DIFFERENTIAL FORMS
APPLIED TO ELECTROMAGNETISM

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Raymond Cunningham Murphy, S.B. (M.I.T.),
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ABSTRACT

Exterior differential forms are introduced and explained in connection with R^3 differential algebra. This leads to a systematic development of vector integration formulas. Differential forms are then re-defined for the space/time coordinate system R^3/t in which electromagnetic phenomena are best described. A complete differential structure is presented for these electromagnetic differential forms, and it is shown to be completely consistent with macroscopic electromagnetic theory. A property of commutativity in this differential structure is examined, leading to a distinction between the mathematical behavior of electric-source and magnetic-source electromagnetism. Direct exterior product relations are investigated, permitting elegant derivations of power and energy formulas as well as reciprocity relations. Finally, we discuss a one-dimensional inverse scattering problem, and dispute the claim that a particular variety of integration theorem leads to its solution.

RESUMÉ

Cette thèse présente d'abord des formes différentielles extérieures et les explique en fonction de l'algèbre différentielle R^3. Ceci conduit à l'élaboration systématique de formules d'intégration vectorielle. L'utilisation du système de coordonnées espace/temps, R^3/t, qui se prête mieux à la description des phénomènes électromagnétiques, permet de redéfinir ces formes différentielles. Puis, cette thèse présente un système différentiel complet, adapté à ces formes différentielles électromagnétiques, et montre qu'il est entièrement en accord avec la théorie électromagnétique macroscopique. L'examen d'une propriété de commutativité de cette structure différentielle conduit à l'établissement d'une distinction entre le comportement mathématique de l'électromagnétisme dont la source est électrique et celui de l'électromagnétisme dont la source est magnétique. L'étude des relations entre les produits extérieurs directs permet de faire d'élégantes dérivations de formules de puissance et d'énergie ainsi que de relations de reciprocité. Enfin, la discussion d'un problème de diffraction inverse uni-dimensionnel nous amène à douter de l'affirmation selon laquelle un type particulier de théorème d'intégration permet de le résoudre.

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My deepest appreciation goes to Professor T.J.F. Pavlasek, my research supervisor. He granted me the flexibility to pursue this largely mathematical subject within a basically experimental research program, and encouraged me to make it accessible as a potential engineering tool. I certainly hope that the development of these ideas can continue for I feel that they offer considerable insight into the fundamental structure of electromagnetic field theory.

I also deeply appreciate my association with the other graduate students in the antenna and field measurement group, as well as with Professor P. Silvester and those who have worked for him during the time I have been at McGill. In particular, I would like to thank Dr. H.S. Cabayan for introducing to me many aspects of modern mathematical analysis, including differential forms.

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CONTENTS

INTRODUCTION

In the theory of differential equations, an "inverse problem" is any problem involving the determination of the coefficients or right-hand side of a differential equation from certain functionals of its solution (LAVRENTIEV, ROMANOV, VASILIEV (1970)). Most differential equations together with their boundary conditions may be reformulated to give a single integral equation (MARGENAU and MURPHY (1956), §14.1). In a certain sense, every integral equation is an "inversion problem" since part or all of the unknown function occurs within the integral. In operational form, integral equations can usually be written as

$$Tf = g \qquad \text{I.1}$$

where g is known (in physical problems, primarily from measured data) and T is a transformational operator on the unknown f. In most instances, the solution of (I.1), which involves finding the inverse operator T^{-1}, is an "improperly posed" problem because of difficulties regarding the existence, uniqueness and stability of the solution f that corresponds to a particular g, g_i. Regularization techniques (DESCHAMPS and CABAYAN (1972)) frequently permit the solution of (I.1) under certain constraints. Because T is completely specified, this problem is known as the "identification" problem: an unknown f_i is identified which yields g_i under the transformation T.

A more fundamental class of problems exists for (I.1) - the class of "synthesis" problems in which both T and f are unknown. That additional constraints are necessary to make the synthesis problem viable can be seen even in the one-dimensional algebraic analog to (I.1), Ohm's law:

$$Ri = e \qquad \text{I.2}$$

Clearly, some engineering judgment is required to constrain the 2 unknowns R and i in order to uniquely produce a given e_δ. Typically, one might constrain the power (a product quantity) and specify an acceptable range of current values.

In the integral equation formulation of antenna problems, g might represent the radiation pattern, f a function describing the source (perhaps current distributions), and T the integral operator whose domain covers the region of source interest (antenna dimensions) and whose kernel contains the propagational characteristics (Green's function). Boundary condition information is information on propagation in space (and time) - reflection, absorption, change in propagation constant, etc. - and is clearly contained in the kernel. The typical synthesis problem in antenna design is approached by choosing a structure (determining T) and treating the problem as an identification problem. Hopefully, some T_δ and f_δ can be found to produce the given g_δ in a stable manner.

One well-known variety of problems in mathematical physics is known as "inverse scattering". The object is to determine the physical characteristics of a scattering object from scattered field measurements. In electromagnetism, this is understood to mean determining the passive sources of a scattered electromagnetic field. The assumption is made that a known illumination interacts with unknown objects, from which a suitable "inversion" technique will provide information on the location, shape and electrical characteristics of the scattering objects (AHLUWALIA and BOERNER (1973)). It should be obvious that under this definition, inverse scattering is a variety of the general synthesis problem, and as such, additional information (e.g., a priori assumptions) is necessary for its precise solution. In fact, in order to yield what is essentially boundary information in a scattering system, it is necessary to determine aspects of the kernel of the appropriate integral equation from the possible field measurements.

While the inverse scattering process involves a known illumination (incident radiation or incident field), it is not a process dependent on information redundancy, such as holography or error - correcting coding, where the variable information is placed in a new state together with known information prior to an operation such as transmission or storage. In inverse scattering, the known illumination merely allows certain assumptions to be made about the scattering process, thereby helping to constrain the synthesis aspect of the problem. The inverse scattering and inverse radiation problems are fundamentally identical (BOJARSKI (1973), §2); both are synthesis problems.

Unfortunately, the usual approach to inverse scattering problems has been situation-by-situation. In electromagnetism, it would ultimately be advantageous to develop a more general understanding, one specifying the sort of additional information that must be supplied in order to reconstruct the kernel and sources in the synthesis problem. Because the electromagnetic field is described via a series of interrelated partial differential equations, and because the space/time domain of these equations can be described geometrically, a mathematics involving both at its foundational level should prove basic to further developments. In fact, we will show that differential vector algebra (where the relationship between geometry and partial differential equations is workable but cumbersome) can be supplanted by the more general algebra of multilinear forms on a differentiable manifold, or exterior differential algebra. We shall see that by using this exterior differential algebra, all conventional results in vector analysis can be demonstrated quite elegantly.

The important objective, however, is the development of a multilinear algebra for a 4-dimensional Cartesian space/time suitable for the partial differential equations of electromagnetism. Following such earlier authors as DESCHAMPS (1970), this has been accomplished. We show that the equations of electromagnetism expressed in the language of differential forms exist as part of a remarkably simple structure. Except for relations explicitly involving conductivity (and the restrictions necessarily implied by this (STRATTON (1941), §1.7)), all fundamental formulas of electromagnetism are immediately evident. The structural relationships may be visualized by means of diagrams, and the utility of these diagrams is carefully demonstrated. Some discussion is directed to the question of why the differential structure of electromagnetic forms has its specific character. The concept of product forms is introduced, leading to energy and power relations and reciprocity formulas. Finally, some of the flexibility gained by the use of exterior differential algebra in electromagnetism is applied to investigating the contention made by BOJARSKI (1973) that certain integral formulas can be used in the solution of inverse scattering problems.

General Outline and Contributions

The first two chapters of this thesis are tutorial. In Chapter 1, we introduce differential forms and show the precise relationship between these new quantities and vectors. We find that the algebra of differential forms, exterior differential algebra, can be tremendously advantageous because of its compactness. This is particularly true when the geometric correspondence to differentiation is outlined, allowing a systematic development of complicated vector integration formulas entirely from basic principles. Although exterior differential algebra is not new, our systematic development of it in relation to vector analysis has resulted in a number of points that should be noted:

1. The introduction of unit differential forms (§2.A.ii) which permit an extended development of vector integration formulas from the general version of Stokes' theorem (1.i.1).

2. The discovery of a derivation-type formula for the codifferential of a product ((1.f.3); also (4.d.1)).

3. A complete and fundamental development of integration formulas symmetric (in various senses) in 2 variables, leading in particular to the general vector Green's theorem (Table 2.10b) and the symmetric vector integration theorems (Table 2.11).

4. A notation which preserves the axial and polar vector identity of vector quantities, and which permits integration without the

use of additional geometric integration variables. The interchangeability between differential forms and vectors is stressed for its utility (§1.A and §2.A).

5. A method, perfected from DESCHAMPS (1970), of displaying in diagrams the relationships between differential forms due to the various differential operators (§1.L).

6. A demonstration that complex variable theory may be based on multilinear algebra provided the definitions of derivatives are made properly (§1.L).

Chapters 3 and 4 make up that unit of the thesis in which the techniques of exterior differential algebra are applied to electromagnetism. The analysis involves potentials, fields, charges, and currents, all considered as functionally real, time dependent quantities. The permeability and dielectric constant are considered constant in a local sense. The 4-dimensional differential structure of the electromagnetic forms also permits the systematic development of the 3-dimensional integration theorems for the electromagnetic quantities. In Chapter 4, we investigate the product quantities found by taking exterior products of the electromagnetic differential forms. We show that there is a concise 4-dimensional derivation leading to relations for energy, power, and momentum. We also derive several reciprocity formulas. The following points should be considered as the contributions of these chapters:

1. The incorporation of dimensions so that the local and global dimensional character of a differential form may be considered separately. Also, the consideration of the * operator as a dimensioned operator (§3.B).

2. The development of a complete differential structure for the electromagnetic forms. This structure contains all of the electromagnetic field equations, and clearly shows all interrelationships (§3.C).

3. An outline of the direct correspondence between the differential and integral vector equations through the introduction of a differential projection concept (§3.D).

4. The development of a technique for the solution of electromag-

netic wave equations by extending the Green's function technique to differential forms (§3.F).

5. Proof that electromagnetic phenomena are possible only when sources are present somewhere in space and time (§3.F).

6. A demonstration that there is a commutative property relating forms in the differential structure, and that this property differs for the electromagnetism manifested by electric and magnetic sources (§4.A).

7. The development of various direct exterior products between electromagnetic differential forms, their interpretation, and explanations of their interrelationships (§4.B → §4.E).

8. A development of product relations leading to formulas involving energy, power, momentum, and equivalent mass (§4.G).

9. A development of product relations leading to reciprocity formulas (§4.H).

In the final section of the thesis, Chapter 5, we look at several short topics, including the inverse scattering problem that prompted the mathematical investigation that constitutes the important work. Included are:

1. A discussion of the implications of conductivity in the electromagnetic structure when certain time dependencies are assumed for all quantities (§5.A).

2. The derivation and interpretation of an integral equation said to be the starting point for the solution of inverse scattering problems (§5.B).

3. A specific investigation of one-dimensional inverse scattering involving a dielectric interface (§5.C).

As a summary to this introductory section, we shall list what are considered to be the most important contributions of the thesis:

A. The presentation of a complete differential structure for electromagnetic differential forms (§3.C).

B. The demonstration of the commutativity properties of this structure, including the differences for electric and magnetic sources (§4.A).

C. The development of energy and power formulas, and the derivation of reciprocity relations (§4.G and §4.H).

D. The systematic and complete development of R^3 (vector and scalar) integration formulas (Chapter 2).

Before proceeding to Chapter 1, where we will introduce multivector algebra and differential forms, we should point out that relevant aspects of the history of this subject (as it applies to electromagnetic theory) are found in the summary to Chapter 3. Also, at the end of the thesis, there is a note concerning the symbolic notation used in the various chapters.

CHAPTER I

INTRODUCTION TO EXTERIOR DIFFERENTIAL ALGEBRA

Anticommutativity under multiplication is a property of certain algebraic systems; most familiar is the anticommutative "cross product" in R^3† vector algebra. When the postulates of vector algebra are generalized by introducing anticommutative behavior to the products of the fundamental basis elements, it is possible to construct "multivector" algebras in which there is no ambiguity about the identity of the vector quantities (In R^3, the identity of polar vectors and axial vectors is maintained). In particular, this allows the systematic construction of multivector algebras on metric spaces of arbitrary signature. In this chapter, we start with the fundamental postulate and develop the mathematics basic to multivector algebra on differentiable manifolds. The fundamental operations $*$ (star), d (differential), δ (adjoint differential or codifferential) and Δ (Laplace-Beltrami) are introduced and explained. The behavior of d, δ and Δ on products of R^3 differential forms is shown in detail. Stokes' theorem, a generalization of the Fundamental Theorem of Calculus (WARNER (1971), §4.7), is introduced after a brief discussion of the utility of simplex manifolds and the integration of differential forms. Stokes' theorem is basic to all integration theorems involving differential forms on manifolds.

In Chapter 3, the laws of electromagnetism are described in exterior differential forms. As a prelude to this, the differential structure for an analytic form in C (the complex line) is given in the final section of this chapter. It provides a simplified description of the differential character of complex manifolds and indicates the direction for a complete and basic development of an "analytic" description of electromagnetism, along the line proposed by TYPALDOS and POGORZELSKI (1973).

Mathematically, the developments in this chapter are outlines. We

†: R^3 is a real 3-dimensional (Cartesian) metric space such that the distance between any 2 distinct points is >0.

refer to the textbooks by FLANDERS (1963), DESCHAMPS (1970) and WARNER (1971) for comprehensive detail and treatment. However, in order to stress the advantage of exterior differential algebra as a tool for use in physical problems, the R^3 (vector) cases are worked out in detail. A notation is introduced for vector quantities which permits their simultaneous identity as vectors and as differential forms, making it possible to concurrently utilize the computational benefits of exterior differential algebra with the physical insight offered by traditional geometric pictures in vector algebra.

A. Vector Spaces and Differentiable Vector Spaces

Let us take an n-dimensional vector space L (basis: $(\sigma_1, \sigma_2, \ldots, \sigma_n)$) over the field of real numbers R, and postulate the existence of an anticommutative product (called the exterior product†) between the basis elements of L:

$$\left.\begin{array}{l} \sigma_i \wedge \sigma_j = -\sigma_j \wedge \sigma_i \\ \sigma_i \wedge \sigma_i = 0 \end{array}\right\} \text{for all } i,j. \qquad 1.a.1$$

From this simple condition we are able to develop an algebra for each of the (n+1) vector spaces $\{p = 0,1,\ldots,n\}$, where the basis of the space of p-vectors is the set of possible forms constructed by taking (p-1) exterior products of the original basis elements of L. For each algebra of p-vectors $\Lambda^p L$, the dimension of the basis is given by (FLANDERS, §2.1):

$$\dim \Lambda^p L = \binom{n}{p} \qquad 1.a.2$$

To illustrate the above points, let us describe the bases and dimen-

†: The name comes from Grassman's use of such products to describe the "extension" - that is, the area or volume - of geometric figures defined by edge vectors (DESCHAMPS, §4.3).

sions of the p-algebras for the Cartesian coordinate space R^3:

P-ALGEBRAS IN R^3 TABLE 1.1

Algebra	Basis Set	Dimension	Name of Elements Constructed from Basis
$\Lambda^0 L$	1	1	0-Vectors
$\Lambda^1 L$	σ_1; σ_2; σ_3	3	1-Vectors
$\Lambda^2 L$	$\sigma_2\wedge\sigma_3$; $\sigma_3\wedge\sigma_1$; $\sigma_1\wedge\sigma_2$	3	2-Vectors
$\Lambda^3 L$	$\sigma_1\wedge\sigma_2\wedge\sigma_3$	1	3-Vectors

Now it is possible to construct vector quantities on a differential basis (DESCHAMPS, §5.1). This involves a change in the notion of the metric, or distance measure. For the Cartesian space R^3, the macroscopic distance measure is based on the Pythagorean relation,

$$[d(X;X')]^2 = (x-x')^2 + (y-y')^2 + (z-z')^2 \qquad \text{1.a.3a}$$

Equally valid is the differential distance measure based on limit arguments familiar from calculus:

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 \qquad \text{1.a.3b}$$

In R^3, a differential 1-form α can be written as

$$\alpha = \sum_{i=1}^{3} A_i dx^i \,{}^{\dagger} \qquad \text{1.a.4}$$

The basis term, a spatial differential, is subject to the exterior multiplication postulate (1.a.1). The coefficients A_i are usually considered as being suitably differentiable functions of position (x^1, x^2, x^3). However, it is also possible for the A_i to be distributions, such as $\delta(X-X')$, making the form α distributional rather than

†: The use of superscript notation for the differential elements follows from their tensor properties: they are "linear alternating forms" (DESCHAMPS, §4.4).

functional in character. Naturally, the A_i may be constant.

The 1-forms (1.a.4) satisfy the requirements for a vector space, and consequently $\{dx^1, dx^2, dx^3\}$ can be taken as its basis (DESCHAMPS, §5.1). Using this differential basis, we shall show that the standard R^3 vector relations can be systematically derived without further geometric arguments.

The introduction of the differential basis at this time is a matter of convenience, enabling us to use a common notation throughout the chapter. Strictly speaking, it should be associated with the introduction of a differentiable vector space (together with a derivative operation) in order to be mathematically justified. Nevertheless, having introduced the differential basis, we can construct multivectors with it; these will be the higher-order differential forms. In R^3, the following equivalence exists between the differential p-forms and vector quantities:

EXTERIOR DIFFERENTIAL FORMS IN R^3 TABLE 1.2

Differential Form Quantity	Vector Quantity
0-Form: $\kappa = K$	K - invariant scalar
1-Form: $\lambda = L_1dx^1 + L_2dx^2 + L_3dx^3$	**L** - polar vector
2-Form: $\mu = M_1dx^{23} + M_2dx^{31} + M_3dx^{12}$	**M** - axial vector
3-Form: $\nu = Ndx^{123}$	[N]- variant scalar

The names of the vector quantities in Table 1.2 are consistent with STRATTON (1941), §1.19. Note that in the definition of the forms μ and ν, we have saved some space by using the compact notation $dx^{ij} = dx^i \wedge dx^j$.

B. Direct Products of Exterior Differential Forms

From the exterior product postulate (1.a.1), it is a simple matter to show that for two forms α and β of degree p and q respectively,

$$\text{degree}\ (\alpha \wedge \beta) = (p+q) \qquad \text{1.b.1a}$$

$$\alpha \wedge \beta = 0 \quad \text{if } (p+q) > n \qquad \text{1.b.1b}$$

$$\alpha \wedge \beta = (-1)^{pq}\ \beta \wedge \alpha \qquad \text{1.b.1c}$$

In addition, the exterior product of 3 forms is associative:

$$\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma \qquad \text{1.b.2}$$

Calculating the exterior products involving the R^3 forms listed in Table 1.2, we have the following possibilities:

R^3 PRODUCTS — TABLE 1.3

Order of Forms		Product		Components of Product
0	0	$\kappa_a \wedge \kappa_b$	=	$K_a K_b$
0	1	$\kappa \wedge \lambda$	=	$KL_1 dx^1 + KL_2 dx^2 + KL_3 dx^3$
0	2	$\kappa \wedge \mu$	=	$KM_1 dx^{23} + KM_2 dx^{31} + KM_3 dx^{12}$
0	3	$\kappa \wedge \nu$	=	$KN\ dx^{123}$
1	1	$\lambda_a \wedge \lambda_b$	=	$(L_{a2}L_{b3} - L_{a3}L_{b2})dx^{23} + (L_{a3}L_{b1} - L_{a1}L_{b3})dx^{31} + (L_{a1}L_{b2} - L_{a2}L_{b1})dx^{12}$
1	2	$\lambda \wedge \mu$	=	$(L_1M_1 + L_2M_2 + L_3M_3)dx^{123}$

The term-by-term expansion on the right demonstrates the equivalence between the R^3 exterior products and the products of R^3 vector algebra. Commutativity or anticommutativity can be found by calculating (1.b.1c) for the 6 distinct R^3 products:

COMMUTATIVITY OF R^3 PRODUCTS — TABLE 1.4

Vector Product	Form Product		Reversed Form Product	Reversed Vector Product
$K_a K_b$	$\kappa_a \wedge \kappa_b$	=	$\kappa_b \wedge \kappa_a$	$K_b K_a$
$K\mathbf{L}$	$\kappa \wedge \lambda$	=	$\lambda \wedge \kappa$	$\mathbf{L}K$
$K\mathbf{M}$	$\kappa \wedge \mu$	=	$\mu \wedge \kappa$	$\mathbf{M}K$
$K[N]$	$\kappa \wedge \nu$	=	$\nu \wedge \kappa$	$[N]K$
$\mathbf{L}_a \times \mathbf{L}_b$	$\lambda_a \wedge \lambda_b$	=	$-\lambda_b \wedge \lambda_a$	$-\mathbf{L}_b \times \mathbf{L}_a$
$\mathbf{L} \cdot \mathbf{M}$	$\lambda \wedge \mu$	=	$\mu \wedge \lambda$	$\mathbf{M} \cdot \mathbf{L}$

From this we see that the anticommutativity of the vector cross product follows directly from the postulate (1.a.1). It is not necessary to invoke any other arguments.

The development of exterior differential algebra is outlined in Figure 1.1. In the following sections, operations will be introduced which substantially increase the usefulness of this algebra.

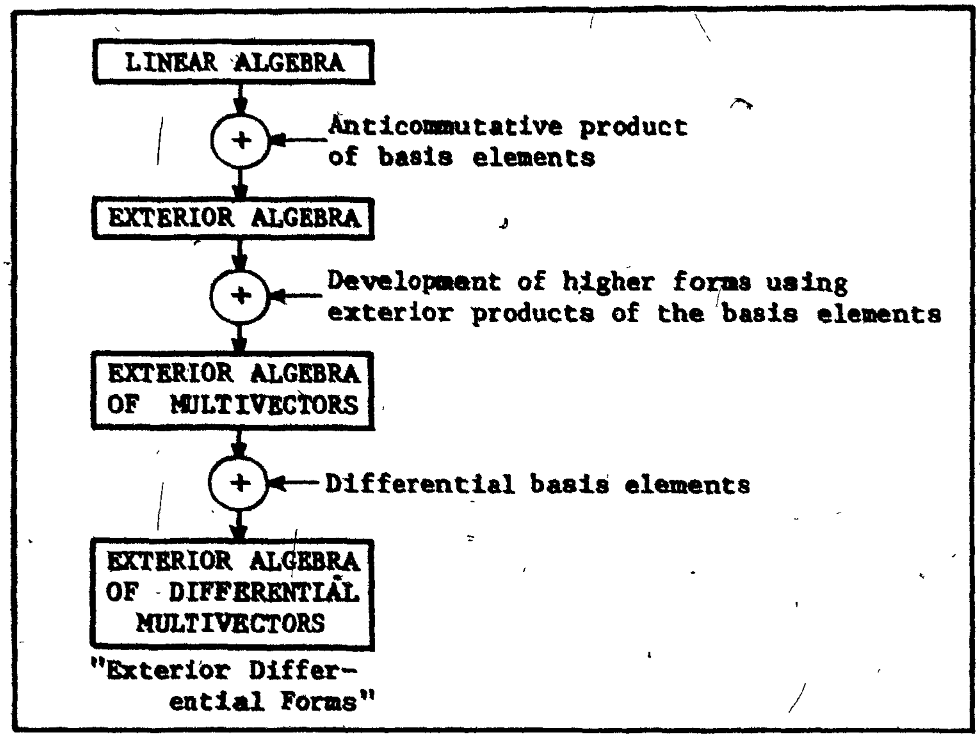

Figure 1.1 : Development of Exterior Differential Algebra.

C. Inner Products in Exterior Differential Algebra

An inner product is a real-valued function which satisfies the following conditions:

1. It is linear in each variable.
2. It is symmetric: $(\alpha,\beta) = (\beta,\alpha)$

} Bilinearity.

3. It is non-degenerate: if for fixed α, $(\alpha,\beta) = 0$ for all β, then $\alpha = 0$. } Positivity. $(\alpha,\alpha) > 0$, $\alpha \neq 0$.

These conditions by themselves do not imply a specific physical interpretation of an inner product; it is the mathematical relation satisfying the conditions which may be interpreted. For example, in R^3 the vector dot product satisfies properties (1) - (3), and this is interpreted as the product of the projected components of the 2 vectors. In exterior differential algebra, we shall employ an inner product which involves the integration of an n-form over a closed n-dimensional manifold. This introduces a conceptual difficulty, because while we can imagine a closed surface in R^3, a closed volume in R^3 is a non-Euclidean entity.

Of basic importance is the fundamental theorem that every inner product space has an orthonormal basis (FLANDERS, §2.5). Since an arbitrary vector may be regarded as a unique linear construction on its basis elements, we shall find it convenient to calculate inner products using the basis elements alone. Basis element orthonormality may be expressed as

$$(\sigma^i,\sigma^j) = \pm\delta^{ij} \qquad 1.c.1$$

δ^{ij} is the Kronecker δ. In an arbitrary n-dimensional orthonormal coordinate system, the complete set of inner products (1.c.1) yields r +'s and s -'s; $r + s = n$. The metric space signature is defined as $t = r - s$. For R^3, $r = 3$ and $s = 0$, hence $t = 3$. For 4-dimensional space-time (sometimes called Minkowski space), identified as R^3/t in this thesis, $r = 3$, $s = 1$, and $t = 2$.

For p-vectors, it is useful to know that the $(\text{norm})^2$ of a basis element of $\Lambda^p L$ can be resolved into a product of the squared norms of those basis elements of $\Lambda^1 L$ included in the $\Lambda^p L$ element. Let $\sigma = \sigma^1 \wedge \sigma^2 \wedge \ldots \wedge \sigma^p$. Then

$$(\sigma,\sigma) = (\sigma^1,\sigma^1)(\sigma^2,\sigma^2)\ldots(\sigma^p,\sigma^p) \qquad 1.c.2$$

Another useful property arises from the inner product: let f be a linear functional on L. Then there is a unique vector β in L such that

$$f(\alpha) = (\alpha, \beta) \qquad \text{1.c.3}$$

(FLANDERS, §2.5). This will be used in proving the uniqueness of the Hodge star operator, which follows.

D. The Hodge Star Operator

For a vector space on which an inner product is defined, it is possible to show that there is a linear operation which uniquely transforms a p-vector into an (n-p)-vector. We take a specific orientation of the inner product space L. Suppose we have 2 vectors, $\alpha \in \Lambda^p L$ and $\beta \in \Lambda^{n-p} L$. Then the transformation $\beta \to \alpha \wedge \beta$ is equivalent to $\Lambda^{n-p} L \to \Lambda^n L$, and since $\Lambda^n L$ is a one-dimensional space with basis $\sigma = \sigma^1 \wedge \sigma^2 \wedge \dots \wedge \sigma^n$, we can write

$$\alpha \wedge \beta = f_\alpha(\beta)\sigma \qquad \text{1.d.1}$$

where $f_\alpha(\beta)$ is a linear functional on $\Lambda^{n-p} L$. From (1.c.3), this in turn implies that

$$f_\alpha(\beta)\sigma = (*\alpha, \beta)\sigma \qquad \text{1.d.2}$$

where $*\alpha \in \Lambda^{n-p} L$, defining the $*$-mapping, which is evidently a linear transformation $\Lambda^p L \to \Lambda^{n-p} L$ (FLANDERS, §2.7). For a specific situation, it is sufficient to compute the $*$-mapping of the ordered basis elements. FLANDERS (§2.7) shows that

$$*\sigma^A = (\sigma^B, \sigma^B)\sigma^B \qquad \text{1.d.3}$$

where $(A,B) = (1,2,\ldots,n)^{\sigma_{perm}}$. For the ordered differential basis elements of R^3, the $\star$-operation is calculated as follows:

$\star$-MAPPING OF R^3 BASIS ELEMENTS — TABLE 1.5

Basis Element		Permu-tation	Inner Product	Complementary Element		Result
$*dx^1dx^2dx^3$	=	(+1)	--	1	=	1
$*dx^2dx^3$	=	(+1)	(+1)	dx^1	=	dx^1
$*dx^3dx^1$	=	(+1)	(+1)	dx^2	=	dx^2
$*dx^1dx^2$	=	(+1)	(+1)	dx^3	=	dx^3
$*dx^1$	=	(+1)	(+1) (+1)	dx^2dx^3	=	dx^2dx^3
$*dx^2$	=	(+1)	(+1) (+1)	dx^3dx^1	=	dx^3dx^1
$*dx^3$	=	(+1)	(+1) (+1)	dx^1dx^2	=	dx^1dx^2
$*1$	=	(+1)	(+1) (+1) (+1)	$dx^1dx^2dx^3$	=	$dx^1dx^2dx^3$

E. The Real Partial Differential Operator d; Differentials of Products

Once the algebraic characterization of differential forms has been completed, we can consider linear operations which shift forms into higher dimensions. Recall that the star operator defined a linear mapping from p-forms to (n-p)-forms. A series of possibly useful operational structures is:

$$\begin{aligned} O_0 &= A \\ O_1 &= B_1dx^1 + B_2dx^2 + B_3dx^3 \\ O_2 &= C_1dx^{23} + C_2dx^{31} + C_3dx^{12} \\ O_3 &= Ddx^{123} \end{aligned} \qquad 1.e.1$$

The "coefficients" are considered to have operational rather than functional characteristics. In this thesis, we are concerned only with a type O_1 operator. O_1, when it is written as

$$d = \frac{\partial}{\partial x^1}dx^1 + \frac{\partial}{\partial x^2}dx^2 + \frac{\partial}{\partial x^3}dx^3 \qquad 1.e.2$$

is known as the real partial differential operator. Its existence, uniqueness and linearity are well-established (FLANDERS, §3.2). (The $\frac{\partial}{\partial x^i}$ form a vector field dual to the differential form dx^i (LANG (1962), App. 2, §1)). When applied to a p-form, the differential terms in d form exterior products with the differential terms of the p-form, while the partial derivatives operate on the coefficients. The result is a (p+1)-form. Applied to the R^3 differential forms found in Table 1.2, we have:

DIFFERENTIALS OF R^3 FORMS — TABLE 1.6

Operation	Expansion	Vector Notation
$d\kappa = d\wedge\kappa =$	$\frac{\partial K}{\partial x^1}dx^1 + \frac{\partial K}{\partial x^2}dx^2 + \frac{\partial K}{\partial x^3}dx^3$	$= \nabla K$: 1-form
$d\lambda = d\wedge\lambda =$	$\left(\frac{\partial L_3}{\partial x^2} - \frac{\partial L_2}{\partial x^3}\right)dx^{23} + \left(\frac{\partial L_1}{\partial x^3} - \frac{\partial L_3}{\partial x^1}\right)dx^{31} + \left(\frac{\partial L_2}{\partial x^1} - \frac{\partial L_1}{\partial x^2}\right)dx^{12}$	$= \nabla\times\mathbf{L}$: 2-form
$d\mu = d\wedge\mu =$	$\left(\frac{\partial M_1}{\partial x^1} + \frac{\partial M_2}{\partial x^2} + \frac{\partial M_3}{\partial x^3}\right)dx^{123}$	$= \nabla\cdot\mathbf{M}$: 3-form
$d\nu = d\wedge\nu =$	0	— —

The operator d is known as a derivative (specifically, an "anti-derivation" (WARNER, §2.11)) because the differential of a product takes the form

$$d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^p\,\alpha\wedge d\beta \qquad \text{1.e.3}$$

where p is the order of the form α (DESCHAMPS, §5.3 and §5.6). In R^3, this formula yields several fundamental relations of vector analysis. For the 6 R^3 products listed in Table 1.3, we have:

R^3 PRODUCT DIFFERENTIALS — TABLE 1.7

Operation	Expansion	Vector Notation
$d(\kappa_a\wedge\kappa_b)$ =	$d\kappa_a\wedge\kappa_b + \kappa_a\wedge d\kappa_b$	$\nabla(K_aK_b) = (\nabla K_a)K_b + K_a(\nabla K_b)$
$d(\kappa\wedge\lambda)$ =	$d\kappa\wedge\lambda + \kappa\wedge d\lambda$	$\nabla\times(K\mathbf{L}) = \nabla K\times\mathbf{L} + K(\nabla\times\mathbf{L})$

TABLE 1.7 (continued)

$d(\kappa\wedge\mu)$	$= d\kappa\wedge\mu + \kappa\wedge d\mu$	$\nabla\cdot(K\mathbf{M})$	$= \nabla K\cdot\mathbf{M} + K(\nabla\cdot\mathbf{M})$
$d(\kappa\wedge\nu)$	$= 0$		
$d(\lambda_a\wedge\lambda_b)$	$= d\lambda_a\wedge\lambda_b - \lambda_a\wedge d\lambda_b$	$\nabla\cdot(\mathbf{L}_a\times\mathbf{L}_b)$	$= (\nabla\times\mathbf{L}_a)\cdot\mathbf{L}_b - \mathbf{L}_a\cdot(\nabla\times\mathbf{L}_b)$
$d(\lambda\wedge\mu)$	$= 0$		

F. The Adjoint Differential Operator δ; Codifferentials of Products

The inner product between a p-form α and the derivative of a (p-1)-form β, $(\alpha,d\beta)$, is used to define the codifferential δ, the adjoint of the operator d:

$$(\alpha,d\beta) = (\delta\alpha,\beta) \qquad 1.f.1$$

In effect, the operator δ shifts the p-form into a (p-1)-form. δ can be written explicitly in terms of $*$ and d (DESCHAMPS, §6.6): If α is a p-form, then

$$\delta\alpha = *^{-1}d*(-1)^p\,\alpha \qquad 1.f.2$$

Calculating the codifferentials of the R^3 forms, we find:

CODIFFERENTIALS OF R^3 FORMS

Operation		Vector Notation		
$\delta\kappa = 0$				
$\delta\lambda = -*^{-1}d*\lambda$	=	$-(\nabla\cdot\mathbf{L})$	:	0-form
$\delta\mu = *^{-1}d*\mu$	=	$(\nabla\times\mathbf{M})$	:	1-form
$\delta\nu = -*^{-1}d*\nu$	=	$-(\nabla[N])$	:	2-form

TABLE 1.8

The codifferentials of products are somewhat more involved than the product differentials. The following formula, which has some properties similar to the anti-derivation (1.e.3), gives the correct results in R^3, although it does not appear to be true for all metric spaces (In

Chapter 4, we shall see that the correct R^3/t formula has + signs preceeding the 3rd and 4th terms):

$$\delta(\alpha\wedge\beta) = \delta\alpha\wedge\beta + (-)^{p_\alpha}\,\alpha\wedge\delta\beta - (*^{-1}(*\alpha\wedge d))\wedge\beta - (-)^{p_\alpha p_\beta}\,(*^{-1}(*\beta\wedge d))\wedge\alpha \qquad \text{1.f.3}$$

(Please see the note on page 34). For the R^3 products, we have:

R^3 PRODUCT CODIFFERENTIALS — TABLE 1.9

a. Operation b. Vector Notation	
$\delta(\kappa_a\wedge\kappa_b)$	$\equiv 0$
$\delta(\kappa\wedge\lambda)$	$= \kappa\wedge\delta\lambda - (*^{-1}(*\lambda\wedge d))\wedge\kappa$
$-\nabla\cdot(K\mathbf{L})$	$= -K(\nabla\cdot\mathbf{L}) - (\mathbf{L}\cdot\nabla)K = -K(\nabla\cdot\mathbf{L}) - \mathbf{L}\cdot\nabla K$
	Identity: $(\mathbf{L}\cdot\nabla)K = \mathbf{L}\cdot\nabla K$
$\delta(\kappa\wedge\mu)$	$= \kappa\wedge\delta\mu - (*^{-1}(*\mu\wedge d))\wedge\kappa$
$\nabla\times(K\mathbf{M})$	$= K(\nabla\times\mathbf{M}) - (\mathbf{M}\times\nabla)K = K(\nabla\times\mathbf{M}) + \nabla K\times\mathbf{M}$
	Identity: $-(\mathbf{M}\times\nabla)K = \nabla K\times\mathbf{M}$
$\delta(\kappa\wedge\nu)$	$= \kappa\wedge\delta\nu - (*^{-1}(*\nu\wedge d))\wedge\kappa$
$-\nabla(K[N])$	$= -K(\nabla[N]) - ([N]\nabla)K = -K(\nabla[N]) - [N]\nabla K$
	Identity: $([N]\nabla)K = [N]\nabla K$
$\delta(\lambda_a\wedge\lambda_b)$	$= \delta\lambda_a\wedge\lambda_b - \lambda_a\wedge\delta\lambda_b - (*^{-1}(*\lambda_a\wedge d))\wedge\lambda_b + (*^{-1}(*\lambda_b\wedge d))\wedge\lambda_a$
$\nabla\times(\mathbf{L}_a\times\mathbf{L}_b)$	$= -(\nabla\cdot\mathbf{L}_a)\mathbf{L}_b + \mathbf{L}_a(\nabla\cdot\mathbf{L}_b) - (\mathbf{L}_a\cdot\nabla)\mathbf{L}_b + (\mathbf{L}_b\cdot\nabla)\mathbf{L}_a$
$\delta(\lambda\wedge\mu)$	$= \delta\lambda\wedge\mu - \lambda\wedge\delta\mu - (*^{-1}(*\lambda\wedge d))\wedge\mu - (*^{-1}(*\mu\wedge d))\wedge\lambda$
$-\nabla(\mathbf{L}\cdot\mathbf{M})$	$= -(\nabla\cdot\mathbf{L})\mathbf{M} - \mathbf{L}\times(\nabla\times\mathbf{M}) - (\mathbf{L}\cdot\nabla)\mathbf{M} - (\mathbf{M}\times\nabla)\mathbf{L}$
	$= -\mathbf{L}\times(\nabla\times\mathbf{M}) - \mathbf{M}\times(\nabla\times\mathbf{L}) - (\mathbf{L}\cdot\nabla)\mathbf{M} - (\mathbf{M}\cdot\nabla)\mathbf{L}$
	Identity: $-(\nabla\cdot\mathbf{L})\mathbf{M} - (\mathbf{M}\times\nabla)\times\mathbf{L} = -\mathbf{M}\times(\nabla\times\mathbf{L}) - (\mathbf{M}\cdot\nabla)\mathbf{L}$

In most cases a vector identity is required to convert the the strict vector translation of (1.f.3) into the most familiar vector form.

In computing the codifferential of a product where the terms are reversed from the examples in Table 1.9, the leading 2 terms in (1.f.3) may be reversed in accordance with the commutation properties. The

$(*^{-1}(*\alpha \wedge d)) \wedge \beta$ terms remain as they are.

G. The Self-Adjoint Laplace-Beltrami Operator Δ; Laplacians of Products

Recalling the defining relationship between the differential and codifferential operators (1.f.1), we can easily show that the new operator

$$\Delta = d\delta + \delta d \qquad 1.g.1$$

is self-adjoint:

$$(\alpha, \Delta\beta) = (\Delta\alpha, \beta) \qquad 1.g.2$$

The operator Δ is known as the Laplace-Beltrami operator. In R^3, Δ is simply the negative Laplacian:

LAPLACIANS OF R^3 FORMS — TABLE 1.10

Laplace-Beltrami Operation	Vector Notation	Laplacian Form	
$\Delta\kappa = d\delta\kappa + \delta d\kappa$	$= -\nabla\cdot\nabla K$	$= -\nabla^2 K$	: 0-form
$\Delta\lambda = d\delta\lambda + \delta d\lambda$	$= -\nabla(\nabla\cdot\mathbf{L}) + \nabla\times\nabla\times\mathbf{L}$	$= -\nabla^2\mathbf{L}$	: 1-form
$\Delta\mu = d\delta\mu + \delta d\mu$	$= \nabla\times\nabla\times\mathbf{M} - \nabla(\nabla\cdot\mathbf{M})$	$= -\nabla^2\mathbf{M}$	: 2-form
$\Delta\nu = d\delta\nu + \delta d\nu$	$= -\nabla\cdot\nabla[N]$	$= -\nabla^2[N]$	: 3-form

The operator Δ, applied to a p-form α, yields a p-form. Applied to a product, Δ has a distributive property but it also yields an additional term. As we shall see in the R^3 case, this distributivity is not always immediately obvious: rather complicated identities are required to reduce the number of terms.

R^3 PRODUCT LAPLACIANS

TABLE 1.11

a. Laplace-Beltrami Operation
b. Vector Notation

$$\Delta(\kappa_a\wedge\kappa_b) = \Delta\kappa_a\wedge\kappa_b + \kappa_a\wedge\Delta\kappa_b - (*^{-1}(*d\kappa_a\wedge d))\wedge\kappa_b - (*^{-1}(*d\kappa_b\wedge d))\wedge\kappa_a$$
$$-\nabla^2(K_aK_b) = -(\nabla^2K_a)K_b - K_a(\nabla^2K_b) - 2(\nabla K_a\cdot\nabla K_b)$$

Identity: $(\nabla K_a\cdot\nabla)K_b = \nabla K_a\cdot\nabla K_b$

$$\Delta(\kappa\wedge\lambda) = \Delta\kappa\wedge\lambda + \kappa\wedge\Delta\lambda - d\{(*^{-1}(*\lambda\wedge d))\wedge\kappa\} + (*^{-1}(*\lambda\wedge d))\wedge d\kappa - (*^{-1}(*d\kappa\wedge d))\wedge\lambda - (*^{-1}(*d\lambda\wedge d))\wedge\kappa$$
$$-\nabla^2(K\mathbf{L}) = -(\nabla^2K)\mathbf{L} - K(\nabla^2\mathbf{L}) - 2(\nabla K\cdot\nabla)\mathbf{L}$$

Identity: $\nabla[(\mathbf{L}\cdot\nabla)K] = (\nabla K\cdot\nabla)\mathbf{L} + (\mathbf{L}\cdot\nabla)\nabla K - [(\nabla\times\mathbf{L})\times\nabla]K$

$$\Delta(\kappa\wedge\mu) = \Delta\kappa\wedge\mu + \kappa\wedge\Delta\mu - d\{(*^{-1}(*\mu\wedge d))\wedge\kappa\} - (*^{-1}(*d\kappa\wedge d))\wedge\mu - (*^{-1}(*\mu\wedge d))\wedge d\kappa - (*^{-1}(*d\mu\wedge d))\wedge\kappa$$
$$-\nabla^2(K\mathbf{M}) = -(\nabla^2K)\mathbf{M} - K(\nabla^2\mathbf{M}) - 2(\nabla K\cdot\nabla)\mathbf{M}$$

Identity: $\nabla\times[(\mathbf{M}\times\nabla)K] = (\nabla K\cdot\nabla)\mathbf{M} - (\mathbf{M}\times\nabla)\times\nabla K - [(\nabla\cdot\mathbf{M})\nabla]K$

$$\Delta(\kappa\wedge\nu) = \kappa\wedge\Delta\nu + d\kappa\wedge\delta\nu - d\{(*^{-1}(*\nu\wedge d))\wedge\kappa\}$$
$$-\nabla^2(K[N]) = -K(\nabla^2[N]) - (\nabla^2K)[N] - 2(\nabla K\cdot\nabla[N])$$

Identity: $\nabla\cdot[([N]\nabla)K] = \nabla[N]\cdot\nabla K + [N](\nabla^2K)$

$$\Delta(\lambda_a\wedge\lambda_b) = \Delta\lambda_a\wedge\lambda_b + \lambda_a\wedge\Delta\lambda_b - d\{(*^{-1}(*\lambda_a\wedge d))\wedge\lambda_b\} + d\{(*^{-1}(*\lambda_b\wedge d))\wedge\lambda_a\} - (*^{-1}(*d\lambda_a\wedge d))\wedge\lambda_b - (*^{-1}(*\lambda_b\wedge d))\wedge d\lambda_a + (*^{-1}(*\lambda_a\wedge d))\wedge d\lambda_b + (*^{-1}(*d\lambda_b\wedge d))\wedge\lambda_a$$
$$-\nabla^2(\mathbf{L}_a\times\mathbf{L}_b) = -(\nabla^2\mathbf{L}_a)\times\mathbf{L}_b - \mathbf{L}_a\times(\nabla^2\mathbf{L}_b) - 2\{\mathbf{A}\}$$

where $\{\mathbf{A}\} = \sum_{i=1}^{3}\left\{\left(\frac{\partial\mathbf{L}_a}{\partial x^i}\times\frac{\partial\mathbf{L}_b}{\partial x^i}\right)\right\}$.

Example: dx^{23} term: $$\left\{\frac{\partial L_{a2}}{\partial x^1}\frac{\partial L_{b3}}{\partial x^1} + \frac{\partial L_{a2}}{\partial x^2}\frac{\partial L_{b3}}{\partial x^2} + \frac{\partial L_{a2}}{\partial x^3}\frac{\partial L_{b3}}{\partial x^3} - \frac{\partial L_{a3}}{\partial x^1}\frac{\partial L_{b2}}{\partial x^1} - \frac{\partial L_{a3}}{\partial x^2}\frac{\partial L_{b2}}{\partial x^2} - \frac{\partial L_{a3}}{\partial x^3}\frac{\partial L_{b2}}{\partial x^3}\right\}$$

Identity: $\nabla\times[(\mathbf{L}_a\cdot\nabla)\mathbf{L}_b] - \nabla\times[(\mathbf{L}_b\cdot\nabla)\mathbf{L}_a] + [(\nabla\times\mathbf{L}_a)\times\nabla]\times\mathbf{L}_b - [(\nabla\times\mathbf{L}_b)\times\nabla]\times\mathbf{L}_a + (\mathbf{L}_b\cdot\nabla)(\nabla\times\mathbf{L}_a) - (\mathbf{L}_a\cdot\nabla)(\nabla\times\mathbf{L}_b) = 2\{\mathbf{A}\}$

$$\Delta(\lambda\wedge\mu) = d\delta\lambda\wedge\mu + \delta\lambda\wedge d\mu - d\lambda\wedge\delta\mu + \lambda\wedge d\delta\mu - d\{(*^{-1}(*\lambda\wedge d))\wedge\mu\} - d\{(*^{-1}(*\mu\wedge d))\wedge\lambda\}$$
$$-\nabla^2(\mathbf{L}\cdot\mathbf{M}) = -(\nabla^2\mathbf{L})\cdot\mathbf{M} - \mathbf{L}\cdot\nabla^2\mathbf{M} - 2\{B\}$$

where $\{B\} = \sum_{i=1}^{3}\sum_{j=1}^{3}\left\{\frac{\partial L_j}{\partial x^i}\frac{\partial M_j}{\partial x^i}\right\}$

Identity: $-(\nabla\cdot\mathbf{L})(\nabla\cdot\mathbf{M}) - (\nabla\times\mathbf{L})\cdot(\nabla\times\mathbf{M}) - \nabla\cdot[(\mathbf{L}\cdot\nabla)\mathbf{M}] - \nabla\cdot[(\mathbf{M}\times\nabla)\times\mathbf{L}] = -\mathbf{L}\cdot(\nabla(\nabla\cdot\mathbf{M})) + (\nabla\times\nabla\times\mathbf{L})\cdot\mathbf{M} - 2\{B\}$

In Chapter 2, we shall see that the operations d, δ and Δ on products of differential forms allow the development of a variety of integration theorems. However, it is first necessary to develop those aspects of geometry which are necessary for integration. Differential forms themselves can be considered as local quantities, defined for a certain differential region. Through the introduction of some geometrical concepts, integration theorems, in which global quantities are formed from the local ones, can be developed. Obviously, this is important for any application of exterior differential algebra to physics.

H. Manifolds and Simplex Chains; The Boundary Operator ∂†

The fundamental utility of differential forms is that they are integrable over certain domains. Integration itself requires a quantitative description of geometric figures such as lines, surfaces and volumes. In particular, we are concerned with the integration of differential forms on manifolds, geometric structures where the points in every neighborhood can be described in terms of a local, orientable coordinate system. We may think of our manifolds as being composed of simplex elements: in one dimension, the straight line element; in 2, the triangle; and in 3, the tetrahedron. As an example, the surface of the earth may be mapped onto a covering mesh of triangular regions. This does not imply that simplex structures themselves satisfy the requirements for every purpose - our most common way of dividing space is rectangular - it is just that simplex elements have the most elementary quantitative geometrical behavior, permitting the encoding of a regular manifold in a systematic way.

A "chain" is defined as an oriented sequence of simplex elements. Figure 1.2 shows a chain of 1-dimensional elements in R^2, approximating a continuous R^2 curve:

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†: Not to be confused with the partial differential symbol.

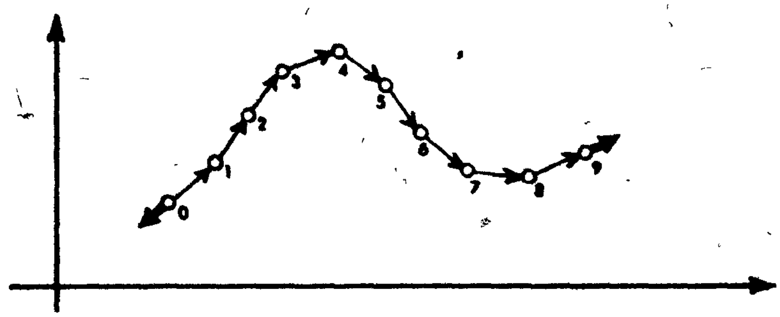

Figure 1.2 : An oriented 1-chain in R^2. The heavy arrows indicate the oriented boundary of this chain.

Each segment of this 1-chain is oriented towards the higher index. The chain has two oriented boundary points, 0 and 9, and the orientation of this boundary with respect to the chain is indicated by the heavy arrows. A chain is called open or closed depending on whether or not it has a boundary. A sphere (expressed as a chain of R^3 elements) has a surface (expressed as a chain of R^2 elements) which is closed.

Let us define an operator ∂ which produces the boundary of a chain c. The vertices, edges and surfaces of simplex elements can be unambiguously labeled so that ∂ provides the proper orientation of the elemental boundaries (FLANDERS, §5.5, §5.6). As a result, the common boundary of contiguous chain elements has two opposing orientations which cancel when the boundary contributions are summed under the operation ∂. The remaining contributions are those of the "external"† boundaries of the chain. In Figure 1.2, this means the endpoints, as indicated by the heavy arrows.

Figure 1.3 shows the possible types of contours in R^3 which may be described as a chain c of simplex elements - the line, surface and volume (drawn as simply-connected domains). The oriented boundary, ∂c, is indicated for each. A "cycle" is defined as a chain c whose boundary, ∂c, vanishes. It is a basic property of the boundary operator that

†: For a multiply-connected domain, "external" and true exterior are not synonymous.

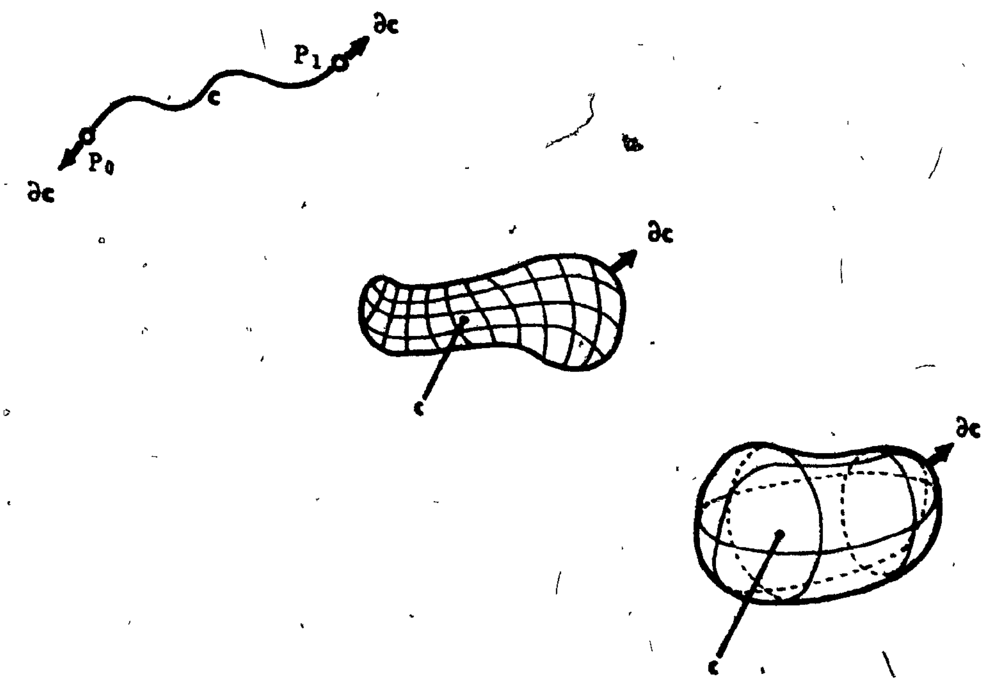

Figure 1.3 : Simply-connected R^3 contours, with oriented boundaries indicated.

each boundary is a cycle, that is,

$$\partial(\partial c) = 0 \qquad 1.h.1$$

In the Euclidean situation, this is easily demonstrated (FLANDERS, §5.6).

I. Stokes' Theorem

All of the integral relationships involving differential forms and their derivatives are either specific varieties of Stokes' theorem, or they may be constructed from it in a straightforeward manner. Stokes' theorem is the general relation between the integral of the derivative of a form over a manifold (expressed as a simplex chain c) and the integral of the form itself around the oriented manifold boundary.

It can be written as (FLANDERS, §5.8)

$$\oint_{\partial c} \alpha = \int_{c} d\alpha \qquad 1.i.1$$

where c is the simplex chain and ∂ is the boundary operator. The dimensionality of all quantities is consistent when α is a p-form and c a (p+1)-chain. By (1.h.1), the boundary chain ∂c is closed (it is a cycle). The closed integral symbol indicates this directly. (1.i.1) is the most general form of Stokes' theorem. There is a variation when either $d\alpha = 0$,

$$\oint_{\partial c} \alpha = 0 \qquad 1.i.2$$

or when $\partial c = 0$:

$$\oint_{c} d\alpha = 0 \qquad 1.i.3$$

The potential of this concise relationship will be demonstrated in Chapter 2.

J. The Poincaré Lemma and the de Rham Theorem

The Poincaré lemma (whose proof in a simple sense is a direct calculation relying only on the equality of mixed partial derivatives), states that for any differential form α,

$$d(d\alpha) = 0 \qquad 1.j.1$$

The two R^3 vector formulas which follow are:

$$dd\kappa = 0 \; : \; \nabla\times\nabla K = 0$$
$$dd\lambda = 0 \; : \; \nabla\cdot\nabla\times L = 0 \qquad 1.j.2$$

On a differentiable manifold M, a p-form α is called "closed" if $d\alpha = 0$. α is called "exact" if there exists a (p-1)-form β such that $\alpha = d\beta$. The Poincaré lemma states that every exact form is closed. (1.j.1) indicates that there is a parallel behavior between the differential operator d (with respect to differential forms) and the boundary operator ∂ (with respect to geometric elements)[1.h.1]. "Closed forms" and "cycles" are both annihilated by their respective operators, and "exact forms" and "boundaries" are both in the image of their operators.

The converse of the Poincaré lemma is used to show the existence of potentials. However, it generally has only a local validity, even on restricted manifolds (DESCHAMPS, §5.4). We can explain this using a geometric analogy: a simple closed surface boundary (cycle) in R^3 is the boundary of a unique R^3 volume, but a simple closed line boundary (cycle) in R^3 can be the boundary of an infinity of possible R^2 surfaces embedded in R^3. Likewise, unique potentials for closed forms exist only when certain conditions are met.

Global properties concerning the existence of potentials follow from de Rham's theorem. The parallel behavior between d and ∂ is best expressed as a mathematical correspondence between the following two groups, both of which concern the invariants of the manifold M with respect to the operators d and ∂ (WARNER, §4.13, §4.16):

> The "p^{th} de Rham cohomology group of M" is defined as the quotient space of the real vector space of closed p-forms on M modulo the subspace of exact p-forms on M:

$$H^p_{deR}(M) = \frac{\{\text{closed p-forms}\}}{\{\text{exact p-forms}\}} \qquad 1.j.3$$

The "p^{th} differentiable singular homology group of M with real coefficients" is defined by

$$_{\infty}H_p(M;R) = \frac{\text{kernel } (\partial_p)}{\text{image } (\partial_{p+1})} \qquad 1.j.4$$

and is a real vector space. The elements of the kernel (∂_p) are called differentiable p-cycles, and the elements of the image (∂_{p+1}) are called differentiable p-boundaries.

The de Rham theorem states that the mapping

$$H^p_{de\,R}(M) \rightarrow {}_{\infty}H_p(M;R)^* \qquad 1.j.5$$

(* indicates the dual space of the real differentiable singular homology) is isomorphic and that for a closed p-form α and a p-cycle z, a real number is determined by the integral

$$\int_z \alpha \qquad 1.j.6$$

(independent of the choice of either α or z by (1.i.2) and (1.i.3) respectively) which is the manifestation of the isomorphism (1.j.5). These real numbers are known as periods. Stokes' theorem states that the periods of an exact form are zero; de Rham's theorem supplies the converse: if the periods of a closed form α are all zero, then it is exact. In addition, if a real number per(z) is assigned to each z on M such that

$$\begin{aligned} &\text{per}(az_1 + z_2) = a\,\text{per}(z_1) + \text{per}\,(z_2) \\ &\text{per(boundary)} = 0, \end{aligned} \qquad 1.j.7$$

then there is a closed form α on M which has the assigned periods (WARNER, §4.17):

$$\int_z \alpha = \text{per}(z) \qquad 1.j.8$$

K. Riemannian Manifolds; The Hodge Decomposition Theorem

In §1.F, we introduced the codifferential δ without specifically describing the inner product space used in its definition. We shall turn our attention toward this now. Let us take a compact (closed), oriented, n-dimensional Riemannian manifold M. Then, for 2 p-forms α and β defined on M, the integral

$$(\alpha,\beta) = \int_M \alpha \wedge *\beta \qquad 1.k.1$$

satisfies the conditions of bilinearity and positivity in §1.C, making it a suitable inner product between α and β on M (FLANDERS, §8.4).

The proof of the codifferential relation (1.f.2) follows immediately. Let α = p-form and β = (p+1)-form. Then the differential of a product (1.e.3) is written:

$$d(\alpha \wedge *\beta) = d\alpha \wedge *\beta + (-)^p \alpha \wedge d*\beta \qquad 1.k.2$$

The terms in (1.k.2) are n-forms. Integrating over M, the left-hand side of (1.k.2) is zero by Stokes' theorem (1.i.3):

$$\int_M d(\alpha \wedge *\beta) = \int_{\partial M} (\alpha \wedge *\beta) = 0 = \int_M d\alpha \wedge *\beta + (-)^p \int_M \alpha \wedge d*\beta \qquad 1.k.3$$

Therefore,

$$\int_M d\alpha \wedge *\beta = (-)^{p+1} \int_M \alpha \wedge **^{-1} d*\beta \qquad 1.k.4$$

and

$$(d\alpha,\beta) = (\alpha, (-)^{p+1} *^{-1} d*\beta) = (\alpha, \delta\beta) \qquad 1.k.5$$

defining the codifferential δ.

The inner product involving the Laplace-Beltrami operator Δ can be expanded as follows:

$$(\Delta\alpha,\alpha) = (\delta\alpha,\delta\alpha) + (d\alpha,d\alpha) \qquad 1.k.6$$

Should α be a harmonic p-form ($\Delta\alpha = 0$), then

$$(\delta\alpha,\delta\alpha) + (d\alpha,d\alpha) = 0 \qquad 1.k.7$$

By the condition of positivity for an inner product (§1.C, (c)), each term in (1.k.7) is ≥ 0. Therefore, if α is harmonic,

$$\begin{aligned} \delta\alpha &= 0 \\ d\alpha &= 0 \end{aligned} \qquad 1.k.8$$

This proof will be used in §3.F to make a remark concerning the global properties of harmonic electromagnetic fields.

One of the most important relations for differential forms defined on a compact Riemannian manifold is the Hodge decomposition theorem. Let $E^p(M)$ denote the space of smooth p-forms on M, and let H^p denote the subspace of harmonic p-forms on M:

$$H^p = \{\omega \in E^p(M) : \Delta\omega = 0\} \qquad 1.k.9$$

Then (WARNER, §6.8),

> For each integer, with $0 \leq p \leq n$, H^p is finite dimensional, and we have the following orthogonal direct sum decompositions of the space $E^p(M)$:

$$\begin{aligned} E^p(M) &= \Delta(E^p) + H^p \\ &= d\delta(E^p) + \delta d(E^p) + H^p \\ &= d(E^{p-1}) + (E^{p+1}) + H^p \end{aligned} \qquad 1.k.10$$

> Consequently, the equation $\Delta\omega = \alpha$ has a solution $\omega \in E^p(M)$ if and only if the p-form α is orthogonal to the space of harmonic p-forms.

In other words, if ω is any p-form, then the decomposition

$$\omega = d\alpha + \delta\beta + \gamma \qquad 1.k.11$$

is unique, with α = (p-1)-form, β = (p+1)-form and γ = a harmonic p-form.

Following the proof of uniqueness for the Hodge decomposition theorem (FLANDERS, §8.4), we can show that if ω is closed ($d\omega = 0$), the term $\delta\beta$ in the decomposition is zero. Let (1.k.11) be the decomposition of ω. Then,

$$d\omega = 0 = dd\alpha + d\delta\beta + d\gamma \qquad 1.k.12$$

Therefore, $d\delta\beta = 0$. Forming the inner product,

$$(d\delta\beta, \beta) = (\delta\beta, \delta\beta) = 0 \qquad 1.k.13$$

Consequently, $\delta\beta = 0$ by the condition of positivity. In the same manner, if the dual of ω is closed ($\delta\omega = 0$), the term $d\alpha$ is zero.

L. C Example; Differential Structure of an Analytic C Form

The primary objective of this thesis is a discussion of the subject of electromagnetism in which the physical quantities are expressed as exterior differential forms. We shall find it useful, however, to first discuss the application of differential forms to complex variable theory, because the space C and the space R^3/t each have a negative metric component. Furthermore, there are some fundamental differences concerning the differential operator on these two spaces, and the development in exterior differential forms makes these differences quite evident. Finally, it allows us to introduce a diagrammatic representation of the "differential structure" - the relationship between differential forms of various orders arising from the operations d, δ, Δ and *.

A geometric description of the complex line **C** is a two-dimensional metric space with differential coordinates $\{dx, idy\}$ such that $(dx,dx) = +1$; $(idy,idy) = -1$. Computing the Hodge * operator, we find:

$$\begin{aligned} * \, i \, dx \wedge dy &= +1 \\ * \, dx &= -i \, dy \\ * \, dy &= +i \, dx \\ * \, 1 &= -i \, dx \wedge dy \end{aligned} \qquad 1.1.1$$

Let us take a 1-form $\omega = Adx + Bidy$ as the most general 1-form on the complex line. The form ω will be harmonic when $\Delta\omega = 0$, also implying that $d\omega = 0$ and $\delta\omega = 0$ by (1.k.8). Following the derivation of the directional derivative on the complex line (HILDEBRAND, §10.4), we can introduce a complex 1-form differential operator,

$$d'^{\dagger} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x}(dx - dy) + i\,\frac{\partial}{\partial y}(dx - dy)\right) \qquad 1.1.2$$

Taking the Hermitian adjoint when computing the codifferential δ',

$$\delta'\omega = *^{-1}\bar{d}'*(-)^{1}\,\omega \qquad 1.1.3$$

where

$$\bar{d}' = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x}(dx - dy) - i\,\frac{\partial}{\partial y}(dx - dy)\right) \qquad 1.1.4$$

we find that the conditions $d'\omega = 0$ and $\delta'\omega = 0$ each yield for the real and imaginary parts

---

†: The complex differential operators on **C** are indicated with a prime in order to avoid confusion with their real counterparts.

$$\left(\frac{\partial A}{\partial x} = \frac{\partial B}{\partial y}\right) \quad \text{and} \quad \left(\frac{\partial B}{\partial x} = -\frac{\partial A}{\partial y}\right), \qquad 1.1.5$$

the Cauchy-Riemann equations. For the equivalent Laplace-Beltrami operator $\Delta'\omega = d'\delta'\omega + \delta'd'\omega$, we find

$$\Delta'\omega = -\left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2}\right)dx - i\left(\frac{\partial^2 B}{\partial x^2} + \frac{\partial^2 B}{\partial y^2}\right)dy, \qquad 1.1.6$$

and since $\Delta'\omega = 0$ (reversing the argument (1.k.8)), A and B each satisfy the Laplacian equation. In other words, when the 1-form ω on $\mathbb{C}$ is harmonic under the definition of the differential operator d' (1.1.2), A and B each satisfy Laplace's equation and are inter-related by the Cauchy-Riemann equations. Thus in these circumstances, harmonic $\leftrightarrow$ analytic.

The above relationships are most simply illustrated with a diagram. Figure 1.4 shows the operations d', δ' and Δ' on a 1-form ω in $\mathbb{C}$. This will be called the "structural diagram" for ω:

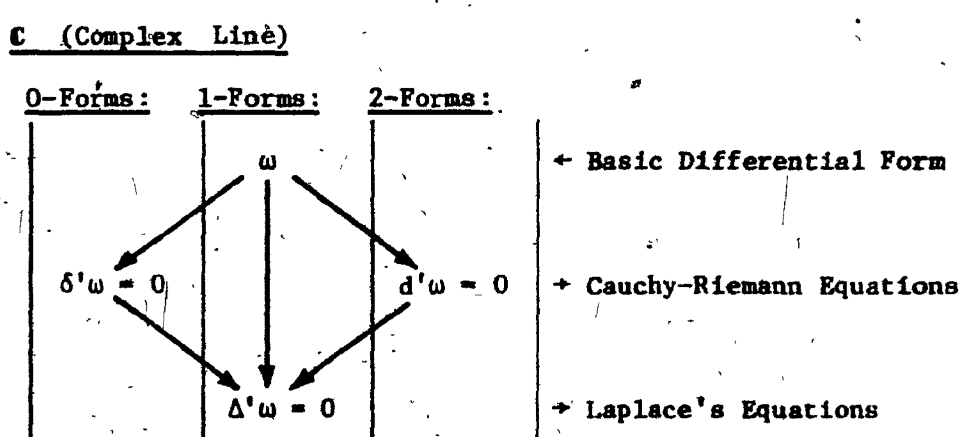

Figure 1.4 : Structural diagram for an analytic $\mathbb{C}$ differential form.

In this type of diagram, the operation d (d') is represented by a right - directed arrow, δ (δ') by a left - directed arrow, and Δ (Δ')

by a downward arrow. It can be shown that under the conditions of analyticity ($d'\omega = \delta'\omega = 0$), ω is not derivable from a potential by the Hodge decomposition theorem (see §1.K). Applying Stokes' theorem to the closed 1-form ω, we have the Cauchy Integral Theorem. EBERLEIN (1975) discusses the classical approach to this. The Cauchy Integral Formula will follow from a suitable calculation of the period of a $\mathbb{C}$ contour enclosing the singularity of the form defined for this integral.

TYPALDOS and POGORZELSKI (1973) discuss certain parallel relationships between the Cauchy-Riemann equations in complex variable theory and the Maxwell equations of electromagnetism, suggesting that Maxwell's electrodynamics is a limited 4-dimensional analytic theory possibly containing an analog to the analytic continuation process. Comparing the structural diagram for an analytic form in $\mathbb{C}$ (Figure 1.4) with the structural diagram for the electromagnetic forms in R^3/t (Figure 3.1), we see that this idea has a certain merit but the substantial differences in the differential operator will not permit such a direct comparison. The only possible comparison by analogy would be for harmonic electromagnetic fields in R^3/t. We shall see in §3.F that the existence of such fields raises certain questions. One possible action would be the definition of an equivalent "complex" R^3/t differential operator similar to (1.1.2). Then the investigation of R^3/t differential structures could closely parallel that for $\mathbb{C}$.

<u>Summary</u>

This chapter forms the first part of the mathematical development required for the application of exterior differential algebra to electromagnetism. We have developed multivector algebra in connection with an R^3 differentiable manifold, and have introduced the differential and star operators. We have noted the parallel character of the differential and boundary operators, and have presented Stokes' theorem and the Hodge theorem, both of which will see further application. We shall now proceed to give an extended exposition of Stokes' theorem, leading to the derivation of a wide variety of R^3 vector and scalar integration formulas.

Note Concerning the Codifferential of a Product (1.f.3)

The $(*^{-1}(*\alpha \wedge d)) \wedge \beta$ terms in (1.f.3) are meant to be calculated as $(*^{-1}((*\alpha) \wedge d)) \wedge \beta$ with the entire parenthetical term acting as an operator. There is no commutativity of terms within the operator. As in the FORTRAN computer programming language, the operation contained in the innermost parenthesis is performed first. It should be noted that the partial differential operators do commute with the differentials (although not with scalar functions), enabling the differential terms of $(*\alpha) \wedge d$ to be collected and then operated on by $*^{-1}$.

CHAPTER II

INTEGRATION THEOREMS

The general integration formula (1.i.1), Stokes' theorem, is essential to the physical application of exterior differential algebra, since it relates integrated quantities defined on a differentiable manifold. Green's theorems, which are a specialized development of Stokes' theorem, are perhaps the most valuable mathematical tool available for the solution of electromagnetic boundary-value problems. They are used to establish the Green's function solution techniques - the construction of particular solutions from a distribution of point-source solutions, each of which satisfies the boundary conditions of the problem. In this chapter, we show the systematic development of R^3 vector integration theorems that is possible from (1.i.1) using the techniques of exterior differential algebra. Although many of the results are familiar integration formulas in vector analysis, the pattern of their development manifests simplicity and completeness, especially since no geometric proofs are involved.

A number of developments in this chapter deserve comment. By introducing a "unit differential form" (a form with constant coefficients, analogous to a unit vector), we increase the flexibility of (1.i.1) for our purposes. We continue our use of a vector notation which is interchangeable with differential forms. We shall see the economy of this notation, eliminating the need for para-geometric integration variables. Finally, we devote considerable attention to the development of integration formulas symmetric (or antisymmetric) in two variables. We establish the general pattern for R^3 Green's theorems and show how the familiar scalar and less-familiar vector (STRATTON, §4.14) Green's theorems are related to it.

A. Integration Formulas Based on Stokes' Theorem

i. Basic varieties of Stokes' theorem

We repeat from §1.I the generalized version of Stokes' theorem:

$$\int_{\partial c^{p}} \alpha_{p} = \int_{c^{p+1}} (d\alpha)_{p+1} \qquad 2.a.1$$

The subscripts and superscripts refer to the order of the differential forms and chains respectively. ∂c refers to the oriented boundary of the chain c. Recall that the best way of defining the orientation is to express the chain c in terms of ordered simplex elements.

As in Chapter 1, we choose "standard" R^3 forms κ, λ, μ, ν for $p = 0, 1, 2, 3$. For these standard forms, (2.a.1) can be expressed as:

STOKES' THEOREMS FOR BASIC R^3 DIFFERENTIAL FORMS — TABLE 2.1a

Order	Stokes' Theorem
p=0	$\sum^{+}_{\partial \bar{c}} (\kappa)_0 = \int_c (d\kappa)_1$
p=1	$\oint_{\partial c} (\lambda)_1 = \iint_c (d\lambda)_2$
p=2	$\oiint_{\partial c} (\mu)_2 = \iiint_c (d\mu)_3$
p=3	$d\nu = 0$ and ∂c is meaningless.

In each case, the orders of the chain c and its boundary ∂c match the order of the differential form quantities. In terms of the vector notation, Table 2.1a is written

BASIC R^3 STOKES' THEOREMS:
VECTOR NOTATION TABLE 2.1b

Order	Stokes' Theorem
p=0	$\sum_{\partial c}^{+}(K)_0 = \int_c (\nabla K)_1$
p=1	$\oint_{\partial c}(\mathbf{L})_1 = \iint_c (\nabla\times \mathbf{L})_2$
p=2	$\oiint_{\partial c}(\mathbf{M})_2 = \iiint_c (\nabla\cdot \mathbf{M})_3$

The subscripts on the various terms indicate the order. This notation provides for correct and unambiguous integration provided the chain is properly oriented. Note the comparison with traditional notation:

BASIC R^3 STOKES' THEOREMS: NOTATION COMPARISON TABLE 2.1c

New Notation	Traditional Vector Notation
$\sum_{\partial c}^{+}(K)_0$	$= -K(P-) + K(P+)$ (P refers to an endpoint of c).
$\oint_{\partial c}(\mathbf{L})_1$	$= \oint \overline{L}\cdot\overline{ds} = \oint \overline{L}\cdot\overline{dl}$
$\oiint_{\partial c}(\mathbf{M})_2$	$= \oiint (\overline{M}\cdot\hat{n})\,da = \oiint \overline{M}\cdot\overline{dS} = \oiint (\hat{n}\cdot\overline{M})\lvert\overline{dS}\rvert$
$\int_c (\nabla K)_1$	$= \int \overline{\nabla K}\cdot\overline{ds} = \int \overline{\nabla K}\cdot\overline{dl}$
$\iint_c (\nabla\times\mathbf{L})_2$	$= \iint (\overline{\nabla}\times\overline{L})\cdot\hat{n}\,da = \iint \hat{n}\cdot(\overline{\nabla}\times\overline{L})\lvert\overline{dS}\rvert$
$\iiint_c (\nabla\cdot\mathbf{M})_3$	$= \iiint (\overline{\nabla}\cdot\overline{M})\,dv$

It is clear that once the integration manifold has been expressed as an ordered chain of simplex elements, an economy of notation results, and it is not necessary to further stipulate line elements, surface elements and unit vectors in order to integrate the vector quantity in question.

ii. Stokes' theorems involving unit differential forms

Besides the three R^3 Stokes' theorems listed in Tables 2.1, we can obtain several other versions. These are easily derived once we have introduced a special set of forms called "unit forms":

$$\begin{aligned} u_0 &= 1 \\ u_1 &= dx^1 + dx^2 + dx^3 \\ u_2 &= dx^{23} + dx^{31} + dx^{12} \\ u_3 &= dx^{123} \end{aligned} \qquad 2.a.2$$

Actually, any R^3 constant form may be used. Note, however, that

$$\begin{aligned} {*u_0} &= u_3 \\ {*u_1} &= u_2 \\ {*u_2} &= u_1 \\ {*u_3} &= u_0 \end{aligned} \qquad 2.a.3$$

Since the coefficients of the u_i are 1, $du_i = 0$. Then, provided $d(\alpha \wedge u_i)$ exists, the product derivative (1.e.3)

$$\begin{aligned} d(\alpha \wedge u_i) &= d\alpha \wedge u_i + (-)^{p_\alpha} \alpha \wedge d(u_i) \\ &= d\alpha \wedge u_i \end{aligned} \qquad 2.a.4$$

allows consideration of Stokes' theorems written in the form

$$\int_{\partial c^{p_\alpha + i}} (\alpha \wedge u_i)_{p_\alpha + i} = \int_{c^{p_\alpha + i + 1}} (d\alpha \wedge u_i)_{p_\alpha + i + 1} \qquad 2.a.5$$

These will be referred to as the "raised order" or "raised" Stokes' theorems. In R^3, the following are possible:

RAISED ORDER STOKES' THEOREMS IN R^3 — TABLE 2.2a

$$\oint_{\partial c} (\kappa \wedge u_1)_1 = \iint_c (d\kappa \wedge u_1)_2$$
$$\oiint_{\partial c} (\kappa \wedge u_2)_2 = \iiint_c (d\kappa \wedge u_2)_3$$
$$\oiint_{\partial c} (\lambda \wedge u_1)_2 = \iiint_c (d\lambda \wedge u_1)_3$$

Proposing a notation for the vector-type unit forms, $u_1 = (\mathbf{n})_1$; $u_2 = (\mathbf{n})_2$†, the equations in Table 2.2a become:

RAISED ORDER THEOREMS: VECTOR NOTATION — TABLE 2.2b

$$\oint_{\partial c} (K\mathbf{n})_1 = \iint_c (\nabla K\times\mathbf{n})_2$$
$$\oiint_{\partial c} (K\mathbf{n})_2 = \iiint_c (\nabla K\cdot\mathbf{n})_3$$
$$\oiint_{\partial c} (\mathbf{L}\times\mathbf{n})_2 = \iiint_c [(\nabla\times\mathbf{L})\cdot\mathbf{n}]_3$$

Again, we compare with traditional notation:

RAISED R^3 STOKES' THEOREMS: NOTATION COMPARISON — TABLE 2.2c

$$\oint_{\partial c} (K\mathbf{n})_1 = \oint K\,\overline{ds} = \oint K\,\overline{dl}$$

†: In R^3, u_1 and u_2 are identical under * (2.a.3), so the use of a different notation (such as $(\mathbf{m})_2$) for u_2 is unnecessary.

TABLE 2.2c (continued)

$\oiint_{\partial c} (\mathbf{K}\mathbf{n})_2$	$= \oiint K\,\hat{n}\,da$	$= \oiint K\,\overline{dS}$	$= \oiint \hat{n}\,K\lvert\overline{dS}\rvert$
$\oiint_{\partial c} (\mathbf{L}\times\mathbf{n})_2$	$= \oiint (\hat{n}\times\overline{L})\,da$	$= -\oiint \overline{L}\times\overline{dS}$	$= \oiint (\hat{n}\times\overline{L})\,\lvert\overline{dS}\rvert$
$\iint_c (\nabla\mathbf{K}\times\mathbf{n})_2$	$= \iint (\hat{n}\times\overline{\nabla K})\,da$	$= \iint (\hat{n}\times\overline{\nabla K})\,\lvert\overline{dS}\rvert$	
$\iiint_c (\nabla\mathbf{K}\cdot\mathbf{n})_3$	$= \iiint (\overline{\nabla K})\,dv$		
$\iiint_c [(\nabla\times\mathbf{L})\cdot\mathbf{n}]_3$	$= \iiint (\overline{\nabla}\times\overline{L})\,dv$		

Here also we see the benefit of the new notation.

<u>iii. Dual versions of Stokes' theorem</u>

So far we have developed the 6 varities of Stokes' theorem usually found in the appendices of books containing vector algebra. However, in traditional vector analysis, no distinction is made concerning the order of a vector quantity. Yet our basic forms κ, λ, μ, ν are definitely ordered. Is the traditional mathematics more flexible? We will now see that the formulas for $(\mathrm{K})_0$, $(\mathbf{L})_1$ and $(\mathbf{M})_2$ in Tables 2.1 and 2.2 are reflexive with formulas for $[\mathrm{N}]_0$, $(\mathbf{M})_1$ and $(\mathbf{L})_2$, and we proceed by developing Stokes' theorem for the duals of κ, λ, μ, ν.

For the duals of the basic forms, we can write

$$d{*}\alpha = {*}{*}^{-1}d{*}\alpha = (-)^{p}{*}\delta\alpha \qquad 2.a.6$$

Therefore, we can construct the following formulas from (2.a.1):

STOKES' THEOREM FOR DUALS OF BASIC R^3 FORMS — TABLE 2.3a

Order	Stokes' Theorem
p=0	$d{*}\kappa \equiv 0$ and ∂c is meaningless.
p=1	$\oint\!\!\oint_{\partial c} ({*}\lambda)_2 = -\iiint_c ({*}\delta\lambda)_3$
p=2	$\oint_{\partial c} ({*}\mu)_1 = +\iint_c ({*}\delta\mu)_2$
p=3	$\sum_{\partial c^-}^{+} ({*}\nu)_0 = -\int_c ({*}\delta\nu)_1$

In vector notation, Table 2.3a becomes:

BASIC DUAL THEOREMS: VECTOR NOTATION — TABLE 2.3b

$$\oint\!\!\oint_{\partial c} (\mathbf{L})_2 = \iiint_c (\nabla\cdot\mathbf{L})_3$$

$$\oint_{\partial c} (\mathbf{M})_1 = \iint_c (\nabla\times\mathbf{M})_2$$

$$\sum_{\partial c^-}^{+} ([N])_0 = \int_c (\nabla[N])_1$$

Using the unit forms, we can raise the order of the duals of the basic forms κ, λ, μ, ν. Then (1.e.3) can be written

$$d({*}\alpha \wedge u_i) = (-)^{p+1}{*}\delta\alpha \wedge u_i\,, \qquad 2.a.7$$

so that the following vector integration formulas are produced:

RAISED ORDER DUAL R^3
STOKES' THEOREMS TABLE 2.4

$$\oint_{\partial c} ([N]\mathbf{n})_1 = \iint_c (\nabla[N]\times\mathbf{n})_2$$

$$\oiint_{\partial c} ([N]\mathbf{n})_2 = \iiint_c (\nabla[N]\cdot\mathbf{n})_3$$

$$\oiint_{\partial c} (\mathbf{M}\times\mathbf{n})_2 = \iiint_c [(\nabla\times\mathbf{M})\cdot\mathbf{n}]_3$$

A comparison of Table 2.3b with Table 2.1b, and Table 2.4 with Table 2.2b, immediately establishes the reflexive similarity between {K, **L**, **M**} and {[N], **M**, **L**} with respect to the group of the 6 "usual" Stokes' theorems. However, these are not the only possible varieties of Stokes' theorem. We can construct still further examples using the duals of the raised-order forms.

From the codifferential expansions in Table 1.9, we find:

$$\begin{aligned}
d{*}(\kappa\wedge u_1) &= {*}[({*}^{-1}({*}u_1\wedge d))\wedge\kappa] \\
d{*}(\kappa\wedge u_2) &= -{*}[({*}^{-1}({*}u_2\wedge d))\wedge\kappa] \\
d{*}(\lambda\wedge u_1) &= {*}[({*}^{-1}({*}u_1\wedge d))\wedge\lambda] + {*}[\delta\lambda\wedge u_1] \\
d{*}(\lambda\wedge u_2) &= {*}[({*}^{-1}({*}u_2\wedge d))\wedge\lambda] - {*}[\delta\lambda\wedge u_2] \\
d{*}(\mu\wedge u_1) &= {*}[({*}^{-1}({*}u_1\wedge d))\wedge\mu] - {*}[\delta\mu\wedge u_1]
\end{aligned} \qquad 2.a.8$$

These allow us to write the following vector integration formulas:

STOKES' THEOREMS FOR THE
DUALS OF RAISED FORMS TABLE 2.5

$$\oiint_{\partial c} (K\mathbf{n})_2 = \iiint_c {}^{*}[(\mathbf{n}\cdot\nabla)K]_3$$

$$\oint_{\partial c} (K\mathbf{n})_1 = \iint_c [(\mathbf{n}\times\nabla)K]_2$$

TABLE 2.5 (continued)

$$\oint_{\partial c} (\mathbf{L}\times\mathbf{n})_1 = \iint_c \{-[(\nabla\cdot\mathbf{L})\mathbf{n}] + [(\mathbf{n}\cdot\nabla)\mathbf{L}]\}_2$$

$$\sum_{\partial c^-}^{+} (\mathbf{L}\cdot\mathbf{n})_0 = \iint_c \{ [(\nabla\cdot\mathbf{L})\mathbf{n}] + [(\mathbf{n}\times\nabla)\times\mathbf{L}]\}_1$$

$$\sum_{\partial c^-}^{+} (\mathbf{M}\cdot\mathbf{n})_0 = \iint_c \{-[(\nabla\times\mathbf{M})\times\mathbf{n}] + [(\mathbf{n}\cdot\nabla)\mathbf{M}]\}_1$$

Subsitution of $*\nu$, $*\mu$, $*\lambda$ for κ, λ, μ in (2.a.8) leads to the dual set of equations:

STOKES' THEOREMS FOR THE DUALS OF RAISED DUAL FORMS — TABLE 2.6

$$\oint\!\!\oint_{\partial c} ([N]\mathbf{n})_2 = \iiint_c [(\mathbf{n}\cdot\nabla)[N]]_3$$

$$\oint_{\partial c} ([N]\mathbf{n})_1 = \iint_c [(\mathbf{n}\times\nabla)[N]]_2$$

$$\oint_{\partial c} (\mathbf{M}\times\mathbf{n})_1 = \iint_c \{-[(\nabla\cdot\mathbf{M})\mathbf{n}] + [(\mathbf{n}\cdot\nabla)\mathbf{M}]\}_2$$

$$\sum_{\partial c^-}^{+} (\mathbf{M}\cdot\mathbf{n})_0 = \int_c \{ [(\nabla\cdot\mathbf{M})\mathbf{n}] + [(\mathbf{n}\times\nabla)\times\mathbf{M}]\}_1$$

$$\sum_{\partial c^-}^{+} (\mathbf{L}\cdot\mathbf{n})_0 = \int_c \{-[(\nabla\times\mathbf{L})\times\mathbf{n}] + [(\mathbf{n}\cdot\nabla)\mathbf{L}]\}_1$$

Our development of the integration formulas in Tables 2.1 through 2.6 has proceeded using differential relations from R^3 exterior differential algebra. It is also possible to use a more traditional route for some of the later cases. The first two relations in Tables 2.5 and 2.6 may be derived from formulas in Tables 2.2 and 2.4 by using standard vector identities on the right-hand side. The remaining three are derived from the formulas in Tables 2.1 and 2.3 by subsituting into the integrands of these equations products of vectors and unit vectors, such as $\lambda \rightarrow *^{-1}(*\mu \wedge u_i)$: $(\mathbf{L})_i \rightarrow (\mathbf{M} \times n)_i$, and reducing the right-hand side with vector identities. Obviously, the differential algebra route is less cumbersome.

Figure 2.1 illustrates the pattern for the development of the Stokes' theorems found in Tables 2.1 through 2.6.

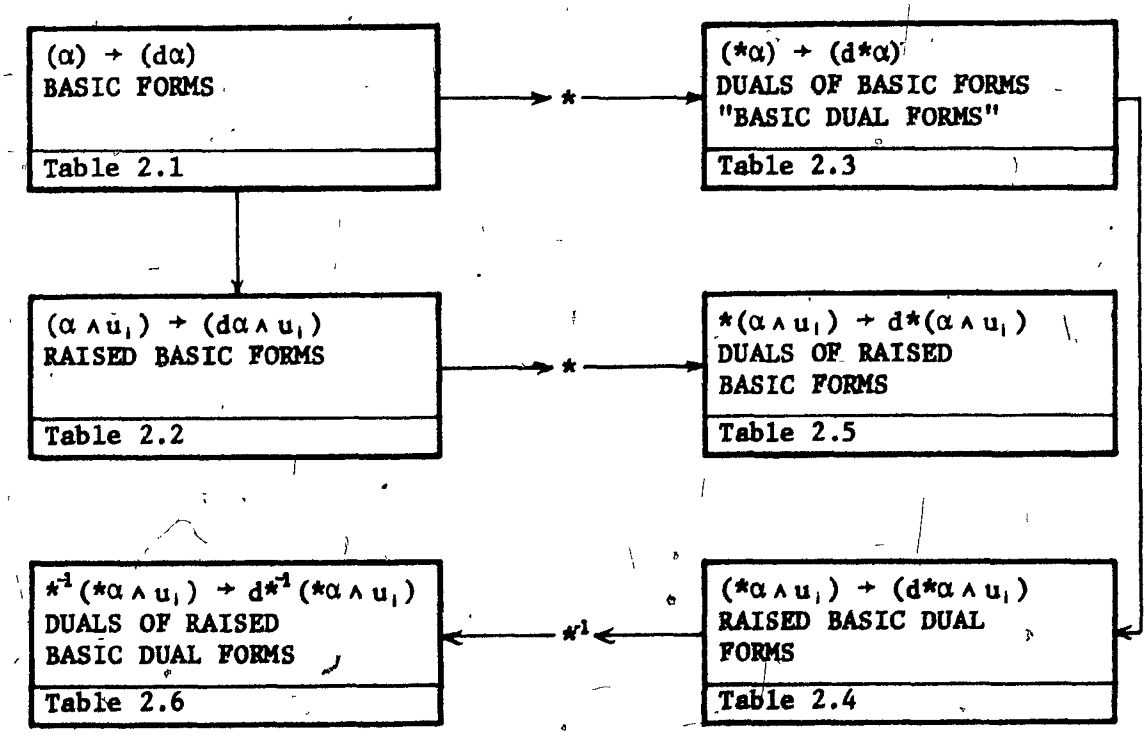

Figure 2.1 : Development of differential relations for the Stokes' theorems in Tables 2.1 through 2.6.

iv. Stokes' theorems involving 2nd derivatives

Following the pattern in Figure 2.1, and by using the Poincaré lemma (1.j.1) to eliminate certain constructions from consideration, we find 18 Stokes' theorems involving the 2^{nd} derivatives of the R^3 basic forms. The fundamental derivatives are:

$$
\begin{aligned}
d(*d\kappa)_2 &= [d*d\kappa]_3 \\
d(*d\lambda)_1 &= [d*d\lambda]_2 \\
d(*d\mu)_0 &= [d*d\mu]_1 \\
d(*d*\lambda)_0 &= [d*d*\lambda]_1 \\
d(*d*\mu)_1 &= [d*d*\mu]_2 \\
d(*d*\nu)_2 &= [d*d*\nu]_3
\end{aligned} \qquad 2.a.9
$$

The fundamental raised derivatives are:

$$
\begin{aligned}
d((*d\lambda) \wedge u_1)_2 &= [(d*d\lambda) \wedge u_1]_3 \\
d((*d\mu) \wedge u_1)_1 &= [(d*d\mu) \wedge u_1]_2 \\
d((*d\mu) \wedge u_2)_2 &= [(d*d\mu) \wedge u_2]_3 \\
d((*d*\lambda) \wedge u_1)_1 &= [(d*d*\lambda) \wedge u_1]_2 \\
d((*d*\lambda) \wedge u_2)_2 &= [(d*d*\lambda) \wedge u_2]_3 \\
d((*d*\mu) \wedge u_1)_2 &= [(d*d*\mu) \wedge u_1]_3
\end{aligned} \qquad 2.a.10
$$

Finally, the duals of the raised derivatives are:

$$
\begin{aligned}
d[*((*d\lambda) \wedge u_1)]_1 &= *[(*^{-1}(*u_1 \wedge d) \wedge (*d\lambda)]_2 \\
d[*((*d\mu) \wedge u_1)]_2 &= *[(*^{-1}(*u_1 \wedge d) \wedge (*d\mu)]_3 \\
d[*((*d\mu) \wedge u_2)]_1 &= -*[(*^{-1}(*u_2 \wedge d) \wedge (*d\mu)]_2 \\
d[*((*d*\lambda) \wedge u_1)]_2 &= *[(*^{-1}(*u_1 \wedge d) \wedge (*d*\lambda)]_3 \\
d[*((*d*\lambda) \wedge u_2)]_1 &= -*[(*^{-1}(*u_2 \wedge d) \wedge (*d*\lambda)]_2 \\
d[*((*d*\mu) \wedge u_1)]_1 &= *[(*^{-1}(*u_1 \wedge d) \wedge (*d*\mu)]_2
\end{aligned} \qquad 2.a.11
$$

In vector notation, we have for (2.a.9),

2^{nd} ORDER STOKES' THEOREMS
FOR BASIC R^3 FORMS — TABLE 2.7

$$\oiint_{\partial c} (\nabla K)_2 = \iiint_c (\nabla\cdot\nabla K)_3$$

$$\oint_{\partial c} (\nabla\times \mathbf{L})_1 = \iint_c (\nabla\times\nabla\times \mathbf{L})_2$$

$$\sum_{\partial c^-}^{+} (\nabla\cdot \mathbf{M})_0 = \int_c (\nabla\nabla\cdot \mathbf{M})_1$$

$$\sum_{\partial c^-}^{+} (\nabla\cdot \mathbf{L})_0 = \int_c (\nabla\nabla\cdot \mathbf{L})_1$$

$$\oint_{\partial c} (\nabla\times \mathbf{M})_1 = \iint_c (\nabla\times\nabla\times \mathbf{M})_2$$

$$\oiint_{\partial c} (\nabla[N])_2 = \iiint_c (\nabla\cdot\nabla[N])_3$$

for (2.a.10),

2^{nd} ORDER STOKES' THEOREMS
FOR RAISED BASIC FORMS — TABLE 2.8

$$\oiint_{\partial c} [(\nabla\times \mathbf{L})\times \mathbf{n}]_2 = \iiint_c [(\nabla\times\nabla\times \mathbf{L})\cdot \mathbf{n}]_3$$

$$\oint_{\partial c} [(\nabla\cdot \mathbf{M})\mathbf{n}]_1 = \iint_c [(\nabla\nabla\cdot \mathbf{M})\times \mathbf{n}]_2$$

$$\oiint_{\partial c} [(\nabla\cdot \mathbf{M})\mathbf{n}]_2 = \iiint_c [(\nabla\nabla\cdot \mathbf{M})\cdot \mathbf{n}]_3$$

$$\oint_{\partial c} [(\nabla\cdot \mathbf{L})\mathbf{n}]_1 = \iint_c [(\nabla\nabla\cdot \mathbf{L})\times \mathbf{n}]_2$$

$$\oiint_{\partial c} [(\nabla\cdot \mathbf{L})\mathbf{n}]_2 = \iiint_c [(\nabla\nabla\cdot \mathbf{L})\cdot \mathbf{n}]_3$$

$$\oiint_{\partial c} [(\nabla\times \mathbf{M})\times \mathbf{n}]_2 = \iiint_c [(\nabla\times\nabla\times \mathbf{M})\cdot \mathbf{n}]_3$$

and for (2.a.11):

2nd ORDER STOKES' THEOREMS FOR DUALS OF RAISED BASIC FORMS TABLE 2.9

$$\oint_{\partial c} [(\nabla\times \mathbf{L})\times \mathbf{n}]_1 = \iint_c [(\mathbf{n}\cdot\nabla)(\nabla\times \mathbf{L})]_2$$

$$\oiint_{\partial c} [(\nabla\cdot \mathbf{M})\mathbf{n}]_2 = \iiint_c [(\mathbf{n}\cdot\nabla)(\nabla\cdot \mathbf{M})]_3$$

$$\oint_{\partial c} [(\nabla\cdot \mathbf{M})\mathbf{n}]_1 = -\iint_c [(\mathbf{n}\times\nabla)(\nabla\cdot \mathbf{M})]_2$$

$$\oiint_{\partial c} [(\nabla\cdot \mathbf{L})\mathbf{n}]_2 = \iiint_c [(\mathbf{n}\cdot\nabla)(\nabla\cdot \mathbf{L})]_3$$

$$\oint_{\partial c} [(\nabla\cdot \mathbf{L})\mathbf{n}]_1 = -\iint_c [(\mathbf{n}\times\nabla)(\nabla\cdot \mathbf{L})]_2$$

$$\oint_{\partial c} [(\nabla\times \mathbf{M})\times \mathbf{n}]_1 = \iint_c [(\mathbf{n}\cdot\nabla)(\nabla\times \mathbf{M})]_2$$

These formulas can also be derived from the earlier cases. The subsitution $\kappa \to *^{-1}(d*\lambda)$: $K \to (\nabla\cdot\mathbf{L})$ into line 1, Table 2.5 yields line 4, Table 2.9.

v. Stokes' theorems involving products of forms

Stokes' theorems involving exterior products may be developed by considering the following subsitutions in Tables 2.1 through 2.9:

$$\begin{array}{lll}
\kappa \to \kappa_a \wedge \kappa_b \;;\; *\nu_a \wedge *\nu_b & K \to K_a K_b \;;\; [N_a][N_b] & \\
\lambda \to \kappa \wedge \lambda \;;\; *\nu \wedge *\mu & \mathbf{L} \to K\mathbf{L} \;;\; [N]\mathbf{M} & \\
\mu \to \kappa \wedge \mu \;;\; *\nu \wedge *\lambda & \mathbf{M} \to K\mathbf{M} \;;\; [N]\mathbf{L} & \\
\nu \to \kappa \wedge \nu \;;\; *\nu \wedge *\kappa & [N] \to K[N] \;;\; [N]K & \quad 2.a.12 \\
\mu \to \lambda_a \wedge \lambda_b ;\; *\mu_a \wedge *\mu_b & \mathbf{M} \to \mathbf{L}_a\times\mathbf{L}_b ;\; \mathbf{M}_a\times\mathbf{M}_b & \\
\nu \to \lambda \wedge \mu \;;\; *\mu \wedge *\lambda & [N] \to \mathbf{L}\cdot\mathbf{M} \;;\; \mathbf{M}\cdot\mathbf{L} &
\end{array}$$

For example, subsitution of line 5, (2.a.12) into line 6, Table 2.8 yields

$$\oint\!\!\oint_{\partial c} [(\nabla\times(L_a\times L_b))\times n] = \iiint_c [\nabla\times\nabla\times(L_a\times L_b)]\cdot n \qquad 2.a.13$$

It is clear that numerous other product relations are possible.

Our development of Stokes' theorems has been a rather complicated one, pursuing those possibilities available to us from exterior differential algebra. However, the pattern of this section's development (Figure 2.1), as expressed in the algebra, indicates the underlying simplicity of what we have done. We shall now leave single integrand formulas, and proceed to develop formulas symmetric and antisymmetric in two variables.

B. Green's Theorems

1. Basic varieties of Green's theorem

Green's theorems are derived from an expansion of the Laplacian structure

$$\alpha*\Delta\beta - \beta*\Delta\alpha \qquad 2.b.1$$

Note that when α and β are not of the same order, one or the other of the terms in (2.b.1) is zero by (1.b.1b), so that further development is of no particular significance. In other words, Green's theorems involve complementary products of equal-order forms and their derivatives. If α and β are both p-forms, then (FLANDERS, §2.7)

$$\alpha\wedge*\beta = \beta\wedge*\alpha \qquad 2.b.2$$

(This is the basis of the bilinearity of the inner product (1.k.1).

Consequently, if we desire, we can re-write (2.b.1) with these changes:

$$\alpha^*\Delta\beta \rightarrow \Delta\beta^*\alpha$$
$$\beta^*\Delta\alpha \rightarrow \Delta\alpha^*\beta \qquad 2.b.3$$

Another result of (2.b.2) is

$$d\alpha^*d\beta = d\beta^*d\alpha$$
$$\delta\alpha^*\delta\beta = \delta\beta^*\delta\alpha \qquad 2.b.4$$

provided α and β have the same order.

Expanding (2.b.1) for the basic R^3 forms κ, λ, μ, ν, we have:

$$\begin{aligned}
\kappa_a{}^*\Delta\kappa_b - \kappa_b{}^*\Delta\kappa_a &= -\kappa_a(d^*d\kappa_b) + \kappa_b(d^*d\kappa_a) \\
\lambda_a{}^*\Delta\lambda_b - \lambda_b{}^*\Delta\lambda_a &= \lambda_a(d^*d\lambda_b) - \lambda_b(d^*d\lambda_a) - \lambda_a({}^*d^{*-1}d^*\lambda_b) + \lambda_b({}^*d^{*-1}d^*\lambda_a) \\
\mu_a{}^*\Delta\mu_b - \mu_b{}^*\Delta\mu_a &= -\mu_a(d^*d\mu_b) + \mu_b(d^*d\mu_a) + \mu_a({}^*d^{*-1}d^*\mu_b) - \mu_b({}^*d^{*-1}d^*\mu_a) \\
\nu_a{}^*\Delta\nu_b - \nu_b{}^*\Delta\nu_a &= \qquad - \nu_a({}^*d^{*-1}d^*\nu_b) + \nu_b({}^*d^{*-1}d^*\nu_a)
\end{aligned} \qquad 2.b.5$$

Using (2.b.4), (2.b.5) can be reduced to either

$$\begin{aligned}
d(\kappa_a{}^*d\kappa_b - \kappa_b{}^*d\kappa_a) &= \kappa_b{}^*\Delta\kappa_a - \kappa_a{}^*\Delta\kappa_b \\
d(\lambda_a{}^*d\lambda_b - \lambda_b{}^*d\lambda_a) &= \lambda_b{}^*\Delta\lambda_a - \lambda_a{}^*\Delta\lambda_b + \lambda_b({}^*d^{*-1}d^*\lambda_a) - \lambda_a({}^*d^{*-1}d^*\lambda_b) \\
d(\mu_a{}^*d\mu_b - \mu_b{}^*d\mu_a) &= \mu_b{}^*\Delta\mu_a - \mu_a{}^*\Delta\mu_b - \mu_b({}^*d^{*-1}d^*\mu_a) + \mu_a({}^*d^{*-1}d^*\mu_b) \\
0 &= \nu_b{}^*\Delta\nu_a - \nu_a{}^*\Delta\nu_b + \nu_b({}^*d^{*-1}d^*\nu_a) - \nu_a({}^*d^{*-1}d^*\nu_b)
\end{aligned} \qquad 2.b.6a$$

or

$$\begin{aligned}
d(\kappa_a{}^*d\kappa_b - \kappa_b{}^*d\kappa_a) &= \kappa_b{}^*\Delta\kappa_a - \kappa_a{}^*\Delta\kappa_b \\
d(\lambda_a{}^*d\lambda_b - \lambda_b{}^*d\lambda_a) + d(\delta\lambda_a \wedge {}^*\lambda_b - \delta\lambda_b \wedge {}^*\lambda_a) &= \lambda_b{}^*\Delta\lambda_a - \lambda_a{}^*\Delta\lambda_b \\
d(\mu_a{}^*d\mu_b - \mu_b{}^*d\mu_a) + d(\delta\mu_a \wedge {}^*\mu_b - \delta\mu_b \wedge {}^*\mu_a) &= \mu_b{}^*\Delta\mu_a - \mu_a{}^*\Delta\mu_b \\
d(\delta\nu_a \wedge {}^*\nu_b - \delta\nu_b \wedge {}^*\nu_a) &= \nu_b{}^*\Delta\nu_a - \nu_a{}^*\Delta\nu_b
\end{aligned} \qquad 2.b.6b$$

providing us with two possible conversions into vector integration formulas. Let us look at each in turn.

The fourth relation in (2.b.6a) can be manipulated into a derivative formula with

$$\begin{aligned} {}^*\nu &= [N]_0 \\ {}^*\nu \wedge u_3 &= \nu \\ \nu &= [N]_0 \wedge u_3 \end{aligned} \qquad 2.b.7$$

Then

$$-\nu_a({}^*d{}^{*-1}d{}^*\nu_b) + \nu_b({}^*d{}^{*-1}d{}^*\nu_a) = [N_a]_0(d{}^*d[N_b]_0) + [N_b]_0(d{}^*d[N_a]_0) \qquad 2.b.8$$

and

$$d([N_a]_0{}^*d[N_b]_0 - [N_b]_0{}^*d[N_a]_0) = \nu_b{}^*\Delta\nu_a - \nu_a{}^*\Delta\nu_b \qquad 2.b.9$$

demonstrating the equivalence between the derivative formulas (and consequently the Green's theorems) for 0-forms and 3-forms in R^3. No such equivalence exists for the 1-form and 2-form relations in (2.b.6a) because the matrix describing the transformation

$$(\mu)_2 = (\lambda)_1 \wedge u_1 \qquad 2.b.10$$

is singular, preventing inversion. The number of distinct Green's theorems that follow from (2.b.6a) is therefore 3, based on these vector formulas:

$$\begin{aligned} \nabla\cdot(K_a\nabla K_b - K_b\nabla K_a)_2 &= (K_a\nabla^2K_b - K_b\nabla^2K_a)_3 \\ \nabla\cdot(\mathbf{L}_a\times(\nabla\times\mathbf{L}_b) - \mathbf{L}_b\times(\nabla\times\mathbf{L}_a))_2 &= (\mathbf{L}_b\cdot(\nabla\times\nabla\times\mathbf{L}_a) - \mathbf{L}_a\cdot(\nabla\times\nabla\times\mathbf{L}_b))_3 \\ \nabla\cdot(\mathbf{M}_a(\nabla\cdot\mathbf{M}_b) - \mathbf{M}_b(\nabla\cdot\mathbf{M}_a))_2 &= (\mathbf{M}_a\cdot(\nabla\nabla\cdot\mathbf{M}_b) - \mathbf{M}_b\cdot(\nabla\nabla\cdot\mathbf{M}_a))_3 \\ \nabla\cdot([N_a]\nabla[N_b] - [N_b]\nabla[N_a])_2 &= ([N_a]\nabla^2[N_b] - [N_b]\nabla^2[N_a])_3 \,. \end{aligned} \qquad 2.b.11a$$

Since these equations are all of the type d(2-form) = 3-form, we may apply Stokes' theorem immediately, finding:

R^3 GREEN'S THEOREMS — TABLE 2.10a

$$\oint\!\!\oint_{\partial c} (K_a \nabla K_b - K_b \nabla K_a) = \iiint_c (K_a \nabla^2 K_b - K_b \nabla^2 K_a)$$

$$\oint\!\!\oint_{\partial c} (\mathbf{L}_a \times (\nabla \times \mathbf{L}_b) - \mathbf{L}_b \times (\nabla \times \mathbf{L}_a)) = \iiint_c (\mathbf{L}_b \cdot (\nabla \times \nabla \times \mathbf{L}_a) - \mathbf{L}_a \cdot (\nabla \times \nabla \times \mathbf{L}_b))$$

$$\oint\!\!\oint_{\partial c} (\mathbf{M}_a (\nabla \cdot \mathbf{M}_b) - \mathbf{M}_b (\nabla \cdot \mathbf{M}_a)) = \iiint_c (\mathbf{M}_a \cdot (\nabla\nabla \cdot \mathbf{M}_b) - \mathbf{M}_b \cdot (\nabla\nabla \cdot \mathbf{M}_a))$$

$$\oint\!\!\oint_{\partial c} ([N_a] \nabla [N_b] - [N_b] \nabla [N_a]) = \iiint_c ([N_a] \nabla^2 [N_b] - [N_b] \nabla^2 [N_a])$$

The vector formulas resulting from (2.b.6b) involve the entire Laplacian. In this circumstance the derivative formulas for the R^3 1-forms and 2-forms are equivalent:

$$\nabla \cdot (K_a \nabla K_b - K_b \nabla K_a)_2 = (K_a \nabla^2 K_b - K_b \nabla^2 K_a)_3$$

$$\nabla \cdot (\mathbf{L}_a \times (\nabla \times \mathbf{L}_b) - \mathbf{L}_b \times (\nabla \times \mathbf{L}_a) + \mathbf{L}_a (\nabla \cdot \mathbf{L}_b) - \mathbf{L}_b (\nabla \cdot \mathbf{L}_a))_2 = (\mathbf{L}_a \cdot \nabla^2 \mathbf{L}_b - \mathbf{L}_b \cdot \nabla^2 \mathbf{L}_a)_3$$

$$\nabla \cdot (\mathbf{M}_a (\nabla \cdot \mathbf{M}_b) - \mathbf{M}_b (\nabla \cdot \mathbf{M}_a) + \mathbf{M}_a \times (\nabla \times \mathbf{M}_b) - \mathbf{M}_b \times (\nabla \times \mathbf{M}_a))_2 = (\mathbf{M}_a \cdot \nabla^2 \mathbf{M}_b - \mathbf{M}_b \cdot \nabla^2 \mathbf{M}_a)_3$$

$$\nabla \cdot ([N_a] \nabla [N_b] - [N_b] \nabla [N_a])_2 = ([N_a] \nabla^2 [N_b] - [N_b] \nabla^2 [N_a])_3$$

2.b.11b

Applying Stokes' theorem, we have this alternative set of integration formulas:

R[3] GREEN'S THEOREMS — TABLE 2.10b

$$\oiint_{\partial c} (K_a \nabla K_b - K_b \nabla K_a) = \iiint_c (K_a \nabla^2 K_b - K_b \nabla^2 K_a)$$

$$\oiint_{\partial c} (\mathbf{L}_a \times (\nabla \times \mathbf{L}_b) - \mathbf{L}_b \times (\nabla \times \mathbf{L}_a) + \mathbf{L}_a(\nabla \cdot \mathbf{L}_b) - \mathbf{L}_b(\nabla \cdot \mathbf{L}_a))$$

$$= \iiint_c (\mathbf{L}_a \cdot \nabla^2 \mathbf{L}_b - \mathbf{L}_b \cdot \nabla^2 \mathbf{L}_a)$$

$$\oiint_{\partial c} (\mathbf{M}_a(\nabla \cdot \mathbf{M}_b) - \mathbf{M}_b(\nabla \cdot \mathbf{M}_a) + \mathbf{M}_a \times (\nabla \times \mathbf{M}_b) - \mathbf{M}_b \times (\nabla \times \mathbf{M}_a))$$

$$= \iiint_c (\mathbf{M}_a \cdot \nabla^2 \mathbf{M}_b - \mathbf{M}_b \cdot \nabla^2 \mathbf{M}_a)$$

$$\oiint_{\partial c} ([N_a] \nabla [N_b] - [N_b] \nabla [N_a]) = \iiint_c ([N_a] \nabla^2 [N_b] - [N_b] \nabla^2 [N_a])$$

The integration formulas in Tables 2.10 are the complete set of Green's theorems for R^3 vector quantities. The first and fourth formulas are the familiar scalar Green's theorem. The second formula in Table 2.10a is the so-called "vector analog" to Green's theorem (STRATTON, §4.14). As we can see, the third formula in Table 2.10a is also an antisymmetric integration formula for vector quantities. The two "partial" vector integration formulas combine to give what is in fact a complete vector Green's theorem, equivalent for R^3 polar vectors and axial vectors (1-forms and 2-forms). This is the second or third formula in Table 2.10b.

ii. Green's theorems involving unit forms

Further development of the Green's theorems is possible. By using the unit forms (2.a.2), any given pair of quantities may be raised to an order suitable for the application of one or more of the above Green's theorems. The left-hand diagram in Figure 2.2 shows the possibilities for raising equal-order pairs of forms; the right-hand diagram, mixed-order pairs. This leads to 16 variations of the basic

Example: A 1-form and 2-form may be placed in the 2-form Green's theorem by first multiplying the 1-form by u_1.

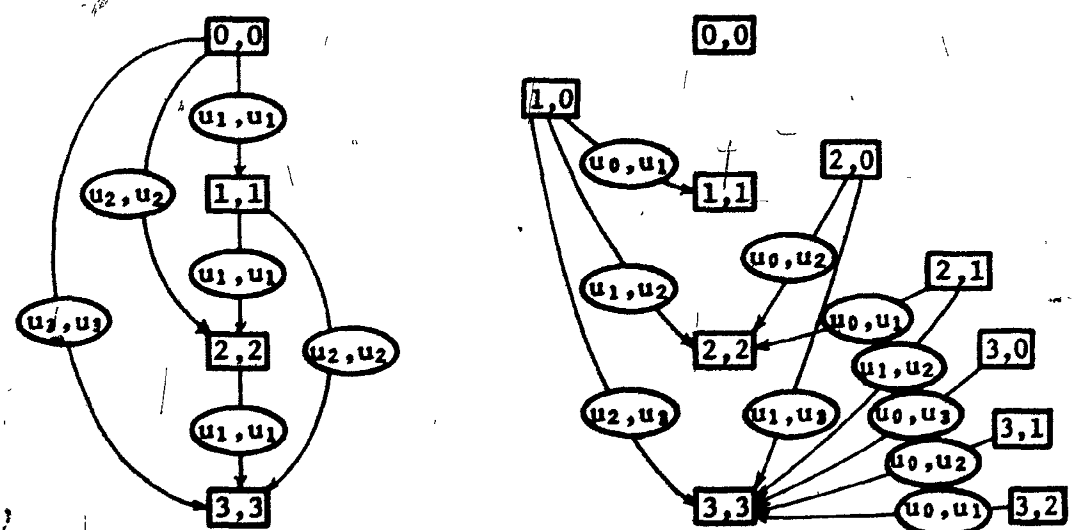

Figure 2.2 : Pattern for the development of raised-order Green's theorems.

Green's theorems, more than is useful to work out in detail. However, one example can illustrate the technique: let α_1 and β_1 be 1-forms such that $\alpha_1 = \alpha_0 \wedge u_1$ and $\beta_1 = \beta_0 \wedge u_1$. Then the 1-form Green's theorem in Table 2.10a is written

$$\nabla\cdot[\alpha n\times(\nabla\beta\times n) - \beta n\times(\nabla\alpha\times n)] = -\alpha n\cdot[-(\nabla\cdot\nabla\beta)n + (n\cdot\nabla)\nabla\beta] + \beta n\cdot[-(\nabla\cdot\nabla\alpha)n + (n\cdot\nabla)\nabla\alpha] \qquad 2.b.13$$

and this has separable left and right sides:

$$\begin{aligned}\nabla\cdot[-\alpha(n\cdot\nabla\beta)n + \beta(n\cdot\nabla\alpha)n] &= -\alpha n\cdot[(n\cdot\nabla)\nabla\beta] + \beta n\cdot[(n\cdot\nabla)\nabla\alpha]\\ \nabla\cdot[\ \alpha(n\cdot n)\nabla\beta - \beta(n\cdot n)\nabla\alpha] &= \alpha n\cdot[(\nabla\cdot\nabla\beta)n] - \beta n\cdot[(\nabla\cdot\nabla\alpha)n]\end{aligned} \qquad 2.b.14$$

Taking $(n\cdot n) = 3$, the lower formula in (2.b.14) is equivalent to the scalar Green's theorem multiplied by the constant 3.

iii. Symmetric integration formulas analogous to Green's theorem

The final development regarding Green's theorems concerns the possibility of utilizing the equations derivable from the Laplacian structure

$$\alpha * \Delta\beta + \beta * \Delta\alpha \qquad 2.b.14$$

Then, proceeding from (2.b.6a) but with the appropriate change of sign, we have these vector integration formulas:

R^3 SYMMETRIC INTEGRATION THEOREMS TABLE 2.11

$$\oint\!\!\oint_{\partial c} (K_a \nabla K_b + K_b \nabla K_a) = \iiint_c (K_a \nabla^2 K_b + K_b \nabla^2 K_a) + \iiint_c 2(\nabla K_a \cdot \nabla K_b)$$

$$\oint\!\!\oint_{\partial c} (L_a \times (\nabla \times L_b) + L_b \times (\nabla \times L_a)) = \iiint_c (-L_a \cdot (\nabla \times \nabla \times L_b) - L_b \cdot (\nabla \times \nabla \times L_a)) + \iiint_c 2(\nabla \times L_a) \cdot (\nabla \times L_b)$$

$$\oint\!\!\oint_{\partial c} (M_a (\nabla \cdot M_b) + M_b (\nabla \cdot M_a)) = \iiint_c (M_a \cdot (\nabla\nabla \cdot M_b) + M_b \cdot (\nabla\nabla \cdot M_a)) + \iiint_c 2(\nabla \cdot M_a)(\nabla \cdot M_b)$$

$$\oint\!\!\oint_{\partial c} ([N_a] \nabla [N_b] + [N_b] \nabla [N_a]) = \iiint_c ([N_a] \nabla^2 [N_b] + [N_b] \nabla^2 [N_a]) + \iiint_c 2(\nabla [N_a] \cdot \nabla [N_b])$$

Proceeding from (2.b.6b), we would find that formulas 2 and 3 in Table 2.11 combine. Formulas 1 and 4 remain unchanged.

C. Integration Formulas Involving Laplacians and Product Laplacians

The relationship (2.b.2) is remarkably useful, and can be used to develop integration formulas involving the Laplacian operator. With (2.b.2), the 4 basic R^3 forms κ, λ, μ, ν satisfy

$$\begin{aligned} \kappa \wedge u_3 &= (*\kappa) \wedge u_0 \\ \lambda \wedge u_2 &= (*\lambda) \wedge u_1 \\ \mu \wedge u_1 &= (*\mu) \wedge u_2 \\ \nu \wedge u_0 &= (*\nu) \wedge u_3 \end{aligned} \qquad 2.c.1$$

Applying the Laplace-Beltrami operator to κ, λ, μ, ν, we find that when all quantities are raised to the order 3,

$$\begin{aligned} \Delta(\kappa) \wedge u_3 &= d(-*d\kappa) \\ \Delta(\lambda) \wedge u_2 &= d(-*^{-1}d*\lambda \wedge u_2) + d(\ *d\lambda \wedge u_1) \\ \Delta(\mu) \wedge u_1 &= d(\ *^{-1}d*\mu \wedge u_1) + d(-*d\mu \wedge u_2) \\ \Delta(\nu) \quad &= d(-*^{-1}d*\nu) \end{aligned} \qquad 2.c.2$$

In vector notation, we have the following Stokes' theorems:

R^3 STOKES' THEOREMS
INVOLVING LAPLACIANS TABLE 2.12

$\oiint_{\partial c} (\nabla K)$	=	$\iiint_c (\nabla^2 K)$
$\oiint_{\partial c} ((\nabla\cdot \mathbf{L})\mathbf{n} - (\nabla\times\mathbf{L})\times\mathbf{n})$	=	$\iiint_c (\nabla^2 \mathbf{L})\cdot\mathbf{n}$
$\oiint_{\partial c} (-(\nabla\times\mathbf{M})\times\mathbf{n} + (\nabla\cdot\mathbf{M})\mathbf{n})$	=	$\iiint_c (\nabla^2 \mathbf{M})\cdot\mathbf{n}$
$\oiint_{\partial c} (\nabla[N])$	=	$\iiint_c (\nabla^2 [N])$

These integration formulas can be used to develop another group involving pairs of forms. In §1.G, we found that the Laplacian of a product has this structure:

$$\Delta(\alpha \wedge \beta) = (\Delta\alpha \wedge \beta) + (\alpha \wedge \Delta\beta) - 2\{--\} \qquad 2.c.3$$

For specific R^3 products, the brackets are listed in Table 1.11. (2.c.3) may be raised to an n-form by the use of unit forms. If $(\alpha \wedge \beta)$ is a p-form,

$$\Delta(\alpha \wedge \beta)_p \wedge u_{n-p} = (\Delta\alpha \wedge \beta) \wedge u_{n-p} + (\alpha \wedge \Delta\beta) \wedge u_{n-p} - 2\{--\} \wedge u_{n-p} \qquad 2.c.4$$

is an n-form which may be converted into a derivative expression by (2.c.2). We have the following vector integration formulas:

R^3 INTEGRATION THEOREMS INVOLVING LAPLACIANS OF PRODUCTS TABLE 2.13

$$\oiint_{\partial c} [\nabla(K_a K_b)] = \iiint_c [(\nabla^2 K_a)K_b + K_a(\nabla^2 K_b) + 2(\nabla K_a \cdot \nabla K_b)]$$

$$\oiint_{\partial c} [(\nabla\cdot(K\mathbf{L}))\mathbf{n} - (\nabla\times(K\mathbf{L}))\times\mathbf{n}]$$

$$= \iiint_c [(\nabla^2 K)\mathbf{L}\cdot\mathbf{n} + K(\nabla^2\mathbf{L})\cdot\mathbf{n} + 2\,[(\nabla K\cdot\nabla)\mathbf{L}]\cdot\mathbf{n}]$$

$$\oiint_{\partial c} [(\nabla\cdot(K\mathbf{M}))\mathbf{n} - (\nabla\times(K\mathbf{M}))\times\mathbf{n}]$$

$$= \iiint_c [(\nabla^2 K)\mathbf{M}\cdot\mathbf{n} + K(\nabla^2\mathbf{M})\cdot\mathbf{n} + 2\,[(\nabla K\cdot\nabla)\mathbf{M}]\cdot\mathbf{n}]$$

$$\oiint_{\partial c} [\nabla(K[N])] = \iiint_c [(\nabla^2 K)[N] + K(\nabla^2[N]) + 2(\nabla K\cdot\nabla[N])]$$

$$\oiint_{\partial c} [(\nabla\cdot(\mathbf{L}_a\times\mathbf{L}_b))\mathbf{n} - (\nabla\times(\mathbf{L}_a\times\mathbf{L}_b))\times\mathbf{n}]$$

$$= \iiint_c [((\nabla^2\mathbf{L}_a)\times\mathbf{L}_b)\cdot\mathbf{n} + (\mathbf{L}_a\times(\nabla^2\mathbf{L}_b))\cdot\mathbf{n} + 2(\{\mathbf{A}\}\cdot\mathbf{n})]$$

$$\oiint_{\partial c} [\nabla(\mathbf{L}\cdot\mathbf{M})] = \iiint_c [(\nabla^2\mathbf{L})\cdot\mathbf{M} + \mathbf{L}\cdot(\nabla^2\mathbf{M}) + 2\{B\}]$$

We can derive a dual set of equations as well.

D. Integration Formulas Involving Sums and Differences of Laplacians

Since the two forms $(\alpha \wedge \Delta\beta)$ and $(\Delta\alpha \wedge \beta)$ are of the same order, it may be possible to develop further integration formulas from them. In particular, we will look at the following two structures:

$$\alpha \wedge \Delta\beta - \Delta\alpha \wedge \beta \qquad 2.d.1a$$

$$\alpha \wedge \Delta\beta + \Delta\alpha \wedge \beta \qquad 2.d.1b$$

For κ, λ, μ, ν, (2.d.1a) involves investigating

$$\begin{aligned}
&1.\ \kappa_a \wedge \Delta\kappa_b - \kappa_b \wedge \Delta\kappa_a \\
&2.\ \kappa \wedge \Delta\lambda - \lambda \wedge \Delta\kappa \\
&3.\ \kappa \wedge \Delta\mu - \mu \wedge \Delta\kappa \\
&4.\ \kappa \wedge \Delta\nu - \nu \wedge \Delta\kappa \\
&5.\ \lambda_a \wedge \Delta\lambda_b + \lambda_b \wedge \Delta\lambda_a \\
&6.\ \lambda \wedge \Delta\mu - \mu \wedge \Delta\lambda
\end{aligned} \qquad 2.d.2$$

Lines 4 and 6 of (2.d.2) can be directly related to the Green's theorem structure (2.b.1) by re-defining one of the variables as a dual. For example,

$$\lambda_a \wedge \Delta{*\lambda_b} - {*\lambda_b} \wedge \Delta\lambda_a = \lambda_a \wedge {*\Delta\lambda_b} - \lambda_b \wedge {*\Delta\lambda_a} \qquad 2.d.3$$

By expanding the Laplace-Beltrami operator in (2.d.2) and re-grouping the separate parts in terms of $d(\alpha \wedge \delta\beta)$, $\delta(\alpha \wedge d\beta)$, $d(\beta \wedge \delta\alpha)$ and $\delta(\beta \wedge d\alpha)$, we can construct the following integration formulas:

R^3 LAPLACIAN DIFFERENCE INTEGRATION FORMULAS TABLE 2.14

1. $$\oiint_{\partial c} (K_a \nabla K_b - K_b \nabla K_a) = \iiint_c (K_a \nabla^2 K_b - K_b \nabla^2 K_a)$$

2. $$\sum_{\partial c}^{+} K(\nabla \cdot \mathbf{L}) = \int_c (K(\nabla^2 \mathbf{L}) - (\nabla^2 K)\mathbf{L})$$
$$+ \int_c [2(\nabla K)(\nabla \cdot \mathbf{L}) + \underline{\nabla \times (K(\nabla \times \mathbf{L}) + \mathbf{L} \times (\nabla K))} + [(\nabla \times \mathbf{L}) \times \nabla]K + (\mathbf{L} \cdot \nabla)\nabla K - (\nabla K \cdot \nabla)\mathbf{L}]$$

3. $$\oint_{\partial c} K(\nabla \times \mathbf{M}) = \iint_c ((\nabla^2 K)\mathbf{M} - K(\nabla^2 \mathbf{M}))$$
$$+ \iint_c [2(\nabla K) \times (\nabla \times \mathbf{M}) + \underline{\nabla(K(\nabla \cdot \mathbf{M}) - \mathbf{M} \cdot (\nabla K))} - [(\nabla \cdot \mathbf{M})\nabla]K + (\mathbf{M} \times \nabla) \times (\nabla K) + (\nabla K \cdot \nabla)\mathbf{M}]$$

4. $$\oiint_{\partial c} (K\nabla [N] - [N](\nabla K)) = \iiint_c (K(\nabla^2 [N]) - (\nabla^2 K)[N])$$

5. $$\oint_{\partial c} (\mathbf{L}_a(\nabla \cdot \mathbf{L}_b) + \mathbf{L}_b(\nabla \cdot \mathbf{L}_a)) = \iint_c ((\nabla^2 \mathbf{L}_a) \times \mathbf{L}_b - \mathbf{L}_a \times (\nabla^2 \mathbf{L}_b))$$
$$+ \iint_c [2(\nabla \times \mathbf{L}_a)(\nabla \cdot \mathbf{L}_b) + 2(\nabla \times \mathbf{L}_b)(\nabla \cdot \mathbf{L}_a) - \underline{\nabla(\mathbf{L}_a \cdot (\nabla \times \mathbf{L}_b) + \mathbf{L}_b \cdot (\nabla \times \mathbf{L}_a))} + (\mathbf{L}_a \cdot \nabla)(\nabla \times \mathbf{L}_b) + (\mathbf{L}_b \cdot \nabla)(\nabla \times \mathbf{L}_a) + [(\nabla \times \mathbf{L}_b) \times \nabla] \times \mathbf{L}_a + [(\nabla \times \mathbf{L}_a) \times \nabla] \times \mathbf{L}_b]$$

6. $$\oiint_{\partial c} (\mathbf{L} \times (\nabla \times \mathbf{M}) + \mathbf{L}(\nabla \cdot \mathbf{M}) - \mathbf{M}(\nabla \cdot \mathbf{L}) - \mathbf{M} \times (\nabla \times \mathbf{L}))$$
$$= \iiint_c (\mathbf{L} \cdot \nabla^2 \mathbf{M} - \mathbf{M} \cdot \nabla^2 \mathbf{L})$$

Formulas 1 and 4 are equivalent to the scalar Green's theorem; 6 is the complete vector Green's theorem. Formulas 2, 3 and 5 are rather complicated relations involving an analog to the Green's theorems when the order of the two variable quantities is not identical or dual-identical.

Note that for formulas 2, 3 and 5, a derivative relation can be found for the dual. This would result in a similarly complicated integration formula involving the underlined quantity in Table 2.14 as the left-hand boundary integral.

For κ, λ, μ, ν, (2.d.1b) includes

$$
\begin{aligned}
&1.\ \kappa_a \wedge \Delta\kappa_b + \kappa_b \wedge \Delta\kappa_a \\
&2.\ \kappa \wedge \Delta\lambda + \lambda \wedge \Delta\kappa \\
&3.\ \kappa \wedge \Delta\mu + \mu \wedge \Delta\kappa \\
&4.\ \kappa \wedge \Delta\nu + \nu \wedge \Delta\kappa \\
&5.\ \lambda_a \wedge \Delta\lambda_b - \lambda_b \wedge \Delta\lambda_a \\
&6.\ \lambda \wedge \Delta\mu + \mu \wedge \Delta\lambda
\end{aligned}
\tag{2.d.4}
$$

By expanding (2.d.4), we can derive integration formulas as we did for Table 2.14. However, it is more productive to subsitute the expanded relations into the product Laplacian formulas of Table 1.11. We get the following:

INTEGRATION FORMULAS BASED ON R^3 LAPLACIAN SUMS — TABLE 2.15

1a.	$\oiint_{\partial c} (K_a \nabla K_b + K_b \nabla K_a)$	$= \iiint_c \nabla^2 (K_a K_b)$
1b.		$= \iiint_c (\nabla^2 K_a) K_b + K_a (\nabla^2 K_b) + \iiint_c 2(\nabla K_a)\cdot(\nabla K_b)$
2.	$\sum^{+}_{\partial \bar{c}} (\mathbf{L}\cdot\nabla) K$	$= \int_c [(\nabla K\cdot\nabla)\mathbf{L} + (\mathbf{L}\cdot\nabla)\nabla K - [(\nabla\times\mathbf{L})\times\nabla]K]$
3.	$\oint_{\partial c} (\mathbf{M}\times\nabla) K$	$= \iint_c [(\nabla K\cdot\nabla)\mathbf{M} - (\mathbf{M}\times\nabla)\times\nabla K - [(\nabla\cdot\mathbf{M})\nabla]K]$
4.	$\oiint_{\partial c} (K\nabla[N] + [N]\nabla K)$	$= \iiint_c \nabla^2 (K[N])$
5.	$\oint_{\partial c} ((\mathbf{L}_b\cdot\nabla)\mathbf{L}_a - (\mathbf{L}_a\cdot\nabla)\mathbf{L}_b)$	$= \iint_c [(\mathbf{L}_b\cdot\nabla)(\nabla\times\mathbf{L}_a) - (\mathbf{L}_a\cdot\nabla)(\nabla\times\mathbf{L}_b) + [(\nabla\times\mathbf{L}_a)\times\nabla]\times\mathbf{L}_b - [(\nabla\times\mathbf{L}_b)\times\nabla]\times\mathbf{L}_a - 2\{\mathbf{A}\}]$

TABLE 2.15 (continued)

6a. $$\oiint_{\partial c}(\mathbf{L}(\nabla\cdot\mathbf{M}) + \mathbf{M}(\nabla\cdot\mathbf{L}) - (\mathbf{L}\cdot\nabla)\mathbf{M} - (\mathbf{M}\cdot\nabla)\mathbf{L}) = \iiint_c [2(\nabla\times\mathbf{L})\cdot(\nabla\times\mathbf{M}) + 2(\nabla\cdot\mathbf{L})(\nabla\cdot\mathbf{M}) - 2\{B\}]$$

6b. $$\oiint_{\partial c}((\nabla\cdot\mathbf{L})\mathbf{M} + (\nabla\cdot\mathbf{M})\mathbf{L} + \mathbf{L}\times(\nabla\times\mathbf{M}) + \mathbf{M}\times(\nabla\times\mathbf{L})) = \iiint_c ((\nabla^2\mathbf{L})\cdot\mathbf{M} + \mathbf{L}\cdot(\nabla^2\mathbf{M})) + \iiint_c 2[(\nabla\cdot\mathbf{L})(\nabla\cdot\mathbf{M}) + (\nabla\times\mathbf{L})\cdot(\nabla\times\mathbf{M})]$$

6c. $$= \iiint_c \nabla^2(\mathbf{L}\cdot\mathbf{M}) + \iiint_c [2(\nabla\cdot\mathbf{L})(\nabla\cdot\mathbf{M}) + 2(\nabla\times\mathbf{L})\cdot(\nabla\times\mathbf{M}) - 2\{B\}]$$

Aside from those formulas which are direct integrations of vector identities which we proved earlier, we have merely derived special combinations of the formulas in Tables 2.11 and 2.13. It appears that this is the limit of possible useful developments, and so we conclude this chapter.

Summary

With this extended development of R^3 integration formulas, we complete the introductory chapters. The most notable derivation is in §2.B.1, where we discuss the scalar and vector Green's theorems. We will now leave developments in R^3 in order to work in R^3/t, a space suitable for the definition of electromagnetic differential forms.

CHAPTER III

THE APPLICATION OF EXTERIOR DIFFERENTIAL ALGEBRA TO ELECTROMAGNETIC FIELD THEORY

The equations of electromagnetism, including Maxwell's equations, form an interrelated group of partial differential equations which is most simply expressed in 4-dimensional space-time R^3/t. The original derivation of the electromagnetic field equations in such a coordinate system is due to Minkowski (WEYL (1920), §21). Since that time, tensor methods have been the primary mathematical tool involved in R^3/t physics. However, they are difficult to apply. FLANDERS (§1.2) states that tensor calculus is not nearly as suitable as differential forms for many applications to physics, mainly because there is a better correspondence between differential forms and the precepts regarding descriptions of physical phenomena. The structure of the group of electromagnetic equations in exterior differential forms manifests a basic simplicity while preserving all the information necessary for physical interpretation and the specific solution of properly formulated problems. We shall demonstrate this in this chapter.

The application of differential forms to electromagnetism proceeds by first considering the physical dimensions (units) of R^3/t forms representing electromagnetic quantities. The R^3/t * operation is calculated; we find it convenient to include in it the permittivity ε and the permeability μ. Thus the duality under * is no longer purely algebraic. We present a structural diagram for the R^3/t electromagnetic differential forms and we demonstrate the completeness of the relationships included in this structure. This is a primary contribution of the thesis.

It is our intention to show that the description of electromagnetism in differential forms, besides being mathematically elegant, has practical value as well. Obviously the connection between differential forms and geometry (as discussed in Chapter 1) will play an important role. We develop a procedure for converting R^3/t differential relations into R^1 (time) and R^3 (space) integration formulas. Unit forms are introduced for R^3/t. A procedure is developed for handling the defining integrals of distributions. Finally, we calculate Green's theorems for the electromagnetic differential forms.

A. The Four-Dimensional Metric Space R^3/t

For the development of a structural description of electromagnetism, we choose the coordinate system R^3/t, a space-time metric space with R^3 as the spatial component. Based on the space-time interval,

$$(dx^1)^2 + (dx^2)^2 + (dx^3)^2 - c^2(dt)^2 = \text{Differential Distance Measure} \qquad 3.a.1$$

The metric in R^3/t has both positive and negative coefficients. The inner products of the elementary basis elements are:

$$\begin{aligned} (dx^i, dx^j) &= \delta^{ij} \quad (i,j = 1,2,3) \\ (dx^i, cdt) &= 0 \quad (i = 1,2,3) \\ (cdt, cdt) &= -1 \end{aligned} \qquad 3.a.2$$

δ^{ij} is the Kronecker delta. Consequently, the signature of R^3/t is $+3 - 1 = 2$.

Let us take a basic set of differential forms for R^3/t and develop a notation which reflects the R^3 vector notation. We define κ, λ, μ, ν, ζ:

EXTERIOR DIFFERENTIAL FORMS IN R^3/t — TABLE 3.1a

$\kappa = K$
$\lambda = L_1dx^1 + L_2dx^2 + L_3dx^3 + L_4dx^4$
$\mu = MA_1dx^{23} + MA_2dx^{31} + MA_3dx^{12} + MB_1dx^{14} + MB_2dx^{24} + MB_3dx^{34}$
$\nu = N_1dx^{234} + N_2dx^{314} + N_3dx^{124} + N_4dx^{123}$
$\zeta = Zdx^{1234}$

Here $dx^4 = cdt$. Normally, we will use the differential term dx^4, separating out the time dependence only in special circumstances. The "exact" notation in Table 3.1a can be condensed by using vector notation wherever appropriate. Secondary subscripts† will now indicate the R^3 order of the coefficient quantity they accompany. A "1", "2" and "3" will indicate a polar space vector, axial space vector, and variant scalar respectively. Occasionally, we will find it convenient to indicate invariant scalars by the subscript "0". In the reduced notation, the equations in Table 3.1a become:

R^3/t DIFFERENTIAL FORMS:
REDUCED NOTATION TABLE 3.1b

Form	R^3 Vector Notation
κ =	K
λ =	$L_1 + L_4 dx^4$
μ =	$\mathbf{MA}_2 + \mathbf{MB}_1 dx^4$
ν =	$\mathbf{N}_2 dx^4 + (N_4)_3$
ζ =	$(Z)_3 dx^4$

Boldface type indicates R^3 vector quantities. Now both the space and time parts of the basic R^3/t forms are expressed in terms of the R^3 notation, but the exact form in Table 3.1a can easily be recovered from Table 3.1b. No information has been suppressed.

The real partial differential operator for R^3/t, analogous to "d" of §1.E, is written as

$$d = \frac{\partial}{\partial x^1}dx^1 + \frac{\partial}{\partial x^2}dx^2 + \frac{\partial}{\partial x^3}dx^3 + \frac{1}{c}\frac{\partial}{\partial t}dx^4 \qquad 3.a.3$$
$$= \nabla_1 + \frac{1}{c}\frac{\partial}{\partial t}dx^4$$

†: We will try to avoid confusion with the usual vector component identification by placing such order-indicating subscripts only on boldface or parenthetical quantities.

Then we can readily calculate the differentials of κ, λ, μ, ν, ζ:

DIFFERENTIALS OF R^3/t FORMS — TABLE 3.2

Operation	R^3 Vector Notation
$d\kappa$	$= (\nabla K)_1 + \frac{1}{c}\left(\frac{\partial K}{\partial t}\right)_0 dx^4$
$d\lambda$	$= (\nabla \times L)_2 - \frac{1}{c}\left(\frac{\partial L}{\partial t}\right)_1 dx^4 + (\nabla L_4)_1 dx^4$
$d\mu$	$= (\nabla \cdot MA)_3 + \frac{1}{c}\left(\frac{\partial MA}{\partial t}\right)_2 dx^4 + (\nabla \times MB)_2 dx^4$
$d\nu$	$= (\nabla \cdot N)_3 dx^4 - \frac{1}{c}\left(\frac{\partial N_4}{\partial t}\right)_3 dx^4$
$d\zeta$	$= 0$

R^3/t codifferentials, essential to the equations of electromagnetism, involve the R^3/t * operation, which will be explained in the next section. R^3/t product differentials and codifferentials will be developed in Chapter 4.

B. Units and Dimensional Analysis; * Operator for Electromagnetism

The fundamental difference between the development of exterior differential forms as an algebraic structure and the application of this algebraic structure to the physics of electromagnetism is in the assignment of physical dimensions to both the coefficients of the differential forms as well as the differential terms themselves. In other words, a properly expressed differential form will now contain systematic information on its geometric behavior (as a vector, variant scalar, etc.) and its physical behavior in both a local (coefficient) and a global (differential form; integrated differential form) sense. The physical character of the coefficient (e.g., a charge density) will be separable from the character of the differential form itself (the charge contained in a differential region), and therefore from that of the integrated differential form. The * operator also becomes dimensioned since it now transforms dimensioned differential terms. In addition, we find it convenient to require the * operator to assign the media characteristics ϵ and μ, which are themselves dimensioned quantities.

The central feature of this chapter, to be outlined in §3.C, is that the entire ensemble of electromagnetic partial differential equations follows from a simple R^3/t structure involving 3 related exterior differential forms and their duals. The fundamental forms are called α, β and γ; the duals, α', β' and γ':

FUNDAMENTAL ELECTROMAGNETIC FORMS IN R^3/t — TABLE 3.3

Fundamental Forms:		
α	=	Potential 1-Form
β	=	Field 2-Form
γ	=	Charge-Current 1-Form
Fundamental Dual Forms:		
α'	=	Potential 3-Form
β'	=	Field 2-Form
γ'	=	Charge-Current 3-Form

In the exact notation, using standard MKS terminology,

$$\begin{aligned}
\alpha &= A_1 dx^1 + A_2 dx^2 + A_3 dx^3 - \frac{1}{c}\Phi\, dx^4 \\
\beta &= \frac{1}{c}E_1 dx^{14} + \frac{1}{c}E_2 dx^{24} + \frac{1}{c}E_3 dx^{34} + B_1 dx^{23} + B_2 dx^{31} + B_3 dx^{12} \\
\gamma &= \mu J_1 dx^1 + \mu J_2 dx^2 + \mu J_3 dx^3 - \frac{\epsilon}{c}\rho\, dx^4
\end{aligned} \qquad 3.b.1$$

Note that the subscripts refer to the usual vector components. In the reduced notation, (3.b.1) becomes:

$$\begin{aligned} \alpha &= \mathbf{A}_1 \quad - \frac{1}{c}\,\phi\, dx^4 \\ \beta &= \frac{1}{c}\,\mathbf{E}_1 dx^4 + \mathbf{B}_1 \\ \gamma &= \mu\, \mathbf{J}_1 \quad - \frac{\varepsilon^{-1}}{c}\,\rho\, dx^4 \end{aligned} \qquad 3.b.2$$

The subscripts in (3.b.2) refer to the R^3 vector order. **A**, Φ, **E**, **B**, **J**, ρ, ε and μ are all standard notation for the electromagnetic quantities. We should also note that

$$\varepsilon\mu = \frac{1}{c^2} \qquad 3.b.3$$

In order to write out the expanded equations for α', β' and γ', it is necessary to calculate the * operator for R^3/t. The mathematics follows from §1.D. As we mentioned earlier, we will modify the * operator to account for ε and μ. Consistent results are established when:

*-MAPPING OF R^3/t BASIS ELEMENTS TABLE 3.4

Basis Element		Dual Element
$* dx^1dx^2dx^3dx^4$	=	$c\varepsilon$
$* dx^1dx^2dx^3$	=	$-\mu^{-1}c^{-1}\ dx^4$
$* dx^2dx^3dx^4$	=	$-c\varepsilon\ dx^1$
$* dx^3dx^1dx^4$	=	$-c\varepsilon\ dx^2$
$* dx^1dx^2dx^4$	=	$-c\varepsilon\ dx^3$
$* dx^2dx^3$	=	$-\mu^{-1}c^{-1}\ dx^1dx^4$
$* dx^3dx^1$	=	$-\mu^{-1}c^{-1}\ dx^2dx^4$
$* dx^1dx^2$	=	$-\mu^{-1}c^{-1}\ dx^3dx^4$
$* dx^1dx^4$	=	$c\varepsilon\ dx^2dx^3$
$* dx^2dx^4$	=	$c\varepsilon\ dx^3dx^1$
$* dx^3dx^4$	=	$c\varepsilon\ dx^1dx^2$
$* dx^1$	=	$-\mu^{-1}c^{-1}\ dx^2dx^3dx^4$
$* dx^2$	=	$-\mu^{-1}c^{-1}\ dx^3dx^1dx^4$
$* dx^3$	=	$-\mu^{-1}c^{-1}\ dx^1dx^2dx^4$
$* dx^4$	=	$-c\varepsilon\ dx^1dx^2dx^3$
$* 1$	=	$-\mu^{-1}c^{-1}\ dx^1dx^2dx^3dx^4$

As a result, the dual forms α', β' and γ' are written

$$\alpha' = -\frac{\mu^{-1}}{c} A_2 dx^4 + \epsilon(\Phi)_3$$
$$\beta' = -\frac{\mu^{-1}}{c} B_1 dx^4 + \epsilon E_2 \qquad 3.b.4$$
$$\gamma' = -\frac{1}{c} J_2 dx^4 + (\rho)_3$$

Defining $H = \mu^{-1} B$ and $D = \epsilon E$, $\beta' = -\frac{1}{c} H_1 dx^4 + D_2$, which is perhaps a more familiar notation.

Using the * operation as listed in Table 3.4, the codifferential can be found using (1.f.2).

All electromagnetic quantities have physical dimensions, and we must incorporate these dimensions into the algebra. One commonly accepted set of dimensions for the coefficient quantities is found in the Handbook of Chemistry and Physics (1961):

DIMENSIONS OF ELECTROMAGNETIC COEFFICIENT QUANTITIES — TABLE 3.5

Quantity	Units	Common Name
$[A]$ =	$[\epsilon^{-1/2}\ m^{1/2}\ 1^{-1/2}]$ -	Vector Potential
$[\Phi]$ =	$[\epsilon^{-1/2}\ m^{1/2}\ 1^{1/2}\ t^{-1}]$ -	Potential
$[E]$ =	$[\epsilon^{-1/2}\ m^{1/2}\ 1^{-1/2}\ t^{-1}]$ -	Electric Field Strength
$[B]$ =	$[\epsilon^{-1/2}\ m^{1/2}\ 1^{-3/2}]$ -	Magnetization Intensity
$[J]$ =	$[\epsilon^{1/2}\ m^{1/2}\ 1^{-1/2}\ t^{-2}]$ -	Current Density
$[\rho]$ =	$[\epsilon^{1/2}\ m^{1/2}\ 1^{-3/2}\ t^{-1}]$ -	Charge Density
$[\epsilon^{-1/2}\ \mu^{-1/2}]$ =	$[1\ t^{-1}]$	

The brackets [--] indicate a dimension. m, 1 and t refer to mass, length and time. By accounting for both the coefficient and differential term dimensions, the "total" dimensions of the fundamental forms are

DIMENSIONS OF THE FUNDAMENTAL ELECTROMAGNETIC FORMS — TABLE 3.6

Quantity	Units	Common Name
$[\alpha]$ =	$[\epsilon^{-1/2}\ m^{1/2}\ 1^{1/2}]$	Magnetic Flux or Pole Strength (α, β)
$[\beta]$ =	$[\epsilon^{1/2}\ m^{1/2}\ 1^{1/2}]$	Magnetic Flux or Pole Strength (α, β)
$[\gamma]$ =	$[\epsilon^{1/2}\ m^{1/2}\ 1^{-3/2}]$ -	Magnetization Intensity

TABLE 3.6 (continued)

$[\alpha']$	=	$[\varepsilon^{1/2}\ m^{1/2}\ l^{7/2}\ t^{-1}]$	– ?
$[\beta']$	=	$[\varepsilon^{1/2}\ m^{1/2}\ l^{3/2}\ t^{-1}]$	} Charge
$[\gamma']$	=	$[\varepsilon^{1/2}\ m^{1/2}\ l^{3/2}\ t^{-1}]$	} Charge

These will also be the dimensions of the global quantity determined by integrating the differential form in question. We can write the dimensional behavior of the transformation of global quantities under the * operation as follows:

$$
\begin{aligned}
[{*\kappa}] &= [\varepsilon\ l^{5}\ t^{-1}][\kappa] \text{ or } [\mu^{-1}\ l^{3}\ t][\kappa] \\
[{*\lambda}] &= [\varepsilon\ l^{3}\ t^{-1}][\lambda] \text{ or } [\mu^{-1}\ l\ \ t][\lambda] \\
[{*\mu}] &= [\varepsilon\ l\ \ t^{-1}][\mu] \text{ or } [\mu^{-1}\ l^{-1}\ t][\mu] \\
[{*\nu}] &= [\varepsilon\ l^{-1}\ t^{-1}][\nu] \text{ or } [\mu^{-1}\ l^{-3}\ t][\nu] \\
[{*\zeta}] &= [\varepsilon\ l^{-3}\ t^{-1}][\zeta] \text{ or } [\mu^{-1}\ l^{-5}\ t][\zeta]
\end{aligned}
\qquad 3.b.5
$$

This enables simplified calculation of the total dimensions of dual forms. Note that in Tables 3.5 and 3.6, we could equally well use a dimension involving μ because of the relation $[\varepsilon^{-1/2}\ \mu^{-1/2}] = [l\ t^{-1}]$.

Under the derivative operation, although the order is shifted by +1 (§1.E), the total dimension does not change:

$$[d\omega] = [d][\omega] = \left[\frac{\partial}{\partial x^{i}}\right][dx^{i}][\omega] = [l^{-1}][l][\omega] = [\omega] \qquad 3.b.6$$

The quantity c has dimensions $[l\ t^{-1}]$, so that $[dx^{4}] = [cdt] = [l]$, which is the dimension of the spatial differential terms. For the codifferential and Laplace-Beltrami operators,

$$
\begin{aligned}
[\delta\omega] &= [l^{-2}][\omega] \\
[\Delta\omega] &= [l^{-2}][\omega]
\end{aligned}
\qquad 3.b.7
$$

In the strictly algebraic formulation of exterior differential forms,

$$****(\omega) = \omega \qquad 3.b.8$$

whereas in electromagnetism, where the properties of the medium are included in the * operator,

$$****(\omega) = \frac{\varepsilon^2}{\mu^2}\ \omega$$
$$[****(\omega)] = [\varepsilon^2\ \mu^2][\omega] \qquad 3.b.9$$

By recognizing the dimensional character of R^3/t and its dual space, we will be able to select the correct dimensions for any quantity.

C. The Differential Structure of the Electromagnetic Field Equations

For the R^3/t differential forms α, β and γ, we have stated that a simple relationship exists which includes all of the fundamental partial differential equations of electromagnetic field theory. We present in this section a diagram outlining this relationship, and we shall demonstrate that it contains all the information claimed. Recall that in these structural diagrams, a differential is indicated by an arrow directed to the right, and a codifferential by one directed to the left. The Laplace-Beltrami operator is represented by a downward arrow. For α, β and γ then, we propose the following regular and dual space structures:

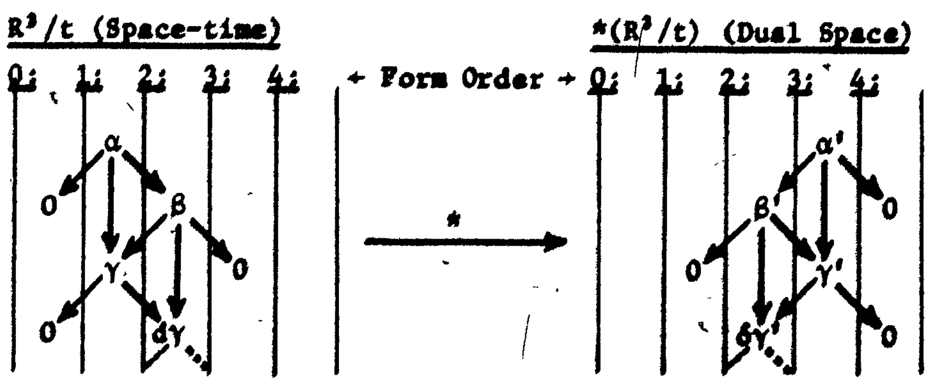

Figure 3.1 : Structural diagram for the electromagnetic forms.

In the space R^3/t, the relationship between the electromagnetic forms under d and δ is illustrated by the alternating pattern between the 1-forms and 2-forms. If R^3/t is a compact manifold, by using the Hodge decomposition theorem (1.k.11), we can find forms extending this alternating pattern under certain conditions. However, because of the definite assignment of zero to the 0-forms and 3-forms in the R^3/t structure, the discussion in §1.K concerning the decomposition of closed and co-closed forms shows that it is impossible for 0-form and 3-form potentials to have any effect on the alternating structure of the electromagnetic 1-forms and 2-forms. 0-form and 3-form potentials, in order to conform to the structure already present, can at best be harmonic, and since for a harmonic form ω, $d\omega = \delta\omega = 0$ (1.k.8), they cannot influence the already present relationships.

In general, it is possible to find forms which continue the pattern in Figure 3.1. We require that derivatives into the spaces of 0-forms and 3-forms be zero. For example, a form ξ such that $\delta\xi = \alpha$ and $\Delta\xi = \beta$ exists. For if $\xi = 0$, $\delta\xi = \Delta\xi = 0$, and therefore $\alpha = \beta = 0$. An identically zero quantity at any point in the structure implies that the entire structure below it is also identically zero. Conversely, it implies that all forms above it are at best harmonic. Let $\gamma = 0$. Then $\delta\beta = d\beta = 0$, implying that β is harmonic (or zero). From this it follows that α is harmonic (or zero), and so on. In Chapter 4, we will return to the mathematical analysis of the electromagnetic structure in Figure 3.1.

While we have shown that other forms exist which continue the alternating pattern in Figure 3.1, the relationships between α, β and γ and these new forms generally fall outside the range of the electromagnetic equations, which are partial differential equations of 1^{st} and 2^{nd} order. Consequently, we will limit our discussion to α, β, γ and $d\gamma$.

Not counting the derivative formulas which are identically zero by the Poincaré lemma ($dd\omega = 0$; $\delta\delta\omega = 0$), each space in Figure 3.1 contains 3 homogeneous 1^{st} order relations, 2 inhomogeneous 1^{st} order relations, and 2 Laplacian (2^{nd} order) relations:

PARTIAL DIFFERENTIAL EQUATIONS OF ELECTROMAGNETISM — TABLE 3.7a

A. Homogeneous 1^{st} Order Relations:

(R^3/t):

1. $\delta\alpha = 0$: Lorentz condition on potentials.
2. $d\beta = 0$: Faraday's law; Magnetic pole law.
3. $\delta\gamma = 0$: Charge-current continuity.

$*(R^3/t)$:

1' $d\alpha' = 0$
2' $\delta\beta' = 0$
3' $d\gamma' = 0$

} Same identification as above.

B. Inhomogeneous 1^{st} Order Relations:

(R^3/t):

1. $d\alpha = \beta$: Relation between potentials and fields.
2. $\delta\beta = \gamma$: Ampère's law; Coulomb's law.

$*(R^3/t)$:

1' $\delta\alpha' = \beta'$
2' $d\beta' = \gamma'$

} Same identification as above.

C. Laplacian (2^{nd} Order) Relations:

1. Left-hand side in unmodified form.

(R^3/t):

1. $\delta d\alpha = \gamma$: Relation between potentials and charge-current.
2. $d\delta\beta = d\gamma$: Wave equation for the fields.

$*(R^3/t)$:

1' $d\delta\alpha' = \gamma'$
2' $\delta d\beta' = \delta\gamma'$

} Same identification as above.

TABLE 3.7a (continued)

ii. Left-hand side in full Laplacian form.			
(R^3/t):			
1.	$\Delta\alpha = \gamma$	:	Relation between potentials and charge-current.
2.	$\Delta\beta = d\gamma$	:	Wave equation for the fields.
$*(R^3/t)$:			
1'	$\Delta\alpha' = \gamma'$	}	Same identification as above.
2'	$\Delta\beta' = \delta\gamma'$	}	

The following table lists the detailed calculations for the relations in Sections A, B and C of Table 3.7a. The calculations are for the R^3/t formulas only, since the dual space formulas yield identical results. Direct conversion can be made using the * operation.

PARTIAL DIFFERENTIAL EQUATIONS OF ELECTROMAGNETISM: VECTOR NOTATION

TABLE 3.7b

A.1.	$\delta\alpha = 0$	: $-(\nabla\cdot\mathbf{A})_0 - \varepsilon\mu\left(\frac{\partial\phi}{\partial t}\right)_0 = 0$	
		$\rightarrow \nabla\cdot\mathbf{A} = -\varepsilon\mu\left(\frac{\partial\phi}{\partial t}\right)$	: Lorentz condition: Relation between vector and scalar potentials.
A.2.	$d\beta = 0$	: $\frac{1}{c}(\nabla\times\mathbf{E})_2 dx^4 + \frac{1}{c}\left(\frac{\partial\mathbf{B}}{\partial t}\right)_2 dx^4 + (\nabla\cdot\mathbf{B})_3 = 0$	
		Time $\rightarrow \nabla\times\mathbf{E} = -\left(\frac{\partial\mathbf{B}}{\partial t}\right)$	: Faraday's law (Maxwell equation).
		Space $\rightarrow \nabla\cdot\mathbf{B} = 0$	: Non-existence of magnetic poles (Maxwell equation).

TABLE 3.7b (continued)

A.3.	$\delta\gamma = 0 \;:\; -\frac{1}{c}(\nabla\cdot\mathbf{J})_0 - \frac{1}{c}\left(\frac{\partial\rho}{\partial t}\right)_0 = 0$	
	$\rightarrow \nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t}$	: Equation of continuity: Relation between charge and current.
B.1.	$d\alpha = \beta \;:\; (\nabla\times\mathbf{A})_2 - \frac{1}{c}\left(\frac{\partial\mathbf{A}}{\partial t}\right)_1 dx^4 - \frac{1}{c}(\nabla\Phi)_1 dx^4 = \frac{1}{c}\mathbf{E}_1 dx^4 + \mathbf{B}_2$	
	Time $\rightarrow \mathbf{E} = -\left(\frac{\partial\mathbf{A}}{\partial t}\right) - \nabla\Phi$	: Relation between electric field and potentials.
	Space $\rightarrow \mathbf{B} = \nabla\times\mathbf{A}$	: Relation between magnetic field and vector potential.
B.2	$\delta\beta = \gamma \;:\; \frac{1}{c}(\nabla\cdot\mathbf{E})_0 dx^4 - \varepsilon\mu\left(\frac{\partial\mathbf{E}}{\partial t}\right)_1 + (\nabla\times\mathbf{B})_1 = \mu\,\mathbf{J}_1 - \frac{\varepsilon^{-1}}{c}\rho\, dx^4$	
	Time $\rightarrow \nabla\cdot\mathbf{D} = \rho$	: Coulomb's law (Maxwell equation).
	Space $\rightarrow \nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}$	: Ampère's law (Maxwell equation).
	Where $\mathbf{D} = \varepsilon\mathbf{E}\,;\; \mathbf{H} = \mu^{-1}\mathbf{B}$.	
C.1.	$\delta d\alpha = \gamma \;:\; (\nabla\times\nabla\times\mathbf{A})_1 + \varepsilon\mu\left(\frac{\partial^2\mathbf{A}}{\partial t^2}\right)_1 + \varepsilon\mu\frac{\partial}{\partial t}(\nabla\Phi)_1 + \frac{1}{c}\left(\nabla\cdot\frac{\partial\mathbf{A}}{\partial t}\right)_0 dx^4 + \frac{1}{c}(\nabla\cdot\nabla\Phi)_0 dx^4 = \mu\,\mathbf{J}_1 - \frac{\varepsilon^{-1}}{c}\rho\, dx^4$	
	Relations between the potentials and charge/current:	
	Time $\rightarrow \nabla\cdot\nabla\Phi + \left(\nabla\cdot\frac{\partial\mathbf{A}}{\partial t}\right) = -\varepsilon^{-1}\rho$	
	Space $\rightarrow \nabla\times\nabla\times\mathbf{A} + \varepsilon\mu\left(\frac{\partial^2\mathbf{A}}{\partial t^2}\right) + \varepsilon\mu\frac{\partial}{\partial t}(\nabla\Phi) = \mu\mathbf{J}$	

TABLE 3.7b (continued)

C.1. $\Delta\alpha = \gamma$: $-(\nabla^2 \mathbf{A})_1 + \varepsilon\mu\left(\frac{\partial^2 \mathbf{A}}{\partial t^2}\right)_1 + \frac{1}{c}(\nabla^2\phi)_0 dx^4 - \frac{\varepsilon\mu}{c}\left(\frac{\partial^2\phi}{\partial t^2}\right)_0 dx^4$

$$= \mu \mathbf{J}_1 - \frac{\varepsilon^{-1}}{c}\rho\, dx^4$$

$$\text{Time} \rightarrow \nabla^2\phi - \varepsilon\mu\left(\frac{\partial^2\phi}{\partial t^2}\right) = -\varepsilon^{-1}\rho$$

$$\text{Space} \rightarrow \nabla^2\mathbf{A} - \varepsilon\mu\left(\frac{\partial^2\mathbf{A}}{\partial t^2}\right) = -\mu\mathbf{J}$$

C.2. $d\delta\beta = d\gamma$: $-\frac{1}{c}(\nabla\nabla\cdot\mathbf{E})_1 dx^4 + \frac{\varepsilon\mu}{c}\left(\frac{\partial^2\mathbf{E}}{\partial t^2}\right)_1 dx^4 - \frac{1}{c}\frac{\partial}{\partial t}(\nabla\times\mathbf{B})_1 dx^4$

$$-\varepsilon\mu\left(\nabla\times\left(\frac{\partial\mathbf{E}}{\partial t}\right)\right)_2 + (\nabla\times\nabla\times\mathbf{B})_2 = \mu(\nabla\times\mathbf{J})_2 - \frac{\mu}{c}\left(\frac{\partial\mathbf{J}}{\partial t}\right)_1 dx^4$$

$$-\frac{\varepsilon^{-1}}{c}(\nabla\rho)_1 dx^4$$

$$\text{Time} \rightarrow -\frac{1}{c}(\nabla\nabla\cdot\mathbf{E}) + \frac{\varepsilon\mu}{c}\left(\frac{\partial^2\mathbf{E}}{\partial t^2}\right) - \frac{1}{c}\frac{\partial}{\partial t}(\nabla\times\mathbf{B}) = -\frac{\mu}{c}\left(\frac{\partial\mathbf{J}}{\partial t}\right) - \frac{\varepsilon^{-1}}{c}(\nabla\rho)$$

$$\text{Space} \rightarrow (\nabla\times\nabla\times\mathbf{B}) - \varepsilon\mu\left(\nabla\times\left(\frac{\partial\mathbf{E}}{\partial t}\right)\right) = \mu(\nabla\times\mathbf{J})$$

C.2. $\Delta\beta = d\gamma$: $-\frac{1}{c}(\nabla^2\mathbf{E})_1 dx^4 + \frac{\varepsilon\mu}{c}\left(\frac{\partial^2\mathbf{E}}{\partial t^2}\right)_1 dx^4 - (\nabla^2\mathbf{B})_2 + \varepsilon\mu\left(\frac{\partial^2\mathbf{B}}{\partial t^2}\right)_2$

$$= \mu(\nabla\times\mathbf{J})_2 - \frac{\mu}{c}\left(\frac{\partial\mathbf{J}}{\partial t}\right)_1 dx^4 - \frac{\varepsilon^{-1}}{c}(\nabla\rho)_1 dx^4$$

Wave equations for the fields:

$$\text{Time} \rightarrow \frac{1}{c}\nabla^2\mathbf{E} - \frac{\varepsilon\mu}{c}\left(\frac{\partial^2\mathbf{E}}{\partial t^2}\right) = \frac{\mu}{c}\left(\frac{\partial\mathbf{J}}{\partial t}\right) + \frac{\varepsilon^{-1}}{c}(\nabla\rho)$$

$$\text{Space} \rightarrow \nabla^2\mathbf{B} - \varepsilon\mu\left(\frac{\partial^2\mathbf{B}}{\partial t^2}\right) = -\mu(\nabla\times\mathbf{J})$$

By subsituting B.2 and A.2, these become:

$$\text{Time} \rightarrow \frac{1}{c}\nabla\times\nabla\times\mathbf{E} + \frac{\varepsilon\mu}{c}\left(\frac{\partial^2\mathbf{E}}{\partial t^2}\right) = -\frac{\mu}{c}\left(\frac{\partial\mathbf{J}}{\partial t}\right)$$

$$\text{Space} \rightarrow \nabla\times\nabla\times\mathbf{B} + \varepsilon\mu\left(\frac{\partial^2\mathbf{B}}{\partial t^2}\right) = \mu(\nabla\times\mathbf{J})$$

The equations in Sets A and B of Table 3.7b include all of the fundamental relations of electromagnetism: Maxwell's equations, the Lorentz condition, the equation of continuity between charge and current, and the defining relation between the potentials and the fields. The 2nd order relations in Set C.1 can be found in STRATTON, §1.9, Equations 10, 11, 17 and 15. BOJARSKI (1973), Appendix 2, derives the wave equations found in the part of Set C.2 that involves the complete Laplacian.

In our development of the equations of electromagnetism, we have not included the concept of conductivity, ordinarily represented by the equation $J = \sigma E$. Such a relationship, which on the surface seems to involve the equality of the 2-form β and the 1-form γ, can only be realized when a specific time dependence is applied to the structure of electromagnetic differential forms. The derivations in Table 3.7b have been made without any time behavior postulated at all. Consequently these equations represent the most general form of electromagnetic activity. The introduction of specific time dependence, constant (electrostatic and magnetostatic behavior), exponential (conductive behavior) and complex exponential (time-harmonic behavior), reduces their generality.

D. Projected Integration; Integral Form of the Electromagnetic Equations

When we introduce a mathematical formalism for the purpose of simplifying a description of physical phenomena, it is wise to insure that we have not simplified our description beyond the point of utility. In R^3/t exterior differential algebra, the equations of electromagnetism exist as a structure not really suited to exploitation by the usual methods handling the spatial behavior separately from that of time. It must be possible to transform the simple structure into some working form, which in this case means standard vector algebra. What we develop now is the concept of a differential projection, in the sense that the space projection of a form $\omega = Adx^{1234}$ is Adx^{123}, while the time projection of the same form is Adx^4.(In the development of Table 3.7b, the vector formulas were separated from the complete R^3/t derivative formulas by this consideration). This enables us to construct the electromagnetic equations in the familiar integral form.

We start with these premises:

1. A 0-form is not integrable over a manifold of dimension > 0.

2. A 1-form Bdx^1 is not integrable with respect to dx^2.

3. A coefficient quantity not integrable with respect to a certain differential will be integrable with respect to this same differential in the *-dual space.

The implication is that in the formation of strictly spatial integration formulas from the space projection of an R^3/t differential form, we need not be concerned with the discarded time differential. The converse is true in the development of time integration formulas. A further implication of the above premises is that the spatial forms of different order resulting from such a projection may be integrated separately.

We will now show that the projection of the R^3/t differential (not codifferential) electromagnetic relations, together with the appropriate R^3 (for space) or R^1 (for time) integration formulas, produces the familiar vector integration formulas of electromagnetic theory. The relations we work with are:

$$\begin{aligned} d\alpha' &= 0 \\ d\alpha &= \beta \\ d\beta &= 0 \\ d\beta' &= \gamma' \\ d\gamma' &= 0 \end{aligned} \qquad 3.d.1$$

Referring to Table 3.7b for the projected components of (3.d.1), we can construct the following space and time integration formulas by applying R^3 and R^1 integration theorems:

INTEGRAL FORM OF THE BASIC ELECTROMAGNETIC EQUATIONS — TABLE 3.8

$d\alpha' = 0$:	Space Integral:	$\oiint_{\partial V} A_2 = -\varepsilon\mu \iiint_V \frac{\partial\Phi}{\partial t}\, dx^{123}$	
	Time Integral:	$\int_t (\nabla\cdot A)dt = -\varepsilon\mu \sum_{t-}^{t+} \Phi(t)$	(1 component)
$d\alpha = \beta$:	Space Integrals:	$\iint_s B_2 = \oint_{\partial s} A_1$	
		$\iint_x \left(E_1 + \frac{\partial A_1}{\partial t}\right) = -\sum_{x-}^{x+} \Phi(x)$	
	Time Integral:	$\int_t (E + \nabla\Phi)dt = -\sum_{t-}^{t+} A(t)$	(3 components)
$d\beta = 0$:	Space Integrals:	$\oiint_{\partial V} B_2 = 0$	(Maxwell Eq.)
		$\oint_{\partial s} E_1 = -\iint_s \left(\frac{\partial B_2}{\partial t}\right)$	(Maxwell Eq.)
	Time Integral:	$\int_t (\nabla\times E)dt = -\sum_{t-}^{t+} B(t)$	(3 components)
$d\beta' = \gamma'$:	Space Integrals:	$\oiint_{\partial V} D_2 = \iiint_V \rho\, dx^{123}$	(Maxwell Eq.)
		$\oint_{\partial s} H_1 = \iint_s \left(J_2 + \left(\frac{\partial D_2}{\partial t}\right)\right)$	(Maxwell Eq.)
	Time Integral:	$\int_t (\nabla\times H - J)dt = \sum_{t-}^{t+} D(t)$	(3 components)
$d\gamma' = 0$:	Space Integral:	$\oiint_{\partial V} J_2 = -\iiint_V \frac{\partial\rho}{\partial t}\, dx^{123}$	(Continuity)
	Time Integral:	$\int_t (\nabla\cdot J)dt = -\sum_{t-}^{t+} \rho(t)$	(1 component)

The integrals referred to as "space integrals" form the usual set of formulas of electromagnetism, including Maxwell's equations, expressed in the vector integral form. The "time integrals" are related to those space integrals that involve a time derivative, but they are normally not studied. The time integrals, it can be seen, each involve a space derivative.

E. Unit Differential Forms in R^3/t

In Chapter 2, we introduced unit forms in order to complete the development of the R^3 integration formulas. This was facilitated by the interrelationship of the R^3 unit forms under the * operation. In R^3/t, the behavior of the * operation is more complicated. In addition, we are concerned with the physical dimension of all quantities. Therefore we have a problem concerning the best definition of an R^3/t unit form. The answer appears to involve the R^3/t inner product. For example, the norm of a 1-form α is based on the exterior product

$$\alpha \wedge {*\alpha} \qquad 3.e.1$$

Dimensionally, this product can be written

$$[\epsilon^{1/2}\, m^{1/2}\, l^{1/2}] \wedge [\epsilon\, l^{3}\, t^{1}][\epsilon^{1/2}\, m^{1/2}\, l^{1/2}] = [m\, l^{4}\, t^{1}] \qquad 3.e.2$$

Consequently, a suitable choice of dimensions for the u_i will be when

$$[(\alpha \wedge u_i) \wedge *(\alpha \wedge u_i)] = [m\, l^{4}\, t^{1}] \qquad 3.e.3$$

Using (3.b.5), we quickly find

$$[u_i] = [l^{i}] \qquad 3.e.4$$

This implies that the dimensions of R^3/t unit forms satisfying (3.e.3) will be simply the dimensions of their differential terms. In other words, the coefficient 1 is dimensionless.

As we mentioned, the dimensional behavior of the R^3/t * operation prevents simple relations between the unit forms and their duals. We therefore use a slightly different criterion in their definition. The set of R^3/t unit forms shown below is designed to facilitate operations involving the fundamental electromagnetic forms α, β and γ on one hand, and α', β' and γ' on the other:

UNIT DIFFERENTIAL FORMS FOR R^3/t AND $*(R^3/t)$ — TABLE 3.9

Unit forms for (R^3/t):

$$u_0 = 1$$
$$u_1 = dx^1 + dx^2 + dx^3 - dx^4$$
$$u_2 = dx^{23} + dx^{31} + dx^{12} + dx^{14} + dx^{24} + dx^{34}$$
$$u_3 = dx^{234} + dx^{314} + dx^{124} - dx^{123}$$
$$u_4 = dx^{1234}$$

Unit forms for $*(R^3/t)$:

$$u_0' = -1$$
$$u_1' = -dx^1 - dx^2 - dx^3 + dx^4$$
$$u_2' = dx^{23} + dx^{31} + dx^{12} - dx^{14} - dx^{24} - dx^{34}$$
$$u_3' = -dx^{234} - dx^{314} - dx^{124} + dx^{123}$$
$$u_4' = -dx^{1234}$$

As we can see from (3.b.2) and (3.b.4), the signs of u_1, u_2, u_2' and u_3' have been chosen to correspond with the signs of the fundamental electromagnetic forms. The selection of the signs on the remaining forms is arbitrary.

F. Harmonic Equations and Wave Equations in Electromagnetism

In the differential structure presented in §3.C, equations that involve the Laplace-Beltrami operator are wave equations, with the right-hand side representing an effective source. If the source term is zero, the equation is harmonic. In particular, the differential form that is the operand of Δ is a harmonic form. Paradoxically, the assumption that the Hodge theorem applies in R^3/t has an interesting physical implication (The Hodge theorem does not apply in R^4/t because the metric is not Riemannian). Forming the inner product (β,β), we find

$$(\beta,\beta) = (d\alpha,d\alpha) = (\alpha,\delta d\alpha) = (\alpha,\gamma) = 0 \qquad 3.f.1$$

provided γ is identically zero. Then, assuming the validity of (1.k.6) in this case,

$$\beta = 0 \qquad 3.f.2$$

By repeating the proof, we would find that all forms in the structure are identically zero, implying a global property to electromagnetism: electromagnetic phenomena can be exhibited only when sources (charge and current) are present somewhere in space and time. Of course, we must deal with unbounded manifolds in order to include effective sources at infinity.

In electromagnetism, the two R^3/t wave equations are:

$$\Delta\alpha = \gamma \qquad \text{Duals:} \quad \Delta\alpha' = \gamma' \qquad 3.f.3$$
$$\Delta\beta = \delta\gamma \qquad \qquad \Delta\beta' = \delta\gamma'$$

The solution of boundary-value problems for these wave equations relies upon an exploitation of the reciprocity that is characteristic of the Green's theorems. Almost all familiar problems involve the "scalar" Green's theorem, where the solution is developed from a "Green's function" satisfying a scalar wave equation that has a scalar distribution for a source (The distribution is commonly the impulse distribution known as the Dirac δ-function). Although little distinction is made between distributions and functions at the practical level, the two cannot be considered identical. The "δ-function" is properly defined through an integral relation (PAPOULIS (1962), §I.1):

$$\int_{-\infty}^{+\infty}\delta(x-x')g(x)dx = g(x') \qquad 3.f.4$$

In order to solve wave equations by Green's function techniques, it is necessary to include in the exterior differential form structure of electromagnetism those aspects of the theory of distributions relevant to the solution of boundary-value problems. Obviously, care is essential, since in exterior differential algebra the concept of integrability has been carefully introduced in connection with the differential nature of the form quantities themselves. Because the very definition of a δ-function involves an integral, we must study the behavior of the wave equations (3.f.3) in order to see how ideas from the theory of distributions may be grafted onto the functional/algebraic theory at this point in its development.

R^3/t Green's theorems, developed for the 1-form α and the 2-form β of the electromagnetic structure, can be written in the following separated form:

$$\begin{aligned}\alpha{*}\Delta g_1 - g_1{*}\Delta\alpha &= d(g_1\wedge{*}d\alpha - \alpha\wedge{*}dg_1) + d(\delta g_1\wedge{*}\alpha - \delta\alpha\wedge{*}g_1)\\ \beta{*}\Delta g_2 - g_2{*}\Delta\beta &= d(g_2\wedge{*}d\beta - \beta\wedge{*}dg_2) + d(\delta g_2\wedge{*}\beta - \delta\beta\wedge{*}g_2)\end{aligned} \qquad 3.f.5$$

The form representing the Green's function, g_i, has the same order as its reflexive counterpart in (3.f.5). We define g_1 and g_2 as follows:

$$\begin{aligned} g_1 &= G_1 - \frac{1}{c}G_4dx^4\\ g_2 &= \frac{1}{c}G_1dx^4 + G_2\end{aligned} \qquad 3.f.6$$

By expanding (3.f.5) into its complete component form, a specific identification of the related right-hand side and left-hand side terms leads to the following space and time vector integration formulas:

GREEN'S THEOREMS FOR THE ELECTROMAGNETIC POTENTIAL FORM IN R^3/t: VECTOR NOTATION

TABLE 3.10a

A.1.
$$\iiint \frac{\mu^{-1}}{c}(\mathbf{A}\cdot\nabla^2\mathbf{G} - \mathbf{G}\cdot\nabla^2\mathbf{A})_3 = \oiint \frac{\mu^{-1}}{c}[(\nabla\cdot\mathbf{G})\mathbf{A} - (\nabla\cdot\mathbf{A})\mathbf{G} - \mathbf{G}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{G})]_2$$

A.2.
$$\iiint \frac{\varepsilon}{c}(G_4\nabla^2\phi - \phi\nabla^2 G_4)_3 = \oiint \frac{\varepsilon}{c}(G_4\nabla\phi - \phi\nabla G_4)_2$$

A.3.
$$\int_t \frac{\varepsilon}{c}[(\mathbf{G}\cdot\ddot{\mathbf{A}}) - (\mathbf{A}\cdot\ddot{\mathbf{G}})]dt = \sum_{t-}^{t+} \frac{\varepsilon}{c}[(\mathbf{G}\cdot\dot{\mathbf{A}}) - (\mathbf{A}\cdot\dot{\mathbf{G}})]$$

A.4.
$$\int_t \frac{\varepsilon^2\mu}{c}(\phi\ddot{G}_4 - G_4\ddot{\phi})dt = \sum_{t-}^{t+} \frac{\varepsilon^2\mu}{c}(\dot{G}_4\phi - \dot{\phi}G_4)$$

A.5.
$$\iiint -\frac{\varepsilon}{c}\frac{\partial}{\partial t}[\mathbf{G}\cdot(\nabla\phi) - \mathbf{A}\cdot(\nabla G_4) + (\nabla\cdot\mathbf{G})\phi - (\nabla\cdot\mathbf{A})G_4]_3 = \oiint \frac{\varepsilon}{c}[G_4\dot{\mathbf{A}} - \phi\dot{\mathbf{G}} + \dot{G}_4\mathbf{A} - \dot{\phi}\mathbf{G}]_2$$

A.6.
$$\int_t \frac{\varepsilon}{c}[\nabla\cdot(G_4\dot{\mathbf{A}} - \phi\dot{\mathbf{G}} + \dot{G}_4\mathbf{A} - \dot{\phi}\mathbf{G})]dt = \sum_{t-}^{t+} -\frac{\varepsilon}{c}[\mathbf{G}\cdot(\nabla\phi) - \mathbf{A}\cdot(\nabla G_4) + (\nabla\cdot\mathbf{G})\phi - (\nabla\cdot\mathbf{A})G_4]$$

GREEN'S THEOREMS FOR THE ELECTROMAGNETIC FIELD FORM IN R^3/t: VECTOR NOTATION

TABLE 3.10b

B.1.
$$\iiint \frac{\varepsilon}{c}(\mathbf{G}\cdot\nabla^2\mathbf{E} - \mathbf{E}\cdot\nabla^2\mathbf{G})_3 = \oiint \frac{\varepsilon}{c}[(\nabla\cdot\mathbf{E})\mathbf{G} - (\nabla\cdot\mathbf{G})\mathbf{E} - \mathbf{E}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{E})]_2$$

B.2.
$$\iiint \frac{\mu^{-1}}{c}(\mathbf{B}\cdot\nabla^2\mathbf{Q} - \mathbf{Q}\cdot\nabla^2\mathbf{B})_3 = \oiint \frac{\mu^{-1}}{c}[\mathbf{B}(\nabla\cdot\mathbf{Q}) - \mathbf{Q}(\nabla\cdot\mathbf{B}) - (\nabla\times\mathbf{Q})\times\mathbf{B} + (\nabla\times\mathbf{B})\times\mathbf{Q}]_2$$

TABLE 3.10b (continued)

B.3.	$\int_t \frac{\varepsilon^2\mu}{c}(E\cdot\ddot{G} - G\cdot\ddot{E})dt = \sum_{t^-}^{t^+} \frac{\varepsilon^2\mu}{c}(\dot{G}\cdot E - \dot{E}\cdot G)$
B.4.	$\int_t \frac{\varepsilon}{c}(Q\cdot\ddot{B} - B\cdot\ddot{Q})dt = \sum_{t^-}^{t^+} \frac{\varepsilon}{c}(Q\cdot\dot{B} - B\cdot\dot{Q})$
B.5.	$\iiint -\frac{\varepsilon}{c}\frac{\partial}{\partial t}[(\nabla\times B)\cdot G - (\nabla\times Q)\cdot E + Q\cdot(\nabla\times E) - B\cdot(\nabla\times G)]; = \oiint \frac{\varepsilon}{c}[\dot{G}\times B - \dot{E}\times Q + G\times\dot{B} - E\times\dot{Q}]$
B.6.	$\int_t \frac{\varepsilon}{c}[\nabla\cdot(\dot{G}\times B - \dot{E}\times Q + G\times\dot{B} - E\times\dot{Q})]dt = \sum_{t^-}^{t^+} \frac{\varepsilon}{c}[(\nabla\times B)\cdot G - (\nabla\times Q)\cdot E + Q\cdot(\nabla\times E) - B\cdot(\nabla\times G)]$

In each section of Table 3.10, the fifth and sixth formulas involve the remaining terms on the right-hand side of the expanded (3.f.5), which, of course, sum to zero. In the Tables 3.10, we see that the standard vector and scalar R^3 Green's theorems apply to the vector and scalar components of the differential forms α and β.

The solution of (3.f.3) by the Green's function technique consists of finding solutions to the equations

$$\Delta g_1 = \delta_1$$
$$\Delta g_2 = \delta_2 \qquad 3.f.7$$

where δ_1 and δ_2 refer to some 1-form and 2-form involving distributional coefficients. The fact that the δ-function is defined by an integral relation in the first place, and that we occasionally "project" R^3/t forms into R^3 or t, presents us with a difficulty when trying to precisely specify the composition of δ_1 and δ_2. This appears to be resolvable by introducing two premises, somewhat analogous to the premises in §3.D:

1. A $\delta(x_1-x_1')$ distributional coefficient on a form lacking the dx^1 differential term is not meaningful since the defining integral is non-existent.

2. We will consider a distributional form such as $\delta(x_1-x_1')\cdot\delta(x_2-x_2')dx^{123}$ to be incomplete with respect to R^3 integration, but complete with respect to projected integration in the 2-dimensional subspace of R^3 covered by the differential dx^{12}. In the subspace, the defining integral is complete. Incomplete distributional integrations will not be performed.

We can now construct a procedure for solving the Green's theorems (3.f.5). If the distributional source forms of (3.f.7) are defined as

$$\begin{aligned} \delta_1 &= \{\delta\text{-set}\}\, u_1 \\ \delta_2 &= \{\delta\text{-set}\}\, u_2 \end{aligned} \qquad 3.f.8$$

where u_1 and u_2 are the previously defined R^3/t unit forms and the $\{\delta\text{-set}\}$ is a collection of the applicable multi-dimensional δ-functions, consideration of the above premises at the stage of problem development immediately prior to integration gives the correct results. For a Cartesian R^3/t where the Jacobian of the R^3/t "volume element" transformation is 1, the $\{\delta\text{-set}\}$ is the following collection:

$$\{\delta\text{-set}\} = \left\{\begin{matrix} \delta(x_1-x_1');\ \delta(x_2-x_2');\ \delta(x_3-x_3');\ \delta(x_4-x_4') \\ \delta(x_2-x_2')\delta(x_3-x_3');\ \delta(x_3-x_3')\delta(x_1-x_1');\ \delta(x_1-x_1')\delta(x_2-x_2') \\ \delta(x_1-x_1')\delta(x_4-x_4');\ \delta(x_2-x_2')\delta(x_4-x_4');\ \delta(x_3-x_3')\delta(x_4-x_4') \\ \delta(x_2-x_2')\delta(x_3-x_3')\delta(x_4-x_4');\ \delta(x_3-x_3')\delta(x_1-x_1')\delta(x_4-x_4') \\ \delta(x_1-x_1')\delta(x_2-x_2')\delta(x_4-x_4');\ \delta(x_1-x_1')\delta(x_2-x_2')\delta(x_3-x_3') \\ \delta(x_1-x_1')\delta(x_2-x_2')\delta(x_3-x_3')\delta(x_4-x_4') \end{matrix}\right\} \qquad 3.f.9$$

Let us illustrate these ideas by solving a standard wave equation. We will work with the 1-form case in (3.f.5). The appropriate R^3/t

equations are:

$$\Delta\alpha = \gamma$$
$$\Delta g_1 = \{\delta\text{-set}\}\ u_1 \qquad 3.f.10$$

From Table 3.7b, we find the components of (3.f.10):

$$-(\nabla^2 \mathbf{A})_1 + \varepsilon\mu \ddot{\mathbf{A}}_1 = \mu J_1$$
$$\frac{1}{c}(\nabla^2\phi)dx^4 - \frac{\varepsilon\mu}{c}\ddot{\phi}\, dx^4 = -\frac{\mathcal{E}^1}{c}\rho\, dx^4$$
$$-(\nabla^2 \mathbf{G})_1 + \varepsilon\mu \ddot{\mathbf{G}}_1 = \{\delta\text{-set}\}n_1 \qquad 3.f.11$$
$$\frac{1}{c}(\nabla^2 G_4)dx^4 - \frac{\varepsilon\mu}{c}\ddot{G}_4 dx^4 = -\frac{1}{c}\{\delta\text{-set}\}dx^4$$
$$(n_1 = dx^1 + dx^2 + dx^3)$$

The space projection of the ϕ and G_4 terms leads to a true scalar in R^3:

$$\nabla^2\phi - \varepsilon\mu\ddot{\phi} = -\mathcal{E}^1\rho$$
$$\nabla^2 G_4 - \varepsilon\mu\ddot{G}_4 = -\{\delta\text{-set}\} \qquad 3.f.12$$

For spatial integration, the Green's theorem A.2 of Table 3.10a is appropriate:

$$\iiint_{V}\frac{\mathcal{E}}{c}(G_4\nabla^2\phi - \phi\nabla^2 G_4)_3 = \oint\!\!\oint_{\partial V}\frac{\mathcal{E}}{c}(G_4\nabla\phi - \phi\nabla G_4)_2 \qquad 3.f.13$$

Subsituting (3.f.12) → (3.f.13),

$$\iiint_V(-G_4\mathcal{E}^1\rho + G_4\varepsilon\mu\ddot{\phi} + \phi\{\delta\text{-set}\} - \phi\,\varepsilon\mu\,\ddot{G}_4)_3 = \oint\!\!\oint_{\partial V}(G_4\nabla\phi - \phi\nabla G_4)_2 \qquad 3.f.14$$

Now the proper member of the {δ-set} may be identified because the integral is completely prepared: since we have an R^3 volume integral, the {δ-set} element (which is the only element we consider, following premises 1 and 2) is

$$\delta(x_1-x_1')\delta(x_2-x_2')\delta(x_3-x_3') \qquad 3.f.15$$

At this point, it is a simple matter to find either the time-harmonic or static reduction of (3.f.14), resulting in the familiar Kirchhoff or Poisson formulas respectively. The important consideration is the generality of the method, applicable to any wave equation from the differential structure of electromagnetism.

Summary

The application of exterior differential forms to electromagnetism has a short history. Apparently the first developments occurred during the 1930's. MISNER and WHEELER (1957) discussed parallels between differential calculus and differential geometry (recall our discussion in §1.K) and described the charge-free R^3/t electromagnetic field using differential forms. In the notation of this thesis, they evolved the following structure:

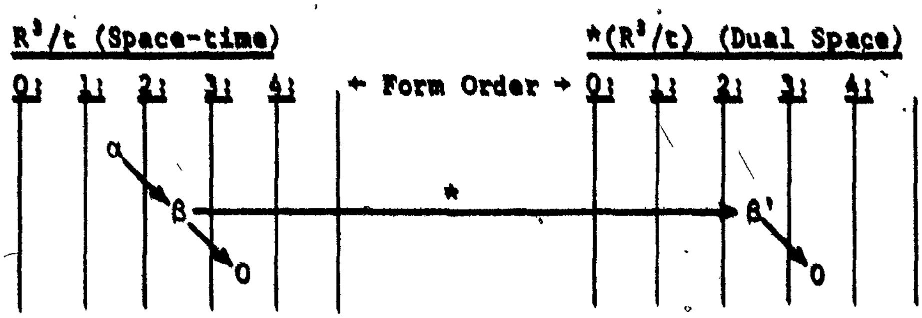

Figure 3.2 : Differential structure of MISNER and WHEELER (1957).

Within a few years, FLANDERS (1963) found that Maxwell's equations could be partially included in their generality, although it remained for DESCHAMPS (1970) to note that a modification to the * operator could account for the material characteristics ε and μ. Deschamps, as well as MISNER, THORNE and WHEELER (1970), introduced the use of diagrams for presenting differential structures. By 1970, the following structure had developed:

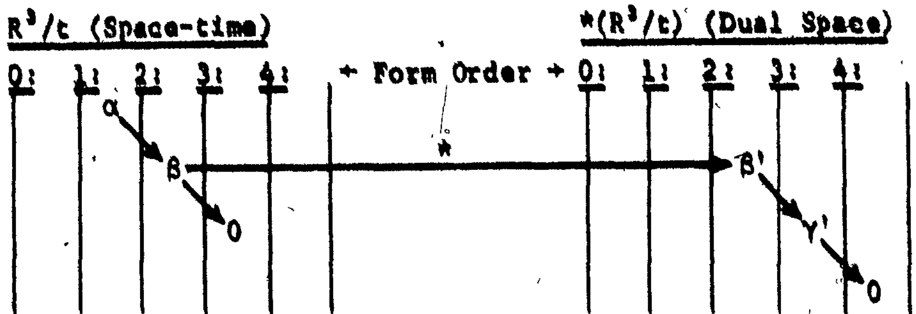

Figure 3.3 : Differential structure of DESCHAMPS (1970) and MISNER, THORNE and WHEELER (1970).

Figure 3.1 (in §3.C) presents the complete differential structure for electromagnetism with electric charge and current sources. In the following chapter, we will discuss the requirements for an electromagnetic field involving magnetic sources. Concerning charge-free electromagnetic fields, we have shown in §3.F that the global charge-free situation is a null situation.

OHKURO (1970) discusses the differential form structure of Maxwell's equations. He notes that there are certain problems with the dimensions of electromagnetic quantities, especially those which rely upon an integral for their definition. In §3.F, we discussed the matter of δ-functions in electromagnetism. The separation of the dimensions of the coefficient from the total dimension of a form seems to resolve this problem.

BALASUBRAMANIAN, LYNN and GUPTA (1970) also derive the Maxwell equations in differential forms, but their book is introductory and oriented towards teaching practical mathematical tools. FRANKEL (1974) makes the relationship between Maxwell's equations and differential forms a rigorous one, but he does not attempt to complete the structure of Figure 3.3. GAMBLIN (1969), using differential forms to study the "electrodynamics" of electric and magnetic charged particles, makes some interesting observations on the types of interactions available to magnetic monopoles, and how they are not the interactions ordinarily expected.

STUCK (1974) introduces two valuable concepts. First, he includes the media characteristics ϵ and μ in a matrix separate from a purely algebraic * operation, and this allows him to deal with anisotropic media. He also introduces the concept of commutativity in the differential structure, where the reference is to the equivalence of parallel sequences of operations. In Chapter 4, we will find that electric-source electromagnetism has a fundamentally different commutative behavior from that of magnetic-source electromagnetism.

CHAPTER IV

ANALYTICAL DEVELOPMENTS APPLIED TO ELECTROMAGNETISM

In the application of exterior differential algebra to electromagnetism, we have found that a suitably dimensioned differential structure in R^3/t contains all of the fundamental partial differential equations of electromagnetic field theory. However, many of the important relations in electromagnetism are product formulas; for example, the Lorentz force law and the Poynting theorem. Indeed, formulas of this type express the electrodynamic interactions whose study founded electromagnetic field theory in the first place. We intend in this chapter to continue the analysis of electromagnetism as it is expressed in differential forms, stressing the analysis of exterior product relations. In particular, we will deal with the Lorentz force law, the Poynting theorem, the reciprocity theorem and other relations involving force, energy and power. We shall also investigate several other important subjects: inner product relations in electromagnetism, the integration formulas available to the electromagnetic product forms, and the commutativity of operational sequences in the R^3/t differential structure. We will demonstrate that there is a fundamental difference between electric source and magnetic source electromagnetism.

A. Commutative Properties of the Electromagnetic Differential Structure

Let us derive a number of relations involving the * operator. From Table 3.4, we find that for a p-form ω,

$$**\omega_p = (-)^{p+1}\frac{\varepsilon}{\mu}\ \omega_p$$

$$*^{-1}\omega_p = (-)^{p+1}\frac{\mu}{\varepsilon} *\omega_p \qquad 4.a.1$$

Using (4.a.1), we can show that for 2 differential forms α (order p_α)

and β (order p_β),

$$**(\alpha \wedge \beta) = (-)^{p_\beta} (**\alpha) \wedge \beta$$
$$= (-)^{p_\alpha} \alpha \wedge (**\beta) \qquad 4.a.2$$

In particular, for the fundamental electromagnetic field 2-form β,

$$**(\beta \wedge \beta) = \beta \wedge (**\beta) = *\beta \wedge *\beta = \beta' \wedge \beta' \qquad 4.a.3$$

These relations are specific to R^3/t, and are not necessarily valid in other metric spaces. The following formulas, however, are true in general.

Using the standard expression for the codifferential operator (1.f.2), we can show

$$\delta * \omega_p = (-)^{n+p-1} * d\omega_p$$
$$* \delta \omega_p = (-)^p d * \omega_p \qquad 4.a.4$$

From these, the following formulas are quickly proven:

$$**d\omega_p = -d**\omega_p$$
$$**\delta\omega_p = -\delta**\omega_p$$
$$*\Delta\omega_p = \Delta*\omega_p \qquad 4.a.5$$
$$**\Delta\omega_p = \Delta**\omega_p$$

We can now analyze the R^3/t and $*(R^3/t)$ structural diagrams for α, β and γ (Figure 3.1). Relations (4.a.4) and (4.a.5) point out that the differential structure of the electromagnetic forms has + −commutative behavior with respect to the operations $*$, $*^{-1}$, d, δ and Δ, because the differentials which in general would not share the + −commutative behavior are zero:

$$
\begin{aligned}
{}^*d\alpha &= \delta{}^*\alpha \rightarrow d\alpha,\ \delta\alpha' \neq 0\\
{}^*\delta\beta &= d{}^*\beta \rightarrow \delta\beta,\ d\beta' \neq 0\\
{}^*d\gamma &= \delta{}^*\gamma \rightarrow d\gamma,\ \delta\gamma' \neq 0\\
\\
{}^*\delta\alpha &= -d{}^*\alpha \rightarrow \delta\alpha,\ d\alpha' = 0\\
{}^*d\beta &= -\delta{}^*\beta \rightarrow d\beta,\ \delta\beta' = 0\\
{}^*\delta\gamma &= -d{}^*\gamma \rightarrow \delta\gamma,\ d\gamma' = 0
\end{aligned}
\qquad 4.a.6
$$

In the figure below, + -commutative behavior means that the operational sequences specified by any two paths are identical:

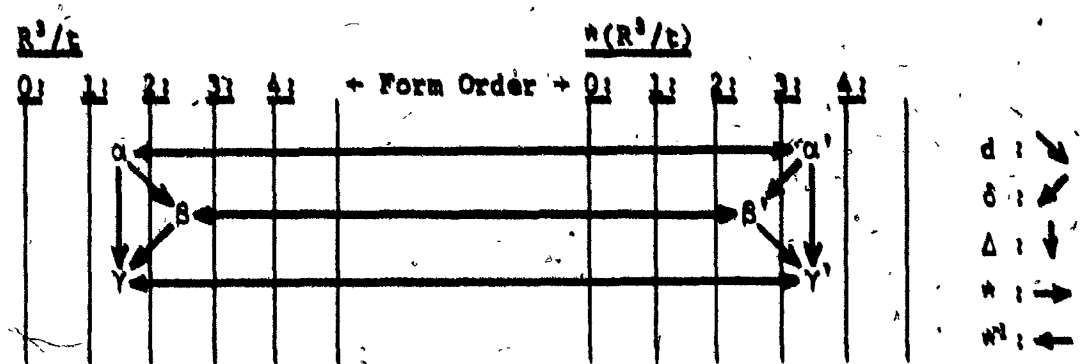

Figure 4.1 : Commutation diagram for the inhomogeneous electromagnetic relations. The commutativity is positive.

The implication that the + -commutative pattern is satisfied only when certain differentials are identically zero means for α and β:

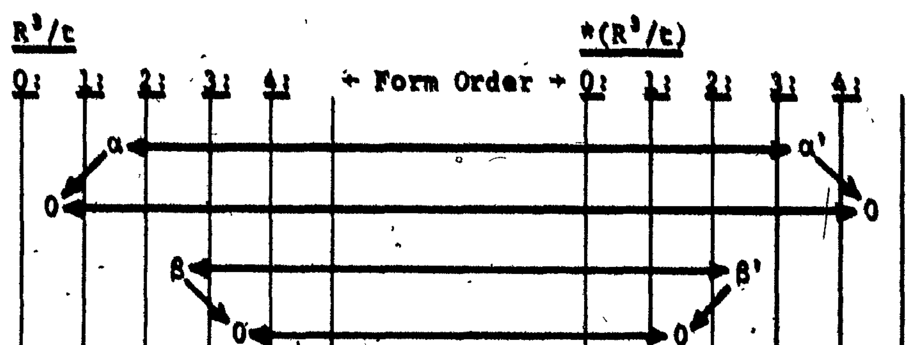

Figure 4.2 : Commutation diagram for the homogeneous relations.

Consequently, we may describe the electromagnetic differential structure as a set of differential forms in the metric space R^3/t, which, together with its derivatives, has a + -commutative behavior under the * operation. A fundamental implication of (4.a.4) is that a similar differential structure consisting of the complementary R^3/t "magnetic source" differential forms

a = Potential 3-Form
b = Field 2-Form   4.a.7
c = Magnetic Charge/Current 3-Form

would have to have - -commutative behavior under the * operation:

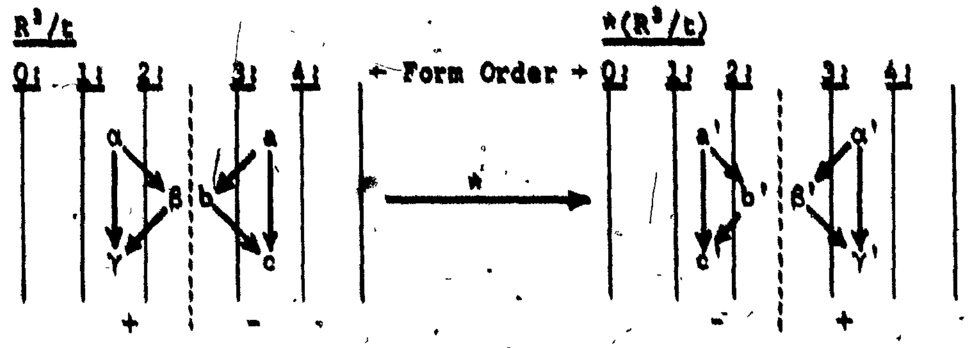

Figure 4.3 : The electric-source structure has + -commutativity under *; the magnetic-source structure has - -commutativity.

We see that both differential structures share the common ground of the 2-form electromagnetic field, but have opposite commutative behavior. Although this difference in commutativity points out a mathematical dissimilarity between the electromagnetism of electric and magnetic sources, we cannot be certain about its deeper significance without a study of the geometric relationships involved in the * operator, and their physical implications. In §4.G, we will briefly analyze the R^3/t differential structure from the point of view of rotation and inversion symmetry, but our discussion will be limited to the energy and power relations.

B. Direct Products of Electromagnetic Differential Forms

Let us consider two distinct electromagnetic differential structures on a given R^3/t manifold (We want the behavior of ε and μ to be the same for both situations). Exterior multiplication of the fundamental forms α, β and γ and their duals determines three categories of direct products, arranged in accordance with the product manifold that is involved:

<u>1. Products in $(R^3/t) \wedge (R^3/t)$:</u>

$\alpha_A \wedge \alpha_B$; $\alpha_A \wedge \beta_B$; $\alpha_A \wedge \gamma_B$; $\beta_A \wedge \beta_B$; $\beta_A \wedge \gamma_B$; $\gamma_A \wedge \gamma_B$

<u>2. Products in $(R^3/t) \wedge *(R^3/t)$:</u>

$\alpha_A \wedge \alpha_B'$; $\alpha_A \wedge \beta_B'$; $\alpha_A \wedge \gamma_B'$; $\beta_A \wedge \beta_B'$; $\gamma_A \wedge \alpha_B'$; $\gamma_A \wedge \beta_B'$; $\gamma_A \wedge \gamma_B'$

<u>3. Products in $*(R^3/t) \wedge *(R^3/t)$:</u>

$\beta_A' \wedge \beta_B'$ 4.b.1

The subscripts A and B refer to the forms from the two differential structures. Note that all of these products satisfy the standard exterior product relations. In particular,

$$\alpha_A \wedge \gamma_B' = \gamma_B \wedge \alpha_A' \qquad (2.b.2)$$
$$\alpha_A \wedge \alpha_B = -\alpha_B \wedge \alpha_A \qquad (1.b.1c)$$
$$\beta_A \wedge \beta_B' = \beta_B' \wedge \beta_A = \beta_B \wedge \beta_A' = \beta_A' \wedge \beta_B \qquad (1.b.1c;\ 2.b.2)$$
$$\beta_A' \wedge \beta_B' = **(\beta_A \wedge \beta_B) = -\frac{\varepsilon}{\mu}(\beta_A \wedge \beta_B) \qquad (2.b.2;\ 4.a.2)$$
4.b.2

When we discuss products of the forms from a single differential structure, we have

$$\alpha \wedge \alpha = 0 \qquad (1.b.1c)$$
$$\gamma \wedge \gamma = 0 \qquad (1.b.1c)$$
4.b.3

$$\alpha \wedge \gamma' = \gamma \wedge \alpha' \qquad (2.b.2)$$

$$\beta' \wedge \beta' = **(\beta \wedge \beta) = -\frac{\varepsilon}{\mu}(\beta \wedge \beta) \qquad (4.a.3) \qquad 4.b.3$$

For the electromagnetic forms from the same structure, there are eleven non-zero direct products:

DIRECT PRODUCTS OF THE ELECTROMAGNETIC DIFFERENTIAL FORMS TABLE 4.1

$(R^3/t) \wedge (R^3/t)$			
1.	$\alpha \wedge \beta$	$= \frac{1}{c}(A \times E)_2 dx^4 + (A \cdot B)_3 - \frac{1}{c}\phi B_2 dx^4$	$[\varepsilon^{-1}\ m\ l]$
2.	$\alpha \wedge \gamma$	$= \mu(A \times J)_2 - \frac{\varepsilon^{-1}}{c}\rho A_1 dx^4 + \frac{\mu}{c}\phi J_1 dx^4$	$[\varepsilon^{-1}\ m\ l^{-1}]$
3.	$\beta \wedge \beta$	$= \frac{2}{c}(E \cdot B)_3 dx^4$	$[\varepsilon^{-1}\ m\ l]$
4.	$\gamma \wedge \beta$	$= \frac{\mu}{c}(J \times E)_2 dx^4 + \mu(J \cdot B)_3 - \frac{\varepsilon^{-1}}{c}\rho B_2 dx^4$	$[\varepsilon^{-1}\ m\ l^{-1}]$
$(R^3/t) \wedge *(R^3/t)$			
5.	$\alpha \wedge \alpha'$	$= -\frac{\mu^{-1}}{c}(A \cdot A)_3 dx^4 + \frac{\varepsilon}{c}\phi^2 dx^{1234}$	$[m\ l^4\ t^{-1}]$
6.	$\alpha \wedge \beta'$	$= \varepsilon(A \cdot E)_3 - \frac{\mu^{-1}}{c}(A \times B)_2 dx^4 - \frac{\varepsilon}{c}\phi E_2 dx^4$	$[m\ l^2\ t^{-1}]$
7.	$\alpha \wedge \gamma' = \gamma \wedge \alpha'$	$= -\frac{1}{c}(A \cdot J)_3 dx^4 + \frac{1}{c}\rho\phi\, dx^{1234}$	$[m\ l^2\ t^{-1}]$
8.	$\beta \wedge \beta'$	$= \frac{\varepsilon}{c}(E \cdot E)_3 dx^4 - \frac{\mu^{-1}}{c}(B \cdot B)_3 dx^4$	$[m\ l^2\ t^{-1}]$
9.	$\gamma \wedge \beta'$	$= \mu\varepsilon(J \cdot E)_3 - \frac{1}{c}(J \times B)_2 dx^4 - \frac{1}{c}\rho E_2 dx^4$	$[m\ t^{-1}]$
10.	$\gamma \wedge \gamma'$	$= -\frac{\mu}{c}(J \cdot J)_3 dx^4 + \frac{\varepsilon^{-1}}{c}\rho^2 dx^{1234}$	$[m\ t^{-1}]$
$*(R^3/t) \wedge *(R^3/t)$			
11.	$\beta' \wedge \beta'$	$= -\frac{2\varepsilon}{\mu c}(E \cdot B)_3 dx^4$	$[\varepsilon\ m\ l^3\ t^{-2}]$

By considering the total dimensions of these quantities, we can infer a structural order for them. Let us first recall Figure 3.1, the structural diagram for α, β and γ:

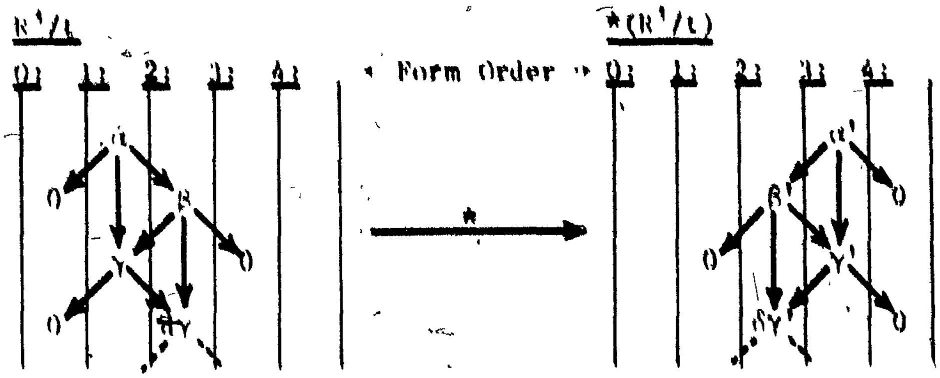

Figure 4.4 : The electromagnetic structural diagram.

The total dimensions of the forms in this structure are indicated below in a pattern directly corresponding with the above diagram:

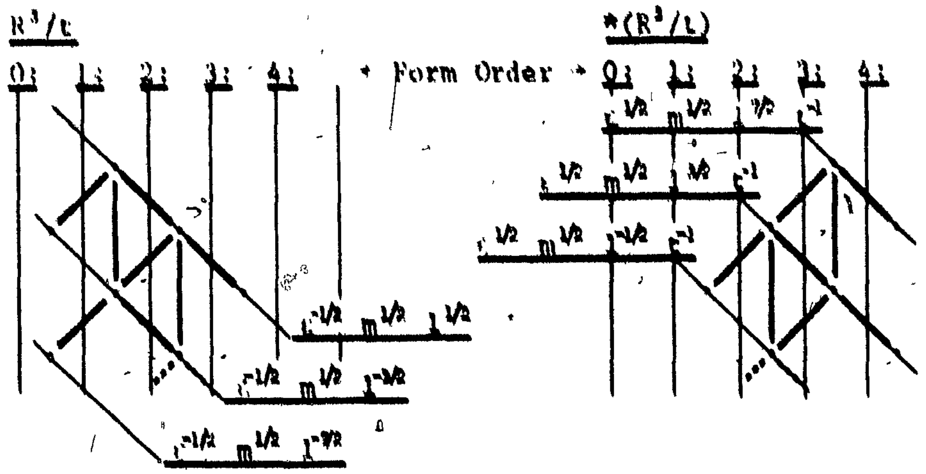

Figure 4.5 : Dimensional nature of the electromagnetic structural diagram. The dimension along a diagonal remains constant.

Along the indicated diagonals in Figure 4.5, the total dimension of the various differential forms is the same. Using (3.b.5), the dimensional characteristic of the * operation on p-forms, we can easily see the correspondence in terms of dimensions between a form in R^3/t and its dual in $*(R^3/t)$.

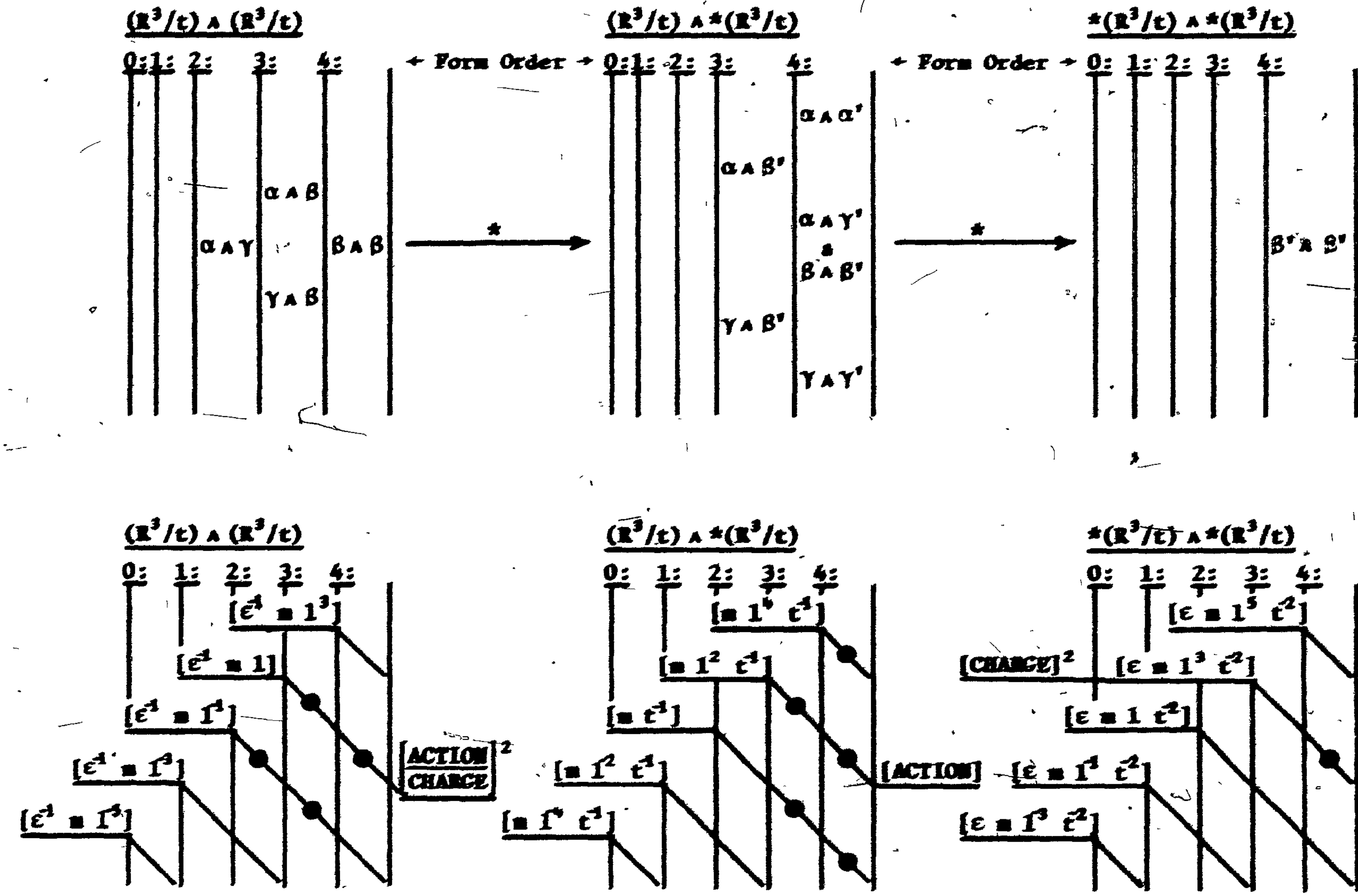

Figure 4.6 : Dimensional character of the R^3/t product spaces, and the ordering of the electromagnetic direct products in these spaces.

Now we will analyze the dimensional order of the product forms. The direct products listed in (4.b.1) and Table 4.1 have been divided into three categories: those in the product space $(R^3/t) \wedge (R^3/t)$, those in $(R^3/t) \wedge *(R^3/t)$, and those in $*(R^3/t) \wedge *(R^3/t)$. Since the total dimension of a product is the product of the total dimensions of the individual forms involved, each product quantity in Table 4.1 can be placed at a particular position in one of the dimensioned multilinear spaces shown in Figure 4.6. This position corresponds to its total dimension as determined by the above statement (Note that we are not developing any interrelationships among the product quantities at this time).

One of the interesting dimensional relationships that follows from (4.a.1) and the total dimensions listed in Figure 4.6 is

$$\left[\frac{\varepsilon}{\mu}\right] = \left[\frac{CHARGE^4}{ACTION^2}\right] \qquad 4.b.4$$

Referring to (3.b.5), we see that the product spaces are consistent with respect to the * operation in the sense that

$$(R^3/t) \wedge (R^3/t) \xrightarrow{*} (R^3/t) \wedge *(R^3/t) \xrightarrow{*} *(R^3/t) \wedge *(R^3/t) \qquad 4.b.5$$

In other words, any structure that may exist in one of the product spaces will have dual behavior in the others. An important observation is that there is a space of purely "mechanical" dimensional character where the dimensions do not involve the electromagnetic media quantities ε and μ. This is the product space $(R^3/t) \wedge *(R^3/t)$.

C. Interpretation of Product Forms

The total dimension of a product form is easy to calculate when the total dimensions of the component forms are known. A systematic dimensional analysis of the product form will then yield the coefficient dimensions for each term. We shall now proceed with this type of analysis for the products in Table 4.1 that involve β and γ, both of which are observable electromagnetic quantities.

In the covariant formulation of electromagnetism, two field quantities are invariant under the Poincaré group of transformations (including translations, spacelike rotations, and timelike rotations or Lorentz transformations). MISNER, THORNE and WHEELER (§4.3), using tensor notation for the electromagnetic field, write them as follows:

$$\tfrac{1}{4} E_{\alpha\beta}\, {}^{*}F^{\alpha\beta} = \mathbf{E}\cdot\mathbf{B}$$

$$\tfrac{1}{2} E_{\alpha\beta}\, F^{\alpha\beta} = \mathbf{B}^2 - \mathbf{E}^2 \qquad 4.c.1$$

We are concerned not with the tensor development of electromagnetism, but with the fact that the two invariants are substantially the product forms of the electromagnetic field, $\beta\wedge\beta$ and $\beta\wedge\beta'$:

$$\beta\wedge\beta = \frac{2}{c}(\mathbf{E}\cdot\mathbf{B})_{,}dx^4 \qquad [t^{-1} m\, 1]$$

$$\beta\wedge\beta' = \frac{\varepsilon}{c}(\mathbf{E}\cdot\mathbf{E})_{,}dx^4 - \frac{1}{\mu c}(\mathbf{B}\cdot\mathbf{B})_{,}dx^4 \qquad [m\, 1^2\, t^{-1}] \qquad 4.c.2$$

Let us compute the coefficient dimensions for the two product forms in (4.d.2). The 4-form differential term dx^{1234} has the dimension $[1^4]$. Therefore,

$$\left[\frac{1}{c}(\mathbf{D}\cdot\mathbf{E} - \mathbf{B}\cdot\mathbf{H})\right] = [m\, 1^2\, t^{-1}\cdot 1^{-4}] = \left[\frac{\text{ACTION}}{1^4}\right] \text{ or } \left[\frac{\text{MOMENTUM}}{1^3}\right]$$

$$[(\mathbf{D}\cdot\mathbf{E}) - (\mathbf{B}\cdot\mathbf{H})] = [1\, t^{-1}\cdot m\, 1^2\, t^{-1}\cdot 1^{-4}] = \left[\frac{m\, 1^3\, t^{-2}}{1^4}\right] \text{ or } \left[\frac{\text{ENERGY}}{1^3}\right] \qquad 4.c.3$$

The possibility $[\text{MOMENTUM}/1^3]$ does not seem reasonable because the coefficient quantity is a scalar (in the R^3 sense), and momentum itself is a vector quantity. Continuing,

$$\left[\frac{2\varepsilon^{-1}}{c}(\mathbf{D}\cdot\mathbf{B})\right] = [\varepsilon^{-1}\, m\, l\cdot l^{-4}] \qquad \text{4.c.4a}$$

Therefore,

$$\left[\frac{2}{c}(\mathbf{D}\cdot\mathbf{B})\right] = [m\, l\cdot l^{-4}] = \left[\frac{m\, l}{l^4}\right] \text{ or } \left[\frac{m}{l^3}\right] \qquad \text{4.c.4b}$$

$$[2\,(\mathbf{D}\cdot\mathbf{B})] = [l\, t^{-1}\cdot m\, l\cdot l^{-4}] = \left[\frac{\text{ACTION}}{l^4}\right] \text{ or } \left[\frac{\text{MOMENTUM}}{l^3}\right]$$

Again, we criticize the possibility [MOMENTUM/l^3] because of the scalar character of the coefficient quantity. The dimension [m/l^3] (a mass density) for $\frac{2}{c}(\mathbf{D}\cdot\mathbf{B})$ indicates that this is a mass equivalent for an electromagnetic field.

$\gamma\wedge\beta'$ is a product 3-form of particular interest:

$$\gamma\wedge\beta' = \mu\varepsilon\,(\mathbf{J}\cdot\mathbf{E})_3 - \frac{1}{c}(\mathbf{J}\times\mathbf{B})_2 dx^4 - \frac{1}{c}\rho\,\mathbf{E}_2 dx^4 \qquad [m\, t^{-1}] \qquad \text{4.c.5}$$

Defining the coefficient quantities

$$W = (\mathbf{J}\cdot\mathbf{E})$$
$$\mathbf{F} = \mathbf{J}\times\mathbf{B} - \rho\,\mathbf{E}, \qquad \text{4.c.6}$$

we inquire about their physical dimensions. Recall that $[dx^{123}] = [l^3] = [dx^{ij4}]$.

$$[\mu\varepsilon][W][dx^{123}] = [l^{-2}\, t^2][l^2\, t^{-2}\cdot m\, t^{-1}\cdot l^{-3}][l^3] \qquad \text{4.c.7}$$
$$[W] = [m\, l^{-1}\, t^{-3}] = \left[\frac{\text{POWER}}{l^3}\right]$$

In other words, W is the rate of electromagnetic energy dissipation.

$$\left[\frac{1}{c}\right][\mathbf{F}][dx^{ij4}] = [l^{-1}\, t][l\, t^{-1}\cdot m\, t^{-1}\cdot l^{-3}][l^3] \qquad \text{4.c.8}$$

$$[\mathbf{F}] = [m\, l^{-2}\, t^{-2}] = \left[\frac{\text{FORCE}}{l^3}\right] \qquad 4.c.8$$

F is the Lorentz force, which in this case is a "self-force" term involving the fields and charge/currents from the same differential structure.

In the context of force and power, it is interesting to speculate about the meaning of the global dimension $[m\, t^{-1}]$. Perhaps this is the mass rate of change equivalent to the electromagnetic energy dissipation.

The second product 3-form involving γ and β is

$$\gamma \wedge \beta = \frac{\mu}{c}(\mathbf{J} \times \mathbf{E})_2 dx^4 + \mu(\mathbf{J} \cdot \mathbf{B})_3 - \frac{\varepsilon^{-1}}{c}\rho \mathbf{B}_2 dx^4 \qquad [\varepsilon^{-1}\, m\, l^{-1}] \qquad 4.c.9$$

In an analysis similar to that for $\gamma \wedge \beta'$, we find

$$[\mathbf{J} \cdot \mathbf{B}] = \left[\frac{m\, l\, t^{-2}}{l^3}\right]$$

$$[\mathbf{J} \times \mathbf{E}\, ;\, \frac{1}{\varepsilon\mu}\rho \mathbf{B}] = \left[\frac{m\, l^2\, t^{-3}}{l^3}\right] \qquad 4.c.10$$

For the 4-form $\gamma \wedge \gamma'$,

$$\gamma \wedge \gamma' = -\frac{\mu}{c}(\mathbf{J} \cdot \mathbf{J})_3 dx^4 + \frac{\varepsilon^{-1}}{c}\rho^2 dx^{1234} \qquad [m\, t^{-1}] \qquad 4.c.11$$

we find for the coefficients the dimensions

$$[\mu(\mathbf{J} \cdot \mathbf{J})\, ;\, \frac{1}{\varepsilon}\rho^2] = \left[\frac{m\, l\, t^{-2}}{l^4}\right] \text{ or } \left[\frac{m\, t^{-2}}{l^3}\right] \qquad 4.c.12$$

By these techniques, we can find the coefficient dimensions of any differential form whose total dimension is known.

D. Product Space Structure I: Differentials and Codifferentials

Having established in §4.B that there is a positional order for the dimensioned differential forms in the multilinear product spaces, we now investigate the possibility of a structural interrelationship for these product forms similar to the R^3/t structure for α, β and γ. This will involve an investigation of the derivative and * operations on the products listed in (4.b.1).

In the R^3/t product spaces, the derivative can be computed for any form whose order is ≤ 3 by (1.e.3). The following derivatives are therefore available:

R^3/t PRODUCT DIFFERENTIALS — TABLE 4.2

1.	$d(\alpha_A \wedge \alpha_B)$	$= d\alpha_A \wedge \alpha_B - \alpha_A \wedge d\alpha_B$	$=$	$\beta_A \wedge \alpha_B - \alpha_A \wedge \beta_B$
2.	$d(\alpha_A \wedge \beta_B)$	$= d\alpha_A \wedge \beta_B - \alpha_A \wedge d\beta_B^{*}$	$=$	$\beta_A \wedge \beta_B$
3.	$d(\alpha_A \wedge \gamma_B)$	$= d\alpha_A \wedge \gamma_B - \alpha_A \wedge d\gamma_B$	$=$	$\beta_A \wedge \gamma_B - \alpha_A \wedge \Delta\beta_B$
4.	$d(\gamma_A \wedge \beta_B)$	$= d\gamma_A \wedge \beta_B - \gamma_A \wedge d\beta_B^{*}$	$=$	$\Delta\beta_A \wedge \beta_B$
5.	$d(\gamma_A \wedge \gamma_B)$	$= d\gamma_A \wedge \gamma_B - \gamma_A \wedge d\gamma_B$	$=$	$\Delta\beta_A \wedge \gamma_B - \gamma_A \wedge \Delta\beta_B$
6.	$d(\alpha_A \wedge \beta_B')$	$= d\alpha_A \wedge \beta_B' - \alpha_A \wedge d\beta_B'$	$=$	$\beta_A \wedge \beta_B' - \alpha_A \wedge \gamma_B'$
7.	$d(\gamma_A \wedge \beta_B')$	$= d\gamma_A \wedge \beta_B' - \gamma_A \wedge d\beta_B'$	$=$	$\Delta\beta_A \wedge \beta_B' - \gamma_A \wedge \gamma_B'$

Note that when both forms are from the same structure ($A = B$), relations 1 and 5 are identically zero on both the right-hand side and left-hand side.

Calculating the codifferentials is not as simple. However, for products of R^3/t differential forms, the following formula has been found to work:

$$\delta(\omega \wedge \eta) = \delta\omega \wedge \eta + (-)^{p_\omega}\, \omega \wedge \delta\eta + (*^{-1}(*\omega \wedge d)) \wedge \eta + (-)^{p_\omega p_\eta}\, (*^{-1}(*\eta \wedge d)) \wedge \omega \qquad 4.d.1$$

(Recall the similar R^3 codifferential formula in §1.F). The codifferential may be applied to all but 0-forms, so it may be applied to each type of R^3/t product:

R³/t PRODUCT CODIFFERENTIALS — TABLE 4.3

1.	$\delta(\alpha_A \wedge \beta_B)$	=	$\delta\alpha_A \wedge \beta_B$	$- \alpha_A \wedge \delta\beta_B$	+	$(*^{-1}(*\alpha_A \wedge d)) \wedge \beta_B$	+	$(*^{-1}(*\beta_B \wedge d)) \wedge \alpha_A$
		=		$- \alpha_A \wedge \gamma_B$	+		+	
2.	$\delta(\alpha_A \wedge \gamma_B)$	=	$\delta\alpha_A \wedge \gamma_B$	$- \alpha_A \wedge \delta\gamma_B$	+	$(*^{-1}(*\alpha_A \wedge d)) \wedge \gamma_B$	−	$(*^{-1}(*\gamma_B \wedge d)) \wedge \alpha_A$
		=			+		−	
3.	$\delta(\beta_A \wedge \beta_B)$	=	$\delta\beta_A \wedge \beta_B$	$+ \beta_A \wedge \delta\beta_B$	+	$(*^{-1}(*\beta_A \wedge d)) \wedge \beta_B$	+	$(*^{-1}(*\beta_B \wedge d)) \wedge \beta_A$
		=	$\gamma_A \wedge \beta_B$	$+ \beta_A \wedge \gamma_B$	+		+	
4.	$\delta(\gamma_A \wedge \beta_B)$	=	$\delta\gamma_A \wedge \beta_B$	$- \gamma_A \wedge \delta\beta_B$	+	$(*^{-1}(*\gamma_A \wedge d)) \wedge \beta_B$	+	$(*^{-1}(*\beta_B \wedge d)) \wedge \gamma_A$
		=		$- \gamma_A \wedge \gamma_B$	+		+	
5.	$\delta(\alpha_A \wedge \alpha'_B)$	=	$\delta\alpha_A \wedge \alpha'_B$	$- \alpha_A \wedge \delta\alpha'_B$	+	$(*^{-1}(*\alpha_A \wedge d)) \wedge \alpha'_B$	−	$(*^{-1}(*\alpha'_B \wedge d)) \wedge \alpha_A$
		=		$- \alpha_A \wedge \beta'_B$	+		−	
6.	$\delta(\alpha_A \wedge \beta'_B)$	=	$\delta\alpha_A \wedge \beta'_B$	$- \alpha_A \wedge \delta\beta'_B$	+	$(*^{-1}(*\alpha_A \wedge d)) \wedge \beta'_B$	+	$(*^{-1}(*\beta'_B \wedge d)) \wedge \alpha_A$
		=			+		+	
7.	$\delta(\alpha_A \wedge \gamma'_B)$	=	$\delta\alpha_A \wedge \gamma'_B$	$- \alpha_A \wedge \delta\gamma'_B$	+	$(*^{-1}(*\alpha_A \wedge d)) \wedge \gamma'_B$	−	$(*^{-1}(*\gamma'_B \wedge d)) \wedge \alpha_A$
		=		$- \alpha_A \wedge \Delta\beta'_B$	+		−	
	$\delta(\gamma_A \wedge \alpha'_B)$	=	$\delta\gamma_A \wedge \alpha'_B$	$- \gamma_A \wedge \delta\alpha'_B$	+	$(*^{-1}(*\gamma_A \wedge d)) \wedge \alpha'_B$	−	$(*^{-1}(*\alpha'_B \wedge d)) \wedge \gamma_A$
		=		$- \gamma_A \wedge \beta'_B$	+		−	
8.	$\delta(\beta_A \wedge \beta'_B)$	=	$\delta\beta_A \wedge \beta'_B$	$+ \beta_A \wedge \delta\beta'_B$	+	$(*^{-1}(*\beta_A \wedge d)) \wedge \beta'_B$	+	$(*^{-1}(*\beta'_B \wedge d)) \wedge \beta_A$
		=	$\gamma_A \wedge \beta'_B$		+		+	
9.	$\delta(\gamma_A \wedge \beta'_B)$	=	$\delta\gamma_A \wedge \beta'_B$	$- \gamma_A \wedge \delta\beta'_B$	+	$(*^{-1}(*\gamma_A \wedge d)) \wedge \beta'_B$	+	$(*^{-1}(*\beta'_B \wedge d)) \wedge \gamma_A$
		=			+		+	
10.	$\delta(\gamma_A \wedge \gamma'_B)$	=	$\delta\gamma_A \wedge \gamma'_B$	$- \gamma_A \wedge \delta\gamma'_B$	+	$(*^{-1}(*\gamma_A \wedge d)) \wedge \gamma'_B$	−	$(*^{-1}(*\gamma'_B \wedge d)) \wedge \gamma_A$
		=		$- \gamma_A \wedge \Delta\beta'_B$	+		−	
11.	$\delta(\beta'_A \wedge \beta'_B)$	=	$\delta\beta'_A \wedge \beta'_B$	$+ \beta'_A \wedge \delta\beta'_B$	+	$(*^{-1}(*\beta'_A \wedge d)) \wedge \beta'_B$	+	$(*^{-1}(*\beta'_B \wedge d)) \wedge \beta'_A$
		=			+		+	

Note that when both forms are from the same structure,

$$3. \rightarrow \delta(\beta \wedge \beta) = 2\gamma \wedge \beta + 2(*^{-1}(*\beta \wedge d)) \wedge \beta$$

$$4. \rightarrow \delta(\gamma \wedge \beta) = (*^{-1}(*\gamma \wedge d)) \wedge \beta + (*^{-1}(*\beta \wedge d)) \wedge \gamma \qquad 4.d.2$$

$$11. \rightarrow \delta(\beta' \wedge \beta') = 2(*^{-1}(*\beta' \wedge d)) \wedge \beta'$$

By studying these differentials and codifferentials, we find that the product forms in (4.b.1) are interrelated, but the relationships are more complicated than those for α, β and γ. In the next section, we will examine several situations in which the complexity of some of these relations is reduced.

E. Product Space Structure II: Special Circumstances

1. Algebraic decompositions of differential forms

When the quantities of a differential form are precisely specified, it may be possible to affect a decomposition as follows:

$$\omega_p = \alpha_q \wedge \beta_{p-q} \qquad 4.e.1$$

Here the p-form ω has been transformed into a product of a q-form and a (p-q)-form, where $q < p$. It is clear that there exists a trivial case where $q = 0$, $\alpha = k$ (a real number), and $\beta = (1/k)\omega_p$. Recalling that for any 1-form α,

$$\alpha \wedge \alpha = 0 \qquad 4.e.2$$

if the electromagnetic p-forms can be decomposed by (4.e.1) to include a 1-form in the product, a number of the product relationships of §4.D will be simplified. For example, suppose the 2-form β is "simple", that is, it can be thought of as a product $\beta = \alpha_A \wedge \alpha_B$, where the α's are 1-forms. Then

$$\beta \wedge \beta = 0 \qquad [(1.b.2);(1.b.1c)] \qquad 4.e.3a$$

If $\beta_A = \alpha_I \wedge \alpha_J$ and $\beta_B = \alpha_I \wedge \alpha_K$, then

$$\beta_A \wedge \beta_B = 0 \qquad [(1.b.2);(1.b.1c)] \qquad 4.e.3b$$

We must note that the operation d, which is <u>similar</u> to a 1-form, does not satisfy (1.b.1c). Otherwise, $\beta = d\alpha$, and $\beta \wedge \beta = (d \wedge \alpha) \wedge (d \wedge \alpha) = 0$!

From Tables 4.2 and 4.3, it is easy to see that if β is simple, many of the derivative relations for the product forms are reduced in complexity. However, the electromagnetic field situation is a special one. The conditions of the above example imply that $\mathbf{E}$ and $\mathbf{B}$ are perpendicular, since by Equation 3, Table 4.1,

$$\beta \wedge \beta = 0 \rightarrow \mathbf{E} \cdot \mathbf{B} = 0 \qquad 4.e.4$$

ii. Eigenvalue situations for the electromagnetic differential structure

If we have a differential structure where there is a characteristic solution to the Laplace-Beltrami operation, the product space relations are greatly simplified (See WARNER, §6, Exercise 16, for a discussion of the eigenvalues of the Laplacian). For example, let

$$\Delta\alpha = k\alpha = \gamma \qquad 4.e.5$$

where k is a dimensioned characteristic value (Recall (3.b.7)). Then

$$\begin{aligned} \alpha \wedge \alpha &= 0 \\ \alpha \wedge \gamma &= \alpha \wedge k\alpha = k(\alpha \wedge \alpha) = 0 \\ \gamma \wedge \gamma &= 0 \end{aligned} \qquad 4.e.6a$$

$$\Delta\beta = \Delta d\alpha = d\Delta\alpha = k\beta \qquad 4.e.6b$$

$$\begin{aligned} \gamma \wedge \gamma' &= k^2(\alpha \wedge \alpha') \\ \alpha \wedge \gamma' &= \gamma \wedge \alpha' = k(\alpha \wedge \alpha') \\ \gamma \wedge \beta' &= k(\alpha \wedge \beta') \\ \gamma \wedge \beta &= k(\alpha \wedge \beta) \end{aligned} \qquad 4.e.6c$$

$$\begin{aligned} d(\alpha \wedge \beta) &= \beta \wedge \beta \\ d(\gamma \wedge \beta) &= \Delta\beta \wedge \beta = k(\beta \wedge \beta) \end{aligned} \qquad 4.e.6d$$

For this eigenvalue problem, many of the product forms become simply related.

Now let us investigate the possibility of a true differential structure on the product space when $\Delta\beta = k\beta$. From Tables 4.2 and 4.3,

$$\Delta(\beta\wedge\beta) = 2(\Delta\beta\wedge\beta) + 2d[(*^{-1}(*\beta\wedge d))\wedge\beta] \qquad 4.e.7$$

Recalling the product Laplacian development in §1.G, we find

$$\Delta(\beta\wedge\beta) = 2(\Delta\beta\wedge\beta) - \frac{4}{c}\sum_{i=1}^{3}\sum_{j=1}^{3}\left\{\frac{\partial B_i}{\partial x^j}\frac{\partial E_i}{\partial x^j}\right\} + \frac{4\varepsilon\mu}{c}\sum_{i=1}^{3}\left\{\frac{\partial B_i}{\partial t}\frac{\partial E_i}{\partial t}\right\} \qquad 4.e.8$$

$$= 2k(\beta\wedge\beta) - \qquad '' \qquad + \qquad ''$$

The R^3/t eigenvalue problem $\Delta\beta = k\beta$ does not provide us with the information necessary to immediately solve the product space eigenvalue problem, because we cannot determine the characteristic solution for the summation terms in (4.e.8) with respect to the product $(\beta\wedge\beta)$ from it. This is the major problem in finding a true differential structure for the product space.

iii. Enforced + -commutative product space behavior

We found in §4.A that the fundamental electromagnetic forms α, β and γ had a + -commutative behavior with respect to the operations d, δ, Δ, $*$ and $*^{-1}$. We will now see the implications of assuming that such commutative behavior exists in the product spaces. Two possibilities arise. In the first, we presume the space $(R^3/t)\wedge(R^3/t)$ would behave as R^3/t, since for two closed forms ω and η ($d\omega = 0$, $d\eta = 0$), the product is closed:

$$d(\omega\wedge\eta) = d\omega\wedge\eta + (-)^{p}\,\omega\wedge d\eta = 0 \qquad 4.e.9$$

Then, $(R^3/t)\wedge *(R^3/t)$ would behave as $*(R^3/t)$ because of (4.b.5), and

$*(R^3/t) \wedge *(R^3/t)$ would behave as R^3/t again. This would mean that the following derivative relations are identically zero:

$$\begin{aligned} d(\alpha \wedge \gamma) &\equiv 0 \\ \delta(\alpha \wedge \beta) &\equiv 0 \\ \delta(\gamma \wedge \beta) &\equiv 0 \\ d(\alpha \wedge \beta') &\equiv 0 \\ d(\gamma \wedge \beta') &\equiv 0 \\ \delta(\alpha \wedge \alpha') &\equiv 0 \\ \delta(\alpha \wedge \gamma') &\equiv 0 \\ \delta(\gamma \wedge \alpha') &\equiv 0 \\ \delta(\beta \wedge \beta') &\equiv 0 \\ \delta(\gamma \wedge \gamma') &\equiv 0 \end{aligned} \qquad 4.e.10$$

This is evidently not meaningful, and we elaborate on two of the relations in (4.e.10) to explain why:

1. One of the electromagnetic Green's theorems, re-written from (3.f.5), is

$$d(\alpha_A \wedge \beta'_B - \alpha_B \wedge \beta'_A) = \alpha_B \wedge \gamma'_A - \alpha_A \wedge \gamma'_B \qquad 4.e.11$$

From line 4, (4.e.10), the left-hand side of (4.e.11) would be identically zero, making this Green's theorem inconsequential.

2. Expanding $\delta(\beta \wedge \beta') = 0$, we have

$$-\bar{\nabla}[(\mathbf{E}\cdot\mathbf{D}) - (\mathbf{B}\cdot\mathbf{H})]_2 dx^k - \frac{\varepsilon\mu}{c}\frac{\partial}{\partial t}[(\mathbf{E}\cdot\mathbf{D}) - (\mathbf{B}\cdot\mathbf{H})]_3 = 0 \qquad 4.e.12$$

Separating into space and time components,

$$\begin{aligned} \varepsilon\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{E}) &= \mu\frac{\partial}{\partial t}(\mathbf{H}\cdot\mathbf{H}) \\ \varepsilon\,\nabla(\mathbf{E}\cdot\mathbf{E}) &= \mu\,\nabla(\mathbf{H}\cdot\mathbf{H}) \end{aligned} \qquad 4.e.13$$

which can only be satisfied if $\mathbf{E}\cdot\mathbf{E}$ and $\mathbf{H}\cdot\mathbf{H}$ have identical space and time behavior, obviously not a very general situation. Therefore, it appears that the conjecture leading to (4.e.10) is not valid.

The other possibility involves the assumption that any electromagnetic direct product has + -commutative behavior with respect to the space of its definition. In other words, for an $(R^3/t) \wedge *(R^3/t)$ product, the space $(R^3/t) \wedge *(R^3/t)$ is considered to behave as R^3/t in terms of commutative character. We have the following relations:

$$\begin{aligned} d(\alpha \wedge \gamma) &= 0 \\ \delta(\alpha \wedge \beta) &= 0 \\ \delta(\gamma \wedge \beta) &= 0 \\ \delta(\alpha \wedge \beta') &= 0 \\ \delta(\gamma \wedge \beta') &= 0 \end{aligned} \qquad 4.e.14$$

One example convinces us that this is more realistic. Using the notation (4.c.6), we have for the product $\gamma \wedge \beta'$,

$$\delta(\gamma \wedge \beta') = 0 = -\frac{1}{c}(\nabla \times F)_1 dx^4 - \epsilon\mu(\nabla W)_2 - \epsilon\mu\left(\frac{\partial F}{\partial t}\right)_2 \qquad 4.e.15$$

Separating the space and time components,

$$\begin{aligned} \nabla \times F &= 0 \\ \nabla W &= -\frac{\partial F}{\partial t} \end{aligned} \qquad 4.e.16$$

The first equation in (4.e.16) indicates that the Lorentz force is conservative. The second is the time derivative of the work-force relation written as a differential equation (FEYNMAN, LEIGHTON and SANDS (1963), §14).

F. R^3/t Inner Products

The R^3/t inner product is defined on a closed manifold. Although such a manifold is beyond our physical perception, The discussion of §3.C and §3.F indicates that there is a certain correspondence with electromagnetic theory. In this section, we will discuss inner products in relation to the electromagnetic differential structure.

For the R^3/t p-forms defined on the closed manifold M,

$$(\omega,\eta) = \oint\!\!\oint\!\!\oint\!\!\oint_M \omega \wedge {*\eta} \qquad 4.f.1$$

Whenever the integrand is exact, the inner product is zero:

$$\oint\!\!\oint\!\!\oint\!\!\oint_M \omega \wedge {*\eta} = \oint\!\!\oint\!\!\oint\!\!\oint_M d\xi = \oint\!\!\oint\!\!\oint_{\partial M} \xi = 0 \qquad 4.f.2$$

A closed manifold has no boundary.

For inner products of electromagnetic differential forms, we must be careful not to compare the integrals based on different product spaces. For example,

$$\begin{aligned}(\alpha,\alpha) &= \oint\!\!\oint\!\!\oint\!\!\oint_{M_1}\left\{-\frac{1}{\mu c}(A\cdot A) + \frac{\varepsilon}{c}\phi^2\right\}dx^{1234} \geq 0\\ (\alpha',\alpha') &= \oint\!\!\oint\!\!\oint\!\!\oint_{M_2}\left\{\frac{\varepsilon}{\mu^2 c}(A\cdot A) - \frac{\varepsilon^2}{\mu c}\phi^2\right\}dx^{1234} \geq 0\end{aligned} \qquad 4.f.3$$

which appears to be a proof that $\varepsilon/\mu < 0$. However, the first integral is in $(R^3/t)\wedge *(R^3/t)$, while the second is in $*(R^3/t)\wedge **(R^3/t)$. By (4.a.1), we see

$$**(\alpha\wedge *\alpha) = (*\alpha \wedge **\alpha) = \alpha'\wedge *\alpha' \qquad 4.f.4$$

Therefore,

$$(\alpha',\alpha') = -\frac{\varepsilon}{\mu}(\alpha,\alpha) \qquad 4.f.5$$

In other words, the manifolds M_1 and M_2 make a difference.

The following is a list of the inner product relations satisfied by the differential forms in the electromagnetic structure:

$$
\begin{aligned}
&1.\ (\alpha,\gamma) = (\alpha,\delta\beta) = (d\alpha,\beta) = (\beta,\beta) \\
&2.\ (\alpha,\gamma) = (\alpha,\Delta\alpha) = (\Delta\alpha,\alpha) = (\gamma,\alpha) \\
&3.\ (\gamma,\gamma) = (\gamma,\delta\beta) = (d\gamma,\beta) = (\Delta\beta,\beta) \\
&4.\ (\beta,\beta') = (d\alpha,\delta\alpha') = (dd\alpha,\alpha') = 0 \\
&5.\ (\beta,\Delta\beta') = (\beta,d\delta\beta' + \delta d\beta') = (d\beta,d\beta') = 0
\end{aligned}
\tag{4.f.6}
$$

Note that relations 1, 3, 4 and 5 in (4.f.6) can be proven using (4.f.2), since these derivatives hold:

$$
\begin{aligned}
d(\alpha\wedge\beta') &= d\alpha\wedge\beta' - \alpha\wedge d\beta' = \beta\wedge\beta' - \alpha\wedge\gamma' \\
d(\gamma\wedge\beta') &= d\gamma\wedge\beta' - \gamma\wedge d\beta' = \Delta\beta\wedge\beta' - \gamma\wedge\gamma' \\
d(\alpha\wedge\beta) &= d\alpha\wedge\beta - \alpha\wedge d\beta = \beta\wedge\beta \\
d(\gamma\wedge\beta) &= d\gamma\wedge\beta - \gamma\wedge d\beta = \Delta\beta\wedge\beta
\end{aligned}
\tag{4.f.7}
$$

G. Relations for Force, Energy, and Power

In the list of direct products for the fundamental forms α, β and γ, we do not find the important product quantity $\mathbf{E}\times\mathbf{H}$. Certain vector products can be constructed from differential forms only by proceeding indirectly. Taking into account the R^3/t differential structure for α, β and γ, we write the following two product relations in $(R^3/t)\wedge *(R^3/t)$:

$$\beta\wedge\delta\beta' - \beta'\wedge\delta\beta = -\beta'\wedge\gamma \qquad [m\ t^{-1}] \tag{4.g.1a}$$

$$*^{-1}(\beta'\wedge\delta\beta') - *(\beta\wedge\delta\beta) = -*(\beta\wedge\gamma) \qquad [m\ l^{2}\ t^{-1}] \tag{4.g.1b}$$

It is also possible to form a similar pair of relations where all signs are +, but the same reduction of the components does not occur. Therefore, we will only consider the above pair. The total dimension given in

(4.g.1) is useful in identifying the physical character of all terms in the expanded versions of these formulas. Proceeding with expansion, we have

$$-\frac{1}{c}[D\times(\nabla\times E)]_2 dx^4 - \frac{\epsilon\mu}{c}[E\times\dot{H}]_2 dx^4 + \epsilon\mu[H\cdot(\nabla\times E)]_3 + \frac{1}{c}[H(\nabla\cdot B)]\,dx^4$$
$$+\ \epsilon\mu[H\cdot\dot{B}]_3 + \frac{1}{c}[E(\nabla\cdot D)]_2 dx^4 + \epsilon\mu[E\cdot\dot{D}]_3 - \epsilon\mu[E\cdot(\nabla\times H)]_3$$
$$+\frac{\epsilon\mu}{c}[H\times\dot{E}]_2 dx^4 - \frac{1}{c}[B\times(\nabla\times H)]_2 dx^4$$
$$= -\ \epsilon\mu(J\cdot E)_3 + \frac{1}{c}(J\times B)_2 dx^4 + \frac{1}{c}\rho E_2 dx^4$$

4.g.2a

$$-\frac{1}{c}[D\cdot(\nabla\times E)]dx^4 - \frac{1}{c}[D\cdot\dot{B}]dx^4 - [D(\nabla\cdot B)]_1 - \epsilon\mu[H\times(\nabla\times E)]_1$$
$$-\ \epsilon\mu[H\times\dot{B}]_1 - \frac{1}{c}[B\cdot\dot{D}]dx^4 + \frac{1}{c}[B\cdot(\nabla\times H)]dx^4 + \epsilon\mu[E\times\dot{D}]_1$$
$$-\ \epsilon\mu[E\times(\nabla\times H)]_1 - [B(\nabla\cdot D)]_1$$
$$=\ \epsilon\mu(J\times E)_1 - \rho B_1 + \frac{1}{c}(J\cdot B)dx^4$$

4.g.2b

Separating the space and time components of (4.g.2a) gives us the following two formulas:

$$\text{Space} \rightarrow\ \epsilon\mu[H\cdot(\nabla\times E)]_3 + \epsilon\mu[H\cdot\dot{B}]_3 + \epsilon\mu[E\cdot\dot{D}]_3 - \epsilon\mu[E\cdot(\nabla\times H)]_3 + \epsilon\mu(J\cdot E)_3 = 0$$

$$\text{Time} \rightarrow \frac{\epsilon\mu}{c}[H\times\dot{E}]_2 dx^4 - \frac{1}{c}[B\times(\nabla\times H)]_2 dx^4 - \frac{1}{c}(J\times B)_2 dx^4 - \frac{1}{c}\rho E_2 dx^4$$
$$-\frac{1}{c}[D\times(\nabla\times E)]_2 dx^4 - \frac{\epsilon\mu}{c}[E\times\dot{H}]_2 dx^4 + \frac{1}{c}[H(\nabla\cdot B)]_2 dx^4$$
$$+\frac{1}{c}[E(\nabla\cdot D)]_2 dx^4 = 0$$

4.g.3

Defining 5 coefficient quantities with the indicated dimensions,

$$\begin{aligned}
S &= E\times H && [\ m\ t^{-3}\ ] = [\text{ENERGY FLUX}]^{\dagger} \\
U &= \tfrac{1}{2}(H\cdot B + E\cdot D) && [m\ l^{-1}\ t^{-2}] = [\text{ENERGY/VOLUME}] \\
W &= J\cdot E && [m\ l^{-1}\ t^{-3}] = [\text{POWER/VOLUME}] \\
F &= J\times B + \rho E && [m\ l^{-2}\ t^{-2}] = [\text{FORCE/VOLUME}] \\
X &= (H\cdot\nabla)B + H(\nabla\cdot B) \\
&\quad + (E\cdot\nabla)D + E(\nabla\cdot D) && [m\ l^{-2}\ t^{-2}] = [\text{FORCE/VOLUME}]
\end{aligned}$$

4.g.4

the separated space and time formulas (4.g.3) become:

$$\text{Space} \rightarrow \varepsilon\mu[\nabla\cdot S + \frac{\partial U}{\partial t} + W]_3 = 0 \qquad = 0$$

Bracket: Scalar Terms, $[m\ l^{-1}\ t^{-3}]$ = [POWER/VOLUME]

$$\text{Time} \rightarrow \frac{1}{c}[-\varepsilon\mu\frac{\partial S}{\partial t} - \nabla U + X - F]_2 dx^4 = 0$$

Bracket: Vector Terms, $[m\ l^{-2}\ t^{-2}]$ = [FORCE/VOLUME]

4.g.5

By making either direct or dual space projections of (4.g.5), we get four integral formulas:

$$\oiint_{\partial V} S_2 = \iiint_V -\left(W + \frac{\partial U}{\partial t}\right)_3 \qquad 4.g.6$$

†: The dimensional analysis for S proceeds from (4.g.3). $[\nabla\cdot S]$ = $[m\ l^{-1}\ t^{-3}]$, which for the space 3-form is $[m\ l^{2}\ t^{-3}/l^{3}]$, or [POWER/VOLUME]. Because S is a space 2-form, we could say $[S] = [m\ l^{2}\ t^{-3}/l^{2}]$, or [POWER/AREA]. However, S is a directed quantity, and the conventional $[m\ l^{2}\ t^{-2}/l^{2}\ t]$ = [ENERGY/AREA TIME] = [ENERGY FLUX] indicates this. Properly handled the differential algebra provides the correct dimensional analysis.

$$\sum_{t^-}^{t^+} U(t) = \int_t -(W + \nabla\cdot\dot{S})dt \qquad \text{(1 component)}$$

$$\sum_{t^-}^{t^+} S_i = \int_t -(\nabla U - \dot{X} + F)_i dt \qquad \text{(3 components)} \qquad 4.g.6$$

$$\sum_{x^-}^{x^+} U(x) = \int_x -\left(\varepsilon\mu\frac{\partial S}{\partial t} - X + F\right)_i$$

The space relation in (4.g.5) is the Poynting theorem, essentially an equation of continuity for power and energy. In the time relation (since the bracketed terms are force densities), the term involving $\dot{S}$ leads to the explanation that $\varepsilon\mu S$ is the momentum density of the electromagnetic field. Also note that the electromagnetic field energy density U behaves as a potential in this equation.

Separating the space and time components of (4.g.2b), we have

$$\text{Space} \rightarrow -[D(\nabla\cdot B)]_i - \varepsilon\mu[H\times(\nabla\times E)]_i - \varepsilon\mu[H\times\dot{B}]_i + \varepsilon\mu[E\times\dot{D}]_i$$
$$- \varepsilon\mu[E\times(\nabla\times H)]_i - [B(\nabla\cdot D)]_i - \varepsilon\mu(J\times E)_i + \rho B_i = 0$$

$$\text{Time} \rightarrow -\frac{1}{c}[D\cdot(\nabla\times E)]dx^4 - \frac{1}{c}[D\cdot\dot{B}]dx^4 - \frac{1}{c}[B\cdot\dot{D}]dx^4$$
$$+ \frac{1}{c}[B\cdot(\nabla\times H)]dx^4 - \frac{1}{c}(J\cdot B)dx^4 = 0$$

4.g.7

Defining 4 quantities with the indicated dimensions,

$$M = D\cdot B \qquad [m\ l^2\ t^{-1}]$$
$$F = J\cdot B \qquad [m\ l^2\ t^{-2}]$$
$$W = J\times E - \frac{1}{\varepsilon\mu}\rho B \qquad [m\ l^1\ t^{-3}]$$
$$Y = (B\cdot\nabla)D + (D\cdot\nabla)B - D(\nabla\cdot B) - B(\nabla\cdot D) \qquad [m\ l^3\ t^{-1}]$$

4.g.8

the separated space and time formulas (4.g.7) become:

$$\text{Space} \rightarrow [\nabla M + \epsilon\mu(\mathbf{H}\times\dot{\mathbf{B}}) - \epsilon\mu(\mathbf{E}\times\dot{\mathbf{D}}) + \epsilon\mu\mathbf{W} - \mathbf{Y}]_1 = 0$$

Bracket: Vector Terms, $[m\ l^{-3}\ t^{-1}]$

$$\text{Time} \rightarrow \frac{1}{c}\left(\frac{\partial M}{\partial t} + \mathbf{D}\cdot(\nabla\times\mathbf{E}) - \mathbf{B}\cdot(\nabla\times\mathbf{H}) + F\right)_0 = 0 \qquad 4.g.9$$

Bracket: Scalar Terms, $[m\ l^{-2}\ t^{-2}]$

Recalling the interpretation we gave to (M/c) in (4.c.4b), the space relation of (4.g.9) can be converted into the gradient of a mass density, and the time relation is the rate of change of this mass density. Two integral formulas follow:

$$\sum_{x-}^{x+}\frac{1}{c}M(x) = -\int_{x}\frac{1}{c}[\epsilon\mu(\mathbf{H}\times\dot{\mathbf{B}}) - \epsilon\mu(\mathbf{E}\times\dot{\mathbf{D}}) + \epsilon\mu\mathbf{W} - \mathbf{Y}]_1$$

$$\sum_{t-}^{t+}\frac{1}{c}M(t) = -\int_{t}\frac{1}{c}[\mathbf{D}\cdot(\nabla\times\mathbf{E}) - \mathbf{B}\cdot(\nabla\times\mathbf{H}) + F]dt \qquad 4.g.10$$

The structure of the two relations (4.g.1) may involve a more fundamental consideration than is at first obvious. Symmetry properties of the differentiable manifold R^3/t can influence their form. Let us look at some relevant aspects of coordinate system transformations. For Cartesian frames in R^3/t, the following are possible:

Inversion: For the space coordinates, reflection.
For the time coordinate, reversal.

Fixed Rotation: For the space coordinates, spatial rotation.
For the space-time coordinates, Lorentz transformation.

Translation: For the space coordinates, shift of position.
For the time coordinate, shift in time.

Scale transformations of the coordinates.

Relative translational motion (constant velocity): Equivalent to a space-time rotation (Lorentz transformation).

Relative rotational motion in space (constant angular velocity): Accelerated motion.

Accelerated translational motion.

Accelerated rotational motion.

In classical mechanics, the principles of conservation of momentum and conservation of energy follow from the invariance of the physics under space and time translations (LANDAU and LIFSHITZ (1960), §6 and §7). On a differentiable manifold, the local coordinate system represented by the differential 1-forms dx^i can only be significantly affected by inversion and rotation, as translation does not have meaning and scale transformations will only introduce constants into the vector results. We will not consider transformations involving accelerations. From the differential algebra, we find that the R^3/t physics (as it is expressed in the differential relations of electromagnetism) is invariant with respect to rotation and inversion operations in the sense that a general transformation operator T satisfies

$$T(d) \wedge T(\omega)_p = T(d\omega)_{p+1} \qquad 4.g.11$$

In particular, the operation involving the inversion of all coordinates, I: $(dx^1, dx^2, dx^3, dx^4) \rightarrow (-dx^1, -dx^2, -dx^3, -dx^4)$, takes the following simple form:

$$I(\omega)_p = (-)^p \omega_p \qquad 4.g.12$$

I and ** (as well as I·I and ****) are therefore not unrelated algebraically. For I, we find

$$I(\omega_p) \wedge I(\omega_q) = (-)^{p+q} \omega_p \wedge \omega_q = I(\omega_p \wedge \omega_q) \qquad 4.g.13$$

This leads us to conjecture a possible underlying structure behind (4.g.1). For electromagnetism, the strictly kinematic action quantities are found in the product space $(R^3/t) \wedge *(R^3/t)$. Perhaps, in a manner analogous to determining the magnitude of a complex quantity by forming a product between it and its conjugate, certain R^3/t magnitude quantities can be found by considering products involving either the two orientations of R^3/t (WARNER, §2, Ex. 13), or the space and its inverse image. Let us look at the second possibility. The $(R^3/t) \wedge *(R^3/t)$ direct product relations

$$\begin{aligned} \beta \wedge \delta\beta' + \beta' \wedge \delta\beta &= \beta' \wedge \gamma \\ *^{-1}(\beta' \wedge \delta\beta') + *(\beta \wedge \delta\beta) &= *(\beta \wedge \gamma) \end{aligned} \qquad 4.g.14$$

become in the new product space $I(R^3/t) \wedge *(R^3/t)$:

$$\begin{aligned} (+\beta) \wedge \delta\beta' + \beta' \wedge (-\delta\beta) &= \beta' \wedge (-\gamma) \\ *^{-1}(\beta' \wedge \delta\beta') + *((+\beta) \wedge (-\delta\beta)) &= *((+\beta) \wedge (-\gamma)) \end{aligned} \qquad 4.g.15$$

We see that (4.g.15) is identical to (4.g.1).

In closing this section, it must be stressed that these last developments merely hint at the possibility of a deeper mathematical basis for the energy and power relations. Future work is necessary to explore the relationship between the * operation and the various transformations available to the electromagnetic differential forms.

H. Reciprocity Relations in Electromagnetism

Developments in the preceeding section were predicated on the electromagnetic differential forms in (4.g.1) being part of the same differential structure. By expanding formulas similar to (4.g.1), but with forms from two differential structures (ε and μ are assumed identical in each), we derive reciprocity relations. These are antisymmetric formulas involving electromagnetic product quantities. Let us first look at a relation based on (4.g.1a):

$$\beta_A \wedge \delta\beta_B' - \beta_B' \wedge \delta\beta_A - \beta_B \wedge \delta\beta_A' + \beta_A' \wedge \delta\beta_B = - \beta_B' \wedge \gamma_A + \beta_A' \wedge \gamma_B \qquad 4.h.1$$

Defining

$$\begin{aligned} S_{AB} &= E_A \times H_B \\ W_{AB} &= J_B \cdot E_A \\ F_{AB} &= J_B \times B_A + \rho_B E_A \end{aligned} \qquad 4.h.2$$

we find for the space projection of (4.h.1)

$$\varepsilon\mu\, \nabla \cdot (S_{BA} - S_{AB})_2 = - \varepsilon\mu (W_{BA} - W_{AB})_3 \qquad 4.h.3a$$

and for the space projection of the time terms,

$$\begin{aligned} &\frac{\varepsilon\mu}{c}\frac{\partial}{\partial t}(S_{BA} - S_{AB})_2 \\ &\quad + \frac{1}{c}(B_A \times (\nabla \times H_B)) - \frac{1}{c}(B_B \times (\nabla \times H_A)) + \frac{1}{c} H_A(\nabla \cdot B_B) - \frac{1}{c} H_B(\nabla \cdot B_A) \\ &\quad - \frac{1}{c}(D_A \times (\nabla \times E_B)) + \frac{1}{c}(D_B \times (\nabla \times E_A)) - \frac{1}{c} E_A(\nabla \cdot D_B) + \frac{1}{c} E_B(\nabla \cdot D_A) \\ &\qquad = - \frac{1}{c}(F_{BA} - F_{AB})_2 \end{aligned} \qquad 4.h.3b$$

When all terms in (4.h.3a) are given a time dependence of exp(jωt), the resulting integral formula is the Lorentz reciprocity theorem (DESCHAMPS, §8.5). (4.h.3a) and (4.h.3b) are general reciprocity formulas for electromagnetism, and do not depend on the specification of a time behavior for the electromagnetic quantities.

Now let us develop reciprocity relations based on (4.g.1b). We shall expand the following formula:

$$*^{-1}(\beta_A' \wedge \delta\beta_B') - *(\beta_B \wedge \delta\beta_A) - *^{-1}(\beta_B' \wedge \delta\beta_A') + *(\beta_A \wedge \delta\beta_B)$$
$$= - *(\beta_B \wedge \gamma_A) + *(\beta_A \wedge \gamma_B) \qquad 4.h.4$$

Defining

$$\begin{aligned} M_{AB} &= D_A \cdot B_B \\ F_{AB} &= J_A \cdot B_B \\ W_{AB} &= J_B \times E_A - \frac{1}{\varepsilon\mu}\rho_B B_A \\ Y_{AB} &= (B_B \cdot \nabla)D_A + (D_A \cdot \nabla)B_B - D_B(\nabla \cdot B_A) - B_A(\nabla \cdot D_B) \end{aligned} \qquad 4.h.5$$

we have for the space projection of (4.h.4)

$$\nabla(M_{AB} - M_{BA})_0 + \varepsilon\mu \frac{\partial}{\partial t}[\mu(H_B \times H_A) + \varepsilon(E_B \times E_A)]_1 + (Y_{BA} - Y_{AB})_1$$
$$= \varepsilon\mu\,(W_{BA} - W_{AB})_1 \qquad 4.h.6a$$

and for the space projection of the time terms

$$-\frac{1}{c}\frac{\partial}{\partial t}(M_{AB} - M_{BA})_0 + \frac{1}{c}\nabla\cdot[\mu(H_B \times H_A) + \varepsilon(E_B \times E_A)]_1 = \frac{1}{c}(F_{AB} - F_{BA})_0 \qquad 4.h.6b$$

This completes the development of reciprocity formulas from (4.g.1).

I. R^3/t Integration Formulas

Let us close this chapter with a few remarks about the integration of electromagnetic differential forms. The development patterns of the R^3 integration formulas (Chapter 2) can be applied to the R^3/t forms, but since in any applications, the R^3/t integration formulas are projected into space and time (Recall §3.D, §3.F), ultimately we are concerned only with R^3 and R^1 integral formulas. However, the R^3/t development does provide us with a general view of integration formula development in electromagnetism.

i. Stokes' theorems

Except for the use of the R^3/t unit forms (Table 3.9), 1-variable integration formulas for electromagnetic differential forms are developed exactly as shown in Figure 2.1. Projection into either R^3 or R^1 proceeds as in §3.D. Let us look at one example of a raised differential. We convert the derivative formula

$$d(\beta' \wedge u_1') = d\beta' \wedge u_1' = \gamma' \wedge u_1' \quad \text{(4-form)} \qquad 4.i.1$$

into vector notation:

$$-\frac{1}{\mu c}[\nabla\cdot(\mathbf{B}\times\mathbf{n})]_3 dx^4 + \frac{\varepsilon}{c}\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{n})_3 dx^4 + \varepsilon(\nabla\cdot\mathbf{E})_3 dx^4 = -\frac{1}{c}(\mathbf{J}\cdot\mathbf{n})_3 dx^4 + \rho\, dx^{1234} \qquad 4.i.2$$

Here we have used $u_1' = -\mathbf{n}_1 + dx^4$. Projecting into space ($dx^{1234} \rightarrow dx^{123}$), the following integral relation is obtained:

$$\oint\!\!\!\oint_{\partial V}[(\mathbf{H}\times\mathbf{n}) - c\mathbf{D}]_2 = \iiint_V \left[\mathbf{J}\cdot\mathbf{n} + \left(\frac{\partial\mathbf{D}}{\partial t}\right)\cdot\mathbf{n} - c\rho\right]_3 \qquad 4.i.3$$

This is the combined integrated form of Coulomb's law and Ampère's law. With the aid of the integral form of Maxwell's equations (Table 3.8), this may be separated into two independent parts:

$$\oint\!\!\!\oint_{\partial V}(\mathbf{H}\times\mathbf{n})_2 = \iiint_V\left(\mathbf{J}\cdot\mathbf{n} + \frac{\partial\mathbf{D}}{\partial t}\cdot\mathbf{n}\right)_3$$

$$\oint\!\!\!\oint_{\partial V}\mathbf{D}_2 = \iiint_V \rho\, dx^{123} \qquad 4.i.4$$

The development of other raised integral formulas in R^3/t proceeds in a similar manner.

ii. Integrals involving Laplacians

Using the material in Tables 3.4 and 3.9, we can prove the following relations:

$$\begin{aligned}(\Delta\alpha)\wedge u_3 &= (d\delta\alpha + \delta d\alpha)\wedge u_3 = -\frac{1}{c\varepsilon}\,\delta d\alpha\wedge(*u_\gamma)\\ &= -\frac{1}{c\varepsilon}\,u_1\wedge *\delta d\alpha = -\frac{1}{c\varepsilon}\,u_1\wedge d{*d\alpha} = \frac{1}{c\varepsilon}\,d{*d\alpha}\wedge u_1\\ &= \frac{1}{c\varepsilon}\,d(*d\alpha\wedge u_1)\end{aligned} \qquad 4.1.5$$

$$(\Delta\beta)\wedge u_2 = (d\delta\beta + \delta d\beta)\wedge u_2 = d(\delta\beta\wedge u_2) \qquad 4.1.6$$

$$(\Delta\alpha')\wedge u_1' = (d\delta\alpha' + \delta d\alpha')\wedge u_1' = d(\delta\alpha'\wedge u_1') \qquad 4.1.7$$

$$\begin{aligned}(\Delta\beta')\wedge u_2' &= (d\delta\beta' + \delta d\beta')\wedge u_2' = \frac{1}{c\varepsilon}\,(\delta d\beta')\wedge *u_2\\ &= -\frac{1}{c\varepsilon}\,u_2\wedge d{*d\beta'} = -\frac{1}{c\varepsilon}\,d{*d\beta'}\wedge u_2\\ &= -\frac{1}{c\varepsilon}\,d(*d\beta'\wedge u_2)\end{aligned} \qquad 4.1.8$$

These formulas permit the application of Stokes' theorem to the integration of the R^3/t Laplacians.

iii. Integrals based on product derivatives

Any differential relation in R^3/t is capable of producing an integral formula. The same is true for the product differential relations. As an example, let us look at Equation 7, Table 4.2 (We use the notation given in (4.c.6)):

$$d(\gamma\wedge\beta') = \Delta\beta\wedge\beta' - \gamma\wedge\gamma' \qquad 4.1.9$$

In vector notation, this is written as

$$d(-\frac{1}{c}F_2dx^4 + \mu\varepsilon W_3) = -\frac{1}{c}(\nabla\cdot F)_3dx^4 - \frac{\mu\varepsilon}{}\frac{\partial W}{\partial t}dx^{1234}$$
$$= -\frac{1}{c}((\nabla^2 E)\cdot D)_3dx^4 + \frac{\varepsilon\mu}{c}(\ddot{E}\cdot D)_3dx^4 + \frac{1}{c}((\nabla^2 B)\cdot H)_3dx^4$$
$$-\frac{\varepsilon\mu}{c}(\ddot{B}\cdot H)_3dx^4 + \frac{\mu}{c}(J\cdot J)_3dx^4 - \frac{1}{\varepsilon c}\rho^2\, dx^{1234} \qquad 4.i.10$$

In this case, the terms on the left-hand side will be measureable from the force and power densities.

iv. Green's theorems

In §3.F, we discussed the two R^3/t Green's theorems applicable to the electromagnetic differential forms. In that development, we completely expanded the Laplacian structure. It is also possible to immediately apply the identically zero differential relations ($\delta\alpha = 0$, $d\beta = 0$) to the expansion, reducing the number of terms:

$$\alpha{*}\Delta g_1 - g_1{*}\Delta\alpha = d(g_1 \wedge {*}d\alpha - \alpha \wedge {*}dg_1) \qquad 4.i.11a$$

$$\beta{*}\Delta g_2 - g_2{*}\Delta\beta = d(\delta g_2 \wedge {*}\beta - \delta\beta \wedge {*}g_2) \qquad 4.i.11b$$

Expanding (4.i.11) (using (3.f.6) for the g_i), and identifying vector formulas based on complete expansions in the left-hand side and right-hand side terms, we produce only part of the group in Tables 3.10a and 3.10b. For example, (4.i.11a) yields Equations A.2 and A.3 of Table 3.10a, together with

$$\iiint \frac{\mu^{-1}}{c}[(A\cdot(\nabla^2 G)) - (G\cdot(\nabla^2 A))]_3 + \frac{\varepsilon^2\mu}{c}[\phi\ddot{G}_4 - G_4\ddot{\phi}]_3$$
$$-\frac{\varepsilon}{c}\frac{\partial}{\partial t}[G\cdot(\nabla\phi) - A\cdot(\nabla G_4)]_3$$
$$= \oint\!\!\oint \frac{\mu^{-1}}{c}[(A\times(\nabla\times G)) - (G\times(\nabla\times A))]_2 + \frac{\varepsilon}{c}[G_4\dot{A} - \phi\dot{G}]_2 \qquad 4.i.12a$$

$$\int_t \left\{ \frac{\mu^{-1}}{c} [\mathbf{A} \cdot (\nabla^2 \mathbf{G}) - \mathbf{G} \cdot (\nabla^2 \mathbf{A})] + \frac{\varepsilon^2 \mu}{c} [\phi \ddot{G}_4 - G_4 \ddot{\phi}] + \nabla \cdot \left(\frac{\mu^{-1}}{c} [\mathbf{G} \times (\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla \times \mathbf{G})] + \frac{\varepsilon}{c} [\phi \dot{\mathbf{G}} - G_4 \dot{\mathbf{A}}] \right) \right\} dt = \sum_{t-}^{t+} \frac{\varepsilon}{c} [\mathbf{G} \cdot (\nabla \phi) - \mathbf{A} \cdot (\nabla G_4)]$$

4.i.12b

Of course, Equations A.1 - A.6 inclusive in Table 3.10a may be derived from (4.i.11a) by adding terms to complete the exact differentials in the expansion of the right-hand side.

v. Symmetric integration formulas analogous to Green's theorems

Expansion of the symmetric Laplacian structures

$$\alpha^* \Delta g_1 + g_1^* \Delta \alpha \qquad 4.i.13a$$

$$\beta^* \Delta g_2 + g_2^* \Delta \beta \qquad 4.i.13b$$

results in integration formulas similar to those R^3 formulas listed in Table 2.11. As an example, we shall expand (4.i.13a):

$$\frac{\mu^{-1}}{c} \left\{ \iiint [\mathbf{A} \cdot (\nabla^2 \mathbf{G}) + \mathbf{G} \cdot (\nabla^2 \mathbf{A})] + 2 \iiint [(\nabla \times \mathbf{G}) \cdot (\nabla \times \mathbf{A}) + (\nabla \cdot \mathbf{G})(\nabla \cdot \mathbf{A})] \right\}_3 = \frac{\mu^{-1}}{c} \oiint [\mathbf{A} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{A}) + (\nabla \cdot \mathbf{G}) \mathbf{A} + (\nabla \cdot \mathbf{A}) \mathbf{G}]_2$$

4.i.14a

$$-\frac{\varepsilon}{c} \left\{ \iiint [\phi (\nabla^2 G_4) + G_4 (\nabla^2 \phi)] + 2 \iiint [\nabla \phi \cdot \nabla G_4] \right\}_3 = -\frac{\varepsilon}{c} \oiint [\phi \nabla G_4 + G_4 \nabla \phi]_2$$

4.i.14b

$$-\frac{\varepsilon}{c} \left\{ \int_t [\mathbf{A} \cdot \ddot{\mathbf{G}} + \mathbf{G} \cdot \ddot{\mathbf{A}}] dt + 2 \int_t \dot{\mathbf{A}} \cdot \dot{\mathbf{G}} \, dt \right\} = -\frac{\varepsilon}{c} \sum_{t-}^{t+} [\mathbf{A} \cdot \dot{\mathbf{G}} + \mathbf{G} \cdot \dot{\mathbf{A}}]$$

4.i.14c

$$\frac{\varepsilon^2 \mu}{c} \left\{ \int_t [\phi \ddot{G}_4 + G_4 \ddot{\phi}] dt + 2 \int_t [\dot{\phi} \dot{G}_4] dt \right\} = \frac{\varepsilon^2 \mu}{c} \sum_{t-}^{t+} [\phi \dot{G}_4 + G_4 \dot{\phi}] \qquad 4.i.14d$$

The remaining terms on the right-hand side of (4.i.13a) sum to zero:

$$
\begin{aligned}
&-\frac{\varepsilon}{c}\nabla\cdot(\phi\dot{\mathbf{G}}) - \frac{\varepsilon}{c}\nabla\cdot(G_4\dot{\mathbf{A}}) + \frac{\varepsilon}{c}\nabla\cdot(\dot{G}_4\mathbf{A}) + \frac{\varepsilon}{c}\nabla\cdot(\dot{\phi}\mathbf{G}) \\
&-\frac{\varepsilon}{c}\frac{\partial}{\partial t}(\mathbf{A}\cdot\nabla G_4) - \frac{\varepsilon}{c}\frac{\partial}{\partial t}(\mathbf{G}\cdot\nabla\phi) + \frac{\varepsilon}{c}\frac{\partial}{\partial t}((\nabla\cdot\mathbf{G})\phi) + \frac{\varepsilon}{c}\frac{\partial}{\partial t}((\nabla\cdot\mathbf{A})G_4) \\
&+2\frac{\varepsilon}{c}\dot{\mathbf{A}}\cdot(\nabla G_4) + 2\frac{\varepsilon}{c}\dot{\mathbf{G}}\cdot(\nabla\phi) - 2\frac{\varepsilon}{c}(\nabla\cdot\mathbf{A})\dot{G}_4 - 2\frac{\varepsilon}{c}(\nabla\cdot\mathbf{G})\dot{\phi} = 0
\end{aligned}
\qquad 4.i.15
$$

By regrouping the ∇ or $\partial/\partial t$ terms on one side, two further integral formulas can be constructed.

vi. List of Laplacian relations

The 4-form Laplacian relations

$$
\begin{aligned}
&\alpha_A \wedge \Delta\alpha'_B - \Delta\alpha_A \wedge \alpha'_B \\
&\beta_A \wedge \Delta\beta_B - \Delta\beta_A \wedge \beta_B \\
&\beta_A \wedge \Delta\beta'_B - \Delta\beta_A \wedge \beta'_B \\
&\beta'_A \wedge \Delta\beta'_B - \Delta\beta'_A \wedge \beta'_B
\end{aligned}
\qquad 4.i.16
$$

all yield Green's theorem derivative equations. The first and third equations in (4.i.16) are directly convertible, and the other two can be manipulated by redefining one of the β's as a * variable. Substituting a + sign in (4.i.16), we get the relations discussed in the previous sub-section.

The 3-form Laplacian relations

$$
\begin{aligned}
&\alpha \wedge \Delta\beta - \Delta\alpha \wedge \beta \\
&\alpha \wedge \Delta\beta' - \Delta\alpha \wedge \beta' \\
&\alpha \wedge \Delta\beta + \Delta\alpha \wedge \beta \\
&\alpha \wedge \Delta\beta' + \Delta\alpha \wedge \beta'
\end{aligned}
\qquad 4.i.17
$$

and the 2-form Laplacian relations

$$\begin{aligned} &\alpha_A \wedge \Delta\alpha_B - \Delta\alpha_A \wedge \alpha_B \\ &\alpha_A \wedge \Delta\alpha_B + \Delta\alpha_A \wedge \alpha_B \end{aligned} \qquad 4.i.18$$

when expanded, produce the derivative relations listed in Table 4.2.

Summary

Sections 4.A, 4.C, 4.G and 4.H contain the most important material in this chapter. The discovery of the commutative properties in the differential structure of electromagnetic forms, and the development of the various relations involving electromagnetic product quantities clearly show the value of R^3/t differential algebra for working within the mathematical structure underlying electromagnetic theory. This completes our R^3/t analytical development. In the following chapter, we shall derive differential structures for time-dependent electromagnetic quantities, and we will discuss the applicability of integral formulas in the solution of a simple inverse problem.

CHAPTER V

RELATED APPLICATIONS-ORIENTED SUBJECTS

The original motivation for investigating exterior differential algebra involved its potential for simplifying complicated derivations in electromagnetic field theory. In particular, the mathematics appeared to provide a basis for discussing a so-called inverse scattering integral equation introduced by BOJARSKI (1973). Now we find that although this algebra is helpful in illuminating the structure of Bojarski's integral equation, discussion on the more critical points (i.e., whether the integral equation in fact leads to a solution having anything to do with an inverse scattering problem) relys primarily on analytical aspects of the physics associated with the structure of specific scattering situations. The algebra by itself does not deal with the matter of boundaries, for example. In this chapter, we will look at the simplest inverse scattering problem, one-dimensional reflection from a dielectric interface. We shall see what is offered by the Bojarski technique. Preceeding this will be a short discussion on the application of certain time dependencies to the entire system of electromagnetic partial differential equations.

A. Specific Time Dependence in the Electromagnetic Structure

Almost all electromagnetic engineering is concerned with the application of Maxwell's equations in the time-harmonic case, which is clearly a simplification of the most general case. In this section, we shall take the general structure of electromagnetic partial differential equations developed in Chapter 3 and investigate the following three circumstances: static, exponential, and complex exponential time behavior for all quantities. In particular, we shall look at the mathematical nature of the constitutive relation $J = \sigma E$ (Ohm's law) in these three time-specific situations.

i. Assumption of static time dependence

Let us assume that all time derivatives in Table 3.7b are zero. Then the basic relations from this table can be written as

$$
\begin{array}{ll}
\text{A.1:} & \nabla\cdot\mathbf{A} = 0 \\
\text{A.2:} & \nabla\times\mathbf{E} = 0 \\
 & \nabla\cdot\mathbf{B} = 0 \\
\text{A.3:} & \nabla\cdot\mathbf{J} = 0 \\
\text{B.1:} & \mathbf{E} = -\nabla\Phi \\
 & \mathbf{B} = \nabla\times\mathbf{A} \\
\text{B.2:} & \nabla\cdot\mathbf{D} = \rho \\
 & \nabla\times\mathbf{H} = \mathbf{J}
\end{array}
\qquad 5.a.1
$$

Using this group of equations, it is a simple matter to show that the wave equations C.1 and C.2 in Table 3.7b are valid (although certain terms are zero). Note that the Poincaré lemma applies rigorously in R^3:

$$
\begin{array}{l}
\nabla\cdot\nabla\times\mathbf{H} = \nabla\cdot\mathbf{J} = 0 \\
\nabla\times\nabla\Phi = -\nabla\times\mathbf{E} = 0
\end{array}
\qquad 5.a.2
$$

In terms of R^3 structural diagrams, the static equations in (5.a.1) can be divided into two independent parts:

$\underline{R^3}$

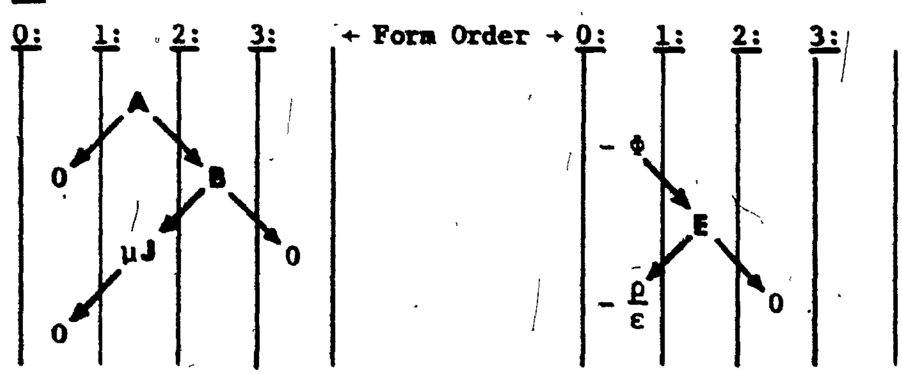

Figure 5.1 : R^3 structural diagrams for Equations (5.a.1).

We can now see that the constitutive relation $J = \sigma E$ is incompatible with Equations (5.a.1) because it requires ρ to be identically zero, implying that E is harmonic in the R^3 sense. This in turn implies that, in sequence, Φ, B, and A are harmonic in R^3. In other words, if we can consider our local space to be part of a closed Riemannian manifold, enforcing the constitutive relation globally decouples all of the electromagnetic differential relationships. That such a situation is completely null follows from the physical understanding that charge and current are interrelated, so that an identically zero ρ implies that J is also identically zero. Consequently, the R^3/t 4-form γ must be identically zero, and in §3.F we proved that this nullifies the whole electromagnetic structure.

ii. Assumption of exponential time dependence

We now assume that all quantities in Table 3.7b have the time behavior $\exp(-(\sigma/\varepsilon)t)$. The basic equations then reduce to the following:

$$\begin{aligned}
\text{A.1:}&\quad \nabla\cdot A = \mu\sigma\Phi\\
\text{A.2:}&\quad \nabla\times E = \frac{\sigma}{\varepsilon}B\\
&\quad \nabla\cdot B = 0\\
\text{A.3:}&\quad \nabla\cdot J = \frac{\sigma}{\varepsilon}\rho\\
\text{B.1:}&\quad E = \frac{\sigma}{\varepsilon}A - \nabla\Phi\\
&\quad B = \nabla\times A\\
\text{B.2:}&\quad \nabla\cdot D = \rho\\
&\quad \nabla\times H = J - \frac{\sigma}{\varepsilon}D
\end{aligned} \qquad \text{5.a.3}$$

Of course, the exponential time dependence is now assumed for each of above equations. The relationships in the set (5.a.3) can be outlined in terms of the two inter-dependent structural diagrams found in Figure 5.2. Now there is a direct implication that the constitutive relation $J = \sigma E$ is valid. The right-hand diagram is simply a multiple of the left-hand one:

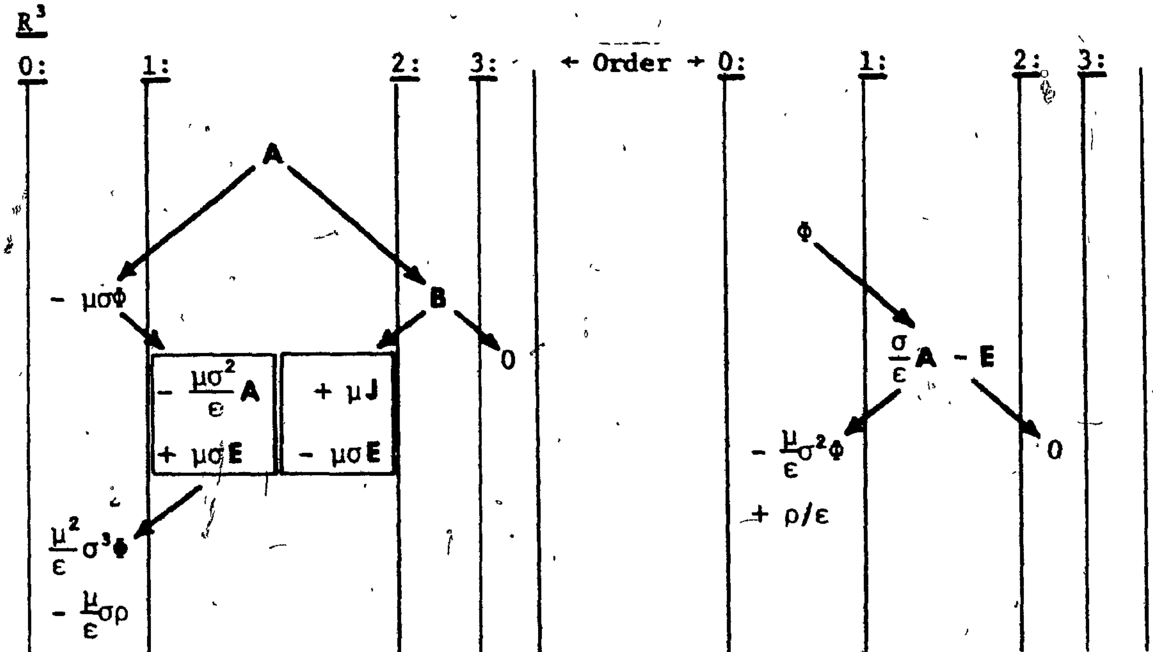

Figure 5.2 : R^3 structural diagrams for Equations (5.a.3).

The validity of the constitutive relation also implies that

$$\nabla \times \mathbf{B} = \mu \mathbf{J} - \mu\sigma \mathbf{E} = \mu\sigma \mathbf{E} - \mu\sigma \mathbf{E} \equiv 0 \qquad 5.a.4$$

Consequently, **B** displays the behavior of a harmonic quantity, and in a global sense may be considered decoupled from the structure of Φ, **A**, **E**, and ρ. Note that this is a stronger condition than that of saying **B** is source-free because its divergence is zero.

iii. Assumption of complex exponential time dependence

By imposing on all quantities in Table 3.7b the time behavior exp(jωt), and by permitting the real quantities in the electromagnetic structure to be the real projection of a complete complex quantity (see HARRINGTON, §1.8), we derive the following set:

$$
\begin{aligned}
&\text{A.1:} && \nabla\cdot\mathbf{A} = -\,\varepsilon\mu j\omega\Phi \\
&\text{A.2:} && \nabla\times\mathbf{E} = -\,j\omega\mathbf{B} \\
& && \nabla\cdot\mathbf{B} = 0 \\
&\text{A.3:} && \nabla\cdot\mathbf{J} = -\,j\omega\rho \\
&\text{B.1:} && \mathbf{E} = -\,j\omega\mathbf{A} - \nabla\Phi \\
& && \mathbf{B} = \nabla\times\mathbf{A} \\
&\text{B.2:} && \nabla\cdot\mathbf{D} = \rho \\
& && \nabla\times\mathbf{H} = \mathbf{J} + j\omega\mathbf{D}
\end{aligned}
\qquad \text{5.a.5}
$$

These relationships may be outlined in the following inter-dependent structural diagrams:

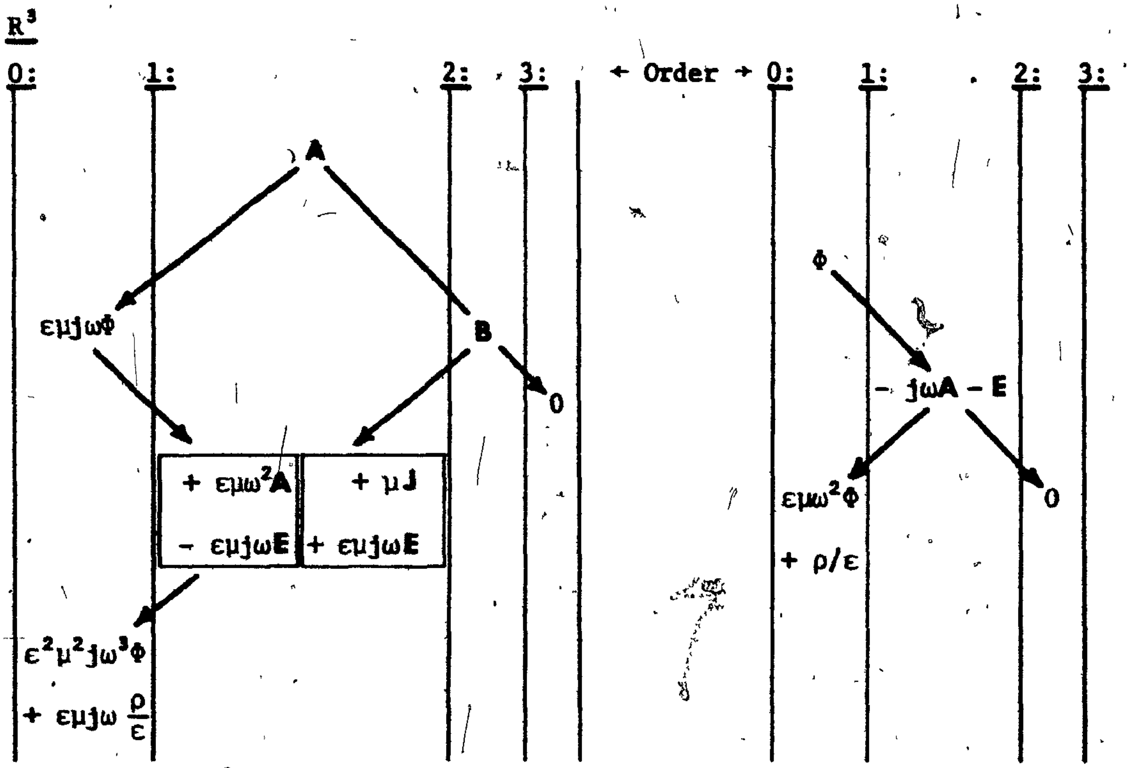

Figure 5.3 : R^3 structural diagrams for Equations (5.a.5).

As in §5.a.ii, there is a directly implied relation between **J** and **E** :

$$\mathbf{J} = -\,j\omega\varepsilon\mathbf{E} \qquad \text{5.a.6}$$

Thus the right-hand diagram may again be seen as a multiple of the left-hand one. Also,

$$\nabla\times\mathbf{B} = \mu\mathbf{J} + j\omega\epsilon\mu\mathbf{E} = - j\omega\epsilon\mu\mathbf{E} + j\omega\epsilon\mu\mathbf{E} = 0 \qquad 5.a.7$$

again globally decoupling $\mathbf{B}$ from the rest of the electromagnetic structure.

The most general situation encountered in practice is actually a combination of (ii) and (iii), where

$$\mathbf{J} = (\sigma - j\omega\epsilon)\mathbf{E} \qquad 5.a.8$$

The quantity in parentheses is known as the total conductivity of the medium supporting the electromagnetic phenemona.

B. Interpretation of Scalar Integration Theorems

This section should be viewed as a prelude to §5.C. We derive the Kirchhoff integral formula by subsituting time-reduced wave equations† into the R^3 scalar Green's theorem. Our objective is to use the Kirchhoff formula for manipulating the similar integral formula derived from the symmetric integration formula analogous to the scalar Green's theorem. It is this second integral formula that BOJARSKI (1973) indicates can be applied to the solution of inverse scattering problems.

i. The Kirchhoff integral for the time-reduced wave equation

The scalar Green's theorem

$$\oiint_{\partial V}(g\nabla\phi - \phi\nabla g) = \iiint_V (g\nabla^2\phi - \phi\nabla^2 g) \qquad 5.b.1$$

†: In a time-reduced wave equation, all quantities have the time behavior $\exp(j\omega t)$, which is suppressed.

together with the time-reduced wave equations for Φ and g (where $k = \omega\sqrt{\varepsilon\mu}$)

$$\nabla^2\Phi(x) + k^2\Phi(x) = -\rho(x) \tag{5.b.2a}$$

$$\nabla^2 g(x-x') + k^2 g(x-x') = -\delta(x-x') \tag{5.b.2b}$$

yield the following formula:

$$\oiint_{\partial V}(g\nabla\Phi - \Phi\nabla g) = -\iiint_V g(x-x')\rho(x) + \iiint_V \Phi(x)\delta(x-x') \tag{5.b.3}$$

The quantities are defined as follows:

$\Phi(x)$ = Scalar field at x.
$\rho(x)$ = Sources at x.
x' = Location of an elementary source.
$\delta(x-x')$ = The elementary source distribution, centered at x'.
$g(x-x')$ = The Green's function (propagation characteristic) between x' and x. (5.b.4)

Consequently, the wave equation (5.b.2a) shows the relation of the field at x to the sources at x, and (5.b.2b) shows the relation of the field at x to an elementary source at x'. The complete interpretation of (5.b.3) proceeds as follows: we assume that the point x' is located within V:

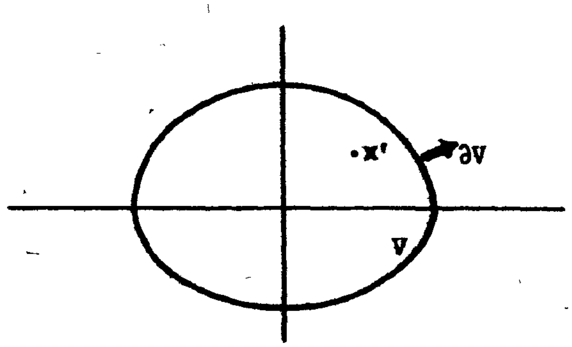

Figure 5.4 : Domain of integration for Equation (5.b.3).

Equation (5.b.3) can now be written

$$\Phi(x')\Big|_{x' \in V} = \iiint_V \Phi(x)\,\delta(x-x') = \iiint_V g(x-x')\rho(x) + \oint\!\!\oint_{\partial V} (g\nabla\Phi - \Phi\nabla g) \qquad 5.b.5$$

It is clear that the 1st term on the right-hand side is a contribution to the field at x' by the sources contained within the volume V. Recalling the discussion in §3.F on the impossibility of globally harmonic fields in electromagnetism, the field contributions at x' will be from those sources inside V and from those sources external to V only. Can we show that the 2nd term on the right-hand side of (5.b.5) is due solely to exterior sources? To answer this, we proceed as follows: let us consider a second surface in Figure 5.4, located at infinity:

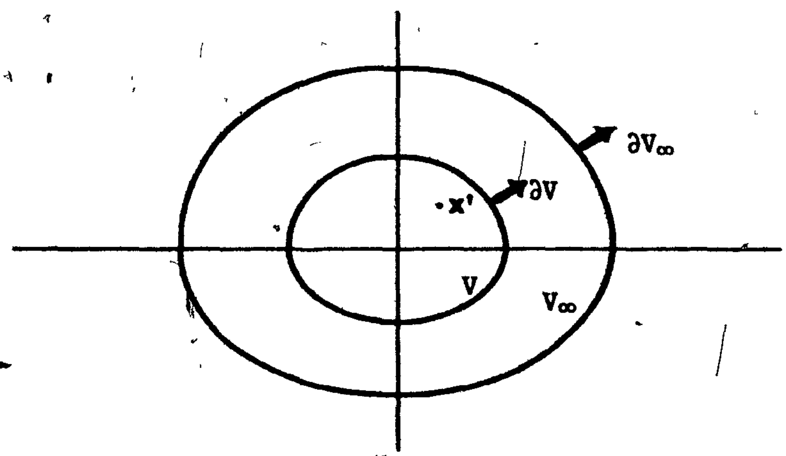

Figure 5.5 : A second surface is located at infinity.

We temporarily assume that all sources are contained within V (The field under this condition will be called Φ_1). We find

$$\Phi_1(x')\Big|_{x' \in V_\infty} = \Phi_1(x')\Big|_{x' \in V} = \iiint_{V_\infty} g(x-x')\rho(x) + \oint\!\!\oint_{\partial V_\infty} (g\nabla\Phi_1 - \Phi_1\nabla g)$$

(Continued)

$$= \iiint_V g(x-x')\rho(x) + \oint\!\!\oint_{\partial V_\infty} (g\nabla\Phi_1 - \Phi_1\nabla g)$$

$$= \iiint_V g(x-x')\rho(x) + \oint\!\!\oint_{\partial V} (g\nabla\Phi_1 - \Phi_1\nabla g) \qquad 5.b.6$$

Consequently,

$$\oint\!\!\oint_{\partial V} (g\nabla\Phi_1 - \Phi_1\nabla g) = \oint\!\!\oint_{\partial V_\infty} (g\nabla\Phi_1 - \Phi_1\nabla g) \qquad 5.b.7$$

Now the condition of regularity at infinity (the radiation condition) prescribes that the right-hand side of (5.b.7) must be identically zero. Thus in the situation where all sources are contained in V, from (5.b.5) we find

$$\Phi_1(x')\Big|_{x' \in \mathring{V}} = \iiint_V g(x-x')\rho(x) + \oint\!\!\oint_{\partial V} (g\nabla\Phi_1 - \Phi_1\nabla g) \qquad 5.b.8$$

As a result,

$$\iiint_V g(x-x')\rho(x) \qquad 5.b.9$$

is the sole interior source contribution to the field. It follows that in the general situation, with sources present on either side of the boundary ∂V, that

$$\oint\!\!\oint_{\partial V} (g\nabla\Phi - \Phi\nabla g) \qquad 5.b.10$$

is the sole contribution to the internal field by the external sources. The surface integral (5.b.10) is a relationship between external sources and internal fields. A similar development for x' outside V, but with sources within V, shows that the surface integral relates the internal sources to external fields. Bojarski is correct in his assessment of these surface integrals.

By considering a field $\Phi_1(X')$ due to internal sources alone, and a second field $\Phi_2(X')$ due only to external sources, we can write

$$\Phi_T(X') = \Phi_1(X') + \Phi_2(X') \qquad 5.b.11$$

Then, for $X' \in V$, we find from (5.b.8) and (5.b.10)

$$\begin{aligned}\Phi_T(X') &= \iiint_V g(X-X')\rho(X) + \cancel{\oint\!\!\oint_{\partial V}(g\nabla\Phi_1 - \Phi_1\nabla g)} + \oint\!\!\oint_{\partial V}(g\nabla\Phi_2 - \Phi_2\nabla g)\\ &= \iiint_V g(X-X')\rho(X) + \oint\!\!\oint_{\partial V}(g\nabla\Phi_T - \Phi_T\nabla g)\end{aligned}$$

5.b.12

Equation (5.b.12) is therefore identical in form to (5.b.5). The surface integral involving the total field is equivalent to the integral involving the externally applied field alone. Of course, the volume integral term does not change because only that part of the distribution of sources inside V is integrated.

ii. The symmetric integral formula for the time-reduced wave equation

The symmetric integration formula analogous to the Green's theorem (5.b.1) is written (see §2.B.iii):

$$\oint\!\!\oint_{\partial V}(g\nabla\Phi + \Phi\nabla g) = \iiint_V (g\nabla^2\Phi + \Phi\nabla^2 g) + \iiint_V 2(\nabla g)\cdot(\nabla\Phi) \qquad 5.b.13$$

By subsituting in that information already gained in the preceeding subsection, we find for the time-reduced case:

$$\iiint_V \nabla^2(g\Phi) = \iiint_V 2((\nabla g)\cdot(\nabla\Phi) - k^2 g\Phi) - \iiint_V g\rho - \iiint_V \delta\Phi = \oint\!\!\oint_{\partial V}(g\nabla\Phi + \Phi\nabla g)$$

5.b.14

When both X' and all of the sources are inside V, the two right-hand terms in the expanded volume integral of (5.b.14) become (see (5.b.8) and (5.b.5)):

$$- \iiint_V g(X-X')\rho(X) = - \Phi_1(X')\Big|_{X' \in V}$$
$$- \iiint_V \delta(X-X')\Phi_1(X) = - \Phi_1(X')\Big|_{X' \in V} \qquad 5.b.15$$

Therefore, (5.b.14) can be written

$$\Phi_1(X')\Big|_{X' \in V} - \iiint_V (\nabla g \cdot \nabla\Phi_1 - k^2 g\Phi_1) = - \tfrac{1}{2} \oiint_{\partial V} (g\nabla\Phi_1 + \Phi_1\nabla g) = - \oiint_{\partial V} (\Phi_1 \nabla g) \qquad 5.b.16$$

where the final transition is permitted by the regularity condition on Φ_1. By a similar procedure, when $X' \in V$ but all sources are external to V, we can show

$$\Phi_2(X')\Big|_{X' \in V} - \iiint_V (\nabla g \cdot \nabla\Phi_2 - k^2 g\Phi_2) = - \oiint_{\partial V} (\Phi_2 \nabla g) \qquad 5.b.17$$

For the combined case, with $\Phi_T = \Phi_1 + \Phi_2$,

$$\Phi_T(X')\Big|_{X' \in V} - \iiint_V (\nabla g \cdot \nabla\Phi_T - k^2 g\Phi_T) = - \tfrac{1}{2} \oiint_{\partial V} (g\nabla\Phi_1 + \Phi_1\nabla g + 2\Phi_2\nabla g)$$
$$= - \tfrac{1}{2} \oiint_{\partial V} (\Phi_T\nabla g + g\nabla\Phi_1 + \Phi_2\nabla g)$$
$$= - \oiint_{\partial V} (\Phi_T \nabla g) \qquad 5.b.18$$

where once again the final transition is permitted by the regularity condition on Φ_1. This derivation parallels that made by BOJARSKI (1973), but there are a few differences in the result. The above derivation is more precise about the matter of superposition, for example. Bojarski claims that the integral equations (5.b.16) → (5.b.18) can be solved if the field is known over the boundary surface, yielding values of $\Phi(X')$ for points contained within the surface. Furthermore, calculation of the closed surface integral on the right-hand side of the above equations is considered to be the critical step, since knowledge of this surface integral is supposed to lead to a unique solution of the integral equation involved. Presumably, if the field can be found everywhere in $\bar{V}$ from values measured on ∂V, the nature, shape, and location of its sources and influences can be inferred, at least to a degree. We shall investigate this claim in the next section.

C. A One-Dimensional Inverse Scattering Problem

We will now discuss a simple one-dimensional example of inverse scattering. In particular, we shall investigate the usefulness of the integral formula (5.b.18) in obtaining the solution for the desired quantities. We propose to interrogate a one-dimensional dielectric structure† with a known incident wave. When there is a single interface, the relative dielectric constant and the position of the interface with respect to a reference position must be determined from field measurements. This requires the analysis of an R^1 time-reduced wave equation, and this equation can be developed from the R^3/t differential structure through projection (taking into consideration the symmetry involved) and time reduction, or it may be developed directly from an R^1/t differential structure. We are assuming that the source of the incident radiation produces a monochromatic continuous wave. Note that this reflection problem parallels the acoustical reflection problem because we are dealing with scalar fields.

The investigation of the inverse scattering aspects to this problem will rely on the following arrangement of source and measurement positions relative to the interface:

†: We consider only perfect dielectrics.

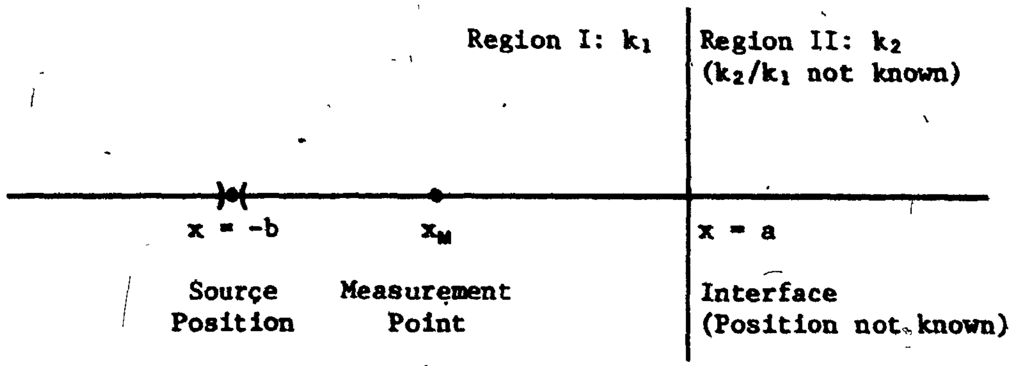

Figure 5.6 : Features of the 1-dimensional inverse scattering problem (Reflection from a plane interface).

We shall make it a convention for the source (located at the distinct point $x = -b$) to radiate equal-amplitude waves into each half-space.† In Figure 5.6, the source wave directed to the left propagates to $x = -\infty$, so we need not consider it in the region $x > -b$.

The measurement point x_M (located in the vicinity of $x = 0$ in Figure 5.6) is the location of a probe capable of measuring some characteristic of the wave motion. For acoustical reflection, a probe could measure instantaneously the field (pressure) at x_M. On the other hand, a typical microwave measurement (E-field probe) would yield an RMS figure. Let us examine this in more detail.

Assuming that the amplitude of the right-directed wave at the source is $A \exp(-j\omega t)$, we can write the total field at x_M:

$$\Phi_T(x_M) = \Phi_R(x_M) + \Phi_L(x_M) = \left[A\, e^{jk_1x_M} + A''\, e^{-jk_1x_M + jk_12a}\right] e^{jk_1b - j\omega t} \qquad 5.c.1$$

The subscripts T, R, and L indicate the total, right-directed, and left-

†: Analogous to the excitation of a junction between a light and heavy string, this situation is presumed to hold for a source located on an interface. Of course, the spatial frequency (and consequently the energy density) would differ in the two media.

directed fields. The first term on the right-hand side of (5.c.1) refers to the wave directly incident from the source. The second, or reflected, term has an apparent origin at $x = 2a + b$, the location of a mirror image. Note that the value of the distance "a" is presumed unknown in the inverse scattering problem.

An RMS measurement of the total field at x_M gives this result:

$$\Phi_{RMS}(x_M) = \sqrt{\Phi_T \bar{\Phi}_T}^{\,\dagger} = [AA + A''A'' + 2AA'' \cos 2k_1(x_M - a)]^{1/2} \qquad 5.c.2$$

A and A" can be determined from this RMS measurement by moving the probe in the vicinity of x_M, finding a maximum and minimum:

$$\begin{aligned} \Phi_{MAX} &= [AA + A''A'' - 2AA'']^{1/2} = [A - A''] \\ \Phi_{MIN} &= [AA + A''A'' + 2AA'']^{1/2} = [A + A''] \end{aligned} \qquad 5.c.3$$

Then

$$\begin{aligned} A &= \frac{\Phi_{MAX} + \Phi_{MIN}}{2} \\ A'' &= \frac{\Phi_{MIN} - \Phi_{MAX}}{2} \end{aligned} \qquad 5.c.4$$

This gives us a method of resolving the wave amplitudes from RMS measurements.

In order to determine k_2 relative to k_1, it is necessary to explicitly figure out the physics of the wave interaction at the interface. For perfect dielectrics, continuity of the wave and its first derivative across the interface leads to

$$\begin{aligned} A + A'' &= A' \\ k_1 A - k_1 A'' &= k_2 A' \end{aligned} \qquad 5.c.5$$

†: The bar indicates the complex conjugate.

where A' is the amplitude of the wave transmitted into the second medium. These equations reduce to:†

$$A'' = \left[\frac{k_1 - k_2}{k_1 + k_2}\right] A \qquad 5.c.6a$$

$$A' = \left[\frac{2k_1}{k_1 + k_2}\right] A \qquad 5.c.6b$$

The first inverse problem is then solved since (5.c.6a) can be written:

$$\frac{k_2}{k_1} = \left[\frac{1 - A''/A}{1 + A''/A}\right] = \left[\frac{A - A''}{A + A''}\right] = \left[\frac{\Phi_{MAX}}{\Phi_{MIN}}\right] \qquad 5.c.7$$

Unfortunately, the above measurements do not unambiguously determine the location of the interface. Presuming the dielectric to be independent of frequency over a certain range of interest, the interface position can be found by tracking a particular point on the RMS standing wave during a predetermined shift in source frequency. For example, we can track the minimum in Figure 5.7 through a shift to a lower frequency:

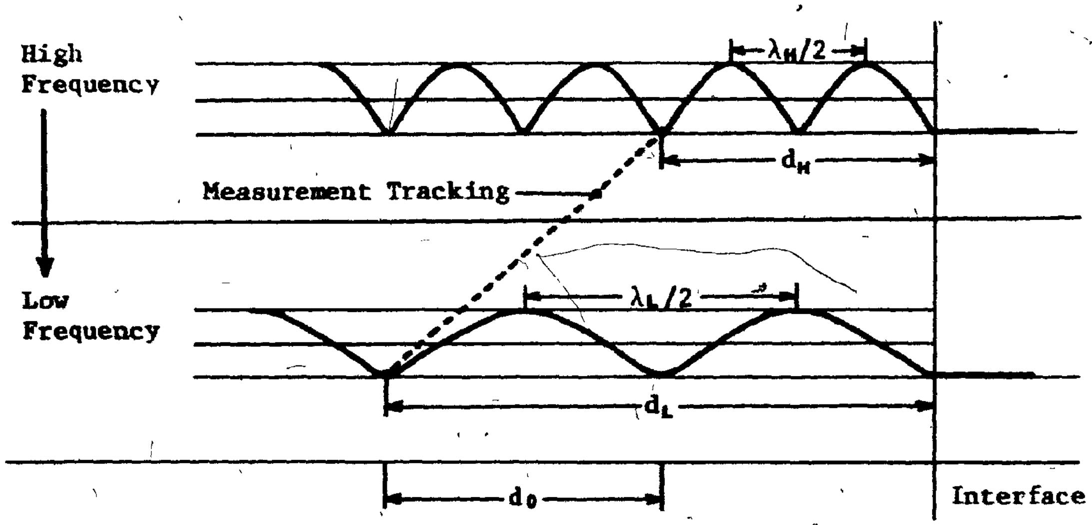

Figure 5.7 : Frequency shift determines interface location.

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†: For an optical problem, $k = \omega n/c$. Therefore, we may subsitute indices of refraction for k in these formulas.

The tracking distance d_0 is related to the two unknown distances by:

$$d_0 = (d_L - d_H) \qquad 5.c.8$$

Since

$$\frac{d_H}{\lambda_H} = \frac{d_L}{\lambda_L} \qquad 5.c.9$$

we can find either d_H or d_L in terms of the measured d_0. For example,

$$d_L = d_0\left[\frac{\lambda_L}{\lambda_L - \lambda_H}\right] \qquad 5.c.10$$

which in terms of the wavenumbers of the first medium, k_{1H} and k_{1L} ($\lambda = 2\pi/k$), can be written

$$d_L = d_0\left[\frac{k_{1H}}{k_{1H} - k_{1L}}\right] \qquad 5.c.11$$

Thus in this ideal one-dimensional situation, techniques exist for determining from field measurements both quantities necessary for reconstructing the scattering situation.

Now let us consider an interval in Figure 5.6 and apply the Green's theorem (5.b.1) over it:

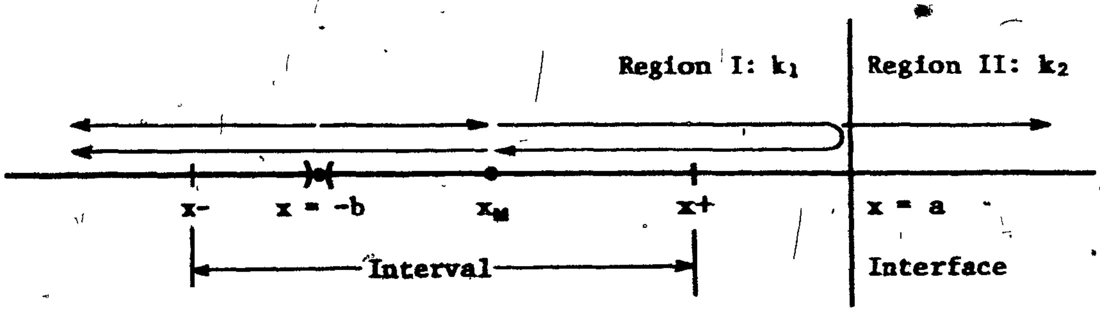

Figure 5.8 : Location of an interval for the scalar Green's theorem.

We have placed the interval entirely in Region I. The total field at x_M is the sum of the field directly incident from the source plus the field reflected from the interface at $x = a$. In Figure 5.8, the choice of an interval means that the reflected component has an effective external source (In fact, the image at $x = 2a + b$). From (5.b.5), we can write the field at x_M:

$$\Phi_T(x_M) = \int_{x-}^{x+} g(x - x_M)\rho(x)dx + \sum_{x-}^{x+} (g\nabla\Phi - \Phi\nabla g) \quad 5.c.12$$

With

$$\rho(x) = -2jk_1A\, e^{-j\omega t}\, \delta(x + b) \quad 5.c.13a$$

and

$$g(x - x_M) = -\frac{1}{2jk_1} e^{-jk_1(-b - x_M)} \quad 5.c.13b$$

where (5.c.13) refers to the sole true source, (5.c.12) can be solved yielding the expression for $\Phi_T(x_M)$ given in (5.c.1). The first term in (5.c.12)

$$\Phi_1(x_M) = \int_{x-}^{x+} g(x - x_M)\rho(x)dx = A\, e^{jk_1(x_M + b) - j\omega t} \quad 5.c.14a$$

is the contribution from the source inside the interval, and

$$\Phi_2(x_M) = \sum_{x-}^{x+} (g\nabla\Phi - \Phi\nabla g) = A''\, e^{jk_1(2a + b) - jk_1x_M - j\omega t} \quad 5.c.14b$$

is the contribution due to the external image source. In other words, when we have complete information on the Green's functions and fields, Green's theorem tells us precisely what we expect it should. However, in the inverse problem we cannot know the complete Green's function beforehand because it involves one significant unknown quantity - the position of the interface. Furthermore, we will not know the relative amplitudes of the waves without performing a measurement such as that discussed earlier. When these things are known, the inverse problem is essentially solved. The utility of Green's theorem is in the construction of a general solution by the superposition of known point-source solutions. Obviously, each point-source solution includes as its determinants geometric and electric structural parameters of the various bodies involved.

Using the known expressions for the fields and Green's functions in the reflection problem, it is not difficult to show that the symmetric integral formula (5.b.18) is also a valid mathematical statement. Can it have anything to do with solving the inverse aspect of the problem ? Presumably, if (5.b.18) is basic to an inverse solution, it will lead us directly to the same information developed earlier by analytical methods.

The one-dimensional time-reduced version of (5.b.18) is:

$$\Phi_T(x_M) - \int_{x-}^{x+} (\nabla g \cdot \nabla \Phi_T - k_1^2 g \Phi_T)\,dx = \sum_{x-}^{x+} (\Phi_T \nabla g) \qquad 5.c.15$$

BOJARSKI ((1973), §2) claims that the right-hand side of this equation is a measureable quantity, and its knowledge leads directly to a general solution for Φ_T over the interval (x-,x+). However, in the inverse problem, Φ_T is not the only unknown. Bojarski himself (§4) tries to solve the integral equation by assuming g to be the free-space Green's function, but since we know that even in the simple one-dimensional interface situation g is considerably more complicated than the free-space Green's function, this whole inverse scattering solution technique lacks practicality. Furthermore, the physics of the interaction must be known in order to interpret any field pattern. If, for example, RMS values of $\Phi_T(x_M)$ could be accurately determined throughout the interval (x-,x+), a physical argument would still be required to relate this field to the objects that in fact influenced it. Equation (5.c.15) offers no help in this matter.

We should point out that field measurements at the boundary of (x-,x+) can be used to determine the unknown quantities in the inverse reflection problem, but the method is related to the earlier analysis. Consider the RMS fields in the two intervals shown below:

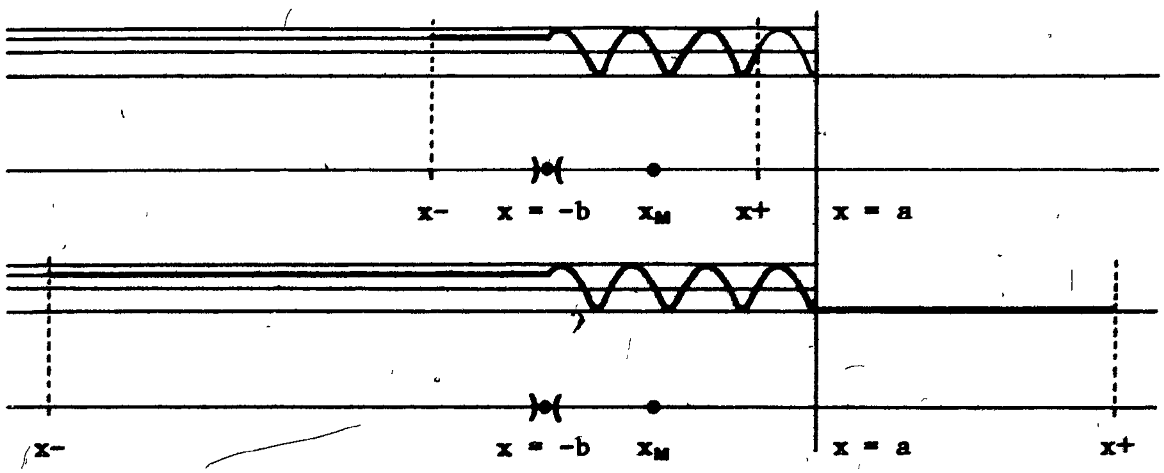

Figure 5.9 : RMS fields for the one-dimensional reflection problem.

In either instance (presuming linearity and frequency independence), a sufficient shift in the source frequency will cause the value of $\Phi_{RMS}(x-)$ to go through a maximum and a minimum, permitting us to find the relative dielectric constant using (5.c.7). Furthermore, if enough of the phase vs. frequency characteristic of the standing wave can be obtained (and this is not a simple problem because of the complicated shape of this standing wave), the source-interface distance can be found. If the source position is known with respect to the origin, the interface position can be referred to it as well. Note that the frequency shift is necessary for a unique inference of the two unknowns, and note that even with this shift, a measurement of $\Phi_{RMS}(x+)$ to the right of the interface remains constant, yielding no information other than the value A'.

Summary

Since the material in this chapter is not directly related to the main theme of the thesis, it must be considered as supplementary. However, it does offer detailed treatment of a few patterns of formulas that occur naturally when using differential forms. The material in the first section shows how the concept of structural diagrams can be used in the widely-studied static and time-harmonic electromagnetic field situations. The discussion of the inverse problem in sections 2 and 3 leads to the conclusion that the physics of the interface is of crucial importance, and that the proposed measurement and integration scheme still contains too many unknown quantities to be considered as a method that will lead to a correct inverse solution.

CONCLUSION

In this thesis, we have seen an investigation of some of the underlying structure of electromagnetism at the classical level. This has been approached by introducing differential forms and developing an exterior algebra for electromagnetic quantities expressed as differential forms. In an applied mathematical sense, the outstanding advantage is the ease with which complicated manipulations in electromagnetic theory may be referred to basic principles, making an assessment of their logic much simpler. The material on power and energy relations in Chapter 4 is a particularly clear example of this, as is the material on Green's theorems in Chapter 2. Other subjects of special interest that have been examined include the completeness of the differential structure (§3.C) and the commutativity of this structure with respect to sequences of the defined operations d, δ, Δ, $*$ and $*^{-1}$ (§4.A).

Besides a direct continuation of the present ideas, one can envision two other directions for this work to follow. The first involves the possibility that relationships between electromagnetism and geometry can be studied to provide a more fundamental understanding of each, so that the meaning of such properties as commutativity and product structure behavior will become clear. The second concerns the possibility that a wide variety of field phenomena can be understood using exterior differential algebra. Misner, Thorne, and Wheeler, in their book on gravitation (1970), make extensive use of differential forms as a basic tool in field theory. Considering those properties of the electromagnetic differential structure which become evident when using differential forms, it seems likely that similar properties will be found for other systems of partial differential equations representing physical phenomena.

SYMBOL AMBIGUITY

Because of conventional practices in notation, certain symbols have 2 or even 3 distinct meanings, depending on their usage. The following list should resolve any ambiguities that arise in this thesis:

δ : Codifferential symbol (used throughout).
δ : Kronecker delta (pages 14 and 62).
δ : Dirac δ-function (page 10, §§3.F, 5.B and 5.C).

σ : Vector space basis elements (§§1.A, 1.C and 1.D).
σ : Permutation group (page 16 only).
σ : Electrical conductivity (page 75 and §5.A).

d : Differential symbol (used throughout).
d : Distance Measure (Equation (1.a.3a) only).

∂ : Partial differential symbol (when used as $\partial/\partial x$).
∂ : Boundary operator symbol (when used as ∂c).

c : Speed of light (used throughout).
c : A general chain of simplex elements (Chapters 1 and 2).

μ : A general 2-form (Chapters 1 and 2, and §§3.A and 3.B).
μ : Permeability (§3.B on, usually together with the permittivity symbol ε).

α,β,γ : Arbitrary R^3 differential forms (Chapters 1 and 2).
α,β,γ : Fundamental electromagnetic differential forms (Chapters 3 and 4).

ω : Arbitrary R^3/t differential form (Chapters 3 and 4).
ω : Angular frequency (Chapters 3 and 5 in time-harmonic functions).

Φ : Scalar potential (Chapters 3 and 4, and §5.A).
Φ : Scalar field (§§5.B and 5.C).

ρ : Charge density (Chapters 3 and 4, and §5.A).
ρ : Source density (§§5.B and 5.C).

$\nearrow$: Indicates a zero quantity (used throughout).

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