotropy group of R_1 in the invariance group of the KP equation.

A) Abelian algebras.

A.1) $R_1 = X(1) = T$. We take $R_2 = X(f) + Y(g) + Z(h)$. Requiring that $[R_1, R_2] = 0$ be satisfied and using the commutation relations (2.3), we find $f'(t) = g'(t) = h'(t) = 0$. Hence $R_2 = aX(1) + bY(1) + bZ(1)$. We then replace R_2 by $R_2' = R_2 - aR_1$, this has the effect of setting $a = 0$ in R_2. Conjugating by $e^{[\lambda Y(t) + \mu Z(t)]}$, if $b \neq 0$, we can arrange for c to vanish; if $b = 0$, then we put $R_2 = Z(1)$. We thus obtain two distinct algebras, namely $\{X(1), Y(1)\}$ and $\{X(1), Z(1)\}$.

A.2) $R_1 = Y(1) = Y$. We again take $R_2 = X(f) + Y(g) + Z(h)$. The condition $[R_1, R_2] = 0$ implies $f'(t) = g'(t) = 0$, hence $R_2 = aX(1) + bY(1) + Z(h)$. We are forced to set $a = 0$, or we would reobtain the precedent case. We can arrange for b to vanish by redefining R_2 through a linear combination with R_1, and we therefore obtain another algebra (actually a family of them), namely $\{Y(1), Z(h)\}$. It should be noted that the remaining freedom in the KP symmetry algebra, namely the invariance of $\{Y(1)\}$ under dilations and time translation still could be used with the consequence of giving arbitrarily chosen values to any two of the Taylor coefficients of the function h(t). Hereafter we choose not to lift such trivial redundancies for the equivalence classes of subalgebras labeled by arbitrary functions.

A.3) $R_1 = Z(1) = Z$. We take R_2 as before. Requiring that it commutes with R_1, we find that it must take the form $R_2 = aX(1) + Y(g) + Z(h)$. If $a \neq 0$ we reobtain case (A.1); if $a = 0$ and $g(t) \neq 0$, we then recover case (A.2). The final possibility is $a = 0$ and $g(t) = 0$, for which we obtain the algebras $\{Z(1), Z(h) \mid h'(t) \neq 0\}$.

B) Non-Abelian algebras.

B.1) $R_1 = X(1) = T$. We take $R_2 = X(f) + Y(g) + Z(h)$. Requiring $[R_1, R_2] = R_1$ implies that $f'(t) = 1$ and $g' = h' = 0$. Conjugating by $e^{[\lambda Y(1) + \mu Z(1)]}$, we can transform away the functions g and h. We thus obtain a single algebra, namely $\{X(1), X(t)\}$.

B.2) $R_1 = Y(1) = Y$. Imposing the appropriate commutation relation we find that R_2 must be of the form $R_2 = \frac{3}{2} X(t) + aX(1) + bY(1) + Z(h)$. We can eliminate b by redefining R_2 through a linear combination with R_1, and transform a and h(t) into 0 through a conjugation by $e^{[\lambda X(1) + Z(H)]}$. The new algebra is hence $\{Y(1), X(\frac{3}{2}t)\}$.

B.3) $R_1 = Z(1) = X$. The commutation relation $[Z(1), R_2] = Z(1)$ yields $R_2 = X(3t) + aX(1) + Y(g) + Z(h)$. We then conjugate by $e^{[\lambda X(1) + Y(G) + Z(H)]}$; this permits to make a, g(t), and h(t) vanish, and we therefore get the algebra $\{Z(1), X(3t)\}$.

Let us summarize the results. Every two-dimensional subalgebra of L_{KP} is conjugate under G_{KP} to precisely one of the following algebras (with the reservation that any two functions h(t) and $e^{\alpha}h(t - \beta)$, where α and β are real constants, give equivalent algebras):

1. Abelian algebras:

$$\begin{aligned} L_{2,1} &= \{X(1), Y(1)\}, \\ L_{2,2} &= \{X(1), Z(1)\}, \\ L_{2,3}{}^{h} &= \{Y(1), Z(h)\}, \\ L_{2,4}{}^{h} &= \{Z(1), Z(h) \mid h'(t) \neq 0\}; \end{aligned} \tag{2.18}$$

2. Non-Abelian algebras (satisfying $[R_1, R_2] = R_1$):

$$\begin{aligned} L_{2,5} &= \{X(1), X(t)\}, \\ L_{2,6} &= \{Y(1), X(\tfrac{3}{2}t)\}, \\ L_{2,7} &= \{Z(1), X(3t)\}. \end{aligned} \tag{2.18}$$

2.3 - Classification of the three-dimensional subalgebras of L_{KP}

It is a well known that a real three-dimensional Lie algebra (JAC1) can be either simple or solvable. We will consider these two cases separately.

A) Simple Lie Subalgebras.

Let us first allow for complex coefficients in the vector fields and construct the algebra sl(2,C). This algebra has a two-dimensional non-Abelian subalgebra $\{R_1,R_2\}$. The commutation relations of sl(2,C) can be written as $[R_1,R_2] = R_1$, $[R_2,R_3] = R_3$, and $[R_1,R_3] = -2R_2$. We shall identify $\{R_1,R_2\}$ with one of the non-Abelian algebras in (2.18), i.e., consider it to be in standard form.

A.1) $L_{2,5} : R_1 = X(1), R_2 = X(t)$. We take $R_3 = X(f) + Y(g) + Z(h)$. Imposing the sl(2,C) commutation relations, we find $R_3 = X(t^2)$.

A.2) $L_{2,6} : R_1 = Y(1), R_2 = X(\frac{3}{2}t)$. We take R_3 as above. It is easily seen that the commutation relation $[R_1, R_3] = -2R_2$ cannot be satisfied.

A.3) $L_{2,7} : R_1 = Z(1), R_2 = X(3t)$. We again take R_3 as above and, once more, the commutation relation $[R_1, R_3] = -2R_2$ cannot be satisfied.

We thus have obtained a single class of sl(2,C) algebras, represented by $\{X(1), X(t), X(t^2)\}$ (see also the physical subalgebra L_P in Section 1). Restricting to real coefficients, we recover the algebra sl(2,**R**), but not su(2).

B) Solvable Lie Subalgebras

A solvable three-dimensional Lie algebra always will have a two-dimensional Abelian ideal (see Refs. MUB1 and PAT2 for a classification of Lie algebras of dimension ≤ 5 into isomorphy classes). Unless the three-dimensional algebra is Abelian or nilpotent, this ideal is unique, up to conjugacy under inner automorphisms. We assume that the ideal $\{R_1, R_2\}$ is already in standard form, as given by the Abelian algebras in (2.18), and look for a third element $R_3 = X(f) + Y(g) + Z(h)$ that acts upon the ideal according to the following equation:

$$\begin{bmatrix} [R_1, R_3] \\ [R_2, R_3] \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}, \qquad M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2.19}$$

The real matrix M in (2.19) can, by a change of the basis of the $\{R_1, R_2\}$ space, be taken into a standard form and further simplification can be achieved by rescaling R_3. Finally, R_3 can itself be simplified by transformations in G_{KP} that leave the algebra $\{R_1, R_2\}$, as a vector space, invariant. Let us run through this procedure for each two-dimensional Abelian subalgebra taken from the list (2.18).

B.1) $L_{2,1}$: $R_1 = X(1), R_2 = Y(1)$. We first impose equation (2.19); using the L_{KP} commutation relations (2.3), we then deduce $d = \frac{2}{3}a$, $b = c = 0$, and thus find $R_3 = aX(t) + \alpha X(1) + \beta Y(1) + \gamma Z(1)$. Since X(1) and Y(1) are elements of the algebra we can always set $\alpha = \beta = 0$. If $a \neq 0$ we apply $e^{\lambda Z(1)}$ in order to transform γ away; we therefore obtain a diagonal action in (2.19). If $a = 0$ we can always choose $\gamma = 1$ and obtain the abelian algebra $\{X(1), Y(1), Z(1)\}$.

B.2) $L_{2,2}$: $R_1 = X(1), R_2 = Z(1)$. From the commutation relations (2.3) and (2.19) we first obtain $R_3 = aX(t) + bZ(t) + \gamma Y(1)$ [up to linear combinations with X(1) and Z(1)]. If $a \neq 0$ we apply $e^{\lambda Y(1) + \mu Z(t)}$ and transform away b and γ; we obtain $R_3 = X(t)$ and a diagonal action in (2.19). If $a = 0$ we must have $b \neq 0$ in order not to recover case A above. If $\gamma = 0$ we obtain the nilpotent algebra $\{X(1), Z(1), Z(t)\}$. If $\gamma \neq 0$ we apply $e^{\lambda X(t)}$ and the discrete symmetry R_y from (2.8) to obtain a second nilpotent algebra, namely $\{X(1), Z(1), Z(t) + Y(1)\}$.

B.3) $L_{2,3}{}^H : R_1 = Y(1), R_2 = Z(H)$. The commutation relations (2.19) imply $c = 0$ and thus $R_3 = aX(\frac{3}{2}t) + \alpha X(1) + Y(g) + Z(h)$, with the following constraints on g(t) and H(t):

$$g'(t) + \tfrac{3}{2}\sigma bH(t) = 0\,, \quad (3at + 2\alpha)H'(t) + (2d - a)H(t) = 0. \tag{2.20}$$

In this case we consider each normal form of the matrix M in (2.19) separately.

(B.3.1) $[a = 0, b = 0, c = 0, d = 0]$. M is the null matrix. The constraints (2.20) then yield the Abelian algebra $\{Y(1), Z(H), Z(h)\}$ (with the restriction that the functions h(t) and H(t) be linearly (functionally) independent) if $\alpha \neq 0$, and we recover the subalgebra $\{X(1), Y(1), Z(1)\}$ for $\alpha = 0$.

(B.3.2) $[a = 0, c = 0, d \neq 0]$ or $[a \neq 0, c = 0, d = 0]$. In this case the action of R_3 on the ideal $\{R_1, R_2\}$ is decomposable, but not Abelian. Consider first the case $a = 0, d \neq 0$. Then $\alpha \neq 0$ and (2.20) implies

$$f = \alpha, \quad H(t) = e^{-dt/\alpha}, \quad g(t) = g_0 + (3\sigma ab/2d)e^{-dt/\alpha}.$$

Changing the basis in the ideal $\{R_1 - (b/d)R_2, R_2\}$ we diagonalize M. Performing then a conjugation by $e^{\lambda Y(G) + \mu Z(K)}$ with appropriately chosen functions G(t), K(t), and constants λ, μ, we get the decomposable algebra $\{Y(1), Z(e^{-t}), X(1)\}$. Similarly, putting $a \neq 0, d = 0$, we obtain, after some rather tedious calculations, a further (inequivalent) decomposable algebra, namely $\{Y(1), Z(t^{1/3}), X(\frac{3}{2}t)\}$.

(B.3.3) $[a = 0, b = 1, c = 0, d = 0]$. M is a nilpotent matrix. If $\alpha \neq 0$ we then obtain $H(t) = 1$ and $g(t) = g_0 - \frac{3}{2}\sigma t$. Performing a conjugation by a group element of the form $e^{\lambda Z(K)}$ with the function Z(K) appropriately chosen, we obtain the nilpotent algebra $\{Y(1), Z(1), X(1) - \frac{3}{2}\sigma Y(t)\}$. If $\alpha = 0$, then we find $H(t) = -\frac{2}{3}\sigma g'(t)$, g(t) and h(t) being arbitrary; we obtain a family of nilpotent algebras $\{Y(1), Z(-\frac{2}{3}\sigma g'), Y(g) + Z(h)) \mid g'(t) \neq 0\}$. It is to be noted that the dependence of g(t) and h(t) cannot be transformed away and each pair $\{g(t), h(t)\}$ therefore provides a different conjugacy class of algebras.

(B.3.4) $[a \neq 0, d \neq a, d \neq 0]$. M is diagonalizable. We begin by putting $a = 1$ and transform away α by using a translation along the variable t. This yields $f = \frac{3}{2}t$ and the constraints $3tH'(t) = (1 - 2d)H(t)$, $g'(t) = -\frac{3}{2}\sigma bH(t)$. Conjugating in an appropriate way by $e^{\lambda Y(G) + \mu Z(K)}$, we obtain a one-parameter class of algebras, namely $\{Y(1), Z(t^{(1-2d)/3}), X(\frac{3}{2}t)\}$.

(B.3.5) $[a = 1, b = 1, c = 0, d = 1]$. M has a Jordan form (since $c = 0$, this is the only remaining possibility). This last case, after conjugating by an appropriate group element of the form $e^{\lambda Z(K)}$, gives the algebra $\{Y(1), Z(t^{-2/3}), X(\frac{3}{2}t) - \frac{9}{2}\sigma Y(t^{1/3})\}$.

B.4) $L_{2,4}^{H}$: $R_1 = Z(1)$, $R_2 = Z(H)$. The commutation relations (2.19), in this case, yield the following constraints on f(t) and H(t):

$$f'(t) - 3[a + bH(t)] = 0\,, \qquad f'(t)H(t) - 3f(t)H'(t)] = 3[c + dH(t)]. \tag{2.21}$$

Moreover, we can take linear combinations of Z(1) and Z(H) that will take the matrix M into some of its standard forms; these combinations are $Z_1(H) = \alpha Z(1) + \beta Z(H)$ and $Z_2(H) = \gamma Z(1) + \delta Z(H)$, with $\alpha\delta - \beta\gamma \neq 0$. A further conjugation by some well chosen group element $e^{\lambda X(F)}$ will then take Z_1 into Z(1), and Z_2 into Z(H'). We thus assume that M is already in standard form, and examine each of these forms separately.

(B.4.1) $[a = 0, b = 0, c = 0, d = 0]$. M is the null matrix. We obtain one new Abelian algebra, namely $\{Z(1), Z(h), Z(H)\}$ with the constraint that 1, h(t), and H(t) be linearly independent functions.

(B.4.2) $[a = 1, b = 0, c = 0, d = 0]$. M has a decomposable form. This yields to one new decomposable algebra, $\{Z(1), Z(t^{1/3}), Z(3t)\}$.

(B.4.3) $[a = 0, b = 0, c = 1, d = 0]$. No new algebras are obtained from this case.

(B.4.4) $[a = 1, b = 0, c = 0]$. M again has a decomposable form. We find one new one-parameter class of algebras, namely $\{Z(1), Z(t^{(1-d)/3}), X(3t)\}$.

(B.4.5) $[b = 1, c = -1, d = a]$. This case corresponds to a *complex* action on the ideal, i.e. M could be diagonalized over **C** but not over **R**. From the constraints (2.21) we see that we must have $f(t) \neq 0$. To solve these constraints we depart from our usual procedure and first perform a transformation $e^{X(F)+Y(G)+Z(K)}$ in order to take the elements R_1, R_2, and R_3 into some new elements $S_1 = Z(h_1)$, $S_2 = Z(h_2)$, and $S_3 = Z(1)$, respectively, where $h_1(t)$ and $h_2(t)$ are arbitrary functions. The commutation relations $[S_1, S_3] = aS_1 + S_2$ and $[S_2, S_3] = aS_2 - S_1$ now imply

$$h'(t) + ah_1(t) + h_2(t) = 0, \quad h_2'(t) - h_1(t) + ah_2(t) = 0.$$

Solving and performing an appropriate time translation, we obtain the algebras $\{Z(e^{-at}\cos(t)), Z(e^{-at}\sin(t)), X(1)\}$.

(B.4.6) $[a = 1, b = 0, c = 1, d = 1]$. M has a Jordan form. We obtain a single class of Lie algebras, represented by $\{Z(1), Z(-\frac{1}{3}\ln(t)), X(3t)\}$.

To conclude this section we present in a unified manner a list of representatives of all conjugacy classes of three-dimensional subalgebras of L_{KP}, ordered by their isomorphy class. The solvable algebras are all given in the order $\{R_1, R_2, R_3\}$, where $N = \{R_1, R_2\}$ is an Abelian ideal and the action of R_3 on N is given in (2.19). In each case we specify the composition of the matrix M.

1. Abelian algebras : $[a = 0, b = 0, c = 0, d = 0]$

$$\begin{aligned} L_{3,1} &= \{X(1), Y(1), Z(1)\}, \\ L_{3,2}^{h,H} &= \{Y(1), Z(h), Z(H) \mid h(t) \neq \lambda H(t)\}, \\ L_{3,3}^{h,H} &= \{Z(1), Z(h), Z(H) \mid 1, h(t), H(t) \text{ are independent}\}; \end{aligned} \tag{2.22a}$$

2. Decomposable non-Abelian algebras : $[a = 1, b = 0, c = 0, d = 0]$

$$\begin{aligned} L_{3,4} &= \{Z(e^{-t}), Y(1), X(1)\}, \\ L_{3,5} &= \{Y(1), Z(t^{1/3}), X(\tfrac{3}{2}t)\}, \\ L_{3,6} &= \{Z(1), Z(t^{1/3}), X(3t)\}; \end{aligned} \tag{2.22b}$$

3. Nilpotent algebras : $[a=0, b=1, c=0, d=0]$

$$L_{3,7} = \{Z(t), -Z(1), X(1)\},$$
$$L_{3,8} = \{Z(t)+Y(1), -Z(1), X(1)\}, \qquad (2.22c)$$
$$L_{3,9}{}^{h,g} = \{Y(1), Z(-\tfrac{2}{3}\sigma g'), Y(g)+Z(h(t)) \mid g'(t)\neq 0\};$$

4. Diagonal action on ideal : $[a=1, b=0, c=0, d\neq 0]$

$$L_{3,10} = \{X(1), Y(1), -X(t)\},\ d=\tfrac{2}{3},$$
$$L_{3,11} = \{X(1), Z(1), -X(t)\},\ d=\tfrac{1}{3}, \qquad (2.22d)$$
$$L_{3,12}{}^{d} = \{Y(1), Z(t^{(1-2d)/3}), X(\tfrac{3}{2}t)\},$$
$$L_{3,13}{}^{d} = \{Z(1), Z(t^{(1-d)/3}), X(3t)\},\ d\neq 1;$$

5. Complex action on ideal : $[a\geq 0, b=1, c=-1, d=a]$

$$L_{3,14}{}^{a} = \{Z[e^{-at}\cos(t)], Z[e^{-at}\sin(t)], X(1)\}; \qquad (2.22e)$$

6. Jordan action on ideal : $[a=1, b=1, c=0, d=1]$

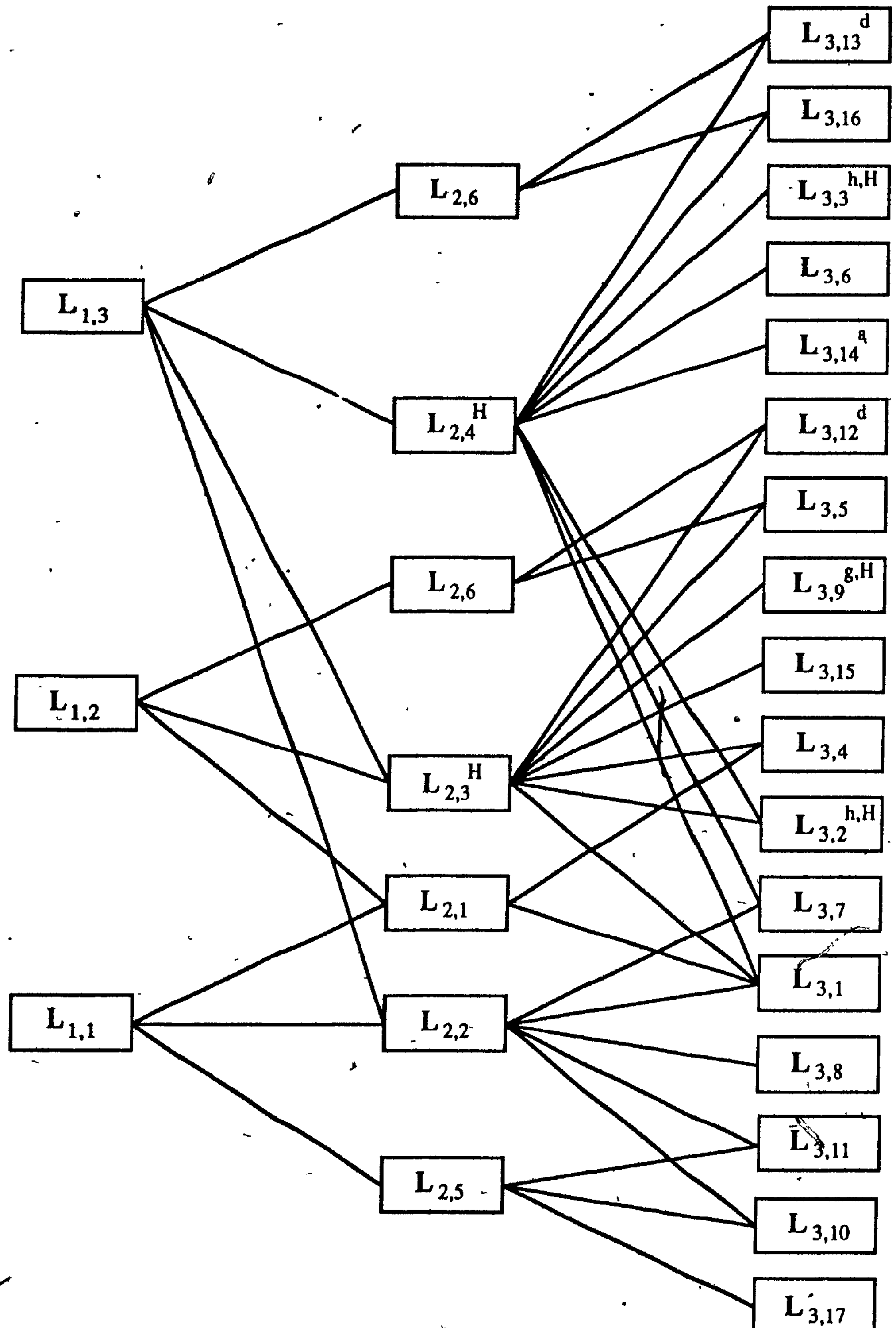
$$L_{3,15} = \{Y(1), Z(t^{-2/3}), X(\tfrac{3}{2}t) - Y(\tfrac{9}{2}\sigma t^{1/3})\}, \qquad (2.22f)$$
$$L_{3,16} = \{Z(1), Z[-\tfrac{1}{3}\ln(t)], X(3t)\};$$

7. The simple Lie algebra sl(2,R)

$$L_{3,18} = \{X(1), X(t), X(t^2)\}. \qquad (2.22g)$$

Thus all isomorphy classes of three-dimensional Lie algebras, except su(2), are represented in the list of subalgebras of the KP symmetry algebra.

The figure on next page is a diagram showing all the one-, two-, and three dimensional subalgebras of L_{KP}. The lines represent how each subalgebra of dimensions two and three were obtained from subalgebras of dimensions one and two, respectively. We point out that the line between the subalgebras $\mathbf{L}_{1,3}$ and $\mathbf{L}_{2,3}{}^{H}$ actually represents two lines. Indeed, when we constructed two-dimensional non-Abelian subalgebras from the one-dimensional subalgebra $\{Z(1)\}$, we omitted the algebra $\mathbf{K} = \{Z(1), Y(g(t))\}$ because it is in fact conjugate to $\mathbf{L}_{2,3}{}^{H} = \{Y(1), Z(H)\}$ through an appropriate conjugation by some group element of the form $e^{\lambda X(F)}$. Hence the line from $\mathbf{L}_{1,3}$ to $\mathbf{L}_{2,3}{}^{H}$ should first pass by $\mathbf{K}$.



$L_{3,13}^{d}$
$L_{3,16}$
$L_{2,6}$
$L_{3,3}^{h,H}$
$L_{3,6}$
$L_{1,3}$
$L_{3,14}^{a}$
$L_{2,4}^{H}$
$L_{3,12}^{d}$
$L_{3,5}$
$L_{2,6}$
$L_{3,9}^{g,H}$
$L_{3,15}$
$L_{1,2}$
$L_{3,4}$
$L_{2,3}^{H}$
$L_{3,2}^{h,H}$
$L_{2,1}$
$L_{3,7}$
$L_{3,1}$
$L_{1,1}$
$L_{2,2}$
$L_{3,8}$
$L_{3,11}$
$L_{2,5}$
$L_{3,10}$
$L_{3,17}$

It is easy to verify for each case that there are indeed no redundancies in the above list except for the possibility of "renormalizing" one of the arbitrary functions in an algebra by the transformation $h(t) \rightarrow e^{\alpha}h(t-\beta)$. The different Abelian algebras are mutually non-conjugate since a general element of $\mathbf{L}_{3,1}$ can be conjugate to X(1), Y(1), or Z(1), one of $\mathbf{L}_{3,2}{}^{h,H}$ is conjugate to Y(1) or Z(1), and an element of $\mathbf{L}_{3,3}{}^{h,H}$ is always conjugate to Z(1). The functions h(t) and H(t) cannot be changed in either case without destroying the standard form of Y(1) or Z(1), respectively. In the decomposable case the ideal $\{R_1, R_2\}$ and the two-dimensional solvable subalgebra $\{R_1, R_3\}$ are well defined and distinguish between the three cases. In the nilpotent case the centre R_3 is uniquely defined. It distinguishes $L_{3,9}{}^{g,H}$ from the other two. An element of $L_{3,7}$ can be conjugate to X(1) or Z(1); and an element of $L_{3,8}$ can be conjugate to X(1), Y(1), or Z(1). In all other solvable cases the Abelian ideal is uniquely defined and therefore suffices to distinguish between different mutually isomorphic cases.

In the next section we shall use these low-dimensional subalgebras of the KP symmetry algebra to perform symmetry reduction on the KP equation and find invariant solutions of this equation under the several subgroups of G_{KP} corresponding to the L_{KP} subalgebras that we just constructed above.

3. Invariant solutions of the Kadomtsev-Petviashvili equation obtained by symmetry reduction

In this section we will apply our knowledge of the KP symmetry group and its subgroups to construct *all* the solutions of the KP equation that are invariant under the action of a one-dimensional subgroup. We will also briefly look at solutions invariant under the action of the two- and three-dimensional subgroups. In doing so, we also complete the proof of the assertion in section 2.1, namely that there exist precisely three orbits of one-dimensional subalgebras of the KP algebra. The one-dimensional subgroups all have codimension three in the space coordinatized by (t,x,y,u). The method provides solutions that depend on three, two, or one arbitrary functions of the variable t, in addition to the arbitrary functions that may appear in the solutions of the reduced equations, which are themselves partial differential equations in two (rather than three) variables. We will proceed as indicated in Chapter one, namely we first pick up a representative of an equivalence class of conjugate subalgebras, spanned by vector fields $\{X_i\}$, and find a basis of functional invariants $\{I_\alpha\}$ by solving the system $X_i I = 0$. For our one-dimensional subalgebras of L_{KP} we find three independent invariants, two symmetry variables $\xi(t,x,y)$ and $\eta(t,x,y)$, and in all cases it is seen that the solution to the KP equation takes the form

$$u(t,x,y) = \alpha(t,x,y)q(\xi,\eta) + \beta(t,x,y), \tag{2.23}$$

where ξ, η, as well as α and β, are known functions. The function $q(\xi,\eta)$, itself a subgroup invariant, is *a priori* not known and is subject to a partial differential equation in the variables ξ and η, obtained by substituting (2.23) back into the KP equation (2.1): this equation is known as the reduced equation (see Chapter one). The entire symmetry group can then be applied to the solution (2.23) to obtain a larger class of solutions.

Two equivalent approaches can be adopted, at this point, for symmetry reduction by a one-dimensional subgroup of G_{KP}. A first one is to make use of the classification of one-dimensional subalgebras of L_{KP} which we established in section 2 of the present chapter. We then go through the above procedure using the representatives of each conjugacy class of elements. Thus the subalgebra $L_{1,1}$, generated by the vector field $X = \partial_t$, implies that $\xi = x$, $\eta = y$, $\alpha = 1$, $\beta = 0$, and we find that $u(t,x,y) = q(x,y)$, where q satisfies the Boussinesq equation. Similarly, for the subalgebra $L_{1,2}$, generated by $X = \partial_y$, we find $\xi = x$, $\eta = t$, $\alpha = 1$, $\beta = 0$, and deduce that $u(t,x,y) = q(x,t)$, where q satisfies the KdV equation. Finally, for $L_{1,3}$, with $X = \partial_x$, we obtain $\xi = t$, $\eta = y$, $\alpha = 1$, $\beta = 0$, and $u(t,x,y) = q(t,y)$, satisfies the linear equation $q_{yy} = 0$. In each of these cases we then apply the transformation (2.7) to get all solutions invariant under the action of a one-dimensional subgroup of the KP symmetry group. The solutions depend on up to three arbitrary functions. A second completely equivalent procedure is to perform the same reduction using a general element of L_{KP}, $V = X(f) + Y(g) + Z(h)$, and, as usual, considering separately the three cases (1) $f(t) = g(t) = 0$ but $h(t) \neq 0$, (2) $f(t) = 0$ with $g(t) \neq 0$, and (3) $f(t) \neq 0$. No further group transformation is necessary in this case. We shall apply the second procedure, mainly because it will confirm the results established when classifying the one-dimensional subalgebras of the KP symmetry algebra; namely all the equations obtained when reducing by the generator of a one-dimensional subalgebra under a transformation of the KP symmetry group are equivalent either to the Boussinesq equation, a once differentiated KdV equation, or a linear equation of second order. Let us list the results in each cases.

<u>Case 1</u>: $f(t) = g(t) = 0$, $h(t) \neq 0$. Solving the equation $Z(h)I = 0$, we find the following set of three subgroup invariants:

$$I_1 = t, \quad I_2 = y, \quad I_3 = u - 2xh'(t)/3h(t).$$

We thus have the two symmetry variables $\xi = t$ and $\eta = y$, and the form of I_3 yields

$$u(t,x,y) = \phi(t,y) + 2xh'(t)/3h(t). \tag{2.24a}$$

Substituting (2.24a) in the KP equation, we find that u is a solution of the KP equation, for a sufficiently smooth function h(t), if, and only if, the function $\phi(t,y)$ satisfies the second order linear equation

$$3\sigma\phi_{yy} + 8h''(t)/3h(t) = 0. \tag{2.24b}$$

Redefining ϕ as $q(t,y) - 4\sigma y^2 h''(t)/9h(t)$, (2.24a) and (2.24b) reduce to

$$u(t,x,y) = \phi(t,y) + 2xh'(t)/3h(t) - 4\sigma y^2 h''(t)/9h(t), \tag{2.24c}$$

$$u_{yy} = 0. \tag{2.24d}$$

Integrating (2.24d) we obtain a family of solutions of the KP equation depending on three arbitrary functions of the variable t:

$$u(t,x,y) = \tfrac{2}{3}xh(t)^{-1}h'(t) - \tfrac{4}{9}\sigma y^2 h(t)^{-1}h''(t) + K(t) + L(t). \tag{2.24e}$$

<u>Case 2</u>: $f(t) = 0$, $g(t) \neq 0$. Solving the equation $[Y(g) + Z(h)]I = 0$, we find the following three invariants:

$$I_1 = t, \quad I_2 = x - yh(t)/g(t) + \tfrac{1}{3}\sigma y^2 g'(t)/g(t),$$

$$I_3 = u - \tfrac{2}{3}yh'(t)/g(t) + \tfrac{2}{9}\sigma y^2 g''(t)/g(t).$$

We thus have the two symmetry variables $\eta = I_1 = t$ and $\xi = I_2$, and we write the function u(t,x,y) as

$$u(t,x,y) = \phi(\xi,\eta) + \tfrac{2}{3}yh'(t)/g(t) - \tfrac{2}{9}\sigma y^2 g''(t)/g(t)\,. \tag{2.25a}$$

Substituting (2.25a) in the KP equation, we find that u is a solution of the KP equation, for sufficiently smooth functions g(t) and h(t), if, and only if, the function $\phi(\xi,\eta)$ satisfies the following nonlinear equation: ✓

$$[4\phi_{\xi\eta} + 6(\phi\phi_\xi) + \phi_{\xi\xi\xi\xi}] + 3\sigma(h/g)^2\,\phi_{\xi\xi} + 2(g'/g)\phi_\xi - 4\sigma g''/3g = 0\,. \tag{2.25b}$$

This equation is a once-differentiated KdV equation with additional extra terms. It is possible, as in the precedent case, to get rid of these terms by slightly modifying the basis of invariants. In fact, redefining ξ, η, and ϕ as

$$\begin{gathered}\xi \to g^{-1/2}(t)\xi\,, \qquad \eta = t \to \int_0^t g^{-3/2}(s)ds\,,\\ \phi = g^{-1/2}(t)q(\xi,\eta) + \frac{6[xg(t) - yh(t)]g'(t) + 2\sigma g'(t)^2 - 9\sigma h(t)^2}{18g(t)^2}\,,\end{gathered} \tag{2.25c}$$

the expression for u(t,x,y) and the reduced equation (2.25b) simplify to a pure KdV equation:

$$u(t,x,y) = g^{-1/2}(t)q(\xi,\eta) + \frac{xg'}{3g} - \frac{\sigma h^2}{2g^2} + \frac{(2gh' - g'h)y}{3g^2} + \frac{\sigma(g'^2 - 2gg'')y^2}{9g^2}\,, \tag{2.25d}$$

$$[q_\eta + \tfrac{3}{2}qq_\xi + \tfrac{1}{4}q_{\xi\xi\xi}]_\xi = 0\,. \tag{2.25e}$$

<u>Case 3</u>: $f(t) \neq 0$. We proceed as before. As the calculations are very tedious for this case, we just present the results. Solving the equation $[X(f) + Y(g) + Z(h)]I = 0$, we find the invariants, the symmetry variables, the reduction transformation for the solution u(t,x,y), and substitute this information in the KP equation. The result is a Boussinesq equation with, here again, *parasitic* terms. It is possible to redefine the basis of invariants and we finally obtain

$$\eta = yf(t)^{-2/3} - \int_0^t g(s)f(s)^{-5/3}\,ds\,,$$

$$\xi = f(t)^{-1/3}[x + \tfrac{2}{3}\sigma y f(t)^{-1} g(t) + \tfrac{2}{9}\sigma y^2 f(t)^{-1} f(t)] - \int_0^t [\tfrac{2}{3}\sigma g(s)^2 f(s)^{-5/3} + h(s) f(s)^{-4/3}]ds\,,$$

$$\begin{aligned} u(t,x,y) = {} & f(t)^{-2/3} q(\xi,\eta) + \tfrac{2}{9}\sigma f(t)^{-2} g(t)^2 + \tfrac{2}{3} f(t)^{-1} h(t) + \tfrac{2}{9} f(t)^{-1} f'(t) x \\ & + \tfrac{4}{27}\sigma f(t)^{-2}[2f'(t)g(t) - 3f(t)g'(t)]y + \tfrac{4}{81}\sigma f(t)^{-2}[2f'(t)^2 - 3f(t)f''(t)]y^2\,, \end{aligned} \tag{2.26a}$$

with the function $q(\xi,\eta)$ satisfying the pure Boussinesq equation:

$$\sigma q_{\eta\eta} + (q^2)_{\xi\xi} + \tfrac{1}{3} q_{\xi\xi\xi\xi} = 0\,. \tag{2.26b}$$

The classes of solutions of the KP equation presented in (2.24), (2 25), and (2.26) depend on one, two, or three arbitrary functions of t, in addition to the arbitrary functions possibly appearing in the solutions of the reduced equations, as in (2.24e), for instance. In general, they will diverge at infinity unless the functions f(t), g(t), an h(t) are appropriately restricted.

We shall now proceed to list some special cases of the above solutions that are of physical interest and illustrate how f(t), g(t), and h(t) ought to be chosen so as to preserve decay at infinity. Boiti and Pempinelli (see BOI3) have shown that the similarity solutions of the potential Boussinesq equation,

$$\sigma w_{\eta\eta} + (w^2{}_{\xi})_{\xi} + \tfrac{1}{3} w_{\xi\xi\xi\xi} = 0\,,$$

obtained by setting $q = w_\xi$ in the Boussinesq equation (2.26b), are of the form

$$w(\xi,\eta) = \varphi[\xi - \eta\sqrt{\sigma/3}\,] - \tfrac{1}{6}\sigma g_1 \eta^2 + h_0 \eta\sqrt{\sigma/3} + \tfrac{1}{6}\xi + k\,, \tag{2.27}$$

where g_1, h_0, and k are arbitrary constants. The equation satisfied by w reduces to an ordinary differential equation for the function φ, which is equivalent to the first Painlevé transcendent equation if $g_1 \neq 0$ and to the equation for the Weierstrass $\wp$-function if $g_1 = 0$. It follows from (2.27) that

$$q = w_\xi = \tfrac{1}{6} + \varphi'[\xi - \eta\sqrt{\sigma/3}\,] \tag{2.28}$$

satisfies the Boussinesq equation (2.26b) and therefore that the function u(t,x,y), as given by (2.26a), is a solution of the KP equation. Furthermore, it is also clear from (2.26a) that if we start with a solution of the Boussinesq equation which is bounded at infinity, then the corresponding solution of the KP equation will possess the same property only if the functions f(t) and g(t) reduce to constants, say $f_0 \neq 0$ and g_0, while the function h(t) remains arbitrary. We thus obtain two classes of solutions of the KP equation depending on one arbitrary function h(t) by performing a *Galilei-like* transformation, involving an arbitrary function h(t) on the solutions of the Boussinesq equation arising from similarity solutions of the potential Boussinesq equation. They are explicitly given by

$$\begin{aligned} u(t,x,y) &= -2(-g_1/2)^{2/5} f_0^{-2/3} P_I(\xi) + \tfrac{2}{9}\sigma f_0^{-2} g_0^{2} + \tfrac{2}{3} f_0^{-1} h(t) + \tfrac{1}{6}, \\ \xi &= (-g_1/2)^{1/5}\Big[x f_0^{-1/3} + \tfrac{2}{3}\sigma y g_0 f_0^{-4/3} - f_0^{-4/3}\int_0^t h(s)ds - \tfrac{2}{3}\sigma t g_0^{2} f_0^{-7/3} \\ &\quad - \sqrt{-\sigma/3}\,(y f_0^{-2/3} - t g_0 f_0^{-5/3}) - \tfrac{2}{3}\sigma t g_0^{2} f_0^{-7/3} - \sqrt{-\sigma/3}\,(y f_0^{-2/3} - t g_0 f_0^{-5/3}) + g_2/g_1\Big], \end{aligned} \tag{2.29a}$$

where $P_I(\xi)$ is the first Painlevé transcendent function, and

$$\begin{aligned} u(t,x,y) &= -2 f_0^{-2/3}\,\wp(\chi, g_2, g_3) + \tfrac{2}{9}\sigma f_0^{-2} g_0^{2} + \tfrac{2}{3} f_0^{-1} h(t) + \tfrac{1}{6}, \\ \chi &= x f_0^{-1/3} + \tfrac{2}{3}\sigma y g_0 f_0^{-4/3} - f_0^{-4/3}\int_0^t h(s)ds - \tfrac{2}{3}\sigma t g_0^{2} f_0^{-7/3} - \sqrt{-\sigma/3}\,(y f_0^{-2/3} - t g_0 f_0^{-5/3}), \end{aligned} \tag{2.29b}$$

where $\wp(\chi, g_2, g_3)$ is the Weierstrass elliptic function. Notice that certain restrictions must be imposed upon the constants g_0, g_1, g_2, and g_3 in order to ensure that we shall obtain real solutions of the KP equation. In particular the above solutions only can be real for the KPII equation, i.e. when $\sigma \equiv -1$. In addition, *lump-type* solutions of the KPII equation are obtained from (2.29b) when $g_2 = g_3 = 0$, since we then have $\wp(\chi, 0, 0) = \chi^{-2}$. From (2.25d) we see that if we start from a solution $q(\xi,\eta)$ of the once-differentiated KdV equation that is bounded, the corresponding solution of the KP equation will share the same property if, and only if, the functions g(t) and h(t) are constants, say $g_0 \neq 0$ and h_0. This solution is given by

$$\begin{aligned} u(t,x,y) &= g_0^{-1/2} q(\xi,\eta) - \tfrac{1}{2}\sigma g_0^{-2} h_0^{2}, \\ \xi &= g_0^{-1/2}(x - y h_0/g_0), \quad \eta = g_0^{-3/2} t. \end{aligned} \tag{2.30}$$

Solutions of the once-differentiated KdV equation (2.25e) can be *rotated* into solutions of the KP equation. This property has been extensively used by Segur (see FIN1, SEGU1) in his construction of KP solutions of genus 1 (see Introduction), which he obtains by rotating cnoidal wave solutions of the KdV equation according to (2.30). Of course, soliton and similarity solutions of the KdV equation also may be transformed into solutions of the KP equation, having physical significance. For example, the similarity solution of the KdV equation (BOI2)

$$q(\xi,\eta) = -(\tfrac{3}{4}t)^{-2/3}[V^2(z,\mu) + V_z(z,\mu)] ,$$

where $z = 2^{2/3}\xi(3\eta)^{-1/3}$, and $V(z,\mu)$ is the second Painlevé transcendent function satisfying

$$V_{zz}(z,\mu) = 2V^3(z,\mu) + zV(z,\mu) + \mu - \tfrac{1}{2} ,$$

gives rise to solution of the KP equation via (2.30). In view of the transformations (2.25) and (2.26) it is quite possible to obtain bounded solutions of the KP equation from solutions of the Boussinesq or KdV equations that diverge asymptotically. One way of obtaining such solutions is to perform a different choice of symmetry variables than the ones described above and thus to reduce to a different partial differential equation in two variables. We have proven that any choice is equivalent under the action of the KP symmetry group to one of the three choices discussed above. It is however possible for the group to transform bounded solutions of an equation into unbounded solutions of an equivalent one. For example, let us choose the following symmetry variables:

$$\tau = \tfrac{1}{9}\ln[(g_0 + tg_1)/g_0] ,$$
$$\xi = [g_1/(g_0 + tg_1)]^{1/3}\left[x - \frac{3\sigma h_0^2}{4g_1(g_0 + tg_1)} - \frac{yh_0}{g_0 + tg_1} + \frac{\sigma y^2 g_1}{3(g_0 + tg_1)}\right] , \tag{2.31a}$$

where g_0, g_1, and h_0 are constants; this choice is different from, but equivalent to, (2.26). It is easy to show that

$$u(t,x,y) = -[g_1/(g_0 + tg_1)]^{2/3}\vartheta(\xi,\tau) \tag{2.31b}$$

will satisfy the KP equation if, and only if, $\vartheta(\xi,\tau)$ satisfies the following nonlinear evolution equation:

$$\vartheta_\tau + \vartheta_{\xi\xi\xi} - 6\vartheta\vartheta_\xi - 4\xi\vartheta_\xi - 2\vartheta = 0 . \tag{2.31c}$$

Bounded solutions of this equation have been obtained by Calogero and Degasperis (see CAL1) using the inverse scattering method. They are given by

$$\begin{aligned}
&\vartheta(\xi,\tau) = \vartheta^*(\xi - z^*(\tau), \rho^*(\tau)), \\
&\vartheta^*(p,q) = 2q[2Ai'(p)Ai(p) + q\{Ai(p)\}^4 G(p,q)]G(p,q), \\
&G(p,q) = (1 + q[Ai'(p)]^2 - pq[Ai(p)]^2)^{-2}, \\
&z^*(\tau) = z_0^* e^{-4\tau}, \quad \rho^*(\tau) = \rho_0^* e^{-4\tau},
\end{aligned} \tag{2.31d}$$

where Ai(q) denotes an Airy function. The solutions of the KP equation defined by (2.31) contain the solutions obtained by Nakamura (see NAK3) as a special case.

We have shown that the use of one-dimensional subalgebras of L_{KP} makes it possible to generate large classes of solutions of the KP equation. For this particular equation the higher dimensional subalgebras are of lesser use. Indeed, consider the two-dimensional subalgebras, all of which are listed in (2.18). Performing symmetry reduction with any such algebra $N = \{R_1, R_2\}$, we obtain, when looking for functional invariants, a system of two linear first-order partial differential equations:

$$R_1 I(t,x,y,u) = 0, \quad R_2 I(t,x,y,u) = 0. \tag{2.32}$$

Typically this system yields a *single* symmetry variable ξ and an expression for the solution of the KP equation, similar to (2.23), having the form

$$u(t,x,y) = \alpha(t,x,y)q(\xi) + \beta(t,x,y), \tag{2.33}$$

where α, β, and $\xi(t,x,y)$ are explicitly known. Substituting (2.33) into the KP equation we actually obtain an ordinary differential equation for $q(\xi)$. The solution (2.33) can then be transformed, by a general transformation of the KP symmetry group, into a more general solution. However, one of the two operators in (2.32), say R_1, will always coincide with one of those used above to reduce the KP equation to the Boussinesq equation, (2.26b) the once-differentiated KdV equation (2.25e), or the linear equation (2.24d). The other operator R_2 then provides a further reduction. In other words, we do not obtain new solutions but particular cases of those discussed above. As for symmetry reduction by three-dimensional subalgebras of L_{KP}, it is clear that we only obtain trivialities, namely that u(t,x,y) adopts a constant form or the like. Let us only give one example of a reduction by a two-dimensional subalgebra, namely $L_{2,1}$; this particular reduction neatly yields the solutions expressible in terms of the first Painlevé transcendent and Weierstrass elliptic functions. Equations (2.32) then writes

$$\partial_t I(t,x,y,u) = 0\ , \qquad \partial_y I(t,x,y,u) = 0\ . \tag{2.34a}$$

From these we deduce that $u(t,x,y) = -2q(x)$, where the factor "-2" is introduced for convenience. Upon substituting this into the KP equation, we obtain an ordinary differential equation which can be integrated twice to this other one:

$$q'' = 6q^2 + \mu q + \nu\ . \tag{2.34b}$$

Three independent types of pairs (μ,ν) are to be considered. The first one is $(\mu = 0,\ \nu = 0)$; equation (2.34b) then reduces to $q'' = 6q^2$ and has the solution $\wp(x - k_1, 0, k_2)$, where k_1 and k_2 are arbitrary real constants. As a second case we consider $\mu = 0$ and $\nu \neq 0$; equation (2.34b) is then amenable to the form $q'' = 6q^2 + \frac{1}{2}$ and has the solution $\wp(x - k_1, 1, k_2)$, where k_1 and k_2 are again arbitrary real constants. Finally, the third case corresponds to choosing $\mu \neq 0$; equation (2.34b) then reduces to the well-known equation $q'' = 6q^2 + x$ which has the solution $P_I(x)$, i.e. the first Painlevé transcendent function.

In part 2 of this chapter, which follows, we shall reconsider the problem of symmetry reduction for the simultaneous system made of the KP equation (in potential form) together with its Bäcklund transformation.

II. A generalized Bäcklund transformation and its simultaneous symmetry reduction with the Kadomtsev-Petviashvili equation

As previously mentioned, the purpose of this second part is to combine together two of the important features of the KP equation. The first of these is the existence of an infinite-dimensional symmetry group G_{KP} of point transformations leaving the equation invariant; this is essentially what we have studied in part I above. The second feature is the existence of a Bäcklund transformation for the KP equation (see CHE1 and LEV1) that can be used for the purpose of generating new solutions from known ones; in particular, the soliton solutions of the KP equation are obtained this way from the trivial solution. We shall begin by showing how the symmetry group of the equation can be used to actually generalize the Bäcklund transformation in a non-trivial manner, in particular by introducing two arbitrary functions of the time variable "t" in its formulation, and ultimately we shall show how to practically use this generalization for constructing a variety of solutions, specifically by applying the symmetry reduction technique, as before, but with the essential difference that, here, we shall reduce *both* the equation *and* its associated Bäcklund transformation. As seen in the Introduction, a Bäcklund transformation is always written as a set of relations which involve the derivatives of the concerned functions. In our case these relations form an over-determined system of partial differential equations for a function $u(t,x,y)$ such that its x-derivative $w(t,x,y) = u_x(t,x,y)$ does satisfy the KP equation. This system is rather tedious to deal with and consequently, it will be to our advantage to rather deal with an equation which is closely related to it, namely the so-called *potential Kadomtsev-Petviashvili* (PKP) equation which we write as

$$[4u_t + 3u_x{}^2 + u_{xxx}]_x + 3s^2u_{yy} = 0, \tag{2.35}$$

where $\sigma = s^2 = \pm 1$ is defined as before. Thus we may get solutions of the KP equation (2.1) by simply differentiating solutions of the PKP equation (2.35) with respect to the variable x. The Bäcklund transformation associated to this PKP equation can be written in the following form (see LEV1, BOI3):

$$
\begin{aligned}
&s(u-v)_y - (u+v)_{xx} - (u-v)(u-v)_x = 0, \\
&4(u-v)_t + (u-v)_{xxx} + 3s(u+v)_{xy} + 3(u-v)(u+v)_{xx} \\
&\qquad\qquad + 3(u-v)_x\,[(u-v)^2 + (u+v)_x] = 0,
\end{aligned}
\tag{2.36}
$$

If one of the functions in (2.36), say $u(t,x,y)$, is a solution of the PKP equation, then the two equations in (2.36) are compatible only if the other function $v(t,x,y)$ is also a solution. Thus equations (2.36) *transform* a solution u into a solution v; see Chapter one for the exact meaning of that. If we set $u(t,x,y) = 0$, then $v(t,x,y)$ will typically be a soliton-type solution.

The symmetry algebra L_{PKP} of the PKP equation exhibits some similarity, in its structure, with the symmetry algebra L_{KP} of the KP equation. We obtained it, using the same computer package as before (see CHA1); a general element $V \in L_{PKP}$ has the form

$$
\begin{aligned}
V &= X(f) + Y(g) + Z(h) + P(k) + Q(l), \\
X(f) &= f\partial_t + \tfrac{2}{3}yf'\partial_y + [\tfrac{1}{3}xf' - \tfrac{2}{9}s^2y^2f'']\partial_x \\
&\qquad - [\tfrac{1}{3}uf' - \tfrac{1}{9}x^2f'' + \tfrac{4}{27}s^2xy^2f''' - \tfrac{4}{243}y^4f'''']\partial_u, \\
Y(g) &= g\partial_y - \tfrac{2}{3}s^2yg'\partial_x - [\tfrac{4}{9}s^2xyg'' - \tfrac{4}{81}y^3g''']\partial_u, \\
Z(h) &= h\partial_x + [\tfrac{2}{3}xh' - \tfrac{4}{9}s^2y^2h'']\partial_u, \\
P(k) &= yk\partial_u, \\
Q(l) &= l\partial_u,
\end{aligned}
\tag{2.37}
$$

where $f(t)$, $g(t)$, $h(t)$, $k(t)$, and $l(t)$ are arbitrary smooth functions over some open neighbourhood of 0. The commutation rules in L_{PKP} are easily calculated; one obtains

$$
\begin{aligned}
&[X(f_1), X(f_2)] = X(f_1f_2' - f_1'f_2), && [X(f), Y(g)] = Y(fg' - \tfrac{2}{3}f'g), \\
&[X(f), Z(h)] = Z(fh' - \tfrac{1}{3}f'h), && [X(f), P(k)] = P(fk' + f'k), \\
&[X(f), Q(l)] = Q(fl' + \tfrac{1}{3}f'l), && [Y(g_1), Y(g_2)] = -\tfrac{2}{3}s^2Z(g_1g_2' - g_1'g_2), \\
&[Y(g), Z(h)] = -\tfrac{4}{9}s^2P(2gh'' + g'h' - g''h), &&
\end{aligned}
\tag{2.38}
$$

$$
\begin{aligned}
&[Y(g), P(k)] = Y(gk), && [Y(g), Q(l)] = 0, \\
&[Z(h_1), Z(h_2)] = \tfrac{2}{3}Q(h_1 h_2' - h_1' h_2), && [Z(h), P(k)] = 0, \\
&[Z(h), Q(l)] = 0, && [P(k_1), P(k_2)] = 0, \\
&[P(k), Q(l)] = 0, && [Q(l_1), Q(l_2)] = 0.
\end{aligned}
\tag{2.38}
$$

The subalgebra $\{X(f), Y(g), Z(h)\}$, with $k(t) = l(t) = 0$, is to be compared with L_{KP}; the additional vector fields $P(k)$ and $Q(l)$ are characteristic of the fact that we are considering the *potential* form of the KP equation. The general action of the corresponding symmetry group G_{PKP} is quite complicated and we do not need it for our purposes, so we shall not present it. However we list the action of each one-parameter subgroup associated to the subalgebras spanned by $\{X(f)\}$, $\{Y(g)\}$, $\{Z(h)\}$, and the two-parameter subgroup associated to the subalgebra $\{P(k), Q(l)\}$; λ and μ stand for real group parameters, independent of t, x, y, and u.

1) $\{X(f)\}$:

$$
\begin{aligned}
&t^* = \Phi^{-1}[\lambda + \Phi(t)], && \Phi(t) = \int_0^t f(z)^{-1} dz, \\
&y^* = e^{-2H(t)/3} y, && H = \ln[f(t)/f(t^*)], \\
&x^* = e^{-H(t)/3} x + \tfrac{2}{9} s^2 y^2 H''(t) e^{-4H(t)/3}, && \\
&u^*(t^*, x^*, y^*) = e^{H(t)/3} u[t(t^*), x(t^*, x^*, y^*), y(t^*, y^*)] - \tfrac{1}{9}(x^*)^2 H'(t^*) && \\
&\qquad + \tfrac{4}{81} s^2 x^* (y^*)^2 [3H''(t^*) - H'(t^*)^2] && \\
&\qquad - \tfrac{4}{2187}[9H'''(t^*) - 9H'(t^*)H''(t^*) + H'(t^*)^3], &&
\end{aligned}
\tag{2.39a}
$$

where:

$$
\begin{aligned}
t(t^*) &= \Phi^{-1}[\Phi(t^*) - \lambda], \\
y(t^*, y^*) &= e^{2H/3} y^*, \\
x(t^*, x^*, y^*) &= e^{H/3}[x^* - \tfrac{2}{9} s^2 (y^*)^2.
\end{aligned}
$$

In particular, setting $f(t) = 1$ or $f(t) = t$, we obtain time translations or space-time dilatations, respectively.

2) $\{Y(g)\}$:

$$t^* = t,$$

$$y^* = y + \lambda g,$$

$$x^* = x - \tfrac{2}{3}\lambda s^2 y g' - \tfrac{1}{3}\lambda^2 s^2 g g', \tag{2.39b}$$

$$u^*(t^*, x^*, y^*) = u[t^*, x^* + \tfrac{2}{3}\lambda s^2 y^* g' - \tfrac{1}{3}\lambda^2 s^2 g g', y^* - \lambda g]$$

$$- \tfrac{2}{9}\lambda s^2 x^*(2y^* + \lambda g)g'' + \tfrac{8}{81}\lambda (y^*)^3 g''' - \tfrac{4}{27}\lambda^2 (y^*)^2 (g'g'' + gg''')$$

$$+ (\tfrac{4}{27}y^* - \tfrac{1}{27}\lambda g)\lambda^3 g(g'g'' + \tfrac{2}{3}gg'''),$$

where $g = g(t) = g(t^*)$. In particular, we can obtain translations in the y-direction and so-called *quasirotations* by putting $g(t) = 1$ and $g(t) = t$, respectively.

3) $\{Z(h)\}$:

$$t^* = t,$$

$$y^* = y,$$

$$x^* = x + \lambda h, \tag{2.39c}$$

$$u^*(t^*, x^*, y^*) = u[t^*, x^* - \lambda h, y^*] + \tfrac{2}{3}\lambda x^* h' - \tfrac{4}{3}\lambda s^2 (y^*)^2 h'' - \tfrac{1}{3}\lambda^2 h h',$$

where $h = h(t) = h(t^*)$ In particular, $h(t) = 1$ and $h(t) = t$ correspond to translational invariance and Galilei invariance in the x-direction, respectively.

4) $\{P(k), Q(l)\}$:

$$t^* = t,$$

$$y^* = y,$$

$$x^* = x, \tag{2.39d}$$

$$u^*(t^*, x^*, y^*) = u t^*, x^*, y^*) + \lambda y^* k(t) + \mu l(t).$$

This last case is characteristic of the potential form of the KP equation. From the commutation relations (2.38) we can analyse the structure of the PKP symmetry algebra. We are not deeply interested in that since, once again, from the KP equation point of view, we gain nothing new with respect to the results already obtained in part I. Let us just mention a few points. First, L_{PKP} admits a Levi decomposition, just as L_{KP} does, in the form $L_{PKP} = S \oplus R$, where $S = \{X(f(t))\}$ is simple, and $R = \{Y(g(t)), Z(h(t)), P(k(t)), Q(l(t))\}$ is a nilpotent ideal. The other point worth mentioning is that we find again a loop structure, obtained in the same manner as precedently. Indeed, developing the functions f(t), g(t), h(t), k(t), and l(t) into formal Laurent series, we get an infinite-dimensional subalgebra L_π'. We consider the set of vector fields formed by taking the coefficients of the diverse powers of the variable t in the expressions for X(f), Y(g), Z(h), P(k), and Q(l); this set is a 13-dimensional Lie algebra, with a 12-dimensional nilpotent subalgebra (the nilradical) and an 8-dimensional maximal abelian ideal, and can be embedded into the Lie algebra sl(9,**R**). This implies, in analogy with a similar result for L_π, that L_π' is then a subalgebra of the affine loop algebra $A_9^{(1)}$ without its centre: $L_\pi' \subset \{\mathbf{R}[t,t^{-1}] \otimes sl(9,\mathbf{R})\} \oplus \mathbf{R}[t,t^{-1}]d/dt$.

We now study how G_{PKP} acts on the Bäcklund transformation (2.36), by composing all the transformations from (2.39); by construction, all these transformations leave the PKP equation invariant. The subgroup corresponding to $\{Z(h), P(k), Q(l)\}$ also leaves invariant the Bäcklund transformation. However the group action induced by Y(g) and X(f) does not. In fact the subgroup corresponding to these two infinitesimal generators change (2.36) into the following expressions.

$$
\begin{aligned}
&s(u-v+p)_y - (u+v)_{xx} - (u-v+p)(u-v+p)_x = 0, \\
&4(u-v+p)_t + (u-v+p)_{xxx} + 3s(u+v)_{xy} + 3(u-v+p)(u+v)_{xx} \qquad (2.40a) \\
&\qquad + 3(u-v+p)_x[(u-v+p)^2 + (u+v)_x] = 0,
\end{aligned}
$$

where $p = R(t) + yS(t)$, and R(t), S(t) are two arbitrary functions of the time variable that can be expressed completely in terms of the arbitrary functions f(t) and g(t) in the vector fields X(f) and Y(g), respectively:

$$
\begin{aligned}
&R(t) = -\tfrac{2}{3}\lambda s^3 g'(t^*(t))e^{-H/3}, \\
&S(t) = \tfrac{4}{9}s^3 H'e^{-H}, \qquad (2.40b) \\
&H = \ln[f(t)/f(t^*(t))].
\end{aligned}
$$

A point to note is that this introduction of arbitrary functions into the expressions for the Bäcklund transformation by means of Lie transformations that leave the original equation invariant, but actually modify the Bäcklund transformation, is completely analogous to the introduction of arbitrary parameters in the Bäcklund transformations for some integrable equations in 1+1 dimensions. Thus the Lorentz invariance of the Sine-Gordon equation gives rise to a real parameter in the corresponding Bäcklund transformation; similarly, the Galilei and dilatation invariances of the nonlinear cubic Schrödinger equation are jointly at the origin of a complex constant occuring in the Bäcklund transformation associated with this equation (see HAR1). A second point of importance is that a permutabilty theorem (see Chapter one) does hold for the Bäcklund transformation (2.40a) (see BOI3 and NAK1). Starting with a solution $u_0(t,x,y)$ we can perform two chains of consecutive transformations depending upon two arbitrary functions $p_1(t)$ and $p_2(t)$:

$$u_0 \xrightarrow{B(p_1)} u_1 \xrightarrow{B(p_2)} u\,,$$

$$u_0 \xrightarrow{B(p_2)} u_2 \xrightarrow{B(p_1)} u'\,.$$

The permutability theorem tells us that $u' = u$, identically. Writing down all four transformations corresponding to the above diagrams and eliminating the y and t derivatives, we obtain that u is uniquely expressed in terms of u_0, u_1, and u_2 as

$$\left.\begin{aligned} &u(t,x,y) = u_1 + u_2 - u_0 + \frac{(u_1 - u_2)_x}{u_1 - u_2 + p_1 - p_2}\,, \\ &p_i = R_i(t) + yS_i(t)\,, \qquad i = 1,2, \end{aligned}\right\} \tag{2.41}$$

where u_0 is an arbitrary solution of the PKP equation, and u_1 and u_2 are generated from u_0 by Bäcklund transformations with functional labels p_1 and p_2, respectively. Formula (2.41) generalizes a result of ref. BOI3, having the same form, with p_1 and p_2 being real constants.

We now set $v = 0$ identically in the Bäcklund transformation (2.40a),

$$\begin{aligned} &s(u+p)_y - u_{xx} - (u+p)(u+p)_x = 0\,, \\ &4(u+p)_t + (u+p)_{xxx} + 3su_{xy} + 3(u+p)u_{xx} + 3(u+p)_x[(u+p)^2 + u_x] = 0\,, \end{aligned} \tag{2.42}$$

and look for solutions. Since we are not able to solve directly this overdetermined system (when brought to a system for the function $u(t,x,y)$ alone, this is even more complicated than the PKP equation itself), we shall therefore proceed in an indirect manner through the use of the symmetry reduction technique. First we find the *joint* symmetry algebra L_J of the PKP equation with its associated Bäcklund transformation (2.42) (in fact the first equation suffices, because of the built-in compatibility between the two "components" of the transformation). This is easy to do, using again the MACSYMA package, and we obtain the following set of generators,

$$X_1 = \partial_x, \qquad X_2 = \partial_y - S(t)\partial_u,$$
$$X_3 = \partial_t - [R'(t) + yS'(t)]\partial_u, \qquad X_4 = t\partial_y - \tfrac{2}{3}s^2 y\partial_x - [tS(t) - \tfrac{2}{3}s^3]\partial_u, \tag{2.43}$$

with commutation relations

$$[X_1, X_2] = [X_1, X_3] = [X_1, X_4] = [X_2, X_3] = 0,$$
$$[X_2, X_4] = -\tfrac{2}{3}s^2 X_1, \quad [X_3, X_4] = X_2;$$

$\{X_1, X_2, X_3\}$ is an abelian subalgebra. The functions $R(t)$ and $S(t)$ are defined as before. L_J is a subalgebra of L_{PKP}; indeed we have

$$X_1 = Z(1), \qquad X_2 = Y(1) - Q(S),$$
$$X_3 = X(1) - Q(R') - P(S'), \qquad X_4 = Y(t) - Q(tS - \tfrac{2}{3}s^3).$$

The action of the corresponding symmetry group G_J is easily constructed. One finds the following actions for the one-parameter subgroups.

$$X_1: \quad t^* = t, \quad x^* = x + \lambda, \quad y^* = y, \quad u^*(t^*, x^* - \lambda, y^*);$$
$$X_2: \quad t^* = t, \quad x^* = x, \quad y^* = y + \lambda, \quad u^*(t^*, x^*, y^* - \lambda) - \lambda S(t^*);$$
$$X_3: \quad t^* = t + \lambda, \quad x^* = x, \quad y^* = y,$$
$$u^*(t^* - \lambda, x^*, y^*) - [R(t^*) + y^*S(t^*)] + [R(t^* - \lambda) + y^*S(t^* - \lambda)];$$
$$X_4: \quad t^* = t, \quad x^* = x - \tfrac{2}{3}(\lambda y + \tfrac{1}{2}\lambda^2 t), \quad y^* = y + \lambda t,$$
$$u^*[t^*, x^* + \tfrac{2}{3}(\lambda y^* - \tfrac{1}{2}\lambda^2 t^*); y^* - \lambda t^*] - \lambda(t^* - \tfrac{2}{3}s^3).$$

Arbitrary one- and two-dimensional subalgebras of the Lie algebra (2.43) are easily shown to be conjugate under the adjoint action of G_j to precisely one of the following algebras, respectively (with $a \in R$):

1-dimensional subalgebras : $M_{1,1}{}^{a} = \{X_4 + aX_3\}$,
$M_{1,2}{}^{a} = \{X_3 + aX_1\}$,
$M_{1,3} = \{X_2\}$,
$M_{1,4} = \{X_1\}$;

2-dimensional subalgebras : $M_{2,1}{}^{a} = \{X_4 + aX_3, X_1\}$,
$M_{2,2}{}^{a} = \{X_3 + aX_1, X_2\}$,
$M_{2,3} = \{X_3, X_1\}$,
$M_{2,4} = \{X_2, X_1\}$.

We shall perform symmetry reduction of the PKP equation simultaneously with the Bäcklund transformation, under the subgroups corresponding to the one-dimensional subalgebras in the above list (symmetry reduction under two-dimensional subgroups does not provide us with anything new), using a representative Y of each class. In each case, symmetry reduction yields a solution in the form

$$u(t,x,y) = \alpha F(\xi,\eta) + \beta, \tag{2.44}$$

where the symmetry variables ξ and η, as well as the functions α and β are functions of the variables t, x, and y, completely determined by invariance requirements.

1) $Y = X_1$. We find $\xi = y$, $\eta = t$, and $\alpha = 1$, $\beta = 0$; thus (2.44) becomes $u(t,x,y) = F(y,t)$. The PKP equation is reduced to the linear equation $F_{yy} = 0$, and the (first component of the) Bäcklund transformation to $F_y + S(t) = 0$, the function $S(t)$ being defined as above. These two reduced equations are easily solved and we finally obtain

$$u(t,x,y) = H(t) - yS(t), \tag{2.45}$$

where H(t) is an arbitrary function.

2) $Y = X_2 + aX_1$, $a \in \mathbf{R}$. This representative is conjugate to X_2. Here, we first find that (2.44) takes the form $u(t,x,y) = F(\xi,t) - yS(t)$, where $\xi = x - ay$. Substituting this into the Bäcklund transformation, we see that the function $F(\xi,t)$ must satisfy a Riccati equation:

$$F_\xi + \tfrac{1}{2}F^2 + RF + H(t) = 0. \tag{2.46a}$$

where H(t) is an arbitrary function. Solving this equation, resubstituting the result into the PKP equation, and adjusting so that the latter be satisfied, we finally find that

$$u(t,x,y) = \kappa \tanh\left\{ \tfrac{1}{2}\kappa\left[x - ay - \left(\tfrac{3}{4}a^2s^2\kappa^{-2} + \tfrac{1}{4}\kappa^2\right)t + b\right]\right\} - R(t) - yS(t), \tag{2.46b}$$

where a, b, and κ are constants, solves the PKP equation; this solutions represents a "kink". The corresponding solution of the KP equation is the usual soliton solution:

$$w(t,x,y) = u_x = \tfrac{1}{2}\kappa^2 \operatorname{sech}^2\left\{ \tfrac{1}{2}\kappa\left[x - ay - \left(\tfrac{3}{4}a^2s^2\kappa^{-2} + \tfrac{1}{4}\kappa^2\right)t + b\right]\right\}. \tag{2.46c}$$

3) $Y = X_4 + aX_2 + bX_1$, $a,b \in \mathbf{R}$. This representative is conjugate to X_4. Here, we first find that (2.44) takes the form

$$\begin{aligned} u(t,x,y) &= [3(t+a)]^{-1/3}F(\xi,t) - yS(t) + 2s^3y[3(t+a)]^{-1}, \\ \xi &= \left[x + (s^2y^2 - 3by)[3(t+a)]^{-1}\right][3(t+a)]^{-1/2}. \end{aligned} \tag{2.47a}$$

The Bäcklund transformation and the PKP equation respectively reduce to

$$F_\xi = \tfrac{1}{2}F^2 - [3(t+a)]^{1/3}\left[R(t) + bs(t+a)^{-1}\right]F - 2[\xi + h(t)], \tag{2.47b}$$

$$F_t = -\tfrac{27}{4}b^2s^2[3(t+a)]^{-7/3}F_\xi - \tfrac{1}{12}(t+a)^{-1}[F_{\xi\xi\xi} - 3F_\xi{}^2 - 4\xi F_\xi + 2F] - \Phi(t), \tag{2.47c}$$

where h(t) and $\Phi(t)$ are functions of t. Their form is determined by requiring that equations (2.47b) and (2.47c) be compatible. Once this is done we can then solve (2.47b,c) and hence the PKP equation in terms of Airy functions (see ABR1). Notice, in this context, that the Riccati equation (2.47b) can be converted into a Schrödinger equation with a linear (in the variable ξ) potential and that (2.47c), upon the substitution $F = v_\xi$, reduces to a member of the class of nonlinear evolution equations, studied by Calogero and Degasperis (see CAL1), containing the cylindrical Korteweg-de Vries equation. The corresponding solution of the PKP equation is thus given by (2.47a) with

$$\begin{aligned} &F = -2W_\xi W^{-1} + [3(t+a)]^{-1/3}\left[R(t) + bs(t+a)^{-1}\right], \\ &W = \mu Ai(z) + \nu Bi(z), \\ &z = \xi + \kappa\left[\frac{\tau + a}{t+a}\right]^{1/3} + \frac{3^{2/3}b^2s^2}{4\sqrt{t+a}}\left[\frac{1}{t+a} - \frac{1}{\tau+a}\right], \end{aligned} \tag{2.47d}$$

where $Ai(z)$ and $Bi(z)$ are two independent solutions of the Airy equation $W_{zz} - zW = 0$, and μ, ν, a, b, κ, and τ are arbitrary constants.

4) $Y = X_3 + aX_1 + bX_2$, $a,b \in \mathbf{R}$. This representative is conjugate to $X_3 + a_0X_1$. In this case, it proves convenient to write (2.44) in the form

$$\begin{aligned} &u(t,x,y) = 2W_\xi(\xi,\eta)/W - yS(t) - R(t), \\ &\xi = x - at, \quad \eta = y - bt. \end{aligned} \tag{2.48a}$$

where the Bäcklund transformation and the PKP equation imply that W satisfies the heat equation and also a third-order linear equation:

$$W_{\xi\xi} - sW_\eta = 0, \tag{2.48b}$$

$$W_{\xi\xi\xi} - bs^3W_{\xi\xi} - aW_\xi = [\xi h_1(\eta) + h_2(\eta)]W, \tag{2.48c}$$

where $h_1(\eta)$ and $h_2(\eta)$ are arbitrary functions. In order to obtain analytical solutions of (2.48) we consider the special case $h_1(\eta) = 0$, $h_2(\eta) =$ constant. The system (2.48) then has constant coefficients and can be solved to yield three distinct types of solutions, depending on whether the characteristic equation for (2.48c) has 3, 2, or 1 distinct roots. These solutions are, respectively

$$W(\xi,\eta) = \sum_{i=1}^{3} A_i e^{k_i(\xi + s^3k_i\eta)}, \tag{2.49a}$$

$$W(\xi,\eta) = A_1 e^{k_1(\xi + s^3 k_1 \eta)} + [A_2 + A_3(\xi + 2s^3 k_2 \eta)]\, e^{k_2(\xi + s^3 k_2 \eta)}, \tag{2.49b}$$

$$W(\xi,\eta) = \{A_1 + A_2(\xi + 2s^3 k\eta) + A_3[(\xi + 2s^3 k\eta)^2 + 2s^3\eta]\}\, e^{k(\xi + s^3 k\eta)}, \tag{2.49c}$$

where A_i, k_i, and k are constants. The corresponding solutions of the KP equation are respectively given by

$$w(\xi,\eta) = 2\sum_{i<j}^{3} A_i A_j (k_i - k_j)^2 e^{\alpha_i + \alpha_j} \Big/ \Big(\sum_i A_i e^{\alpha_i}\Big)^2, \tag{2.50a}$$

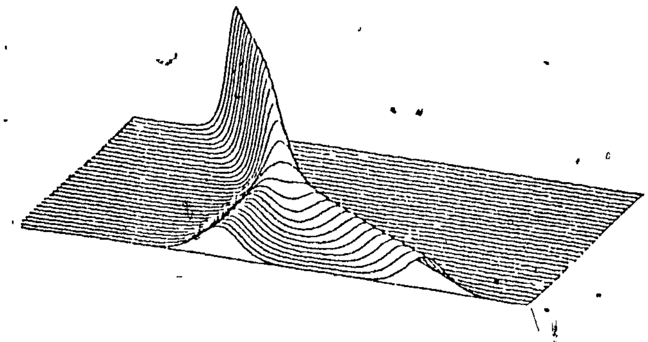
$$w(\xi,\eta) = \frac{2(k_2 - k_1)A_1\{2A_3 + (k_2 - k_1)[A_2 + A_3\beta_2]\}e^{\alpha_1 + \alpha_2} - 2A_3{}^2 e^{2\alpha_2}}{\{A_1 e^{\alpha_1} + (A_2 + A_3\beta_2)e^{\alpha_2}\}^2} \tag{2.50b}$$

$$w(\xi,\eta) = \frac{2A_3[A_1 + A_2\beta + A_3(\beta^2 + 2s^3\eta)] - 2(A_2 + 2A_3\beta)^2}{\{A_1 + A_2\beta + A_3(\beta^2 + 2s^3\eta)\}^2}, \tag{2.50c}$$

where $\alpha_i = k_i(\xi + s^3 k_i \eta)$, $\beta_i = \xi + 2s^3 k_i \eta$, and ξ,η are defined as in (2.48a). The expression (2.50a) leads to interesting solutions (see Figure 1 below) that we term *splittons*. Such a solution is constituted of three connected solitary branches whose relative orientations are preserved through time; thus this is a "monoblock" object propagating in the physical xy-plane. These splittons are related in some way to so-called *soliton resonances*, obtained as special cases of two-soliton solutions, by several authors for the Boussinesq equation, the KP equation and others (see HIR3, OHK1, MUS1). We point out that we obtained these solutions by applying a Bäcklund transformation to a zero solution; they are hence on the same footing as single usual solitons. A quite similar phenomenon has been observed for some 1+1 dimensional nonlinear evolution equations (see AIY1 and AIY2). The expression (2.50c) leads to singular solutions of the KP equation.

Figure 1

The splitton solution, as given by formula (2.50b) with $A_1 = A_2 = A_3 = 1$, $k_1 = -1, k_2 = 1, k_3 = 3$, $\xi = x - t, \eta = y - t$. The plot gives the splitton at time $t = 0$.



5) $Y = X_4 + aX_3 + bX_2 + cX_1$, $a,b,c \in \mathbf{R}$. This representative is conjugate to $X_4 + a_0X_3$. In this last case, invariance under the subgroup corresponding to Y yields

$$u(t,x,y) = 2W_\xi(\xi,\eta)/W - yS(t) - R(t) + \tfrac{2}{3}a^{-1}s^3t,$$
$$\xi = x - ct/a + \tfrac{2}{3}a^{-1}s^2yt - \tfrac{2}{9}a^{-2}s^2t^3, \tag{2.51a}$$
$$\eta = y - bt/a - \tfrac{1}{2}a^{-2}t^2,$$

where the function W again satisfies two linear equations: the heat equation (2.48b) and the following third-order linear equation:

$$W_{\xi\xi\xi} - ba^{-1}s^3W_{\xi\xi} - a^{-1}[c - \tfrac{2}{3}s^2\eta]W_\xi = [\xi h_1(\eta) + h_2(\eta)]W, \tag{2.51b}$$

where $h_1(\eta)$ and $h_2(\eta)$ are arbitrary functions. We have not obtained any analytic solutions of the system (2.48b), (2.51b).

The above results may naturally be combined with those obtained in the first part of the chapter. In fact, we have constructed a net of solutions for the KP equation and one may obtain infinitely many solutions by applying, in any order and in any combination (in principle), Bäcklund transformations, symmetry reduction on the KP equation alone or together with its associated Bäcklund transformation, group or subgroup actions on solutions; in addition, one may also obviously use the results obtained by applying symmetry reduction to the reduced equations associated to the KP equation, e.g., the Boussinesq equation and the Korteweg-de Vries equation.

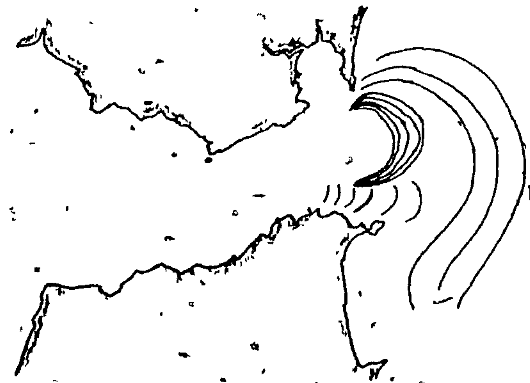
CHAPTER THREE

Integrable nonlinear equations for water waves in channels of varying depth and width

As we saw earlier in the introductory chapter, water waves provide us with excellent examples of solitary waves and experiments clearly show that the shape of such waves (by this we understand the form of the curve defined by the wavecrests), is much dependent on the topography of the oceanic bottom. Over flat bottoms, waves are usually encountered in groups of essentially straight fronts moving with a relatively constant speed; such is the case, for instance, for the wave fronts occuring in the Andaman Sea and described in OSB1. Theories based on a pure constant coefficient KdV equation, or slight perturbation thereof, provide good enough models for this kind of geophysical situation. These exhibit some validity also for waves propagating through channel-like configurations, with possibly some modulation in depth as long as the variations progress along the "axis" of the channel, i.e. along the direction of propagation of the waves. The KdV equation is all the more interesting for the above purpose because, from the analytical perspective, it also is the prototype of a quite large family of so-called nonlinear soliton equations which one can solve exactly through, say, the inverse scattering transform procedure. Straight solitary wave fronts are however a very special phenomenon and it is certainly not uncommon to observe many wave fronts which *do* exhibit some curvature. A typical example is that of the Strait of Gibraltar as shown in Figure 2, below. This is a rough drawing based upon observations made from photographs taken on october 11, 1984, by the U.S. Space Shuttle crew of Mission STS 41-G. The most prominent surface features on the photographs were crests bowing eastward into the Mediterranean Sea. These disturbances (i.e. the lines in Figure 2), are the surface manifestations of a packet of several *internal* waves moving with an approximate speed of 1.5 meters per second, also eastward; their amplitudes are deduced (recall from ART2 that the amplitude of an internal wave is related to that of its associated surface wave) to be more than 200 meters.

Figure 2

Surface manifestation of internal solitary waves emerging from the Strait of Gibraltar into the Mediterranean Sea with an approximate speed of 1.5 m/s.



In order to model such waves, one cannot use the KdV equation anymore because of the one-dimensional character of this equation. This problem can be circumvented by considering its two-dimensional generalization, namely the Kadomtsev-Petviashvili equation. This other equation, however, also treats an idealized situation, namely waves moving in a shallow two-dimensional fluid of *constant* depth. In fact the two-dimensional character gained with existing models based on this equation is only a liberty on the direction of propagation of *straight* wave fronts, even though the KP equation *do* have solutions with nontrivial behaviour in both space directions that can be obtained by using the symmetry group G_{KP}. The alternative for inducing genuine two- dimensionality is to allow the KP equation to have *non-constant*-coefficients; this is also a problem since the obtained equation then becomes non integrable for a general geophysical topography. The purpose of the present chapter is to attack this problem and see if we cannot characterize the *integrable* cases for realistic geophysical situations by examining the applicability of certain nonlinear integrable equations which will be obtained from a more general KP-like equation. Our approach differs from previous works in the following aspects. First, we start from the basic equations of hydrodynamics, but allow for arbitrary vorticity and consider channels (or straits) of arbitrarily varying shape; in principle this means that possible configurations include meandering rivers. Second, we shall consider the propagation of waves, in particular of solitary waves, in straits or channels, somewhat wider than their depth (for instance, note that the width-to-depth ratio for the Strait of Gibraltar is of about 20), characterized by varying depth and width, when this propagation is essentially *non* one-dimensional. It may thus be important to take into account perturbations which propagate in the transverse direction but are of much longer wavelength than the main wave longitudinal wavelength. The model that we shall derive, it must be pointed out, is also valid when the side boundaries are removed (i.e., brought away to infinity) and is thus applicable for wave propagation in plain ocean or other unlimited bodies of fluid of varying depth. Third, our basic approximating hypothesis will be chosen in such a way that we can take as initial condition for our resulting nonlinear equation a solitary wave and thus be able to follow its evolution from the shallow water unidimensional situation in a strait into the deeper and wider open ocean. We also wish to remark that although we restrict the model to *surface* waves for a single layer of constant density fluid, the same ideas could also be applied to stratified fluids so as to be able to describe internal waves; we shall come back to that point later.

In the first section, we begin from the basic equations of hydrodynamics, the Euler system, allowing for arbitrary vorticity, but *no* viscosity (one then has to start out from the Navier-Stokes system and things get more complicated), and arbitrary boundary conditions. We make use of the standard multiscale (or stretching) perturbation theory (see FRI1) and restrict ourselves to the type of situation that leads, in the limit of constant depth, no vorticity, and removed side boundaries, to the pure (by this we mean that its coefficients are pure real constants) KP equation, i.e. the equation for long waves of small amplitudes moving predominantly along a single direction in shallow water. Under quite general assumptions, namely quasi one-dimensional long waves in shallow water, we shall then proceed to derive a rather complicated nonlinear wave equation that we have termed the *generalized Kadomtsev-Petviashvili* (or GKP for short) equation, together with some boundary conditions that must be satisfied on the side boundaries of the channel. Naturally, these conditions are absent if the medium of propagation has infinite extension; these conditions also vanish if the side boundaries vary *very slowly* in the direction of propagation so that, there too, we recover the case of an infinite body of water: in the following, we shall typically refer to this situation as the *no boundaries situation.* The GKP equation differs radically from the pure KP equation in that it features some additional terms, but also in having *variable* coefficients that depend on the space coordinates via the functions describing the depth and the vorticity. The GKP equation is thus generically non integrable. Note, moreover, that the boundary conditions would interfere with the integrability, even for the KP equation itself.

Instead of trying to solve the obtained equation and the associated boundary conditions through some numerical scheme, although this would be of interest by itself, we look for conditions under which it reduces to one of the following *known to be* integrable equations: the pure KP equation, the KdV equation, or the cylindrical KdV (cKdV for short) equation, this is the object of the second section. We do this by performing certain rather general transformations of *both* the dependent and independent variables, and making physical assumptions whenever necessary. In general, it will be seen that integrability is recovered at a price. When the side boundaries are absent, the requirement of integrability will impose certain constraints on the vorticity function as well. When the side boundaries are involved, as in the case of straits, we shall be forced to impose constraints relating the functions describing the depth and the width, and the vorticity functions will then be determined in terms of the boundaries. Moreover, if we wish to reduce the boundary conditions to a form that can be dealt with, satisfied analytically, and that does not interfere with the usual form of integrability, we must, at least in some cases, introduce secular terms in the perturbation expansion. The constraints that we must impose in order to obtain integrable equations can be viewed as predictions. Indeed, the integrable geophysical topographies that they prescribe are just the conditions under which we should expect solitons to be observed in straits or to emerge from straits into oceans. In section 3, we shall consider a few of these integrable topographies and construct the *explicit and exact* solutions that correspond to them. Among these, let us mention the cases when the depth function is quadratic, logarithmic, or hyperbolic-tangent in the longitudinal coordinates. The first two situations could model waves emerging from straits into the sea (the Strait of Gibraltar, for example) and the third one could be applied to waves crossing an area where a more or less steep depression (or elevation) occurs.

The fourth, and final, section deals with a more mathematical topic. It will be spent on the construction and analysis of the conservation laws associated to the GKP equation. In contrast with the pure KP equation which is known to possess an infinite number of local conservation laws, we shall see that the GKP equation only allows for a few. That is not very surprising since this equation is, in some sense, *conditionally* integrable. Moreover, the conserved quantities that we shall derive will generically exhibit some dependence on the geophysical characteristics of the medium of propagation.

I. Derivation of the generalized KP equation and the associated boundary conditions

We consider a channel (or strait) of arbitrarily varying depth and width, with bottom and sides described by a given function $\Xi(x_1, x_2, x_3) = 0$ (see Figure 3). The channel contains a homogeneous incompressible inviscid ideal fluid subject to a gravity force only. This fluid is characterized by a constant density ρ, a pressure field p^*, and an Eulerian velocity field (in the following, *vector* fields will be denoted by *boldface* characters) $\mathbf{v}^* = (v_1^*, v_2^*, v_3^*)$ with its curl given by $\zeta^* = \nabla^* \times \mathbf{v}^*$. This last quantity is the local vorticity, measuring a local rigid-body rotation with angular velocity $\frac{1}{2}|\zeta^*|$. Under these assumptions the mass conservation equation is

$$\nabla^* \cdot \mathbf{v}^* = 0, \tag{3.1a}$$

and the Euler equations of motion governing the dynamics of the flow are

$$\rho \partial \mathbf{v}^*/\partial t^* + \rho(\mathbf{v}^* \cdot \nabla^*)\mathbf{v}^* + \nabla^* p^* + \rho \mathbf{g} = 0, \tag{3.1b}$$

where $\mathbf{g} = (0,0,g)$, g being the gravity constant. The symbol ∇^* stands for the gradient operator in the variables x_1, x_2, and x_3 (in that order). Equations (3.1a) and (3.1b) form a nonlinear coupled system defining the velocity field and the pressure within the fluid. In addition to these dynamical equations, certain boundary conditions must be satisfied. First, the boundary layer on the free surface does not experience any mechanical stress; this implies no pressure jump and vanishing tangential velocity on the free surface. Thus we have (the variable η stands for the elevation of the fluid level with respect to its undisturbed height),

$$\left. \begin{aligned} & p^* \Big|_{x_3 = \eta^*(x_1, x_2, t^*)} = 0, \\ & \left(\frac{\partial \eta^*}{\partial t^*} + v_1^* \frac{\partial \eta^*}{\partial x_1} + v_2^* \frac{\partial \eta^*}{\partial x_2} - v_2^* \right) \Bigg|_{x_3 = \eta^*(x_1, x_2, t^*)} = 0. \end{aligned} \right\} \tag{3.2a}$$

Second, the impermeability condition on the side and bottom boundaries is taken into account by requiring

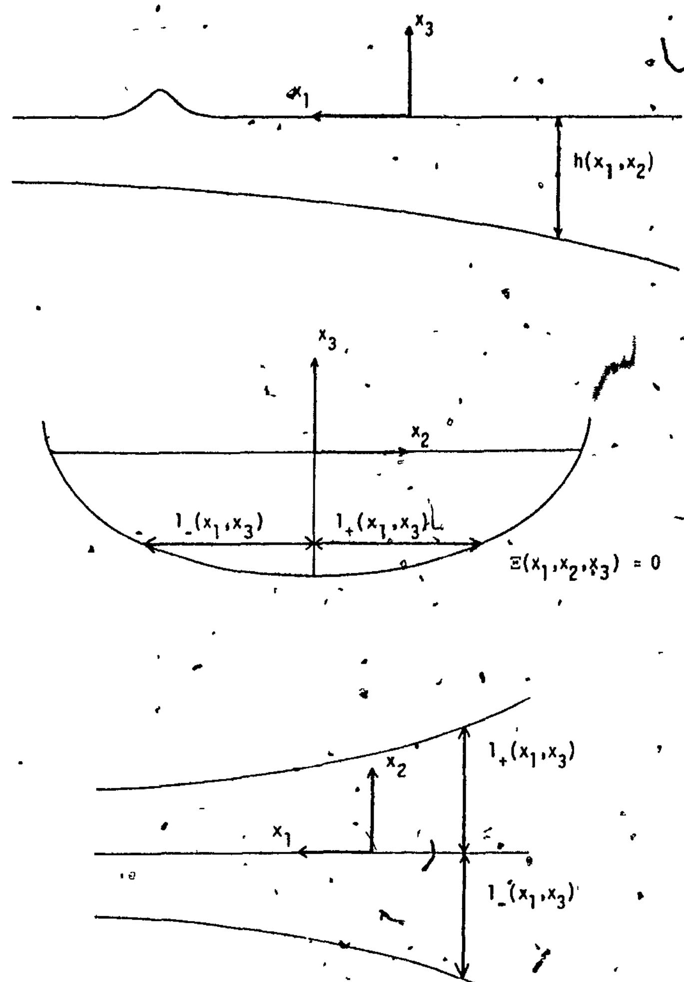


Figure 3

The three projections of the channel (or strait). The quantities $h\,(x_1, x_2)$ and $l_{\pm}(x_1, x_3)$ characterize the depth and the width, respectively. The cross-section of the channel is given by the functional relation $\Xi(x_1, x_2, x_3) = 0$.

$$(v^{*}\cdot\nabla^{*}\,\Xi)\big|_{\Xi=0} = 0. \tag{3.2b}$$

We aim at describing a situation in which solitons are observed; the perturbative approach should hence yield equations which have soliton-like solutions, such as the KdV equation, the KP equation, or similar equations (see ABL1, CAL1). In order to define the perturbative parameter we introduce dimensionless variables and equations. Let H_0 be an average measure of the depth of the channel; the average velocity of the gravity waves will then be $c_0 = \sqrt{gH_0}$. Further, let N_0 be a characteristic average wave amplitude, L_x an average wavelength (or width of a solitary wave), and L_y a length over which perturbations of the wave front manifest themselves in a direction perpendicular to that of the wave propagation. Given these quantities, we introduce nondimensional variables by the following rescalings:

$$\begin{aligned}
&p^* = \rho g H_0\, p, && \eta^* = N_0 \eta, && t^* = c_0^{-1} L_x t, \\
&x_1 = L_x x, && x_2 = L_y y, && x_3 = H_0 z, \\
&v_1^* = c_0 L_x^{-2} H_0^2 v_1, && v_2^* = c_0 L_x^{-1} L_y^{-1} H_0^2 v_2, && v_3^* = c_0 L_x^{-1} H_0 v_3, \\
&\zeta_1^* = c_0 L_x^{-1} L_y^{-1} H_0 \zeta_1, && \zeta_2^* = c_0 L_x^{-2} H_0 \zeta_2; && \zeta_3^* = c_0 L_x^{-2} L_y^{-1} H_0^{-2} \zeta_3 .
\end{aligned} \tag{3.3}$$

The approximation that leads to a KP equation is obtained by considering long waves with small amplitudes in shallow water, with wave crests that vary slowly in the perpendicular direction. More specifically, this can be achieved by assuming

$$(H_0/L_x)^2 = \alpha\varepsilon, \qquad (H_0/L_y)^2 = \alpha\beta\varepsilon^2, \qquad N_0/H_0 = \gamma\varepsilon, \tag{3.4}$$

where ε is a small parameter and α, β, γ are constants of the order of the unity. The implied condition $L_y >> H_0$ imposes that for our approach to be valid, the width of the channel must be larger than its depth. The fundamental equations (3.1) and boundary conditions (3.2) can be rewritten in terms of the rescaled dimensionless variables defined through (3.3) as follows:

$$\begin{aligned}
&v_{3,z} + \alpha\varepsilon v_{1,x} + \alpha\beta\varepsilon^2 v_{2,y} = 0, \\
&p_{,x} + \alpha\varepsilon(v_{1,t} + v_3 v_{1,z}) + \alpha^2\varepsilon^2 v_1 v_{1,x} + \alpha^2\beta\varepsilon^3 v_2 v_{1,y} = 0, \\
&p_{,y} + \alpha\varepsilon(v_{2,t} + v_3 v_{2,z}) + \alpha^2\varepsilon^2 v_1 v_{2,x} + \alpha^2\beta\varepsilon^3 v_2 v_{2,y} = 0, \\
&p_{,z} + \alpha\varepsilon(v_{3,t} + v_3 v_{3,z}) + \alpha^2\varepsilon^2 v_1 v_{3,x} + \alpha^2\beta\varepsilon^3 v_2 v_{3,y} = -1,
\end{aligned} \tag{3.5}$$

and

$$p\,\big|_{z=\gamma\varepsilon\eta} = 0, \tag{3.6a}$$

$$[v_3 - \gamma\varepsilon\eta_{,t} - \alpha\gamma\varepsilon^2 v_1 \eta_{,x} - \alpha\beta\gamma\varepsilon^3 v_2 \eta_{,y}]\big|_{z=\gamma\varepsilon\eta} = 0, \tag{3.6b}$$

$$[\Xi_{,z} v_3 + \alpha\varepsilon\Xi_{,x} v_1 + \alpha\beta\varepsilon^2 \Xi_{,y} v_2]\big|_{\Xi(x,y,z)=0} = 0, \tag{3.6c}$$

$$\zeta = \nabla \times v\,, \tag{3.6d}$$

where $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$. Further restrictions must be imposed on the function Ξ. In the linear approximation of the perturbative expansion we require that the above system should yield a wavelike solution, moving in the x-direction, i.e. the longitudinal direction of the channel or strait. In order to ensure such behaviour we must solve the equation $\Xi(x,y,z)=0$ for z and then require that z be a slowly varying function of x, and for future need, also of y. More specifically, the depth should be given in the form

$$z = -h(\varepsilon x, \varepsilon y)\,.$$

For a genuine channel with steep sides we must divide the cross-sectional area into three parts, as illustrated in Figure 3. The above form for z will apply everywhere, except close to the side boundaries. There, on the contrary, we shall assume that we have steep walls and that we can solve the equation $\Xi = 0$ for the variable y to obtain explicit expressions for the functions $y = l_{\pm}(x,z)$ that describe both walls. Since the wavelike solution should depend explicitly on the form of the side boundaries, we must also require that the functions $l_{\pm}$ should be slowly varying functions in the region of the channel which is very close to the sides. Specifically, we ask that

$$l_{\pm} = l_{\pm}(\varepsilon x, \varepsilon^2 z)\,.$$

Under these restrictions we can rewrite the boundary condition (3.6c) as

$$\begin{aligned} &\{\alpha[\beta v_2 - v_1 l_{\pm,\varepsilon x}] - v_3 l_{\pm,\varepsilon^2 z}\}\big|_{y=l_{\pm}(\varepsilon x,\varepsilon^2 z)} = 0, \\ &\{v_3 + \alpha\varepsilon^2 v_1 h_{\varepsilon x} + \alpha\beta\varepsilon^3 v_2 h_{\varepsilon y}\}\big|_{z=-h(\varepsilon x,\varepsilon y)} = 0. \end{aligned} \tag{3.7}$$

In view of the perturbative approach that we are undertaking in order to get a wave equation we must pay care about the time domain (or range) of validity of the resulting equation. This question is crucial and demand that we choose an *appropriate* system of coordinates. With our present choice, the origin is fixed; an observer positioned at this point will therefore look at a wave going away from him: as time goes on an increasing lack of precision will thus occur. In order to palliate to this problem we shall introduce a *moving* frame. To do so, let us first express $\mathbf{v}$, p, and η as formal power series in the small parameter ε:

$$\mathbf{v}(x, y, z, t) = \sum_{i=0}^{\infty} \mathbf{v}^i(x, y, z, t)\varepsilon^i,$$

$$p(x, y, z, t) = \sum_{i=0}^{\infty} p^i(x, y, z, t)\varepsilon^i, \tag{3.8}$$

$$\eta(x, y, t) = \sum_{i=0}^{\infty} \eta^i(x, y, t)\varepsilon^i.$$

We then substitute (3.8) into (3.5), (3.6a), (3.6b), (3.7), and proceed to solve these iteratively up to first order. We find that the wave amplitude η^0 must satisfy the wave equation

$$\eta^0_{tt} = h(\varepsilon x, \varepsilon y)\eta^0_{xx} \tag{3.9a}$$

together with the boundary condition

$$\beta\eta^0_y - L_{\pm,\varepsilon x}\eta^0_x \Big|_{y = L_{\pm}(\varepsilon x, \varepsilon^2 z)} = 0. \tag{3.9b}$$

A general solution of equation (3.9a) valid at least up to order ε^2 is obtained in the form

$$\eta^0(x, y, t) = h^{1/4}(\varepsilon x, \varepsilon y)[f^+(X^+) + f^-(X^-)], \tag{3.10a}$$

$$X^{\pm} = R(x, y, \varepsilon) \pm C(\varepsilon x, y, \varepsilon)t, \tag{3.10b}$$

$$R(x, y, \varepsilon) = \int_{x_0}^{x} ds \frac{C(\varepsilon s, y, \varepsilon)}{\sqrt{h(\varepsilon s, \varepsilon y)}}, \tag{3.10c}$$

$$C(\varepsilon x, y, \varepsilon) = 1 + \sum_{i=1}^{\infty} C_i(\varepsilon x, y)\varepsilon^i. \tag{3.10d}$$

If $\partial C/\partial(\varepsilon x) \neq 0$, then secular terms will appear in the solution of (3.9a) at order ε^2, or at some higher order. Thus, in particular, if we have $\partial C_1/\partial(\varepsilon x) \neq 0$, then the perturbation series (3.8) is viewed as meaningful for times $t \ll 1/\varepsilon^2$ only. Therefore, in the future, whenever possible, we shall set the auxiliary function $C(\varepsilon x, y, \varepsilon) = 1$. As we shall see, this *a priori* arbitrary function is needed in order to reduce the equations describing wave propagation in straits of nonconstant cross-section to integrable equations with boundary conditions that can be explicitly solved.

The functions $f^{\pm}$ are thus far arbitrary functions which remain to be determined by the initial conditions. We also observe that (3.9a) remains valid as long as the higher-order corrections, including the nonlinear interaction terms in ε in the expressions for v, do not play any role. This is the case as long as $t \ll \varepsilon^{-1}$. For a long time scale $t \sim \varepsilon^{-1}$ we must take into account the interaction of the waves with themselves. To do so we introduce the *wave frame* through the following transformation of the coordinates:

$$
\begin{aligned}
&X = R(x, y, \varepsilon) - \sigma C(\varepsilon x, y, \varepsilon)t, \\
&Y = y, \qquad Z = z, \qquad T = \varepsilon x,
\end{aligned}
\tag{3.11}
$$

where $\sigma = \pm 1$ for right- and left-going waves, respectively. We shall choose $\sigma = 1$ but note that the results for $\sigma = -1$ are recovered by changing the signs of the the velocity field components in all formulas. This wave frame is a coordinate frame that follows the zeroeth approximation of the wave (its linear component). It should be stressed here that the transformation (3.11) is so chosen that the physical variables x and t go over to T and X, respectively. The variable T, although proportional to the physical variable x, will formally play the role of time in the evolution-type equations that we shall be deriving. In particular this will allow us to obtain KP- or KdV-like equations with variable coefficients (depending on T). The Cauchy problem for these equations can in many cases be solved analytically [ABL1, CAL1, ECK1]. The Cauchy data at $T = T_0$ will correspond to data measured over some period of physical time t^* at one point x_1^0 This corresponds to the usual physical case when the data corresponding to surface or internal waves are measured using instruments installed at one fixed position in space.

We now expand the auxiliary quantity $C(\varepsilon x, y, \varepsilon)$, the depth function $h(\varepsilon x, \varepsilon y)$, and the side functions $l_{\pm}(\varepsilon x, \varepsilon^2 z)$, expressing everything in terms of the new variables (3.11):

$$C(Y,T,\varepsilon) = 1 + \sum_{i=1}^{\infty} C_i(Y,T)\varepsilon^i, \tag{3.12a}$$

$$h(\varepsilon Y,T) = \sum_{i=0}^{\infty} h_i(T)Y^i\varepsilon^i, \tag{3.12b}$$

$$l_\pm(\varepsilon^2 Z,T) = \sum_{i=0}^{\infty} l_{\pm i}(T)Z^i\varepsilon^{2i}. \tag{3.12c}$$

This provides us with an expansion for the function $R(x, y, \varepsilon)$ in (3.10c). Notice that these quantities are such that $R(x, y, \varepsilon)$ is of order ε^{-1} whereas $\partial R/\partial x$ and $\partial R/\partial y$ are of order ε^0; this is the reason why we are setting $h = h(\varepsilon Y,T)$. Thus we obtain the following vector field transformations:

$$\begin{aligned}
\partial_x &= h_0^{-1/2}\partial_X + \varepsilon[\partial_T + B_0(Y,T)\partial_X] + \varepsilon^2 B_1(X,Y,T)\partial_X + O(\varepsilon^3),\\
\partial_y &= \partial_Y + A_0(Y,T)\partial_X + \varepsilon A_1(X,Y,T)\partial_X + \varepsilon^2 A_2(X,Y,T)\partial_X + O(\varepsilon^3),\\
\partial_z &= \partial_Z,\\
\partial_t &= -[1 + \varepsilon C_1(Y,T) + \varepsilon^2 C_2(Y,T)]\partial_X + O(\varepsilon^3).
\end{aligned} \tag{3.13}$$

The quantities $A_i(X,Y,T)$ and $B_i(X,Y,T)$ are all expressed in terms of the $C_i(Y,T)$ and the $h_i(T)$; below we shall need only the explicit expressions for A_0 and B_0:

$$\begin{aligned}
B_0(Y,T) &= h_0^{-1/2}C_1(Y,T) - \tfrac{1}{2}h_0^{-3/2}h_1 Y,\\
A_0(Y,T) &= \int_{T_0}^{T} h_0^{-1/2}\left(C_{1,Y}(Y,s) - \tfrac{1}{2}h_0^{-1}h_1\right)ds.
\end{aligned} \tag{3.14}$$

It should be pointed out that B_1, A_1, etc., contain an explicit dependence on the variable X, due to the secular terms discussed above. Performing the transformation, we obtain a new system of equations written down with respect to the wave frame coördinates; to order ε^2 we have

$$v_{3,Z} + \alpha\varepsilon h_0^{-1/2}v_{1,X} + \alpha\varepsilon^2[v_{1,T} + B_0 v_{1,X} + \beta A_0 v_{2,X} + \beta v_{2,Y}] = 0,$$

$$h_0^{-1/2} p_X + \varepsilon[B_0 p_X + p_T - \alpha v_{1,X} + \alpha v_3 v_{1,Z}] \\ + \varepsilon^2 [B_1 p_X - \alpha C_1 v_{1,X} + \alpha^2 h_0^{-1/2} v_1 v_{1,X}] = 0, \qquad (3.15a)$$

$$p_Y + A_0 p_X + \varepsilon[A_1 p_X - \alpha v_{2,X} + \alpha v_3 v_{2,Z}] \\ + \varepsilon^2 [A_2 p_X - \alpha C_1 v_{2,X} + \alpha^2 h_0^{-1/2} v_1 v_{2,X}] = 0,$$

$$p_Z + 1 + \alpha\varepsilon[v_3 v_{3,Z} - v_{3,X}] + \alpha\varepsilon^2 [\alpha h_0^{-1/2} v_1 v_{3,X} - C_1 v_{3,X}] = 0.$$

The boundary conditions transform into

$$p\Big|_{Z=\varepsilon\gamma\eta} = 0,$$

$$[v_3 + \varepsilon\gamma\eta_X + \varepsilon^2\gamma(C_1 - \alpha h_0^{-1/2} v_1)\eta_X]\Big|_{Z=\varepsilon\gamma\eta} = 0,$$

$$[v_3 + \alpha\varepsilon^2 v_1 h_0']\Big|_{Z=-h(T,\varepsilon Y)} = 0, \qquad (3.15b)$$

$$[\alpha\beta v_2 - \alpha v_1 l_{\pm,0}' - v_3 l_{\pm,1} - Z\varepsilon^2(\alpha l_{\pm,1}' + 2 v_3 l_{\pm,2})]\Big|_{Y=l_\pm(T,\varepsilon^2 Z)} = 0.$$

Note that here, and below, primes denote differentiation for a function of a single variable. To derive a nonlinear evolution equation, we expand $\mathbf{v}$, p, and η as in (3.8), substitute into (3.15), and solve up to order ε^2, inclusive. The order ε^0 yields

$$p^0 = -Z, \qquad v_3^0 = 0,$$

$$[\beta v_2^0 - v_1^0 l_{\pm,0}']\Big|_{Y=l_{\pm,0}} = 0.$$

From terms of order ε we obtain

$$p^1 = \gamma\eta^0(X,Y,T),$$

$$v_1^0 = \alpha^{-1}\gamma h_0^{-1/2}[\eta^0(X,Y,T) + \phi_0(Y,Z,T)],$$

$$v_2^0 = \alpha^{-1}\gamma\Big[A_0(Y,T)\eta^0 + \int_{X_0}^{X} \eta^0_Y(s,Y,T)\,ds + \psi_0(Y,Z,T)\Big], \qquad (3.16)$$

$$v_3^1 = -\gamma h_0^{-1}(T)[Z + h_0]\eta^0_X.$$

In the above expressions, the quantities ϕ_0 and ψ_0 have appeared as integration "constants." By definition they are independent of X. In view of the transformation of coordinates (3.11) these functions are therefore independent of the physical time variable t; also in view of (3.11), X_0 corresponds to some arbitrarily chosen instant of the physical time. Below we shall choose X_0 such that the wave amplitude vanishes there, i.e. $\eta^0(X_0,Y,T)=0$. With such a choice the components $v_i^0(X_0,Y,Z,T)$, $i=1,2$, are seen to be proportional to ϕ_0 and ψ_0, respectively. Finally, from the ε^2 terms, we obtain p^2, v_1^1, v_2^1, and v_3^2 in terms of η^0 and η^1. In order to calculate η^1 we would have to proceed to higher order in ε. We can however eliminate η^1 from the different expressions of order ε^2 and obtain the wave amplitude equation and boundary condition for $\eta^0(X,Y,T)$. Dropping all details, we present the resulting equations:

$$\eta^0{}_T + \tfrac{3}{2}\gamma h_0^{-3/2}(T)\eta^0\eta^0{}_X + \tfrac{1}{6}\alpha h_0^{1/2}\eta^0{}_{XXX} + \tfrac{1}{2}\beta h_0^{1/2}\int_{X_0}^{X}\eta^0{}_{YY}(s,Y,T)ds$$
$$+ \beta h_0^{-1/2}A_0\eta^0{}_Y + M_1(Y,T)\eta^0 + M_2(Y,T)\eta^0{}_X + M_3(Y,T) = 0, \qquad (3.17a,b)$$
$$\left[\beta\left(\psi_0(Y,Z,T) + A_0\eta^0 + \int_{X_0}^{X}\eta^0{}_Y(s,Y,T)ds\right) - h_0^{-1/2}[\eta^0 + \phi_0(Y,Z,T)]l'_{\pm,0}\right]\Big|_{Y=l_{\pm,0}} = 0.$$

The quantity $A_0(Y,T)$ was defined before. The other quantities appearing in (3.17a) are

$$M_1(Y,T) = \tfrac{1}{4}h_0^{-1}h_0' + \tfrac{1}{2}\beta h_0^{1/2}A_{0,Y}(Y,T),$$

$$M_2(Y,T) = \tfrac{1}{2}\beta h_0^{1/2}A_0^2 + \gamma h_0^{-5/2}\int_{-h_0}^{0}\phi_0(Y,Z,T)dZ - C_{1,T}\int_{T_0}^{T}h_0^{1/2}(s)ds, \qquad (3.17c)$$

$$M_3(Y,T) = \tfrac{1}{2}h_0\left[h_0'\phi_0(Y,Z=-h_0,T) + h_0^{1/2}\int_{-h_0}^{0}\left[(\phi_0/h_0^{1/2})_T + \beta\psi_{0,Y}\right]dZ\right].$$

Note that second order also yields expressions for v_1^1, v_2^1, and v_3^2; these quantities however explicitly depend on the next correction η^1 of the wave amplitude. We shall call equation (3.17a) the integral form of the *generalized Kadomtsev-Petviashvili* (GKP) equation. It does coincide with the pure KP equation whenever $h_0(T)=1$, $h_1(T)=0$, $\phi_0(Y,Z,T)=\psi_0(Y,Z,T)=0$, and $C_1(Y,T)=0$ (see KAD1). To lowest order in ε we obtain from (3.6d), (3.11), and

(3.16) that the physical components of the vorticity field in the x, y, and z directions are

$$\zeta_1{}^0 = -\gamma\alpha^{-1}\psi_{0,Z}\,, \qquad \zeta_2{}^0 = \gamma\alpha^{-1}h_0{}^{-1/2}\phi_{0,Z}\,, \qquad \zeta_3{}^0 = -\gamma\alpha^{-1}h_0{}^{-1/2}\phi_{0,Y}\,. \tag{3.17d}$$

Let us now make some general comments on the situation described by equation (3.17a).

1. The depth function $h(\varepsilon Y,T)$ was expanded into a formal series (3.12b). If the bottom happens to be symmetric in the x_2 direction (the Y direction), i.e. if $h(\varepsilon Y,T) = h(-\varepsilon Y,T)$, then we have $h_1(T) = 0$. Looking at equation (3.17a), at the expression for A_0, and at (3.17b)-(3.17d), we notice that the equation and boundary conditions are in this case insensitive to any variations of the bottom in the x_2 direction. For nonsymmetric bottoms, the function $h_1(T)$ figures explicitly in the expression for $A_0(Y,T)$ and hence in $M_1(Y,T)$ and $M_2(Y,T)$. Thus, if we are interested, for instance, in waves propagating parallel to a shore along a sloping beach, then the contribution of the function $h_1(T)$ may be important.

2. The sides of the strait only figure in the boundary condition (3.17b). Moreover, only the leading term $L_{\pm 0}(T)$ in the expansion (3.12c) figures in the equation (3.17b). This is a consequence of the approximations (3.4), which include the assumption that the width of the strait is much larger than its depth. The variation of the width as a function of the depth is thus immaterial. This contrasts with the usual situation in a channel, where the width and depth are comparable and the boundary conditions enter mainly through the effective cross-sectional area of the channel (see GRI1).

3. The vorticity functions ϕ_0 and ψ_0 figuring in the components $v_1{}^0$ and $v_2{}^0$ of the velocity field [see (3.16)] only enter in equation (3.17a) integrated over the entire depth of the fluid, or via their value at the bottom of the strait [see (3.17c)]. On the other hand the functions ϕ_0 and ψ_0 figure directly in the side boundary condition (3.17b). Their Z dependence leads to a decoupling of these boundary conditions into separate conditions for the wave amplitude η^0 and for the vorticity functions.

4. The bottom and side boundaries play different roles in our treatment. The shape of the bottom has entered via the depth functions $h_0(T)$ and $h_1(T)$ into the coefficients of equation (3.17a) and into the boundary condition (3.17b). The side boundaries, on the other

hand, only figure in the boundary condition (3.17b). This makes it possible to consider two different physical problems. The first corresponds to studying the integrated GKP equation without any boundary conditions; this describes long waves of small amplitude in an infinite two-dimensional body of water with varying bottom and nonvanishing vorticity. The second problem corresponds to considering the integrated GKP equation together with the boundary condition (3.17b). This provides a model for long waves in a shallow strait with varying sides and bottom, the strait being allowed to possibly meander along some given direction and change its cross-section, either broadening or narrowing.

5. The coefficients of the integrated GKP equation depend on physical quantities such as the depth functions $h_0(T)$ and $h_1(T)$, describing the bottom, and the vorticity functions $\phi_0(Y, Z,T)$ and $\psi_0(Y, Z,T)$. In addition, they depend on the auxiliary function $C_1(Y,T)$, figuring in A_0 and M_2. This function, as mentionned earlier, remains at our disposal and will be chosen in each case to simplify the results, taking into account however our previous comments on secular terms which may occur for certain specific situations.

The integrated GKP equation remains meaningful also when all the Y dependence is discarded, i.e. when we set $\eta^0 = \eta^0(X,T)$, $\phi_0 = \phi_0(Z,T)$, $h_1 = \psi_0 = 0$, $C_1 = C_1(T)$, so that $A_0 = 0$; in this case, when the bottom is flat ($h_0 = 1$), we recover the results of Benjamin [BEN1], namely that solitary waves are possible even in the case of a fluid with nonvanishing vorticity. The physical situation that we are interested in is one in which we have bounded solutions at $x_1 = x_1^0$, i.e. $T = T_0$, for instance a soliton. The boundary conditions for the Euler equations (3.1) and (3.2) are completely specified by giving, at x_1^0 and for all t^* and x_2, the surface amplitude η^* as well as the velocity field $\mathbf{v}^*$. In view of (3.16) this means that we must specify, up to order ε^2, the initial conditions (at $T = T_0$) for the amplitude η^0 and the vorticity functions ϕ_0, ψ_0. Given these initial conditions it is clear that the integrated GKP equation, being an equation for the amplitude η^0 alone, is in principle not sufficient to determine completely the flow at an arbitrary point of space. On the other hand. we can rewrite the integrated GKP equation equivalently as a coupled system of equations, obtained, on one hand, by differentiating (3.17a) with respect to X, and on the other hand by evaluating (3.17a) at $X = X_0$:

$$[\eta^0{}_T + \tfrac{3}{2}\gamma h_0^{-3/2}(T)\eta^0\eta^0{}_X + \tfrac{1}{6}\alpha h_0^{1/2}\eta^0{}_{XXX}]_X + \tfrac{1}{2}\beta h_0^{1/2}\eta^0{}_{YY} + [\beta h_0^{-1/2}A_0\eta^0{}_Y + M_1\eta^0 + M_2\eta^0{}_X]_X = 0, \tag{3.18a}$$

$$\eta^0{}_T(X_0) + \tfrac{3}{2}\gamma h_0^{-3/2}\eta^0(X_0)\eta^0{}_X(X_0) + \tfrac{1}{6}\alpha h_0^{1/2}\eta^0{}_{XXX}(X_0) + \beta h_0^{-1/2}A_0\eta^0{}_Y(X_0) + M_1\eta^0(X_0) + M_2\eta^0{}_X(X_0) + M_3(Y,T) = 0. \tag{3.18b}$$

The boundary condition (3.17b) can be treated in the same manner to obtain

$$\left[\beta A_0\eta^0 + \eta^0{}_Y - h_0^{-1/2}\eta^0{}_X\, l_{\pm,0}'\right]\Big|_{Y=l_{\pm,0}} = 0, \tag{3.19a}$$

$$\left[\beta\,[\psi_0 + A_0\eta^0(X_0)] - h_0^{-1/2}\,[\eta^0(X_0) + \phi_0]\,l_{\pm,0}'\right]\Big|_{Y=l_{\pm,0}} = 0. \tag{3.19b}$$

For an arbitrary fixed X_0 we have a coupled system of equations for $\eta_0(X,Y,T)$ and the vorticity functions $\phi_0(Y,Z,T)$, $\psi_0(Y,Z,T)$, to be solved for some given initial conditions.

A judicious choice of X_0 in (3.18b) and (3.19b) makes it possible to decouple the system. Indeed, let us choose X_0 as a point where the amplitude $\eta_0(X_0, Y, T)$ vanishes. This is possible if the perturbation is always bounded, as required by physical considerations. If we have

$$\lim_{X\to-\infty} \eta_0(X,Y,T) = 0 \tag{3.20}$$

then the choice $X_0 = -\infty$ is appropriate. Equations (3.18b) and (3.19b) then reduce to

$$M_3(Y,T) = \tfrac{1}{2}h_0\left[h_0'\phi_0(Y,Z=-h_0,T) + h_0^{1/2}\int_{-h_0}^{0}\left[(\phi_0/h_0^{1/2})_T + \beta\psi_{0,Y}\right]dZ\right] = 0 \tag{3.21a}$$

with boundary condition

$$\left[\beta\psi_0 - h_0^{-1/2}\phi_0\, l_{\pm,0}'\right]\Big|_{Y=l_{\pm,0}} = 0, \tag{3.21b}$$

a linear system relating the vorticity functions. With this choice, ϕ_0 and ψ_0 represent the stream velocity of the fluid [see equation (3.16)].

Equation (3.18a) is then simply a nonlinear partial differential equation for the wave amplitude $\eta^0(X,Y,T)$ with initial conditions $\eta^0(X,Y,T_0)$ and boundary condition (3.19a). We shall use the term *generalized Kadomtsev-Petviashvili equation* for (3.18a) (this is the differential form of the GKP equation). In general equation (3.18a) is *not* integrable, in the sense that no analytical procedure is available for obtaining solutions of the corresponding Cauchy problem. However, for a function of three variables $q = q(\xi, \theta, \tau)$, the Kadomtsev-Petviashvili (KP) equation

$$[q_\tau - 6qq_\xi + q_{\xi\xi\xi}]_\xi + \sigma q_{\theta\theta} = 0, \qquad \sigma = \pm 1, \tag{3.22a}$$

and for a function of two variables $q = q(\xi, \tau)$, the Korteweg-de Vries (KdV) equation

$$q_\tau - 6qq_\xi + q_{\xi\xi\xi} = 0 \tag{3.22b}$$

and the cylindrical Korteweg-de Vries (cKdV) equation

$$q_\tau - 6qq_\xi + q_{\xi\xi\xi} - 4\xi q_\xi - 2q = 0 \tag{3.22c}$$

belong to a quite large class of equations that are infinite-dimensional integrable Hamiltonian systems. Powerful analytical techniques are available for solving the Cauchy problem for these equations, such as the *inverse spectral transform technique* in one of its various forms (see, e.g., ABL1 and CAL1). Infinite classes of exact solutions exist, such as solitons and multi-solitons, periodic and quasi-periodic solutions, rational solutions, etc., obtained through the means of Bäcklund transformations, algebraic geometry, and many other techniques. More generally, perturbation methods have been applied to variable-coefficient KdV (VC-KdV) equations, namely

$$q_\tau + F(\tau)qq_\xi + q_{\xi\xi\xi} = 0, \tag{3.23a}$$

and have provided physically significant results (see ECK1, GRI2, JOH2). Similar perturbative methods could also be applied to the variable-coefficient KP (VC-KP) equation

$$[q_\tau + F(\tau)qq_\xi + q_{\xi\xi\xi}]_\xi + G(\tau)q_{\theta\theta} = 0, \tag{3.23b}$$

and the variable-coefficient cKdV (VC-cKdV) equation

$$q_\tau - 6qq_\xi + q_{\xi\xi\xi} - 4\xi q_\xi + H(\tau)q = 0. \tag{3.23c}$$

Our technique in the following section will be to find transformations of the dependent and independent variables that have the effect to reduce the GKP equation (3.18a) to the VC-KP equation (3.23b), the VC-KdV equation (3.23a), or the VC-cKdV equation (3.23c). In special cases, under certain restrictions on the shape of the channel, we reduce directly to one of the integrable equations (3.22), i.e. the pure KP, KdV, or cKdV equations.

If we are considering the second problem, namely waves in channels with boundaries, we must also transform the boundary conditions (3.19a). In order to be able to solve them, we reduce them to a *standard form*

$$q_\theta \Big|_{\theta=\theta(l_{\pm 0})} = 0, \tag{3.24a}$$

which is solved trivially by imposing

$$q = q(\xi,\tau) \tag{3.24b}$$

for all values of θ. It is easy to verify that the most general transformation of variables that will not introduce unwanted terms in (3.18a), i.e. extra terms which do not appear in equations (3.22) and (3.23), and can be used to transform the GKP equation into a VC-KP, VC-KdV, or VC-cKdV equation is

$$\begin{aligned} \eta^0(X,Y,T) &= R(T)q(\xi,\theta,\tau) + S(X,Y,T), \\ \xi &= a(T)X + K(Y,T), \\ \theta &= U(T)Y + V(T), \\ \tau &= \int_{T_0}^{T} P(s)ds. \end{aligned} \tag{3.25}$$

If we wish to use known results on solitons and other bounded solutions of the KdV, cKdV, and KP equations, we must also require

$$q(\xi,\theta,\tau) \to 0 \quad \text{as} \quad |\xi| \to \infty. \tag{3.26a}$$

For physical reasons we also wish (3.20) to be true; these two boundary conditions imply

$$S(X,Y,T) = 0. \tag{3.26b}$$

In some cases the requirement (3.26b) will turn out to be too restrictive if imposed for all values of the variable T. We shall then impose a weaker condition, namely

$$\lim_{T\to\infty} S(X,Y,T) = 0 \tag{3.26c}$$

II. Reduction of the GKP equation to integrable equations

In this section we shall carry out the reduction of the GKP equation to one of the equations (3.22), (3.23). The section will be subdivided in two parts. Within the first two subsections below (II.1 and II.2) we shall consider the problem of reducing the GKP equation to a KP- or KdV-type equation without taking into account the side boundaries, i.e. considering the fluid to have an infinite extension in the y-direction (as well as in the x one). In the second part (subsections II.3 and II.4) we transform the boundary conditions into the standard form (3.24a) and reduce the GKP equation to a KdV or cKdV-type equation.

1 - Reduction of the GKP equation to the VC-KP and KP equations

If no boundary conditions are imposed, we can always choose

$$S(X,Y,T) = 0 \tag{3.27a}$$

in the transformation of variables (3.25). In order to arrive at one of the equations (3.22), (3.23) for the new function $q(\xi,\theta,\tau)$ we must always impose a constraint on the vorticity function ϕ_0, namely

$$\int_{-h_0}^{0} \phi_{0,YYY}(Y,Z,T)dZ = 0\,, \tag{3.27b}$$

i.e., we can write

$$\int_{-h_0}^{0} \phi_0 dZ = \beta^{-1}\gamma^{-1}h_0^{\,2}(T)[Q_0(T) + Q_1(T)Y + Q_2(T)Y^2]\,, \tag{3.27c}$$

where the moments Q_0, Q_1, and Q_2 are completely determined by the Z-integral of the vorticity function $\phi_0(Y,Z,T)$. On the other hand, taking into account (3.18a), (3.27a), and the fact that we are considering bounded solutions of the GKP equation, we can assert that, in all cases under consideration, equation (3.21a) reduces to

$$\int_{-h_0}^{0} \psi_{0,Y}\, dZ = -\beta^{-1}\int_{-h_0}^{0} (\phi_0 / h_0^{1/2})_T\, dZ - \beta^{-1} h_0^{-1/2} h_0' \phi_0(Y, Z=-h_0, T), \tag{3.27d}$$

which defines, up to an arbitrary T-dependent function, the integral over all Z of the vorticity function ψ_0. Moreover, it turns out that if no side boundaries are considered, then the auxiliary function $C_1(Y,T)$ serves no useful physical purpose and we therefore can always set $C_1(Y,T)=0$, so that

$$A_0(Y,T) = A_0(T) = -\tfrac{1}{2}\int_{T_0}^{T} h_0^{-3/2} h_1\, ds, \tag{3.28a}$$

and we can always set

$$a(T) = 1, \qquad P(T) = \tfrac{1}{6}\alpha h_0^{1/2}(T). \tag{3.28b}$$

Putting

$$R(T) = h_0^{-1/4}(T) R_0 U^{1/2}(T), \qquad R_0 = \text{constant},$$

$$K(Y,T) = \frac{-1}{\beta}\left\{ \frac{U'/U}{2h_0^{1/2}} Y^2 + \left[\frac{V'/U}{h_0^{1/2}} - \tfrac{1}{2}\beta\int_{T_0}^{T} h_0^{-3/2} h_1\, ds\right] Y \right. \tag{3.28c}$$

$$\left. + \tfrac{1}{2}\int_{T_0}^{T} \frac{(V'/U)^2}{h_0^{1/2}} ds + \int_{T_0}^{T} h_0^{1/2} Q_0\, ds \right\}$$

in (3.25), we obtain a VC-KP equation (3.23b) in the form

$$[q_\tau + 9\alpha^{-1}\gamma R_0 h_0^{-9/4}(T) U^{1/2}(T) q q_\xi + q_{\xi\xi\xi}]_\xi + 3\alpha^{-1}\beta U^2(T) q_{\theta\theta} = 0. \tag{3.29a}$$

The function U(T) is determined, in terms of the variable bottom $h_0(T)$ and of the second moment $Q_2(T)$ of the vorticity function ϕ_0, by a Riocati equation:

$$(U'/U)' = (U'/U)^2 + \tfrac{1}{2}(h_0' U'/h_0 U) + 2Q_2. \tag{3.29b}$$

The function V(T) is then obtained by solving a linear equation, in terms of the variable bottom $h_0(T)$, $h_1(T)$ and of the first moment of the vorticity function ϕ_0:

$$(V'/U)' = [(U'/U) + \tfrac{1}{2}(h_0'/h_0)](V'/U) + \tfrac{1}{2}\beta(h_1/h_0) + Q_1 . \qquad (3.29c)$$

It is worthwhile to notice that if $Q_2(T)=0$, then the Riccati equation (3.29b) is explicitly solvable and implies

$$U(T) = U_0 \left[1 + U_1 \int_{T_0}^{T} h_0^{1/2}\, ds\right]^{-1},$$

where U_0 and U_1 are constants. If we require the coefficients of the VC-KP equation (3.29a) to reduce to constants, in order to obtain a pure KP equation (3.22a), we must impose

$$h_0(T) = 1, \qquad Q_2(T) = 0, \qquad U_1 = 0, \qquad (3.30a)$$

and hence the depth is given by

$$h(Y,T) = 1 + \varepsilon Y h_1(T) + O(\varepsilon^2) .$$

Since $\alpha > 0$, $\beta > 0$, the KP equation which we thus obtain corresponds to $\sigma = +1$, which is the KP equation admitting stable soliton solutions (see ABL1, CAL1). In this case we have

$$R(T) = -2\alpha/3\gamma = \text{constant},$$

$$U(T) = (\alpha/3\beta)^{1/2} = \text{constant},$$

$$V'(T) = \tfrac{1}{2}\beta\, U \int_{T_0}^{T} h_1(s)ds + U \int_{T_0}^{T} Q_1(s)ds, \qquad (3.30b)$$

$$K(Y,T) = -\tfrac{1}{2} Y \int_{T_0}^{T} Q_1(s)ds - \tfrac{1}{2}\beta^{-1} \int_{T_0}^{T} (V'/U)^2\, ds - \beta^{-1} \int_{T_0}^{T} Q_0(s)ds .$$

Notice that even if the bottom is completely flat, i.e. if $h_1(T)=0$, the KP equation (3.22a) is not obtained for a generic vorticity field. The requirement of reducing the GKP equation to a KP equation implies that the vorticity field can have a very limited variation in the Y-direction, and in particular the integral along Z of $\zeta_3^{\,0}$ is constant in Y [see (3.27c)].

2 - Reduction of the GKP equation to the VC-KdV and KdV equations

Let us first consider the reduction to a VC-KdV equation (3.23a). We put

$$U(T) = 1, \qquad V(T) = 0, \tag{3.31a}$$

and thus we obtain

$$K(Y,T) = \frac{-1}{\beta\sqrt{h_0}}\Big[\frac{R'}{R} + \frac{h_0'}{4h_0}\Big]Y^2 + \Big[K_2 + \tfrac{1}{2}\int_{T_0}^{T} h_0^{-3/2}\, h_1\, ds\Big]Y$$
$$- \tfrac{1}{2}\beta\int_{T_0}^{T} h_0^{1/2} K_2^2(s)ds - \beta^{-1}\int_{T_0}^{T} h_0^{-1/2} Q_0\, ds, \tag{3.31b}$$

where the function R(T) satisfies a Riccati equation

$$(R'/R)' = 2(R'/R)^2 + \tfrac{3}{2}(R'/R)(h_0'/h_0) + \tfrac{1}{4}(h_0'/h_0)^2 - \tfrac{1}{4}(h_0'/h_0)' + Q_2, \tag{3.31c}$$

and $K_2(T)$ the linear equation

$$K_2' = 2K_2[(R'/R) + \tfrac{1}{4}(h_0'/h_0)] - \tfrac{1}{2}h_0^{-3/2}h_1 - \beta^{-1}h_0^{-1/2}Q_1. \tag{3.31d}$$

The transformation (3.25), satisfying (3.27a), (3.28a), (3.28b) and (3.31), reduces the GKP equation (3.18a) to the following equation for $q(\xi, \theta, \tau)$:

$$[q_\tau + 9\alpha^{-1}\gamma h_0^{-2}(T)R(T)qq_\xi + q_{\xi\xi\xi}]_\xi + 3\alpha^{-1}\beta\, q_{\theta\theta} + 6\alpha^{-1}\beta\, K_2(T)q_{\theta\xi} = 0, \tag{3.31e}$$

In particular equation (3.31e) admits solutions that are independent of the variable θ; if as initial condition we put $q = q_0(\xi,\tau)$, then (3.31e) reduces to a VC-KdV equation for $q = q(\xi,\tau)$. If we require that $q(\xi,\tau)$ should satisfy a pure KdV equation (3.22b), we have to impose, in addition to the equations (3.31a), that

$$R(T) = -(2\alpha/3\gamma)h_0^2(T) \tag{3.32a}$$

so that equation (3.31b) reduces to

$$K(Y,T) = \frac{-9h_0'}{4\beta\, h_0^{3/2}}Y^2 + \Big[K_2 + \tfrac{1}{2}\int_{T_0}^{T} h_0^{-3/2}h_1 ds\Big]Y \tag{3.32b}$$

$$-\tfrac{1}{2}\beta\int_{T_0}^{T} h_0^{1/2}K_2^{2}(s)ds - \beta^{-1}\int_{T_0}^{T} h_0^{-1/2}Q_0\,ds\,, \qquad (3.32b)$$

and (3.31d) to

$$K_2' = \tfrac{9}{2}(h_0'/h_0)K_2 - \tfrac{1}{2}h_0^{-3/2}h_1 - \beta^{-1}h_0^{-1/2}Q_1\,. \qquad (3.32c)$$

Equation (3.31c) then becomes a constraint relating the shape of the bottom with the second moment of the vorticity function ϕ_0 :

$$(h_0'/h_0)' = 5(h_0'/h_0)^2 + \tfrac{4}{9}Q_2\,. \qquad (3.32d)$$

If the vorticity is assumed to vanish, then the behaviour of the depth function $h_0(T)$ is determined by (3.32d):

$$(h_0'/h_0)' = 5(h_0'/h_0)^2\,,$$

i.e.

$$h_0(T) = \left[\frac{c_1 - T_0}{c_1 - T}\right]^{1/5},$$

where c_1 is an arbitrary constant. The above equation shows that as $T \to \infty$, $h_0 \to 0$, so that the depth is a decreasing function; also, for $T = c_1$ the bottom is infinitely deep. A similar result has been obtained by Grimshaw (see GRI2) when considering a two-dimensional irrotational fluid over a variable bottom. The presence of the vorticity terms allow us to get a physically more reasonable behaviour of the bottom, i.e., by introducing an appropriate vorticity field we can model, in principle, *any* behaviour of the bottom. We finally note that the presence of some vorticity is absolutely essential in order to get a KdV equation: indeed, if $Q_0 = Q_1 = Q_2 = 0$, then we are no more able to go over to the KdV equation for the function q.

3 - Solution of the boundary conditions and reduction to a KdV equation

Let us now concentrate on the GKP equation (3.18a) together with the boundary condition (3.18a). To obtain the boundary condition in the standard form (3 24a) we require

$$A_0(Y = l_{\pm,0}, T) = \beta^{-1} h_0^{-1/2} l_{\pm,0}' \tag{3.33}$$

together with (3.21b). The boundary condition for the wave amplitude $\eta^0(X,Y,T)$ is then

$$\eta^0{}_Y(X,Y,T)\,\big|_{Y = l_{\pm,0}} = 0 .$$

When performing the transformation (3.25), we wish to obtain the boundary condition for q in the form (3.24a). In order to do this we require $S_Y = 0$, $K_Y = 0$, $U(T) = 1$, $V(T) = 0$ in (3.25). We shall solve the boundary condition (3.24a) in a trivial manner, namely by requiring that the transformed wave function $q(\xi, \theta, \tau)$ should be independent of the variable θ in the entire strait, not just on the boundary. We thus set

$$q = q(\xi, \tau) . \tag{3.34}$$

Note that this does not imply that the physical amplitude η is independent of y. The variable X, and hence also ξ, depends on y via the function $C(\varepsilon x, y, \varepsilon)$ in (3.11) and (3.10c).

The assumption (3.34) is compatible with the GKP equation (3 18a) only if M_1 and M_2 are independent of Y. Using (3.17c), our expression (3.14) for A_0 in terms of C_1, and the boundary condition (3.33), we then find that $A_0(Y,T)$ and $C_1(Y,T)$ are completely determined. Introducing the total *width* $\Delta(T)$ and a further characteristic $\Sigma(T)$ of the strait by putting

$$l_{\pm,0}(T) = \tfrac{1}{2}[\Sigma(T) \pm \Delta(T)] ,$$

we find

$$A_0(Y,T) = \beta^{-1} h_0^{-1/2} \{(\Delta'/\Delta)Y + \tfrac{1}{2}\Sigma[(\Sigma'/\Sigma) - (\Delta'/\Delta)]\} \tag{3.35}$$

Since we must have $A_0(Y, T_0) = 0$ [see (3.14)], we obtain

$$\Delta'(T_0) = \Sigma'(T_0) = 0 , \tag{3.36a}$$

and by an appropriate choice of the origin of the coordinates (i.e. in the middle of the channel at some point $x_1 = x_1^{\,0}$) we make the normalization

$$\Sigma(T_0) = 0. \tag{3.36b}$$

Equations (3.35) and (3.14) then define $C_1(Y,T)$, up to an arbitrary function of T, as

$$C_1(Y,T) = C_0(T) + \tfrac{1}{2}\beta^{-1}\{h_0^{1/2}[h_0^{-1/2}\Delta'/\Delta]'Y^2 + \beta h_0^{-1}h_1 + h_0^{1/2}[h_0^{-1/2}\Delta(\Sigma/\Delta)']'\}. \tag{3.37a}$$

This equation gives a well-defined result for $C_1(Y,T)$ at $T = T_0$ only if

$$\Delta''(T_0) = \Sigma''(T_0) = 0, \tag{3.37b}$$

in addition to the conditions (3.36) and $h_0(T_0) = 1$, $h_1(T_0) = 0$. Now, we must also ensure that the function $M_2(Y,T)$ in (3.17c) will be independent of the variable Y. This will be so only if the vorticity function $\phi_0(Y, Z,T)$ satisfies (3.27c). Moreover, the second and first moments are seen to be completely determined in terms of the geometry of the strait as follows:

$$Q_2(T) = -\tfrac{1}{2}(\Delta'/\Delta)', \tag{3.38a}$$

$$Q_1(T) = -\tfrac{1}{2}\Delta'(\Sigma/\Delta)'. \tag{3.38b}$$

In particular this implies that

$$Q_1(T_0) = 0, \qquad Q_2(T_0) = 0. \tag{3.38c}$$

In order to obtain a VC-KdV equation we specialize equation (3.25) to

$$\begin{aligned}
&\eta^0(X,T) = R(T)q(\xi,\tau) - \tfrac{2}{3}\gamma^{-1}h_0^{3/2}\,[a'(T)/a(T)]X, \\
&\xi = a(T)X + K(T), \\
&\theta = Y, \\
&\tau = \tfrac{1}{6}\alpha\int_{T_0}^{T} a^3(s)h_0^{1/2}(s)ds,
\end{aligned} \tag{3.39a}$$

where

$$R(T) = a(T)\Delta^{-1/2} h_0^{-1/4},$$

$$K'(T) = \frac{3\gamma}{2\alpha\beta}\int_{T_0}^{T} \Delta^{-1/2} h_0^{-11/4}\{Q_0 + \tfrac{1}{8}\Delta^2[(\Sigma/\Delta)']^2\}\,ds, \tag{3.39b}$$

$$a(T) = a_0\left[1 - a_1\int_{T_0}^{T}\Delta^{-1/2} h_0^{-7/4}\,ds\right],$$

and a_0 and a_1 are arbitrary constants. Under the assumptions made above, transformations (3.39a) reduce the GKP equation (3.18a) to the VC-KdV equation

$$q_\tau + \frac{9\gamma}{\alpha\,\Delta^{1/2} h_0^{9/4}(T)\,a(T)}\, q q_\xi + q_{\xi\xi\xi} = 0. \tag{3.40}$$

If we make the physically plausible assumption that

$$\phi_0(Y, T, Z = -h_0) = 0, \tag{3.41a}$$

i.e. that the mainstream motion in the x_1-direction be vanishing on the bottom, then we can solve (3.27c) and the boundary condition (3.21b) for the remaining moments of the vorticity functions. They can be expressed in terms of the geometry of the channel as follows:

$$\begin{aligned} Q_0 = &-\tfrac{1}{2}Q_1\Sigma - \tfrac{1}{12}Q_2(3\Sigma^2 + \Delta^2) + h_0^{-9/2}\Delta^{-5/3}\{Q_{00} \\ &+ \tfrac{3}{2}h_0'(T_0)\Delta(T_0)\int_{T_0}^{T} h_0^{1/2}\Delta^{2/3}[\tfrac{1}{12}Q_2(3\Sigma^2 + \Delta^2) + \tfrac{1}{2}Q_1\Sigma - \tfrac{1}{4}[\Delta(\Sigma/\Delta)']^2]\,ds\} \end{aligned} \tag{3.41b}$$

$$\int_{-h_0}^{0}\psi_0\,dZ = -\gamma^{-1}\beta^{-2}[R_0(T) + R_1(T)Y + R_2(T)Y^2 + R_3(T)Y^3],$$

$$\begin{aligned} R_3 &= \tfrac{1}{3}(h_0^{3/2}Q_2)', \\ R_2 &= \tfrac{1}{2}(h_0^{3/2}Q_2)', \\ R_1 &= (h_0^{3/2}Q_2)' + 3h_0'(T_0)\Delta(T_0)h_0^{-5/2}(T)\Delta^{-1}(T)\{2Q_0 + \tfrac{1}{4}[\Delta(\Sigma/\Delta)']^2\}, \\ R_0 &= \tfrac{1}{2}[\tfrac{3}{4}h_0'(T_0)\Delta(T_0)h_0^{5/2}(T)\Delta^{-1}(T)\{2Q_0 + \tfrac{1}{4}[\Delta(\Sigma/\Delta)']^2\} \\ &\quad - h_0^{3/2}(T)\{\tfrac{1}{12}Q_2\Sigma(\Sigma^2 + 3\Delta^2) + \tfrac{1}{4}Q_1(\Sigma^2 + \Delta^2) + Q_0\Sigma\}], \end{aligned} \tag{3.41c}$$

where Q_{00} is some (irrelevant) constant. The VC-KdV equation (3.40) reduces to a pure KdV equation whenever we have

$$\frac{9\gamma}{\alpha\Delta^{1/2}\,h_0^{9/4}(T)a(T)} = -6\,. \tag{3.42a}$$

This corresponds to quite specific geophysical circumstances. The above equation (3.42a) can be solved in order to relate $\Delta(T)$ and $h_0(T)$ directly. We actually find

$$a_0 = -\tfrac{3}{2}\alpha^{-1}\gamma\Delta^{-1/2}(T_0), \qquad a_1 = -\tfrac{9}{4}\Delta^{1/2}(T_0)h_0'(T_0), \tag{3.42b}$$

and

$$\Delta(T) = \Delta(T_0)h_0^{-9/2}(T)\left[1+\tfrac{9}{2}h_0'(T_0)\int_{T_0}^{T} h_0^{1/2}(s)ds\right], \tag{3.42c}$$

or equivalently

$$h_0(T) = \left[1+4h_0'(T_0)\int_{T_0}^{T}[\Delta(T_0)/\Delta(s)]^{1/9}\right]^{1/4}\left[\Delta(T_0)/\Delta(T)\right]^{2/9}, \tag{3.42d}$$

which, due to the conditions (3.37b), implies

$$h_0''(T_0) + 3[h_0'(T_0)]^2 = 0\,. \tag{3.42e}$$

Notice also that for $h_0'(T_0) = 0$ we have $a_1 = 0$ and (3.42a) reduces to

$$\Delta(T) = \Delta(T_0)h_0^{-9/2}(T)\,;$$

in particular this means that as the strait widens, the water gets shallower. On the other hand, for $h_0'(T_0) \neq 0$ there exists solutions of (3.42a) that correspond to the usual situation, i.e. that the strait gets deeper as it widens, e.g., as it opens up into the ocean.

The crucial role played by the vorticity moments for the derivation of the integrable equations is worth stressing. For instance, if we impose

$$Q_2(T) = 0\,,$$

then we obtain from (3.38a) that the width of the strait must be constant,

$$\Delta(T) = \Delta(T_0)\,,$$

i.e. that the strait must be of constant width. In addition, to obtain a pure KdV equation, we require that the depth function should satisfy

$$h_0(T) = [1 + 4(T - T_0)h_0'(T_0)]^{1/4}\,.$$

Notice that in order to obtain a KdV equation while assuming realistic conditions on the geometry of the strait, we needed $a_1 \neq 0$ in equations (3.39b) and (3.42b). Hence we have

$$S = -\tfrac{2}{3}\gamma^{-1}h_0^{3/2}(a'/a)X \neq 0$$

in (3.39a). Thus, bounded solutions $q(\xi,\tau)$ lead to unbounded amplitudes $\eta^0(X,T)$ for $X \rightarrow \infty$. The condition (3.26c), namely

$$-\tfrac{2}{3}\gamma^{-1}h_0^{3/2}(a'/a)X \rightarrow 0 \quad \text{as} \quad T \rightarrow \infty\,,$$

is satisfied if $\Delta^{1/2}(T)h_0^{1/4}(T)$ increases with T. The appearance of a nonzero function S(X,T) can be avoided by reducing to a cKdV equation, rather than to the KdV equation itself.

4 - Solution of the boundary conditions and reduction to a cKdV equation

We solve the boundary condition exactly as in the previous subsection, so equations (3.33) to (3.38) apply again. Instead of (3.39a) we perform the transformation

$$\begin{aligned}
&\eta^0(X,T) = R(T)q(\xi,\tau)\,,\\
&\xi = a(T)X + K(T)\,,\\
&\theta = Y\,,\\
&\tau = \tfrac{1}{6}\alpha\int_{T_0}^{T} a^3(s)h_0^{1/2}(s)ds\,.
\end{aligned} \tag{3.43a}$$

In order to obtain a variable-coefficient cKdV (VC-cKdV) equation we put

$$R(T) = -\tfrac{2}{3}\alpha\gamma^{-1}a^2(T)h_0^{\,2}(T),$$

$$K(T) = -\beta^{-1}a(T)\int_{T_0}^{T} h_0^{-1/2}\Big[Q_0 + \tfrac{1}{8}\Sigma^2\Big(\frac{\Sigma'}{\Sigma} - \frac{\Delta'}{\Delta}\Big)^2\Big], \tag{3.43b}$$

$$a(T) = \tfrac{9}{4}\alpha^{-1}h_0'(T_0)^{1/3}\Big[1 - \tfrac{9}{2}h_0'(T_0)\int_{T_0}^{T} h_0^{1/2}\,ds\Big]^{-1/3}.$$

With this choice we have

$$\tau = \tfrac{1}{12}\ln\Big[1 - \tfrac{9}{2}h_0'(T_0)\int_{T_0}^{T} h_0^{1/2}\,ds\Big]. \tag{3.43c}$$

The transformation (3.43a), together with the assumption that $q = q(\xi,\tau)$, reduces the GKP equation to

$$q_\tau - 6qq_\xi + q_{\xi\xi\xi} - 4\xi q_\xi - \Big[8 - \frac{3}{\alpha a^3 h_0^{1/2}}\Big(\frac{9h_0'}{2h_0} - \frac{\Delta'}{\Delta}\Big)\Big]q = 0. \tag{3.43d}$$

This VC-cKdV equation reduces to a pure cKdV equation (3.22c) if we impose

$$8 - \frac{3}{\alpha a^3 h_0^{1/2}}\Big(\frac{9h_0'}{2h_0} - \frac{\Delta'}{\Delta}\Big) = 2.$$

In view of (3.43b), this is just a relation between the width and the depth of the strait. It can easily be solved and yields the relations (3.42c) and (3.42d), expressing the function $\Delta(T)$ in terms of the depth function $h_0(T)$, or vice versa. The pure cKdV equation (3.22c) can be solved, e.g. by the inverse spectral transform method, and the bounded solutions $q(\xi,\tau)$ of this equation can then be transformed by (3.43a) into bounded wave amplitudes $\eta^0(X,T)$.

III. Examples of exact solutions

In this part we aim at constructing, in an explicit manner, a few *exact* solutions of the GKP equation by using some of the reductions, devised in part II, taking this equation into simpler integrable equations of KdV type. The GKP solutions that we shall thus obtain below can be viewed as *deformations* of functions satisfying these integrable equations. The practical motivation of what we present here is to show, through these examples, that the GKP equation, within its theoretical limitations, provides us with a realistic description of solitary surface waves propagating over geophysical regions characterized by non-flat topographies. In view of the many measurements and observations conducted throughout the oceans by many oceanographers (see Introduction and references mentioned therein), the GKP equation indeed proves to be of particular interest since, by construction, its solutions represent waves that exhibit some nontrivial *curvature*, transversally to the direction of propagation; such a behaviour, as we previously explained, is not allowed by theories based on the KdV equation.

We recall that reductions of the GKP equation were performed through the following transformations of variables [equation (3.25)]:

$$\begin{aligned} \eta^0(X,Y,T) &= R(T)q(\xi,\theta,\tau) + S(X,Y,T)\,, \\ \xi &= a(T)X + K(Y,T)\,, \\ \theta &= U(T)X + V(T)\,, \\ \tau &= \int_{T_0}^{T} P(s)ds\,. \end{aligned} \tag{3.44}$$

The quantities R(T), S(X,Y,T), a(T), K(Y,T), U(T), V(T), and P(T) are functions specific for each reduction and well defined in terms of the oceanographic topography; we shall specify them later. We also recall from (3.26) that we ask that the functions q and S be vanishing asymptotically with respect to the variables ξ and T, respectively. These requirements are necessary in order to guarantee that the solutions be asymptotically converging, as any physical solution should be. In fact, we shall be interested in solutions q of the KdV equations that are of solitary wave type.

In what follows, we shall consider situations corresponding to three distinct types of geophysical topographies, coresponding to three different behaviours of the depth function h_0, namely when its shape is quadratic, tangent hyperbolic, and logarithmic in the longitudinal variable x. We always assume that $h_1(T)$ is zero; if, for instance, we would want to describe waves propagating along a sloping beach, then we would set it differently. For the first two cases, we shall assume that the side boundaries are *absent*. In view of the nature of our perturbation scheme, it is to be noted that this does not exclude the actual presence of these boundaries, i.e. if the channel is sufficiently wide as well as shallow and the depth is slowly increasing near the sides, then the side boundaries have no measurable influence on the behaviour of the wave within the validity range admitted by the perturbation scheme, namely at and sufficiently near to the x-axis. For the logarithmic shaped bottom, we shall treat both cases where the boundaries may or may not be present. In the latter case, recall that there then exists a quite restrictive constraint between the depth function and the width functions (actually the total width $\Delta(T)$ of the channel) which must be satisfied in order that the reduction to the KdV equation be possible.

We point out that even though the formulas describing the reductions above considered were well defined, there arises a practical problem when *inverting* the reductions. Indeed the transformation formulas involve integrals which are typically *difficult* to perform because the integrands are typically complicated functions expressed in terms of the depth function, and therefore the class of depth functions for which we can get results analytically (i.e. without using numerical recipes) proves to be quite limited; the above choices fall within that class.

Let us begin with the case of no side boundaries. Note that our results will be expressed in terms of the variables X, Y, and T, the wave frame coordinates, related to the *physical* variables by equation (3.11); recall that T and X really play the roles of space and time, respectively. Recall also that the function S(X,Y,T) can be chosen, with no restriction, to be vanishing identically. Further, the function C_1 may be set equal to zero as well; thus C(Y,T) becomes independent of Y (C = 1) and non desirable secular terms disappear from the perturbation scheme. Referring to section II.2, the remaining functions appearing in

(3.44) are

$$\begin{aligned} &U(T) = 1, \qquad V(T) = 0, \qquad a(T) = 1, \qquad P(T) = \tfrac{1}{6}\alpha h_0^{1/2}(T), \\ &R(T) = -(2\alpha/3\gamma)h_0^2(T), \qquad K(Y,T) = -(9h_0'/4\beta h_0^{3/2})Y^2, \end{aligned} \tag{3.45}$$

where it is implicitly assumed that the function K_2 and the vorticity moment Q_1 vanish, and Q_0 is set to zero as it does not play any important physical role; the moment Q_2 is determined by the constraint (3.32d). We point out that the reduction generated by (3.45) is consequently not the most general one that yields the KdV equation, but it is sufficient for our purposes. The generic one-soliton solution of the KdV equation is (see, e.g., CAL1)

$$q(\xi,\tau) = -2\nu^2 \operatorname{sech}^2[\nu(\xi - \xi_0) - 4\nu^3\tau],$$

where ν is some real constant. Performing the transformation (3.44) we get the following deformed soliton solution of the GKP equation:

$$\begin{aligned} &\eta^0(X,Y,T) = (4\alpha\nu^2/3\gamma)h_0^2(T)\operatorname{sech}^2[\nu H(X,Y,T) - \nu\xi_0], \\ &H(X,Y,T) = X - (9h_0'/4\beta h_0^{3/2})Y^2 - \tfrac{2}{3}\alpha\nu^2\int_{T_0}^{T} h_0^{1/2}(s)ds. \end{aligned} \tag{3.46}$$

From now on, without loss of generality, we set the phase factor ξ_0 to be zero. If the depth function $h_0(T)$ is smooth enough (actually slowly varying) then we observe that the wave amplitude attains a local maximum when $H = 0$. Thus, solving this equation for fixed values of X (i.e. "time"), we obtain an equation for the crest of the deformed soliton solution of the GKP equation:

$$Y^2 = \tfrac{4}{9}\beta h_0^{3/2}(h_0')^{-1}\left(X - \tfrac{2}{3}\alpha\nu^2\int_{T_0}^{T} h_0^{1/2}(s)ds\right). \tag{3.47}$$

When h_0 is a constant, (3.47) is no longer valid, but $H = 0$ then just describes a straight line, as H is independent of Y, and we thus recover the description of the usual standard KdV soliton. Note that our transformation (3.44), (3.45) can be used with *any* KdV solution $q(\xi,\tau)$; here we have chosen q to be the one-soliton solution but we could have considered quasi-periodic solutions (cnoidal functions) as well.

1 - Case of a parabolic depth function

Our first example of oceanographic topography is that of a depth $h_0(T)$ given as a quadratic function of the variable T,

$$h_0(T) = 1 + a^2 T^2, \tag{3.48a}$$

where "a" is a constant characterizing the rate of steepness of the bottom; it is physically reasonable to choose it to be relatively small, especially if we want to use formula (3.47). Without any loss of generality we can set $T_0 = 0$. We have chosen the form of $h_0(T)$ such that $h_0(0) = 1$ and, more important, such that $h_0'(0) = 0$; thus we are able to *connect* this solution, at $T = 0$, to a *straight* solitary wave satisfying the pure KdV equation in the domain $T \leq 0$. According to equation (3.47), the wavecrests are then described by

$$Y^2 = \frac{2\beta(1+a^2T^2)^{3/2}X}{9a^2T} + \frac{\alpha v^2}{3a}\left[\, aT(1+a^2T^2)^{1/2} + \ln(aT + (1+a^2T^2)^{1/2})\right]. \tag{3.48b}$$

The curves defined by (3.48b) are illustrated on the upper hand graphs of Figures 4 and 5, where we have plotted Y as a function of the variable T, for fixed values of X. In these figures, and the others that will follow, the lower hand graphs represent the geophysical configuration of the bottom; note however that the scales of Y and $h_0(T)$ are *not* equal. We have chosen the following constant values: $a = 0.05$ and $\alpha = \beta = v = 1$. These curves represent the time evolution of a wavecrest for positive values of T, beginning at the instant T = 0. Note that for $X = 0$, the crest is given by $T = 0$ and is therefore a straight line identified with the Y-axis; this is consistent with the fact that the GKP solution is a pure KdV solitonic front for negative values of T. Figure 4 shows that as "time" X increases, the wavecrest begins to acquire some curvature and quickly takes the form of a *bell*. Eventually, as seen on Figure 5 which is drawn for values of X taken at a much larger scale, the wavecrest gets pinched and evolves into a *horseshoe* curve. This pinching effect, which occurs for large enough values of X, is *not* physical; it simply reflects the limitation of the validity range of the solution along the Y direction. As time increases, this lateral validity range decreases and it is thus a sign that the solution (3.47) is a good description only for a limited period of time; equivalently, we can say that this solution lacks precision as the depth gets larger and larger. We also point out that the fact that the non flat features of this solution are essentially localized around $Y=0$ is a natural consequence of the perturbation scheme yielding the GKP equation: the expansion is done in a sufficiently small domain around $Y=0$. It is however worth mentioning that these horseshoe-like solitons are very reminiscent of the type of solitary waves seen to emerge from the Strait of Gibraltar into the Mediterranean Sea (see Figure 2 and LAV1).

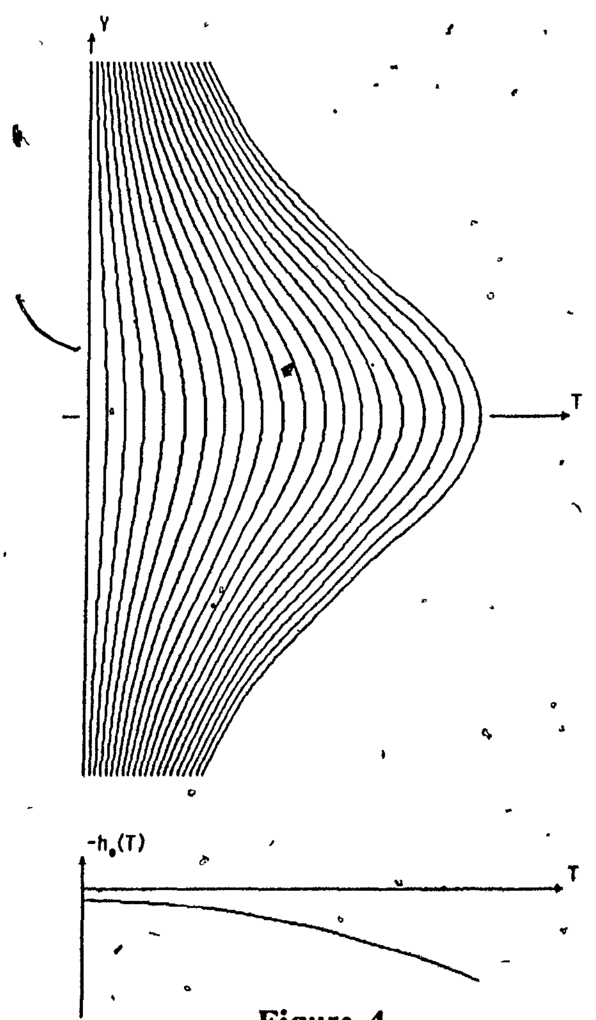


Figure 4

Solution of the GKP equation associated with a bottom of the form $1+(aT)^2$ with $a=0.05$ and $\alpha=\beta=\nu=1$. The curves represent the "time" evolution of the shape of the wavecrest from $X=0$ to $X=0.1$. The last curve meets the T-axis at about $T=0.15$ and Y ranges from -14.5 to 14.5.

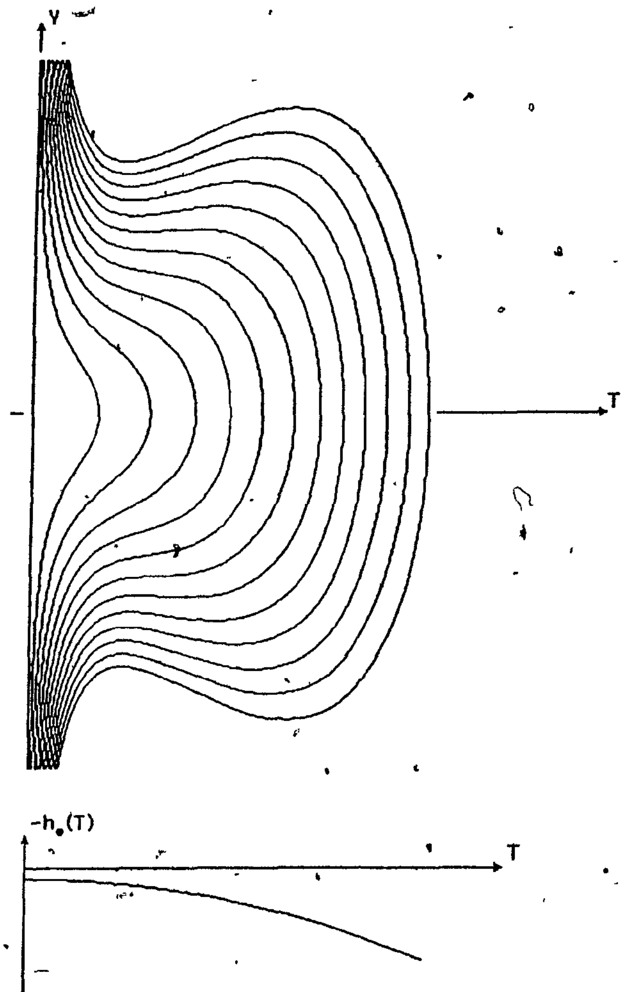


Figure 5

Solution of the GKP equation associated with a bottom of the form $1+(aT)^2$ with $a = 0.05$ and $\alpha = \beta = \nu = 1$. The curves represent the "time" evolution of the shape of the wavecrest from $X = 0$ to $X = 120$. The last curve meets the T-axis at about $T = 86$ and Y ranges from -54.5 to 54.5.

2 - Case of a hyperbolic tangent depth function

As our second example, let us consider a depth function shaped as a hyperbolic tangent of the form

$$h_0(T) = A + B\tanh[\mu(T-\kappa)]. \tag{3.49a}$$

For the sake of physical plausibility, we shall specify this formula a little bit more. We actually require that the following conditions be satisfied:

$$h_0(0) = 1, \qquad \lim_{T\to\infty} h_0(T) = \delta, \qquad \lim_{|T|\to\infty} h_0'(T) = 0.$$

The last condition is an identity which is already satisfied; the two others determine the values of the constants A and B. We thus get

$$h_0(T) = \frac{1+\delta\tanh(\mu\kappa) + (\delta-1)\tanh[\mu(T-\kappa)]}{1+\tanh(\mu\kappa)}. \tag{3.49b}$$

Finally, making a further change of variables $r = e^{2\mu(T-\kappa)}$, with $r_0 = e^{-2\mu\kappa}$, and introducing the quantity $\omega = 1+(1-\delta)e^{-2\mu\kappa}$, the depth function then writes as

$$h_0(r) = (\omega+\delta r)/(r+1). \tag{3.49c}$$

In this form, h_0 describes a bottom which varies from depth ω, when $T \to -\infty$ (i.e. $r \to 0$), down to δ when $T \to \infty$ (i.e $r \to \infty$), the variation being essentially localized around $T = \kappa$ with a relative steepness characterized by the constant μ. In view of the calculation of the integral defining the variable τ, it is *necessary* to require that $\omega > 0$, this is consistent with physical reality, namely that the bottom remains always submerged under water. Observe that we shall not connect the solution corresponding to this type of geophysical situation with an *initial* KdV solitonic front, due to the form of the depth function, it is clear that the GKP solution will asymptotically tend to such soliton solutions (with two distinct amplitudes), being essentially different from them only for values of T near of κ. Furthermore, it is easy to show that $h_0'(T=0) < 2\mu/[\delta(\delta-1)]$; thus if the rate of steepness is sufficiently small with respect to the final depth, then the GKP solution is near to the KdV solution even at $T = 0$. It is consequently reasonable to choose, here also, $T_0 = 0$. Note that other (negative) values for

T_0 are also permitted; $T_0 \to \infty$ is however forbidden because the integral defining the variable τ then becomes singular. Performing the transformation of variables (3.44), we then obtain that the wavecrests are defined by the following expression:

$$Y^2 = \frac{2\beta(r+1)^{1/2}(\omega+\delta r)^{3/2}}{9\mu(\delta-\omega)r}\Big[X - \frac{2\alpha v^2}{3\mu}\Big(\delta^{1/2}\ln\{[\delta(r+1)]^{1/2}+(\omega+\delta r)^{1/2}\}\Big|_{r_0}^{r}$$
$$- \omega^{1/2}\ln\{[\omega(1+1/r)]^{1/2}+(\delta+\omega/r)^{1/2}\}\Big|_{r_0}^{r}\Big)\Big]. \tag{3.49c}$$

The curves defined by this equation are illustrated in Figures 6 and 7. As in the previous case, we have plotted Y as a function of T, for several fixed values of the variable X. We have fixed $\delta = 2$ and $\alpha = \beta = \mu = v = \kappa = 1$. As predicted above, for values of T near zero, Figure 6 clearly shows (see the curve furthest to the left) that the wavecrest is almost a vertical straight line and therefore very near to a pure soliton; the same is true for large enough values of T. As a matter of fact, when a solitonic front departs from $T = 0$ rightwards, its curvature increases and attains a maximum when the wave is crossing the line of steepest descent of the bottom, at $T = \kappa$. We observe that the magnitude of this curvature is quite small, except when Y becomes sufficiently large: then it gets dramatically large and induces a catastrophe similar to that which occured for the parabolic bottom, namely a pinch of the wavecrest occurs (Figure 7). For values of T larger than κ, we see that the wave now loses its curvature and asymptotically recovers it flatness; qualitatively speaking, it gets flat again relatively rapidly. The pinching seen in Figure 7, where curves are drawn for a larger domain of the variable Y (the X-range is of the same order), expresses the fact that our solution remains valid in a limited range of values for the transverse variable Y. Within this range, however, it behaves reasonably for all positive values of T; in other words, it gives a good description for an unlimited range of the physical time. This also was to be expected a priori from the form of the depth function and is related to the fact that the solution evolves smoothly between two distinct KdV solutions. From an oceanographical point of view, this solution presents some interest. Indeed, it models typical solitary waves with similar behaviour when crossing elevations, e.g. the border of a plateau (see for instance FU1).

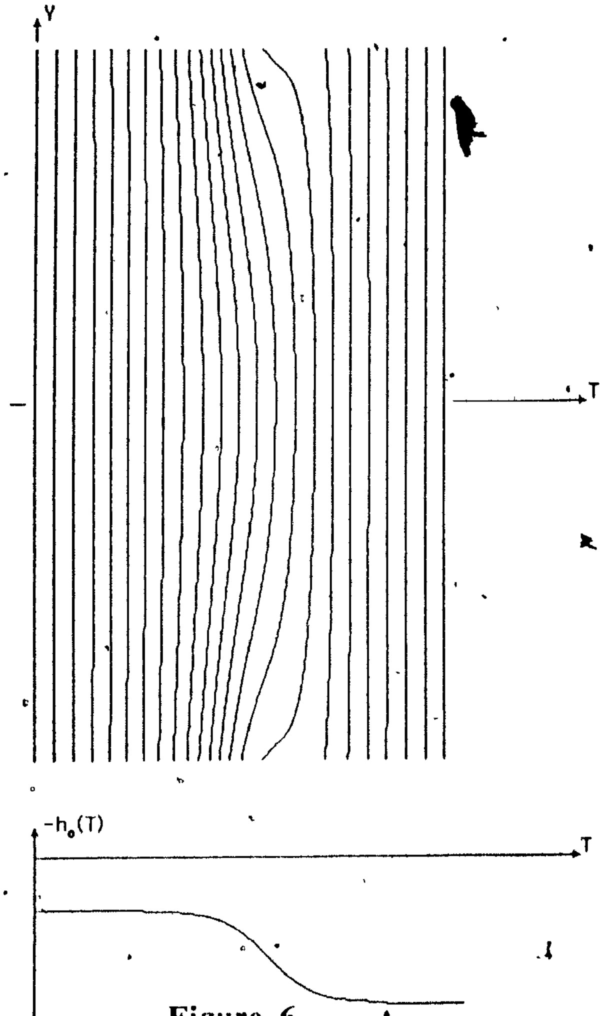


Figure 6

Solution of the GKP equation associated with a hyperbolic tangent shaped bottom given by (3.49b) with $\delta = 2$ and $\alpha = \beta = \mu = \nu = \kappa = 1$. The curves show the "time" evolution of the shape of the wavecrest from X = -1.26 to X = 5.34. The last curve meets the T-axis at about T = 6 and Y ranges from -1.09 to 1.09.

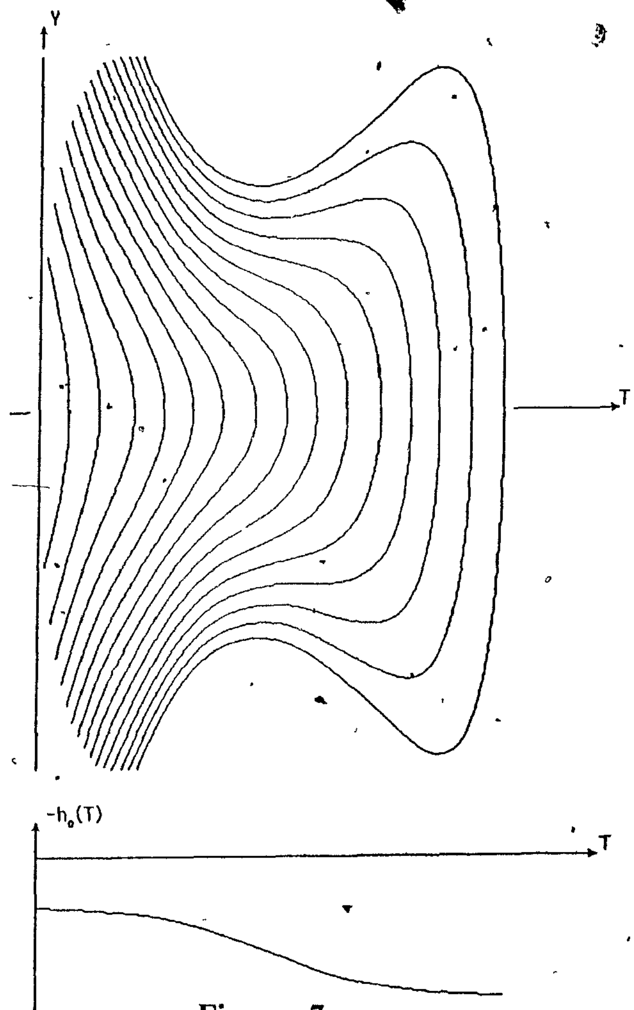


Figure 7

Solution of the GKP equation associated with a hyperbolic tangent shaped bottom given by (3.49b) with $\delta = 2$ and $\alpha = \beta = \mu = \nu = \kappa = 1$. The curves show the "time" evolution of the shape of the wavecrest from $X = -0.47$ to $X = 2.52$. The last curve meets the T-axis at about $T = 3$ and Y ranges from -2.04 to 2.04.

3 - Case of a logarithmic depth function

Our third example is that of a bottom exhibiting a logarithmic behaviour; we represent it by the following depth function,

$$h_0(T) = 1 + \tfrac{1}{3} \ln(1+cT)\,, \tag{3.50a}$$

where the constant "c" is assumed to be relatively small. The factor $\frac{1}{3}$ was chosen for future convenience, but can be replaced by another constant. Here also, since $h_0'(T)$ never vanishes except for $T \to \infty$, the GKP solution associated with this prescription for the bottom cannot be matched to a KdV solution; initial data would rather consist of some slightly curved disturbance that could be generated, for instance, by dropping a mass of appropriate shape at $T_0 = 0$. For this case, it is not possible to integrate in closed form for τ, the solution can however be expressed in terms of an exact formal power series. Performing the transformation (3.44) and calculating (3.47) accordingly yields the following formula for the wavecrests:

$$Y^2 = \tfrac{4}{3}(\beta/c)(1+cT)[1+\tfrac{1}{3}\ln(1+cT)]^{3/2}\,(X - \tfrac{4}{9}\alpha v^2 L/c)\,, \tag{3.50b}$$

$$L = 3^{1/2}e^{-3}\sum_{j=0}^{\infty}\frac{[3+\ln(1+cT)]^{j+3/2} - 3^{j+3/2}}{(2j+3)\,j!}\,. \tag{3.50c}$$

The time evolution of a crest is illustrated in Figure 8. Notice that we have approximated the function L by truncating the infinite sum: we kept the first six terms; this is reasonable because an elementary numerical analysis of the first partial sums shows that their magnitudes decrease rapidly. Also note that the curvature of a crest does not exhibit a great variation as the wave propagates and that no pinching phenomenon occurs.

Let us now consider, for the same depth function, the case when side boundaries are present; this situation genuinely deals with waves moving within a channel. The reduction of the GKP equation is then more intricate. First we recall that we must have (3.42c) satisfied, namely the depth function must be such that $h_0''(T_0) + 3[h_0'(T_0)]^2 = 0$, the choice (3.50a) made above trivially verifies this condition for any T, and also satisfies the constraint (3.26c) requiring that the function S(X,Y,T) in (3.44) be asymptotically vanishing for large values of

T. Once the depth function is fixed, the total width $\Delta(T)$ of the channel is completely determined; using (3.42c) we find

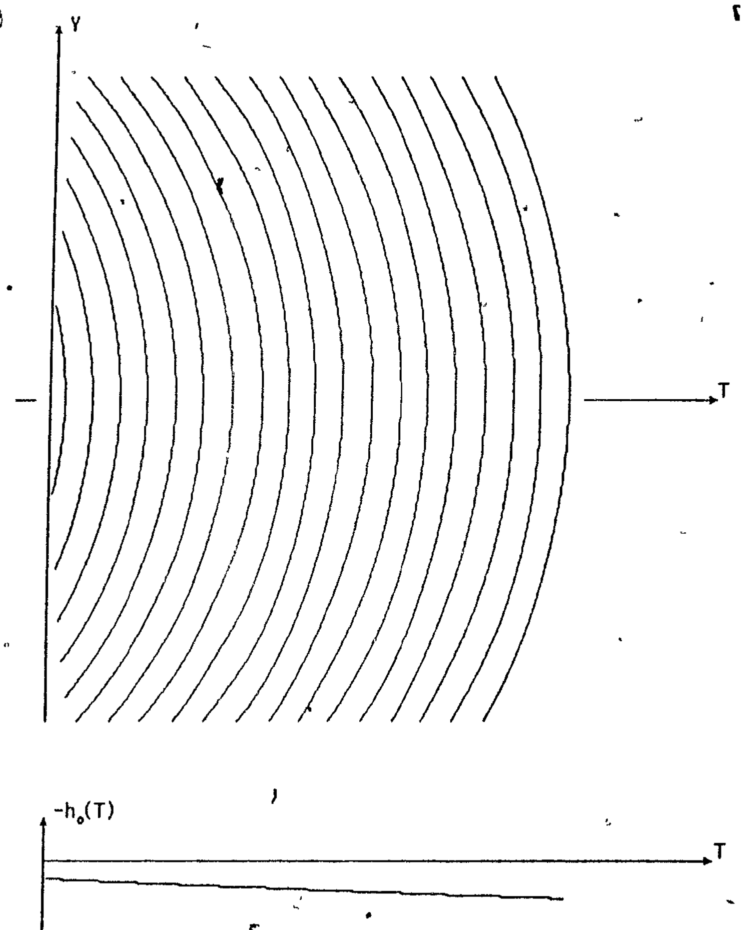


Figure 8

Solution of the GKP equation associated with a logarithmic shaped bottom given by equation (3.50a) with $c = 0.1$ and $\alpha = \beta = \nu = 1$. The curves show the "time" evolution of the shape of the wavecrest from $X = 0$ to $X = 8$. The last curve meets the T-axis at about $T = 6$ and Y ranges from -5 to 5.

$$\Delta(T) = \delta h_0^{-9/2}[1+L(T)], \tag{3.50d}$$

where L(T) is defined by (3.50c), and $\delta = \Delta(T_0)$. We then proceed to calculate the functions defining the transformation of variables (3.44), following the prescriptions of section II.3. We get

$$U(T) = 1, \qquad a(T) = \frac{-3\gamma}{2\alpha\delta^{1/2}(1+L)^{1/2}}, \qquad \tau = \frac{3\gamma^3[(1+L)^{-1/2} - 1]}{4c\alpha^2\delta^{3/2}},$$

$$V(T) = 0, \qquad S(T) = \frac{ch_0^2 X}{2\gamma(1+L)}, \qquad R(T) = \frac{-3\gamma h_0^2}{2\alpha\,\delta(1+L)},$$

$$K(Y,T) = \frac{3\gamma}{2\alpha\beta}\int_0^T \frac{Q_0(s)ds}{\Delta^{1/2}h_0^{11/4}}, \tag{3.50e}$$

$$Q_0 = \tfrac{1}{24}\Delta'^2 + h_0^{-9/2}\Delta^{-5/3}\{Q_{00} + \tfrac{1}{48}c\int_{T_0}^{T} h_0^{1/2}\Delta^{2/3}\Delta'^2 ds.$$

Let us introduce the constants

$$B_0 = 3\gamma v^2/2\alpha\,\delta, \qquad B_1 = c\alpha\,\delta/3\gamma^2 v^2, \qquad B_2 = 3\gamma/2\alpha\delta^{1/2}, \tag{3.50f}$$

and the following additional quantities:

$$M = 3^{1/2}c^{-1}e^{-3}\sum_{j=0}^{\infty}\frac{[3+\ln(1+cT)]^{j+1/2} - 3^{j+1/2}}{(j+\frac{1}{2})\,j!}, \tag{3.50g}$$

Recall that the Y dependence of the solution is hidden in the variable X only; in order to make it explicit we must revert to the *physical* variables x, y, and t through the transformation (3.11). Thus $T = \varepsilon x$; calculating the function C(Y,T) up to order ε, we then find that X writes as .

$$X = \frac{M}{\varepsilon} - t + \frac{y^2\Delta'}{2\beta h_0^{1/2}\Delta} + \int_0^T h_0^{-1/2}C_0 ds. \tag{3.50h}$$

Using (3.39a), we finally obtain the solution of the GKP equation in the form

$$\eta^0 = B_0 h_0^2(1+L)^{-1}\left(\mathrm{sech}^2\,[B_2(1+L)^{-1/2}H] + B_1X\right),$$

$$H = \frac{M}{\varepsilon} - t + \frac{y^2\Delta'}{2\beta h_0^{1/2}\Delta} + G(T). \tag{3.50i}$$

where

$$G(T) = B_0^{-1}(v\xi_0 + K)(1+L)^{-1} + 4 - \int_0^T h_0^{-1/2}C_0(s)ds.$$

As the function $C_0(T)$ is arbitrary, it proves convenient to choose it so as to make the function G(T) to vanish. The curves representing the wavecrests cannot be expressed in closed form; indeed the equations defining the local maximum of (3.50i) are transcendental in the variable X. These equations can however be reasonably well solved through a numerical scheme. In view of the explicit use of a non-constant form for the auxiliary function C(Y,T) which implies that secular terms do occur in the perturbation expansion yielding the GKP equation, it must be pointed out that the solution (3.50i) gives a good description only in a sufficiently small time interval [$t < \varepsilon^{-2}$; see discussion following (3.10)]. We present the behaviour of the solution in the pictures of Figure 9 and 10. Figure 9 was done for a relatively small interval along the positive longitudinal direction. As in our precedent example, this figure does consist in a succession of *snapshots* that follow a wavecrest along the positive region of the physical x-axis. The time evolution starts at $t = 0$ when the crest, by construction, consists of a straight line identified with the y-axis. As time increases, the crest begins to exhibit some curvature, the rate of which is increasing. In fact, the magnitude of this curvature is connected to the rate at which the straits gets wider, i.e., to $\Delta'(\varepsilon x)$. It grows and diminishes with this quantity; thus it is maximal when Δ' is itself maximal. This is better illustrated by Figure 10 which shows the evolution of the crest for a longer time. Asymptotically, the crest regains its straightness. This example is to be compared with the waves seen in existing straits, for instance the Straits of Gibraltar and Messina, which do indeed behave as the example presented here.

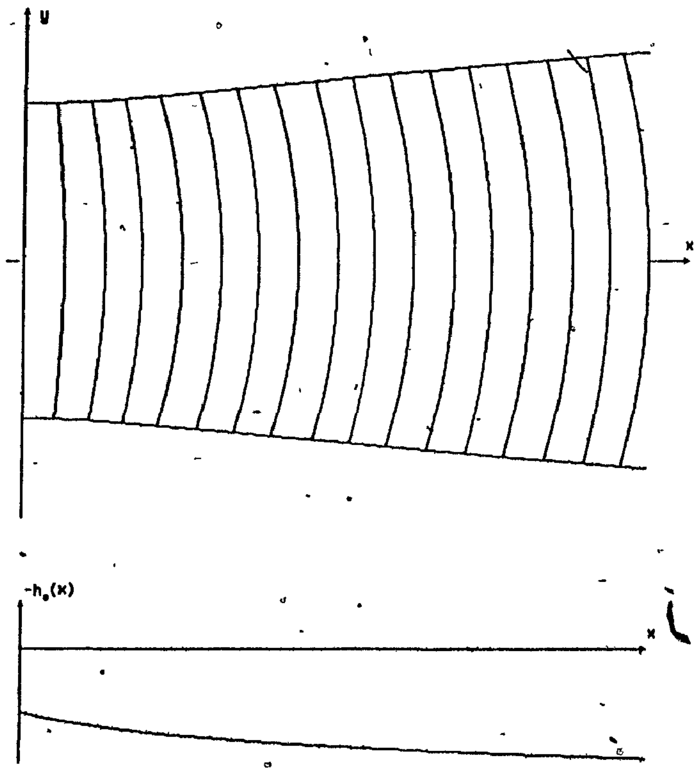


Figure 9

Solution of the GKP equation associated with a logarithmic shaped bottom given by equation (3.50i) with $c = 32$ and $\alpha = \beta = \nu = 1$. The curves show the time evolution of the shape of a wavecrest enclosed in a strait from $x = 0$ to $x = 5$. The halfwidth of the strait ranges from 0.5 to about 0.658.

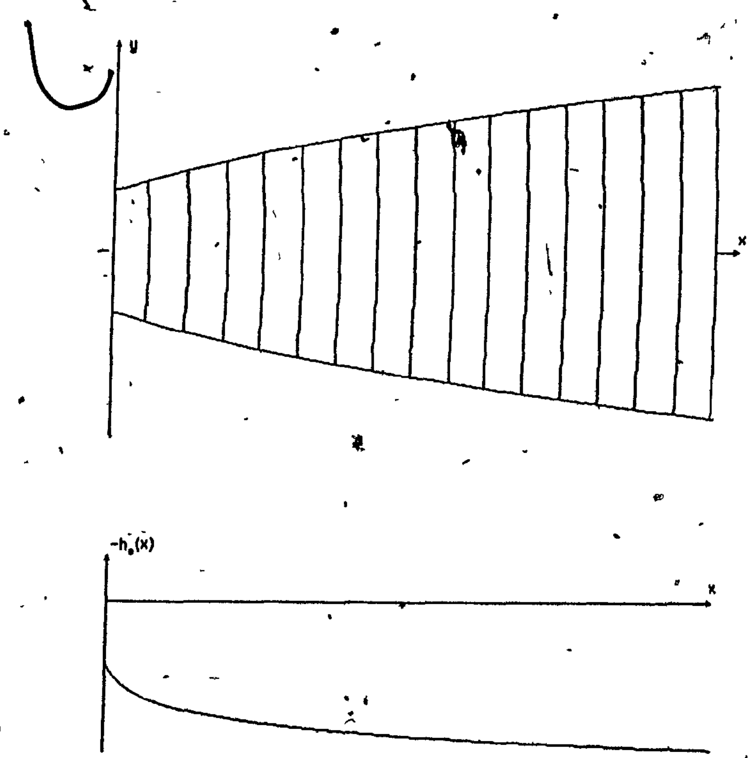


Figure 10

Solution of the GKP equation associated with a logarithmic shaped bottom given by equation (3.50i) with $c = 32$ and $\alpha = \beta = \nu = 1$. The curves show the time evolution of the shape of a wavecrest enclosed in a strait from $x = 0$ to $x = 40$. The halfwidth of the strait ranges from 0.5 to about 1.353.

IV. Conservation laws associated with the GKP equation

The GKP equation (3.16a) which we derived in part II admits several time dependent conserved quantities of physical interest. These can be obtained directly from the equation and their form depends on whether we are taking the side boundaries into account or not; we shall consider these two cases separately. Also, in contrast with the pure KP equation for which it is well known that infinitely many conserved quantitites do exist, the GKP equation possesses only a finite number of them.

1 - Conserved quantities for the GKP equation without boundary conditions

In this first case we are considering a wave with amplitude specified by $\eta^0(X,Y,T)$, propagating on the surface of an infinite extension body of water. We restrict ourselves to the class of solutions of the GKP equation that decay sufficiently rapidly in all directions:

$$\eta^0(X,Y,T) \to 0 \qquad \text{as } |X| \to \infty \ \text{ or } \ |Y| \to \infty .$$

We assume that this behaviour holds for the amplitude $\eta^0(X,Y,T)$ and for $\eta^0{}_X$, $\eta^0{}_{XX}$, and $\eta^0{}_Y$ as well. We then have $M_3(Y,T) = 0$ in (3.16a) [see (3.20a)]. Choosing $C_1(Y,T) = 0$, as we always can when no boundaries are involved, we obtain

$$A_0 = A_0(T) = -\tfrac{1}{2}\int_{T_0}^{T} h_0^{-3/2}\, h_1\, ds ,$$

$$M_1 = M_1(T) = h_0'/4h_0 .$$

We now introduce the following quantities:

$$M(T) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \eta^0(X,Y,T)\, dXdY ,$$

$$E(T) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\eta^0(X,Y,T)]^2\, dXdY ,$$

$$I_X(T) = M^{-1}(T)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X\eta^0(X,Y,T)\, dXdY ,$$

$$I_Y(T) = M^{-1}(T)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Y\eta^0(X,Y,T)\,dX\,dY\,,$$

corresponding, respectively, to the mass, the energy, and the X and Y coordinates of the centre of inertia of the propagating wave. From the GKP equation we can determine the time evolution of these quantities; we find:

$$M(T) = M(T_0)h_0^{-1/4}(T)\,, \tag{3.51a}$$

$$E(T) = E(T_0)h_0^{-1/2}(T)\,, \tag{3.51b}$$

$$I_X(T) = I_X(T_0) + \tfrac{3}{4}\gamma M^{-1}(T_0)E(T_0)\int_{T_0}^{T} h_0^{-7/4}\,ds$$
$$+\int_{T_0}^{T}\Big[\tfrac{1}{8}\beta h_0^{1/2}\Big(\int_{T_0}^{s} h_0^{-3/2} h_1\,dr\Big)^2 + \beta^{-1}h_0^{-1/2}Q_0\Big]\,ds\,, \tag{3.51c}$$

$$I_Y(T) = I_Y(T) - \tfrac{1}{2}\beta\int_{T_0}^{T} ds\, h_0^{1/2}\int_{T_0}^{s} h_0^{-3/2}h_1\,dr\,. \tag{3.51d}$$

These quantities are obtained by first integrating, over the whole XY-plane, the GKP equation multiplied by 1, η^0, X, and Y, respectively: this step gives differential equations which may then be integrated to yield the above results. For instance, integrating the GKP equation over X and Y yields the differential equation $\partial M(T)/\partial T + \frac{1}{4}(h_0'/h_0)M(T) = 0$, which obviously has the solution (3.51a). We note that the expression (3.51c) for $I_X(T)$ holds only if M_2, in (3.16a) and (3.16c), is independent of Y, i.e. if we have

$$\int_{T_0}^{0} \phi_0(Y,Z,T)\,dZ = (\beta\gamma)^{-1}h_0^{2}(T)Q_0(T)\,,$$

i.e, if $Q_1(T)$, $Q_2(T)$, and all higher moments vanish. The conserved quantities are $M(T_0)$, $E(T_0)$, $I_X(T_0)$, and $I_Y(T_0)$. The mass M(T) and the energy E(T) are only conserved in the special case when the depth is constant along the physical y-axis, or, in other words when $h_0(T) = 1$ in

$$h(Y,T) = h_0(T) + \varepsilon Y h_1(T) + O(\varepsilon^2)\,.$$

The centre of inertia of the wave moves along the X-axis with constant velocity only when $h_0(T) = 1$, $h_1(T) = 0$, and $Q_0(T) = 0$, implying constant depth and vanishing vorticity. We then have

$$I_X(T) = I_X(T_0) + \tfrac{3}{4}\gamma M^{-1}(T_0)E(T_0)(T - T_0),$$
$$I_Y(T) = I_Y(T_0).$$

If the depth function $h_0(T)$ increases with $T = \varepsilon x$, e.g. if a wave moves outwards into sea, we see from (3.51a) and (3.51b) that its mass $M(T)$ and energy $E(T)$ will decrease accordingly: the wave dissipates. From (3.51c) and (3.51d) we see that if $h_1(T) \neq 0$, i.e. if the shape of the bottom is Y-dependent, then the centre of inertia of the wave moves away from the X-axis, in general with nonconstant velocity.

2 - Conserved quantities for the GKP equation with boundary conditions

In this second case, we consider the GKP equation (3.16a) with the boundary condition (3.16b). Similarly as in subsections 3 and 4 of part II, we assume that we can reduce this boundary condition to the standard form

$$\eta^0{}_{YY}(X,Y,T)\Big|_{Y = l_{\pm,0}} = 0$$

and then solve it by assuming

$$\eta^0 = \eta^0(X,T) \tag{3.52}$$

throughout the entire channel, so that the physical y-direction enters only in an implicit way via the variable X in (3.14). As we have already established, the above asumption (3.52) is consistent only if $C_1(Y,T)$ is chosen as in equation (3.37a), so that $A_0(Y,T)$ has the form (3.35). The physical assumptions made are that the vorticity satisfies (3.26c) with $Q_2(T)$ and $Q_1(T)$ given by (3.38a) and (3.38b), respectively. The GKP equation is thus reduced to the equation

$$\eta^0{}_T + \tfrac{3}{2}\gamma h_0^{-3/2}(T)\eta^0\eta^0{}_X + \tfrac{1}{6}h_0^{1/2}(T)\eta^0{}_{XXX} + M_1(T)\eta^0 + M_2(T)\eta^0{}_X = 0 \tag{3.53}$$

with M_1 and M_2 given by

$$M_1 = \tfrac{1}{4}(h_0'/h_0 + 2\Delta'/\Delta),$$

$$M_2 = \frac{1}{\beta\sqrt{h_0}}\left[Q_0 + \tfrac{1}{8}\Sigma^2\left(\frac{\Sigma'}{\Sigma} - \frac{\Delta'}{\Delta}\right)^2\right].$$

Solutions of the GKP equation (3.53) satisfying (3.52) and vanishing sufficiently rapidly, namely

$$\begin{aligned} &\eta^0(X,T) \to 0 \quad \text{as} \quad |X| \to \infty, \\ &\eta^0{}_X(X,T) \to 0 \quad \text{as} \quad |X| \to \infty, \\ &\eta^0{}_{XX}(X,T) \to 0 \quad \text{as} \quad |X| \to \infty, \end{aligned} \tag{3.54}$$

will again be subject to conservation laws, following directly from (3.53). This time let us define

$$m(T) = \int_{-\infty}^{\infty} \eta^0(X,T)\, dX,$$

$$e(T) = \int_{-\infty}^{\infty} [\eta^0(X,T)]^2\, dX,$$

$$i(T) = \int_{-\infty}^{\infty} X\eta^0(X,T)\, dX,$$

again representing the mass, energy, and position of the centre of inertia of the wave. From (3.53) and (3.54) we have

$$m(T) = m(T_0)h_0^{-1/4}(T)\Delta^{-1/2}(T), \tag{3.55a}$$

$$e(T) = e(T_0)\Delta(T_0)h_0^{-1/2}(T)\Delta^{-1}(T), \tag{3.55b}$$

$$\begin{aligned} i(T) = i(T_0) &+ \tfrac{3}{4}\gamma m^{-1}(T_0)e(T_0)\Delta^{1/2}(T_0)\int_{T_0}^{T} h_0^{-7/4}\Delta^{-1/2}\, ds \\ &+ \tfrac{1}{8}\beta^{-1}\int_{T_0}^{T} h_0^{-1/2}\left[\Sigma^2\left(\frac{\Sigma'}{\Sigma} - \frac{\Delta'}{\Delta}\right)^2 + 8Q_0\right]. \end{aligned} \tag{3.55c}$$

These quantities are obtained in a similar way as that which gave us (3.51), namely by integrating the GKP equation (3.53), multiplied by an appropriate factor, and then by integrating the resulting differential equation. Again, we find that $m(T_0)$, $e(T_0)$, and $i(T_0)$ are conserved quantities. The mass $m(T)$ and energy $e(T)$ are only conserved if the geometry of the channel is such that

$$h_0^{1/2}(T)\Delta(T) = c_0 = \text{constant}.$$

The centre of inertia undergoes uniform rectilinear motion only if both integrands in (3.55c) are constants.

The integrable equation of part II, i.e. the pure KP, KdV, and cKdV equations, admit infinitely many conservation laws, expressed in terms of the function $q(\xi, \theta, \tau)$ or $q(\xi, \tau)$ (see NOV1, CAL1). These, in general, do not necessarily go over conservation laws for the amplitude η^0. The transformation of variable may introduce new τ-dependence and may also cause the integrals for η^0 to become divergent. For a discussion of conservation laws for the VC-KdV equation see also references JOH1 and MIL2.

CONCLUSIONS

The present thesis presents the results obtained from the investigations about two main problems, connected by the fact that both are related to the Kadomtsev-Petviashvili equation. In this final chapter we summarize the results obtained in this work, emphasizing the most important ones, and point out additional precisions in some cases. We also state several problems which arise from our work: some being already under investigation and thus excluded from this thesis because of incompleteness, the others being open at this date.

Chapter two orbits around the method of invariant solutions, or symmetry reduction technique. We analysed the symmetry structure of the pure KP equation, found several classes of special solutions (part I), and presented a new way of applying the symmetry reduction technique by means of an example on the system made of the KP equation coupled with its associated Bäcklund transformation (part II). The starting point for this chapter, as it is when studying the group-theoretical aspects of any given differential equation, is provided by giving the Lie algebra L_{KP} of symmetries of the KP equation. This algebra was already known since some time (see SCH1) but remained completely unexploited except for deducing some very special invariant solutions of the equation, namely the so-called similarity solutions.

Our first practical goal, in part I, is inspired by three facts. The first is that many of the important nonlinear partial differential equations of modern physics turn out to have infinite-dimensional symmetry groups; such is the case with the KP equation, as well as others such as the Jimbo-Miwa equation and the Davey-Stewartson [or (2+1)-dimensional Schrödinger system]. The second one is that methods have been developed for classifying subgroups of finite-dimensional Lie groups that constitute the basic ingredients for performing the symmetry reduction technique; it turns out, as we have shown in this chapter and as was shown in subsequent works (e.g. CHA2), that these methods can also be applied to infinite-dimensional Lie groups. The third fact concerns the recent developments in the theory of infinite-dimensional Lie algebras, in particular Kac-Moody algebras, and the realization of the important role that these play in the study of integrable dynamical systems.

Our very first result was to observe that L_{KP} possesses a Levi decomposition [see (2.4)], even though it is infinite-dimensional; this property is useful in view of classifying its low-dimensional subalgebras. Our second result consisted in the integration of L_{KP}, i.e. in constructing the explicit group action of the most general 1-parameter subgroup of the corresponding Lie group G_{KP} on the solution manifold coordinatized by (t,x,y,u), where $u(t,x,y)$ solves the KP equation. We provided three different group actions [see (2.5), (2.6) and (2.7)], for the three cases $f(t) = g(t) = 0 \neq h(t)$, $f(t) = 0 \neq g(t)$, and $f(t) \neq 0$, where $f(t)$, $g(t)$, and $h(t)$ are arbitrary smooth functions which parametrize the elements of the symmetry algebra (or infinitesimal generators). By construction, this group action gives us a means of constructing new solutions of the KP equation from given ones. For instance, let us consider the infinitesimal group generator $e^{-t}\partial_y + \frac{2}{3}\lambda\sigma y e^{-t}\partial_x - \frac{4}{9}\sigma y e^{-t}\partial_u$, corresponding to the particular choice $f(t) = h(t) = 0$ and $g(t) = e^{-t}$; then formulas (2.6) tell us that if $u = 0$ is a solution of the KP equation then the same is true of $\frac{2}{9}\lambda\sigma e^{-t}[\lambda e^{-t} - 2y]$. We point out, however, that the group action is *linear* with respect to the dependent variable u, and therefore it does not permit us to recover all solutions, for instance the *multi-soliton* solutions, these are rather obtained through the use of Bäcklund transformations. Such transformations, in general, do not generate a group, they do, however, when the associated nonlinear differential equation can be written as the compatibility condition of a first order matrix system of Zakharov Shabat type: one can easily show that the space of Bäcklund transformations is then isomorphic to a matrix group whose elements are known as the "dressing matrices" (see Introduction)

After explicitating the particular six dimensional subalgebra L_P of L_{KP} obtained by restricting the three arbitrary functions $f(t)$, $g(t)$, and $h(t)$ to be linear in their argument [see (2.9)] and noting that this is the subalgebra which has been extensively used by several authors in order to get specific invariant solutions of the KP equation, we proceeded to examine another special subalgebra L_π of L_{KP}, obtained by restricting $f(t)$, $g(t)$, and $h(t)$ to admit Laurent expansion. It was shown that this algebra possess a Kac-Moody loop structure:

$$L_\pi = (R[t,t^{-1}] \otimes L_0) \oplus R[t,t^{-1}]d/dt,$$

where $L_0 \subset sl(5,R)$ is an eight-dimensional Lie algebra [see (2.11)]. The grading in this loop structure is provided. This property of the symmetry algebra of the KP equation is quite remarkable. Indeed it was realized *a posteriori* that this property is also shared by other (2+1)-dimensional integrable nonlinear equations of evolution type, for instance the Davey-Stewartson system; however no (3+1)-dimensional equations, to our knowledge, have this property. This is therefore something *very special* and is certainly worth of further investigation; in particular, one would be interested in knowing *what* are the necessary structural conditions on the equation that will ensure the existence of this property.

In section 2, we attacked the problem of classifying the low-dimensional subalgebras of L_{KP}. As mentioned above, this is the first step that must be taken before actually performing the symmetry reduction of the KP equation. As explained in chapter one, the classification of these subalgebras (into optimal systems) is done by grouping together in the same equivalence class all those subalgebras that are conjugate under the adjoint action of the group G^{l}_{KP} We also identify the isomorphy class of each subalgebra. It was interesting to observe, as we mentioned earlier, that the tools and the techniques for constructing this classification, although developed for finite-dimensional algebras, also work for L_{KP}, even though this algebra is infinite-dimensional. The classification was done for subalgebras of dimension lower than or equal to 3, but it would prove interesting, from the group-theoretical point of view, to provide the classification for higher dimensional subalgebras. This would allow us to approach, in a very systematic manner, the question of *symmetry breaking* for the KP equation, i.e., the construction of related equations, invariant under *subgroups* of the KP symmetry group, rather than under the *entire* group; this remains an open problem as yet. The classes of 1-dimensional subalgebras were first derived and we found that there are three of them: $\{X(1)\}$, $\{Y(1)\}$, and $\{Z(1)\}$ [see (2.17)]. These are the most useful ones in view of symmetry reduction. The classes of 2-dimensional subalgebras were then found. There are two infinite families of Abelian ones, plus five others, two of these being also Abelian [see (2.18)]. We finally constructed the 3-dimensional ones and found more diversity: 3 doubly infinite families, 3 simply infinite families, and 11 other specific classes. These are listed in (2.22) according to seven types: Abelian, decomposable, nilpotent, diagonal action, complex action, Jordan action, and simple; there was only a single class of the latter kind, namely the class with sl(2,R) as a representative, given by $\{X(1), X(t), X(t^2)\}$ [see (2.22g)].

In the third section of chapter one we applied the symmetry reduction technique, using representatives of each conjugacy class of 1-dimensional subalgebras of L_{KP}. Using subalgebras of dimension 1, the method provides us with reductions to 2-dimensional equations for a function $q(\xi,\eta)$. Note, as explained in chapter one, that this is generic; the decrease in dimension is related to the rank of the optimal systems which are used: in our case, our representatives of the equivalence classes are obviously of maximal rank

Because we found three distinct conjugacy classes of nonconjugate 1-dimensional subalgebras, we therefore expected three distinct reductions. As a matter of fact, it turned out that the KP solution $u(t,x,y)$ is always related to the solution $q(\xi,\eta)$ of the reduced equation in the same way, namely u is linear with respect to q [see (2.23)]. Among the reductions that we derived, one is almost trivial: we obtained $u = q$ and the reduced equation is $\partial^2 u/\partial y^2 = 0$, the corresponding invariant solutions are parametrized by two arbitrary functions [see (2.24)]. The other reductions are more interesting. One yields a once-differentiated KdV equation [see (2.25)] and the other gave the Boussinesq equation [see (2.26)]. After having established these reductions and the corresponding formulas for the invariant solutions, we proceeded to give some examples. For the reduction to the Boussinesq equation, we started from a class of solutions of this equation obtained by Boiti and Pempinelli (see BO12) and constructed two invariant solutions of the KP equation which are expressed in terms of the first Painlevé transcendent and the Weierstrass elliptic function, respectively [see (2.29a) and (2.29b)]; the latter was seen to reduce, in a certain limit, to the so-called lump solution. We wish to point out that we interested ourselves in finding solutions that remain asymptotically bounded, since these are the type of solutions which are of genuine physical interest. As for the reduction to the KdV equation, in view of the preceding remark, we considered two distinct situations corresponding to the fact that the KdV solution from which one reconstructs a KP solution may be either bounded or unbounded. In the first case we deduced that the KP solution is bounded provided that a certain constraint is satisfied. As an example, we gave KP solutions which can be *rotated* from a similarity solution of the KdV equation expressed in terms of the second Painlevé transcendent [see (2.30) and the following paragraph]. For the second case, it proved convenient to modify the class representative which yields the reduction to the KdV equation. This new choice was such that the reduced equation was the cylindrical KdV equation (2.31c) with solution (2.31d) given in terms of Airy functions; the corresponding KP solutions are then given by (2.31b).

We closed the section by briefly discussing symmetry reduction using 2- and 3-dimensional subalgebras. In the first case, we remarked that it always amounted to first performing one of the above reductions by using a 1-dimensional subalgebra, followed by a second reduction on the reduced equation; thus KP solutions may be reconstructed via two consecutive reductive transformations. Moreover, as the reductions of both the KdV and the Boussinesq equation have been studied to some extent by others, we restricted ourselves to a single simple example of a direct reduction, that which yields the first Painlevé equation. In the second case, we mentioned that all reductions performed by using a 3-dimensional subalgebra were trivial and thus present no real interest. Let us mention that the reconstruction of KP solution from solutions of a reduced equation can, of course, be coupled with the group action induced by G_{KP} on the solution manifold of the KP solution. Thus, by combining the results of sections 1 and 3, we arrive at a very large set of KP solutions.

The other problem treated in chapter two deals with the method of symmetry reduction again. The motivation was that it should make sense to use this method on a given soliton equation together with its associated Bäcklund transformation, rather than on the equation alone; then one should be inclined to think that the solitonic character would appear naturally. From a practical point of view, one would like to combine together reductive and Bäcklund transformations; this is what we undertook to do with the KP equation. We first remarked that it proves useful to use the KP equation in its potential form, the PKP equation (2.35), because the associated Bäcklund transformation (2.36) is indeed known for the latter. We gave the symmetry algebra L_{PKP} of this equation and stated that it is quite similar, from the structural point of view, to L_{KP} [see (2.37) and (2.38)]; we observed, in particular, that a loop structure also exists for the "Laurent restriction" of this algebra. We constructed the group action for the several 1-parameter Lie groups corresponding to each single generator of the symmetry algebra, and then examined how the group acts on the Bäcklund transformation; three generators leave it invariant but the remaining two others generalize its form by introducing time dependent functions in it [see (2.40)]. Naturally, this implied that the form of the permutability theorem was to be accordingly modified. Our next step was to find the symmetry algebra of the system constituted by the PKP equation and the generalized

Bäcklund transformation; the result was a four-dimensional algebra [see (2.43)] for which we constructed the corresponding 1-parameter subgroup actions.

We then proceeded to examine the question of symmetry reduction. First, we found out that there are four classes of 1-dimensional subalgebras of the above 4-dimensional algebra, two of them being actually parametrized families of classes. It is important to point out that the reduction will naturally give us two objects: a reduced equation and a reduced Bäcklund transformation expressed as a differential equation. One may then solve this system and reconstruct a solution of the PKP equation; the corresponding KP solution can then be obtained by performing a further differentiation on the PKP solution. We realized that five distinct representatives are sufficient to exhaust all possible reductions. Among these, let us first mention that one yields a *kink* solution for the PKP equation, thus a soliton solution for the KP equation. Another yields an oscillatory solution that can be written in terms of Airy functions. The most interesting reduction, in one limit case, gave us a quite special KP solution which we termed *splitton*. This object was graphically seen to be a sort of fork-shaped solitonic front which rigidly translates in the xy-plane; it is related to a certain type of solutions, called soliton resonances, which were previously found and studied by several authors for the Boussinesq equation. We emphasized that this type of solution must be considered on the same footing as the usual soliton solutions in the sense that they also originate from applying a Bäcklund transformation to a zero solution. We recall that for one of the reductions we could not construct any PKP solutions because we were not able to find analytic solutions to the reduced system [see (2.48b) and (2.51b)], this system merits deeper investigation. More generally, we point out that the work presented here could be repeated for other equations; in particular, the idea of joint symmetry reduction seems most promising.

The principal practical outcome of chapter two is that we gave means of generating infinitely many solutions of the KP equations. We may start from a given KP solution and find another one by acting with the KP symmetry group. We can also reconstruct some from solutions of one of the equations that are obtained by symmetry reduction. Let us point out that these reduced equations, for instance the KdV equation, are well studied and have their own symmetry groups which one may use to generate new solutions. Further, they admit reductions to ordinary differential equations whose solutions also yield new invariant solutions, and sometimes even Bäcklund transformations that may be used to the same purpose through nontrivial nonlinear correspondences. Noting the connection between the KP and the PKP equation, we may take a KP solution, map it into a PKP solution, apply the Bäcklund transformation, and map back the resulting PKP solution into a new KP solution. New solutions may also be constructed directly from a PKP solution which may result from another PKP solution via the PKP symmetry group, or by applying the symmetry reduction technique on the PKP equation alone or on the system consisting of the PKP equation and its associated Bäcklund transformation.

In chapter three we investigated the possibility of using certain known integrable equations of water wave theory in order to describe nontrivial solitonic wave behaviour. By this we mean solitonic wavefronts which exhibit some transverse curvature. This has a physical interest to the extent that experimental observations confirm that such objects indeed exist in oceans, for instance tidal waves going through and out of marine straits. The basic objects of our model are the *generalized Kadomtsev-Petviashvili* equation (3.16a), together with the boundary condition (3.16b), which describe the propagation of gravity waves on the surface of a fluid in a channel, or a strait, characterized by varying depth and width. This equation was derived under the following assumptions:

1) the fluid has zero viscosity coefficient and constant density,
2) the only force present is gravity,
3) the quantities $(H_0/L_x)^2$, H_0/L_y, N_0/H_0 are all small and of the same degree [see (3.4)],
4) the waves propagate predominantly in one direction chosen to be the x-axis,
5) we restricted to waves moving in a sufficiently wide, slowly varying channel.

The GKP equation has variable coefficients that depend on the variables $Z=z$, $Y=y$, and $T=\epsilon x$ via the physical functions

$$h_0(T), \quad h_1(T), \quad l_{\pm 0}(T), \quad \phi_0(Y,Z,T), \quad \psi_0(Y,Z,T),$$

describing the geometry of the strait and the vorticity in the fluid; ϕ_0 and ψ_0 enter either in integrals over all values of Z, or for $Z=-h_0(T)$. The coefficients also depend on an auxiliary function $C_1(Y,T)$ which we chose to be zero if no side boundary conditions are imposed. The choice of $C_1(Y,T)$ given by equation (3.27a), in terms of the geometrical functions $h_0(T)$, $h_1(T)$, $\Delta = l_{+0} - l_{-0}$, and $\Sigma = l_{+0} + l_{-0}$, leads to a standard form of the boundary conditions. It also leads, generically, to secular terms in the perturbative expansion.

The GKP equation (3.16a), together with its boundary condition (3.16b), was seen to form a coupled system for the unknown wave amplitude $\eta^0(X,Y,T)$ and the vorticity functions $\phi_0(Y,Z,T)$, $\psi_0(Y,Z,T)$. If we choose $X_0 = -\infty$ in these equations and make the additional physically reasonable assumptions (3.41a) (on the vorticity at the bottom) and (3.19) (on the behaviour of the perturbation, e.g. satisfied by solitary waves), we can partially decouple the equations. For the vorticity functions we obtained the linear relations (3.20), and for η^0 the GKP equation (3.17a) with boundary condition (3.18a). Notice that the function $M_2(Y,T)$ in (3.17a) depends on the vorticity function $\phi_0(Y,Z,T)$. Additional information is needed to determine this function completely. For instance, let us assume a polynomial dependence on the variable Y:

$$\int_{h_0}^{0} \phi_0 \, dZ = \sum_{k=0}^{N} Q_k(T) Y^k. \tag{c.1}$$

We then find that equations (3.17a) and (3.20) imply

$$\int_{h_0}^{0} \psi_0 \, dZ = R_0(T) - \beta^{-1} \sum_{k=0}^{N} (Q_k / h_0^{1/2})' Y^{k+1}/(k+1), \tag{c.2}$$

where

$$R_0(T) = \beta^{-1} \sum_{k=0}^{N} [Q_k h_0^{-1/2} (l_{+0})^{k+1}]'/(k+1)$$

and

$$\sum_{k=0}^{N} [Q_k h_0^{-1/2} \{(l_{+0})^{k+1} - (l_{-0})^{k+1}\}]'/(k+1) = 0. \tag{c.3}$$

Thus, the moments $Q_k(T)$ are subject to one constraint, namely (c.3), and then they determine the integrals (c.1) and (c.2) completely. These moments, in the approximation considered, must be taken as part of the input into the problem, together with the functions determining the bottom and the sides of the channel.

The GKP equation admits physically interesting conservations laws, discussed in part IV. The mass, energy, and centre of inertia of the considered wave behave in a physically reasonable manner. In particular, in straits for which the quantity

$$\chi(T) = h_0^{1/2}(T)\Delta(T) \tag{c.4}$$

increases (e.g. a strait opening up into the ocean and getting deeper), we saw that the mass and energy, as defined in section IV.2, gradually dissipate. If $\chi(T)$ decreases with T, as happens when a wave from the ocean enters the strait, the mass and energy increase and eventually our approximation breaks down (i.e. higher order terms in ε must then be taken into account).

An open natural problem would be that of extending our model in order to describe internal waves propagating on the interface between two layers of fluid. It is already known, for a KdV based theory, that these waves are proportional to the surface waves through a coupling between the equations that describe them. We should expect something similar if we would use the same kind of assumptions as in our present treatment of surface waves. In fact, we have done some calculations that we did not present here since they are incomplete. Let us only mention that some complications occur because, for instance, one has to consider as distinct cases situations when the upper layer have greater, equal, or smaller thickness than that of the lower layer. This problem is presently under investigation.

Part II of chapter three was spent on the effective reduction of the GKP equation to completely integrable equations with soliton solutions, under additional hypotheses on the channel and the vorticity; reductions were achieved by performing appropriate transformations of the variables of the form (3.24). First, a pure KP equation was obtained only if the side boundaries are absent, if the bottom is flat at least in the middle of the channel $[h_0(T) = 1]$, and if the second vorticity moment vanishes $[Q_2(T) = 0]$. The mass and energy are then conserved, as we observed in part IV, and in addition the KP equation admits infinitely-many further conserved quantities. Also for the case of no side boundaries, a pure KdV equation was obtained under different assumptions, namely that the initial wave amplitude is independent of the variable θ, that the vorticity satisfies (3.26), and that the depth function $h_0(T)$ and vorticity moment $Q_2(T)$ are related by (3.32d), a Riccati equation for the function h_0'/h_0. For both of the above reductions, the function $S(X,Y,T)$ in (3.24) was chosen to be zero; hence bounded solutions of the KP and KdV equations (in particular solitons, multisolitons, or periodic solutions of the KdV equation) provide bounded solutions of the GKP equation and therefore bounded physical wave amplitudes, as long as the function $R(T)$ in (3.24) remains finite.

When the side boundaries are taken into account, we could again obtain a KdV equation (section II.3), and managed to satisfy the boundary conditions in a trivial manner, by simply getting rid of the θ-dependence in the function q. We point out that further study of these conditions in order to solve them in a less trivial manner, leading to more general results, is certainly worth pursuing. The necessary conditions for reducing to the KdV equation are that the vorticity satisfies (3.26b), (3.38), (3.41a), and moreover that the depth and the width are related by (3.42c,d). In this case the function $S(X,Y,T)$ cannot be set equal to zero; instead we have

$$S = -\tfrac{2}{3}\gamma^{-1}a_1\Delta^{-1/2}h_0^{-1/4}\left[1 - a_1\int_{T_0}^{T}\Delta^{-1/2}h_0^{-1/4}\,ds\right]^{-1}X\,.$$

Thus bounded solutions of the KdV equation, i.e. for q bounded in (3.24), lead to amplitudes η^0 that increase linearly in X. As long as the quantity $\chi(T)$ [see (c.4)] is an increasing function of T, then condition (3.25c) is verified, i.e. the term linear in the variable X gradually dies out as T increases. From the point of view of mass and energy conservation (see section IV.2) this corresponds to the case $m(T) \to 0$, $e(T) \to 0$ as $T \to \infty$. It is worth pointing out that always, in this case, at the initial position $x_1 = x_1^0$, the functions describing the sides of the strait, together with their first derivatives, are continuous functions of x_1. An alternative approach to the GKP equation with boundary conditions was taken in section II.4, where we reduced to the cylindrical KdV equation. The assumptions on the shape of the strait and on the vorticity were exactly the same as for the reduction to the pure KdV equation; however the transformation of variables was different [see (3.47a,b,c)], and in particular we had $S(X,Y,T) = 0$ in (3.24). This proves interesting, as it implies that bounded solutions of the cKdV equation provide bounded wave amplitudes $\eta^0(\xi,\tau)$. The cKdV equation is known to have bounded solutions, expressed in terms of Airy functions (note that these can be transformed into unbounded solutions of the KdV equation). These solutions are clearly of physical interest; they have the form of trains of solitary waves with total zero area and could be used to fit oceanographic data on internal waves, observed at several places in the oceans.

Let us consider again the question of the dependence of the wave amplitude on the renormalized physical variables x, y, and t for the case when η^0 is assumed to be independent of Y. We start at some given point x_0 of the channel (i.e. at $T_0 = \varepsilon x_0$) and give the form of the initial perturbation which, by assumption, is independent of y, for instance a soliton (or other localized disturbance) in a straight section of the channel. At $T = T_0$ we have $C_1(Y,T_0) = 0$ [see (3.37a)], and the transformation (3.11) is hence simply

$$X = -t, \qquad Y = y, \qquad Z = z, \qquad T_0 = \varepsilon x_0 .$$

The initial physical amplitude $\eta^*(x_1{}^0,t^*)$ provides the initial amplitude for the GKP equation, namely

$$\eta^0(X,T_0) = \eta^*(x_1{}^0,t^*)/N_0$$

[see (3.3)]. The characteristics of the channel (h_θ, h_1, Σ, and Δ) are given, and we can, at least in principle, solve the Cauchy problem for the GKP equation, or one of the equations derived from it in part II. We then know $\eta^0(X,T)$, $C_1(Y,T)$, and can return to the physical amplitude $\eta^*(x_1, x_2, t^*)$.

A problem which definitely merits some further investigation is that of solving the boundary conditions. More precisely, one would be interested in trying to solve them in a less trivial way; recall that we simply got rid of them by requiring that the function $q(\xi,\theta,\tau)$ be independent of the variable θ; that obviously forced us to eliminate interesting situations such as solutions characterized by a standing wave behaviour along the direction transverse to that of propagation (the Y direction).

Part III of chapter three was spent on the explicit construction of a few exact solutions of the GKP equation. We actually restricted ourselves to examples illustrating the reduction of the GKP equation to the KdV equation; the GKP solutions thus obtained were therefore taking the shape of solitonic wave fronts characterized by some curvature induced by the reduction transformation [see equation (3.44)]. The first three examples were treated in the case when the side boundaries are absent; a fourth example was presented for the other case; these examples were characterized by specifying the shape of the bottom.

We first considered a bottom described by a depth function $h_0(T)$, flat for $T \leq 0$ and parabolic for $T \geq 0$; this kind of situation could describe waves going out of a strait (not near the coast however). We deduced that the GKP solution associated to this geophysical configuration is a wave whose crest is initially a straight line (because the solution is then a *pure* KdV soliton) which gradually becomes bell-shaped as time goes on and eventually deforms into a horseshoed curve. When this happens, it is indicative that the solution is propagating into a physical region, or time domain, where our perturbation expansion yielding the GKP equation is no longer valid. In particular, we noted that the solution has a finite domain of validity in the transverse direction; this is in fact characteristic of all of our examples. We then treated the case of a bottom specified by a depth function described by a scaled translated hyperbolic tangent; this could describe the dynamics of solitary waves when going over a region where the bottom gets deeper (or shallower) to a certain degree. In fact, we found that the GKP solution which we obtained for this kind of bottom has the following behaviour. In the shallower region, the crest of the wave is essentially rectilinear. When getting nearer to the localized region where depth changes, it gets curved and then eventually regains its flatness as the wave gets far in the deeper region. Our third example was that of a logarithmic shaped bottom. The main observation was that the wavecrest has a relatively constant curvature. Our last example treated the same kind of bottom as our third; however we included side boundaries, i.e. the wave is moving *in* a channel. In fact, this channel has a very definite form, described by its total width function $\Delta(T)$ which is completely determined in terms of the depth function, up to a multiplicative factor; we have assumed it to be symmetric with respect to the Y-direction [$\Sigma(T) = 0$]. Due to the necessity of allowing secular terms in the perturbation expansion in order to have nontrivial behaviour in the transverse direction, the corresponding GKP solution was seen to be valid only in a relatively small time interval. We also observed that the wave crest has some definite varying curvature and is indeed representative of the kind of solitary waves that are seen to emerge through existing marine straits.

Further examples must be looked at, particularly when side boundaries are present. For this case it proves quite hard to find depth functions that satisfy the constraint (3.42e) *as well as* the further condition that the quantity $h_0^{1/4}(T)\Delta^{1/2}(T)$ be monotonic increasing function (recall that Δ is determined by h_0); this is why we choose our depth function as we did above by requiring that (3.42e) be identically satisfied for any T. Let us also remark that it would be interesting to solve the Cauchy problem for the GKP equation by using numerical tools; despite the fact that the GKP equation is rather complicated, good algorithms have been developed in the recent years for solving such complex problems, based, for instance, upon time changing triangular grids.

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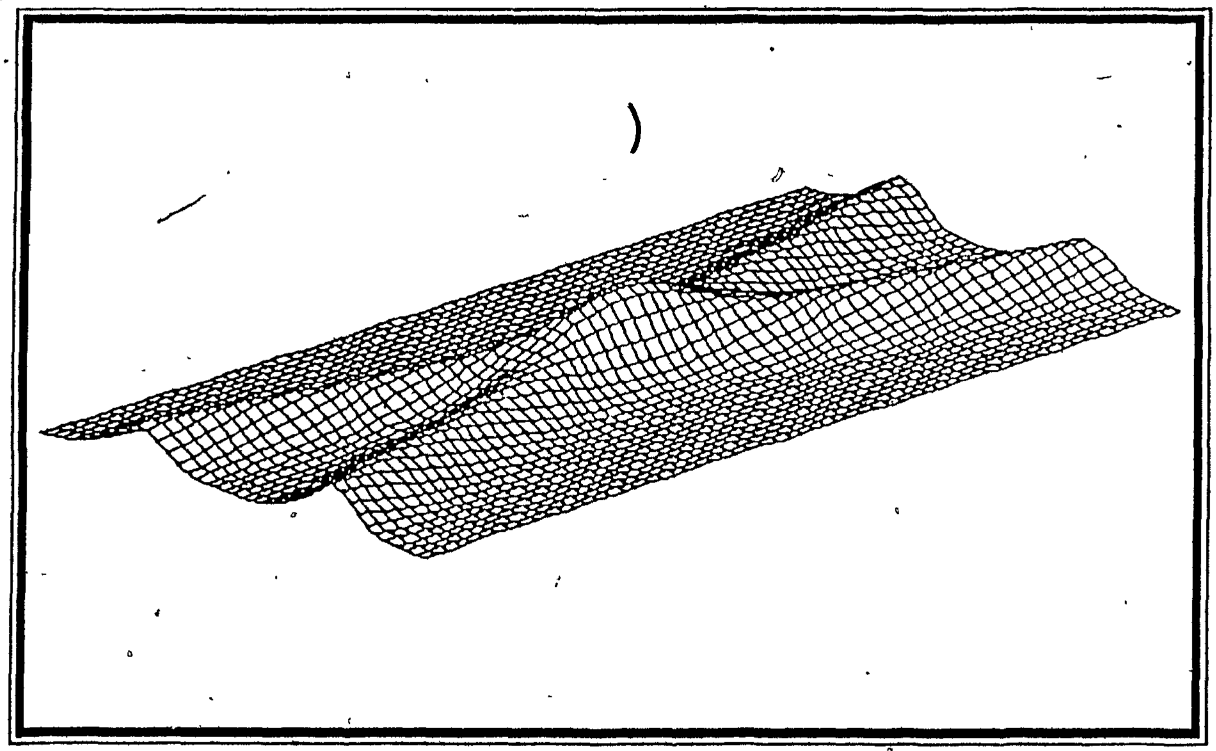
APPENDIX

Some Hidden Secrets of the KP Equation ... in Twenty Minutes

Lecture given at oral examination



Introduction

The subject of this afternoon's talk is the Kadomtsev-Petviashvili (thereafter called the KP) equation, an old favourite among the small world of integrable nonlinear partial differential equations. More specifically, I would like to present two main collections of results about two different aspects of this equation. The first results pertain to the possibility of using this equation, or rather a generalized form of it, as a model for the description of solitary waves, and their propagation, in oceans or through marine straits, or channels if you will (the terms "strait" and "channel" will be used interchangeably to denote a body of water appropriately bounded from below and from the sides). I shall show that this KP-like equation, although not integrable in its actual form, can be reduced to other equations from which one can build solutions which are indeed interesting from the geophysical point of view as they represent real curved wavefronts that are actually observed in the oceans. The second part of the talk will be devoted to group theoretical aspects of the KP equation. We shall look at its symmetries and use them, in concomitance with the so-called "method of symmetry reduction", to generate many special solutions that could be quite difficult to obtain by using standard tools, and even impossible to guess. Details have been reduced to a minimum and can be found in the thesis or our publications.

1. Solitary waves in oceans and marine straits

One certainly interesting and indeed quite common geophysical phenomenon in oceans is the occurence of solitary waves. Figure A1 is a reproduction of a quite famous picture representing such waves in the Andaman Sea. Notice the regularity of the waves. Figures A2 and A3 provide another example; Figure A2 shows solitary waves emerging out of the Strait of Gibraltar and Figure A3 sketches similar waves over a topographic map. Marine solitons do not only occur in exotic places: Canada has its own ones, for instance in the Georgia Strait, B.C., as depicted by Figure A4. Figure A5, reproduced from a military weather satellite's data taken over the Sulu Sea, is interesting as it shows how a few wave packets (dashed lines) evolved through a time period of about 4 hours (continuous lines). The picture is about 600 kms wide and we can extrapolate the displacements for a period of 12 hours. Roughly, one gets a displacement of about 100 kms; this is quite consistent with the fact that the waves propagate at a speed of about 2.4 m/s. Typical internal waves have amplitudes of the order of 50 m. Surface waves have much smaller amplitudes but they are more easily observed, for instance on photographs taken from satellites. As for internal waves, these are rather deduced from measurements taken by chains of thermistors placed at different depths in the water. These are known to be generated, directly or indirectly, through tidal processes. As an example, consider the Strait of Gibraltar. One of the tidal processes that is tought to cause the waves is an indirect one, namely that there occurs a semi-diurnal reversal of one internal strata current; at the entry of the Strait, it interacts with the incoming flow from the Atlantic Ocean by pushing upwards the top layer. In fact, there is experimental evidence that oceanic solitons always occur in pairs of internal and surface waves and it should be mentioned, to dissipate one's expected belief, that an internal wave have a negative amplitude with respect to that of a surface wave, i..e. the disturbance is a depression rather than a lump. The model to be presented here describes surface waves only, for a constant density fluid; it must however be pointed out that it is also valid as a model for internal solitary waves moving in multilayered fluid to the extent that both kind of waves are governed by a pair of two equations, each of them being the same as the one which will be used here, up to numerical factors in the coefficients that can be scaled away.

Figure A1
Packet of solitary waves in the Andaman Sea

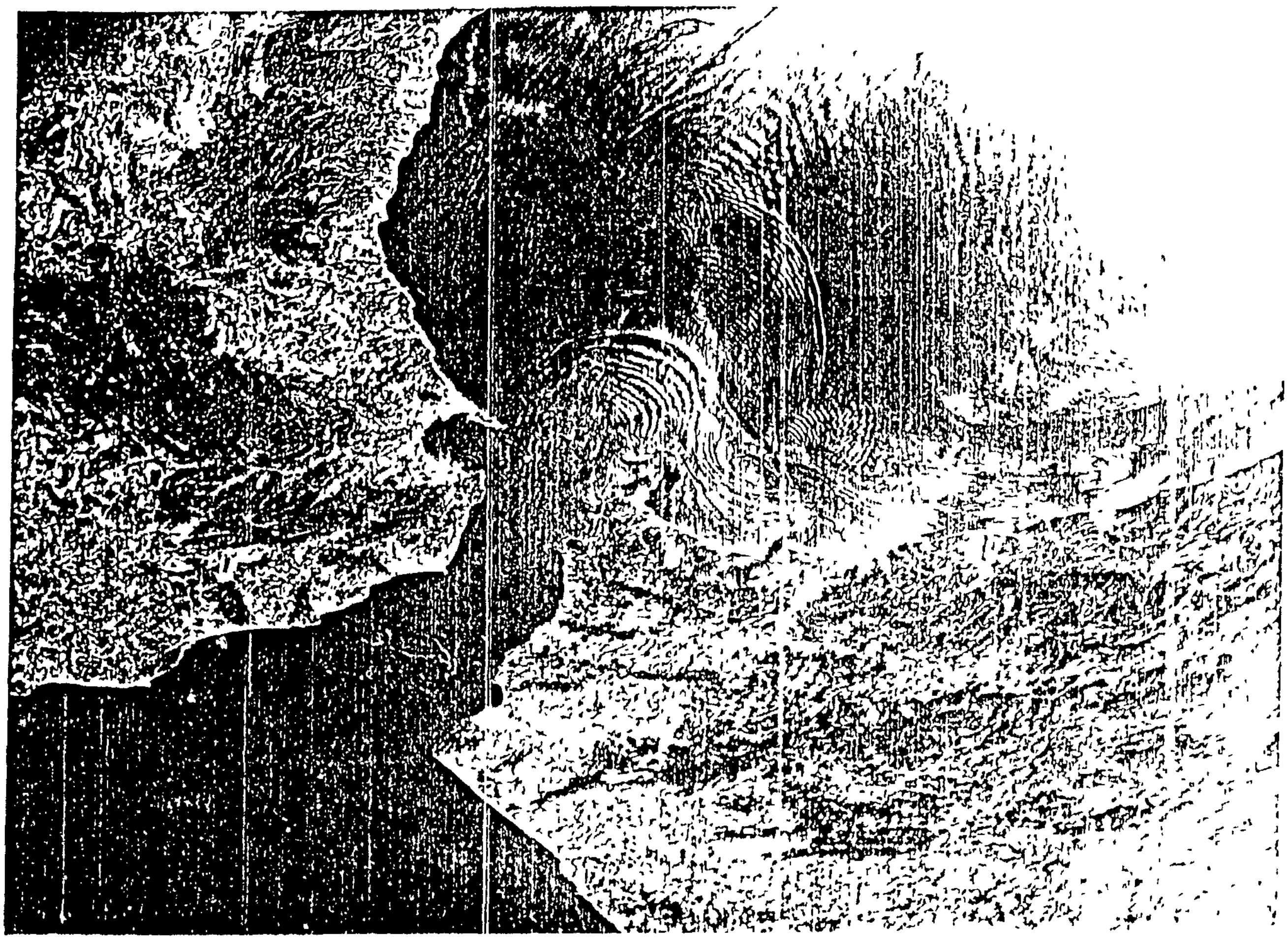


Figure A2

Solitons emerging out from the Strait of Gibraltar into the Mediterranean Sea

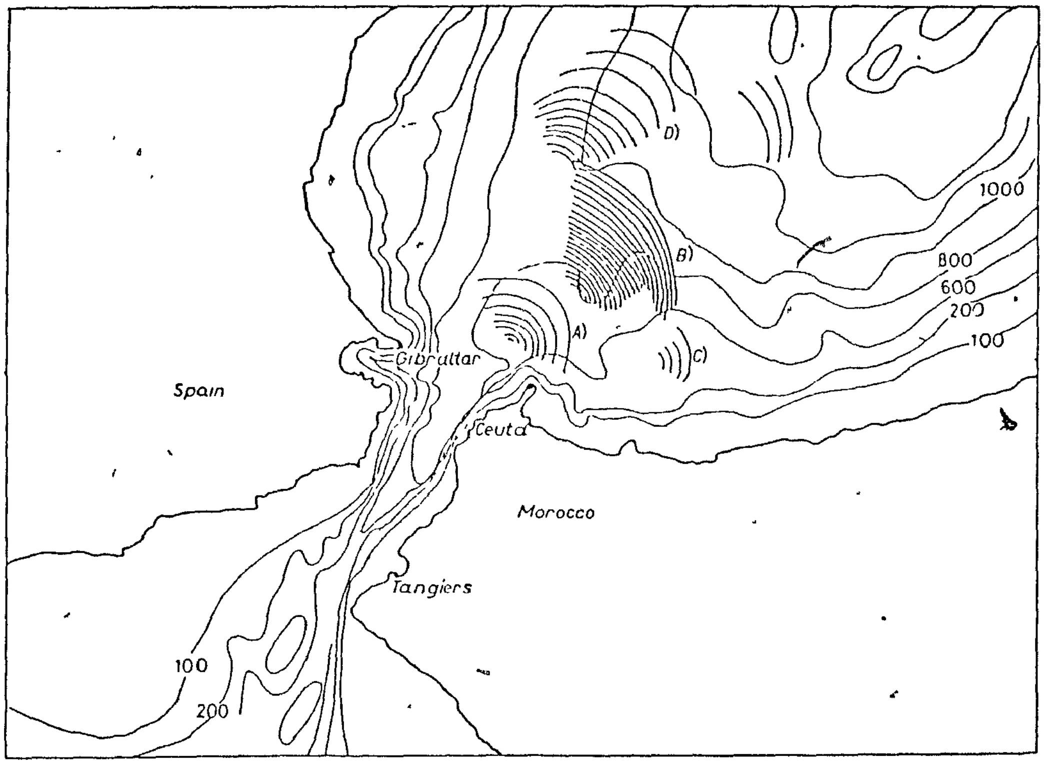


Figure A3

Topography of the entry of Mediterranean Sea

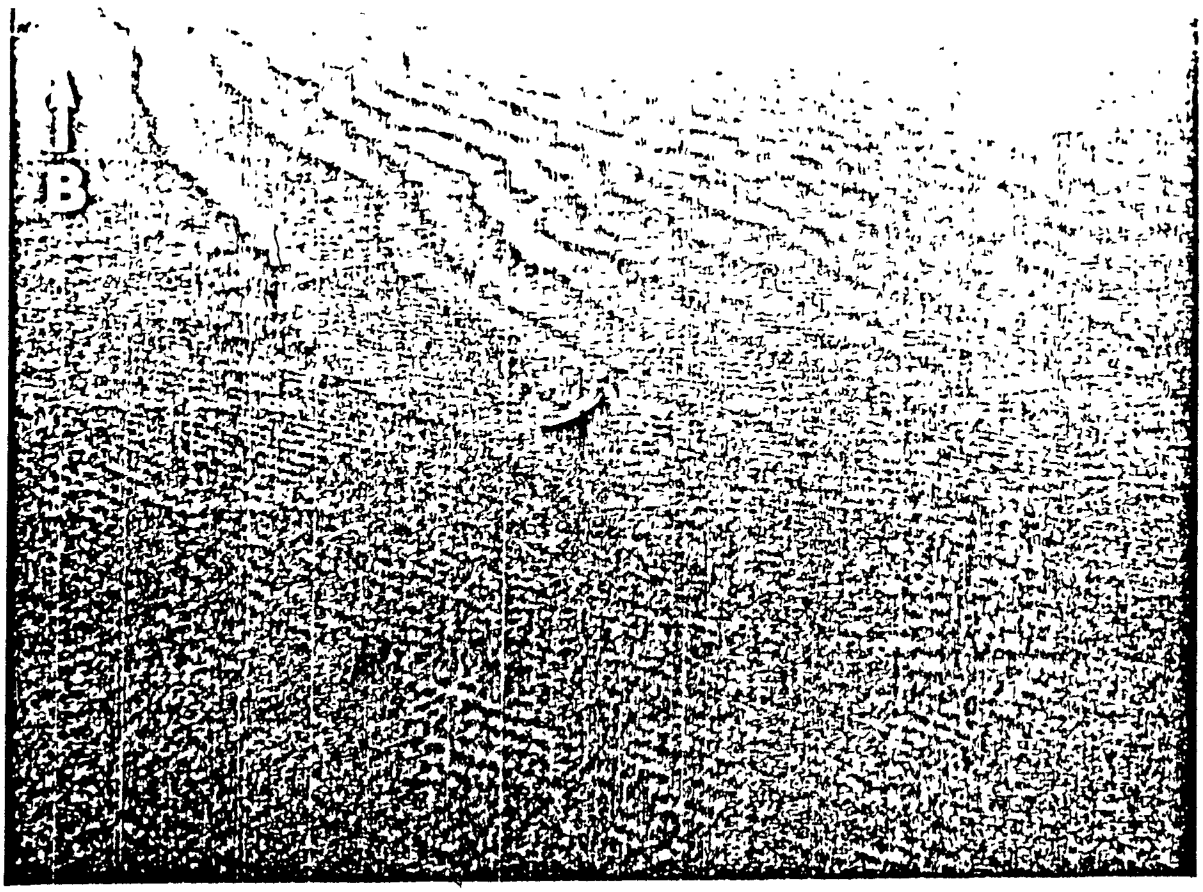


Figure A4

Surface manifestation of internal waves in the Georgia Strait, B.C.

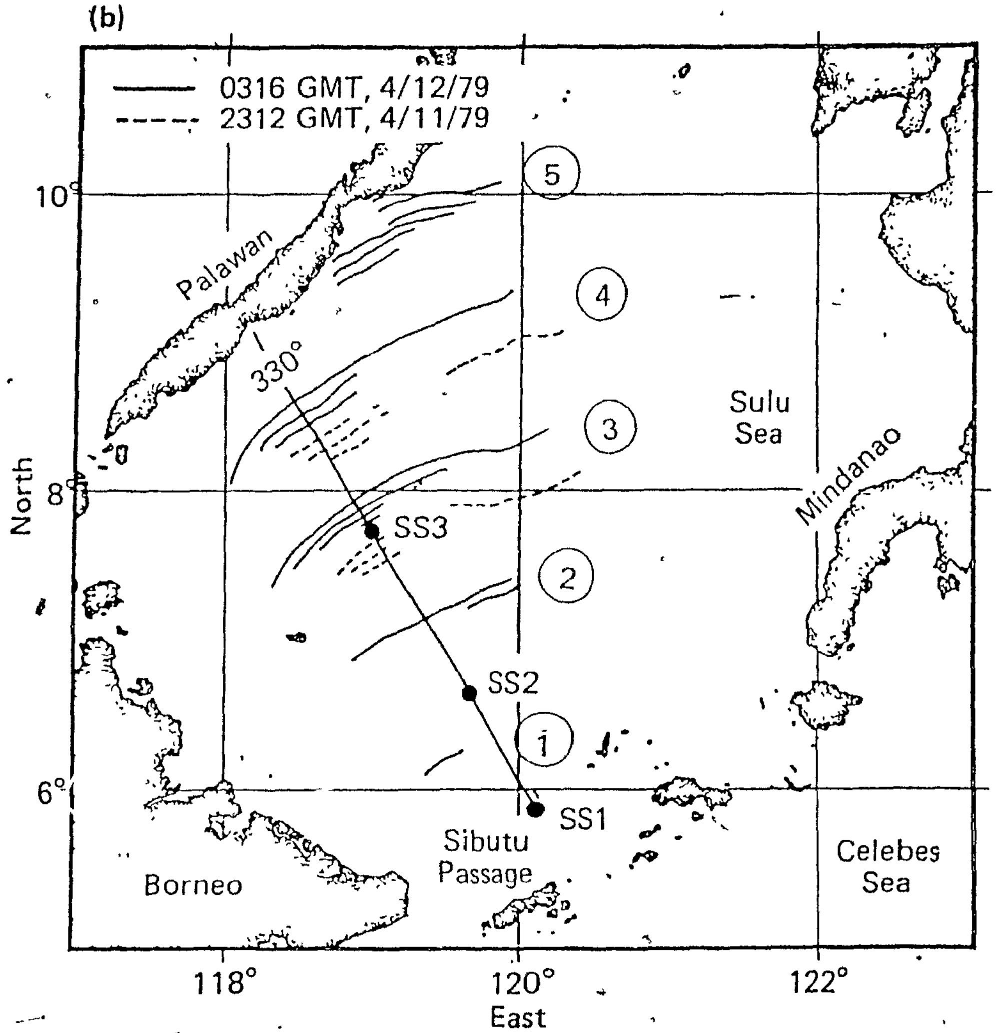


Figure A5

Time evolution of solitary waves in the Sulu Sea

A quite noticeable feature of oceanic solitary wavetrains is that each single wavefront, as it comes out of a strait, has its crest defined by some "non-straight" line whose curvature continually changes as it propagates, to eventually straighten later on. Solitons photographed over the Strait of Gibraltar (see Figures A2 and A3) as well as over the Strait of Messina are nice examples of this behaviour. As examplified by Figure A3, this is due to the fact that the bottom of the ocean, and the side boundaries in the case of a channel, are non-flat surfaces. The work presented here is an attempt to describe such curved waves.

From an historical point of view, attempts of description of solitary water waves were initiated back in last century and the first mathematically correct formulation using a nonlinear partial differential equation, namely the famous KdV equation, is due to Korteweg and de Vries. The KdV equation describes solitons moving over a perfectly rectangular channel: this is nice but unrepresentative of what occurs in Nature in the sense that real straits are not rectangular channels and that Nature is dissipative. Subsequent studies and models essentially made use of slight variations of this equation by allowing for variable, i.e., time dependent, coefficients. As for the KP equation, it was introduced in 1970; it was derived not as an hydrodynamical equation but through stability analysis of nonlinear disturbances in plasmas. However, its analytic form indicated that it could be useful as a water wave equation; incidentally, its original name was the "2-dimensional KdV equation" (for obvious reasons). Besides its similarity with the KdV equation (and because of it...) the KP equation is also famous because it was the very first example of a 2+1 dimensional nonlinear equation solvable through the celebrated "inverse spectral transform" method, a nonlinear analogue of the Fourier method for solving the Cauchy problem for linear equations. Hence, the KP equation also has soliton type solutions with all their nice properties, an associated Bäcklund transformation, an infinite set of local conserved quantities, and so forth. However, as a water wave theory it does not yet bring us to Heaven, for it describes solitary waves which are trivial extensions, as far as their shape is concerned, of the KdV theory; this was to be expected because of the way in which the transverse variable appears into the equation, i.e., linearly (its nonlinear terms are purely longitudinal ones). Fortunately, the Purgatory is not too terrible. Curved solitary waves can be obtained as deformations of pure KdV-KP solitons by accordingly considering deformations of the KP equation, by allowing some of its numerical coefficients to be replaced by functional quantities.

Our initial motivation was based on following fact. Let us derive a KP-like equation from the basic hydrodynamical equations, i.e., the usual Euler system and mass conservation equation in a way analogous to that which yields variable coefficient KdV equations, but incorporating two-dimensional ingredients. Obviously, this will yield a variable coefficient KP equation and the chances are that it will be non-integrable as such. The question was then: is that generalized KP equation amenable, through some well chosen mappings parametrized by the pertinent geophysical data [for instance the depth function and the vorticity (which is time independent in our treatment as no viscosity is present)], to 2-dimensional, also nonlinear, equations whose solutions could be mapped back to solutions of the generalized KP equation of the sort we want, i.e., which exhibit nontrivial transverse behaviour? We indeed succeded in answering this question positively, with the KdV, cylindrical KdV, and KP equations as "reduced equations". I shall now proceed to describe the model, with a minimum of details, and give examples of what kind of solutions can be obtained.

We consider a channel of a priori arbitrarily varying depth and width, containing a homogeneous incompressible inviscid ideal fluid subject to a gravity force **g** (with constant density ρ, pressure field p, and Eulerian velocity field **v**), whose boundary is determined by the vanishing of an appropriately chosen function $\Xi(\mathbf{x})$. This actually means that we want this function to represent a slight deformation of a rectangular channel; in other words, the bottom of the channel is a slowly varying function $h_0(x, y)$ and its sides are given by a pair of also slowly varying functions $l_{\pm}(x, z)$. Moreover, we shall require that the depth of the strait be much smaller than its width. As I just mentioned out, our starting point are the Euler equations and the mass conservation, together with appropriate boundary conditions (the variable η stands for the elevation of the fluid surface with respect to its undisturbed height). These are then rescaled and adimensionalized, so as to be written in the form

$$v_{3,z} + \alpha\varepsilon v_{1,x} + \alpha\beta\varepsilon^2 v_{2,y} = 0$$

$$p_x + \alpha\varepsilon(v_{1,t} + v_3 v_{1,z}) + \alpha^2\varepsilon^2 v_1 v_{1,x} + \alpha^2\beta\varepsilon^3 v_2 v_{1,y} = 0$$

$$p_y + \alpha\varepsilon(v_{2,t} + v_3 v_{2,z}) + \alpha^2\varepsilon^2 v_1 v_{2,x} + \alpha^2\beta\varepsilon^3 v_2 v_{2,y} = 0$$

$$p_z + \alpha\varepsilon(v_{3,t} + v_3 v_{3,z}) + \alpha^2\varepsilon^2 v_1 v_{3,x} + \alpha^2\beta\varepsilon^3 v_2 v_{3,y} = -1$$

$$p(z=\gamma\varepsilon\eta) = 0$$

$$[v_3 - \gamma\varepsilon\eta_t - \alpha\gamma\varepsilon^2 v_1\eta_x - \alpha\beta\gamma\varepsilon^3 v_2\eta_y]\big|_{z=\gamma\varepsilon\eta} = 0$$

$$\{\alpha[\beta v_2 - v_1 l_{\pm,\varepsilon x}] - v_3 l_{\pm,\varepsilon^2 z}\}\big|_{y=l_{\pm}(\varepsilon x,\, \varepsilon^2 z)} = 0$$

$$\{v_3 + \alpha\varepsilon^2 v_1 h_{\varepsilon x} + \alpha\beta\varepsilon^3 v_2 h_{\varepsilon y}\}\big|_{z=-h(\varepsilon x,\, \varepsilon y)} = 0$$

$$\zeta = \nabla \times \mathbf{v}$$

where ζ is the vorticity field and α, β, γ are constants of order 1. We then go to the approximation of almost one-dimensional "long" waves with "small" amplitudes in "shallow" water with wavecrests that are assumed to be slowly varying along the perpendicular direction by letting ε be a small parameter. Thus our approach is perturbative and care must be paid about the time domain of validity of whatever equation will result; this is in fact crucial and demand that we choose an appropriate system of coordinates. To do so, we do as follows. We assume that v, p, and h are expressible as formal power series in the parameter ε. Then we solve the above equations to get the zeroeth approximation equation for the wave amplitude η in the form $\eta^0_{tt} = h\eta^0_{xx}$. Solving this equation to order 2 in the parameter ε yields the following "waveframe" coordinates (for rightgoing waves):

$$X = \int_{x_0}^{x} h^{-1/2} C(\varepsilon x, y, \varepsilon)\, ds - C(\varepsilon x, y, \varepsilon)t, \qquad C(\varepsilon x, y, \varepsilon) = 1 + \sum_{i=1}^{\infty} C_i(\varepsilon x, y)\varepsilon^i$$

$$Y = y, \qquad Z = z, \qquad T = \varepsilon x$$

These coordinates follow the wave and will ensure that the perturbative approach will be valid for reasonable time intervals. The function C is a priori arbitrary and will be needed when reducing our resulting variable coefficient equation to more simple equations when boundaries are present. In absence of boundaries we shall set it equal to 1, identically. The appearance of this function however has a cost: secularity occurs at second order whenever it is not constant and has the effect that the results will be meaningful for times $t < \varepsilon^{-2}$. It will be of interest later to note that the role of time and space are inverted through some extent; this will permit us to obtain an equation whose coefficients vary only in terms of Y and "T"; in effect, we want terms like $F(T)\eta\eta_X$ but not like $F(x)\eta\eta_x$ (see equation below) since we could do nothing with them. This causes no difficulty but one must think of the Cauchy problem in an unusual meaning (which actually is pleasing for experimentalists because of the way measures are taken). The next step is to perform this change of coordinates on the differential system and the boundary conditions. To derive a nonlinear evolution equation from the resulting system, we again expand the dependent variables (pressure, velocity, velocity components) and geometrical variables (the depth function h and the side functions $l_\pm$) as formal power series in ε, substitute in the system, and proceed to solve. We have to go up to order 2, included, as the equation for the zeroeth approximation η^0 of the wave amplitude will appear as a compatibility condition for the 1st order approximation η^1. The resulting equation and boundary conditions are:

$$\eta^0_T + \tfrac{3}{2}\gamma h_0^{-3/2}(T)\eta^0\eta^0_X + \tfrac{1}{6}\alpha h_0^{1/2}\eta^0_{XXX} + \tfrac{1}{2}\beta h_0^{1/2}\int_{X_0}^{X}\eta^0_{YY}(s,Y,T)ds$$

$$+\beta h_0^{-1/2}A_0(Y,T)\eta^0_Y + M_1(Y,T)\eta^0 + M_2(Y,T)\eta^0_X + M_3(Y,T) = 0$$

$$\left[\beta\{\psi_0(Y,Z,T) + A_0\eta^0 + \int_{X_0}^{X}\eta^0_Y(s,Y,T)ds\} - h_0^{-1/2}[\eta^0 + \phi_0(Y,Z,T)]l_{\pm 0}'\right]\Big|_{Y=l_{\pm 0}} = 0$$

where M_1, M_2, M_3, and A_0 are some complicated functions of Y and T involving the function $C_1(Y,T)$; ϕ_0 and ψ_0 are vorticity potential functions. The above system is what we call the "generalized Kadomtsev-Petviashvili" (or GKP for short) system. It does coincide with the pure KP equation whenever $h_0(T) = 1$, $h_1(T) = 0$, $\phi_0 = \psi_0 = 0$, and $C_1(Y,T) = 0$, i.e., in the situation of a flat bottom with absence of vorticity.

At this point, let me make a few remarks. If the bottom is symmetric in the transverse (Y) direction then $h_1(T) = 0$, and the equation, as well as the boundary conditions, are then insensitive to any Y-variation of the bottom. However, if for instance we would be interested in waves propagating parallel to a shore along a sloping beach, then we would need the function $h_1(T)$. Second, it is to be noticed that the side functions only enter at order zero: this follows from the assumption that the strait is much larger than its depth. The bottom and side boundaries plays different role in our treatment, as the depth function appears in both the equation and the boundary condition, whereas the sides do only occur in the boundary conditions. This makes it possible for us to consider two different physical problems. A first one consists in analyzing the GKP system without considering any boundary conditions: this provides a description of long waves of small amplitudes in an essentially infinite body of water, ... and also in a channel! The second problem is to study the GKP system together with the boundary conditions. This yields a model for long waves propagating in a shallow strait of variable geometry, in principle allowing for exotic behaviours such as a meandering channel. Finally, it is worth pointing out that the GKP equation is of use even if we brutally discard all Y dependence from it. In this case we recover the flat bottom situation with $\phi_0 = \phi_0(T,Z)$ and can confirm a previous result of Benjamin, i.e., that solitary waves are possible even in the case of a fluid with nonvanishing vorticity.

The situation which is of interest to us is that for which we have bounded solutions (for instance solitons) at $T = T_0$ (i.e., at $x = x_0$). In fact, the Cauchy data (remember that we are considering a space evolution problem) for our problem is the specification of the wave amplitude η^0 and vorticity components ϕ_0, ψ_0 at $T = T_0$. Given these data, it still remains that the equation in the GKP system, being an equation for the amplitude alone, is not sufficient to completely determine the flow. On the other hand, the above GKP system can be shown to be equivalent to the following equivalent system:

$$[\eta^0{}_T + \tfrac{3}{2}\gamma h_0{}^{-3/2}\eta^0\eta^0{}_X + \tfrac{1}{6}\alpha h_0{}^{1/2}\eta^0{}_{XXX}]_X + \tfrac{1}{2}\beta h_0{}^{1/2}\eta^0{}_{YY} + [\beta h_0{}^{-1/2}A_0\eta^0{}_Y + M_1\eta^0 + M_2\eta^0{}_X]_X = 0 \qquad \text{(GKP)}$$

$$\eta^0{}_T(X_0) + \tfrac{3}{2}\gamma h_0{}^{-3/2}\eta^0(X_0)\eta^0{}_X(X_0) + \tfrac{1}{6}\alpha h_0{}^{1/2}\eta^0{}_{XXX}(X_0) + \beta h_0{}^{-1/2}A_0\eta^0{}_Y(X_0) + M_1\eta^0(X_0) + M_2\eta^0{}_X(X_0) + M_3(Y,T) = 0 \qquad (*)$$

$$\left[\beta A_0\eta^0 + \eta^0{}_Y - h_0{}^{-1/2}\eta^0{}_X l_{\pm 0}'\right]\Big|_{Y = l_{\pm 0}} = 0$$

$$\left[\beta\{\psi_0 + A_0\eta^0(X_0)\} - h_0{}^{-1/2}[\eta^0(X_0) + \phi_0]l_{\pm 0}'\right]\Big|_{Y = l_{\pm 0}} = 0 \qquad (*)$$

This is a coupled system for η^0, and the vorticity functions ϕ_0 and ψ_0. A judicious choice of

X_0 makes it possible to decouple the above system, namely the point where the wave amplitude vanishes: this is possible if the perturbation is always bounded, as required by physical considerations. Effectively, we choose $X_0 \to -\infty$, which then implies that the (*) equations reduce to $M_3(Y,T) = 0$ and

$$\left[\beta\psi_0 - h_0^{-1/2}\phi_0 l_{\pm 0}'\right]\Big|_{Y = l_{\pm 0}} = 0,$$

a linear system relating the vorticity functions. As mentioned earlier, the (GKP) equation is not integrable, in the sense that no direct analytical procedure is available for obtaining solutions of the corresponding Cauchy problem. Therefore we solved it in an indirect manner, namely by reducing it to simpler equations such as the KP, KdV, and cKdV equations:

$$q_\tau - 6qq_\xi + q_{\xi\xi\xi} + \sigma\int^\xi q_{\theta\theta}d\xi = 0, \quad \sigma = \pm 1; \qquad \text{(KP)}$$

$$q_\tau - 6qq_\xi + q_{\xi\xi\xi} = 0, \qquad \text{(KdV)}$$

$$q_\tau - 6qq_\xi + q_{\xi\xi\xi} - 4\xi q_\xi - 2q = 0. \qquad \text{(cKdV)}$$

These are equations which belong to a large class of equations that are integrable by powerful techniques such as the inverse spectral transform method and are shown to possess infinite classes of exact interesting solutions such as (multi-) solitons, (quasi-) periodic solutions, rational solutions, etc.

The technique that we used was quite simple. It consisted in finding invertible transformations of the independent (as well as the dependent) variables that have the effect of reducing (GKP) to the above mentioned simpler equations. I shall restrict to the case of reduction to the KdV equation for the case of a symmetric strait (in which case $h \equiv h_0$). Our strategy was as follows. First, we determined that the most general transformation of the variables that did not introduce "unwanted" terms, i.e., terms that implies non-integrability or other undesirable naughty effects, is the following one:

$$\eta^0(X,Y,T) = R(T)q(\xi,\tau,\theta) + S(X,Y,T)$$

$$\xi = a(T)X + K(Y,T), \quad \theta = U(T)Y + V(T), \qquad \tau = \int_{T_0}^{T} P(s)ds.$$

Also, when considering the case with boundary conditions, we had to solve the boundary conditions. We did it by first reducing them to a "standard form"

$$q_\theta \big|_{\theta = \theta(L_{\pm 0})} = 0$$

which we solved trivially by imposing $q = q(\xi, \tau)$. For the purpose of constructing solutions to the GKP equation, this function q was chosen to be the usual KdV 1-soliton:

$$q(\xi, \tau) = -2\nu^2 \mathrm{sech}^2[\nu(\xi - \xi_0) - 4\nu^3\tau].$$

It is to be pointed out that other solutions, such as cnoidal, quasi-periodic, or rational (algebraic lumps) solutions could also be used. For physical reasons, we also want the above function S to be (at least asymptotically) vanishing to preserve boundedness of the solutions.

1°. Reduction in absence of boundary conditions

In this case, the function S can be set to zero, identically; we also ask that the function C_1 be vanishing as well since it serves no purpose at all. It also proves necessary for the vorticity function ϕ_0 to satisfy the following constraint:

$$\int_{-h_0}^{0} \phi_0 dZ = \beta^{-1}\gamma^{1} h_0^{2}(T)[Q_0(T) + Q_1(T)Y + Q_2(T)Y^2],$$

where we can choose $Q_0' = Q_1 = 0$ (but not Q_2 as the three vorticity moments cannot simultaneously vanish in order to be able to do the reduction), and Q_2 has to verify

$$(h_0'/h_0)' = 5(h_0'/h_0)^2 + \tfrac{4}{9}Q_2.$$

I am presenting here two examples depicted in Figure A6. The top illustrations correspond to the choice of a quadratic depth function and the lower ones deal with the case when the bottom behaves as a hyperbolic tangent. Both solutions, I emphasize, are *exact* and are representative of common geometries occuring in Nature (deep oceanic depressions, continental shelf borders).

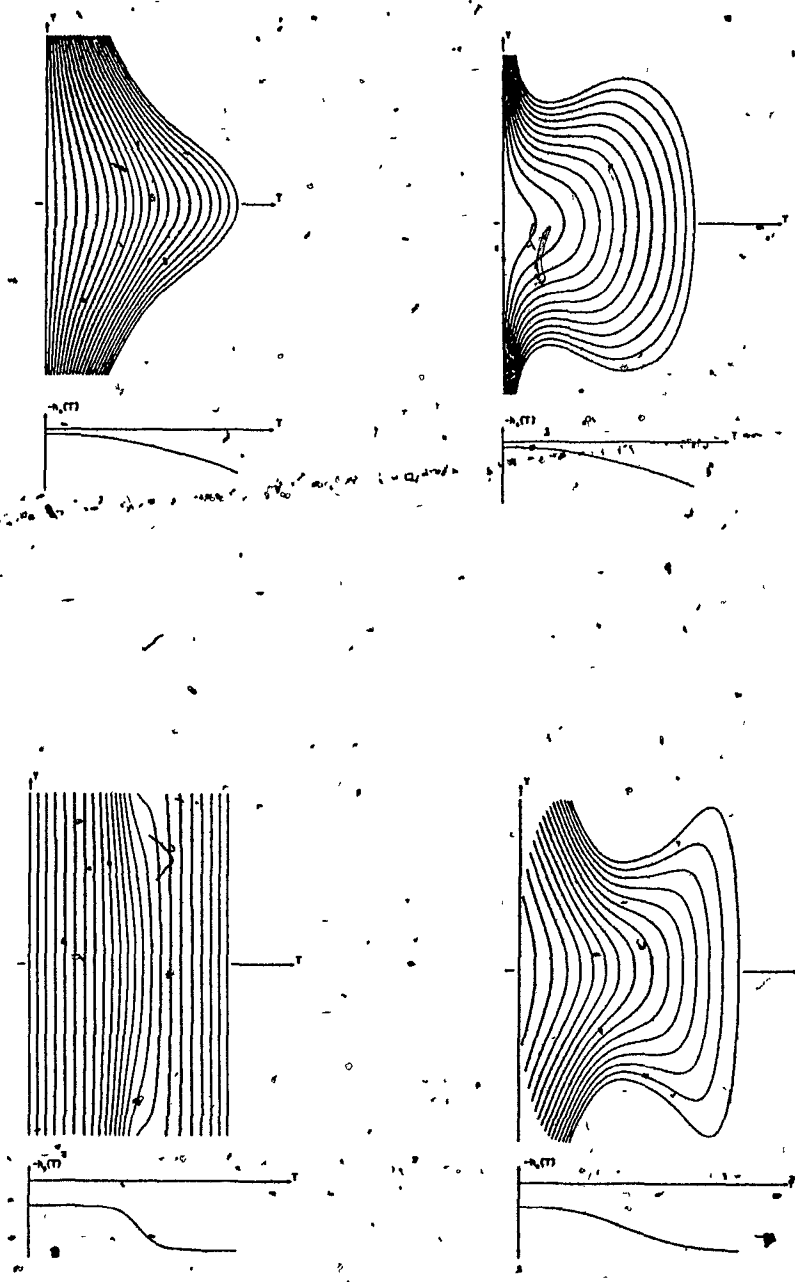


Figure A6

Examples of simple exact solutions of the GKP equation (no boundaries)

These illustrations represent maxima of the solutions at several consecutive "times" X (remember that time and space inversion through the wave frame transformation); we may think of them as cumulated "snapshots" of a wavecrest as it evolves through time. In both cases, the initial condition is a pure KP soliton whose crest is therefore a *straight* line identified with the Y-axis. The left illustrations are small scale ones, i.e., the drawings are made for "sufficiently" small intervals near the origin. It is to be noticed that the wavecrest, having departed from the initial condition, acquires some curvature, due to the fact that the depth is non-constant. The pictures on the right ones are made for much larger scales and show what goes on eventually near the T-axis, i.e., the *physical* x-axis, as well as the limitations of the model. Note the *horseshoe* form of the crest for long times. This is an indication that the solutions are valid only within a *sufficiently small* interval along the Y direction. They are also limited in the longitudinal direction, although this is not evident from the pictures. The reason is that the function R (from the above transformation for reducing the GKP equation) is a multiplicative factor for the wave amplitude. Thus, for the quadratic bottom, the solution asymptotically blow up. In the case of the hyperbolic tangent bottom, the problem is not apparent because R is taking values in a finite interval; note also, in this particular example, that the correlation between the depth and the crest's curvature is very clear, especially in the small scale illustration: the more the depth is varying, the more the curvature is pronounced.

2. Reduction with boundary conditions

In this case, the function S cannot be set equal to zero anymore. I also remind that secularity is then the price to pay when reducing the boundary condition to the standard form by an appropriate choice of the function C(Y,T). The reduction is somewhat more complex than in the previous case and I thus skip over the details (see our papers). It is however important to mention that the width of the channel is constrained by the depth function (or vice-versa). Some components of the vorticity are also determined once the geometry of the strait is given. We have treated the case of a logarithmically shaped bottom. The time behaviour of a crest, initially straight, is shown on Figure A7. Note that its curvature increases and decreases respectively with the width of the strait.

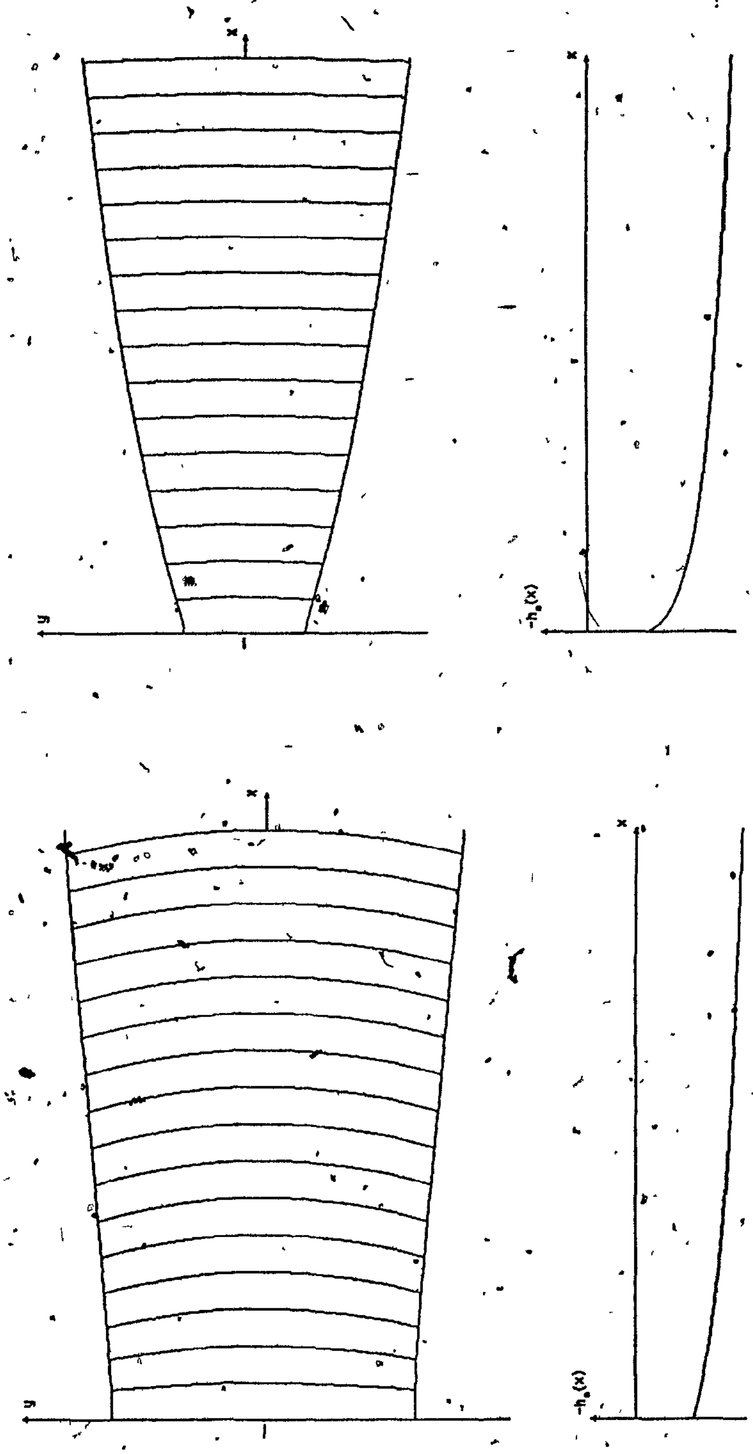


Figure A7

Evolution of a solitary wave through a strait with logarithmic depth

2. Symmetries and special solutions of the KP equation

The classical KP equation,

$$[4u_t + 6uu_x + u_{xxx}]_x + 3\sigma u_{yy} = 0,$$

is quite interesting from the group theoretical point of view. The fundamental fact is that it has an infinite-dimensional invariance (or symmetry) group. Other (systems of) equations also have such groups, for instance the Euler system from hydrodynamics, but the KP equation is special. Our initial motivation was simply to perform symmetry reduction on this equation (see later) but we soon realized that its symmetry algebra had a peculiar property. Let us then proceed without any further ado. This symmetry algebra, denoted L_{KP}, is the vector space generated by vector fields of the form

$$X[f(t)] + Y[g(t)] + Z[h(t)],$$

$$X(f) = f\partial_t + [\tfrac{1}{3}xf' - \tfrac{2}{9}\sigma y^2 f'']\partial_x + \tfrac{2}{3}yf'\partial_y - [\tfrac{4}{27}\sigma y^2 f''' - \tfrac{2}{9}xf'' + \tfrac{2}{3}uf']\partial_u,$$

$$Y(g) = g\partial_y - \tfrac{2}{3}\sigma yg'\partial_x - \tfrac{4}{9}\sigma yg''\partial_u,$$

$$Z(h) = h\partial_x + \tfrac{2}{3}h'\partial_u,$$

where f, g, and h are arbitrary smooth functions, i.e., of class C^∞, defined in some open subset of **R**. The structural commutation relations of this algebra are

$$[X(f_1), X(f_2)] = X(f_1f_2' - f_1'f_2), \qquad [X(f), Y(g)] = Y(fg' - \tfrac{2}{3}f'g),$$

$$[X(f), Z(h)] = Z(fh' - \tfrac{1}{3}f'h), \qquad [Y(g_1), Y(g_2)] = \tfrac{2}{3}\sigma Z(g_1g_2' - g_1'g_2),$$

$$[Y(g), Z(h)] = 0, \qquad [Z(h_1), Z(h_2)] = 0.$$

Associated to L_{KP}, is the symmetry group of the KP equation, denoted G_{KP}, whose elements are quite complicated mappings which I will not present; suffice it to say that they take the form

$$\phi: \mathbf{R}^3 \times \mathbf{R} \to \mathbf{R}^3 \times \mathbf{R}, \quad (x, u) \to (x^*, u^*) \qquad x^* = \psi(x), \quad u^* = \chi(x, u),$$

(G_{KP} acts projectably), and take KP solutions into other KP solutions. We shall work mainly with L_{KP}, hence our results are *local* ones (n.b.: local does not mean "incredibly near" of zero, but in a "sufficiently" small open set around the identity element of G_{KP}, so that global results are *a priori* not excluded).

Let us consider the subalgebra L_π obtained by restricting the functions f, g, and h to be Laurent polynomials in the variable t, spanned by all vector fields $X(t^n)$, $Y(t^n)$, and $Z(t^n)$, $n \in \mathbf{Z}$. L_π is characterized by the commutation relations

$$[X(t^n), X(t^m)] = (m-n)X(t^{n+m-1}), \qquad [X(t^n), Y(t^m)] = (m-\tfrac{2}{3}n)Y(t^{n+m-1}),$$

$$[X(t^n), Z(t^m)] = (m-\tfrac{1}{3}n)Z(t^{n+m-1}), \qquad [Y(t^n), Y(t^m)] = \tfrac{2}{3}\sigma(n-m)Z(t^{n+m-1}),$$

$$[Y(t^n), Z(t^m)] = 0, \qquad [Z(t^n), Z(t^m)] = 0.$$

The interesting fact is that $L_\pi \subset \{R[t,t^{-1}] \otimes sl(5, \mathbf{R})\} \oplus R[t,t^{-1}]d_t$, i.e., it can be embedded into the affine loop algebra $A_4^{(1)}$ without its centre. This remarkable property was to be later observed for other 2+1 dimensional integrable nonlinear equations, for instance the Davey-Stewartson and the modified KP equations. Interestingly, no equations, to our knowledge, in 1+1 nor 3+1 have this property. At the present time, we have no idea of why 2+1 dimensional equations are so special, but we are investigating the question.

We then proceeded to classify the low-dimensional (≤ 3) subalgebras of L_{KP}. This is rather standard; the methods for doing that are defined for finite-dimensional algebras, yet they prove to be applicable even if L_{KP} is infinite-dimensional (the difference being that we have equations for the functions labeling the group elements rather than algebraic conditions for parameters). This classification is done by identifying together all subalgebras that are conjugate under the adjoint action of G_{KP}. Recall that two vector fields V_1 and V_2 of L_{KP} are said to be conjuate if there exists an element g of L_{KP} such that $V_2 = e^g V_1 e^{-g}$. We have established that there are exactly three classes of non-conjugate 1-dimensional subalgebras; they are:

$$L_{1,1} = \{X(1)\}, \qquad L_{1,2} = \{Y(1)\}, \qquad L_{1,3} = \{Z(1)\}.$$

The 2-dimensional subalgebras are also easy to classify; they are:

$$L_{2,1} = \{X(1), Y(1)\}, \quad L_{2,2} = \{X(1), Z(1)\}, \quad L_{2,3} = \{Y(1), Z(h)\},$$
$$L_{2,4} = \{Z(1), Z(h) \mid h'(t) \neq 0\};$$
$$L_{2,5} = \{X(1), X(t)\}, \quad L_{2,6} = \{Y(1), X(\tfrac{3}{2}t)\}, \quad L_{2,7} = \{Z(1), X(3t)\}.$$

The first four of them are Abelian. Finally, we found eighteen non-conjugate 3-dimensional subalgebras which I do not list here; let me just mention that all isomorphy classes of three-dimensional Lie algebras are represented, except su(2).

Symmetry reduction is a very powerful, yet simple, method by which one can obtain solutions of *difficult* differential equations. In essence, it consists in using invariants of the symmetry group of the equation in order to define a new *reduced* equation which has fewer variables. One may then use the solutions of the reduced equation to construct solutions of the original equation. Note that this will not yield *all* the solutions, but only those which are invariant under a specific symmetry, or set of symmetries. The method is particularly simple, from the computational point of view, the the considered symmetry group acts projectably on the solution manifold, i.e., the group transformations map the old independent variables to new ones which do not depend on the old dependent variables; such is the case for the KP equation. To find invariant solutions for the KP equation is not recent business. In fact, many authors have written an appreciable quantity of literature on this subject, constructing so-called *similarity* solutions. The point is that this literature is highly self-intersecting in the sense that many of these solutions are redundant; by this, I mean solutions that fall within a same class, or, in other words, solutions that can be mapped one to eachother through some invertible transformations. Our contribution was to make a *clean up*. Indeed, one can perform symmetry reduction in an exhaustive way by considering only the non-conjugate classes of subalgebras of L_{KP}. Practically speaking, this means that it is sufficient to build one invariant solution for one representative of each conjugacy class; all other invariant solutions may then be obtained by letting the group act on them.

The fundamental reductions to be considered here are those using the 1-dimensional subalgebras of L_{KP}. These will yield reduced equations depending on two variables rather than three. Naturally, reductions using two-dimensional are permissible. However, I do not consider them since they amount to further reduce the reduced equations obtained from the

1-dimensional subalgebras: these further reductions are well studied. It was easy to show that the KP equation exactly reduces to three different equations. The subalgebra $L_{1,1}$ yields the Boussinesq equation: $3\sigma q_{\eta\eta} + 3(q^2)_{\xi\xi} + q_{\xi\xi\xi} = 0$. The subalgebra $L_{1,2}$ implies a once differentiated KdV equation: $[4q_\eta + 6qq_\xi + q_{\xi\xi\xi}]_\xi = 0$. Finally, the subalgebra $L_{1,3}$ reduces the KP equation to the linear equation $u_{yy} = 0$. Numerous types of special KP solutions can be obtained from these reductions. I shall present one of them which is new. The reduction class which contains the Boussinesq equation also contains the following equation:

$$\vartheta_\tau + \vartheta_{\xi\xi\xi} - 6\vartheta\vartheta_\xi - 4\xi\vartheta_\xi - 2\vartheta = 0.$$

This is the cylindrical KdV equation. It has been extensively studied by Calogero and Degasperis and has bounded solutions which yields to the following bounded KP solutions:

$$\xi = -[v_1/(v_0 + v_1 t)]^{1/3}\left\{x + \frac{\frac{1}{3}v_1\sigma y^2 - v_2 y - \frac{3}{4}\sigma v_2^2/v_1}{v_0 + v_1 t}\right\}, \quad \eta = \ln[(v_0 + v_1 t)/v_0],$$

$$u = [v_1/(v_0 + v_1 t)]^{2/3}\left\{2q^2[Ai(p)]^4 + \frac{4qAi(p)Ai'(p)}{\{1 + q[Ai'(p)]^2 - pq[Ai(p)]^2\}^2}\right\},$$

$$p = \xi - v_3 e^{-4\tau}, \qquad q = v_4 e^{-4\tau},$$

where $v_0, \ldots, v_4$ are constants and Ai denotes an Airy function. This solution contains, as a particular case, a solution obtained by Nakamura.

An interesting property shared by most of integrable nonlinear partial differential equations is that they possess a *Bäcklund transformation* (BT). Such a transformation is in fact a correspondence rule between first derivatives of solutions of a same equation. A well known example is given by the Cauchy-Riemann conditions: if u solves the Laplace equation on the complex plane, then so is v provided that $v_x = -u_y$ and $v_y = u_x$. The KP equation does have its own. In fact it is best expressed for the *potential* KP (PKP) equation:

$$[4u_t + 3u_x^2 + u_{xxx}]_x + 3s^2 u_{yy} = 0,$$

where $\sigma = s^2$. If u solves the PKP equation, then $w = u_x$ solves the KP equation. The BT for the PKP equation is defined through the following relations:

$$s(u-v)_y - (u+v)_{xx} - (u-v)(u-v)_x = 0,$$
$$4(u-v)_t + (u-v)_{xxx} + 3s(u+v)_{xy} + 3(u-v)(u+v)_{xx} + 3(u-v)_x[(u-v)^2 + (u+v)_x] = 0.$$

Our goal was to implement a new way of applying symmetry reduction, namely by applying this technique on a system formed by a given partial differential equation together with its associated BT. As an example, we chose to do that with the (P)KP equation. The PKP equation also has an infinite-dimensional Lie group of point symmetries which has properties very similar to that of the KP equation; I shall not elaborate on that. For our purposes, we are interested in how it acts on the BT. Not surprisingly, it turns out that not all the group leaves it invariant. In fact the BT is transformed to

$$s(u-v+p)_y - (u+v)_{xx} - (u-v+p)(u-v+p)_x = 0,$$
$$4(u-v+p)_t + (u-v+p)_{xxx} + 3s(u+v)_{xy} + 3(u-v+p)(u+v)_{xx} + 3(u-v+p)_x[(u-v+p)^2 + (u+v)_x] = 0,$$

where $p = R(t) + yS(t)$, and R, S are expressible in terms of arbitrary functions labeling the group elements. Note that the introduction of such an arbitrary function in the BT is analogous to the introduction of arbitrary constants into BT's associated to 1+1 dimensional integrable equations (for instance the Lorents invariance of the Sine-Gordon equation gives rise to a real-valued parameter). The simplest way to use a BT is when the input solution is trivial; choosing $v = 0$, the BT reduces to

$$\begin{aligned} &s(u+p)_y - u_{xx} - (u+p)(u+p)_x = 0, \\ &4(u+p)_t + (u+p)_{xxx} + 3su_{xy} + 3(u+p)u_{xx} + 3(u+p)_x[(u+p)^2 + u_x] = 0. \end{aligned} \tag{*}$$

We now look for the joint symmetry group of the PKP equation and (*). It has an algebra generated by the following vector fields:

$$X_1 = \partial_x, \qquad X_2 = \partial_y - S(t)\partial_u,$$
$$X_3 = \partial_t - [R'(t) + yS'(t)]\partial_u, \qquad X_4 = t\partial_y - \tfrac{2}{3}s^2y\partial_x - [tS(t) - \tfrac{2}{3}s^3]\partial_u.$$

There are 4 classes of non-conjugate 1-dimensional subalgebras: $\{X_1\}$, $\{X_2\}$, $\{X_4 + aX_3\}$, and $\{X_3 + aX_1\}$, where a is a constant. These can be used to perform symmetry reduction. I shall give only one example. Consider the representative $X_3 + aX_1 + bX_2$, belonging to the same class as $X_3 + aX_1$. Performing the reduction, it is found that the KP solution has the solution $w = 2\partial_x[\Phi^{-1}\Phi_\xi]$ where $\Phi(\xi, \eta)$ obeys the following equations ($\xi = x - at$, $\eta = y - bt$):

$$\Phi_{\xi\xi} - s\Phi_\eta = 0,$$

$$\Phi_{\xi\xi\xi} - bs^3\Phi_{\xi\xi} - a\Phi_\xi = [h_1(\eta)\xi + h_2(\eta)]\Phi.$$

The first equation comes from the BT and the second one from the PKP equation. In order to obtain analyti solutions, we consider the special case $h_1 = 0$, $h_2 =$ constant. One then finds that the above system has three different types of solutions, depending whether the characteristic equation for the second has 1, 2, or 3 distinct roots. The only case yielding bounded solutions is when there are 3 roots. The corresponding KP solution is

$$u = \frac{2\sum' A_i A_j (k_i - k_j)^2 \exp(\alpha_i + \alpha_j)}{[\sum A_i \exp(\alpha_i)]^2},$$

where the $\sum'$ is taken for $i < j$, k_i is a constant, and $\alpha_i = k_i(\xi + s^3 k_i \eta)$. We have termed this solution *splitton* and it is shown on Figure A8. As one can see, this solution is an object made of three semi-infinite soliton-like fronts merging at some point. The relative angles between each branches are constant and the whole object translates in the xy-plane. This solution is to be considered on the same footing as the soliton as it also is obtained from the BT acting on the trivial solution.

To resume the situation with this symmetry business applied on the KP equation, let me say that we have now tools for constructing a quite large amount of special solutions to this equation. One may start from a KP solution and get another one by letting the KP symmetry group act on it. Solutions can also be obtained through the action of the BT on PKP solutions that can be mapped back to KP solutions. One can also use the solutions of equations obtained by reducing the KP equation. Note that one can also perform symmetry reduction on the reduced equations to get further reduced equations; these will be ordinary differential equations whose solutions can be mapped back to solutions of the equations obtained by the first level of reduction, and these to KP solutions. All this is schematized on Figure A9.

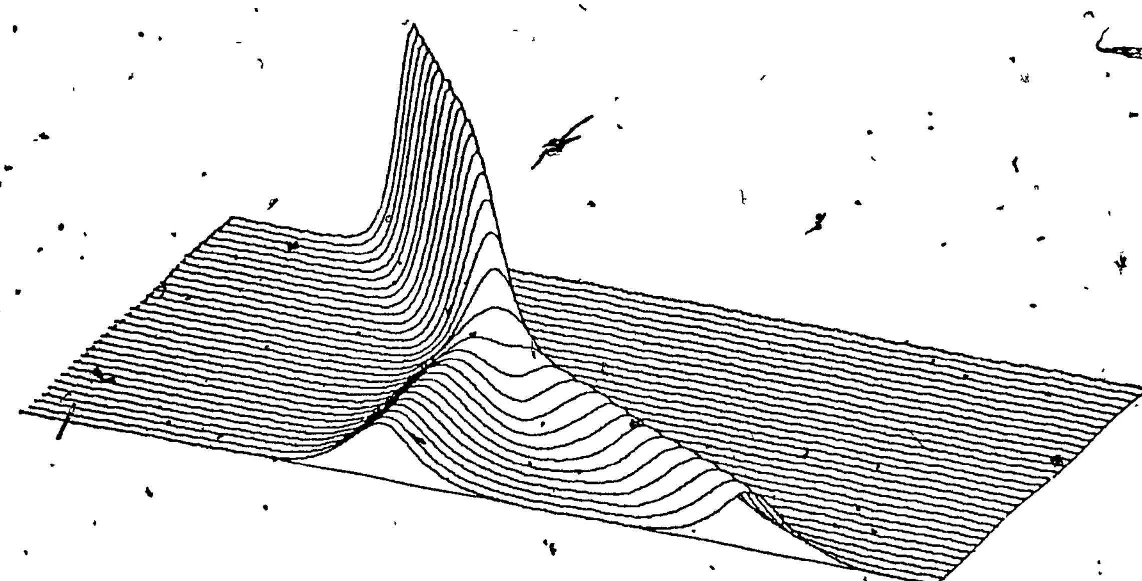


Figure A8
The Splitton solution of the KP equation

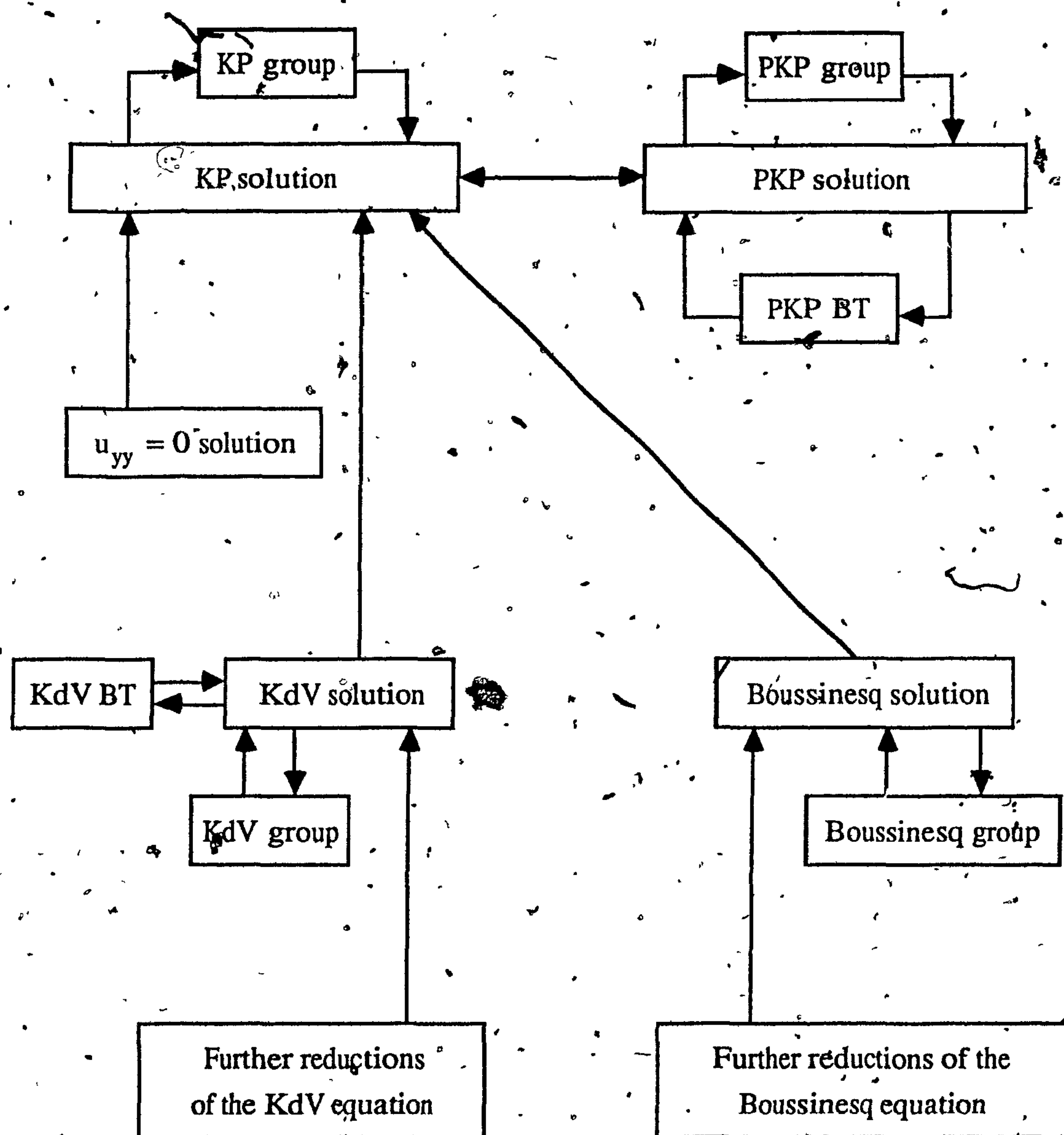


Figure A9

How to construct special solutions of the KP equation