On Admissible Square Roots of Non-negative $C^{2,2\alpha}$ Functions

Huangchen Zhou

A thesis submitted to McGill University
in partial fulfillment of the requirements of the degree of
Master of Science.

Department of Mathematics and Statistics
McGill University, Montreal

March 2023

Acknowledgements

I am very thankful to my supervisor, Professor Pengfei Guan for his guidance and support. The thesis will never be finished without the valuable discussions with him. I would like to thank the Department of Mathematics and Statistics for the financial support. I am also thankful to my friend, Yanghao Zhou for helping me translate the abstract into French. Finally, I want to thank my parents for their infinite love. This thesis is dedicated to them.

Abstract

Motivated by an isometric embedding problem in the graph setting, we discuss the $C^{1,\alpha}$ regularity of the admissible square root of a non-negative $C^{2,2\alpha}$ function.

Résumé

Motivé par un problème de prolongement isométrique dans le cadre des graphes, on discute de la régularité $C^{1,\alpha}$ de la racine carrée admissible d'une fonction positive de classe $C^{2,2\alpha}$.

Contents

1 Introduction

The thesis concerns a fundamental problem regarding non-negative functions: decomposition of non-negative functions as sum of squares and the regularity of square root of $C^{2,2\alpha}$ non-negative functions.

Sum of squares problem can be traced back to Hilbert's seventeenth problem: Given a non-negative real-valued polynomial, can it be represented as a sum of squares of rational functions? Counterexample exists. The Motzkin polynomial, $m(X,Y) = X^4Y^2 + X^2Y^4 - 3X^2Y^2 + 1$ is a non-negative polynomial, but it cannot be written as sum of squares of elements in $\mathbb{R}[X,Y]$.

In 1978, Fefferman and Phong [4] stated that any non-negative C^∞ (in fact, $C^{3,1}$) function in $\mathbb{R}^n$ can be written as a sum of squares of $C^{1,1}$ functions. A detailed proof was given in [6] which was communicated by Fefferman. Guan used this decomposition to obtain C^2 a priori estimates for degenerate Monge-Ampère equations.

In the same spirit, one may also ask the question of optimal regularity of square root of non-negative functions. Nirenberg-Trèves' [14] gradient estimate for non-negative $C^{1,1}(\mathbb{R}^n)$ functions implies square roots of these functions are Lipschitz. This estimate plays important roles in analysis of linear and nonlinear PDEs (e.g., [8], [1]). For functions of one variable, Glaeser [5] proved that if $0 \le f \in C^2(\mathbb{R})$ is 2-flat on its zeroes (i.e., $f(x) = 0$ implies $f''(x) = 0$), then $f^{1/2} \in C^1(\mathbb{R})$. Mandai [13] proved that for any $0 \le f \in C^2(\mathbb{R})$, f always has an admissible square root $g \in C^1(\mathbb{R})$. In [3], Bony, Broglia, Colombini and Pernazza considered higher regularity. They found a necessary and sufficient condition for a non-negative function $f \in C^4(\mathbb{R})$ to have an admissible square root in $C^2(\mathbb{R})$, which is only related to the non-zero local minimum points of f.

The main focus of this thesis is the optimal regularity of square roots of $C^{2,2\alpha}(\mathbb{R})$ non-negative functions. We establish a necessary and sufficient condition for the existence of $C^{1,\alpha}(\mathbb{R})$ admissible square root g [9]. Below is the statement of the main theorem.

Theorem 1.1. *Let $0 \le f \in C^{2,2\alpha}(\mathbb{R})$ with $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \le 1$. $0 < \alpha \le 1$. Define the set*

$$\mathcal{A} = \{x_0 \in \mathbb{R} : f(x_0) > 0, f'(x_0) = 0, f''(x_0) > 0\}.$$

Then $f = g^2$ for some $g \in C^{1,\alpha}(\mathbb{R})$ if and only if there is a constant $M > 0$ such that

$$f''(x_0) \le M \cdot (f(x_0))^{\frac{\alpha}{1+\alpha}} \quad \forall x_0 \in \mathcal{A}. \tag{1.1}$$

Moreover, if (1.1) is satisfied, then $\|g\|_{C^{1,\alpha}(\mathbb{R})} \le C$ for some universal $C > 0$, depending only on α and M.

Remark 1.2. *We adopt the following notation. If $1/2 < \alpha \leq 1$, $C^{2,2\alpha}(\mathbb{R})$ means $C^{3,2\alpha-1}(\mathbb{R})$.*

The main theorem is motivated by the isometric embedding problem. Guan and Li [7] showed that if g is a C^4 Riemannian metric on S^2 with Gauss curvature $K \geq 0$, then there exists a $C^{1,1}$ isometric embedding $X : (S^2, g) \to (\mathbb{R}^3, g_{Eucl})$. A natural question is, can the embedding X be improved to $C^{2,1}$? Jiang [12] gave positive answer in the graph setting, under the assumption X takes the form $X(x,y) = (x, y, u(x,y))$ in local coordinates. [12] relies on a square root regularity for square of monotone functions. It is a special case of Theorem 1.1 where $\alpha = 1$ and $\mathcal{A} = \varnothing$, which can be stated as follows.

Corollary 1.3. *Let $I = [-1/2, 1/2]$. Assume $0 \leq f \in C^{3,1}(I)$ with $\|f\|_{C^{3,1}(I)} \leq 1$. The zero set of f in I is a closed interval $N := [x_0', x_0]$ (possibly $x_0' = x_0$). f is non-increasing in $[-1/2, x_0')$ and f is non-decreasing in $(x_0, 1/2]$. Then $\exists g \in C^{1,1}(I)$ such that $f = g^2$ in I, g is non-decreasing in I and $\|g\|_{C^{1,1}(I)} \leq C$ for some universal constant $C > 0$.*

The thesis is organized as follows. In section 2, 3, 4, we prove the main theorem 1.1, which is developed in collaboration with Guan [9]. In section 5, we prove a sum of squares theorem, which is a weaker version of Bony [2]. Section 6 is the conclusion. Notions of Hölder continuous functions and basic calculus theorems are included in the appendix.

2 A Calderón-Zygmund decomposition

Our approach follows [6], using a Calderón-Zygmund decomposition, a powerful tool in harmonic analysis. See for example Stein [15].

Lemma 2.1. *Let $0 \leq f \in C^{2,2\alpha}(\mathbb{R})$ with $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \leq 1$.*

If $1/2 < \alpha \leq 1$, then there is a countable collection of cubes $\{Q_\nu\}_{\nu \geq 1}$ taking the form of $(a, b]$, whose interiors are disjoint, such that

(1) *$\mathbb{R} = \mathcal{F} \cup (\cup_\nu Q_\nu)$ and $\mathcal{F} \cap (\cup_\nu Q_\nu) = \varnothing$, where*

$$\mathcal{F} := \{x \in \mathbb{R} : f(x) = \nabla f(x) = \nabla^2 f(x) = 0\}. \quad (\nabla^k f := f^{(k)})$$

(2) *Let $\delta_\nu = diam(Q_\nu)$. Then for any ν, $\delta_\nu \leq 1$ and*

$$\inf_{x \in Q_\nu} \left(\sum_{k=0}^{3} \delta_\nu^{k-(2+2\alpha)} |\nabla^k f(x)| \right) > 4. \tag{2.1}$$

If $0 < \alpha \leq 1/2$, then there is a countable collection of cubes $\{Q_\nu\}_{\nu \geq 1}$ taking the form of $(a, b]$, whose interiors are disjoint, such that

(1') $\mathbb{R} = \mathcal{F} \cup (\cup_\nu Q_\nu)$ *and* $\mathcal{F} \cap (\cup_\nu Q_\nu) = \varnothing$.

(2') *Let* $\delta_\nu = diam(Q_\nu)$. *Then for any* ν, $\delta_\nu \leq 1$ *and*

$$\inf_{x \in Q_\nu} \left(\sum_{k=0}^{2} \delta_\nu^{k-(2+2\alpha)} |\nabla^k f(x)| \right) > 3. \tag{2.2}$$

Proof. We prove first for the case where $1/2 < \alpha \leq 1$. We decompose $\mathbb{R}$ into a mesh of equal cubes $(a_n, b_n]$, whose interiors are disjoint, and whose common diameter is so large that

$$\inf_{x \in Q'} \left(\sum_{k=0}^{3} (diam(Q'))^{k-(2+2\alpha)} |\nabla^k f(x)| \right) \leq 4$$

for every cube Q' in this mesh. Note $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \leq 1$, so the common diameter can be chosen to be 1.

Let Q' be a fixed cube in this mesh. By bisecting each of the sides of Q', we divide Q' into 2 congruent cubes. Let Q'' be one of these new cubes.

(i) If
$$\inf_{x \in Q''} \left(\sum_{k=0}^{3} (diam(Q''))^{k-(2+2\alpha)} |\nabla^k f(x)| \right) > 4,$$
then we don't sub-divide Q'' any further, and Q'' is selected as one of the cubes Q_ν.

(ii) If
$$\inf_{x \in Q''} \left(\sum_{k=0}^{3} (diam(Q''))^{k-(2+2\alpha)} |\nabla^k f(x)| \right) \leq 4,$$
then we proceed with the sub-division of Q'', and repeat this process until we are forced to the case (i).

If $0 < \alpha \leq 1/2$, the stopping condition is then (2.2). □

Let $N(\alpha) = 2$ if $0 < \alpha \leq 1/2$, $N(\alpha) = 3$ if $1/2 < \alpha \leq 1$.

Lemma 2.2. *Let* $C = 1000$. *Let* $3Q_\nu$ *be the cube of diameter* $3\delta_\nu$, *with the same center at* Q_ν, *then*

$$\sum_{k=0}^{N(\alpha)} \delta_\nu^{k-(2+2\alpha)} |\nabla^k f(x)| \leq C \quad \forall x \in 3Q_\nu. \tag{2.3}$$

Proof. We prove first for the case where $1/2 < \alpha \leq 1$.

Let $\tilde{Q}_\nu$ be the step before we get Q_ν. Then $Q_\nu \subset \tilde{Q}_\nu$ and diameter of $\tilde{Q}_\nu$ is $2\delta_\nu$. Since we didn't stop at $\tilde{Q}_\nu$, there is $x_0 \in \tilde{Q}_\nu \subset 3Q_\nu$ such that $\sum_{k=0}^{3}(2\delta_\nu)^{k-(2+2\alpha)}|\nabla^k f(x_0)| \leq 4$. That is

$$|\nabla^k f(x_0)| \leq 4(2\delta_\nu)^{(2+2\alpha)-k}, \quad k = 0, 1, 2, 3. \tag{2.4}$$

Using $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \leq 1$ and $dist(x, x_0) \leq 3\delta_\nu$, we get

$$|\nabla^3 f(x)| \leq |\nabla^3 f(x_0)| + 1 \cdot |x - x_0|^{2\alpha-1} \leq 4(2\delta_\nu)^{2+2\alpha-3} + (3\delta_\nu)^{2\alpha-1} \leq 11\delta_\nu^{2\alpha-1} \quad \forall x \in 3Q_\nu. \tag{2.5}$$

Using (2.4) and (2.5), we get

$$|\nabla^2 f(x)| \leq \sup_{3Q_\nu}|\nabla^3 f| \cdot |x - x_0| + |\nabla^2 f(x_0)| \leq 11\delta_\nu^{2\alpha-1} \cdot 3\delta_\nu + 4(2\delta_\nu)^{(2+2\alpha)-2} \leq 49\delta_\nu^{2\alpha} \quad \forall x \in 3Q_\nu.$$

Going backwards, we get $|\nabla f(x)| \leq 179\delta_\nu^{1+2\alpha}$ and $|f(x)| \leq 601\delta_\nu^{2+2\alpha} \quad \forall x \in 3Q_\nu$.

If $0 < \alpha \leq 1/2$, similarly, let $\tilde{Q}_\nu$ be the step before we get Q_ν. Then $Q_\nu \subset \tilde{Q}_\nu$ and diameter of $\tilde{Q}_\nu$ is $2\delta_\nu$. Since we didn't stop at $\tilde{Q}_\nu$, there is $x_0 \in \tilde{Q}_\nu \subset 3Q_\nu$ such that $\sum_{k=0}^{2}(2\delta_\nu)^{k-(2+2\alpha)}|\nabla^k f(x_0)| \leq 3$. That is

$$|\nabla^k f(x_0)| \leq 3(2\delta_\nu)^{(2+2\alpha)-k}, \quad k = 0, 1, 2. \tag{2.6}$$

Using $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \leq 1$ and $dist(x, x_0) \leq 3\delta_\nu$, we get

$$|\nabla^2 f(x)| \leq |\nabla^2 f(x_0)| + 1 \cdot |x - x_0|^{2\alpha} \leq 3(2\delta_\nu)^{2+2\alpha-2} + (3\delta_\nu)^{2\alpha} \leq 9\delta_\nu^{2\alpha} \quad \forall x \in 3Q_\nu. \tag{2.7}$$

Using (2.6) and (2.7), we get

$$|\nabla f(x)| \leq \sup_{3Q_\nu}|\nabla^2 f| \cdot |x - x_0| + |\nabla f(x_0)| \leq 9\delta_\nu^{2\alpha} \cdot 3\delta_\nu + 3(2\delta_\nu)^{(2+2\alpha)-1} \leq 39\delta_\nu^{1+2\alpha} \quad \forall x \in 3Q_\nu.$$

Going backwards, we get $|f(x)| \leq 141\delta_\nu^{2+2\alpha} \quad \forall x \in 3Q_\nu$.

$\square$

Lemma 2.3. *Let $\epsilon_0 = (\frac{1}{10^5})^{1/(2\alpha-1)}$ if $1/2 < \alpha \leq 1$, $\epsilon_0 = (\frac{1}{10^5})^{1/(2\alpha)}$ if $0 < \alpha \leq 1/2$. Let $c_0 = 1/10$. Let Q_ν^* be the cube of diameter $(1 + \epsilon_0)\delta_\nu$, with the same center at Q_ν, then*

$$\inf_{x \in Q_\nu^*}\Big(\sum_{k=0}^{N(\alpha)} \delta_\nu^{k-(2+2\alpha)}|\nabla^k f(x)|\Big) \geq c_0. \tag{2.8}$$

Proof. We first deal with the case where $1/2 < \alpha \leq 1$.

Assume not, then $\exists x_0 \in Q_\nu^*$ such that $\sum_{k=0}^{3} \delta_\nu^{k-(2+2\alpha)}|\nabla^k f(x_0)| < c_0$. Let $B := \{x \in \mathbb{R} :$

$dist(x,x_0)\le \epsilon_0\delta_\nu\}$. Using $\|f\|_{C^{2,2\alpha}(\mathbb{R})}\le 1$ and mean value theorem, we get

$$|\nabla^3 f(x)| \le |\nabla^3 f(x_0)| + 1\cdot |x-x_0|^{2\alpha-1} \le (c_0+1)\delta_\nu^{2\alpha-1} \quad \forall x\in B.$$

$$|\nabla^2 f(x)| \le \sup_B |\nabla^3 f|\cdot |x-x_0| + |\nabla^2 f(x_0)| \le (2c_0+1)\delta_\nu^{2\alpha} \quad \forall x\in B.$$

Going backwards, we get $|\nabla f(x)|\le (3c_0+1)\delta_\nu^{1+2\alpha}$ and $|f(x)|\le [(3c_0+1)\epsilon_0+c_0]\delta_\nu^{2+2\alpha}$. Note $\epsilon_0<\frac{1}{10^5}$, so for any $x\in B$, $\sum_{k=0}^3 \delta_\nu^{k-(2+2\alpha)}|\nabla^k f(x)|<4$, contradicting with (2.1).

If $0<\alpha\le 1/2$, assume that $\exists x_0\in Q_\nu^*$ such that $\sum_{k=0}^2 \delta_\nu^{k-(2+2\alpha)}|\nabla^k f(x_0)|<c_0$. Let $B:=\{x\in\mathbb{R} : dist(x,x_0)\le\epsilon_0\delta_\nu\}$. Using $\|f\|_{C^{2,2\alpha}(\mathbb{R})}\le 1$ and mean value theorem, we get

$$|\nabla^2 f(x)| \le |\nabla^2 f(x_0)| + 1\cdot |x-x_0|^{2\alpha} \le (c_0+1)\delta_\nu^{2\alpha} \quad \forall x\in B.$$

$$|\nabla f(x)| \le \sup_B |\nabla^2 f|\cdot |x-x_0| + |\nabla f(x_0)| \le (2c_0+1)\delta_\nu^{2\alpha+1} \quad \forall x\in B.$$

$$|f(x)| \le \sup_B |\nabla f|\cdot |x-x_0| + |f(x_0)| \le [(2c_0+1)\epsilon_0+c_0]\delta_\nu^{2+2\alpha} \quad \forall x\in B.$$

Note $\epsilon_0<\frac{1}{10^5}$, so for any $x\in B$, $\sum_{k=0}^2 \delta_\nu^{k-(2+2\alpha)}|\nabla^k f(x)|<3$, contradicting with (2.2). □

Lemma 2.4. *Let $\lambda=\epsilon_0/2$. Let Q_ν^+ be the cube of diameter of $(1+\lambda)\delta_\nu$, with the same center at Q_ν.*

Then for $z\in Q_\nu^+$, either

(i)

$$f(z)\ge \tilde{c}\delta_\nu^{2+2\alpha}, \tag{2.9}$$

or (ii)

$$f(z)<\tilde{c}\delta_\nu^{2+2\alpha} \text{ and } |\nabla^2 f(z)|\ge \tilde{c}\delta_\nu^{2\alpha}, \tag{2.10}$$

where $\tilde{c}=\frac{1}{10^4}\cdot(\frac{1}{10^5})^{4/(2\alpha-1)}$ if $\alpha>1/2$, $\tilde{c}=\frac{1}{10^3}\cdot(\frac{1}{10^5})^{3/(2\alpha)}$ if $\alpha\le 1/2$.

Proof. We prove the case where $1/2<\alpha\le 1$.

By translation we assume $z=0$. Assume that

$$f(0)<\tilde{c}\delta_\nu^{2+2\alpha} \quad \text{and} \quad |\nabla^2 f(0)|<\tilde{c}\delta_\nu^{2\alpha}. \tag{2.11}$$

Then by Taylor expansion of f near 0,

$$f(x)=f(0)+f'(0)x+\frac{1}{2}f''(0)x^2+\frac{1}{6}f'''(0)x^3+\frac{1}{6}\frac{f'''(\xi)-f'''(0)}{|\xi-0|^{2\alpha-1}}|\xi-0|^{2\alpha-1}x^3, \tag{2.12}$$

where ξ lies in between 0 and x.

Let $c>0$ small such that $2c\delta_\nu < (diam(Q_\nu^*) - diam(Q_\nu^+))/2$. By (2.11), (2.12) and $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \leq 1$, for any $|x| < 2c\delta_\nu$,

$$0 \leq f(x) \leq \tilde{c}\delta_\nu^{2+2\alpha} + f'(0)x + \frac{1}{2}\tilde{c}\delta_\nu^{2\alpha}x^2 + \frac{1}{6}f'''(0)x^3 + \frac{1}{6}|x|^{2+2\alpha}. \tag{2.13}$$

Taking x and $-x$ in (2.13), for any $|x| < 2c\delta_\nu$,

$$|f'(0)x + \frac{1}{6}f'''(0)x^3| \leq \tilde{c}\delta_\nu^{2+2\alpha} + \frac{1}{2}\tilde{c}\delta_\nu^{2\alpha}x^2 + \frac{1}{6}|x|^{2+2\alpha}. \tag{2.14}$$

In particular, for any $|x| < c\delta_\nu$,

$$|f'(0)x + \frac{1}{6}f'''(0)x^3| \leq \tilde{c}\delta_\nu^{2+2\alpha} + \frac{1}{2}\tilde{c}\delta_\nu^{2\alpha}(c\delta_\nu)^2 + \frac{1}{6}|c\delta_\nu|^{2+2\alpha} =: A\delta_\nu^{2+2\alpha}. \tag{2.15}$$

On the other hand, by substituting x with $2x$ in (2.14), for any $|x| < c\delta_\nu$,

$$\begin{aligned}|f'(0)(2x) + \frac{1}{6}f'''(0)(2x)^3| &\leq \tilde{c}\delta_\nu^{2+2\alpha} + \frac{1}{2}\tilde{c}\delta_\nu^{2\alpha}(2x)^2 + \frac{1}{6}|2x|^{2+2\alpha} \\ &\leq \tilde{c}\delta_\nu^{2+2\alpha} + \frac{1}{2}\tilde{c}\delta_\nu^{2\alpha}(2c\delta_\nu)^2 + \frac{1}{6}|2c\delta_\nu|^{2+2\alpha} \\ &=: B\delta_\nu^{2+2\alpha}. \end{aligned} \tag{2.16}$$

Combining (2.15) and (2.16), we obtain for any $|x| < c\delta_\nu$,

$$|f'(0)x| \leq \frac{1}{6}(8A+B)\delta_\nu^{2+2\alpha},$$

$$|\frac{1}{6}f'''(0)x^3| \leq \frac{1}{6}(2A+B)\delta_\nu^{2+2\alpha}.$$

Thus $|f'(0)| \leq \frac{8A+B}{6c}$ and $|f'''(0)| \leq \frac{2A+B}{c^3}$. If $c = \epsilon_0/10$, $\tilde{c} = c^4$, then

$$\frac{8A+B}{6c} = [9c^3 + 6c^5 + \frac{1}{6}c^{1+2\alpha}(8 + 2^{2+2\alpha})]/6 < 0.01,$$

$$\frac{2A+B}{c^3} = 3c + 3c^3 + \frac{1}{6}c^{2\alpha-1}(2 + 2^{2+2\alpha}) < 0.07.$$

Hence

$$\sum_{k=0}^{3}\delta_\nu^{k-(2+2\alpha)}|\nabla^k f(0)| \leq \tilde{c} + \frac{8A+B}{6c} + \tilde{c} + \frac{2A+B}{c^3} < 0.01^4 + 0.01 + 0.01^4 + 0.07 < c_0,$$

contradicting with (2.8).

If $0 < \alpha \le 1/2$, by translation we assume $z = 0$, with

$$f(0) < \tilde{c}\delta_\nu^{2+2\alpha} \quad \text{and} \quad |\nabla^2 f(0)| < \tilde{c}\delta_\nu^{2\alpha}. \tag{2.17}$$

Then by Taylor expansion of f near 0,

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{2}\frac{f''(\xi) - f''(0)}{|\xi - 0|^{2\alpha}}|\xi - 0|^{2\alpha}x^2, \tag{2.18}$$

where ξ lies in between 0 and x.

Let $c > 0$ small such that $2c\delta_\nu < (diam(Q_\nu^*) - diam(Q_\nu^+))/2$. By (2.17), (2.18) and $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \le 1$, for any $|x| < 2c\delta_\nu$,

$$0 \le f(x) \le \tilde{c}\delta_\nu^{2+2\alpha} + f'(0)x + \frac{1}{2}\tilde{c}\delta_\nu^{2\alpha}x^2 + \frac{1}{2}|x|^{2+2\alpha}. \tag{2.19}$$

If $c = \epsilon_0/10$, $\tilde{c} = c^3$, setting $x = \pm c\delta_\nu$ in (2.19) yields

$$|f'(0)| \le (c^2 + \frac{1}{2}c^4 + \frac{1}{2}c^{1+2\alpha})\delta_\nu^{1+2\alpha} < 0.01\delta_\nu^{1+2\alpha}.$$

Hence

$$\sum_{k=0}^{2} \delta_\nu^{k-(2+2\alpha)}|\nabla^k f(0)| \le \tilde{c} + 0.01 + \tilde{c} < 0.01^3 + 0.01 + 0.01^3 < c_0,$$

contradicting with (2.8).

□

Next we prove a version of Fefferman-Phong's lemma(see [4] and Lemma 18.6.9 of [10]).

Lemma 2.5. *Let $C = 1000$ and $I = [-1/2, 1/2]$, with $N(\alpha)$ and $\tilde{c}$ defined as before. If $0 \le \phi \in C^{2,2\alpha}(I)$ such that*

$$|\phi^{(k)}(t)| \le C \quad \forall t \in I \text{ for } k = 0, 1, \cdots, N(\alpha), \quad [\phi]_{C^{2,2\alpha}(I)} \le 1 \tag{2.20}$$

$$\text{and} \quad \max\{\phi(0), |\phi''(0)|\} \ge \tilde{c}. \tag{2.21}$$

Then there is a universal small constant $r_0 > 0$ and a universal small constant $c_2 > 0$ such that, for $t \in (-r_0, r_0)$, either

(I)

$$c_2 \le \phi(t) \le C, \tag{2.22}$$

and $\sqrt{\phi(t)}$ is in $C^{1,\alpha}((-r_0, r_0))$, or

(II)

$$c_2 \leq \phi''(t) \leq C, \tag{2.23}$$

$$\phi(t) = \phi(T) + (t-T)^2 \int_0^1 \phi''(t + s(T-t))s\, ds, \tag{2.24}$$

where $t = T$ is the unique strict local minimum point of the function ϕ in $(-r_0, r_0)$.

Moreover, $g(t) \coloneqq (t-T)(\int_0^1 \phi''(t+s(T-t))s\, ds)^{1/2}$ is in $C^{1,\alpha}((-r_0, r_0))$.

Remark 2.6. *If $1/2 < \alpha \leq 1$, $[\phi]_{C^{2,2\alpha}(I)} \leq 1$ means $[\phi]_{C^{3,2\alpha-1}(I)} \leq 1$.*

Proof. We first prove the case for $1/2 < \alpha \leq 1$.

(i)If $\phi(0) \geq \tilde{c}$, by Taylor expansion and (2.20) , for $|t| < \mu$, $\mu \coloneqq \frac{\tilde{c}}{3C}$,

$$\begin{aligned} \phi(t) &\geq \tilde{c} - C \cdot \mu - \frac{1}{2} C \cdot \mu^2 - \frac{1}{6} C \cdot \mu^3 - \frac{1}{6} \cdot 1 \cdot \mu^{2+2\alpha} \\ &\geq \tilde{c} - \frac{1}{3}\tilde{c} - \frac{1}{2} \cdot \frac{1}{3}\tilde{c} - \frac{1}{6} \cdot \frac{1}{3}\tilde{c} - \frac{1}{6} \cdot \frac{1}{3}\tilde{c} > \frac{1}{3}\tilde{c}. \end{aligned} \tag{2.25}$$

So by (2.20), (2.25), and mean value theorem, for $|t_1| < \mu$ and $|t_2| < \mu$, $t_1 \neq t_2$,

$$|(\sqrt{\phi})'(t_1)| = |\frac{\phi'(t_1)}{2\sqrt{\phi(t_1)}}| \leq \frac{C}{2\sqrt{\frac{1}{3}\tilde{c}}} =: b, \tag{2.26}$$

$$\begin{aligned} &2|(\sqrt{\phi})'(t_1) - (\sqrt{\phi})'(t_2)|/|t_1 - t_2|^\alpha = |\frac{\phi'(t_1)}{\sqrt{\phi(t_1)}} - \frac{\phi'(t_2)}{\sqrt{\phi(t_2)}}|/|t_1 - t_2|^\alpha \\ &\leq |\frac{\phi'(t_1)}{\sqrt{\phi(t_1)}} - \frac{\phi'(t_2)}{\sqrt{\phi(t_1)}}|/|t_1 - t_2|^\alpha + |\frac{\phi'(t_2)}{\sqrt{\phi(t_1)}} - \frac{\phi'(t_2)}{\sqrt{\phi(t_2)}}|/|t_1 - t_2|^\alpha \\ &\leq \frac{1}{\sqrt{\frac{1}{3}\tilde{c}}} \cdot \frac{|\phi''(\xi_1)||t_1 - t_2|}{|t_1 - t_2|^\alpha} + C \cdot \frac{|\phi'(\xi_2)||t_1 - t_2|}{2(\frac{1}{3}\tilde{c})^{3/2}|t_1 - t_2|^\alpha} \\ &\leq \frac{1}{\sqrt{\frac{1}{3}\tilde{c}}} \cdot C(2\mu)^{1-\alpha} + C \cdot \frac{C}{2(\frac{1}{3}\tilde{c})^{3/2}} \cdot (2\mu)^{1-\alpha} =: C_1, \end{aligned} \tag{2.27}$$

where $b, C_1 > 0$ are universal constants.

(ii)Assume $|\phi''(0)| \geq \tilde{c}$.

(a) If $\phi''(0) \leq -\tilde{c}$, then by Taylor expansion of $\phi'' \in C^{1,2\alpha-1}(I)$ near 0, we get,

$$\phi''(t) = \phi''(0) + \phi'''(0)t + \frac{\phi'''(\xi) - \phi'''(0)}{|\xi - 0|^{2\alpha-1}}|\xi - 0|^{2\alpha-1}t. \tag{2.28}$$

Hence for $|t| < \mu$, $\mu = \frac{\tilde{c}}{3C}$,

$$\phi''(t) \leq -\tilde{c} + C \cdot \mu + 1 \cdot |\mu|^{2\alpha} \leq -\frac{1}{3}\tilde{c}.$$

For any $|t_0| < \frac{1}{2}\mu$, by Taylor expansion of ϕ near t_0,

$$0 \leq \phi(t_0 + h) \leq \phi(t_0) + \phi'(x_0)h + \frac{1}{2} \cdot (-\frac{1}{3}\tilde{c}) \cdot h^2 + \frac{1}{6}\phi'''(t_0)h^3 + \frac{1}{6}|h|^{2+2\alpha},$$

$$0 \leq \phi(t_0 - h) \leq \phi(t_0) - \phi'(t_0)h + \frac{1}{2} \cdot (-\frac{1}{3}\tilde{c}) \cdot h^2 - \frac{1}{6}\phi'''(t_0)h^3 + \frac{1}{6}|h|^{2+2\alpha}.$$

Combing the above two equations, and letting $h = \frac{1}{2}\mu$, we get, for $|t_0| < \frac{1}{2}\mu$,

$$\phi(t_0) \geq \frac{1}{6}\tilde{c}h^2 - \frac{1}{6}|h|^{2+2\alpha} \geq \frac{1}{48}\mu^2\tilde{c}.$$

Similar to case (i), we have $\sqrt{\phi} \in C^{1,\alpha}((-\mu/2, \mu/2))$.

(b) If $\phi''(0) \geq \tilde{c}$ and $\phi(0) < c_1$, where $c_1 > 0$ is a small and universal constant to be determined, then by Taylor expansion of $\phi' \in C^{2,2\alpha-1}(I)$ near 0, we get

$$\phi'(t) = \phi'(0) + \phi''(0)t + \frac{1}{2}\phi'''(0)t^2 + \frac{1}{2}\frac{\phi'''(\xi) - \phi'''(0)}{|\xi - 0|^{2\alpha-1}}|\xi - 0|^{2\alpha-1}t^2. \tag{2.29}$$

Using (2.29) and (2.20), we obtain

$$|\phi'(t) - \phi'(0) - \phi''(0)t| \leq \frac{1}{2}Ct^2 + \frac{1}{2} \cdot 1 \cdot |t|^{2\alpha+1}, \quad \forall |t| < \frac{1}{2}. \tag{2.30}$$

In particular, (2.30) shows that $\phi'(r) > 0$ and $\phi'(-r) < 0$ if

$$\phi''(0)r > |\phi'(0)| + \frac{1}{2}Cr^2 + \frac{1}{2}r^{2\alpha+1}. \tag{2.31}$$

Fix $r > 0$ so small ($r = \frac{\tilde{c}}{3C}$ would work) that $\frac{1}{2}\tilde{c}r > \frac{1}{2}Cr^2 + \frac{1}{2}r^{2\alpha+1}$. We expand $\phi'' \in C^{1,2\alpha-1}(I)$ near 0. By (2.28) and (2.20), for $|t| \leq r$,

$$\phi''(t) \geq \tilde{c} - Cr - 1 \cdot r^{2\alpha} \geq \frac{1}{3}\tilde{c}. \tag{2.32}$$

On the other hand, following the idea of Nirenberg-Trèves [14] (see also Lemma 7.7.2 of [11]), near 0,

$$0 \leq \phi(0) + \phi'(0)t + \frac{\phi''(\xi)}{2}t^2 \leq c_1 + \phi'(0)t + \frac{1}{2}Ct^2.$$

Letting $t = \pm\sqrt{c_1/C}$, we have $|\phi'(0)| \leq \frac{3}{2}\sqrt{C \cdot c_1}$. Hence (2.32) and (2.31) hold for $c_1 = \frac{\tilde{c}^4}{(9C)^2 \cdot C}$ and $r = \frac{\tilde{c}}{3C}$, and by intermediate value theorem, $\phi'(t) = 0$ has a unique solution $t = T$ in B_r. By Taylor expansion of ϕ near $t = T$, we obtain in B_r,

$$\phi(t) = \phi(T) + (t-T)^2 \int_0^1 \phi''(t + s(T-t))s\,ds. \tag{2.33}$$

We note $t = T$ is the unique strict local minimum point of the function ϕ in B_r.

We will estimate Hölder seminorm of g'. Assume without loss of generality that $\phi(T) = 0$. Then in B_r, $g(t) = \sqrt{\phi(t)}$ if $t \geq T$ and $g(t) = -\sqrt{\phi(t)}$ if $t < T$. By Taylor expansion,

$$\lim_{t \to T^+} \frac{g(t) - g(T)}{t - T} = \lim_{t \to T^+} \frac{\sqrt{\frac{1}{2}\phi''(T)(t-T)^2 + O(|t-T|^3)} - \sqrt{0}}{t - T} = \sqrt{\frac{1}{2}\phi''(T)}.$$

We obtain the same value for the left limit and then $g'(T) = \sqrt{\frac{1}{2}\phi''(T)}$.

If $t \neq T$, then by Taylor expansion, for some ξ_1, ξ_2, ξ_3 lying in between t and T,

$$\phi(t) = \frac{1}{2}\phi''(T)(t-T)^2 + \frac{1}{6}\phi'''(\xi_1)(t-T)^3 =: A + B, \tag{2.34}$$

$$\phi'(t) = \phi''(T)(t-T) + \frac{1}{2}\phi'''(\xi_2)(t-T)^2 =: E + F, \tag{2.35}$$

$$\phi''(t) = \phi''(T) + \phi'''(\xi_3)(t-T). \tag{2.36}$$

By (2.32), (2.20) and $|t - T| \leq 2r$,

$$|B| \leq \frac{2}{3}A \quad \text{and} \quad |F| \leq |E|. \tag{2.37}$$

By (2.34), (2.35), (2.37) and $\phi''(t) \sim 1$ in B_r, there exists a universal $b > 0$ such that, for any $t \in B_r$,

$$|g'(T) - g'(t)| \leq b \cdot |T - t|^\alpha. \tag{2.38}$$

Below is the proof for the case $t > T$. Proof is the same for $t < T$. We won't repeat.

$$\begin{aligned}
|g'(T) - g'(t)| &= \left|\sqrt{\frac{1}{2}\phi''(T)} - \frac{\phi'(t)}{2\sqrt{\phi(t)}}\right| \\
&\le \left|\sqrt{\frac{1}{2}\phi''(T)} - \frac{\phi'(t)}{2\sqrt{A}}\right| + \left|\frac{\phi'(t)}{2\sqrt{A}} - \frac{\phi'(t)}{2\sqrt{A+B}}\right| \\
&= \left|\frac{-\frac{1}{2}\phi''(\xi_2)(t-T)}{\sqrt{2\phi''(T)}}\right| + \frac{1}{2}|\phi'(t)|\left|\frac{B}{\sqrt{A}\cdot\sqrt{A+B}\cdot(\sqrt{A+B}+\sqrt{A})}\right| \\
&\le \frac{C/2}{\sqrt{2\cdot\frac{1}{3}\tilde{c}}}|T-t| + \frac{1}{2}\cdot|2E|\cdot\left|\frac{B}{\sqrt{A}\cdot\sqrt{\frac{1}{3}A}\cdot(\sqrt{\frac{1}{3}A}+\sqrt{A})}\right| \\
&\le b\cdot|T-t| \le b\cdot|T-t|^{\alpha},\ \text{since } |T-t|<1.
\end{aligned}$$

By (2.34), (2.35), (2.36), and $\phi''(t) \sim 1$ in B_r, there exists a universal $c > 0$ such that, for any $t \in B_r$,

$$|\phi''(t)\cdot\phi(t) - \frac{1}{2}\phi'(t)^2| = O(|t-T|^3),$$

$$|g''(t)| = \frac{1}{2}\left|\frac{\phi''(t)\cdot\phi(t) - \frac{1}{2}\phi'(t)^2}{\phi(t)^{3/2}}\right| \le \frac{1}{2}\left|\frac{O(|t-T|^3)}{(\frac{1}{3}A)^{3/2}}\right| \le c. \tag{2.39}$$

Let $t_1, t_2 \in B_r$, $t_1 < t_2$.

If $t_1 = T$ ($t_2 = T$ is similar), then $|g'(t_1) - g'(t_2)| \le b\cdot|t_1 - t_2|^{\alpha}$ by (2.38).

If $t_1 < T < t_2$, then by (2.38),

$$|g'(t_1) - g'(t_2)| \le |g'(t_1) - g'(T)| + |g'(T) - g'(t_2)| \le b|t_1 - T|^{\alpha} + b|T - t_2|^{\alpha} \le 2b\cdot|t_1 - t_2|^{\alpha}.$$

If $T < t_1 < t_2$ ($t_1 < t_2 < T$ is similar), then by mean value theorem, (2.39) and $|t_1 - t_2| \le 1$, $\exists \xi \in (t_1, t_2)$ such that

$$|g'(t_1) - g'(t_2)| = |g''(\xi)||t_1 - t_2| \le c\cdot|t_1 - t_2| \le c\cdot|t_1 - t_2|^{\alpha}.$$

(c) If $c_1 \le \phi(0) < \tilde{c}$, then similar to case (i), we have $\sqrt{\phi} \in C^{1,\alpha}((-\frac{c_1}{3C}, \frac{c_1}{3C}))$.

To summarize, case (i), (ii)(a) and (ii)(c) lead to (I). Case (ii)(b) leads to (II).

Next, we prove the case where $0 < \alpha \le 1/2$.

(i') If $\phi(0) \geq \tilde{c}$, by Taylor expansion and (2.20), for $|t| < \mu$, $\mu := \frac{\tilde{c}}{3C}$,

$$\begin{aligned}\phi(t) &\geq \tilde{c} - C \cdot \mu - \frac{1}{2} C \cdot \mu^2 - \frac{1}{2} \cdot 1 \cdot \mu^{2+2\alpha} \\ &\geq \tilde{c} - \frac{1}{3}\tilde{c} - \frac{1}{2} \cdot \frac{1}{3}\tilde{c} - \frac{1}{2} \cdot \frac{1}{3}\tilde{c} \geq \frac{1}{3}\tilde{c}.\end{aligned}$$

Following the computation of (2.26) and (2.27), $\sqrt{\phi} \in C^{1,\alpha}((-r, r))$.

(ii') Assume $|\phi''(0)| \geq \tilde{c}$.

(a) If $\phi''(0) \leq -\tilde{c}$, then for $|t| < \mu_2$, $\mu_2 := (\frac{\tilde{c}}{3})^{\frac{1}{2\alpha}}$,

$$\phi''(t) \leq \phi''(0) + 1 \cdot |t - 0|^{2\alpha} \leq -\tilde{c} + \mu_2^{2\alpha} \leq -\frac{2}{3}\tilde{c}, \quad \text{since } [\phi]_{C^{2,2\alpha}(I)} \leq 1.$$

For any $|t_0| < \frac{1}{2}\mu_2$, by Taylor expansion of ϕ near t_0,

$$0 \leq \phi(t_0 + h) \leq \phi(t_0) + \phi'(x_0)h + \frac{1}{2} \cdot (-\frac{2}{3}\tilde{c}) \cdot h^2 + \frac{1}{2}|h|^{2+2\alpha},$$

$$0 \leq \phi(t_0 - h) \leq \phi(t_0) - \phi'(t_0)h + \frac{1}{2} \cdot (-\frac{2}{3}\tilde{c}) \cdot h^2 + \frac{1}{2}|h|^{2+2\alpha}.$$

Combing the above two equations, and letting $h = \frac{1}{2}\mu_2$, we get, for $|t_0| < \frac{1}{2}\mu_2$,

$$\phi(t_0) \geq \frac{1}{3}\tilde{c}h^2 - \frac{1}{2}|h|^{2+2\alpha} \geq h^2(\frac{1}{3}\tilde{c} - \frac{1}{2}|h|^{2\alpha}) \geq h^2(\frac{1}{3}\tilde{c} - \frac{1}{2} \cdot \frac{\tilde{c}}{3}) \geq \frac{1}{24}\mu_2^2 \cdot \tilde{c}.$$

Similar to case (i'), we have $\sqrt{\phi} \in C^{1,\alpha}((-\mu_2/2, \mu_2/2))$.

(b) If $\phi''(0) \geq \tilde{c}$ and $\phi(0) < c_1$, where $c_1 > 0$ is a small and universal constant to be determined, then by Taylor expansion of $\phi' \in C^{1,2\alpha}(I)$ near 0, we get

$$\phi'(t) = \phi'(0) + \phi''(0)t + \frac{\phi''(\xi) - \phi''(0)}{|\xi - 0|^{2\alpha}}|\xi - 0|^{2\alpha}t. \tag{2.40}$$

Using (2.40) and (2.20), we obtain

$$|\phi'(t) - \phi'(0) - \phi''(0)t| \leq 1 \cdot |t|^{2\alpha+1}, \quad \forall |t| < \frac{1}{2}. \tag{2.41}$$

In particular, (2.41) shows that $\phi'(r) > 0$ and $\phi'(-r) < 0$ if

$$\phi''(0)r > |\phi'(0)| + r^{2\alpha+1}. \tag{2.42}$$

Fix $r > 0$ so small ($r = \frac{1}{2} \cdot (\frac{\tilde{c}}{3})^{\frac{1}{2\alpha}}$ would work) that $\frac{1}{2}\tilde{c}r > r^{2\alpha+1}$. By $[\phi]_{C^{2,2\alpha}(I)} \leq 1$, for $|t| \leq r$,

$$\phi''(t) \geq \phi''(0) - 1 \cdot |t-0|^{2\alpha} \geq \tilde{c} - \frac{1}{3}\tilde{c} \geq \frac{2}{3}\tilde{c}. \tag{2.43}$$

On the other hand, near 0,

$$0 \leq \phi(0) + \phi'(0)t + \frac{\phi''(\xi)}{2}t^2 \leq c_1 + \phi'(0)t + \frac{1}{2}Ct^2.$$

Letting $t = \pm\sqrt{c_1/C}$, we have $|\phi'(0)| \leq \frac{3}{2}\sqrt{C \cdot c_1}$. Hence (2.43) and (2.42) hold for $c_1 = \frac{1}{4C} \cdot (\frac{\tilde{c}}{3})^{2+\frac{1}{\alpha}}$ and $r = \frac{1}{2} \cdot (\frac{\tilde{c}}{3})^{\frac{1}{2\alpha}}$, and by intermediate value theorem, $\phi'(t) = 0$ has a unique solution $t = T$ in B_r. By Taylor expansion of ϕ near $t = T$, we obtain in B_r,

$$\phi(t) = \phi(T) + (t-T)^2 \int_0^1 \phi''(t + s(T-t))s\, ds. \tag{2.44}$$

We note $t = T$ is the unique strict local minimum point of the function ϕ in B_r.

We will estimate Hölder seminorm of g'. Assume without loss of generality that $\phi(T) = 0$. Then in B_r, $g(t) = \sqrt{\phi(t)}$ if $t \geq T$ and $g(t) = -\sqrt{\phi(t)}$ if $t < T$. By Taylor expansion,

$$\lim_{t \to T^+} \frac{g(t) - g(T)}{t-T} = \lim_{t \to T^+} \frac{\sqrt{\frac{1}{2}\phi''(T)(t-T)^2 + O(|t-T|^{2+2\alpha})} - \sqrt{0}}{t-T} = \sqrt{\frac{1}{2}\phi''(T)}.$$

We obtain the same value for the left limit and then $g'(T) = \sqrt{\frac{1}{2}\phi''(T)}$.

If $t \neq T$, then by Taylor expansion,

$$\phi(t) = \frac{1}{2}\phi''(T)(t-T)^2 + O_1(|t-T|^{2+2\alpha}) =: A + B, \tag{2.45}$$

$$\phi'(t) = \phi''(T)(t-T) + O_2(|t-T|^{1+2\alpha}) =: E + F, \tag{2.46}$$

$$\phi''(t) = \phi''(T) + O_3(|t-T|^{2\alpha}). \tag{2.47}$$

By (2.43), (2.20) and $|t - T| \leq 2r$,

$$|B| \leq \frac{1}{2}A \quad \text{and} \quad |F| \leq \frac{1}{2}|E|. \tag{2.48}$$

By (2.45), (2.46), (2.48) and $\phi''(t) \sim 1$ in B_r, there exists a universal $b > 0$ such that, for any $t \in B_r$,

$$|g'(T) - g'(t)| \leq b \cdot |T - t|^\alpha. \tag{2.49}$$

Below is the proof for the case $t > T$. Proof is the same for $t < T$. We won't repeat.

$$
\begin{aligned}
|g'(T) - g'(t)| &= \left| \sqrt{\frac{1}{2}\phi''(T)} - \frac{\phi'(t)}{2\sqrt{\phi(t)}} \right| \\
&\leq \left| \sqrt{\frac{1}{2}\phi''(T)} - \frac{\phi'(t)}{2\sqrt{A}} \right| + \left| \frac{\phi'(t)}{2\sqrt{A}} - \frac{\phi'(t)}{2\sqrt{A+B}} \right| \\
&= \left| \frac{-O_2(|t-T|^{1+2\alpha})}{\sqrt{2\phi''(T)} \cdot (t-T)} \right| + \frac{1}{2}|\phi'(t)| \left| \frac{B}{\sqrt{A} \cdot \sqrt{A+B} \cdot (\sqrt{A+B} + \sqrt{A})} \right| \\
&\leq \frac{O_2(|t-T|^{2\alpha})}{\sqrt{2 \cdot \frac{2}{3}\tilde{c}}} + \frac{1}{2} \cdot |\frac{3}{2}E| \cdot \left| \frac{B}{\sqrt{A} \cdot \sqrt{\frac{1}{2}A} \cdot (\sqrt{\frac{1}{2}A} + \sqrt{A})} \right| \\
&\leq b \cdot |T-t|^{2\alpha} \leq b \cdot |T-t|^{\alpha}, \text{ since } |T-t| < 1.
\end{aligned}
$$

By (2.45), (2.46), (2.47), and $\phi''(t) \sim 1$ near 0, for any $t \in B_r$ there exists a universal $c > 0$ such that

$$
|\phi''(t) \cdot \phi(t) - \frac{1}{2}\phi'(t)^2| = O(|t-T|^{2+2\alpha}),
$$

$$
|g''(t)| = \frac{1}{2} \left| \frac{\phi''(t) \cdot \phi(t) - \frac{1}{2}\phi'(t)^2}{\phi(t)^{3/2}} \right| \leq \frac{1}{2} \left| \frac{O(|t-T|^{2+2\alpha})}{(\frac{1}{2}A)^{3/2}} \right| \leq c \cdot |T-t|^{2\alpha-1}. \tag{2.50}
$$

Let $t_1, t_2 \in B_r$, $t_1 < t_2$.

If $t_1 = T$ ($t_2 = T$ is similar), then $|g'(t_1) - g'(t_2)| \leq b \cdot |t_1 - t_2|^{\alpha}$ by (2.49).

If $T < t_1 < t_2$ ($t_1 < t_2 < T$ is similar) and $|t_2 - t_1| \geq \frac{1}{2}|t_1 - T|$, then

$$
|t_2 - T| \leq |t_2 - t_1| + |t_1 - T| \leq |t_2 - t_1| + 2|t_2 - t_1| = 3|t_2 - t_1|. \tag{2.51}
$$

By (2.49) and (2.51)

$$
\begin{aligned}
|g'(t_1) - g'(t_2)| &\leq |g'(t_1) - g'(T)| + |g'(T) - g'(t_2)| \leq b \cdot |t_1 - T|^{\alpha} + b \cdot |T - t_2|^{\alpha} \\
&\leq b \cdot (2|t_2 - t_1|)^{\alpha} + b \cdot (3|t_2 - t_1|)^{\alpha} \leq 5b \cdot |t_2 - t_1|^{\alpha}. \qquad (2.52)
\end{aligned}
$$

If $T < t_1 < t_2$ ($t_1 < t_2 < T$ is similar) and $|t_2 - t_1| < \frac{1}{2}|t_1 - T|$, then by mean value theorem and (2.50), $\exists \xi \in (t_1, t_2)$ such that

$$
\begin{aligned}
|g'(t_1) - g'(t_2)| &= |g''(\xi)||t_1 - t_2| \leq c \cdot |\xi - T|^{2\alpha-1} \cdot |t_1 - t_2| \\
&= c \cdot |\xi - T|^{\alpha} \cdot |t_1 - t_2|^{\alpha} \cdot (\frac{|t_1 - t_2|}{|\xi - T|})^{1-\alpha} \leq c \cdot |t_1 - t_2|^{\alpha}. \qquad (2.53)
\end{aligned}
$$

If $t_1 < T < t_2$, then by (2.49),

$$|g'(t_1) - g'(t_2)| \leq |g'(t_1) - g'(T)| + |g'(T) - g'(t_2)| \leq b|t_1 - T|^\alpha + b|T - t_2|^\alpha \leq 2b \cdot |t_1 - t_2|^\alpha.$$

(c)If $c_1 \leq \phi(0) < \tilde{c}$, then similar to case (i'), we have $\sqrt{\phi} \in C^{1,\alpha}((-\frac{c_1}{3C}, \frac{c_1}{3C}))$.

□

For any $z \in Q_\nu^+$, we apply the above lemma to the function $\phi(t) := \delta_\nu^{-(2+2\alpha)} \cdot f(z + t\delta_\nu)$.

Corollary 2.7. *Let $C = 1000$. For $z \in Q_\nu^+$, there is a universal constant $r_0 > 0$ and a universal small constant $c_2 > 0$ such that, for $x \in (z - r_0\delta_\nu, z + r_0\delta_\nu)$, either*

(I)

$$c_2\delta_\nu^{2+2\alpha} \leq f(x) \leq C\delta_\nu^{2+2\alpha}, \tag{2.54}$$

and $\sqrt{f(x)}$ is in $C^{1,\alpha}((z - r_0\delta_\nu, z + r_0\delta_\nu))$, or

(II)

$$c_2\delta_\nu^{2\alpha} \leq f''(x) \leq C\delta_\nu^{2\alpha}, \tag{2.55}$$

$$f(x) = f(X) + (x - X)^2 \int_0^1 f''(x + t(X - x))t\, dt, \tag{2.56}$$

where $x = X$ is the unique strict local minimum point of the function f in $(z - r_0\delta_\nu, z + r_0\delta_\nu)$.

Moreover, $g(x) := (x - X)(\int_0^1 f''(x + t(X - x))t\, dt)^{1/2}$ is in $C^{1,\alpha}((z - r_0\delta_\nu, z + r_0\delta_\nu))$.

3 Proof of sufficiency

Let $0 \leq f \in C^{2,2\alpha}(\mathbb{R})$ with $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \leq 1$.

3.1 Construction of g

$\mathcal{F} = \{x \in \mathbb{R} : f(x) = \nabla f(x) = \nabla^2 f(x) = 0\}$ is a closed set in $\mathbb{R}$. We write $\mathbb{R} \setminus \mathcal{F}$ as a countable union of disjoint open intervals, so that $\mathbb{R} \setminus \mathcal{F} = \cup_{k=1}^\infty I_k$. Note if $\exists x_0 \in I_k$ with $f(x_0) = 0$, then $f''(x_0) \neq 0$. (If $0 < \alpha \leq 1/2$, by Lemma 7.6, $|f(x_0)|$ and $|f''(x_0)|$ dominate $|f'(x_0)|$. If $1/2 < \alpha \leq 1$, by Lemma 7.7, $|f(x_0)|$ and $|f''(x_0)|$ dominate $|f'(x_0)|$ and $|f'''(x_0)|$.) For each $m, k \in \mathbb{N}$, we define $I_{k,m} = \{x \in I_k : dist(x, \mathcal{F}) > \frac{1}{m}\}$ and $B = \{x \in \mathbb{R} : f(x) = 0, f''(x) \neq 0\}$, then

Lemma 3.1. *$I_k \cap B$ is at most countable for each k, and*

$$I_k \cap B = \{\cdots x_{-2} < x_{-1} < x_0 < x_1 < x_2 \cdots\}.$$

Proof. $\forall N > 0$, we claim that $I_{k,m} \cap B \cap [-N, N]$ is finite for each $m, k \in \mathbb{N}$. Assume $I_{k,m} \cap B \cap [-N, N]$ is infinite, then $\exists x_0 \in \mathbb{R}$ such that x_0 is an accumulation point of $I_{k,m} \cap B$. So there is a sequence $\{x_n\}$ in B such that $\lim_{n\to\infty} x_n = x_0$, and $f(x_0) = \lim_{n\to\infty} f(x_n) = 0$. Note $f \geq 0$, so $f'(x_0) = 0$.

If $f''(x_0) \neq 0$, then $x = x_0$ is a strict local minimum point of f. However, by construction, near x_0 there is a point $x_1 \in B$, so that $f(x_1) = 0$, contradicting with strict local minimality.

If $f''(x_0) = 0$, then $x_0 \in \mathcal{F}$. However, $(x_0 - \frac{1}{2m}, x_0 + \frac{1}{2m}) \cap I_{k,m} = \varnothing$, contradiction.

Now since I_k is an interval and $I_{k,m} \subset I_{k,m+1}$. Points in $I_{k,m+1} \setminus I_{k,m}$ is either on the left or right of $I_{k,m}$. The points in $I_k \cap B \cap [-N, N]$ can be ordered. The lemma follows by letting $N \to \infty$. $\square$

We define the function g as follows. If $x \in \mathcal{F}$, set $g(x) := 0$. For each k, if $I_k \cap B = \varnothing$ in I_k, then define $g(x) := \sqrt{f(x)}$ for $x \in I_k$. Otherwise,

$$I_k \cap B = \{\cdots x_{-2} < x_{-1} < x_0 < x_1 < x_2 \cdots\}.$$

Define $g(x) := (-1)^i \sqrt{f(x)}$ for $x \in [x_{i-1}, x_i]$. Note that g changes sign when crossing each x_i in I_k.

3.2 C^0, C^1 regularity of g

g is continuous in each $I_k = (a_k, b_k)$. It suffices to discuss the continuity at $x_0 \in \mathcal{F}$. By Taylor expansion of f near x_0, $f(x) = O(|x-x_0|^{2+2\alpha})$, so that $|\pm\sqrt{f(x)}| = O(|x-x_0|^{1+\alpha}) \to 0$ as $x \to x_0$ and $\lim_{x\to x_0} g(x) = 0$.

We discuss the C^1 regularity of g.

Lemma 3.2. *$g \in C^1(I_k)$ for each k.*

Proof. If $I_k \cap B = \varnothing$, then $g' = \frac{f'}{2\sqrt{f}} \in C^0(I_k)$. If $I_k \cap B \neq \varnothing$, then for each $x_i \in I_k \cap B$, $x_i \in Q_\nu$ for some $\nu = \nu(x_i)$. By Corollary 2.7, only (II) holds and near x_i, f can locally be written as

$$f(x) = (x - x_i)^2 \int_0^1 f''(x + t(x_i - x))t\, dt,$$

with $\int_0^1 f''(x + t(x_i - x))t\, dt \sim \delta_\nu^{2\alpha}$. By definition of g, near x_i, $g(x) = \pm(x - x_i)(\int_0^1 f''(x + t(x_i - x))t\, dt)^{1/2}$(the sign depends only on the choice of sign of g near x_0), so that g changes sign when crossing x_i. By Corollary 2.7, g' is continuous at x_i. $\square$

The next is a key lemma to obtain uniform estiamte for g' under (1.1).

Lemma 3.3. *Assume condition (1.1) is satisfied. There exists a universal constant $C_2 > 0$ such that, for any $x_0 \in I_k$ with $x_0 \in Q_\nu$ for some $\nu = \nu(x_0)$, then*

$$|g'(x_0)| \le C_2 \delta_\nu^\alpha. \tag{3.1}$$

Proof. By Corollary 2.7, if (I) holds, then by (2.54) and (2.3),

$$|(\sqrt{f})'(x_0)| = |\frac{f'(x_0)}{2\sqrt{f(x_0)}}| \le \frac{C\delta_\nu^{1+2\alpha}}{2\sqrt{c_2\delta_\nu^{2+2\alpha}}} \lesssim \delta_\nu^\alpha.$$

If (II) holds, then for $x \in (x_0 - r_0\delta_\nu, x_0 + r_0\delta_\nu)$, $f''(x) \sim \delta_\nu^{2\alpha}$ and

$$f(x) = f(X) + (x - X)^2 \int_0^1 f''(x + t(X - x))t\, dt, \tag{3.2}$$

where $x = X$ is the unique strict local minimum point of the function f in $(x_0 - r_0\delta_\nu, x_0 + r_0\delta_\nu)$.

If $f(X) = 0$, then $g(x) = \pm(x - X)(\int_0^1 f''(x + t(X - x))t\, dt)^{1/2}$. By (2.55), local Hölder continuity of g', and $g'(X) = \sqrt{\frac{1}{2}f''(X)}$, there is universal $b > 0$ such that,

$$|g'(x)| \le |g'(X)| + b|x - X|^\alpha \le \sqrt{\frac{1}{2}C\delta_\nu^{2\alpha}} + b\delta_\nu^\alpha \le C_2\delta_\nu^\alpha, \quad \forall x \in (x_0 - r_0\delta_\nu, x_0 + r_0\delta_\nu).$$

If $f(X) \ne 0$, then by (1.1) and (2.55),

$$M \cdot (f(X))^{\frac{\alpha}{1+\alpha}} \ge f''(X) \ge c_2\delta_\nu^{2\alpha}.$$

So that (3.2) reads

$$f(x) \ge f(X) \ge (\frac{c_2}{M})^{\frac{1+\alpha}{\alpha}} \cdot \delta_\nu^{2+2\alpha}.$$

By (2.3), $f(x) \sim \delta_\nu^{2+2\alpha}$ and computation is reduced to case (I). □

Corollary 3.4. *Assume $I_k = (a_k, b_k)$, where $b_k < +\infty$. Then*

$$\lim_{x \to b_k^-} g'(x) = 0.$$

Similarly, if $a_k > -\infty$, then $\lim_{x \to a_k^+} g'(x) = 0$.

Proof. By Corollary 2.7, for each $x \in I_k$, $(x - r_0\delta_{\nu(x)}, x + r_0\delta_{\nu(x)}) \subset I_k$. Hence $\lim_{x \to b_k^-} \delta_{\nu(x)} = 0$. By (3.1),

$$|g'(x)| \le C_2\delta_{\nu(x)}^\alpha \to 0 \text{ as } x \to b_k^-.$$

□

Corollary 3.5. *For any $x_0 \in \mathcal{F}$, $g'(x)$ is continuous at x_0, with*

$$\lim_{x \to x_0} g'(x) = g'(x_0) = 0.$$

Proof. By Taylor expansion of f near x_0, $f(x) = O(|x - x_0|^{2+2\alpha})$, so that

$$\left| \frac{g(x) - g(x_0)}{x - x_0} \right| = \left| \frac{\pm\sqrt{f(x)}}{x - x_0} \right| = O(|x - x_0|^{\alpha}) \to 0 \text{ as } x \to x_0.$$

If x_0 has a neighbourhood which is contained in $\mathcal{F}$, then the result is trivial. Otherwise, x_0 is the boundary point of some interval $I_k = (a_k, b_k)$. Without loss of generality we assume $x_0 = b_k < +\infty$.

If x_0 is discrete, then x_0 is the boundary point of two consecutive intervals I_k and I_{k+1}, with $a_k < b_k = x_0 = a_{k+1} < b_{k+1}$. By Corollary 3.4,

$$\lim_{x \to b_k^-} g'(x) = \lim_{x \to a_{k+1}^+} g'(x) = 0.$$

Otherwise, $x_0 \in [x_0, a_{k+1}] \subset \mathcal{F}$ for some a_{k+1}. By Corollary 3.4 again,

$$\lim_{x \to b_k^-} g'(x) = \lim_{x \to x_0^+} g'(x) = 0.$$

□

To summarize, $g \in C^1(\mathbb{R})$, with $|g'(x)| \leq C_2$, $\forall x \in \mathbb{R}$, since $\delta_\nu \leq 1$.

3.3 Global Hölder estimate

Let $x, y \in \mathbb{R}$ with $x \neq y$.

1. If $\exists z \in \mathbb{R} \setminus \mathcal{F}$ such that x and y are both contained in $(z - r_0\delta_{\nu(z)}, z + r_0\delta_{\nu(z)})$, then by Corollary 2.7, the Hölder estimate is trivial if case (I) holds or case (II) holds with $f(X) = 0$. If case (II) holds with $f(X) \neq 0$, then by (1.1) and (2.55),

$$M \cdot (f(X))^{\frac{\alpha}{1+\alpha}} \geq f''(X) \geq c_2 \delta_\nu^{2\alpha}.$$

   So that (2.56) reads

$$f(x) \geq f(X) \geq (\frac{c_2}{M})^{\frac{1+\alpha}{\alpha}} \cdot \delta_\nu^{2+2\alpha} := K\delta_\nu^{2+2\alpha} \text{ and } f(y) \geq K\delta_\nu^{2+2\alpha}. \tag{3.3}$$

So by (2.3), (3.3), and mean value theorem, $\exists \xi_1, \xi_2$ lying in between x, y such that

$$
\begin{aligned}
&2|(\sqrt{f})'(x) - (\sqrt{f})'(y)|/|x-y|^\alpha = |\frac{f'(x)}{\sqrt{f(x)}} - \frac{f'(y)}{\sqrt{f(y)}}|/|x-y|^\alpha \\
&\le |\frac{f'(x)}{\sqrt{f(x)}} - \frac{f'(y)}{\sqrt{f(x)}}|/|x-y|^\alpha + |\frac{f'(y)}{\sqrt{f(x)}} - \frac{f'(y)}{\sqrt{f(y)}}|/|x-y|^\alpha \\
&\le \frac{1}{\sqrt{K\delta_\nu^{2+2\alpha}}} \cdot \frac{|f''(\xi_1)||x-y|}{|x-y|^\alpha} + C\delta_\nu^{1+2\alpha} \cdot \frac{|f'(\xi_2)||x-y|}{2(K\delta_\nu^{2+2\alpha})^{3/2}|x-y|^\alpha} \\
&\le \frac{1}{\sqrt{K\delta_\nu^{2+2\alpha}}} \cdot C\delta_\nu^{2\alpha}\delta_\nu^{1-\alpha} + C\delta_\nu^{1+2\alpha} \cdot \frac{C\delta_\nu^{1+2\alpha}}{2(K\delta_\nu^{2+2\alpha})^{3/2}} \cdot \delta_\nu^{1-\alpha} = \frac{C}{\sqrt{K}} + \frac{C^2}{2K^{3/2}},
\end{aligned}
$$

which is a constant depending only on M and α.

2. Assume $\nexists z \in \mathbb{R} \setminus \mathcal{F}$ such that x and y are both contained in $(z - r_0\delta_{\nu(z)}, z + r_0\delta_{\nu(z)})$.

   (a) If $x \in \mathcal{F}$ and $y \in \mathcal{F}$, then by Corollary 3.5, $|g'(x) - g'(y)| = |0 - 0| = 0$.

   (b) If $x \notin \mathcal{F}$ and $y \in \mathcal{F}$, then $x \in Q_\nu$ for some $\nu = \nu(x)$ and $|x - y| \ge r_0\delta_\nu$. By (3.1) and Corollary 3.5,

$$|g'(x) - g'(y)| = |g'(x)| \le C_2\delta_\nu^\alpha \le \frac{C_2}{r_0^\alpha} \cdot |x-y|^\alpha.$$

   (c) If $x \notin \mathcal{F}$ and $y \notin \mathcal{F}$, then $x \in Q_{\nu(x)}$ and $x \in Q_{\nu(y)}$, with $|x - y| \ge r_0\delta_{\nu(x)}$ and $|x - y| \ge r_0\delta_{\nu(y)}$. By (3.1),

$$|g'(x) - g'(y)| \le |g'(x)| + |g'(y)| \le C_2\delta_{\nu(x)}^\alpha + C_2\delta_{\nu(y)}^\alpha \le \frac{2C_2}{r_0^\alpha} \cdot |x-y|^\alpha.$$

Remark 3.6. *In fact, if the condition (1.1) is satisfied, then the $C^{1,\alpha}$ estimate of g doesn't depend on the choice of sign of g in each interval I_k.*

4 Proof of necessity

We prove the necessity of Theorem 1.1.

Proof. Assume (1.1) doesn't hold and $f = g^2$ for some $g \in C^{1,\alpha}(\mathbb{R})$, then there is a sequence x_n in $\mathcal{A}$ such that

$$f''(x_n) \ge n f^{\frac{\alpha}{1+\alpha}}(x_n) \quad \forall n \in \mathbb{N}. \tag{4.1}$$

$f(x_n) > 0$, so $x_n \in Q_\nu$ for some $\nu = \nu(n)$.

In case (i) of Lemma 2.4, $f(x_n) \geq \tilde{c}\delta_\nu^{2+2\alpha}$ and $f''(x_n) < C\delta_\nu^{2\alpha}$. By (4.1),

$$C\delta_\nu^{2\alpha} \geq n(\tilde{c}\delta_\nu^{2+2\alpha})^{\frac{\alpha}{1+\alpha}}, \tag{4.2}$$

so that δ_ν get cancelled. Letting $n \to \infty$ in (4.2), contradiction.

In case (ii) of Lemma 2.4, $f(x_n) < \tilde{c}\delta_\nu^{2+2\alpha}$ and $f''(x_n) \geq \tilde{c}\delta_\nu^{2\alpha}$. Define $s_n = \sqrt{\frac{f(x_n)}{f''(x_n)}}$. By (4.1) and $\delta_\nu \leq 1$,

$$s_n = \sqrt{\frac{f(x_n)}{f''(x_n)}} \leq \sqrt{\frac{f(x_n)}{nf^{\frac{\alpha}{1+\alpha}}(x_n)}} = \frac{f^{\frac{1}{2+2\alpha}}(x_n)}{\sqrt{n}} \leq \tilde{c}^{\frac{1}{2+2\alpha}} \cdot \frac{\delta_{\nu(n)}}{\sqrt{n}} \to 0 \text{ as } n \to \infty. \tag{4.3}$$

If $1/2 < \alpha \leq 1$, by Taylor expansion and $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \leq 1$,

$$f(x_n + s_n) \geq f(x_n) + \frac{1}{2}f''(x_n)s_n^2 - \frac{1}{6}|f'''(x_n)|s_n^3 - \frac{1}{6}s_n^{2+2\alpha}.$$

$$f(x_n + s_n) \leq f(x_n) + \frac{1}{2}f''(x_n)s_n^2 + \frac{1}{6}|f'''(x_n)|s_n^3 + \frac{1}{6}s_n^{2+2\alpha}.$$

By (2.3) and (4.3),

$$|f'''(x_n)|s_n^3 = |f'''(x_n)|s_n \cdot \frac{f(x_n)}{f''(x_n)} \leq C\delta_\nu^{2+2\alpha-3} \cdot \tilde{c}^{\frac{1}{2+2\alpha}} \cdot \frac{\delta_\nu}{\sqrt{n}} \cdot \frac{f(x_n)}{\tilde{c}\delta_\nu^{2\alpha}} \leq \frac{1}{2}f(x_n) \text{ for large } n,$$

$$s_n^{2+2\alpha} = s_n^{2\alpha} \cdot \frac{f(x_n)}{f''(x_n)} \leq (\tilde{c}^{\frac{1}{2+2\alpha}} \cdot \frac{\delta_\nu}{\sqrt{n}})^{2\alpha} \cdot \frac{f(x_n)}{\tilde{c}\delta_\nu^{2\alpha}} \leq \frac{1}{2}f(x_n) \text{ for large } n.$$

So $4f(x_n) \geq f(x_n + s_n) \geq f(x_n) > 0$ for large n. By mean value theorem,

$$\begin{aligned}
2|g'(x_n + s_n) - g'(x_n)| &= |\pm\frac{f'(x_n + s_n)}{\sqrt{f(x_n + s_n)}} - (\pm\frac{f'(x_n)}{\sqrt{f(x_n)}})| \\
&= |\frac{f'(x_n + s_n) - f'(x_n)}{\sqrt{f(x_n + s_n)}}| = \frac{|f''(\xi_n)| \cdot s_n}{\sqrt{f(x_n + s_n)}},
\end{aligned}$$

where $\xi_n \in (x_n, x_n + s_n)$. By Taylor expansion of f'', for large n,

$$f''(\xi_n) \geq f''(x_n) - |f'''(x_n)|s_n - s_n^{2\alpha} \geq \frac{1}{2}f''(x_n).$$

If $0 < \alpha \leq 1/2$, by expansion to the second order and $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \leq 1$, we have

$$f(x_n) + \frac{1}{2}f''(x_n)s_n^2 + \frac{1}{2}s_n^{2+2\alpha} \geq f(x_n + s_n) \geq f(x_n) + \frac{1}{2}f''(x_n)s_n^2 - \frac{1}{2}s_n^{2+2\alpha}.$$

By (4.3),

$$s_n^{2+2\alpha} = s_n^{2\alpha} \cdot \frac{f(x_n)}{f''(x_n)} \leq (\tilde{c}^{\frac{1}{2+2\alpha}} \cdot \frac{\delta_\nu}{\sqrt{n}})^{2\alpha} \cdot \frac{f(x_n)}{\tilde{c}\delta_\nu^{2\alpha}} \leq \frac{1}{2} f(x_n) \text{ for large } n.$$

So $4f(x_n) \geq f(x_n + s_n) \geq f(x_n) > 0$ for large n. By mean value theorem,

$$\begin{aligned}
2|g'(x_n + s_n) - g'(x_n)| &= | \pm \frac{f'(x_n + s_n)}{\sqrt{f(x_n + s_n)}} - (\pm \frac{f'(x_n)}{\sqrt{f(x_n)}})| \\
&= |\frac{f'(x_n + s_n) - f'(x_n)}{\sqrt{f(x_n + s_n)}}| = \frac{|f''(\xi_n)| \cdot s_n}{\sqrt{f(x_n + s_n)}},
\end{aligned}$$

where $\xi_n \in (x_n, x_n + s_n)$. By $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \leq 1$, for large n,

$$f''(\xi_n) \geq f''(x_n) - s_n^{2\alpha} \geq \frac{1}{2} f''(x_n).$$

Therefore, for any $0 < \alpha \leq 1$, by (4.1),

$$\begin{aligned}
2|g'(x_n + s_n) - g'(x_n)| &= \frac{|f''(\xi_n)| \cdot s_n}{\sqrt{f(x_n + s_n)}} \geq \frac{\frac{1}{2} f''(x_n) \cdot s_n}{\sqrt{4f(x_n)}} = \frac{1}{4}\sqrt{f''(x_n)} \\
&= \frac{1}{4} s_n^\alpha \cdot \frac{f''(x_n)^{1/2}}{s_n^\alpha} = \frac{1}{4} s_n^\alpha \cdot \left(\frac{f''(x_n)}{f(x_n)^\alpha} \cdot f''(x_n)^\alpha \right)^{1/2} \\
&= \frac{1}{4} s_n^\alpha \cdot \left(\frac{f''(x_n)^{1+\alpha}}{f(x_n)^\alpha} \right)^{1/2} \geq \frac{1}{4} s_n^\alpha \cdot \sqrt{n}.
\end{aligned}$$

Hence

$$|g'(x_n + s_n) - g'(x_n)|/s_n^\alpha \geq \frac{1}{8}\sqrt{n} \to \infty \text{ as } n \to \infty.$$

Contradiction. □

5 A sum of squares theorem

Theorem 5.1. *Let $0 \leq f \in C^{2,2\alpha}(\mathbb{R})$ with $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \leq 1$. $0 < \alpha \leq 1$. Then there is a positive integer N, depending only on α, and there are functions $g_1, \cdots, g_N$ with $\|g_j\|_{C^{1,\alpha}(\mathbb{R})} \leq C$, such that $f = \sum_{j=1}^N g_j^2$, where $C > 0$ is a universal constant, depending only on α.*

Proof. The proof exactly follows Guan [6], with slightly more details. We prove the case $\alpha > 1/2$, and $\alpha \leq 1/2$ is the same.

By Corollary 2.7, for any $z \in Q_\nu^+$,

$$f = g_{1,z}^2 + g_{2,z}^2 \quad \text{in } (z - r_0\delta_\nu, z + r_0\delta_\nu),$$

with $g_{i,z} \in C^{1,\alpha}((z - r_0\delta_\nu, z + r_0\delta_\nu))$, and $|g_{i,z}| \lesssim \delta_\nu^{1+\alpha}$, $|g'_{i,z}| \lesssim \delta_\nu^\alpha$ for $i = 1, 2$.

Choosing a (finite) partition of unity for Q_ν^+, denoted by $\{\rho_{\nu,z_j}, j = 1, \cdots, \tilde{N}(\alpha)\}$, such that

$$\sum_{j=1}^{\tilde{N}(\alpha)} \rho_{\nu,z_j}^2 = 1 \text{ in } Q_\nu^+, \text{ with } \operatorname{supp}(\rho_{\nu,z_j}) \subset (z_j - r_0\delta_\nu, z_j + r_0\delta_\nu),$$

we can write

$$f = f \cdot (\sum_{j=1}^{\tilde{N}(\alpha)} \rho_{\nu,z_j}^2) = \sum_{j=1}^{\tilde{N}(\alpha)} (g_{1,z_j} \cdot \rho_{\nu,z_j})^2 + (g_{2,z_j} \cdot \rho_{\nu,z_j})^2 =: \sum_{j=1}^{N} g_{\nu,j}^2 \text{ in } Q_\nu^+, \tag{5.1}$$

where $N := 2\tilde{N}(\alpha)$. Then $g_{\nu,j} \in C^{1,\alpha}(Q_\nu^+)$, with $|g_{\nu,j}| \lesssim \delta_\nu^{1+\alpha}$, $|g'_{\nu,j}| \lesssim \delta_\nu^\alpha$.

By (2.3) and (2.8), if the boundaries of Q_ν and $Q_{\nu'}$ touch, then

$$\sum_{k=0}^{3} (\frac{\delta_\nu}{\delta_{\nu'}})^{2+2\alpha-k} \geq \frac{c_0}{C}, \tag{5.2}$$

which implies

$$\frac{\delta_\nu}{\delta_{\nu'}} > \lambda. \tag{5.3}$$

Similarly, $\frac{\delta_\nu}{\delta_{\nu'}} < 1/\lambda$. Therefore, we conclude, for each Q_ν^+, there are only finite (universal) number of $Q_{\nu'}^+$ intersecting with it. Let $\tilde{\rho}(x) \in C_0^\infty(\mathbb{R})$, $\tilde{\rho}(x)$ such that $\tilde{\rho}(x) \geq 0$, $\tilde{\rho}(x) = 0$ outside of the cube $(1+\lambda)Q$, where Q is the unit cube in $\mathbb{R}^n$ centered at the origin. Define $\tilde{\rho}_\nu(x) = \rho(\frac{x - x_\nu}{\delta_\nu})$, where x_ν is the center of Q_ν. Let $\rho_\nu^2 = \frac{\tilde{\rho}_\nu^2}{\sum_\nu \tilde{\rho}_\nu^2}$. We produce a partition of unity ρ_ν^2 of $\cup_\nu Q_\nu^+$, such that

$$\sum_\nu \rho_\nu^2 = 1 \text{ in } \cup_\nu Q_\nu^+, \text{ and } \operatorname{supp}\rho_\nu \subset Q_\nu^+ \subset Q_\nu^*.$$

We will group Q_ν. Let $\mathcal{A}_j$ be the collection of cubes Q_ν with diameter $\delta_j = \frac{1}{2^j}$. Choose $m > 0$ large enough such that $\frac{1}{2^m} < (\frac{1}{10^5})^{1/(2\alpha-1)}$ (If $0 < \alpha \leq 1/2$, choose $m > 0$ such that $\frac{1}{2^m} < (\frac{1}{10^5})^{1/(2\alpha)}$). By (2.3) and (2.8), if $Q_{\nu_1} \in \mathcal{A}_{j_1}$, $Q_{\nu_1} \in \mathcal{A}_{j_2}$ with $|j_1 - j_2| \geq m$, then

$$Q_{\nu_1}^* \cap Q_{\nu_2}^* = \varnothing. \tag{5.4}$$

For any j, we can divide $\mathcal{A}_j$ into two collections

$$\mathcal{A}_j = \mathcal{A}_j^1 \cup \mathcal{A}_j^2$$

such that if $Q_{\nu_1}, Q_{\nu_2} \in \mathcal{A}_j^k$, $k \in \{1, 2\}$ and if $Q_{\nu_1} \neq Q_{\nu_2}$, then

$$Q_{\nu_1}^* \cap Q_{\nu_2}^* = \varnothing. \tag{5.5}$$

Let

$$\mathcal{B}_1^k = \bigcup_{j=1}^{\infty} \mathcal{A}_{1+mj}^k, \cdots, \mathcal{B}_m^k = \bigcup_{j=1}^{\infty} \mathcal{A}_{m+mj}^k, \quad k = 1, 2.$$

By (5.4) and (5.5), $\forall Q_{\nu_1}, Q_{\nu_2} \in \mathcal{B}_i^k$, if $Q_{\nu_1} \neq Q_{\nu_2}$, then

$$Q_{\nu_1}^* \cap Q_{\nu_2}^* = \varnothing. \tag{5.6}$$

Therefore, by (5.1) and (5.6),

$$\begin{aligned} f(x) = \sum_{\nu} \rho_\nu^2 f(x) &= \sum_{i=1}^{m} \sum_{k=1}^{2} \left(\sum_{Q_\nu \in \mathcal{B}_i^k} \rho_\nu^2 f(x) \right) \\ &= \sum_{i=1}^{m} \sum_{k=1}^{2} \left(\sum_{Q_\nu \in \mathcal{B}_i^k} \rho_\nu^2 (\sum_{j=1}^{N} g_{\nu,j}^2) \right) \\ &= \sum_{i=1}^{m} \sum_{k=1}^{2} \sum_{j=1}^{N} \left(\sum_{Q_\nu \in \mathcal{B}_i^k} (\rho_\nu g_{\nu,j})^2 \right) \\ &= \sum_{i=1}^{m} \sum_{k=1}^{2} \sum_{j=1}^{N} \left(\sum_{Q_\nu \in \mathcal{B}_i^k} \rho_\nu g_{\nu,j} \right)^2 . \end{aligned}$$

$\sum_{Q_\nu \in \mathcal{B}_i^k} \rho_\nu g_{\nu,j}$ is in $C^{1,\alpha}(\mathbb{R})$ for fixed i, j, k. The result follows. □

6 Conclusion

Starting with a Calderón-Zygmund decomposition, we proved an adapted version of Fefferman-Phong's lemma, which shows the local $C^{1,\alpha}$ regularity of admissible square root. The sufficiency is given by a globalization argument based on this local lemma, and the sum of squares theorem is a natural consequence. With a proper choice of the step s_n, the necessity of the main theorem 1.1 is established.

We are wondering if such Fefferman-Phong type lemma also exists for $C^{4,\alpha}$ non-negative functions in $\mathbb{R}$, where $0 < \alpha \leq 1$. Then we have to consider the case where the fourth order derivative dominates. This might be a totally different problem, since we don't have a nice theory for non-negative polynomials of degree four. We are also seeking for applications of the main theorem 1.1, especially in isometric embedding problems.

7 Appendix

Definition 7.1 (Big-oh notation)**.** We write

$$f = O(g) \quad \text{as } x \to x_0,$$

if there exists a constant $C > 0$ such that

$$|f(x)| \leq C|g(x)|$$

for all x sufficiently close to x_0.

Definition 7.2 (Vinogradov notation)**.** We write $X \lesssim Y$ if there exists a constant $C > 0$ such that

$$X \leq C \cdot Y.$$

We write $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$.

Definition 7.3. Let $k \geq 0$ be an integer and $0 < \alpha \leq 1$. We say a function $f : \Omega \subset \mathbb{R} \to \mathbb{R}$ is in $C^k(\Omega)$ if $f', f'', \cdots, f^{(k)}$ all exist and are continuous over Ω.

The Hölder space $C^{k,\alpha}(\Omega)$ consists of all functions $f \in C^k(\Omega)$ for which the norm

$$\begin{aligned} \|f\|_{C^{k,\alpha}(\Omega)} &\coloneqq \left(\sum_{i=1}^{k} \|f\|_{C^i(\Omega)} \right) + [f]_{C^{k,\alpha}(\Omega)} \\ &\coloneqq \left(\sum_{i=1}^{k} \sup_{x \in \Omega} |f^{(i)}(x)| \right) + \sup_{x,y \in \Omega, x \neq y} \frac{|f^{(k)}(x) - f^{(k)}(y)|}{|x-y|^\alpha} \end{aligned}$$

is finite.

Theorem 7.4 (Mean Value theorem)**.** *If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a number ξ in (a, b) such that*

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 7.5 (Taylor's theorem)**.** *Suppose that $f', f'', \cdots, f^{(n+1)}$ are defined on $[a, x]$, and that $R_{n,a}(x)$ is defined by*

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n,a}(x).$$

Then

$$R_{n,a}(x) = \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n(x-a) \text{ for some } \xi \text{ in } (a,x).$$
$$R_{n,a}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \text{ for some } \xi \text{ in } (a,x).$$

Moreover, if $f^{(n+1)}$ is integrable on $[a, x]$, then

$$R_{n,a}(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n \, dt.$$

The following two lemmas follow Tataru [16].

Lemma 7.6 (Even dominate odd, $C^{2,\alpha}$)**.** *Let $0 < \alpha \leq 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2 non-negative function such that $[f]_{C^{2,\alpha}(\mathbb{R})} \leq 1$. Then*

$$|f'(x)| \leq \frac{3}{2}|f(x)|^{\frac{1+\alpha}{2+\alpha}} + \frac{1}{2}|f''(x)| \cdot f(x)^{\frac{1}{2+\alpha}} + f(x)^{\frac{\alpha}{2+\alpha}} \cdot |f''(x)|^{\frac{1}{\alpha}} \quad \forall x \in \mathbb{R}. \tag{7.1}$$

Proof. If $f(x) = 0$, then $f'(x) = 0$ since $f \geq 0$. Otherwise, by Taylor expansion, $\forall x \in \mathbb{R}$,

$$\begin{aligned} 0 \leq f(x+h) &= f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{2}\frac{f''(\xi) - f''(x)}{|\xi - x|^\alpha}|\xi - x|^\alpha h^2 \\ &\leq f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{2}|h|^{2+\alpha}. \end{aligned}$$

So by replacing h with $\pm h$,

$$|f'(x)h| \leq f(x) + \frac{1}{2}|f''(x)|h^2 + \frac{1}{2}|h|^{2+\alpha}. \tag{7.2}$$

Setting $h = \frac{f(x)^{\frac{2}{2+\alpha}}}{f(x)^{\frac{1}{2+\alpha}} + |f''(x)|^{\frac{1}{\alpha}}}$ in (7.2) and using $h \leq f(x)^{\frac{1}{2+\alpha}}$, we obtain (7.1).

□

Lemma 7.7 (Even dominate odd, $C^{3,\alpha}$)**.** *Let $0 < \alpha \leq 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^3 non-negative* *function such that* $[f]_{C^{3,\alpha}(\mathbb{R})} \le 1$. *Then*

$$|f'(x)| \le \frac{13}{6} f(x)^{\frac{2+\alpha}{3+\alpha}} + \frac{3}{2} f(x)^{\frac{1+\alpha}{3+\alpha}} \cdot |f''(x)|^{\frac{1}{1+\alpha}} + f(x)^{\frac{1}{3+\alpha}} \cdot |f''(x)|, \quad \forall x \in \mathbb{R}. \tag{7.3}$$

$$|f'''(x)| \le 6 f(x)^{\frac{\alpha}{3+\alpha}} + 6|f''(x)|^{\frac{\alpha}{1+\alpha}} \quad \forall x \in \mathbb{R}. \tag{7.4}$$

Proof. By Taylor expansion, $\forall x \in \mathbb{R}$,

$$0 \le f(x+h) \le f(x) + f'(x)h + \frac{1}{2} f''(x)h^2 + \frac{1}{6} f'''(x)h^3 + \frac{1}{6}|h|^{3+\alpha}. \tag{7.5}$$

So by replacing h with $\pm h$,

$$|f'(x)h + \frac{1}{6} f'''(x)h^3| \le f(x) + \frac{1}{2}|f''(x)|h^2 + \frac{1}{6}|h|^{3+\alpha} =: A. \tag{7.6}$$

Replacing h by $2h$ in (7.6), we have

$$|2 \cdot f'(x)h + 8 \cdot \frac{1}{6} f'''(x)h^3| \le f(x) + \frac{1}{2}|f''(x)|(2h)^2 + \frac{1}{6}|2h|^{3+\alpha} =: B. \tag{7.7}$$

Combining (7.6) and (7.7), we have

$$|f'(x)h| \le \frac{8A+B}{6}, \tag{7.8}$$

$$|\frac{1}{6} f'''(x)h^3| \le \frac{2A+B}{6}. \tag{7.9}$$

If $f(x) = 0$, then $f'(x) = 0$ since $f \ge 0$. Otherwise, setting $h = \frac{f(x)^{\frac{2}{3+\alpha}}}{f(x)^{\frac{1}{3+\alpha}} + |f''(x)|^{\frac{1}{1+\alpha}}}$ in (7.8) and using $h \le f(x)^{\frac{1}{3+\alpha}}$, we have

$$\begin{aligned} |f'(x)| &\le \frac{1}{6}\left(9 \cdot \frac{f(x)}{h} + 6 \cdot |f''(x)|h + 4 \cdot |h|^{2+\alpha}\right) \\ &\le \frac{1}{6}\left(9 \cdot f(x)^{\frac{1+\alpha}{3+\alpha}}(f(x)^{\frac{1}{3+\alpha}} + |f''(x)|^{\frac{1}{1+\alpha}}) + 6 \cdot |f''(x)| \cdot f(x)^{\frac{1}{3+\alpha}} + 4 \cdot f(x)^{\frac{2+\alpha}{3+\alpha}}\right). \end{aligned}$$

Thus (7.3) holds.

If $f(x) = f''(x) = 0$, then $f'''(x) = 0$ by (7.5).

Otherwise, setting h to be $\max\{f(x)^{\frac{1}{3+\alpha}}, |f''(x)|^{\frac{1}{1+\alpha}}\}$ and using $\max\{a,b\} \le a+b$ in (7.9), we obtain

$$
\begin{aligned}
|f'''(x)| &\leq \frac{3f(x)}{h^3} + \frac{3|f''(x)|}{h} + (\frac{1}{3} + \frac{1}{6} \cdot 2^{3+\alpha}) \cdot |h|^\alpha \\
&\leq 3f(x)^{\frac{\alpha}{3+\alpha}} + 3|f''(x)|^{\frac{\alpha}{1+\alpha}} + 3 \cdot \left| f(x)^{\frac{1}{3+\alpha}} + |f''(x)|^{\frac{1}{1+\alpha}} \right|^\alpha \\
&\leq 3f(x)^{\frac{\alpha}{3+\alpha}} + 3|f''(x)|^{\frac{\alpha}{1+\alpha}} + 3 \cdot \left(f(x)^{\frac{\alpha}{3+\alpha}} + |f''(x)|^{\frac{\alpha}{1+\alpha}} \right).
\end{aligned}
$$

(Note $(a+b)^\alpha \leq a^\alpha + b^\alpha$ for $a, b \geq 0$ and $0 < \alpha \leq 1$.) Thus (7.4) holds. $\square$

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