RECURRENCE RELATIONS IN MOMENTS OF ORDER STATISTICS

BY

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ABSTRACT

Much work has been done in the field of Order Statistics concerning the calculation of their moments. Of special interest are recurrence relations between the moments of order statistics and certain general approaches giving bounds and approximations to the moments of order statistics.

In this thesis, an extensive review of recurrence relations involving moments of order statistics is given and discussed. These consist of recurrence relations of moments of order statistics and recurrence relations among product-moments. Also discussed are recurrence relations among quasi-ranges, other type of relations among moments of order statistics and some bounds and approximations for moments of order statistics.

RESUME

Dans le domaine de la statistique d'ordre beaucoup de travail a été fait dans le calcul des moments. Plus spécifiquement les relations de récurrences entre les moments de la statistique d'ordre et certaines techniques générales donnent des limites et approximations des moments de la statistiques d'ordre.

Ce mémoire fait une revue extensive des relations de récurrence à propos des moments de la statistique d'ordre. Cela consiste en relations de récurrence des moments de la statistique d'ordre et les relations de récurrence parmi des moments mixtes. On traite aussi les relations de récurrence parmi les quasi étendues, autres types de relations parmi les moments des statistiques d'ordre et quelques limites et approximations des moments de la statistique d'ordre.

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INTRODUCTION

Within the past fifty years, a large number of papers have been written on statistics based on ordered observations. Systematic statistics or order statistics are now being increasingly used in new statistical procedures, since a considerable amount of new statistical inference theory can be established assuming nothing stronger than continuity of the cumulative distribution function of the population. Furthermore, statistical inference theory based on order statistics, generally, make the statistical procedures themselves very simple and broadly applicable and also permit very simple solutions of some of the more important parametric problems of statistical estimation and testing of hypothesis. Problems on ranges, quasi-ranges, tolerance limits, estimation of location and scale-like parameters, censored samples, selection and ranking, generally make extensive use of order statistics.

Many authors have studied recurrence relations between the moments of order statistics, usually with the principal aim of reducing the number of independent calculations required for the evaluation of the moments. One of the aims of this thesis is to attempt to provide an extensive review of all the recurrence relations involving moments of order statistics. The review shall basically consist of five subsections: (1) Recurrence relations of moments of order statistics; (2) Recurrence relations among product moments; (3) Recurrence relations among quasi-ranges; (4) other relations (recurrent or otherwise); (5) Some bounds and approximations for moments of order statistics.

CHAPTER I

RELATIONS BETWEEN MOMENTS OF ORDER STATISTICS

In this chapter the relations between moments of order statistics, most of them of the recurrence type will be reviewed and discussed. Many of the derivations and proofs of the recurrence relations to be discussed in this chapter can be also applied and sometimes derived from the relations of product moments of order statistics. However, in an attempt to simplify and clarify this presentation, only those recurrence relations between actual moments shall be discussed, with appropriate reference to other chapters when recurrence relations can be applied to product moments and vice versa.

For an arbitrary continuous distribution

$$(1.1) \qquad \mu^{(k)}_{r+i:n} = n\binom{n-1}{r+i-1} \sum_{j=0}^{n-r} (-1)^{n-r}\binom{i}{j}\ \mu^{(k)}_{r:n-j}/(n-j)\binom{n-j-1}{r-1}$$

$$(1.2) \qquad \mu^{(k)}_{r:n-i} = (n-i)\binom{n-i-1}{r-1} \sum_{j=0}^{n-r} \binom{i}{j}\ \mu^{(k)}_{r+j:n}/n\binom{n-1}{r+j-1}$$

$$0 < i < n-r$$

$$\binom{i}{j} = 0 \text{ if } j > i$$

$$k = 1,2,3,\ldots;\ r = 1,2,\ldots,n-1$$

Relations (1.1) and (1.2) were obtained by Cole (1951) by multiplying moments of order statistics by certain factors and putting them into matrix form. Furthermore by letting $i=1$ in (1.2) one gets

$$(1.2.1) \quad (n-r)\mu_{r:n}^{(k)} + r\mu_{r+1:n}^{(k)} = n\mu_{r:n-1}^{(k)}$$

Relation (1.2.1) enables one to compute the expected values of all order statistics, their squares etc. from the expected values of the first order statistics, their squares etc. Relation (1.2.1) was proved by Melnick (1964) for the discrete case. David (1970) develops a proof that holds for both the discrete and continuous case. Let

$$\text{(a)} \quad \mu_{r:n}^{(k)} = \int_{-\infty}^{\infty} x^k \frac{d}{dx} I_{p(x)}(r,n-r+1)dx$$

and

$$\text{(b)} \quad \mu_{r:n}^{(k)} = \sum_{x=0}^{\infty} x^k \Delta I_{p(x)}(r,n-r+1)$$

where $\Delta I_{p(x)}(a,b) = I_{p(x)}(a,b) - I_{p(x-1)}(a,b)$. Taking the recurrence formula for the incomplete beta function $aI_y(a+1,b) + bI_y(a,b+1) = (a+b)\ I_y(a,b)$ and letting $a=r$, $b=n-r$, $y=P(x)$, relation (1.2.1) can be easily established. Using (1.2.1) the moments for a sample of size $n-1$ can be obtained by summing adjacent pairs of moments for a sample of size n. Also, by the simple operation of addition, the moments for all samples of sizes less than n can be obtained from the moments for a sample of size n. By reversing the process and knowing $\mu_{1:n}, \mu_{1:n-1}, \ldots, \mu_{1:1}$, the moments for all samples of sizes not greater than n can be determined by successive differencing. This method, however, is not very good for large n. However, as Harter (1961) and Srikantan (1962) point out, if (1.2.1) is written as $\mu_{i:n-1} = (i/n)\mu_{i+1:n} + \{(n-i)/n\}\mu_{i:n}$, $(i = 1,\ldots,n-1)$. Then it can be used for working "downwards", i.e. going from a larger n to a smaller n, with no serious accumulation of rounding errors. A related relation, which may be used as a check for accuracy (Govindarajulu, 1962) is

$$\text{(1.2.1a)} \quad \mu_{1:n} = \sum_{i=1}^{n} (-1)^{i-1}\binom{n}{i}\mu_{i:i}$$

Relation (1.2.1a) is derived by writing $\mu_{1:n}$ as an integral, expanding $[1 - F(x)]^{n-1}$ as a binomial series and integrating term-wise.

Taking (1.2.1) again, for even n and letting $r = \frac{n}{2}$, one gets

$$\text{(1.2.1.1)} \quad \frac{1}{2}(\mu^{(k)}_{n/2+1:n} + \mu^{(k)}_{n/2:n}) = \mu^{(k)}_{n/2:n-1}$$

From (1.2.1.1) one can see by letting k=1 that the expected values of the median in samples of n (even) and n-1 are equal. If the parent distribution is symmetric about the origin and n is even, directly from (1.2.1.1), substituting $\mu^{(k)}_{(n+1)/2:n} = (-1)^k \mu_{n/2:n}$ one gets

$$\text{(1.2.1.2)} \quad \mu^{(k)}_{n/2:n} = \mu^{(k)}_{n/2:n-1} \qquad k \text{ even}$$

$$= 0 \qquad k \text{ odd}$$

and

(1.2.1.3) $\mu_{i:n} = -\mu_{n-i+1:n}$

This result for normal ordered statistics has been given by Jones (1948). The general result follows from the definition of $\mu_{i:n}$ and $F(-x) = 1 - F(x)$.

Another known and easily verified relationship (Hoeffding, 1953) is

$$(1.2.2) \qquad \sum_{i=1}^{n} \mu_{i:n}^{(k)} = n\, E(X^k)$$

Looking at the proof of relation (1.2.1) one sees that it depends only on the recurrence property of the incomplete beta function. Thus it is clear that the same recurrence relation also links the p.d.f.'s, c.d.f.'s and (if they exist) the expected values of a given function, say, $g(X_{r:n})$ (Srikantan, 1962), that is,

$$(1.2.1a) \qquad (n-r)\, Eg(X_{r:n}) + r\, Eg(X_{r+1:n}) = n\, Eg(X_{r:n-1}).$$

Similar generalizations can be applied to any relation whose proof can be shown to be dependent on a property of the beta function.

Recurrence relation (1.2.1a) expresses the expected value of a given function of the (r+1)-st order statistic in a sample of size n in terms of the expected values of the same function of the r-th order statistic in samples of sizes n and n-1. By induction it follows that the expected value of a given function of any order statistic in a sample of size n can be expressed in terms of the expected values of the same function of the first order statistics in samples up to size n. A similar result in terms of the largest order statistics was obtained by Cole (1951) for "normalized" moments of order statistics. By using (1.2.1) and induction Srikantan (1962) got the following explicit solution for recurrence relation (1.2.1):

$$(1.2.1.4) \quad \mu_{r:n}^{(k)} = \sum_{i=n-r+1}^{n} \binom{i-1}{n-r}\binom{n}{i}(-1)^{i-n+r-1}\mu_{1:i}^{(k)}$$

Relation (1.2.1.4) or (1.2.1a) enables one to compute the expected values of the first order statistics, their squares, etc., as well as the tabulation of the c.d.f.'s of order statistics. Govindarajulu (1961) has also mentioned that moments of order statistics can be obtained from those of the lowest order statistics. David (1970) gets a corresponding result in terms of the greatest order statistics in samples up to size n, by using the incomplete beta function and expanding $(1-t)^{n-r}$ in the equation $I_{p(x)}(r,n-r+1) = \int_0^{f(x)}$ $\{n!/(r-1)!(n-r)!\}\ t^{r-1}(1-t)^{n-r}$ dt. The integrand on the RHS becomes $\{n!/(r-1)!(n-r!)\} \sum_{j=0}^{n-r} \binom{n-r}{j}(-1)^j\ t^{r-1+j}$. Putting $i = j+r$, the RHS equals $\sum_{i=r}^{n} \binom{i-1}{r-1}\binom{n}{i}(-1)^{i-r}\cdot it^{i-1}$ and substituting this expression in (a) or (b) (after equation (1.2.1)), one gets

$$(1.2.1.5) \quad \mu_{r:n}^{(k)} = \sum_{i=r}^{n} \binom{i-1}{r-1}\binom{n}{i}^{i-r} \mu_{i:i}^{(k)}$$

It can be seen that relation (1.2.1.5) is similar to the result of Cole (1951), i.e. that "normalized" moments of order statistics can be obtained by successive differencing of those of the largest order statistics. By repeated application of relation (1.2.1) (David, 1970), the following recurrence relation holds for any arbitrary distribution:

$$(1.2.1.6) \quad (n-r)^{(m)}\mu_{r:n}^{(k)} = \sum_{i=0}^{m} (-r)^{(i)}(n)^{(m-i)}\binom{m}{i}\mu_{r+i:n-m+i}^{(k)}$$

If one substitutes m=n-r in (1.2.1.6) one gets relation (1.2.1.5) as a special case.

Using as pivots the expected values of the function of the median statistics in even samples, i.e. $E[g(x_{r:2r})]$ and $E[g(x_{r+1:2r})]$, $r = 1(1)[[(n+1)/2]]$, Srikantan (1962) gets

$$(1.2.3) \quad E[g(x_{r:s})] = r\binom{s}{r} \sum_{j=0}^{[[(s-2r+1)/2]]} (-1)^j \Big\{\binom{s-2r-j+1}{j} \cdot E\, g(x_{r+j:2r+2j}) \Big/ \binom{2r+2j}{r+j}(r+j)$$

$$+ \binom{s-2r-j}{j-1} E[g(x_{r+j+1:2r+2j})] \Big/ \Big[\binom{2r+2j}{r+j+1}(r+j+1)\Big]\Big\},$$

$[[\lambda]]$ denotes greatest integer not exceeding λ

$$\binom{s-2r}{-1} = 0,$$

$$1 < r < [[s/2]] \quad \text{and} \quad s = 1,2,\ldots,n$$

Letting $\mu_{i:n} = E[g(x_{i:n})]$, one gets the equality (1.2.3) with moments. Using the similar procedure as above, defining for a symmetric population $\Psi(i) = (\mu^{(k)}_{i+1:2i} - \mu^{(k)}_{i:2i})/\binom{2i}{i}$ and setting in relation (1.2.3),

$$\mu^{(k)}_{i+1:2i} = -\mu^{(k)}_{i:2i} = (\tfrac{1}{2})\{\mu^{(k)}_{i+1:2i} - \mu^{(k)}_{i:2i}\}$$
$$= (\tfrac{1}{2})\binom{2k}{k}\Psi(k)$$

where moments are taken about the mean, one gets the same but more general result derived by Godwin (1949), i.e.

$$\text{(1.2.4a)} \quad \mu_{i:n} = \frac{1}{2}\binom{n}{i} \sum_{j=0}^{\frac{1}{2}(n-2i)} (-1)^j \int_{-\infty}^{\infty} F^{j+i}(x)[1-F(x)]^{j+i}\, dx$$
$$\cdot \left\{2\binom{n-2i-j}{j} - \binom{n-2i-j-1}{j}\right\}$$

One can obtain various identities by means of expression (1.2.4a). For example, the mean ranges for odd sample sizes can be found from the mean ranges for even sample sizes. The identity, $\chi_{n,p} = \frac{1}{2}\binom{n}{p}(-1)^{p+1}\,\Delta^p\,\omega_{n-p}$ (see Chapter III), suggested by "student" and proved by E.S. Pearson (1926), connecting $\mu_{i:n}$ with mean ranges, can also be obtained from the last relation (1.2.4a). However, it is not very useful for computation, owing to the large number of additions and subtractions involved.

Another solution for (1.2.3), with the expected values of the function g of order statistics in the largest sample as pivots, and proved by induction, is, in terms of moments (i.e. letting $x_{i:n} = g(x_{i:n})$);

$$\begin{aligned}
\mu_{i:j} = \{1/(j+1)(j+2)\ldots n\}\{ & \tbinom{n-j}{0}(j-i+1)(j-i+2)\ldots(n-i)\mu_{i:n} \\
& + \tbinom{n-j}{1}\, i(j-i+1)(j-i+2)\ldots(n-i-1)\ \mu_{i+1:n} \\
(1.2.5) \qquad & + \tbinom{n-j}{2}\, i(i+1)(j-i+1)(j-i+2)\ldots(n-i-2)\ \mu_{i+2:n} \\
& + \ldots + \tbinom{n-j}{n-j}\, i(i+1)(i+2)\ldots(i+n-j-1)\ \mu_{i+n-j:n}\} \\
& (1\leq i\leq j\leq n)
\end{aligned}$$

Srikantan (1962) discusses relations (1.2.3) and (1.2.5) and their application to the preparation of tables of c.d.f.'s, expected values, variances, etc.

Sillitto (1964) developed a relation which establishes a connection between moments of order statistics in samples of different sizes from any continuous population.

$$(1.2.6) \qquad \mu_{m:r} = \sum_{i=0}^{n-r} \binom{n-m-i}{r-m}\binom{r+i-1}{i}\ \mu_{m+i:n}/\binom{n}{r}, \quad (n>r)$$

By summation under the integral sign relation (1.2.6) is proved as follows:

$$\begin{aligned}
\text{R.H.S.} = \int_0^1 x \sum_{i=0}^{n-r} & \frac{(n-m-i)!}{(r-m)!(n-r-i)!} \cdot \frac{(m+i-1)!}{i!(m-1)!} \cdot \\
& \cdot \frac{n!}{(m+i-1)!(n-m-i)!} \cdot \frac{r!(n-r)!}{n!}\, F^{m+i-1}(1-F)^{n-m-i}\, dF
\end{aligned}$$

$$= \int_0^1 x\, m\binom{r}{m} F^{m-1}(1-F)^{r-m} \sum_{i=0}^{n-r} \frac{(n-r)!}{(n-r-i)!\,i!}$$

$$\cdot F^{i}(1-F)^{n-r-i}dF$$

$$= \int_0^1 x\, m\binom{r}{m} F^{m-1}(1-F)^{r-m}\{F + (1-F)\}^{n-r}dF$$

$$= m\binom{r}{m} \int_0^1 x\, F^{m-1}(1-F)^{r-m}dF = \text{L.H.S.}$$

Downton (1966) developed some relations* related to (1.2.6), by noting that the first two moments of linear functions are linear functions of moments $\mu_{i:n}$, and under certain circumstances these linear functions of moments may be considerably simplified; as for example

$$\sum_{i=1}^{n} (i-1)^{(k)}(n-i)^{(e)}\mu_{i:n} = \int_{-\infty}^{\infty} x \sum_{i=k+1}^{n-e} \frac{n!}{(i-k-1)!(n-e-i)!}$$

$$F^{i-1}(x)\{1 - F(x)\}^{n-i}dF(x)$$

(1.2.6a)

$$= n^{(k+e+1)} \int_{-\infty}^{\infty} xF^{k}(x)\{1-F(x)\}^{e}dF(x)$$

$$k!e!\binom{n}{k+e+1}\mu_{k+1:k+e+1}$$

Identity (1.2.6a) also holds for $\mu_{i:n}^{(2)}$, the second order moments.

Gupta (1960) has considered ordered statistics from the

*See other chapters.

standardized gamma distribution with the parameter t defined on the positive integers (i.e. from the chi-distribution with even degrees of freedom) and has derived expressions for the r-th moments of an order statistic* (about the origin). Letting the gamma p.d.f. be $g_t(x) = e^{-x} x^{t-1}/\Gamma(t)$, $0<x<\infty$, (t, positive integer). Then the cumulative distribution function of x, $G_t(x)$ can be written as probabilities in a Poisson distribution, $G_t(x) = \sum_{i=t}^{\infty} e^{-x} x^i/i!$ and the k-th moment about the origin of the r-th order statistic from a sample of size n is given as

$$\mu_{r:n}^{(k)} = \frac{n!}{(r-1)!(n-r)!} \int_0^\infty [1 - \sum_{j=0}^{t-1} \frac{e^{-y}(y)^j}{j!}]^{r-1} [\sum_{j=0}^{t-1} \frac{e^{-y}(y)^j}{j!}]^{n-r}$$

$$\times \{(e^{-y} y^{(t+k-1)})/\Gamma(t)\}dy$$

(which reduces to)

$$\mu_{r:n}^{(k)} = \frac{n!}{(r-1)!(n-r)!\Gamma(t)} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i}$$

$$\sum_{m=0}^{(t-1)(n-r+i)} a_m(t,n-r+i) \frac{\Gamma(t+k+m)}{(n-r+i+1)^{t+k+m}}$$

where $a_m(t,i)$ is the coefficient of S^m in the expansion of $(\sum_{j=0}^{t-1} \frac{s^j}{j!})^i$. Thus the k-th moment of the r-th order statistic may be expressed in terms of the k-th moments of the 1-st order statistic:

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*And also the covariance between two order statistics; see other chapter.

$$(1.2.9) \qquad \mu_{r:n}^{(k)} = \frac{n!}{(r-1)!(n-r)!} \sum_{i=0}^{r-1} \{(-1)^i \binom{r-1}{i} \mu_{1:n-r+i+1}^{(k)} / (n-r+i+1)\}$$

Letting $\nu_{r:n}^{(k)} = \{(r-1)!(n-r)!\Gamma(t)/n!\}\mu_{r:n}^{(k)}$, then the following recursion formula is satisfied:

$$(1.2.10) \qquad \nu_{r:n}^{(k)} = \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} \nu_{r+i:r+i}^{(k)}$$

A slightly more generalized form of (1.2.10) is

$$(1.2.10.1) \qquad \nu_{r:n}^{(k)} = \sum_{i=0}^{\beta} (-1)^i \binom{\beta}{i} \nu_{r+i:n-\beta-i}^{(k)}$$

(β positive integer $\leq n-r$)

By letting $\beta=1$ we get (1.2.1).

Govindarajulu (1962) has investigated some relationships between moments of order statistics from chi (1 d.f.) ($\nu_{r:n}^{(k)}$) and the standard normal distributions ($\eta_{r:n}^{(k)}$)*.

When n is even,

$$(1.3) \qquad \nu_{n:n}^{(1)} = \sum_{i=0}^{n-2} (-1)^i \, 2^{n-1-i} \binom{n}{i} \eta_{n-i:n-i}^{(1)}$$

Relation (1.3) is proven by taking the expression, (A), $2^n n \int_0^\infty v\phi(v) [\Phi(v) - \frac{1}{2}]^{n-1} dV$ and letting $2\,\Phi(v) - 1 = F(v)$.

*See Chapter II for related work on mixed moments.

Then (A) becomes $n\int_0^\infty v\, f(v)[F(v)]^{N-1}\,dV = \nu_{n:n}$. On the other hand, if N is even, the integrand in (A) is an even function and can be written as,

$$n\ 2^{n-1}\int_{-\infty}^{\infty} V\phi(v)[\Phi(v) - \tfrac{1}{2}]^{n-1}\,dV =$$

$$n\ 2^{n-1}\sum_{i=0}^{n-1}(-1)^i\binom{N-1}{i}\ 2^{-i}\int_{-\infty}^{\infty} v\phi(v)\ \Phi^{N-1-i}(v)\ dV$$

which simplifies to the form given in (1.3). If n+k is odd, then symbolically letting $\nu_{n:n}^{(k)} = \frac{1}{2}(2\ \ {}^{(k)}_{1:1} - 1)^n$, and if powers of $\eta_{1:1}^{(k)}$ say $(\eta_{1:1}^{k})^{i}$ are replaced by $\eta_{i:i}^{(k)}$ for i>1 (for i=0, define $\mu_{0:0}^{(k)} = 0$) we get

$$(1.3.1)\qquad \nu_{n:n}^{(k)} = \frac{1}{2}\sum_{i=0}^{n-1}(-1)^i\ 2^{n-i}\binom{n}{i}\ \eta_{n-i:n-i}^{(k)}.$$

Jones (1948), Godwin (1949), and Bose and Gupta (1949) have expressed, for n<5, the moments (and product moments) of order statistics of the normal distribution in terms of elementary functions. Following the last authors, let

$$(A)\ \ I_n(a) = \int_{-\infty}^{\infty}[\Phi(ax)]^n\ e^{-x^2}dx$$

where $I_0(a) = \pi^{\frac{1}{2}}$

Now $\int_{-\infty}^{\infty}[\Phi(ax) - \frac{1}{2}]^{2m+1}\ e^{-x^2}dx = 0,\ m = 0,1,2,\ldots$ since the integrand is an odd function of x. Therefore,

$$\frac{I_{2m+1}(a) = \sum_{i=1}^{2m+1} (-1)^{i+1}\binom{2m+1}{i} I_{2m-i+1}(a)}{2^i}$$

In particular $I_1(a) = \frac{1}{2} I_0(a) = \frac{1}{2}\pi^{\frac{1}{2}}$ and

$$I_3(a) = \frac{3}{2} I_2(a) - \frac{3}{4} I_1(a) + \frac{1}{8} I_0(a)$$

$$= \frac{3}{2} I_2(a) - \frac{1}{4} I_0(a)$$

Differentiating (A) with respect to a, one obtains for n=2

$$(2\pi)^{\frac{1}{2}} I_2'(a) = \int_{-\infty}^{\infty} \Phi(ax) - 2x\, e^{-\frac{1}{2}x^2(a^2+2)} dx$$

and integration by parts gives $I_2'(a) = (\pi)^{-\frac{1}{2}} a(a^2+2)^{-1}(a^2+1)^{-\frac{1}{2}}$ so that

$$I_2(a) = (\pi)^{-\frac{1}{2}} \times \arctan[(a^2+1)^{\frac{1}{2}}]$$

and

$$I_3(a) = 1.5\ \pi^{-\frac{1}{2}} \times \arctan[(a^2+1)^{\frac{1}{2}}] - \frac{1}{4}\pi^{\frac{1}{2}}.$$

With the help of these results the ordinary moments of order statistics can be evaluated. Furthermore, the authors in the same paper have developed a more general relation for the moments of order statistics from a normal distribution, in which the k-th moment of $X_{r:n}$ is expressed in terms of lower moments of order k-2i $(i = 1,2,\ldots,\frac{1}{2}k$ or $\frac{1}{2}(k-1)$ and the integral

$$\int_{-\infty}^{\infty} P_{k+1}(x)\ e^{-\frac{1}{2}(k+1)x^2}dx$$

where $P_{k+1}(x)$ for $k>0$ is defined by $P_{k+1}(x) = r\binom{n}{r} \frac{d^k}{d\Phi^k(x)}$ $[\Phi^{r-1}(x)(1-\Phi(x))^{n-r}]$, i.e. $P_x(x)$ is a polynomial of degree $(n-k)$ in $\Phi(x)$, $k\leq n$ and zero if $k>n$.

Hastings et al. (1947) developed certain relations by working with a special distribution, which they called the "representing function" r(u), a monotone function such that $Pr\{r(u_1) < x < r(u_2)\} = u_2 - u_1,\ u_2>u_1$. Thus for example if u has a uniform (= rectangular on [0,1]) distribution then $x = r(u)$ defines a variate with the given distribution. The means*, then can be written as follows:

$$\mu_{n-i+1:n} = E[r(\mu_{n-i+1:n}] = i\binom{n}{i} \int_0^1 r(u)\mu^{i-1}(I-u)^{n-i}du$$

Letting $E_s = \int_0^1 r(u)\ u^s du$ one has

(1.3.2) $$\mu_{n-i+1:n} = i\binom{n}{i} \sum_k (-1)^k\binom{n-i}{k} E_{i-1+k}$$

and similarly

*See chapter for variance and covariance.

(1.3.2a) $\mu^{(2)}_{n-i+1:n} = i\binom{n}{i} \sum_k (-1)^k \binom{n-i}{k} E_{i-1+k:i-1+k}$

where $E_{ss} = \int_0^1 r^2(u)\, u^s du$.

Thus for the uniform distributions the well known relations follow:

(1.3.3) $\mu_{i:n} = (n-i+1)/n+1$

(1.3.3a) $\mu^{(2)}_{i:n} = i(n-i+1)/(n+1)^2(n+2)$

When the "representing function" is $x = r(u) = 1/(1-u)^\lambda - 1/u^\lambda$ $(\lambda>0)$, one obtains a symmetrical distribution with long tails. (For the normal distribution $r(u) = 0(\ln u)$ as $u\to 0$). Thus the integrals wanted are

$$E_s = \int_0^1 \{(1-u)^{-\lambda} - u^{-\lambda}\} u^s du$$

$$E_{ss} = \int_0^1 \{(1-u)^{-\lambda} - u^{-\lambda}\}$$

which can be expressed as

$$E_s = A_s(\lambda) - B_s(\lambda)$$

$$E_{ss} = A_{s,s}(\lambda) - 2\, B_{s,s}(\lambda) + C_{s,s}(\lambda)$$

where

$$A_s(\lambda) = \int_0^1 (1-u)^{-\lambda} u^s du = b(-\lambda, s)$$

$$B_s(\lambda) = \int_0^1 u^{-\lambda} u^s du = 1/(s+1-\lambda)$$

$$A_{ss}(\lambda) = \int_0^1 (1-u)^{-2\lambda} u^s du = b(-2\lambda, s)$$

$$B_{ss}(\lambda) = \int_0^1 (1-u)^{-\lambda} u^{-\lambda} u^s du = b(-\lambda, s-\lambda)$$

$$C_{ss}(\lambda) = \int_0^1 u^{-2\lambda} u^s du = 1/(s+1-2\lambda)$$

where

$$b(p,q) = p!q!/(p+q+1)!$$

Using the preceding formulae along with (1.3.2) makes the means of this special distribution readily available.

Weiss (1962), Margolin and Winokur (1967) have presented formulae for the first two moments of the order statistics in closed form, from a geometric distribution: $p(x) = q_i^{x_i-1} p_i$, $q_i = 1 - p_i$, $x_i = 1,2,\ldots.$ They developed first the following recurrence relation, for the discrete case, using properties of the incomplete beta function;

$$(1.1a) \qquad \mu_{r:n}^{(k)} = n\binom{n-1}{r-1} \sum_{i=0}^{r-1} \frac{(-1)^i \binom{r-1}{i}}{(n-r+i+1)} \mu_{i:n-r+i+1}^{(k)}$$

Result (1.1a) has been presented in a slightly varied form for the continuous distribution by Cole (relation (1.2)). For the special case of the geometric distribution, (1.3.4a) $\mu^{(1)}_{1:n-r+i+1} = 1/(1 - q^{n-r+i+1})$

and

(1.3.5b) $\mu^{(2)}_{1:n-r+i+1} = (1 + q^{n-r+i+1})/(1 - q^{n-r+i+1})^2$ and

since $X_{(1:n-r+i+1)}$ is geometrically distributed with parameter $1 - q^{n-r+i+1}$. Then,

$$(1.3.4) \qquad \mu^{(1)}_{r:n} = n\binom{n-1}{r-1} \sum_{i=0}^{r-1} \frac{(-1)^i\binom{r-1}{i}}{(n-r+i+1)} \cdot (1 - q^{n-r+i+1})^{-1}$$

$$(1.3.5) \qquad \mu^{(2)}_{r:n} = n\binom{n-1}{r-1} \sum_{i=0}^{r-1} \frac{(-1)^i\binom{r-1}{i}}{(n-r+i+1)} \cdot \frac{1+q^{n-r+i+1}}{(1-q^{n-r+i+1})^2}$$

for the geometric distribution.

Young (1970) has developed some recurrence relations of moments of order statistics of independent random variables. By using the probability generating function of $X_{r:n}$ and the condition that $X_{r:n}$ is bounded, he obtained the following two general recurrence equations.

$$(1.4) \qquad \mu^{(k)}_{r:n} = \sum_{i=0}^{r-1} \sum_{j=0}^{i} \binom{n}{i}\binom{i}{j}(-1)^j \mu^{(k)}_{1:n-i-j}$$

$$(1.4.1) \qquad \mu^{(k)}_{r:n} = \sum_{i=r}^{n} \sum_{j=0}^{i} \binom{n}{i}\binom{i}{j}(-1)^{j+1} \mu^{(k)}_{1:n-i+j}$$

$j = 0,1,2,\ldots,i \ (i<n)$

$j = 1,2,\ldots,i \ (i=n)$

Equations (1.4) is useful for lower values of n and (1.4.1) is useful for the high order statistics. Through the use of these two fundamental equations Young has provided tables of the expected values of the order statistics of the negative binomial distribution.

In another paper Young (1973) has presented a general recurrence relation for moments of order statistics of n+1 independent random variables when n of the variables are identically distributed (i.e. a slippage configuration) and then applied it to the case when the random variables have the negative binomial distribution. Letting $X_1,\ldots,X_{n+1}$ be the n+1 independent random variables and letting the first n variables be identically distributed with $G(x) = pr(X_i > x)$ $(i = 1,\ldots,n)$, $G^s(x) = pr(X_{n+1} > x)$; then when $r>1$;

$$pr\{X^{(s)}_{r:n+1} > x\} = \sum_{i=0}^{r-2} \binom{n}{i}\{G(x)\}^{n-i}\{1 - G(x)\}^i$$
$$+ \binom{n}{r-1} G^s(x)\{G(x)\}^{n-r+1}\{1 - G(x)\}^{r-1}$$

Expressing the sum of binomial probabilities in terms of the incomplete beta function, we obtain

$$pr(X^{(s)}_{r:n+1} > x) = \frac{\Gamma(n+1)}{\Gamma(n-r+2)\Gamma(r-1)} \int_0^{G(x)} u^{n-r+1}(1-u)^{r-2}du$$
$$+ \binom{n}{r-1}\ G^{(s)}(x) \sum_{j=0}^{r-1} \binom{r-1}{j}(-1)^j\{G(x)\}^{n-r+j+1}$$
$$= \binom{n}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j}(-1)^j\{G(x)\}^{n-r+j+1}$$

$$\{G^{(s)}(x) - \frac{j}{n-r+j+1}\}$$

(a) $$= \binom{n}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j [\Pr(X^{(s)}_{1:n-r+j+2} > x) - \frac{j}{n-r+j+1} \Pr(X_{1:n-r+j+1} > x)]$$

Thus if the variables are discrete and integer valued, the relation for moments about the origin follow directly from (a), i.e.

(1.5) $$\mu^{s(k)}_{r:n+1} = \binom{n}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j [\mu^{s(k)}_{1:n-r+j+2} - \frac{j}{n-r+j+1} \mu^{s(k)}_{1:n-r+j+1}]$$

Relation (1.5) also holds for continuous variables.

CHAPTER II

RECURRENCE RELATIONS AMONG PRODUCT MOMENTS

This chapter shall attempt to outline all the recurrence relations among products, and some of the important proofs, and relevant derivations. Only those derivations and proofs that are not already used and shown in Chapter I shall be outlined in this chapter to avoid repetitions. Appropriate reference shall be made where proofs or derivations for product moment relations are similar to those relations which are alreay given for simple moments.

For an arbitrary continuous distribution Govindarajulu (1962) developed the following product-moment relation:

$$(2.1)\qquad (i-1)\mu_{i,j:n} + (j-i)\mu_{i-1,j:n} + (n-j+1)\mu_{i-1,j-1:n} = n\mu_{i-1,j-1:n-1} \qquad (1<i<j<n)$$

Relation (2.1) is derived by taking the integral defining $\mu_{i-1,j-1:n-1}$, and splitting up the integral as the sum of three similar integrals according to the partition $1 = F(x) + [F(y) - F(x)] + [1 - F(y)]$. Relation (2.1) holds also for the discrete case and can be proved by using a similar approach. However as pointed out in the first chapter for relation (1.2.1), relation (2.1) has some problems as far as accuracy is concerned, namely that reported applications for working "upwards" causes a loss of accuracy of 1 to 3 units in the last decimal place. If, however, as Harter (1961) and Srikantan (1962) point out, relation (2.1) is rewritten as

$$\text{(2.1a)} \qquad \mu_{i-1,j-1:n-1} = \{\frac{i-1}{n}\}\mu_{i,j:n} + \{\frac{j-1}{n}\}\mu_{i-1,j:n} + \{\frac{n-j+1}{n}\}\mu_{i-1,j-1:n}$$

Then, they can be used for working "downwards" with no serious accumulation of rounding errors. However, for this procedure one has to evaluate all the first, second and mixed moments of order statistics in an arbitrary large sample size n, which may not always be possible. Srikantan (1962) gives an explicit solution of (2.1) in terms of the expected values of the function of the first order statistics paired with all other order statistics in each sample size:

$$\mu_{i,j:k} = (-1)^{i-1} \sum_{t=0}^{i-1} (-1)^{t} \binom{k}{t} \sum_{s=t}^{i-1} \binom{k-j+s-t}{s-t} \binom{j-2-s}{i-1-s} \mu_{1,j-s:k-t}$$

$$(1<i<j<k<n)$$

This relation enables one to compute the $n(n^2-1)/6$ expected values $\mu_{i,j:n}$ $(1<i<j<k<n)$ in terms of the $n(n-1)/2$ expected values $\mu_{i,j:k}$ $(1<i<j<k<n)$. If one lets i=1, and j=i+1 we

get as a special case, relation (1.2.1). Therefore for any arbitrary distribution symmetric about zero, the number of distinct and independent constraints among the distinct $\mu_{i,j:n}$ $(i=j)$ imposed by (2.1) is $\{(n-1)^2 - 1\}/4$ if n is even and $(n-1)^2/4$ if n is odd.

Downton (1966) by using the definitions of $\mu_{r,s:n}$ and following similar arguments that he used for (1.2.6a) developed the following two relations, which hold for any continuous distribution.

$$(2.2) \quad \sum_{i=1}^{n} (i-1)^{(k)} (n-i)^{(\ell)} \mu_{i,i:n} = k!\ell!\binom{n}{k+\ell+1}\mu_{k+1,k+1:k+\ell+1}$$

$$(2.3) \quad \sum_{i<j}\sum (i-1)^{(k)} (n-j)^{(\ell)} \mu_{i,j:k} = k!\ell!\binom{n}{k+\ell+2}\mu_{k+1,k+2:k+\ell+2}$$

Then using the relations $(a+b)^{(m)} = \overbrace{\sum_{r=0}^{m} \binom{m}{r} a^{(r)} b^{(m-r)}}^{(a)}$ and $(a-b)^{(m)} = \underbrace{\sum_{r=0}^{m} (-1)^r \binom{m}{r} (a-r)^{(m-r)} b^{(r)}}_{(b)}$ together with (2.2) and (2.3) Downton got

$$\mu_{k+1:k+\ell+1}\mu_{p+1:p+q+1} = 1/\binom{n}{k+\ell+1} \cdot 1/\binom{n}{p+q+1} \cdot 1/p!q!k!\ell!$$

$$\left\{ \sum_{r=0}^{p} \sum_{s=0}^{q} \binom{p}{r}\binom{q}{s} k^{(p-r)(q-s)} \right.$$

$$\times\ (k+r)!(\ell+s)!\binom{n}{k+\ell+r+s+1}\mu_{k+r+1,k+r+1:k+\ell+r+s+1}$$

$$+ \sum_{r=0}^{\ell} \sum_{s=0}^{p} (-1)^{r+s} \binom{\ell}{r} \binom{p}{s} (n-k-r-1)^{(\ell-r)} (n-q-s-1)^{(p-s)} (k+r)!(q+s)!$$

$$\times \binom{n}{k+q+r+s+2} \mu_{k+r+1,k+r+2:k+q+r+s+2}$$

$$+ \sum_{r=0}^{q} \sum_{s=0}^{k} (-1)^{r+s} \binom{q}{r} \binom{k}{s} (n-p-r-1)^{(q-r)} (n-\ell-s-1)^{(k-s)} (p+r)!(\ell+s)!$$

$$\times \binom{n}{p+\ell+r+s+2} \mu_{p+r+1,p+r+2:p+\ell+r+s+2} \}.$$

This relation, because of its generality, is rather complicated, however, in many applications considerable simplification is possible. It may be noted that with (2.4), there is no need to compute the complete variance matrix associated with the random variable $x_{i:m}$ as the covariance of $x_{k+1:k+\ell+1}$ and $x_{p+q:p+q+1}$ defined on the terms of the form $\mu_{i,i:m}$ and $\mu_{i,i+1:m}$ for $m < k+\ell+p+q+2$, of relatively small variance matrices.

Again using identities (a) and (b) and relations (2.2) and (2.3) Downton (1966) gets

$$\sum_{r=1}^{n} \sum_{s=1}^{n} (r-1)^{(i)} (s-1)^{(j)} \mu_{r,s:n} = \sum_{r=1}^{n} (r-1)^{(i)} (r-1)^{(j)} \mu_{r,r:n}$$

$$+ \sum_{r<s} \{(r-1)^{(i)} (s-1)^{(j)} + (s-1)^{(i)} (r-1)^{(j)}\} \mu_{r,s:n}$$

$$= \sum_{t=0}^{i} \binom{i}{t} j^{(i-t)} (j+t)! \binom{n}{j+t+1} \mu_{j+t+1,j+t+1:j+t+1}$$

$$+ \sum_{t=0}^{i} (-1)^{t} \binom{i}{t} (n-t-1)^{(i-t)} j!t! \binom{n}{j+t+2} \mu_{j+1,j+2:j+t+2}$$

$$+ \sum_{t=0}^{i} (-1)^t \binom{j}{t} (n-t-1)^{(j-t)} i!t! \binom{n}{i+t+2} \mu_{i+1,i+2:i+t+2}.$$

Because of symmetry the first sum in this expression could have been written with i in place j and vice versa. It may also be noted that the sample size n enters into these sums only in terms of factorial powers. Downton also expresses (2.4.1) in a form using factorial powers: i.e.

$$\text{(2.4.1a)} \quad \sum_{r=1}^{n} \sum_{s=1}^{n} (r-1)^{(i)} (s-1)^{(j)} \mu_{r,s:n} = \sum_{t=j+1}^{i+j+2} a_{ij:t}\, n^{(t)}$$

where

$$a_{ij:i+j+2} = \mu_{i+1:i+1}\mu_{j+1:j+1}/((i+1)(j+1))$$

and for

$$j+1 < s < i+j+1$$

$$\begin{aligned} a_{ij:s} &= i!j!\mu_{s,s:s}/\{(i+j+1-s)!(s-i-1)!(s-j-1)!\} \\ &+ i!j!(j+1)^{(i+j+2-s)} \sum_{r=0}^{s-j-2} (-1)^r \mu_{j+1,j+2:j+2+r}(i-j-2-s)!/ \\ \text{(2.4.1b)} \qquad & \qquad (j+r+2)!(s-j-2-r)!\} \\ &- i!j!(i+1)^{(i+j+2-s)} \sum_{r=0}^{i} (-1)^r \mu_{s-i-1,s-i:s-i+r}/ \\ & \qquad (i+j+2-s)!(i-r)!(s-i+r)!\} \\ &+ i!j!\mu_{i+1:i+1}\mu_{s-i-1:s-i-1}/\{(i+j+2-s)!(s-j-1)!(s-j-1)!\} \end{aligned}$$

The second term vanishes when s=j+1, while the third and fourth terms vanish when i=j and s=j+1.

Govindarajulu (1963) developed some recurrent relations for an arbitrary distribution:

$$(2.5)\qquad \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mu_{i:n}^{r}\, \mu_{j:n}^{s} = \binom{n}{2} \mu_{1:2}^{r} \mu_{2:2}^{s} \qquad (r,s>0)$$

The proof of (2.5) follows by letting

$$\text{L.H.S.} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{n!}{(i-1)!(j-i-1)!(n-j)!}$$

$$\iint_{-\infty<x<y<\infty} x^r y^s F^{i-1}(x)\,[F(y)-F(x)]^{j-i-1}\,[1-F(y)]^{n-j}$$

$$dF(x)\ dF(y)$$

$$= \sum_{i=1}^{n-1} \frac{n!}{(i-1)!(n-i-1)!} \iint_{x<y} x^r y^s F^{i-1}(x)\,[1-F(x)]^{n-i-1}$$

$$d(Fx)\ dF(y)$$

$$= n(n-1) \iint_{x<y} x^r y^s\ dF(x)\ dF(y) = \text{R.H.S.}$$

By noting that $\mu_{1:2}^{r}\, \mu_{2:2}^{r} = (\mu^{r})^{2}$ and using (2.5) and (1.2.2) one gets

$$(2.5.1)\qquad \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mu_{i,j:n}^{(r)} = \binom{n}{2} (\mu^{r})^{2}$$

with r=1 (2.5.1) becomes

$$(2.5.2)\qquad \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mu_{i,j:n} = \binom{n}{2} \mu^{2}$$

For an arbitrary distribution and even n,

$$\mu_{1,n:n} = \sum_{i=1}^{(n-2)/2} (-1)^{i-1} \binom{n}{i} \mu_{i:i}\mu_{n-i:n-i} + (\tfrac{1}{2})(-1)^{(n-2)/2} \binom{n}{n/2} \mu^2_{n/2:n/2} \tag{2.6}$$

Govindarajulu (1963) proves it by considering the symmetrical integral

$$\mu_{1,n:n} = n(n-1) \iint_{-\infty<x<y<\infty} xy\,[F(y)-F(x)]^{n-2} dF(x)\,dF(y)$$

$$= \tfrac{1}{2}n(n-1) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\,[F(y)-F(x)]^{n-2} dF(x)\,dF(y)$$

Expanding $[F(y)-F(x)]^{n-2}$ in powers of $F(y)$ and $F(x)$, and integrating on x and y one obtains

$$\mu_{1,n:n} = \frac{1}{2} \sum_{i=0}^{n-2} (-1)^{i} \binom{n}{i+1} \mu_{i+1:i+1}\mu_{n-i-1:n-i-1}$$

$$= \sum_{i=0}^{(n-4)/2} (-i)^{j} \binom{n}{i+1} \mu_{i+1:i+1}\mu_{n-i-1:n-i-1} + \frac{1}{2}(-1)^{(n-2)/2} \binom{n}{n/2} \mu^2_{n/2:n/2}$$

$$= \text{R.H.S. of (2.6)}$$

Ruben (1956) uses a similar proof for normal order statistics. Relation (2.6) is also very similar to one used by Teichroew (1956) for normal order statistics. Govindarajulu (1963) also developed a recurrence relation for any arbitrary continuous distribution for which $f'(x) = xf(x)$:

(2.7) $$\mu^{(2)}_{i:n} = 1 + \frac{n!}{(i-1)!(n-i)!} \sum_{m=0}^{n-i} (-1)^m \frac{1}{i+m}\binom{n-i}{m} \mu_{i+m-1,i+m:i+m}$$

if i=n, then

(2.7.1) $$\mu^{(2)}_{n:n} = 1 + \mu_{n-1,n:n}$$

Using (1.2.1.2) and (1.2.1.3) with (2.7.1) one obtains

(2.7.2) $$\mu^{(2)}_{1:n} = 1 + \mu_{1,2:n}$$

Setting i=n-1, one gets

(2.7.3) $$\mu^{(2)}_{n-1:n} = 1 + n\mu_{n-2,n-1:n-1} - (n-1)\mu_{n-1,n:n}$$

Also setting i=1, one gets

$$\mu^{(2)}_{1:n} = 1 + N \sum_{m=0}^{n-1} (-1)^m (m+1)^{-1} \binom{n-1}{m} \mu_{m,m+1:m+1}$$

$$= 1 + \sum_{m=0}^{n-1} (-1)^m \binom{n}{m+1} \mu_{m,m+1:m+1}$$

$$= 1 + \sum_{m=2}^{n} (-1)^{m-1} \binom{n}{m} \mu_{m-1,m:m}$$

Using (2.7.3) and (2.7.4) Govindarajulu (1963) gets

(2.7.5) $$\mu_{1,2:n} = \sum_{m=2}^{n} (-1)^{m-1} \binom{n}{m} \mu_{1,2:m}$$

With these formulae one now has a systematic procedure for evaluating the first, second, and mixed (linear) moments of the normal order statistics for any given n given these moments of order statistics for all sample sizes up to and including n-1. First evaluate $\mu_{1:n}$ and solve for the rest of $\mu_{i:n}$ using the $\mu_{i:n-1}$ and the recurrence formulae (1.2.1) with k=1 (assume n is even). Then relation (1.2.1) gives us $\mu^{(2)}_{n/2:n} = \mu^{(2)}_{n/2:n-1}$ and therefore $\mu^{(2)}_{n/2:n}$ is known and the rest of $\mu^{(2)}_{i:n}$ will be known. Now $\mu_{1,n:n}$ is available from relation (2.6). From relation (2.1) one gets $\mu^{(2)}_{1:n} = 1 + \mu_{1,2:n}$ and $\mu_{1,2:n}$ is known. Then evaluate any (n-4)/2 of the rest of the distinct $\mu_{i,j:n}$ $(i \neq j)$ and use the recurrence relation (2.1) with i=1, and j = 2,3,...,n-1, i=2 and j = 3,4,...,n-2 and i=3 and j = 4,5,...,n-3 etc. until the total number of these relationships is n(n-2)/4 and one then solves for the rest of the distinct $\mu_{i,j:n}$. If n is odd, then $\mu_{(n+1)/2} = 0$. Using (1.2.1) the rest of $\mu_{i:n}$ are known. Then evaluate one $\mu^{(2)}_{i:n}$ and then using (1.2.1) with k=2, the rest of the $\mu^{(2)}_{i:n}$ can be solved. From (2.1) $\mu_{1,2:n}$ is known. Then one must evaluate (n-3)/2 of the rest of the distinct $\mu_{i,j:n}$ $(i \neq j)$, using recurrence relation (2.1), which should total $(n-1)^2/4$. Then one solves for the rest of the distinct $\mu_{i,j:n}(i \neq j)$. This type of computational scheme is useful especially if it is possible to evaluate exactly the lower moments of some order statistics for each n. However, for some distributions, especially the normal distribution, it will be difficult to evaluate these exactly for small samples.

Govindarajulu (1962) has derived some relationships among moments of order statistics in samples drawn from chi-population (1 d.f.) ($\nu_{i,j:n}$), and also some relationships between the moments of order statistics from the chi-distribution (1 d.f.) and the standard normal distribution.

Letting

$$\nu_{n-1,n:n} = n(n-1) \iint_{0<x<y<\infty} xyf(x)\ f(y)\ [F(x)]^{n-2}dxdy$$

and integrating with respect to y one gets

$$\nu_{n-1,n:n} = n(n-1) \int_0^\infty xf^2(x)\ [F(x)]^{n-2}dx.$$

Then writing

$$f(x)\ [F(x)]^{n-2} = d/dx\,[F^{n-1}(x)/(n-1)]$$

and integrating by parts Govindarajulu gets

(2.8) $$\nu^{(2)}_{n:n} = 1 + \nu_{n-1,n:n}$$

In a similar manner one gets the following relationship:

(2.9) $$\nu^{(2)}_{1,n} = 1 + \nu_{1,2:n} - n(2/\pi)^{\frac{1}{2}}\nu_{1:n-1}$$

Letting

$$\nu_{1,n:n} = n(n-1) \iint_{0<x<y<\infty} xyf(x)f(y)\ [F(y)-F(x)]^{n-2}dxdy$$

(for even n)

$$= n(n-1)/2 \int_0^{\infty} \int_0^{\infty} xyf(x)f(y)[F(y)-F(x)]^{n-2}dxdy$$

and expanding $[F(y)-F(x)]^{n-2}$ and integrating term-wise one gets (n even)

$$(2.10) \qquad \nu_{1,n:n} = \sum_{i=1}^{(n-2)/2} (-1)^{i-1}\binom{n}{i} \nu_{i:i}\nu_{n-i:n-i} + (-1)^{(n-2)/2}\left(\tfrac{1}{2}\right)\binom{n}{n/2}\nu^2_{n/2:n/2}$$

When n is odd, following the proof of (2.8):

$$\nu_{n-1:n:n} = n(n-1) \int_0^{\infty} xf^2(x)F^{n-2}(x)dx$$

$$= 2^n n(n-1) \int_0^{\infty} x\,\phi^2(x)\,[\Phi(x) - \tfrac{1}{2}]^{n-2}dx$$

$$= 2^{n-1}n(n-1) \int_{-\infty}^{\infty} x\phi^2(x) \qquad [\Phi(x) - \tfrac{1}{2}]^{n-2}dx,$$

and expanding $[\Phi(x) - \frac{1}{2}]^{n-2}$ and integrating term by term, one gets the following relationship:

$$(2.11) \qquad \nu_{n-1,n:n} = \sum_{i=0}^{n-2} (-1)^i 2^{n-1-i}\binom{n}{i}\mu_{n-i-1,n-i:n-i} \quad \text{(n odd)}.$$

With formulae (2.8), (2.9), (2.10), (2.11) one can now evaluate the first, second and mixed moments of order statistics in a sample of size n, given these moments in samples to size n-1, and a table of these moments of normal order statistics to size n.

Defining

$$\nu_{1,2:n} = n(n-1) \iint_{0<x<y<\infty} xyf(x)f(y)[1 - F(x)]^{n-2}dxdy$$

and integrating with respect to y one gets

$$\nu_{1,2:n} = - n(n-1) \int_0^\infty xf^2(x)[1 - F(x)]^{n-2}dx$$

$$+ n(n-1)(2/\pi)^{\frac{1}{2}} \int_0^\infty xf(x)[1 - F(x)]^{n-2}dx;$$

then expanding $[1 - F(x)]^{n-2}$ and integrating term-wise Govindarajulu (1962) gets

$$(2.12) \qquad \nu_{1,2:n} = \sum_{i=2}^{n} (-1)^{i-1}\binom{n}{i}\nu_{i-1,i:i} + n(2/\pi)^{\frac{1}{2}} \sum_{j=1}^{n-1} (-1)^{j-1}\binom{n}{j}\nu_{j:j} \qquad (n, \text{ odd})$$

For the case of even n,

$$\nu_{1,n:n} = 2^n n(n-1) \iint_{0<x<y<\infty} xy\phi(x)\phi(y)[\Phi(y)-\Phi(x)]^{n-2}dxdy$$

Note that the integrand is symmetrical with respect to the origin and in x and y. Taking the relation

$$\mu_{1,n:n} = n(n-1) \iint_{-\infty<x<y<\infty} xy\phi(x)\phi(y)\,[\Phi(y)-\Phi(x)]^{n-2}dxdy$$

$$= 2^{-(n-1)}\nu_{1,n:n}+n(n-1) \int_{x=-\infty}^{0} \int_{y=0}^{\infty} xy\phi(x)\phi(y)$$

$$[\Phi(y)-\Phi(x)]^{n-2}dxdy,$$

and expanding $[\Phi(y)-\Phi(x)]^{n-2}$ and changing x to -z, one gets

$$\mu_{1,n:n} = 2^{-(n-1)}\nu_{1,n:n} + \sum_{i=0}^{n-2}(-1)^{i+1}n(n-1)\binom{n-2}{i}$$

$$[\int_0^\infty z\phi(z)\{1 - \Phi(z)\}^{n-2-i}dz]\,[y\phi(y)\Phi^{i}(y)dy]$$

$$= 2^{-(n-1)}\nu_{1,n:n} + \sum_{i:0}^{n-2}(-1)^{i+1}\binom{n-2}{i}n(n-1)$$

$$2^{-n}[\int_0^\infty zf(z)\{1 - F(z)\}^{n-2-i}dz]\,[\int_0^\infty yf(y)\{1+F(y)\}^{i}dy]$$

Then if one expands $[1\pm F]^{\alpha}$ in powers of F and integrates termwise one gets the following relationship (Govindarajulu, 1962):

(2.13) $$2\nu_{1,n:n} = 2^{n}\mu_{1,n:n} + \sum_{i=1}^{n-1}\binom{n}{i}(-1)^{i+1}\nu_{1:n-i}\,[\sum_{j=1}^{i}\binom{i}{j}\nu_{j:j}]$$

Let

$$\sum_{j=2}^{n}\nu_{i,j:n} = \sum_{j=2}^{n}\frac{n!}{(j-2)!\,(n-j)!}$$

$$\iint_{0<x<y<\infty} xyf(x)f(y)\,[F(y)-F(x)]^{j-2}$$

$$[1-F(y)]^{n-j}dxdy$$

$$= n(n-1) \iint_{0<x<y<\infty} xyf(x)f(y)[1-F(x)]^{n-2}dxdy$$

$$= n(n-1) \int_0^\infty xf^2(x)[1-F(x)]^{n-2}dx.$$

Integrating by parts Govindarajulu (1962) gets:

$$(2.14) \qquad \sum_{j=2}^{n} \nu_{1,j:n} = 1 - \nu_{1:n}^{(2)}$$

Using the same method as for (2.13), one gets

$$\sum_{j=1}^{n-1} \nu_{j,n:n} = n(n-1) \iint_{0<x<y<\infty} xyf(x)f(y)[F(y)]^{n-2}dxdy$$

$$\nu_{n-1,n:n} + \sum_{j=1}^{n-1} \nu_{j,n:n} = n(n-1) \int_0^\infty \int_0^\infty xyf(x)f(y)[F(x)]^{n-2}dxdy$$

$$= n(2/\pi)^{\frac{1}{2}} \nu_{n-1:n-1}$$

Then by (2.8)

$$(2.15) \qquad \sum_{j=1}^{n-1} \nu_{j,n:n} = n(2/\pi)^{\frac{1}{2}} \nu_{n-1:n-1} - \nu_{n:n}^{(2)} + 1$$

Again using similar methodology;

$$\sum_{j=i+1}^{n} \nu_{i,j:n} = \frac{n(n-1)\ldots(n-i)}{(i-1)!} \iint_{0<x<y<\infty} xyf(x)f(y) F^{i-1}(x)[1-F(x)]^{n-i-1}dxdy$$

$$= \frac{n!}{(n-i-1)(i-1)!} \int_0^\infty xf^2(x)F^{i-1}(x)[1-F(x)]^{n-i-1}dx.$$

Then letting $(n-i)f(x)[1-F(x)]^{n-i-1} = d[1-F(x)]^{n-i}$ and integrating by parts Govindarajulu (1962) gets:

$$(2.16) \qquad \sum_{j=i+1}^{n} \nu_{i,j:n} - \sum_{j=1}^{n} \nu_{i-1,j:n} = 1 - \nu_{i:n}^{(2)} \qquad (1<i<n-1)$$

Another formulae developed by Govindarajulu (1962), a bit more complicated than the preceding ones, expresses the relationship between $\nu_{i,j:n}$ and $\nu_{n-j+1,n-i+1:n}$ given a table of values of the μ's up to n and the ν's up to n-1:

$$\nu_{i,j:n} = 2^n \sum_{m=0}^{i-1} (-1)^m 2^{-m} \binom{n}{m} \mu_{i-m,j-m:n-m} \qquad (n>0)$$

$$(2.17) \qquad + (-1)^{i+1}\binom{n}{i} \sum_{m=1}^{j-i} i(j-m)^{-1} \binom{n-i}{j-i-m} \nu_{j-m:j-m}\nu_{m:n-j+m}$$

$$+ (-1)^{i} \sum_{m=1}^{i} \sum_{n=1}^{n-j+1} (-1)^{m-1} \binom{n}{j+n-1} \binom{j+n-1}{i-m}$$

$$\nu_{n,j-i+n:j-i+m+n-1}$$

Using relation (2.1) recurrently, one can generate the $\nu_{i:n}^{(k)}$ $(i = 1,2,\ldots,n)$ if the $\nu_{i:n-1}^{(k)}$ $(i = 1,2,\ldots,n-1)$ and any one of the $\nu_{i:n}^{(k)}$ are available. Similarly using formulae (4.1) recurrently one can generate the $\nu_{i,j:n}$ $(i<j,\ i,j = 1,2,\ldots,n)$ if the $\nu_{i,j:n-1}$ $(i<j;\ i,j = 1,2,\ldots,n-1)$ and any n-1 of the $\nu_{i,j:n}$ are available. Formulae (4.1) and (4.3) can be used for checking the computations.

CHAPTER III

RELATIONS BETWEEN QUASI-RANGES

Certain linear systematic statistics such as the range which measures dispersion have expectations which are independent of the size of the sample from which they are calculated. These statistics may be used as indices of skewness and kurtosis for samples from any continuous population, and have the advantages, for instance, when estimates from samples of different sizes are to be combined.

This chapter shall attempt to summarize all the work that has been done in this area to date. One statistic in measuring dispersions is the range, whose probability integral in samples from a normal population was tabulated by E.S. Pearson (1942) and Hartley (1942). Tippet (1925) tabulated its mean value for sample sizes 2-1000, and gave formulae for higher moments, and E.S. Pearson (1926) has calculated second, third, and fourth moments for several sample sizes. Other measures which involve the difference of observations have been proposed, such as the interquartile range, discussed by Hojo (1931) and the differences of quindeciles, suggested by K. Pearson (1920). Mosteller (1946) discussed these and other differences of symmetrically placed ranks which he calls quasi-ranges. Nair has considered the mean deviation from the median and proposes several questions as to the usefulness of this statistic, and Godwin (1949) further discussed this statistic and tabulated the mean, variances and covariances of ranks.

Kendall and Stuart (1969) has shown that the expectation of the range $\bar{\omega}_n$, in a sample of n from any continuous population of variate-values z, which has distribution function F, is

$$\omega_n = \int_{-\infty}^{\infty} [1 - \{F(x)\}^n - \{1 - F(x)\}^n]\, dx.$$

K. Pearson (1902) has shown that if $\chi_{n,p}$ represents the expectation of the difference between the (p+1)-th and the p-th value of z when the sample members are arranged in ascending order of magnitude. Then $\chi_{n,p} = \binom{n}{p} \int F^{n-p}(x)$ $(1-F(x))^p dz$, (a). Letting $\omega_{n-1} = \int\{1 - (1-F)^{n-1} - F^{n-1}\}dz$, the integrals being taken over the whole range of z. Therefore,

$$\omega_n = \int \{(1-F)^{n-1}F + F^{n-1}(1-F)\}dz + \omega_{n-1}$$

$$= (\chi_{n,1}+\chi_{n,n-1})/n + \omega_{n-1}$$

ω_{n-1} may also be expressed in terms of $\chi_{n-1,1}$, $\chi_{n-1,n-2}$ and ω_{n-2} and so on.

From K. Pearson's equality (a), Sillitto (1951) shows that

$$\chi_{n,p} = \binom{n}{p} \int F^{n-p}(1-F)^p dz$$

$$(3.2) \qquad = \int \{ \binom{n}{p} F^{n-p}(1-F)^{p-1} - \binom{n}{p} F^{n-p+1}(1-F)^{p-1} \} dz$$

$$= \frac{n}{p} \chi_{n-1,p-1} - \frac{n-p+1}{p} \chi_{n,p-1}$$

By repeatedly using the right-hand side of relation (3.2), Sillitto (1951) found that

$$(3.3) \qquad \chi_{n,p} = \frac{(\eta)_\nu}{(r)_\nu} \sum_{i=0}^{\nu} (-1)^i \binom{\nu}{i} \frac{(n-p+i)_i}{(\eta-\nu+i)_i} \chi_{n-\nu+i,p-\nu} \qquad (\nu<p-1)$$

By using a similar method of summation under the integral sign, as was used for relation (1.2.6) Sillitto (1964) gets another relation:

$$(3.3.1) \qquad \chi_{n,p} = \binom{m}{q} \sum_{r=0}^{n-m} \binom{n-m}{r} \chi_{n,q+r} / \binom{n}{q+r} \qquad (n>m)$$

The difference between the p-th and the (n-p)-th difference in a sample of n, or more generally a linear combination of such differences, may be considered as a linear measure of skewness. The more familiar measure due to K. Pearson; skewness = (mean-mode)/σ is subject to the inconvenience of determining the mode and therefore is not independent of the population. Sillitto (1951) found that it is possible to choose a linear combination which has an expectation value independent of the size of the sample, that is suitable as a measure of dispersion:

First, an expression

$$na_1(X_{n,1}-X_{n,n-1}) + na_2(X_{n,2}-X_{n,n-2}) + \ldots + na_p(X_{n,p}-X_{n,n-p})$$

$$p = 1,2,\ldots,\tfrac{1}{2}(n-1) \quad p, \text{ odd}$$

$$p = 1,2,\ldots,(\tfrac{1}{2}n-1) \quad p, \text{ even}$$

is needed whose value is independent of sample size. Then this expression must equal $3a_1(X_{3,1}-X_{3,2})$. Then for all permissible values of p, $X_{n,p} - X_{n,n-p}$ is expressed in terms of first differences (by using relation (3.3) and taking note that $X_{2,1} = \omega_2 = \frac{2}{3}\omega_3 = \frac{2}{3}(X_{3,1}-X_{3,2})$, the resulting equations can be solved for $X_{3,1} - X_{3,2}$, i.e.

$$(3.3.1) \qquad S_L = (\tfrac{1}{3}(X_{3,1}-X_{3,2})) = \sum_p [(X_{n,p}-X_{n,n-p}) \{\binom{n-2}{p-1} \frac{n-2p}{n-2}\}/\binom{n}{p}]$$

Thus the statistic S_L is a linear systematic statistic which measures skewness in samples of n from any continuous population, and its expectation $\frac{1}{3}(X_{3,1}-X_{3,2})$ is the expectation of the difference (median-mode) in samples of three from the population.

When $\nu = p-1$, relation (3.3) becomes

$$X_{n,p} = \binom{n}{p} \sum_{i=0}^{p-1} (-1)^i \binom{p-1}{i} \frac{X_{n-p+1+i,1}}{n-p+1+i}$$

By using relation (3.2) in the right-hand side of relation (3.1), other expressions for ω_n in terms of mean differences in samples of size n and smaller can be obtained and from such expressions one can also express $X_{n,p} + X_{n,n-p}$ in terms of mean ranges in samples of size n and less. Sillitto (1951) obtained such a relation:

$$-\sum_{r=0}^{p}(-1)^r\binom{p}{r}\omega_{n-p+r} = \int -\{\Sigma(-1)^r\binom{p}{r} - \Sigma(-1)^r\binom{p}{r}(1-F)^{n-p+r}$$

$$- \Sigma(-1)^r\binom{p}{r}F^{n-p+r}\}dz$$

$$(3.4) \qquad = \int - \{(1-1)^p - (1-F)^{n-p}F^p$$

$$- F^{n-p}(1-F)^p\}dz$$

$$= (p!(n-p)!/n!)(X_{n,p}+X_{n,n-p})$$

E.S. Pearson (1926) proved (3.4) for a symmetrical population, in which $X_{n,p} = X_{n,n-p}$.

If one gives p the values 1,2,...,n-1, (n-1) expressions from relation (3.4) are obtained and (n-1) independent equations relating $\omega_2, \omega_3, \ldots, \omega_n$ and $X_{n,1}, X_{n,2}, \ldots X_{n,n-1}$ arise. These equations are then solved by Sillitto (1951) to derive explicit expressions for ω_n $(2\leq m\leq n)$ in terms of mean differences in a sample of size n:

$$(3.4.1) \qquad \omega_m = \sum_{p=1}^{n-1} X_{n,p} - \frac{(n-m)!}{n!}\sum_{p=1}^{n-m}\frac{(n-p)!}{(n-m-p)!}(X_{n,p}+X_{n,n-p})$$

$$(2\leq m\leq n).$$

Relation (3.4.1) was first obtained and proved in another way by E.S. Pearson (1926). This general expression can also be directly verified from the integral expressions for ω_m and

$\chi_{n,p}$.

When m=2, Sillitto (1951), after some reduction obtains

$$(3.4.2) \qquad \omega_2 = \frac{2}{n(n-1)} \{(n-1)\chi_{n,1} + 2(n-2)\chi_{n,2} + \ldots + p(n-p)\chi_{n,p} + \ldots + (n-1)\chi_{n,n-1}\}$$

The right-hand side can be recognized as the form taken by the expectation of g, Gini's coefficient of mean difference (Kendel & Stuart, 1969), when it is expressed in terms of successive differences of the ranked variate-values.

It is sometimes necessary to estimate the mean range in a sample of m from a sample of n (>m) observations; for instance in quality control work or when it is proposed to use ω_m as an index of dispersion. Relation (3.4.1) could be used for this purpose, if the values $\chi_{n,p}$ of the p-th successive difference ($p = 1,2,\ldots,n-1$) between the ranked members of a sample of n observations are inserted on the right-hand side of this equation instead of their expectations. The quantities obtained from (3.4.1) are generalizations of Gini's coefficient of mean difference and their expectations are independent of the number of observations in the sample from which they are calculated. Sillitto (1951) compares their efficiency as estimates of ω_m in the case of a normal population, with some other methods.

To define kurtosis one needs to have some standard population as a reference. The virtue of the normal population for this purpose is not involved except when one uses measures derived from moments, and in considering continuous populations in general, the most natural one to take as a standard appears to be the rectangular, which is flat throughout its range. The smallest sample which could give an indication of the kurtosis is that of four values. In such a sample, an index of kurtosis relative to the rectangular population would be $\frac{1}{2}(X_{4,1}-2X_{4,2}+X_{4,3})$, which has expected value zero in the rectangular population. Sillitto (1951) suggests that one could take as a measure of kurtosis a linear combination of the differences between the values in the sample, which has the same expectation as this expression, independently of the size of the sample. Since $\frac{1}{2}(X_{4,1}+2X_{4,2}+X_{4,3}) = 5\omega_4 - 9\omega_2$ and the mean difference in any sample of more than four can be expressed using relation (3.4.1) in terms of ω_4 and ω_2; a linear systematic statistic suggested for measuring kiertosis is

$$K_L = 5\omega_4 - 9\omega_2 .$$

If one wishes to restrict oneself to linear statistics one would have to bring them to the same scale by dividing them by a measure of the dispersion such as the standard deviation, or ω_2.

In relation (3.4), if one takes p=1, and then p=n-1, n odd, Sillitto (1951) gets

$$\omega_n - \omega_{n-1} = (1!(n-1)!/n!)(X_{n,1}+X_{n,n-1})$$

$$= \omega_n + \binom{n-1}{1}\omega_{n-1} - \binom{n-1}{2}\omega_{n-2} + \dots + \binom{n-1}{1}\omega_2$$

$$= \frac{(n-1)!1!}{n!}(\chi_{n,n-1} + \chi_{n,1})$$

which leads to

$$(3.5) \qquad \omega_n = \frac{1}{2}\{\omega_{n-1} + \sum_{i=1}^{n-2}(-1)^{i+1}\binom{n-1}{i}\omega_{n-i}\}$$

Other relations between mean ranges can be obtained by giving other values to P.

Romanovsky (1933) developed the relation which involved mean ranges in samples taken from a normal population:

$$(3.6) \qquad \frac{1}{2}\sum_{r=0}^{P}(-1)^r\binom{P}{r}\frac{\omega_{2p+1-r}}{2p+1-r} = 0$$

Relation (3.6), as Sillitto (1951) pointed out, by means of relation (3.4) actually holds for any continuous population.

Robbins (1944) using elementary probability theory showed that the expected value of the range for n=3 is always three-halves that for n=2, i.e.

$$(3.6.1) \qquad \omega_3 = \frac{3}{2}\omega_2$$

Relation (3.6.1) is only a special case of relation (3.6).

It is often possible, in certain types of mass production, to use a large sample of articles for simple routine inspection and to find with ease the articles with more extreme values of the characteristics measured. In these cases, a great deal of labour can be saved, if the dispersion is estimated from these extreme values, which may comprise only about 5% of the total. Such an estimate of dispersion may be used in controlling variability by specifying limits for this estimate. One method of specifying the variability, which avoids the complication of subdividing the sample, is to lay down limits for the difference between the sum of the r highest and r lowest values observed in the sample of n ($1<r<\frac{1}{2}n$ or $\frac{1}{2}(n-1)$ according as n is even or odd. Nair (1950) has proposed the symbol $y_{(r)}$ for the above mentioned difference, suggested by Jones (1946) as a measure of dispersion in large samples. Sillitto (1951) uses $J_{(r)}$ to denote this statistic in a sample of n, so that

$$\begin{aligned} nJ_{(r)} &= (z_n+z_{n-1}+\ldots+z_{n-r+1}) - (z_r+z_{r-1}+\ldots+z_1) \\ &= (z_n-z_1) + (z_{n-1}-z_2)+\ldots+(z_{n-r+1}-z_r) \end{aligned}$$

and therefore

$$\begin{aligned} E(nJ_{(r)}) = \omega_n + (\omega_n-\chi_{n,n-1}-\chi_{n,1})+\ldots+\{\omega_n - (\chi_{n,n-1}+\chi_{n,1}) \\ - (\chi_{n,n-2}+\chi_{n,2}) -\ldots- (\chi_{n,n-r+1}+\chi_{n,r-1})\} \end{aligned}$$

By means of relation (3.4) the right-hand side can be expressed as a series of mean ranges and the coefficients of each of the ω_{n-s} ($0<s<r-1$) can then be summed, giving

$$(3.7) \qquad E(nJ_{(r)}) = \sum_{s=0}^{r-1} (-1)^{r-s-1} \binom{n-s-2}{r-s-1} \binom{n}{s} \omega_{n-s}$$

The mean deviation from the median in the sample of n, given that $\chi_{n,p}$ is the p-th difference between the ranked individuals, if n is odd, is:

$$m'_n = \frac{1}{n}\{\chi_{n,1}+2\chi_{n,2} + \ldots + \frac{1}{2}(n-1)\chi_{n,\frac{1}{2}(n-1)} + \frac{1}{2}(n-1)\chi_{n,\frac{1}{2}(n+1)} + \ldots + 2\chi_{n,n-2}+\chi_{n,n-1}\},$$

and if n is even:

$$m'_n = \frac{1}{n}\{\chi_{n,1}+2\chi_{n,2} + \ldots + (\frac{1}{2}n-1)\chi_{n,\frac{1}{2}n-1} + \frac{1}{2}n\chi_{n,n/2} + (\frac{1}{2}n-1)\chi_{n,\frac{1}{2}n+1} + \ldots + 2\chi_{n,n-2}+\chi_{n,n-1}\}$$

By taking expectations and using relation (3.2) Sillitto gets

$$(3.8) \qquad E(m'_{2k-1}) = E(m'_{2k}) + (\frac{1}{2(2k-1)})\chi_{2k,k}$$

and

$$(3.8.1) \qquad E(m'_{2k+1}) = E(m'_{2k}),$$

(the latter also having been proved by Godwin (1949)), and thus gets the following result, which can also be expressed in terms of mean ranges by use of relation (3.4):

$$(3.8.2) \qquad E(m'_{2k+1}) = E(m'_{2k}) = \frac{1}{2}\sum_{r=1}^{k} \frac{1}{2r-1}\chi_{2r,r}$$

Letting $F_{n,p}$ represent the c.d.f. of $Z_{n,p}$. Then (Kendal & Stuart, 1969)

(a) $$dF_{n,p} = p\binom{n}{p}F^{p-1}(1-F)^{n-p}dF$$

Integrating by parts one gets the two following alternative expressions (Sillitto, 1964)

(b) $$F_{n,p} = \sum_{r=0}^{n-p}\binom{n}{p+r}F^{p+r}(1-F)^{n-p+r}$$

(c) $$F_{n,p} = 1 - \sum_{r=1}^{p}\binom{n}{p-r}F^{p-r}(1-F)^{n-p+r}$$

Therefore by means of these expressions, certain properties of q-th members of samples of size m can be expressed in terms of order statistics of samples of size n(n>m). Taking note that $F_{m,m}=F^m$ by (b) and $(1-F_{m,m}) = \sum_{r=1}^{m}\binom{m}{m-r}F^{m-r}(1-F)^r$ by (c), one finds that by using expression (3.0), that the expectation of the difference between powers of largest-members-of-m is (Kendall & Stuart, 1969)

$$\omega_2(Z_{m,n}) = \int\{1 - (1-F_{m,m})^2 - F^2_{m,m}\}dz$$

$$= 2\int F_{m,m}(1-F_{m,m})dz$$

(3.9) $$= 2\int \sum_{r=1}^{m}\binom{m}{r}F^{2m-r}(1-F)^r dz$$

$$= 2\sum_{r=1}^{m}\binom{m}{r}\chi_{2m,2m-r}/\binom{2m}{r}$$

i.e., the expectation of the difference between pairs of largest-members-in-samples-of-m may be expressed in terms of the expectations of differences between successive members in samples of 2m and therefore by using relation (3.3.1), in terms of expectations of differences between successive members of samples of size greater than 2m.

In a similar manner

$$(3.9.1) \qquad \omega_2(Z_{m,1}) = 2 \sum_{r=1}^{m} \binom{m}{r} \chi_{2m,r} / \binom{2m}{r}$$

Using similar methods, though with more algebraic labour, Sillitto (1964) finds the expectation of the difference between pairs of medians-of-samples-of-(2t-1) to be:

$$(3.10) \qquad \omega_2(Z_{2t-1,t}) = 2[\sum_{r=0}^{t-2} \sum_{u=0}^{r} \binom{t+r}{u} \binom{3t-2-r}{2t-1-u} / \binom{4t-2}{2t-1}] \{\chi_{4t-2,t+r} + \chi_{4t-2,3t-2-r}\} + \chi_{4t-2,2t-1}$$

The expectation of the difference between pairs, i.e. the expectation of the range in samples of two is a measure of the dispersion of a variate, being the expectation of Gini's coefficient of mean difference. By using relation (3.9.1) and relation (3.4.2), Sillitto (1964) points out that it is possible to determine whether a population is such, that the distribution of the medians of samples of 2t-1 has a smaller dispersion than the population dispersion; and that the expectation of the range in samples of two observations is related to the successive differences in an ordered sample of 4t-2 by

$$\omega_{2(z)} = \frac{2}{(4t-2)(4t-3)} \sum_{s=1}^{4t-3} s(4t-2-s)X_{4t-2,s}$$

Pearson (1931) developed a general formulae for moments of the distribution of the distance from origin to the q-th rank of the rectangular distribution:

$$(3.11) \qquad \mu'_s = \frac{b(q+s-1)}{(n+s)} \mu'_{s-1}$$

where b equals the "length" of the distribution. Also, the s-th moment of the range between the q-th and q'-th ranks was developed:

$$(3.12) \qquad \mu'_s = \frac{b(q'-q-1+s)}{(n+s)} \mu'_{s-1} \qquad \text{where } \mu'_0=1$$

Pearson (1931) also developed a formulae for the s-th moment coefficient of the centre of the range, where the moment-coefficient is taken about the start of the rectangle:

$$(3.13) \qquad \mu'_s = (\tfrac{1}{2}b)^{s} \sum_{t=0}^{s} \frac{s!n!}{(n+s)(s-t)!(n+t-1)}$$

and then transferring to the mean Sillitto gets the following relation:

$$\mu_{2i} = \left(\tfrac{1}{2}b\right)^{2i} \frac{n!(2i)!}{(n+2i)} \qquad (\mu_3=0=\mu_s=\ldots=\mu_{2i+1})$$

In random samples of n from a population with density function p(x) and distribution function P(x), the probability density of the range $f_n(\omega)$ is given by

$$f_n(\omega) = n(n-1) \int_{-\infty}^{\infty} p(x)p(x+\omega)[P(x+\omega) - P(x)]^{n-2}dx.$$

For the exponential population $p(x) = e^{-x}$ $(x>0, p(x)=0, x<0)$, the preceding equation (Maguire et al., 1952) thus becomes

$$f_n(\omega) = (n-1)(1-e^{-\omega})^{n-2} e^{-\omega}$$

with the moment generating function of the range being

$$M_n(t) = \int_0^{\infty} e^{t\omega} f_n(\omega)d\omega$$

$$= \Gamma(n) \; \Gamma(t+1)/\Gamma(n+t)$$

Cox (1954) then obtains the cumulants of the range by taking logs and expanding in powers of T, and, utilizing certain properties of the digamma and trigamma functions, then gets the mean and variance of the range:

$$\text{mean of range} = \sum_{r=1}^{n-1} 1/r$$

$$\text{variance of range} = \sum_{r=1}^{n-1} 1/r^2.$$

The range is a useful tool in estimating rapidly a population standard deviation if the sample size is not too much greater than 20. Above 20 the efficiency of such an estimate, when compared with one based on sample deviation falls off rapidly. For larger samples the ratio of mean sample range to population standard deviation depends rather critically on the form of the tails of the point distribution. A large sample also increases the probability of "esoteric" observations, thus giving an unusual large value for the range. It is because of these disadvantages that lead to the idea of possible use of ω_r $(= X_{n-r}-X_{r+1})$ calling these statistics, quasi-ranges. For example since ω_1 does not depend on the values of the extreme observations, it is likely to be less affected by departures from normality or by possible presence of an occasional "esoteric" observation. Thus ω_1 should be preferable to the range beyond a certain sample size.

Quasi-ranges may also be used for the estimation of standard deviation. For example, Godwin (1949) determined the optimum linear combination of $\omega_0, \omega_1, \omega_2, \ldots,$ using all the possible quasi-ranges and for n=10 gives an efficiency of 99%. Mosteller (1946) considers certain unweighted sums of two values, in order to make the estimate of the standard deviation quicker. His investigation is restricted to large samples, where the ω_r are replaced by interquantile distances. Nair (1950) considers the sum of the first k quasi-ranges for sample sizes up to 10, while Jones (1946) investigated it in the large sample case.

Godwin (1949, 1949) includes in his papers, the first two moments of ω_r, for values of n up to 10. His method depends on a series of double quadratures and this method becomes very laborious as n increases. Cadwell (1953) developed a method of evaluating the probability density function of the i-th quasi-range in a sample from a normal population. Rubin (1956) expressed the odd moments of the normal sample range when N is odd and its even moments when N is even, as linear functions of the expectations of the extreme order statistics. Chu and Hotelling (1955) and Chu (1957) gave some uses of quasi-ranges. Harter (1959) has discussed estimates in terms of sample quasi-ranges, of the standard deviation in rectangular, exponental and normal populations and has also tabulated the expected values and variances of sample quasi-ranges for i=0(1)8 and N=(2i+2)(1)100, accurate to 5 and 6 decimal places. Leon et al. (1961) has used sample quasi-ranges in setting up confidence intervals for the population standard deviation. Govindarajulu (1963) has expressed the expected values, variances and covariances of quasi-ranges in samples from any population symmetric about zero, in terms of expected values, variances, and covariances of order statistics in the sample. Furthermore simple recurrence formulae among the expected values of sample quasi-ranges from an arbitrary population were obtained. Following up in more detail; notation-wise, letting

$$W_{i:n} = X_{n-i:n} - X_{i+1:n} \qquad (i=0,1,\ldots,(n-2)/2)$$

$$\omega_{i:n} = E(W_{i:n}) = \mu_{n-1:n} - \mu_{i+1:n} \qquad (i=0,1,\ldots,(n-2)/2)$$

$$d_{i,j:n} = E[W_{i:n}W_{j:n}] \qquad 0<i<j<(n-2)/2$$

$$d^{(2)}_{i:n} = d_{i,i:n} \qquad i=0,1,\ldots,(n-2)/2$$

$\rho_{i,j:n}$ = correlation between $X_{i:n}$ and $X_{j:n}$ $\quad 1<i<j<n$

Govindarajulu (1963) gets the following results.

$$(3.14) \qquad (n-i)\omega_{i-1:n} + i\omega_{i:n} = n\omega_{i-1:n-1} \quad i=0,1,\ldots,(n-2)/2$$

Equation (3.14) is obtained by using relation (1.2.1) and letting k=1. Changing i to (n-i) one obtains $(n-i)\mu_{n-i+1:n} + i\mu_{n-i:n} = N\mu_{n-i:n-1}$ $(i = 1,2,\ldots,n-1)$. Using the "i" and "n-i" equations one gets $(n-i)(\mu_{n-i+1:n}-\mu_{i:n}) + i(\mu_{n-i:n} - \mu_{i+1:n}) = n(\mu_{n-i:n-i}-\mu_{i:n-1})$. Using the definition of $\omega_{i:n}$, relation (3.14) follows. By dividing both sides of the recurrence formula (3.14) by n, one can then use it for working "downwards" in numerical evaluation of the expected values of the simple quasi-ranges, without serious accumulation of rounding errors.

Govindarajulu gets also the following easily proven results, for distributions symmetric about zero and $0<i<j<(n-2)/2$:

$$(3.15) \qquad d_{i,j:n} = 2[\mu_{i+1,j+1:n} - \mu_{i+1,n-j:n}]$$

$$(3.16) \qquad \mathrm{Var}(W_{i:n}) = 2\,\mathrm{Var}(X_{n-i:n})(1-\rho_{i+1,n-i:n})$$

Also letting

$$\begin{aligned} \mathrm{Cov}(W_{i:n}, W_{j:n}) &= \mathrm{Cov}(X_{n-i:n} - X_{i+1:n}, X_{n-j:n} - X_{j+1:n}) \\ &= \mathrm{Cov}(X_{n-i:n}, X_{n-j:n}) \\ &\quad - \mathrm{Cov}(X_{n-i:n}, X_{j+1:n} \\ &\quad - \mathrm{Cov}(X_{i+1:n}, X_{n-j:n}) \\ &\quad + \mathrm{Cov}(X_{i+1:n}, X_{j+1:n}) \end{aligned}$$

and because of symmetry one gets

$$\text{(3.17)} \quad \mathrm{Cov}(W_{i:n}, W_{j:n}) = 2[\mathrm{Cov}(X_{i+1:n}, X_{j+1:n}) - \mathrm{Cov}(X_{i+1:n}, X_{n-j:n})] \qquad 0<i<j<(n-2)/2$$

Using relations (3.15) and (3.16) Govindarajulu (1963) obtains

$$\begin{aligned} \rho(W_{i:h}, W_{j:n}) &= \frac{\mathrm{Cov}(x_{i+1:n}, x_{j+1:n}) - \mathrm{Cov}(x_{i+1:n}, x_{n-j:n})}{[\mathrm{Var}(X_{n-i:n})\mathrm{Var}(X_{n-j:n})(1-\rho_{i+1,n-j:n})]^{\frac{1}{2}}} \\ &\quad \times \frac{1}{[(1-\rho_{j+1,n-j:n})]^{\frac{1}{2}}} \\ &= \frac{\rho_{i+1,j+1:n} - \rho_{i+1,n-j:n}}{[(1-\rho_{i+1,n-i:n})(1-\rho_{j+1,n-j:n})]^{\frac{1}{2}}} \end{aligned}$$

The preceding three equations enable one to prepare tables of the expected values, variances and covariances of quasi-ranges in samples drawn from populations symmetric about zero, provided tables of these for the corresponding order statistics are available.

CHAPTER IV

SPECIAL TYPES OF RELATIONS

There are some interesting relations that do not fall under the headings of the previous three chapters, but are nevertheless of some interest.

The following four basic formulae are true for an arbitrary distribution, for which the corresponding integrals converge:

$$(4.1) \qquad \sum_{i=1}^{n} \mu_{i:n} = nE(x)$$

$$(4.2) \qquad \sum_{i=1}^{n} \mu_{i:n}^{(2)} = nE(x^2)$$

$$(4.3) \qquad \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mu_{i,j:n} = \frac{1}{2} n(n-1) [E(x)]^2$$

$$(4.4) \qquad \sum_{i=1}^{n} \sum_{j=1}^{n} \mathrm{Cov}(x_{i:n}, x_{j:n}) = nE[X-E(x)]^2$$

Formulae (4.1)-(4.4) can be derived by writing every term on the left side of each formula as an integral and summing underneath the integral sign. Formulae (4.4) follows by considering the variance of $(x_{1:n} + \ldots + x_{n:n})/n$.

Govindarajulu (1968) obtains some interesting relations that he subsequently uses to obtain bounds for variance of estimators of location and scale parameters based on censored samples. Letting $S^* = \Sigma F(Z_{i:n})$, where the summation is from $1+r_1$ to $n-r_2$ and where $Z_{1+r_1:n} < Z_{2+r_1:n} < \ldots < Z_{n-r_2:n}$ denotes the available portion of a random sample of size n drawn from the population having F(z) for its cumulative distribution function, then

$$E(S^*) = (n-r_1-r_2)(n-r_2+r_1+1)/2(n+1) \tag{4.5}$$

Relation (4.5) follows from the fact that $F(Z_{i:n})$, $i = 1,2,\ldots,n$, constitute order statistics in a random sample of size n drawn from the uniform distribution on (0,1) and $EF(Z_{i:n}) = i/(n+1)$. Letting

$$\text{Var } S^* = \sum_{i=1+r_1}^{n-r_2} \text{Var } F(Z_{i:n}) + 2 \sum_{i+r_1 \leq i<j \leq n-r_2} \sum \text{Cov}\{F(Z_{i:n}), F(Z_{j:n})\}.$$

Therefore

$$(n+1)^2(n+2) \text{ Var } S^* = \Sigma\, i(n-i+1) + 2 \sum_{i<j} \Sigma\, i(n-j+1)$$

$$= (n+1)\, \Sigma\, i - \Sigma\, i^2 - 2 \sum_{i<j} \Sigma\, ij + 2(n+1) \sum_{i<j} \Sigma\, i$$

$$= (n+1)\, \Sigma\, i - (\Sigma\, i)^2 + 2(n+1) \sum_{i=1+r_1}^{n-r_2-1} (n-r_2-i)i$$

$$= (n+1)(1+2n-2r_2)\, \Sigma\, i - (\Sigma\, i)^2 - 2(n+1)\, \Sigma\, i^2$$

where the summation on i is from $1+r_2$ to $n-r_2$. Substituting

$$\int_x^\infty [F(y)-F(x)]^{j-i-1}[1-F(y)]^{n-j+1}dF(y)$$

$$= [1-F(x)]^{n-i+1} \int_0^1 u^{n-j+1}(1-u)^{j-i-1}du$$

and hence

$$E\{Z_{i:n}(1-F(Z_{j:n}))\} = (n+1-j)\mu_{i:n+1}/(n+1)$$

and

$$E\{Z_{i:n}F(Z_{j:n})\} = \mu_{i:n} - (n+1-j)\ \mu_{i:n+1}/(n+1),\ i<j \tag{4.9}$$

Taking the identity

$$E[\{a_i Z_{i:n}\}\ \{\Sigma F(Z_{i:n})\}] = \sum_{i<j}\sum a_j E\{Z_{j:n}F(Z_{i:n})$$

$$+ \sum_{i<j}\sum a_i E\ Z_{i:n}F(Z_{j:n})\}$$

where the summations are from $1+r_1$ to $n-r_2$ and the a_i are some constants, and using relations (4.8) and (4.9), Govindarajulu (1968) obtains

$$(n+1)\{L.H.S.\} = \sum_{j=2+r_1}^{n-r_2} a_j\mu_{j+1:n+1}\ (\sum_{i=1+r_1}^{j} i) + (n+1)$$

$$\sum_{i=1+r_1}^{n-r_2-1} a_i\mu_{i:n}\ (\sum_{j=1+i}^{n-r_2} 1) - \sum_{i=1+r_1}^{n-r_2-1} a_i\ \mu_{i:n+1}\{\sum_{j=1+i}^{n-r_2}(n+1-j)\}$$

$$= \Sigma\ ia_i\mu_{i+1:n+1} + \frac{1}{2}\sum_j (j-r_1)(j+r_1+1)a_j\mu_{j+1:n+1}$$

$$+ (n+1)\ \Sigma(n-r_2-i)a_i\ \mu_{i:n}$$

$$- \frac{1}{2}\ \Sigma(n-i-r_2)(n-i+r_2+1)a_i\mu_{i:n+1}\ .$$

After expansion of the product under the second and fourth summations in powers of i and combining similar terms, one obtains

$$E[\{\Sigma a_i Z_{i:n}\}\{\Sigma F(Z_{i:n})\}] = \{2(n+1)\}^{-1}[\Sigma i^2(\mu_{i+1:n+1}-\mu_{i:n+1})a_i$$
$$+ \Sigma i a_i \mu_{i+1:n+1} + (2n+1)\ \Sigma i a_i \mu_{i:n+1} - 2(n+1)\ \Sigma i a_i \mu_{i:n}$$
$$- r_1(r_1+1)\ \Sigma a_i \mu_{i+1:n+1} + 2(n+1)(n-r_2)\ \Sigma a_i \mu_{i:n}$$
$$- (n-r_2)(n+r_2+1)\ \Sigma a_i \mu_{i:n+1}]$$

Lieblein (1953), and further discussed by Downton (1966) have developed explicit closed formulas for moments of order statistics in samples from the extreme-value distribution; $F(x) = \exp(-e^{-y})$, $y = \frac{x-\mu}{\beta}$, $-\infty<x<\infty$, which involves only tabulated functions. For this distribution, the first moments for the "reduced" distribution are given by

$$\mu_{r:n} = \frac{n!}{(r-1)!(n-r)!} \int_{-\infty}^{\infty} xe^{-x-re^{-x}}[1-e^{-e^{-x}}]^{n-r}\,dx$$

$$(4.2.7) \qquad = \frac{n!}{(r-1)!(n-r)!} \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} \int_{-\infty}^{\infty} xe^{-x-(r+i)e^{-x}}\,dx$$

$$= \frac{n!}{(r-1)!(n-r)!} \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} [\gamma + \log(r+i)]/(r+i)$$

where

γ is Euler's constant

$\gamma = -\Gamma'(1) = .5772156649$

Therefore the equation for the first moments reduces to

(4.2.7.1) $\mu_{n:n} = \gamma + \log n$

Similarly the second moment is given by

$$\mu^{(2)}_{r:n} = \frac{n!}{(r-1)!(n-r)!} \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} \int_{-\infty}^{\infty} x^2 e^{-x-(r+i)e^{-x}} dx$$

(4.2.8) $$= \frac{n!}{(r-1)!(n-r)!} \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} [\tfrac{1}{6}\pi^2$$

$$+ (\gamma + \log(r+i))^2]/(r+i)$$

In particular when i=n

(4.2.8.1) $\mu_{n,n:n} = \frac{1}{6}\pi^2 + (\gamma+\log n)^2$

Liblein (1955) also developed an expression for the first two moments of the order statistics in samples from the Weibull distribution:

$$H(x) = \begin{cases} 0 & x<0 \\ 1 - \exp(-x^m) & x>0 \\ & m>0 \end{cases}$$

Example,

(4.2.8.1) $$\mu_i^{(k)} = \frac{n!}{(i-1)!(n-i)!} \Gamma(1+\tfrac{k}{m}) \sum_{\mu=0}^{i-1} (-1)^{\mu} \binom{i-1}{\mu}$$

$$(n+\mu-i+1)^{-1-k/m}$$

$(m>0,\ i = 1,2,\ldots n;\ k = 0,1,2,\ldots$

Liblein (1955) also found the following expression for the product moment of the Weibull distribution:

$$\mu_{ij:n}^{(k)} = m^2 \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \sum_{r=0}^{i-1} \sum_{s=0}^{j-i-1} (-1)^{r+s} \binom{i-1}{r}$$

$$m^{-2} [(j-i+r-s)(n-j+s+1)]^{-(\frac{1}{m}+1)} \cdot$$

$$\Gamma(2(1+\frac{1}{m}) \; \beta_{\frac{j-i+r-s}{n-i+r+1}}(1+\frac{1}{m}, \; 1+\frac{1}{m})$$

$$(m>0; \; i<j; \; i,j = 1,2,\ldots,n)$$

The Weibull distribution can be used in applications to breaking strength and fatigue problems. An important application of the moments of order statistics of this distribution is in finding minimum-variance unbiased estimators by means of linear functions of these statistics.

Gupta (1960) developed the product moment between the m-th and n-th order statistics from the gamma distribution (see Chapter I (1.2.9)); first let

$$\mu_{rs:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \int_0^\infty \int_x^\infty xy \; G_t^{m-1}(x) [G_t(y) - G_t(x)]^{s-r-1}$$

$$[1 - G_t(y)]^{n-s} g_t(x) g_t(y) dxdy$$

$$= \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \sum_{\alpha\beta} (-1)^{\alpha+\beta} \binom{r-1}{\alpha} \binom{s-r-1}{\beta}$$

$$\int_0^\infty e^{-qu} [\sum_{j=0}^{t-1} \frac{u^j}{j!}]^{q-1} \frac{u^t}{\Gamma(t)} du \cdot \int_0^\infty e^{-k\nu} [\sum_{j=0}^{t-1} \frac{\nu^j}{j!}]^{k-1} \frac{\nu^t}{\Gamma(t)} d\nu,$$

where $q = \alpha+s-r-\beta$ and $k = n-s+\beta+1$ and the first summation on the right side is over the positive integers α,β $(0<\alpha<r-1,$ $0<\beta<s-r-1$. Letting $a_b(t,p)$ denote the coefficient of u^t in the expansion for $\left(\sum_{j=0}^{t-1} \frac{u_j}{j!}\right)^p$, then

$$\mu_{rs:n} = \frac{n!(\Gamma(t))^{-2}}{(r-1)!(s-r-1)!(n-s)!} \sum_{\alpha,\beta} (-1)^{\alpha+\beta} \binom{r-1}{\alpha} \binom{s-r-1}{\beta}$$

$$\sum_{t=0}^{(t-1)(k-1)} \frac{a_b(t,k-1)\Gamma(b+t+1)}{k^{b+t+1}}$$

$$\times \sum_{\alpha,\beta}^{b+t} \left(\frac{k^j}{j!}\right) \sum_{p=0}^{(t-1)(q-1)} \frac{\Gamma(j+t+p+1)\,a_p(t,q-1)}{(k+q)^{j+t+p+1}} .$$

$$= \frac{n!(\Gamma(t))^{-2}}{(r-1)!(s-r-1)!(n-s)!} \sum_{\alpha,\beta,b,j,p} (-1)^{\alpha+\beta} \binom{r-1}{\alpha} \binom{s-r-1}{\beta}$$

$$\left[\frac{a_b(t,k-1)\,a_p(t,q-1)\,\Gamma(b+t+1)}{j!\,k^{b+t-j+1}(k+q)^{j+t+p+1}} \times \Gamma(j+t+p+1)\right]$$

The covariance $(X_r X_s)$ can be obtained by subtracting from $E(X_r X_s)$ the product $E(X_r)E(X_s)$.

Recurrence formulae (1.2.7) and (1.2.8) for the first and second moments for the extreme value distribution were briefly mentioned in Chapter I. The covariance terms $\mu_{i,j:n}$, are slightly more complicated, but expansion of the integral in a similar way yields for $i<j$ (Downton, 1966)

$$(4.8) \qquad \mu_{i,j:n} = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \sum_{r=0}^{j-i-1} \sum_{s=0}^{n-j} (-1)^{r+s}$$

$$\binom{j-i-1}{r} \binom{n-j}{s} \phi(i+r,j-i-r+s)$$

where

$$\phi(p,q) = \int_{-\infty}^{\infty} \int_{-\infty}^{y} xy e^{-x-pe^{-x}} e^{-y-qe^{-y}} dxdy$$

(4.8a) $$= \{\frac{1}{2pq(p+q)}\}\{\frac{1}{3} q\pi^2 + (q-p)[\gamma + \log(p+q)]^2 + (p+q)[\gamma + \log p]^2 - 2(p+q)k(q/p)\}$$

where $k(x)$ is the dilogarith, (or Spere's function) of x defined by

$$k(x) = \int_1^{1+x} \frac{\log_e t \, dt}{t-1} .$$

In computing the coefficients $b_{ij:s}$ as defined by (2.4.1c) only special cases of the moments $\mu_{ij:n}$ appear, namely when $j = i+1$. In this case (4.8) reduces to

(4.8.1) $$\mu_{i,i+1:n} = \frac{n!}{(i+1)!(n-i-1)!} \sum_{s=0}^{n-i-1} (-1)^s \binom{n-i-1}{s} \phi(i,s+1)$$

The structure (2.4.1c) that involves the coefficients $b_{ij:s}$ may also be simplified:

For example,

$$\sum_{r=0}^{s-1} (-1)^r \mu_{j+1,j+2:j+2+r}/\{(j+2+r)!(s-1-r)!$$

(4.8.2) $$= \sum_{r=0}^{s-1} (-1)^r \int_{-\infty}^{\infty} \int_{-\infty}^{y} xy e^{-x-(j+1)e^{-x}} e^{-y-e^{-y}} (1-e^{-e^{-y}})^r dxdy$$

$$\times 1/\{j!r!(s-1-r)!\}$$

$$(4.8.2) \quad = \int_{-\infty}^{\infty} \int_{-\infty}^{y} xye^{-x-(j+1)e^{-x}} e^{-y-se^{-y}} dxdy/\{j!(s-1)!\}$$

$$= \phi(j+1,s)/\{j!(s-1)!\}$$

Similarly,

$$\sum_{r=0}^{i} (-1)^r \mu_{s+j-i,s+j-i+1:s+j-i+1+r}/\{(s+j-i+1+r)!(i-r)!\}$$

$$= \phi(s+j-i,i+1)/\{s+j-i-1)!i!\}.$$

The expression (2.4.1c), in the case of the extreme value distribution is

$$b_{ij:s} = \frac{i!j!}{(i+1-s)!(s+j-i)!s!} \{\frac{i+1-s}{s+j+1} \mu_{s+j+1,s+j+1:s+j+1}$$

$$+ \mu_{i+1:i+1} \times [\mu_{s+j-1:s+j-i} - \mu_{j+1:j+1}]$$

$$+ s(j+1)\phi(j+1,s) - (i+1)(s+j-i)\phi(s+j-i,i+1)\}$$

Using equation (1.2.7.1), (1.2.8.1) and (4.8a) the above expression reduces to,

$$b_{ij:s} = \frac{i!j!}{(i+1-s)!(s+j-i)!s!} \{\frac{1}{2}[\log(\frac{i+1}{j+1})]^2$$

$$- \frac{1}{2}[\log(\frac{i+1}{s+j-i})]^2 + k(\frac{s}{j+1}) \qquad (s \neq o,\ i \neq j)$$

that is,

$$b_{ii:0} = i!\{\mu_{i+1,i+1:i+1} - [\mu_{i+1:i+1}]^2\}/(i+1)$$

$$= i!\pi^2/6(i+1) \qquad (s=0,\ i=j)$$

CHAPTER V

SOME BOUNDS AND APPROXIMATIONS FOR MOMENTS OF ORDER STATISTICS

The previous four chapters have dealt with calculations that involved the precise or exact values of moments of order statistics. However, in many cases, only approximations are available and many techniques exist, that greatly simplify solutions to moments of order statistics. In many of these approaches bounds and error terms are also included, so that where approximations are sufficient, these approaches may provide for easier "solutions" than those that calculate the moments exactly.

Bounds on these moments are available with the help of the Schwartz inequality and some of its generalizations. The expected values of the extremes $X_{n:n}$ and $X_{1:n}$, for example, cannot be arbitrarily large, if variate X has a finite variance even if the range of X is unbounded. A bound can usually be found which in the case of the extremes, is attainable for a certain class of cdf's. Symmetrical cdf's usually lend themselves to attaining better bounds. In the case of order statistics other than extremes, bounds obtained in this manner are not attainable in general, but can be improved by the use of a generalized Schwartz inequality. Some other approaches give approximations with known error bounds for the expected values of all order statistics. These will be discussed in this chapter.

Another approach, based on the Taylor expansion in powers of 1/n, which frequently provides reasonable approximations to the means, variances, and covariances of order statistics will also be discussed. The asymptotic result would be the first term of such a series. In the case of E $X_{r:n}$, this is simply the quantile approximation; when n is finite and conditions are suitable, later terms may provide successive improvements. However, these later terms are less easier controlled, and modifications of the quantile approximation for use in finite samples are also briefly discussed.

Van Zwet (1964) has considered conditions under which approximations of expected values of order statistics by appropriate population quantiles, are over-estimates or under-estimates, thereby allowing large sample approximations to be replaced by inequalities which are valid for all sample sizes. Barlow (1965) and Barlow et al. (1963) have also done work in this one. Ali and Chan (1965) shows that if F(x) is a continuous symmetric distribution, and X(i) represents the i-th order statistics from a sample of size n, then for $i > (n+1)/2$.

$$E(X(i)) > G(i/(n+1)) \quad \text{if} \quad F \text{ is unimodal.}$$

and

$$E(X(i)) < G(i/(n+1)) \quad \text{if} \quad F \text{ is U-shaped}$$

where $x = G(u)$ is the inverse function of $F(x)=U$. These inequalities are of interest since Blom (1958) has shown that for sufficiently large n the bound $G(i/(u+1))$ approaches $E(X(i))$. See also Ali (1976) for a geometrical proof of the preceding inequalities.

Plackett (1947) has shown that for certain populations W_n/σ is arbitrarily near zero, while for no population will the ratio exceed a certain value, namely

$$(5.1) \qquad 0 < \frac{W_n}{\sigma} < \left(\frac{n}{(2n-1)!}\{(2n-2)! - [(n-1)!]^2\}\right)^{\frac{1}{2}}$$

This ratio of mean range W_n in samples of n to population standard deviation σ, is often used in control chart work (when the population is assumed normal) to estimate σ from the ranges of a set of small samples. More recently it has also received some attention in techniques of short-cut analysis of variance.

Moriguti (1951), on the other hand, derived the maximum for the mean largest value under the assumption that the distribution from which the maximum is taken is symmetrical:

$$(5.2) \qquad \mu_{n:n} < \frac{n}{\sqrt{2(2n-1}} \left(1 - \frac{[(n-1)!]^2}{(2n-2)!}\right.$$

His mean value turns out to be one-half of the value given by Plackett.

Gumbel (1954) has shown that Plackett's maximum holds for any continuous variate possessing the first two moments.

Hartely and David (1954), by using procedures of the calculus of variation on lines similar to those used by Plackett (1947) and Moriguti (1951) derive the maximum of $\mu_{r:n}$ in the "unrestrained case", i.e. the abandoning of the condition of symmetry in the parential population imposed by Moriguti (1951). Their maximum turns out to be:

$$(5.3) \qquad \mu_{n:n} < (n-1)/(2n-1)^{\frac{1}{2}}$$

If the mean and variance of the parent distribution are μ and σ^2 (rather than 0 and 1), inequality (5.3) simply becomes

$$(5.3.1) \qquad \mu_{n:n} < \mu + \frac{(n-1)\sigma}{(2n-1)^{\frac{1}{2}}}$$

and likewise

$$(5.3.2) \qquad \mu_{1:n} > \mu - \frac{(n-1)\sigma}{(2n-1)^{\frac{1}{2}}}$$

(David, 1970). Using the same procedure, Hartely and David (1954) and Suguira (1962) also show that

$$(5.3.3) \qquad |E(X(m)| < \left(\frac{\beta(2m-1,2n-2m+1)}{[\beta(m,n-m+1)]^2} - 1\right)^{\frac{1}{2}} \qquad (m=1,2,\ldots,n)$$

This upper bound can be obtained by a probability distribution only if m=n (or m=1), as the stationary solution is

$$x \propto \frac{1}{\beta(m,n-m+1)} P^{m-1}(1-P)^{n-m} - 1$$

and this expression for x is monotonic only if m=n or m=1.

Hartley and David (1954) show also that Moriguti's maximum (5.2) applies generally. The same authors give an upper bound of $E(W_n)$ for $-X<x<X$:

$$(5.4) \qquad W_n = n(1-2P_1X^2)(1-P_1)^{n-1} - P_1^{n-1}/X$$
$$+ 2X(1 - (1-P_1)^n - P_1^n)$$

where P_1 is the root of

$$\frac{X^2}{[(2n-1)(1-P_1)^{n-1}-P_1^{n-1}]^2} \{(1-P_1)^{2n-1} - P_1^{2n-1}$$
$$- (2I_{1-P_1}(n,n) - 1)/\binom{2n-2}{n-1}\} + P_1X^2 = \frac{1}{2} .$$

No equally general results are possible for lower bounds since lower bounds may be made arbitrarily close to zero by choice of a parent cdf., P(x) with sufficiently large σ. However more worthwhile lower bounds can be obtained by imposing suitable conditions on P(x) such as, for example, limiting the range of X, say a<X<b (a,b, finite). Hartely and David (1954) have investigated the preceding restriction and have found that the minimizing distribution is a two-parent distribution. With a=-c, b=c they obtained and

provided a shorttable of the lower bound:

$$E\left(\frac{W_n}{\sigma}\right) > \min\left\{\begin{array}{l} 2\left[1 - \left(\frac{1}{2}\right)^{n-1}\right] \\ \left[(1 - p^n - q^n)/(pq)\right]^{\frac{1}{2}} \end{array}\right.$$

where

$$p = c^2/(1+c^2) \quad \text{and} \quad q = 1-p.$$

Rustagi (1957) discusses the case of upper and lower bounds in the case of $-c<X<c$ further.

Moriguti (1951, 1954) considers the extreme, and range, for a symmetrical parent and derives, in both cases, the upper bound for the expectation, and lower bounds for the variance and coefficient of variation for extreme order statistics:

$$(5.6) \qquad |E(X(r)| < \left[\frac{n\binom{2n-2r}{n-r}\binom{2r-2}{r-1}}{\binom{2n-1}{n-1}}\right]^{\frac{1}{2}}$$

Ludwig (1960) developed an inequality for the case of $r<s$:

$$E(X(s)-X(r)) < \sigma\left\{\frac{2n}{\binom{2n}{n}}\left[\binom{2s-2}{s-1}\binom{2n-2s}{n-s} + \binom{2r-s}{r-1}\binom{2n-2r}{n-r} - 2\binom{r+s-2}{r-1}\binom{2n-r-s}{n-s}\right]\right\}^{\frac{1}{2}}$$

of which Plackett's inequality (5.1) is a special case. Another special case of (5.7) is

$$(5.7.1) \qquad \frac{E(X_{(r+1)}-X_{(r)})}{\sigma} < n\left\{\frac{1}{(2n-1)(2n-3)} \frac{\binom{n-1}{r-1}\binom{n-1}{r}}{\binom{2n-4}{2r-2}}\right\}^{\frac{1}{2}}$$

Furthermore,

$$(5.7.2) \qquad \frac{E(X_{(2)}-X_{(1)})}{\sigma} < n\left\{\frac{(n-1)}{(2n-3)(2n-1)}\right\}^{\frac{1}{2}} .$$

Another approach in "approximating" moments, due essentially to K. and M.V. Pearson (1931) is on inverse expansion of X in terms of F. If X(r) is the parent value such that F(X(r)) = r/n+1, one can expand x(r) about X(r) in a Taylor series $x(r) = X_r + h_r X_r' + \frac{1}{2!} h_r^2 X_r'' + \ldots$ where $h_r = F(x_{(r)}) - F(X_r) = F_r - r/(n+1)$ and

$$X_r' = \frac{dX_r}{dF} = \left.\frac{dx}{dF}\right|_{x=X_r}, \quad X_r'' = \left.\frac{d^2x}{dF^2}\right|_{x=X_r}, \text{ etc.}$$

From this series one can express powers of x in series of powers of h; and the expectation of any power of h is easily derived. Provided, then, that the series converge in a suitable manner (or, more generally, give good approximations of an asymptotic kind), one may derive approximations to as many moments as one wishes of any order-statistic. David and Johnson (1954) have pursued this subject systematically and give expansions for cumulants and product-cumulants of order-statistics up to and including the fourth-order cumulants and degree three in 1/(n+1). They chose expansions in terms of

$(n+1)^{-1}$ rather than n^{-1} because of the natural appearance of the former quantity in elementary cases. For example, for the median $x_{r+1:n}$ in samples of $n=2r+1$ observations, to order $(n+2)^{-2}$ the authors show that

$$(5.8) \qquad E(x_{(r+1)}) = X_{r+1} + \frac{1}{8(n+2)} X''_{r+1} + \frac{1}{128(n+2)^2} X^{(iv)}_{r+1}$$

$$(5.9) \qquad \mathrm{Var}\ x_{(r+1)} = \frac{1}{4(n+2)} (X'_{r+1})^2 + \frac{1}{32(n+2)} \{2X'_{r+1}X'''_{r+1} + (X''_{r+1})^2\}$$

$$\Psi_1(x_{(r+1)}) = \frac{3}{2(n+2)} \frac{X''_{r+1}}{X'_{r+1}}, \quad \Psi_2(x_{(r+1)}) = \frac{1}{n+2} \left\{\frac{X'''_{r+1}}{X'_{r+1}} + \frac{3(X''_{r+1})^2}{(X'_{r+1})^2} - 6\right\}$$

Chu (1955) shows that for a continuous symmetrical distribution with frequency function $f(x)$ which possesses an absolute maximum at its median ξ that the variance of the sample median in samples of $n=2r+1$ satisfies the inequality

$$(5.10) \qquad \mathrm{Var}\ x_{(r+1)} > [4\{f(\xi)\}^2(n+2)]^{-1}.$$

The expansions used by David and Johnson need not necessarily be in inverse powers $n+2$ (see Clark and Williams, 1958). Saw (1960) has obtained bounds for the remainder term when the expansion of $E(X_{(r)})$ is terminated after an even number of terms. However, it must be noted from a practical point of view, that the most important feature of the expansion is that convergence may be slow or even nonexistent if r/n is close to 0 or 1. A different approach based on the logistic rather than the uniform distribution has been developed by Plackett (1958). See also Chan (1967). Although a bit more complicated in its approach, Saw (1960) indicates that in the normal case for the same number of terms Plachett's series for $E(X_{(r)})$ is somewhat more accurate than that of David and Johnson.

Formulae like (5.8) and (5.9) along with Pearson's Taylor series are sometimes rather tedious to apply, especially for distributions that, unlike the normal, do not allow dx/dF to be simply expressed. From mean-value theorem considerations Blom (1958) has suggested semi-empirical "α,β-corrections" and writes

$$EX(r) = Q(\pi_r) + R,$$

where $\pi_r = (r-\alpha_r)/(n+1-\alpha_r-\beta_r)$, and R is of order $1/n$. By a suitable choice of α_r and β_r (which generally also depends on n), the remainder R may be made sufficiently small, so that $Q(\pi_r)$ may be used as an approximation to $EX_{(r)}$. This approach simplifies considerably when the present distribution is symmetric. In the normal case Blom finds α_r to be remarkably stable for $n<20$, and all r when α_r takes on values between 0.33 and 0.39. Harter (1961) shows that for larger n ($\leq$ 400), α = 0.4 is better in the range $50<n<400$.

Sugiura (1962, 1964) uses an orthogonal inverse expansion procedure to provide both bounds and approximations for the means, variances, and covariances of order statistics. He shows that if $u=F(x)$ is absolutely continuous with respect to the Lebesque measure with mean μ and variance σ^2 and letting $\Psi_0 = 1, \Psi_1, \ldots \Psi_m$ be any orthonormal system over (0,1) with

$$a_k = \int_0^1 x(u)\Psi_k(u)\,du$$

$$b_k = \frac{1}{B(r,n-r+1)} \int_0^1 u^{r-1}(1-u)^{n-r}\;\Psi_k(u)\,du$$

then

$$(5.11)\quad \left|EX(r) - \mu - \sum_{k=1}^{m} a_k b_k\right| \leq \left(\sigma^2 - \sum_{k=1}^{m} a_k^2\right)^{\frac{1}{2}} \left\{\frac{B(2r-1,2n-2r+1)}{[B(r,n-r+1)]^2} - 1 - \sum_{k=1}^{m} b_k^2\right\}^{\frac{1}{2}}$$

if the distribution is known to be symmetric, then corresponding to (5.11) one obtains

$$(5.11.1)\quad \left|EX_{(r)} - \sum_{k=0}^{m} a_{2k+1}b_{2k+1}\right| \leq \left(\sigma^2 - \sum_{k=0}^{m} a_{2k+1}^2\right)^{\frac{1}{2}} \times \left[\frac{B(2r-1,2n-2r+1) - B(n,n)}{2[B(r,n-r+1)]^2} - \sum_{k=0}^{m} b_{2k+1}^2\right]^{\frac{1}{2}}$$

Furthermore Sugiura (1962) obtains for distribution functions which are symmetric and absolutely continuous with respect to the Lebesque measure with variance 1 the result:

$$(5.12) \qquad |E(X_{(r)})| < \frac{1}{[2B(r,n-r+1)]^{\frac{1}{2}}} \{B(2r-1,2n-r+1) - B(n,n)\}^{\frac{1}{2}} .$$

Inequality (5.12) coincides with the results of Moriguti (1951), though he deals only with the case r=n.

In another paper Sugiura (1964) treats the bivariate case and finds the universal upper bounds and approximation for $E(X^i_{r:n}X^j_{s:n})$ $(i,j = 1,2)$. Using the same procedures as he used to derive (5.11) Sugiura obtains the following result:

Let $X_{i:n}$ be the i-th (smallest) order statistic in a random sample of size n with distribution function F(x) absolutely continuous with respect to the Lebesque measure having mean μ and finite variance σ^2. Let $\{\Psi_\ell(u)\}_{\ell=0,1,\ldots}$ $(\Psi_0(a)=1)$ be any complete orthonormal system over (0,1) and let for any pair r,s $(1<r<s\leq n)$

$$a_k = \int_0^1 x(u)\Psi_k(u)\,du$$

$$b_{k,\ell} = \frac{1}{B(r,s-r,n-s+1)} \iint_{0<u<v<1} u^{r-1}(v-u)^{s-r-1} (1-v)^{n-s}\Psi_k(u)\Psi_\ell(v)\,du\,dv$$

then

$$(5.13)\quad |E(X_{r:n}X_{s:n}) - \mu E(X_{r:n}+X_{s:n}) + \mu^2 - \frac{1}{2}\sum_{k,\ell}^{m} a_k a_\ell (b_{k,\ell}+b_{\ell,k})$$

$$\leq \{\sigma^4 - \sum_{k,\ell=1}^{m} a_k^2 a_\ell^2\}^{\frac{1}{2}} \{\frac{B(2r-1,2s-2r-1,2n-2s+1)}{2[B(r,s-r,n-s+1]^2}$$

$$-\frac{B(2r-1,2n-2r+1)}{2[B(r,n-r+1)]^2} - \frac{B(r+s-1,2n-r-s+1)}{B(r,n-r+1)B(s,n-s+1)}$$

$$-\frac{B(2s-1,2n-2s+1)}{2[B(s,n-s+1)]^2} + 1 - \frac{1}{4}\sum_{k,\ell}^{m}(b_{k,\ell}+b_{\ell,k})^2\}^{\frac{1}{2}}$$

where

$$B(p,q,r) = \frac{\Gamma(p)\Gamma(q)\Gamma(r)}{\Gamma(p+q+r)}$$

For any distribution absolutely continuous with respect to the Lebesque measure with mean zero and variance one and for any r,s $(1\leq r<s\leq n)$, Sugiura (1964) also gets the following inequality:

$$(5.14)\quad |E(X_{r:n}X_{s:n})| \leq \{\frac{B(2r-1,2s-2r,2n-2s+1)}{2[B(r,s-r,n-s+1)]^2}$$

$$-\frac{B(2r-1,2n-2r+1)}{2[B(r,n-r+1)]^2}$$

$$-\frac{B(r+s-1,2n-r-s+1}{B(r,n-r+1)B(s,n-s+1)}$$

$$-\frac{B(2s-1,2n-2s+1)}{2[B(s,n-s+1)]^2} + 1\}^{\frac{1}{2}}$$

In particular, when $r=n-1$ and $s=n$ one gets

$$(5.14.1)\quad E(X_{n-1:n}X_{n:n}) < \frac{n-2}{2}\left(\frac{n-1}{2n-1}\right)^{\frac{1}{2}}$$

If $X_{i:n}$ is the i-th (smallest) order statistic in a random sample of size n, with symmetric distribution $F(x)$, absolutely continuous with respect to the Lebesque measure with finite variance σ^2, and we let $\{\Psi_\ell(u)\}_{\ell=0,1,\ldots}$ $(\Psi_0(u)=1)$ be any complete orthonormal system over $(0,1)$ satisfying $\Psi_\ell(u) = (-1)^\ell\Psi_\ell(1-u)$ pr $\ell = 1,2,\ldots$ while putting a_k and $b_{k,\ell}$ as in "(5.13)", Sugiura gets for any r,s $(1<r<s<n)$ the inequality

$$(5.15)\qquad \Big|E(X_{r:n}X_{s:n}) - \frac{1}{2}\sum_{k,\ell}^{m} a_{2k+1}a_{2\ell+1}(b_{2k+1,2\ell+1} + b_{2\ell+1,2k+1})\Big|$$

$$< \{\sigma^4 - \sum_{k,\ell=0}^{m} a_{2k+1}^2 a_{2\ell+1}^2\}^{\frac{1}{2}}\{A_1 - B_1 - \frac{1}{4}\sum_{k,\ell}^{m}(b_{2k+1,2\ell+1} + b_{2\ell+1,2k+1})^2\}^{\frac{1}{2}}$$

where

$$A_1 = \frac{B(2r-1,2s-2r-1,2n-2s+1) + B(n+r-s,n+r-s,2s-2r-1)}{8[B(r,s-r,n-s+1)]^2}$$

$$B_1 = [I(2r-1,n-s+1,n-s+1,s-r,s-r) + 2I(n+r-s,r,n-s+1,s-r,s-r)$$

$$+ I(2n-2s+1,r,s-r,s-r)]/8[B(r,s-r,n-s+1)]^2$$

where

$$I(a,b,c,d,e) = \sum_{i=0}^{b-1}\sum_{j=0}^{i-1}\binom{b-1}{i}\binom{c-1}{j}$$

$$\times\; B(d+i,e+j,a+b+c-i-j-2)\,2^{-(a+b+c-i-j-2)}$$

It follows quite readily that for any symmetric distribution absolutely continuous with respect to the Lebesque measure with mean zero and variance one that

$$(5.16)\quad E(X_{r:n}X_{s:n}) < (2\sqrt{2}\,B(r,s-r,n-s+1))^{-1} \{B(2r-1,2s-2r-1,2n-2s+1) + B(n+r-s,n+r-s,2s-2r-1) - I(2r-1,n-s+1,n-s+1,s-r,s-r) - 2I(n+r-s,r,n-s+1,s-r,s-r) - I(2n-2s+1,r,r,s-r,s-r)\}^{\frac{1}{2}}$$

where I(a,b,c,d,e) is defined as above. Therefore, for example, the upper bound corresponding to (5.14) for

$$|E(X_{1:n}X_{n:n})| < n\left\{\frac{n-1}{4(2n-3)} - \frac{(n-1)^2}{n^2(2n-1)} - \frac{1}{(2n-1)\binom{2n-2}{n-1}}\right\}^{\frac{1}{2}}$$

and when the distribution is required to be symmetric with mean zero and variance one, the inequality becomes

$$|E(X_{1:n}X_{n:n})| < \frac{n}{2}\left\{\frac{n-1}{2(2n-3)} - \frac{1}{\binom{2n-2}{n-1}}\right\}^{\frac{1}{2}}$$

If $E(X^4) < \infty$, then under the same assumptions (and notation) preceding inequality (5.12) Sugiura gets

$$(5.17)\qquad |E(X_{r:n}^2 X_{s:n}^2) - \sigma^2 E(X_{r:n}^2 + X_{s:n}^2) + \sigma^2$$

$$- \frac{1}{2} \sum_{k,\ell=1}^{\infty} a'_{2k} a'_{2\ell} (b_{2k,2\ell} + b_{2\ell,2k})| < \{(E(X^4) - \sigma^2))^2$$

$$- \sum_{k,\ell=1}^{m} a'^2_{2k} a'^2_{2\ell} \{A_2 + B_2 - C_2 + 1 - \frac{1}{4} \sum_{k,\ell=1}^{m} (b_{2k,2\ell} + b_{2\ell,2k})^2\}^{\frac{1}{2}}$$

where

$$a'_k = \int_0^1 [x(a)]^2 \Psi_k(a)\,du$$

$$A_2 = A_1$$

$$B_2 = B_1$$

$$C_2 = \frac{B(2r-1,2n-2r+1) + B(n,n)}{4[B(r,n-r+1)]^2}$$

$$+ \frac{B(2s-1,2n-2s+1) + B(n,n)}{4[B(s,n-s+1)]^2}$$

$$+ \frac{B(r+s-1,2n-r-s+1) + B(n+r-s,n-r+s)}{2B(r,n-r+1)\,B(s,n-s+1)}$$

Also under the same assumptions (and notation) as holding for (5.17) the following two inequalities hold for any r,s $(1<r<s<n)$

$$(5.18a)\qquad |E(X_{r:n}^2 X_{s:n}) - \sigma^2 E(X_{s:n}) - \sum_{k,\ell=0}^{m} a'_{2k+2} a_{2\ell+1} b_{2k+2,2\ell+1}|$$

$$< \{\sigma^2 E(X^4) - \sigma^2 - \sum_{k,\ell=0}^{m} a'^2_{2k+2} a^2_{2\ell+1}\}^{\frac{1}{2}}$$

$$\{A_3 - B_3 - \sum_{k,\ell=0}^{m} b^2_{2k+2,2\ell+1}\}^{\frac{1}{2}}$$

$$(5.15b)\qquad \left|E(X_{r:n}X_{s:n}^2) - \sigma^2 E(X_{r:n}) - \sum_{k,\ell=0}^{m} a_{2k+1}a'_{2\ell+2}b_{2k+1,2\ell+2}\right|$$

$$< \{\sigma^2 E(X^4) - \sigma^2 - \sum_{k,\ell=0}^{m} a_{2k+1}^2 a'^2_{2\ell+2}\}^{\frac{1}{2}}$$

$$\{A_4 - B_4 - \sum_{k,\ell=0}^{m} b_{2k+1,2\ell+2}^2\}^{\frac{1}{2}}$$

where

$$A_3 = \frac{B(2r-1,2s-2r-1,2n-2s+1)}{4[B(r,s-r,n-s+1)]^2} - \frac{B(2s-1,2n-2s+1)-B(n,n)}{2[B(s,n-s+1)]^2}$$

$$B_3 = \frac{I(2r-1,n-s+1,n-s+1,s-r,s-r) - I(2n-2s+1,r,r,s-r,s-r)}{4[B(r,s-r,n-s+1)]^2}$$

$$A_4 = \frac{B(2r-1,2s-2r-1,2n-2s+1)}{4[B(r,s-r,n-s+1)]^2} - \frac{B(2r-1,2n-2r+1) - B(n,n)}{2[B(r,n-r+1)]^2}$$

$$B_4 = -B_3 .$$

Joghi (1969) discusses further the problem of obtaining approximations and bounds for the moments of order statistics from a continuous parent distribution, and shows that these bounds and approximations depend on the distribution function only through certain moments of order statistics in small samples. For example, he shows that if $\Psi_0=1,\Psi_1,\Psi_2$ is any orthonormal system in $[0,1]$ and if $E(X^2_{2p+1:2p+2q+1}) < \infty$ for some integral $p,q>0$, then for $r = 1,2,\ldots,n$

$$(5.19)\qquad \left|\frac{B(p+r,q+n-r+1)}{B(r,n-r+1)} E(X_{p+r:p+q+n}) - \sum_{k=0}^{t} a_k b_k\right|$$

$$\leq [B(2p+1,2q+1)E(X^2_{2p+1:2p+2q+1}) - \sum_{k=0}^{t} a_k^2]^{\frac{1}{2}}$$

$$\times [\frac{B(2r-1,2n-2r+1)}{B(r,n-r+1)^2} - \sum_{k=0}^{t} b_k^2]^{\frac{1}{2}}$$

where

$$a_k = \int_0^1 f(u)\Psi_k(u)du \quad ; \quad f(u) = x(u)u^p(1-u)^p$$

$$b_k = \int_0^1 g(u)\Psi_k(u)du \quad ; \quad g(u) = \frac{u^{r-1}(1-u)^{n-r}}{B(r,n-r+1)}$$

Another result that Joshi (1969) proves is the following: if a distribution X is continuous and symmetric about x=0 and letting $\Psi_0 = 1, \Psi_1, \Psi_2, \ldots$ be any complete orthonormal system in [0,1] satisfying $\Psi_k(u) = (-1)^k \Psi_k(1-u)$, k = 1,2,... with the further condition that $E(X^2_{2m+1:4m+1}) < \infty$ for some integral $m \geq 0$, then for r = 1,2,...,n;

$$(5.20) \qquad \left|\frac{B(m+r,m+n-r+1)}{B(r,n-r+1)} E(X_{m+r:2m+n}) - \sum_{k=0}^{t} a_{2k+1}b_{2k+1}\right|$$

$$\leq [B(2m+1,2m+1)E(X^2_{2m+1:4m+1}) - \sum_{k=0}^{t} a^2_{2k+1}]^{\frac{1}{2}}$$

$$\times [\frac{B(2r-1,2n-2r+1)-B(n,n)}{2[B(r,n-r+1)]^2} - \sum_{k=0}^{t} b^2_{2k+1}]^{\frac{1}{2}}$$

where a_k and b_k are the same as in (5.19), with p=q=m. If further, $E(X^2_{2m+1:4m+1}) < \infty$, $m \geq 0$, then for r = 1,2,...,n;

$$(5.21)\qquad \left|\frac{B(m+r,m+n-r+1)}{B(r,n-r+1)} E(X^2_{m+r:2m+n}) - \sum_{k=0}^{t} a'_{2k} b_{2k}\right|$$

$$< [B(2m+1,2m+1)E(X^4_{2m+1:4m+1}) - \sum_{k=0}^{t} a'^2_{2k}]^{\frac{1}{2}}$$

$$\times \left[\frac{B(2r-1,2n-2r+1)+B(n,n)}{2[B(r,n-r+1)]^2} - 1 - \sum_{k=1}^{t} b^2_{2k}\right]^{\frac{1}{2}}$$

where $a'_k = \int_0^1 x^2(u)\cdot u^m(1-u)^m \Psi_k(u)\,du$.

Mathai (1975, 1976) obtained bounds for the product moments of an arbitrary finite number of ordered random variables, in both asymmetric and symmetric cases, with the help of a representation of an arbitrary function in terms of a complete orthonormal system in a pre-Hilbert space of square integrable functions defined in a k-dimensional unit cube. He shows, for example that if $X_{i:n}$ is the i-th smallest order statistic in a random sample of size n with distribution function F(x) absolutely continuous with respect to the Lebesgue measure having mean μ and finite variance σ^2 and letting $\{\Psi_\ell(u)\}_{,\ell=0,1,\ldots}$ $(\Psi_0(u)=1)$ be any complete orthonormal system in $L^2(0,1)$, and for $1\le r_1<r_2<\ldots<r_k\le n$. Letting

$$a_\lambda = \int_0^1 x(u)\Psi_\lambda(u)\,du$$

$$b_{\lambda_1,\ldots k} = B^{-1} \int_{0<u_1<\ldots<u_k<1} \cdots \int u_1^{r_1-1}(u_2-u_1)^{r_2-r_1-1}$$

$$\times (u_3-u_2)^{r_3-r_2-1}\ldots(1-u_k)^{n-r_k}\Psi_{\lambda_1}(u_1)\ldots\Psi_{\lambda_k}(u_k)\,du_1\ldots du_k$$

one gets the following inequality:

$$|E(X_{r_1:n}X_{r_2:n}\cdots X_{r_k:n}) + (-1)^k\mu^k + (-1)^{k-1}\mu^{k-1}\Sigma_{i_1=1}^{k}E(X_{r_{i_1}:n})$$

$$+ (-1)^{k-2}\mu^{k-2}\Sigma_{i_1=1}^{k}\Sigma_{i_2=1}^{k}E(X_{r_{i_1}:n}X_{r_{i_2}:n}) + \ldots$$

$$- \mu\,\Sigma_{i_1=1}^{k}\cdots\Sigma_{i_{k-1}=1}^{k}E(X_{r_{i_1}:n}X_{r_{i_2}:n}\cdots X_{r_{i_{k-1}}:n})$$

(5.22) $\quad i_1<i_2<\ldots<i_{k-1}$

$$- \Sigma_{\lambda_1=1}^{t}\cdots\Sigma_{\lambda_k=1}^{t}a_{\lambda_1}\cdots a_{\lambda_k}[\Sigma_{(i_1,\ldots,i_k)}b_{\lambda_{i_1},\ldots,\lambda_{i_k}}/k!]|$$

$$< \{\sigma^{2k}-\Sigma_{\lambda_1=1}^{t}\cdots\Sigma_{\lambda_k=1}^{t}a_{\lambda_1}^2\cdots a_{\lambda_k}^2\}^{\frac{1}{2}}\{B(2r_1-1,2r_2-2r_1-1,$$

$$\ldots,2r_k-2r_{k-1}-1,2n-2r_k+1)/k!B^2 + (-1)^k + [[(-1)^{k-1}/k]$$

$$\times\ \Sigma_{i_1=1}^{k}\Sigma_{j_1=1}^{k}B(r_{i_1}+r_{j_1}-1,2n-r_{i_1}+1)/B(r_{i_1},n-r_{i_1}+1)$$

$$B(r_{j_1},n-r_{j_1}+1)] + [(-1)^{k-2}/k(k-1)]\Sigma_{i_1=1}^{k}\Sigma_{i_2=1}^{k}\Sigma_{j_1=1}^{k}\Sigma_{j_2=1}^{k}[B(r_{i_1}$$

$$i_1<i_2,\ j_1<j_2$$

$$+ r_{j_1}-1,r_{i_2}-r_{i_1}+r_{j_2}-r_{j_1}-1,\ 2n-r_{i_2}-r_{j_2}+1)/B(r_{i_1},r_{i_2}-r_{i_1},$$

$$n-r_{i_2}+1)B(r_{j_1},r_{j_2}-r_{j_1},n+r_{j_2}+1) + \ldots - [1/k!]$$

$$\Sigma_{i_1=1}^{k}\cdots\Sigma_{i_{k-1}=1}^{k}\Sigma_{j_1=1}^{k}\Sigma_{j_{k-1}=1}^{k}$$

$$i_1<\ldots<i_{k-1},\ j_1<\ldots<j_{k-1}$$

$$[B(r_{i_1}+r_{j_1}-1, r_{i_2}-r_{i_1}+r_{j_2}-r_{j_1}-1, \ldots, 2n-r_{i_{k-1}}-r_{j_{k-1}}+1)/$$

$$B(r_{i_1}, r_{i_2}-r_{i_1}, \ldots, r_{i_{k-1}}-r_{i_{k-2}}, n-r_{i_{k-1}}+1)]$$

$$B(r_{j_1}, r_{j_2}, \ldots, n-r_{j_{k-1}}+1)$$

$$- \Sigma_{\lambda_1=1}^{t} \cdots \Sigma_{\lambda_k=1}^{t} [\Sigma_{(i_1,\ldots,i_k)} b_{\lambda_{i_1},\ldots,\lambda_{i_k}}/k!]^2\}^{\frac{1}{2}}$$

where $\Sigma_{(i_j \ldots i_k)}$ denotes the sum of all permutations of the integers $1,2,\ldots,k$, and $B(r_1, r_2-r_1, \ldots, r_k-r_{k-1}, n-r_k+1) = \Gamma(r_1)\Gamma(r_2-r_1)\ldots\Gamma(n-r_n+1)/\Gamma(n+1)$. When $k=1$, and $r_1=r$ the preceding inequality reduces to inequality (5.11) and when $k=2$, and $r_1=r$ one gets inequality (5.13) as special cases.

Furthermore, letting $X_{i:n}$ be the i-th smallest order statistic ina random sample of size n from a population with symmetric distribution $F(x)$ absolutely continuous with respect to the Lebesque measure with finite variance σ^2 and letting $\{\Psi_\ell(u)\}_{\ell=0,1,\ldots}$, $\Psi_0(u)=1$, be any complete orthonormal system in $L^2(0,1)$ satisfying the condition $\Psi_\ell(u) = (-1)^\ell \Psi_\ell(1-u)$ for $\ell = 1,2,\ldots$ with a_λ and $b_{\lambda_1,\ldots,\lambda_k}$ defined as in the preceding result (5.22), then for $1<r_1<r_2<\ldots<r_k<n$, Mathai (1976) gets the following result:

$$|E(X_{r_1:n}\ldots X_{r_k:n}) - \Sigma_{\lambda_1=0}^{t}\ldots\Sigma_{\lambda_k=0}^{t} a_{2\lambda_1+1}\ldots a_{2\lambda_k+1} \tag{5.23}$$

$$\times [_{(i_1,\ldots,i_k)} b_{2\lambda_{i_1}+1,\ldots,2\lambda_{i_k}+1/k!}] | < \{\sigma^{2k}$$

$$- \Sigma^{t}_{\lambda_1=0} \cdots \Sigma^{t}_{\lambda_k=0} a^2_{2\lambda_1+1} \cdots a^2_{2\lambda_k+1}\}^{\frac{1}{2}} \{\Sigma^{\infty}_{\lambda_1=0} \cdots \Sigma^{\infty}_{\lambda_k=0} b^{*2}_{2\lambda_1+1,\ldots,2\lambda_k+1}$$

$$- \Sigma^{t}_{\lambda_1=0} \cdots \Sigma^{t}_{\lambda_k=0} b^{*2}_{2\lambda_1+1,\ldots,2\lambda_k+1}\}^{\frac{1}{2}},$$

where the closed form expression:

$$\Sigma^{\infty}_{\lambda_1=0} \cdots \Sigma^{\infty}_{\lambda_k=0} b^{*2}_{2\lambda_1+1,\ldots,2\lambda_k+1} = \int_{R} \cdots \int h^2(u_1,\ldots,u_k)\,du_1 \ldots du_k$$

with h being defined as:

$$h(u_1,\ldots,u_k) = 2^{-k}\{\xi(u_1,\ldots,u_k) - \xi(1-u_1,u_2,\ldots,u_k) - \cdots$$
$$\cdots - \xi(u_1,\ldots,1-u_k) + \xi(1-u_1,1-u_2,u_3,\ldots,u_k)$$
$$+\cdots+ \xi(u_1,\ldots,1-u_{k-1},1-u_k) - \cdots + (-1)^k \xi(1-u_1,\ldots,1-u_k),$$

and ξ is defined as

$$\xi(u_1,\ldots,u_k) = [2(k!)B]^{-1}[u_{i_1}^{r_1-1}(u_{i_2}-u_{i_1})^{r_2-r_1-1}$$
$$\cdots (1-u_{i_k})^{n-r_k} + (-1)^k (1-u_{i_k})^{r_1-1}(u_{i_k}-u_{i_{k-1}})^{r_2-r_1-1}$$
$$\cdots u_{i_1}^{n-r_k}],$$

for $0<u_{i_1}<\ldots< u_{i_k}<1$ and 0 elsewhere.

Inequalities (5.22) and (5.23) can easily be generalized for the case of product moments of higher orders. See Mathai (1975, 1976) for further details and explanations on evaluating the expressions contained in (5.22) and (5.23).

INDEX OF SYMBOLS AND NOTATIONS

$X_1, X_2, \ldots X_n$	unordered variates
$x_1, x_2, \ldots, x_n$	unordered observations
$X_{(1)} < X_{(2)} < \ldots < X_{(n)}$	ordered variates
	order statistics
$x_{(1)} < x_{(2)} < \ldots < x_{(n)}$	ordered observations
$X_{1:n} < X_{2:n} < \ldots < X_{n:n}$	ordered variates (extensive form)
$P(x) = \Pr\{X<x\}$	cumulative probability function of X (c.d.f.)
$p(x)$	- probability density function for a continuous variate (p.d.f.) - probability function (p.f.) for a discrete variable
$F_r(X)$, $F_{r:n}(X)$	c.d.f. of X(r), $X_{r:n}$ ($r = 1,2,\ldots,n$)
$f_r(x)$, $f_{r:n}(x)$	p.d.f. or p.f. of X(r), $X_{r:n}$
$F_{rs}(x,y) = P_v\{X(r)<x, X(s)<y\}$	joint c.d.f. of X(r) and X(s), ($1<r<s<n$)
$f_{rs}(x,y)$	joint p.d.f. or p.f. of X(r) and X(s)
$W_1, W_n = X_{(n)} - X_{(1)}$	(sample) range
$W(i) = X(n_{(1+i)}) - X_{(i)}$	i-th quasi range (W(1);W)
$W_{rs} = X_{(s)} - X_{(r)}$	
$\mu = EX$, $\sigma^2 = \mathrm{Var}\ X$	mean, variance of X
$\mu_x = EX$, $\mu_y = EY$	means of X, Y (bivariate case)
$\sigma_x^2 = \mathrm{Var}\ X$, $\sigma_y^2 = \mathrm{Var}\ Y$	Variances of X,Y

$\sigma_{xy} = \mathrm{Cov}(X,Y)$, $\rho = \sigma_{xy}/\sigma_x\sigma_y$	covariance, correlation coefficient of X,Y
$\mu_{r:n} = EX_{r:n}$	mean of $X_{r:n}$
$\mu_{r:n}^{(k)} = E(X_{r:n}^k)$	k-th row moment of $X_{r:n}$
$\mu_{rs:n} = E(X_{r:n}X_{s:n})$	joint moments
$\sigma_{r:n}^2 = \mathrm{Var}\ X_{r:n}$	Variance of the r-th order statistic
$\sigma_{rs:n} = \mathrm{Cov}(X_{r:n}, X_{s:n})$ $= E(X_{r:n}-\mu_{r:n})(X_{s:n}-\mu_{s:n})$	
$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$, $(a>0,\ b>0)$	beta function
$I_p(a,b) = \int_0^p t^{a-1}(1-t)^{b-1}dt/B(a,b)$	incomplete beta function
$\beta(a,b)$	beta variate X having c.d.f. $\Pr\{X\ x\}=I_x(a,b)$
χ_ν^2	chi-square variate with ν d.f.
$\phi(x) = (2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}$ $(-\infty<x<\infty)$	unit normal p.d.f.
$\Phi(x) = \int_{-\infty}^{x} \phi(t)\,dt$	unit normal c.d.f.
$N(\mu,\sigma^2)$	normal variate, mean μ, variance σ^2
$n^{(k)} = n(n-1)\ldots(n-k+1)$, $(k = 1,2,\ldots,n)$	
d.f.	degrees of freedom
$\nu_{r:n}^{(k)} = E(V_{r:n}^k)$	k-th row moment of $V_{r:n}$; $f(v) = \begin{cases} (2/\pi)^{\frac{1}{2}}e^{-\frac{1}{2}v^2} & v>0 \\ 0 & v<0 \end{cases}$ (χ^2 with 1 d.f.)
$\eta_{r:n}^{(k)} = E(Z_{r:n}^k)$	$Z \sim N(0,1)$, standard normal

$$X_{n,r} = E[X_{r+1:n} - X_{r:n}] = \binom{n}{r} \int_{-\infty}^{\infty} F^{n-r}(1-F)^r dx, \quad \text{(K. Pearson, 1902)}$$

$$\mu_{r:n} = \int_{-\infty}^{\infty} x f_r(x)\,dx = n\binom{n-1}{r-1} \int_{-\infty}^{\infty} x[P(x)]^{r-1}[1-P(x)]^{n-r} dP(x)$$

$$\mu_{rs:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \int_{-\infty}^{\infty} \int_{-\infty}^{y} xy\,[P(x)]^{r-1}[P(y)P(x)]^{s-r-1}$$

$$[1-F(y)]^{n-s} dP(x)\,dP(y)$$

$$E[W_{0:n}] = \mu_{n:n} - \mu_{1:n} = \omega_{0:n} = \int_{-\infty}^{\infty} 1 - F^n(x) - (1-F(x))^n dx$$

$$nJ(r) = \sum_{i=n-r+1}^{n} Z_i - \sum_{i=1}^{r} Z_i \quad (1<r<\tfrac{1}{2}n,\ n \text{ even};\ 1<r<\tfrac{1}{2}(n-1),\ n \text{ odd})$$
$(Z_i$ ordered $Z_{i+1}>Z_i)$

$\omega_2(Z_{mm})$ = expectation of the difference between pairs of largest members-in-samples-of-m.

$$W_{i:N} = X_{N-i:N} - X_{i+1:n} \qquad (i = 0,1,\ldots,[n-2]/2)$$

$$\omega_{i:N} = E[W_{i:N}, W_{j:N}] \qquad (0<i<j<(N-2)/2)$$

$$\omega^{(2)}_{i:N} = \omega_{i,i:N} \qquad (i = 0,1,\ldots,(N-2)/2)$$

$M_{r:N(k)}$ = expected value of the r-th largest of n medians-of-samples-of-k.

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