A Two-Dimensional Extension of Lambek's Categorical Proof Theory

Joël Ouaknine

Department of Mathematics and Statistics
McGill University, Montréal

July 1997

A thesis submitted to the faculty of Graduate Studies and Research
in partial fulfilment of the requirements of the degree of
Master of Science

0-612-44239-X

A Two-Dimensional Extension of Lambek's Categorical Proof Theory

Joël Ouaknine

Department of Mathematics and Statistics
McGill University, Montréal

July 1997

Abstract

The notion of cartesian closed bicategory is presented, and use of the resulting entities is made to reinterpret, in a uniform and consistent way, the standard work of Lambek and Scott in categorical proof theory [LS]. (Cartesian closed) bicategories are a two-dimensional analogue, or extension, of (cartesian closed) categories. We study them in quite a bit of detail, carefully showing for instance how all the relevant properties of cartesian closed categories can in fact be naturally lifted to their two-dimensional counterparts. They (cartesian closed bicategories) are also shown to allow for a purely algebraic (inference rules based) definition, being models of a certain generalised algebraic theory, in the same way that cartesian closed categories can themselves be entirely specified via Lambek's equational calculus [LS]. After a review of the pertinent pieces of work in categorical logic (mostly based on [LS] and [HM]), we set out to reinterpret all of it within the new framework in a uniform and consistent manner (in the sense that there will be natural injections and projections between the corresponding one- and two-dimensional entities preserving all the relevant features and properties).

Résumé

On introduit la notion de bicatégorie cartésienne fermée, et l'on se sert des entités qui en résultent pour réinterpréter, d'une façon uniforme et consistante, les travaux bien connus de Lambek et Scott en théorie catégorique de la preuve [LS]. Les bicatégories (cartésiennes fermées) sont une sorte d'extension bidimensionnelle des catégories (cartésiennes fermées). On les étudie en détail, en prenant soin par exemple de montrer

comment transposer au niveau des bicatégories, de façon naturelle, toutes les propriétés des catégories cartésiennes fermées qui nous intéressent. On démontre également qu'il est possible de donner une définition purement algébrique (à base de règles d'inférence) de la notion de bicatégorie cartésienne fermée, celle-ci pouvant en fait s'exprimer comme une théorie algébrique généralisée; il en résulte ainsi un parallèle clair avec la manière dont les catégories cartésiennes fermées peuvent elles-mêmes être entièrement définies par l'entremise du calcul équationnel de Lambek [LS]. Après avoir passé en revue les travaux en logique catégorique dont nous voulons traiter (ceux-ci provenant essentiellement de l'un ou l'autre ouvrage [LS] et [HM]), on entreprend d'adapter et de réinterpréter ceux-ci dans le cadre de nos nouvelles structures bidimensionnelles, et ce de façon uniforme et consistante (c'est-à-dire que l'on observe l'existence d'injections et de projections naturelles, entre les diverses entités à une et deux dimensions, qui préservent toutes les propriétés pertinentes).

Acknowledgments

First and foremost, I would like to heartily thank my supervisor, Professor Michael Makkai, for his invaluable help in the preparation of this thesis, as well as for his very thorough and meticulous proof-reading; his many insightful comments have greatly improved, I believe, the final version. Professor Makkai also deserves my gratitude for his constant availability, his patience and his unending willingness to discuss Mathematics. I would also like to thank him for his generous financial support.

I have benefited from many other people during my stay at McGill. I am particularly indebted to Prakash Panangaden, James Loveys and Alexis Maciel for the very stimulating courses they taught, as well as for the countless discussions on the widest range of topics I have had with them. Thanks also to the many other students and staff which have made so pleasant both my academic and personal life throughout my studies.

The constant love, support and encouragement provided by my parents, sisters and brother is something I can never acknowledge enough. I would also like to thank my dear friends James and Duane, as well as Csilla, without whom this thesis would no doubt have been completed much earlier.

Ce qui limite le vrai,
ce n'est pas le faux,
c'est l'insignifiant.

René Thom

Contents

Introduction

Logic as a formal discipline and object of study probably first emerged some 2300 years ago with the school of the Greek philosopher Aristotle, although one usually considers that our modern view was shaped by the famous treatise of Bertrand Russell and Alfred North Whitehead's, *Principia Mathematica* [RW]. A comprehensive and current reference on the subject is [S2].

Category theory, on the other hand, emerged much more recently in the early 1940s, through the papers of Samuel Eilenberg and Saunders Mac Lane of that period (see, for instance, [EM]). It started out as a convenient "unifying" language, but rapidly grew and developed into a large discipline in its own right. The classic text written by Mac Lane himself, *Categories for the Working Mathematician* [CWM], is still considered to be the standard reference on the subject.

Using algebraic structures (boolean algebras, Heyting algebras, etc.), Alfred Tarski and others developed algebraic logic, giving algebraic forms to many results in propositional logic. It was F. William Lawvere who, starting in the early 1960's, initiated the categorical formulation of the basic concepts (foremost amongst them the quantifiers) of logic. A central paper detailing his insight is [L4]. The fact that certain structured categories could be used to model, or give a semantics to, the formal deductions associated with particular theories in propositional logic, was first observed a few years later by Joachim Lambek in the series of papers [L1], [L2] and [L3]. The coupling of category theory and logic, now known as categorical logic, is beautifully expounded in the classic book *Introduction to Higher-Order Categorical Logic* [LS] by Lambek and Scott. Since then, categorical logic has kept on growing, and has given rise to a large body of categorical structures, with connections to many branches of mathematics and theoretical computer science.

As we mentioned above, category theory has also kept on evolving as a discipline in its own right. Bicategories, first introduced by Jean Bénabou [B2] in 1967, are among the several offshoots it has given birth too, as they are a kind of two-dimensional extension of categories. In analogy with the stem work in categorical logic, we carefully develop and investigate in this thesis the notion of cartesian closed bicategory[1], and show how a large

[1]Our notion of cartesian closed bicategory differs only very marginally from that spelled out (for the first time?) by Makkai in [M3]. It is unfortunate that the nearly identical terminology of "cartesian bicategory"

part of the work presented in [LS], mostly that dealing with the categorical proof theory (semantic modelling of formal deductions) of certain propositional logics, can be uniformly and consistently reinterpreted in the new framework. As one would expect, the extra level of structure is also shown to encapsulate more refined information about the objects under consideration, namely formal deductions.

We assume throughout a certain familiarity with basic category theory and standard everyday mathematical concepts. In the first chapter, we go through the construction and definition of the particular structured categories we are interested in, namely those which are cartesian closed; it is intended that the treatment presented there serve to motivate what is to follow. In the second chapter, we start by laying out the basic construct of bicategory, and then proceed to carry over the concepts of chapter 1 to the new entities. This chapter also includes a number of basic results on bicategories and cartesian closed bicategories. The third chapter makes the point of showing that cartesian closed bicategories can be construed as models of a certain generalised algebraic theory, in the sense of Cartmell [Ca]; this represents an extension of Lambek's equational calculus for cartesian closed categories [LS] to cartesian closed bicategories. The explicit syntactic rules listed in this chapter will prove very convenient in chapter 5. The fourth chapter retraces some of the early groundwork in categorical logic, a large part of which is taken from [LS] and [HM]. We have added a certain number of items to the framework, which otherwise tends to focus on proof theory. The fifth chapter is intended to be to the previous one what chapter 2 is to chapter 1 – namely, it is shown there how it is possible to reinterpret in a uniform and consistent way all of the constructions and results of chapter 4, using (cartesian closed) bicategories instead of categories. This is essentially all original work. And finally, the last chapter reflects back upon the whole enterprise, examines some of the alternative definitions we might have chosen along the way, and proposes a number of avenues for future work. We also briefly discuss some of the links and applications of cartesian closed (bi)categories to λ-calculus.

---

had already been claimed almost ten years before by Carboni and Walters in [CW] to denote a very different concept.

Chapter 1

Basic One-Dimensional Concepts and Examples

In this chapter we aim to give an introductory presentation of the notion of cartesian closed category. We will assume that the reader is acquainted with the very basic ideas of partially ordered set (poset), category, functor, natural transformation and the like (for references, see e.g. [CWM]).

We start by giving some definitions. In what follows, we let **C** be a fixed category, and A, B, C, etc., be objects of **C**.

A *(binary) product* of A and B is any object P, together with arrows (known as *projections*) $A \xleftarrow{p} P \xrightarrow{p'} B$ such that, for any other object C and arrows $A \xleftarrow{f} C \xrightarrow{g} B$, there is a unique arrow $h: C \to P$ such that the diagram

$$\begin{tikzcd} & P \arrow[dl, "p"'] \arrow[dr, "p'"] & \\ A & & B \\ & C \arrow[ul, "f"] \arrow[uu, "h"] \arrow[ur, "g"'] & \end{tikzcd}$$

commutes. To avoid confusion, we will at times need to carefully distinguish between the *product* (P) and the *product diagram* ($A \xleftarrow{p} P \xrightarrow{p'} B$). Actually, a given product may have several product diagrams, but when we say "product", although we only mean to designate an object, in general we also implicitly have a particular diagram in mind.

The objects A and B need not have a product, of course. Or they may have more than one. (This is a general phenomenon that occurs whenever something is defined via universal properties, as done above.) However, if they do have a product, then it is the case that any other product (of A and B) will be isomorphic to it in a strong sense (i.e. there will be a unique isomorphism between the two preserving the respective diagrams). That is why many category theorists loosely refer to "the" product (of A and B) instead of "a" product, etc. The underlying philosophy is that, *for all practical* (i.e. category-theoretic) *purposes*, any product is just as good as any other one (they are indistinguishable).

Nevertheless, lacking the uniqueness may still be a problem in some cases, for example when one attempts to give an equational presentation of a category (as we shall do later on). A typical solution is to "hand-pick", or *specify*, a particular product (and diagram) for every pair of objects (something which may require the Axiom of Choice). (There are still some difficulties associated with this setup, however, it being un-aesthetic not the worse of them. In some sense, what we have here is some sort of "fold in the rug", and attempts at ironing it out just cause it to go somewhere else...) But at any rate, this is the method we shall adopt in the present work, not just for products but for all other constructions given by universal properties as well. The reader may rest assured that, in each case, there is always a unique isomorphism between two distinct instances of the concept that preserves the distinguished arrows (i.e. the diagrams).

It is an important fact that the notion of binary product (of A and B) can be formulated equationally by saying that we have a diagram $A \xleftarrow{\pi} A \times B \xrightarrow{\pi'} B$ and, for each object C, a function $\langle\_,\_\rangle^C : \mathrm{Hom}(C,A) \times \mathrm{Hom}(C,B) \to \mathrm{Hom}(C, A \times B)$ satisfying, for every triple $(f: C \to A, g: C \to B, h: C \to A \times B)$, the following equations:

$$\pi\langle f,g\rangle = f, \qquad \pi'\langle f,g\rangle = g, \qquad \langle \pi h, \pi' h\rangle = h.$$

This perspective will take a central place in what lies ahead.

C is said to *have binary products* if every pair of objects has a product. When it exists, we will denote the (specified) product diagram of A and B by $A \xleftarrow{\pi_{A,B}} A \times B \xrightarrow{\pi'_{A,B}} B$. Moreover, as above, the unique arrow h is denoted $\langle f,g\rangle^C_{A,B}$. We may omit the various indices when there is no risk of confusion.

In **Set**, the category of sets and functions, the binary product is the usual cartesian product, with coordinate projections. Of course, any other set of the same cardinality, together with appropriate projections, is also a product (of the same two objects), but the cartesian product (with ordered pairs encoded in some fixed way), is *the* binary product we specify for **Set**.

In a poset (viewed as a category by saying that there is an arrow from a to b iff $a \le b$), the binary product of two elements (if it exists) is just their meet (or greatest lower bound). For that reason, it is usually denoted $a \wedge b$ in this context. We remark that here, there is always at most one product for any given pair of objects (thanks to the antisymmetry law).

A useful piece of notation is the following one: assume that $A \times B$ and $C \times D$ exist and suppose that there are arrows $f: A \to C$ and $g: B \to D$. Then of course they induce a corresponding arrow $f \times g: A \times B \to C \times D$, namely $f \times g = \langle f\pi, g\pi' \rangle$.

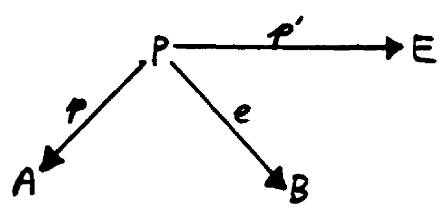
We say that the object T is *terminal* if there is exactly one arrow $A \to T$ for any object A. The (specified) terminal object is denoted by $\mathbf{t}$, and the unique arrow, by $!_A$ (or often just !). In **Set**, the terminal object is a (particular) singleton. In a poset that would be the maximum element (assuming it exists).

Here, the equational formulation reads: for every $f: C \to \mathbf{t}$, $f = !_A$.

We say that **C** is *cartesian* if it has binary products as well as a terminal object. **Set** is thus a cartesian category. A cartesian poset is known as a *meet semi-lattice*.

The definition of binary product generalizes to any number of objects in a natural way. We will leave the precise formulation to the reader. It will easily be seen that a 0-ary product is "essentially the same" as a terminal object, that a unary product (of one object) is "essentially the same" as the object in question, and that ternary, quaternary, etc., products, are "essentially the same" as repeated binary products. By "essentially the same", we mean that there are unique isomorphisms preserving the respective diagrams. In view of this fact, we will sometimes use products of a finite number of objects without explicitly laying out just how they are to be constructed, whether it be via universal properties, repeated binary products in a certain order, or otherwise, since it doesn't really make any difference in the end.

Given objects A and B, an exponential of A and B (in that order), is an object E together with arrows



with $A \xleftarrow{p} P \xrightarrow{p'} E$ an (unspecified) product diagram, and such that for any other diagram

$$\begin{tikzcd} & F \arrow[dl, "q"'] \arrow[dr, "h"] \arrow[rr, "q'"] & & C \\ A & & B & \end{tikzcd}$$

with $A \xleftarrow{q} F \xrightarrow{q'} C$ an (unspecified) product diagram, there is a unique pair $(C \xrightarrow{k} E,\ F \xrightarrow{l} P)$ such that the diagram

$$\begin{tikzcd} & P \arrow[dl, "p"'] \arrow[dr, "e"] \arrow[rr, "p'"] & & E \\ A & & B & \\ & F \arrow[ul, "q"] \arrow[ur, "h"'] \arrow[uu, "l"] \arrow[rr, "q'"'] & & C \arrow[uu, "k"'] \end{tikzcd}$$

commutes. After specifying the various objects and arrows involved, the exponential diagram looks like this:

$$\begin{tikzcd} & A \times B^A \arrow[dl, "\pi"'] \arrow[dr, "\varepsilon_{A,B}"] \arrow[rr, "\pi'"] & & B^A \\ A & & B & \end{tikzcd}$$

In addition, we write $h^{\sim}$ for the unique k given by h. And of course, we agree that $\varepsilon_{A,B}$ (known as the *evaluation arrow*) may simply be written ε when the context is clear.

Here too we can formulate the concept equationally: given the diagram just above, we require, for any object C, that the function $(\_)^{\sim}:\mathrm{Hom}(A \times C, B) \to \mathrm{Hom}(C, B^A)$ satisfy, for any $h: A \times C \to B$ and any $k: C \to B^A$, the two equations

$$\varepsilon\langle \pi_{A,C}, h^{\sim}\pi'_{A,C}\rangle^{A\times C}_{A,B^A} = h \qquad \text{and} \qquad \left(\varepsilon\langle \pi_{A,C}, k\pi'_{A,C}\rangle^{A\times C}_{A,B^A}\right)^{\sim} = k.$$

In **Set**, the exponential of A and B is the set of all functions from A to B, and ε takes the pair $(a \in A, f: A \to B)$ to $f(a) \in B$. In a poset, b^a, usually denoted $a \to b$ in this context, is known as the relative pseudo-complement of a with respect to b. It need not exist but, like the binary meet and the maximum element, if it does, it is unique. It is easy to verify that $a \to b$ is in fact the largest c such that $a \wedge c \leq b$.

If the category **C** has binary products, we say that it *has exponentials* if every pair of objects has an exponential. If in addition **C** has a terminal object, then **C** is said to be *cartesian closed.* **Set** is hence a cartesian closed category, and a cartesian closed poset is known as an *implicational meet semi-lattice*.

A most important example (from both a practical and theoretical point of view) is that of the category **Cat** of all[1] categories. It has categories as objects, and functors as morphisms between these. It is well known that **Cat** is cartesian closed (see, for instance, [CWM]); the binary product of the categories **A** and **B** is the category $\mathbf{A} \times \mathbf{B}$ whose set of objects is the set $\mathrm{Ob}(\mathbf{A}) \times \mathrm{Ob}(\mathbf{B})$, with morphisms between (A, B) and (C, D) all pairs $(f{:}A \to C, g{:}B \to D)$. The projection functors are defined coordinate-wise in the obvious way. The terminal category is a one-object, one-arrow category, and is denoted **1**. Finally, the exponential $\mathbf{B}^{\mathbf{A}}$ of **A** and **B** has as objects all functors from **A** to **B**, and as morphisms all natural transformations between them. The evaluation arrow is again defined in a manner analogous to what it is in **Set**.

Having introduced structures on categories, we now address the question of determining what are the "natural" maps between these. Clearly, they should be structure-preserving functors. There are at this point two main possibilities, arising of course because of the duality between specified and unspecified operations. We present both below in the case of cartesian closed categories, from which one can infer the appropriate treatment for the other cases.

Let **C** and **D** be cartesian closed categories (with or without specified operations). A functor $F{:}\mathbf{C} \to \mathbf{D}$ is said to be *cartesian closed* if F preserves the (unspecified) diagrams for the operations of product, terminal object and exponentiation. For example, we require that, whenever $A \xleftarrow{\pi} A \times B \xrightarrow{\pi'} B$ is a product diagram in **C**, $F(A) \xleftarrow{F(\pi)} F(A \times B) \xrightarrow{F(\pi')} F(B)$ be a product diagram in **D**, etc....

[1] Strictly speaking, it doesn't make sense to talk about the categories of "all" category, as it immediately exposes us to a Russell's paradox-type contradiction; one therefore has to be more circumspect, considering for example only "small" categories, i.e. categories whose underlying collections of objects and morphisms are sets (as opposed to proper classes), etc. We will not be overly concerned with such subtleties in this work. (Mac Lane [CWM] concedes this is yet another "fold in the rug", although terming the matter "esoteric", as some authors have, seems a bit excessive...)

For the second definition, let $\mathbf{C}$ and $\mathbf{D}$ be cartesian closed categories with specified operations. A functor $F:\mathbf{C} \to \mathbf{D}$ is said to be *strict cartesian closed* if F preserves the specified diagrams for the operations of product, terminal object and exponentiation. For example, given the specified product diagram $A \xleftarrow{\pi_{A,B}} A \times B \xrightarrow{\pi'_{A,B}} B$ of the objects A and B in $\mathbf{C}$, it should be the case that $F(A \times B) = F(A) \times F(B)$, $F(\pi_{A,B}) = \pi_{F(A),F(B)}$ and $F(\pi'_{A,B}) = \pi'_{F(A),F(B)}$ (where the new symbols all represent specified objects and arrows in $\mathbf{D}$). One would also want to require that F preserve the functions $\langle \_,\_ \rangle$ and $(\_)^{\sim}$, but in fact this is a consequence of the definition.

It is not difficult to see that a strict cartesian closed functor is indeed cartesian closed, as the name would suggest. The reason is that (unspecified) products, terminal objects and exponentials are always unique up to isomorphism, and functors preserve isomorphisms. In keeping with our commitment to specified operations in general, strict cartesian closed functors will be our preferred choice of map between cartesian closed categories throughout the remainder of the text (we shall therefore from now on drop the "strict" from their name). Before concluding this chapter, let's take the opportunity to define the category **CCC**, whose objects are "all" cartesian closed categories, and arrows all cartesian closed functors between these. (That **CCC** is indeed a category is readily checked.)

Chapter 2

Moving to Two Dimensions: Introducing Bicategories

We extend in this chapter the notions of the preceding one to a certain kind of two-dimensional analogue of categories known as bicategories.

We begin with the definition of bicategory. We essentially follow Bénabou's 1967 introductory paper on them [B2]. As usual, we keep in mind that, throughout this work, we reserve the right to omit any of the indices appearing on the various symbols we introduce.

A *bicategory* **C** is given by the following data:

(i) A collection Ob(**C**) of *0-cells*, or *objects* (usually denoted A, B, C, etc.).
(ii) For every pair (A,B) of objects, a category $\mathrm{Hom}_{\mathbf{C}}(A,B)$ whose objects are called *1-cells*, or *arrows* (and denoted $f,g,h{:}A \to B$ etc.), and whose morphisms are called *2-cells* (denoted $\beta,\gamma,\varphi{:}f \Rightarrow g$ etc.). If $\beta{:}f \Rightarrow g$ and $\gamma{:}g \Rightarrow h$, the composite is written $\gamma \circ \beta$ or often $\gamma\beta$. It is customary to refer to this operation as *vertical composition*. If $f,g{:}A \to B$, the collection of 2-cells between them is denoted $2\mathrm{Hom}(f,g)$.
(iii) For every triple (A,B,C) of objects, a functor $*_{A,B,C}{:}\mathrm{Hom}(A,B) \times \mathrm{Hom}(B,C) \to \mathrm{Hom}(A,C)$. This *horizontal composition* functor is normally written infixed, with its arguments in reverse order ($(\beta,\gamma) \mapsto \gamma * \beta$, etc...), and is often suppressed, when no confusion arises as a result.
(iv) For each object A, a 1-cell $1_A{:}A \to A$, called the *identity arrow* of A.
(v) For every quadruple (A,B,C,D) of objects, a natural isomorphism (i.e. an invertible natural transformation) $\alpha^{A,B,C,D}$ (known as the *associativity isomorphism*) between the following two functors:

$$\mathrm{Hom}(A,B)\times\mathrm{Hom}(B,C)\times\mathrm{Hom}(C,D)\xrightarrow{\bullet_{A,B,C}\times\mathrm{Hom}(C,D)}\ldots$$

$$\ldots\to\mathrm{Hom}(A,C)\times\mathrm{Hom}(C,D)\xrightarrow{\bullet_{A,C,D}}\mathrm{Hom}(A,D)$$

$$[(f,g,h)\mapsto(gf,h)\mapsto h(gf)]$$

$$\Downarrow\alpha^{A,B,C,D}$$

$$[(f,g,h)\mapsto(f,hg)\mapsto(hg)f]$$

$$\mathrm{Hom}(A,B)\times\mathrm{Hom}(B,C)\times\mathrm{Hom}(C,D)\xrightarrow{\mathrm{Hom}(A,B)\times\bullet_{B,C,D}}\ldots$$

$$\ldots\to\mathrm{Hom}(A,B)\times\mathrm{Hom}(B,D)\xrightarrow{\bullet_{A,B,D}}\mathrm{Hom}(A,D)$$

(where $\mathrm{Hom}(C,D){:}\mathrm{Hom}(C,D)\to\mathrm{Hom}(C,D)$, etc., is just the identity functor; the reader should be aware that we often subscribe, throughout this work, to this common practice of identifying objects with identity arrows on them). In general, we distinguish the various components of a natural transformation by their subscripts. For instance, the component of $\alpha^{A,B,C,D}$ at e.g. (f, g, h) is written $\alpha^{A,B,C,D}_{f,g,h}$ or simply $\alpha_{f,g,h}$.

(vi$_a$) For each pair (A, B) of objects, a natural isomorphism $\lambda^{A,B}$ (known as the *left identity isomorphism*) between the following two functors:

$$\mathrm{Hom}(A,B)\times\mathbf{1}\xrightarrow{\mathrm{Hom}(A,B)\times\ulcorner 1_B\urcorner}\mathrm{Hom}(A,B)\times\mathrm{Hom}(B,B)\xrightarrow{\bullet_{A,B,B}}\mathrm{Hom}(A,B)$$

$$[(f,\$)\mapsto(f,1_B)\mapsto(1_Bf)]$$

$$\Downarrow\lambda^{A,B}$$

$$[(f,\$)\mapsto f]$$

$$\mathrm{Hom}(A,B)\times\mathbf{1}\xrightarrow{\pi}\mathrm{Hom}(A,B)$$

(where $\ulcorner 1_B\urcorner{:}\mathbf{1}\to\mathrm{Hom}(B,B)$ stands for the unique functor taking the single object ($\$$) of the category $\mathbf{1}$ to the object $1_B\in\mathrm{Ob}(\mathrm{Hom}(B,B))$).

(vi$_b$) For each pair (A, B) of objects, a natural isomorphism $\rho^{A,B}$ (known as the *right identity isomorphism*) between the following two functors:

$$\mathbf{1}\times\mathrm{Hom}(A,B)\xrightarrow{\ulcorner 1_A\urcorner\times\mathrm{Hom}(A,B)}\mathrm{Hom}(A,A)\times\mathrm{Hom}(A,B)\xrightarrow{\bullet_{A,A,B}}\mathrm{Hom}(A,B)$$

$$[(\$,f)\mapsto(1_A,f)\mapsto f1_A]$$

$$\Downarrow\rho^{A,B}$$

$$[(\$,f)\mapsto f]$$

$$\mathbf{1}\times\mathrm{Hom}(A,B)\xrightarrow{\pi'}\mathrm{Hom}(A,B).$$

In addition, this data is required to satisfy the following two *coherence conditions*:

(1) For any quintuple (A, B, C, D, E) of objects, and any quadruple $(f{:}A \to B,\ g{:}B \to C, h{:}C \to D, k{:}D \to E)$, the following pentagonal diagram should commute:

$$\begin{tikzcd}
k(h(gf)) \arrow[rr, Rightarrow, "k\alpha_{f,g,h}"'] \arrow[d, Rightarrow, "\alpha_{gf,h,k}"'] & & k((hg)f) \arrow[d, Rightarrow, "\alpha_{f,hg,k}"] \\
(kh)(gf) \arrow[dr, Rightarrow, "\alpha_{f,g,kh}"'] & & (k(hg))f \arrow[dl, Rightarrow, "\alpha_{g,h,k}\cdot f"] \\
& ((kh)g)f &
\end{tikzcd}$$

(2) For every triple (A, B, C) of objects, and any pair $(f{:}A \to B, g{:}B \to C)$, the following triangular diagram should commute:

$$\begin{tikzcd}
g(1_B f) \arrow[rr, Rightarrow, "\alpha_{f,1_B,g}"'] \arrow[dr, Rightarrow, "g\lambda_f"'] & & (g1_B)f \arrow[dl, Rightarrow, "\rho_g\cdot f"] \\
& gf &
\end{tikzcd}$$

(λ_f is of course short for $\lambda_{\$,f}$ (where \$ is the single object of **1**), etc...).

Before we go on to some examples, a few comments are in order. The reader will no doubt have noticed that one of the features of this definition is that it makes Ob(**C**) into some sort of "lax" category, i.e. one where composition (mimicked by the object part of $*$) is associative only up to some *2-isomorphisms* (invertible 2-cells), and where composing with an identity arrow (1_X) is akin to doing nothing up to some further 2-isomorphisms. Equalities on 1-cells are thus being replaced by canonical 2-isomorphisms. This is the guiding thread we follow when (later on) we extend the notion to cartesian closedness, etc.: whenever, in the respective one-dimensional case, a diagram was made to commute, in the two-dimensional construction we will simply declare that there should be a canonical natural 2-isomorphism marking this fact. We wind up with a system in which every operation that is performed on the first level is systematically kept track of on the second level, as if some sort of cost were associated to such operations and we wanted to obtain a very detailed "bill"... We also note that, by convention, we always make these (invertible) canonical 2-cells point in the intuitive direction of "reduction".

From this discussion we can already make the following observation (yielding by the same token a plentiful supply of (rather uninteresting) bicategories): any category can be made a (trivial) bicategory by letting the objects and arrows of the category constitute respectively the 0- and 1-cells of the bicategory, and then by declaring that the 2-cells α, λ and ρ should simply be pure identities. (Obviously in this case the canonical 2-cells don't keep track of anything!) Here each Hom(A, B) is made a discrete category with only identity 2-cells. This "inclusion" operation turns out to have a one-sided inverse (a "collapse" map), and together they form an adjunction. This phenomenon will be carefully analyzed in chapter 5.

The definition of bicategory requires only three kinds of 2-cells (that we call canonical) which moreover are to be isomorphisms. Of course, further 2-cells may be locally introduced in a given bicategory to enrich this basic system. However, the coherence theorem (numbered 2.1 below) essentially asserts that, between any two 1-cells, there is at most one "generalized canonical" 2-cell. (Please consult the theorem for the exact statement, as well as for the precise meaning of the expression "generalized canonical", etc....)

Let's briefly look at compositions involving 2-cells. There are three kinds, known as *vertical, horizontal* and *mixed composition.* Consider the following diagram:

$$A\left\{\begin{array}{c}\xrightarrow{f_1}\\ \Downarrow\beta\\ \xrightarrow{f_2}\\ \Downarrow\beta'\\ \xrightarrow{f_3}\end{array}\right\}B\left\{\begin{array}{c}\xrightarrow{g_1}\\ \Downarrow\gamma\\ \xrightarrow{g_2}\\ \Downarrow\gamma'\\ \xrightarrow{g_3}\end{array}\right\}C.$$

The two possible basic vertical compositions are $\beta'\circ\beta : f_1 \Rightarrow f_3$ and $\gamma'\circ\gamma : g_1 \Rightarrow g_3$ (given by the categorical structure of Hom(A,B) and Hom(B,C)), whereas the four basic horizontal compositions are $\gamma*\beta : g_1f_1 \Rightarrow g_2f_2$, $\gamma'*\beta : g_2f_1 \Rightarrow g_3f_2$, $\gamma*\beta' : g_1f_2 \Rightarrow g_2f_3$ and $\gamma'*\beta' : g_2f_2 \Rightarrow g_3f_3$ (given by the functoriality of $*$). (There are in fact more horizontal (and vertical) compositions possible in this diagram, as we could compose e.g. β with $\gamma'\circ\gamma$, etc....) There are a bundle of mixed compositions (those involve a 2-cell and a 1-

cell) but we single out two of them; and rather than making up some new dot to denote this operation, we simply represent it as concatenation: $\gamma f_2 : g_1 f_2 \Rightarrow g_2 f_2$, $g_2\beta : g_2 f_1 \Rightarrow g_2 f_2$. This kind of composition is in fact a special case of horizontal composition, namely it is the same as composing the given 2-cell with the identity 2-cell on the 1-cell involved, but it is important enough that it deserves to be mentioned separately.

As a consequence of the fact that $*$ is a functor (it preserves composition), we have the *interchange law*: $(\gamma' \circ \gamma) * (\beta' \circ \beta) = (\gamma' * \beta') \circ (\gamma * \beta)$. Specialized to mixed compositions, this law reads $(\gamma' \circ \gamma) f_2 = (\gamma' f_2) \circ (\gamma f_2)$, with a similar equation for the second case. Since $*$ preserves identities, we get $\mathrm{Id}_{g_1} * \mathrm{Id}_{f_1} = \mathrm{Id}_{g_1 f_1}$, etc.... One might also like to unravel the meaning of the naturality of the canonical 2-cells. Of course, it essentially just makes canonical 2-cells "commute" with other 2-cells. Naturality of ρ, for example, says that composing vertically in the following two diagrams gives precisely the same 2-cell:

$$A \left\{ \begin{array}{c} \xrightarrow{f_1 1_A} \\ \Downarrow \rho_{f_1} \\ \xrightarrow{f_1} \\ \Downarrow \beta \\ \xrightarrow{f_2} \end{array} \right\} B \qquad \text{and} \qquad A \left\{ \begin{array}{c} \xrightarrow{f_1 1_A} \\ \Downarrow \beta 1_A \\ \xrightarrow{f_2 1_A} \\ \Downarrow \rho_{f_2} \\ \xrightarrow{f_2} \end{array} \right\} B.$$

On a related matter, one could be tempted to ask whether horizontal composition $*$ is, in general, associative on 2-cells (we know of course it isn't on 1-cells). The answer is also no (quite obviously, mind you): to start with, it doesn't even typecheck properly to be associative!

We now give some examples of bicategories. The first case that jumps to mind (although not necessarily the most natural one, as we will see) is, of course, **Cat**. We recall that its objects are "all" categories and its 1-cells all functors between these; its 2-cells are then simply all natural transformations between those functors. Because **Cat** is in fact also a category, α, λ and ρ are identity natural transformations. As a matter of fact, **Cat** is most accurately described as being a 2-category[1].

---

[1]It may sometimes happen when giving examples that we invoke concepts which are foreign to the reader, without fully explaining them anywhere in this work, for lack of space. We apologize in advance, but we assure the reader that when this occurs, the corresponding example may be skipped without prejudice to understanding other parts of the text. In any case, an appropriate reference will always be provided; for 2-cat-

Monoidal categories[2] are in one-to-one correspondence with one-object bicategories: in the notation of [EK] and [CWM], if $\mathbf{V} = (V, \otimes, I, r, l, a)$ is a monoidal category, then $\mathbf{V}$ is made a bicategory with a single object \$ by putting $\mathrm{Hom}(\$,\$) = V$, $* = \otimes$, $1_\$ = I$, $\alpha = a$, $\lambda = l$, $\rho = r$. This procedure has, of course, a straightforward two-sided inverse.

For our third example, let $\mathbf{A}$ be a category with (specified) pullbacks[3]. The bicategory $\mathrm{Sp}(\mathbf{A})$ of *spans over* $\mathbf{A}$ is defined as follows: the objects of $\mathrm{Sp}(\mathbf{A})$ are the objects of $\mathbf{A}$. A 1-cell between A and B in $\mathrm{Sp}(\mathbf{A})$ is a diagram $A \xleftarrow{f} X \xrightarrow{g} B$ in $\mathbf{A}$. Identity 1-cells are those for which both f and g are identities. Given $s = A \xleftarrow{f} X \xrightarrow{g} B$ and $s' = A \xleftarrow{f'} X' \xrightarrow{g'} B$, both objects of $\mathrm{Hom}(A, B)$, a 2-cell $\beta : s \Rightarrow s'$ is a commutative diagram in $\mathbf{A}$:

$$\beta = \left\{ \begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "i"] & \\ A & & B \\ & X' \arrow[ul, "f'"] \arrow[ur, "g'"'] & \end{tikzcd} \right\}$$

Vertical composition of 2-cells is the obvious one, and horizontal composition is given by pullback. Explicitly, given the above diagram, and given further two 1-cells $t, t' : B \to C$ and a 2-cell $\gamma : t \Rightarrow t'$, as illustrated here,

$$\gamma = \left\{ \begin{tikzcd} & Y \arrow[dl, "h"'] \arrow[dr, "k"] \arrow[dd, "j"] & \\ B & & C \\ & Y' \arrow[ul, "h'"] \arrow[ur, "k'"'] & \end{tikzcd} \right\}$$

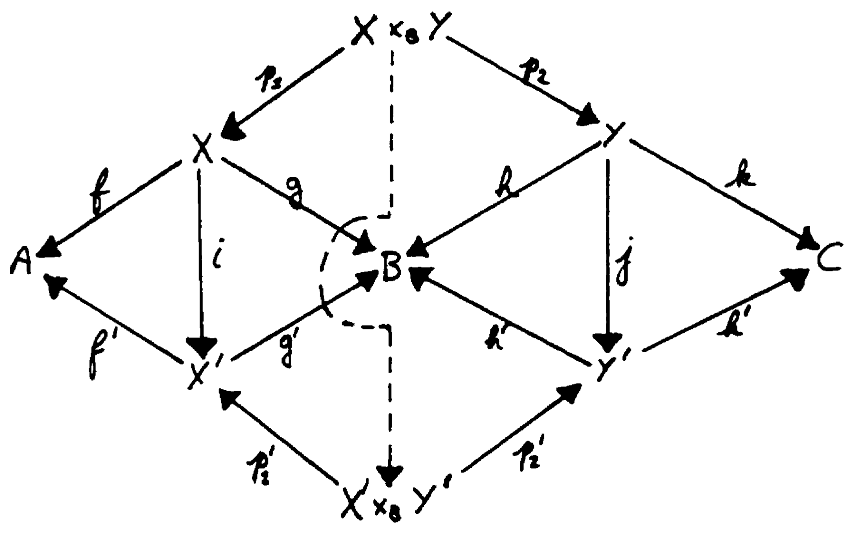
the composites ts and $t's'$ are represented respectively by the top and bottom edges of the outside diamond below, in which two pullbacks were taken:

---

egories, the reader may consult [KS] or even [CWM]. This time, luckily, a simple definition can be provided: a 2-category is just a bicategory in which α, λ and ρ are identity natural transformations.

[2]Monoidal categories, first introduced in [B1] as "catégories avec multiplication", are also examined in [EK] and [CWM]. Historically, they came before bicategories, and have certainly been studied much more extensively.

[3]The well-known notion of pullback is explained in most texts on category theory; see, for instance, [CWM].



The composite $\gamma\beta: ts \Rightarrow t's'$ is given by the arrow $X \times_B Y \to X' \times_B Y'$ in **A** provided by the two arrows $ip_1: X \times_B Y \to X'$, $jp_2: X \times_B Y \to Y'$ and by the universal property of the bottom pullback. Finally, α, λ and ρ are obtained by the usual (and unique) isomorphisms of associativity and identity of pullbacks. A few routine verifications will show that we indeed have a bicategory.

We now present the so-called coherence theorem. Its origins date back to a 1963 paper by Mac Lane ([M1], later recorded in [CWM] as well) where the theorem is stated and proved in the special case of monoidal categories. Bénabou [B3] gave a much more general result in 1968, but the precise theorem that we present here was published in 1985 by Mac Lane and Paré [MP].

We first need to go through some preparations. Namely, we want to ensure that no equalities on objects and 1-cells (such as $gf = g'f'$) hold that are not "strictly necessary", something vital for our purposes. We can either simply assume this to be true of **C**, or else construct a new bicategory **B** that has this disjointness property but is otherwise the "same" as **C**. The latter construction is straightforward; details are in [MP]. So we will just assume that **C** has the required property. Next, we want to call ***generalized canonical***[4] (***g-canonical*** for short) all (well-defined) composites (vertical or horizontal) of instances of the identity 2-cells, α, λ, ρ, and their inverses; i.e., the g-canonical 2-cells are the ones belonging to the smallest set of 2-cells closed under vertical and horizontal composition which contains all instances of the identity 2-cells, the canonical ones as well

---

[4]Mac Lane and Paré use the simple "canonical" instead, but this term already has a meaning for us.

as their inverses.[5] An alternative way of getting at the new notion is to call g-canonical those and only those 2-cells which also belong to the smallest sub-bicategory of **C** having the same objects and same 1-cells. We can now state the coherence theorem:

THEOREM 2.1. To each 1-cell $f{:}A \to B$ in **C** there is associated a 1-cell $\hat{f}{:}A \to B$ and a g-canonical 2-cell $\sigma_f{:}f \Rightarrow \hat{f}$. For $f,g{:}A \to B$, there is at most one g-canonical 2-cell $f \Rightarrow g$ and there is one if and only if $\hat{f} = \hat{g}$.

The proof in [MP] heavily relies on and refers to its earlier sibling in [CWM]. We note, for the record, that some of the missing details (such as the acrobatic exercises required to show that $\lambda_{1_X} = \rho_{1_X}{:}1_X{*}1_X \Rightarrow 1_X$) can be found in [JS]. We will not reproduce the proof here, but we can perhaps give an idea of the construction involved by saying that, to get from f to $\hat{f}$ (the latter being called the *standard form* of f in [MP]), one repeatedly uses instances of α, λ and ρ to shift all parentheses to the left and drop all identity 1-cells. The resulting g-canonical 2-cell is then shown to be unique by virtue of the coherence conditions as well as the naturality of α, λ and ρ.

The term "coherence" stems from the fact that an equivalent statement of the theorem is that, under the same hypotheses, any (well-defined) closed diagram of g-canonical 2-cells commutes. We will also see later on (chapter 5) how to define a notion of *free bicategory on an empty set of 2-cells* – for the moment we just ask the reader to try and imagine for himself what this means, in analogy with the well-known mathematical ideas of free objects on a set of generators, such as groups, etc.... The important point is that the only 2-cells that a free bicategory on an empty set of 2-cells has are the g-canonical ones; and moreover, in a free bicategory as few 2-cells as possible are identified. The appropriate universal property of freeness then entails that the coherence theorem can be restated (we believe more elegantly) by saying that, in an arbitrary free bicategory on an empty set of 2-cells, there is at most one 2-cell between any pair of 1-cells.

We move on to defining further structures on bicategories, corresponding to those introduced in chapter 1. The goal is to equip bicategories with the notions of product, terminal object and exponential, yielding what Makkai calls "cartesian closed bicategories" in [M3]; we are in fact not aware of any work anterior to [M3] introducing this particular

---

[5]It is not necessary of course to specifically require that the identity 2-cells be included in this set (since that will happen automatically), but it would be mandatory for the statement of a result where the assumption of invertibility of the canonical 2-cells were dropped.

idea (whether or not under the same terminology), although we do not claim by any means to have performed a thorough check of the literature. However we do remind the reader that, as explained in the introduction, our "cartesian closed bicategories" have unfortunately nothing to do with Carboni and Walters' "cartesian categories" [CW].

Even though the notions are at heart the same, our treatment differs in certain respects from that of [M3]. First, we have chosen as definitions ones that are as similar in format as possible to that of bicategories. That we are indeed talking about the same thing as [M3] is then established as propositions 2.4 and 2.8. Second, and much more subtly, we have included certain coherence conditions absent from [M3]; interestingly, our definitions still *remain equivalent* to those of [M3]. The (paradoxical) additional requirements play a major rôle, however, when we start discussing a conjecture which would extend (not quite literally, but for all intents and purposes) theorem 2.1 if verified. Later on, the coherence conditions will be seen to influence the construction of free bicategories. We will of course clarify these statements when we reach the appropriate stage. Lastly, we investigate properties and features of these structured bicategories in a certain amount of detail, occasionally motivating definitions and spelling out proofs of "routine" statements, as we are not aware that what follows is fully collected anywhere in the literature.

From now on in the discussion, all objects, 1-cells and 2-cells we will be dealing with are understood to belong to some fixed and unnamed bicategory. We stated earlier that our general philosophy in extending notions such as that of binary product is to record what used to be an equality between two 1-cells (i.e. a commutative diagram) by means of a (canonical natural) 2-isomorphism. The question arises, however, as to how we should "translate" the property that a 1-cell be unique. A moment's thought will quickly show that literal uniqueness is something we have to forgo; for instance, we could not carry over the notion of terminal object, since in general there are either infinitely many 1-cells between two objects or none at all! The whole setup is in fact geared towards identifying as few 1-cells as possible. The solution to our problem, then, is hinted at by realizing that saying that an arrow with certain properties is unique is really saying that a bunch of diagrams all commute. Therefore, instead of pure uniqueness, we shall demand "uniqueness up to 2-isomorphism". (Each of) the 1-cell(s) in question will then be said to be *2-unique*.

This discussion reinforces the idea that the "right" notion of equality between 1-cells is that of 2-isomorphism. Because it is so important, we will adopt the infix symbol $\cong$ to represent this concept that we stop short of calling 2-equality[6]. $\equiv$ is clearly reflexive, symmetric and transitive. If f, g, h, 1_A, 1_B are 1-cells of the appropriate type, then $(hg)f \equiv h(gf)$, $1_B f \equiv f$ and $f 1_A \equiv f$. Moreover, $\equiv$ is preserved by any functor on the Hom-categories, since functors preserve isomorphisms. As an immediate consequence, $\equiv$ is preserved by horizontal composition.

Another point that should be handled carefully is the following. In defining earlier binary products, for example, we noted that it was possible to have two distinct product diagrams for the same two objects, but that when this happened, there was a (unique) isomorphism between the two products preserving the respective projections. We did eventually adopt the method of *specifying* products, of course, thereby rendering the point moot, but the "moral" justification for our doing so was precisely that isomorphism property. (All this according to a general principle of category theory which claims that the only "good" notions are those invariant under isomorphism.) We would therefore like to have some similar feature guaranteeing the "soundness" of our new constructions. Again, it's quite clear that the literal notion of isomorphism (i.e. invertible 1-cell) doesn't work anymore: for example, in general $1_A : A \to A$ isn't even invertible! The natural analogue to the idea of isomorphism between A and B is, clearly, the requirement that there be two 1-cells $f: A \to B$ and $g: B \to A$ with $gf \equiv 1_A$ and $fg \equiv 1_B$. This is in fact the definition of *1-isomorphism*[7] (between objects) we will adopt for bicategories. Of course, we also have to give up the hope that e.g. a 1-isomorphism between binary products should literally preserve the respective projections – instead, we just expect the 1-cells concerned to be 2-isomorphic.

(At last:) A binary product[8] of the objects A and B is an object $A \times B$ together with a diagram $A \xleftarrow{\pi_{A,B}} A \times B \xrightarrow{\pi'_{A,B}} B$ such that, for any C, there is a *pairing* functor $\langle \_, \_ \rangle^C_{A,B} : \mathrm{Hom}(C,A) \times \mathrm{Hom}(C,B) \to \mathrm{Hom}(C, A \times B)$ and three natural isomorphisms $\tau^{A,B,C}$, $\tau'^{A,B,C}$ and $\overline{\tau}^{A,B,C}$ between the following functors:

[6] Actually, we will introduce later an even stronger, if technical, notion incorporating naturality requirements.

[7] This type of property is usually called "equivalence" in the literature.

[8] The term "biproduct" is sometimes seen to denote similar constructions in related contexts, but this otherwise excellent terminology has the bad fortune of clashing with that of a well-established notion from standard one-dimensional category theory.

$$\mathrm{Hom}(C,A)\times\mathrm{Hom}(C,B)\xrightarrow{\langle\_,\_\rangle^{C}_{A,B}}\mathrm{Hom}(C,A\times B)\xrightarrow{\pi_{A,B}*(\_)}\mathrm{Hom}(C,A)$$
$$\left[(f,g)\mapsto\langle f,g\rangle\mapsto\pi\langle f,g\rangle\right]$$
$$\Downarrow\tau^{A,B,C}$$
$$\left[(f,g)\mapsto f\right]$$
$$\mathrm{Hom}(C,A)\times\mathrm{Hom}(C,B)\xrightarrow{\pi_{\mathrm{Hom}(C,A),\mathrm{Hom}(C,B)}}\mathrm{Hom}(C,A)$$

(where $\pi_{A,B}*(\_):\mathrm{Hom}(C,A\times B)\to\mathrm{Hom}(C,A)$ stands for composing on the left with $\pi_{A,B}$),

$$\mathrm{Hom}(C,A)\times\mathrm{Hom}(C,B)\xrightarrow{\langle\_,\_\rangle^{C}_{A,B}}\mathrm{Hom}(C,A\times B)\xrightarrow{\pi'_{A,B}*(\_)}\mathrm{Hom}(C,B)$$
$$\left[(f,g)\mapsto\langle f,g\rangle\mapsto\pi'\langle f,g\rangle\right]$$
$$\Downarrow\tau'^{A,B,C}$$
$$\left[(f,g)\mapsto g\right]$$
$$\mathrm{Hom}(C,A)\times\mathrm{Hom}(C,B)\xrightarrow{\pi'_{\mathrm{Hom}(C,A),\mathrm{Hom}(C,B)}}\mathrm{Hom}(C,B),$$

$$\mathrm{Hom}(C,A\times B)\xrightarrow{\langle\pi_{A,B}*(\_),\pi'_{A,B}*(\_)\rangle^{\mathrm{Hom}(C,A\times B)}_{\mathrm{Hom}(C,A),\mathrm{Hom}(C,B)}}\ldots$$
$$\ldots\to\mathrm{Hom}(C,A)\times\mathrm{Hom}(C,B)\xrightarrow{\langle\_,\_\rangle^{C}_{A,B}}\mathrm{Hom}(C,A\times B)$$
$$\left[h\mapsto(\pi h,\pi' h)\mapsto\langle\pi h,\pi' h\rangle\right]$$
$$\Downarrow\bar{\tau}^{A,B,C}$$
$$\left[h\mapsto h\right]$$
$$\mathrm{Hom}(C,A\times B)\xrightarrow{\mathrm{Hom}(C,A\times B)}\mathrm{Hom}(C,A\times B).$$

(The reader has realized, of course, that the single occurrence of the notation $\langle\_,\_\rangle^{\mathrm{Hom}(C,A\times B)}_{\mathrm{Hom}(C,A),\mathrm{Hom}(C,B)}$ is to be understood in the sense of chapter 1.)

In addition, we require that this data satisfies the following two coherence conditions:

(1) For any object C and arrow $h:C \to A\times B$, the following diagram should commute:

$$
\begin{tikzcd}
{[\pi * h, \pi' * h]} \arrow[rr, Rightarrow, "="', "{[\pi * h, \pi' * h]}"'] \arrow[dr, Rightarrow, "{[\pi * \bar{\tau}_h^{-1}, \pi' * \bar{\tau}_h^{-1}]}"'] & & {[\pi * h, \pi' * h]} \\
& {[\pi * \langle \pi * h, \pi' * h\rangle, \pi' * \langle \pi * h, \pi' * h\rangle]} \arrow[ur, Rightarrow, "{[\tau_{\pi * h, \pi' * h}, \tau'_{\pi * h, \pi' * h}]}"'] &
\end{tikzcd}
$$

(where $[\_,\_]$ denotes the notationally cumbersome functor $\langle \_,\_\rangle^{\mathrm{Hom}(C,A\times B)}_{\mathrm{Hom}(C,A),\mathrm{Hom}(C,B)}$).

(1′) For any object C and arrows $f:C\to A$, $g:C\to B$, the following diagram should commute:

$$
\begin{tikzcd}
\langle f, g\rangle \arrow[rr, Rightarrow, "="', "\langle f, g\rangle"'] \arrow[dr, Rightarrow, "\bar{\tau}^{-1}_{\langle f,g\rangle}"'] & & \langle f, g\rangle \\
& \langle \pi * \langle f, g\rangle, \pi' * \langle f, g\rangle\rangle \arrow[ur, Rightarrow, "{\langle \tau_{f,g}, \tau'_{f,g}\rangle}"'] &
\end{tikzcd}
$$

We insist on the fact that we do not assume at all *a priori* that the objects A and B have a binary product, and therefore our use of the notation $A\times B$, etc., should be simply understood as a "shortcut".[9] We will also unfortunately have to insist at times on the following subtle point, in order to correctly state (and prove) the next few propositions: in the notation above, the *product* is simply the object $A\times B$, the *product diagram* is the data $A \xleftarrow{\pi_{A,B}} A\times B \xrightarrow{\pi'_{A,B}} B$, whereas the *product package* consists of everything mentioned in the definition, i.e., the diagram, plus all the pairing functors $\langle \_,\_\rangle^{C}_{A,B}$, as well as all the natural isomorphisms $\tau^{A,B,C}$, $\tau'^{A,B,C}$ and $\bar{\tau}^{A,B,C}$. Of course, when we say "product", we assume there exists at least one product package of which the product in question is just one of the components, but nothing more (although later we may say "product" to mean "product package", when it is harmless to do so). We may also talk about, e.g., the product *package* $A\times B$, or $A \xleftarrow{\pi_{A,B}} A\times B \xrightarrow{\pi'_{A,B}} B$, etc. In this case, it should be assumed we have a particular package in mind (usually in the notation of the above definition), but that we haven't bothered to explicitly name each of its components.

---

[9]We are using the reserved notation from the start mostly because we believe that the meaning of the natural transformations is significantly more transparent this way, especially for someone who wants to quickly consult them later on.

We would now like to make a few comments on the coherence conditions. First, (1) and (1′) are in fact equivalent (each implies the other); the reason for this will become clear in a short while. We are including both identities for reasons of symmetry and because of the fact that this pair of triangles has a well-known counterpart in standard category theory. More importantly, should we later want to explore alternative constructions by weakening the canonical 2-cells by not requiring they be invertible (as we will do), (1) and (1′) will no longer be equivalent, and both should be required. More surprising is the fact that the definition of binary product (diagram) we have just given is logically *equivalent* to the same definition, minus the two coherence conditions(!) This also will soon become obvious, but for now let us just say that, because the definition is essentially existential in nature (i.e., it only requires the *existence* of certain natural isomorphisms, etc.), what happens is that, if there exist *any* natural isomorphisms τ, τ' and $\bar{\tau}$ between the appropriate functors, not necessarily however satisfying the coherence conditions, then it automatically follows that there *also* exist three natural isomorphisms between the same functors that *do* satisfy them. The reason our definition must be made existential in this sense is that it is technically a definition of product *diagram*, as opposed to product *package*. In this respect (and the same will go for the exponential, to be introduced shortly) we remain in synch with [M3]. The difference will come afterwards, in the form of which 2-cells are marked as canonical: indeed, only ones that do pass the coherence test need apply. This is, we believe, a crucial point (for our coherence conjecture to hold – see later). Of course, when we get round to *specifying* products, we will require that any pair of object have a unique (specified) product *package* (which includes the 2-cells!) associated to it. These 2-cells, which we will then mark as canonical, will obviously satisfy the coherence conditions. Also affected by these conditions is the notion of *free cartesian closed bicategory*, which will come up in chapter 5.

It is perhaps a good idea at this point to pause and come back to the stem one-dimensional instance of the concept of binary product (see chapter 1). We recall that there were two equivalent ways to go about defining binary products. One was to state a certain universal property, and then decree that any diagram satisfying it was *a* binary product diagram. Then, if need be, we would actually choose among all the suitable candidates and therefore specify which diagram was going to be *the* binary product diagram of two given objects. The alternative approach was to designate beforehand a particular diagram, and then require that certain equations between arrows always hold. Either way, it led to the same thing – the choice of which method to use largely remained a question of style.[10]

The question is now, is there likewise an equivalent "universal property"-like way to go about defining binary products in bicategories, and if so, are there any arguments for preferring one method over the other? The first answer is yes, as will be shown shortly. As for the second question, I'm tempted to think it remains once again a matter of personal taste.

We start by investigating some properties of products. We would first like to clearly expose the tight links there are between the present notion and the one introduced in the last chapter. The proofs are very similar to their corresponding one-dimensional counterparts, but we include them for illustrative purposes. We are still always in the context of a fixed bicategory **C**.

PROPOSITION 2.2. Given a product package $A \xleftarrow{\pi} A \times B \xrightarrow{\pi'} B$ (with pairing functors and natural isomorphisms as in the definition above), for any object C and arrows $f: C \to A$, $g: C \to B$, there exists a 2-unique arrow $h: C \to A \times B$ such that $\pi h \cong f$ and $\pi' h \cong g$.

PROOF. Obviously, setting $h = \langle f, g \rangle$ takes care of the existence part of the statement (the 2-isomorphisms between πh and f and between $\pi' h$ and g are components of τ and τ' respectively). Now suppose that for $k: C \to A \times B$, we have $\pi k \cong f$ and $\pi' k \cong g$. Components of $\bar{\tau}$ give us $h \cong \langle \pi h, \pi' h \rangle \cong \langle f, g \rangle \cong \langle \pi k, \pi' k \rangle \cong k$, as required.

The reader will not be surprised to learn that the contents of this proposition is not quite strong enough to fully capture the notion of binary product (as was done in the one-dimensional case). Indeed, it is not hard to imagine a diagram $A \xleftarrow{\pi} A \times B \xrightarrow{\pi'} B$ satisfying the universal property of proposition 2.2 but such that no pairing *functor* $\langle \_, \_ \rangle$ exists. And even when such a functor does exist, there is no reason at all for the isomorphisms τ, τ' and $\bar{\tau}$ to be natural. However, nothing else is missing. But first let us record the following fact:

[10] For instance, Hamik and Makkai [HM] deem the "universal property" formulation of the notion to be preferable, on the grounds that it simply involves less data.

PROPOSITION 2.3. Let $A \xleftarrow{\pi} A\times B \xrightarrow{\pi'} B$ be a product package (with the remainder of the data just as in the definition). If $A \xleftarrow{p} P \xrightarrow{p'} B$ is any other product package, with pairing functors $[\_,\_]^C$:$\mathrm{Hom}(C,A)\times\mathrm{Hom}(C,B)\rightarrow\mathrm{Hom}(C,P)$ (and unnamed natural isomorphisms), then there is a 2-unique 1-isomorphism $i{:}P \xrightarrow{\cong} A\times B$ such that $\pi i \cong p$ and $\pi' i \cong p'$.

PROOF. Let $i=\langle p,p'\rangle^P$. By proposition 2.2, i is 2-unique with the property $\pi i\cong p$ and $\pi' i\cong p'$. It remains to show it is a 1-isomorphism. Let $j{:}A\times B\rightarrow P$ be $j=[\pi,\pi']^{A\times B}$. We claim that $ij\cong 1_{A\times B}$ and $ji\cong 1_P$. Since $A \xleftarrow{p} P \xrightarrow{p'} B$ is a product diagram, $pj\cong\pi$ and $p'j\cong\pi'$. Now we have $\pi(ij)\cong(\pi i)j\cong pj\cong\pi$, and similarly $\pi'(ij)\cong\pi'$. But observe that $1_{A\times B}$ also satisfies these equations (i.e. $\pi 1_{A\times B}\cong\pi$ and $\pi' 1_{A\times B}\cong\pi'$), and thus by 2-uniqueness (proposition 2.2), we get that $ij\cong 1_{A\times B}$. A symmetric argument will establish that $ji\cong 1_P$, and therefore that i is a 1-isomorphism as required.

Later on in this chapter we will officially adopt the method of specifying binary product diagrams (when they exist) in bicategories. Our "moral" justification, of course, is the contents of the last proposition.

Finally, we can state the following proposition, which contains the alternative formulation of the notion of binary product, that put forth in [M3]:

PROPOSITION 2.4. The diagram $A \xleftarrow{p} P \xrightarrow{p'} B$ is a product diagram of A and B if and only if, for any object C, the functor $F_C =_{def} \langle p*(\_),p'*(\_)\rangle^{\mathrm{Hom}(P,A\times B)}_{\mathrm{Hom}(C,A),\mathrm{Hom}(C,B)}$: $\mathrm{Hom}(C,P)\rightarrow\mathrm{Hom}(C,A)\times\mathrm{Hom}(C,B)$ is an equivalence of categories.

PROOF. We do the positive direction first. That is, we assume the given product diagram can be completed into a product package, and seek to establish the equivalence. For clarity, let us rename P to $A\times B$, and p and p' to π and π' respectively. The rest of the data is just as in the definition. So let C be fixed. If $G_C =_{def} \langle\_,\_\rangle^C_{A,B}$: $\mathrm{Hom}(C,A)\times\mathrm{Hom}(C,B)\rightarrow\mathrm{Hom}(C,P)$, it is easy to check that $\eta:\mathrm{Id}_{\mathrm{Hom}(C,A\times B)}\Rightarrow G_C F_C$ is given by $\eta =_{def} (\bar{\tau}^{A,B,C})^{-1}$ and that $\varepsilon: F_C\circ G_C\Rightarrow\mathrm{Id}_{\mathrm{Hom}(C,A)\times\mathrm{Hom}(C,B)}$ is given by $\varepsilon =_{def} \left(\tau^{A,B,C},\tau'^{A,B,C}\right)$ (this takes place in the product of the functor categories $\mathrm{Hom}(C,A)^{\mathrm{Hom}(C,A)\times\mathrm{Hom}(C,B)}\times\mathrm{Hom}(C,B)^{\mathrm{Hom}(C,A)\times\mathrm{Hom}(C,B)}$), as required.

For the other direction, let us fix C again, and let us also rename P, p and p' to the more familiar notation, as above. We assume there is $G_C : \mathrm{Hom}(C,A) \times \mathrm{Hom}(C,B) \to \mathrm{Hom}(C, A \times B)$, with the natural isomorphisms $\eta : \mathrm{Id}_{\mathrm{Hom}(C,A\times B)} \xrightarrow{\cong} G_C \circ F_C$ and $\varepsilon : F_C \circ G_C \xrightarrow{\cong} \mathrm{Id}_{\mathrm{Hom}(C,A)\times\mathrm{Hom}(C,B)}$. Moreover, we may assume in addition that η and ε are respectively the unit and counit of the adjunction $F_C \dashv G_C$ (thanks to the theorem which asserts that any equivalence can in fact be made an *adjoint* equivalence, merely by modifying one of η or ε – see, for instance, [CWM]). Now put $\langle \_,\_ \rangle^{C}_{A,B} =_{def} G_C$, and for any $f{:}C \to A$, $g{:}C \to B$, put $\tau^{A,B,C}_{f,g} =_{def} \pi_{\mathrm{Hom}(C,A),\mathrm{Hom}(C,B)}(\varepsilon_{f,g})$ and $\tau'^{A,B,C}_{f,g} =_{def} \pi'_{\mathrm{Hom}(C,A),\mathrm{Hom}(C,B)}(\varepsilon_{f,g})$, as well as $\bar{\tau}^{A,B,C} =_{def} \eta^{-1}$. Again quoting [CWM], η and ε are respectively unit and counit if and only if they obey the following two triangle identities:

$$\begin{tikzcd} F_C \arrow[rr, "="] \arrow[dr, "F_C\eta"'] & \bullet & F_C \\ & F_C G_C F_C \arrow[ur, "\varepsilon F_C"'] & \end{tikzcd} \qquad \begin{tikzcd} G_C \arrow[rr, "="] \arrow[dr, "\eta G_C"'] & \bullet & G_C \\ & G_C F_C G_C \arrow[ur, "G_C\varepsilon"'] & \end{tikzcd}$$

Unravelling the notation, these are easily seen to precisely be the two coherence conditions. That all of the data indeed constitutes a product package is equally straightforward to verify.

There are thus two different ways to define the notion of product. Moving back and forth between them can prove very fruitful. For instance, it is now plain where the coherence conditions incorporated into our definition came from: all we did was translate the requirement that, for any C, $(F_C, G_C, \eta, \varepsilon)$ not only constitute an equivalence of the categories $\mathrm{Hom}(C,P)$ and $\mathrm{Hom}(C,A) \times \mathrm{Hom}(C,B)$, but an *adjoint equivalence.* It is also clear why the two coherence conditions are in fact equivalent: one may easily check that, when η and ε are isomorphisms, either one of the triangle identities quoted in the proof above implies that the other holds as well. Lastly, we can see why the coherence conditions do not technically change anything to the definition *per se* of binary product: as already mentioned, roughly speaking an equivalence exists if and only if an adjoint one does – since the definition of product is only concerned with the *existence* of an adjoint equivalence, it would have been the same to require the existence of a straight equivalence, that is to say, dispense with the coherence conditions: there is no difference insofar as the resulting product diagram is concerned (but there is a difference for the product *package*!).

We now move on to the next definition. An object $\mathbf{t}$ is said to be *terminal* if, for any object A, there is a 1-cell $!_A : A \to \mathbf{t}$ and a natural isomorphism ξ^A between the following functors:

$$\begin{array}{c} \mathrm{Hom}(A,\mathbf{t}) \xrightarrow{\mathrm{Hom}(A,\mathbf{t})} \mathrm{Hom}(A,\mathbf{t}) \\ [f \mapsto f] \\ \Downarrow \xi^A \\ [f \mapsto \$ \mapsto !_A] \\ \mathrm{Hom}(A,\mathbf{t}) \xrightarrow{!_{\mathrm{Hom}(A,\mathbf{t})}} \mathbf{1} \xrightarrow{\ulcorner !_A \urcorner} \mathrm{Hom}(A,\mathbf{t}) \end{array}$$

(where $!_{\mathrm{Hom}(A,\mathbf{t})}$ is of course meant in the sense of chapter 1).

PROPOSITION 2.5. There is a 2-unique 1-isomorphism between any two terminal objects.

As usual, the proof can be carried over from the one-dimensional case almost verbatim. We will therefore omit it. We remark that $\mathbf{t}$ is a terminal object if and only if the categories $\mathrm{Hom}(A,\mathbf{t})$ and $\mathbf{1}$ are equivalent (and again, this is the definition of terminal object in bicategories adopted in [M3]). We note that in this case, it is not possible to have an equivalence (fully spelled out – two functors and two natural isomorphisms) which is not at the same time an adjoint equivalence!

If the bicategory $\mathbf{C}$ has binary products (for every pair of objects) and terminal objects, we say that $\mathbf{C}$ is *cartesian*. We have already stated that **Cat** was a cartesian *category*. It is also, of course, a cartesian *bicategory*.

The notion of binary product can be generalized to any finite number of objects, either by directly modifying the definition, or by employing the terminal object and repeated binary products. Given a fixed finite set of objects, it is a fact that all products of them generated by instances of these various possible definitions will all be strongly 1-isomorphic to one another, i.e. there will be 2-unique 1-isomorphisms between them preserving the respective projections. (One can prove this exactly as in the one-dimensional case, merely by making good use of proposition 2.2.) It is hence reasonable, in the context of, say, a cartesian bicategory, to talk about a (and later on, *the*) product of several objects without additional justifications.

An exponential of the objects A and B is an object B^A together with a diagram

$$\begin{tikzcd}
 & A \times B^A \arrow[dl, "\pi_{A,B^A}"'] \arrow[dr, "\varepsilon_{A,B}"] \arrow[rr, "\pi'_{A,B^A}"] & & B^A \\
A & & B &
\end{tikzcd}$$

where $A \xleftarrow{\pi_{A,B^A}} A \times B^A \xrightarrow{\pi'_{A,B^A}} B^A$ is a product package (with pairing functors $\langle \_,\_ \rangle^C_{A,B}$ and unlabelled natural transformations), such that, for any product package $A \xleftarrow{\pi_{A,D}} A \times D \xrightarrow{\pi'_{A,D}} D$ (with the rest of the data unnamed), there is a functor (called exponentiation) $(\_)^{\sim}_{A,B,D}: \mathrm{Hom}(A \times D, B) \to \mathrm{Hom}(D, B^A)$ and two natural isomorphisms $\zeta^D (= \zeta^{D,A\times D})$ and $\bar{\zeta}^D (= \bar{\zeta}^{D,A\times D})$ between the following functors:

$$\mathrm{Hom}(A \times D, B) \xrightarrow{\cup^{\sim}} \mathrm{Hom}(D, B^A) \xrightarrow{\langle \pi_{A,D}, \cup \circ \pi'_{A,D} \rangle^{A\times D}_{A,B^A}} \ldots$$

$$\ldots \to \mathrm{Hom}(A \times D, A \times B^A) \xrightarrow{\varepsilon_{A,B} \circ \cup} \mathrm{Hom}(A \times D, B)$$

$$\left[h \mapsto h^{\sim} \mapsto (\pi_{A,D}, h^{\sim}\pi'_{A,D}) \mapsto \varepsilon\langle \pi_{A,D}, h^{\sim}\pi'_{A,D} \rangle \right]$$

$$\Downarrow \zeta^D$$

$$[h \mapsto h]$$

$$\mathrm{Hom}(A \times D, B) \xrightarrow{\mathrm{Hom}(A \times D, B)} \mathrm{Hom}(A \times D, B),$$

$$\mathrm{Hom}(D, B^A) \xrightarrow{(\pi_{A,D}, \cup \circ \pi'_{A,D})} \mathrm{Hom}(A \times D, A \times B^A) \xrightarrow{\varepsilon_{A,B} \circ \cup} \ldots$$

$$\ldots \to \mathrm{Hom}(A \times D, B) \xrightarrow{\cup^{\sim}} \mathrm{Hom}(D, B^A)$$

$$\left[k \mapsto (\pi_{A,D}, k\pi'_{A,D}) \mapsto \varepsilon\langle \pi_{A,D}, k\pi'_{A,D} \rangle \mapsto \left(\varepsilon\langle \pi_{A,D}, k\pi'_{A,D} \rangle \right)^{\sim} \right]$$

$$\Downarrow \bar{\zeta}^D$$

$$[k \mapsto k]$$

$$\mathrm{Hom}(D, B^A) \xrightarrow{\mathrm{Hom}(D, B^A)} \mathrm{Hom}(D, B^A).$$

In addition, we require that this data satisfies the following two coherence conditions:

(1) For any object D and arrow $k: D \to B^A$, the following diagram should commute:

$$\begin{tikzcd}
\varepsilon*(A\times k) \arrow[rr, Rightarrow, "="', "\varepsilon*(A\times k)"] \arrow[dr, Rightarrow, "\varepsilon*(A\times\zeta^{-1})"'] & & \varepsilon*(A\times k) \\
& \varepsilon*(A\times(\varepsilon*(A\times k))^{\sim}) \arrow[ur, Rightarrow, "\zeta_{\varepsilon*(A\times k)}"'] &
\end{tikzcd}$$

(1′) For any object D and arrow $h: A \times D \to B$, the following diagram should commute:

$$\begin{tikzcd}
h^{\sim} \arrow[rr, Rightarrow, "="', "h^{\sim}"] \arrow[dr, Rightarrow, "\zeta^{-1}_{h^{\sim}}"'] & & h^{\sim} \\
& (\varepsilon*(A\times h^{\sim}))^{\sim} \arrow[ur, Rightarrow, "(\zeta_h)^{\sim}"'] &
\end{tikzcd}$$

In both (1) and (1′), we have abbreviated as $A\times(\_)$ the unwieldy functor $\left\langle \pi_{A,D}, (\_)*\pi'_{A,D}\right\rangle^{A\times D}_{A,B^A}: \mathrm{Hom}(D, B^A) \to \mathrm{Hom}(A\times D, A\times B^A)$.

Again, contrary to what our notation may suggest, the products in the above definition are meant to be arbitrary (i.e. unspecified). The same goes for the object B^A, the evaluation arrow $\varepsilon_{A,B}: A\times B^A \to B$, etc. – there is no functional dependence on the objects A and B (for the time being!) – however, when we do specify exponentials shortly, the currently misleading notation will (hopefully!) be much easier to refer to. We observe that, like before, two objects may have several exponentials, or none at all. Lastly, the same convention we had about binary products regarding *diagrams*, *packages*, etc., applies here.

Virtually everything we said concerning the coherence conditions in the definition of product carries over – indeed, the coherence conditions just introduced are simply a translation of the triangle identities the unit and counit of a certain adjunction are known to satisfy. As a consequence, as expected (1) and (1′) are again logically equivalent, and both could moreover have been omitted altogether from the definition without changing it. For this reason, when in the next few propositions we are occasionally faced with the task of having to show that a certain set of data is an exponential package, we will just ignore the coherence conditions and pretend they never existed.

Propositions 2.6 and 2.7 consolidate the connection between the new notion and the old one. As expected, the proofs barely differ from their respective *vis-à-vis*. We only include the first one. Of course, all the remarks we made in the preceding paragraph apply here as well.

PROPOSITION 2.6. Given an exponential package of A and B as in the above definition, for any product package $A \xleftarrow{\pi_{A,D}} A \times D \xrightarrow{\pi'_{A,D}} D$ and arrow $h{:}A \times D \to B$, there are two 2-unique arrows $k{:}D \to B^A$ and $l{:}A \times D \to A \times B^A$ such that the following diagram commutes *up to 2-isomorphism*:

$$\begin{tikzcd}
& A \times B^A \arrow[dl, "\pi_{A,B^A}"'] \arrow[dr, "\varepsilon_{A,B}"] \arrow[rr, "\pi'_{A,B^A}"] & & B^A \\
A & & B & \\
& A \times D \arrow[ul, "\pi_{A,D}"] \arrow[uu, "l"] \arrow[ur, "h"'] \arrow[rr, "\pi'_{A,D}"'] & & D \arrow[uu, "k"']
\end{tikzcd}$$

PROOF. Obviously one may choose $k = h^{\sim}$ and $l = \langle \pi_{A,D}, h^{\sim}\pi'_{A,D} \rangle$ to show existence. Now assume that $\bar{k}{:}D \to B^A$ and $\bar{l}{:}A \times D \to A \times B^A$ constitute another solution. Because $A \times B^A$ is a product, this forces $\bar{l} \equiv \langle \pi_{A,D}, \bar{k}\pi'_{A,D} \rangle$. We also have $\varepsilon\langle \pi_{A,D}, \bar{k}\pi'_{A,D} \rangle \equiv h$. Combining these with the component at $\bar{k}$ of $\bar{\zeta}$, we get $\bar{k} \equiv \left(\varepsilon\langle \pi_{A,D}, \bar{k}\pi'_{A,D} \rangle\right)^{\sim} \equiv h^{\sim} = k$. From this we immediately deduce that $\bar{l} \equiv \langle \pi_{A,D}, \bar{k}\pi'_{A,D} \rangle \equiv \langle \pi_{A,D}, h^{\sim}\pi'_{A,D} \rangle = l$, which concludes the proof.

As expected, this last proposition's conclusion is strictly weaker than the definition, although only in a marginal sense. As for the next proposition, it validates our specifying exponentials in a moment.

PROPOSITION 2.7. Suppose that both packages below represent an exponential of A and B.

$$\begin{tikzcd}
 & A\times B^A \arrow[dl, "\pi_{A,B^A}"'] \arrow[dr, "\varepsilon_{A,B}"] \arrow[rr, "\pi'_{A,B^A}"] & & B^A \\
A & & B & \\
 & \overline{A\times B^A} \arrow[ul, "\overline{\pi_{A,B^A}}"] \arrow[ur, "\overline{\varepsilon_{A,B}}"'] \arrow[uu, dashed, "j"] \arrow[rr, "\overline{\pi'_{A,B^A}}"'] & & \overline{B^A} \arrow[uu, dashed, "i"']
\end{tikzcd}$$

Then there are 2-unique 1-isomorphisms $i:\overline{B^A} \xrightarrow{\cong} B^A$ and $j:\overline{A\times B^A} \xrightarrow{\cong} A\times B^A$ making everything commute.

The proposition below shows that our definition corresponds to that of [M3].

PROPOSITION 2.8. The following diagram is an exponential diagram

$$\begin{tikzcd}
 & A\times E \arrow[dl, "\pi_{A,E}"'] \arrow[dr, "e"] \arrow[rr, "\pi'_{A,E}"] & & E \\
A & & B &
\end{tikzcd}$$

(where $A \xleftarrow{\pi_{A,E}} A\times E \xrightarrow{\pi'_{A,E}} E$ is assumed to be a product diagram), if and only if the functor $e*\left(\pi_{A,D}, (\_)*\pi'_{A,D}\right):\mathrm{Hom}(D,E)\to\mathrm{Hom}(A\times D,B)$ is an equivalence of categories.

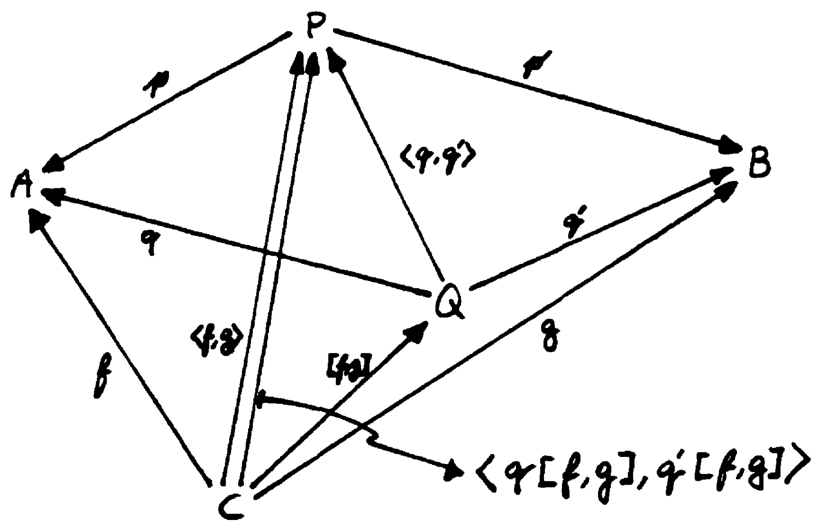
In lieu of proof, we will simply mention that, if the above functor is written F_D, then the functor G_D in the other direction is simply $(\_)^{\sim}$, and the unit and counit of the adjunction $F_D \dashv G_D$ are given respectively by $\eta =_{def} \tilde{\zeta}^{-1} : \mathrm{Id}_{\mathrm{Hom}(D,E)} \Rightarrow G_D F_D$ and $\varepsilon =_{def} \zeta$: $F_D G_D \Rightarrow \mathrm{Id}_{\mathrm{Hom}(A\times D,B)}$. The coherence conditions naturally correspond precisely to the two relevant triangle identities.

The next proposition also has a one-dimensional analogue, but this time it takes a little more work to carry the proof over. We will need this result to specify exponentials.

PROPOSITION 2.9. If the objects A and B have an exponential (denoted B^A), then any product diagram $A \xleftarrow{p} P \xrightarrow{p'} B^A$ of A and B^A can be completed into an exponential package.

The proof is easier to give once we are equipped with the following preliminary notation. Let X, Y, U and V be four objects and let $F, G\colon \mathrm{Hom}(X,Y) \to \mathrm{Hom}(U,V)$ be two functors. Furthermore, let $h\colon X \to Y$. We say that a 2-cell isomorphism $\beta\colon F(h) \Rightarrow G(h)$ is *natural in h* (which we will mark with the symbol $\equiv_h$) to express the fact that there is a (usually self-evident) natural isomorphism from F to G of which β is just the component at h. We will write $\equiv_{f,g}$ to say that the 2-isomorphism in question is natural in both f and g, etc.... We immediately observe that $\equiv_h$ is reflexive, symmetric and transitive. The following equations hold whenever well-defined: $(hg)f \equiv_{f,g,h} h(gf)$, $1_Y h \equiv_h h$ and $h1_X \equiv_h h$. $\equiv_h$ is also preserved by functors (such as horizontal composition) because functors preserve composition and isomorphisms. We note also that, if the functors F and G are constant (with respect to h), then any 2-isomorphism $\beta\colon F(h) \Rightarrow G(h)$ is automatically natural in h. Lastly, we record the following very useful property: let F, G, H be functors of the appropriate type on the Hom-categories, and suppose that it is given to us that $F(k) \equiv_k G(k)$. If now $k = H(h)$, then we may conclude that $F \circ H(h) \equiv_h G \circ H(h)$. We will usually abuse notation and write the ill-typed $F(H(h)) \equiv_{H(h)} G(H(h))$ to stand for $F(k) \equiv_k G(k)$. This property will very frequently be used (often tacitly) in contexts similar to that of the following example: from $((ph)g)f \equiv_{f,g,ph} (ph)(gf)$ (associativity isomorphism), we obtain $((ph)g)f \equiv_{f,g,h} (ph)(gf)$.

LEMMA 2.10. Let A and B be objects, and $A \xleftarrow{p} P \xrightarrow{p'} B$ (with pairing functors $\langle\_,\_\rangle$), $A \xleftarrow{q} Q \xrightarrow{q'} B$ (pairing functors: $[\_,\_]$) be product packages. Then, for any object C and arrows $f\colon C \to A$, $g\colon C \to B$, the following diagram commutes *up to 2-isomorphism natural in f and g*:



PROOF. Obviously, $q[f,g] \equiv_{f,g} f$ (def. of product) and $p\langle q,q'\rangle \equiv_{f,g} q$ (def. of product and constancy w/r to f, g). Since $[f,g] \equiv_{f,g} [f,g]$, by composing we get $p\langle q,q'\rangle[f,g] \equiv_{f,g} q[f,g]$, and hence by transitivity $p\langle q,q'\rangle[f,g] \equiv_{f,g} f$.[11] By symmetry, the corresponding result also holds on the other side of the diagram. This shows that in general (as we would expect), from the commutativity (up to natural 2-isomorphism) of the inner loops of a diagram, we can infer the commutativity (up to natural 2-isomorphism) of the outer loop. We now write $\langle f,g\rangle \equiv_{f,g} \langle p\langle q,q'\rangle[f,g], p'\langle q,q'\rangle[f,g]\rangle$. On the other hand, $\langle p\langle q,q'\rangle[f,g], p'\langle q,q'\rangle[f,g]\rangle \equiv_{\langle q,q'\rangle[f,g]} \langle q,q'\rangle[f,g]$ (def. of product), so $\langle p\langle q,q'\rangle[f,g], p'\langle q,q'\rangle[f,g]\rangle \equiv_{f,g} \langle q,q'\rangle[f,g]$. Combining, $\langle f,g\rangle \equiv_{f,g} \langle q,q'\rangle[f,g]$. Lastly, $\langle f,g\rangle \equiv_{f,g} \langle q[f,g], q'[f,g]\rangle$ is immediate. This completes the proof of the lemma.

PROOF of proposition 2.9. Let the pairing functors for $A \xleftarrow{p} P \xrightarrow{p'} B^A$ be written $[\_,\_]$. Let

$$\begin{tikzcd}
 & A \times B^A \arrow[dl, "\pi"'] \arrow[dr, "\varepsilon"] \arrow[rr, "\pi'"] & & B^A \\
A & & B &
\end{tikzcd}$$

be another exponential package exactly as in the definition (i.e. pairing functors $\langle\_,\_\rangle$, exponential functors $(\_)^{\sim}$, etc....). Let $e = \varepsilon\langle p,p'\rangle$ be the evaluation arrow for the exponential in the making. Given any object D and product diagram $A \xleftarrow{q} A \times D \xrightarrow{q'} D$, let the "new" exponential functor simply be the "old" one: $(\_)^{\sim}:\mathrm{Hom}(A \times D, B) \to \mathrm{Hom}(D, B^A)$. We claim that all this data constitutes an exponential diagram. What has to be checked is just the existence of the required natural 2-isomorphisms. So let $h: A \times D \to B$ be given. We know that $\varepsilon\langle q, h^{\sim}q'\rangle \equiv_h h$. On the other hand the lemma tells us that $\langle p,p'\rangle[q,h^{\sim}q'] \equiv_{q,h^{\sim}q'} \langle q,h^{\sim}q'\rangle$, from which we infer that $\langle p,p'\rangle[q,h^{\sim}q'] \equiv_h \langle q,h^{\sim}q'\rangle$. Combining everything, $e[q,h^{\sim}q'] = \varepsilon\langle p,p'\rangle[q,h^{\sim}q'] \equiv_h \varepsilon\langle q,h^{\sim}q'\rangle \equiv_h h$. The second natural isomorphism is obtained in a similar fashion – details are left to the reader. This completes the proof.

If the bicategory **C** has terminal objects, as well as binary products and exponentials for every pair of objects, we say that **C** is *cartesian closed*. Our old friend **Cat**, for instance, is a cartesian closed bicategory.

[11]We are omitting explicit mention of issues pertaining to associativity to alleviate the notation.

The time has come for the specification of particular instances of the various constructions that have just been laid out. The binary products and terminal objects do not pose any unexpected difficulties, and the specified notation we adopt for them is precisely that used in their respective definitions; thus, any pair of objects is assumed to have a unique product package associated to it – in particular, the natural isomorphisms existentially postulated in the definition are now an integral part of the data; and every instance of them is from now on designated a canonical 2-cell. Naturally, they do obey the coherence conditions. In the case of the exponential the situation is slightly more subtle: according to the definition, one must "look at" *all* possible product packages of some pair of objects (i.e. not just specified ones). This of course runs completely contrary to the very spirit of specification, and would also force an unnecessarily high number of 2-cells in the exponential to be designated canonical. We thus proceed as follows: given a bicategory $\mathbf{C}$ that has (specified) products and (unspecified) exponentials, for every pair of objects A and B, we pick an exponential B^A, we form the already specified product package $A \xleftarrow{\pi} A \times B^A \xrightarrow{\pi'} B^A$ (where the rest of the data is unnamed), and then we pick the rest of the data for the exponential (i.e. an evaluation arrow $\varepsilon: A \times B^A \to B$, etc.) in such a way that the resulting package is indeed exponential (that this may be done is precisely the contents of proposition 2.9). Again, for the purposes of determining which 2-cells will be marked as "canonical", we decree that only *specified* product packages ($A \times D$ etc.) shall be "looked at".

It would seem that we wind up, in essence, with two slightly different definitions for exponentials, the former appearing to be somewhat stronger than the latter. (Strictly speaking, this is not yet the case, since the last definition was stated on the assumption that exponentials (of the original kind) existed to start with. However, the re-engineered concept is but a tiny step away from a definition which would only assume the existence of specified products, and then sanction a diagram as exponential if, roughly speaking, it satisfied the requirements of the original definition, but only as far as *specified* products were concerned.) The next proposition clarifies the situation by showing that, whatever definition is used, nothing is lost; it also provides us with easier means to verify that a given diagram is indeed exponential.

PROPOSITION 2.11. Suppose that the diagram below satisfies the requirements of the definition of exponential *whenever tested with a* specified *product package* $A \xleftarrow{\pi_{A,D}} A \times D \xrightarrow{\pi'_{A,D}} D$. Then in fact it fully is an exponential diagram *as per* the (original) definition.

$$\begin{tikzcd}
 & A\times B^A \arrow[ld, "\pi_{A,B^A}"'] \arrow[rd, "\varepsilon"] \arrow[rr, "\pi'_{A,B^A}"] & & B^A \\
A & & B &
\end{tikzcd}$$

The main idea of the proof, of course, is to show that one can "compose" (in the mixed sense) the 1-isomorphism between an unspecified and a specified product of A and D with the appropriate canonical natural 2-isomorphisms of the exponential, in such a way that the resulting 2-cells are themselves natural 2-isomorphisms; put another way, these new 2-cells simply "factor through" the old ones without compromising the naturality. The principal ingredient is the contents of the following lemma:

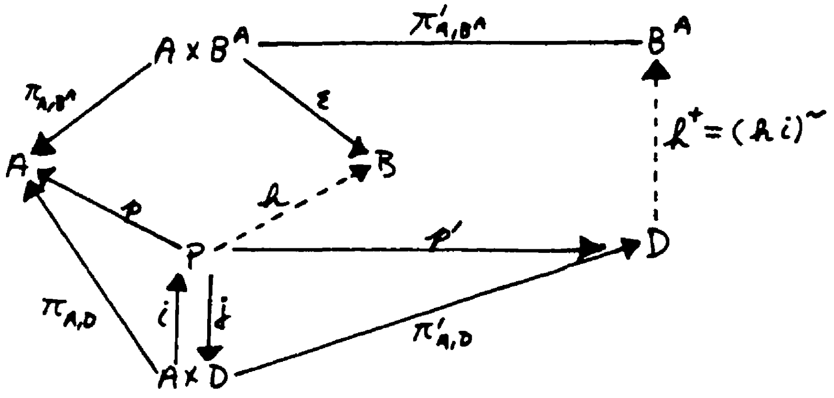
LEMMA 2.12. Let A and B be objects, with product $A \xleftarrow{\pi} A\times B \xrightarrow{\pi'} B$, etc.... Then, for any objects C and D and 1-cells $f{:}C \to A$, $g{:}C \to B$ and $k{:}D \to C$, we have that $\langle fk, gk\rangle^{D}_{A,B} \equiv_{f,g} \langle f,g\rangle^{C}_{A,B}k$ (in fact, naturality in k holds as well, but we won't need this).

PROOF. (In what follows the symbol $\langle\_,\_\rangle$ is used to denote both the functors $\langle\_,\_\rangle^{C}_{A,B}$ and $\langle\_,\_\rangle^{D}_{A,B}$.) We have $\pi\langle f,g\rangle \equiv_{f,g} f$ (def. of product), thus $\pi\langle f,g\rangle k \equiv_{f,g} fk$. We also have $\pi\langle fk,gk\rangle \equiv_{fk,gk} fk$, yielding $\pi\langle fk,gk\rangle \equiv_{f,g} fk$. By transitivity, it follows that $\pi\langle fk,gk\rangle \equiv_{f,g} \pi\langle f,g\rangle k$ (1a). Likewise, $\pi'\langle fk,gk\rangle \equiv_{f,g} \pi'\langle f,g\rangle k$ (1b). We also have $\langle fk,gk\rangle \equiv_{\langle fk,gk\rangle} \langle \pi\langle fk,gk\rangle, \pi'\langle fk,gk\rangle\rangle$ (def. of product), from which it follows that $\langle fk,gk\rangle \equiv_{f,g} \langle \pi\langle fk,gk\rangle, \pi'\langle fk,gk\rangle\rangle$ (2). Similarly (def. of product), since $\langle f,g\rangle k \equiv_{\langle f,g\rangle k}$ $\langle \pi(\langle f,g\rangle k), \pi'(\langle f,g\rangle k)\rangle$, we get $\langle f,g\rangle k \equiv_{f,g} \langle \pi(\langle f,g\rangle k), \pi'(\langle f,g\rangle k)\rangle$ (3). Putting everything together,

$$\begin{aligned}
\langle fk,gk\rangle &\equiv_{f,g} \langle \pi\langle fk,gk\rangle, \pi'\langle fk,gk\rangle\rangle \quad (2)\\
&\equiv_{f,g} \langle \pi\langle f,g\rangle k, \pi'\langle f,g\rangle k\rangle \quad \text{(by (1))}\\
&\equiv_{f,g} \langle f,g\rangle k \quad (3).
\end{aligned}$$

This completes the proof of the lemma.

PROOF of proposition 2.11. Let $A \xleftarrow{p} P \xrightarrow{p'} D$ be an arbitrary product diagram of A and D. Let $i{:}P \xrightarrow{\cong} A\times D$ and $j{:}A\times D \xrightarrow{\cong} P$ be the two 2-unique 1-isomorphisms between P and $A\times D$ preserving (up to 2-isomorphism) each other's projections, *as per* proposition 2.3. We recall that $ij \equiv 1_P$ and $ji \equiv 1_{A\times D}$. Define the functor $(\_)^+ : \mathrm{Hom}(P,B) \to \mathrm{Hom}(D,B^A)$ to be $(\_)^+ = ((\_)*i)^\sim$; i.e., for $h: P \to B$, $h^+ : D \to B^A$ is simply $(hi)^\sim$, etc.... (See diagram below.)



We must show that the new functor $(\_)^+$, together with the data given in the hypothesis, constitute an exponential. What must be ascertained is the existence of appropriate natural transformations. (In what follows we use $\langle\_,\_\rangle$ to denote either $\langle\_,\_\rangle^{P}_{A,B^A}$ or $\langle\_,\_\rangle^{A\times D}_{A,B^A}$.) Our choice of i and j entails $\pi_{A,D}j \equiv p$ and $\pi'_{A,D}j \equiv p'$, so certainly $\pi_{A,D}j \equiv_h p$ and $\pi'_{A,D}j \equiv_h p'$ (here h is assumed to be an arbitrary 1-cell $h: P \to B$). Hence we can write $\varepsilon\langle p,(hi)^\sim p'\rangle \equiv_h \varepsilon\langle \pi_{A,D}j,(hi)^\sim \pi'_{A,D}j\rangle$ (1). We now invoke lemma 2.12 to get $\langle \pi_{A,D}j,(hi)^\sim \pi'_{A,D}j\rangle \equiv_{\pi_{A,D},(hi)^\sim \pi'_{A,D}} \langle \pi_{A,D},(hi)^\sim \pi'_{A,D}\rangle j$. It easily follows that $\varepsilon\langle \pi_{A,D}j,(hi)^\sim \pi'_{A,D}j\rangle \equiv_h \varepsilon\langle \pi_{A,D},(hi)^\sim \pi'_{A,D}\rangle j$ (2). By hypothesis, $\varepsilon\langle \pi_{A,D},(hi)^\sim \pi'_{A,D}\rangle \equiv_{hi} hi$, thus $\varepsilon\langle \pi_{A,D},(hi)^\sim \pi'_{A,D}\rangle j \equiv_h hij$ (3). Of course since $ij \equiv 1_P$, $hij \equiv_h h$ (4). Now we string everything together:

$$\begin{aligned}
\varepsilon\langle p,h^+ p'\rangle &= \varepsilon\langle p,(hi)^\sim p'\rangle \quad &&(\text{def. of } (_)^+)\\
&\equiv_h \varepsilon\langle \pi_{A,D}j,(hi)^\sim \pi'_{A,D}j\rangle &&(1)\\
&\equiv_h \varepsilon\langle \pi_{A,D},(hi)^\sim \pi'_{A,D}\rangle j &&(2)\\
&\equiv_h hij \equiv_h h &&(3,4).
\end{aligned}$$

This establishes the existence of one of the required natural transformations. The second one can now be obtained in a matter of a few additional lines – we leave it to the conscientious reader. This completes proposition 2.11.

We now state our main coherence conjecture. We believe the best way to do this is to invoke free cartesian closed bicategories, even though these are only introduced in chapter 5. We hope the reader will forgive this small glitch in the interest of the conjecture being stated where, far and away, it "morally" belongs...

CONJECTURE. In free cartesian closed bicategory on an empty set of 2-cells, there is at most one 2-cell between any pair of 1-cells.

(This conjecture also has a sibling which considers the case where the bicategory is only cartesian.) We can briefly paraphrase the conjecture in the style of [MP] as follows: assume we are given a cartesian closed bicategory **C** which has the property that "no objects or 1-cells are identified that are not of necessity the same[12]". (In practice, as in [MP], one would just construct such a bicategory from **C** via a straightforward inductive process, in such a way that the new bicategory is "for all practical purposes" equivalent to **C**.) It is also clear what *generalized canonical* (*g-canonical*) should now mean; namely, any element of the smallest set of 2-cells containing all identity and canonical 2-cells, and closed under vertical composition as well as all the relevant functors (such as pairing, etc.). The statement is then that there is at most one g-canonical 2-cell between any pair of 1-cells in **C**. The reader should find it very simple to verify that the two formulations are indeed equivalent after having looked at chapter 5.

Our research into this conjecture has been limited to verifying manually a small number of plausible 2-cell equations. Needless to say, there have been no counter-examples so far, and in fact most equations are relatively easy to deal with. However, it seems pretty clear that the much talked-about coherence conditions of the product and the exponential are vitally needed. At any rate, this conjecture is definitely a very interesting puzzle which no doubt warrants further investigation. It is conceivable that an inductive proof in the style of [MP] could be devised, but it would most likely be substantially more complex.

The time has now come to examine the types of maps one wants to have between bicategories. It turns out there is quite a wide range of possibilities (hardly surprising...), some more useful, some more natural, than others. We will essentially focus on two notions, that of *strict homomorphism* and that of (plain) *homomorphism*. The basic ideas are essentially borrowed from Bénabou [B2], although we have made some small changes.

[12] We are aware that this is not, in its current form, a well-defined statement! It can be made precise however, but this requires significantly more space. Please consult [MP] for details.

Given two (not necessarily cartesian closed) bicategories **C** and **D**, the usual devices of universal algebra yield the following natural notion of morphism F between **C** and **D**:

(i) A set-map $F_0:\mathrm{Ob}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{D})$.

(ii) For each pair (A, B) of objects of **C**, a functor $F_1^{A,B}:\mathrm{Hom}(A,B) \to \mathrm{Hom}(F_0(A), F_0(B))$ preserving identity 1-cells, the composition functors as well as the three canonical isomorphisms, i.e. such that

$$F_1^{A,A}(1_A) = 1_{F_0(A)},$$

$$F_1^{A,C}(g*f) = F_1^{B,C}(g)*F_1^{A,B}(f) \text{ and } F_1^{A,C}(\gamma*\beta) = F_1^{B,C}(\gamma)*F_1^{A,B}(\beta)$$
$$\text{whenever } \beta:f \Rightarrow f':A \to B,\ \gamma:g \Rightarrow g':B \to C, \text{ and}$$

$$F_1^{A,D}\left(\alpha_{f,g,h}^{A,B,C,D}\right) = \alpha_{F_1^{A,B}(f),F_1^{B,C}(g),F_1^{C,D}(h)}^{F_0(A),F_0(B),F_0(C),F_0(D)},\ F_1^{A,B}\left(\lambda_f^{A,B}\right) = \lambda_{F_1^{A,B}(f)}^{F_0(A),F_0(B)} \text{ and}$$
$$F_1^{A,B}\left(\rho_f^{A,B}\right) = \rho_{F_1^{A,B}(f)}^{F_0(A),F_0(B)},$$
$$\text{whenever } A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D.$$

Such a morphism will be called a *strict homomorphism* (of bicategories). It is obvious how to compose two such strict homomorphisms, and it is easy to see that bicategories and strict homomorphisms give rise to a category **Bicat**.

There is, however, another approach possible, which in some sense is more in synch with the philosophy we have so far been following. It rests with the idea that, as far as possible, we never try to identify 1-cells, but rather only ask that there be natural isomorphisms recording what should have otherwise been an equality. Of course, one might be justified in questioning whether this principle should in fact be taken that far – after all, we are *not* adding anything to the construction of bicategories proper anymore, but only trying to define maps between them. And in this respect, the definition of strict homomorphism above parallels those of a vast number of algebraic structures. But, as pointed out by Bénabou, the justification for the proposed alternative "lax" approach to functoriality lies in the wealth of (specific) mathematical examples in which the "weaker" maps are the ones actually present, as well as in the fact that all of Bénabou's "desired results" hold in the more general context. It is less so in our case, but it certainly seems that the notion is of sufficient importance and interest to warrant being studied in its own right. (Actually, Bénabou's main notion of morphism between bicategories is even weaker than the one we will use; this, and other possible variations, will be considered in chapter 6. There is another difference between the concept presented below and Bénabou's, on which we will comment after having given the definition.)

Given two (not necessarily cartesian closed) bicategories **C** and **D**, a *homomorphism* F between **C** and **D** consists in

(i) A set-map $F_0:\mathrm{Ob}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{D})$.

(ii) For each pair (A, B) of objects of **C**, a functor $F_1^{A,B}:\mathrm{Hom}(A,B) \to \mathrm{Hom}(F_0(A), F_0(B))$, such that:

(1) For each object A of **C**, there exists an isomorphism $\varphi^A : F_1^{A,A}(1_A) \Rightarrow 1_{F_0(A)}$ (in **D**).[13]

(2) For every triple (A,B,C) of objects of **C**, there exists a natural isomorphism $\ddot{\varphi}^{A,B,C}$ (i.e., a set of 2-cells in **D** satisfying the required naturality conditions) between the following two functors:

$$\mathrm{Hom}(A,B)\times\mathrm{Hom}(B,C) \xrightarrow{\;\bullet\;} \mathrm{Hom}(A,C) \xrightarrow{F_1^{A,C}} \mathrm{Hom}(F_0(A),F_0(C))$$
$$[(f,g) \mapsto gf \mapsto F_1(gf)]$$
$$\Downarrow \ddot{\varphi}^{A,B,C}$$
$$[(f,g) \mapsto (F_1(f), F_1(g)) \mapsto F_1(g)F_1(f)]$$
$$\mathrm{Hom}(A,B)\times\mathrm{Hom}(B,C) \xrightarrow{F_1^{A,B}\times F_1^{B,C}} \ldots$$
$$\ldots \to \mathrm{Hom}(F_0(A),F_0(B))\times\mathrm{Hom}(F_0(B),F_0(C)) \xrightarrow{\;\bullet\;} \mathrm{Hom}(F_0(A),F_0(C)).$$

In addition, this data is required to satisfy the following coherence conditions:

[13] Of course, (1) can be restated as a requirement that there be a natural isomorphism between two obvious functors from 1 to $\mathrm{Hom}(F_0(A), F_0(A))$. There is little point to complicate things unnecessarily however.

(3) For any objects and 1-cells $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, the following diagram should commute:

$$\begin{tikzcd}
F_1^{A,D}(h(gf)) \arrow[r, Rightarrow, "F_1^{A,D}(\alpha)"] \arrow[d, Rightarrow, "\ddot{\varphi}^{A,C,D}_{gf,h}"'] & F_1^{A,D}((hg)f) \arrow[d, Rightarrow, "\ddot{\varphi}^{A,B,D}_{f,hg}"] \\
F_1^{C,D}(h)\,F_1^{A,C}(gf) \arrow[d, Rightarrow, "F_1^{C,D}(h)\ddot{\varphi}^{A,B,C}_{f,g}"'] & F_1^{B,D}(hg)\,F_1^{A,B}(f) \arrow[d, Rightarrow, "\ddot{\varphi}^{B,C,D}_{g,h}\,F_1^{A,B}(f)"] \\
F_1^{C,D}(h)\left(F_1^{B,C}(g)F_1^{A,B}(f)\right) \arrow[r, Rightarrow, "\alpha"] & \left(F_1^{C,D}(h)F_1^{B,C}(g)\right)F_1^{A,B}(f)
\end{tikzcd}$$

(4) For any objects and 1-cell $A \xrightarrow{f} B$, the following two diagrams should commute:

$$\begin{tikzcd}
F_1^{A,B}(1_B f) \arrow[r, Rightarrow, "F_1^{A,B}(\lambda)"'] \arrow[d, Rightarrow, "\ddot{\varphi}^{A,B,B}_{f,1_B}"] & F_1^{A,B}(f) \\
F_1^{B,B}(1_B)\,F_1^{A,B}(f) \arrow[r, Rightarrow, "\varphi^{B}_{F_1^{A,B}(f)}"] & 1_{F_0(B)}\,F_1^{A,B}(f) \arrow[u, Rightarrow, "\lambda"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F_1^{A,B}(f 1_A) \arrow[r, Rightarrow, "F_1^{A,B}(\rho)"'] \arrow[d, Rightarrow, "\ddot{\varphi}^{A,A,B}_{1_A,f}"] & F_1^{A,B}(f) \\
F_1^{A,B}(f)\,F_1^{A,A}(1_A) \arrow[r, Rightarrow, "F_1^{A,B}(f)\varphi^{A}"] & F_1^{A,B}(f)\,1_{F_0(A)} \arrow[u, Rightarrow, "\rho"']
\end{tikzcd}$$

We pause to make the point that the families of natural isomorphisms identified in conditions (1) and (2) (and which we will abbreviate respectively as φ and $\ddot{\varphi}$), *are not actually "part" of the homomorphism* – we only require *that they should exist.* We stress this because we are actually departing here from Bénabou's definition [B2], in which φ and $\ddot{\varphi}$ actually form an integral part of F. We are conscious of the fact that this deviation might jeopardize a whole body of results and constructs (presumably the "desired results" Bénabou was referring to). Indeed, our notion is somewhat hybrid, being somewhere between the two extremes of making the definition as strict as possible, and the opposite. But we should emphasize that in fact, our preferred notion of morphism is the first one we put forward, namely that of strict homomorphism: we will see in chapter 5 that this is what gets us the nicest results. We insist on considering other possibilities, however, as an exploratory and "litmus-test" tool; in fact if it weren't for considerations of space, we would gladly investigate all sensible alternatives to the paths we are following. But in this case, and for our purposes, the notion of homomorphism we have chosen seems more useful because, contrary to Bénabou's, it does not force one to distinguish between two homomorphisms who behave exactly the same on objects, 1-cells and 2-cells, if their associated φ and $\ddot{\varphi}$ are different. For example, jumping ahead, we can say that our definition allows us to devise a faithful functor between the resulting category of bicategories and homomorphisms, and the category of bicategory presentations (see chapter 5); this result would unfortunately not hold if we had strictly followed Bénabou's definition.

Before moving on, let us give warning that, when one specifies a homomorphism, one usually implicitly has in mind some particular families φ and $\ddot{\varphi}$; we will at times therefore abuse notation, and speak of a homomorphism $(F,\varphi,\ddot{\varphi})$ instead of just F.

It is obvious what the identity homomorphism on a bicategory $\mathbf{C}$ should be. Composition is performed as follows: if $F=(F,\varphi,\ddot{\varphi})\colon\mathbf{C}\to\mathbf{D}$ and $F'=(F',\varphi',\ddot{\varphi}')\colon\mathbf{D}\to\mathbf{E}$ are two homomorphisms, their composite $F'\circ F = F'F = G = (G,\psi,\ddot{\psi})\colon\mathbf{C}\to\mathbf{E}$ is given by

(i) $G_0 = F_0'\circ F_0$.

(ii) For each pair (A,B) of objects of $\mathbf{C}$, $G_1^{A,B} = F_1'^{A,B}\circ F_1^{A,B}$.

(1) For each object A of $\mathbf{C}$, $\psi^A = \varphi'^{F_0(A)}\circ F_1'^{F_0(A),F_0(A)}\left(\varphi^A\right)$.

(2) For every diagram $A\xrightarrow{f}B\xrightarrow{g}C$ in $\mathbf{C}$, $\ddot{\psi}_{f,g}^{A,B,C} = \ddot{\varphi}'^{F_0(A),F_0(B),F_0(C)}_{F_1^{A,B}(f),F_1^{B,C}(g)}\circ \mathrm{F}_1'^{F_0(A),F_0(C)}\left(\ddot{\varphi}_{f,g}^{A,B,C}\right)$ as a component of $\ddot{\psi}^{A,B,C}$.

The following proposition ensures that all of this is well-defined, and that moreover the resulting things can be organized into a category:

PROPOSITION 2.13. *G* as defined above is indeed a homomorphism, and with this composition, bicategories and homomorphisms form a category, denoted $\mathbf{Bicat'}$.

The proof of this, actually in slightly greater generality, can be found in [B2]. It is not difficult by any means, although it does give rise to some very "juicy" diagrams.

So we now have two notions of map between bicategories. We remark, incidentally, that the second subsumes the first, in that a strict homomorphism is simply a homomorphism for which the natural transformations φ and $\ddot{\varphi}$ are identities, hence that $\mathbf{Bicat}$ is a (non-full) subcategory of $\mathbf{Bicat'}$; we will denote the associated faithful inclusion functor

i^b:**Bicat** $\to$ **Bicat$'$**. Our next goal is naturally to extend these concepts to cartesian closed bicategories. Again, we start with strict homomorphisms:

Given two cartesian closed bicategories **C** and **D** (with specified operations), a *cartesian closed strict homomorphism* F:**C** $\to$ **D** is a strict homomorphism which in addition preserves all (specified) product, terminal and exponential packages. Equationally, this means that the following always hold:

$$F_0(A\times B)=F_0(A)\times F_0(B),$$

$$F_1^{A\times B,A}\left(\pi_{A,B}\right)=\pi_{F_0(A),F_0(B)} \text{ and } F_1^{A\times B,A}\left(\pi'_{A,B}\right)=\pi'_{F_0(A),F_0(B)},$$

$$F_1^{C,A\times B}\left(\langle f,g\rangle\right)=\left\langle F_1^{C,A}(f),F_1^{C,B}(g)\right\rangle \text{ and } F_1^{C,A\times B}\left(\langle\beta,\gamma\rangle\right)=\left\langle F_1^{C,A}(\beta),F_1^{C,B}(\gamma)\right\rangle,$$

$$F_1^{C,A}\left(\tau_{f,g}^{A,B,C}\right)=\tau_{F_1^{C,A}(f),F_1^{C,B}(g)}^{F_0(A),F_0(B),F_0(C)},\ F_1^{C,B}\left(\tau'^{A,B,C}_{f,g}\right)=\tau'^{F_0(A),F_0(B),F_0(C)}_{F_1^{C,A}(f),F_1^{C,B}(g)}, \text{ and}$$

$$F_1^{C,A\times B}\left(\overline{\tau}_h^{A,B,C}\right)=\overline{\tau}_{F_1^{C,A\times B}(h)}^{F_0(A),F_0(B),F_0(C)},$$

whenever $\beta{:}f\Rightarrow f'{:}C\to A,\ \gamma{:}g\Rightarrow g'{:}C\to B,$

$$F_0(\mathbf{t})=\mathbf{t},\ F_1^{A,\mathbf{t}}(!_A)=!_{F_0(A)} \text{ and } F_1^{A,\mathbf{t}}\left(\xi_f^A\right)=\xi_{F_1^{A,\mathbf{t}}(f)}^{F_0(A)} \text{ whenever } A\xrightarrow{f}\mathbf{t}, \text{ and}$$

$$F_0\left(B^A\right)=F_0(B)^{F_0(A)},\ F_1^{A\times B^A}\left(\varepsilon_{A,B}\right)=\varepsilon_{F_0(A),F_0(B)},$$

$$F_1^{D,B^A}(h^{\sim})=\left(F_1^{A\times D,B}(h)\right)^{\sim} \text{ and } F_1^{D,B^A}(\beta^{\sim})=\left(F_1^{A\times D,B}(\beta)\right)^{\sim},$$

$$F_1^{A\times D,B}\left(\zeta_h^D\right)=\zeta_{F_1^{A\times D,B}(h)}^{F_0(D)} \text{ and } F_1^{D,B^A}\left(\tilde{\zeta}_k^D\right)=\tilde{\zeta}_{F_1^{D,B^A}(k)}^{F_0(D)},$$

whenever $\beta{:}h\Rightarrow h'{:}A\times D\to B,\ \gamma{:}k\Rightarrow k'{:}D\to B^A.$

Clearly, the composition of two cartesian closed strict homomorphisms is again cartesian closed. We thus have the category **CCBiC** of cartesian closed bicategories and cartesian closed strict homomorphisms. It is of course a subcategory of **Bicat**, but not a full one.

Now, to extend cartesian closedness to homomorphisms, we will require that the pairing functor, the exponential functor and the various distinguished 1-cells be preserved only up to natural isomorphism. Of course, for this it must be the case that product, exponential and terminal *objects* be strictly preserved.

Given two cartesian closed bicategories **C** and **D** (with specified operations), a *cartesian closed homomorphism* F consists of a homomorphism $(F,\varphi,\ddot{\varphi})\colon \mathbf{C} \to \mathbf{D}$ satisfying $F_0(A \times B) = F_0(A) \times F_0(B)$, $F_0(\mathbf{t}) = \mathbf{t}$ and $F_0(B^A) = F_0(B)^{F_0(A)}$ for all objects A, B of **C**. In addition, we require that:

(1) For every pair (A,B) of objects of **C**, there exist isomorphisms $\hat{\tau}^{A,B} : F_1^{A\times B,A}(\pi_{A,B}) \Rightarrow \pi_{F_0(A),F_0(B)}$ and $\hat{\tau}'^{A,B} : F_1^{A\times B,B}(\pi'_{A,B}) \Rightarrow \pi'_{F_0(A),F_0(B)}$.

(2) For every triple (A,B,C) of objects of **C**, there exists a natural isomorphism $\hat{\hat{\tau}}^{A,B,C}$ between the following two functors:

$$\mathrm{Hom}(C,A)\times\mathrm{Hom}(C,B) \xrightarrow{\langle -,- \rangle} \mathrm{Hom}(C,A\times B) \xrightarrow{F_1^{C,A\times B}} \mathrm{Hom}(F_0(C),F_0(A\times B))$$

$$[(f,g)\mapsto \langle f,g\rangle \mapsto F_1(\langle f,g\rangle)]$$

$$\Downarrow \hat{\hat{\tau}}^{A,B,C}$$

$$[(f,g)\mapsto (F_1(f),F_1(g)) \mapsto \langle F_1(f),F_1(g)\rangle]$$

$$\mathrm{Hom}(C,A)\times\mathrm{Hom}(C,B) \xrightarrow{F_1^{C,A}\times F_1^{C,B}} \ldots$$

$$\ldots \to \mathrm{Hom}(F_0(C),F_0(A))\times\mathrm{Hom}(F_0(C),F_0(B)) \xrightarrow{\langle -,- \rangle} \mathrm{Hom}(F_0(C),F_0(A\times B)).$$

(3) For every object A of **C**, there exists an isomorphism $\hat{\xi}^A : F_1^{A,\mathbf{t}}(!_A) \Rightarrow\, !_{F_0(A)}$.

(4) For every pair (A,B) of objects of **C**, there exists an isomorphism $\hat{\zeta}^{A,B} : F_1^{A\times B^A,B}(\varepsilon_{A,B}) \Rightarrow \varepsilon_{F_0(A),F_0(B)}$.

(5) For every triple (A,B,C) of objects of **C**, there exists a natural isomorphism $\hat{\hat{\zeta}}^{A,B,C}$ between the following two functors:

$$\mathrm{Hom}(A\times D,B) \xrightarrow{(\ )^{\sim}} \mathrm{Hom}(D,B^A) \xrightarrow{F_1^{D,B^A}} \mathrm{Hom}(F_0(D),F_0(B)^{F_0(A)})$$

$$[h\mapsto h^{\sim} \mapsto F_1(h^{\sim})]$$

$$\Downarrow \hat{\hat{\zeta}}^{A,B,D}$$

$$[h\mapsto F_1(h^{\sim}) \mapsto (F_1(h))^{\sim}]$$

$$\mathrm{Hom}(A\times D,B) \xrightarrow{F_1^{A\times D,B}} \mathrm{Hom}(F_0(A\times D),F_0(B)) \xrightarrow{(\ )^{\sim}} \mathrm{Hom}(F_0(D),F_0(B)^{F_0(A)}).$$

Again, we insist that the numerous natural isomorphisms identified in the definition above do not actually form part of the cartesian closed homomorphism; i.e., a cartesian closed homomorphism is simply a homomorphism satisfying some additional properties. However, we will at times write $F = \left(F, \varphi, \ddot{\varphi}, \hat{\tau}, \widetilde{\tau'}, \hat{\ddot{\tau}}, \hat{\xi}, \hat{\zeta}, \hat{\dot{\zeta}} \right)$ to designate a particular cartesian closed homomorphism, especially when we have in mind some particular natural transformations.

We have already defined composition of homomorphisms; the fact that this operation preserves cartesian closedness is a consequence of the well-known facts that functors and categorical composition both preserve isomorphisms and natural transformations. We thus have the category **CCBiC′** of cartesian closed bicategories and cartesian closed homomorphisms. Naturally, this is a non-full subcategory of **Bicat′**. The restriction of the functor i^b:**Bicat** $\rightarrow$ **Bicat′** to **CCBiC** gives us another faithful functor (also denoted) i^b:**Bicat** $\rightarrow$ **Bicat′**. Unfortunately, it is rather difficult to manufacture a sensible functor in the other direction.

If one compares the above definitions with their one-dimensional counterpart, one sees that what we have just introduced corresponds to the notion of *strict* cartesian closed functor. Of course, it would have been possible also to devise an analogue to the non-strict version.

Chapter 3

Structured Bicategories as Models of Generalised Algebraic Theories

In the preceding chapter we introduced the notions of bicategory, bicategory with binary products, cartesian closed bicategory, and so forth. The definitions we gave themselves relied on the concepts of category, functor, natural transformation, etc.... The point that we want to make here is that a purely algebraic (equational) formulation could have been given each time, at least in the case where instances of the eventual additional structure are specified. More precisely, we will show that bicategories, bicategories with (specified) products, ..., and cartesian closed bicategories (with specified products, terminal object and exponentials), can all be construed as models of generalised algebraic theories, in the sense of Cartmell [Ca]. In what follows we directly recast the definition of cartesian closed bicategory (with specified operation) in Cartmell's framework, using his notational style throughout. One may recover the various other versions (such as cartesian bicategories, etc.) by appropriately deleting certain rules.

We first give a brief synopsis of the Cartmell formalism – for the full story the reader is referred, of course, to [Ca]. According to Cartmell, "a *generalised algebraic theory*[1] consists of (i) a set of sorts, each with a specified role either as a constant type or else as a variable type varying in some way, (ii) a set of operator symbols, each one with its argument type and its value type specified (the value type may vary as the argument varies), (iii) a set of axioms. Each axiom must be an identity between similar well-formed expressions, either between terms of the same possibly varying type or else between type expressions." We write $t \in \Delta$ to assert the fact that the term t is of type Δ. Rules for constructing types or rules asserting that a given term is of a particular type are always given by a pair (Premisses, Conclusion) in which "Premisses" is a (possibly empty) set of assumptions that certain variables are of a certain type, and "Conclusion" is the assertion that some symbol is to stand for a type, or that some term is of a particular type. The rest of the syntax (including exactly how the pair (Premisses, Conclusion) is to be represented) is rather self-explanatory. Finally, a *model* of a given generalised algebraic theory is a model in the usual sense, i.e. where sorts are interpreted as sets or families of sets,

[1] Emphasis is ours.

and operator symbols as operators on these, in such a way that all axioms become true statements.

The theory of bicategories:

Symbol	***Introductory Rule(s)***
Ob	Ob is a type.
Hom	$A,B\in \mathrm{Ob}:\mathrm{Hom}(A,B)$ is a type.
2Hom	$A,B\in \mathrm{Ob}, f,g\in \mathrm{Hom}(A,B): \mathrm{2Hom}(f,g)$ is a type.
$\circ$	$\dfrac{A,B\in \mathrm{Ob}, f,g,h\in \mathrm{Hom}(A,B), \beta\in \mathrm{2Hom}(f,g), \gamma\in \mathrm{2Hom}(g,h)}{\circ(\beta,\gamma)\in \mathrm{2Hom}(f,h).}$
Id	$A,B\in \mathrm{Ob}, f\in \mathrm{Hom}(A,B): \mathrm{Id}(f)\in \mathrm{2Hom}(f,f).$
$*$	$A,B,C\in \mathrm{Ob}, f\in \mathrm{Hom}(A,B), g\in \mathrm{Hom}(B,C): *(f,g)\in \mathrm{Hom}(A,C);$ $\dfrac{\left\{\begin{matrix} A,B,C\in \mathrm{Ob}, f,f'\in \mathrm{Hom}(A,B), g,g'\in \mathrm{Hom}(B,C), \\ \beta\in \mathrm{2Hom}(f,f'), \gamma\in \mathrm{2Hom}(g,g') \end{matrix}\right\}}{*(\beta,\gamma)\in \mathrm{2Hom}(*(f,g),*(f',g')).}$
1	$A\in \mathrm{Ob}: 1(A)\in \mathrm{Hom}(A,A).$
α	$\dfrac{A,B,C,D\in \mathrm{Ob}, f\in \mathrm{Hom}(A,B), g\in \mathrm{Hom}(B,C), h\in \mathrm{Hom}(C,D)}{\alpha(f,g,h)\in \mathrm{2Hom}\big(*(*(f,g),h),*(f,*(g,h))\big).}$
α^{-1}	$\dfrac{A,B,C,D\in \mathrm{Ob}, f\in \mathrm{Hom}(A,B), g\in \mathrm{Hom}(B,C), h\in \mathrm{Hom}(C,D)}{\alpha^{-1}(f,g,h)\in \mathrm{2Hom}\big(*(f,*(g,h)),*(*(f,g),h)\big).}$
λ	$A,B\in \mathrm{Ob}, f\in \mathrm{Hom}(A,B): \lambda(f)\in \mathrm{2Hom}\big(*(f,1(B)),f\big).$
λ^{-1}	$A,B\in \mathrm{Ob}, f\in \mathrm{Hom}(A,B): \lambda^{-1}(f)\in \mathrm{2Hom}\big(f,*(f,1(B))\big).$
ρ	$A,B\in \mathrm{Ob}, f\in \mathrm{Hom}(A,B): \rho(f)\in \mathrm{2Hom}\big(*(1(A),f),f\big).$
ρ^{-1}	$A,B\in \mathrm{Ob}, f\in \mathrm{Hom}(A,B): \rho^{-1}(f)\in \mathrm{2Hom}\big(f,*(1(A),f)\big).$

Axioms

• <u>Properties of the identity arrow</u>:

$\circ\big(\mathrm{Id}(f),\beta\big)=\beta,$ whenever $A,B\in \mathrm{Ob}, f,g\in \mathrm{Hom}(A,B)$ and $\beta\in \mathrm{2Hom}(f,g)$.

$\circ\big(\beta,\mathrm{Id}(g)\big)=\beta,$ whenever $A,B\in \mathrm{Ob}, f,g\in \mathrm{Hom}(A,B)$ and $\beta\in \mathrm{2Hom}(f,g)$.

• <u>Associativity of composition</u>:

$\circ\big(\circ(\beta,\gamma),\varphi\big)=\circ\big(\beta,\circ(\gamma,\varphi)\big),$ whenever $A,B\in \mathrm{Ob}, f,g,h,k\in \mathrm{Hom}(A,B), \beta\in \mathrm{2Hom}(f,g),$ $\gamma\in \mathrm{2Hom}(g,h)$ and $\varphi\in \mathrm{2Hom}(h,k)$.

• Functoriality of horizontal composition:

$*(\mathrm{Id}(f),\mathrm{Id}(g))=\mathrm{Id}(*(f,g))$, whenever $A,B,C\in\mathrm{Ob}, f\in\mathrm{Hom}(A,B)$ and $g\in\mathrm{Hom}(B,C)$.

$*(\circ(\beta,\beta'),\circ(\gamma,\gamma'))=\circ(*(\beta,\gamma),*(\beta',\gamma'))$, whenever $A,B,C\in\mathrm{Ob}, f,f',f''\in\mathrm{Hom}(A,B)$, $g,g',g''\in\mathrm{Hom}(B,C),\beta\in 2\mathrm{Hom}(f,f'),\beta'\in 2\mathrm{Hom}(f',f'')$, $\gamma\in 2\mathrm{Hom}(g,g')$ and $\gamma'\in 2\mathrm{Hom}(g',g'')$.

• Naturality of α:

$\circ(\alpha(f,g,h),*(\beta,*(\gamma,\varphi)))=\circ(*(*(\beta,\gamma),\varphi),\alpha(f',g',h'))$, whenever $A,B,C,D\in\mathrm{Ob}$, $f,f'\in\mathrm{Hom}(A,B),g,g'\in\mathrm{Hom}(B,C),h,h'\in\mathrm{Hom}(C,D)$, $\beta\in 2\mathrm{Hom}(f,f'),\gamma\in 2\mathrm{Hom}(g,g')$ and $\varphi\in 2\mathrm{Hom}(h,h')$.

• Invertibility of α:

$\circ(\alpha(f,g,h),\alpha^{-1}(f,g,h))=\mathrm{Id}(*(*(f,g),h))$, whenever $A,B,C,D\in\mathrm{Ob},f\in\mathrm{Hom}(A,B)$, $g\in\mathrm{Hom}(B,C)$ and $h\in\mathrm{Hom}(C,D)$.

$\circ(\alpha^{-1}(f,g,h),\alpha(f,g,h))=\mathrm{Id}(*(f,*(g,h)))$, whenever $A,B,C,D\in\mathrm{Ob},f\in\mathrm{Hom}(A,B)$, $g\in\mathrm{Hom}(B,C)$ and $h\in\mathrm{Hom}(C,D)$.

• Naturality of λ:

$\circ(\lambda(f),\beta)=\circ(*(\beta,\mathrm{Id}(1(B))),\lambda(f'))$, whenever $A,B\in\mathrm{Ob},f,f'\in\mathrm{Hom}(A,B)$ and $\beta\in 2\mathrm{Hom}(f,f')$.

• Invertibility of λ:

$\circ(\lambda(f),\lambda^{-1}(f))=\mathrm{Id}(*(f,1(B)))$, whenever $A,B\in\mathrm{Ob}$ and $f\in\mathrm{Hom}(A,B)$.

$\circ(\lambda^{-1}(f),\lambda(f))=\mathrm{Id}(f)$, whenever $A,B\in\mathrm{Ob}$ and $f\in\mathrm{Hom}(A,B)$.

• Naturality of ρ:

$\circ(\rho(f),\beta)=\circ(*(\mathrm{Id}(1(A)),\beta),\rho(f'))$, whenever $A,B\in\mathrm{Ob},f,f'\in\mathrm{Hom}(A,B)$ and $\beta\in 2\mathrm{Hom}(f,f')$.

• Invertibility of ρ:

$\circ(\rho(f),\rho^{-1}(f))=\mathrm{Id}(*(1(A),f))$, whenever $A,B\in\mathrm{Ob}$ and $f\in\mathrm{Hom}(A,B)$.

$\circ(\rho^{-1}(f),\rho(f))=\mathrm{Id}(f)$, whenever $A,B\in\mathrm{Ob}$ and $f\in\mathrm{Hom}(A,B)$.

• Coherence conditions:

$\circ(*(\alpha(f,g,h),\mathrm{Id}(k)),\circ(\alpha(f,*(g,h),k),*(\mathrm{Id}(f),\alpha(g,h,k))))=$
$\circ(\alpha(*(f,g),h,k),\alpha(f,g,*(h,k)))$, whenever $A,B,C,D,E\in\mathrm{Ob},f\in\mathrm{Hom}(A,B)$, $g\in\mathrm{Hom}(B,C),h\in\mathrm{Hom}(C,D)$ and $k\in\mathrm{Hom}(D,E)$.

$\circ(\alpha(f,1(B),g),*(\mathrm{Id}(f),\rho(g)))=*(\lambda(f),\mathrm{Id}(g))$, whenever $A,B,C\in\mathrm{Ob}$, $f\in\mathrm{Hom}(A,B)$ and $g\in\mathrm{Hom}(B,C)$.

Models of the above theory are easily seen upon inspection to be in one-to-one correspondence with bicategories as defined in the previous chapter. We should perhaps point out that our persistent use of prefix notation, which some readers might find annoying, is intended, among other things, to emphasize the omnipresent functional dependence of the various "things" we are dealing with on one another. We have however relaxed this commitment on occasion when expressing the next few rules, so as not to render them completely illegible...

We now expand the theory to capture the notion of bicategory with (specified) binary products:

Symbol	*Introductory Rule(s)*
$\times$	$A,B \in \mathrm{Ob}: \times(A,B) \in \mathrm{Ob}.$
π	$A,B \in \mathrm{Ob}: \pi(A,B) \in \mathrm{Hom}(\times(A,B),A).$
π'	$A,B \in \mathrm{Ob}: \pi'(A,B) \in \mathrm{Hom}(\times(A,B),B).$
$\langle\_,\_\rangle$	$A,B,C \in \mathrm{Ob}, f \in \mathrm{Hom}(C,A), g \in \mathrm{Hom}(C,B): \langle f,g\rangle \in \mathrm{Hom}(C,A\times B);$
	$\dfrac{\left\{\begin{matrix} A,B,C \in \mathrm{Ob}, f,f' \in \mathrm{Hom}(C,A), g,g' \in \mathrm{Hom}(C,B), \\ \beta \in 2\mathrm{Hom}(f,f'), \gamma \in 2\mathrm{Hom}(g,g') \end{matrix}\right\}}{\langle\beta,\gamma\rangle \in 2\mathrm{Hom}(\langle f,g\rangle,\langle f',g'\rangle).}$
τ	$\dfrac{A,B,C \in \mathrm{Ob}, f \in \mathrm{Hom}(C,A), g \in \mathrm{Hom}(C,B)}{\tau(f,g) \in 2\mathrm{Hom}(*(\langle f,g\rangle,\pi(A,B)),f).}$
τ^{-1}	$\dfrac{A,B,C \in \mathrm{Ob}, f \in \mathrm{Hom}(C,A), g \in \mathrm{Hom}(C,B)}{\tau^{-1}(f,g) \in 2\mathrm{Hom}(f,*(\langle f,g\rangle,\pi(A,B))).}$
τ'	$\dfrac{A,B,C \in \mathrm{Ob}, f \in \mathrm{Hom}(C,A), g \in \mathrm{Hom}(C,B)}{\tau'(f,g) \in 2\mathrm{Hom}(*(\langle f,g\rangle,\pi'(A,B)),g).}$
τ'^{-1}	$\dfrac{A,B,C \in \mathrm{Ob}, f \in \mathrm{Hom}(C,A), g \in \mathrm{Hom}(C,B)}{\tau'^{-1}(f,g) \in 2\mathrm{Hom}(g,*(\langle f,g\rangle,\pi'(A,B))).}$
$\bar{\tau}$	$\dfrac{A,B,C \in \mathrm{Ob}, h \in \mathrm{Hom}(C,\times(A,B))}{\bar{\tau}(f,g) \in 2\mathrm{Hom}(\langle *(h,\pi(A,B)),*(h,\pi'(A,B))\rangle,h).}$
$\bar{\tau}^{-1}$	$\dfrac{A,B,C \in \mathrm{Ob}, h \in \mathrm{Hom}(C,\times(A,B))}{\bar{\tau}^{-1}(f,g) \in 2\mathrm{Hom}(h,\langle *(h,\pi(A,B)),*(h,\pi'(A,B))\rangle).}$

Axioms

• Functoriality of pairing functor:

$\langle \mathrm{Id}(f),\mathrm{Id}(g)\rangle = \mathrm{Id}(\langle f,g\rangle)$, whenever $A,B,C \in \mathrm{Ob}, f \in \mathrm{Hom}(C,A)$ and $g \in \mathrm{Hom}(C,B)$.

$\langle \circ(\beta,\beta'),\circ(\gamma,\gamma')\rangle = \circ(\langle\beta,\gamma\rangle,\langle\beta',\gamma'\rangle)$, whenever $A,B,C \in \mathrm{Ob}, f,f',f'' \in \mathrm{Hom}(C,A)$, $g,g',g'' \in \mathrm{Hom}(C,B), \beta \in 2\mathrm{Hom}(f,f'), \beta' \in 2\mathrm{Hom}(f',f'')$, $\gamma \in 2\mathrm{Hom}(g,g')$ and $\gamma' \in 2\mathrm{Hom}(g',g'')$.

• Naturality of τ:

$\circ\big(\tau(f,g),\beta\big)=\circ\big(*(\langle\beta,\gamma\rangle,\mathrm{Id}(\pi(A,B))),\tau(f',g')\big)$, whenever $A,B,C\in\mathrm{Ob}$,
$f,f'\in\mathrm{Hom}(C,A),g,g'\in\mathrm{Hom}(C,B)$,
$\beta\in 2\mathrm{Hom}(f,f')$ and $\gamma\in 2\mathrm{Hom}(g,g')$.

• Invertibility of τ:

$\circ\big(\tau(f,g),\tau^{-1}(f,g)\big)=\mathrm{Id}\big(*(\langle f,g\rangle,\pi(A,B))\big)$, whenever $A,B,C\in\mathrm{Ob},f\in\mathrm{Hom}(C,A)$ and
$g\in\mathrm{Hom}(C,B)$.

$\circ\big(\tau^{-1}(f,g),\tau(f,g)\big)=\mathrm{Id}(f)$, whenever $A,B,C\in\mathrm{Ob},f\in\mathrm{Hom}(C,A)$ and $g\in\mathrm{Hom}(C,B)$.

• Naturality of τ':

$\circ\big(\tau'(f,g),\gamma\big)=\circ\big(*(\langle\beta,\gamma\rangle,\mathrm{Id}(\pi'(A,B))),\tau'(f',g')\big)$, whenever $A,B,C\in\mathrm{Ob}$,
$f,f'\in\mathrm{Hom}(C,A),g,g'\in\mathrm{Hom}(C,B)$,
$\beta\in 2\mathrm{Hom}(f,f')$ and $\gamma\in 2\mathrm{Hom}(g,g')$.

• Invertibility of τ':

$\circ\big(\tau'(f,g),\tau'^{-1}(f,g)\big)=\mathrm{Id}\big(*(\langle f,g\rangle,\pi'(A,B))\big)$, whenever $A,B,C\in\mathrm{Ob}$,
$f\in\mathrm{Hom}(C,A)$ and $g\in\mathrm{Hom}(C,B)$.

$\circ\big(\tau'^{-1}(f,g),\tau'(f,g)\big)=\mathrm{Id}(g)$, whenever $A,B,C\in\mathrm{Ob},f\in\mathrm{Hom}(C,A)$ and $g\in\mathrm{Hom}(C,B)$.

• Naturality of $\bar{\tau}$:

$\circ\big(\bar{\tau}(h),\beta\big)=\circ\big(\big\langle *(\beta,\mathrm{Id}(\pi(A,B))),*(\beta,\mathrm{Id}(\pi'(A,B)))\big\rangle,\bar{\tau}(h')\big)$, whenever $A,B,C\in\mathrm{Ob}$,
$h,h'\in\mathrm{Hom}(C,\times(A,B))$ and $\beta\in 2\mathrm{Hom}(h,h')$.

• Invertibility of $\bar{\tau}$:

$\circ\big(\bar{\tau}(h),\bar{\tau}^{-1}(h)\big)=\mathrm{Id}\big(\big\langle *(h,\pi(A,B)),*(h,\pi'(A,B))\big\rangle\big)$, whenever $A,B,C\in\mathrm{Ob}$ and
$h\in\mathrm{Hom}(C,\times(A,B))$.

$\circ\big(\bar{\tau}^{-1}(h),\bar{\tau}(h)\big)=\mathrm{Id}(h)$, whenever $A,B,C\in\mathrm{Ob}$ and $h\in\mathrm{Hom}(C,\times(A,B))$.

• Coherence conditions:

$\circ\big(*\big(\bar{\tau}^{-1}(h),\pi(A,B)\big),\tau\big(*(h,\pi(A,B)),*(h,\pi'(A,B))\big)\big)=\mathrm{Id}\big(*(h,\pi(A,B))\big)$ whenever
$A,B,C\in\mathrm{Ob}$ and $h\in\mathrm{Hom}(C,\times(A,B))$.

$\circ\big(*\big(\bar{\tau}^{-1}(h),\pi'(A,B)\big),\tau'\big(*(h,\pi(A,B)),*(h,\pi'(A,B))\big)\big)=\mathrm{Id}\big(*(h,\pi'(A,B))\big)$ whenever
$A,B,C\in\mathrm{Ob}$ and $h\in\mathrm{Hom}(C,\times(A,B))$.

$\circ\big(\bar{\tau}^{-1}(\langle f,g\rangle),\langle\tau(f,g),\tau'(f,g)\rangle\big)=\mathrm{Id}(\langle f,g\rangle)$ whenever $A,B,C\in\mathrm{Ob}$ and
$f\in\mathrm{Hom}(C,A),g\in\mathrm{Hom}(C,B)$.

We now augment the theory so that its models have a (specified) terminal object. The rules below are independent of those given just above concerning binary products; incidentally, if these (above) were deleted, the original rules together with the rules below would give the theory of bicategories with a (specified) terminal object.

Symbol	*Introductory Rule*
$\mathbf{t}$	$\mathbf{t} \in \mathrm{Ob}$.
$!$	$A \in \mathrm{Ob}: !(A) \in \mathrm{Hom}(A,\mathbf{t})$.
ξ	$A \in \mathrm{Ob}, f \in \mathrm{Hom}(A,\mathbf{t}): \xi(f) \in 2\mathrm{Hom}(f, !(A))$.
ξ^{-1}	$A \in \mathrm{Ob}, f \in \mathrm{Hom}(A,\mathbf{t}): \xi^{-1}(f) \in 2\mathrm{Hom}(!(A), f)$.

Axioms

• Naturality of ξ:

$\xi(f) = \circ(\beta, \xi(f'))$, whenever $A \in \mathrm{Ob}$, $f, f' \in \mathrm{Hom}(A,\mathbf{t})$ and $\beta \in 2\,\mathrm{Hom}(f, f')$.

• Invertibility of ξ:

$\circ(\xi(f), \xi^{-1}(f)) = \mathrm{Id}(f)$, whenever $A \in \mathrm{Ob}$ and $f \in \mathrm{Hom}(A,\mathbf{t})$.

$\circ(\xi^{-1}(f), \xi(f)) = \mathrm{Id}(!(A))$, whenever $A \in \mathrm{Ob}$ and $f \in \mathrm{Hom}(A,\mathbf{t})$.

Next we give the additional rules for the theory of bicategories with (specified) exponentials. They are independent of the rules dealing with the terminal object, but they rely however on those associated with binary products. To avoid any confusion, we immediately let the reader know that $\exp(A,B)$ is meant to represent the object B^A.

Symbol	*Introductory Rule(s)*
exp	$A, B \in \mathrm{Ob}: \exp(A,B) \in \mathrm{Ob}$.
ε	$A, B \in \mathrm{Ob}: \varepsilon(A,B) \in \mathrm{Hom}(\times(A, \exp(A,B)), B)$.
$(\_)^{\sim}$	$A, B, D \in \mathrm{Ob}, h \in \mathrm{Hom}(\times(A,D), B): h^{\sim} \in \mathrm{Hom}(D, \exp(A,B))$;
	$A, B, D \in \mathrm{Ob}, h, h' \in \mathrm{Hom}(\times(A,D), B), \beta \in 2\mathrm{Hom}(h,h'): \beta^{\sim} \in 2\mathrm{Hom}(h^{\sim}, h'^{\sim})$.
ζ	$\dfrac{A, B, D \in \mathrm{Ob}, h \in \mathrm{Hom}(\times(A,D), B)}{\zeta(h) \in 2\mathrm{Hom}\left(*\left(\left\langle \pi(A,D), *\left(\pi'(A,D), h^{\sim}\right)\right\rangle, \varepsilon(A,B)\right), h\right)}$.
ζ^{-1}	$\dfrac{A, B, D \in \mathrm{Ob}, h \in \mathrm{Hom}(\times(A,D), B)}{\zeta^{-1}(h) \in 2\mathrm{Hom}\left(h, *\left(\left\langle \pi(A,D), *\left(\pi'(A,D), h^{\sim}\right)\right\rangle, \varepsilon(A,B)\right)\right)}$.
$\tilde{\zeta}$	$\dfrac{A, B, D \in \mathrm{Ob}, k \in \mathrm{Hom}(D, \exp(A,B))}{\tilde{\zeta}(k) \in 2\mathrm{Hom}\left(\left(*\left(\left\langle \pi(A,D), *\left(\pi'(A,D), k\right)\right\rangle, \varepsilon(A,B)\right)\right)^{\sim}, k\right)}$.
$\tilde{\zeta}^{-1}$	$\dfrac{A, B, D \in \mathrm{Ob}, k \in \mathrm{Hom}(D, \exp(A,B))}{\tilde{\zeta}^{-1}(k) \in 2\mathrm{Hom}\left(k, \left(*\left(\left\langle \pi(A,D), *\left(\pi'(A,D), k\right)\right\rangle, \varepsilon(A,B)\right)\right)^{\sim}\right)}$.

Axioms

• Functoriality of exponentiation:

$(\mathrm{Id}(h))^{\sim} = \mathrm{Id}(h^{\sim})$, whenever $A, B, D \in \mathrm{Ob}$ and $h \in \mathrm{Hom}(\times(A,D),B)$.

$(\circ(\beta,\beta'))^{\sim} = \circ(\beta^{\sim},\beta'^{\sim})$, whenever $A,B,D \in \mathrm{Ob}, h,h',h'' \in \mathrm{Hom}(\times(A,D),B)$, $\beta \in 2\mathrm{Hom}(h,h')$ and $\beta' \in 2\mathrm{Hom}(h',h'')$.

• Naturality of ζ:

$\circ(\zeta(h),\beta) = \circ\Big(*\big(\big\langle \mathrm{Id}(\pi(A,D)), *\big(\mathrm{Id}(\pi'(A,D)),\beta\big)\big\rangle, \mathrm{Id}(\varepsilon(A,B))\big), \zeta(h')\Big)$, whenever $A,B,D \in \mathrm{Ob}, h,h' \in \mathrm{Hom}(\times(A,D),B)$ and $\beta \in 2\mathrm{Hom}(h,h')$.

• Invertibility of ζ:

$\circ\big(\zeta(h),\zeta^{-1}(h)\big) = \mathrm{Id}\Big(*\big(\big\langle \pi(A,D), *\big(\pi'(A,D),h^{\sim}\big)\big\rangle, \varepsilon(A,B)\big)\Big)$, whenever $A,B,D \in \mathrm{Ob}$ and $h \in \mathrm{Hom}(\times(A,D),B)$.

$\circ\big(\zeta^{-1}(h),\zeta(h)\big) = \mathrm{Id}(h)$, whenever $A,B,D \in \mathrm{Ob}$ and $h \in \mathrm{Hom}(\times(A,D),B)$.

• Naturality of $\tilde{\zeta}$:

$\circ\big(\tilde{\zeta}(k),\beta\big) = \circ\Big(\Big(*\big(\big\langle \mathrm{Id}(\pi(A,D)), *\big(\mathrm{Id}(\pi'(A,D)),\beta\big)\big\rangle, \mathrm{Id}(\varepsilon(A,B))\big)\Big)^{\sim}, \tilde{\zeta}(k')\Big)$, whenever $A,B,D \in \mathrm{Ob}, k,k' \in \mathrm{Hom}(D,\exp(A,B))$ and $\beta \in 2\mathrm{Hom}(k,k')$.

• Invertibility of $\tilde{\zeta}$:

$\circ\big(\tilde{\zeta}(k),\tilde{\zeta}^{-1}(k)\big) = \mathrm{Id}\Big(\Big(*\big(\big\langle \pi(A,D), *\big(\pi'(A,D),k\big)\big\rangle, \varepsilon(A,B)\big)\Big)^{\sim}\Big)$, whenever $A,B,D \in \mathrm{Ob}$ and $k \in \mathrm{Hom}(D,\exp(A,B))$.

$\circ\big(\tilde{\zeta}^{-1}(k),\tilde{\zeta}(k)\big) = \mathrm{Id}(k)$, whenever $A,B,D \in \mathrm{Ob}$ and $k \in \mathrm{Hom}(D,\exp(A,B))$.

• Coherence conditions:

$\circ\Big(*\Big(\big\langle \pi(A,D), *\big(\mathrm{Id}(\pi'(A,D)),\tilde{\zeta}^{-1}(k)\big)\big\rangle, \varepsilon(A,B)\Big), \zeta\Big(*\big(\big\langle \pi(A,D), *\big(\pi'(A,D),k\big)\big\rangle, \varepsilon(A,B)\big)\Big)\Big) =$ $\mathrm{Id}\Big(*\big(\big\langle \pi(A,D), *\big(\pi'(A,D),k\big)\big\rangle, \varepsilon(A,B)\big)\Big)$, whenever $A,B,D \in \mathrm{Ob}$ and $k \in \mathrm{Hom}(D,\exp(A,B))$.

$\circ\big(\tilde{\zeta}^{-1}(h^{\sim}),(\zeta(h))^{\sim}\big) = \mathrm{Id}(h^{\sim})$, whenever $A,B,D \in \mathrm{Ob}$ and $h \in \mathrm{Hom}(\times(A,D),B)$.

The overall collection of rules we have given defines the (generalised algebraic) theory of cartesian closed bicategories (with specified operations). Formally verifying that the two different presentations of the concept are concordant is perhaps a bit tedious, but straightforward; we will therefore not say more about it.

Given two instances of a particular generalised algebraic theory, Cartmell defines a *homomorphism* between them to be, roughly speaking, a set-map between the two instances preserving the sorts and the operators (the precise definition can be found in [Ca]). This corresponds precisely to our own notion of strict homomorphism, as defined in the previous chapter.

Chapter 4

Propositional Logic, the Lambek Calculus and Algebraic Structures

The aim of this chapter is to present the so-called Lambek calculus for a fragment of (classical) propositional logic known as *positive intuitionistic propositional logic*[1], and at the same time to study various algebraic (including categorical) structures modelling this calculus. The story this chapter relates the beginning of which is also known as (categorical) proof theory. The standard reference on this material is the classic book [LS] by Lambek and Scott. A streamlined, yet elegant and thorough introduction to the subject matter is also provided in [HM] – incidentally, a large part of our notation (if not of our treatment!) is taken directly from one or the other of [LS] and [HM]. We will assume familiarity with basic algebraic structures such as graphs and preorders, as well as a working knowledge of the notions of categorical adjunction, fullness and faithfulness of functors (see, e.g. [CWM]). In addition, it would be helpful for the reader to be acquainted with the rudiments of propositional logic, model theory and recursion theory.

We start with a number of definitions. We assume we are given an arbitrary set L (a *language*), the elements of which we call *atomic propositions*, or *atoms*. We immediately derive the familiar concept of *formula*, built (in a standard recursive manner) from the atoms, the nullary (constant) symbol $\mathbf{t}$, and the binary connectives $\wedge$ and $\rightarrow$ (those three operation symbols are precisely what characterizes *positive intuitionistic propositional logic*). In certain contexts, it may be more convenient to assume that only one or two of the above connectives were used in building formulas. We shall generally use the letters $A, B, C, \ldots$ to denote formulas.

The notion of formula then gives rise to that of *entailment*: an entailment is simply an ordered pair of formulas, written (for formulas A and B) $A \succ B$ (read "*A entails B*"). A *theory* is then defined to be any collection of entailments. Here the common underlying assumption, of course, is that of some arbitrary but fixed language L.

A *proof system* is a collection of *rules of inference* for producing ("*deducing*") entailments. The rules of inference are of the following form: they have a finite number of

[1]Terminology as in [LS] – however, "implicational logic" may also be found in the literature.

hypotheses (possibly zero), and one conclusion. Individual hypotheses and conclusion alike consist of single entailments (or schemes of such). In effect, the actual proof systems we will be interested in are far from arbitrary. To specify them, it is useful to list a basic set of rules of inference; they are given below (middle column) in the same syntactic form as the Cartmell rules of the preceding chapter. The column on the right-hand side gives the name of the corresponding rule, whereas data in the left-hand side column (the totality of which we denote D_L) will be used shortly. The letters A, B, C and D stand for arbitrary formulas.

1_A	$\dfrac{}{A \succ A}$	(TAUT)
$*_{A,B,C}$	$\dfrac{A \succ B \quad B \succ C}{A \succ C}$	(CUT)
$\pi_{A,B}$	$\dfrac{}{A \wedge B \succ A}$	($\wedge$LEFT1)
$\pi'_{A,B}$	$\dfrac{}{A \wedge B \succ B}$	($\wedge$LEFT2)
$\langle \_,\_ \rangle^{C}_{A,B}$	$\dfrac{C \succ A \quad C \succ B}{C \succ A \wedge B}$	($\wedge$RIGHT)
$!_A$	$\dfrac{}{A \succ \mathbf{t}}$	(TRUE)
$\varepsilon_{A,B}$	$\dfrac{}{A \wedge (A \rightarrow B) \succ B}$	($\rightarrow$LEFT)
$(\_)^{-}_{A,B,D}$	$\dfrac{A \wedge D \succ B}{D \succ A \rightarrow B}$	($\rightarrow$RIGHT)

In the context of a particular theory T, we also have, for each entailment $x \in T$, the following *axioms* (rules with no hypotheses):

k_x	$\dfrac{}{x}$	(T)

Disregarding rule T for a moment, we remark that for every symbol in the first column having the entailment $A \succ B$ as the conclusion of its associated rule of inference, there is a unique corresponding sorted operation symbol with value type Hom(A,B) in the generalised algebraic theory of cartesian closed bicategories, and each of the rules of inference listed above in the middle column likewise corresponds to the associated typing rule for the operation symbol in question in the generalised algebraic theory.

We are now in a position to specify the proof systems we would like to study. They all share a common feature in having rule T as a rule of inference (not something restrictive since the theory T can always be assumed to be empty). A *deductive system* is any proof system with the rules TAUT and CUT. A *conjunction calculus* is a deductive system having, in addition, the rules $\wedge$LEFT1, $\wedge$LEFT2, $\wedge$RIGHT and TRUE.[2] Finally, a *positive intuitionistic calculus* is a conjunction calculus equipped with the two last remaining rules, $\to$LEFT and $\to$RIGHT. The particular proof system under consideration is often not explicitly identified; in general it can be inferred from the context.

The next notion we want to introduce is that of a *deduction* (or *proof*) (in the context of a particular theory T). This concept is quite standard, but the traditional presentation (in terms of sequences, etc.) is not fully satisfactory for our purposes. The definition we give here is taken almost verbatim from [HM]: a *deduction* is a finite tree with additional data; the nodes are occurrences of entailments; the leaves (nodes without successors) are instances of axioms, the axiom applying attached as a justification label (the same entailment could be an instance of two distinct axioms); every other node has successors, and if it has more than one, the order of the successors is supplied as additional data; every node n is the conclusion of some rule of inference, given as a justification label on n, in which the hypothesis is (hypotheses are) the successor(s) of n (in the given order if there are several hypotheses). The deduction is a deduction *of* the entailment at its root. We say that the theory T *deduces* the entailment $A \succ B$ (and we denote this fact $T \rhd A \succ B$) if there is some deduction (based on T) of $A \succ B$.

Perhaps an example is in order at this point. Here is a deduction of the entailment $(A \to B) \wedge A \succ B$, a variant of the so-called "deduction theorem", over the empty theory $T = \phi$ (of course the deduction is hence valid in *any* theory T):

$$\dfrac{\dfrac{\dfrac{}{(A \to B) \wedge A \succ A}(\wedge\mathrm{LEFT2}) \qquad \dfrac{}{(A \to B) \wedge A \succ (A \to B)}(\wedge\mathrm{LEFT1})}{(A \to B) \wedge A \succ A \wedge (A \to B)}(\wedge\mathrm{RIGHT}) \qquad \dfrac{}{A \wedge (A \to B) \succ B}(\to\mathrm{LEFT})}{(A \to B) \wedge A \succ B}(\mathrm{CUT})$$

Naturally, one may have several distinct deductions, within a particular theory, of a given entailment. We will write $f{:}A \succ B$ to indicate that f is a deduction of $A \succ B$. (This

[2]It is often appropriate, in the context of a conjunction calculus, to require that the connective "$\to$" does not appear in formulas. Similarly, no connectives should normally be allowed within the formulas of a (pure) deductive system – that is to say, in that case, the only formulas are the atomic formulas.

notation does not mention the theory T under consideration, which is usually clear from the context.) To recapitulate: $T \rhd A \succ B$ if and only if there is some deduction f such that $f{:}A \succ B$.

There is no doubt that the tree-form notation used to present deductions is cumbersome; it turns out, however, that a very convenient alternate symbolism is readily available, using the symbols listed on the left of the statements of the rules of inference above (i.e. the symbols in the set $D_L \cup \{k_x : x \in T\}$). The idea (sketched here, but more fully developed in [HM]) is to construe the set of deductions as a multi-sorted algebra, with (finite ordered tuples of) entailments as sorts (the "value-sort", however, is always a single entailment for every operation). For instance, the nullary operation 1_A, representing a particular deduction of the entailment $A \succ A$, has no argument-sort but has value-sort $A \succ A$; the binary operation $*_{A,B,C}$ has argument-sort the ordered pair $(A \succ B, B \succ C)$ and value-sort $A \succ C$, etc.... Each axiom listed is, of course, a deduction in its own right (represented by the corresponding left-hand symbol (element of $D_L \cup \{k_x : x \in T\}$)), whereas instances of the other rules *ipso facto* become genuine deductions as soon as their hypotheses are "replaced" by already existing deductions (i.e. as soon as every entailment in its hypotheses is seen to be the conclusion of some earlier deduction). The representation of the resulting deduction is the left-hand symbol corresponding to the n-ary rule in question, but with each of the n "place-holders" of the symbol replaced (in the appropriate order) by the representations for each of the n "sub-deductions" of the entailments constituting the hypotheses of the rule. It is then easily seen that by repeatedly applying this "composition" process, using only the "basic" representations of the rules we have listed, one may in effect produce any given deduction, and furthermore every (closed) term uniquely denotes a well-formed deduction. Put another way, the deductions are in bijective correspondence with the elements of the absolutely free algebra with signature D_L described above, with generators all k_x.

To illustrate, the deduction given earlier of the entailment $(A \to B) \wedge A \succ B$ would read: $\varepsilon_{A,B} *_{(A\to B)\wedge A, A\wedge(A\to B), B} \langle \pi'_{A\to B,A}, \pi_{A\to B,A} \rangle^{(A\to B)\wedge A}_{A,A\to B}$ (where, among other things, the binary operation $*_{(A\to B)\wedge A, A\wedge(A\to B), B}$ was written infixed, with its first argument on its right and its second argument on its left, etc.).

So what we have achieved so far is a rigorous notational system, or calculus, for deductions. Let us agree to call the associated (free, multi-sorted) algebra of terms a ***pre-Lambek algebra***. Given a theory T over a language L, we denote this algebra by

$pF_L(T)$. (Of course, to be complete the notation should also incorporate information as to which logic is under consideration – in practice however we will mostly be working with positive intuitionistic propositional logic, and in the other cases the logic used will either be stated explicitly or understood from the context.)

The *Lambek calculus* is now a mere stone's throw away: it simply consists in identifying deductions considered to be only "inessentially" different. We give below the appropriate equations (known as *cartesian closed identities*) for the Lambek calculus of positive intuitionistic propositional logic. The other (Lambek) calculi are obtained by disregarding all equations making use of an operation symbol absent from the logic in question. To alleviate the notation, some indices have been omitted, and the various instances of the infix operation $*$ are written as simple juxtaposition. A, B, C, D, naturally, represent arbitrary formulas.

$f1_A = f$, $1_B f = f$, $(hg)f = h(gf)$, whenever $f{:}A \succ B, g{:}B \succ C$ and $h{:}C \succ D$.
$f = !_A$, whenever $f{:}A \succ \mathbf{t}$.
$\pi_{A,B}\langle f,g\rangle = f$, $\pi'_{A,B}\langle f,g\rangle = g$, $\langle \pi_{A,B}h, \pi'_{A,B}h\rangle = h$, whenever $f{:}C \succ A, g{:}C \succ B$ and $h{:}C \succ A \wedge B$.
$\varepsilon_{A,B}\langle \pi_{A,D}, h^{\check{}}\pi'_{A,D}\rangle = h$, $\left(\varepsilon_{A,B}\langle \pi_{A,D}, k\pi'_{A,D}\rangle\right)^{\check{}} = k$, whenever $h{:}A \wedge D \succ B$ and $k{:}D \succ A \rightarrow B$.

We construct the congruence relation on $pF_L(T)$ generated by all possible instances of the above equations. (A congruence relation is an equivalence relation $\equiv$ which in addition satisfies a well-known substitution property, e.g. if $f \equiv f'$ and $g \equiv g'$, then it must be the case that $\langle f,g\rangle \equiv \langle f',g'\rangle$, etc. – this is required to hold of every non-nullary operator, namely each instance of $*$, $\langle \_,\_\rangle$ and $(\_)^{\check{}}$.) By definition, this is the smallest (in the sense of set containment) equivalence relation satisfying the above equations as well as the relevant substitution properties. Its existence is established by the well-known process of taking the set-intersection of all such equivalence relations, etc.... The Lambek calculus is then obtained from the original calculus by simply attaching to it the (fully expanded) set of identities constituting the congruence relation. Naturally, we will want to call the associated algebra a *Lambek algebra*. That is to say, a Lambek algebra is a free multi-sorted algebra whose terms are equivalence classes of deductions, the equivalence relation being the congruence relation construed above. Given a language L and a theory T, we denote this algebra by $F_L(T)$. In general, we do not distinguish notationally between (genuine) deductions and their associated equivalence classes.

Our particular choice of notation increasingly begs the question: "What is the connection between Lambek algebras and cartesian closed categories?". We now turn to the task of addressing this. Let a language L be fixed. Suppose we are given any (multi-sorted) algebra **A** with signature D_L satisfying the cartesian closed identities. We then build a cartesian closed category with specified operations (also denoted **A**) as follows. The objects of **A** are the L-formulas, and, given two objects A and B, the arrows from A to B are the elements of the algebra **A** of sort $A \succ B$. The product $A \times B$ of A and B is the formula $A \wedge B$, the terminal object **t** is the formula **t**, and the exponential B^A is the formula $A \rightarrow B$. The various operations $\pi_{A,B}$, $!_A$, $\langle\_,\_\rangle^C_{A,B}$, etc., of **A** as an algebra, directly correspond to the distinguished arrows of **A** as a category. The category **A** is cartesian closed precisely because we assumed that the algebra **A** satisfies the cartesian closed identities.

The cartesian closed category $F_L(T)$ (obtained via the above process from the Lambek algebra $F_L(T)$) has a very interesting property: it is *free* in the standard algebraic sense; the precise definition follows after the next few preliminaries.

Recall the notion of an (oriented) graph. (For us, "graph" will always mean "oriented graph".) It comes along with the idea of graph homomorphism, which is simply a map on vertices and edges preserving the source and target of edges. We will be interested in a particular kind of graphs, the *cartesian closed graphs*, which form a category **CCG**. Its objects are all graphs with the following property: they have a distinguished vertex **t**, and, given two vertices A and B, they always have distinguished vertices $A \times B$ and B^A. The arrows of **CCG** (known of course as *cartesian closed graph homomorphisms*) are all graph homomorphisms f which in addition preserve the cartesian closed structure of vertices, i.e. $f(\mathbf{t}) = \mathbf{t}$, $f(A \times B) = f(A) \times f(B)$, and $f(B^A) = f(B)^{f(A)}$.[3]

A category may be viewed as a graph in a natural way, by "forgetting" the structure on arrows: the vertices are the objects, whereas the edges are the arrows. The graph so obtained from a category **C** is denoted Gr(**C**). This forgetful map applies to any category **C**, but its restriction to **CCC**, the category of all cartesian closed categories, can be extended in an obvious way to a functor Gr:**CCC** $\rightarrow$ **CCG**.

[3]The terminology used might prompt one to wonder why we haven't also required that there be certain "distinguished edges" (from e.g. $A \times B$ to both A and B, etc.), which the cartesian closed graph homomorphisms should preserve. It turns out that, for our purposes, it wouldn't make the slightest difference – hence our preference for the most economic definition.

We are now ready to define the *free cartesian closed category (with specified operations) over a cartesian closed graph G*: it is any cartesian closed category $\mathrm{Free}^s(G)$, equipped with a cartesian closed graph homomorphism $i{:}G \to \mathrm{Gr}(\mathrm{Free}^s(G))$, such that, for any other cartesian closed category $\mathbf{D}$, and any cartesian closed graph homomorphism $j{:}G \to \mathrm{Gr}(\mathbf{D})$, there is a unique (strict) cartesian closed functor $H{:}\mathrm{Free}^s(G) \to \mathbf{D}$ with the property that $\mathrm{Gr}(H) \circ i = j$. We also say that $\mathrm{Free}^s(G)$ is the *cartesian closed category freely generated by G*.[4]

PROPOSITION 4.1. Given a cartesian closed graph G, there always exists a cartesian closed category $\mathrm{Free}^s(G)$ satisfying the requirements of the above definition; moreover, any two such must be isomorphic. Naturally, this implies that Free^s above is, for all practical purposes, a function, which can be extended to a functor $\mathrm{Free}^s{:}\mathbf{CCG} \to \mathbf{CCC}$, left adjoint to $\mathrm{Gr}{:}\mathbf{CCC} \to \mathbf{CCG}$.

There is no great difficulty in the proof, which we therefore omit. The argument in these kinds of situation rarely varies; its flavour may be gleaned from examining similar statements in [LS] or [HM].

It is perhaps worth mentioning that in general, a cartesian closed category $\mathbf{C}$ is *not* isomorphic, or even equivalent, to $\mathrm{Free}^s(\mathrm{Gr}(\mathbf{C}))$ (so long as the category has more than one object – the reason is that extra arrows will have been added to $\mathrm{Free}^s(\mathrm{Gr}(\mathbf{C}))$); however, it is always possible to construct an equivalence relation on the parallel arrows of $\mathrm{Free}^s(\mathrm{Gr}(\mathbf{C}))$ in such a way that the "resulting" category is actually *isomorphic* to $\mathbf{C}$. This is one of the techniques used by [LS], although their use of bare graphs (as opposed to cartesian closed ones) in order to define freeness, forces them to identify not only arrows but also *objects* in $\mathrm{Free}^s(\mathrm{Gr}(\mathbf{C}))$, to recover the original category. This fact is still dissimulated because they only overtly mention the identification of arrows; the point is, however, that the arrows they require be identified are in general *not* parallel – hence the identification *ipso facto* of their corresponding sources and targets.

We now return to the question of the freeness of $F_L(T)$. (Here L is an arbitrary set, and T an arbitrary L-theory.) We build a cartesian closed graph $G_L(T)$, whose vertices are

[4]There are several other variants on the definition of freeness which could have been invoked to replace the one adopted in the text; for instance, one can have free cartesian closed categories on sets, (bare) graphs, (ordinary) categories, etc.... The definition we chose is very closely related to those of [LS] and [HM], but its main advantage is that it allows us to get to the notion of freeness in a single step, as opposed to having to pass through a two-stage process (objects, then arrows).

all L-formulas, with an edge from A to B whenever there is an axiom in T postulating the entailment $A \succ B$. We have:

PROPOSITION 4.2. The cartesian closed category $F_L(T)$ is (isomorphic to) $\mathrm{Free}^g(G_L(T))$.

(An adaptable proof can be found in [HM].)

Let us pause for an instant to consider what we have accomplished so far. We have succeeded in defining an algebraic structure (a free cartesian closed category) whose objects of interest (i.e. arrows!) are, essentially, proofs, with algebraic operations on them corresponding more or less directly to the traditional syntactic manipulations prescribed by the various rules of inference, etc.... The category, in other words, faithfully reflects everything that "happens" in the proof theory.

There is, however, another approach possible. It is based, roughly speaking, on the consideration that, in logic (and in fact mathematics in general), one is often interested not so much in all the different proofs of a given statement, but rather only in the *existence* of at least one. What we want, in some sense, is to refocus our attention from deductions to (bare) entailments. Our next task, therefore, should be to address the problem of finding the algebraic structures most adequate for this purpose, and of course investigate whatever connections they might have with the various constructions described so far.

We are thus naturally led to the following definition. Let **C** be a category. The *preorder collapse of* **C** is the preorder whose elements are the objects of **C**, and for which $A \leq B$ just in case there is an arrow $f: A \to B$ in **C**. Now consider the associated poset Po(**C**), obtained by identifying elements A and B whenever $A \leq B$ and $B \leq A$. Naturally, we call such a poset the *poset collapse of* **C**.

Recall the notion of implicational meet semi-lattice (*imsl* for short) from chapter 1, which is just a cartesian closed poset, when the latter is viewed as a category. We have the category **IMSL** whose objects are all imsl's, and arrows all cartesian closed functors between them, i.e. maps preserving conjunction, implication and maximum element (such maps are also automatically order-preserving). It is easy to verify that the function Po then gives rise to a functor Po:$\mathbf{CCC} \to \mathbf{IMSL}$.[5] On the other hand, we have the inclusion i^p:$\mathbf{IMSL} \to \mathbf{CCC}$ which takes an imsl to itself, viewed as a cartesian closed category. The two functors are closely linked as the following proposition indicates.

PROPOSITION 4.3. i^p:$\mathbf{IMSL} \to \mathbf{CCC}$ is full and faithful, and is right adjoint to Po:$\mathbf{CCC} \to \mathbf{IMSL}$ (written Po$\dashv i^p$).

We omit the proof, but let us still describe what the unit η of the adjunction does: if $\mathbf{C}$ is in $\mathbf{CCC}$, $\eta_{\mathbf{C}} : \mathbf{C} \to i^p \circ \mathrm{Po}(\mathbf{C})$ is a surjective cartesian closed functor taking any object C of $\mathbf{C}$ to its equivalence class in (the inclusion into $\mathbf{CCC}$ of) the poset collapse of $\mathbf{C}$, with the obvious corresponding effect on arrows of $\mathbf{C}$.

In the case of the cartesian closed category of proofs $F_L(T)$, $\mathrm{Po}(F_L(T))$ is clearly simply the well-known Lindenbaum-Tarski algebra of T for positive intuitionistic propositional logic[6].

Let us now tackle the problem of *presenting* imsl's, following the idea from group theory. An *implicational meet semi-lattice presentation* (*imslPres* for short) is a pair $(L;T)$, where L is a language, and T an L-theory (in positive intuitionistic propositional logic).

Before we show how it is exactly that an imslPres actually "presents" an imsl, we observe that the new objects can naturally be organized in a category. We first need to make the following trivial observation/convention: suppose we have two languages L and L', with a (set-)map $f:L \to L'$. Then f naturally induces a map (also denoted f) from formulas over L to formulas over L', which preserves all connectives, and only applies (the original) f on atoms – we are talking, in other words, about the homomorphism induced by f from the free algebra of L-formulas to the free algebra of L'-formulas.

We define the category **IMSLPres** as follows: its objects are all imslPres's, and given $(L;T)$ and $(L';T')$ two such, an arrow $f:(L;T) \to (L';T')$ is simply a set-map $f:L \to L'$

[5] However, this assertion should not automatically be taken for granted; for instance, the poset collapse of a category with pullbacks does *not* in general have pullbacks.
[6] Lindenbaum-Tarski algebras are a very fundamental concept in logic; see, for instance, [CK].

with the property that, whenever an entailment $A \succ B$ is in T, then it is the case that the entailment $f(A) \succ f(B)$ is in T'. (Here A and B are arbitrary L-formulas.)[7]

A few routine verifications confirm that **IMSLPres** is indeed a category. Now, given an imslPres $P = (L;T)$, we build the *free imsl* $\mathrm{Free}^i(P)$ *over P* as follows: its elements are equivalence classes of formulas, under the equivalence relation of bientailment: A is equivalent to B if and only if $T \rhd A \succ B$ and $T \rhd B \succ A$. Given (the equivalence classes of) two formulas A and B, we put $A \leq B$ if and only if $T \rhd A \succ B$.

PROPOSITION 4.4. Free^i so defined extends to a functor Free^i:**IMSLPres** $\rightarrow$ **IMSL**.

PROOF. In what follows, A, B, A' and B' stand for arbitrary L-formulas. We first observe that the $\leq$ relation on $\mathrm{Free}^i(P)$ is well-defined. Indeed, if A is equivalent to A' and B is equivalent to B' (as defined), then clearly $T \rhd A \succ B$ iff $T \rhd A' \succ B'$, by repeated use of the rule CUT. $\leq$ is reflexive (by TAUT), antisymmetric (by construction), and transitive (by CUT). So $\mathrm{Free}^i(P)$ is a poset. It has a maximum element, the equivalence class of the formula **t**, by virtue of TRUE. Now, if A and B are respectively equivalent to A' and B', then $A \wedge B$ and $A \rightarrow B$ are equivalent, respectively, to $A' \wedge B'$ and $A' \rightarrow B'$ (as is not very difficult to verify). It follows that conjunction and implication, as pure algebraic operations, can be defined in $\mathrm{Free}^i(P)$. That they indeed satisfy the expected property is then a consequence of $\wedge$LEFT1, $\wedge$LEFT2 and $\wedge$RIGHT (for conjunction), and $\rightarrow$LEFT and $\rightarrow$RIGHT (for implication). So $\mathrm{Free}^i(P)$ is an imsl. Lastly, given $f:P = (L;T) \rightarrow P' = (L';T')$ an arrow between two imslPres's, we let $\mathrm{Free}^i(f):\mathrm{Free}^i(P) \rightarrow \mathrm{Free}^i(P')$ be the map that takes (the equivalence class of) A to (the equivalence class of) $f(A)$. We need to show that this is well-defined, i.e. we need to show that, if $T \rhd A \succ A'$, then $T' \rhd f(A) \succ f(A')$. This follows from the fact that the hypothesis $f:(L;T) \rightarrow (L';T')$ implies, for every entailment ("axiom") in T, that there is a corresponding entailment in T'. The desired result is then obtained via a straightforward induction on the deduction of $A \succ A'$. Of course, $\mathrm{Free}^i(f)$ preserves conjunction, implication and maximum element, simply because f does. That Free^i is a functor should now be clear.

[7]We might have called this category "**lazy-IMSLPres**" instead; the "eager" version, by contrast, would only require of an arrow f, in the notation of the definition, that $T' \rhd f(A) \succ f(B)$ whenever $A \succ B \in T$. Computationally, this would give rise to a much more complex entity, simply because the notion of deduction is computationally much more complex than that of set membership. In practice however, it turns out that the results obtained are very similar.

Our next goal is to define a certain functor in the other direction. Again, we first need to make a little observation/convention: given an imsl Q, let $L = |Q|$ be the underlying set of Q; L-formulas then have an obvious interpretation as elements of $|Q|$, precisely because Q, being an imsl, is equipped with operations corresponding to the connectives in L-formulas. If we denote this "evaluation function" by $\mathrm{eval}:\{L\text{-formulas}\} \to |Q|$, we require that eval should take atoms (i.e. elements $|Q|$) to themselves, and preserve conjunction, implication and **t**.

So let an imsl Q be given, and construct an imslPres $\mathrm{Pres}^i(Q) = (L;T)$ as follows: L is taken to be the underlying set of Q ($L = |Q|$), and, for arbitrary L-formulas A and B, put the entailment $A \succ B$ in T if and only if $\mathrm{eval}(A) \le \mathrm{eval}(B)$. We say that T is the *diagram of* Q (written $T = \mathrm{Diag}(Q)$). (This corresponds to the "canonical presentation" (of, e.g., groups) in algebra.)

PROPOSITION 4.5. Pres^i so defined extends to a full and faithful functor $\mathrm{Pres}^i: \mathbf{IMSL} \to \mathbf{IMSLPres}$, right adjoint to $\mathrm{Free}^i: \mathbf{IMSLPres} \to \mathbf{IMSL}$.

PROOF. If Q and Q' are two imsl's, we need to specify the intended effect of Pres^i on an arrow $f: Q \to Q'$. We simply put $\mathrm{Pres}^i(f): (|Q|;\mathrm{Diag}(Q)) \to (|Q'|;\mathrm{Diag}(Q')) = f$. To verify that this is legitimate, let A, B be two $|Q|$-formulas such that $A \succ B \in \mathrm{Diag}(Q)$. Then certainly $\mathrm{eval}_Q(A) \le \mathrm{eval}_Q(B)$. By assumption, f preserves this inequality, and moreover, since f, eval_Q and $\mathrm{eval}_{Q'}$ all preserve the connectives/operations $\wedge$, $\to$ and **t**, they "commute" with one another; we may thus write $\mathrm{eval}_{Q'}(f(A)) \le \mathrm{eval}_{Q'}(f(B))$. Of course, this means that $f(A) \succ f(B) \in \mathrm{Diag}(Q')$. It is then easy to conclude from there that Pres^i is indeed a functor.

Faithfulness of Pres^i is automatic. For fullness, we argue as follows: given $h: (|Q|;\mathrm{Diag}(Q)) \to (|Q'|;\mathrm{Diag}(Q'))$, we let $f = h: |Q| \to |Q'|$. Naturally, all we need do is show that in fact, $f: Q \to Q'$ (it will then ensue that $\mathrm{Pres}^i(f) = h$). We illustrate the truth of the claim by showing that f preserves conjunction. Let A, B belong to $|Q|$. We carefully distinguish between the *element of* $|Q|$ $A \wedge_Q B$, and the $|Q|$*-formula* $A \wedge B$. By construction, both the entailments $A \wedge B \succ A \wedge_Q B$ and $A \wedge_Q B \succ A \wedge B$ are in $\mathrm{Diag}(Q)$. By assumption, then, both $h(A) \wedge h(B) \succ h(A \wedge_Q B)$ and $h(A \wedge_Q B) \succ h(A) \wedge h(B)$ are in $\mathrm{Diag}(Q')$. But this can only be the case if the corresponding inequalities hold in Q', which is to say $h(A \wedge_Q B) = h(A) \wedge_{Q'} h(B)$, i.e. $f(A \wedge_Q B) = f(A) \wedge_{Q'} f(B)$. The preservation of the two other operations is proved similarly. So Pres^i is indeed full.

Lastly, we sketch the proof of the adjunction by briefly commenting on the required bijection between families of arrows. Let Q be an imsl, and $(L;T)$ an imslPres. Given $f:\mathrm{Free}^i(L;T)\to Q$, let $h:(L;T)\to(|Q|,\mathrm{Diag}(Q))$ simply be the restriction of f to L (notwithstanding the fact that f is really a map on *equivalence classes* of L-formulas...). The construction of Diag(Q) ensures that h is a legitimate arrow. It is quite clear that there is a straightforward well-defined inverse to this procedure; a few more routine verifications will show that $\mathrm{Free}^i \dashv \mathrm{Pres}^i$ as required. This concludes the proof.

It follows from the fact that the right adjoint Pres^i is full and faithful that every imsl is free over some imslPres (see, for instance, [CWM]); in other words, every imsl arises as a (Lindenbaum-Tarski) algebra for some theory T, namely, its diagram.

Let's take the opportunity to recast some basic model-theoretic concepts in our framework. We need just a few more preliminaries: let $P=(L;T)$ be an imslPres. A *model M of P* consists of a pair $M=(Q,f)$, with Q an imsl, and f a set-map $f:L\to|Q|$ (extending naturally to L-formulas), such that, for every entailment $A\succ B$ in T, we have $f(A)\le f(B)$ (we say that the entailment $A\succ B$ is *true in M*). Clearly, (Q,f) is a model of $(L;T)$ if and only if $f:(L;T)\to\mathrm{Pres}^i(Q)$ is an imslPres homomorphism. $(\mathrm{Free}^i(P),i)$ (where i is the component at P of the unit of the adjunction $\mathrm{Free}^i\dashv\mathrm{Pres}^i$), is certainly a model of P, actually a universal one, in the sense that an entailment is true in every model of P if and only if it is true in $(\mathrm{Free}^i(P),i)$ (this is a direct consequence of proposition 4.5).

Let a language L be fixed. Given an L-theory T, and an entailment $A\succ B$ of L-formulas, we already have a *syntactic* notion of "truth" of $A\succ B$, namely deducibility: $T\rhd A\succ B$. We can now define a *semantic* counterpart: we say that *$A\succ B$ is true in T*, written $T\rhd\!= A\succ B$, if $A\succ B$ holds in *every* model of $(L;T)$. A result in logic states that $T\rhd A\succ B$ if and only if $T\rhd\!= A\succ B$ (this is known as the general completeness theorem)[8]. To see this, we recall once more that $T\rhd\!= A\succ B$ iff $(\mathrm{Free}^i(L;T),i)$ is a model of $A\succ B$ iff $A\le B$ holds in $\mathrm{Free}^i(L;T)$ iff (by definition of $\mathrm{Free}^i(L;T)$) $T\rhd A\succ B$. (More on this can be found in [M2].) This, of course, is of little use in practice when one is trying to decide if a particular entailment $A\succ B$ really is a consequence of a theory T; such questions in fact lead us straight into the area of decidability and recursion theory, a huge

[8] There are several other "completeness theorems" (depending on the logic under consideration), which weaken the requirements for semantic truth. These theorems then assert that semantic truth still nevertheless equates deducibility – see [CK] or [S2].

subject in its own right (a good basic reference is [S2]). We have briefly restated some of the relevant results below – again, for a fuller discussion, please consult [S2] and [M2].

Let us place ourselves in the context of a particular logic L. If L is a language and T is a theory over L, let us denote by Cons(T) the set of entailments over L that T deduces. (So Cons(T) simply corresponds to the arrows of $\mathrm{Po}(F_L(T))$.) Let's assume that L and T are finite and that Cons(T) is then suitably encoded using any "reasonable" Gödel numbering. We then say that L *has a solvable decision problem* if Cons(T) is recursive for any choice of finite L and T.[9] Classical propositional logic (i.e., the well-known logic extending positive intuitionistic propositional logic with the connectives $\vee$ ("or") and $\neg$ ("not"), and satisfying the famous "law of the excluded middle" $\dfrac{}{A \vee \neg A}$), as well as a few others, have a solvable decision problem. So has intuitionistic propositional logic (the other well-known logic extending its positive sibling, and having the same connectives as classical propositional logic, but without the crucial law of the excluded middle). It turns out that a careful examination of the proof of the latter fact (in [M2]) will show that it can be carried over verbatim to the case of positive intuitionistic propositional logic, i.e. that positive intuitionistic propositional logic also has a solvable decision problem. What this implies is that there exists an algorithm which, given a finite theory T (in positive intuitionistic propositional logic) over some finite language L, and given a further entailment $A \succ B$ over L, will decide in a finite amount of time whether $T \rhd A \succ B$ or not. Very roughly speaking, this algorithm consists in simultaneously searching for a proof of the entailment as well as for a counter-example to it amongst finite imsl's which model T (the collection of which is of course denumerable). This process will provably come to an end (see [M2]). We point out that there actually exist some entailments which are *not* consequences of the empty theory in positive intuitionistic propositional logic, but which *are* provable, from any theory, in classical propositional logic. One such example is the entailment $\mathbf{t} \succ ((A \rightarrow B) \rightarrow A) \rightarrow A$. This will never be the case, however, when the latter logic is merely intuitionistic propositional logic (we say that intuitionistic propositional logic is a *conservative extension* of positive intuitionistic propositional logic). As a consequence, the famous semantics devised by Kripke for intuitionistic propositional logic, based on the idea of "possible worlds", applies to the positive case as well; i.e., the well-

[9] In fact, we needn't limit ourselves to the context of particular logics – the applicability of these ideas is actually quite large. Group theory, for instance, is well-known to have an unsolvable word-problem (see [S2]). That is to say, there exists no algorithm which, given a particular finite group presentation, can decide whether two given terms are actually the same in the presented group or not.

known Kripke completeness theorem for intuitionistic propositional logic is also valid for positive intuitionistic propositional logic. For details, please refer to [M2].

Let us now return to our algebraic constructions. What we would like to do at this point is to define a notion of presentation for the richer class of cartesian closed categories.

So let a language L be given. Recall the signature D_L introduced earlier in this chapter ($D_L = \{1_A, *_{A,B,C}, \pi_{A,B}, \ldots | A, B, C, D \text{ are } L\text{-formulas}\}$). We will ultimately want to think of D_L as a collection of symbols to operate on arrows in a cartesian closed category, but for now, just recall how we viewed D_L as a multi-sorted signature: the argument-sort and value-sort (implicitly) attached to each symbol in D_L are finite tuples of entailments over L, etc.... Next, define a *formal arrow over* L to be any triple (f, A, B), with A, B L-formulas. f is the *name* of the formal arrow, A, its *source*, and B, its *target*. We declare that such an f has no argument-sort, and value-sort $A \succ B$; a more standard way to represent f is, of course, $f{:}A \rightarrow B$. (We will usually identify a formal arrow with its name, as we have done here.) Given an arbitrary collection Ar of formal arrows over L, we consider the absolutely free D_L-algebra of terms generated by Ar. Let us agree to call the elements of this free algebra *Ar-terms*. (So Ar-terms, being closed terms, have no argument-sort, and naturally their value-sort consists of a single entailment over L; we can thus speak of them as having a *source* and *target*, defined by their value-sort. We also extend the above notational convention on the representation of formal arrows to Ar-terms.) We identify, of course, the formal arrows in Ar with their *vis-à-vis* as Ar-terms. Given any set-map $F{:}Ar \rightarrow Ar'$ between two sets of formal arrows, there is a natural extension of F (also denoted F) from Ar-terms to Ar'-terms, which preserves all the D_L-operations, namely the homomorphism of free D_L-algebras induced by F. Two Ar-terms are said to be *parallel* whenever they have the same source and target. A *formal identity of Ar-terms* is defined to be an (ordered) pair of parallel Ar-terms, usually written with the "$\approx$" symbol infixed between the first Ar-term of the pair (on the left), and the second (on the right).

We are now ready to give our main definition: a *cartesian closed category presentation* (*cccPres* for short) is a triple $(L;Ar;\Phi)$ where L is a language, Ar a set of formal arrows over L, and Φ a set of formal identities of Ar-terms. We remark that this gives rise to a calculus of deductions of formal identities. Without going into too much detail, we say that $(L;Ar;\Phi)$ deduces the formal identity $u \approx v$ (recorded $(L;Ar;\Phi) \triangleright u \approx v$), if

$u \approx v$ can be obtained as a theorem (an entailment(!), in which the $\approx$ symbol has replaced the usual $\succ$) from the deduction system having the set Φ and the cartesian closed identities as axioms plus a rule guaranteeing the symmetry of $\approx$ (reflexivity and transitivity, corresponding to the rules CUT and TAUT, are by definition present in any deduction system).

We form the category **CCCPres**, paralleling the construction of **IMSLPres**. Its objects are all cccPres's, and a morphism $F:(L;Ar;\Phi) \to (L';Ar';\Phi')$ between two such consists of two set-maps (both denoted F) $F:L \to L'$ and $F:Ar \to Ar'$, with the following properties:

(1) F preserves the source and target of formal arrows, i.e. if $f:A \to B$ is in Ar, then $F(f):F(A) \to F(B)$ (is in Ar').

(2) Whenever $t \approx u$ is a formal identity of Ar-terms in Φ, then the formal identity of Ar'-terms $F(t) \approx F(u)$ is in Φ'.

CCCPres is readily seen to be a category. We observe that cccPres morphisms preserve deducibility (as a routine induction will show). Next, given a cccPres $P = (L;Ar;\Phi)$, we construct the *free cartesian closed category* $\mathrm{Free}^c(P)$ *over* P as follows: the objects of $\mathrm{Free}^c(P)$ are all L-formulas; given two objects A and B, the arrows from A to B are equivalence classes of Ar-terms with source A and target B, where the equivalence relation is the congruence relation on Ar-terms generated by both the cartesian closed identities (described earlier in this chapter), and the identities in Φ. The identity arrow 1_A on A is the (equivalence class of the) Ar-term 1_A, and composition of compatible arrows is performed by the (various instances of the) operation symbol $*$. By construction, this is well-defined, and the category also has, defined in the obvious way, (all instances of) the operators $\langle \_,\_ \rangle$ and $(\_)^{*}$ acting on its arrows, since the equivalence relation is in fact a congruence relation. Moreover, it is clear that $\mathrm{Free}^c(P)$ is indeed a cartesian closed category (with specified operation).

PROPOSITION 4.6. Free^c so defined extends to a functor Free^c:**CCCPres** $\to$ **CCC**.

PROOF. In what follows, A, B are L-formulas, and $t,u:A \to B$, are Ar-terms. In the course of this proof, we will represent the equivalence class corresponding to t by $[t]$, etc., but afterwards we will generally not distinguish notationally between individual terms and their equivalence classes.

We need only specify the effect of Free^c on cccPres homomorphisms. So let $F: P = (L;Ar;\Phi) \to P' = (L';Ar';\Phi')$. The functor(!) $\mathrm{Free}^c(F){:}\mathrm{Free}^c(P) \to \mathrm{Free}^c(P')$ is (induced by) F on both objects and arrows, i.e. $\mathrm{Free}^c(F)(A) = F(A)$, $\mathrm{Free}^c(F)(B) = F(B)$, and $\mathrm{Free}^c(F)([t]) = [F(t)]$. This is well-defined: certainly the source and target of $[F(t)]$ are legitimate, by definition of F; moreover, if $[t] = [u]$, using the fact that F preserves the identities in Φ, we can show by induction that $[F(t)] = [F(u)]$. $\mathrm{Free}^c(F)$ is a (strict) cartesian closed functor, again because F, by definition, preserves the D_L-operations, i.e. the cartesian closed operations. It is just as plain that Free^c preserves identities and composition, i.e. that it is indeed a functor.

It is interesting to observe that the functors $\mathrm{Free}^c{:}\mathbf{CCCPres} \to \mathbf{CCC}$ and $\mathrm{Free}^s{:}\mathbf{CCG} \to \mathbf{CCC}$ (introduced earlier) are related in the following way: given an L-theory T, construct a set $Ar(T)$ with a single formal arrow from A to B whenever the entailment $A \succ B$ is in T. (For example, put $Ar(T) = \{((A,B),A,B){:}A \succ B \in T\}$). We then have the isomorphisms $\mathrm{Free}^c(L;Ar(T);\phi) \cong \mathrm{Free}^s(G_L(T)) \cong F_L(T)$.

Now suppose the cccPres $(L;Ar;\Phi)$ deduces the formal identity $u \approx v$. Suppose further that we are given a cartesian closed category $\mathbf{C}$, and that we interpret elements of L as objects of $\mathbf{C}$, and elements of Ar as arrows of $\mathbf{C}$, such that the two interpretations are compatible (i.e. the source and target of arrows are preserved). Lastly, suppose that, under this interpretation, every formal identity in Φ happens to be verified in $\mathbf{C}$. Then clearly it will follow that $u \approx v$ will also be interpreted in $\mathbf{C}$ as an equality. Again, we can define the semantic counterpart to the notion of deducibility, by saying that the formal identity $u \approx v$ is *true in* $(L;Ar;\Phi)$ if it is found to hold in any cartesian closed category under any interpretation making true all the formal identities in Φ. We denote this by $(L;Ar;\Phi) \rhd\!= u \approx v$. The general completeness theorem tells us that $(L;Ar;\Phi) \rhd u \approx v$ iff $(L;Ar;\Phi) \rhd\!= u \approx v$. This is again a consequence of the clear fact that $(L;Ar;\Phi) \rhd u \approx v$ iff $[u]=[v]$ in $\mathrm{Free}^c(L;Ar;\Phi)$. We will soon make use of these observations.

Given a cartesian closed category $\mathbf{C}$, let $L = |\mathbf{C}|$ be the underlying set of objects of $\mathbf{C}$. There is then an obvious "evaluation function" $\mathrm{Eval}{:}\{L\text{-formulas}\} \to |\mathbf{C}|$, which is the identity on $L = |\mathbf{C}|$, and preserves the cartesian closed operations. Next, we can define a set $Ar_{\mathbf{C}}$ of formal arrows over L as follows: given any two L-formulas A and B, for every arrow $f{:}\mathrm{Eval}(A) \to \mathrm{Eval}(B)$ in $\mathbf{C}$, we put the formal arrow (f,A,B) in $Ar_{\mathbf{C}}$. If $\mathrm{Arr}(\mathbf{C})$ denotes the set of arrows of $\mathbf{C}$, we have another "evaluation function" $\mathrm{Eval}'{:}\{Ar_{\mathbf{C}}\text{-terms}\} \to \mathrm{Arr}(\mathbf{C})$, with the following properties: Eval' takes the formal arrow (f,A,B) to the arrow $f:\mathrm{Eval}(A) \to \mathrm{Eval}(B)$ of $\mathbf{C}$; moreover, Eval′ takes an Ar-term $t:A \to B$ to a certain arrow $\mathrm{Eval}'(t):\mathrm{Eval}(A) \to \mathrm{Eval}(B)$, and does this in such a way as to uniformly preserve the operation symbols in D_L.

So let $\mathbf{C}$ be a cartesian closed category. We define a cccPres $\mathrm{Pres}^c(\mathbf{C}) = (L;Ar;\Phi)$ with $L = |\mathbf{C}|$, $Ar = Ar_{\mathbf{C}}$, and $\Phi = \{t \approx u : t \text{ and } u \text{ are parallel, and } \mathrm{Eval}'(t) = \mathrm{Eval}'(u)\}$. Φ is said to be the *diagram* of $\mathbf{C}$, denoted $\Phi = \mathrm{Diag}(\mathbf{C})$.

PROPOSITION 4.7. Pres^c so defined extends to a faithful functor $\mathrm{Pres}^c:\mathbf{CCC} \to \mathbf{CCCPres}$, right adjoint to $\mathrm{Free}^c:\mathbf{CCCPres} \to \mathbf{CCC}$.

PROOF. First we show how Pres^c is a functor by specifying its effect on an arbitrary (strict) cartesian closed functor $F:\mathbf{C} \to \mathbf{C}'$ between cartesian closed categories. We give the two components of $\mathrm{Pres}^c(F):(|\mathbf{C}|;Ar_{\mathbf{C}};\mathrm{Diag}(\mathbf{C})) \to (|\mathbf{C}'|;Ar_{\mathbf{C}'};\mathrm{Diag}(\mathbf{C}'))$ separately. For $X \in |\mathbf{C}|$, $\mathrm{Pres}^c(F)(X) = F(X)$; for $f:A \to B \in Ar_{\mathbf{C}}$ (with A, B $|\mathbf{C}|$-formulas), we set $\mathrm{Pres}^c(F)(f) = (F(\mathrm{Eval}'_{\mathbf{C}}(f)), \mathrm{Pres}^c(F)(A), \mathrm{Pres}^c(F)(B))$. A few routine calculations will show that this is well-defined, and that Pres^c is a functor. That it is a faithful one should be plain.

We now tackle the adjunction. Let $(L;Ar;\Phi)$ be a cccPres, and $\mathbf{C}$ be a cartesian closed category. Given a cartesian closed functor $F:\mathrm{Free}^c(L;Ar;\Phi) \to \mathbf{C}$, we will show how to obtain a morphism $G:(L;Ar;\Phi) \to (|\mathbf{C}|;Ar_{\mathbf{C}};\mathrm{Diag}(\mathbf{C}))$ in a natural way: for X in L, put $G(X) = F(X)$, and if $A \xrightarrow{f} B$ is a formal arrow in Ar, put $G(f) = (F([f]), G(A), G(B))$. We need to verify this is well-defined: suppose we have the formal identity $u \approx v$ in Φ. Then $[u] = [v]$ will hold in $\mathrm{Free}^c(L;Ar;\Phi)$, i.e. $F([u]) = F([v])$ in $\mathbf{C}$, so by definition the identity $G(u) \approx G(v)$ will indeed be in $\mathrm{Diag}(\mathbf{C})$. Going in the other direction, suppose G is given. This is how we construct F: if A is an L-formula, $F(A) = \mathrm{Eval}(G(A))$; if $A \xrightarrow{[u]} B$ is an arrow $\mathrm{Free}^c(L;Ar;\Phi)$, set $F([u]) = \mathrm{Eval}'(G(u))$. This is well-defined because if, for instance, we have $[u] = [v]$, then as seen above, it must be the case that $(L;Ar;\Phi) \rhd u \approx v$. Since morphisms preserve deducibility, we obtain $(|\mathbf{C}|;Ar_{\mathbf{C}};\mathrm{Diag}(\mathbf{C})) \rhd G(u) \approx G(v)$, and thus $\mathrm{Eval}'(G(u)) = \mathrm{Eval}'(G(v))$. It is also easy to see that F will preserve identities and composition, as well as the whole cartesian closed structure, i.e. that it is a (strict) cartesian closed functor. We leave it to the reader to convince himself that the double procedure described here is indeed a (natural) bijection, so that $\mathrm{Free}^c \dashv \mathrm{Pres}^c$.

It is worth commenting on the fact that, perhaps contrary to one's expectations, Pres^c is *not* full. The reason is that the objects of the category get "de-specified", in some sense, when we pass to the cccPres. To take a very simple example, imagine a cartesian closed category **C** which has a specified terminal object $\hat{\mathbf{t}}$ and a single other object, $\hat{\mathbf{t}}'$, isomorphic to it. Then both $\hat{\mathbf{t}}$ and $\hat{\mathbf{t}}'$ get recorded as mere "atom-objects" in $\mathrm{Pres}^c(\mathbf{C})$, with appropriate formal identities to ensure that they are both isomorphic not only to each other but also to the new "formula-object" **t**. Notice that in $\mathrm{Pres}^c(\mathbf{C})$, $\hat{\mathbf{t}}$ and $\hat{\mathbf{t}}'$ are indistinguishable, in the sense that the fact that the terminal object in **C** was $\hat{\mathbf{t}}$ and not $\hat{\mathbf{t}}'$ is absolutely not kept track of anywhere. But any (strict) cartesian closed endofunctor on **C** must take $\hat{\mathbf{t}}$ to $\hat{\mathbf{t}}$ (and $\hat{\mathbf{t}}'$ to either $\hat{\mathbf{t}}$ or $\hat{\mathbf{t}}'$). With the mapping of the cccPres to itself, however, it is assumed the formula **t** will go to **t**, but $\hat{\mathbf{t}}$ and $\hat{\mathbf{t}}'$ are free to be mapped to themselves or each other – something **CCC** cannot "keep up" with (mapping $\hat{\mathbf{t}}$ to $\hat{\mathbf{t}}'$ is forbidden because of the strictness of functors). There will therefore in general always be less morphisms between categories than between their presentations. (This problem didn't occur with imsl's because by definition, isomorphic objects are automatically identified in imsl's; therefore no new objects were ever really created like in the present case.)

This state of affairs is slightly annoying, because it implies that some information is lost when passing from a cartesian closed category to its presentation. We will see that this is in fact rather benign, but let us still briefly consider what we could have done to ensure that Pres^c be full. There are two possibilities, based either on modifying **CCC** or **CCCPres**:

It is pretty clear from the discussion above that, if we redefined **CCC** such that its objects were still cartesian closed categories with specified operations, *but* its morphisms were allowed to be non-strict cartesian closed functors, then Pres^c would be full. The question then becomes which of the two alternatives, this hybrid version of **CCC**, or the non-fullness of the Pres^c functor, we dislike the most. In our case, that would be the former.

Looking at altering the definition of cccPres's leads us to two sub-possibilities: the first is simply to allow for formal identities to be recorded between non-parallel formal arrows, with the convention that, when this happens, the sources and targets of the two arrows will be identified. (This is the approach put forth in [LS] in a slightly different context.) Of course, under the appropriate conditions this suffices, because for one thing any two objects A and B can be forcefully collapsed together, by postulating the formal identity $1_A = 1_B$. However, it seems to us this goes against the spirit of type theory, by negating the purpose of having particular sources and targets for arrows in the first place. The second option is to redefine a cccPres to be a quadruple $(L;\Sigma;Ar;\Phi)$, where Σ is a set of formal identities between L-formulas, and Φ is a set of formal identities between parallel Ar-terms, where "parallel" here means that two Ar-terms must have the same source and same target modulo the identities in Σ. Here at least the spirit of type theory is maintained, albeit at the cost of having to deal with a more complex structure. In the final analysis, it all really comes down to a matter of personal preference.

The fact that Pres^c isn't full unfortunately entails that the counit of the adjunction $\mathrm{Free}^c \dashv \mathrm{Pres}^c$ cannot be an isomorphism (see [CWM]). However, we have:

PROPOSITION 4.8. Any cartesian closed category $\mathbf{C}$ is equivalent to $\mathrm{Free}^c(\mathrm{Pres}^c(\mathbf{C}))$.

PROOF. Consider the functor $i:\mathbf{C} \to \mathrm{Free}^c(\mathrm{Pres}^c(\mathbf{C}))$ which takes $A \xrightarrow{f} B$ to $A \xrightarrow{[(f,A,B)]} B$. First we show that every formula B of $\mathrm{Free}^c(\mathrm{Pres}^c(\mathbf{C}))$ is isomorphic to $i(A)$ for some object A in $\mathbf{C}$. Put $A = \mathrm{Eval}(B)$. Then by definition, one has the following four formal arrows in $\mathrm{Pres}^c(\mathbf{C})$: $(\ulcorner 1_A \urcorner, A, A)$, $(\ulcorner 1_B \urcorner, B, B)$, $(\ulcorner 1_A \urcorner, A, B)$ and $(\ulcorner 1_B \urcorner, B, A)$. Here the notation "$\ulcorner \_ \urcorner$" is meant to avoid confusing, e.g., the formal arrow $(\ulcorner 1_A \urcorner, A, A)$ (postulated by the existence of the arrow $1_A : A \to A$ in $\mathbf{C}$), with the $Ar_{\mathbf{C}}$-term $(1_A, A, A)$ (postulated by the D_L nullary operation symbol 1_A). We also caution the reader not to confuse the distinct arrows above which are anti-parallel, but happen to have the same name. By definition, the following formal identities will necessarily be included in $\mathrm{Pres}^c(\mathbf{C})$: $(\ulcorner 1_B \urcorner, B, A) * (\ulcorner 1_A \urcorner, A, B) \approx (\ulcorner 1_A \urcorner, A, A)$ and $(\ulcorner 1_A \urcorner, A, B) * (\ulcorner 1_B \urcorner, B, A) \approx (\ulcorner 1_B \urcorner, B, B)$. The corresponding equalities will therefore hold in $\mathrm{Free}^c(\mathrm{Pres}^c(\mathbf{C}))$, which of course means that $A = i(A)$ and B are isomorphic. Next, we show that i is full and faithful. For faithfulness, we observe that, for $f,g:A \to B$ two arrows in $\mathbf{C}$, $[(f,A,B)] = [(g,A,B)]$ in $\mathrm{Free}^c(\mathrm{Pres}^c(\mathbf{C}))$ iff $\mathrm{Pres}^c(\mathbf{C}) \rhd (f,A,B) \approx (g,A,B)$ iff $(f,A,B) \approx (g,A,B)$ is already a formal identity of $\mathrm{Pres}^c(\mathbf{C})$ iff $f = g$ in $\mathbf{C}$. For fullness, we first note that since distinct objects of $\mathbf{C}$ are never identified in $\mathrm{Free}^c(\mathrm{Pres}^c(\mathbf{C}))$, all the arrows from A to B are of the form $[(f,A,B)]$ for some Ar-term f. But clearly, this is the image under i of the arrow $\mathrm{Eval}'(f):A \to B$ of $\mathbf{C}$, since the formal identity $(f,A,B) \approx (\mathrm{Eval}'(f),A,B)$ must belong to $\mathrm{Pres}^c(\mathbf{C})$. Thus i is indeed full. That it is in fact an equivalence now follows from a standard theorem of category theory (see, e.g., [CWM]). Interestingly, it can be shown that it is the component at $\mathbf{C}$ of the counit of the adjunction $\mathrm{Free}^c \dashv \mathrm{Pres}^c$, a functor $\varepsilon_{\mathbf{C}}:\mathrm{Free}^c\big(\mathrm{Pres}^c(\mathbf{C})\big) \to \mathbf{C}$, which is the equivalence going in the other direction.

The last connections we want to investigate are those occurring between the categories **IMSLPres** and **CCCPres**. There are two interesting ways in which the two structures are related. More precisely, there is, as far as we can see, only one reasonable way to go from right to left, namely, to simply postulate an entailment between two formulas in the imslPres whenever there is a formal arrow between the same two formulas in the cccPres, and to disregard the set of formal identities. In the other direction, however, we have two possibilities. To start with, we postulate a single formal arrow between two formulas whenever there is an entailment between them. The real choice occurs at the level of the set of identities: we can either take the empty set, or the maximal set (i.e., the one in which an identity is postulated between any pair of parallel formal arrows). Both options are interesting, as will be seen below.

We first define a functor Simp^p:**CCCPres** $\to$ **IMSLPres** as follows: given a cccPres $(L;Ar;\Phi)$, we let $\mathrm{Simp}^p(L;Ar;\Phi)$ be the imslPres $(L;\Sigma)$, where, for any L-formulas A and B, $A \succ B \in \Sigma$ if and only if there exists u such that $(u,A,B) \in Ar$. As for morphisms, Simp^p takes a cccPres morphism to its restriction to L. It is trivial to check that Simp^p is indeed a functor.

Now we define a functor Comp^p:**IMSLPres** $\to$ **CCCPres** in the other direction. It takes the imslPres $(L;\Sigma)$ to the cccPres $(L;Ar;\Phi)$, where $Ar = Ar(\Sigma)$ (we recall this was defined to be the set $\{((A,B),A,B):A \succ B \in \Sigma\}$), and $\Phi = \{u = v: u, v \text{ are parallel } Ar\text{-terms}\}$. Comp^p takes a morphism between two imslPres's into its unique extension as a morphism between cccPres's with the same behaviour on L. Again, it is clear this defines a functor.

PROPOSITION 4.9. Comp^p:**IMSLPres** $\to$ **CCCPres** is full and faithful, and is right adjoint to Simp^p:**CCCPres** $\to$ **IMSLPres**.

We omit the very easy proof, only noting that it uses the two following facts: given the L-formulas A and B, there is always at most one formal arrow with source A and target B in $\mathrm{Comp}^p(L;\Sigma)$ (for any theory Σ); and given any cccPres $(L;\Sigma;\Phi)$ and imslPres $(L';\Sigma')$, the conditions on the set of identities between a potential morphism

$f:(L;\Sigma;\Phi) \to \mathrm{Comp}^p(L';\Sigma')$ will always be satisfied, because the set of identities of $\mathrm{Comp}^p(L';\Sigma')$ is maximal.

We now define the functor $\overline{\mathrm{Comp}}^p$:**IMSLPres** $\to$ **CCCPres**. It is the same as Comp^p, except that, in the notation above, we would have $\Phi = \phi$. $\overline{\mathrm{Comp}}^p$ is also easily seen to be full and faithful, but of course, because of the uniqueness of adjoints, we cannot possibly have $\mathrm{Simp}^p \dashv \overline{\mathrm{Comp}}^p$ (as direct inspection will also confirm). However, it is possible to express $\overline{\mathrm{Comp}}^p$ as the composite of two functors who do have adjoints, as we now demonstrate.

We define an *imsl multi-presentation* (*imslMultiPres* for short) to be a pair $(L;Ar)$, where Ar is a set of formal arrows over L. Given two imslMultiPres's $(L;Ar)$ and $(L';Ar')$, a morphism $f:(L;Ar) \to (L';Ar')$ consists of two set maps (both denoted f) $f:L \to L'$ and $f:Ar \to Ar'$, with the latter preserving the source and target of arrows. We organize these things into a category **IMSLMultiPres**.

We have an obvious inclusion functor inc^p:**IMSLPres** $\to$ **IMSLMultiPres** taking an imslPres to the imslMultiPres over the same language, and whose set of arrows contains a single arrow $((A,B),A,B)$ for every entailment $A \succ B$ in the imslPres. It is obvious what the effect of inc^p on morphisms should be. On the other hand, we have a functor Ent^p:**IMSLMultiPres** $\to$ **IMSLPres** which leaves the language fixed, and postulates an entailment between two formulas as long as there is at least one formal arrow between these formulas in the imslMultiPres. Again, the morphism part of the functor is rather obvious.

We define the functor Triv^p:**IMSLMultiPres** $\to$ **CCCPres** taking the imslMultiPres $(L;Ar)$ to the cccPres $(L;Ar;\phi)$, and behaving as the identity on morphisms. We also have the forgetful functor Forg^p:**CCCPres** $\to$ **IMSLMultiPres** in the other direction, which simply drops the set of identities from the cccPres.

PROPOSITION 4.10. inc^p and Triv^p are both full and faithful, and the following two adjunctions hold: $\mathrm{Ent}^p \dashv \mathrm{inc}^p$, $\mathrm{Triv}^p \dashv \mathrm{Forg}^p$. Moreover, $\overline{\mathrm{Comp}}^p = \mathrm{Triv}^p \circ \mathrm{inc}^p$ and $\mathrm{Simp}^p = \mathrm{Ent}^p \circ \mathrm{Forg}^p$.

These facts are easily seen upon inspection. We omit the proof.

The major results of this chapter can be concisely summarized in the following two diagrams:

$$
\begin{tikzcd}
\mathrm{IMSLPres} \arrow[r, shift right, "\overline{\mathrm{Comp}}^p(ff)"'] \arrow[d, shift right, "\mathrm{Free}^p"'] \arrow[r, phantom, "\perp"] \arrow[d, phantom, "\dashv"] & \mathrm{CCCPres} \arrow[l, shift right, "\mathrm{Simp}^p"'] \arrow[d, shift right, "\mathrm{Free}^c"'] \arrow[d, phantom, "\dashv"] \\
\mathrm{IMSL} \arrow[u, shift right, "\begin{matrix}\mathrm{Pres}^p\\(ff)\end{matrix}"'] \arrow[r, shift right, "i^p(ff)"'] \arrow[r, phantom, "\perp"] & \mathrm{CCC} \arrow[l, shift right, "\mathrm{Po}"'] \arrow[u, shift right, "\begin{matrix}\mathrm{Pres}^c\\(\mathrm{faith.})\end{matrix}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{IMSLMultiPres} \arrow[dl, shift right, "\mathrm{Ent}^p"'] \arrow[dl, phantom, "\perp"] \arrow[dr, shift right, "\mathrm{Diag}^p(ff)"'] \arrow[dr, phantom, "\top"] & \\
\mathrm{IMSLPres} \arrow[ur, shift right, "\mathrm{inc}^p(ff)"'] \arrow[rr, shift left, "\overline{\mathrm{Comp}}^p(ff)"] \arrow[d, shift right, "\mathrm{Free}^p"'] \arrow[d, phantom, "\dashv"] & & \mathrm{CCCPres} \arrow[ul, shift right, "\mathrm{Forget}^p"'] \arrow[ll, shift left, "\mathrm{Simp}^p"] \arrow[d, shift right, "\mathrm{Free}^c"'] \arrow[d, phantom, "\dashv"] \\
\mathrm{IMSL} \arrow[u, shift right, "\begin{matrix}\mathrm{Pres}^p\\(ff)\end{matrix}"'] \arrow[rr, shift right, "i^p(ff)"'] \arrow[rr, phantom, "\perp"] & & \mathrm{CCC} \arrow[ll, shift right, "\mathrm{Po}"'] \arrow[u, shift right, "\begin{matrix}\mathrm{Pres}^c\\(\mathrm{faith.})\end{matrix}"']
\end{tikzcd}
$$

They allow us to move back and forth between all the different algebraic structures we have defined. Of course, some information may be lost through some passages, especially when going "left", and to a lesser extent when going "down". More precisely, the full and faithful right adjoints all completely preserve information (i.e., if you compose them with their left adjoint in the other direction, you get the same object (or morphism) back (up to isomorphism)), whereas the faithful functor $\mathrm{Pres}^c : \mathbf{CCC} \to \mathbf{CCCPres}$ loses no information up to equivalence of categories. There are of course also a number of commutative equalities holding, all of which may be established either by direct inspection, or with the help of some standard theorems such as the uniqueness (up to isomorphism) of adjoints and the fact that the composition of two left (respectively right) adjoints is again a left (right) adjoint (see [CWM] for details). We illustrate the precept with the following proposition:

PROPOSITION 4.11. $\mathrm{Free}^p = \mathrm{Po} \circ \mathrm{Free}^c \circ \overline{\mathrm{Comp}}^p : \mathbf{IMSLPres} \to \mathbf{IMSL}$.[10]

[10]*Literal* equality, in fact, will not necessarily hold; the point is that the two functors can really only be guaranteed to be (naturally) isomorphic. What this means in practice is that, for instance, they could map the same imslPres to two isomorphic imsl's, which differ (say) only by the way in which each set-theoretically records the partial order relation on its elements, etc.... One would certainly *feel* that the two imsl's ought to be considered the same, but technically this isn't the case. In fact, there would be no way of ensuring the strict equality of the two functors above short of rewriting just about every definition in this thesis specifically at the lowest set-theoretical level. Of course, as with the vast majority of mathematical structures (like groups, fields, etc...), we are more than happy to call equal two merely isomorphic entities. The proof we give therefore overlooks the subtle distinction between (strict) equality and (mere) isomorphism, and should therefore be viewed as only "morally" correct...

PROOF. First we observe that $\mathrm{inc}^p \circ \mathrm{Pres}^p = \mathrm{Forg}^p \circ \mathrm{Pres}^c \circ i^p : \mathbf{IMSL} \rightarrow \mathbf{IMSLMultiPres}$. This is a simple inspection, but let us spell it out nevertheless. Start with an imsl P. Its presentation is the pair $(L;T)$ where $L = |P|$ and T records once every entailment of P. Then inc^p takes this imslPres to the imslMultiPres $(L;T)$ (the same!), except now that "entailments" (in T) are called "arrows". Going around the other way, we start with i^p which keeps the same objects, but "transforms" an inequality between two objects of P into an arrow. Then Pres^c mutates this into a triple $(L;Ar;\Phi)$, where Ar is the set of arrows of $i^p(P)$. Lastly, Forg^p just drops the third component, giving us $(L;Ar)$, which clearly is the same as $(L;T)$. Similarly, one can see that the two functors will act in exactly the same way on an arbitrary homomorphism $f : P \rightarrow Q$. That is to say, they are indeed equal.

Because they are composites of right adjoints, $\mathrm{inc}^p \circ \mathrm{Pres}^p$ and $\mathrm{Forg}^p \circ \mathrm{Pres}^c \circ i^p$ are themselves right adjoints; their left adjoints are, of course, $\mathrm{Free}^p \circ \mathrm{Ent}^p$ and $\mathrm{Po} \circ \mathrm{Free}^c \circ \mathrm{Triv}^p$ respectively. But adjoints are unique, therefore we have $\mathrm{Free}^p \circ \mathrm{Ent}^p = \mathrm{Po} \circ \mathrm{Free}^c \circ \mathrm{Triv}^p : \mathbf{IMSLMultiPres} \rightarrow \mathbf{IMSL}$. Composing with inc^p, we get $\mathrm{Free}^p \circ \mathrm{Ent}^p \circ \mathrm{inc}^p = \mathrm{Po} \circ \mathrm{Free}^c \circ \mathrm{Triv}^p \circ \mathrm{inc}^p$. But $\mathrm{Ent}^p \circ \mathrm{inc}^p$ is the identity functor since inc^p is a full and faithful right adjoint, and $\mathrm{Triv}^p \circ \mathrm{inc}^p = \overline{\mathrm{Comp}}^p$ by proposition 4.10. Thus $\mathrm{Free}^p = \mathrm{Po} \circ \mathrm{Free}^c \circ \overline{\mathrm{Comp}}^p$ as claimed.

This proposition has a central importance, in that it claims that, if we start with a propositional theory T over a language L, whether we construct directly its Lindenbaum-Tarski algebra (an imsl), or first study its category of proofs (a cartesian closed category), and then only collapse the category into an imsl, *we get exactly the same thing*. This guarantees the consistency and compatibility of the model- and proof-theoretic endeavours.

In the next chapter, we will seek to extend these diagrams to the right, in the same spirit as what was done in this chapter.

Chapter 5

Bicategories as Two-Dimensional Models of the Lambek Calculus

We will attempt here to reinterpret the Lambek calculus for positive intuitionistic logic using certain bicategories, in a uniform and consistent way. As expected, these bicategories will turn out to be cartesian closed, and free is some kind of sense. In view of the fact that a large part of the work carried out here is, in essence, quite similar in format to what was done in the preceding chapter, we will occasionally pick up the pace, and at times cut down on the amount of details provided. We trust the clarity of the exposition will not suffer as a result.

Our first goal is to obtain an adjunction allowing us to move back and forth between **CCBiC** and **CCC**, much like we had one between **CCC** and **IMSL**. We define the functor i^c:**CCC** $\rightarrow$ **CCBiC**, taking a cartesian closed category to itself, viewed as a cartesian closed bicategory (i.e., one in which all the canonical 2-cells are simply identities), and taking a cartesian closed functor to itself viewed as a cartesian closed strict homomorphism (with trivial effect on the identity 2-cells). i^c is clearly full and faithful.

It is a bit trickier to define the functor Coll:**CCBiC** $\rightarrow$ **CCC** in the other direction. Given a cartesian closed bicategory **C**, we define an equivalence relation ~ on 1-cells as follows: ~ is the smallest equivalence relation such that, for parallel 1-cells u and v, u~v if there exists a 2-cell $\beta{:}u \Rightarrow v$ (note that β isn't required to be an isomorphism). Put another way, if we construct a non-oriented graph whose vertices are all 1-cells of **C**, with an edge between two vertices if and only if there is a 2-cell in **C** between them, we immediately get that the equivalence classes of ~ simply correspond to the path-connected components of the above graph. We remark that only parallel 1-cells can belong to the same equivalence class, and also note that all the functors defined on the Hom(_,_) categories (or product thereof) of **C** (such as, for example, the pairing functor $\langle\_,\_\rangle^C_{A,B}{:}\mathrm{Hom}(C,A)\times\mathrm{Hom}(C,B)\rightarrow\mathrm{Hom}(C,A\times B)$), preserve ~, as an easy induction will show. We represent the equivalence class of u as $|u|$ (and reserve the right to sometimes abuse notation and just identify 1-cells with their equivalence classes).

Let **C** be a bicategory. We define a category Coll(**C**) as follows: the objects of Coll(**C**) are the 0-cells of **C**, and, given two objects A and B, the morphisms between them are all equivalence classes $|u|$ with $u{:}A \to B$ in $\mathbf{C}$. The identity morphism on the object A is $|1_A|$, and, given $A \xrightarrow{|f|} B \xrightarrow{|g|} C$ in Coll($\mathbf{C}$), we define their composite as $|g| \circ |f| = |g * f|{:}A \to C$. This is well defined because $*$ is a functor. It is immediate that this function can be extended to a functor Coll:**Bicat** $\to$ **Cat**, by forgetting the effect of strict homomorphisms of 2-cells.

PROPOSITION 5.1 The restriction of Coll to **CCBiC** is a functor (also denoted) Coll:**CCBiC** $\to$ **CCC**.

PROOF. Suppose $\mathbf{C}$ is a cartesian closed bicategory. We must show Coll($\mathbf{C}$) is cartesian closed. To illustrate, we will show that Coll($\mathbf{C}$) has binary products: if A and B are two objects of Coll($\mathbf{C}$), their product diagram is $A \xleftarrow{|\pi_{A,B}|} A \times B \xrightarrow{|\pi'_{A,B}|} B$, where $A \xleftarrow{\pi_{A,B}} A \times B \xrightarrow{\pi'_{A,B}} B$ is of course the product diagram in $\mathbf{C}$. Now, given the further data $A \xleftarrow{|f|} C \xrightarrow{|g|} B$, put $\langle |f|, |g| \rangle = |\langle f, g \rangle|$. This is well defined, since $\langle \_, \_ \rangle$ is a functor. We get $|\pi| \langle |f|, |g| \rangle = |\pi| |\langle f, g \rangle| = |\pi \langle f, g \rangle| = |f|$, and similarly $|\pi'| \langle |f|, |g| \rangle = |g|$. Finally, if $|k|{:}C \to A \times B$ has the property that $|\pi||k| = |f|$ and $|\pi'||k| = |g|$, we get $\langle |f|, |g| \rangle = \langle |\pi||k|, |\pi'||k| \rangle = \langle |\pi k|, |\pi' k| \rangle = |\langle \pi k, \pi' k \rangle| = |k|$, i.e. $\langle |f|, |g| \rangle$ is unique, so we indeed do have a product. (The reader will have noticed that all we had to do here was essentially transcribe the proof of proposition 2.2; the crux of the matter is that, in this case, $\sim$ has all the properties required of $\equiv$ to make the argument go through. Which is why it would be rather pointless to etch out the proofs that Coll($\mathbf{C}$) has a terminal object and exponentials as well – the reader will find them in chapter 2.) Of course, it is quite clear Coll will take a cartesian closed strict homomorphism to a cartesian closed functor, completing the proof.

PROPOSITION 5.2 Coll:**CCBiC** $\to$ **CCC** is left adjoint to the full and faithful functor i^c:**CCC** $\to$ **CCBiC**.

PROOF. If $\mathbf{B}$ is a cartesian closed bicategory and $\mathbf{C}$ is a cartesian closed category, we indicate how to pass from a cartesian closed functor F:Coll($\mathbf{B}$) $\to \mathbf{C}$ to a cartesian closed strict homomorphism $G{:}\mathbf{B} \to i^c(\mathbf{C})$, and vice-versa. If F is given, we set, for a 0-cell A in $\mathbf{B}$, $G(A) = F(A)$; for a 1-cell $A \xrightarrow{f} B$ in $\mathbf{B}$, $G(f) = F(|f|)$; and for a 2-cell $\beta{:}f \Rightarrow g$ in $\mathbf{B}$, $G(\beta) = \mathrm{Id}_{F(|f|)=F(|g|)}$. Going in the other direction, we observe that, by default, G maps all the 2-cells in $\mathbf{B}$ to identities. From this and an easy induction, we conclude that if

$f \sim g$ holds between two 1-cells in **B**, then it must be the case that $G(f) = G(g)$. How one goes about constructing F is now obvious.

It is interesting to note that the component at **B** of the unit of the adjunction is a "surjective" strict cartesian closed homomorphism $\eta_{\mathbf{B}} : \mathbf{B} \to i^c \circ \mathrm{Coll}(\mathbf{B})$ which takes objects to themselves and 1-cells to their equivalence class.

The results stated in the above two propositions only deal with bicategories in which the maps are strict homomorphisms. Of course, similar results hold for bicategories and homomorphisms:

We define a functor $i'^c : \mathbf{CCC} \to \mathbf{CCBiC}'$ "identical" to $i^c : \mathbf{CCC} \to \mathbf{CCBiC}$. It takes a cartesian closed category to itself viewed as a cartesian closed bicategory, and a cartesian closed functor to itself, viewed as a cartesian closed homomorphism. Again, it is full and faithful. We also need a functor $\mathrm{Coll}' : \mathbf{CCBiC}' \to \mathbf{CCC}$; it is essentially defined the same way as $\mathrm{Coll} : \mathbf{CCBiC} \to \mathbf{CCC}$. The only small point to be made is that, even though homomorphisms do not necessarily literally preserve canonical 1-cells and functors, they preserve them up to natural equivalence, which of course is largely sufficient for us, as in the collapsed category, isomorphic 1-cells get identified with one another.

PROPOSITION 5.2′. $\mathrm{Coll}' : \mathbf{CCBiC}' \to \mathbf{CCC}$ is left adjoint to the full and faithful functor $i'^c : \mathbf{CCC} \to \mathbf{CCBiC}'$.

The proof is identical to that of the mirror proposition 5.2.

Our next step is to define the algebraic structure used to present cartesian closed bicategories. We first need some preliminaries. Let a language L and a set Ar of formal arrows over L be given. We want to define an algebra for operating on 2-cells, the signature of which we will call $2D_L$. Recall the generalised algebraic theory of cartesian closed bicategories from chapter 3. Consider the sorted operation symbols which have 2Hom(_,_) as generic value type. The typing rule associated with such a symbol implicitly indexes it with a number of variables, those of generic type Ob or Hom(_,_). If we set Ob to contain all L-formulas, and, for two L-formulas A and B, we set Hom(A,B) to be the collection of all Ar-terms with source A and target B, instantiating the above variables with all possible values naturally gives us, for each of the operation symbols having 2Hom(_,_) as generic value type, an indexed family of operation symbols. We let $2D_L$ simply be the union of all these families.

For example, the symbol τ has the typing rule:

$$\frac{A,B,C \in \mathrm{Ob}, f \in \mathrm{Hom}(C,A), g \in \mathrm{Hom}(C,B)}{\tau(f,g) \in 2\mathrm{Hom}\big(*(\langle f,g\rangle, \pi(A,B)), f\big).}$$

Writing $f^{A,B}$ instead of the usual, but more cumbersome, (f,A,B), to represent an Ar-term with source A and target B, we get that $2D_L$ contains an operation symbol $\tau_{A,B,C,f^{C,A},g^{C,B}}$ for every possible triple (A, B, C) of L-formulas, for every possible Ar-term $f{:}\,C \to A$, and for every possible Ar-term $g{:}\,C \to B$. Because there are no variables with generic type 2Hom(_,_) amongst the premisses of the rule, each such $\tau_{A,B,C,f^{C,A},g^{C,B}}$ will be a nullary, or constant, operation symbol, of *value-type* $\left(\pi\left(f^{C,A}, g^{C,B}\right)\right)^{C,A} \Rightarrow f^{C,A}$. (This last expression simply denotes an ordered pair of Ar-terms – all types under consideration are defined to be of that form.)

As another example, consider the operation symbol $\circ$, whose typing rule is:

$$\frac{A,B \in \mathrm{Ob}, f,g,h \in \mathrm{Hom}(A,B), \beta \in 2\mathrm{Hom}(f,g), \gamma \in 2\mathrm{Hom}(g,h)}{\circ(\beta,\gamma) \in 2\mathrm{Hom}(f,h).}$$

This time, in $2D_L$ we will need to put the operation symbol $\circ_{A,B,f^{A,B},g^{A,B},h^{A,B}}$ for every possible L-formulas A and B, and every possible Ar-terms $f,g,h{:}\,A \to B$. Of course, these $\circ_{A,B,f^{A,B},g^{A,B},h^{A,B}}$ will all be sorted binary operation symbols, with *argument-types* $f^{A,B} \Rightarrow g^{A,B}$ and $g^{A,B} \Rightarrow h^{A,B}$ (in that order), and value-type $f^{A,B} \Rightarrow h^{A,B}$.

Going through the full list, we make the observation that it is sufficient to supply only the Ar-terms as indexing information, as they themselves include all information about the L-formulas. We can thus rewrite the two operation symbols above as $\tau_{f^{C,A},g^{C,B}}$ and $\circ_{f^{A,B},g^{A,B},h^{A,B}}$. Of course, this notation doesn't provide information as to the argument-types (if any), nor the value-type, of the operation symbol in question – this has to be retrieved from the appropriate Cartmell typing rule.

We define a *formal 2-cell* over Ar to be any triple $(\beta, u^{A,B}, v^{A,B})$, where $u,v{:}\,A \to B$ are arbitrary parallel Ar-terms (we will occasionally just write β however). We call u the *source*, and v the *target* of β, and consider β to have no argument-type, and value-type $u \Rightarrow v$. Now, given a set $2Ar$ of formal 2-cells, we consider the absolutely free $2D_L$-algebra of terms generated by $2Ar$. The terms of this algebra we call $2Ar$-terms. Naturally,

$2Ar$-terms are closed terms, and thus have no argument-type, and a single value-type associated to them, giving them a source and target in the obvious way. Two $2Ar$-terms are *parallel* when they have the same source and the same target. A *formal identity of $2Ar$-terms* is defined to be an (ordered) pair of parallel $2Ar$-terms, written with the " $\approx$ " symbol infixed.

There is a certain collection of formal identities of $2Ar$-terms that are very important to us, namely those ensuring that each Hom(_,_) is a category, that certain $2Ar$-terms are invertible, that certain families of $2Ar$-terms globally satisfy some naturality and/or coherence conditions, and that certain operation symbols in $2D_L$ are expected to behave as functors. The (long!) list of all these identities is in fact contained, once again, within the syntax of the presentation of cartesian closed bicategories as generalised algebraic theories: they correspond to *all* the axioms listed there, each of which is a (generic) instance of some identity between certain 2-cells. For example, the axiom

$$\circ\big(\lambda^{-1}(f), \lambda(f)\big) = \mathrm{Id}(f), \quad \text{whenever } A, B \in \mathrm{Ob} \text{ and } f \in \mathrm{Hom}(A, B)$$

which in our new notation would read

$$\lambda_{f^{A,B}} \circ_{f^{A,B},(1_B f^{A,B})^{A,B},f^{A,B}} \lambda^{-1}_{f^{A,B}} \approx \mathrm{Id}^f_{f^{A,B}},$$

asserts half of the fact that $\lambda_{f^{A,B}}$, as well, of course, as $\lambda^{-1}_{f^{A,B}}$, are (meant to represent) isomorphisms. We point out that $\lambda^{-1}_{f^{A,B}}$ is a nullary operation symbol in its own right, of type $f^{A,B} \Rightarrow 1_B f^{A,B}$, entirely distinct from $\lambda_{f^{A,B}}$ (the two are anti-parallel). The nullary operation symbol $\mathrm{Id}^f_{f^{A,B}}$, of type $f^{A,B} \Rightarrow f^{A,B}$, is meant to represent the identity 2-cell on $f^{A,B}$; of course, there are identities elsewhere to ensure that as well. Finally, we remark that the "formal identity" stated above isn't quite one: technically, we require that the L-formulas A, B, and the Ar-term $f^{A,B}$, be instantiated to some particular, fixed, $2Ar$-terms (as opposed to being "generic variables", as they are here). We call the collection of all possible instantiations of all the identities obtained in this way from the axioms of the generalised algebraic theory, the ***cartesian closed bicategorical identities***. It is worth mentioning that these identities do indeed only pair up parallel $2Ar$-terms.[1]

---

[1] It would be a small step at this point to fully turn this whole setup into a "two-dimensional" Lambek calculus, providing us, after the manner of §4, with a notational system for "second order" deductions (usually known as *reductions*), etc. This would lead us to study some further (multi-sorted) propositional logics, and

We are now in a position to give our main definition. A *cartesian closed bicategory presentation* (*ccbicPres* for short) consists of a quadruple $(L;Ar;2Ar;\Psi)$, where L is a language, Ar is a set of formal arrows over L, $2Ar$ is a set of formal 2-cells over Ar, and Ψ is a set of formal identities of $2Ar$-terms. Given $(L;Ar;2Ar;\Psi)$ and $(L';Ar';2Ar';\Psi')$ two ccbicPres's, a morphism $F:(L;Ar;2Ar;\Psi) \to (L';Ar';2Ar';\Psi')$ between the two consists of three set maps (all denoted F) $F:L \to L'$, $F:Ar \to Ar'$ and $F:2Ar \to 2Ar'$ preserving the source and target of both formal arrows and formal 2-cells, as well as the formal identities of $2Ar$-terms. This gives rise to the category **CCBiCPres**.

So let $P=(L;Ar;2Ar;\Psi)$ be a ccbicPres. We construct a cartesian closed bicategory $\mathrm{Free}^b(P)$ as follows. The 0-cells of $\mathrm{Free}^b(P)$ are all L-formulas. The 1-cells are all Ar-terms (between the appropriate objects), and the 2-cells are the congruence classes of $2Ar$-terms (between appropriate 1-cells), where the congruence relation in question is the one generated by both the cartesian closed bicategorical identities and the identities in Ψ. That we require a congruence relation (as opposed to a mere equivalence relation) simply means that the underlying equivalence relation should be preserved by all the $2D_L$ operation symbols (in particular the ones which aren't nullary!).

PROPOSITION 5.3. Free^b is well-defined, and extends to a functor Free^b:**CCBiCPres** $\to$ **CCBiC**.

PROOF. For the purposes of this proof, we will carefully distinguish $2Ar$-terms from their equivalence classes, which we represent (for an arbitrary $2Ar$-term β), as $[\beta]$. It should be relatively clear that, for $P=(L;Ar;2Ar;\Psi)$, $\mathrm{Free}^b(P)$ is indeed a cartesian closed bicategory, since the construction is obviously a Cartmell model of the generalised algebraic theory of cartesian closed bicategories as expounded in chapter 3. What we want to specify now is the effect of Free^b on a morphism $F:P=(L;Ar;2Ar;\Psi) \to P'=(L';Ar';2Ar';\Psi')$ between two ccbicPres's. The strict homomorphism $\mathrm{Free}^b(F):\mathrm{Free}^b(P) \to \mathrm{Free}^b(P')$ is defined as follows: for A a 0-cell (i.e., an L-formula), $\mathrm{Free}^b(F)(A)=F(A)$; for $u:A \to B$ a 1-cell (i.e., an Ar-formula), $\mathrm{Free}^b(F)(u)=F(u)$; and lastly, for $[\beta]:u \to v$ a 2-cell (where β is a $2Ar$-term), $\mathrm{Free}^b(F)([\beta])=[F(\beta)]$. This is shown by induction to be well-defined on account of the fact that F preserves the identities in Ψ. $\mathrm{Free}^b(F)$ is a cartesian closed strict homomor-

yield a proof theory of the original proof theory. We will not, however, explicitly pursue this avenue here, for lack of space.

phism because F preserves L-formulas, Ar-terms, and $2Ar$-terms. Moreover, Free^b plainly preserves identities and composition – in other words, it is indeed a functor.

We can immediately state the mirror proposition regarding **CCBiC′**. The definition of Free'^b is exactly the same as that of Free^b, and the proof is identical.

PROPOSITION 5.3′. Free'^b is a functor Free'^b:**CCBiCPres** → **CCBiC′**.

We recall that the set of 0-cells of a bicategory **C** is denoted Ob(**C**). Assume **C** is cartesian closed, and put $L = \mathrm{Ob}(\mathbf{C})$. We have the usual "evaluation function" $\mathrm{Eval}:\{L\text{-formulas}\} \to \mathrm{Ob}(\mathbf{C})$ which is the identity on $L = \mathrm{Ob}(\mathbf{C})$, and preserves the cartesian closed operations. Next, define a set $Ar_\mathbf{C}$ of formal arrows over L as follows: given two L-formulas A and B, for every 1-cell $f:\mathrm{Eval}(A) \to \mathrm{Eval}(B)$, put the formal arrow (f,A,B) in $Ar_\mathbf{C}$. If Hom(**C**) denotes the set of 1-cells of **C**, we have the other usual "evaluation function" $\mathrm{Eval}':\{Ar_\mathbf{C}\text{-terms}\} \to \mathrm{Hom}(\mathbf{C})$ with the obvious properties. Lastly, we define a set $2Ar_\mathbf{C}$ of formal 2-cells over $Ar_\mathbf{C}$ in the following manner: given two parallel $Ar_\mathbf{C}$-terms $u^{A,B}$ and $v^{A,B}$, for every 2-cell $\beta:\mathrm{Eval}'(u^{A,B}) \to \mathrm{Eval}'(v^{A,B})$, we put the formal 2-cell $(\beta,u^{A,B},v^{A,B})$ in $2Ar_\mathbf{C}$. If 2Hom(**C**) denotes the collection of all 2-cells of **C**, that gives us a third "evaluation function" $\mathrm{Eval}'':\{2Ar_\mathbf{C}\text{-terms}\} \to 2\mathrm{Hom}(\mathbf{C})$ taking the $2Ar_\mathbf{C}$-term $(\beta,u^{A,B},v^{A,B})$ to the 2-cell $\beta:\mathrm{Eval}'(u^{A,B}) \to \mathrm{Eval}'(v^{A,B})$, and preserving all of the $2D_L$ operations.

Let **C** be a cartesian closed bicategory. We define a ccbicPres $\mathrm{Pres}^b(\mathbf{C}) = (L;Ar;2Ar;\Psi)$ with $L = \mathrm{Ob}(\mathbf{C})$, $Ar = Ar_\mathbf{C}$, $2Ar = 2Ar_\mathbf{C}$, and $\Psi = \{\beta \approx \gamma : \beta \text{ and } \gamma \text{ are parallel, and } \mathrm{Eval}''(\beta) = \mathrm{Eval}''(\gamma)\}$. We call Ψ the *2-diagram* of **C**, written $\Psi = 2\mathrm{Diag}(\mathbf{C})$.

PROPOSITION 5.4 Pres^b so defined extends to a faithful functor Pres^b:**CCBiC** → **CCBiCPres**, right adjoint to Free^b:**CCBiCPres** → **CCBiC**.

PROOF. First we need to specify the effect of Pres^b on an arbitrary cartesian closed strict homomorphism $F:\mathbf{C} \to \mathbf{C}'$. Let's write $\mathrm{Pres}^b(\mathbf{C}) = (L;Ar;2Ar;\Psi)$ and $\mathrm{Pres}^b(\mathbf{C}') = (L';Ar';2Ar';\Psi')$. For $A \in L$, $\mathrm{Pres}^b(F)(A) = F_0(A)$; for $(f,A,B) \in Ar$, $\mathrm{Pres}^b(F)(f^{A,B}) = \left(F_1^{\mathrm{Eval}(A),\mathrm{Eval}(B)}\left(\mathrm{Eval}'(f^{A,B})\right), \mathrm{Pres}^b(F)(A), \mathrm{Pres}^b(F)(B)\right)$; and lastly, for $(\beta,u^{A,B},v^{A,B}) \in 2Ar$,

$\mathrm{Pres}^b(F)\left(\beta^{u^{A,B},v^{A,B}}\right) = \left(F_1^{\mathrm{Eval}(A),\mathrm{Eval}(B)}\left(\mathrm{Eval}''\left(\beta^{u^{A,B},v^{A,B}}\right)\right), \mathrm{Pres}^b(F)\left(u^{A,B}\right), \mathrm{Pres}^b(F)\left(v^{A,B}\right)\right)$. It is a matter of a few routine verifications to check that this assignment does indeed make Pres^b a functor. It is clearly faithful.

We now deal with the adjunction; we will show how to transform a cartesian closed strict homomorphism $F{:}\mathrm{Free}^b(L;Ar;2Ar;\Psi) = \mathbf{C} \to \mathbf{C}'$ into a morphism of ccbicPres's $G{:}(L;Ar;2Ar;\Psi) \to \mathrm{Pres}^b(\mathbf{C}') = \left(\mathrm{Ob}(\mathbf{C}') = L';Ar_{\mathbf{C}'};2Ar_{\mathbf{C}'};2\mathrm{Diag}(\mathbf{C}') = \Psi'\right)$, and vice-versa. So suppose F is given as above. For $A \in L$, put $G(A) = F_0(A)$; for $A \xrightarrow{f} B \in Ar$, put $G(f) = (F_1(f), G(A), G(B))$; and for $\beta{:}f^{A,B} \Rightarrow g^{A,B} \in 2Ar$, put $G(\beta) = \left(F_1([\beta]), G(f), G(g)\right)$. Now if the formal identity $\beta = \gamma$ is in Ψ, then of course $[\beta] = [\gamma]$ in $\mathbf{C}$, so $F([\beta]) = F([\gamma])$ in $\mathbf{C}'$, and therefore $G(\beta) = G(\gamma)$ is a formal identity in Ψ'. Going in the other direction, assume G is given as above, and construct F as follows: for A a 0-cell (an L-formula), $F_0(A) = \mathrm{Eval}(G(A))$; for $A \xrightarrow{f} B$ a 1-cell, $F_1(f) = \mathrm{Eval}'(G(f))$; and for $[\beta]{:}f^{A,B} \Rightarrow g^{A,B}$ a 2-cell, $F_1([\beta]) = \mathrm{Eval}''(G(\beta))$. This last step is legitimate for the usual reason, namely, an equality $[\beta] = [\gamma]$ holds only if it was already an implicit consequence of the formal identities in Ψ, which are preserved by G, and then realized through the Eval'' function – details are left to the reader. It is also clear that F is a cartesian closed strict homomorphism, and that what we have defined here is in fact a (natural) bijection, as required.

As expected from the previous chapter, Pres^b is not full, for the same sort of reasons. We therefore must rule out the hope that in general, a given cartesian closed bicategory $\mathbf{C}$ be isomorphic to $\mathrm{Free}^b\left(\mathrm{Pres}^b(\mathbf{C})\right)$. (Nevertheless, the same kind of fixes as those listed in chapter 4, which here would involve modifying either **CCBIC** or the definition of ccbicPres's, would work here as well – of course, we simply prefer the current setup and state of affairs.) One might wonder, however, whether we can get an analogue to proposition 4.8 (which stated that any cartesian closed category $\mathbf{C}$ was equivalent to $\mathrm{Free}^c\left(\mathrm{Pres}^c(\mathbf{C})\right)$) here. The answer is yes, but a full discussion of this would necessitate the introduction of further concepts, such as that of *2-natural transformations* (cf. [B2] or [GPS], for instance), which reasons of space prevent us from doing; we will nevertheless prove the following:

PROPOSITION 5.5. Given a cartesian closed bicategory $\mathbf{C}$, there is a cartesian closed strict homomorphism $I{:}\mathbf{C} \to \mathrm{Free}^b\left(\mathrm{Pres}^b(\mathbf{C})\right) = \mathbf{C}'$ which has the following properties:

(1) For any objects and 1-cell $A \xrightarrow{f} B$ in $\mathbf{C}'$, there is a diagram $\hat{A} \xrightarrow{\hat{f}} \hat{B}$ in $\mathbf{C}$ and 1-isomorphisms $i{:}I_0(\hat{A}) \xrightarrow{\cong} A$, $j{:}I_0(\hat{B}) \xrightarrow{\cong} B$ in $\mathbf{C}'$ such that the two 1-cells $fi{:}I_0(\hat{A}) \to B$ and $jI_1(\hat{f}){:}I_0(\hat{A}) \to B$ are isomorphic.

(2) For any objects A, B in $\mathbf{C}$, the functor $I_1^{A,B}{:}\mathrm{Hom}_{\mathbf{C}}(A,B) \to \mathrm{Hom}_{\mathbf{C}'}(I_0(A), I_0(B))$ is full and faithful.

PROOF. I is defined as follows: I_0 takes the object A in $\mathbf{C}$ to itself in $\mathbf{C}'$, and $I_1^{A,B}$ takes the 1-cell $f{:}A \to B$ in $\mathbf{C}$ to the formal arrow (f, A, B) in $\mathbf{C}'$ (recall the 1-cells of $\mathbf{C}'$ are A r-terms for some set A r); and if $\beta{:}f \Rightarrow g{:}A \to B$ is a 2-cell in $\mathbf{C}$, $I_1^{A,B}(\beta) = \left[\left(\beta, f^{A,B}, g^{A,B}\right)\right]$. That $I_1^{A,B}$ is full and faithful is proved in a manner analogous to that used in the corresponding sub-statement of proposition 4.8, and we will therefore not say more about it.

To show (1), we consider an arbitrary diagram $A \xrightarrow{f} B$ in $\mathbf{C}'$. Let us write $\hat{A} \xrightarrow{\hat{f}} \hat{B}$ for the diagram $\mathrm{Eval}(A) \xrightarrow{\mathrm{Eval}'(f)} \mathrm{Eval}(B)$ in $\mathbf{C}$. Now, because the two diagrams $\mathrm{Eval}(A) \xrightarrow{\mathrm{Eval}'(f)} \mathrm{Eval}(B)$ and $\mathrm{Eval}(\hat{A}) \xrightarrow{\mathrm{Eval}'(\hat{f})} \mathrm{Eval}(\hat{B})$ are actually identical, there will be some formal "1- and 2-isomorphisms" (written $\left(\ulcorner 1_A \urcorner, A, \hat{A}\right)$, $\left(\ulcorner \mathrm{Id}_{f_{A,B}}^{f} \urcorner, f^{A,B}, \hat{f}^{\hat{A},\hat{B}}\right)$, etc.) postulated between the respective components of this diagram, with the appropriate formal identities in $\mathrm{Pres}^b(\mathbf{C})$ to ensure the required commutativity conditions. This completes the proof.

Let us now have a look at what the corresponding situation with **BiCat′** is. First, one defines the functor $\mathrm{Pres}'^b{:}\mathbf{CCBiC}' \to \mathbf{CCBiCPres}$ exactly the same way $\mathrm{Pres}^b{:}\mathbf{CCBiC} \to \mathbf{CCBiCPres}$ was defined. We then have:

PROPOSITION 5.4′. Pres'^b is a faithful functor $\mathrm{Pres}^b{:}\mathbf{CCBiC} \to \mathbf{CCBiCPres}$.

The adjunction (between Free'^b and Pres'^b), unfortunately, does not hold here however. The reason, roughly speaking, is that there are in general too many cartesian closed homomorphisms between two cartesian closed bicategories, because the said homomorphisms do not have to preserve 1-cells on the nose, but just have to preserve them up to (natural) isomorphism; hence the failure of the required bijection between ccbicPres maps and cartesian closed strict homomorphisms. One might think that modifying the definition of **CCBiCPres** could solve the problem, but not at the cost of introducing a rather high level of artificiality and complexity to the definition. One's first attempt could be to require that ccbicPres morphisms preserve the D_L-operations on Ar-terms only up to some formal 2-cell. The problem then would be the opposite, i.e. there would be too many ccbicPres morphisms, since the 2-cells in questions wouldn't in general be invertible, let alone part of a natural family, and the whole collection of such is even less likely to satisfy the other coherence conditions that are required of the corresponding components of homomorphisms. Of course, it would be possible to "force" all these requirements into the definition of ccbicPres morphism, but one would hardly be pleased with the result. We will therefore have to satisfy ourselves with the "humbler" proposition 5.4′.

The corresponding contents of proposition 5.5, however, are completely unaffected, as they have little to do with morphisms and homomorphisms. The proof is the same as well.

PROPOSITION 5.5′. Given a cartesian closed bicategory **C**, there is a cartesian closed homomorphism (in fact, a strict homomorphism) $I:\mathbf{C} \to \mathrm{Free}'^{b}(\mathrm{Pres}'^{b}(\mathbf{C})) = \mathbf{C}'$ which has the following properties:

(1) For any objects and 1-cell $A \xrightarrow{f} B$ in $\mathbf{C}'$, there is a diagram $\hat{A} \xrightarrow{\hat{f}} \hat{B}$ in **C** and 1-isomorphisms $i:I_0(\hat{A}) \xrightarrow{\cong} A$, $j:I_0(\hat{B}) \xrightarrow{\cong} B$ in $\mathbf{C}'$ such that the two 1-cells $fi:I_0(\hat{A}) \to B$ and $jI_1(\hat{f}):I_0(\hat{A}) \to B$ are isomorphic.

(2) For any objects A, B in **C**, the functor $I_1^{A,B}:\mathrm{Hom}_{\mathbf{C}}(A,B) \to \mathrm{Hom}_{\mathbf{C}'}(I_0(A),I_0(B))$ is full and faithful.

We now turn our attention to the connections between **CCCPres** and **CCBiCPres**. We first define a functor Simp^c:**CCBiCPres** → **CCCPres** as follows: given a ccbicPres $(L;Ar;2Ar;\Psi)$, we let $\mathrm{Simp}^c(L;Ar;2Ar;\Psi) = (L;Ar;\Phi)$, where, for any pair of parallel Ar-formulas $f^{A,B}$ and $g^{A,B}$, the formal identity $f^{A,B} \approx g^{A,B}$ is in Φ if and only if there is a formal 2-cell $(\beta, f^{A,B}, g^{A,B})$ between $f^{A,B}$ and $g^{A,B}$ in Ψ. ccbicPres morphisms are simply taken to their restrictions as cccPres morphisms. That we indeed have a functor is easily checked.

We define a functor Comp^c:**CCCPres** → **CCBiCPres** in the other direction. It takes the cccPres $(L;Ar;\Phi)$ to the ccbicPres $(L;Ar;2Ar(\Phi);\Psi)$, where $2Ar(\Phi) =$

$\left\{\left(\left(f^{A,B},g^{A,B}\right),f^{A,B},g^{A,B}\right):f^{A,B}\approx g^{A,B}\in\Phi\right\}$ and $\Psi=\{\beta=\gamma{:}\beta,\gamma\text{ are parallel }2Ar\text{-terms}\}$. Comp^c takes a morphism between two cccPres's into its unique extension as a morphism between ccbicPres's with the same behaviour on L and Ar. Again, it is clear that this defines a functor.

PROPOSITION 5.6. $\mathrm{Comp}^c{:}\mathbf{CCCPres}\to\mathbf{CCBiCPres}$ is full and faithful, and right adjoint to $\mathrm{Simp}^c{:}\mathbf{CCBiCPres}\to\mathbf{CCCPres}$.

The proof of this is very similar to that of proposition 4.9 – in any case, it is rather simple. We therefore omit it.

We now define a functor $\overline{\mathrm{Comp}}^c{:}\mathbf{CCCPres}\to\mathbf{CCBiCPres}$, parallel to Comp^c: it is in fact the same as Comp^c, except that we would instead put $\Psi=\phi$ in the definition above. $\overline{\mathrm{Comp}}^c$ is clearly full and faithful. To factor $\overline{\mathrm{Comp}}^c$ as the composite of two functors which do have adjoints, we first need the following definition:

A *cartesian closed category multi-presentation* (*cccMultiPres* for short) is a triple $(L;Ar;2Ar)$, where Ar is a set of formal arrows over L, and $2Ar$ is a set of formal 2-cells over Ar. Given two cccMultiPres's $(L;Ar;2Ar)$ and $(L';Ar';2Ar')$, a morphism $F{:}(L;Ar;2Ar)\to(L';Ar';2Ar')$ between them consists of three set-maps (all denoted F) $F{:}L\to L'$, $F{:}Ar\to Ar'$ and $F{:}2Ar\to 2Ar'$, with the last two preserving the relevant sources and targets of formal arrows/formal 2-cells. This definition naturally gives us a category **CCCMultiPres**.

We have an obvious inclusion functor $\mathrm{inc}^c{:}\mathbf{CCCPres}\to\mathbf{CCCMultiPres}$ taking a cccPres to the cccMultiPres with the same language and same set of formal arrows, with a single formal 2-cell $\left(\left(f^{A,B},g^{A,B}\right),f^{A,B},g^{A,B}\right)$ for every formal identity $f^{A,B}\approx g^{A,B}$ in the cccPres. It should be equally clear what the effect of inc^c on morphisms should be. We can also define a functor $\mathrm{Ent}^c{:}\mathbf{CCCMultiPres}\to\mathbf{CCCPres}$ which again leaves the language and set of formal arrows fixed, and postulates a formal identity $f^{A,B}\approx g^{A,B}$ between two parallel Ar-terms if and only if there exists at least one formal 2-cell from $f^{A,B}$ to $g^{A,B}$ in the cccMultiPres. The morphism part of this functor is the obvious one.

We define the functor $\mathrm{Triv}^c{:}\mathbf{CCCMultiPres}\to\mathbf{CCBiCPres}$ taking the cccMultiPres $(L;Ar;2Ar)$ to the ccbicPres $(L;Ar;2Ar;\phi)$, and behaving as the identity on morphisms.

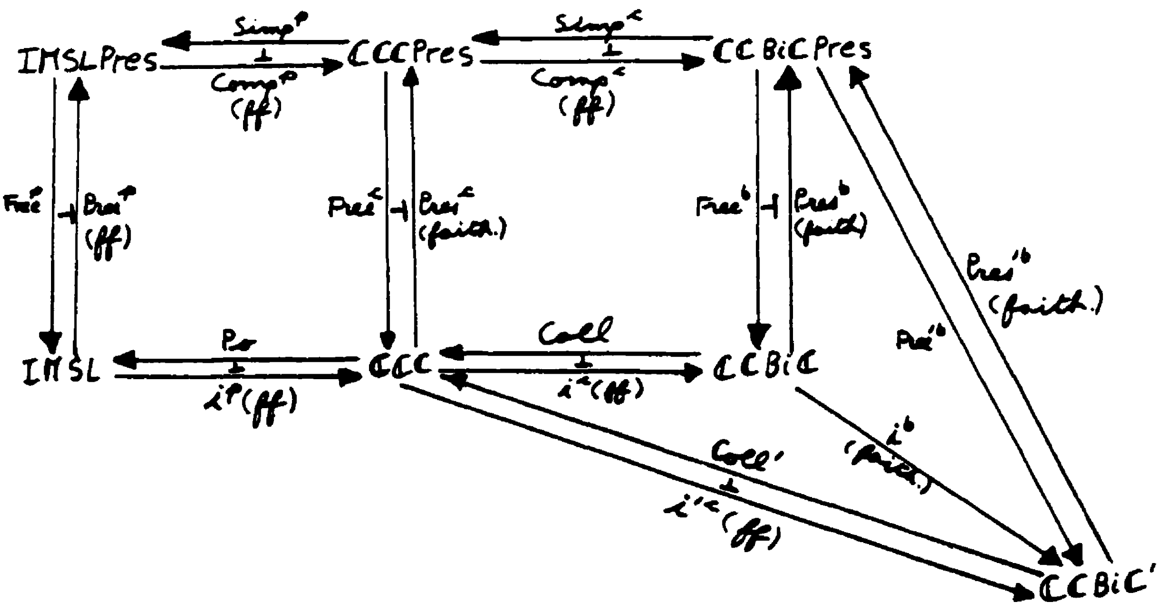
We also have the forgetful functor Forg^c:**CCBiCPres** $\rightarrow$ **CCCMultiPres** in the other direction, which simply drops the set of identities from the cccPres, and acts accordingly on morphisms.

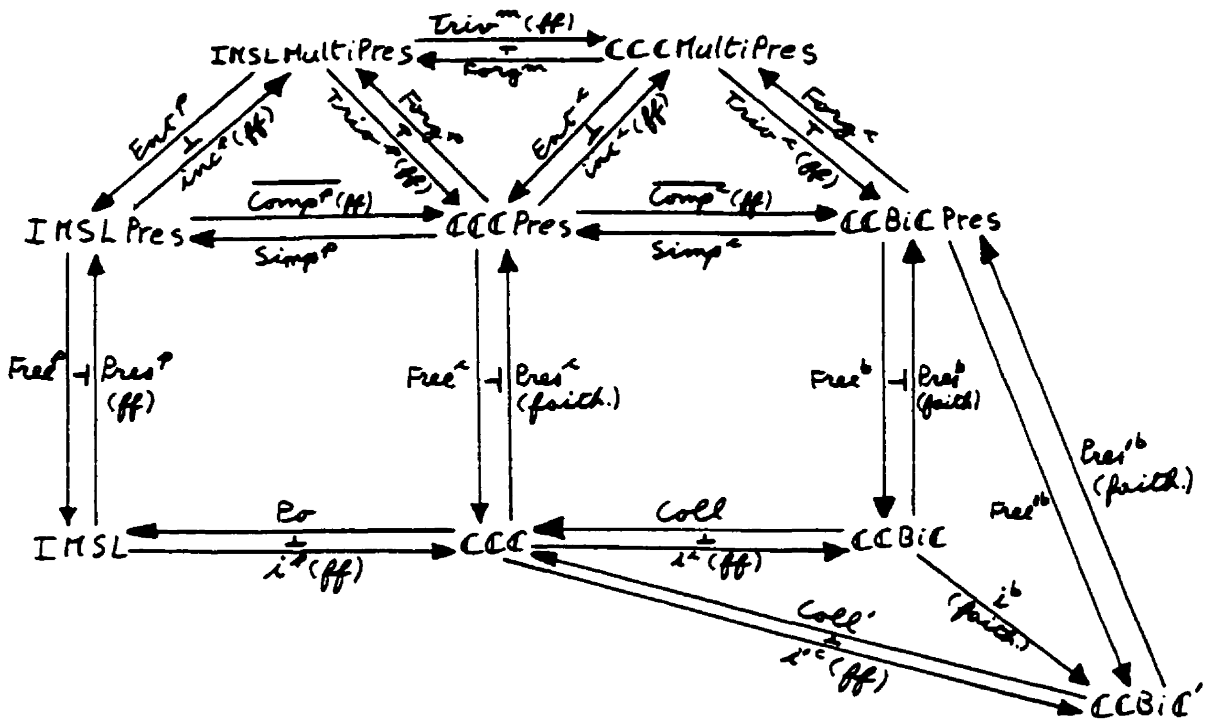
And lastly, we have the functor Triv^m:**IMSLMultiPres** $\rightarrow$ **CCCMultiPres**, taking the imslMultiPres $(L;Ar)$ to the cccMultiPres $(L;Ar;\phi)$, and acting as identity on morphisms, as well as the functor Forg^m:**CCCMultiPres** $\rightarrow$ **IMSLMultiPres**, which drops the set of formal arrows from the cccMultiPres, and drops the $2Ar$-part of morphisms.

PROPOSITION 5.7. inc^c, Triv^c and Triv^m are all full and faithful, and the following adjunctions hold: $\mathrm{ent}^c \dashv \mathrm{inc}^c$, $\mathrm{Triv}^c \dashv \mathrm{Forg}^c$ and $\mathrm{Forg}^m \dashv \mathrm{Triv}^m$. Moreover, $\overline{\mathrm{Comp}}^c = \mathrm{Triv}^c \circ \mathrm{inc}^c$, $\mathrm{Simp}^c = \mathrm{Ent}^c \circ \mathrm{Forg}^c$, $\mathrm{Triv}^m = \mathrm{inc}^c \circ \mathrm{Triv}^p$ and $\mathrm{Forg}^m = \mathrm{Forg}^p \circ \mathrm{ent}^c$.

These facts are easily seen upon inspection. We omit the proof.

We can summarize the essence of our work in the following two diagrams:





Even though we discuss this at greater length in the next chapter, we can briefly recapitulate what we have accomplished so far as follows: we have constructed several mathematical objects suitable to varying extents to study theories in positive intuitionistic propositional logic; we have also introduced a number of maps, all preserving "information" to some degree, allowing us to move back and forth between these objects. Given a language L and a theory T over L in positive intuitionistic propositional logic, we can form the three following entities:

(1) the Lindenbaum-Tarski algebra of T, $\mathrm{Free}^p(L;T)$,
(2) the free cartesian closed category of proofs of T, $\mathrm{Free}^c \circ \overline{\mathrm{Comp}}^p(L;T)$, and
(3) the free cartesian closed bicategory of proofs of T, $\mathrm{Free}^b \circ \overline{\mathrm{Comp}}^c \circ \overline{\mathrm{Comp}}^p(L;T)$.

The main point is that each of these objects collapses into the previous one *in a consistent way*. The chief result is:

THEOREM 5.8. $\mathrm{Free}^c \circ \overline{\mathrm{Comp}}^p = \mathrm{Coll} \circ \mathrm{Free}^b \circ \overline{\mathrm{Comp}}^c \circ \overline{\mathrm{Comp}}^p : \mathbf{IMSLPres} \to \mathbf{CCC}$.

The proof proceeds exactly like that of proposition 4.11, and is therefore omitted.

We insist once more on the fact that this theorem together with proposition 4.11 guarantee the consistency and compatibility of the various algebraic and categorical approaches to the study of propositional theories.

Chapter 6

Discussion, Future Work and Conclusion

In this chapter, we would like to reflect once more upon our whole enterprise, discuss what possible applications this work might have, what alternative setups to the ones we have chosen one might consider, and what possibilities for future work we might envisage.

We should perhaps start by restating what the intended uses of the fundamental three structures we have been dealing with are: in the context of a particular theory, we have three algebraic entities helping us probe the properties of that theory and the proof system associated with it. The first such entity is a free imsl; it allows us to investigate truth and provability questions, as it is concerned solely with the existence of proofs between formulas; it therefore has applications, among other things, to model theory – in particular it is a congenial tool for establishing "hard" (as opposed to "general") completeness theorems; see, e.g., [CK]. The second entity is a free cartesian closed category; because it attempts to distinguish between deductions (up to certain equivalences), it allows us to study the proofs themselves – for instance, how many "genuinely distinct" proofs there might be between two formulas, etc. There is also a huge "hidden side" (i.e., that we haven't mentioned in this work) to this structure, namely its applications to modelling the λ-calculus (see, for instance, [LS] – but we will come back to this point). Naturally, there are also numerous connections with model theory: a very nice example of this is the paper [HM]. The last entity is a free cartesian closed bicategory; it allows us to study proofs proper (i.e., before identification) in much greater detail: for example, it makes explicit how to transform one proof into another, leading to possible attempts at defining "bad" (presumably "long") and "good" ("short") proofs, and perhaps pointing to ways to "improve" a given proof, etc.... Potential applications could therefore include automated theorem provers, for example. And there is here as well a vast terrain of exploration in linking this to higher-order models of the λ-calculus and related computational formalisms (see, e.g., [S1]), as well as model theory. At any rate, it seems to us worthwhile to search for more concrete and natural mathematical models of cartesian closed bicategories: it is very likely that interesting connections and results might be gathered by studying their interaction with logic.

We have of course also defined certain maps between the above entities; the fact that these maps preserve their relevant properties and structure in a consistent manner is of course crucial, and is what makes the higher-dimensional structures interesting.

Considerations of logic and proof theory aside, one can also regard our work as pure investigations in category theory. This point of view combines very well with the preceding to provide a multitude of ideas in which we could modify, refine or pursue the present setup. Another motivator consists in studying the connections with various paradigms of the λ-calculus. We give below a combined list of ideas for possible future work.

An obvious extension could be to add coproducts (corresponding to logical disjunction), and/or an initial object (corresponding to the constant false) – that would give us *bi-cartesian* closed bicategories, modelling intuitionistic propositional logic or classical propositional logic, depending on what 2-cells we choose to include. Other similar constructions are of course possible, aimed either at studying some particular propositional logics, or paralleling standard one-dimensional categorical operations such as limits, etc. Passing to predicate logic, however, is a much bigger challenge. Lawvere [L4] was the first to construct categorical semantics for predicate logic using fibrations in certain categories, and it seems reasonable to intuit that the same could be achieved in the case of bicategories. It turns out that the technicalities of fibrations in bicategories have been worked out since 1980(!) [S3]; this certainly opens up a vast terrain for further research.

Naturally, another obvious extension of bicategories is in the direction of further dimensions. There has been quite a bit of work in this area recently, even if Bénabou had already touched the subject as far back as 1967 [B2]. In fact, the debate is still ongoing as to what are the "good", or "natural", definitions. [GPS] is a very interesting paper giving a coherence theorem for tricategories, apparently introducing them for the first time as well. It is amusing to learn that one of the authors, Ross Street, had in fact written a paper a few years earlier on ω-categories(!) This is certainly an area of active research in category theory; the eventual connections to logic, however, appear a bit nebulous at this point.

The current framework is nevertheless rather satisfactory for a number of purposes. For instance, it would now be a routine matter to carry over such classical constructions as the adjunction of an indeterminate to a given (cartesian closed) bicategory, substitution of an arrow for an indeterminate, and associated results such as functional completeness (see, e.g. [LS]). Questions of coherence are not only very interesting, they are quite important as well. Lambek was the first to "reformulate" the coherence problem (albeit in a different context) as finding an algorithm for deciding when two 2-cells are equal. There has since been quite a bit of work carried out on the subject (the reader is invited to consult [LS] for a list of relevant references).

The connections between cartesian closed categories and the typed λ - calculus (see, e.g., [LS]) have been known for quite some time, and have been rather extensively studied. Very roughly, the idea consists in considering objects in the category as types, and arrows as λ - terms. Under appropriate conditions, this relation turns out in fact to be an isomorphism. [LS] explains how a version of the Church-Rosser theorem has led to the formulation of an algorithm for deciding when two arrows in a cartesian closed category are equal (i.e., when two proofs describe the same arrow). By theorem 5.8, we can immediately reinterpret that as an algorithm for deciding whether there is a 2-cell between two arbitrary parallel 1-cells in a free cartesian closed bicategory. Of course, more efforts have been expanded trying to obtain results *about* λ - calculus *from* cartesian closed categories. Given that the λ - calculus has a very strong computational flavour to it, so that the study of reductions between λ - terms takes on a prominent rôle, it seems to us that (cartesian closed) bicategories are ideally suited to the tasks at hand: the 2-cells can provide us with detailed information about the reductions. Work along these lines has already been carried out by Seely [S1]. We will come back to this in a moment.

We have pointed out throughout this thesis that the foundational nature of our work made it such that we very often had to choose among several likely candidates what particular definition we were going to use when introducing new constructions. Partly for reasons of completeness, and partly because some setups do offer certain advantages over others and vice-versa, we present and comment below on a select few alternatives.

The most obvious alteration would be to drop the requirement, for some or all of the various canonical 2-cells we have introduced, that they be isomorphisms. Doing this with the products, terminal object and exponentials yields what we call a *weak* cartesian closed bicategory. It is important, and not entirely obvious, to note that in this case, we would still have been able to define a "collapse" cartesian closed functor preserving "freeness" from the category of weak cartesian closed bicategories into CCC. We may of course also apply this process of weakening 2-cells to the associativity, left and right identity isomorphisms, giving us the notion of *lax* bicategory. The interest of these setups is that 2-cells would now only point in the direction of "reduction", exclusively transforming "long" ("bad") proofs into "short"("good") ones. Interesting sub-questions then arise: for instance, is there always a unique "terminal" ("shortest", "best") proof, or can there be several, and so on... The problems of coherence also resurface with new twists. Looking in another direction, because of the fact that the so-called β- and η-reductions of λ-terms essentially correspond respectively to the canonical 2-cells ζ and $\tilde{\zeta}$ in a cartesian closed bicategory, we can also make use of this framework to refine the study of reductions in the λ-calculus. In fact, if only certain kinds of reductions are of interest to us, we can weaken the associated canonical 2-cells not to be invertible, while postulating that all the others should be plain identities, etc.... These types of setup allow us to focus our attention on some very specific properties of the λ-calculus, or variants thereof. In [S1], Seely illustrates how 2-categories can be put to good use in such endeavours. He quotes as motivation for using a two-dimensional structure the need not to be forced to equate, in the semantic model, each stage in a computation process with the result of the computation. Bicategories thus seem even better suited to the task in that they identify even less than 2-categories! Pursuing this avenue further certainly appears promising.

References

[B1] J. Bénabou, *Catégories avec multiplication*, Comptes Rendus de l'Académie des Sciences de Paris, vol. 256, 1963, pp. 1887–1890.

[B2] J. Bénabou, *Introduction to Bicategories*, Lecture Notes in Mathematics, vol. 47, Springer-Verlag, Berlin, 1967, pp. 1–77.

[B3] J. Bénabou, *Structures algébriques dans les catégories*, Cahiers de Topologie et Géométrie Différentielle, vol. 10, 1968, pp. 1–126.

[Ca] J. Cartmell, *Generalised Algebraic Theories and Contextual Categories*, Annals of Pure and Applied Logic, vol. 32, North-Holland, Amsterdam, 1986, pp. 209–243.

[CK] C.C. Chang and H.J. Keisler, *Model Theory*, North-Holland, New York, 1977.

[CW] A. Carboni and R.F.C. Walters, *Cartesian Bicategories I*, Journal of Pure and Applied Algebra, vol. 49, North-Holland, Amsterdam, 1987, pp. 11–32.

[CWM] S. Mac Lane, *Categories for the Working Mathematician*, Springer-Verlag, Berlin, 1971.

[EK] S. Eilenberg and G.M. Kelly, *Closed Categories*, Proceedings of the Conference on Categorical Algebra, La Jolla, Springer-Verlag, New York, 1966, pp. 421–562.

[EM] S. Eilenberg and S. Mac Lane, *General Theory of Natural Equivalences*, Transactions of the American Mathematical Society, vol. 58, 1945, pp. 231–294.

[GPS] R. Gordon, A.J. Power and R. Street, *Coherence for Tricategories*, Memoirs of the American Mathematical Society, vol. 117, 1995.

[HM] V. Harnik and M. Makkai, *Lambek's Categorical Proof Theory and Läuchli's Abstract Realizability*, The Journal of Symbolic Logic, vol. 57, Association for Symbolic Logic, 1992, pp. 200–230.

[JS] A. Joyal and R. Street, *Braided Tensor Categories*, Advances in Mathematics, vol. 102, Academic Press, 1993, pp. 20–78.

[KS] G.M. Kelly and R. Street, *Review of the Elements of 2-Categories*, Lecture Notes in Mathematics, vol. 420, Springer-Verlag, Berlin, 1974, pp. 75–103.

[L1] J. Lambek, *Deductive Systems and Categories. I*, Mathematical Systems Theory, vol. 2, 1968, pp. 287–318.

[L2] J. Lambek, *Deductive Systems and Categories. II*, Lecture Notes in Mathematics, vol. 86, Springer-Verlag, Berlin, 1969, pp. 76–122.

[L3] J. Lambek, *Deductive Systems and Categories. III*, Lecture Notes in Mathematics, vol. 274, Springer–Verlag, Berlin, 1972, pp. 57–82.

[L4] F.W. Lawvere, *Adjointness in Foundations*, Dialectica, vol. 23, 1969, pp. 281–296.

[L5] F.W. Lawvere, *Equality in Hyperdoctrines and Comprehension Schema as an Adjoint Functor*, Proceedings of the New York Symposium on Applications of Categorical Algebra, American Mathematical Society, Providence, 1970, pp. 1–14.

[LS] J. Lambek and P.J. Scott, *Introduction to Higher-Order Categorical Logic*, Cambridge University Press, Cambridge, 1986.

[M1] S. Mac Lane, *Natural Associativity and Commutativity*, Rice University Studies in Mathematics, vol. 49, 1963, pp. 28–46.

[M2] M. Makkai, *An Algebraic Look at Propositional Logic*, Aila Preprint, vol. 18, Associazione Italiana di Logica e sue Applicazioni, 1994.

[M3] M. Makkai, *Avoiding the Axiom of Choice in General Category Theory*, Journal of Pure and Applied Algebra, vol. 108, North-Holland, Amsterdam, 1996, pp. 109–173.

[MP] S. Mac Lane and R. Paré, *Coherence for Bicategories and Indexed Categories*, Journal of Pure and Applied Algebra, vol. 37, North-Holland, Amsterdam, 1985, pp. 59–80.

[RW] B. Russell and A.N. Whitehead, *Principia Mathematica I-III*, Cambridge University Press, Cambridge, 1910–1913.

[S1] R.A.G. Seely, *Modelling Computations: a 2-Categorical Framework*, Proceedings of the Symposium on Logic in Computer Science, IEEE, 1993, pp. 65–71.

[S2] J.R. Shoenfield, *Mathematical Logic*, Addison-Wesley, Reading, 1967.

[S3] R. Street, *Fibrations in Bicategories*, Cahiers de Topologie et Géométrie Différentielle, vol. 21, 1980, pp. 111–159.