# ON THE SPECTRUM OF THE ELEMENTS OF $\mathbb{R}[\mathrm{SU}(2)]$

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# DÉDICACE

À ma famille qui m'a toujours encouragé dans mes projets.

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# ABRÉGÉ

Dans ce mémoire, nous explorons le spectre des éléments de l'algèbre de groupe $\mathbb{R}[\mathrm{SU}(2)]$. Établir l'existence d'un sous-groupe ayant ce que l'on appelle un trou spectral fut une étape cruciale dans la résolution du problème de Banach-Ruziewicz. Il est bien connu que la mesure de Lebesgue est la seule mesure $\sigma$-additive sur la sphère $\mathbf{S}^n$ invariante par rotations. Or, Ruziewicz demanda si cela était toujours vrai lorsque la mesure est simplement additive (voir [RL10]). Banach démontra qu'elle n'est pas unique lorsque $n = 1$. Ce n'est que plus tard que Drinfel'd résolut le problème lorsque $n = 2$ ou 3 en démontrant qu'il n'existe aucune autre telle mesure. Indépendamment, Sulivan et Dennis apportèrent une réponse, elle aussi possitive, lorsque $n > 3$. La démarche utilisée pour $n = 2$ et 3 ramena le problème à trouver un sous-groupe de $\mathrm{SO}(3, \mathbb{R})$ pour lequel son graphe de Cayley est un graphe de Ramanujan. Cette dernière propriété, liée au trou spectral, fut difficile à établir. Plus récemment, Gamburd, Jakobson et Sarnak établirent une méthode robuste qui a permis d'exhiber plusieurs exemples de sous-groupes de SU(2) qui ont cette propriété (voir [GJS99]). Ce texte propose une introduction à plusieurs outils utilisés dans l'étude du spectre des éléments de $\mathbb{R}[\mathrm{SU}(2)]$ ainsi que plusieurs théorèmes et démonstrations se rapportant à ces derniers.

## ABSTRACT

In this thesis, we will explore the spectrum of the elements of the group ring $\mathbb{R}[\mathrm{SU}(2)]$. The existence of a subgroup with a spectral gap plays a fundamental role in the resolution of the Banach-Ruziewicz problem. It is well known that the Lebesgue measure is the only rotation-invariant $\sigma$-additive measure defined on every Lebesgue measurable sets of the sphere $\mathbf{S}^n$ (see [RL10]). Ruziewicz then asked wether it is still true when the measure is finitely additive. Banach proved that the Lebesgue measure is not unique when $n = 1$. It was only later that Drinfel'd solved the problem for $n = 2$ or 3. Independently, Sulivan and Dennis found a solution, this one also positive, for $n > 3$. The approach for $n = 2$ and 3 reduced the problem to finding a subgroup of $\mathrm{SO}(3, \mathbb{R})$ for which its Cayley graph is a Ramanujan graph. This last property, linked to the spectral gap, proved difficult to establish. More recently, Gamburd, Jakobson and Sarnak introduced a robust and elementary method that allowed to produce more examples of subgroups of $\mathrm{SU}(2)$ with a spectral gap (see [GJS99]). This text offer an introduction to several tools used to study the spectrum of the elements of $\mathbb{R}[\mathrm{SU}(2)]$, together with theorems and proofs about the latter.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

# CHAPTER 1
# Introduction

The entirety of this document is devoted to the group SU(2), specifically the spectrum of its subgroups. We will make precise in chapter 1 what we mean by the spectrum of a subgroup. The spectrum has applications in analysis, such as the Banach-Ruziewicz problem (see [RL10]), as well as in the theory of quantization (see [Sar97]). It is also closely related to the Hecke operators (see [LPS86] and [LPS87]). Chapter 1 will introduce the special unitary group, as well as other important matrix groups. We will discuss the representations of SU(2) in an appropriate setting, the group rings. The chapter will end with a brief section a on link between SU(2) and $SO(3, \mathbb{R})$, and the Hecke operator.

To better understand the spectrum, we introduce the Cayley graph in chapter 2. The work of Kesten will be presented to establish the existence of the Kesten measure (see [McK81]), which, in some cases, will describe the spectrum of a subgroup of SU(2). We will also prove that for every finitely generated subgroup of SU(2), there exists measures that correspond to counting the eigenvalues.

Finally, in chapter 3, we look at representations of SU(2) from a different point of view, that of random matrices. In the chapter, we define the classical random matrix ensembles (see, for example, [Meh04]), and we will prove that when the number of generators of the subgroup goes to infinity, the resulting matrix is a random ensemble (this is shown in [GJS99]).

# CHAPTER 2
## The Matrix Group SU(2) and Its Representations

Here, we introduce properly the special unitary group. The first section defines tools needed in the document. Afterward, several matrix groups are defined, and the rest of the chapter is dedicated to SU(2).

### 2.1 Algebraic Preliminaries

In this first section, we present the free groups and the group rings. The idea of a free group is to have a structure satisfying no relations, other than the group axioms. This is a simple idea from the outset, but we can see only from this intuition that it has several consequences on the type of group we seek to define. For example, finite groups cannot be considered "free", as any finite group satisfies a relation of the type $g^n = e$. On the other hand, the additive group $\mathbb{Z}$ satisfies no relations, so intuitively, this group could be considered "free". Moreover, the lack of any relation means that a homomorphism is entirely determined by a generating set. For example, any homomorphism $f$ with domain $\mathbb{Z}$ is entirely determined by $f(1)$, as $f(n)$ is given by $f(1) + \cdots + f(1)$ $n$-times.

The group ring of a group $G$ and a ring $R$ is the smallest ring containing an isomorphic copy of $G$ and $R$. In other words, if a ring $A$ contains an injection of $G$ in $A^*$ and $R$ in $A$, then $A$ contains the group ring noted $R[G]$. This idea allows one to add the elements of the group using the addition of the ring. This can be desirable when, for example, one has a group homomorphism into a multiplicative matrix group, but the addition of matrices in the range also has an important meaning.

### 2.1.1 Free groups

We will describe the construction of a free group and outline the important properties. For more details, see [DF04] section 6.3. Let $S$ be any finite or countable set; it will be called the *alphabet*. Let $S'$ be a set disjoint from $S$ and in bijection with it, and let $\varphi\colon S \to S'$ be a bijection. For $a \in S$, we will note the element $\varphi(a) \in S'$ by $a^{-1}$, extending our alphabet. We introduce one last formal symbol $e$ such that the singleton $\{e\}$ is disjoint from $S \cup S'$. Now, we have completed our alphabet with the letters in $\mathcal{A} := S \sqcup S' \sqcup \{e\}$.

We define $\mathbf{w}_S$ to be set of *all possible words* generated by $S$:

$$\mathbf{w}_S := \mathbb{N}^{(\mathcal{A})} = \big\{ f\colon \mathbb{N} \to \mathcal{A} \,|\, \exists N \in \mathbb{N},\ \forall n \geq N,\ f(n) = e \big\}.$$

A *word*, strictly speaking, is a sequence of letters that will become constant to $e$, for $n$ large enough. The *length* of a word will be given by $\min\{n \in \mathbb{N} : \forall m \geq n, f(m) = e\}$. We will use the following notation for a word: given a sequence $(a_1, a_2, a_3, \ldots, a_k, e, e, e, \ldots)$ with $a_i \in \mathcal{A}$ for $1 \leq i < k$ and $a_k \in \mathcal{A} \setminus \{e\}$, we will identify it to $w := a_1 a_2 a_3 \cdots a_k$. The sequence $(e, e, e, e, \ldots)$ will be identitied to $e$ or $\emptyset$, and it will be called the *empty word*.

With the idea in mind that $a^{-1}$ should be the inverse of $a$ and $e$ should be an identity element, we introduce the *reduction of a word*. Given a word $w = s_1 s_2 \ldots s_k$, the word $w' = t_1 t_2 \ldots t_\ell$ is a reduction of $w$ if one of the following four cases is true:

1. $\ell = k$ and $t_i = s_i$ for $1 \leq i \leq k$;

2. $\ell = k - 1$ and $t_i = s_i$ for $1 \leq i < n$, $s_n = e$ and $t_i = s_{i+1}$ for $n \leq i \leq \ell$;

3. $\ell = k - 2$ and $t_i = s_i$ for $1 \leq i < n$, $s_n = a$, $s_{n+1} = a^{-1}$, and $t_i = s_{i+2}$ for $n \leq i \leq \ell$;

4. $\ell = k - 2$ and $t_i = s_i$ for $1 \leq i < n$, $s_n = a^{-1}$, $s_{n+1} = a$, and $t_i = s_{i+2}$ for $n \leq i \leq \ell$.

Case 2 applies to a word with the identity appearing, such as $abaeb$, and it should reduce to $abab$. As for case 3 and 4, it applies for words with a letter and its inverse next to it, for example $bbaa^{-1}b$ and $bba^{-1}ab$ should both reduce to $bbb$.

We can now define an equivalence relation $\sim$ by $w \sim w'$ if and only if $w$ is obtained by a succesion of reduction from $w'$ or vice versa. We will not verify that this is indeed an equivalence relation. For each equivalence class, there is a word that cannot be reduced further. We will call it the *reduced word*. This special word should satisfy these two properties, if it is spelled $s_1 s_2 \cdots s_n$:

1. for all $i$, $s_i$ is not $e$, unless the word is $e$ itself;

2. for all $i$, $s_{i+1} \neq s_i^{-1}$ and $s_{i+1}^{-1} \neq s_i$.

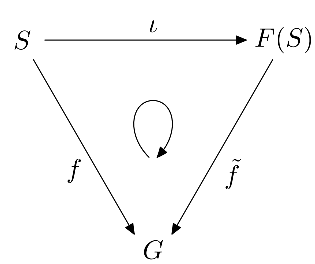
We will now define a binary operation on $\mathbf{w}_S/\sim$. The concatenation of words is a map taking two words $s_1 s_2 \cdots s_k$ and $t_1 t_2 \cdots t_\ell$ and giving the word $s_1 s_2 \cdots s_k t_1 t_2 \cdots t_\ell$. This defines a binary operation on $\mathbf{w}_S$, but it does not form a group. Instead, if we prove that we can concatenate and reduce in any order, this will yield the binary operation $[w] * [w'] := [ww']$, where $[w], [w'] \in \mathbf{w}_S/\sim$ and $[ww']$ is the equivalence class of $ww' \in \mathbf{w}_S$.

The identity will be $[e]$, since $[w][e] = [we] = [w]$ for all words. As for the inverse, we already have $[aa^{-1}] = [e]$ and $[a^{-1}a] = [e]$, by definition of $\sim$, so that $[a^{-1}]$ is the inverse of $[a]$. Therefore, the inverse of any word $[s_1 s_2 \cdots s_n]$ is simply $[s_n^{-1} \cdots s_2^{-1} s_1^{-1}]$. The associativity will not be verified, but it holds. Lastly, for every $s \in S$, there is a corresponding word $[s]$. This is an injection of $S$ in the group, so that we can identify $S$ to that subset in the group, and say that it contains $S$. All this will be summarized in the following theorem. A proof of the theorem is omitted here to alleviate the text, but can be found in [DF04].

**Theorem 2.1.1.** *Let $S$ be a finite or countable set and $\mathbf{w}_S$ be the set of all words over $S$. Let $\sim$ be the relation on $\mathbf{w}_S$ where two words are in relation if they have the same reduced word (see item 1. to 4. above). The binary operation on $\mathbf{w}_S/\sim$ given by $[w][w'] = [ww']$, where $ww'$ is the concatenation of $w$ and $w'$ is a group operation. This defines the free group over $S$, which will be noted $F(S)$.*

Note that in the theorem, we talk about *the* free group over $S$. This group is indeed unique up to isomorphism. This follows from an even more important fact about free groups: *the universal property.* In fact, any group satisfying the universal property is isomorphic to a free group; this quality characterizes free groups.

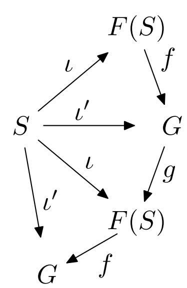
**Theorem 2.1.2** (Universal Property of Free Groups)**.** *Let $S$ be a finite or countable set. Let $F(S)$ be a free group over $S$ with an inclusion $\iota\colon S \to F(S)$; $\iota(s) = [s]$. For any group $G$ and any function $f\colon S \to G$, there exists a unique homomorphism $\tilde{f}\colon F(S) \to G$ such that $\tilde{f}([s]) = f(s)$ for every $s \in S$. This is summarized in the commutative diagram:*



*Proof.* We define $\tilde{f}([e])$ to be the identity of $G$, and $\tilde{f}([s^{-1}]) = f(s)^{-1}$. For every element of $F(S)$, we define $\tilde{f}([s_1 s_2 \cdots s_n]) = f(s_1)f(s_2)\cdots f(s_n)$ for any reprensentative, and this is well defined since reduction in $w$ are cancelations in $G$. $\square$

**Corollary 1.** *The free group $F(S)$ is unique up to isomorphism. In particular, $F(\{a\})$ is isomorphic to the additive group $\mathbb{Z}$.*

*Proof.* Let $G$ be another free group generated by $S$. There are two inclusion map $\iota\colon S \to F(S)$ and $\iota'\colon S \to G$. By the universal property, there exists two unique homomorphisms $f\colon F(S) \to G$ and $g\colon G \to F(S)$ such that the following diagram commute:




Since $g \circ f \circ \iota = \iota$, we have that $g \circ f\big|_S$ is the identity, so the composition extends to the identity homomorphism, and similarly for $f \circ g$.

Lastly, since the additive group $\mathbb{Z}$ is a free group generated by $\{1\}$, we have $\mathbb{Z} \simeq F(\{a\})$. $\square$

### 2.1.2 Group Rings

We begin with the definition of a group ring. It will be given in terms of formal linear combinations, as it will be the more useful notation for us, but the pedantic reader can translate the definition in terms of functions $f\colon G \to R$, where a formal sum $\sum_g r_g g$ represents the function $f(g) = r_g$.

**Definition 1.** Let $G$ be a group, finite or not, and let $R$ be a ring. We define the *group ring* $R[G]$ by

$$R[G] = \left\{ \sum_{g \in G} r_g g \,\middle|\, r_g \in R, r_g \neq 0 \text{ for only a finite number of } g \right\},$$

where the formal sum represent a function $f : G \to R;\ g \mapsto r_g$. The operations $\boldsymbol{+}$ and $\bullet$ are given by

$$\sum_{g \in G} r_g g \boldsymbol{+} \sum_{g \in G} s_g g = \sum_{g \in G} (r_g + s_g) g,$$

and

$$\left( \sum_{a \in G} r_a a \right) \bullet \left( \sum_{b \in G} s_b b \right) = \sum_{g \in G} \left( \sum_{ab = g} r_a s_b \right) g. \tag{2.1}$$

Lastly, we define a *scalar multiplication* by an element of $r \in R$ by

$$r \left( \sum_{g \in G} r_g g \right) = \sum_{g \in G} (r r_g) g.$$

The *support* of $z \in R[G]$ is $\operatorname{supp} z = \{ g \in G \,|\, z_g \neq 0 \}$.

In the next proposition, we verify that a group ring satisfies the axioms of a ring and that the scalar multiplication endows an $R$-module structure.

**Proposition 1.** *Let $G$ be a group, and let $R$ be a ring.*

*1. The group ring $R[G]$ forms a ring and has an $R$-module structure.*

*2. If $G$ and $R$ are commutative, then so is $R[G]$.*

*Proof.* We consider $z_1, z_2, z_3 \in R[G]$, where

$$z_1 = \sum_{g \in G} s_g g, \qquad z_2 = \sum_{g \in G} t_g g, \qquad z_3 = \sum_{g \in G} u_g g.$$

The addition $+$ is well defined, since $\operatorname{supp}(z_1 + z_2) \subseteq \operatorname{supp} z_1 \cup \operatorname{supp} z_2$, and so is the scalar multiplication since $\operatorname{supp} rz \subseteq \operatorname{supp} z$. The commutative group property of $+$ follows directly from that of the addition of the ring. The distributivity of the scalar multiplication follows similarly.

We prove that $z_1 \bullet z_2$ is well-defined. We only need to show $\operatorname{supp}(z_1 \bullet z_2)$ is finite. We use the notation $z_1 \bullet z_2(g)$ for the coefficient

$$\sum_{\substack{(a,b)\in G\times G \\ ab=g}} s_a t_b.$$

If $z_1 \bullet z_2(g)$ is not 0, then there exists $(a, b)$ such that $ab = g$ and $s_a t_b \neq 0$. Therefore, we have $g \in \{ab \,|\, a \in \operatorname{supp} z_1, b \in \operatorname{supp} z_2\}$, a finite set, and $\operatorname{supp}(z_1 \bullet z_2) \subseteq \{ab \,|\, a \in \operatorname{supp} z_1, b \in \operatorname{supp} z_2\}$ follows.

We now look at the associativity of $\bullet$, but first it will useful to note that when $G$ is a group, the product can written

$$z_1 \bullet z_2 = \sum_{g\in G} \sum_{h\in G} s_{gh^{-1}} t_h g,$$

since $\{a \in G \,|\, \exists h, ah = g\}$ and $\{gh^{-1} \,|\, h \in G\}$ are the same. Now tackling the associativity, we have

$$\begin{aligned}
(z_1 \bullet z_2) \bullet z_3(g) &= \sum_{m\in G} z_1 \bullet z_2(gm^{-1}) u_m \\
&= \sum_{m\in G} \left[ \sum_{n\in G} s_{gm^{-1}n^{-1}} t_n \right] u_m \\
&= \sum_{n\in G} \sum_{m\in G} s_{gm^{-1}n^{-1}} t_n u_m && \text{(because the series are finite sums)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\ell \in G} \sum_{m \in G} s_{g\ell^{-1}} t_{\ell m^{-1}} u_m && \text{(with } n = \ell m^{-1}\text{)} \\
&= \sum_{\ell \in G} s_{g\ell^{-1}} \left[ \sum_{m \in G} t_{\ell m^{-1}} u_m \right] \\
&= \sum_{\ell \in G} s_{g\ell^{-1}} (z_2 \bullet z_3)(\ell) \\
&= z_1 \bullet (z_2 \bullet z_3)(g).
\end{aligned}
$$

The distributivity of $\bullet$ over $\boldsymbol{+}$ is seen by expanding the sum and follows from the one on the ring

$$
\begin{aligned}
z_1 \bullet (z_2 \boldsymbol{+} z_3)(g) &= \sum_{m \in G} s_{gm^{-1}} (t_m + u_m) \\
&= \sum_{m \in G} \left( s_{gm^{-1}} t_m + s_{gm^{-1}} u_m \right) \\
&= z_1 \bullet z_2(g) + z_1 \bullet z_3(g) \\
&= (z_1 \bullet z_2 \boldsymbol{+} z_1 \bullet z_3)(g).
\end{aligned}
$$

The right distributivity is analogous to this.

To prove *2*, we simply have

$$
\begin{aligned}
z_1 \bullet z_2(g) &= \sum_{m \in G} s_{gm^{-1}} t_m \\
&= \sum_{\ell \in G} s_\ell t_{\ell^{-1} g} && \text{(with } \ell = gm^{-1}\text{)} \\
&= \sum_{\ell \in G} t_{g\ell^{-1}} s_\ell \\
&= z_2 \bullet z_1(g).
\end{aligned}
$$

This concludes the proof. ☐

*Remark.* If $r$ is an element of the center of $R$, $Z(R)$, then compatibility of the scalar multiplication and $\bullet$ follows from

$$r(z_1 \bullet z_2) = r\left(\sum_{g\in G}\sum_{m\in G} s_{gm^{-1}}t_m\right) = \sum_{g\in G}\sum_{m\in G} rs_{gm^{-1}}t_m,$$

from which we can see that $r(z_1 \bullet z_2) = (rz_1) \bullet z_2 = z_1 \bullet (rz_2)$. In other words, $R[G]$ forms a ring algebra over $Z(R)$.

We list a few more properties of group rings. For a more thorough elaboration, see [DF04]. We will denote the identity of the group by $e$, and when $R$ is a ring with unity, its unity will be denoted by 1.

- If $S$ is subring of $R$, then $S[G]$ is a subring of $R[G]$. If $H$ is a subgroup of $G$, then $R[H]$ is a subring of $R[G]$.
- The center of the ring is in the center of $R[G]$. In particular, if $R$ is commutative, it is in $Z(R[G])$. In this case, we deduce from the remark above that the group ring is an algebra over $R$. When $R$ is a field, we sometimes talk about a *group algebra.*
- There is a subring isomorphic to $R$ in $R[G]$ given through $r \mapsto re$. In a similar way, if $R$ has a unity, we can identify $G$ in $R[G]^\times$ through $g \mapsto 1g$. The operation $\bullet$ is an extension of both the operation of the group and the multiplication of the ring. As such, if $R[G]$ is commutative, then $G$ is abelian and $R$ is commutative. Together with point 2 of the previous proposition, we proved that $R[G]$ *is commutative if and only if both $G$ and $R$ are commutative.*

*Example* 1. The group ring (actually a group algebra) $\mathbb{C}[\mathbb{Z}]$ is the set of all bi-sequences that have finitely many nonzero entries. The multiplication $\bullet$ is simply the convolution of sequences. In this example, we can see that $\mathbb{C}$ is identified with sequences zero everywhere except position 0, and $\mathbb{Z}$ is identified with sequences zero everywhere except at position $n$, with value 1.

Here is slightly different example to illustrate group rings.

*Example* 2. The group algebra $\mathbb{C}[\mathbb{R}]$ is similar to the one of the previous example, but instead of sequences, we have functions $f\colon \mathbb{R} \to \mathbb{C}$ zero everywhere except at finitely many points.

## 2.2 The Matrix Group SU(2)

A matrix group is simply a group where the elements are matrices and the law of composition is the matrix multiplication. Our goal is to introduce the special unitary group, but we will begin by introducing several important matrix groups.

### 2.2.1 Matrix Groups

The first groups are well known, and accessible in the literature (for example see [DF04]). For the symplectic groups described later in the section, we followed the beginning of chapter 3 of [Ros06].

We begin with $\mathrm{GL}(n, \mathbf{F})$ and $\mathrm{SU}(2)$. The *general linear group* $\mathrm{GL}(n, \mathbf{F})$ is the set of invertible matrix with entries in a field $\mathbf{F}$ equipped with matrix multiplication as the group operation. For us, this field will be $\mathbb{C}$ or $\mathbb{R}$. The determinant $\det\colon \mathrm{GL}(n, \mathbf{F}) \to \mathbb{R}^\times$ is a group homomorphism into the multiplicative group of nonzero real numbers, and since $\{1\}$ is a subgroup of $\mathbb{R}^\times$, $\det^{-1}(1)$ is a subgroup of $\mathrm{GL}(n, \mathbf{F})$ called the *special linear group*, $\mathrm{SL}(n, \mathbf{F})$. It is the subgroup of matrices with determinant 1.

The group $\mathrm{SU}(n)$ is called *special unitary group*. The field is not specified because it is implied that the matrices have complex entries. The *adjoint* of a matrix $A$ is noted $A^*$ and is the conjugate transposed of $A$, $\overline{A^t}$; it is the matrix that satisfies $\langle Av, w\rangle = \langle v, A^*w\rangle$ for the standard inner product. We say that $A$ is *unitary* when $AA^* = I$, where $I$ is the identity. Under matrix multiplication, these matrices form a group called *unitary group* $\mathrm{U}(n)$; it is a subgroup of $\mathrm{GL}(n, \mathbb{C})$. The unitary matrices whose determinant is 1 compose $\mathrm{SU}(n)$. We will go into much greater details for the case $n = 2$, as $\mathbb{R}[\mathrm{SU}(2)]$ will be our group ring of interest.

When the underlying field is the real numbers, the adjoint can still be defined; it is simply the transposed. In this case, we will talk about the *orthogonal group* $\mathrm{O}(n, \mathbb{R})$, in lieu of $\mathrm{U}(n)$, for those matrices that satisfy $AA^t = I$, and similarly for $\mathrm{SO}(n, \mathbb{R})$. In fact, the orthogonal group can be defined with any underlying field. The group $\mathrm{O}(n, \mathbf{F})$ will be the matrices of $\mathrm{GL}(n, \mathbf{F})$ that satisfy $AA^t = I$.

One last important group for later will be defined in a similar way to $\mathrm{SU}(n)$, but first, let us introduce the *quaternions*. It is a vector space of real dimension 4 endowed with a product, much like the complex numbers. We will note the space $\boldsymbol{H}(\mathbb{R})$ and the elements are usually denoted

$$\boldsymbol{H}(\mathbb{R}) = \{a + ib + jc + kd \,|\, a, b, c, d \in \mathbb{R}\}.$$

The product is entirely described by the rules $ij = k$, $jk = i$, $ki = j$ and $i^2 = j^2 = k^2 = -1$. It forms a division ring (sometimes *skew field*), the real numbers are in the center, so that it is an $\mathbb{R}$-algebra. It contains an isomorphic copy of $\mathbb{R}$ and $\mathbb{C}$. Note that the complex numbers do not commute with every elements, although they satisfy a commuting formula $zj = j\overline{z}$ for $z \in \mathbb{C}$, and similarly with $k$.

The quaternions have a conjugation similar to complex numbers: for $q = a+ib+jc+kd$, we define $\overline{q}$ by $a - ib - jc - kd$. The inner product of $\mathbb{R}^4$ applies to this ring, so that $\langle q, q' \rangle = aa' + bb' + cc' + dd'$. This defines a norm $N(q)$ which is equal to $q\overline{q} = |q|^2$.

Equipped with an inner product, we can define the adjoint of a matrix of $\mathrm{GL}(n, \boldsymbol{H}(\mathbb{R}))$ as above. We will keep the notation of $*$ for the adjoint here.[1] The set of matrices satisfying $MM^* = I$ is still a group, which is sometimes called *hyperunitary*, and could be noted $\mathrm{U}(n, \boldsymbol{H}(\mathbb{R}))$. However, we will use the notation $\mathrm{USp}(2n, \boldsymbol{H}(\mathbb{R}))$, for symplectic, because we want to reserve SU for the complex numbers. The reason of this notation will be specified below.

Several matrix groups were introduced. For those unfamiliar with them, they may seem a little daunting at first, so below we summarize the ones introduced in Table 2–1.

Other than the general linear group and the special linear group, all these matrix groups were defined with respect to the standard inner product. It is possible to define more general counterparts by replacing the inner product by a bilinear form. We will introduce two of them that will be useful in section 2.3.4.

[1] It is sometimes noted $M^\dagger$ when the underlying division ring is the quaternion.

| | | |
|---|---|---|
| $\mathrm{GL}(n, \mathbf{F})$ | General linear group | Invertible $n \times n$ matrices |
| $\mathrm{SL}(n, \mathbf{F})$ | Special linear group | Matrices of $\mathrm{GL}(n, \mathbf{F})$ with $\det = 1$ |
| $\mathrm{U}(n)$ | Unitary group | Matrices whose adjoint is its inverse |
| $\mathrm{SU}(n)$ | Special unitary group | Matrices of $\mathrm{U}(n)$ with $\det = 1$ |
| $\mathrm{O}(n, \mathbb{R})$ | Orthogonal group | Matrices whose transpose is its inverse |
| $\mathrm{SO}(n, \mathbb{R})$ | Special orthogonal group | Matrices of $\mathrm{O}(n, \mathbb{R})$ with $\det = 1$ |
| $\mathrm{USp}(2n, \boldsymbol{H}(\mathbb{R}))$ | Hyperunitary group | Unitary matrices of $\mathrm{GL}(n, \boldsymbol{H}(\mathbb{R}))$ |

Table 2–1: Matrix groups.

Let $\varphi$ be a bilinear from on $V \times V$, where $V$ is a vector space over $\mathbb{R}$ or $\mathbb{C}$. The form is *non-degenerate* if for every $x_0 \neq 0$, there exists $x$ and $y$ such that $\varphi(x_0, x) \neq 0$ and $\varphi(y, x_0) \neq 0$. We will assume $\varphi$ to be non-degenerate form now on.

If $\varphi$ is symmetric, so that $\varphi(x, y) = \varphi(y, x)$, then we retrieve an orthogonal group. That is, any matrix $A$ that satisfy $\varphi(Ax, Ay) = \varphi(x, y)$ will be orthogonal with respect to $\varphi$ and we will note it $\mathrm{O}(\varphi)$. If $\varphi$ is *sesquilinear*, meaning $\varphi(ax, by) = \overline{a}b\varphi(x, y)$, and selfadjoint, $\varphi(x, y) = \overline{\varphi(y, x)}$, then we retrieve a unitary group in the complex case, noted $U(\varphi)$. When there is a basis on $V$, we can write $\varphi$ as a matrix, and it will be given by $(\varphi(e_i, e_j))_{ij}$. On a suitable basis, $\varphi$ will be the identity matrix, so that the elements $A$ of $\mathrm{O}(\varphi)$ and $\mathrm{U}(\varphi)$ will satisfy $AA^* = I$.

If $\varphi$ is *antisymmetric*, $\varphi(x, y) = -\varphi(y, x)$, the matrices that leave $\varphi$ invariant define a new group that we call *symplectic*. The dimension of $V$ must be even, and when $V = \mathbb{R}^{2m}$, we note the group $\mathrm{Sp}(m, \mathbb{R})$, while $\mathrm{Sp}(m, \mathbb{C})$ is used for the case

$V = \mathbb{C}^{2m}$. With a suitable basis, the matrix of $\varphi$ can have the form

$$\Omega = \begin{pmatrix} 0 & 1 & 0 & 0 & & \\ -1 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & -1 & & \\ 0 & 0 & -1 & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

and a matrix $A \in \mathrm{Sp}(\varphi)$ will satisfy $A^t \Omega A = \Omega$.

We do not have to consider an "anti-selfadjoint" form in the complex case, as multiplication of such form by $i$ gives a selfadjoint form and *vice versa*.

These are all the groups that we will introduce. We will end this section by establishing a link between $\mathrm{USp}(2n, \boldsymbol{H}(\mathbb{R}))$ and the complex symplectic group $\mathrm{Sp}(2n, \mathbb{C})$. Because we can write a quaternion as the sum of two complex numbers $q = z + jw$, we can decompose $\boldsymbol{H}(\mathbb{R})$ in the direct sum $\mathbb{C} + j\mathbb{C}$. This leads to the decomposition of $\mathrm{Mat}(n, \boldsymbol{H}(\mathbb{R}))$ into $\mathrm{Mat}(n, \mathbb{C}) + j\,\mathrm{Mat}(n, \mathbb{C})$. For a vector $x + jy \in \boldsymbol{H}(\mathbb{R})^n$ written as a column vector $(x \quad y)^t$, the left multiplication by $a + jb \in \mathrm{Mat}(n, \boldsymbol{H}(R))$ is realized by the matrix

$$\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}. \tag{2.2}$$

This way, we can see a matrix of $\mathrm{Mat}(n, \boldsymbol{H}(R))$ as a matrix of $\mathrm{Mat}(2n, \mathbb{C})$. When the matrix is unitary in $\boldsymbol{H}(\mathbb{R})$, that is for an element $M \in \mathrm{USp}(2n, \boldsymbol{H}(\mathbb{R}))$, the corresponding matrix of $\mathrm{Mat}(2n, \mathbb{C})$ will be symplectic, simply because $M$ preserves the inner product of $\boldsymbol{H}(R)^n$, $\langle Mv, Mw \rangle = \langle v, w \rangle$, but with the ealier notation of

$v = (v_1 \;\; v_2)^t, w = (w_1 \;\; w_2)^t \in \mathbb{C}^{2n}$, this inner product is an antisymmetric form on $\mathbb{C}^{2n}$. We can also see that the matrix in (2.2) is unitary in complex field. Conversely, a matrix $M \in \mathrm{U}(2n, \mathbb{C}) \cap \mathrm{Sp}(2n, \mathbb{C})$ is a unitary matrix of $\boldsymbol{H}(\mathbb{R})^n$, that is $M \in \mathrm{USp}(2n, \boldsymbol{H}(\mathbb{R}))$.

### 2.2.2 The Matrix Group SU(2)

We will now describe the group SU(2) in more detail. First, for $A, B \in \mathrm{SU}(2)$, we have $(AB)^* = B^* A^*$ so that $AB(AB)^* = I$, therefore SU(2) forms a subgroup of $\mathrm{Mat}(2, \mathbb{C})$. The property of this group is more rigid than it could seem from the outset. With it, we can derive an explicit form for its matrices, which we describe now.

If $A \in \mathrm{SU}(2)$ is given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then $A^*$ is $\left(\begin{smallmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{smallmatrix}\right)$ and they satisfy the relation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which gives us the equations

$$|a|^2 + |b|^2 = 1,$$
$$|c|^2 + |d|^2 = 1,$$
$$a\overline{c} + b\overline{d} = 0.$$

Since $\det A = 1$, we have a final equation $ad - bc = 1$. This system leads to $d = \overline{a}$ and $b = -\overline{c}$. The group SU(2) can now be written

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \in \mathrm{Mat}(2, \mathbb{C}) \;\middle|\; |z|^2 + |w|^2 = 1 \right\}. \qquad (2.3)$$

With an explicit expression, we can easily find the eigenvalues of $A \in \mathrm{SU}(2)$. We have

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} z - \lambda & -\overline{w} \\ w & \overline{z} - \lambda \end{vmatrix} \\
&= (z - \lambda)(\overline{z} - \lambda) + |w|^2 \\
&= |z|^2 - \lambda z - \lambda \overline{z} + \lambda^2 + |w|^2 \\
&= \lambda^2 - 2\lambda \Re z + 1 = 0.
\end{aligned}$$

This equation can be solved with the quadratic formula, giving

$$\lambda = \Re z \pm \sqrt{(\Re z)^2 - 1}. \tag{2.4}$$

Since $|z|^2 + |w|^2 = 1$, we deduce that $|z|^2 \leq 1$, and therefore $|\Re z| \leq 1$ with equality if and only if $z = \pm 1$ and $w = 0$, that is if and only if $A$ is the identity matrix. Otherwise, $\lambda$ is a complex number so that $A$ has the two eigenvalues $\lambda$ and $\overline{\lambda}$. Moreover, we can see that $|\lambda|^2 = (\Re z)^2 + (1 - (\Re z)^2) = 1$, so we can write $\lambda = e^{i\theta}$ for some real $\theta$. This angle is related to $z$ by $\theta = \arccos(\Re z)$, which can be seen from equation (2.4). It follows that every matrix of $\mathrm{SU}(2)$ is conjugate to

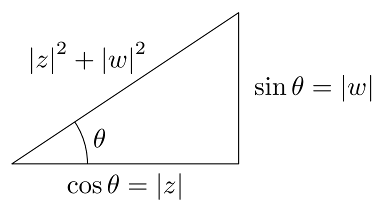
$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \tag{2.5}$$

for some real $\theta$.

### 2.2.3 SU(2) As a 3-Sphere and Its Haar Measure

Equation (2.3) shows us that a matrix of SU(2) is determined by two complex numbers $z$ and $w$. Since $\mathbb{C}^2$ is of real dimension 4, and because $|z|^2 + |w|^2 = 1$, the couple $(z, w)$ lies on $S^3$ seen in $\mathbb{R}^4$. This gives a homeomorphism; the sphere being compact, we deduce SU(2) is compact with its induced topology in $\mathbb{C}^4$. We can refer to SU(2) as being a *compact group*.

We can parametrize SU(2) with the following coordinates. From the relation $|z|^2 + |w|^2 = 1$, we see that there is an angle $0 \leq \theta \leq \frac{\pi}{2}$ such that




$$|z| = \cos\theta;$$

$$|w| = \sin\theta.$$

There are then two angles $0 \leq \varphi \leq 2\pi$ and $0 \leq \chi \leq 2\pi$ such that $z = \cos\theta e^{i\varphi}$ and $w = \sin\theta e^{i\chi}$. This parametrization allows us to define an integral on SU(2). We can naturally integrate in $\mathbb{C}^2$ seen as $\mathbb{R}^4$ using the described parametrization plus a radius $0 \leq r < \infty$ and the Jacobian matrix, giving

$$\int_{\mathbb{R}^4} f(x_1, x_2, x_3, x_4)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4 = \int_{r=0}^{\infty}\int_{\theta=0}^{\frac{\pi}{2}}\int_{\varphi,\chi=0}^{2\pi} f(r\cos\theta e^{i\varphi}, r\sin\theta e^{i\chi})\frac{r^3}{2}\sin 2\theta\,\mathrm{d}\chi\,\mathrm{d}\varphi\,\mathrm{d}\theta\,\mathrm{d}r.$$

To integrate on SU(2), we require $r = 1$. A function over SU(2) given in this parametrization will be integrated by $\frac{1}{2}\int_S f(\theta, \varphi, \chi)\sin 2\theta\,\mathrm{d}\chi\,\mathrm{d}\varphi\,\mathrm{d}\theta$ over a measurable set $S$. If the function is 1, we find the "hyper-area" of SU(2), which is $2\pi^2$. Thence, we will equip SU(2) with what we call the normalized *Haar measure*

$$\int_{\mathrm{SU}(2)} f(g)\,\mathrm{d}g := \frac{1}{4\pi^2}\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{2\pi} f(\theta,\varphi,\chi)\sin 2\theta\,\mathrm{d}\chi\,\mathrm{d}\varphi\,\mathrm{d}\theta. \tag{2.6}$$

We call it the Haar measure, because it is left-invariant under the product by SU(2). That is $\int_S f(gh)\,\mathrm{d}h = \int_{gS} f(h)\,\mathrm{d}h$ for any $g \in \mathrm{SU}(2)$, similar to how the Lebesgue measure is translation-invariant. This follows from the fact that multiplying by an element of SU(2) preserves distances in $\mathbb{C}^2$. Given $\left(\begin{smallmatrix} z & -\overline{w} \\ w & \overline{z} \end{smallmatrix}\right)$ in $\mathbb{C}^2$ and $U \in \mathrm{SU}(2)$, the product $U\left(\begin{smallmatrix} z & -\overline{w} \\ w & \overline{z} \end{smallmatrix}\right) = \begin{pmatrix} z' & -\overline{w'} \\ w' & \overline{z'} \end{pmatrix}$ is at the same distance from the origin; indeed, we have $|z|^2 + |w|^2 = |z'|^2 + |w'|^2$, which we obtain by taking the determinant of the product. The Lebesgue measure being invariant by isometry, we deduce that the measure defined is invariant by left and right multiplication of elements of SU(2).

### 2.2.4 SU(2) As Rotations of a 2-Sphere

There are quite a few things we can say about rotations. We will begin by describing them, and then discuss their relation with SU(2).

A *rotation* of the Euclidean space $\mathbb{R}^3$ is a linear map that preserves distance and orientation. In a vector space, this is described by the inner product and the determinant. Given a basis $(e_1, e_2, e_3)$, the orientation is preserved by a matrix $R$ if $\det(Re_1, Re_2, Re_3)$ has the same sign has $\det(e_1, e_2, e_3)$, in other words, $\det R > 0$. If $v$ and $w$ are two vectors of $\mathbb{R}^3$, the distance is preserved if $\langle Rv, Rw\rangle = \langle v, w\rangle$, so that it is orthogonal. Since an orthogonal matrix has determinant $\pm 1$, we conclude that a rotation is an orthogonal matrix with determinant 1. The composition of two rotations is still a rotation, thence all the rotations about the origin is the group $\mathrm{SO}(3,\mathbb{R})$.

There is a two-to-one homomorphism $\varphi\colon \mathrm{SU}(2) \to \mathrm{SO}(3,\mathbb{R})$. This homomorphism will be discussed in section 2.2.4. It is often useful to see a special unitary matrix as a rotation of $\mathbb{R}^3$, especially for motivational purposes. For instance, one can be interested in different groups generated by different finite subsets SU(2). If $S$ is a finite subset, the subgroup is obtained by taking every words formed by concatenating the letters in $S$, where concatenation is the product of SU(2); it is the smallest subgroup containing $S$. The group obtained, call it $\Gamma_S$, has an action on the 2-sphere: given a word $w = \gamma_1\gamma_2\cdots\gamma_n \in \Gamma_S$ and a point $x \in \mathbf{S}^2$, we define $w.x = \varphi(\gamma_1) \circ \cdots \circ \varphi(\gamma_n)(x)$. Although not the main subject of this thesis, this is an important topic that we will briefly discuss in section 2.4.

### 2.2.5 The Group Ring $\mathbb{R}[\mathrm{SU}(2)]$

It is time to introduce the group ring $\mathbb{R}[\mathrm{SU}(2)]$. This object is the set of all functions $f\colon \mathrm{SU}(2) \to \mathbb{R}$ zero everywhere except at finitely many points. It is not commutative, since SU(2) is not abelian, but because the real numbers form a field, it has the structure of an algebra over $\mathbb{R}$. Given an element $f$ of $\mathbb{R}[\mathrm{SU}(2)]$, we can see SU(2) as a "basis" of a vector space, so that we can write the function $f$ as the "vector"

$$v := \sum_{g\in \mathrm{SU}(2)} f(g)g.$$

There is no question of convergence here, since $f(g) = 0$ for all $g$ except for a finite number of them. From now on, for an element $z \in \mathbb{R}[\mathrm{SU}(2)]$, the notation

$$z = \sum_{g\in G} z_g g$$

will be interpreted as the function defined by $f(g) = z_g \in \mathbb{R}$.

The product was defined by equation (2.1) in section 2.1.2. There are several ways to write this formula, and we introduce one here which will be useful later in chapter 3. For $z$ and $w$, two elements of $\mathbb{R}[\mathrm{SU}(2)]$, we have

$$\begin{aligned} z \bullet w &= \sum_{g \in G} \left( \sum_{ab=g} z_a w_b \right) g \\ &= \sum_{g \in G} \sum_{h \in G} z_{gh^{-1}} w_h g && \text{(with } h = b \text{ and } a = gh^{-1}) \\ &= \sum_{k \in G} \sum_{h \in G} z_k w_h kh && \text{(with } k = gh^{-1}) \end{aligned}$$

and when $w = z$, we get a formula for $z^{\bullet n} := \underbrace{z \bullet z \bullet \cdots \bullet z}_{n \text{ times}}$ :

$$z^{\bullet n} = \sum_{g_1, \ldots, g_n \in G} z_{g_1} \cdots z_{g_n} g_1 \cdots g_n. \tag{2.7}$$

Lastly, as mentioned in section 2.1.2, our group ring contains an isomorphic copy of $\mathrm{SU}(2)$ in $\mathbb{R}[\mathrm{SU}(2)]^\times$ and $\mathbb{R}$ in $\mathbb{R}[\mathrm{SU}(2)]$. The group homomorphism $\varphi$ and the ring homomorphism $\psi$ are given by

$$\begin{aligned} \varphi\colon\ \mathrm{SU}(2) &\to \mathbb{R}[\mathrm{SU}(2)] & \psi\colon \mathbb{R} &\to \mathbb{R}[\mathrm{SU}(2)] \\ g &\mapsto 1g & x &\mapsto x \cdot e. \end{aligned}$$

The product of two elements $g$ and $h$ of $\mathrm{SU}(2)$ seen is $\mathbb{R}[\mathrm{SU}(2)]$ is simply the element $gh$ of $\mathbb{R}[\mathrm{SU}(2)]$.

## 2.3 Representations of SU(2)

A representation of a group is a homomorphism $\varphi$ from a group $G$ to $\hom(V)$, where $V$ is a vector space over the complex numbers, or equivalently to $\mathrm{GL}(n, \mathbb{C})$, when $V$ has dimension $n$, after choosing a basis. The representations of $\mathrm{SU}(2)$ will be the main object of study in this document, especially the spectrum. In this section, we will describe the irreducible representations of SU(2), their character, and the spectrum of an element.

### 2.3.1 Finite Groups Against Infinite Compact Groups

Often, one is interested in the representation theory of finite groups. Finite groups have averaging operators that are useful when talking about representations. In our case, we are mostly interested in the matrix group SU(2), which is uncountable. However, as mentioned in section 2.2.3, SU(2) is a compact subset of $\mathbb{C}^2$. This allows us to take the Haar measure as our averaging operator. Equipped so, several results concerning representation of finite groups can be adapted to compact groups.

### 2.3.2 Irreducible Representations of SU(2)

We will be interested in the irreducible representation of SU(2). We start describing them now by the use of a new tool; we introduce vector spaces called *k-th symmetric power*.

**Definition 2.** Let $V$ be a vector space over a field $k$ of dimension $n$. We define the *$N$-th symmetric power of $V$* by the quotient

$$\operatorname{sym}^N V = V^{\otimes N} \Big/ \left\{ v_1 \otimes \cdots \otimes v_N - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(N)} \,\middle|\, \sigma \in S_N \right\}.$$

We will use the notation $v \vee w$ for a simple tensor of $\operatorname{sym}^N V$. From the definition, we see that $v \vee w = w \vee v$. Also, the sets appearing in the quotient are vector spaces, so the quotient itself is still a vector space. The dimension of $V^{\otimes N}$ is $n^N$ and it has a canonical basis $\{e_{i_1} \otimes \cdots \otimes e_{i_N} \,;\, 1 \leq i_j \leq n\}$, where the $\{e_i\}_{1 \leq i \leq n}$ is the canonical basis of $V$. The basis of $\operatorname{sym}^N V$ will be those elements which are not a permutation of one another. Therefore, the number of basis elements is the number of ways to choose $N$ objects from $n$ objects with replacement[2] . This number is given by $\binom{n+N-1}{N}$ and it is also the number of homogeneous polynomials of degree $N$ with $n$ indeterminates. This gives us an isomorphism with the vector space

$$\operatorname{Span}\left\{x_1^{\ell_1} x_2^{\ell_2} \cdots x_N^{\ell_N} \mid \textstyle\sum_i \ell_i = n\right\}.$$

In the case of $\dim V = 2$, the dimension of $\operatorname{sym}^N V$ is $N+1$ and is spanned by $y^N, xy^{N-1}, \ldots, x^N$. We will note it $W_{N+1}$.

The importance of this vector space comes from an action of SU(2) on the homogeneous polynomials of two indeterminates. This action is obtained by replacing occurrences of $x$ and $y$ by

$$(x,\ y)\begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} = (\alpha x + \beta y,\ -\overline{\beta} x + \overline{\alpha} y).$$

This describes a linear application on $W_{N+1}$, namely

$$x^k y^{N-k} \mapsto (\alpha x + \beta y)^k (-\overline{\beta} x + \overline{\alpha} y)^{N-k},$$

and every element of SU(2) can be associated to such an application, so that we have a homomorphism $\varphi\colon\ \mathrm{SU}(2) \to \hom(W_{N+1})$. To see a proof that all the irreducible representing of SU(2) are given this way, for some $W_{N+1}$, we refer to [Hal03].

[2] tirage avec remise

### 2.3.3 A Formula for the Characters

We described all the irreducible representations of SU(2). To every representation, we can introduce a function called the *character*, $\chi_\pi\colon \mathrm{SU}(2) \to \mathbb{R}$, given by $\chi_\pi(g) = \mathrm{tr}(\pi(g))$. A representation is entirely described by its character, and since the trace is the sum of the eigenvalues of $\pi(g)$, it is intimately related to the spectrum of the latter. When $g$ is diagonal, it is easy to compute its matrix, $\pi_N(g)$. Since every special unitary matrix is diagonalizable, the following lemma will be useful. Note that when $g = e$, then $\chi_{\pi_N}(g) = N+1$, and if $g = -e$, then $\chi_{\pi_N}(g) = (-1)^N(N+1)$.

**Lemma 2.3.1.** *Let $\pi_N$ be the irreducible representation of* SU(2) *in $W_{N+1}$ and let $g$ be the diagonal matrix $\left(\begin{smallmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{smallmatrix}\right)$ in* SU(2)*. If $g$ is not $\pm e$, then we have*

$$\chi_{\pi_N}(g) = \frac{\sin(N+1)t}{\sin t}. \tag{2.8}$$

*Proof.* For a diagonal element of SU(2), it is possible to find explicitly $\pi_N(g)$. Remember that the action is given by $x^{N-k}y^k \mapsto (\alpha x + \beta y)^{N-k}(-\overline{\beta}x + \overline{\alpha}y)^k$, so if $g = \left(\begin{smallmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{smallmatrix}\right)$, the action is simply $x^{N-k}y^k \mapsto e^{it(N-k)}x^{N-k}e^{-itk}y^k$, so the matrix $\pi_N(g)$ is diagonal, with diagonal elements $(e^{it(N-2(k-1))})_{kk}$. The trace is then simply

$$\begin{aligned}
\mathrm{tr}(\pi_N(g)) &= \sum_{k=0}^{N} e^{it(N-2k)} \\
&= e^{itN}\sum_{k=0}^{N}\left(e^{-2it}\right)^k \\
&= e^{itN}\frac{1-e^{-2it(N+1)}}{1-e^{-2it}} \\
&= e^{itN}\frac{e^{-it(N+1)}\left(e^{it(N+1)}-e^{-it(N+1)}\right)}{e^{-it}\left(e^{it}-e^{-it}\right)}
\end{aligned}$$

$$= \frac{2i \sin(N+1)t}{2i \sin t}.$$

□

### 2.3.4 To Define the Spectrum of An Element of $\mathbb{R}[\mathrm{SU}(2)]$

For an element $z = x_1 g_1 + x_2 g_2 + \cdots + x_k g_k \in \mathbb{R}[\mathrm{SU}(2)]$, we can apply to each summand the representation $\pi_N$. That is, we define

$$\widehat{z}(\pi_N) = x_1 \pi_N(g_1) + \cdots + x_k \pi_N(g_k),$$

where the sum is the usual addition of matrices. This defines a ring homomorphism of $\mathbb{R}[\mathrm{SU}(2)]$ into $\mathrm{Mat}(W_{N+1})$.

The eigenvalues of $\widehat{z}(\pi_N)$ are what interest us. To better understand them, we introduce the bilinear form on the space $W_{N+1}$ of homogeneous polynomial of degree $N$ with basis $e_j = x^j y^{N-j}$

$$\langle e_j, e_k \rangle = \binom{N}{j}^{-1} (-1)^j \delta_{j,N-k}.$$

The matrix of this form $(\langle e_i, e_j \rangle)_{ij}$ only has nonzero elements on the anti-diagonal, so the determinant is the product of these elementss, adjusted by a sign, and therefore this determinant is nonzero. This says that the form is non-degenerate. Also, when $N$ is even, the form is symmetric, while when it is odd, it is antisymmetric. Lastly, it is preserved by $\pi_N$: $\langle \pi_N(g)v, \pi_N(g)w \rangle = \langle v, w \rangle$. Indeed, it is even preserved by the representation of $\mathrm{SL}(2, \mathbb{C})$.

To prove the latter, we use the fact that

$$\mathrm{SL}(2,\mathbb{C}) = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \middle| \, t \in \mathbb{C}, x \in \mathbb{C}^* \right\}.$$

We also have the following equality for an element of $\mathrm{SL}(2,\mathbb{C})$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & a/c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1/c & 0 \\ 0 & -c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -d/c \\ 0 & 1 \end{pmatrix},$$

which can be proved by direct computation and used to prove the previous equality. We only need to prove that the bilinear form is preserved by the generating elements of $\mathrm{SL}(2,\mathbb{C})$. Let $u(t) = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$. The action of $u(t)$ on $(x \;\; y)$ gives $(x \;\; tx+y)$ and if we apply it to $e_j = x^j y^{N-j}$, we obtain, after using the binomial expansion,

$$\pi_N(u(t))(e_j) = \sum_{k=0}^{N-j} t^k e_{j+k}.$$

Afterward, we do the computation with the bilinear form. We get

$$\langle \pi_N(u(t))e_j, \pi_N(u(t))e_\ell\rangle = \sum_{m=0}^{N-j}\sum_{n=0}^{N-\ell}\binom{N-j}{m}\binom{N-\ell}{n} t^m t^n \langle e_{m+j}, e_{n+\ell}\rangle,$$

we write it as a convolution and we know from the definition it is nonzero only when $j = N-k$, so that we have

$$\begin{aligned}
&= \sum_{n=0}^{N-j-\ell}\binom{N-j}{\ell+n}\binom{N-\ell}{n} t^{N-j-\ell}\langle e_{N-\ell-n}, e_{n+\ell}\rangle \\
&= (-1)^{N-\ell} t^{N-\ell-n} \sum_{n=0}^{N-j-\ell} (-1)^n \frac{\binom{N-j}{\ell+n}\binom{N-\ell}{n}}{\binom{N}{\ell+n}} \\
&= (-1)^{N-\ell} t^{N-\ell-n} \frac{(N-j)!(N-\ell)!}{N!(N-j-\ell)!} \sum_{n=0}^{N-j-\ell} (-1)^n \binom{N-j-\ell}{n}
\end{aligned}$$

the sum is zero, unless $j+\ell = N$, so that

$$= (-1)^j \binom{N}{N-j}^{-1} \delta_{j,N-\ell} = \langle e_j, e_\ell\rangle.$$

The other two matrices generating $\mathrm{SL}(2,\mathbb{C})$ are done similarly, but the computations are shorter and we will take them for granted. Now, we know that $\langle\cdot,\cdot\rangle$ is preserved by $\pi_N$ extended to $\mathrm{SL}(2,\mathbb{C})$, so in particular to the original $\pi_N$ of $\mathrm{SU}(2)$.

When $N$ is even, $\langle\cdot,\cdot\rangle$ is symmetric, so the $\pi_N(g)$ is orthogonal with respect to this bilinear form. With a suitable complex change of basis, we can arrange for the matrix of $\langle\cdot,\cdot\rangle$ to be the identity. For a $g \in \mathrm{SU}(2)$, we can arrange for $\pi_N(g)$ to be unitary, so that $\pi_N(g)$ is a real matrix with $\pi_N(g)^t = \pi_N(g)^{-1}$. If we consider an element of the form $\pi_N(g)+\pi_N(g^{-1})$, it will be selfadjoint. We can extend this to any element of $z \in \mathbb{R}[\mathrm{SU}(2)]$, so that if $z$ is of the form $x_1(g_1+g_1^{-1})+\cdots+x_k(g_k+g_k^{-1})$, then $\widehat{z}(\pi_N)$ will be selfadjoint. It follows that the eigenvalues of $\widehat{z}(\pi_N)$ will be real and there will be $N+1$ of them, counting multiplicity. This is the spectrum we are interested in.

When $N$ is odd, the bilinear form is antisymmetric. With a change basis, the form can be written

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & & \\ -1 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 1 & & \\ 0 & 0 & -1 & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}.$$

The matrix $\pi_N(g)$, in this basis, satisfy $\pi_N(g)^t J \pi_N(g) = J$ and, for $g \in \mathrm{SU}(2)$, we can arrange for $\pi_N(g)$ to be unitary, so that it belongs to $\mathrm{USp}((N+1)/2, \boldsymbol{H}(\mathbb{R}))$. In other words, with $2M = (N+1)$, it is a $M \times M$ unitary matrix with quaternions entries. It follows that $\pi_N(g)$ belong to the real linear space $\mathcal{H}$ of dimension $N+1$ of matrices $H$ such that $H = H^*$ and $J^t H J = H$. These matrices have $2M = N+1$ real eigenvalues each of multiplicity two.

### 2.4 Further Topics on Representations of SU(2)

This section is devoted to other important topics closely related to the main subject of this document. We begin by describing with more details the relation between the special unitary group and the rotation of the space. Afterward, we will proceed to describing a representation of $\mathrm{SO}(3,\mathbb{R})$. Due to the strong link between the two groups, the representations of $\mathrm{SO}(3,\mathbb{R})$ are related to those of $\mathrm{SU}(2)$, but since $\mathrm{SO}(3,\mathbb{R})$ has a geometric interpretation, rotation of the space, it is sometimes convenient to work with this group. Lastly, we will introduce the Hecke operator, an endomorphism of a function space that has interesting questions about its spectrum.

### 2.4.1 Homomorphism of SU(2) onto SO(3, ℝ)

We have discussed before that there is a two-to-one homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3,\mathbb{R})$. This is a representation of SU(2) in $\mathbb{R}^3$.

The link between SU(2) and $\mathrm{SO}(3,\mathbb{R})$ is best explained using the quaternions, which were introduced in section 2.2.1. Remember that the quaternions come equipped with a norm, induced by the inner product of $\mathbb{R}^4$, given by $N(a+ib+jc+kd) = a^2+b^2+c^2+d^2$. A quaternion $q$ with norm one is called a *unit quaternion*. Let $\boldsymbol{H}^1(\mathbb{R})$ be the multiplicative subgroup of unit quaternions. There is an isomorphism between SU(2) and $\boldsymbol{H}^1(\mathbb{R})$ given by

$$\begin{pmatrix} a+ib & -c+id \\ c+id & a-ib \end{pmatrix} \mapsto a+ib+jc+kd.$$

A unit quaternion has an action on $\mathbb{R}^3$: a vector $\vec{v} = (v_1, v_2, v_3)$ can be embedded in $\boldsymbol{H}(\mathbb{R})$ by $\vec{v} \mapsto v_1 i + v_2 j + v_3 k$, and then, for a unit $q$, one can check that $q\vec{v}\,\overline{q}$, with the product taken in $\boldsymbol{H}(\mathbb{R})$ is still of the form $v_1' i + v_2' j + v_3' k$. We define a map $R_q$ by $R_q(\vec{v}) = q\vec{v}\,\overline{q} \in \mathbb{R}^3$. Since the quaternion norm coincide with the norm on $\mathbb{R}^3$, we have $|R_q(\vec{v})| = |q\vec{v}\,\overline{q}| = |\vec{v}|$. This map is an isometry. With some computation, one can find that $\det\big(R_q(e_1), R_q(e_2), R_q(e_3)\big) = 1$. We define

$$\begin{aligned} \varphi\colon \boldsymbol{H}^1(\mathbb{R}) &\to \mathrm{SO}(3,\mathbb{R}) \subset \mathrm{GL}(3,\mathbb{R}) \\ q &\mapsto R_q. \end{aligned}$$

It is a homomorphism, as we can see $\varphi(q_1q_2)(\vec{v}) = R_{q_1q_2}(\vec{v}) = q_1q_2\vec{v}\,\overline{q}_2\overline{q}_1 = R_{q_1} \circ R_{q_2}(\vec{v})$. To summarize, we have

$$\mathrm{SU}(2) \xrightarrow{\simeq} \boldsymbol{H}^1(\mathbb{R}) \twoheadrightarrow \mathrm{SO}(3,\mathbb{R}).$$

### 2.4.2 A Word on Representations of $\mathrm{SO}(3,\mathbb{R})$

We give representations of $\mathrm{SO}(3,\mathbb{R})$ in a Hilbert space. This is taken from [Hal03]. Recall that a *Hilbert space* is vector space, possibly infinite dimensional, equipped with a scalar product that makes it a complete metric space. We will denote such a space $\mathcal{H}$. Since it has a scalar product, we can define the adjoint of an operator $T\colon \mathcal{H} \to \mathcal{H}$ by the operator $T^*$ that satisfies $\langle Tv, w\rangle = \langle v, T^*w\rangle$. As before, an operator is *unitary* if $TT^* = id$. The set of all unitary operators will be denoted with little surprise $U(\mathcal{H})$. The infinite dimensionality of a vector space can bring some complication at a topological level. In finite dimension, every linear function, that is the elements of $\mathrm{GL}(V)$, are continuous. If $\mathcal{H}$ is infinite dimensional, a linear operator need not be continuous. This leads to the *continuity condition*: If $G$ is a group and $\varphi$ a representation over $\mathcal{H}$, then for every sequences $A_n$ of $G$ that converge to $A$ in $G$, we ask that

$$\varphi(A_n)v \to \varphi(A)v \qquad (n \to \infty) \quad \text{for every } v \in \mathcal{H}.$$

The representation for $\mathrm{SO}(3,\mathbb{R})$ is in the Hilbert space $L^2(\mathbb{R}^3)$, the space of square integrable[3] functions $f\colon \mathbb{R}^3 \to \mathbb{R}$. For $R \in \mathrm{SO}(3,\mathbb{R})$, we define the operator $\varphi(R)$ by

$$\varphi(R)f(x) = f(R^{-1}x).$$

It is a homomorphism, as seen by

$$[\varphi(RS)f](x) = f(S^{-1}R^{-1}x) = [\varphi(S)f](R^{-1}x) = \varphi(R)[\varphi(S)f](x).$$

The continuity property requires some knowledge of $L^p$ spaces, see for example [Rud87]. In particular, the needed result is that the completion of $C_c(\mathbb{R}^d)$ under the norm $\|\cdot\|_p$ is $L^p$ itself. Let $R_n$ be a sequence of $\mathrm{SO}(3,\mathbb{R})$ and $R \in \mathrm{SO}(3,\mathbb{R})$, its limit. Let $f \in L^2(\mathbb{R}^3)$, and let $\{g_n\} \subset C_c(\mathbb{R}^3)$ be a sequence converging to $f$ in norm $\|\cdot\|_2$. We have

$$\|\varphi(R_n)f-\varphi(R)f\| \leq \|\varphi(R_n)f-\varphi(R_n)g_m\|+\|\varphi(R_n)g_m-\varphi(R)g_m\|+\|\varphi(R)g_m-\varphi(R)f\|.$$

For $m$ large enough, the first and third terms on the right-hand side are smaller than some $\varepsilon_m > 0$ which is a $o_m(1)$, and this, independently of $n$. The middle term requires closer inspection. If we let $n \to \infty$ in that term, by the dominated convergence theorem (each $g_m$ is bounded since it has a compact support), it tends

[3] A *square integrable function* is a function $f\colon X \to \mathbb{R}$ defined on a measure space $X$ such that $\int_X |f|^2 \,\mathrm{d}\mu$ converges. The set of all these functions is noted $L^2(X)$ and it forms a vector space.

to 0. Therefore, we have

$$\lim_{n\to\infty} \|\varphi(R_n)f - \varphi(R)f\| \le o_m(1)$$

and letting $m \to \infty$ gives the result.

An interesting property of this representation is that the image of $\varphi$ is contained in $\mathrm{U}(\mathcal{H})$. In other words, for every rotation $R \in \mathrm{SO}(3,\mathbb{R})$, the operator $\varphi(R)$ is unitary. This follows from the invariance under rotation of the Lebesgue measure. More explicitly, we have, setting $T = \varphi(R)$,

$$\langle Tf, g\rangle = \int_{\mathbb{R}^3} f(R^{-1}x)g(x)\,\mathrm{d}x = \int_{\mathbb{R}^3} f(y)g(Ry)\,\mathrm{d}y = \langle f, T^{-1}g\rangle. \qquad (\text{with } y = R^{-1}x)$$

To summarize, we showed that we have a homomorphism $\varphi\colon\ \mathrm{SO}(3,\mathbb{R}) \to \mathrm{U}(L^2(\mathbb{R}^3))$.

### 2.4.3 Hecke Operators

Here, we will have brief discussion about Hecke operators. For more information, the reader may consult [LPS86]. In the previous section, we described a unitary representation for $\mathrm{SO}(3,\mathbb{R})$. A *Hecke operator* is an averaging operator over a finite set of $\mathrm{SO}(3,\mathbb{R})$.

We will describe the operator in terms of $\mathrm{SU}(2)$. This as the advantage of allowing us to use all the tools we developed in this chapter. Let $z \in \mathbb{R}[\mathrm{SU}(2)]$ be of the form $z = x_1(g_1 + g_1^{-1}) + \cdots + x_k(g_k + g_k^{-1})$. We established an epimorphism of $\mathrm{SU}(2)$ over $\mathrm{SO}(3,\mathbb{R})$, so that the representation described earlier can be used on $z$ by identifying $g \in \mathrm{SU}(2)$ to the rotation it represents. We define $T_z$ by

$$T_z f := \frac{1}{2k} \sum_{g\in \operatorname{supp} z} f(gx) = \frac{1}{2k} \sum_{n=1}^{k} \left[ f(g_n.x) + f(g_n^{-1}.x) \right].$$

It is selfadjoint, and the constant functions are eigenvectors, with eigenvalue 1. It is the highest eigenvalue of $T_z$, and $\|T_z\| = 1$. We will denote the eigenvalues by $|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_{n-1}| \leq \lambda_n = 1$, and we will give the second to greatest eigenvalue $|\lambda_{n-1}|$ the notation $\lambda$.

The operator helps to determine whether the $2k$ points $\{g_1x, g_1^{-1}x, \dots, g_kx, g_k^{-1}x\}$ are well distribution on the sphere, in some sense. The idea is that if they are well distributed, the averaging operator $T_zf$ should be close to the average of $f$, $1/(4\pi)\int_{\mathbf{S}^2} f\,\mathrm{d}m$, for every function. If we apply $T_z$ several times, the number of points will increase, and it would reasonable to hope that the averaging operator be a more accurate approximation of the average. The composition $T_z^n f$ can be expressed in terms of words of a free group with for letters, the rotations $g_1, \dots, g_k$,

$$T_z^n f = \frac{1}{(2k)^n} \sum_{g_1,\dots,g_k \in \operatorname{supp} z} f(g_1 \cdots g_k x).$$

A sufficient condition, called the *spectral gap*, to determine that the points are well distributed has been established and well studied. We will give here an idea of how to find this condition. The first step is to prove that the operator $T_z$ commutes with the Laplacian, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$. It is well known that the space of spherical harmonics of degree $n$, $H_n$, is an eigenspace of $\Delta$ and that the space of functions $L^2(\mathbf{S}^2)$ is decomposed in

$$L^2(\mathbf{S}^2) = H_0 \oplus H_1 \oplus \cdots \oplus H_n \oplus \cdots.$$

Since $T_z$ and $\Delta$ commute, $T_z$ leaves those spaces invariant. Therefore, we can write $f \in L^2(\mathbf{S}^2)$ as a sum of spherical harmonics $c_0 + c_1 s_1 + \cdots + c_n s_n + \cdots$. For each $s_k$, we know that $|T_z^n s_k| \leq \lambda^n s_k$. Remember that $\lambda$ is the second to greatest eigenvalue $|\lambda_{n-1}| \leq 1$. Note that the constant term of $f$ is exactly its average, $c_0 = 1/(4\pi) \int_{\mathbf{S}^2} f \, \mathrm{d}m$. Now, we will approximate how close $T_z^n f$ is to its average $c_0$:

$$\begin{aligned} |T_z^n f - c_0| &\leq |c_1 T_z^n s_1| + \cdots + |c_k T_z^n s_k| + \cdots \\ &\leq |c_1 \lambda^n s_1| + \cdots + |c_k \lambda^n s_k| + \cdots \\ &\leq C\lambda^n. \end{aligned}$$

If $\lambda < 1$, that is, if $|\lambda_{n-1}| < \lambda_n$, then the limit when $n \to \infty$ of the above approximation will go to 0. This condition is what we call the *spectral gap*. The size of the spectral gap $1 - \lambda$ is important in the rate of convergence. The spectral gap is a property of the subgroup $\Gamma_z$ generated by $\operatorname{supp} z$. It is not obvious that such groups exist, so we refer the interested reader to [GJS99]. For more information on the Hecke operator, we invite the reader to consult [LPS86].

# CHAPTER 3
# Spectral Theory on Graphs

Starting with linear algebra, with the simple idea of finding a number and a vector trivializing the product of a given matrix, the spectral theory is concerned with finding eigenvalues of endomorphisms in a multitude of contexts. In graph theory, we can associate a matrix, the adjacency matrix, to a graph. The eigenvalues, as will be seen, then take the interpretation of counting the number of closed walks in the graph. Depending on the context from which the graph arises, these closed walks can have different interpretations. For instance, when the graph is generated from a free group, a closed walk is a word from the group which spells the identity.

In this chapter, we will link together spectral theory and group theory, using the Cayley graph. The beginning section will introduce this tool, and expose some examples. We will then move on to the spectral theory on regular graphs, more specifically we will introduce the work of Kesten and the explicit formula found for the measure counting the eigenvalues. Lastly, for an element of $\mathbb{R}[\mathrm{SU}(2)]$, we will examine the eigenvalues of the matrices obtained from its representations, and apply the previous results on the Cayley graph of the support of this element, provided it is a free subgroup.

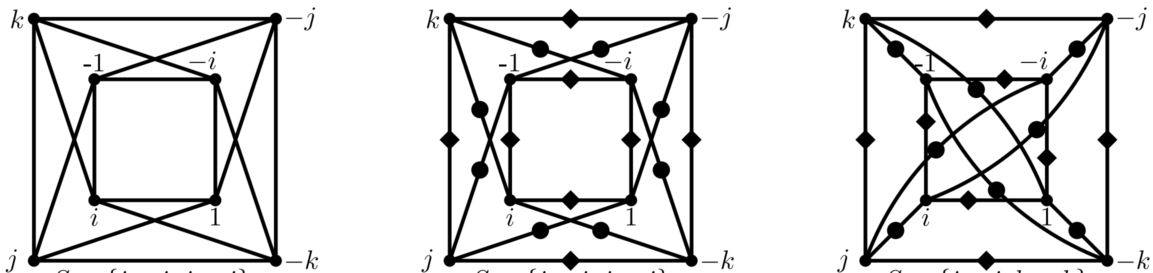
## 3.1 Cayley Graphs

A Cayley graph for a group is a display of the elements of the group, the vertices, and the product of the group, the edges. This idea can be carried out to a group action: if a group $G$ acts on a set $\Omega$, each element of $\Omega$ would be a vertex, and an edge would connect two vertices if one element can be written as a product in terms of another element. The composition law being an action on the group itself, we can consider the following definition as special case of this idea; however, it is sufficient for the purpose of this chapter.

**Definition 3.** Let $G$ be a group and let $S$ be a subset of $G$. We say that $S$ is *symmetric* if for every $s \in S$, we have $s^{-1} \in S$. If $S$ is symmetric and generates $G$, then we define the *Cayley graph*, noted $\mathrm{Cay}(G, S)$, as the graph with vertex set $G$ and the edge set $\big\{\{g, gs\} \bigm| g \in G, s \in S\big\}$.

In the Cayley graph, we have an edge between the vertices $g$ and $h$ if there is an element $s$ in $S$ such that $h = gs$. The symmetry of $S$ is equivalent to having no orientation on the edges; if $(g, h)$ was an edge oriented from $g$ towards $h$, then we would have the edge $(h, hs^{-1})$ in the opposite direction. Furthermore, in the definition, we ask for $S$ to generate $G$, which ensures that the Cayley graph is connected. As a last remark, we note that we can obtain a coloured graph by assigning a colour to each pair $s$ and $s^{-1}$ of $S$, and paint the edges in a consequential way. See Figure 3–1 for examples, with diamonds and dots replacing the use of colour. This, however, is negligible for our exposition.

A group can have several generating sets and the graph obtained usually depends on the set chosen. As we can see in Figure 3–1, the two graphs of the quaternion are isomorphic. However on Figure 3–2, this is not the case. On the left, we have a 3-regular graph, whilst on the right, it is 4-regular. When an element in the set $S$ presents a relation, for example $s^n = e$, it will manifest itself in the graph in the




$S = \{i, -i, j, -j\}$.

$S = \{i, -i, j, -j\}$
with diamonds for $i$, dots for $j$.

$S = \{i, -i, k, -k\}$
with diamonds for $i$, dots for $j$.

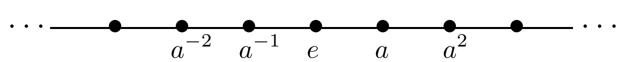
Figure 3–1: Cayley graph of $Q_8$ with different colours and generating sets.

form of a cycle. In Figure 3–2 on the left, the element (123) is of order three, and we can see several cycle of length three appear in the graph, labeled with diamonds.

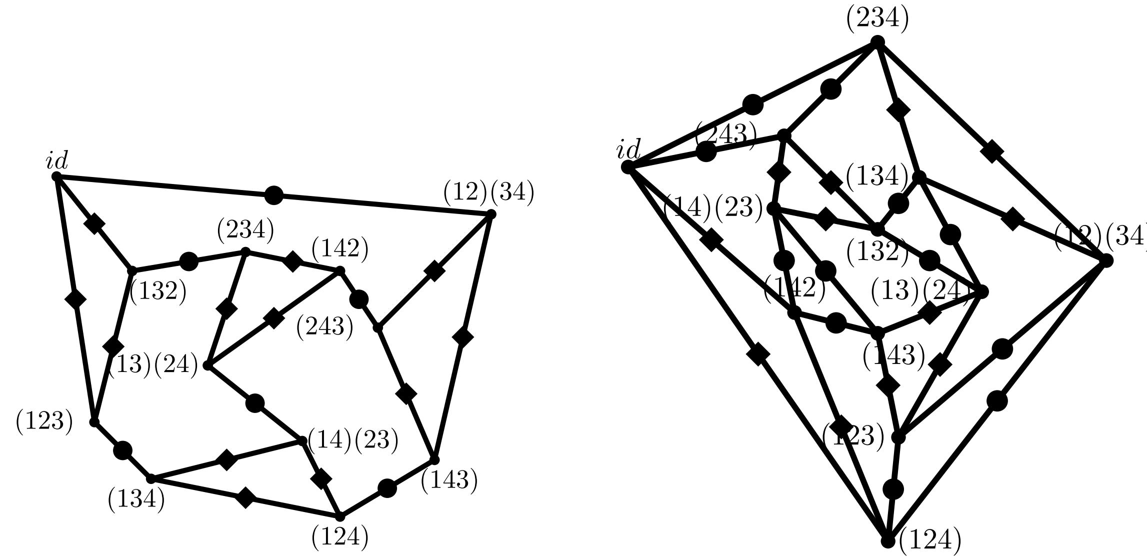
### 3.1.1 Cayley Graphs of Free Groups

Given a free group $G = \langle a_1, a_2, \ldots, a_n \rangle$, we're interested in its Cayley graph. Such a group satisfies no relation, therefore no cycle should appear in the graph. A connected undirected graph with no loop or cycle is called a *tree*, and sometimes a *forest* if the graph is disconnected. However, if our set $S$ generates $G$, then the graph will be connected.

For example, when $G$ is generated by a single element, say $a$, then we get the graph




The Cayley graph of the free group with two generators $G = \langle a, b \rangle$ can be seen on Figure 3–3.




$S = \{(123), (132), (12)(34)\}$
with diamonds for (123) and dots for (12)(34).

$S = \{(234), (243), (124), (142)\}$
with dots for (234) and diamonds for (124).

Figure 3–2: Two Cayley graphs of $A_4$.

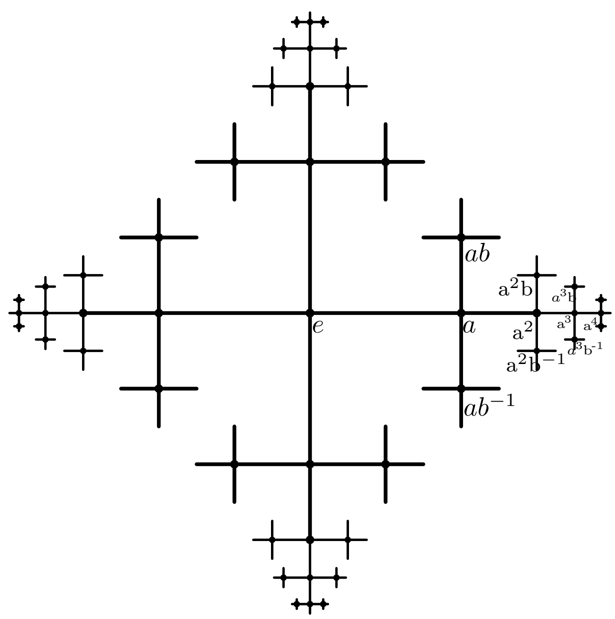



Figure 3–3: Cayley graph of the free group with two generators.

### 3.2 Kesten Measure

For now, we are concerned with the adjacency matrix of a graph, and not the context in which the graph arises, but we should keep in mind that Cayley graphs are an interesting source of graphs.

For a non-oriented regular graph with no loop $X$, let $n(X)$ be the number of vertices and $c_k(X)$ be number of cycles of length $k$. Let $A(X)$ denote the adjacency matrix of $X$, that is $a_{ij}$ is 1 if there is an edge between the $i^{\text{th}}$ and $j^{\text{th}}$ vertex and 0 otherwise. This matrix is symmetric, therefore it has $n(X)$ real eigenvalues, counting multiplicity. Define a function $F(X,x)$ that gives the proportion of eigenvalues of $A(X)$ lying in $(-\infty, x]$. We can write this function as

$$F(X,x) = \frac{1}{n(X)} \sum_{i=1}^{n(X)} H(x - \lambda_i), \tag{3.1}$$

where $H$ is the *Heaviside step function*[1] that we define here as (the value at $x = 0$ is important!)

$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

Here are some properties of $F$. It is weakly[2] increasing and right continuous. If $X$ has degree $v$, then the greatest eigenvalue of $A(X)$ in absolute value is $v$, so that for $x \geq v$ it has value 1, and for $x < -v$, value 0. We are interested in the behaviour

[1] In distribution theory, the Heaviside step function has the Dirac's delta function as derivative. Therefore, we can write $f(X,x) = F'(X,x) = \frac{1}{n(X)} \sum \delta_{\lambda_i}(x)$ in the sense of distribution.

[2] By weakly increasing, we mean nondecreasing.

of $F(X,x)$ when we let the graph $X$ grow. More specifically, consider a sequence of graphs $X_1, X_2, X_3, \ldots$ such that $X_i$ is $v$-regular for every $i$. Moreover, when $i \to \infty$, the sequence should satisfy:

1. $n(X_i) \to \infty$;
2. $\dfrac{c_k(X_i)}{n(X_i)} \longrightarrow 0$, where $c_k(X)$ is the number of cycles of length $k$ in $X$.

To study this limiting process, understanding the adjacency matrix is essential. The powers of $A(X)$ give the number of walks in $X$. More precisely, the entry $b_{ij}$ of $A^n$ gives the number of walks of length $n$ starting at the $i^{\text{th}}$ vertex and ending at the $j^{\text{th}}$ one. To dismiss any ambiguity, here a walk a is path which may appeal to the same edge multiple times. An entry on the diagonal of $A^n$ gives the number of closed walks of length $n$ at that vertex. The trace of $A^n$ is therefore the total number of closed walks in the graph; recall that it is also the sum of the eigenvalues of $A$ to the power $n$. The following theorem will motivate why we are interested in counting the number of closed walks. We will not prove this theorem, but one can find a proof in [McK81]. The proof mostly uses basic analysis tools; it is not pertinent to the discussion.

**Theorem 3.2.1.** *Let $F_1$, $F_2, \ldots : \mathbb{R} \to \mathbb{R}$ be a sequence of functions such that for every $i$:*

1. *$F_i(x) = 0$ if $x < a$ for a real number $a$;*
2. *$F_i(x)$ is constant if $x > b$ for a real number $b > a$;*
3. *$F_i(x)$ is right continuous for all $x$;*
4. *the total variation of $F_i$ is bounded by $M$.*

*If for every integer $r \geq 0$, $\int x^r \, \mathrm{d}F_i \to \mu(r)$ when $i \to \infty$, then there exists a unique function $F$ satisfying* 1. *to* 4. *such that $\int x^r \, \mathrm{d}F = \mu(r)$ and $F_i \to F$ wherever $F_i$ is continuous.*

Considering that the integral $\int x^r \, \mathrm{d}F(X_i, x)$, where the Lebesgue-Stieltjes measure $\mathrm{d}F(X_i, x)$ is derived from (3.1), is the average of closed walks of length $r$ over the number of vertices[3] in the graph $X$, determining the number of closed walk will be an important step in proving the following main result.

**Theorem 3.2.2** (Kesten's Theorem)**.** *Let $X_1, X_2, \dots$ be a sequence of regular graphs, each with vertices $\{1, 2, \dots, n(X_i)\}$ and of degree $v \geq 2$. If the following conditions are met:*

1. *$n(X_i) \to \infty$ as $i \to \infty$,*
2. *for each $k \geq 3$, $c_k(X_i)/n(X_i) \to 0$ as $i \to \infty$;*

*then for every $x$, $F(X_i, x) \to F(x)$ as $i \to \infty$, where $F(x)$ is the function defined as follows:*

$$F(x) = \begin{cases} 0 & \text{if } x \leq -2\sqrt{v-1}, \\ \displaystyle\int_{-2\sqrt{v-1}}^{x} \frac{v\sqrt{4(v-1)-t^2}}{2\pi(v^2-t^2)} \, \mathrm{d}t & \text{if } -2\sqrt{v-1} < x < 2\sqrt{v-1}, \\ 1 & \text{if } x \geq 2\sqrt{v-1}. \end{cases}$$

[3] This is most easily seen using the distribution theory point of view. If $F(X, x)$ is as in (3.1), then

$$\int x^r \, \mathrm{d}F = \frac{1}{n(X)} \sum_{i=1}^{n(X)} \int x^r \delta_{\lambda_i}(x) \, \mathrm{d}x = \frac{1}{n(X)} \sum_{i=1}^{n(X)} \lambda_i^r.$$

*The integral appearing can be evaluated to*

$$\frac{1}{2} + \frac{1}{2\pi}\left[v \arcsin \frac{x}{2\sqrt{v-1}} - (v-2)\arctan \frac{(v-2)x}{v\sqrt{4(v-1)-x^2}}\right].$$

The function $F$ is differentiable almost everywhere, so that $\mathrm{d}F = f\,\mathrm{d}m$, where $f = F'$. This defines the measure that we call *the Kesten measure*. We interpret the number $\int_a^b f\,\mathrm{d}m$ as the proportion of eigenvalues of the limiting graph $X$ falling in the interval $(a, b]$.

### 3.2.1 Closed Walks in a Regular Tree

Recall that a tree is a connected graph with no loop or cycle. Let us suppose that the tree is $v$-regular. We want to count the number of closed walks of length $r'$ starting at a given vertex $\nu_0$. A closed walk is a sequence of $r'$ consecutive vertices $(\nu_0, v_1, v_2, \ldots, v_{r'-2}, \nu_{r'-1})$ with $\nu_{r'-1} = \nu_0$ and where each pair $\{v_i, v_{i+1}\}$ is an edge, for $0 \leq i < r' - 1$.

We can associate to a walk a sequence of $r'$ numbers $\varepsilon_i$, where each $\varepsilon_i$ is $-1$ or $+1$, depending on whether at the step $i$ you move closer or farther respectively from the origin $\nu_0$. The partial sum $s(\ell) = \sum_i \varepsilon_i$ gives the distance from the origin at step $\ell$. Since a closed walk must return to the starting point, we know that $\sum_i \varepsilon_i = 0$. From this, it follows that $r'$ is even, that is, there are no closed walks of odd length. We will note $r'$ by $2r$ from here on out. This sequence that we associate to a walk can be thought of as a walk on a 2-regular tree, in other words at each step, you can move forward or backward. We will first attempt to count these such walks, and from there, we can easily obtain all the closed walks on a $v$-regular tree by multiplying by the number direction you can move at each step.

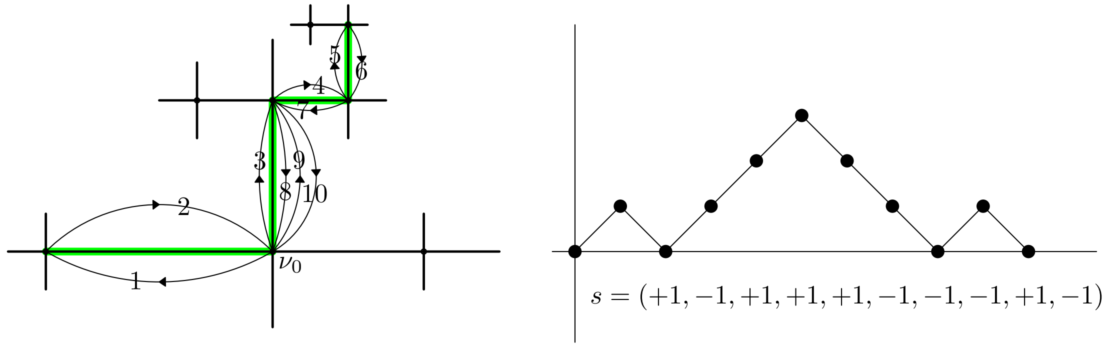



Figure 3–4: A walk on a graph, and its graph of polygonal segments.

We can represent a sequence $s = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2r})$ by a curve of polygonal segments that have slope 1 on the interval $[i, i+1]$ if $\varepsilon_i$ is 1 and slope $-1$ otherwise (see Figure 3–4). It is the graph of $s(\ell)$ to which you add arcs.

Following [McK81] lemma 2.2 and using the approach of [Fel66] chapter 3, section 7 (theorem 4), our goal is to count the number of ways to draw curves of polygonal segments always above or on $x = 0$ and returning to $x = 0$ at $i = 2r$. We say the walk *returns to the origin* when at some step, it is back at the starting point. This can be seen graphically by the polygonal line touching the abscissa. Figure 3–4 is an example of a walk with three returns to the origin. If we forget the connection to the graph and just think of a walk as a sequence of $\varepsilon_i$ (which in turn, can be thought of as a sequence of games of head or tail), we can count the number of walks returning exactly $k$ times to the origin while never crossing the abscissa by the following proposition, which is a restatement of theorem 4 from [Fel66], chapter 3 section 7, and it follows from theorem 2 and the proof theorem 4.

**Proposition 2.** *The number of walks never crossing the abscissa with a $k$-th return at the origin at step $2r$ is*

$$\frac{k}{2r-k}\binom{2r-k}{r}.$$

To prove this, we will use the following lemma.

**Lemma 3.2.3.** *The number of walks starting at $0$ and ending at $m$ after $n$ steps is given by*

$$N(n,m)=\binom{n}{\frac{n+m}{2}}.$$

*Proof of lemma.* The walk is a sequence of $+1$ and $-1$. Let $p$ denote the number of $+1$ and $q$, the number of $-1$. It follows from this notation that $n=p+q$ and $m=p-q$. The number of walks is then the number of sequences of length $n$ with $p$ coordinates being $+1$, that is $\binom{p+q}{p}$, from which the result follows, together with $p=\frac{n+m}{2}$. ☐

*Proof of proposition.* Suppose the graph of the walk is always below the abscissa and has $k$ points on it, without counting the starting vertex. The idea is to count the number of ways you can choose $+1$ or $-1$ in a sequence of $\varepsilon_i$'s. When we return to the origin, we have no choice of the next step, since we must not cross the abscissa. If we return $k$ times to the origin, it means that there $k$ steps we did not have a choice of going forward or backward ($+1$ or $-1$), therefore we can associate to each of these walks a walk of length $2r-k$ in a unique way (they are in bijection). We draw the new walk by following the previous one, except that we ignore the segments leaving the abscissa. The new walks will all end with a first visit to $k$, that is they will attain height $k$ for the first time at step $2r-k$ (see Figure 3–5), so we reduced the

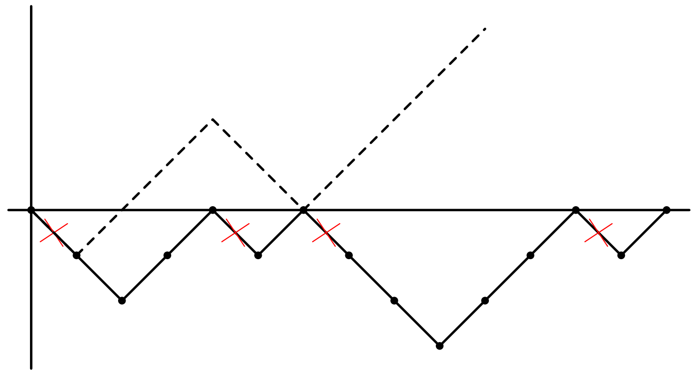


Figure 3–5: To obtain a first visit to 4 (dashed) from four returns to the origin.

problem to counting those walks. We will use $n := 2r - k$ from now on to alleviate the notation.

A walk with first visit to $k$ at step $n$ is described by the fact that at the previous step, it is of height $k-1$. In other word, it is a walk of length $n-1$ with a maximum of $k-1$. All the paths starting at 0 and ending at $k-1$ after the step $n-1$ are counted by $N(n-1, k-1)$, by the previous lemma, and they have a maximum of at least $k-1$. Since we want a maximum of *exactly* $k-1$, we subtract the walks with a maximum greater or equal to $k$. This number is given by the idea of a *reflection*.

The walks from the origin to a point $A$ are in bijection with the walks from the origin to $A'$, where $A'$ is obtained by a reflection from a certain horizontal line. If we reflect a walk from the origin to $(p, q)$ with a maximum of at least $m$ through the axis $x = m$, we obtain a walk from the origin to $(p, 2m - q)$ with a maximum of at least $2m - q$ (see Figure 3–6 for an example). In other words, all these walks are counted by $N(p, 2m - q)$. Therefore, if we reflect $(n - 1, k - 1)$ with respect to $x = k$, we obtain the point $(n - 1, k + 1)$, so the number we want is $N(n - 1, k + 1)$.

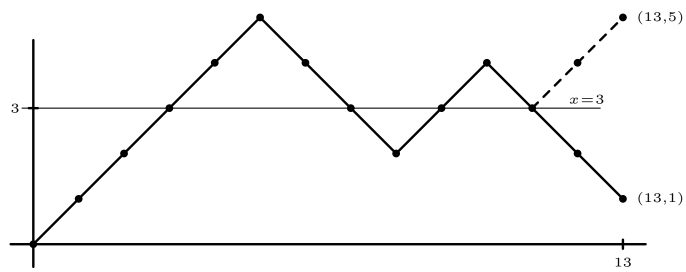



Figure 3–6: Reflection about the axis $x = 3$ of a walk with a maximum $\geq 3$.

We conclude with a simple computation

$$
\begin{aligned}
&N(n-1,k-1) \\
&\quad - N(n-1,k+1) = \binom{n-1}{\frac{n-1+k-1}{2}} - \binom{n-1}{\frac{n-1+k+1}{2}} \\
&\qquad = \binom{n-1}{\frac{2r-2}{2}} - \binom{n-1}{\frac{2r}{2}} \qquad \text{(replacing } n \text{ by } 2r-k) \\
&\qquad = \frac{(n-1)!}{(r-1)!\,(n-1-r+1)!} - \frac{(n-1)!}{r!\,(n-1-r)!} \\
&\qquad = \frac{r}{n}\frac{n!}{r!\,(n-r)!} - \frac{n-r}{n}\frac{n!}{r!\,(n-r)!} \\
&\qquad = \frac{r-n+r}{n}\frac{n!}{r!\,(n-r)!} \\
&\qquad = \frac{k}{n}\binom{n}{r}.
\end{aligned}
$$

□

Now, if we return to the graph, for a walk with $k$ returns to the origin, there will be $k$ steps where we will have $v$ directions to choose from, $r-k$ steps with $v-1$, since we can't go back on those ones, and the remaining steps are the ones moving closer to the origin, and there is only one choice of direction for them. Therefore, the total number of closed walks of length $2r$ at a given vertex is

$$\theta(2r) := \sum_{k=1}^{r} \frac{k}{2r-k} \binom{2r-k}{r} v^k (v-1)^{r-k}, \tag{3.2}$$

where the sum is over the number of returns to the origin. This discussion proves the following lemma.

**Lemma 3.2.4** ( [McK81],lemma 2.1)**.** *Suppose $X$ is a regular graph of degree $v$. Let $\nu_0$ be a vertex of $X$ and suppose the subgraph of vertex at distance at most $r$ forms a tree, then the number of closed walks of length $2r$ is given by* (3.2)*. Also, note that $\theta(s) = 0$ for $s$ odd.*

This next lemma allows to conclude the existence of the limiting function $F$ in theorem 3.2.2.

**Lemma 3.2.5.** *Let $X_1, X_2, \ldots$ be a sequence of graphs as in theorem 3.2.2. For $r \geq 0$, $i \geq 1$, if $\varphi_r(X_i)$ denote the total number of closed walks of length $r$ in $X_i$, then for each $r$, we have $\varphi_r(X_i)/n(X_i) \to \theta(r)$ as $i \to \infty$.*

*Proof.* If there are $n_r(X_i)$ vertices in $X_i$ such that the vertex together with vertices of distance at most $r$ form a tree (that is, the number of vertex satisfying lemma 3.2.4), then each of these vertex have $\theta(r)$ closed walks starting there. For the other $n(X_i) - n_r(X_i)$ vertices, the number of closed walks is bounded by $v^r$, and we will note it by $\overline{\theta}_r(X_i)$. We have

$$\lim_{i\to\infty} \frac{\varphi_r(X_i)}{n(X_i)} = \lim_{i\to\infty} \frac{n_r(X_i)\theta(r) + [n(X_i) - n_r(X_i)]\overline{\theta}_r(X_i)}{n(X_i)} = \theta(r),$$

since $n_r(X_i)/n(X_i) \to 1$ as $i \to \infty$, because of condition 2 of theorem 3.2.2. □

The lemma says that $\int x^r \,\mathrm{d}F(X_i, x) \to \theta(r)$ as $i \to \infty$. By theorem 3.2.1, we know there exists a function $F$ such that $\int x^r \,\mathrm{d}F = \theta(r)$ and $F(X_i, x) \to F(x)$ as $i \to \infty$ on points of continuity.

### 3.2.2 The support of the Kesten measure

Let $f$ be the derivative of $F$ appearing in theorem 3.2.2. As mentioned earlier, $f\,\mathrm{d}m$ is called the *Kesten Measure*. Here, we will highlight the reason why this measure is supported in $[-2\sqrt{v-1}, 2\sqrt{v-1}]$. Recall that $\theta(r)$, defined by equation (3.2), is the number of closed walks of length $r$ starting at a given vertex, and that $\int x^r \,\mathrm{d}F$, called the $r$-th moment, is equal to $\theta(r)$.

An asymptotic expression of $\theta(r)$ is given by (see Lemma 3.1 of [McK81])

$$\theta(2s) \sim \frac{4^s v(v-1)^{s+1}}{s(v-2)^2\sqrt{\pi s}}. \tag{3.3}$$

Define $\omega$ as $\sup\{|x| : 0 < F(x) < 1\}$. Since $\int x^{2s+2}\,\mathrm{d}F \leq \omega^2 \int x^{2s}\,\mathrm{d}F$, we have

$$\limsup_{s\to\infty} \frac{\int x^{2s+2}\,\mathrm{d}F}{\int x^{2s}\,\mathrm{d}F} \leq \omega^2.$$

Now, for $0 < \alpha < \beta < \omega$, we have

$$\int\limits_{|x|\geq\alpha} x^{2s}\,\mathrm{d}F \geq \int\limits_{|x|\geq\beta} x^{2s}\,\mathrm{d}F \geq \int\limits_{|x|\geq\beta} \beta^{2s}\,\mathrm{d}F = \beta^{2s}\big[1 - F(\beta) + F(-\beta)\big]$$

and

$$\int_{|x|\leq\alpha} x^{2s}\,\mathrm{d}F \leq \alpha^{2s}\big[F(\alpha) - F(-\alpha)\big],$$

so that

$$\lim_{s\to\infty} \frac{\int_{|x|\geq\alpha} x^{2s}\,\mathrm{d}F}{\int_{|x|\leq\alpha} x^{2s}\,\mathrm{d}F} = 0.$$

We find that

$$\frac{\displaystyle\int x^{2s+2}\,\mathrm{d}F}{\displaystyle\int x^{2s}\,\mathrm{d}F} = \frac{\displaystyle\int_{|x|\geq\alpha} x^{2s+2}\,\mathrm{d}F + \int_{|x|\leq\alpha} x^{2s+2}\,\mathrm{d}F}{\displaystyle\int_{|x|\geq\alpha} x^{2s}\,\mathrm{d}F + \int_{|x|\leq\alpha} x^{2s}\,\mathrm{d}F}$$

$$\geq \left(\frac{\displaystyle\int_{|x|\geq\alpha} x^{2s+2}\,\mathrm{d}F}{\displaystyle\int_{|x|\geq\alpha} x^{2s}\,\mathrm{d}F}\right) \cdot \frac{\left(1 + \frac{\displaystyle\int_{|x|\leq\alpha} x^{2s+2}\,\mathrm{d}F}{\displaystyle\int_{|x|\geq\alpha} x^{2s+2}\,\mathrm{d}F}\right)}{\left(1 + \frac{\displaystyle\int_{|x|\leq\alpha} x^{2s}\,\mathrm{d}F}{\displaystyle\int_{|x|\geq\alpha} x^{2s}\,\mathrm{d}F}\right)},$$

and since $\int_{|x|\geq\alpha} x^{2s+2}\,\mathrm{d}F \geq \alpha^2 \int_{|x|\geq\alpha} x^{2s}\,\mathrm{d}F$, it follows that

$$\liminf_{s\to\infty} \frac{\displaystyle\int x^{2s+2}\,\mathrm{d}F}{\displaystyle\int x^{2s}\,\mathrm{d}F} \geq \alpha^2.$$

We can let $\alpha$ be arbitrarily close to $\omega$. We conclude that $\omega^2 = \lim_{s\to\infty} \theta(2s+2)/\theta(2s)$. The limit is $4(v-1)$, computed from the asymptotic expression (3.3).

To obtain the precise formula for the function, Tchebysheff polynomials are used. We will not discuss this problem any further, but the reader is invited to consult section 3 of [McK81].

### 3.3 Spectral Measure of Elements of $\mathbb{R}[\mathrm{SU}(2)]$

Let $z \in \mathbb{R}[\mathrm{SU}(2)]$ be a selfadjoint element, that is $z = x_1 g_1 + x_1 g_1^{-1} + \cdots + x_k g_k + x_k g_k^{-1}$. Remember that we define $\widehat{z}(\pi_N)$ by $\sum_i [x_i \pi_N(g_i) + x_i \pi_N(g_i)^{-1}]$. This matrix is also selfadjoint, so that it has a real spectrum $\operatorname{spec}(\widehat{z}(\pi_N)) \subset [-2k, 2k]$. We will denote the $N+1$ eigenvalues by $\left\{\lambda_i\big(\widehat{z}(\pi_N)\big)\right\}_{i=1}^{N+1}$. We are interested in the proportion of eigenvalues lying in $(-\infty, x]$, similar to what we do in section 3.2. This defines a measure[4] , which we will call the *spectral measure* of $z$:

$$\mu_N(z) = \frac{1}{N+1} \sum_{i=1}^{N+1} \delta_{\lambda_i(\hat{z}(\pi_N))}. \tag{3.4}$$

The behaviour of $\mu_N$ when $N \to \infty$ will be the subject of the next theorem. There are two cases to consider: when $N$ is odd, and when it is even. Looking at the proof, the reason for this is because $\chi_{\pi_N}(-e) = (-1)^N (N+1)$, as stated by lemma 2.3.1. It will be proved that the limit exists in both cases. Moreover, when $\Gamma_z$ is free, we can see through the proof that the limit will converge to the Kesten measure if the coefficients of $z$ are all 1, as the moments of $\mu_N$ will be nothing more than the number of walks in the Cayley graph of $\Gamma_z$.

For the sake of clarity, we will drop the $z$ of $\mu_N(z)$, but it should not be forgotten that the definition of $\mu_N$ depends on $z$.

**Theorem 3.3.1** ( [GJS99]). *Let $z = \sum_g x_g g \in \mathbb{R}[\mathrm{SU}(2)]$ such that $z = z^*$ and with $\#\operatorname{supp} z = 2k$. Let $\mu_N$ be the density of the spectrum of $z$, defined by equation* (3.4). *There are two measures $\nu^{even}$ and $\nu^{odd}$ such that for every continuous*

[4] This equality is to be understood in the sense that for every measurable set $\mathcal{A}$, we have $\mu_N(z)(\mathcal{A})$ given by $\sum_i \delta_{\lambda_i(\hat{z}(\pi_N))}(\mathcal{A})$, where $\delta_x(\mathcal{A}) = 1$ if and only if $x \in \mathcal{A}$.

*functions* $f \in C(\mathbb{R})$

$$\lim_{n\to\infty}\int_{\mathbb{R}} f\,\mathrm{d}\mu_n = \int_{\mathbb{R}} f\,\mathrm{d}\nu^{even} \qquad \text{for } n = 2N;$$

$$\lim_{n\to\infty}\int_{\mathbb{R}} f\,\mathrm{d}\mu_n = \int_{\mathbb{R}} f\,\mathrm{d}\nu^{odd} \qquad \text{for } n = 2N+1.$$

*Moreover, if* $\Gamma_z$*, the subgroup generated by* $\operatorname{supp}(z)$*, is free, and* $z$ *is of the form* $g_1 + g_1^{-1} + \cdots + g_k + g_k^{-1}$*, then the* $\nu^{odd}$ *and* $\nu^{even}$ *are supported on* $[-2\sqrt{2k-1}, 2\sqrt{2k-1}]$ *and*

$$\mathrm{d}\nu^{odd} = \mathrm{d}\nu^{even} = \frac{2k\sqrt{4(2k-1)-x^2}}{2\pi(4k^2-x^2)}\,\mathrm{d}x \qquad \text{on the support.}$$

*Proof.* The goal is to compute the moments of $\mu_N$. We know that $\operatorname{tr}(\widehat{z}(\pi_N))$ is the sum of eigenvalues of the matrix $z$ represented in $\mathrm{GL}(N+1,\mathbb{C})$, but it is also given by $\sum_g x_g \operatorname{tr}(\pi_N(g)) = \sum_g x_g \chi_{\pi_N}(g)$. This yields, for the first moment,

$$(N+1)\int_{\mathbb{R}} x\,\mathrm{d}\mu_N = \operatorname{tr}(\widehat{z}(\pi_N)) = \sum_{g\in\mathrm{SU}(2)} x_g \chi_{\pi_N}(g).$$

More generally, for the $m$-th moment, $\operatorname{tr}(\widehat{z^{\bullet m}}(\pi_N))$ represent the sum of the $m$-th power of eigenvalues of $\widehat{z}(\pi_N)$, therefore, by equation (2.7) of chapter 2, it is equal to

$$(N+1)\int_{\mathbb{R}} x^m\,\mathrm{d}\mu_N = \operatorname{tr}(\widehat{z^{\bullet m}}(\pi_N)) = \sum_{g_1,\ldots,g_m\in\mathrm{SU}(2)} x_{g_1}\cdots x_{g_m}\chi_{\pi_N}(g_1\cdots g_m).$$

By lemma 2.3.1, we know $\chi_{\pi_N}(g_1\cdots g_m) = \big(\sin(N+1)t\big)/(\sin t)$ for some real $t$ if $g_1\cdots g_m \neq \pm e$. In this case, it follows that ${\chi_{\pi_N}(g)}/{(N+1)} \longrightarrow 0$ when $N\to\infty$. If $g = e$, then $\chi_{\pi_N}(g) = N+1$, and if $g = -e$, $\chi_{\pi_N}(g) = (-1)^N(N+1)$. There are two limit points: when $N$ is odd and when it is even. Every terms of the sum in the previous equation vanishes when $N \to \infty$, except those terms for which $g_1 \cdots g_m = \pm e$, so we have

$$\lim_{n\to\infty} \int_{\mathbb{R}} x^m \,\mathrm{d}\mu_n = \sum_{\substack{g_1,\dots,g_m \in G \\ g_1\cdots g_m = \pm e}} x_{g_1} \cdots x_{g_m} \qquad \text{for } n = 2N;$$

$$\lim_{n\to\infty} \int_{\mathbb{R}} x^m \,\mathrm{d}\mu_n = \sum_{\substack{g_1,\dots,g_m \in G \\ g_1\cdots g_m = e}} x_{g_1} \cdots x_{g_m} - \sum_{\substack{g_1,\dots,g_m \in G \\ g_1\cdots g_m = -e}} x_{g_1} \cdots x_{g_m} \qquad \text{for } n = 2N+1.$$

We will use Theorem 3.2.1. We define $F_n^{odd}(x) = \int_{-\infty}^x \mathrm{d}\mu_{2n+1}$ and $F_n^{even}(x) = \int_{-\infty}^x \mathrm{d}\mu_{2n}$. These two functions satisfy the hypothesis of theorem 3.2.1, with $\operatorname{supp} F_n^{even}, \operatorname{supp} F_n^{odd} \subseteq [-\|z\|, \|z\|]$ for all $n$, where $\|z\| = \sum_g |x_g|$, and with moments given above, so there exists $F^{odd}$ and $F^{even}$ such that $F_n^{odd} \to F^{odd}$ and $F_n^{even} \to F^{even}$ when $n \to \infty$. The remaining of the argument has to be made for $F^{odd}$ and $F^{even}$, but it is the same one for both, so we will simply write $F_n$ to mean either one. Note that $\int_{\mathcal{A}} \mathrm{d}F_n = \int_{\mathcal{A}} \mathrm{d}\mu_{n'}$ for every measurable set $\mathcal{A}$, where $n'$ is $2n$ or $2n+1$ respectively for $F^{even}$ or $F^{odd}$.

Now, every continuous functions of $C(\mathbb{R})$ can be approximated uniformly by polynomials on the support of $F_n$ because the latter is compact, so that for $g$ continuous and every $\varepsilon > 0$, there is a polynomial $p_n$ of degree $n$ such that $\sup_x |g(x) - p(x)| < \varepsilon$. We then have

$$\left| \int_{\mathbb{R}} g(\,\mathrm{d}F_k - \,\mathrm{d}F) \right| = \left| \int_{\mathbb{R}} g - p_n + p_n(\,\mathrm{d}F_k - \,\mathrm{d}F) \right|$$
$$\leq \int_{[-2k,2k]} \sup |g - p_n| (\,\mathrm{d}F_k + dF) + \left| \int_{\mathbb{R}} p_n(\,\mathrm{d}F_k - \,\mathrm{d}F) \right|,$$

the total variation of $F$ and $F_k$ is bounded, so $\int_{\mathbb{R}}(\,\mathrm{d}F_k + \mathrm{d}F) \leq M < \infty$ and

$$\leq M\varepsilon + \left|\sum_{\ell=0}^{n}\int_{\mathbb{R}} x^{\ell}(\,\mathrm{d}F_k - \,\mathrm{d}F)\right|$$

$$= M\varepsilon + \left|\sum_{\ell=0}^{n}\sum_{\substack{g_1,\dots,g_\ell\in G\\ g_1\cdots g_\ell\neq\pm e}} x_1\cdots x_\ell \frac{\chi_{\pi_k}(g_1\cdots g_\ell)}{k+1}\right|$$

$$\leq M\varepsilon + \left|\sum_{\ell=0}^{n} R\frac{\sin(k+1)t}{(k+1)\sin t}\right|$$

$$\leq M\varepsilon + \left|\frac{nR}{(k+1)}\frac{\sin(k+1)t}{\sin t}\right|$$

and with $k = \lceil (n+1)R(\sin t)^{-1}/\varepsilon\rceil$

$$\leq (M+1)\varepsilon.$$

We conclude $\int g\,\mathrm{d}F_k \to \int g\,\mathrm{d}F$.

Lastly, if $\Gamma_z$ is free, and $x_g = 1$ for every $g \in \operatorname{supp} z$, then the sums in the moments are counting the number of closed walks. Indeed, we have

$$\sum_{\substack{g_1,\dots,g_m\in G\\ g_1\cdots g_m=e}} x_{g_1}\cdots x_{g_m} = \sum_{\substack{g_1,\dots,g_m\in G\\ g_1\cdots g_m=e}} 1,$$

so this sum is counting the number of words of $m$ letters in $\Gamma_z$ that reduces to $e$. This corresponds exactly to a walk in the Cayley Graph of $\Gamma_z$ of length $m$ starting and ending at $e$. Moreover, because $\Gamma_z$ is free, the element $-e$ does not belong to the group, as it satisfies $(-e)(-e) = e$, so there are no words of $m$ letters that reduces to $-e$. Hence, the formula for the moments of $\mu_{2n}$ and $\mu_{2n+1}$ both simplify to the number of closed walks starting at $e$, which is also the average of closed walks, since every vertex has the same number of closed walks. Therefore, their moments coincide with those of the Kesten measure. $\square$

# CHAPTER 4
# Random Matrices

We begin the chapter with some review of probability theory in the context of measure theory. Focusing mainly on real-valued random variables, the beginning section follows the first few sections of chapter one in [Lam11]. The subsequent sections introduce vector-valued random variables, in particular matrix-valued, which is the object of interest here. We refer the interested reader to [Meh04] which also offer a physical context to the ensembles introduced below, and to [Tao13] for a video of a conference on Wigner matrices.

## 4.1 Measures and Random Variables

We expect the reader to be familiar with the concept of real measures, especially those *finite* and $\sigma$*-finite*. We refer the reader to [Rud87]. As a point of reference, we will take the existence of the *Lebesgue measure*, the Lebesgue-Radon-Nykodym decomposition, and the Dominated Convergence Theorem for granted. However, we will recall important aspects when needed. Let us begin by defining an important tool for probability theory.

**Definition 4.** Let $(X, \mathcal{A}, \mu)$ be a measure space and $(Y, \Sigma)$ a measurable space. Given a measurable function $f\colon X \to Y$, we define the *image measure* $f_*\mu$ by

$$f_*\mu(\mathcal{E}) = \mu(f^{-1}(\mathcal{E})).$$

It can be shown that $f_*\mu$ is indeed a measure and it simply follows from the definition and the property of $f^{-1}$ with respect to unions and intersections of sets. Now, the triple $(Y, \Sigma, f_*\mu)$ forms a measure space, and we see that $\int_Y \mathbf{1}_\mathcal{E} \,\mathrm{d}f_*\mu = \int_X \mathbf{1}_{f^{-1}(\mathcal{E})} \,\mathrm{d}\mu$, where $\mathbf{1}_\mathcal{E}$ is the indicating function: equal to 1 if $x \in \mathcal{E}$, 0 otherwise. By the usual simple function argument, we have that for every measurable $g\colon Y \to \mathbb{R}$ such that $g \circ f$ is integrable, the equality $\int_\mathcal{E} g \,\mathrm{d}f_*\mu = \int_{f^{-1}(\mathcal{E})} g \circ f \,\mathrm{d}\mu$ holds.

We continue this discussion by recalling some definitions and facts about random variables in terms of the natural settings of measure spaces. Firstly, a *probability space* is a measure space $(\Omega, \mathcal{F}, \boldsymbol{P})$ such that $\boldsymbol{P}(\Omega) = 1$. We call the elements of $\mathcal{F}$ *events*. A *random variable* is a measurable function $X\colon \Omega \to \mathbb{R}$. There is an important distinction to make here: the *measure* determines the probability of each events in $\mathcal{F}$, while the random variable can be thought as giving a weight to each event.

For a random variable $X$, the image measure $X_*\boldsymbol{P}$ is called the *distribution* of $X$ and is noted $\boldsymbol{P}_X$. This defines a new probability space on $\mathbb{R}$. The *distribution function* associated is

$$F_X(t) := \int_{-\infty}^{t} \mathrm{d}\boldsymbol{P}_X,$$

from which we have that $F_X(t) = \boldsymbol{P}_X\big((-\infty, t]\big) = \boldsymbol{P}(\{\omega : X(\omega) \leq t\})$. We see that the distribution function is weakly increasing and has its range in $[0, 1]$. The *probability density function*, when it exists, is the weight $f\colon \mathbb{R} \to [0, \infty)$ that allows to express $\boldsymbol{P}_X$ in terms of the more familiar *Lebesque measure* $m$: $\boldsymbol{P}_X(\mathcal{A}) = \int_\mathcal{A} f \,\mathrm{d}m$. A sufficient condition for this weight to exist is *absolute continuity* of $\boldsymbol{P}_X$ with respect to $m$, noted $\boldsymbol{P}_X \ll m$. Recall that $\boldsymbol{P}_X \ll m$ if $\boldsymbol{P}_X(\mathcal{A}) = 0$ whenever $m(\mathcal{A}) = 0$. This function $f$ is called the *Radon-Nikodym derivative* and is sometimes noted $\mathrm{d}\boldsymbol{P}_X/\,\mathrm{d}m$.

The *expectation* is defined by

$$\boldsymbol{E}(X) := \int_\Omega X(\omega)\,\mathrm{d}\boldsymbol{P}(\omega);$$

it is the average weight given to each event. The *law of large numbers* states that it is the value $X$ is expected to take *on average*. The $n$-th moment is defined as $\boldsymbol{E}(X^n)$. The moments can help understand the behaviour of certain random variables. For example, if the moments of two variables $X$ and $Y$ agree for every natural number $n$, then we can conclude that they have the same distribution function. Compare with theorem 3.2.1 of section 2. Note that the expectation is the first moment. If $f_X$ is the density function of $X$, the $n$-th moment can be computed by $\int_\mathbb{R} x^n f\,\mathrm{d}m$. This relation is a consequence of the the following theorem.

**Theorem 4.1.1.** *Let $(\Omega, \mathcal{F}, \boldsymbol{P})$ be a probability space, $(\Omega', \mathcal{F}')$ be a measurable space, and let $\Phi\colon \Omega \to \Omega'$ and $X'\colon \Omega' \to \mathbb{R}$ be measurable functions. The composition $X = X' \circ \Phi\colon \Omega \to \mathbb{R}$ is a real random variable and*

$$\int_\Omega X(\omega)\,\mathrm{d}\boldsymbol{P}(\omega) = \int_{\Omega'} X'(t)\,\mathrm{d}\boldsymbol{P}_\Phi(t) \tag{4.1}$$

*where $\boldsymbol{P}_\Phi(A') = \boldsymbol{P}(\Phi^{-1}(A'))$ for every events $A' \in \mathcal{F}'$.*

**Corollary 2.** *If $X$ is a real valued random variable and if $g\colon \mathbb{R} \to \mathbb{R}$ is a measurable function, then*

$$\boldsymbol{E}(g(X)) = \int_\mathbb{R} g(t)\,\mathrm{d}\boldsymbol{P}_X(t).$$

*Moreover, if $\boldsymbol{P}_X$ is absolutely continuous with respect to the Lebesgue measure, then*

$$\boldsymbol{E}(g(X)) = \int_{-\infty}^{\infty} g(x) f(x) \, \mathrm{d}m(x),$$

*where $f$ is the Radon-Nikodym derivative, $\mathrm{d}\boldsymbol{P}_X / \mathrm{d}m$.*

*Proof of corollary.* We take $\Omega' = \mathbb{R}$ and $X' = id$, so that $X = \Phi$ and

$$\int_\Omega X \, \mathrm{d}\boldsymbol{P} = \int_{\mathbb{R}} id \, \mathrm{d}\boldsymbol{P}_X = \int_{-\infty}^{\infty} t \, \mathrm{d}\boldsymbol{P}_X(t).$$

The corollary then follows from the property of the image measure $\int_\Omega g \circ X \, \mathrm{d}P = \int_{\mathbb{R}} g \circ id \, \mathrm{d}\boldsymbol{P}_X$. □

*Proof of theorem.* First, if $X'$ is the indicating function $\mathbf{1}_{A'}$, then $\int_{\Omega'} X' \, \mathrm{d}\boldsymbol{P}_\Phi = \boldsymbol{P}_\Phi(A')$. Also, $X = \mathbf{1}_{\Phi^{-1}(A')}$, so that $\boldsymbol{E}(X) = \boldsymbol{P}(\Phi^{-1}(A'))$ and (4.1) holds. We see with little trouble that it holds also for simple functions. Now suppose that $X'$ is positive and consider an increasing sequence of simple functions $s'_n$ converging to $X'$. We have that $s_n = s'_n \circ \Phi$ is also an increasing sequence of simple functions, so that we have

$$\begin{aligned} \int_\Omega X \, \mathrm{d}\boldsymbol{P} &= \lim_{n\to\infty} \int_\Omega s_n \, \mathrm{d}\boldsymbol{P} \\ &= \lim_{n\to\infty} \int_{\Omega'} s'_n \, \mathrm{d}\boldsymbol{P}_\Phi \quad \text{(by monotone convergence theorem)} \\ &= \int_{\Omega'} X' \, \mathrm{d}\boldsymbol{P}_\Phi \end{aligned}$$

For a general $X'$, we consider the positive and the negative part. □

In the particular case that $X$ is a real random variable with density function $f = \mathrm{d}P_X / \mathrm{d}m$ for which the $n$-th moment exists, taking the measurable function

$g(x) = x^n$, we obtain the formula for the moments

$$\boldsymbol{E}(X^n) = \int_{-\infty}^{\infty} x^n f(x)\,\mathrm{d}m(x).$$

Before proceeding to the next section, we will summarize all the technical terms we have introduced in the next definition.

**Definition 5.** Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $(Y, \Sigma)$ be a measurable space.

1. For a measurable function $f\colon \Omega \to Y$, we define the *image measure* by $f_*\mu(\mathcal{E}) = \mu(f^{-1}(\mathcal{E}))$.
2. The measure space $(\Omega, \mathcal{A}, \mu)$ is a *probability space* if $\mu(\Omega) = 1$.

We suppose from now that $\Omega$ is a probability space.

3. A *random variable* is a measurable function $X\colon \Omega \to \mathbb{R}$.
4. The *distribution* of $X$ is the image measure $\boldsymbol{P}_X := X_*\boldsymbol{P}$.
5. The *distribution function* of $X$ is defined by $F(t) := \int_{-infty}^{t} \boldsymbol{P}_X$.
6. The *probability density function*, if it exists, is the function $f\colon \mathbb{R} \to \mathbb{R}$ such that $\boldsymbol{P}_X(\mathcal{E}) = \int_E f\,\mathrm{d}m$, or equivalently, such that $f = F'$.
7. The expectation is $\boldsymbol{E}(X) := \int_\Omega X\,\mathrm{d}\boldsymbol{P}$.
8. The $n$-th moment is $\boldsymbol{E}(X^n) = \int_\Omega X^n\,\mathrm{d}\boldsymbol{P}$.

### 4.2 Random Matrices

We wish to generalize slightly random variables. Instead of having range $\mathbb{R}$, we allow them to take values in a real vector space $\mathbb{R}^n$. Note that this includes spaces of square matrices seen as $\mathbb{R}^{n^2}$. For the following subsections, we will partly follow chapter 2 of [Meh04]. The book motivates the theory by physical approaches which we will not discuss, but the reader is invited to consult the book to learn more of those physical concepts. The discussion here is more centered around the context of measure theory and probability spaces.

A random variable $X\colon \Omega \to \mathbb{R}^n$ can be written

$$X(\omega) = \begin{pmatrix} X_1(\omega) \\ X_2(\omega) \\ \vdots \\ X_n(\omega) \end{pmatrix},$$

so that it is really just a vector of $n$ real random variables. Here, the expectation will be a vector simply containing the expectation of each entry. As such, the theorem of the previous section can be generalize by applying it to each component of the vector.

We will be mostly concerned with *matrix-valued* random variables. The *eigenvalues* are what set the random matrices apart from the random vectors. If $X\colon \Omega \to \mathrm{Mat}_n(\mathbf{F})$ is a random variable, then we can write $X(\omega) = (\xi_{ij}(\omega))_{i,j=1}^n$. If we restrict the matrices enough, they will have eigenvalues $\lambda_1(\omega), \ldots, \lambda_n(\omega)$ which will themselves be random variables.

The setting of our discussion is around *Wigner matrices*. Let $\mathcal{H}$ denote the space of selfadjoint[1] $n \times n$ matrices with independent entries. Since our matrices are selfadjoint, they have $n$ real eigenvalues $\lambda_1(H) \leq \lambda_2(H) \leq \cdots \leq \lambda_n(H)$. If we let

[1] Recall that a matrix is *selfadjoint* or *hermitian* if $M = M^*$, where $M^*$ is the adjoint, and that, in this case, $M$ has $\operatorname{tr} I$ real eigenvalues, counting multiplicities.

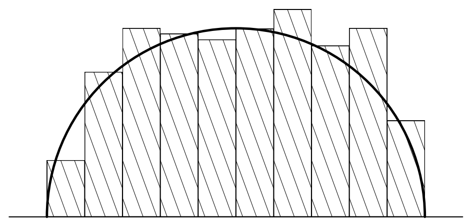


Figure 4–1: An example of histogram for $\frac{1}{n}N_I$.

$n \to \infty$, the *Wigner semi-circle law* tells us that the eigenvalues are distributed in a semi-circle, if properly normalized (see [Meh04]).

More precisely, for every interval $I \subseteq \mathbb{R}$, let $N_I$ denote the number of eigenvalues in $I$. For a large $n$, if we draw the histogram of $\frac{1}{n}N_I$, we could get something such as in Figure 4–1. The Wigner semi-circle law then states that

$$\frac{1}{n}N_I = \frac{1}{2\pi}\int_I \sqrt{4-x^2}\,\mathrm{d}x + o(1).$$

### 4.2.1 Gaussian Orthogonal Ensemble

The Gaussian Orthogonal Ensemble (GOE) is a set of Wigner matrices. Formally, consider a random variable $X\colon \Omega \to \mathcal{H}$ with $\Omega$ a probability space and $\mathcal{H}$ the space of a real symmetric $n \times n$ matrices and let $\boldsymbol{P}_X$ be the image measure of $X$. The set $\mathcal{H}$ is a real vector space of dimension $n(n+1)/2$, so it is equipped with the Lebesgue measure $\mathrm{d}m$. We ask that $\boldsymbol{P}_X$ be absolutely continuous with respect to $\mathrm{d}m$. To form a Gaussian orthogonal ensemble, $\boldsymbol{P}_X$ must be invariant under conjugation by orthogonal matrices, meaning that for every measurable set $\mathcal{A} \in \mathcal{H}$ and every orthogonal matrix $O$, we have

$$\boldsymbol{P}_X(O\mathcal{A}O^t) = \boldsymbol{P}_X(\mathcal{A}).$$

The independence condition can be translated to

$$\frac{\mathrm{d}\boldsymbol{P}_X}{\mathrm{d}m} = \prod_{1 \le i \le j \le n} f_{ij}$$

for $n(n+1)/2$ functions $f_{ij} \colon \mathbb{R} \to [0, \infty)$.

From here, we can ponder on how restricted the choice of the probability density functions are. As showed in [Meh04], the choice is quite limited and will be briefly discussed in section 4.2.4.

### 4.2.2 Gaussian Unitary Ensemble

The Gaussian Unitary Ensemble (GUE) is defined similarly to the orthogonal one: here the underlying field is $\mathbb{C}$ instead of $\mathbb{R}$, so that orthogonal matrices are now unitary matrices and the space $\mathcal{H}$ will be selfadjoint matrices with respect to the standard inner product on $\mathbb{C}^n$. Remember that a *selfadjoint matrix* is a matrix such that $H = H^*$. The definition of $H^*$ used here is the matrix such that $\langle Hv, w\rangle = \langle v, H^* w\rangle$, so that $H^*$ depend on the bilinear form and the underlying field. For example, $H^* = H^t$ in the case of real numbers and $H^* = \overline{H^t}$ for $\mathbb{C}$.

A selfadjoint matrix has $n$ real numbers on the diagonal and $n(n-1)/2$ complex numbers on the upper triangle portion of the matrix. The real dimension of $\mathcal{H}$ is $n + n(n-1) = n^2$. We can write

$$\frac{\mathrm{d}\boldsymbol{P}_X}{\mathrm{d}m} = \prod_{1 \le i \le j \le n} f_{ij},$$

where $f_{ii}$ is defined on $\mathbb{R}$ and $f_{ij}$ for $i < j$ is defined on $\mathbb{C}$. Note that the functions $f_{ij}$ defined on the complex plane should further factorize to a product: $f_{ij}(z) = p_{ij}(x)q_{ij}(y)$ for a complex number $z = x + iy$, $x, y \in \mathbb{R}$. This effectively gives us a product of $n + n(n-1)$ functions defined on $\mathbb{R}$, that is the $f_{ii}$ and the $g_{ij}h_{ij}$ for $1 \le i < j \le n$.

### 4.2.3 Gaussian Symplectic Ensemble

The Gaussian Symplectic Ensemble (GSE) is defined similarly like its orthogonal and unitary counterparts, but this time the underlying field is $\mathbf{H}(\mathbb{R})$ with with quaternion conjugation for $q = a + ib + jc + kd$ defined by $\overline{q} = a - ib - jc - kd$. The real dimension is $n + 2n(n-1) = 2n^2 - n$. It is called *symplectic* because in the case the field is the quaternion, the *symplectic matrix group* of the complex numbers contains the unitary matrices with the decomposition of $\mathrm{Mat}(2n, \boldsymbol{H}(\mathbb{R}))$ into $\mathrm{Mat}(n, \mathbb{C}) + j\,\mathrm{Mat}(n, \mathbb{C})$, as discussed in section 2.2.1. Note that the operation $-^*$ uses the quaternion conjugation in this case. The Radon-Nikodym derivative will now be a product of $f_{ii}$ for $i$ from 1 to $n$ and $p_{ij}q_{ij}r_{ij}s_{ij}$ defined on $\mathbb{R}$.

### 4.2.4 Joint Probability Density Function

The GOE, GUE and GSE are very similar in certain aspects. Here, we will describe them in a more uniform way. We use $\mathbf{F}$ to mean the field $\mathbb{R}$, $\mathbb{C}$, or $\mathbf{H}(\mathbb{R})$. Let $\mathcal{H}(\mathbf{F})$ denote the real vector space of selfadjoint matrices over the standard inner product on $\mathbf{F}$. The random variable $X\colon \Omega \to \mathcal{H}(\mathbf{F})$ with independent entries is one of the three Gaussian ensembles depending on the field if $P_X$ is absolutely continuous with respect to the Lebesgue measure on $\mathcal{H}$ and the probability density function $f$ is invariant under conjugation by unitary matrices $\mathrm{U}(n, \mathbf{F})$. The next theorem from [Meh04] displays the rigidity of these two conditions.

**Theorem 4.2.1.** *In all of the above three cases, the form of* $\mathrm{d}\boldsymbol{P}_X / \mathrm{d}m$ *is automatically restricted to*

$$\frac{\mathrm{d}\boldsymbol{P}_X}{\mathrm{d}m}(H) = \exp\left(-a \operatorname{tr} H^2 + b \operatorname{tr} H + c\right), \tag{4.2}$$

*where* $a$ *is real and positive, and* $b$ *and* $c$ *are any real.*

*Sketch of proof.* We will focus on the case of GUE, but the other two ensembles are very similar.

We have the defining relation

$$H = UH'U^*. \tag{4.3}$$

If $U$ depends on a parameter $\theta$ in a differentiable way, by differentiating (4.3) we get

$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}\theta} &= \frac{\mathrm{d}U}{\mathrm{d}\theta}H'U^* + UH'\frac{\mathrm{d}U^*}{\mathrm{d}\theta} \\ &= \frac{\mathrm{d}U}{\mathrm{d}\theta}U^*H + HU\frac{\mathrm{d}U^*}{\mathrm{d}\theta} \\ &= AH + HA^*,\end{aligned}$$

where $A = \frac{\mathrm{d}U}{\mathrm{d}\theta}U^*$. For the special case where

$$U = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & \cdots & 0 \\ \sin\theta & \cos\theta & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

we find

$$A = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Since $f := \mathrm{d}\boldsymbol{P}/\,\mathrm{d}m$ is invariant under conjugation by unitary matrices, we have $f(H) = f(H')$ and $(\mathrm{d}/\mathrm{d}\theta)f(H') = 0$. Combining all this, we have the following

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}\theta}(H(\theta)) &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\prod_{1\le i\le j\le n} f_{ij}(H(\theta))\right) \\ &= \sum_{1\le i\le j\le n}\left(\frac{\partial}{\partial h_{ij}} f_{ij}(h_{ij}(\theta))\frac{\mathrm{d}}{\mathrm{d}\theta}h_{ij}(\theta)\prod_{(k,\ell)\neq(i,j)} f_{k\ell}(h_{k\ell}(\theta))\right).\end{aligned}$$

Remember that for $i < j$, $f_{ij}$ is a product $p_{ij}q_{ij}$. If we expand the sum, we will find different sums that each depend on different variables, so we can derive equations for $3 \le k \le n$ such as (see [Meh04] for more details)

$$-\frac{h_{2k}}{p_{1k}}\frac{\mathrm{d}p_{1k}}{\mathrm{d}h_{1k}} + \frac{h_{1k}}{p_{2k}}\frac{\mathrm{d}p_{2k}}{\mathrm{d}h_{2k}} = C_k.$$

If we divide both side by $h_{1k}h_{2k}$, we obtain

$$-\frac{1}{h_{1k}p_{1k}}\frac{\mathrm{d}p_{1k}}{\mathrm{d}h_{1k}} + \frac{1}{h_{2k}p_{2k}}\frac{\mathrm{d}p_{2k}}{\mathrm{d}h_{2k}} = \frac{C_k}{h_{1k}h_{2k}}. \tag{4.4}$$

Afterward, we use the following lemma:

**Lemma 4.2.2.** *If three differentiable functions $f_1, f_2, f_3$ satisfy*

$$f_1(xy) = f_2(x) + f_3(y),$$

*then they are of the form* $a \log x + b_k$ *for* $k = 1, 2, 3$ *with* $b_1 = b_2 + b_3$.

Since in the equation (4.4) we cannot conclude that those functions are of the form $a \log(x) + b$, we conclude that the constant must be 0. Since both summand on the left-hand side of (4.4) depends on different variables, we must have

$$\frac{1}{h_{1k} p_{1k}} \frac{\mathrm{d}p_{1k}}{\mathrm{d}h_{1k}} = \frac{1}{h_{2k} p_{2k}} \frac{\mathrm{d}p_{2k}}{\mathrm{d}h_{2k}} = -2a \qquad \text{(some constant)}$$

and integrating yields

$$p_{1k}(h_{1k}) = \exp(-a {h_{1k}}^2 + c).$$

We find a similar equation for $k = 2$, while for $k = 1$ we get

$$p_{11}(h_{11}) = \exp(-a {h_{11}}^2 + h_{11} b + c).$$

Finally, this last lemma allows us to conclude.

**Lemma 4.2.3.** *All the invariants of an* $n \times n$ *matrix* $H$ *under conjugation*

$$H \mapsto H' = AHA^{-1}$$

*can be expressed in terms of the traces of the first* $N$ *powers of* $H$. □

There are a few comments that should be mentioned. First, let us look more closely to the trace of $H^2$. Since the trace is invariant by the conjugation $X \mapsto AXA^{-1}$, and $H$ is diagonalizable, it is simply the sum of the squared eigenvalues of $H$. On the other hand, if we compute the elements on the diagonal of the product, we find

$$h_{ii} = \sum_{k=1}^{n} h_{ik} h_{ki}.$$

Using the fact that $H$ is selfadjoint, we have that $h_{ik}h_{ki} = h_{ik}\overline{h}_{ik} = |h_{ik}|^2$, so that the trace become

$$\operatorname{tr} H^2 = \sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} |h_{ij}|^2 = \sum_{i=1}^{n} h_{ii}^2 + 2 \sum_{1\le i<j\le n} |h_{ij}|^2,$$

where the last equality uses again the fact that $H = \overline{H^t}$. Let us write the constant $c$ of (4.2) as a sum of $c_{ij}$, so that this equation becomes

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{P}_X}{\mathrm{d}m}(H) &= \exp\left(-a\sum_{i=1}^{n} {h_{ii}}^2 - 2a\sum_{1\le i<j\le n} |h_{ij}|^2 + b\sum_{i=1}^{n} h_{ii} + \sum_{1\le i\le j\le n} c_{ij}\right)\\ &= \prod_{i=1}^{n} e^{-a{h_{ii}}^2 + bh_{ii} + c_{ii}} \prod_{1\le i<j\le n} e^{-2a|h_{ij}|^2 + c_{ij}}.\end{aligned}$$

Recall that a Gaussian distribution has a probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2},$$

and the expectation is $\mu$ and the variance, $\sigma^2$. We can deduce that the probability density function $f$ of $\boldsymbol{P}_X$ is a product of normal distributions.

### 4.3 Level Spacings

We mentioned earlier that the eigenvalues of the selfadjoint $N \times N$ matrix $H$ are real and they are random variables $\lambda_i \colon \mathcal{H} \to \mathbb{R}$. We are now concerned with their joint probability density function and what we refer to as "level spacing", that is the value $|\lambda_{i+1} - \lambda_i|$ or some other statistics. In the chapter 3 of [Meh04], we have the following theorem for the joint probability density function.

**Theorem 4.3.1** ( [Meh04]chap. 3.1)**.** *The joint probability density function for the eigenvalues of the matrices from a Gaussian orthogonal, Gaussian unitary or Gaussian symplectic ensemble with a probability density function given by* (4.2) *is given by*

$$\boldsymbol{P}_{N_\beta}(x_1, x_2, \dots, x_N) = C_{N_\beta} \exp\left(-\frac{1}{2}\beta \sum_{j=1}^{N} x_j^2\right) \prod_{j<k\leq N} |x_j - x_k|^\beta,$$

*where* $\beta = 1$ *if the ensemble is orthogonal,* $\beta = 2$ *if unitary, and* $\beta = 4$ *if symplectic. The density has been centered with* $\lambda_j = (1/\sqrt{2a}\,)x_j + b/2a$*.*

Here is an outline of the procedure to prove this. Equation (4.2) can be written in terms of the eigenvalues, since $\operatorname{tr} H^k = \sum_i \lambda_i^k$, but a selfadjoint matrix is described with more parameters than this, so we introduce $\ell := \dim \mathcal{H} - N$ parameters $p_i$. The joint probability density function of (4.2) can written

$$\exp\left(-a \operatorname{tr} H^2 + b \operatorname{tr} H + c\right) \,\mathrm{d}m(H) = \exp\left(-a\sum_{i=1}^{N} \lambda_i^2 + b\sum_{i=1}^{N} \lambda_i + c\right) |J(\theta, p)| \,\mathrm{d}m(\theta, p),$$

where $J(\theta, p)$ is the Jacobian matrix

$$|J(\theta, p)| = \left|\frac{\partial(h_{11}, h_{12}, \dots, h_{NN})}{\partial(\theta_1, \dots, \theta_N, p_1, \dots, p_\ell)}\right|.$$

Afterward, the joint probability density function is the marginal distribution, that is, we integrate with respect to the extra parameters.

The density of the proportion of eigenvalues lying in set can be derived from the previous theorem; it is what we call *the Wigner semi-circle law.* As mentioned earlier, if the ensemble is on matrices of size $N$, then the proportion $C(I)/N$, where $C(I)$ is counting the number of eigenvalues in the set $I$, converges to the area under the part supported by $I$ of a semi-circle.

**Theorem 4.3.2.** *Let $\mathcal{A}$ be a measurable set of $\mathbb{R}$. For a Gaussian Orthogonal, Gaussian Unitary, and Gaussian Symplectic Ensemble of $N \times N$ matrices, let $\{\lambda_i\}_{i=1}^N$ denote the eigenvalues. We define $C(\mathcal{A}) := \#\{i; \lambda_i \in \mathcal{A}\}$, and we have*

$$\frac{C(\mathcal{A})}{N} = \frac{1}{2\pi} \int_{\mathcal{A}} \rho_{sc}(x) \, \mathrm{d}x + o(1),$$

*where the density $\rho_{sc}(x)\,\mathrm{d}x$ is given by*

$$\rho_{sc}(x) = \begin{cases} \sqrt{4N - x^n} & \text{if } |x|^2 < 4N; \\ 0 & \text{if } |x|^2 \geq 4N. \end{cases}$$

### 4.4 Random Ensembles of Elements of $\mathbb{R}[\mathrm{SU}(2)]$

Let $z \in \mathbb{R}[\mathrm{SU}(2)]$ be a group ring element of the form $z = g_1 + g_1^{-1} + \cdots + g_k + g_k^{-1}$. In the previous chapter, we look at the behaviour of the spectral measure of[2] $\widehat{z}(\pi_N)$ as $N \to \infty$. Interestingly, when we equip SU(2) with its normalized Haar measure, we obtain a matrix-valued random variable. This statement is made more precise in the next proposition. The random matrix is obtained by a sum of independent and identically distributed random variables on the generators. When we let the number of generators grow, appealing to the multivariate central limit theorem, we obtain a normal distribution.

**Proposition 3.** *Let $G = \mathrm{SU}(2)$ with the Haar measure $\mathrm{d}g$ normalized so that $\int_G \mathrm{d}g = 1$. Let $\mathcal{H}$ denote the real linear space of selfadjoint $(N+1) \times (N+1)$*

[2] Remember that $\widehat{z}(\pi_N)$ is the matrix of $\mathrm{GL}(N+1, \mathbb{C})$ defined by $\pi_N(g_1) + \pi_N(g_1^{-1}) + \cdots + \pi_N(g_k) + \pi_N(g_k^{-1})$.

*matrices (cf. section 2.3.4). Consider the product measure* $\mathrm{d}\mu = \mathrm{d}g_1 \cdots \mathrm{d}g_k$ *on* $G^{(k)}$ *and the measurable map*

$$\varphi\colon G^{(k)} \to \mathcal{H}$$
$$(g_1, \ldots, g_k) \mapsto \left(\frac{1}{\sqrt{k}}(g_1 + g_1^{-1} + \cdots + g_k + g_k^{-1})\right)^{\wedge}(\pi_N)$$

*to define* $\nu_{N,k}$ *as* $\nu_{N,k}(\mathcal{A}) = \mu(\varphi^{-1}(\mathcal{A}))$*. Then,* $\nu_{N,k}$ *is a probability measure on* $\mathcal{H}$ *and if* $k \to \infty$*,* $\nu_{N,k}$ *converges to a GOE measure if* $N$ *is even and to a GSE if* $N$ *is odd.*

*Proof.* We do the proof for $N$ even. For the case $N$ odd, the reader can look at [GJS99], proposition 1. In the odd case, the proof is very similar, but the matrix has a different form, which generates more computations.

For the case $N$ even, the random variable $\varphi$ is a sum of independent and identically distributed random variables, because we can see it as adding $k$ times the measurable function $\frac{1}{\sqrt{k}}\left(\pi_N(g) + \pi_N(g^{-1})\right)$. The strategy is to use the central limit theorem for vector-valued random variables. Recall that for a sequence $X_1, X_2, \ldots, X_n$ of vector-valued independent and identical random variables, $\sqrt{n}(\overline{X} - \mu) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Sigma)$ in distribution, where $\Sigma$ is the *covariance matrix*, that is $\Sigma = \boldsymbol{E}(X_1 X_1^t)$.

We have to show that the expectation of $\varphi$ is zero and compute the covariance matrix of $H(g) := \pi_N(g) + \pi_N(g)^{-1}$ seen as a vector. We get the expectation by

$$\int_G H(g)\,\mathrm{d}g = \left(\int_G h_{ij}(g)\,\mathrm{d}g\right)_{ij}$$

and for the covariance matrix, it will be necessary to compute

$$\int_G h_{ij}(g) h_{rs}(g) \, \mathrm{d}g. \tag{4.5}$$

Since $\pi_N$ is an irreducible representation, we can use it to deduce the following.

The map $1\colon G \to \mathbb{C}; g \mapsto 1$ is a class function, so the inner product $\langle \pi_N, 1 \rangle$ defines an intertwining operator[3] and, by Schur's lemma, it is an homothety.[4] The ratio $\lambda$ of this homothety is given by $\langle \chi_{\pi_N}, 1 \rangle$, which is zero since the representation 1 and $\pi$ are orthogonal, that is $\langle \chi_{\pi_N}, \chi_1 \rangle = 0$ and $(\dim_{\mathbb{C}} V) \cdot \langle \chi_{\pi_N}, 1 \rangle = \langle \chi_{\pi_N}, \chi_1 \rangle$. This proves that

$$\int_G H(g) \, \mathrm{d}g = \frac{1}{\sqrt{k}} \int_G \left[ \pi_N(g) + \pi_N(g)^{-1} \right] \mathrm{d}g = 0.$$

Here is another consequence of Schur's lemma ( from [SS12] section 2.2, corollary 3.)

$$\int_G \pi_N(i,j)(g) \pi_N(m,n)(g^{-1}) \, \mathrm{d}g = \frac{\delta_{in}\delta_{jm}}{N+1}.$$

It is derived as follow. Consider an endomorphism $h$ of $W_{N+1}$ with matrix element $\left(a_{ij}\right)_{ij}$ and define

$$h_0 = \int_G \pi(g^{-1}) h \pi(g) \, \mathrm{d}g, \tag{4.6}$$

[3] An intertwining operator is a linear map $\theta\colon V \to V$ such that $\pi \circ \theta = \theta \circ \pi'$.

[4] A matrix of the form $\lambda I$.

where from now on, $\pi$ represent $\pi_N$. This is also an endomorphism and it satisfies $\pi(g)h_0 = h_0\pi(g)$ since

$$\begin{aligned}
\pi(g^{-1})h_0\pi(g) &= \int_G \pi(g^{-1})\pi(t^{-1})h\pi(t)\pi(g)\,\mathrm{d}t \\
&= \int_G \pi(tg)^{-1}h\pi(tg)\,\mathrm{d}t \\
&= h_0.
\end{aligned}$$

By Shur's Lemma, $h_0$ is a homothety of ratio $\frac{1}{N+1}\operatorname{tr} h$. The coefficient at $(i,j)$ in the matrix of $h_0$ is of the form $\frac{1}{N+1}\delta_{ij}\operatorname{tr} h = \frac{1}{N+1}\delta_{ij}\sum_{k,\ell}\delta_{k\ell}a_{k\ell}$. If we expand each element of the matrices in equation (4.6), we find

$$\frac{1}{N+1}\sum_{k,\ell}\delta_{ij}\delta_{k\ell}a_{k\ell} = \sum_{k,\ell}\int_G \pi(i,k)(g^{-1})a_{k,\ell}\pi(\ell,j)(g)\,\mathrm{d}g$$

and this is true for every choice of $h$, so equating the coefficient $a_{k,\ell}$ gives

$$\frac{\delta_{ij}\delta_{k\ell}}{N+1} = \int_G \pi(i,k)(g^{-1})\pi(\ell,j)(g)\,\mathrm{d}g.$$

Returning to the computation of equation (4.5), since $\pi_N$ is real orthogonal, the previous equation is equivalent to

$$\int_G \pi(k,i)(g)\pi(\ell,j)(g)\,\mathrm{d}g = \frac{\delta_{ij}\delta_{k\ell}}{N+1}.$$

Finally, we find

$$\begin{aligned}
\int_G h_{ij}(g)h_{rs}(g)\,\mathrm{d}g &= \int_G \pi(i,j)(g)\pi(r,s)(g) + \pi(i,j)(g)\pi(s,r)(g) \\
&\qquad + \pi(j,i)(g)\pi(r,s)(g) + \pi(j,i)(g)\pi(s,r)(g)\,\mathrm{d}g \\
&= \frac{2}{(N+1)}(\delta_{is}\delta_{jr} + \delta_{ir}\delta_{js}).
\end{aligned}$$

From this, we see that

$$\int_G h_{ij} h_{mn} \, \mathrm{d}g = \begin{cases} \frac{2}{N+1} & \text{if either } (m,n) = (i,j) \text{ or } (m,n) = (j,i); \\ \frac{4}{N+1} & \text{if } i = j = m = n; \\ 0 & \text{otherwise.} \end{cases}$$

The next step is to compute the covariance matrix. Since $\mathcal{H}$ is $(N+1)(N+2)/2$ dimensional real vector space, we define the $(N+1)(N+2)/2$ vector $X^t = (h_{11}, h_{12}, \ldots, h_{1N+1}, h_{22}, \ldots, h_{2N+1}, h_{33}, \ldots, h_{N+1N+1})$. Note that the element $h_{ij}$ has position $n = (i-1)(N+1-i/2)+j$. We are interested in computing $\boldsymbol{E}(XX^t)$, and we have, for $1 \leq a, b \leq (N+1)(N+2)/2$,

$$\int x_a x_b \, \mathrm{d}g \neq 0 \iff a = b,$$

and in that case, for $a = b = (i-1)(N+1-i/2)+j$, if $i = j$, the integral is $4/(N+1)$ and otherwise it is $2/(N+1)$. The matrix $\Sigma = \boldsymbol{E}(XX^t)$ is therefore a diagonal matrix and its inverse is simply the same matrix whose non-zero entries are the reciprocal.

Finally, recall that the density distribution of a Gaussian $\mathcal{N}(0, \Sigma)$ is given by

$$f(x_1, \ldots, x_n) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \exp\left(-\frac{1}{2} \boldsymbol{x}^t \Sigma^{-1} \boldsymbol{x}\right).$$

Here, $n = (N+1)(N+2)/2$ and $\boldsymbol{x}$ will be a vector of the form

$$\boldsymbol{x}^t = (x_{11}, x_{12}, \ldots, x_{1N+1}, x_{22}, \ldots, x_{N+1N+1}).$$

This also describe a matrix $X$ of $\mathcal{H}$ defined by $X = (x_{ij})_{ij}$ where $x_{ji} := x_{ij}$ for $1 \leq i < j \leq N+1$. Now, we get for the product

$$\begin{aligned}
\boldsymbol{x}^t \Sigma^{-1} \boldsymbol{x} &= (N+1) \sum_{i=1}^{N+1} \frac{x_{ii}^2}{4} + (N+1) \sum_{1 \leq i < j \leq N+1} \frac{x_{ij}^2}{2} \\
&= (N+1) \sum_{i=1}^{N+1} \frac{x_{ii}^2}{4} + (N+1) \sum_{\substack{i,j=1 \\ i \neq j}}^{N+1} \frac{x_{ij}^2}{4} \\
&= \frac{(N+1)}{4} \sum_{i,j=1}^{N+1} x_{ij}^2 \\
&= \frac{(N+1)}{4} \operatorname{tr} X^2.
\end{aligned}$$

We conclude that $\nu_{N,k}$ converge in distribution to

$$C e^{\operatorname{tr} H^2} \, \mathrm{d}H,$$

so that it forms a Gaussian orthogonal ensemble over $\mathcal{H}$. □

# CHAPTER 5
# Conclusion

In this thesis, we have discussed some results on the spectrum of elements of $\mathbb{R}[\mathrm{SU}(2)]$. On the one hand, we looked at the measure counting the eigenvalues, while on the other hand, we looked at densities of the eigenvalues when the number of generators approaches infinity.

There are still many questions we could ask, and the theory is still a work in progress. For instance, it was recently proved by Bourgain and Gamburd that a condition we did not introduce here, the diophantine condition, is sufficient to get the spectral gap (see [BG08]). This condition, introduced in [GJS99], says that $g_1, \dots, g_k \in \mathrm{SU}(2)$ are diophantine if there is a constant $B$ such that for every word $R_m \neq \pm e$ of length $m$ with letters the $g_i$'s, we have $\|R_m \pm e\| \geq B^{-m}$, where

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\| = |a|^2 + |b|^2 + |c|^2 + |d|^2.$$

It can be shown that if $g_1, \dots, g_k \in \mathrm{Mat}(2, \overline{\mathbb{Q}})$, that is they have algebraic number entries, then they are diophantine. We do not know whether non-algebraic elements can have a spectral gap. The spectral gap is also being researched for $\mathrm{SU}(d)$. See for instance [BG11], [BG10].

One of the questions we could try to answer in the future is if we can get the convergence of $\pi_N(z)$ to GOE or GSE when we let both $N$ and $k$ increase simultaneously, or even better, if $k$ is fixed. Looking at the proof of proposition 3, the central limit theorem plays a central role, so that we must look at the technicalities of its proof. Another approach that could yield perhaps a more satisfying answer would be to look at higher moments of $\pi_N$.

Finally, a subject that we did not investigate is the eigenvectors. Can we describe their behaviour? Given a $z \in \mathbb{R}[\mathrm{SU}(2)]$, how are they distributed when $N \to \infty$ after a proper normalization? If we let, again, $N$ and $k$ grow simultaneously, we could hope to extract some information when comparing with the work of Tao and Vu on random ensembles (see [TV11]).

The spectrum of the elements of $\mathbb{R}[\mathrm{SU}(2)]$ has numerous applications and links to differents fields. Perhaps one day, its many mysteries will be uncovered.

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