Groups Elementary Equivalent to a Free 2-Nilpotent Group of Arbitrary Finite Rank

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Abstract

In this thesis we discuss the characterization of groups elementary equivalent to a free 2-nilpotent group G of arbitrary finite rank. We find a characterization and verify it using two different lines of argument. The first one goes through using a construction very similar to the famous Mal'cev correspondence. This strategy is very much in the same spirit as the work of O. V. Belegradek on unitriangular groups. The second method, we call the method of bilinear mappings, is due to Alexei Miasnikov. A bilinear map f_G is associated to the nilpotent group G. Then a commutative associative ring $P(f_G)$ is recovered via the bilinear mapping f_G. This ring is the maximal ring relative to which f_G remains bilinear. Under some reasonable conditions the ring $P(f_G)$ is absolutely interpretable in G. Then we use this construction to give a second proof for the characterization.

Résumé

Dans ce mémoire nous trouvons une caractérisation des groupes équivalent élémentaires à un groupe, G, 2-nilpotent libre de type fini que nous prouvons de deux manières différentes. le premier argument utilise une construction similaire à la correspondance de Mal'cev. Cette stratégie est dans la même ligne de pensée qu' O.V. Belegradek utilisa pour ses travaux sur les groupes de matrices unitriangulaires. Le second argument, que nous appelons la méthode des fonctions bilinéaires est, dû à Alexei Miasnikov. Nous faisons correspondre à chaque groupe nilpotent G une application bilinéaire f_G. Un anneau commutatif et associatif $P(f_G)$ est construit à l'aide de l'application bilinéaire f_G. Cet anneau a la propriété d'être l'anneau maximal pour lequel f_G dememre bilinéaire. Sous certaines hypothèses raisonnables, l'anneau $P(f_G)$ est interprétable absolument dans G. Finalement, nous utilisons cette construction pour donner une nouvelle preuve de la caractérisation de départ.

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My wife, Roya has been the special one to me. Without her support, patience, kindness and understanding every thing would be much harder for me.

If this work is worth dedicating, it is dedicated to the memory of my uncle, Mr. Yadollah Sohrabi. His life gave a lot more meaning to my life, his loss the deepest sorrow and sadness.

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Contents

Introduction

0.1 A bit of history of model theory of groups

Here we survey some results in the model theory of groups. Since a comprehensive survey goes well beyond the scope of this work we only discuss those closely related to the content of this thesis.

There are a number of problems concerning a class $\mathfrak{C}$ of groups, considered to be the most important. Among them two are of special interest to us:

- Classification of groups in $\mathfrak{C}$ up to elementary equivalence,
- Characterization of groups elementary equivalent to a given group in class $\mathfrak{C}$.

Elementary theory of a structure is the set of first order sentences true in the structure. Two structures are said to be elementary equivalent if they have the same elementary theories.

The class of abelian groups and some of its subclasses attracted a lot of attention from model theorists. Tarski worked out the classification of free abelian groups of finite rank up to elementary equivalence. He proved that two such groups are elementary equivalent if an only if they have the same rank. Following Tarski's work W. Szmielew [17] gave a complete classification of abelian groups up to elementary equivalence. Eklof and Fischer [5] gave another proof for the problem. The model theory of abelian groups is a very sophisticated subject now.

Mal'cev [8] did the pioneering work in the model theory of nilpotent groups. He studied a correspondence between rings with unit and the group of 3×3 upper unitriangular matrices, $UT_3(R)$. He proved that the ring R is interpretable in an enrichment of $UT_3(R)$. Using the same method he showed that the ring of integers is interpretable in a non-abelian free nilpotent group. Thus he proved that the theory of non-abelian free nilpotent groups is undecidable . Ershov [6] used similar interpretations to extend the Mal'cev result on non-abelian free nilpotent groups to non abelian finitely generated nilpotent groups. We will see how Mal'cev's work influenced the study of the problems formulated above for the class of nilpotent groups.

Like the case of abelian groups, classification of finitely generated nilpotent groups up to elementary equivalence has a complete solution now. Kargapolov conjectured that two finitely generated nilpotent groups are elementary equivalent if and only if they are isomorphic. Zilber refuted Kargapolov's conjecture. Miasnikov [9] proved that if G and H are elementary equivalent finitely generated nilpotent groups such that the center of G sits inside the commutator subgroup of G then Kargapolov's conjecture holds. It was F. Oger [14] who came up with the final solution. He proved two finitely generated nilpotent groups G and H are elementary equivalent if and only if $G \times \mathbb{Z}$ and $H \times \mathbb{Z}$ are isomorphic, when $\mathbb{Z}$ is an infinite cyclic group.

There are also some recent advances in the model theory of free groups and hyperbolic groups. Kharlampovich and Miasnikov [7] proved that the elementary theory of free groups is decidable. Kharlampovich and Miasnikov and Sela independently proved that two free non-abelian groups of finite rank are elementary equivalent. Sela also announced the classification of torsion free hyperbolic groups up to elementary equivalence[1].

The model theory of the class of unitriangular groups UT_n has a rich history itself (see Belegradek [3]). $UT_n(R)$ is the group of $n \times n$ all upper unitriangular

[1]Most of the events mentioned in the above paragraphs came to my attention through a conversation with the thesis supervisor, Professor Alexei Miasnikov. Unfortunately I wasn't able to find the corresponding references for some of them.

matrices over a ring R with unit. UT_n is the class of all $UT_n(R)$ groups, when R runs through the class of all rings with unit. Belegradek [1], influenced by Mal'cev's idea mentioned above, proved that the class UT_3 is not axiomatizable. He gave a characterization for groups elementary equivalent to a UT_3 group. Later on he [2] extended Mal'cev's ideas to the case of UT_n and proved that this class is not axiomatizable for any n and gave a characterization for groups elementary equivalent to a UT_n group.

Since the present work is influenced by that of Belegradek's on UT_3 groups we explain his work in more detail. Let R be a ring with unit. The group $UT_3(R)$ is isomorphic to the group of triples (α, β, γ), $\alpha, \beta, \gamma \in R$ with the multiplication:

$$(\alpha, \beta, \gamma)(\alpha', \beta', \gamma') = (\alpha + \alpha', \beta + \beta', \gamma + \gamma' + \alpha\beta').$$

It was Mal'cev who proved that ring R is interpretable in the enriched group $(UT_3(R), e_1, e_2)$ when $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$. Let $f_1, f_2 : R^+ \times R^+ \to R$ be two symmetric 2-cocycles from the additive group R^+ of R into itself. Now a new multiplication on UT_3 can be defined by

$$(\alpha, \beta, \gamma) \odot (\alpha', \beta', \gamma') = (\alpha + \alpha', \beta + \beta', \gamma + \gamma' + \alpha\beta' + f_1(\alpha, \alpha') + f_2(\beta, \beta')).$$

The new group is called a quasiunitriangular group over R and denoted by $UT_3(R, f_1, f_2)$. The class $QUT_3(\mathcal{R})$ when $\mathcal{R}$ is a class rings with unit, consists of all groups $UT_3(R, f_1, f_2)$ when R runs through the class $\mathcal{R}$ and f_1, f_2 are arbitrary symmetric 2-cocycles described above. If $\mathcal{R}$ is the class of all rings with unit then $QUT_3(\mathcal{R})$ is denoted by QUT_3. Belegradek proved that the class UT_3 is not axiomatizable but the class QUT_3 is so. He specified the exact subclass of QUT_3 which is the elementary closure of UT_3.

0.2 The present work

The group $UT_3(\mathbb{Z})$ is of special interest to us since it is a free 2-nilpotent group of rank 2. Belegradek constructed a group H which is elementary equivalent to

$UT_3(\mathbb{Z})$ and not a UT_3 group. However H is a QUT_3 group. Belegradek actually gave a better description of H. It is isomorphic to some $UT_3(R, f_1, f_2)$ for a ring R which is elementary equivalent to $\mathbb{Z}$.

It is time now to describe the results in this thesis. We give a characterization of groups elementary equivalent to a free 2-nilpotent of arbitrary finite rank. Our work is mostly, except in the last chapter, in the same spirit as that of Belegradek's for UT_3 groups. Let R be a ring with unit. Consider the set $N_{2,n}(R)$ of all $n+\frac{n(n+1)}{2}$ tuples $((\alpha_i)_{1\leq i\leq n}, (\gamma_{1\leq i<j\leq n}))$ of elements of R. Define a multiplication

$$\begin{aligned} xy &= (((\alpha_i)_{1\leq i\leq n}, (\gamma_{ij})_{1\leq i<j\leq n})((\beta_i)_{1\leq i\leq n}, (\gamma')_{1\leq i<j\leq n}) \\ &= ((\alpha_i + \beta_i)_{1\leq i\leq n}, (\gamma_{ij} + \gamma'_{ij} + \alpha_i\beta_j)_{1\leq i<j\leq n}). \end{aligned}$$

The set $N_{2,n}(R)$ is a group under the multiplication defined above. Our main interest in $N_{2,n}$ groups is that $N_{2,n}(\mathbb{Z})$ is a free 2-nilpotent group of rank n. We notice that $N_{2,2}(R)$ is isomorphic to $UT_3(R)$. Let $f^i : R^+ \times R^+ \times R^{\binom{n}{2}}$, $1 \leq i \leq n$, be some symmetric 2-cocycles. Each $f^i = (f^i_{ls})_{1\leq l<s\leq n}$. Note that each $f^i_{ls} : R^+ \times R^+ \to R$, $1 \leq l < s \leq n$, is also a symmetric 2-cocycle. Now define a new multiplication on the set $N_{2,n}(R)$ by

$$\begin{aligned} xy &= ((\alpha_i)_{1\leq i\leq n}, (\gamma_{ij})_{1\leq i<j\leq n}) \odot ((\beta_i)_{1\leq i\leq n}, (\gamma'_{ij})_{1\leq i<j\leq n}) \\ &= ((\alpha_i + \beta_i)_{1\leq i<j\leq n}, (\gamma_{ij} + \gamma'_{ij} + \alpha_i\beta_j + \sum_{k=1}^{n} f^k_{ij}(\alpha_k, \beta_k))_{1\leq i<j\leq n}). \end{aligned}$$

We denote an isomorphic copy of this group by $N_{2,n}(R, f^1, \ldots, f^n)$ and call it a $QN_{2,n}$ group over R. We prove that the class of $QN_{2,n}$ groups over the class of all associative rings with unit is axiomatizable. We continue with proving that if H is a group elementary equivalent to a free 2-nilpotent group of rank n, then H is isomorphic to $N_{2,n}(R, f^1, \ldots, f^n)$ for a ring R elementary equivalent to $\mathbb{Z}$ and symmetric 2-cocycles described above.

Now we describe contents of the chapters. In Chapter 1 we present the basic facts and definitions from the theory of nilpotent groups, the theory of group extensions and model theory. $N_{2,n}$ and $QN_{2,n}$ groups and some related concepts are introduced in Chapter 2 and several properties of them are discussed. The central concepts of this chapter are basis and strong basis. We give a characterization theorem for the class of $QN_{2,n}$ over associative rings with unit in Section 2.3. The chapter will close with the main result, Theorem 2.4.6, of this thesis in which we characterize groups elementary equivalent to a free 2-nilpotent group of arbitrary rank.

Chapter 3 contains some elements of model theory of bilinear mappings. All the results and definitions in this chapter are taken from [11] except those in the last section. This section contains an alternate proof of Theorem 2.4.6. In this chapter we discuss the bilinear mapping f_G associated to a nilpotent group G. To every such bilinear mapping there is a maximal associative commutative ring $P(f_G)$ associated, relative to which the mapping f_G remains bilinear. We represent a proof that under some reasonable circumstances the ring $P(f_G)$ is absolutely interpretable in the group G (for the original proofs see [11] and [12]). This technique, due to Alexei Miasnikov, has already found applications in model theory of some structures other then nilpotent groups (see [10]and [13]). This technique can be considered as a general Mal'cev correspondence. Using this correspondence we give an alternate proof to Theorem 2.4.6 in Section 3.2, which will be the final section of this thesis.

Finally we fix some notations. We use "$\cong$" for isomorphism of structures and "$\equiv$" for elementary equivalence. We use the symbols, "$\wedge$", "$\rightarrow$" and "$\leftrightarrow$" for logical connectives meaning *and*, *implies* and *if and only if* respectively. The symbols "$\forall$" and "$\exists$" are intended to mean *for all* and *there exists* respectively. We use "$\Leftrightarrow$" for the meta-linguistic *if and only if*.

0.3 Future research

We intend to extend the results in this work to more general situations. Characterizing groups elementary equivalent to a free nilpotent group of arbitrary class and finite rank seems to be the next step. Then we hope to extend the results to torsion free finitely generated nilpotent groups and finally to arbitrary finitely generated groups. We already have some insight to the above problems. First of all we understand that the methods used in Chapter 2 are very hard and in some cases impossible to generalize to the new situations and they should be replaced. More clearly we shall replace the concept basis as described in chapter 2 with the so-called *Mal'cev basis* and use the method of bilinear mappings described in chapter 3. We guess a situation"similar" to that of free 2-nilpotent groups holds in general. More clearly, though still rough, if G is a finitely generated nilpotent group and H is a group such that $G \equiv H$, then H has a "ring" elementary equivalent to the "ring" of G and multiplication in H is the multiplication in G twisted by some 2-cocycles.

Chapter 1

Preliminaries

In this chapter we discuss the basic concepts and tools we'll need later. Definitions of lower and upper central series of a group and nilpotent groups are given in Section 1.1. We also discuss a bit of theory of group extensions in Section 1.2. We are only concerned with abelian and central extensions. All the relevant material can be found in the standard group theory texts such as [15] or [16]. A good reference for nilpotent groups is [18]. We also introduce model theoretic concepts and tools we use, the most important of all interpretability of one structure in an other one in Section 1.3. For general model theory the reader may refer to [4] but our approach to interpretations is that of [10].

1.1 Nilpotent groups

Let G be a group with a series of subgroups:

$$G = G_1 \trianglerighteq G_2 \trianglerighteq \ldots G_n \trianglerighteq G_{n+1} = 0,$$

where each G_{i+1} is a normal subgroup of G_i and each factor G_i/G_{i+1} is an abelian group. Let G act on each factor G_i/G_{i+1} by conjugation, i.e.

$$g.xG_{i+1} =_{df} g^{-1}xgG_{i+1}.$$

If the above action of G on all the factors is trivial then the above series is called a *central series* and any group G with such a series is called a *nilpotent* group.

1.1.1 Lower and upper central series

For elements x and y of a group G let $[x, y] = x^{-1}y^{-1}xy$. $[x, y]$ is called the *commutator* of the elements x and y. The subgroup $[G, G]$ is the subgroup of G generated by all $[x, y]$, $x, y \in G$. In general for H and K subgroups of G, $[H, K]$ is the subgroup of G generated by commutators $[x, y]$, $x \in H$ and $y \in K$. Let us define a series $\Gamma_1(G), \Gamma_2(G), \ldots$ of subgroups of G by setting

$$G = \Gamma_1(G), \quad \Gamma_{n+1}(G) = [\Gamma_n(G), G] \quad \text{for all } n > 1.$$

It can be easily checked that the above series is a central series. If c is the least number that $\Gamma_{c+1}(G) = 0$ then G is said to be a nilpotent group of *class c* or simply a *c-nilpotent* group. We call the series above the *lower central series* of the group G.

Let $Z(G)$ denote the center of a group G. We define a series of subgroups $Z_i(G)$ of G by setting

$$Z_1(G) = Z(G), \quad Z_{i+1}(G) = \{x \in G : xZ_i \in Z(G/Z_i(G))\}, \quad i > 1.$$

This series is also a central series and called the *upper central series* of the group G. If $Z_{n+1}(G) = G$ for some finite number n and c is the least such number then G is provably a c-nilpotent group.

Let $F(n)$ be the free group on n generators. Let G be a group isomorphic to the factor group $F(n)/\Gamma_{c+1}(F(n))$. Then G is called a *free nilpotent group of rank c*. In category theoretical terms the group G is a free object in the category of c-nilpotent groups over n generators. These groups are in the center of our attention in this thesis.

1.2 Central and abelian extensions

Let A and B be abelian groups. Consider the short exact sequence:

$$0 \to A \xrightarrow{\mu} E \xrightarrow{\nu} B \to 0.$$

Let $\tau : B \to E$ be a function such that $\nu \circ \tau = Id$ and $\tau(0) = 1$ when E is written multiplicatively. Such a function is called a *transversal function*. Define an action of B on A by:

$$\mu(x.a) =_{df} (\tau(x))^{-1}\mu(a)\tau(x).$$

In our case, where A is an abelian group the action is independent of the choice of the function τ. The group E is called an *abelian extension* of A by B if E is an abelian group. E is said to be a *central extension* of A by B if $\mu(A)$ sits inside the center of E, i.e. the action defined above is trivial. Obviously every abelian extension is central. It can be easily seen that every central extension of two abelian groups is a 2-nilpotent group. An extension:

$$0 \to A \xrightarrow{\mu'} E' \xrightarrow{\nu'} B \to 0$$

is *equivalent* to the extension above if there is an isomorphism $\eta : E \to E'$ such that $\nu' \circ \eta = \nu$ and $\eta \circ \mu = \mu'$. The relation "equivalence" defines an equivalence relation on the set of all central extensions of the abelian groups A and B.

We now review the relation between equivalence classes of central extensions of an abelian group A by an abelian group B and the group called the *second cohomology* group, $H^2(B, A)$, when the action of B on A, described above is trivial. Let

$$0 \to A \xrightarrow{\mu} E \xrightarrow{\nu} B \to 0$$

be a central extension. Let $\tau : B \to E$ be a transversal function such that $\tau(0) = 1$, the group E written multiplicatively. We note that for $x, y \in B$, $\tau(x + y)$ and $\tau(x)\tau(y)$ fall in the same coset so that we can define a function $f : B \times B \to A$ by setting

$$\tau(x + y) = \mu(f(x, y))\tau(x)\tau(y).$$

Actually f makes up for τ not being a group homomorphism in general. Notice that $f(0, x) = f(x, 0) = 0$ for every x in B. Moreover the associativity of addition in the group B imposes a restriction on the function f. As a result f satisfies the identity:

$$f(x + y, z) + f(x, y) = f(x, y + z) + f(y, z)$$

for x, y and z in B. When the action of B on A is trivial any function satisfying the above identity is called a *2-cocycle.*

Now two questions come into mind. First how the 2-cocycle f changes if we pick a transversal function τ' different from τ. Second, if E' and E are equivalent as central extensions of A by B how do the "corresponding" 2-cocycles differ. It turns out that answers to both questions are the same. The two 2-cocycle differ by a special kind of 2-cocycles called 2-coboundaries where a *2-coboundary* $g : B \times B \to A$ is a function defined by an identity:

$$\psi(x + y) = \mu(g(x, y))\psi(x)\psi(y)$$

when $\psi : B \to A$ is a function from B into A. Here is how it happens. Let $\tau, \tau' : B \to E$ be two transversal functions and $f, f' : B \times B \to A$ be the corresponding 2-cocycles. Functions τ and τ' both being transversal functions means that for any $x \in B$, $\nu \circ \tau(x) = x = \nu \circ \tau'(x)$. Thus $\nu(\tau(x)\tau'(x)^{-1}) = 0$. Meaning that $\tau(x)\tau'(x)^{-1} \in \mu(A)$. So we can define a function $\psi : B \to A$ by setting $\psi(x) = \mu^{-1}(\tau(x)\tau'(x)^{-1})$. It can be easily checked that the 2-coboundary $g : B \times B \to A$ arising from the function ψ is actually the difference between the 2-cocycles f and f'. We can make the set $B^2(B, A)$ of all 2-cocycles and the set $Z^2(B, A)$ of all 2-coboundaries into abelian groups by letting addition of the corresponding functions be the point-wise addition. Clearly $Z^2(B, A)$ is a subgroup of $B^2(B, A)$. Now to see why the second question above has the same answer as the first one let E and E' be two equivalent central extensions of A by B and $\eta : E \to E'$ be the group isomorphism establishing the equivalence of the two extensions. Let $\tau : B \to E$ and $\tau' : B \to E'$ be two transversal functions and $f, f' : B \times B \to A$ be two corresponding

2-cocycles respectively. We can always choose a transversal $\tau'' : B \to E'$ such that $\eta \circ \tau'' = \tau$. If $f'' : B \times B \to A$ is the 2-cocycle corresponding to τ'', f' and f'' differ by a 2-coboundary. We can easily check that f and f'' also differ by a 2-coboundary. As a result f and f' differ only by a 2-coboundary.

We have now assigned to every equivalence class of central extensions of A by B a unique element of the factor group $H^2(B,A) \cong B^2(B,A)/Z^2(B,A)$. For the converse let $f : B \times B \to A$ be a 2-cocycle. Define a group $E(f)$ by $E(f) = B \times A$ as sets with the multiplication

$$(b_1, a_1)(b_2, a_2) = (b_1 + b_2, a_1 + a_2 + f(b_1, b_2)) \quad a_1, a_2 \in A, \quad b_1, b_2 \in B.$$

The above operation is a group operation and the resulting extension is central. Moreover it can be verified that if $f, f' : B \times B \to A$ are two 2-cocycles differing only by a 2-coboundary then the extensions $E(f)$ and $E(f')$ are equivalent. Therefore there is a bijection between the equivalent classes of central extensions and elements of the group $H^2(B,A)$.

A 2-cocycle $f : B \times B \to A$ is *symmetric* if it also satisfies the identity:

$$f(x,y) = f(y,x) \quad \text{for all } x, y \in B.$$

Actually the 2-cocycle f is symmetric if and only if it arises from an abelian extension of A by B. As it can be easily imagined there is a one to one correspondence between the equivalent classes of abelian extension and the factor group $Ext(B,A) = S^2(B,A)/S^2(B,A) \cap Z^2(B,A)$. Here $S^2(B,A)$ denotes the group of symmetric 2-cocycles. Note that $Ext(B,A) \cong (Z^2(B,A) + S^2(B,A))/Z^2(B,A)$, meaning $Ext(B,A)$ is a subgroup of $H^2(B,A)$.

1.3 Structures, signatures and interpretations

1.3.1 Structures and signatures

A *structure* $\mathfrak{A}$ is an object with the following four ingredients:

1. A set of objects $|\mathfrak{U}|$ called the *universe* of the structure.

2. A set of constants from the universe of the structure each named by a *constant symbol.*

3. For each positive integer n a set of n-ary relations (predicates) on $|\mathfrak{U}|$ (subsets of the product $|\mathfrak{U}|^n$) each named by an n-ary *relation symbol* (*predicate symbol*).

4. For each positive integer n a set of n-ary functions from $|\mathfrak{U}|^n$ to $|\mathfrak{U}|$ each named by an n-ary *function symbol.*

The *signature* of the structure $\mathfrak{U}$ is given by the set of constant symbols, for each positive integer n the set of n-ary relation symbols and n-ary function symbols. Thus a structure fixes its signature uniquely. Suppose a structure and its signature are fixed. Any new constants added to the structure are called *parameters.* We usually let the parameters name themselves, i.e. we don't distinguish between the parameters as elements of the universe and parameters as symbols in the signature. The new structure obtained by adding parameters is called an *enriched* structure.

Sometimes we denote a structure $\mathfrak{U}$ by a tuple $\langle|\mathfrak{U}|, \ldots, \ldots, \ldots\rangle$. For example by $\langle\mathbb{R}, +, -, ., 0, 1\rangle$ is meant a structure whose universe is the set of real numbers $\mathbb{R}$, whose binary functions are $+$, $.$, named by the symbols $+$ and $.$ respectively. The unary function of the structure is $-$ named by $-$. It also contains 0 and 1 as constants named by 0 and 1 respectively. We call this signature the *signature of rings.* A group G is considered to be the structure $\langle|G|, ., ^{-1}, 1\rangle$ where $.$, $^{-1}$ and 1, name multiplication, inverse operation and the trivial element of the group respectively. We consider this signature as the *signature of groups.* We use $[x, y]$ as an abbreviation for $x^{-1}.y^{-1}.x.y$.

By an *algebraic structure* we mean a structure including functions only, constants aside. Strangely enough in this thesis sometimes we assume that algebraic structures consist only of predicates in addition to constant symbols. But in a sense what we mean is clear. Algebraic operations are considered as relations rather than functions.

Let $\mathfrak{A}$ be a structure and $\phi(x_1, \ldots, x_n)$ be a first order formula of the signature of $\mathfrak{A}$ with $x_1,\ldots,x_n$ free variables. Let $(a_1, \ldots, a_n) \in |\mathfrak{A}|^n$. We denote such a tuple by $\bar{a}$. The notation $\mathfrak{A} \models \phi(\bar{a})$ is intended to mean that the tuple $\bar{a}$ satisfies $\phi(\bar{x})$ when $\bar{x}$ is an abbreviation for the tuple $(x_1, \ldots, x_n)$ of variables[1].

Given a structure $\mathfrak{A}$ and a first order formula $\phi(x_1, \ldots, x_n)$ of the signature of $\mathfrak{A}$, $\phi(\mathfrak{A}^n)$ refers to $\{\bar{a} \in |\mathfrak{A}|^n : \mathfrak{A} \models \phi(\bar{a})\}$. Such a relation or set is called *first order definable without parameters.* If $\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is a first order formula of the signature of $\mathfrak{A}$ and $\bar{b}$ an m-tuple of elements of $\mathfrak{A}$ then $\psi(\mathfrak{A}^n, \bar{b})$ means $\{\bar{a} \in |\mathfrak{A}|^n : \mathfrak{A} \models \psi(\bar{a}, \bar{b})\}$. A set or relation like this is said to be *first order definable with parameters.*

Let $\mathfrak{A}$ be a structure of signature Σ. The *theory* $Th(\mathfrak{A})$ of the structure $\mathfrak{A}$ is the set:

$$\{\phi : \mathfrak{A} \models \phi, \phi \text{ a first order sentence of signature } \Sigma\}.$$

Finally two structures $\mathfrak{A}$ and $\mathfrak{B}$ of the signature Σ are *elementary equivalent* if $Th(\mathfrak{A}) = Th(\mathfrak{B})$.

1.3.2 Interpretations

Let $\mathfrak{B}$ and $\mathfrak{A}$ be algebraic structures of signatures Δ and Σ respectively not having function symbols. The structure $\mathfrak{A}$ is said to be *interpretable* in $\mathfrak{B}$ with parameters $\bar{b} \in |\mathfrak{B}|^n$ or *relatively interpretable* in $\mathfrak{B}$ if there is a set of first order formulas

$$\Psi = \{A(\bar{x}, \bar{y}), E(\bar{x}, \bar{y^1}, \bar{y^2}), \psi_\sigma(\bar{x}, \bar{y^1}, \ldots, \bar{y^{t_\sigma}}) : \sigma \text{ a predicate of signature } \Sigma\}$$

of the signature Δ such that

1. $A(\bar{b}) = \{\bar{a} \in |\mathfrak{B}|^n : \mathfrak{B} \models A(\bar{b}, \bar{a})\}$ is not empty,

2. $E(\bar{x}, \bar{y^1}, \bar{y^2})$ defines an equivalence relation $\epsilon_{\bar{b}}$ on $A(\bar{b})$,

[1]For definitions of a formula of a signature, free variables and satisfaction the reader should refer to [4].

3. if the equivalent class of a tuple of elements $\bar{a}$ from $A(\bar{b})$ modulo the equivalence relation $\epsilon_{\bar{b}}$ is denoted by $[\bar{a}]$, for every n-ary predicate σ of signature Σ, predicate P_σ is defined on $A(\bar{b})/\epsilon_{\bar{b}}$ by

$$P_\sigma([\bar{b}], [\bar{a^1}], \dots [\bar{a^n}]) \Leftrightarrow_{df} \mathfrak{B} \models \psi_\sigma(\bar{b}, \bar{a^1}, \dots \bar{a^n}),$$

4. the structures $\mathfrak{U}$ and $\Psi(\mathfrak{B}, \bar{b}) = \langle A(\bar{b})/\epsilon_{\bar{b}}, P_\sigma : \sigma \in \Sigma \rangle$ are isomorphic.

Let $\phi(x_1, \dots, x_n)$ be a first order formula of the signature Δ and $\bar{b} \in \phi(\mathfrak{B}^n)$ be as above. If $\mathfrak{U}$ is interpretable in $\mathfrak{B}$ with the parameters $\bar{b}$ and $\mathfrak{B} \models \phi(\bar{b})$ then $\mathfrak{U}$ is said to be *regularly interpretable* in $\mathfrak{B}$ with the help of formula ϕ. If the tuple $\bar{b}$ is empty, $\mathfrak{U}$ is said be *absolutely interpretable* in $\mathfrak{B}$.

Now we give a few examples some of which will be used later. In all the examples we follow the notation introduced above.

Example 1.3.1. $\mathfrak{U} = \langle \mathbb{Q}, +_{\mathbb{Q}}, \cdot_{\mathbb{Q}}, 1, 0 \rangle$ is absolutely interpretable in $\mathcal{B} = \langle \mathbb{Z}, +_{\mathbb{Z}}, \cdot_{\mathbb{Z}}, 1, 0 \rangle$. Here we treat multiplication and addition as 3-ary relations. For example $+(x, y, z)$ is true if and only if $x + y = z$. Let For any variable v, $\bar{v}$ denote the tuple (v_1, v_2). Thus:

1. $A(y_1, y_2)$ is given by $y_2 \neq 0$. Therefore the set A is constituted of those couples from $\mathbb{Z}$ whose second coordinates are not zero.

2. $E(\bar{y^1}, \bar{y^2})$ is given by the formula:

$$\forall x_1 x_2 (\cdot_{\mathbb{Z}}(y_1^1, y_2^2, x_1) \wedge \cdot_{\mathbb{Z}}(y_2^1, y_1^2, x_2) \rightarrow x_1 = x_2)$$

3. $\psi_{\cdot_{\mathbb{Q}}}(\bar{y^1}, \bar{y^2}, \bar{y^3})$ is given by $\cdot_{\mathbb{Z}}(y_1^1, y_1^2, y_1^3) \wedge \cdot_{\mathbb{Z}}(y_2^1, y_2^2, y_2^3)$ and $\psi_{+_{\mathbb{Q}}}(\bar{y^1}, \bar{y^2}, \bar{y^3})$ is the formula $\forall x_1, x_2(\cdot_{\mathbb{Z}}(y_1^1, y_2^2, x_1) \wedge \cdot_{\mathbb{Z}}(y_2^1, y_1^2, x_2) \rightarrow +_{\mathbb{Z}}(x_1, x_2, y_1^3) \wedge \cdot_{\mathbb{Z}}(y_2^1, y_2^2, y_2^3))$

4. Now the predicates $P_{\cdot_{\mathbb{Q}}}$ and $P_{+_{\mathbb{Q}}}$ can be defined on A/ϵ and the isomorphism of $\mathfrak{U}$ and $\Psi(\mathfrak{B})$ can be easily proved.

In the next example we show for a group G and a definable normal subgroup K of G the factor group G/K is absolutely interpretable in G

Example 1.3.2. Let $\langle |G|, ., ^{-1}, 1\rangle$ be a group with a definable (without use of parameters) normal subgroup K. Let ϕ be the formula of the signature of groups defining K in G. Instead of using $.(x, y, z)$ we just use the more familiar notation $x.y = z$. Let $G/K = H$ and we denote the multiplication in H by $._H$ and the inverse operation by $^{-1_H}$.

1. $A(y)$ is given by $y = y$.

2. $E(y_1, y_2)$ is given by

$$\forall x_1 x_2 (y_1^{-1} = x_1 \wedge x_1.y_2 = x_2 \rightarrow \phi(x_2)).$$

3. $\psi_{._H}(y_1, y_2, y_3)$ is given by

$$\forall x_1 x_2 x_3 (y_1.y_2 = x_1 \wedge y_3^{-1} = x_2 \wedge x_2.x_1 = x_3 \rightarrow \phi(x_3)).$$

   And $\psi_{-1_H}(y_1, y_2)$ is the formula

$$\forall x_1 x_2 (y_1^{-1} = x_1 \wedge x_1.y_2 = x_2 \rightarrow \phi(x_2)).$$

4. Now the predicates $P_{._H}$ and P_{-1_H} can be introduced on A/ϵ which is really the factor set G/K. The predicates are well-defined by normality of the subgroup K obviously.

Example 1.3.3. **Interpreting a ring R in the group $UT_3(R)$ (Mal'cev)**

Let R be a ring with unit. We can represent any upper unitriangular matrix

$$\begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}$$

over R by a triple (α, β, γ). The the multiplication is defined by

$$(\alpha_1, \beta_1, \gamma_1)(\alpha_2, \beta_2, \gamma_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2, \gamma_1 + \gamma_2 + \alpha_1\beta_2).$$

Mal'cev [8] showed that the ring R is interpretable in the group $UT_3(R)$ with parameters $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$. We repeat Mal'cev's construction in Lemma 2.3.3 but for a larger class of groups which will be introduced later.

1.3.3 Multi-sorted vs. one-sorted structures

An n-sorted structure, $n \geq 1$, is a structure with n universes, $M_1, \ldots, M_n$, a set of constant elements from the universes, a set of sorted relations and a set of sorted functions. What we mean by a sorted m-ary relation $P(M_{i_1}, \ldots, M_{i_m})$ is a subset of $M_{i_1} \times \ldots \times M_{i_m}$ and by a sorted m-ary function $f(M_{i_1}, \ldots, M_{i_m}, S(M))$,

$$f(M_{i_1}, \ldots, M_{i_m}, S(M)) : M_{i_1} \times \ldots \times M_{i_n} \rightarrow S(M)$$

when $S(M)$ is the collection of all the universes M_i.

To every n-sorted structure

$$\langle M_1, \ldots M_n, \ldots \rangle,$$

$n > 1$, it can be associated a one sorted structure

$$\langle M, P_{M_1}, \ldots, P_{M_n}, \ldots \rangle$$

where $M = \bigcup_{i=1}^{n} M_n$ and each P_{M_i} is the unary predicate separating the set M_i in M.

We will use various multi-sorted structures later and we discuss the interpretability of these structures in one another. Since we didn't define the interpretability of multi-sorted structures whenever we say a multi-sorted structure is interpretable in another multi-sorted structure it means that the corresponding one-structure of the first structure is interpretable in the corresponding one-sorted structure of the second one.

Chapter 2

Characterization of $QN_{2,n}$ groups

This chapter includes four sections. In the first on we introduce $N_{2,n}$ groups. The rationale for introducing these groups is that $N_{2,n}(\mathbb{Z})$ is a free 2-nilpotent of rank n (see Proposition 2.1.2). $QN_{2,n}$ groups are introduced in Section 2.2. In Section 2.3 the concepts basis and strong basis are introduced and an algebraic characterization of $QN_{2,n}$ groups over associative rings with unit is given. It is only in Section 2.4 that the reason behind the work done in this chapter up to that point becomes clear. It will be proved that the class $QN_{2,n}(\mathcal{R})$ is axiomatizable if $\mathcal{R}$ is an axiomatizable class of associative rings with unit. This alone proves that a group elementary equivalent to a $N_{2,n}(R)$ is a $QN_{2,n}$ group. Lemma 2.4.5 states that if the ring R is associative commutative with unit then a group elementary equivalent to $N_{2,n}(R)$ is a $QN_{2,n}$ group over some ring S such $S \equiv R$. Then one can easily give a characterization for groups elementary equivalent to a free 2-nilpotent of finite rank. Theorem 2.4.6 gives such a characterization.

2.1 $N_{2,n}$ groups

2.1.1 Definition of $N_{2,n}$ groups

Let R be a ring with unit and for an arbitrary natural number $n \geq 2$ consider the set of all $n + \binom{n}{2}$-tuples $((a_i)_{1\leq i\leq n}, (d_{ij})_{1\leq i<j\leq n})$ of elements of R when by $((-), (-))$ is meant a concatenation of two tuples. We denote this set by $N_{2,n}(R)$. We drop the subscripts and denote the tuple only by $((a_i), (d_{ij}))$. Always $(\bar{0})$ means that all the coordinates are 0. Define a multiplication on this set by:

$$((a_i), (d_{ij}))((b_i), (d'_{ij})) =_{def} ((a_i + b_i), (d_{ij} + d'_{ij} + a_i b_j)), \quad a_i, b_i, d_{ij}, d'_{ij} \in R. \quad (2.1)$$

Lemma 2.1.1. *The set $N_{2,n}(R)$ is a group with respect to the multiplication defined in* (2.1).

Proof. Let $x = ((a_i), (d_{ij}))$, $y = ((b_i), (d'_{ij}))$ and $z = ((c_i), (d''_{ij}))$ be elements of $N_{2,n}(R)$. Then

$$\begin{aligned}
(xy)z &= ((a_i + b_i), (d_{ij} + d'_{ij} + a_i b_j))z \\
&= (((a_i + b_i) + c_i), (((d_{ij} + d'_{ij}) + d''_{ij} + a_i b_j + (a_i + b_i)c_j)) \\
&= ((a_i + (b_i + c_i)), ((d_{ij} + (d'_{ij}) + d''_{ij}) + a_i(b_j + c_j) + b_i c_j) \\
&= ((a_i), (d_{ij}))((b_i + c_i), (d'_{ij} + d''_{ij} + b_i c_j)) \\
&= x(yz),
\end{aligned}$$

which proves the associativity of the operation. The identity element is clearly $((\bar{0}), (\bar{0}))$ and if x is as above then $x^{-1} = ((-a_i), (a_i a_j - d_{ij}))$. So $N_{2,n}(R)$ is a group. □

An isomorphic copy of $N_{2,n}(R)$ is called an *$N_{2,n}$ group over R*. If $\mathcal{R}$ is a class of rings, $N_{2,n}(\mathcal{R})$ is the class of all groups G such $G \cong N_{2,n}(R)$ for some ring R in $\mathcal{R}$. If $\mathcal{R}$ is the class of all rings a member of the class $N_{2,n}(\mathcal{R})$ is called an $N_{2,n}$ group. We note that $N_{2,2}(R) \cong UT_3(R)$ (see Example 1.3.3).

Next proposition shows our main interest in $N_{2,n}$ groups. We postpone the proof to the end of Subsection 2.1.4.

Proposition 2.1.2. *If $\mathbb{Z}$ is the ring of integers then $N_{2,n}(\mathbb{Z})$ is a free 2-nilpotent group of rank n.*

2.1.2 Commutator subgroup and center of a $N_{2,n}$ group

Let G be a $N_{2,n}$ group over some ring R with unit. We compute the commutator, [x,y], of two element x and y of G. Let $x((a_i),(d_{ij}))$ and $y = ((b_i),(d'_{ij}))$. We have $x^{-1}y^{-1} = ((-a_i - b_i),(a_ia_j + b_ib_j - d_{ij} - d'_{ij} + a_ib_j))$. So

$$\begin{aligned}[x,y] = ((-a_i - b_i + a_i + b_i),(a_ia_j + b_ib_j - d_{ij} - d'_{ij} \\ + a_ib_j + d_{ij} + d'_{ij} + a_ib_j + (-a_i - b_i)(a_j + b_j)) \\ = ((\bar{0}),(a_ib_j - b_ia_j))\end{aligned}$$

Now we can study the relation between the commutator subgroup $[G,G]$ and the center $Z(G)$ of G.

Lemma 2.1.3. *Let G be a $N_{2,n}$ group. Then $Z(G) = [G,G]$.*

Proof. By the equation for commutators obtained above it is clear that $[G,G]$ is the set of elements of the form $x = ((\bar{0}),(d_{ij}))$. It is clear that for such x, $[x,y] = 1$ for every $y \in G$. So $[G,G] \subseteq Z(G)$. For the converse let $x = ((a_i),(d_{ij})) \in Z(G)$. If $y = ((b_i),(d'_{ij}))$ is an arbitrary element of G then we must have $[x,y] = ((\bar{0}),(a_ib_j - b_ia_j)) = 1 = ((\bar{0}),(\bar{0}))$. Since this equality holds for all elements b_i and b_j of R it also holds if $b_j = 1$ and $b_i = 0$, for each $1 \leq i < j \leq n$. So all $a_i = 0$, $1 \leq i \leq n-1$. Setting $b_{n-1} = 1$ and $b_n = 0$ will prove that $a_n = 0$. So $x \in [G,G]$. □

We note that as a consequence of Lemma 2.1.3 a $N_{2,n}$ group is 2-nilpotent.

2.1.3 Standard basis for a $N_{2,n}$ group

When all coordinates of an element $x = ((a_i),(d_{ij}))$ of $N_{2,n}(R)$ are zero except possibly the i-th coordinate then x is denoted by $g_i^{a_i}$. If every coordinate of x is zero except possibly the ij-th coordinate x is denoted by $g_{ij}^{d_{ij}}$. In particular $g_i^0 = g_{ij}^0 = 1$.

We also assume that $g_i^1 = g_i$ for all $1 \le i \le n$ and $g_{ij}^1 = g_{ij}$ for all $1 \le i < j \le n$. By what has been shown above:

$$[g_i^a, g_j^b] = g_{ij}^{ab}, \quad a, b \in R,$$

and $[g_i, g_j] = g_{ij}$. So $[g_i^a, g_j^b] = g_{ij}^{ab} = [g_i, g_j]^{ab}$. Thus given an element $x = ((a_i), (d_{ij}))$ it is clear that

$$x = g_n^{a_n} g_{n-1}^{a_{n-1}} \dots g_1^{a_1} g_{12}^{d_{12}} \dots g_{1n}^{d_{1n}} g_{23}^{d_{23}} \dots g_{n-1,n}^{d_{n-1,n}}. \tag{2.2}$$

As the discussions above indicate we also have the following representation for x:

$$x = g_n^{a_n} g_{n-1}^{a_{n-1}} \dots g_1^{a_1} [g_1, g_2]^{d_{12}} \dots [g_1, g_n]^{d_{1n}} [g_2, g_3]^{d_{23}} \dots [g_{n-1}, g_n]^{d_{n-1,n}}. \tag{2.3}$$

Thus the set $\{g_i^a | 1 \le i \le n, a \in R\}$ is a generating set for $N_{2,n}(R)$. Moreover it should be clear that every element x of $N_{2,n}(R)$ has a unique representation of the form given in the equations (2.2) and (2.3). We call the elements $g_1, \dots, g_n$ a *standard basis* for the group $N_{2,n}(R)$.

2.1.4 Centralizers of elements of the standard basis of a $N_{2,n}$ group

Consider a ring R with unit and a $N_{2,n}$ group G over R. Let $C_G(x)$ denote the centralizer of an element x of G in G. Now Set:

$$G_i =_{df} C_G(g_i), \quad 1 \le i \le n,$$

where $g_1, \dots, g_n$ constitute the standard basis for G. Let $g_i^R = \{g_i^\alpha : \alpha \in R\}$. We first prove that $G_i = g_i^R.Z(G)$.

Lemma 2.1.4. *For each* $1 \le i \le n$, $G_i = g_i^R.Z(G)$.

Proof. Let $x = ((a_i), (d_{ij}))$ be an arbitrary element of G. By the discussion in

Subsection 2.1.3, $x = g_n \dots g_1 v$ when v is an element of the center $Z(G)$. Then

$$\begin{aligned} [g_i, x] &= [g_i, g_n^{a_n} \dots g_1^{a_1} v] \\ &= [g_i, g_n \dots g_1][g_i, v] \\ &= [g_i, g_n \dots g_1] \\ &= g_{1i}^{-a_i} \dots g_{i-1,i}^{-a_{i-1,i}} g_{i,i+1}^{a_{i,i+1}} \dots g_{jn}^{a_n} \end{aligned}$$

For x to be in G_i it is necessary that $[g_i, x] = 1$. So by the above equality $a_j = 0$ for $i \neq j$. So x must have the form $g_i^a v$ for some a in R and v in $Z(G)$. □

Corollary 2.1.5. *Each G_i, $1 \leq i \leq n$ is abelian.*

Proof. Clear by Lemma 2.1.4. □

Next define subgroups G_{ij} of G by

$$G_{ij} =_{df} [g_i, G_j], \quad 1 \leq i < j \leq n. \tag{2.4}$$

Lemma 2.1.6. *The equalities:*

$$G_{ij} = [G_i, g_j] = [G_i, G_j], \quad 1 \leq i < j \leq n,$$

hold, when G_{ij} are defined in Equation (2.4).

Proof. That is enough to show that the generators of one lie in another one. let first prove that $G_{ij} = [G_i, g_j]$. Let x be an element of G_j. By Lemma 2.1.4 $x = g_j^a v$ for some $a \in R$ and $v \in Z(G)$. Then,

$$\begin{aligned} [g_i, x] &= [g_i, g_j^a v] = [g_i, g_j^a] \\ &= [g_i, g_j]^a = [g_i^a, g_j] \in [G_i, g_j]. \end{aligned}$$

The converse inclusion follows similarly.

It remains to prove $G_{ij} = [G_i, G_j]$ for each $1 \leq i < j \leq n$. The direction $\subseteq$ is obvious. For the converse let $x \in G_i$ and $y \in G_j$. By Lemma 2.1.4, $x = g_i^a v$ and

$y = g_j^b v'$ for some $a, b \in R$ and $v, v' \in Z(G)$. Thus,

$$\begin{aligned}[g_i^a v, g_j^b v'] &= [g_i^a, g_j^b] \\ &= [g_i, g_j]^{ab} \\ &= [g_i, g_j^{ab}] \in G_{ij}.\end{aligned}$$

□

The next thing we can verify is that

$$G_i \cap G_j = Z(H), \quad 1 \leq i, j \leq n. \tag{2.5}$$

Lemma 2.1.7. *Equations* (2.5) *hold in for the subgroups G_i, G_j and $Z(H)$ for each $1 \leq i, j \leq n$.*

Proof. Let the element x of G be such that $x \in G_i$ and $y \in G_j$. By Lemma 2.1.4, $x = g_i^a v = g_j^b v'$ for some $a, b \in R$ and $v, v' \in Z(G)$, implying that $a = b = 0$. Therefore $x \in Z(G)$. The other direction is clear. □

We assemble the lemmas and corollaries above in a single proposition.

Proposition 2.1.8. *Let G be a $N_{2,n}$ group over a ring R with unit. Suppose $g_1, \ldots, g_n$ constitute the standard basis for G. Let $G_i = C_G(g_i)$, $1 \leq i \leq n$, and $G_{ij} = [g_i, G_j]$, $1 \leq i < j \leq n$. Then the following statements hold,*

1. $G_{ij} = [G_i, g_j] = [G_i, G_j], \quad 1 \leq i < j \leq n,$
2. $G_i \cap G_j = Z(G), \quad 1 \leq i < j \leq n,$
3. $[G_i, G_i] = 1, \quad 1 \leq i \leq n,$
4. $[G, G] = Z(G),$
5. *every element of G can be written as $u_n u_{n-1} \ldots u_1 v$ where each $u_i \in G_i$ and $v \in Z(G)$ and each u_i is unique modulo the center. Moreover each $v \in Z(G)$ can be uniquely written as $u_{12} \ldots u_{1n} u_{23} \ldots u_{n-1,n}$ when $u_{ij} \in G_{ij}$.*

Proof. See Subsection 2.1.3, Lemmas 2.1.6, 2.1.7 and Corollary 2.1.5. □

Now we are in a good set up to prove Proposition 2.1.2.

Proof. (**Proof of Proposition 2.1.2**) Let $F(n)$ be the free group on generators $u_1, \ldots, u_n$. Let $\Gamma_3(F(n))$ denote the third term of the lower central series of $F(n)$. Let $g_1, \ldots, g_n$ be elements of the standard basis for $N_{2,n}(\mathbb{Z})$. Note that $\{g_1, \ldots, g_n\}$ is a generating set for $N_{2,n}(\mathbb{Z})$. The mapping:

$$F(n)/\Gamma_3(F(n)) \longrightarrow N_{2,n}(\mathbb{Z}), \quad u_i\Gamma_3(F(n)) \mapsto g_i, \quad 1 \leq i \leq n,$$

is a well defined homomorphism of groups since $\Gamma_3(F(n))$ is generated by the simple commutators $[[u_{i_1}, u_{i_2}], u_{i_3}]$, and $[[g_{i_1}, g_{i_2}], g_{i_3}] = 1$ holds in $N_{2,n}(\mathbb{Z})$. It is also a surjection since $g_1, \ldots, g_n$ generate $N_{2,n}(\mathbb{Z})$.

We prove that it is also an injection. Notice that for every integer k and m,

$$[u_{i_1}^m\Gamma_3(F(n)), u_{i_2}^k\Gamma_3(F(n))] = [u_{i_1}, u_{i_2}]^{mk}\Gamma_3(F(n)) \tag{2.6}$$

holds for each pair u_{i_1}, u_{i_2} of elements in $\{u_1, \ldots, u_n\}$. So every element u in $F(n)/\Gamma_3(F(n))$ can be brought to the form in (2.3), g_i substituted by u_i for each $1 \leq i \leq n$. This can be done using the so-called elimination process, applying the relations (2.6) and using the fact that all the commutators in $F(n)/\Gamma_3(F(n))$ belong to the center of the group. For example if u_{i_1} and u_{i_2} are such that $i_1 < i_2$ and m and k are integers then:

$$\begin{aligned} u_{i_1}^m u_{i_2}^k \Gamma_3(F(n)) &= u_{i_2}^k u_{i_1}^m [u_{i_1}^m, u_{i_2}^k]\Gamma_3(F(n)) \\ &= u_{i_2}^k u_{i_1}^m [u_{i_1}, u_{i_2}]^{mk}\Gamma_3(F(n)). \end{aligned}$$

By repeating this process finitely many times and moving the commutators to the right hand side we arrive at the indicated form for any element of $F(n)/\Gamma_3(F(n))$ (see [18], Section 6). Therefore under the mapping defined above u gets mapped to an element g of $N_{2,n}(\mathbb{Z})$ with a representation exactly like what appears in (2.3). This form is unique so the element g is trivial in $N_{2,n}(\mathbb{Z})$ if and only if all the exponents in (2.3) are zero if and only if u is trivial in $F(n)/\Gamma_3(F(n))$. And we are done. □

2.2 $QN_{n,2}$ groups

2.2.1 Definition of $QN_{n,2}$ groups

Let $f : R^+ \times R^+ \longrightarrow R^{\binom{n}{2}}$ be a symmetric 2-cocycle, when R is a ring with unit. Such a 2-cocycle has $\binom{n}{2}$ coordinates $f_{jk} : R^+ \times R^+ \longrightarrow R$, $1 \leq j < k \leq n$, each of which is a symmetric 2-cocycle. Now for each i, $1 \leq i \leq n$, let $f^i : R^+ \times R^+ \longrightarrow R^{\binom{n}{2}}$ be a symmetric 2-cocycle with components f^i_{jk}, $1 \leq j < k \leq n$.

We define a new multiplication $\odot$ on $N_{2,n}(R)$ by

$$((a_i), (d_{ij})) \odot ((b_i), (d'_{ij})) = ((a_i + b_i), (d_{ij} + d'_{ij} + a_i b_j + \sum_{k=1}^{n} f^k_{ij}(a_k, b_k))). \tag{2.7}$$

Lemma 2.2.1. *The set $N_{2,n}(R)$ is a group with respect to the multiplication $\odot$ defined in* (2.7).

Proof. Let $x = ((a_i), (d_{ij}))$, $y = ((b_i), (d'_{ij}))$ and $z = ((c_i), (d''_{ij}))$ be elements of $N_{2,n}(R)$. Then,

$$\begin{aligned}
(x \odot y) \odot z &= ((a_i + b_i), (d_{ij} + d'_{ij} + a_i b_j + \sum_{k=1}^{n} f^k_{ij}(a_k, b_k))) \odot z \\
&= (((a_i + b_i) + c_i), ((d_{ij} + d'_{ij}) + d''_{ij} + a_i b_j + (a_i + b_i)c_j \\
&\quad + \sum_{k=1}^{n} (f^k_{ij}(a_k, b_k) + f^k_{ij}(a_k + b_k, c_k)))) \\
&= ((a_i + (b_i + c_i)), (d_{ij} + (d'_{ij} + d''_{ij}) + a_i(b_j + c_j) + b_i c_j \\
&\quad + \sum_{k=1}^{n} (f^k_{ij}(a_k, b_k + c_k) + f^k_{ij}(b_k, c_k)))) \\
&= ((a_i), (d_{ij})) \odot ((b_i + c_i), (d'_{ij} + d''_{ij} + b_i c_j + \sum_{k=1}^{n} f^k_{ij}(b_k, c_k))) \\
&= x \odot (y \odot z)
\end{aligned}$$

The identity element is $((\bar{0}), (\bar{0}))$ and inverse $x^{(-1)}$ of an element $x = ((a_i), (d_{ij}))$ is given by

$$x^{(-1)} = ((-a_i), (-d_{ij} + a_i a_j - \sum_{k=1}^{n} f^k_{ij}(a_k, -a_k))))$$

So the new multiplication is also a group operation. ◻

We denote the new group by $N_{2,n}(R, f^1 \dots f^n)$. If $\mathcal{R}$ is a class of rings with unit, by $QN_{2,n}(\mathcal{R})$ we mean the class of all groups G such that $G \cong N_{2,n}(R, f^1, \dots, f_n)$ for some ring R in $\mathcal{R}$ and symmetric 2-cocycles $f^i : R^+ \times R^+ \to R^{\binom{n}{2}}$, $i = 1, \dots, n$. Such a group G is called a $QN_{2,n}$ group over R. If $\mathcal{R}$ is the class of all rings a member of the class $QN_{2,n}(\mathcal{R})$ is called a $QN_{2,n}$ group.

2.2.2 Commutator subgroup and center of a $QN_{2,n}$ group

Let G be a $QN_{2,n}$ group over a ring R with unit. To give a formula for the commutator of two elements we need to verify a basic fact about symmetric 2-cocycles.

Lemma 2.2.2. *Let $f : R^+ \times R^+ \longrightarrow R$ be a symmetric 2-cocycle from the additive group of R to itself. For every a and b in R the following holds:*

$$f(a,b) + f(-a,-b) - f(a,-a) - f(b,-b) + f(-a-b, a+b) = 0$$

Proof. Clear by considering:

$$f(-a-b, a+b) = f(-a,a) + f(-b, a+b) - f(-a,-b)$$

and,

$$f(a+b, -b) = f(b,-b) - f(a,b)$$

◻

Now let $x = ((a_i), (d_{ij})$ and $y = ((b_i), (d'_{ij}))$ be in G then

$$\begin{aligned} x^{(-1)} \odot y^{(-1)} \odot x \odot y = ((\bar{0}), (d_{ij} + d'_{ij} - d_{ij} - d'_{ij} \\ + a_i a_j + b_i b_j + a_i b_j + a_i b_j + (-a_i - b_i)(a_j + b_j) \\ - \sum_{k=i}^{n} (f^k_{ij}(a_k, -a_k) - f^k_{ij}(b_k, -b_k) \\ + f^k_{ij}(a_k, b_k) + f^k_{ij}(-a_k - b_k, a_k + b_k)) \\ = ((\bar{0}), (a_i b_j - b_i a_j)) \end{aligned}$$

by the above lemma. Thus commutators in $QN_{2,n}$ and $N_{2,n}$ groups coincide. So we have the lemma:

Lemma 2.2.3. *In a $QN_{2,n}$ group G, $Z(G) = [G, G]$.*

Proof. The proof goes through exactly like that of Lemma 2.1.3. □

2.2.3 Standard basis for a $QN_{2,n}$ group

Again as in the $N_{2,n}$ groups we denote an element $((a_i), (d_{ij}))$ which has zeros everywhere except possibly at the i-th position by $g_i^{a_i}$ and the one which has zeros everywhere except possibly at ij-th position by $g_{ij}^{d_{ij}}$. We call the set $\{g_1, \ldots, g_n\}$ the *standard basis* of the group $QN_{2,n}(R)$. Let us note that for a $QN_{2,n}$ group G over a ring R with unit and the standard basis $\{g_1, \ldots, g_n\}$, the quotient $G/Z(G)$ is a free module over R of rank n generated by $\{g_1Z(G), \ldots, g_nZ(G)\}$. Moreover $Z(G) = [G, G]$ is a free R-module of rank $\frac{n(n+1)}{2}$ generated by the $g_{ij} = [g_i, g_j]$, $1 \leq i < j \leq n$.

Proposition 2.2.4. *Let G be a $QN_{2,n}$ group over a ring R with unit, G_i for each $1 \leq i \leq n$ and G_{ij} for each $1 \leq i < j \leq n$ be defined as in proposition 2.1.8. Then all the conditions (1)-(5) in proposition 2.1.8 are also true in the group G.*

Proof. Similar to the proof of Proposition 2.1.8. □

2.2.4 Generators and relations for a $QN_{2,n}$ group

Here we specify a set of generators and relations for a $QN_{2,n}$ group.

Lemma 2.2.5. *The group $G = N_{2,n}(R, f^1, \ldots, f^n)$ is generated by*

$$\{g_i^a, g_{kl}^b : 1 \leq i \leq n, 1 \leq k < l \leq n, \quad \alpha, \beta \in R\},$$

and defined by the relations:

(a) $[g_i^\alpha, g_j^\beta] = g_{ij}^{\alpha\beta}$, *for all* $1 \leq i < j \leq n$, $\alpha, \beta \in R$

(b) $[g_i^\alpha, g_{kl}^\beta] = 1$, *for all* $1 \le i \le n$ *and* $1 \le k < l \le n$, $\alpha, \beta \in R$

(c) $g_i^\alpha \odot g_i^\beta = g_i^{(\alpha+\beta)} g_{12}^{f_{12}^i(\alpha,\beta)} \cdots g_{n-1,n}^{f_{n-1,n}^i(\alpha,\beta)}$, *for all* $1 \le i \le n$, $\alpha, \beta \in R$

(d) $g_{ij}^\alpha \odot g_{ij}^\beta = g_{ij}^{\alpha+\beta}$, *for all* $1 \le i < j \le n$, $\alpha, \beta \in R$.

Proof. clearly the set

$$\mathcal{G} = \{g_i^a, g_{kl}^b | 1 \le i \le n, 1 \le k < l \le n, \quad \alpha, \beta \in R\},$$

is a generating set for G. Let F be the free group generated by the set $\mathcal{G}$ and $\mathcal{R}$ be the normal subgroup of F generated by the relations (a)-(d) above, multiplication $\odot$ taken to be concatenation. Now consider the group $\langle \mathcal{G} | \mathcal{R} \rangle$, the quotient of F by $\mathcal{R}$. Consider the mapping:

$$\langle \mathcal{G} | \mathcal{R} \rangle \longrightarrow G, \quad g_i^\alpha \mapsto g_i^\alpha, \quad g_{kl}^\alpha \mapsto g_{kl}^\alpha$$

for every $\alpha \in R$, $1 \le i \le n$ and $1 \le k < l \le n$. The map is a well-defined homomorphism since all the relations (a)-(d) hold in G. Every word W in $\langle \mathcal{G} | \mathcal{R} \rangle$ is equivalent to a word with the form given in (2.2), multiplication taken to be concatenation. this element gets mapped to an element g with the same form in the group G, multiplication taken to be $\odot$. This form is unique by Proposition 2.2.4. So g is trivial in G if and only if W is trivial in $\langle \mathcal{G} | \mathcal{R} \rangle$. The proposition is proved. □

2.3 Characterization of $Q_{2,n}$ groups

2.3.1 groups with a basis

Definition 2.3.1 (Basis). *Let H be a group with distinct nontrivial elements h_1, $h_2, \ldots, h_n$, $h_{12}, \ldots, h_{n-1,n}$ where $[h_i, h_j] = h_{ij}$ holds for every $1 \le i < j \le n$. Let H_1, H_2, $\ldots$, H_n and H_{12}, $\ldots$, $H_{n-1,n}$ be subgroups of H satisfying the following conditions:*

1. $H_i = C_H(h_i)$ *and* $H_{ij} = [h_i, H_j] = [H_i, h_j] = [H_i, H_j]$, $1 \le i < j \le n$,

2. $H_i \cap H_j = Z(H)$, $1 \leq i < j \leq n$,

3. $[H_i, H_i] = 1$, $1 \leq i \leq n$,

4. $[H, H] \subseteq Z(H)$,

5. (a) *every element of H can be written as $u_n u_{n-1} \dots u_1 v$ where each $u_i \in H_i$ and $v \in Z(H)$ and each u_i is unique modulo the center,*

   (b) *each $v \in Z(H)$ can be uniquely written as $u_{12} \dots u_{1n} u_{23} \dots u_{n-1,n}$ when $u_{ij} \in H_{ij}$.*

Then $\mathcal{B} = \{h_1, h_2, \dots, h_n\}$ is called to be a basis for H.

Lemma 2.3.2. *Let H be a group with elements $h_1, \dots, h_n$ constituting a basis for H. Then the subgroups $Z(H)$, H_i, $1 \leq i \leq n$, H_{ij}, $1 \leq i < j \leq n$, and $[H, H]$ are first order definable in the enriched group $(H, \mathcal{B})$. Thus all the conditions (1)-(5) of the definition of basis can be expressed by first order formulas of the signature of groups.*

Proof. The center is defined by the formula

$$\phi_{Z(H)}(x) : \forall y [x, y] = 1.$$

For each $1 \leq i \leq n$, H_i is defined by:

$$\phi_{H_i}(\bar{h}, x) : [h_i, x] = 1.$$

For each $1 \leq i < j \leq n$, the subgroup H_{ij} is generated by the set $\{[h_i, y] : y \in H_j\}$. So for every element x of H_{ij}, for some fixed $1 \leq i < j \leq n$, can be written as a product

$$x = [h_i, y_1] \dots [h_i, y_m], \quad y_1, \dots, y_n \in H_j.$$

Since H is a 2-nilpotent group, by condition (4) we can rewrite x as $x = [h_i, y_1 \dots y_n]$. So the subgroup H_{ij} is defined by the formula:

$$\phi_{H_{ij}}(\bar{h}, x) : \exists y (x = [h_i, y] \wedge \phi_{H_j}(y)).$$

By (4), $[H, H]$ sits inside the center. Therefore every element x of $[H, H]$ has the form mentioned in (5)-(b). Conversely if some arbitrary element $x \in H$ has the form indicated in (5)-(b), then $x \in [H, H]$ since $H_{ij} \subseteq [H, H]$ for each $1 \leq i < j \leq n$. Thus $[H, H]$ is defined in $(H, \mathcal{B})$ by the formula:

$$\exists y_{12} \dots y_{n-1,n}\big(\bigwedge_{1 \leq i < j \leq n} \phi_{H_{ij}}(y_{ij}) \wedge x = y_{12} \dots y_{n-1,n}\big).$$

It is now clear how to formulate conditions (1),(2),(3) and (5). Condition (4) is simply given by:

$$\forall x, y, z([x, y].z = z.[x, y]).$$

□

2.3.2 Characterization theorem

To give a first order characterization of $QN_{2,n}$ groups we proceed by a series of lemmas and definitions.

Lemma 2.3.3. *Let H be group with a basis $\mathcal{B} = \{h_1, \dots, h_n\}$. Then for any fixed $1 \leq i < j \leq n$, there is a ring $R_{ij} = \langle H_{ij}, \boxplus_{ij}, \boxdot_{ij} \rangle$ interpretable in the enriched group $(H, \mathcal{B})$.*

Proof. We define a ring R_{ij} with unit on the subgroup H_{ij} using the standard Mal'cev construction. let $R_{ij} = H_{ij}$ as sets. Let the addition $\boxplus_{ij}$ on R_{ij} be the multiplication on H_{ij}. Next define the homomorphisms:

$$\tau_1 : H_i \xrightarrow{[-, h_j]} H_{ij},$$

and

$$\tau_2 : H_j \xrightarrow{[h_i, -]} H_{ij}$$

These homomorphisms are surjective by (1) of Definition 2.3.1. Now define the multiplication $\boxdot_{ij}$ by

$$\tau_i(x) \boxdot_{ij} \tau_j(y) = [x, y].$$

The multiplication is well defined by condition (2) of the definition of a basis. The distributivity of the multiplication on addition follows from the fact that τ_1 and τ_2 are homomorphisms. The unit of the ring is h_{ij} since

$$h_{ij} \boxdot_{ij} \tau_2(y) = \tau_1(h_i) \boxdot_{ij} \tau_2(y) = [h_i, y] = \tau_2(y).$$

The dual equality follows in the same manner.

We now follow the notation of Subsection 1.3.2 to show that the ring R_{ij} is interpretable in $(H, \mathcal{B})$. For simplicity we just use $\boxdot_{ij}$ and $\boxplus_{ij}$ for operations (or the predicates describing the operation) in R_{ij} and its interpretation in $(H, \mathcal{B})$.

1. $A(\bar{h}, x) : \phi_{H_{ij}}(\bar{h}, x)$, where $\phi_{H_{ij}}$ is defined in Lemma 2.3.2
2. $E(\bar{h}, x_1, x_2) : x_1 = x_2$
   - $\psi_{\boxplus_{ij}}(\bar{h}, x_1, x_2, x_3) : x_1x_2 = x_3$, and for $a_1, a_2, a_3 \in A(\bar{h})$,

     $$a_1 \boxplus_{ij} a_2 = a_3 \Leftrightarrow (H, \mathcal{B}) \models \psi_{\boxplus_{ij}}(\bar{h}, a_1, a_2, a_3)$$

   - $\psi_{\boxdot_{ij}}(\bar{h}, x_1, x_2, x_3)$:

     $$\forall y_1 y_2([y_1, h_j] = x_1 \wedge [h_i, y_2] = x_2 \rightarrow [y_1, y_2] = x_3,$$

     and for $a_1, a_2, a_3 \in A(\bar{h})$

     $$a_1 \boxdot_{ij} a_2 = a_3 \Leftrightarrow (H, \mathcal{B}) \models \psi_{\boxdot_{ij}}(\bar{h}, a_1, a_2, a_3).$$

Thus R_{ij} is interpretable in $(H, \mathcal{B})$. □

Remark 2.3.4. The ring R_{ij} recovered above is neither commutative nor associative in general. Let us have a look at the case of $H = UT_3(R)$, the group of upper unitriangular matrices over the ring R with unit. We follow the notation of Example 1.3.3. The elements e_1 and e_2 constitute a basis for H and $Z(H) = H_{12} = [H, H]$. Notice that $R_{12} = Z(H) = (0, 0, R)$ as sets. Since there is only one pair $1 \leq i < j \leq n$ one can substitute $\boxdot_{ij}$ and $\boxplus_{ij}$ by $\boxdot$ and $\boxplus$ respectively. Define a mapping

$$\eta : R \to R_{12}, \quad \gamma \mapsto (0, 0, \gamma).$$

The mapping above is an isomorphism of rings since $\eta(0) = (0,0,0)$, $\eta(1) = (0,0,1) = h_{12}$ and

$$\begin{aligned}\eta(\gamma_1 + \gamma_2) &= (0,0,\gamma_1+\gamma_2)\\ &= (0,0,\gamma_1)(0,0,\gamma_2)\\ &= (0,0,\gamma_1) \boxplus (0,0,\gamma_2)\\ &= \eta(\gamma_1) \boxplus \eta(\gamma_2),\end{aligned}$$

and

$$\begin{aligned}\eta(\gamma_1\gamma_2) &= (0,0,\gamma_1\gamma_2)\\ &= [(\gamma_1,0,0),(0,\gamma_2,0)]\\ &= (0,0,\gamma_1) \boxdot (0,0,\gamma_2)\\ &= \eta(\gamma_1) \boxdot \eta(\gamma_2).\end{aligned}$$

This proves that the rings R and R_{ij} are isomorphic. Therefore the ring R_{12} is commutative (associative) if the ring R is. We denote the recovered ring by $Ring(UT_3(R), e_1, e_2)$.

Definition 2.3.5. *Let $\mathfrak{B}$ be an algebraic structure. Let $\mathfrak{U}$ and $\mathfrak{U}'$ be interpretable in some enrichment $\mathfrak{B}^*$ of $\mathfrak{B}$. An isomorphism $\eta : \mathfrak{U} \to \mathfrak{U}'$ is definable in $\mathfrak{B}^*$ if there is a formula $\psi(x,y)$ of signature of $\mathfrak{B}^*$ such that $\eta(b) = b'$, $b \in \mathfrak{B}$ and $b' \in \mathfrak{B}'$, if and only if $\mathfrak{B} \models \psi(b,b')$.*

Remark 2.3.6. Let $\eta : G \to H$ be an isomorphism of groups and suppose H has a right (left) R-module structure for some ring R. It can be easily checked that the group G has a right (left) R-module structure defined by:

$$h^\alpha =_{df} \eta((\eta^{-1}(h))^\alpha).$$

Let $\langle R, M, \delta \rangle$ be a two-sorted structure where R is a ring, M is an R-module and the predicate δ determines the action of R on M. We denote this structure with M_R.

Definition 2.3.7. *Let M_R be a structure as described above and $\mathfrak{B}$ be an arbitrary algebraic structure. The action of the ring R on the module M is absolutely (regularly, relatively) interpretable in $\mathfrak{B}$ if the structure M_R is absolutely (regularly, relatively) interpretable in $\mathfrak{B}$ (see Subsection 1.3.3).*

Lemma 2.3.8. *Let H be a group with a basis $h_1, \ldots, h_n$ and R_{ij} be the ring recovered in Lemma 2.3.3. If R_{ij} is associative then the quotients $H_k/Z(H)$, $1 \leq k \leq n$, and the subgroups H_{ls}, $1 \leq l < s \leq n$, are all cyclic R_{ij}-modules. Moreover all the module structures defined are interpretable in $(H, \mathcal{B})$.*

Proof. Let $R = R_{ij}$ for the moment. Here we do not assume that R is commutative so we have to distinguish between left and right R-module structures. Let x^α denote an element x of H_{ij} acted upon by an element α of R. Since it is impossible to read from our notation whether the action is a left or right one we will be clear about it whenever there is a possibility of confusion. Since R and H_{ij} have the same underlying set, for x as above there exists an element β of R such that $x = \beta$. Now the left action is given by $x^\alpha =_{df} \alpha \,\Box_{ij}\, \beta$ and the right action by $x^\alpha =_{df} \beta \,\Box_{ij}\, \alpha$. Actually H_{ij} is a cyclic left-right R-module generated by h_{ij}. The action is interpretable in $(H, \mathcal{B})$ since the group H_{ij} is definable in $(H, \mathcal{B})$, the ring R is also interpretable in $(H, \mathcal{B})$. The right action is defined by:

$$x^\alpha = y \Leftrightarrow \psi_{\Box_{ij}}(\bar{h}, x, \alpha, y).$$

The left action can be defined by $\psi_{\Box_{ij}}(\bar{h}, \alpha, x, y)$. So the action of R on H_{ij} is interpretable in $(H, \mathcal{B})$. Let the formulas $\phi_{Z(H)}$, ϕ_{H_i} and $\phi_{H_{ij}}$ be the ones introduced in Lemma 2.3.2. For any $x \in H$ let us denote $xZ(H)$ by $[x]$. There is an isomorphism of groups:

$$\eta : H_i/Z(H) \to H_{ij}, [x] \mapsto [x, h_j].$$

The isomorphism η is defined in $(H, \mathcal{B})$ by the formula:

$$\phi_1(\bar{h}, x_1, x_2) : \forall y_1 (\phi_{H_1}(\bar{h}, x_1) \wedge \phi_{H_{ij}}(\bar{h}, x_2) \wedge \phi_{Z(H)}(x_1^{-1}.y_1) \to x_2 = [y_1, h_j]).$$

We can define a right (left) R-module structure on $H_i/Z(H)$ via the definable isomorphism η by setting

$$\eta([x]^\alpha) =_{df} \eta([x])^\alpha, \quad x \in H_i$$

considering H_{ij} as a right (left) R-module. Thus $H_i/Z(H)$ is a cyclic left-right R-module generated by $[h_i]$. If we choose the right action on H_{ij} then for $x_1, x_2 \in H_i$ and $\alpha \in R$ the following holds:

$$\begin{aligned} [x_1]^\alpha = [x_2] \Leftrightarrow & \forall y_1 y_2 y_3 z_1 z_2 (\bigwedge_{i=1}^{2} (\phi_{H_i}(\bar{h}, x_i) \wedge \phi_{H_i}(\bar{h}, z_i) \wedge \phi_{Z(H)}(x_i^{-1} z_i)) \\ & \wedge \phi_1(z_1, y_3) \wedge \phi_1(z_2, y_2) \wedge \phi_{H_{ij}}(\bar{h}, \alpha) \wedge \psi_{\Box_{ij}}(\bar{h}, y_3, \alpha, y_1) \\ & \rightarrow y_1 = y_2). \end{aligned} \tag{2.8}$$

The ring R and the quotient $H_i/Z(H)$ are interpretable in $(H, \mathcal{B})$. This fact together with (2.8) proves that the right action of R on $H_i/Z(H)$ is interpretable in $(H, \mathcal{B})$. The interpretability of the left action follows in a similar way.

Let us define an action of R on H_i by setting:

$$[x^\alpha] =_{df} [x]^\alpha, \quad x \in H_i, \alpha \in R.$$

Obviously the action is well-defined only modulo the center $Z(H)$ of H.

For $k \neq j$ we make H_{ik} into an R-module. An element of H_{ik} is of the form $[x, h_k]$ for some $x \in H_i$. An element of H_{ij} is also of the form $[x, h_j]$, for some $x \in H_i$. Consider the mapping:

$$\varphi : H_{ij} \rightarrow H_{ik}, \quad [x, h_j] \mapsto [x, h_k].$$

Since H is 2-nilpotent, for every x, y and z in H the identities

$$[x, z][y, z] = [xy, z], \tag{2.9}$$

and

$$[x, y^k] = [x^k, y] = [x, y]^k, \quad k \in \mathbb{Z} \tag{2.10}$$

hold. By (2.10), $[x, z] = [y, z]$ holds whenever $[xy^{-1}, z] = 1$. On the other hand if $[x, h_j] = [y, h_j]$ for $x, y \in H_i$ then $xy^{-1} \in H_i$ and also $xy^{-1} \in H_j$ by definition of

H_j. Hence $xy^{-1} \in Z(H)$. Thus $[x, h_k] = [y, h_k]$, which proves that φ is well-defined. Identity 2.9 proves directly that φ is a homomorphism of groups. The surjectivity is clear. For injectivity assume $[x, h_k] = 1$, $x \in H_i$. So $x \in H_i \cap H_j = Z(H)$. Therefore $[x, h_j] = 1$. Hence φ is an isomorphism of groups.

The isomorphism φ is also definable in $(H, \mathcal{B})$ by the formula:

$$\begin{aligned}\psi(\bar{h}, y_1, y_2) :& \forall x(\phi_{H_i}(\bar{h}, x) \wedge \phi_{H_{ij}}(\bar{h}, y_1) \wedge \phi_{H_{ik}}(\bar{h}, y_2)\\ &\wedge [x, h_j] = y_1 \rightarrow [x, h_k] = y_2),\end{aligned}$$

of the signature of groups. Now we define a right action of R on H_{ik} via the isomorphism φ by setting

$$\begin{aligned}y_1^{\alpha} = y_2 \Leftrightarrow & \forall x_1, x_2 (\bigwedge_{i=1}^{2} \phi_{H_{ij}}(\bar{h}, x_i) \bigwedge_{i=1}^{2} \phi_{H_{ik}}(\bar{h}, y_i)\\ & \wedge \psi(\bar{h}, x_1, y_1) \wedge \phi_{H_{ij}}(\bar{h}, \alpha) \wedge \psi_{\Box_{ij}}(\bar{h}, x_1, \alpha, x_2)\\ & \rightarrow \psi(\bar{h}, x_2, y_2)).\end{aligned} \tag{2.11}$$

This together with the fact that R and H_{ij} are interpretable in $(H, \mathcal{B})$ proves that the right action of R on H_{ij} is interpretable in $(H, \mathcal{B})$. Substituting $\psi_{\Box_{ij}}(\bar{h}, x_1, \alpha, x_2)$ with $\psi_{\Box_{ij}}(\bar{h}, \alpha, x_1, x_2)$ in the above formula proves the interpretability of the left action.

We can continue the process above to make every H_{ls}, $1 \leq l < s \leq n$, and $H_k/Z(H)$, $1 \leq k \leq n$ an R-module, each with an action interpretable in $(H, \mathcal{B})$. The lemma is proved. □

Remark 2.3.9. In the proof of Lemma 2.3.8 we made every H_{ls}, $1 \leq l < s \leq n$, and $H_k/Z(H)$, $1 \leq k \leq n$ a right and/or left cyclic R-module. Let us notice that the right and left actions of an element α of R on the generators of the cyclic modules coincide. That is because

$$\underbrace{h_{ij}^{\alpha} = 1 \Box_{ij} \alpha}_{\text{right action}} = \alpha = \underbrace{\alpha \Box_{ij} 1 = h_{ij}^{\alpha}}_{\text{left action}}.$$

Remark 2.3.10. If G is a group with basis of cardinality n, $n \geq 3$, the ring R_{ij} recovered in Lemma 2.3.3 is dependent to the choice of i and j, i.e. if $l \neq i$ or $s \neq j$ for some $1 \leq l < s \leq n$ then it might happen that $R_{ij} \ncong R_{ls}$. An example follows.

Let R be an associative non-commutative ring with unit. Let us define a new multiplication on the set $N_{2,3}(R)$ by

$$\begin{aligned} xy =& (\alpha_1, \alpha_2, \alpha_3, \gamma_1, \gamma_2, \gamma_3)(\beta_1, \beta_2, \beta_3, \gamma_1', \gamma_2', \gamma_3') \\ =& (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \alpha_3 + \beta_3, \\ & \gamma_1 + \gamma_1' + \alpha_1\beta_2, \gamma_2 + \gamma_2' + \beta_3\alpha_1, \gamma_3 + \gamma_3' + \alpha_2\beta_3). \end{aligned}$$

It can be easily checked that $N_{2,3}(R)$ is a group G with respect to this multiplication. The commutator $[x, y]$ of elements x and y as above is given by:

$$(0, 0, 0, \alpha_1\beta_2 - \beta_1\alpha_2, \beta_3\alpha_1 - \alpha_3\beta_1, \alpha_2\beta_3 - \beta_2\alpha_3).$$

Let $g_1 = (1, 0, 0, 0, 0, 0)$ and g_2 and g_3 be defined correspondingly. It can be checked that the elements g_1, g_2 and g_3 constitute a basis for G.

Let the opposite ring R^{op} of R be a ring with the same additive group as R with the multiplication

$$\alpha.\beta =_{df} \beta\alpha, \quad \alpha, \beta \in R$$

when the multiplication on the right hand side is that of R. Let G^{ij} be the subgroup of G generated by the set $\{g_i^\alpha, g_j^\beta : \alpha, \beta \in R\}$. It is easy to check that $G^{12} \cong UT_3(R)$ and $G^{13} \cong UT_3(R^{op})$. On the other hand $R_{12} = Ring(G^{12}, g_1, g_2) \cong R$ and $R_{13} = Ring(G^{13}, g_1, g_3) \cong R^{op}$ (see Remark 2.3.4). Since R is not commutative:

$$R_{12} = R \ncong R^{op} = R_{13}.$$

In the following definition we aim to define a new class of groups which have a basis strong enough to make the definition of the ring R_{ij} of Lemma 2.3.3 independent of the choice of i and j. In this class the ring R_{ij} is also associative. We justify our definition which looks a bit odd in a lemma right after the definition.

Definition 2.3.11 (Strong basis). *Let H be a group with element $h_1,\dots,h_n$ and $h_{ls} = [h_l, h_s]$, $1 \le l < s \le n$. The set of elements $\mathcal{B} = \{h_1, \dots, h_n\}$ is a strong basis for the group H if*

1. *$\mathcal{B}$ is basis for H,*
2. *the ring R_{ij} recovered in Lemma 2.3.3 is associative,*
3. *for each $1 \le i < j, k \le n$ the following condition holds.*
   for any elements $x_1, x_2, x_3 \in H_i$, $y_1 \in H_k$ and $y_2 \in H_j$ if
   - $[x_2, h_k] = [x_1, y_1]$,
   - $[x_3, h_k] = [h_i, y_1]$,
   - $[h_i, y_2] = [x_3, h_j]$,

   then $[x_1, y_2] = [x_2, h_j]$.

Lemma 2.3.12. *Let H be a group with a strong basis $\{h_1, \dots, h_n\}$. Then for each $1 \le i < j, k \le n$, $R = R_{ij} \cong R_{ik}$, where R_{ij} and R_{ik} are obtained as in Lemma 2.3.3.*

Proof. Let us denote the ring multiplication of R_{ij} by $\boxdot_{ij}$ and that of R_{ik} by $\boxdot_{ik}$. In the proof of Lemma 2.3.8 we proved that the mapping:

$$\varphi : H_{ik} \to H_{ij}, \quad [x, h_k] \mapsto [x, h_j], \quad x \in H_i,$$

is an isomorphism of groups. So the same mapping is a bijection between R_{ik} and R_{ij} which is an additive isomorphism. It also takes unit to unit obviously. Let us compute $\varphi(\alpha \boxdot_{ik} \beta)$ and $\varphi(\alpha) \boxdot_{ij} \varphi(\beta)$ for $\alpha, \beta \in R_{ik}$. By definition of $\boxdot_{ik}$:

$$\alpha \boxdot_{ik} \beta = [x_1, y_1], \text{ when } [x_1, h_k] = \alpha, \text{ and } [h_i, y_1] = \beta.$$

Then:

$$\begin{aligned} \varphi(\alpha \boxdot_{ik} \beta) &= \varphi([x_1, y_1]) \\ &= [x_2, h_j] \quad (\text{for } [x_2, h_k] = [x_1, y_1]). \end{aligned} \tag{2.12}$$

On the other hand,

$$\begin{aligned}\varphi(\alpha) \boxdot_{ij} \varphi(\beta) &= \varphi([x_1, h_k]) \boxdot_{ij} \varphi([h_i, y_1]) \\ &= [x_1, h_j] \boxdot_{ij} \varphi([x_3, h_k]) \quad (\text{for } [x_3, h_k] = [h_i, y]) \\ &= [x_1, h_j] \boxdot_{ij} [x_3, h_j] \\ &= [x_1, h_j] \boxdot_{ij} [h_i, y_2] \quad (\text{for } [h_i, y_2] = [x_3, h_j]) \\ &= [x_1, y_2].\end{aligned} \tag{2.13}$$

The existence of x_1, x_2, x_3, y_1 and y_2 is guarantied by the assumption that $\{h_1, \ldots, h_n\}$ is a basis for H. Comparing Equations (2.12) and (2.13) and the condition (3) of strong basis it is clear that

$$\varphi(\alpha \boxdot_{ik} \beta) = \varphi(\alpha) \boxdot_{ij} \varphi(\beta).$$

Thus,

$$R_{ij} \cong R_{ik}, \quad 1 \leq i < j, k \leq n.$$

Therefore in a group with a strong basis the ring R_{ij} constructed in Lemma 2.3.3 is independent of the choice of ij such that $1 \leq i < j \leq n$. We denote the ring R_{ij} by R and the multiplication and addition on R by $\boxdot$ and $\boxplus$ respectively.

□

Lemma 2.3.13. *Let H be a group with a set of elements $\mathcal{B} = \{h_1, \ldots, h_n\}$. There is a first order formula $Stbasis(x_1, \ldots, x_n)$ of the signature of groups such that $\mathcal{B}$ is a strong basis for the group H if and only if:*

$$H \models Stbasis(h_1, \ldots, h_n).$$

Proof. By Lemma 2.3.2 there is a formula $Basis(x_1, \ldots, x_n)$ of the signature of groups such that elements $h_1, \ldots, h_n$ of a group H is a basis for H if and only if $Basis(\bar{h})$ holds in H.

For $\alpha_1, \alpha_2, \alpha_3 \in R$,

$$(\alpha_1 \boxdot \alpha_2) \boxdot \alpha_3 = \alpha_1 \boxdot (\alpha_2 \boxdot \alpha_3)$$

holds if and only if $H \models Assoc(\bar{h}, \alpha_1, \alpha_2, \alpha_3)$ for the first order formula $Assoc(\bar{h}, t_1, t_2, t_3)$ of the signature of groups as following:

$$\begin{aligned}
\forall x_1 \ldots x_4 y_1 \ldots y_4 z_1 z_2 (\bigwedge_{k=1}^{4} (\phi_{H_i}(\bar{h}, x_k) \wedge \phi_{H_j}(\bar{h}, x_k))\\
&\bigwedge_{k=1}^{3} \phi_{H_{ij}}(\bar{h}, t_k) \bigwedge_{k=1}^{2} \phi_{H_{ij}}(\bar{h}, z_k)\\
&\wedge [x_1, h_j] = t_1 \wedge [h_i, y_1] \wedge [x_2, h_j] = [x_1, y_1]\\
&\wedge [h_i, y_2] = t_3 \wedge [x_2, y_2] = z_1\\
&\wedge [x_3, h_j] = t_2 \wedge [h_i, y_3] = t_3 \wedge [h_i, y_4] = [x_3, y_3]\\
&\wedge [x_4, h_j] = t_1 \wedge [x_4, y_4] = z_2\\
&\rightarrow z_1 = z_2)
\end{aligned}$$

Conditions (3) of the strong basis are clearly formalizable in terms of first order formulas of signature of groups.

Conjunction of all the formulas whose existence proved above is the desired formula. Notice that there are only a finite number of formulas for each condition.

□

Theorem 2.3.14 (Characterization theorem). *Let $h_1, \ldots, h_n$ be some elements of H. Then the following are equivalent:*

(a) The group H has a strong basis, $\mathcal{B} = \{h_1, \ldots, h_n\}$;

(b) There is an associative ring R with unit and symmetric 2-cocycles

$$f^i : R^+ \times R^+ \rightarrow R^{\binom{n}{2}},$$

such that

$$H \cong N_{2,n}(R, f^1, \ldots, f^n).$$

If (a) holds each symmetric cocycle $f^i : R^+ \times R^+ \rightarrow R^{\binom{n}{2}}$, $1 \leq i \leq n$ is constructed such that each $H_i = C_H(h_i)$ is an abelian extension of $R^{\binom{n}{2}}$ by R^+ via the symmetric 2-cocycle f^i.

Proof. (b) $\Rightarrow$ (a) By Proposition 2.2.4 and Lemma 2.2.5.
(a)$\Rightarrow$ (b) We follow the notation of Lemma 2.3.8. For every $1 \leq i < j \leq n$ recover a ring R_{ij} as in 2.3.3. By Lemma 2.3.12, there is a ring R such $R \cong R_{ij}$ for every $1 \leq i < j \leq n$. So let us assume that $R = R_{ij}$ for all $1 \leq i < j \leq n$ and denote the multiplication of R by $\boxdot$.

We prove that the relations (a)-(d) of Lemma 2.2.5 hold in H with a suitable choice for h_i^α among the representatives $[h_i^\alpha]$ for $\alpha \in R$ and $1 \leq i \leq n$.

For each $1 \leq i < j, k \leq n$, $H_{ik} \cong H_{ij} \cong R^+$ by Lemma 2.3.8. Thus there is an isomorphism $\mu_1^i : H_i/Z(H) \to R^+$ for each $1 \leq i \leq n$. Let $\mu_2^i : H_i \to H_i/Z(H)$ be the canonical surjection and $\mu_i = \mu_1^i \circ \mu_2^i$. By condition (5)-(b) of Definition 2.3.1, $Z(H) \cong R^{\binom{n}{2}}$ as groups. Therefore the sequence,

$$0 \longrightarrow R^{\binom{n}{2}} \xrightarrow{id} H_i \xrightarrow{\mu_i} R \longrightarrow 0$$

is an exact sequence of abelian groups for each $1 \leq i \leq n$. Let for each $1 \leq i \leq n$, $f^i : R^+ \times R^+ \to R^{\binom{n}{2}}$ be the 2-cocycle corresponding to the extension above (see Subsection 1.2). Each f^i is clearly a symmetric 2-cocycle since H is abelian by condition (3) of Definition 2.3.1, hence H_i is an abelian extension of $R^{\binom{n}{2}}$ by R via the 2-cocycle f^i. Therefore $H_i \cong R^{\binom{n}{2}+1}$, $1 \leq i \leq n$, as groups when the multiplication:

$$\begin{aligned} xy &= (\alpha, \gamma_{12}, \ldots, \gamma_{n-1,n})(\beta, \gamma'_{12}, \ldots, \gamma'_{n-1,n}) \\ &= (\alpha + \alpha', \gamma_{12} + \gamma'_{12} + f^i_{12}(\alpha, \alpha'), \ldots, \\ &\qquad \gamma_{n-1,n} + \gamma'_{n-1,n} + f^i_{n-1,n}(\alpha, \alpha')), \end{aligned}$$

is assumed on $R^{\binom{n}{2}+1}$ and $f^i = (f^i_{12}, \ldots, f^i_{n-1,n})$. Suppose $\tau_i : R^{\binom{n}{2}+1} \to H_i$ be the group isomorphism whose existence established above. Now for each $1 \leq i \leq n$ and $\alpha \in R$ let $h_i^\alpha \in H_i$ be the element of the equivalence class $[h_i^\alpha]$ such that

$$h_i^\alpha = \tau_i(\alpha, \underbrace{0, \ldots, 0}_{\binom{n}{2}-\text{times}}).$$

Firstly notice that $h_i^\alpha = 1$ if and only if $\alpha = 0$. Moreover it is clear that for each $1 \leq i \leq n$ and $\alpha, \beta \in R$:

$$h_i^\alpha h_i^\beta = h_i^{\alpha+\beta} h_{12}^{f_{12}^i(\alpha,\beta)} \dots h_{n-1,n}^{f_{n-1,n}^i(\alpha,\beta)}.$$

Thus the relations (c) of Lemma 2.2.5 hold between h_i^α, $1 \leq i \leq n$ and $\alpha \in R$. We Note that,

$$[h_i^\alpha, h_j^\beta] = h_{ij}^{\alpha \Box_{ij} \beta} = h_{ij}^{\alpha \Box \beta}, \quad 1 \leq i < j \leq n, \alpha, \beta \in R,$$

which proves that relations (a) hold. Relations (b) are true in H since each g_{ij}^α, $1 \leq i < j \leq n$ and $\alpha \in R$, is central. Relations (d) hold also in H by the fact that each H_{ij} is an R-module.

By Lemma 2.3.8 the set,

$$\mathcal{H} = \{h_i^\alpha, h_{kl}^\beta : 1 \leq i \leq n, 1 \leq k < l \leq n, \quad \alpha, \beta \in R\},$$

generates H as a group. Let F be the free group on $\mathcal{H}$. Let $\mathcal{R}'$ be the normal closure of the relations in the lemma 2.2.5 in F, g_i and g_{ij} substituted by h_i and h_{ij} and the exponents come from the ring R defined here and $\odot$ taken to be concatenation. Let $\langle \mathcal{H} | \mathcal{R}' \rangle$ be F modulo the normal subgroup $\mathcal{R}'$. Consider the mapping:

$$\langle \mathcal{H} | \mathcal{R}' \rangle \longrightarrow H, \quad h_i^\alpha \mapsto h_i^\alpha, \quad h_{kl}^\alpha \mapsto h_{kl}^\alpha$$

for $\alpha \in R$, $1 \leq i \leq n$ and $1 \leq k < l \leq n$. The map is a well defined homomorphism of groups since as proved above the relations (a)-(d) of 2.2.5 hold also in H. The map is also surjective since $\mathcal{H}$ generates H. Every word W in $\langle \mathcal{H} | \mathcal{R}' \rangle$ is equivalent to a word of the form $h_n^{\alpha_n} \dots h_1^{\alpha_1} h_{12}^{\alpha_{12}} \dots h_{n-1,n}^{\alpha_{n-1,n}}$. So W maps to an element h of H of this form. The uniqueness of this form for h in H is guarantied by (5) of 2.3.1. So if h is trivial in H then all the exponents in the above form are zero so the word W is trivial in $\langle \mathcal{H} | \mathcal{R}' \rangle$. So $\langle \mathcal{H} | \mathcal{R}' \rangle \cong H$. But by Lemma 2.2.5, $\langle \mathcal{H} | \mathcal{R}' \rangle \cong N_{2,n}(R, f^1, \dots f^n)$, hence

$$N_{2,n}(R, f^1, \dots, f^n) \cong H.$$

□

2.4 $QN_{2,n}$ groups over commutative rings

2.4.1 Bilinear map of a nilpotent group and ring of a bilinear map

Let G be a nilpotent group of class n. The map

$$f_G : \Gamma_1(G)/\Gamma_2(G) \times \Gamma_{n-1}(G)/\Gamma_n(G) \longrightarrow \Gamma_n(G),$$

$f_G(x\Gamma_2(G), y\Gamma_n(G)) = [x, y]$, for $x \in \Gamma_1(G)$ and $y \in \Gamma_{n-1}(G)$ is a bilinear map. We call this map the *bilinear map of the nilpotent group G*. Note that in a $QN_{2,n}$ group $\Gamma_2(G) = \Gamma_n(G) = Z(G)$.

Let G_1, G_2 and G_0 be abelian groups. Let

$$f : G_1 \times G_2 \longrightarrow G_0$$

be a bilinear map of abelian groups. Consider the triples (ϕ_1, ϕ_2, ϕ_0) in the ring $END = End(G_1) \times End(G_2) \times End(G_0)$, which satisfy the identity,

$$f(\phi_1(x), y) = f(x, \phi_2(y)) = \phi_0(f(x, y)) \quad x \in G_1, y \in G_2.$$

The set of all such triples is a subring of END denoted by $P(f)$.

Lemma 2.4.1. *Let G be a $QN_{2,n}$ group over an associative ring R with unit. Let $(\phi_1, \phi_2, \phi_0) \in P(f_G)$ and $x, y \in G$. Then $\phi_1(xZ(G)) = x^\gamma Z(G)$, $\phi_2(yZ(G)) = y^\gamma Z(G)$ and $\phi_0(f_G(x, y)) = f_G(x, y)^\gamma$ for some $\gamma \in Z(R)$.*

Proof. Let $\{g_1, g_2, \ldots, g_n\}$ be the standard basis for G and $g_{ij} = [g_i, g_j]$ for $1 \le i < j \le n$. Since $g_i Z(G)$, $1 \le i \le n$, generate $G/Z(G)$ and g_{ij}, $1 \le i < j \le n$ generate $Z(G)$ it is enough to study the action of the ϕ_i on powers of basis elements.

Fix $1 \le i < j \le n$. We assume that $\phi_1(g_i Z(G)) = g_1^{\alpha_1} g_2^{\alpha_2} \ldots g_n^{\alpha_n} Z(G)$ and

$\phi_2(g_jZ(G)) = g_1^{\beta_1}\dots g_n^{\beta_n}Z(G)$ for some $\alpha_k, \beta_k \in R$, $1 \le k \le n$. Then

$$\begin{aligned} g_{1j}^{\alpha_1}\cdots g_{j-1,j}^{\alpha_{j-1}}g_{j,j+1}^{\alpha_{j+1}}\cdots g_{jn}^{\alpha_n} &= [g_1^{\alpha_1}\dots g_n^{\alpha_n}] \\ &= f_G(\phi_1(g_iZ(G)), g_jZ(G)) \\ &= \phi_0(g_{ij}) = f_G(g_iZ(G), \phi_2(g_jZ(G))) \\ &= [g_i, g_1^{\beta_1}\dots g_n^{\beta_n}] \\ &= g_{1i}^{-\beta_1}\cdots g_{i-1,i}^{-\beta_{i-1}}g_{i,i+1}^{\beta_{i+1}}\cdots g_{i,n}^{\beta_n}. \end{aligned}$$

So $\alpha_k = 0$, $k \neq i,j$, $\beta_l = 0$, $l \neq i,j$ and $\alpha_i = \beta_j$. It remains to determine β_i and α_j. For α_j it is enough to consider $f_G(\phi_1(g_iZ(G)), g_iZ(G)) = \phi_0([g_i, g_i]) = 1$ which proves that $\alpha_j = 0$. We also have $\beta_i = 0$ since $f_G(g_jZ(G), \phi_2(g_jZ(G))) = \phi_0([g_j, g_j]) = 1$. Assume $\gamma = \alpha_i = \beta_j$. Thus $\phi_1(g_i) = g_i^\gamma$, $\phi_2(g_j) = g_j^\gamma$ and $\phi_0(g_ij) = g_{ij}^\gamma$.

Next we prove that for every $1 \le s < t \le n$, the actions of ϕ_1, ϕ_2 on g_t and g_s and the action of ϕ_0 on g_{st} are the same as the action of the element γ of the ring R obtained above. Without loss of generality we can assume $i \le s \le j$. Suppose $\phi_1(g_s) = g_1^{\delta_1}\dots g_n^{\delta_n}$ and $\phi_2(g_s) = g_1^{\mu_1}\dots g_n^{\mu_n}$ then,

$$\begin{aligned} g_{is}^\gamma &= [g_i^\gamma, g_s] \\ &= f_G(\phi_1(g_iZ(G)), g_sZ(G)) = \phi_0(g_{is}) = f_G(g_i, \phi_2(g_s)) \\ &= g_{1i}^{\mu_2}\cdots g_{i-1,i}^{\mu_{i-1}}g_{i,i+1}^{\mu_{i+1}}\cdots g_{in}^{\mu_n}. \end{aligned}$$

So $\mu_s = \gamma$ and $\mu_k = 0$ if $k \neq s, i$. The identity,

$$f_G(g_sZ(G), \phi_2(g_sZ(G))) = \phi_0([g_s, g_s]) = 1,$$

proves that $\mu_i = 0$. Similar considerations show that $\delta_s = \gamma$ and $\delta_k = 0$ if $k \neq s$. Therefore for each t, $1 \le t \le n$,

$$\phi_1(g_t) = \phi_2(g_t) = g_t^\gamma.$$

It is clear that

$$\phi_0(g_{st}) = g_{st}^\gamma,$$

for $1 \leq s < t \leq n$.

Next let $\alpha \in R$ and fix $1 \leq i < j \leq n$. Suppose $\phi_1(g_i^\alpha Z(G)) = g_1^{\alpha_1} \dots g_n^{\alpha_n} Z(G)$ and $\phi_2(g_j^\alpha Z(G)) = g_1^{\beta_1} \dots g_n^{\beta_n} Z(G)$ then:

$$\begin{aligned} g_{1j}^{\alpha_1} \cdots g_{j-1,j}^{\alpha_{j-1}} g_{j,j+1}^{-\alpha_{j+1}} \cdots g_{jn}^{-\alpha_n} &= [g_1^{\alpha_1} \dots g_n^{\alpha_n}, g_j] \\ &= f_G(\phi_1(g_i^\alpha Z(G)), g_j Z(G)) = \phi_0(g_{ij}^\alpha) \\ &= f_G(g_i^\alpha Z(G), \phi_2(g_j Z(G))) \\ &= [g_i^\alpha, g_j^\gamma] = g_{ij}^{\alpha\gamma} \end{aligned}$$

So $\alpha_i = \alpha\gamma$ and $\alpha_k = 0$ if $k \neq i, j$. To prove $\alpha_j = 0$ it is enough to consider $f_G(\phi_1(g_i^\alpha, g_i)) = 1$. We also have:

$$\begin{aligned} g_{1i}^{-\beta_1} \cdots g_{i-1,i}^{-\beta_{i-1}} g_{i,i+1}^{\beta_{i+1}} \cdots g_{in}^{\beta_n} &= [g_i, g_1^{\beta_1} \dots g_n^{\beta_n}] \\ &= f_G(g_i Z(G), \phi_2(g_j^\alpha Z(G))) = \phi_0(g_{ij}^\alpha) \\ &= f_G(\phi_1(g_i Z(G)), g_j^\alpha Z(G)] \\ &= [g_i^\gamma, g_j^\alpha] = g_{ij}^{\gamma\alpha} \end{aligned}$$

Therefore $\beta_j = \gamma\alpha$ and $\beta_i = 0$ if $i \neq j$. Also from the equations above $g_{ij}^{\alpha\gamma} = \phi_0(g_{ij}^\alpha) = g_{ij}^{\gamma\alpha}$ which implies $\alpha\gamma = \gamma\alpha$. Hence $\gamma \in Z(R)$. □

Proposition 2.4.2. *Let R be a commutative associative ring with unit and G be a $QN_{2,n}$ group over R. Then $P(f_G) \cong R$*

Proof. Define a mapping

$$\eta : P(f_G) \to R, \quad (\phi_0, \phi_1, \phi_2) \mapsto \gamma_\phi$$

where $\phi_1(x) = \phi_2(x) = x^{\gamma_\phi}$ for $x \in G/Z(G)$ and $\phi_0(y) = y^{\gamma_\phi}$ for $y \in Z(G)$. Such a γ_ϕ exists by Lemma 2.4.1. The mapping is well defined since if $x^{\gamma_1} = x^{\gamma_2}$ for any $x \in G/Z(G)$ then $g_i^{\gamma_1} = g_i^{\gamma_2}$ implies $\gamma_1 = \gamma_2$ since g_i is a generator of a cyclic module over ring R with unit and so there is an R-module isomorphism taking g_i to the unit of R.

Let γ be an element of the ring R. Define a triple (ϕ_0, ϕ_1, ϕ_2) where $\phi_1, \phi_2 \in End(G/Z(G))$ and $\phi_0 \in End(Z(G))$ by setting $\phi_1(x) = \phi_2(x) = x^\gamma$, for $x \in G/Z(G)$

and $\phi_0(y) = y^\gamma$, for $y \in Z(G)$. We show that $(\phi_0, \phi_1, \phi_2) \in P(f_G)$. Let $\{g_1, \dots, g_n\}$ be the standard basis for G, $g_{ij} = [g_i, g_j]$, $1 \leq i < j \leq n$, $xZ(G) = g_1^{\alpha_1} \dots g_n^{\alpha_n} Z(G)$ and $yZ(G) = g_1^{\beta_1} \dots g_n^{\beta_n}$ for $x, y \in G$. Then by associativity and commutativity of R,

$$\begin{aligned} f_G(x^\gamma Z(G), yZ(G)) &= f_G(g_1^{\alpha_1} \dots g_n^{\alpha_n} Z(G), g_1^{\beta_1} \dots g_n^{\beta_n} Z(G)) \\ &= g_{12}^{((\alpha_1\gamma)\beta_2 - \beta_1(\alpha_2\gamma))} \dots g_{n-1,n}^{((\alpha_{n-1}\gamma)\beta_n - \beta_{n-1}(\alpha_n\gamma))} \\ [x,y]^\gamma &= (g_{12}^{(\alpha_1\beta_2 - \beta_1\alpha_2)} \dots g_{n-1,n}^{(\alpha_{n-1}\beta_n - \beta_{n-1}\alpha_n)})^\gamma \\ &= g_{12}^{(\alpha_1(\beta_2\gamma) - \beta_1(\gamma\alpha_2))} \dots g_{n-1,n}^{(\alpha_{n-1}(\beta_n\gamma) - (\beta_{n-1}\gamma)\alpha_n)} \\ &= f_G(xZ(G), y^\gamma Z(G)). \end{aligned}$$

So $(\phi_0, \phi_1, \phi_2) \in P(f_G)$. This proves the surjectivity of η. If $(\phi_0, \phi_1, \phi_2) \in P(f_G)$ maps to the zero of the ring R under the mapping η, it means that all the ϕ_i are zero endomorphisms. Hence (ϕ_0, ϕ_1, ϕ_2) is the zero of $P(f_G)$. Hence the mapping η is injective.

To prove that η is an additive homomorphism note that

$$\eta((\phi_0, \phi_1, \phi_2) + (\psi_0, \psi_1, \psi_2)) = \eta((\phi_0 + \psi_0, \phi_1 + \psi_1, \phi_2 + \psi_2))$$

is an element γ of R such that for every $x \in G/Z(G)$, $x^\gamma = \phi_1(x)\psi_1(x) = \phi_2(x)\psi_2(x)$ and for every $y \in Z(G)$, $y^\gamma = \phi_0(y)\psi_0(y)$. But $\phi_1(x)\psi_1(x) = \phi_2(x)\psi_2(x) = x^{\gamma_\phi} x^{\gamma_\psi}$ and $\phi_0(y)\psi_0(y) = y^{\gamma_\phi} y^{\gamma_\psi}$. Thus $\gamma = \gamma_\phi + \gamma_\psi$, hence η is an additive homomorphism. On the other hand identities $\phi_1 \circ \psi_1(x) = x^{\gamma_\psi \gamma_\phi} = x^{\gamma_\phi \gamma_\psi}$ and $\phi_0 \circ \psi_0(y) = y^{\gamma_\psi \gamma_\phi} = y^{\gamma_\phi \gamma_\psi}$ imply the multiplicative linearity of η. The proposition is proved. □

Theorem 2.4.3. *Let R be a an associative ring with unit. If $N_{2,n}(R, f_1, \dots f_n) \cong N_{2,n}(S, q_1, \dots q_n)$ then $Z(R) \cong Z(S)$. In particular if R is commutative, S is also commutative.*

Proof. The proof is exactly like the theorem 1.13 of [2] just instead of using Proposition 1.12 of that paper we should use Lemma 2.4.1 above. □

2.4.2 Groups elementary equivalent to a free 2-nilpotent group of arbitrary finite rank

Theorem 2.4.4. *If $\mathcal{R}$ is a (finitely) axiomatizable class of associative rings then the class $QN_{2,n}(\mathcal{R})$ is (finitely) axiomatizable.*

Proof. By Lemma 2.3.13 there is a first order formula $Stbasis(x_1, \dots x_n)$ of the signature of groups such that $\mathcal{B} = \{g_1, \dots, g_n\}$ is a strong basis for G if and only if

$$(G, \mathcal{B}) \models Stbasis(g_1, \dots, g_n).$$

Let G be a $QN_{2,n}$ group over a ring R with unit. The ring R is interpretable in $(G, \mathcal{B})$ by Lemma 2.3.3. So for every first order formula $\phi(y_1, \dots, y_m)$ of the signature of rings there is a first order formula $\phi^*(x_1, \dots x_n, y_1, \dots, y_m)$ of the signature of groups such that

$$R \models \phi(\alpha_1, \dots, \alpha_m) \Leftrightarrow (G, \mathcal{B}) \models \phi^*(\bar{g}, \alpha_1, \dots, \alpha_m),$$

where $\alpha_i \in R = H_{ls}$, $1 \leq i \leq m$, $1 \leq l < s \leq n$. Now let $\phi \in Th(\mathcal{R})$ and ψ_ϕ be the sentence:

$$\exists x_1 \dots x_n (Stbasis(\bar{x}) \wedge \phi^*(\bar{g}, \bar{x})).$$

The sentences ψ_ϕ when ϕ runs through $Th(\mathcal{R})$ axiomatize the class $QN_{2,n}(\mathcal{R})$. □

Lemma 2.4.5. *Let R be a commutative associative ring with unit. Suppose $G \equiv N_{2,n}(R, f_1, \dots f_n)$. Then $G \cong N_{2,n}(S, q_1, \dots q_n)$ for some ring S such that $R \equiv S$.*

Proof. By Theorem 2.4.4 the group G has the form $N_{2,n}(S, q_1, \dots q_n)$ for some ring S such that $S \models Th(R)$. So in particular $S \equiv R$. Suppose there is another ring S' with this property. By Theorem 2.4.3, $S \cong S'$, since both of them are commutative as R is. So $G \cong N_{2,n}(S, q_1, \dots q_n)$ where S is unique up to isomorphism. □

We are now able to prove the main result of this thesis as a corollary of Lemma 2.4.5

Theorem 2.4.6. *Let G be a free 2-nilpotent group of rank n. Let H be a group such that $G \equiv H$. then H has the form $N_{2,n}(R, f_1, \dots f_n)$ for some ring $R \equiv \mathbb{Z}$.*

Proof. By Proposition 2.1.2, $G \cong N_{2,n}(\mathbb{Z})$. By Lemma 2.4.5, H has the form indicated in the statement of the theorem. □

Remark 2.4.7. This question can come into mind that whether the result in Theorem 2.4.6 can be enhanced. For example is the group H, keeping the notation of Theorem 2.4.6, of the form $N_{2,n}(R)$ for some ring $R \equiv \mathbb{Z}$. In [1], Belegradek constructed a group elementary equivalent to $UT_3(\mathbb{Z})$ which is not a UT_3 group (see also [2]). So the class of all unitriangular groups is not axiomatizable. He actually specifies exactly which subclass of QUT_3, the class of all quasiunitriangular groups ($QN_{2,2}$ groups for us) is the elementary closure of the class of all unitriangular groups. Here we just assumed that the same thing is true, namely, the class of all $N_{2,n}$ groups is not elementary closed. We will try to carry over his result to $N_{2,n}$ groups in a future work.

Chapter 3

The method of bilinear mappings

In this chapter we use the concepts of bilinear mapping f_G of a nilpotent group G and its maximal commutative ring $P(f_G)$ introduced in Subsection 2.4.1 in a much more substantial way. We introduce $P(f_G)$ in a bit different from the construction in the referred subsection which is suitable for model theoretic purposes, though they happen to be the same in the end. Then we present a proof that the ring $P(f_G)$ is absolutely interpretable in G providing that G is a finitely generated group in which the center and the commutator subgroup coincide. This leads to an alternate proof for Theorem 2.4.6. We review the required material from [11].

3.1 Some model theory of bilinear mappings

Let M and N be exact R-modules for some commutative ring R. An R-module M is *exact* if $rm = 0$ for $r \in R$ and $0 \neq m \in M$ imply $r = 0$. Let's recall that an R-bilinear mapping $f : M \times M \to N$ is called *non-degenerate* in both variables if $f(x, M) = 0$ or $f(M, x) = 0$ implies $x = 0$. We call the bilinear map f, *"onto"* if N is generated by $f(x, y)$, $x, y \in M$. We associate two many sorted structures to every bilinear mapping described above. One of them

$$\mathfrak{U}_R(f) = \langle R, M, N, \delta, s_M, s_N \rangle,$$

where the predicate δ describes the mapping and s_M and s_N describe the actions of R on the modules M and N respectively. The other one,

$$\mathfrak{U}(f) = \langle R, M, N, \delta \rangle,$$

contains only a predicate δ describing the mapping f. It can be easily seen that the structure $\mathfrak{U}(f)$ is absolutely interpretable in $\mathfrak{U}_R(f)$. We intend to show that there is a ring $P(f)$ such that $\mathfrak{U}_{P(f)}(f)$ is absolutely interpretable in $\mathfrak{U}(f)$. Moreover this ring is the maximal ring relative to which f remains bilinear.

3.1.1 Regular Vs. absolute interpretability

In this subsection we discuss the relation between regular and absolute interpretability. We are mostly concerned with this question that under what circumstances regular interpretability implies the absolute interpretability. The concepts regular, relative and absolute interpretability were introduced in Subsection 1.3.2. We denote the regular interpretation of the structure $\mathfrak{U}$ in the structure $\mathfrak{B}$ of signature Δ with formula Φ of signature Δ by $\Psi(\mathfrak{B}, \Phi)$. Let $\Phi(\mathfrak{B}^n) = \{\bar{a} \in |\mathfrak{B}|^n : \mathfrak{B} \models \Phi(\bar{a})\}$. Then $\Psi(\mathfrak{B}, \bar{b})$ introduced in Subsection 1.3.2 for $\bar{b} \in \Phi(\mathfrak{B})$ will be denoted by $\mathfrak{U}(\bar{b})$.

Definition 3.1.1. *A system of isomorphisms $\theta_{\bar{b},\bar{c}} : \mathfrak{U}(\bar{b}) \to \mathfrak{U}(\bar{c})$ is connecting if $\theta_{\bar{b},\bar{c}} \circ \theta_{\bar{c},\bar{d}} = \theta_{\bar{b},\bar{d}}$ holds for any $\bar{b}, \bar{c}, \bar{d} \in \Phi(\mathfrak{B}^n)$. A connecting isomorphism $\theta_{\bar{b},\bar{c}}$ of interpretation $\Psi(\mathfrak{B}, \Phi)$ is said to be definable if there is a formula $Is(\bar{x}, \bar{y}, \bar{z}_1, \bar{z}_2)$ of signature Δ such that $\mathfrak{B} \models Is(\bar{b}, \bar{c}, \bar{a}_1, \bar{a}_2)$ for $\bar{a}_1 \in \Phi(\bar{b})$ and $\bar{a}_2 \in \Phi(\bar{c})$ if and only if $\theta_{\bar{b},\bar{c}}([\bar{a}_1]) = [\bar{a}_2]$.*

Lemma 3.1.2. *Suppose the structure $\mathfrak{U}$ is regularly interpretable in a structure $\mathfrak{B}$ of signature Δ with formula Φ of the signature Δ. If the connecting isomorphisms of the interpretation $\Psi(\mathfrak{B}, \Phi)$ are definable in $\mathfrak{B}$ then $\mathfrak{U}$ is absolutely interpretable in $\mathfrak{B}$.*

Proof. We follow the notation of Subsection 1.3.2. Suppose all the connecting isomorphisms of interpretation $\Psi(\mathfrak{B}, \Phi)$ are definable in $\mathfrak{B}$. First we make all the sets

$A(\bar{b})$ disjoint by adjoining the tuple $\bar{b}$ to the right of each tuple $\bar{a} \in A(\bar{b})$. Now set

$$A = \bigcup_{\bar{b}\in\Phi(\mathfrak{B})} A(\bar{b}) = \{\bar{a} \in A(\bar{B}) : \bar{b} \in \Phi(\mathfrak{B}^n), A(\bar{a}, \bar{b})\}.$$

Now define a predicate $Id(\bar{x}, \bar{y})$ on A by:

$$Id(\bar{x}, \bar{y}) \Leftrightarrow \exists \bar{z}_1 \bar{z}_2 \Phi(\bar{z}_1) \wedge \Phi(\bar{z}_2) \wedge Is(\bar{x}, \bar{y}, \bar{z}_1, \bar{z}_2).$$

This means that elements of the set A are in the relation Id if and only if there is a connecting isomorphism of the interpretation $\Psi(\mathfrak{B}, \Phi)$ taking one element to the other. Thus Id is a definable equivalence relation on a definable subset A. Let us fix $\bar{b} \in \Phi(\mathfrak{B}^n)$. There is an injection $A(\bar{b}) \to A$ which induces a bijection

$$\eta_{\bar{b}} : A(\bar{b})/\epsilon_{\bar{b}} \to A/Id.$$

Let Σ' be the signature introduced for $A(\bar{b})/\epsilon_{\bar{b}}$ as a result of interpreting $\mathfrak{U}$ in $\mathfrak{B}$. Now we can introduce a signature Σ'' for A/Id consisting of predicate symbols $P^{\eta_{\bar{b}}}$ for each predicate symbol P of signature Σ'. Let σ be a s-ary predicate symbol of signature Σ for the structure $\mathfrak{U}$ and $\psi(\bar{x}, \bar{y}^1, \ldots, \bar{y}^s)$ be the formula of signature Δ defining the predicate P_σ on $A(\bar{b})/\epsilon_{\bar{b}}$. Now we define a structure $\mathfrak{U}_0$ on A/Id by letting $P_\sigma^{\eta_{\bar{b}}}([\bar{a}_1]^{\eta_{\bar{b}}}, \ldots, [\bar{a}_s]^{\eta_{\bar{b}}})$, $\bar{a}_i \in A(\bar{b})$, $1 \leq i \leq s$, if and only if there are $\bar{c}_1, \ldots, \bar{c}_s$ in $A(\bar{b})$ and connecting isomorphisms θ_i such that for each $1 \leq i \leq s$, $\theta([\bar{c}_i]) = [\bar{a}_i]$ and $P_\sigma([\bar{c}_1], \ldots, [\bar{c}_s])$ holds in $\mathfrak{U}(\bar{b})$. If $\bar{c} \in \Phi(\mathfrak{B}^n)$ be tuple different from $\bar{b}$ and $\theta : \mathfrak{U}(\bar{b}) \to \mathfrak{U}(\bar{c})$ be a connecting isomorphism then the diagram

$$\begin{array}{ccc} A(\bar{b})/\epsilon_{\bar{b}} & \xrightarrow{\eta_{\bar{b}}} & A/Id \\ \theta\downarrow & & \downarrow Id \\ A(\bar{c})/\epsilon_{\bar{c}} & \xrightarrow[\eta_{\bar{c}}]{} & A/Id \end{array}$$

is commutative. Therefore $[\bar{a}_i]^{\eta_{\bar{b}}} = [\bar{a}_i]^{\eta_{\bar{c}}}$ and the definition above is independent of the choice $\bar{b} \in \Phi(\mathfrak{B}^n)$ and we can drop the subscript $\bar{b}$ from $\eta_{\bar{b}}$ in the definitions. Now it is clear that

$$P_\sigma^\eta([\bar{a}_1]^\eta, \ldots, [\bar{b}_s]^\eta) \Leftrightarrow \mathfrak{B} \models \exists \bar{x}, \bar{y}_1, \ldots, \bar{y}_s \Phi(\bar{x}) \bigwedge_{1\leq i\leq s} A(\bar{x}, \bar{y}_i) \bigwedge_{1\leq i\leq s} Id(\bar{x}, \bar{y}_i) \wedge \Psi_\sigma(\bar{x}, \bar{y}_1, \ldots, \bar{y}_s).$$

Therefore all the predicates of $\mathfrak{U}_0$ are definable in $\mathfrak{B}$. The isomorphism of $\mathfrak{U}$ and $\mathfrak{U}_0$ is also clear. Therefore $\mathfrak{U}$ is absolutely interpretable in $\mathfrak{B}$. □

Lemma 3.1.3. *Let M_R be regularly interpretable in the algebraic structure $\mathfrak{B}$ of signature Δ with the help of a formula Φ of the signature Δ such that the abelian group M is absolutely interpretable in $\mathfrak{B}$. Then M_R is absolutely interpretable in $\mathfrak{B}$.*

Proof. We prove that the connecting isomorphisms of the interpretation $\Psi(\mathfrak{B}, \Phi)$ are definable in $\mathfrak{B}$ (see Lemma 3.1.2).

Let $\bar{b_1}, \bar{b_2} \in \Phi(\mathfrak{B}^n)$. Each connecting isomorphism $\theta : M_R(\bar{b_1}) \to M_R(\bar{b_2})$ has two components $\theta_1 : M(\bar{b_1}) \to M(\bar{b_2})$ and $\theta_2 : R(\bar{b_1}) \to R(\bar{b_2})$. Since M is absolutely interpretable in $\mathfrak{B}$, $M(\bar{b_1}) = M(\bar{b_2})$ and θ_1 is the identity mapping. Therefore definability of θ reduces to definability of θ_2. Let $\bar{\alpha} \in R(\bar{b_1})$. The action of $\bar{\alpha}$ induces an endomorphism $\phi_{\bar{\alpha}} : M(\bar{b_1}) \to M(\bar{b_1})$. Let $\psi_\delta(\bar{x}, \bar{y}, \bar{z_1}, \bar{z_2})$ be the formula of the signature Δ defining the above mentioned action in the interpretation $\Psi(\mathfrak{B}, \Phi)$, i.e. $\phi_{\bar{\alpha}}(\bar{u}) = \bar{v} \Leftrightarrow \psi_\delta(\bar{b_1}, \bar{\alpha}, \bar{u}, \bar{v})$. On the other hand $\theta_2(\bar{\alpha}) = \bar{\beta}$, for $\bar{\alpha} \in R(\bar{b_1})$ and $\bar{\beta} \in R((\bar{b_2})$ holds if and only if $\phi_{\bar{\alpha}} = \phi_{\bar{\beta}}$, since the only predicate in M_R, δ, describes the action of the ring R on the module M. The later equality holds if and only if the formula

$$\mathfrak{B} \models \forall \bar{x}, \bar{y}(\psi_\delta(\bar{b_1}, \bar{\alpha}, \bar{x}, \bar{y}) \leftrightarrow \psi_\delta(\bar{b_2}, \bar{\beta}, \bar{x}, \bar{y})),$$

and the definability of θ_2 is proved. □

3.1.2 Enrichments of bilinear mappings

Let M be an R-module and let $\mu : R \to P$ be an inclusion of rings. Then the P-module M is an *P-enrichment* of the R-module M with respect to μ if for every $r \in R$ and $m \in M$, $rm = \mu(r)m$. Let us denote the set of all R endomorphisms of the R-module M by $End_R(M)$. Suppose the R-module M admits a P-enrichment with respect to the inclusion of rings $\mu : R \to P$. Then every $\alpha \in P$ induces an R-endomorphism, $\phi_\alpha : M \to M$ of modules defined by $\phi_\alpha(m) = \alpha m$ for $m \in M$. This in turn induces an injection $\phi_P : P \to End_R(M)$ of rings. Thus we associate a subring of the ring $End_R(M)$ to every ring P with respect to which there is an enrichment of the R-module M.

Definition 3.1.4. *Let $f : M \times M \to N$ be an R-bilinear "onto" mapping and $\mu : R \to P$ be an inclusion of rings. The mapping f admits P-enrichment with respect to μ if the R-modules M and N admit P enrichments with respect to μ and f remains bilinear with respect to P. We denote such an enrichment by $E(f, P)$.*

We define an ordering $\leq$ on the set of enrichments of f by letting $E(f, P_1) \leq E(f, P_2)$ if and only if f as an P_1 bilinear mapping admits a P_2 enrichment with respect to inclusion of rings $P_1 \to P_2$. The largest enrichment $E_H(f, P(f))$ is defined in the obvious way. We shall prove existence of such an enrichment for a large class bilinear mappings.

Proposition 3.1.5. *If $f : M \times M \to N$ is a non-degenerate "onto" R-bilinear mapping over a commutative ring R, f admits the largest enrichment.*

Proof. An R-endomorphism A of the R-module M is called *symmetric* if $f(Ax, y) = f(x, Ay)$ for every $x, y \in M$. Let us denote the set of all such endomorphisms by $Sym_f(M)$. Set $Z = \{B \in Sym_f(M) : A \circ B = B \circ A, \quad \forall A \in Sym_f(M)\}$. This set is actually an R-subalgebra of $End_R(M)$. Let for each n, Z_n be the set of all endomorphisms A in $Sym_f(M)$ that satisfy the formula

$$S_n(A) \Leftrightarrow \forall x_i, y_i, u_i, v_i \sum_{i=1}^{n} f(x_i, y_i) = \sum_{i=1}^{n} f(u_i, v_i) \to$$
$$\sum_{i=1}^{n} f(Ax_i, y_i) = \sum_{i=1}^{n} f(Au_i, v_i).$$

Each Z_n is also an R-subalgebra of Z. Now set $P(f) = \cap_{i=1}^{\infty} Z_n$. The identity mapping is in every Z_n so $P(f)$ is not empty. Since the mapping f is "onto" for every $x \in N$ there are x_i and y_i, $1 \leq i \leq n$, in M such that $x = \sum_{i=1}^{n} f(x_i, y_i)$ for some n. Thus we can define the action of $P(f)$ on N by setting $Ax = \sum_{i=1}^{n} f(Ax_i, y_i)$. The action is clearly well-defined since A satisfies all the $S_n(A)$.

In order to prove that the ring $P(f)$ is the largest ring of scalars, we prove that for any ring P with to respect to which f is bilinear, $\phi_P(P) \subseteq P(f)$. Since f is P bilinear $\phi_P(P) \subseteq Sym_f(M)$. Let $\alpha \in P$ then for $A \in Sym_f(M)$ and $x, y \in M$,

$$\begin{aligned} f(A \circ \phi_\alpha(x), y) &= f(\phi_\alpha(x), Ay) \\ &= \alpha f(x, Ay) = \alpha f(Ax, y) \\ &= f(\phi_\alpha \circ A(x), y). \end{aligned}$$

The degeneracy of f implies that $\phi_\alpha \circ A = A \circ \phi_\alpha$. Therefore $\phi_P(P) \subseteq Z$. It is clear that ϕ_α belongs to every Z_n by bilinearity of f with respect to P. Therefore $\phi_P(P) \subseteq P(f)$, hence $E(f, P) \leq E(f, P(f))$. □

3.1.3 Interpretability of the $P(f)$ structure

Let $f : M \times M \to N$ be a non-degenerate "onto" R-bilinear mapping for some commutative ring R. The mapping f is said to have *finite width* if there is a natural number S such that for every $u \in N$ there are x_i and y_i in M we have

$$u = \sum_{i=1}^{n} f(x_i, y_i).$$

The least such number, w(f), is the *width* of f.

A set $E = \{e_1, \ldots e_n\}$ is a complete system for f if $f(x, E) = f(E, x) = 0$ for $x \in M$ implies $x = 0$. The cardinality of a complete system with minimal cardinality is denoted by $c(f)$.

Type of a bilinear mapping f, denoted by $\tau(f)$, is the couple $(w(f), c(f))$. The mapping f is said to be of finite type if $c(f)$ and $w(f)$ are both finite numbers.

Now we state the main theorem of this subsection:

Theorem 3.1.6. *Let $f : M \times M \to$ be non-degenerate "onto" bilinear mapping of finite type. Then the structure $\mathfrak{U}_{P(f)}(f)$ is absolutely interpretable in $\mathfrak{U}(f)$*

We proceed by proving two lemmas.

Lemma 3.1.7. *Let f be a bilinear mapping as in the statement of Theorem 3.1.6. The abelian group $Sym_f(M)$ and its action on M are regularly interpretable in M.*

Proof. Firstly let us notice that any endomorphism in $Sym_f(M)$ is determined by its action on any complete system for f. Let $A, B \in Sym_f(M)$ and $E = \{e_1, \dots e_n\}$ be a complete system for f and $x \in M$. Suppose also $Ae_i = Be_i$ for each $i = 1, \dots, n$. Then for each i,

$$f(Ax, e_i) = f(x, Ae_i) = f(x, Be_i) = f(Bx, e_i).$$

Similarly $f(e_i, Ax) = f(e_i, Bx)$ for each $i = 1, \dots n$. Thus the completeness of E and non-degeneracy of f imply that $Ax = Bx$. Therefore $A = B$. Now let A be a symmetric endomorphism of M and E a complete system as above. Let $Ae_i = a_i$, $i = 1, \dots, n$, and $\bar{a} = (a_1, \dots a_n)$. By discussion above the element $y = Ax$ is determined uniquely by the formula

$$S_0(x, y, \bar{a}, E) \Leftrightarrow_{df} \bigwedge_{i=1}^{n} (f(x, a_i) = f(y, e_i) \wedge f(a_i, x) = f(e_i, y)).$$

The symmetry of A is describable by the formula

$$S_1(\bar{a}, E) \Leftrightarrow_{df} \bigwedge_{i=1}^{2} (S_0(x_1, x_2, \bar{a}, E)) \to f(x_1, y_2) = f(y_1, x_2).$$

Clearly $\bar{a}$ satisfies the condition:

$$S(\bar{a}, E) \Leftrightarrow_{df} \forall x \exists y \quad S_1(\bar{a}, E) \wedge S_0(x, y, \bar{a}, E).$$

Conversely suppose a tuple $\bar{a}$ satisfies $S(\bar{x}, E)$. The formula S_0 determines a unique mapping $A : M \to M$ such that $Ae_i = a_i$, $i = 1, \dots, n$. The mapping A satisfies $S_1(\bar{a}, E)$, hence A is symmetric. We show that it is also a homomorphism. Let $x, y \in M$. By symmetry of A and bilinearity of f,

$$\begin{aligned} f(A(x+y), e_i) =& f(x+y, a_i) = f(x, a_i) + f(y, a_i) \\ &= f(Ax, e_i) + f(Ay, e_i) = f(Ax + Ay, e_i) \end{aligned}$$

for each $i = 1, \dots n$. The identity $f(e_i, A(x+y)) = f(e_i, Ax + Ay)$ can be obtained in a similar manner. The two identities with completeness of E and non-degeneracy of f imply that $A(x+y) = Ax + Ay$. The R-linearity can also be obtained easily.

Thus the subset $S = \{\bar{a} \in M^n : \mathfrak{U}(f) \models S(\bar{a}, E)\}$ is a subgroup of M^n isomorphic to $Sym_f(M)$ via the mapping:

$$A \mapsto (Ae_1, \dots Ae_n).$$

Therefore the group $Sym_f(M)$ is regularly interpretable in $\mathfrak{U}_f$ with the help of the formula:

$$\forall x(\bigwedge_{i=1}^{n}(f(x, y_i) = 0 \wedge f(y_i, x) = 0)) \to x = 0,$$

which defines the complete systems of cardinality n for the mapping f with $c(f) = n$ in $\mathfrak{U}_f$. The action of $Sym_f(M)$ on M is also defined by the formula S_0 described above. The lemma is proved. □

Lemma 3.1.8. *Let the bilinear mapping f of Theorem 3.1.6 have width s. Then for any $n \geq s+1$, $Z_n = Z_{n+1}$.*

Proof. $Z_{n+1} \subseteq Z_n$ clear by the definition of Z_n.

For the converse let $A \in Z_{s+1}$. Let first prove that for x_i, y_i, u_i and v_i, $1 \leq i \leq n$, in M

$$\sum_{i=1}^{n} f(x_i, y_i) = \sum_{i=1}^{s} f(u_i, v_i) \tag{3.1}$$

implies

$$\sum_{i=1}^{n} f(Ax_i, y_i) = \sum_{i=1}^{s}(Au_i, v_i). \tag{3.2}$$

If $n = s+1$ it is true by the assumption that $A \in Z_{s+1}$. So suppose $n > s+1$. We proceed by induction on n. Since f has width s,

$$\sum_{i=1}^{n-1} f(x_i, y_i) = \sum_{i=1}^{s} f(x'_i, y'_i) \tag{3.3}$$

for some x' and y' in M. Equation (3.3) and the induction hypothesis imply

$$\sum_{i=1}^{n-1} f(Ax_i, y_i) = \sum_{i=1}^{s} f(Ax'_i, y'_i). \tag{3.4}$$

On the other hand from (3.1) and (3.3) we have

$$\sum_{i=1}^{s} f(x_i', y_i') + f(x_n, y_n) = \sum_{i=1}^{s} f(u_i, v_i),$$

which along with the assumption $A \in Z_{n+1}$ implies

$$\sum_{i=1}^{s} f(Ax_i', y_i') + f(Ax_n, y_n) = \sum_{i=1}^{s} f(Au_i, v_i). \tag{3.5}$$

Equations (3.4) and (3.5) entail (3.2), which is the desired result. □

Proof. **(Proof of Theorem 3.1.6)**

By Lemma 3.1.7 the abelian group $Sym_f(M)$ and its action on M are regularly interpretable in $\mathfrak{U}_f$. The algebra Z is definable in $Sym_f(M)$ without parameters, hence is regularly interpretable in $\mathfrak{U}_f$ (see proof of the Proposition 3.1.5). For each n, Z_n is definable in Z which guaranties the regular interpretability of each Z_n in $\mathfrak{U}_f$. By Lemma 3.1.8 and definition of $P(f)$ for the mapping f of width s, $P(f) = \cap_{n=1}^{\infty} Z_n = \cap_{n=1}^{s} Z_n$, which proves that $P(f)$ is regularly interpretable in $\mathfrak{U}_f$. The regular interpretability of the action of $P(f)$ on M is clear. Interpretability of the action of $P(f)$ on N is easily proved by interpretability of action $P(f)$ on M.

We have proved that the structures $M_{P(f)} = \langle P(f), M, \delta_M \rangle$ and $N_{P(f)} = \langle P(f), N, \delta_N \rangle$ where δ_M and δ_N describe the action of $P(f)$ on M and N respectively are regularly interpretable in $\mathfrak{U}(f)$. The abelian groups M and N are absolutely interpretable in $\mathfrak{U}(f)$ obviously. Lemma 3.1.3 implies that both structures $M_{P(f)}$ and $N_{P(f)}$ are absolutely interpretable in $\mathfrak{U}(f)$. Consequently $\mathfrak{U}_{P(f)}(f)$ will be absolutely interpretable in $\mathfrak{U}(f)$. And we are done. □

Remark 3.1.9. If we scrutinize the proofs of Theorem 3.1.6 and the lemmas proceeding it we realize that the formulas needed to interpret the maximal ring $P(f)$ in $\mathfrak{U}(f)$ only depend on the type $\tau(f)$ of the mapping f. Therefore if g is a bilinear mapping with a type less than that f (assuming the lexicographical order on τ) then $P(g)$ is interpretable in $\mathfrak{U}(g)$ with the same formulas which interpret $P(f)$ in $\mathfrak{U}(f)$. The same thing is true in the case of the action of the ring $P(g)$ on the corresponding modules.

3.2 Groups elementary equivalent to free 2-nilpotent of arbitrary finite rank revisited

Now we prove that ring $P(f_H)$ where H is a group with basis is absolutely interpretable in H. Then we use this fact to characterize groups elementary equivalent to a free 2-nilpotent group of finite rank. We actually give an alternate proof of Theorem 2.4.6.

Lemma 3.2.1. *Let G be group with elements $g_1, \dots, g_n$ constituting a basis for G. Then the ring $P(f_G)$ is absolutely interpretable in G.*

Proof. f_G has width at most $\frac{n(n+1)}{2}$. Moreover the set $\{g_1, \dots g_n\}$ is a finite complete system for f_G. So by Theorem 3.1.6 the structure

$$\mathfrak{U}_{P(f_G}(f_G) = \langle P(f_G), G/Z(G), Z(G), s_{G/Z(G)}, s_{Z(G)}, \delta_{f_G} \rangle,$$

where $s_{G/Z(G)}$ and $s_{Z(G)}$ describing the action of $P(f_G)$ on $G/Z(G)$ and $Z(G)$ respectively is absolutely interpretable in

$$\mathfrak{U}(f_G) = \langle P(f_G), G/Z(G), Z(G), \delta_{f_G} \rangle.$$

The factor group $G/Z(G)$ is absolutely interpretable in G (see Example 1.3.2. The subgroup $Z(G)$ is clearly definable without parameters. There is a formula of signature of groups describing the bilinear mapping f_G, since f_G is defined just by commutators. So $\mathfrak{U}(f_G)$ is absolutely interpretable in G. In turn $(U)_{P(f_G)}(f)$ is absolutely interpretable in G. □

Proof. **(An alternate proof for Theorem 2.4.6)** Let $G = N_{2,n}(\mathbb{Z})$. So the group G has a basis (see Proposition 2.1.8). So $P(f_G)$ and its action on $G/Z(G)$ and $Z(G)$ are absolutely interpretable in G by Lemma 3.2.1. By Proposition 2.4.2, $\mathbb{Z} \cong P_{f_G}(f_G)$. The factor group $G/Z(G)$ is generated by the basis elements modulo the center and the subgroup $Z(G)$ is generated by commutators of the basis elements. So $G/Z(G)$ and $Z(G)$ are generated as $P(f_G)$ modules by the elements described above. Since the action of $P(f_G)$ is interpretable in G we can describe the above fact by some formula of the signature of groups.

Now let H be a group such that $H \equiv G$. We prove that H is a $QN_{2,n}$ group over the ring $P(f_H)$. Elementary equivalence of H with G implies that H has a basis of the same cardinality as that of G. So $\tau(f_H) \leq \tau(f_G)$ (see Remark 3.1.9). This means that $P(f_H)$ and its action on $H/Z(H)$ and $Z(H)$ are absolutely interpretable in H with the same formulas interpreting $P(f_G)$ an its action on $G/Z(G)$ and $Z(G)$ in G. Thus $P(f_H) \equiv P(f_G) \cong \mathbb{Z}$. So the basis elements modulo $Z(H)$ generate $H/Z(H)$ and their commutators generate $Z(H)$ as $P(f_H)$ modules. If $\{h_1, \ldots, h_n\}$ is the basis and $h_{ij} = [h_i, h_j]$ then its clear that the set

$$\{h_i^\alpha, h_{kl}^\beta : 1 \leq i \leq n, 1 \leq k \leq l \leq n, \quad \alpha, \beta \in P(f_H)\},$$

generate H, where each h_i^α is chosen as in the proof of Theorem 2.3.14. The relations (a) of lemma 2.2.5 are readily verified here by the definition of $P(f_H)$. The rest of the proof goes through just like the $(a) \Rightarrow (b)$ of Theorem 2.3.14. □

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