Estimation and Testing in Quantitative Linear Models with Autocorrelated Errors

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DEDICATED TO

My grandmother

FATMA SÜNDÜR

And

My Mother

MERYEM ALPARGU

Abstract

The efficiency of estimation procedures and the validity of testing procedures in simple and multiple quantitative linear models with autocorrelated errors have been studied in this thesis. The importance of the nature of the explanatory variable(s), fixed and trended versus purely random or following a first-order autoregressive [AR(1)] process, has been emphasized in Monte Carlo studies. The estimation procedures were compared on the basis of different measures of efficiency, relative to OLS or GLS, depending on the context. The estimation procedures studied include the Ordinary Least Squares (OLS), Generalized Least Squares (GLS), estimated GLS, Maximum Likelihood (ML), Restricted Maximum Likelihood (REML), First Differences (FD) and original First-Difference Ratios (FDR). The derived testing procedures were compared on the basis of a condition of strict validity as well as a criterion taking the variability of empirical significance levels into account.

In a preliminary step, the conflicting statements made in the literature concerning estimation in quantitative linear models with autocorrelated errors were sorted out. Unlike the efficiency of estimation procedures, the validity of testing procedures had been studied less extensively before. One of the main results of this thesis is that the more efficient of two estimators does not necessarily provide a more valid testing procedure for the parameter of interest. First, FD and FDR are highly inefficient relative to OLS, but they generally provide a valid test for the combinations of sample size and error autocorrelation parameter considered, whatever the nature of the explanatory variable(s) may be. Second, almost all the testing procedures, including the classical t-test and some modified t-tests of the slope, satisfy the criterion of validity in simple linear regression when the explanatory variable is purely random and the errors follow an AR(1) process.

An explanation in terms of effective sample size is given. Third, ML and REML are equally efficient for large sample sizes, and at the same time REML provides a test of the slope that is more valid than the ML testing procedures. These features are illustrated in an application to environmental data.

Résumé

Cette thèse a pour but l'étude de l'efficacité de certaines estimations et certains tests statistiques appliqués à des modèles linéaires simples et multiples, dont les erreurs sont autocorrelées. Plus précisément, on applique la méthode de Monte-Carlo où l'on met en évidence les différents types de variable indépendante x; soit fixée, soit purement aléatoire ou encore, aléatoire et suivant un processus AR(1). Ceci, nous amène à comparer les procédures d'estimations étudiées à l'aide des méthodes des moindres carrés, ordinaire (OLS) ou généralisée (GLS) selon le contexte, en termes de leur efficacité relative. Parmis les procédures d'estimations considérées, nous retrouvons les méthodes OLS et GLS, GLS estimée, maximum de vraisamblance (ML), maximum de vraisamblance restraint (REML), différence première (FD) et notre propre méthode, différence première de rapports (FDR). Ces mêmes procédures d'estimations sont alors comparées entre elles à partir de conditions strictes de validité et de critères tenant compte de la variabilité des niveaux de signification empirique.

Nous obtenons ainsi des résultats nous permettant de faire resortir certaines contradictions présentes dans plusieurs articles traitant des estimations sur des modèles linéaires quantitatifs dont les erreurs sont autocorrelées. Contrairement aux procédures d'estimations, peu de travaux sur l'analyse de la validité des tests ont été réalisés. L'un des principaux résultats contenus dans cette thèse, nous permet de conclure qu'une meilleure efficacité d'un estimateur donné ne mème pas nécéssairement à une meilleure validité de la procédure d'estimation pour le paramètre en question. Ainsi, les procédures FD et FDR sont tout-à-fait inefficaces comparées à la méthode OLS, mais conduisent à des tests valides pour les différentes valeurs de taille d'échantillon et d'erreurs autocorrelés, quelque soit le type de variable indépendante considérée. De plus, une grande majorité de tests, incluant les test classiques "t-test" et "t-test modifié" sur la pente, satis-

Contributions of Authors

The results of this thesis are presented in five chapters numbered 2 to 6 and an appendix, from which a number of manuscripts will be derived for publication in scientific journals. All of these manuscripts will be co-authored by Gülhan Alpargu and Pierre Dutilleul. Gülhan Alpargu has carried out all the Monte Carlo studies and has prepared a first draft of each manuscript. Pierre Dutilleul has reviewed every step in each part of the project and has edited all the manuscripts. Both authors have participated in the design of each study.

The first manuscript originally titled "Efficiency Analysis of Ten Estimation Procedures for Quantitative Linear Models with Autocorrelated Errors" (Chapter 2) has been published in Volume 69 of the *Journal of Statistical Computation and Simulation* in July 2001. Chapter 2, in which the focus is on estimation aspects, serves as a basis for the following chapters, in which the focus is rather on testing aspects (especially Chapters 4 and 5).

Chapter 3 is entitled "Efficiency and Validity Analyses of Two-Stage Estimation and Testing Procedures in Quantitative Linear Models with AR(1) Errors". The manuscript is presently in revision for possible publication in *Communications in Statistics–Simulation and Computation*. In this chapter, the mathematical proof is given for a new estimator of the error autocorrelation parameter. The two resulting two-stage estimation procedures and six others available in the literature are assessed for their efficiency relative to Generalized Least Squares, and the derived testing procedures are assessed for their validity.

Chapter 4 is entitled "Is the Classical t-Test of the Slope Really Invalid in Linear Regression Models with Autocorrelated Errors?". The manuscript will be revised and submitted to the Canadian Journal of Statistics for publication. The explanatory variable in this chapter is purely random. This study originated from conflicting statements made in the literature about the validity of testing procedures in correlation analysis with time series and the invalidity of testing procedures in regression analysis with spatial data.

A revised version of "To Be or Not To Be Valid in Testing the Significance of Slopes in Quantitative Linear Models with Autocorrelated Errors" (Chapter 5) will be submitted to *Computational Statistics and Data Analysis* for publication very soon. The explanatory variable here is fixed and trended or random and autocorrelated.

The extension to multiple linear quantitative models is made in Chapter 6. The nine combinations of two explanatory variables [i.e., both fixed, both purely random, both AR(1), and the six mixed cases] are considered in the efficiency analysis of estimation procedures and validity analysis of testing procedures. The simulation results for the mixed cases, along with the example with environmental data that motivated this part of the project, will be presented in a fifth manuscript to be submitted to the *Journal of Agricultural, Biological and Environmental Statistics*. The results of the Monte Carlo study in which the two explanatory variables are both fixed, both purely random, or both AR(1) are reported in an appendix, and will provide the material for a sixth manuscript to be submitted to the *Journal of Statistical Planning and Inference* or the *Journal of Statistical Computation and Simulation*.

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Chapter 1

Introduction

In this chapter, a literature review on the analysis of quantitative linear models with autocorrelated errors is presented, and the main properties of classical estimation and testing procedures are described.

1.1 Literature Review

Consider a quantitative linear model

$$y = X\beta + \varepsilon, \tag{1.1}$$

where y is an $n \times 1$ observable random vector; β is a $q \times 1$ unknown vector to be estimated; X is an $n \times q$ matrix whose first column is a column of ones; ε is an $n \times 1$ unobservable random vector of errors with expected value 0 and variance-covariance matrix $\text{Cov}(\varepsilon) = \Sigma$. Throughout this thesis, any matrix X is assumed to be of full rank, $q < n$.

Many particular cases, depending on the distribution of ε and X as well as the rank of X, have been studied in quantitative linear models. The columns of X may be all non-random, all random, or mixed random and non-random. Also, the errors in (1.1) may be correlated or not. If they are uncorrelated and the variances are the same, then the Ordinary Least Squares (OLS) method provides the Best Linear Unbiased Estimator (BLUE) of β. If there is autocorrelation among the errors and/or their variances are unequal, then the Generalized Least Squares (GLS) estimator is the BLUE. The Maximum Likelihood (ML) and Restricted Maximum Likelihood (REML) methods are used to estimate β when the probability distribution of the errors is known. First differencing (FD) reduces the autocorrelation among errors, prior to estimating the slope parameter(s) on the first differences. In the literature, the efficiency of the different estimators has been studied to some extent when there is temporal or spatial dependency among the errors and/or the explanatory variables.

Rao and Griliches (1969) studied the efficiency of the OLS and GLS estimators and of two-stage estimators such as the non-linear estimator and the Cochrane-Orcutt (CO), Durbin, and Prais-Winsten (PW) estimators (see Section 1.7), when the errors and the explanatory variables in (1.1) follow a first-order autoregressive or AR(1) process. They concluded that none of the procedures performed unilaterally better than the others over the range of parameter values considered, but the two-stage procedures performed better than the others when the value of the autocorrelation parameter of the errors, ρ, is greater than or equal to 0.3 in absolute value (i.e., for moderate to strong autocorrelation). They concluded that the non-linear estimator was not more efficient than the other two-stage estimators.

Patterson and Thompson (1971) introduced the method of REML to obtain unbiased estimates of the variance components in a general linear model. In general, the ML and REML estimators provide very similar results. However, if they differ substantially, the REML estimator is to be preferred. Harville (1974) gave a Bayesian interpretation of REML. Tunnicliffe-Wilson (1989) used it under the name of "marginal likelihood" in time series analysis. He showed that REML coped much better than ML when the variance-covariance of the errors, Σ, is close to singularity. More recently, Cullis and McGilchrist (1990) and Verbyla and Cullis (1990) applied it to longitudinal data. Diggle *et al.* (1996, section 4.5) gave a very interesting summary.

Martin (1974) discussed the use of the OLS estimator in terms of bias in the presence of positive spatial dependency among the errors and/or the explanatory variables. He also examined the efficiency of the spatial FD procedure. He referred to "Student" (1914) in correlation analysis and Lebart (1969) in factor analysis for published formal studies of the use of differencing to reduce the effect of spatial dependency. Following his Monte Carlo study, Martin (1974) concluded that the spatial FD procedure reduces the rate of false statement of the significance of the parameter of interest when errors and/or explanatory variables are positively autocorrelated in space.

Maeshiro (1976) studied the properties of the OLS and CO estimators when the independent variable is trended, or not, and the random errors follow an AR(1) process. From his simulation study, Maeshiro drew conclusions that contradicted previous findings when ρ is known. First, he found that the CO procedure had reduced instead of increased efficiency in many cases. Moreover, the author disagreed with the advice given in econometrics and statistics texbooks, according to which first differences may be used in a regression model only when ρ is close to 1.

Beach and Mackinnon (1978) argued that the first transformed data should not be disregarded in the ML procedure, which is equivalent to the PW procedure in regression models with AR(1) errors. Note that whereas the first transformed datum is disregarded in the CO procedure, the other $n-1$ transformed data are included in both the ML procedure and the CO procedure. Beach and Mackinnon developed a computationally efficient technique for maximizing the full likelihood function, in which the first observation is taken into account. Also, the stationarity condition $|\rho| < 1$ is included as an a priori condition in the evaluation of the likelihood function. By means of theoretical arguments, the authors showed that their estimator was superior to the CO estimator.

Spitzer (1979) basically replicated the study of Rao and Griliches (1969), including the ML estimator. His simulation results were in conflict with those of Rao and Griliches, as the non-linear and ML estimators appeared to be efficient for the sample size considered (i.e., 20). According to Spitzer, the discrepancy between his results and those of Rao and Griliches might be due to the fact that (1) Rao and Griliches' ML estimator was not efficient for small sample sizes because the Jacobian term was ignored, and/or (2) Rao and Griliches' non-linear estimation procedures had problems of convergence that might have been caused by a programming error or by the use of second derivatives that were not computed analytically.

Park and Mitchell (1980) studied the small-sample properties of the OLS and GLS estimators and the CO and PW estimators, with and without iteration in the estimation of ρ, in linear regression models with AR(1) errors and trended explanatory variables. They concluded that the PW procedures performed better than the CO procedures and that the iterated PW procedure was the best among the estimation procedures considered. They also noticed that previous Monte Carlo studies used a wrong estimator of the autocorrelation parameter in the CO and PW procedures.

Cook and Pocock (1983) suggested the examination of OLS residuals as an ad hoc procedure for finding the parametric structure of autocorrelated errors. They illustrated the application of their method on a data set from the British Regional Heart Study (BRHS). Their results showed that the classical t-test overstated the significance of the regression coefficients, whereas their method, which incorporates an autocorrelation structure of the errors, substantially reduced the statistical significance of the coefficients. At the time of its development, the authors' procedure required substantial computing resources for large data sets.

In their landmark book, Upton and Fingleton (1985, pp. 282-283) claimed that the autocorrelation among errors invalidate the classical t- and F-tests in linear regression models because the division of the slope estimator by an underestimate of its standard error inflates the Type I error risk of the t-test, for instance. In claiming this, the authors did not specify the nature and type of explanatory variables.

This was an overview of the estimation and testing procedures available for the analysis of linear quantitative models with autocorrelated errors. In the following sections, these procedures are reviewed in greater detail.

1.2 Ordinary Least Squares

Assume the Gauss-Markov properties (Graybill 1976) are satisfied in (1.1). A necessary condition for $\varepsilon'\varepsilon$ to be minimized is $\partial\varepsilon'\varepsilon/\partial\beta = 0$. The OLS estimator of β and its covariance matrix are

$$\hat{\beta}_{\mathrm{OLS}} = (X'X)^{-1}X'y \quad \text{and} \quad \mathrm{Cov}(\hat{\beta}_{\mathrm{OLS}}) = \sigma^2(X'X)^{-1},$$

where σ^2 is estimated by $\hat{\sigma}^2_{\mathrm{OLS}} = (y - X\hat{\beta}_{\mathrm{OLS}})'(y - X\hat{\beta}_{\mathrm{OLS}})/(n-q)$.

$\hat{\beta}_{\mathrm{OLS}}$ and $\hat{\sigma}^2_{\mathrm{OLS}}$ are unbiased estimators of β and σ^2, respectively. Moreover, the *Gauss-Markov theorem* ensures that $\hat{\beta}_{\mathrm{OLS}}$ is the best linear unbiased estimator (BLUE) of β.

1.3 Generalized Least Squares

If there are correlations among all the errors and/or all the errors do not have the same variances in (1.1), then the covariance matrix of the errors is in the form of $\Sigma = \sigma^2 V$, where V is a symmetric positive definite matrix (Aitken 1935). Then, the BLUE of β and its covariance matrix are

$$\hat{\beta}_{\mathrm{GLS}} = (X'V^{-1}X)^{-1}X'V^{-1}y \quad \text{and} \quad \mathrm{Cov}(\hat{\beta}_{\mathrm{GLS}}) = \sigma^2(X'V^{-1}X)^{-1},$$

where σ^2 estimated by $\hat{\sigma}^2_{\mathrm{GLS}} = (y - X\hat{\beta}_{\mathrm{GLS}})'(y - X\hat{\beta}_{\mathrm{GLS}})/(n-q)$.

Another way to obtain $\hat{\beta}_{\mathrm{GLS}}$ and $\mathrm{Cov}(\hat{\beta}_{\mathrm{GLS}})$ is to premultiply (1.1) by P such that $V^{-1} = P'P$, and use the transformed variable in the OLS formulas.

1.4 Maximum Likelihood

Assume $\varepsilon \sim N_n(0, \sigma^2 I)$ in (1.1). Then, the joint likelihood function of ε is

$$L = 1/(2\pi\sigma^2)^{n/2} \exp\{-\varepsilon'\varepsilon/(2\sigma^2)\},$$

or similarly,

$$\log L = -(n/2)\log 2\pi - (n/2)\log\sigma^2 - \{1/(2\sigma^2)\}(y'y - 2\beta'X'y + \beta'X'X\beta).$$

Differentiating $\log L$ with respect to β and σ^2 and then equating the derivatives to zero yields

$$\hat{\beta}_{\mathrm{ML}} = (X'X)^{-1}X'y \quad \text{and} \quad \hat{\sigma}^2_{\mathrm{ML}} = \{1/n\}(y'y - 2\beta'X'y + \beta'X'X\beta).$$

It follows that the OLS and ML estimators of β are the same, which is not the case for σ^2. Note that $\hat{\sigma}^2_{\mathrm{ML}}$ is a biased estimator of σ^2.

Now, assume that ε is spatially correlated in (1.1) (known as *the spatial disturbance model*):

$$\varepsilon = \rho W\varepsilon + u \quad \text{or} \quad u = (I - \rho W)\varepsilon = A\varepsilon, \tag{1.2}$$

where $A = (I - \rho W)$, W is an $n \times n$ known weight matrix, and $u \sim N_n(0, \sigma^2 I)$. Then, the likelihood function of ε is

$$\log L = k - (n/2)\log\sigma^2 - (y'A'Ay - 2\beta'X'A'Ay + \beta'X'A'AX\beta)/(2\sigma^2) + \log|A|,$$

where $k = -n/2\log 2\pi$, $|A|$ is the absolute value of the determinant of the matrix A, known as the *Jacobian of the transformation*. The ML estimators of β and σ^2

are obtained from the equations.

$$\partial \log L/\partial\beta = -(-2X'A'Ay + 2X'A'AX\beta)/(2\sigma^2) = 0,$$

$$\partial \log L/\partial\sigma^2 = -n/(2\sigma^2) + (y'A'Ay - 2\beta'X'A'Ay + \beta'X'A'AX\beta)/2\sigma^4 = 0.$$

It follows that

$$\hat{\beta}_{\text{ML}} = (X'A'AX)^{-1}X'A'Ay \quad \text{and} \quad \hat{\sigma}^2_{\text{ML}} = (Ay)'P(Ay)/n,$$

where $P = I - (AX)\{(AX)'(AX)\}^{-1}(AX)'$.

If ρ in (1.2) is unknown, then the estimate $\hat{\rho}_{\text{ML}}$ of ρ is found by maximizing $\log L$ or, equivalently, minimizing

$$M = \log\hat{\sigma}^2 - (2/n)\log|A| \quad \text{or} \quad M^* = \log(Ay)'P(Ay) - (2/n)\log|A|.$$

For further discussion, see Cliff and Ord (1981), Doreian (1980), Ord (1975), and Ripley (1981).

In order to find the asymptotic variance-covariance matrix F of the estimators $\hat{\beta}_{\text{ML}}$, $\hat{\sigma}^2_{\text{ML}}$ and $\hat{\rho}_{\text{ML}}$, the expected value of the second partial derivatives of $\log L$ are needed. Then, $F = I^{-1}(\theta)$, where $I(\theta) = -\text{E}(\partial^2 \log L/\partial\theta_r\partial\theta_s)$ with θ_r and θ_s as the parameters to be estimated. $I(\theta)$ is called the information matrix of θ. The first partial derivative of $\log L$ with respect to ρ is

$$\partial \log L/\partial\rho = \partial \log|A|/\partial\rho - (1/2\sigma^2)\partial(\varepsilon'A'A\varepsilon)/\partial\rho, \tag{1.3}$$

where

$$\partial \log|A|/\partial\rho = \partial/\partial\rho\{\sum_{i=1}^{n}\log(1-\rho\lambda_i)\} = -\sum_{i=1}^{n}\lambda_i/(1-\rho\lambda_i),$$
$$\partial(\varepsilon'A'A\varepsilon)/\partial\rho = \partial/\partial\rho\{\varepsilon'(I-\rho W')(I-\rho W)\varepsilon\} = -2u'W\varepsilon,$$

with λ_i the eigenvalues of W. Also, the second partial derivatives of $\log L$ are

$$\partial^2 \log L/\partial(\sigma^2)^2 = n/2\sigma^4 - (1/\sigma^6)\varepsilon'A'A\varepsilon = -n/2\sigma^4,$$

$$\partial^2 \log L/\partial\rho\partial\sigma^2 = (1/2\sigma^2)\partial(\varepsilon'A'A\varepsilon)/\partial\rho = (-1/\sigma^4)\, u'W\varepsilon,$$

$$\partial^2 \log L/\partial\rho\partial\beta = (1/2\sigma^2)\partial/\partial\beta\{u'W(y-X\beta)\} = (-1/2\sigma^2)u'WX,$$
$$\partial^2 \log L/\partial\rho^2 = \alpha - (\varepsilon'W'W\varepsilon)/\sigma^2,$$
$$\partial^2 \log L/\partial\beta^2 = (-1/\sigma^2)X'A'AX,$$
$$\partial^2 \log L/\partial\beta\partial\sigma^2 = 0,$$

where $\alpha = -\sum_{i=1}^{n} \lambda_i^2/(1-\rho\lambda_i)^2$. Then, their expected values are

$$\mathrm{E}(\partial^2 \log L/\partial\beta^2) = (-1/\sigma^2)X'A'AX,$$
$$\mathrm{E}(\partial^2 \log L/\partial(\sigma^2)^2) = -n/2\sigma^4,$$
$$\mathrm{E}(\partial^2 \log L/\partial\rho^2) = \alpha - \mathrm{tr}(B'B),$$
$$\mathrm{E}(\partial^2 \log L/\partial\rho\partial\sigma^2) = (-1/\sigma^2)\mathrm{tr}B,$$
$$\mathrm{E}(\partial^2 \log L/\partial\rho\partial\beta) = \mathrm{E}(\partial^2 \log L/\partial\beta\partial\sigma^2) = 0,$$

where $B = WA^{-1}$. Hence,

$$F(\hat{\sigma}^2_{\mathrm{ML}},\ \hat{\rho}_{\mathrm{ML}},\ \hat{\beta}_{\mathrm{ML}}) = \sigma^4 \begin{pmatrix} n/2 & \sigma^2\mathrm{tr}(B) & 0' \\ \sigma^2\mathrm{tr}(B) & \sigma^4(\mathrm{tr}(B'B)-\alpha) & 0' \\ 0 & 0 & \sigma^2(AX)'AX \end{pmatrix}^{-1},$$

and if $\rho = 0$, then

$$F(\hat{\sigma}^2_{\mathrm{ML}},\ \hat{\beta}_{\mathrm{ML}}) = \sigma^4 \begin{pmatrix} n/2 & 0' \\ 0 & \sigma^2X'X \end{pmatrix}^{-1}. \tag{1.4}$$

The ML estimator has some desirable properties for finite or large sample sizes. First, if there exists an estimator which reaches the minimum variance bound (MVB), then it is the ML estimator. Second, if the MVB is not attained, then the ML estimator will have the minimum variance among the other (linear or nonlinear; unbiased or biased) estimators. Third, the ML estimator is often an unbiased estimator. Fourth, for any sample size, it holds the invariant property, i.e., if $\hat{\theta}$ is the ML estimator of θ, then $h(\hat{\theta})$ is the ML estimator of the one-to-one function $h(\theta)$. For large sample sizes, under some regularity conditions (Schmidt

1976), it is consistent, asymptotically normally distributed and asymptotically efficient. Thus, the ML estimator reaches the Cramér–Rao lower bound with its asymptotic variance-covariance matrix, i.e., $\hat{\theta} \sim \mathrm{AN}(\theta, I^{-1}(\theta))$, where AN stands for "asymptotically normally distributed".

Depending on the distribution of the explanatory variables and the errors, the OLS, GLS and ML methods may provide unbiased, consistent or efficient estimators. In the following sections, we study the asymptotic properties of these estimators.

1.5 Non-Random Explanatory Variables

Let X be a non-random matrix and $\Sigma = \sigma^2 I$ in (1.1) with $\lim_{n\to\infty} X'X/n = Q$, a $q \times q$ finite and nonsingular matrix. $\hat{\beta}_{\mathrm{OLS}}$ is then a consistent estimator because it is unbiased and its covariance matrix vanishes asymptotically, i.e.,

$$\lim_{n\to\infty}(X'X)^{-1} = \lim_{n\to\infty}(1/n)(\lim_{n\to\infty} X'X/n)^{-1} = 0.$$

An alternative proof of the consistency of $\hat{\beta}_{\mathrm{OLS}}$ uses the probability limit plim. Since $\mathrm{E}(X'\varepsilon/n) = 0$ and $\lim_{n\to\infty} \mathrm{E}\{(X'\varepsilon/n)(X'\varepsilon/n)'\} = 0$, that is, $\mathrm{plim}\,(X'\varepsilon/n) = 0$, it follows that

$$\mathrm{plim}\ \hat{\beta}_{\mathrm{OLS}} = \beta + (\mathrm{plim}\ X'X/n)^{-1}\mathrm{plim}\ (X'\varepsilon/n) = \beta.$$

The consistency of $\hat{\sigma}^2_{\mathrm{OLS}}$ can be shown similarly.

The following theorem (Schmidt 1976) states the conditions in which the asymptotic normal distribution of $\hat{\beta}_{\mathrm{OLS}}$ can be achieved when the distribution of ε in (1.1) is unknown.

Theorem 1.1 *Let $\varepsilon_i \quad (i = 1, \ldots, n)$ be independently and identically distributed random variables with mean zero and finite variance σ^2. Let the elements of X be uniformly bounded and $Q = \lim_{n\to\infty} X'X/n$ be finite and nonsingular. Then,*

$X'\varepsilon/\sqrt{n}$ converges in distribution to a normal distribution with mean zero and covariance matrix $\sigma^2 Q$, or equivalently,

$$X'\varepsilon/\sqrt{n} \sim \mathrm{AN}(0, \sigma^2 Q). \tag{1.5}$$

See Schmidt (1976, pp. 56-60) for the proof.

Theorem 1.2 *$\sqrt{n}(\hat{\beta}_{\mathrm{OLS}} - \beta)$ is asymptotically normally distributed with asymptotic mean* 0 *and asymptotic covariance matrix $\sigma^2 Q^{-1}$.*

Proof: (1.5) and $\sqrt{n}(\hat{\beta}_{\mathrm{OLS}} - \beta) = (X'X/n)^{-1}(X'\varepsilon/\sqrt{n})$ prove that

$$\sqrt{n}(\hat{\beta}_{\mathrm{OLS}} - \beta) \sim \mathrm{AN}(0, \sigma^2 Q^{-1}) \quad \text{or} \quad \hat{\beta}_{\mathrm{OLS}} \sim \mathrm{AN}(\beta, \sigma^2 (X'X)^{-1}).$$

On the basis of the previous theorem, all conventional t- and F-tests based on $\hat{\beta}_{\mathrm{OLS}}$ are valid asymptotically. Therefore, the significance of each individual slope estimator and the overall significance of the vector of slope estimators can be assessed in the classical way when the distribution of ε in (1.1) is unknown, provided the sample size is sufficiently large.

The OLS (or the ML) estimator of β reaches the MVB. However, neither the OLS nor the ML estimator of σ^2 does. Nevertheless, using the following lemma, we show that $\hat{\sigma}^2_{\mathrm{ML}}$ has a smaller variance than $\hat{\sigma}^2_{\mathrm{OLS}}$. Actually, $\hat{\sigma}^2_{\mathrm{ML}}$ is the smallest.

Lemma 1.1 *Let A be an $n \times n$ symmetric and idempotent matrix and $\varepsilon \sim N(0, \sigma^2 I)$. Then $\varepsilon' A \varepsilon/\sigma^2 \sim \chi^2(v)$, where $v = \mathrm{tr}A$.*

Applying the above lemma to $\hat{\sigma}^2_{\mathrm{OLS}}$ and $\hat{\sigma}^2_{\mathrm{ML}}$, and using $\mathrm{Var}(\chi^2(v)) = 2v$, we get

$$\mathrm{Var}(\hat{\sigma}^2_{\mathrm{OLS}}) = 2\sigma^4/(n-q) \quad \text{and} \quad \mathrm{Var}(\hat{\sigma}^2_{\mathrm{ML}}) = 2\sigma^4(n-q)/n^2.$$

Clearly, $\mathrm{Var}(\hat{\sigma}^2_{\mathrm{ML}})$ is smaller than $\mathrm{Var}(\hat{\sigma}^2_{\mathrm{OLS}})$.

So far, we have assumed that $\Sigma = \sigma^2 I$. Let us consider here that $\Sigma = \sigma^2 V$, $\lim_{n\to\infty} X'X/n = Q$ is a $q \times q$ finite and nonsingular matrix, $\lim_{n\to\infty} X'VX/n = R$ is a $q \times q$ finite matrix, and $\lim_{n\to\infty} X'V^{-1}X/n$ is finite and nonsingular. Then,

$$\mathrm{Cov}(\hat{\beta}_{\mathrm{OLS}}) = (X'X)^{-1}X'\Sigma X(X'X)^{-1}.$$

$(X'X)^{-1}X'\Sigma X(X'X)^{-1} - (X'\Sigma^{-1}X)^{-1}$ is a non-negative definite matrix since $\hat{\beta}_{\text{GLS}}$ is the BLUE.

$\hat{\beta}_{\text{OLS}}$ does not require the knowledge of Σ, as does $\hat{\beta}_{\text{GLS}}$. Therefore, many researchers have been looking for conditions on X and Σ, in which $\hat{\beta}_{\text{OLS}}$ and $\hat{\beta}_{\text{GLS}}$ are equally efficient, or in which they are the same (Zyskind 1967). It is important to realize that $\hat{\beta}_{\text{OLS}} = \hat{\beta}_{\text{GLS}}$ does not imply that $\Sigma = \sigma^2 I$.

Zyskind (1967) has proved that $\hat{\beta}_{\text{OLS}}$ and $\hat{\beta}_{\text{GLS}}$ are the same if and only if the span of X is invariant under the matrix V (see also Kruskal 1968 and Krämer 1980). Furthermore, McElroy (1967) has shown that OLS estimators are the best linear unbiased estimators if and only if the errors have the same variance and the same non-negative coefficient of correlation between each pair. He has proved that a necessary and sufficient condition for $X'\Sigma^{-1}y = 0$ for all X is that Σ be of the form $(1-\rho)I + \rho ee'$, where $e' = (1, 1, \ldots, 1)$.

It is well known that $\hat{\beta}_{\text{OLS}}$ and $\hat{\beta}_{\text{GLS}}$ are equal with probability one if and only if their covariance matrices are equal, i.e., $(X'X)^{-1}X'\Sigma X(X'X)^{-1} = (X'\Sigma^{-1}X)^{-1}$ (see Puntanen and Styan 1989; Baksalary, Puntanen and Styan 1991). Bloomfield and Watson (1975) have given the lower bound for the relative efficiency of GLS estimators with respect to OLS estimators as being

$$\frac{|X'X|^2}{|X'\Sigma X| \cdot |X'\Sigma^{-1}X|} \geq \prod_{i=1}^{k} \frac{4\lambda_i\lambda_{n-i+1}}{(\lambda_i + \lambda_{n-i+1})^2}, \tag{1.6}$$

where $\lambda_1 \geq \cdots \geq \lambda_n > 0$ are the eigenvalues of the positive definite matrix Σ and X is an $n \times q$ matrix of rank q with $n \geq 2q$. Alpargu *et al.* (1997) followed the proof given by Bloomfield and Watson (1975) with a modification due to Drury. The inequality (1.6) is known as a special case of the Kantrovich Inequality (see Alpargu 1996a; Alpargu and Styan 1996b-1996d; Watson *et al.* 1997).

The unbiasedness and consistency of $\hat{\beta}_{\text{OLS}}$ are still valid when $\Sigma = \sigma^2 V$. $\hat{\beta}_{\text{OLS}}$ is consistent since

$$\lim_{n\to\infty}(X'X)^{-1}X'VX(X'X)^{-1} = (\lim_{n\to\infty} X'X/n)^{-1}(\lim_{n\to\infty} X'VX/n^2)(\lim_{n\to\infty} X'X/n)^{-1}$$

equals zero. Unfortunately, $\hat{\sigma}^2_{\text{OLS}}$ is biased because $\text{tr}(MV) \neq n - q$ in general, whereas

$$\text{E}(\hat{\sigma}^2_{\text{OLS}}) = c\,\text{E}(\varepsilon' M\varepsilon) = c\,\text{E}(\text{tr}(\varepsilon' M\varepsilon)) = c\,\sigma^2\text{tr}(MV)$$

where $c = 1/(n-q)$ and $M = I - X(X'X)^{-1}X'$. $\hat{\sigma}^2_{\text{OLS}}$ is inconsistent because

$$\text{plim}\ \hat{\sigma}^2_{\text{OLS}} = \text{plim}\ \text{E}(\text{tr}(\varepsilon' M\varepsilon))/(n-q) = \text{plim}\ (\sigma^2/n)\text{tr}(MV) \neq \sigma^2. \tag{1.7}$$

In the previous sections, we have shown that if $\Sigma = \sigma^2 I$, then $\hat{\beta}_{\text{OLS}}$ and $\hat{\sigma}^2_{\text{OLS}}$, as well as $\hat{\beta}_{\text{GLS}}$ and $\text{Cov}(\hat{\beta}_{\text{GLS}})$ which can be obtained by applying the OLS formulas to the transformed variables $\Sigma^{-1/2}y$, are unbiased, consistent and efficient estimators. Moreover, the asymptotic distribution of $\hat{\beta}_{\text{GLS}}$ is

$$\hat{\beta}_{\text{GLS}} \sim \text{AN}(\beta, \sigma^2(X'V^{-1}X)^{-1}).$$

1.6 Random Explanatory Variables

Explanatory variables can not be always controlled by the observer or the data collector. This means they are a realization of some stochastic system or process. In this section, we assume that X possesses a multivariate density function $h(X)$, which does not involve the parameters of the linear model. For example, $h(X)$ in (1.1) is not a function of β and σ^2, and X and ε are independently distributed. The estimation formulas of the OLS, GLS and, conditional on X, ML estimators are the same as those given when X is fixed. The properties of the estimators are also the same, provided the errors are not autocorrelated. For example, $\hat{\beta}_{\text{OLS}}$ is an unbiased estimator of β since

$$\text{E}(\hat{\beta}_{\text{OLS}}) = \beta + \text{E}\{(X'X)^{-1}X'\varepsilon\} = \beta + \text{E}\{(X'X)^{-1}X'\}\text{E}(\varepsilon) = \beta.$$

The covariance matrix of $\hat{\beta}_{\text{OLS}}$ is

$$\text{Cov}(\hat{\beta}_{\text{OLS}}) = \text{E}\{(X'X)^{-1}X'\varepsilon\varepsilon' X(X'X)^{-1}\}$$

of Restricted Maximum Likelihood (REML) to produce unbiased estimators of variance components in a general linear model.

Assume $\varepsilon \sim N_n(0, \Sigma)$, where $\Sigma = \sigma^2 V \neq \sigma^2 I$ in (1.1). Hence, the probability density function (pdf) of y in (1.1), conditional on X if random, is

$$f(y) = (2\pi)^{-\frac{1}{2}n}|\Sigma|^{-\frac{1}{2}} \exp\{-\frac{1}{2}(y - X\beta)'\Sigma^{-1}(y - X\beta)\}.$$

The ML estimator of β for a known Σ is the GLS estimator

$$\hat{\beta}_{\text{GLS}} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y$$

with pdf

$$f(\hat{\beta}_{\text{GLS}}) = (2\pi)^{-\frac{1}{2}q}|X'\Sigma^{-1}X|^{\frac{1}{2}} \exp\{-\frac{1}{2}(\hat{\beta}_{\text{GLS}} - \beta)'(X'\Sigma^{-1}X)(\hat{\beta}_{\text{GLS}} - \beta)\}.$$

The REML estimator is defined as the ML estimator computed on a linearly transformed set of data $y^* = My$ such that the distribution of y^* does not depend on β. We define $M = I - X(X'X)^{-1}X'$ so that y is transformed to OLS residuals. However, y^* has a singular multivariate normal distribution. Therefore, we choose any $n-q$ linearly independent columns of M to ensure a non-singular distribution to y^*. Let $z = G'y$, where $G'G = I$ and $GG' = M$ with I the $(n-q) \times (n-q)$ identity matrix.

It is easy to show that $\text{E}(z) = 0$, and z and $\hat{\beta}$ are independent. Then, the pdf of z is proportional to

$$\frac{f(y)}{f(\hat{\beta})} = (2\pi)^{-\frac{1}{2}(n-q)}|\Sigma|^{-\frac{1}{2}}|X'\Sigma^{-1}X|^{-\frac{1}{2}} \exp\{-\frac{1}{2}(y - X\hat{\beta})'\Sigma^{-1}(y - X\hat{\beta})\}.$$

Therefore, the REML estimators maximize

$$L^* = -\frac{1}{2}\log|\Sigma| - \frac{1}{2}\log|X'\Sigma^{-1}X| - \frac{1}{2}(y - X\hat{\beta})'\Sigma^{-1}(y - X\hat{\beta}),$$

whereas the ML estimators maximize

$$L = -\frac{1}{2}\log|\Sigma| - \frac{1}{2}(y - X\hat{\beta})'\Sigma^{-1}(y - X\hat{\beta}).$$

The difference between $L^\star$ and L is the extra term $\frac{1}{2}\log|X'\Sigma^{-1}X|$ in $L^\star$. Since the matrix $X'\Sigma^{-1}X$ is of order q, the ML and REML estimators will be different when q is large. In general, the two estimation methods are asymptotically equivalent as n tends to infinity for fixed q. When q gets larger, the REML estimators are known to be better than the ML estimators (Diggle *et al.* 1996).

1.8 First Differencing

Let us reconsider

$$y = X\beta + \varepsilon \quad \text{with} \quad \varepsilon = \rho W\varepsilon + u \quad \text{or} \quad \varepsilon = (I - \rho W)^{-1}u = A^{-1}u, \tag{1.8}$$

where X is a non-random or random matrix, W is the matrix of the spatial or temporal lag operator, and $u \sim N(0,\ \sigma^2 I)$. A is a positive definite matrix, but it is not necessarily symmetric because W is not symmetric with time series data, and may not be symmetric with spatial data.

The First-Difference (FD) procedure assumes $\rho \approx 1$. Premultiplying $y = X\beta + \varepsilon$ by $I - W$, we obtain

$$(I-W)y = (I-W)X\beta + (I-W)\varepsilon \quad \text{or} \quad y^\star = X^\star\beta + \varepsilon^\star,$$

where $y^\star = (I-W)y$, $X^\star = (I-W)X$ and $\varepsilon^\star = (I-W)\varepsilon$. Since $\mathrm{E}(\varepsilon^\star) = 0$ and $\mathrm{Cov}(\varepsilon^\star) = \sigma^2 I$, the transformed variables $y^\star$ and $X^\star$ can be incorporated into the computation of $\hat\beta_{\mathrm{OLS}}$ and $\mathrm{Cov}(\hat\beta_{\mathrm{OLS}})$, which provides $\hat\beta_{\mathrm{FD}}$ and $\mathrm{Cov}(\hat\beta_{\mathrm{FD}})$. It is clear that the FD transformation aims at filtering the data in order to obtain independent and identical errors prior to fitting the model.

We know that the OLS estimators are unbiased if $\rho = 0$, whereas this is the case for the FD estimators only if $\rho \approx 1$.

1.9 Two–Stage Estimators

Consider a simple linear model

$$y_t = a + bx_t + \varepsilon_t \quad \text{with} \quad \varepsilon_t = \rho\varepsilon_{t-1} + u_t \quad (t = 1, 2, \ldots, n),$$

where $u_t \sim N(0, \sigma_u^2)$, $|\rho| < 1$, $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$ where $\sigma_\varepsilon^2 = \sigma_u^2/(1-\rho^2)$, and x_t and ε_t are independently distributed.

The matrix form of the above model is

$$y = X\beta + \varepsilon, \tag{1.9}$$

where y, β, ε are $n \times 1$ vectors, and X is an $n \times 2$ matrix. The two-stage estimation procedures transform the model (1.9) to

$$Ty = TX\beta + T\varepsilon, \tag{1.10}$$

with a conformable matrix T, called transformation matrix.

1.9.1 Cochrane–Orcutt Estimator

In this procedure, the transformation matrix

$$T_1 = \begin{pmatrix} -\rho & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\rho & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & -\rho & 1 \end{pmatrix} \tag{1.11}$$

is used in (1.12). Then, the OLS method is performed on the transformed variables. If the autoregressive parameter of the errors, ρ, is unknown, then it is estimated by the sample autocorrelation coefficient at lag 1, $\hat{\rho} = \sum_{t=2}^{n} e_t e_{t-1} / \sum_{t=1}^{n-1} e_t^2$, where e_t $(t = 1, 2, \ldots, n)$ are the OLS residuals of (1.9).

1.9.2 Durbin Estimator

In this two-stage estimation procedure, an OLS estimator of ρ is estimated first, as the coefficient of y_{t-1} in the model

$$y_t = \rho y_{t-1} + (1-\rho)\alpha + \beta(x_t - \rho x_{t-1}) + \varepsilon_t - \rho\varepsilon_{t-1}.$$

or

$$y_t = \rho y_{t-1} + (1-\rho)a + bx_t - \beta\rho x_{t-1} + \varepsilon_t - \rho\varepsilon_{t-1}. \tag{1.12}$$

Thereafter, the transformation matrix T_1 is used with that estimator of ρ. It is known that the OLS estimator of ρ is consistent. Moreover, Durbin (1960) proved that the estimator of β obtained at the second stage of the estimation procedure is also consistent and asymptotically efficient.

1.9.3 Prais–Winsten Estimator

The transformation matrix

$$T_2 = \begin{pmatrix} \sqrt{1-\rho^2} & 0 & 0 & 0 & \cdots & 0 \\ -\rho & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\rho & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & -\rho & 1 \end{pmatrix}$$

is used in (1.10). If ρ is unknown, then it is estimated by $\tilde{\rho} = \sum_{t=2}^{n} e_t e_{t-1} / \sum_{t=2}^{n-1} e_t^2$. Another two-stage estimation procedure is called Durbin-Prais-Winsten. This uses the Durbin estimator of ρ in T_2.

Another two-stage estimator is called Durbin–Prais–Winsten which uses the same Durbin estimate of ρ in T_2.

1.9.4 A Non-Linear Estimator

This estimation procedure estimates ρ and β simultaneously with the non-linear constraint $\widehat{b\rho} = \hat{b}\hat{\rho}$ in (1.12). Thus, the residual sum of squares is minimized with respect to β and ρ. The resulting estimators are ML estimators and hence they are asymptotically efficient.

1.10 Hypothesis Testing

In the previous sections, the focus was on estimation. Another aspect of interest is that of testing whether pre-designated values of the parameters are indeed acceptable. In the decision process, two kinds of error can be made: the Type I error when the null hypothesis is rejected while, in fact, it is true, and the Type II error when the null hypothesis is not rejected while, in fact, it is false. The risk associated with the Type I error is called the level of significance or size of the test.

Definition 1.1 *(Valid Test) A test T is said to be valid if its actual level of significance is less than or equal to a predetermined α.*

1.10.1 Classical Tests

The significance of the individual parameters β_i in (1.1) can be assessed by a t-test. For example, if the estimator of β in (1.1) is $\hat{\beta} \sim N_q(\beta, \Sigma_{\hat{\beta}})$, then $H_0 : \beta_i = \beta_o$ versus $H_a : \beta_i \neq \beta_o$ can be tested using

$$t_{\text{obs}} = (\hat{\beta}_i - \beta_o)/\sqrt{(\widehat{\text{Var}(\hat{\beta}_i)}},$$

where $\widehat{\text{Var}(\hat{\beta}_i)}$ is the ith diagonal element of an appropriate estimator of $\Sigma_{\hat{\beta}}$, such that t_{obs} follows a t-distribution with $n-q$ degrees of freedom.

Another classical test is based on the likelihood function of the vector of observations y, $L(\theta : y)$ with $\theta \in \Omega$. If $H_0 : \theta \in \omega$ versus $H_a : \theta \in \Omega - \omega$ are the where $\bar{\sigma}_{ii}$ is the mean of the variances, $\bar{\sigma}_{i.}$ is the mean of the ith row or column of the covariance matrix, and $\bar{\sigma}_{..}$ is the mean of all elements of the covariance matrix. Note that in this section specifically, ε denotes Box's epsilon instead of the random vector of errors in (1.1). The expression of ε in matrix notation (Greenhouse and Geisser 1959) is

$$\varepsilon = \{\text{tr } (C'\Sigma C)\}^2/\{(p-1)\text{tr}(C'\Sigma C)^2\}, \tag{1.13}$$

where C is a $p \times (p-1)$ matrix of $(p-1)$ orthonormal contrasts of dimension p, i.e., $C'C = I_{p-1}$ and $CC' = I - (1/p)J$, with J the $p \times p$ matrix of ones, and Σ is the $p \times p$ population covariance matrix. In terms of eigenvalues

$$\varepsilon = (\sum_{i=1}^{p-1} \lambda_i)^2/\{(p-1)\sum_{i=1}^{p-1} \lambda_i^2\},$$

where λ_i $(i = 1, \ldots, p-1)$ are the eigenvalues of $C'\Sigma C = \Sigma B$ with $B = I - (1/p)J$.

In theory, ε ranges between $1/(p-1)$ and 1 inclusively. If $\varepsilon = 1$ (i.e., the circularity condition is satisfied, and hence, no adjustment of the numbers of degrees of freedom is required), then the classical F-test is appropriate for hypothesis testing. If $\varepsilon = 1/(p-1)$, then the strongest reduction of the numbers of degrees of freedom is applied. In practice, an estimate of ε, $\hat{\varepsilon}$ is obtained by replacing Σ in (1.13) by the sample covariance matrix $\hat{\Sigma}$. Huynh and Feldt (1976) showed that $\hat{\varepsilon}$ is seriously biased when the theoretical ε is greater than 0.75 and n is less than $2p$. Therefore, they suggested another correction factor

$$\tilde{\varepsilon} = \{n(p-1)\hat{\varepsilon} - 2\}/[(p-1)\{n-1-(p-1)\hat{\varepsilon}],$$

which is less biased than $\hat{\varepsilon}$ when the conditions on ε and n above are satisfied. It is easy to see that for any value of n and p, $\tilde{\varepsilon} \geq \hat{\varepsilon}$, given the equality $\hat{\varepsilon} = 1/(p-1)$. Whenever either correction factor $\hat{\varepsilon}$ or $\tilde{\varepsilon}$ exceeds 1, it is set to 1.

In correlation analysis, Clifford and Richardson (1985) suggested a procedure for testing the significance of the product-moment correlation coefficient, r, of two spatially autocorrelated processes. Their procedure is based on an estimate of the variance of r, which is used to calculate an effective sample size that takes the spatial autocorrelation of the two processes into account. This effective sample size is used to adjust the number of degrees of freedom of the t-test. The authors assessed the validity of their modified t-test by doing some simulation. Clifford *et al.* (1989) gave an expanded presentation of the procedure introduced by Clifford and Richardson (1985), including an extensive Monte Carlo study, various test statistics and an epidemiology application.

The adjusted number of degrees of freedom recommended by Clifford and Richardson (1985) and Clifford *et al.* (1989) is

$$\sigma_r^{-2} - 1 \quad \text{with} \quad \sigma_r^2 = \frac{\text{tr}(\Sigma_X \Sigma_Y)}{\text{tr}(\Sigma_X)\text{tr}(\Sigma_Y)},$$

where Σ_X and Σ_Y are the theoretical autocovariance matrices of partial realizations of processes X and Y, respectively. In practice, estimated autocovariance matrices are used in the formula.

Dutilleul (1993) gave the mathematical proof for the correction of a small-sample approximation in the adjusted number of degrees of freedom of the modified t-test used by Clifford et al. (1989). Dutilleul's adjusted number of degrees of freedom is

$$\sigma_r^{-2} - 1 \quad \text{with} \quad \sigma_r^2 = \frac{\text{tr}(B\Sigma_X B\Sigma_Y)}{\text{tr}(B\Sigma_X)\text{tr}(B\Sigma_Y)},$$

where Σ_X and Σ_Y are defined as before, and $B = I-(1/R)J$ with R the number of time or space sampling points (i.e., the size of the partial realizations of processes X and Y), I the $R \times R$ identity matrix, and J the $R \times R$ matrix of ones. In practice, estimated autocovariance matrices are used in the formula.

1.11 Thesis Objectives

Because of the conflicting statements available in the literature concerning the estimation aspects in quantitative linear models with autocorrelated errors (see

Section 1.1), it seemed timely to sort out the reported efficiency analyses and possibly shed new light on these estimation aspects by including the three cases of fixed and trended x, purely random x, and autocorrelated x in the same study. This defined a first objective for my thesis, as a prerequisite to any study regarding test statistics since these are based on slope estimators.

A second objective was to inquire into the testing aspects in quantitative linear models with autocorrelated errors, for which reported validity analyses are much less numerous than efficiency analyses on the estimation side. In particular, would it be possible to incorporate variants of modified tests used in repeated measures ANOVA and correlation analysis with autocorrelated sample data, and have them satisfy the validity condition in quantitative linear models with autocorrelated errors? Furthermore, would there be a testing procedure robust enough to be valid, whether the regressor is fixed or random? For instance, could such a procedure be based on data transformation? Eventually, could one find a condition that allows valid unmodified testing with autocorrelated errors, as does the circularity condition in repeated measures ANOVA?

In both estimation and testing, it was equally important, from the perspective of the user of the procedures, to define the limits of efficient estimation and valid testing in terms of required sample size, range of autocorrelation parameter values and nature of the regressor. The performance of estimation and testing procedures established on the basis of asymptotic arguments, when used with small samples, was of particular interest. Drawing such practical limits of efficiency and validity defined a third objective.

Finally, it was important to extend the efficiency and validity analyses to the case of multiple quantitative linear models with autocorrelated errors, and to illustrate the estimation and testing procedures that were found to be the most efficient and valid with an application to a real data set. After extensive Monte Carlo studies and a few mathematical proofs, such an application is a sort of "happy end" to any applied statistics project. In this study in particular, the application to real data was not merely an illustration, since the extension to multiple quantitative linear models actually originated from that application.

Chapter 2

Efficiency Analysis of Eleven Estimation Procedures for Quantitative Linear Models with Autocorrelated Errors

ABSTRACT

Many estimation procedures for quantitative linear models with autocorrelated errors have been proposed in the literature. A number of these procedures have been compared in various ways for different sample sizes and autocorrelation parameter values and for structured or random explanatory variables. In this paper,[1] we revisit three situations that were considered to some extent in previous studies, by comparing eleven estimation procedures: Ordinary Least Squares

[1]A slightly different version of this chapter will be published in the July 2001 issue (vol. 69/ no. 3) of *Journal of Statistical Computation and Simulation*, under the title "Efficiency Analysis of Ten Estimation Procedures for Quantitative Linear Models with Autocorrelated Errors". [Results for the REML procedure were not available at the time that manuscript was accepted for publication.]

(OLS), Generalized Least Squares (GLS), estimated Generalized Least Squares (six procedures), Maximum Likelihood (ML), Restricted Maximum Likelihood (REML), and First Differences (FD). The six estimated GLS procedures and the ML and REML procedures differ in the way the error autocovariance matrix is estimated. The three situations can be defined as follows: in Case 1, the explanatory variable x in the simple linear regression is fixed; in Case 2, x is purely random; and in Case 3, x is first-order autoregressive. Following a theoretical presentation, the eleven estimation procedures are compared in a Monte Carlo study conducted in the time domain, where the errors are first-order autoregressive in Cases 1-3. The measure of comparison for the estimation procedures is their efficiency relative to OLS. It is evaluated as a function of the time series length and the magnitude and sign of the error autocorrelation parameter. Overall, knowledge of the model of the time series process generating the errors enhances efficiency in estimated GLS. Differences in the efficiency of the estimation procedures between Case 1 and Cases 2 and 3 as well as differences in efficiency among procedures in a given situation are observed and discussed.

Keywords: Autocorrelated errors; First differences; Least squares; Linear models; Maximum likelihood; Restricted maximum likelihood; Monte Carlo study; Relative efficiency; Structured *versus* random explanatory variable

1. INTRODUCTION

In general terms, statistical linear models can be classified as quantitative models or qualitative models. Following Graybill (1976, p. 143), the quantitative linear models are mainly composed of the general linear model and the linear regression model, whereas the qualitative linear models are represented by the design model and the components-of-variance model. In this paper, we consider quantitative linear models fitted to time series data, although we will also mention some work in spatial statistics. For simplicity, we will refer to these models as "linear models", except when explicitly stated otherwise.

The main difference between a general linear model and a linear regression model is that the matrix X in (2.1) below consists of non-random variables under the former, whereas it consists of random variables under the latter, that is, the matrix X is structured or not, respectively. Consider

$$y = X\beta + \varepsilon, \quad \varepsilon = \rho W \varepsilon + u, \tag{2.1}$$

where y is an $n \times 1$ observable random vector, β is a $k \times 1$ unknown vector to be estimated, X is an $n \times k$ matrix of rank $k < n$, ε is an $n \times 1$ unobservable vector of random errors with zero expected value, $-1 < \rho < 1$ and $u \sim N_n(0, \sigma^2 I)$ with I the $n \times n$ identity matrix and σ^2 an unknown positive constant. In the linear regression model, X and ε are assumed to be uncorrelated. The matrix W is a weight matrix in which the weights are dependent on the lag between observations. For equally spaced observations in time, weights $w_{ii'} = 1$ if $|i - i'| = 1$, and 0 otherwise, define a first-order autoregressive process.

The Ordinary Least Squares (OLS) estimator of β in (2.1) is

$$\hat{\beta}_{\text{OLS}} = (X'X)^{-1}X'y. \tag{2.2}$$

If no autocorrelation of the errors is assumed (i.e., $\rho = 0$), then the covariance matrix of $\hat{\beta}_{\text{OLS}}$ is

$$\text{Cov}(\hat{\beta}_{\text{OLS}}) = \sigma^2 (X'X)^{-1}. \tag{2.3}$$

For any value of ρ, $\hat{\beta}_{\text{OLS}}$ is an unbiased estimator of β. It has minimum variance among the linear unbiased estimators if $\rho = 0$.

A general covariance structure for ε in a linear model was considered by Aitken (1935), among others. It is defined by $\Sigma = \sigma^2 V$, where the matrix V is positive definite. If V is known, then the Best Linear Unbiased Estimator (BLUE) of β

is the Generalized Least Squares (GLS) estimator

$$\hat{\beta}_{\text{GLS}} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y, \tag{2.4}$$

with the covariance matrix

$$\text{Cov}(\hat{\beta}_{\text{GLS}}) = (X'\Sigma^{-1}X)^{-1}. \tag{2.5}$$

If $\text{Cov}(\varepsilon) = \Sigma$ in a linear model, then the OLS estimator of β remains (2.2), but its covariance matrix is

$$\text{Cov}(\hat{\beta}_{\text{OLS}}) = (X'X)^{-1}X'\Sigma X(X'X)^{-1}. \tag{2.6}$$

The OLS estimator of β in a linear model with autocorrelated errors is known to be unbiased but inefficient (Graybill, 1976; Schmidt, 1976). On the other hand, the use of the GLS method is limited in practice because it requires the covariance matrix of the errors, Σ, which is generally unknown. When the family of distribution of the errors is known and a structure of autocovariance is postulated, then the methods of maximum likelihood (ML) and restricted maximum likelihood (REML) can be applied to estimate β and the variance and autocorrelation parameters. Actually, when there is dependency among the data, the question is whether one should incorporate it in the estimation procedure or remove it from the data prior to fitting the model. The ML and REML methods tend to belong to the former approach of incorporating dependencies, whereas the GLS method pertains to both approaches since Σ is used in (2.4) and GLS is nothing else but OLS applied to $\Sigma^{-1/2}y$.

The ML method assumes that the probability density function of the observations y_i or, equivalently, of the errors ε_i is known. The ML estimators of the parameters of model (2.1) are:

$$\hat{\beta} = (X'A'AX)^{-1}X'A'Ay \quad \text{and} \quad \hat{\sigma}^2 = (Ay)'P(Ay)/n, \tag{2.7}$$

where $A = (I - \hat{\rho}W)$ and $P = I - (AX)\{(AX)'(AX)\}^{-1}(AX)'$, and $\hat{\rho}$ minimizes (Upton and Fingleton, 1985)

$$M^* = \log(Ay)'P(Ay) - (2/n)\log|A|. \tag{2.8}$$

The REML procedure introduced by Patterson and Thompson (1971) is a simple modification of the ML procedure. Namely, the REML estimators maximize

$$L^* = -\frac{1}{2}\log|\Sigma| - \frac{1}{2}\log|X'\Sigma^{-1}X| - \frac{1}{2}(y - X\hat{\beta})'\Sigma^{-1}(y - X\hat{\beta}),$$

whereas the ML estimators maximize

$$L = -\frac{1}{2}\log|\Sigma| - \frac{1}{2}(y - X\hat{\beta})'\Sigma^{-1}(y - X\hat{\beta}).$$

The difference between L^* and L is the extra term $\frac{1}{2}\log|X'\Sigma^{-1}X|$ in L^*. As the sample size increases for a fixed number of columns of X, the ML and REML procedures provide similar estimators of the variance-covariance parameters. Otherwise, REML is to be preferred.

Definitively, the next estimation method, called differencing, aims at removing the dependency from the data prior to fitting the model. It assumes that $\rho \approx 1$ in (2.1) so that the vector of transformed errors, $(I - W)\varepsilon$, is $N_n(0, \sigma^2 I)$. The model on differences is

$$(I - W)y = (I - W)X\beta + (I - W)\varepsilon. \tag{2.9}$$

The differencing method combines the linear transformation of the data by $(I - W)$ with the OLS method performed on the transformed data. This transformation is inspired by the differencing operator used for stationarity purposes in time series analysis (Box *et al.*, 1994). A more complete comparison of the transformation $(I - W)$ and the differencing operator of time series analysis is made in the Methods section. Differencing can be used with time series data as well as with one- or two-dimensional spatial data (Martin, 1974).

Many estimation procedures for linear regression models with autocorrelated errors have been proposed in the literature. In the reported studies, several procedures have been compared for various sample sizes and autocorrelation parameter values and for structured or random explanatory variables. For instance, Rao and Griliches (1969) studied the efficiency of the OLS, GLS and ML procedures as well as some two-stage estimation procedures (e.g., Cochrane–Orcutt, Durbin, Prais–Winsten), with AR(1) explanatory variables and AR(1) errors in a temporal context. They concluded that the efficiency of the two-stage estimation procedures was superior to that of the other procedures for moderate and strong autocorrelation of the errors, and slightly lower otherwise. Martin (1974) discussed the unbiasedness conditions of the OLS procedure and studied the differencing procedure in a spatial context with purely random or autocorrelated explanatory variables. At the end of his Monte Carlo study, he concluded that the first spatial differencing procedure substantially reduced the rate of false statements of significance concerning β in the case of positive autocorrelation. The statement of Rao and Griliches (1969) about the asymptotic lack of efficiency of the OLS procedure for $\rho \neq 0$ in all cases was challenged later by Maeshiro (1976), who also challenged other reported studies. His argument was based on the distinction between trended and non-trended explanatory variables that we call structured and random explanatory variables, respectively. Spitzer (1979) replicated the study of Rao and Griliches (1969) and discussed the results that they had reported for ML. In a study similar to Rao and Griliches (1969), Park and Mitchell (1980) gave the estimate of ρ to be used in two-stage estimation procedures in order to minimize the sum of squares of the errors. Moreover, many authors have discussed the relative merits of the ML and REML estimators of variance-covariance parameters. For example, Tunnicliffe-Wilson (1989) showed that REML coped much better than ML when the covariance matrix of the errors Σ was close to singularity. This is only a sample of reported studies, but one may already retain from it that some attention must be paid to the extent to which some of the conclusions hold. It is very possible that some of them are not as general as they might seem to be.

In this paper, we revisit and complement three situations that were considered to some extent in previous studies, by comparing eleven estimation procedures for quantitative linear models with autocorrelated errors in the same Monte Carlo study. The objective is to shed some light on aspects that have not yet been investigated. This study on estimation aspects was designed as a preliminary step to studies and articles on the testing aspects that will follow.

2. METHODS

Eleven procedures derived from the OLS, GLS, ML and REML methods as well as the first-difference (FD) method are considered to estimate the slope parameter in a simple linear model with errors following a stationary AR(1) process in time. As explained below, all these procedures differ somehow in the way the autocorrelation of errors is handled. Three types of explanatory variable x are considered: 1) structured (i.e., non-random, fixed or trended); 2) purely random; and 3) following a stationary AR(1) process in time.

The covariance matrix of an $n \times 1$ random vector ε under the stationary AR(1) model in time is

$$\Sigma = \sigma^2/(1-\rho^2) \begin{pmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{n-1} \\ \rho & 1 & \rho & \cdots & \rho^{n-2} \\ \vdots & & & & \vdots \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \cdots & 1 \end{pmatrix} = \sigma_\varepsilon^2 V, \qquad (2.10)$$

where $-1 < \rho < 1$.

Using (2.4) for GLS estimation requires that ρ in (2.10) be known. Otherwise, ρ may be estimated by using the sample autocorrelation coefficient of the errors at lag 1 or some other estimator (Beach and Mackinnon, 1978). The estimated zero beyond a certain lag. Therefore, the significance of $\rho(i)$ ($i = 1, 2, \ldots, m = \text{INT}(n/4)$) can be assessed by an approximate z-test. Namely, if the approximate z-test lies between -2 and 2, then the hypothesis $\rho(i) = \rho(i+1) = \cdots = \rho(m) = 0$ was not rejected at an approximate significance level of 5%.

For comparison purposes, we have iterated the estimated GLS. The iterations were stopped when successive estimates of the slope differed by 0.001 or less.

The last three estimation procedures that we have included in our comparative study are the ML, REML and FD procedures. In the ML procedure, W was defined as $w_{ij} = 1$ if $j = i - 1$, and $w_{ij} = 0$, otherwise. As a result,

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\hat{\rho} & 1 & 0 & \cdots & \vdots \\ 0 & -\hat{\rho} & 1 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\hat{\rho} & 1 \end{pmatrix}. \tag{2.13}$$

The (1,1) element of A was changed to $\sqrt{1-\hat{\rho}^2}$ according to Beach and Mackinnon (1978) and Spitzer (1979), and the estimated value of ρ was evaluated to the nearest 0.001 following Upton and Fingleton (1985). The REML method was carried out with PROC MIXED of SAS (SAS Institute Inc., 1997). The FD procedure studied here used the same transformation matrix as in the Cochrane–Orcutt procedure of Rao and Griliches (1969), except that ρ was not estimated but replaced by 1.

In some instances, no estimated GLS slope estimate was available because the estimated covariance matrix of the errors was not positive definite. We overcame this problem by using the following theorem from Graybill (1969, p. 329): Let C be an $n \times n$ symmetric matrix. Then there exists a scalar t such that $B = C + tI$ is positive definite. Since C is symmetric, there is an orthogonal matrix P that diagonalizes C, i.e., $P'CP = D$. We used $t = \max(d_{ii}) - \min(d_{ii})$, where d_{ii} are the diagonal elements of D (and also are the eigenvalues of C).

3. MONTE CARLO STUDY

The model used for simulation was

$$y_i = a + bx_i + \varepsilon_i \text{ with } \varepsilon_i = \rho\varepsilon_{i-1} + u_i \quad (i = 1, 2, \ldots, n), \tag{2.14}$$

where a and b were fixed at 1 and 0, respectively; the u_is were i.i.d. $N(0, 1)$; and the value of ρ ranged from -0.9 to 0.9 by steps of 0.2, in addition to $\rho = 0$. The generation of autocorrelated errors followed a procedure similar to that of Dutilleul and Legendre (1992). Three situations were considered for the matrix X:

Case 1: $X = [1, x]$, where $x = (1, 2, \ldots, n)'$.

Case 2: $X = [1, x]$, where the elements of x were i.i.d. $N(0, 1)$ observations.

Case 3: $X = [1, x]$, where the x-entries originated from an AR(1) process in time

$$x_i = \lambda x_{i-1} + v_i \quad (i = 1, 2, \ldots, n), \tag{2.15}$$

where the v_is were i.i.d. $N(0, 1)$.

In the three situations, 1 is a column vector of ones. In Cases 2 and 3, x and ε were independently distributed. In Case 3, the autocorrelation parameters ρ and λ were fixed at the same value. The slope estimates were evaluated for 1000 simulation runs for sample sizes $n = 10, 20, 30, 50$, and 100 for each value of ρ in each of the three situations. Following Park and Mitchell (1980), the mean squared error (MSE) was calculated for each procedure as 0.001 times the sum of squares of the slope estimates because the theoretical value of the slope parameter was zero in our Monte Carlo study. Recall that MSE is a combined measure of the bias and variance of the slope estimates, since MSE $=$ bias^2 $+$ variance.

We used our own computer programs written in SAS/IML language (SAS Institute Inc., 1997) to implement all the procedures, except REML. The generation of $N(0, 1)$ observations was carried out with the random number function RANNOR of SAS (SAS Institute Inc., 1997).

some exceptions detailed below. Fifth, the knowledge of the model of the time series process generating the errors appears to be a real advantage in estimated GLS, as the efficiencies of $\Sigma_{\hat{\rho}1}$ and $\Sigma_{\hat{\rho}2}$ are generally close to Σ_{ρ} and very different from $\hat{\Sigma}_{13}$, $\hat{\Sigma}_{14}$, $\hat{\Sigma}_{23}$, and $\hat{\Sigma}_{24}$. Last but not the least, FD is less efficient than all the other procedures, including OLS, when $\rho < 0$. The differences between ML and REML were generally small, with no evidence of one of the two procedures prevailing unilaterally over the other. Because of repeated lack of convergence of the REML algorithm in PROC MIXED of SAS when $n = 10$, results for this sample size are not reported in Tables 2.1-2.3.

Case 1: With the exception of FD (discussed at the end of this paragraph), differences in efficiency are small when $\rho > 0$. When $\rho < 0$, differences are large for $n = 10$ and then decrease with increasing n; this decrease is associated with a general increase in efficiency of all procedures, except FD, relative to OLS. Differences among the four estimated GLS procedures that do not require the knowledge of a time series model for the errors are small in Case 1. The only ρ and n values for which FD is more efficient than OLS in this situation are $\rho = 0.7$ when $n = 10$ and $\rho = 0.9$ when $n = 10, 20, 30$, and 50. For all other values, FD is less efficient than OLS, and the lack of efficiency of the former over the latter increases with increasing n.

Cases 2 and 3: We grouped the results specific to Cases 2 and 3, as these are very similar. Compared to Case 1, OLS suffers from a more severe lack of efficiency when the error autocorrelation is strong, whether positive or negative. Larger n values worsen the efficiency of OLS instead of improving it here. Another difference with Case 1 is a greater symmetry between efficiencies for positive and negative autocorrelation; in particular, efficiencies for $\rho = -0.7$ are almost equal to those for $\rho = 0.7$ when $n = 50$ and 100, as are those for $\rho = -0.9$ and 0.9. Furthermore, the differences between estimated GLS with and without the test of significance of the sample autocorrelation coefficients are important when

$n = 10$ and 20, and tend to decrease with larger n values in Cases 2 and 3. The option of not performing the test of significance on the sample autocorrelation coefficients is to be preferred. Slight differences between estimated GLS with and without iteration are observed, which favors the no-iteration option in practice. When $\rho = 0.9$, FD is among the most efficient estimation procedures with GLS, ML and REML ($n = 10$ excepted), whether x is purely random or first-order autoregressive, for all series lengths considered here. These results for FD are in agreement with those obtained by Martin (1974) in space. When $\rho > 0$ in Cases 2 and 3, ML, REML and FD can be recommended in practice because they are the most efficient just after GLS.

5. CONCLUSIONS

With the exception of GLS, which is useless in practice since it requires the complete knowledge of the error autocorrelation matrix, and of OLS, which tends to have a greater efficiency compared to estimated GLS, ML, REML and FD when $-0.1 \leq \rho \leq 0.1$, no estimation procedure was unilaterally superior to all the others in Cases 1, 2 and 3 for all series lengths and other values of ρ. To some extent, ML and REML approached such a criterion for larger sample sizes ($n \geq 50$) and strong ($|\rho| \geq 0.5$) autocorrelation of the errors. For $\rho = 0.9$, FD was more efficient than ML or REML when x was purely random and was more efficient than both of them when x was first-order autoregressive. Recall that ML and REML, unlike FD, requires knowledge of the model of the time series process generating the errors. The two estimated GLS procedures that required the same kind of knowledge performed almost as well as ML and REML. Larger sample sizes helped the performance of OLS only in the x fixed case, in particular when $\rho < 0$. Slight differences between estimated GLS with and without iteration were observed. Among the four estimated GLS procedures that did not require the knowledge of a time series model for the errors, those in which the test of significance of sample autocorrelation coefficients was not performed were more efficient than the others. The differences among these four procedures tended to decrease with increasing n, especially when x was purely random or first-order autoregressive. Since the eleven estimation procedures compared were expected to be theoretically unbiased, if not in finite samples at least asymptotically, it may be argued that the observed differences in efficiency were mainly due to differences in the variance of the slope estimators.

Our results have shown that extreme care must be taken when discussing the efficiency of estimation procedures for quantitative linear models with autocorrelated errors and that due attention must be paid to the nature, structured or random, of the explanatory variable when drawing conclusions. Although the results reported here are limited to time and simple linear regression, they provide reliable guidelines to analysts of autocorrelated sample data as to which estimation procedure to use, or not to use, in a given situation for a given sample size and autocorrelation level. These results on estimation aspects in quantitative linear models with autocorrelated errors also provide useful information for future studies on testing aspects in these models.

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Table 2.1: Efficiency of the different estimation procedures relative to OLS when x is fixed, as a function of the sample size n and the error autocorrelation coefficient ρ. See the text for other notations.

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
					n=10						
Σ_ρ	0.2851	0.6854	0.8874	0.9541	0.998	1	0.9989	0.9775	0.9254	0.9126	0.9088
$\Sigma_{\beta 1}$	0.2949	0.6984	0.9156	0.9548	1.0188	1.0336	1.0237	1.0048	1	0.9783	0.9765
$\Sigma_{\beta 2}$	0.2929	0.6971	0.917	0.9548	1.0221	1.041	1.0289	1.0061	1.0013	0.9748	0.9705
$\hat{\Sigma}_{13}$	0.685	0.9013	0.9381	0.9956	1.0664	1.0656	1.0237	1.0133	1.0818	1.0116	0.9976
$\hat{\Sigma}_{14}$	0.6781	0.8914	0.985	0.9946	1.0004	0.9992	1.0007	1.0002	1.0014	1.0003	1.0004
$\hat{\Sigma}_{23}$	0.6391	0.845	0.9432	0.9976	1.0302	1.0774	1.0291	1.0044	1.0281	1.0109	1.0001
$\hat{\Sigma}_{24}$	0.6738	0.9036	0.9834	0.9945	1.0043	0.9995	1.0003	1.0062	0.9997	1.0035	0.983
ML	0.2884	0.6971	0.9363	0.946	1.0625	1.0547	0.9946	0.9972	1.0359	0.9534	0.9819
FD	10.9466	6.8959	4.4942	3.0745	2.3525	2.1247	1.8904	1.373	1.0976	0.9499	0.9111
					n=20						
Σ_ρ	0.4084	0.7732	0.9155	0.985	0.9924	1	0.9978	0.9901	0.9212	0.8802	0.799
$\Sigma_{\beta 1}$	0.4114	0.7828	0.9207	0.9846	0.9973	1.011	1.0071	1.0051	0.9573	0.9284	0.8961
$\Sigma_{\beta 2}$	0.411	0.7821	0.9204	0.9847	0.9977	1.0123	1.0082	1.0082	0.9553	0.9262	0.8767
$\hat{\Sigma}_{13}$	0.719	0.8723	1.0264	1.088	1.0578	1.0213	1.0405	1.0095	1.0203	1.006	0.9884
$\hat{\Sigma}_{14}$	0.7302	0.8965	0.9595	0.9891	1.0251	1.0154	1.0423	1.0288	0.9953	0.991	0.9903
$\hat{\Sigma}_{23}$	0.7043	0.8746	0.9921	1.0262	1.0698	1.0372	1.0431	1.0211	1.018	1.0015	0.9787
$\hat{\Sigma}_{24}$	0.729	0.8973	0.9566	0.9896	1.0311	1.003	1.0336	1.0095	1.0126	0.987	0.9956
ML	0.4058	0.7714	0.9203	1	0.9893	1.008	1.0384	0.9964	0.9437	0.9619	0.8666
REML	0.4061	0.7732	0.8945	0.9506	1.0652	1.0392	0.997	1.1803	0.9237	0.9815	0.8217
FD	30.1912	16.8393	9.7296	6.8957	4.9895	3.671	3.211	2.444	1.4044	1.1018	0.7988
					n=30						
Σ_ρ	0.5473	0.8381	0.9569	0.9714	0.9968	1	1.0015	0.979	0.9556	0.9	0.7644
$\Sigma_{\beta 1}$	0.5512	0.8453	0.9578	0.9808	0.9997	1	1.0071	0.9963	0.9669	0.9293	0.8523
$\Sigma_{\beta 2}$	0.5512	0.8448	0.9583	0.9805	0.9997	1	1.0073	0.9984	0.9684	0.9304	0.8285
$\hat{\Sigma}_{13}$	0.7855	1.259	1.0178	1.0989	1.0726	1.0477	1.0289	1.0392	1.0098	0.9946	0.9631
$\hat{\Sigma}_{14}$	0.7994	0.9419	0.9893	0.9965	1.0161	1.0096	0.9996	1.3762	1.0135	1.1293	0.9775
$\hat{\Sigma}_{23}$	0.7863	0.982	1.0058	1.0484	1.0463	1.0282	1.0347	1.0453	1.0064	0.9906	0.9599
$\hat{\Sigma}_{24}$	0.7979	0.9394	0.9791	1.0056	1.0024	1.0197	0.9931	1.0134	0.9903	0.9821	0.9895
ML	0.5488	0.8392	0.9707	0.9819	0.9877	1.0114	0.9969	0.9831	0.9792	0.9252	0.8
REML	0.5488	0.8385	0.9684	0.9404	1.0437	1.0168	0.9575	1.0706	1.0004	1.1396	0.8012
FD	49.8928	24.9558	13.6674	9.2201	6.8078	5.571	4.3953	2.7728	2.1429	1.3582	0.7669
					n=50						
Σ_ρ	0.658	0.8959	0.9639	0.9791	1	1	0.9991	0.9795	0.9671	0.9065	0.8126
$\Sigma_{\beta 1}$	0.658	0.8985	0.9661	0.9843	1.0037	1.0021	1.0009	0.9805	0.9717	0.9198	0.8611
$\Sigma_{\beta 2}$	0.658	0.8985	0.9661	0.9843	1.0037	1.0021	1.0009	0.9805	0.972	0.9183	0.853
$\hat{\Sigma}_{13}$	0.8504	1.1396	1.0361	1.6475	1.2145	1.0986	1.0967	1.0835	1.0359	1.0024	1.0205
$\hat{\Sigma}_{14}$	0.8527	0.9619	1.0226	1.0017	0.9988	1.0092	1.0185	1.0113	0.988	0.9831	0.9788
$\hat{\Sigma}_{23}$	0.848	0.9645	1.0271	1.0401	1.0486	1.0554	1.0633	1.0502	1.035	0.9988	0.9838
$\hat{\Sigma}_{24}$	0.8575	0.9594	1.0226	1.0105	1.0012	1.0082	1.0246	1.0072	0.9896	0.9645	0.9775
ML	0.6627	0.8934	0.9752	0.9791	0.995	0.9949	1.0193	0.938	1.0144	0.8923	0.8491
REML	0.6643	0.9026	1.0159	1.0246	1.015	1.0021	1.0132	0.9615	1.0236	0.9464	0.7883
FD	102.0166	42.0025	25.5688	16.5131	10.0486	8.654	7.7882	4.4428	3.4611	1.6732	0.9182
					n=100						
Σ_ρ	0.7909	0.9568	0.9726	0.9973	1.0013	1	0.9928	0.9874	0.9785	0.9231	0.8404
$\Sigma_{\beta 1}$	0.7927	0.9565	0.9736	0.9967	1.0031	1.0088	1	0.9874	0.9785	0.9333	0.8667
$\Sigma_{\beta 2}$	0.7927	0.9565	0.9736	0.9967	1.0031	1.0088	1	0.9874	0.9785	0.9333	0.8644
$\hat{\Sigma}_{13}$	0.9256	1.0889	1.0024	1.0435	1.0314	1.0177	1.1377	1.0628	1.0343	0.9945	0.9813
$\hat{\Sigma}_{14}$	0.9131	0.9967	1.0071	1.0236	1.007	1.0177	1.0072	1	0.9914	0.9788	0.9788
$\hat{\Sigma}_{23}$	0.9255	0.9859	1.0365	1.0299	1.0339	1.0265	1.0362	1.0251	1.0236	0.9945	0.9831
$\hat{\Sigma}_{24}$	0.9123	0.9953	1.0117	1.0265	1.0112	1.0265	1.0145	1	0.9893	0.9843	0.9678
ML	0.7928	0.9498	0.9836	0.9866	1.0325	1.0177	0.9855	1.0084	0.9571	0.9224	0.8748
REML	0.8152	1.0544	0.8677	0.89	1.0627	1.0455	1.0357	0.9542	0.9447	0.8703	0.8604
FD	261.7438	92.3215	48.1149	28.4605	21.903	18.5752	15.3841	9.4477	5.5408	3.0714	1.2111

Table 2.3: Efficiency of the different estimation procedures relative to OLS when x follows an AR(1) process with autocorrelation coefficient λ, as a function of the sample size n and the error autocorrelation coefficient ρ (i.e., $\lambda = \rho$). See the text for other notations.

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
					n=10						
Σ_ρ	0.2328	0.4912	0.7389	0.919	0.9751	1	0.987	0.8965	0.7766	0.5868	0.4389
$\Sigma_{\beta 1}$	0.5085	0.6771	0.8098	0.9496	1.0153	1.04	1.0209	1.0301	0.995	0.8464	0.8109
$\Sigma_{\beta 2}$	0.4186	0.6232	0.8033	0.9768	1.0642	1.0836	1.0531	1.0939	1.0182	0.8316	0.7593
$\hat{\Sigma}_{13}$	0.7474	0.9105	0.8959	1.005	1.0434	1.0561	1.0373	1.0887	1.0464	0.9267	0.8954
$\hat{\Sigma}_{14}$	0.9316	0.9555	0.9852	0.9925	0.9998	0.9992	0.9961	0.9991	0.9989	0.993	0.9951
$\hat{\Sigma}_{23}$	0.8192	0.8431	0.9011	1.0868	1.1418	1.1614	1.0595	1.1793	1.0993	0.9397	0.853
$\hat{\Sigma}_{24}$	1.1483	1.069	1.0471	1.0312	1.0008	1.016	0.9539	1	0.9939	1.0867	1.122
ML	0.3098	0.5625	0.9375	0.9972	1.2057	1.1813	1.0111	1.1966	1.0153	0.8851	0.6825
FD	1.8259	1.4764	2.1181	1.6093	1.608	1.4197	1.1569	1.2856	0.812	0.6686	0.4416
					n=20						
Σ_ρ	0.1545	0.4195	0.6369	0.8849	0.9901	1	0.9723	0.8692	0.6615	0.4625	0.2806
$\Sigma_{\beta 1}$	0.2664	0.5058	0.7215	0.9644	1.037	1.0622	1.0607	0.9441	0.7917	0.6427	0.4495
$\Sigma_{\beta 2}$	0.1918	0.4654	0.713	0.9807	1.051	1.1058	1.1053	0.954	0.7657	0.5792	0.359
$\hat{\Sigma}_{13}$	0.6825	0.7486	0.9148	1.0698	1.1668	1.1299	1.1436	1.0736	0.9159	0.8109	0.727
$\hat{\Sigma}_{14}$	0.8532	0.8616	0.9726	1.0235	1.035	1.0165	0.9983	1.0228	0.9903	0.9523	0.8812
$\hat{\Sigma}_{23}$	0.7078	0.749	0.9184	1.1858	1.3213	1.3006	1.3646	1.257	0.9293	0.7906	0.6996
$\hat{\Sigma}_{24}$	0.923	0.8903	0.9889	1.0648	1.039	1.0066	1.0152	1.0372	0.9485	0.9581	1.0472
ML	0.1824	0.4292	0.7976	0.9428	1.1634	1.107	1.0862	0.9683	0.7779	0.4839	0.3584
REML	0.1853	0.3956	0.7711	0.8505	1.119	1.1241	1.024	0.9838	0.6796	0.464	0.3105
FD	1.3441	1.4373	1.7057	1.5256	1.7946	1.448	1.3682	1.0073	0.7748	0.4473	0.3
					n=30						
Σ_ρ	0.1318	0.3762	0.6233	0.88	0.9837	1	0.9781	0.8529	0.637	0.4641	0.1979
$\Sigma_{\beta 1}$	0.2027	0.4169	0.6897	0.9178	1.0336	1.0606	1.0149	0.9123	0.7339	0.5398	0.3169
$\Sigma_{\beta 2}$	0.1497	0.3958	0.6755	0.9286	1.0524	1.0771	1.0248	0.9121	0.7082	0.5071	0.2478
$\hat{\Sigma}_{13}$	0.6656	0.7505	0.8939	1.0572	1.0951	1.158	1.2654	1.0317	0.9391	0.8864	0.7707
$\hat{\Sigma}_{14}$	0.8128	0.8145	0.9596	0.9946	1.0759	1.0398	1.0228	1.0442	0.9591	0.8893	0.8297
$\hat{\Sigma}_{23}$	0.6989	0.7476	0.931	1.2322	1.3017	1.3631	1.2932	1.1811	0.9397	0.833	0.7346
$\hat{\Sigma}_{24}$	0.8062	0.8919	1.0103	1.0471	1.1119	1.0567	1.0229	1.0781	0.9169	0.9019	0.9315
ML	0.1423	0.4094	0.714	0.9415	0.9668	1.0983	1.0651	0.8696	0.6649	0.5294	0.2478
REML	0.14	0.4243	0.8087	1.0523	1.069	1.0427	1.0777	0.8738	0.6651	0.444	0.2208
FD	1.3047	1.5095	1.8166	1.8283	1.4859	1.4425	1.4117	1.0256	0.6874	0.5	0.2114
					n=50						
Σ_ρ	0.1231	0.3819	0.6536	0.8484	0.9831	1	0.9818	0.9095	0.6566	0.3883	0.1506
$\Sigma_{\beta 1}$	0.1625	0.413	0.6811	0.8809	1.0293	1.0432	1.007	0.9436	0.7059	0.4245	0.2054
$\Sigma_{\beta 2}$	0.1312	0.4008	0.6845	0.8813	1.0388	1.0469	1.0091	0.955	0.6977	0.4122	0.1669
$\hat{\Sigma}_{13}$	0.6805	0.76	0.9197	1.0559	1.1576	1.1582	1.2914	1.1473	0.9655	0.8037	0.7545
$\hat{\Sigma}_{14}$	0.7957	0.893	0.9363	1.0316	1.0496	1.0489	1.0542	1.0263	0.9472	0.8368	0.7935
$\hat{\Sigma}_{23}$	0.6959	0.7629	0.9166	1.1503	1.3536	1.3391	1.2995	1.2216	0.9922	0.7723	0.7374
$\hat{\Sigma}_{24}$	0.8025	0.8086	0.8793	1.0262	1.0809	1.1273	1.0923	1.026	0.9321	0.813	0.8141
ML	0.1309	0.4052	0.6881	0.8554	1.0889	1.051	1.0175	0.98	0.6974	0.4077	0.1425
REML	0.1308	0.4158	0.6894	0.8155	1.0361	1.0539	0.9651	0.9463	0.6802	0.4177	0.1406
FD	1.1748	1.4817	1.673	1.59	1.6418	1.5221	1.3189	1.1896	0.7938	0.4285	0.1356
					n=100						
Σ_ρ	0.1243	0.3627	0.6174	0.8225	0.9699	1	0.9779	0.8627	0.6642	0.3749	0.116
$\Sigma_{\beta 1}$	0.1359	0.3672	0.6203	0.8343	0.9871	1.0297	0.988	0.8783	0.6781	0.3821	0.135
$\Sigma_{\beta 2}$	0.1279	0.3642	0.6203	0.8355	0.9882	1.0324	0.9894	0.8782	0.6786	0.3796	0.1212
$\hat{\Sigma}_{13}$	0.7323	0.7977	1.0167	1.0438	1.2188	1.1779	1.4521	1.1068	0.9319	0.8415	0.7292
$\hat{\Sigma}_{14}$	0.7723	0.7568	0.8762	0.9422	1.0563	1.0777	1.0823	1.0317	0.8735	0.7672	0.7565
$\hat{\Sigma}_{23}$	0.7265	0.7916	0.9254	1.101	1.2703	1.2342	1.2784	1.1559	0.937	0.8085	0.7182
$\hat{\Sigma}_{24}$	0.7737	0.7828	0.8599	0.9676	1.05	1.0902	1.0686	0.9725	0.8756	0.7698	0.7691
ML	0.1265	0.3623	0.6481	0.7761	1.0417	1.0485	1.0109	0.8617	0.6269	0.416	0.1177
REML	0.1279	0.3593	0.728	0.7757	1.0389	1.0508	0.9703	0.9423	0.6288	0.4173	0.1322
FD	1.1408	1.4458	1.6807	1.432	1.6826	1.5409	1.4282	1.0701	0.7392	0.4339	0.1148

Chapter 3

Efficiency and Validity Analyses of Two–Stage Estimation and Testing Procedures in Quantitative Linear Models with AR(1) Errors

ABSTRACT

In a quantitative linear model with errors following a stationary Gaussian, first-order autoregressive or AR(1) process, Generalized Least Squares (GLS) on raw data and Ordinary Least Squares (OLS) on prewhitened data are efficient methods of estimation of the slope parameters if the autocorrelation parameter of the error AR(1) process, ρ, is known. When ρ is unknown, which is generally the case in practice, the Prais–Winsten (PW) procedure is an established two–stage estimation method in which ρ is estimated first before being used in the estimation of the slope parameters. Different estimators of ρ have been considered in

previous studies of the PW procedure.

In this chapter, we assess the efficiency of six variants of the PW procedure and two variants of the Cochrane-Orcutt (CO) procedure relative to GLS. Six of them are based on three estimators of ρ that have been considered previously. We propose a new estimator provided by the sample autocorrelation coefficient of the OLS residuals at lag 1, denoted $r(1)$. We use the four estimators of ρ with or without iteration on $\hat{\beta}$ or, equivalently, on $\hat{\rho}$ in a Monte Carlo study. Furthermore, we investigate the validity of the testing procedures derived from the GLS and the eight two-stage estimation procedures. Three types of explanatory variable x in the quantitative linear model with AR(1) errors are considered in the time domain: Case 1, x is fixed; Case 2, x is purely random; and Case 3, x follows an AR(1) process with the same autocorrelation parameter value as the error AR(1) process. The efficiency of the estimation procedures and the validity of the derived testing procedures are discussed in terms of the sample size and the value of the autocorrelation parameter of the errors. In particular, the two-stage estimation procedures based on the new estimator of ρ are shown to be more efficient than the other two-stage estimation procedures for small to moderate values of ρ and any of the sample sizes considered here. Differences among Cases 1, 2 and 3 are also discussed.

Key Words: AR(1) errors; Cochrane–Orcutt procedure; generalized least squares estimation; valid hypothesis testing; Prais–Winsten procedure; efficient estimation; fixed versus random explanatory variable.

1. INTRODUCTION

When the errors follow a stationary Gaussian, first-order autoregressive or AR(1) process in a quantitative linear model, several estimators of the autocorrelation parameter ρ have been proposed in the literature. As we shall see below, the estimation of ρ varies depending on the transformation matrix used and on whether the evaluation of the estimate is iterative or not. The corresponding methods of estimation of the regression coefficients are called *two-stage* because ρ is estimated first and then substituted in the transformation matrix to perform the OLS method on the transformed data. The procedures of Cochrane and Orcutt (1949) (CO), Prais and Winsten (1954) (PW), and Durbin (1960) (D) are the most commonly used two-stage estimation procedures in quantitative linear models with AR(1) errors (Rao and Griliches, 1969; Spitzer, 1979).

Consider

$$y = X\beta + \varepsilon, \tag{3.1}$$

where y is an $n \times 1$ observable random vector, β is a $k \times 1$ unknown vector to be estimated, X is an $n \times k$ matrix of rank $k < n$, ε is an $n \times 1$ unobservable vector of random errors with zero expected value, and the explanatory variables contained in X and the errors are assumed to be uncorrelated when the explanatory variables are random.

Let ε in (3.1) follow an AR(1) process

$$\varepsilon_i = \rho\varepsilon_{i-1} + u_i \quad (i = 1, 2, \ldots, n), \tag{3.2}$$

where $-1 < \rho < 1$ and the u_is are i.i.d. $N(0, \sigma^2)$ with σ^2 an unknown positive constant. Then, the variance-covariance matrix of the n-variate random vector ε is

$$\Sigma = \sigma^2/(1-\rho^2) \begin{pmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{n-1} \\ \rho & 1 & \rho & \cdots & \rho^{n-2} \\ \vdots & & & & \vdots \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \cdots & 1 \end{pmatrix} = \sigma_\varepsilon^2 V \quad \text{(Graybill, 1983).}$$

If ρ is known, then the Best Linear Unbiased Estimator (BLUE) of β in (3.1) is the Generalized Least Squares (GLS) estimator or Aitken's (1935) estimator

$$\hat{\beta} = (X'V^{-1}X)^{-1}X'V^{-1}y, \tag{3.3}$$

where

$$V^{-1} = \frac{1}{(1-\rho^2)} \begin{pmatrix} 1 & -\rho & 0 & \cdots & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & \cdots & 0 & 0 \\ 0 & -\rho & 1+\rho^2 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1+\rho^2 & -\rho \\ 0 & 0 & 0 & \cdots & -\rho & 1 \end{pmatrix}.$$

The variance-covariance matrix of (3.3) is

$$\mathrm{Cov}(\hat{\beta}_{\mathrm{GLS}}) = \sigma^2 (X'V^{-1}X)^{-1}. \tag{3.4}$$

From Graybill (1976), we know that there exists a unique nonsingular lower triangular $n \times n$ matrix T_1 such that $V^{-1} = T_1'T_1$. One can also find an $(n-1) \times n$ matrix T_2 such that pre-multiplying (3.1) by T_2 yields a model with independent (i.e., prewhitened) and identically distributed errors. When ρ is known, commonly used transformation matrices are

$$T_1 = \begin{pmatrix} \sqrt{1-\rho^2} & 0 & 0 & \cdots & 0 \\ -\rho & 1 & 0 & \cdots & 0 \\ 0 & -\rho & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & -\rho & 1 \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} -\rho & 1 & 0 & \cdots & 0 \\ 0 & -\rho & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & -\rho & 1 \end{pmatrix}.$$

Note that $T_2'T_2$ matches V^{-1}, except that the $(1,1)$-element of $T_2'T_2$ is ρ^2 instead of 1.

Pre-multiplying (3.1) by T_j $(j = 1, 2)$ yields the transformed model

$$y_j^* = X_j^*\beta + \varepsilon_j^*,$$

where $y_j^* = T_j y$, $X_j^* = T_j X$ and $\varepsilon_j^* = T_j\varepsilon$ with $\mathrm{Cov}(\varepsilon_1^*) = \sigma^2 T_1 V T_1' = \sigma^2 I_n$ and $\mathrm{Cov}(\varepsilon_2^*) = \sigma^2 T_2 V T_2' = \sigma^2 I_{n-1}$. The OLS estimator of β is then

$$\hat{\beta}_{\mathrm{OLS}_j} = (X_j^{*\prime} X_j^*)^{-1} X_j^{*\prime} y_j^* \quad (j = 1, 2) \tag{3.5}$$

with covariance matrix

$$\mathrm{Cov}(\hat{\beta}_{\mathrm{OLS}_j}) = \sigma^2 (X_j^{*\prime} X_j^*)^{-1}. \tag{3.6}$$

In practice, ρ is generally unknown and hence, needs to be estimated. The PW estimator

$$\hat{\rho}_{\mathrm{PW}} = \sum_{i=2}^{n} e_i e_{i-1} / \sum_{i=2}^{n-1} e_i^2 \tag{3.7}$$

replaces ρ in T_1, and the CO estimator

$$\hat{\rho}_{\mathrm{CO}} = \sum_{i=2}^{n} e_i e_{i-1} / \sum_{i=1}^{n-1} e_i^2 \tag{3.8}$$

replaces ρ in T_2, where the e_is are the OLS residuals of the untransformed model (3.1). The non-iterative PW and CO estimation procedures are defined by their respective transformation matrix and ρ estimator in (3.5). The iterative PW and CO estimation procedures are defined as follows: i) obtain the OLS residuals of the untransformed model (3.1); ii) calculate $\hat{\rho}_{\mathrm{PW}}$ or $\hat{\rho}_{\mathrm{CO}}$; iii) use $\hat{\rho}_{\mathrm{PW}}$ in T_1 or $\hat{\rho}_{\mathrm{CO}}$ in T_2, and evaluate $\hat{\beta}_{\mathrm{OLS}_j}$ in (3.5); iv) use $\hat{\beta}_{\mathrm{OLS}_j}$ to obtain new residuals and go back to step ii); v) repeat steps ii)-iv) until successive $\hat{\rho}_{\mathrm{PW}}$ or $\hat{\rho}_{\mathrm{CO}}$ estimates differ by less than a fixed infinitesimal quantity.

Rao and Griliches (1969) compared the small-sample properties of the OLS, GLS and non-linear least-squares estimators of the slope parameters with a number of two-stage estimators (i.e., PW, D, CO, and PW with Durbin's estimator of ρ) in a linear regression model with AR(1) explanatory variables and AR(1) errors in the time domain. Their Monte Carlo study showed that none of these estimators was unilaterally superior to the others over the range of parameter values considered. Nevertheless, a significant gain in efficiency was observed for the two-stage estimators when $|\rho| > 0.3$. Spitzer (1979) partially replicated the Monte Carlo study of Rao and Griliches (1969), including the maximum likelihood estimator. His results were not consistent with those of Rao and Griliches.

Park and Mitchell (1980) studied the small-sample properties of the OLS estimator, along with the iterative and non-iterative PW and CO two-stage estimators in a linear model with trended explanatory variables and AR(1) errors. They concluded that the iterative PW procedure was superior to the CO procedures, but none of the test statistics derived from the estimators was valid. They pointed out that previous Monte Carlo studies used an estimator of ρ,

$$\hat{\rho}_{W} = \sum_{i=2}^{n} e_i e_{i-1} / \sum_{i=2}^{n} e_i^2, \tag{3.9}$$

which does not minimize the sum of squares of the errors.

In the next section, following a proof from Anderson (1971, p. 354), we show that (3.7) and a new estimator provided by the sample autocorrelation coefficient of the OLS residuals at lag 1

$$r(1) = \sum_{i=2}^{n} e_i e_{i-1} / \sum_{i=1}^{n} e_i^2 \tag{3.10}$$

approximate the maximum likelihood estimator of ρ in (3.2) for small to moderate sample sizes. The objective of this chapter is twofold: to assess the efficiency of the iterative and non-iterative versions of four two-stage estimation procedures based on (3.7), (3.8), (3.9) and $r(1)$ relative to GLS and to investigate the validity of the testing procedures derived from the GLS and the eight two-stage estimation procedures. We consider three types of explanatory variable x in a quantitative linear model with AR(1) errors in the time domain: Case 1, x is fixed; Case 2, x is purely random; and Case 3, x follows an AR(1) process. Efficiency and validity are discussed in terms of the sample size and the autocorrelation parameter value.

2. NEW ESTIMATOR

Let

$$A = \sum_{i=2}^{n-1} \varepsilon_i^2, \quad B = \sum_{i=2}^{n} \varepsilon_i \varepsilon_{i-1} \quad \text{and} \quad C = \varepsilon_1^2 + \varepsilon_n^2.$$

The natural logarithm of the likelihood function of (3.2) is

$$\begin{aligned} \ln L &= (1/2)\ln(1-\rho^2) - (n/2)\ln 2\pi - (n/2)\ln\sigma^2 \\ &- \{1/(2\sigma^2)\}\{C - 2\rho B + (1+\rho^2)A\}. \end{aligned} \quad (3.11)$$

By taking the first derivatives of $\ln L$ with respect to ρ and σ^2 and setting these to zero, we obtain

$$\begin{aligned} \hat{\sigma}^2 &= (1/n)\{C - 2\hat{\rho}B + (1+\hat{\rho}^2)A\} \qquad \text{and} \\ f(\hat{\rho}) &= \{(n-1)/n\}A\hat{\rho}^3 - \{(n-2)/n\}B\hat{\rho}^2 \\ &- \{((n+1)/n)A + (1/n)C\}\hat{\rho} + B = 0. \end{aligned} \quad (3.12)$$

Note that $f(-1) < 0$, $f(1) > 0$ and $f(0) = B$. There is one zero root if $B = 0$, one root in $(-1, 0)$ if $B > 0$, and one root in $(0, 1)$ if $B < 0$. For any value of B, there is one root that is less than -1 and another that is greater than 1. The roots can be estimated by maximum likelihood (Beach and MacKinnon, 1978). For large sample sizes (i.e., $n \to \infty$), (3.12) becomes

$$g(\hat{\rho}) = \hat{\rho}^3 - (B/A)\hat{\rho}^2 - \hat{\rho} + (B/A) = 0. \quad (3.13)$$

It is easy to verify that $g(\pm 1) = 0$ and $g(B/A) = 0$. In view of the solution to question 78 in Anderson (1971, p. 369), $\hat{\rho} = (B/A)(1 - 1/n)$ is a solution of (3.12) to order $1/n$. Hence, B/A or

$$B/(A+C) = \sum_{i=2}^{n} \varepsilon_i \varepsilon_{i-1} / \sum_{i=1}^{n} \varepsilon_i^2 \quad (3.14)$$

is an approximate solution to (3.12).

Compared to (3.7), (3.8) and (3.9), $r(1)$ always provides an estimate of ρ in $(-1, 1)$. When estimates of ρ evaluated by (3.7), (3.8) or (3.9) exceed 1 in absolute value, they are replaced in practice by $-1+p$ or $1-p$, where p is a small positive quantity. This drawback and its possible effect on the efficiency of the corresponding two-stage estimation procedures are addressed in the Results and

Discussion section. The time series length is generally recommended to be 50 or more to obtain better estimates of the true autocorrelation parameters (Box *et al.*, 1994).

3. MONTE CARLO STUDY

The model used for simulation was

$$y_i = a + bx_i + \varepsilon_i \text{ with } \varepsilon_i = \rho\varepsilon_{i-1} + u_i \quad (i = 1, 2, \ldots, n),$$

where a and b were fixed at 1 and 0, the u_is were i.i.d. $N(0,1)$, and the value of ρ ranged from -0.9 to 0.9 by steps of 0.2, in addition to $\rho = 0$. The generation of autocorrelated errors followed a procedure similar to that of Dutilleul and Legendre (1992). Three cases were considered for the X matrix:

Case 1: $X = [1, x]$, where $x = (1, 2, \ldots, n)'$.

Case 2: $X = [1, x]$, where the entries of x were pseudo-random $N(0,1)$ observations.

Case 3: $X = [1, x]$, where the entries of x originated from a stationary Gaussian AR(1) process in the time domain

$$x_i = \lambda x_{i-1} + v_i \quad (i = 1, 2, \ldots, n),$$

where the v_is were i.i.d. $N(0,1)$.

In all cases, 1 was a column vector of ones. In Case 3, the autocorrelation parameters ρ and λ were fixed at the same value, but x and ε were independently distributed. The slope estimates were evaluated for 1000 simulation runs for sample sizes $n = 10, 20, 30$ and 50 for each value ρ in the three cases. Following Park and Mitchell (1980), the mean squared error (MSE) was calculated as 0.001 times the sum of squares of the slope estimates, since the theoretical value of the slope was zero in our Monte Carlo study. Whenever (3.7), (3.8) or (3.9) exceeded ± 1, we set the estimates to ± 0.99 and kept track of how many times this occurred.

The efficiency of eight two-stage estimation procedures relative to GLS was calculated on the basis of mean squared errors. For example, the efficiency of the so-called procedure 1 was calculated as $\text{Eff}(\Sigma_{\hat{\rho}1}) = \text{MSE}(\Sigma_{\hat{\rho}1})/\text{MSE}(\text{GLS})$, where $\Sigma_{\hat{\rho}1}$ denotes procedure 1; notations are defined below. In the testing procedures, the standard error of the GLS slope estimator was calculated as the positive square root of the $(2, 2)$-entry of (3.4), whereas the positive square root of $(2, 2)$-entry of (3.6) was used for procedures 1-8, by replacing σ^2 with the error mean square of the corresponding estimation procedure. The empirical significance level was calculated as 0.001 times the number of rejections of the hypothesis of a zero value for the slope b in 1000 t-tests with $n-2$ degrees of freedom (df) performed at a theoretical significance level of 5%. Under the binomial distribution model, the standard deviation of the empirical significance level p is $\sigma_{\text{p}} = \sqrt{p(1-p)/s}$, where s is the number of simulation runs. An approximate 95% confidence interval for the true significance level was calculated as $p \pm 2\sigma_{\text{p}}$. 0.065 was the maximum value of p, for which the theoretical significance level of 0.05 fell within $p \pm 2\sigma_{\text{p}}$.

In addition to GLS, eight two-stage estimation procedures were included in our Monte Carlo study. Using notations that refer to the estimator of ρ as used in the transformation matrix T_1 or T_2, these two-stage estimation procedures can be defined as follows. 1: $\Sigma_{\hat{\rho}1}$, ρ is replaced by $r(1)$ in T_1, no iteration on $\hat{\beta}$. 2: $\Sigma_{\hat{\rho}2}$, same as procedure 1, except there was iteration on $\hat{\beta}$. 3: $\Sigma_{\hat{\rho}\text{W}1}$, ρ is replaced by (3.9) in T_1, no iteration on $\hat{\beta}$. 4: $\Sigma_{\hat{\rho}\text{W}2}$, same as procedure 3, except there was iteration on $\hat{\beta}$. 5: $\Sigma_{\hat{\rho}\text{CO}1}$, ρ is replaced by (3.8) in T_2, no iteration on $\hat{\beta}$. 6: $\Sigma_{\hat{\rho}\text{CO}2}$, same as procedure 5, except there was iteration on $\hat{\beta}$. 7: $\Sigma_{\hat{\rho}\text{PW}1}$, ρ is replaced by (3.7) in T_1, no iteration on $\hat{\beta}$. 8: $\Sigma_{\hat{\rho}\text{PW}2}$, same as procedure 7, except there was iteration on $\hat{\beta}$. These notations are used in Tables 3.1-3.7.

We used our own computer programs written in SAS/IML language (SAS Institute Inc., 1997) to implement all estimation and testing procedures. The generation of pseudo-random $N(0, 1)$ observations was carried out with the random number function RANNOR of SAS (SAS Institute Inc., 1997).

4. RESULTS AND DISCUSSION

Table 3.1 reports the number of estimates of ρ that exceeded 1 in absolute value in procedures 1, 3, 5 and 7. Tables 3.2, 3.4 and 3.6 present the efficiency of procedures 1-8 relative to GLS in Cases 1, 2 and 3, respectively. For interpretation purposes, procedure 1 is said to be more (less) efficient than GLS if $\mathrm{Eff}(\Sigma_{\hat{\rho}1})$ is smaller (greater) than 1 and more (less) efficient than procedure 2, for instance, if $\mathrm{Eff}(\Sigma_{\hat{\rho}1})$ is smaller (greater) than $\mathrm{Eff}(\Sigma_{\hat{\rho}2})$. Tables 3.3, 3.5 and 3.7 present the empirical significance levels observed for a theoretical significance level of 5% in the three cases. For interpretation purposes, a testing procedure is said to be valid when it satisfactorily controls the Type I error, that is, when the empirical significance level is at most equal to the theoretical significance level of 5% used in the Monte Carlo study or when the approximate 95% confidence interval $p \pm 2\sigma_{\mathrm{p}}$ contains 0.05, otherwise. In other words, a testing procedure is said to be valid here if the empirical significance level is at most equal to 0.065.

A number of general comments hold for Cases 1-3. First, the number of inadmissible (i.e., exceeding 1 in absolute value) estimates of ρ in $\Sigma_{\hat{\rho}_{\mathrm{PW1}}}$ is equal to or greater than those in $\Sigma_{\hat{\rho}_{\mathrm{W1}}}$ and $\Sigma_{\hat{\rho}_{\mathrm{CO1}}}$ for all values n and ρ, whereas $\Sigma_{\hat{\rho}_{\mathrm{W1}}}$ and $\Sigma_{\hat{\rho}_{\mathrm{CO1}}}$ produced similar numbers of inadmissible estimates of ρ overall. Second, inadmissibility of $\hat{\rho}$ decreases with increasing n, and is more severe when the autocorrelation parameter of the AR(1) error process is negative. Third, GLS, which requires the knowledge of ρ, has the greatest efficiency with very few exceptions. Fourth, none of the testing procedures derived from the two-stage estimation procedures has an empirical significance level equal to or smaller than the theoretical significance level of 5% when $\rho = 0$ and 0.1.

Case 1: In view of Table 3.1, it appears that the problem of inadmissible estimates of ρ occurs in particular when $\rho = -0.9$ and the sample size is small to moderate (i.e., $n = 10, 20, 30$) in the x fixed case. In this case, some inadmissible estimates of ρ are also observed for $\rho = 0.9$ when $n > 10$ and for negative values of ρ other than -0.9 when $n = 10$. Note that $\Sigma_{\hat{\rho}_{PWI}}$ produces many inadmissible estimates of ρ when $\rho = -0.9$ for $n = 10$.

Overall, the observed differences in efficiency of the two-stage estimation procedures relative to GLS are small when x is fixed. With the exceptions of the two CO procedures when $\rho > 0$, the differences observed in the efficiency of the two-stage estimation procedures are small in Case 1. Excluding the two CO procedures, the largest differences (i.e., between 0.06 and 0.13) in the relative efficiency of the other two-stage estimation procedures relative to GLS are observed when $\rho \geq 0.5$ for $n = 10$ and when $\rho = 0.9$ for $n = 20$ and 30. Again, excluding CO, there seems to be a slight advantage in favor of the iterative procedures when $\rho = \pm 0.9$, especially for $n = 10$ and 20. The two procedures using the new estimator of ρ appear to be more efficient than the others when ρ is small to moderate (i.e., $-0.5 \leq \rho \leq 0.5$), and less efficient than the PW and $\hat{\rho}_W$-based procedures when $\rho = \pm 0.9$. The differences in efficiency among the estimation procedures decrease with increasing n overall, but the differences of CO with the other procedures remain important even when $n = 50$.

The testing procedure based on the GLS estimator of the slope tends to be valid the most often. In fact, the empirical significance level it provides is below 5% for all negative values of ρ considered here when $n = 10$, for all values of ρ when $n = 20$, for all non-zero values of ρ when $n = 30$, and for ρ equal to or greater than 0.3 in absolute value when $n = 50$. The eight other estimation procedures without exception are valid when ρ is strong and negative, that is, when $\rho \leq -0.5$ for $n = 10$ and when $\rho \leq -0.3$ for $n = 20, 30$ and 50. Note that the testing procedures derived from the two CO estimation procedures are also valid when $\rho = -0.3$ for $n = 10$ and when $\rho = -0.1$ for $n = 20$. Iteration does not seem to help the case of the testing procedures derived from the estimation procedures 2, 4, 6 and 8. In fact, their empirical significance levels are equal to or slightly greater than those of their non-iterative counterpart when ρ is negative, and slightly smaller but still far above 5% when ρ is positive.

Cases 2 and 3: The following observations apply when x is purely random (Case 2) or follows an AR(1) process (Case 3). Compared to the x fixed case and excluding CO, the differences in efficiency among the estimation procedures are much larger (Tables 3.4 and 3.5). For $\Sigma_{\hat{\rho}_{PW1}}$, this observation extends to all the values of ρ considered in our study. For the same procedure, a large number of inadmissible estimates of the autocorrelation parameter are observed when $\rho = 0.9$ for $n = 20$ and 30. Compared to the x fixed case and excluding CO, much larger differences in the efficiency of estimation procedures are observed (Tables 3.4 and 3.5). In particular, differences in efficiency relative to GLS can reach 100% when $\rho = -0.9$ for $n = 10$. Compared to Case 1, the rate of decrease of the differences in efficiency with increasing n is lower. Procedure 1 is generally more efficient than the other two-stage estimation procedures when $-0.3 \leq \rho \leq 0.3$. In contrast with Case 1, iteration appears to help the case of procedures 2, 4, 6 and 8 with a noticeable gain in efficiency when $|\rho| \geq 0.7$ for all values of n. In Cases 2 and 3, the testing procedure based on the GLS estimator of the slope is valid for most non-zero values of ρ and all values of n, and is more rarely valid when $\rho = 0$. When $n = 50$, the testing procedures based on two-stage estimation procedures 1-8 are valid for $\rho \leq -0.5$ and $\rho \geq 0.3$ (Tables 3.5 and 3.7).

The main differences between Case 2 and Case 3 are the following. Concerning the question of admissibility of the estimates of ρ, more estimates of ρ were found to exceed 1 when $\rho = 0.9$ in Case 2 than in Case 3 (Table 3.1). As for the estimation aspects concerning the slope b, the values of relative efficiency reported in Table 3.6 are larger than those in Table 3.4, especially when $\rho = \pm 0.9$. The rate of decrease of the relative efficiency values with increasing sample size appears to be higher when x is purely random than when x follows an AR(1) process. In view of Tables 3.5 and 3.7, none of the testing procedures derived from the two-stage estimation procedures is valid for $n = 10$ when x follows an AR(1) process (Case 3), whereas all of them are valid for $\rho \leq -0.5$ and some of them are valid for $\rho \geq 0.3$ and $n = 10$ when x is purely random (Case 2). Differences between Cases 2 and 3 decrease with increasing n. When $n = 20$, the testing procedures derived from estimation procedures 1-8 are generally valid for $\rho \leq -0.5$ in Case 3, whereas they are generally valid for $|\rho| > 0.3$ in Case 2. When $n = 30$, these testing procedures are generally valid for $\rho \leq -0.1$ and often valid for $\rho \geq 0.5$ in Case 3, whereas they are valid for $|\rho| \geq 0.3$ in Case 2. When $n = 50$, the only main difference in validity between Cases 2 and 3 is observed for $\rho = -0.3$. Overall, the empirical significance levels reported in Table 3.5 are smaller than those in Table 3.7. In particular, the testing procedures derived from estimation procedures 1-8 perform equally well in Case 2 when $n = 30$ and 50. Only slight differences are observed in this case when $n = 10$ and 20; these differences are in favor of the two CO procedures and the non-iterative procedures based on the new estimator of ρ and the $\hat{\rho}_W$ estimator. In Case 3, some differences are observed for $\rho = -0.9$ when $n = 20$ and for $\rho \geq 0.5$ when $n = 30$. These differences are rather in favor of the PW procedures and the iterative version of the other procedures.

Procedures 3-8: The following results are specific to the two-stage estimation procedures that were already available in the literature. Concerning the two-stage estimation procedures available in the literature (procedures 3-8) more specifically, the following can be said. As for estimation aspects, both CO procedures are eclipsed by procedures 1-4 and 7-8 when $\rho \geq 0$ for all values of n in the x fixed case. In fact, the two CO procedures are as efficient as GLS and the other two-stage estimation procedures only for $\rho = -0.9$ when $n = 20, 30$ and 50 in Case 1. In contrast, the efficiency of the iterative CO procedure is satisfactory for $|\rho| > 0.5$ when $n \geq 20$ in Case 2 and for $|\rho| > 0.5$ when $n = 50$ in Case 3. Otherwise, the PW procedure, whether iterative or non-iterative depending on the situation, tends to be more efficient than the other two-stage estimation procedures. The efficiency of procedures 3 and 4 is similar to that of procedures 1 and 2, respectively. As for validity aspects, the testing procedures derived from the two-stage estimation procedures 3-8 perform equally well. Moreover, the testing procedures derived from the two CO estimation procedures do not suffer from the lack of efficiency that characterizes the CO estimators of the slope in some instances.

5. CONCLUSIONS

Our Monte Carlo study showed that the efficiency of two-stage estimation procedures and the validity of derived testing procedures in quantitative linear models with AR(1) errors may vary with the nature, fixed or random, of the explanatory variable, the sample size or the value of the autocorrelation parameter of the error AR(1) process. In particular, the two-stage estimation procedures involving the new estimator of ρ, $r(1)$, were shown to be efficient only when the value of ρ was small to moderate, but for any value of n and whether the explanatory variable x was fixed or random. This result of our Monte Carlo study confirmed, to some extent, the theoretical argument that led us to consider the sample autocorrelation coefficient of the OLS residuals as a new estimator of ρ in our study. Another interesting result is the good performance of the testing procedures derived from the two CO estimation procedures, despite the fact that the two CO estimation procedures were eclipsed by the other two-stage estimation procedures on a number of occasions, especially when the explanatory variable is fixed.

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Table 3.1: Number of times the estimates of ρ exceeded 1 in absolute value in 1000 simulation runs, as a function of the sample size n and the autocorrelation parameter ρ.

	n=10			n=20			n=30			n=50		
ρ	$\Sigma_{\hat\rho W1}$	$\Sigma_{\hat\rho CO1}$	$\Sigma_{\hat\rho PW1}$	$\Sigma_{\hat\rho W1}$	$\Sigma_{\hat\rho CO1}$	$\Sigma_{\hat\rho PW1}$	$\Sigma_{\hat\rho W1}$	$\Sigma_{\hat\rho CO1}$	$\Sigma_{\hat\rho PW1}$	$\Sigma_{\hat\rho W1}$	$\Sigma_{\hat\rho CO1}$	$\Sigma_{\hat\rho PW1}$
						x fixed						
-0.9	17	21	102	15	15	82	5	3	33	0	0	3
-0.7	2	0	15	0	0	2	0	0	1	0	0	0
-0.5	0	0	2	0	0	0	0	0	0	0	0	0
-0.3	0	0	1	0	0	0	0	0	0	0	0	0
-0.1	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0.1	0	0	0	0	0	0	0	0	0	0	0	0
0.3	0	0	0	0	0	0	0	0	0	0	0	0
0.5	0	0	1	0	0	0	0	0	0	0	0	0
0.7	0	0	0	0	0	0	0	0	0	0	0	0
0.9	0	0	0	0	1	6	0	0	6	0	0	1
						x purely random						
-0.9	52	50	231	15	16	78	7	2	26	0	0	0
-0.7	10	12	60	1	0	6	0	0	0	0	0	0
-0.5	7	6	35	0	0	0	0	0	0	0	0	0
-0.3	3	2	11	0	0	0	0	0	0	0	0	0
-0.1	1	2	7	0	0	0	0	0	0	0	0	0
0	2	0	6	0	0	0	0	0	0	0	0	0
0.1	0	0	3	0	0	0	0	0	0	0	0	0
0.3	3	0	5	0	0	0	0	0	0	0	0	0
0.5	2	0	13	0	0	0	0	0	0	0	0	0
0.7	5	2	25	0	0	4	0	0	0	0	0	0
0.9	12	6	64	7	2	42	0	5	19	4	3	7
						x AR(1)						
-0.9	44	38	173	12	15	64	4	4	19	0	0	1
-0.7	13	9	58	0	1	4	0	0	0	0	0	0
-0.5	3	3	15	0	0	0	0	0	0	0	0	0
-0.3	1	2	7	0	0	0	0	0	0	0	0	0
-0.1	0	2	8	0	0	0	0	0	0	0	0	0
0	0	1	3	0	0	0	0	0	0	0	0	0
0.1	2	0	5	0	0	0	0	0	0	0	0	0
0.3	0	0	4	0	0	0	0	0	0	0	0	0
0.5	2	2	7	0	0	0	0	0	0	0	0	0
0.7	3	2	13	1	0	3	0	0	0	0	0	0
0.9	4	7	41	2	7	25	3	3	14	0	1	1

Table 3.2: Efficiency of the two-stage estimation procedures relative to GLS when x is fixed, as a function of the sample size n and the autocorrelation parameter ρ. The relative efficiencies reported were obtained from 1000 simulation runs. See the text for other notations.

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
						n=10					
$\Sigma_{\hat\beta 1}$	1.0329	1.0238	1.0291	1.002	1.0156	1.0307	1.021	1.0278	1.0845	1.0654	1.0882
$\Sigma_{\hat\beta 2}$	1.0261	1.0214	1.0307	1.0023	1.0183	1.0378	1.0268	1.027	1.0849	1.0627	1.0808
$\Sigma_{\hat\beta W1}$	1.0168	1.0189	1.0309	1.0041	1.0225	1.0364	1.0229	1.0299	1.09	1.0627	1.0844
$\Sigma_{\hat\beta W2}$	1.0098	1.0216	1.0372	1.008	1.0304	1.0482	1.0307	1.0336	1.0899	1.062	1.0782
$\Sigma_{\hat\beta CO1}$	1.0711	1.0956	1.1898	1.1859	1.2777	1.4729	1.4311	1.4964	1.7354	1.7652	1.6772
$\Sigma_{\hat\beta CO2}$	1.0747	1.1085	1.2481	1.4098	1.3526	1.6395	2.0598	2.1084	2.4811	3.0232	2.7583
$\Sigma_{\hat\beta PW1}$	1.0153	1.022	1.0373	1.0054	1.0279	1.0486	1.0338	1.0316	1.0901	1.0617	1.0787
$\Sigma_{\hat\beta PW2}$	1.0161	1.0302	1.0511	1.0128	1.043	1.07	1.0457	1.0401	1.095	1.062	1.0764
						n=20					
$\Sigma_{\hat\beta 1}$	1.0076	1.013	1.0077	0.9978	1.0055	1.0106	1.0093	1.018	1.0423	1.0563	1.1264
$\Sigma_{\hat\beta 2}$	1.0067	1.0121	1.0075	0.9981	1.0059	1.0121	1.0104	1.0214	1.0411	1.0483	1.1043
$\Sigma_{\hat\beta W1}$	1.0006	1.01	1.0074	1.0005	1.0053	1.0135	1.0116	1.022	1.0449	1.0495	1.1109
$\Sigma_{\hat\beta W2}$	1	1.009	1.0074	1.0016	1.0056	1.0146	1.013	1.0246	1.0439	1.0429	1.0943
$\Sigma_{\hat\beta CO1}$	1.0101	1.061	1.0728	1.1299	1.1287	1.1446	1.1408	1.2733	1.4092	1.9677	5.2957
$\Sigma_{\hat\beta CO2}$	1.0097	1.0608	1.0732	1.1322	1.1312	1.1545	1.1474	1.3111	1.4517	2.3347	6.0906
$\Sigma_{\hat\beta PW1}$	0.994	1.0061	1.0081	1.0025	1.0065	1.0167	1.0137	1.0254	1.0404	1.0484	1.0899
$\Sigma_{\hat\beta PW2}$	0.9934	1.0049	1.0088	1.0043	1.007	1.0174	1.0157	1.0281	1.0404	1.0448	1.0823
						n=30					
$\Sigma_{\hat\beta 1}$	1.0069	1.0074	1.0009	1.01	1.0033	1	1.0035	1.0173	1.0179	1.0321	1.1172
$\Sigma_{\hat\beta 2}$	1.0067	1.0071	1.001	1.0099	1.0033	1	1.0038	1.0193	1.0187	1.0349	1.0922
$\Sigma_{\hat\beta W1}$	1.0014	1.0063	1.0016	1.0093	1.0036	1.0005	1.0048	1.02	1.019	1.036	1.0915
$\Sigma_{\hat\beta W2}$	1.0008	1.006	1.0019	1.0092	1.0038	1.0006	1.0052	1.0216	1.0201	1.0371	1.0768
$\Sigma_{\hat\beta CO1}$	0.9991	1.0207	1.0357	1.0445	1.0489	1.0695	1.0943	1.234	1.3208	1.6381	2.6244
$\Sigma_{\hat\beta CO2}$	0.9995	1.0207	1.0359	1.0447	1.0492	1.0705	1.0955	1.2367	1.3276	1.7801	3.0486
$\Sigma_{\hat\beta PW1}$	1.0023	1.0043	1.0036	1.009	1.0037	1.0007	1.0055	1.0215	1.0197	1.0372	1.0704
$\Sigma_{\hat\beta PW2}$	1.0019	1.0039	1.0044	1.0089	1.0038	1.0008	1.0059	1.0225	1.0207	1.0371	1.0649
						n=50					
$\Sigma_{\hat\beta 1}$	1.0016	1.0013	1.0026	1.0039	1.0043	1.002	1.0023	1.003	0.9989	1.0293	1.0611
$\Sigma_{\hat\beta 2}$	1.0015	1.0013	1.0026	1.0038	1.0043	1.002	1.0024	1.003	0.999	1.0265	1.0478
$\Sigma_{\hat\beta W1}$	0.9999	1.0016	1.0026	1.0038	1.0045	1.002	1.0025	1.0034	1.0002	1.0262	1.0464
$\Sigma_{\hat\beta W2}$	0.9998	1.0017	1.0026	1.0038	1.0046	1.0021	1.0026	1.0034	1.0003	1.0243	1.0406
$\Sigma_{\hat\beta CO1}$	1.0002	1.0084	1.0265	1.0171	1.0645	1.0414	1.084	1.1523	1.1394	1.3682	1.8394
$\Sigma_{\hat\beta CO2}$	1.0002	1.0084	1.0265	1.0171	1.0646	1.0416	1.0844	1.1528	1.1407	1.3717	1.9183
$\Sigma_{\hat\beta PW1}$	1.0002	1.0014	1.0021	1.0034	1.0047	1.0023	1.0031	1.0029	0.9998	1.0215	1.043
$\Sigma_{\hat\beta PW2}$	1.0001	1.0015	1.0021	1.0034	1.0048	1.0023	1.0032	1.0029	1	1.0208	1.0406

Table 3.3: Empirical significance level of the testing procedures derived from the GLS and the eight two-stage estimation procedures for a theoretical significance level of 5% when x is fixed, as a function of the sample size n and the autocorrelation parameter ρ. The empirical significance levels reported were obtained from 1000 simulation runs. A t distribution with $n-2$ df was used as the theoretical distribution of the test statistics. See the text for other notations.

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
n=10											
Σ_{ρ}	0.004	0.015	0.031	0.047	0.04	0.053	0.052	0.066	0.058	0.058	0.052
$\Sigma_{\hat{\beta}1}$	0.002	0.007	0.027	0.052	0.069	0.097	0.082	0.152	0.202	0.295	0.384
$\Sigma_{\hat{\beta}2}$	0.002	0.007	0.027	0.052	0.069	0.097	0.083	0.155	0.196	0.287	0.369
$\Sigma_{\hat{\beta}W1}$	0.002	0.007	0.028	0.056	0.073	0.098	0.085	0.158	0.199	0.293	0.371
$\Sigma_{\hat{\beta}W2}$	0.004	0.009	0.029	0.06	0.078	0.101	0.089	0.159	0.196	0.284	0.362
$\Sigma_{\hat{\beta}CO1}$	0.002	0.008	0.022	0.047	0.067	0.076	0.067	0.128	0.17	0.232	0.298
$\Sigma_{\hat{\beta}CO2}$	0.002	0.008	0.023	0.049	0.069	0.079	0.066	0.128	0.17	0.223	0.284
$\Sigma_{\hat{\beta}PW1}$	0.002	0.008	0.028	0.062	0.082	0.104	0.091	0.157	0.193	0.282	0.359
$\Sigma_{\hat{\beta}PW2}$	0.004	0.013	0.033	0.066	0.089	0.112	0.094	0.161	0.196	0.278	0.36
n=20											
Σ_{ρ}	0.004	0.014	0.023	0.033	0.042	0.041	0.045	0.039	0.047	0.033	0.021
$\Sigma_{\hat{\beta}1}$	0.002	0.008	0.021	0.032	0.055	0.074	0.076	0.095	0.147	0.225	0.338
$\Sigma_{\hat{\beta}2}$	0.002	0.008	0.021	0.032	0.055	0.074	0.077	0.095	0.144	0.209	0.308
$\Sigma_{\hat{\beta}W1}$	0.002	0.008	0.024	0.032	0.057	0.074	0.079	0.098	0.143	0.207	0.314
$\Sigma_{\hat{\beta}W2}$	0.002	0.008	0.025	0.032	0.056	0.075	0.08	0.098	0.14	0.199	0.293
$\Sigma_{\hat{\beta}CO1}$	0.001	0.005	0.024	0.03	0.044	0.073	0.071	0.091	0.134	0.187	0.259
$\Sigma_{\hat{\beta}CO2}$	0.001	0.005	0.024	0.03	0.044	0.072	0.071	0.091	0.134	0.183	0.25
$\Sigma_{\hat{\beta}PW1}$	0.002	0.008	0.025	0.035	0.059	0.076	0.08	0.098	0.139	0.193	0.276
$\Sigma_{\hat{\beta}PW2}$	0.002	0.008	0.026	0.036	0.058	0.077	0.081	0.099	0.135	0.189	0.265
n=30											
Σ_{ρ}	0	0.014	0.026	0.034	0.044	0.051	0.045	0.048	0.036	0.022	0.009
$\Sigma_{\hat{\beta}1}$	0	0.01	0.023	0.037	0.057	0.073	0.074	0.081	0.108	0.146	0.228
$\Sigma_{\hat{\beta}2}$	0	0.01	0.023	0.037	0.058	0.073	0.074	0.081	0.106	0.139	0.201
$\Sigma_{\hat{\beta}W1}$	0	0.01	0.023	0.037	0.058	0.073	0.074	0.081	0.106	0.138	0.204
$\Sigma_{\hat{\beta}W2}$	0	0.01	0.023	0.037	0.058	0.073	0.074	0.08	0.105	0.132	0.192
$\Sigma_{\hat{\beta}CO1}$	0	0.009	0.016	0.043	0.055	0.067	0.064	0.084	0.101	0.119	0.195
$\Sigma_{\hat{\beta}CO2}$	0	0.009	0.016	0.043	0.056	0.066	0.064	0.084	0.101	0.115	0.19
$\Sigma_{\hat{\beta}PW1}$	0	0.011	0.023	0.037	0.058	0.073	0.075	0.079	0.103	0.127	0.181
$\Sigma_{\hat{\beta}PW2}$	0	0.011	0.025	0.037	0.058	0.074	0.075	0.078	0.103	0.121	0.178
n=50											
Σ_{ρ}	0.002	0.014	0.021	0.033	0.053	0.051	0.051	0.043	0.033	0.021	0.006
$\Sigma_{\hat{\beta}1}$	0.001	0.009	0.02	0.044	0.066	0.065	0.07	0.073	0.068	0.089	0.168
$\Sigma_{\hat{\beta}2}$	0.001	0.009	0.02	0.044	0.066	0.065	0.07	0.073	0.067	0.082	0.159
$\Sigma_{\hat{\beta}W1}$	0.001	0.01	0.02	0.044	0.066	0.065	0.07	0.073	0.065	0.079	0.154
$\Sigma_{\hat{\beta}W2}$	0.001	0.01	0.021	0.044	0.066	0.065	0.07	0.072	0.065	0.078	0.149
$\Sigma_{\hat{\beta}CO1}$	0	0.006	0.02	0.042	0.065	0.065	0.07	0.082	0.06	0.08	0.122
$\Sigma_{\hat{\beta}CO2}$	0	0.007	0.02	0.041	0.065	0.065	0.07	0.082	0.06	0.08	0.123
$\Sigma_{\hat{\beta}PW1}$	0.001	0.011	0.022	0.044	0.068	0.065	0.07	0.071	0.064	0.068	0.134
$\Sigma_{\hat{\beta}PW2}$	0.001	0.011	0.022	0.044	0.068	0.065	0.07	0.071	0.064	0.068	0.13

Table 3.4: Efficiency of the two-stage estimation procedures relative to GLS when x is purely random, as a function of the sample size n and the autocorrelation parameter ρ. The relative efficiencies reported were obtained from 1000 simulation runs. See the text for other notations.

n=10

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
$\Sigma_{\hat\beta 1}$	2.0229	1.3126	1.0844	1.0915	1.1171	1.0523	1.0917	1.1536	1.2107	1.3801	1.6265
$\Sigma_{\hat\beta 2}$	1.4003	1.1749	1.0965	1.1477	1.2227	1.1205	1.1782	1.2142	1.2844	1.2734	1.4116
$\Sigma_{\hat\beta W1}$	1.9005	1.2927	1.0743	1.1071	1.1348	1.0657	1.1056	1.16	1.207	1.3582	1.5469
$\Sigma_{\hat\beta W2}$	1.3463	1.1607	1.1205	1.1827	1.268	1.1759	1.2142	1.2324	1.3236	1.2987	1.3699
$\Sigma_{\hat\beta CO1}$	1.8217	1.3791	1.1916	1.2393	1.3164	1.2261	1.2325	1.2974	1.3855	1.4844	1.6182
$\Sigma_{\hat\beta CO2}$	1.3761	1.334	1.2802	1.3739	1.5217	1.4238	1.4118	1.4367	1.6121	1.4498	1.5593
$\Sigma_{\hat\beta PW1}$	1.7608	1.2821	1.0742	1.1252	1.1541	1.0864	1.1202	1.1679	1.2109	1.3398	1.4767
$\Sigma_{\hat\beta PW2}$	1.3089	1.1824	1.1446	1.2185	1.3246	1.2547	1.2689	1.2829	1.3729	1.2882	1.3732

n=20

$\Sigma_{\hat\beta 1}$	1.1314	1.0735	1.0401	1.0927	1.0402	1.0606	1.0681	1.1029	1.1195	1.1151	1.2401
$\Sigma_{\hat\beta 2}$	1.0178	1.0313	1.0269	1.1113	1.0678	1.098	1.0994	1.1221	1.0966	1.058	1.0653
$\Sigma_{\hat\beta W1}$	1.0699	1.0649	1.0378	1.0955	1.0475	1.0642	1.073	1.1057	1.1166	1.101	1.2011
$\Sigma_{\hat\beta W2}$	1.0061	1.0246	1.0273	1.1213	1.0835	1.109	1.1088	1.1313	1.0949	1.0504	1.0439
$\Sigma_{\hat\beta CO1}$	1.1055	1.0507	1.062	1.1466	1.1261	1.0969	1.1179	1.182	1.1407	1.1113	1.1459
$\Sigma_{\hat\beta CO2}$	1.0033	1.0447	1.0537	1.1845	1.1632	1.1739	1.1698	1.2194	1.1278	1.0732	1.063
$\Sigma_{\hat\beta PW1}$	1.0393	1.0525	1.0356	1.1011	1.0535	1.0701	1.0776	1.1098	1.1109	1.0877	1.1393
$\Sigma_{\hat\beta PW2}$	1.0014	1.0212	1.0278	1.135	1.0975	1.1192	1.1175	1.1382	1.0924	1.0435	1.0341

n=30

$\Sigma_{\hat\beta 1}$	1.0588	1.0342	1.0465	1.0298	1.0287	1.042	1.0362	1.0459	1.0278	1.0619	1.0586
$\Sigma_{\hat\beta 2}$	1.0083	1.0201	1.0395	1.0333	1.0453	1.0577	1.0508	1.0592	1.023	1.0272	1.0127
$\Sigma_{\hat\beta W1}$	1.0444	1.0271	1.0464	1.03	1.0303	1.0429	1.0381	1.0458	1.0264	1.0572	1.0421
$\Sigma_{\hat\beta W2}$	1.0005	1.0151	1.0419	1.0364	1.0482	1.0592	1.0548	1.0656	1.0234	1.0257	1.0087
$\Sigma_{\hat\beta CO1}$	1.0373	1.0355	1.0678	1.0617	1.0406	1.0792	1.0669	1.0863	1.0488	1.0641	1.0552
$\Sigma_{\hat\beta CO2}$	0.9986	1.0258	1.0678	1.0707	1.0627	1.1016	1.0884	1.1108	1.0501	1.0361	1.0207
$\Sigma_{\hat\beta PW1}$	1.03	1.0237	1.0444	1.0307	1.0336	1.045	1.0414	1.0459	1.0252	1.0506	1.0282
$\Sigma_{\hat\beta PW2}$	0.998	1.014	1.0419	1.0394	1.0527	1.0624	1.0603	1.0672	1.0239	1.022	1.0027

n=50

$\Sigma_{\hat\beta 1}$	1.0116	1.0049	1.0474	1.0262	1.0318	1.0116	1.0389	1.0358	1.0249	1.0307	1.0171
$\Sigma_{\hat\beta 2}$	0.9981	0.9985	1.0356	1.026	1.0424	1.0169	1.0462	1.0337	1.0181	1.0133	1.0073
$\Sigma_{\hat\beta W1}$	1.0115	1.0027	1.0483	1.0272	1.0329	1.012	1.0397	1.035	1.0211	1.0257	1.0162
$\Sigma_{\hat\beta W2}$	0.9992	0.9976	1.0381	1.0277	1.0442	1.0177	1.0474	1.0333	1.015	1.0097	1.0071
$\Sigma_{\hat\beta CO1}$	1.015	1.0145	1.0794	1.0292	1.0561	1.0388	1.0585	1.0541	1.038	1.0333	1.0139
$\Sigma_{\hat\beta CO2}$	1.0064	1.01	1.0692	1.0319	1.0706	1.0472	1.0685	1.0538	1.032	1.0184	1.0079
$\Sigma_{\hat\beta PW1}$	1.0089	1.0008	1.0484	1.0279	1.0344	1.0136	1.0406	1.0346	1.0187	1.0235	1.013
$\Sigma_{\hat\beta PW2}$	0.9985	0.9969	1.0395	1.0292	1.0464	1.0198	1.0486	1.0333	1.0131	1.0091	1.0055

Table 3.5: Empirical significance level of the testing procedures derived from the GLS and the eight two-stage estimation procedures for a theoretical significance level of 5% when x is purely random, as a function of the sample size n and the autocorrelation parameter ρ. The empirical significance levels reported were obtained from 1000 simulation runs. A t distribution with $n-2$ df was used as the theoretical distribution of the test statistics. See the text for other notations.

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
n=10											
Σ_{ρ}	0.004	0.012	0.019	0.052	0.049	0.049	0.053	0.049	0.043	0.044	0.023
$\Sigma_{\beta 1}$	0.02	0.031	0.034	0.058	0.069	0.057	0.063	0.061	0.039	0.047	0.044
$\Sigma_{\beta 2}$	0.013	0.024	0.037	0.064	0.075	0.062	0.071	0.07	0.049	0.053	0.042
$\Sigma_{\hat{\rho}_{W1}}$	0.018	0.028	0.033	0.058	0.068	0.06	0.063	0.062	0.045	0.048	0.043
$\Sigma_{\hat{\rho}_{W2}}$	0.013	0.026	0.038	0.066	0.078	0.067	0.076	0.073	0.062	0.057	0.041
$\Sigma_{\hat{\rho}_{CO1}}$	0.017	0.028	0.03	0.055	0.059	0.051	0.053	0.045	0.037	0.034	0.032
$\Sigma_{\hat{\rho}_{CO2}}$	0.014	0.024	0.032	0.066	0.068	0.059	0.061	0.055	0.05	0.047	0.032
$\Sigma_{\hat{\rho}_{PW1}}$	0.019	0.028	0.037	0.061	0.07	0.06	0.068	0.064	0.049	0.053	0.044
$\Sigma_{\hat{\rho}_{PW2}}$	0.012	0.026	0.042	0.07	0.084	0.075	0.08	0.078	0.066	0.06	0.041
n=20											
Σ_{ρ}	0.008	0.012	0.034	0.034	0.049	0.041	0.051	0.033	0.044	0.026	0.008
$\Sigma_{\beta 1}$	0.005	0.013	0.034	0.049	0.061	0.053	0.061	0.04	0.044	0.023	0.009
$\Sigma_{\beta 2}$	0.005	0.012	0.034	0.05	0.066	0.06	0.063	0.042	0.042	0.02	0.006
$\Sigma_{\hat{\rho}_{W1}}$	0.005	0.013	0.034	0.05	0.062	0.055	0.062	0.04	0.044	0.02	0.009
$\Sigma_{\hat{\rho}_{W2}}$	0.006	0.011	0.036	0.053	0.068	0.066	0.065	0.045	0.044	0.02	0.006
$\Sigma_{\hat{\rho}_{CO1}}$	0.006	0.006	0.034	0.048	0.066	0.052	0.058	0.043	0.045	0.015	0.008
$\Sigma_{\hat{\rho}_{CO2}}$	0.006	0.008	0.034	0.048	0.07	0.059	0.06	0.049	0.046	0.012	0.005
$\Sigma_{\hat{\rho}_{PW1}}$	0.006	0.014	0.035	0.052	0.064	0.06	0.062	0.041	0.044	0.02	0.009
$\Sigma_{\hat{\rho}_{PW2}}$	0.007	0.011	0.037	0.053	0.071	0.067	0.065	0.046	0.046	0.02	0.005
n=30											
Σ_{ρ}	0	0.01	0.019	0.03	0.05	0.062	0.049	0.038	0.037	0.018	0.009
$\Sigma_{\beta 1}$	0	0.011	0.02	0.035	0.057	0.077	0.054	0.044	0.036	0.016	0.004
$\Sigma_{\beta 2}$	0	0.011	0.02	0.036	0.059	0.081	0.056	0.044	0.036	0.013	0.004
$\Sigma_{\hat{\rho}_{W1}}$	0	0.011	0.021	0.035	0.058	0.078	0.055	0.044	0.037	0.015	0.004
$\Sigma_{\hat{\rho}_{W2}}$	0	0.01	0.02	0.036	0.06	0.083	0.056	0.045	0.036	0.013	0.005
$\Sigma_{\hat{\rho}_{CO1}}$	0	0.009	0.019	0.03	0.051	0.077	0.058	0.043	0.035	0.016	0.004
$\Sigma_{\hat{\rho}_{CO2}}$	0	0.009	0.019	0.031	0.052	0.081	0.061	0.045	0.037	0.012	0.004
$\Sigma_{\hat{\rho}_{PW1}}$	0	0.012	0.022	0.035	0.057	0.079	0.055	0.045	0.036	0.015	0.003
$\Sigma_{\hat{\rho}_{PW2}}$	0	0.011	0.021	0.036	0.061	0.083	0.056	0.046	0.036	0.014	0.004
n=50											
Σ_{ρ}	0.001	0.006	0.016	0.046	0.043	0.049	0.057	0.04	0.022	0.008	0.005
$\Sigma_{\beta 1}$	0	0.005	0.021	0.046	0.051	0.057	0.06	0.044	0.021	0.006	0.002
$\Sigma_{\beta 2}$	0	0.005	0.019	0.047	0.054	0.059	0.062	0.046	0.021	0.006	0.001
$\Sigma_{\hat{\rho}_{W1}}$	0	0.005	0.021	0.047	0.051	0.058	0.06	0.044	0.021	0.006	0.002
$\Sigma_{\hat{\rho}_{W2}}$	0	0.005	0.019	0.047	0.054	0.059	0.063	0.046	0.021	0.006	0.001
$\Sigma_{\hat{\rho}_{CO1}}$	0	0.004	0.021	0.047	0.051	0.059	0.062	0.044	0.023	0.006	0.003
$\Sigma_{\hat{\rho}_{CO2}}$	0	0.004	0.02	0.047	0.054	0.06	0.064	0.043	0.023	0.006	0.002
$\Sigma_{\hat{\rho}_{PW1}}$	0	0.005	0.021	0.047	0.052	0.058	0.06	0.044	0.022	0.006	0.003
$\Sigma_{\hat{\rho}_{PW2}}$	0	0.005	0.019	0.047	0.054	0.059	0.067	0.045	0.022	0.006	0.002

Table 3.6: Efficiency of the two-stage estimation procedures relative to GLS when x follows an AR(1) process, as a function of the sample size n and the autocorrelation parameter ρ. The relative efficiencies reported were obtained from 1000 simulation runs. See the text for other notations.

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
						n=10					
$\Sigma_{\beta 1}$	2.6737	1.3657	1.1731	1.06	1.0292	1.0551	1.1138	1.0744	1.3416	1.4823	1.9669
$\Sigma_{\beta 2}$	1.698	1.2087	1.149	1.1094	1.0812	1.1143	1.2111	1.1363	1.3645	1.4357	1.6928
$\Sigma_{\beta W1}$	2.3914	1.3227	1.1654	1.0697	1.0456	1.0644	1.1267	1.0753	1.3546	1.4693	1.8984
$\Sigma_{\beta W2}$	1.5127	1.2263	1.1834	1.1369	1.1261	1.1532	1.2397	1.1563	1.3923	1.442	1.609
$\Sigma_{\beta CO1}$	2.4965	1.5013	1.3888	1.1871	1.1949	1.1781	1.2656	1.1968	1.5103	1.6142	1.9701
$\Sigma_{\beta CO2}$	1.8135	1.4958	1.5253	1.3466	1.4255	1.3368	1.4976	1.3539	1.7621	1.6927	1.8275
$\Sigma_{\beta PW1}$	2.1118	1.3014	1.1591	1.0902	1.0616	1.0773	1.1502	1.0834	1.3565	1.4626	1.8124
$\Sigma_{\beta PW2}$	1.436	1.2991	1.214	1.1878	1.179	1.2086	1.2919	1.1896	1.416	1.4768	1.5386
						n=20					
$\Sigma_{\beta 1}$	2.0326	1.243	1.093	1.1184	1.046	1.0591	1.0596	1.1207	1.1575	1.34	1.9649
$\Sigma_{\beta 2}$	1.294	1.1246	1.0911	1.1702	1.0639	1.101	1.0767	1.1315	1.1484	1.1745	1.4036
$\Sigma_{\beta W1}$	1.826	1.2087	1.0852	1.1289	1.0473	1.0625	1.0634	1.1197	1.1598	1.3046	1.8409
$\Sigma_{\beta W2}$	1.2171	1.1199	1.0993	1.1901	1.0743	1.1098	1.0857	1.1393	1.1632	1.1532	1.301
$\Sigma_{\beta CO1}$	1.8496	1.2872	1.1723	1.196	1.1167	1.1114	1.138	1.1855	1.2213	1.3704	1.8949
$\Sigma_{\beta CO2}$	1.2793	1.2288	1.2069	1.2694	1.1577	1.1829	1.1832	1.2157	1.2348	1.2138	1.4502
$\Sigma_{\beta PW1}$	1.6475	1.1909	1.0842	1.1386	1.0522	1.0685	1.0654	1.1227	1.1552	1.2751	1.7266
$\Sigma_{\beta PW2}$	1.1667	1.1193	1.1181	1.2116	1.0845	1.1252	1.0938	1.1544	1.1734	1.1316	1.2614
						n=30					
$\Sigma_{\beta 1}$	1.745	1.1695	1.105	1.0562	1.0598	1.0603	1.052	1.0556	1.151	1.2455	1.8043
$\Sigma_{\beta 2}$	1.1617	1.0546	1.0914	1.0679	1.0858	1.0779	1.0686	1.0614	1.1161	1.1242	1.3055
$\Sigma_{\beta W1}$	1.556	1.1494	1.1046	1.0563	1.0639	1.0638	1.0553	1.0563	1.1453	1.2213	1.6808
$\Sigma_{\beta W2}$	1.0944	1.0464	1.1024	1.07	1.0968	1.0828	1.0732	1.0635	1.1141	1.1059	1.2275
$\Sigma_{\beta CO1}$	1.6109	1.1859	1.1744	1.0929	1.107	1.1327	1.0958	1.0898	1.1819	1.2286	1.6152
$\Sigma_{\beta CO2}$	1.1668	1.0948	1.1709	1.114	1.1476	1.1566	1.1223	1.1033	1.1573	1.1279	1.1987
$\Sigma_{\beta PW1}$	1.4084	1.1291	1.102	1.0583	1.0696	1.0687	1.0584	1.0567	1.1405	1.2031	1.556
$\Sigma_{\beta PW2}$	1.0754	1.0448	1.1084	1.0745	1.1077	1.0903	1.0783	1.0658	1.1124	1.0968	1.1467
						n=50					
$\Sigma_{\beta 1}$	1.3468	1.1127	1.0507	1.0379	1.0371	1.0441	1.0507	1.0239	1.0555	1.1188	1.4829
$\Sigma_{\beta 2}$	1.0876	1.0591	1.0396	1.0446	1.0474	1.0484	1.0563	1.0253	1.0608	1.0613	1.1308
$\Sigma_{\beta W1}$	1.2778	1.1032	1.0501	1.0382	1.0386	1.0456	1.0519	1.0242	1.0528	1.1099	1.4049
$\Sigma_{\beta W2}$	1.072	1.0565	1.0432	1.0454	1.0493	1.0502	1.0579	1.0262	1.0603	1.0556	1.0885
$\Sigma_{\beta CO1}$	1.283	1.1085	1.0693	1.0543	1.0482	1.083	1.0751	1.0389	1.0839	1.1559	1.3738
$\Sigma_{\beta CO2}$	1.075	1.068	1.0672	1.0646	1.0597	1.0892	1.0831	1.0421	1.0953	1.1057	1.0675
$\Sigma_{\beta PW1}$	1.2267	1.0893	1.0487	1.0386	1.0397	1.0473	1.0536	1.0245	1.051	1.1007	1.3317
$\Sigma_{\beta PW2}$	1.0664	1.0489	1.0459	1.0473	1.0508	1.0522	1.06	1.0268	1.0641	1.0503	1.0446

Table 3.7: Empirical significance level of the testing procedures derived from the GLS and the eight two-stage estimation procedures for a theoretical significance level of 5% when x follows an AR(1) process, as a function of the sample size n and the autocorrelation parameter ρ. The empirical significance levels reported were obtained from 1000 simulation runs. A t distribution with $n-2$ df was used as the theoretical distribution of the test statistics. See the text for other notations.

n=10

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
Σ_{ρ}	0.011	0.019	0.041	0.036	0.047	0.057	0.047	0.068	0.048	0.05	0.03
$\Sigma_{\beta 1}$	0.204	0.089	0.073	0.06	0.057	0.072	0.062	0.08	0.098	0.131	0.174
$\Sigma_{\beta 2}$	0.151	0.07	0.064	0.059	0.057	0.077	0.067	0.084	0.101	0.126	0.162
$\Sigma_{\hat{\beta}W1}$	0.181	0.08	0.073	0.058	0.059	0.074	0.065	0.081	0.099	0.13	0.17
$\Sigma_{\hat{\beta}W2}$	0.121	0.07	0.065	0.064	0.063	0.08	0.074	0.088	0.107	0.124	0.152
$\Sigma_{\hat{\beta}CO1}$	0.153	0.078	0.067	0.051	0.053	0.06	0.053	0.073	0.085	0.108	0.136
$\Sigma_{\hat{\beta}CO2}$	0.115	0.067	0.068	0.054	0.059	0.073	0.063	0.079	0.096	0.108	0.126
$\Sigma_{\hat{\beta}PW1}$	0.148	0.076	0.069	0.06	0.061	0.075	0.069	0.087	0.1	0.133	0.164
$\Sigma_{\hat{\beta}PW2}$	0.096	0.068	0.066	0.067	0.07	0.089	0.081	0.091	0.111	0.131	0.145

n=20

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
Σ_{ρ}	0.008	0.016	0.028	0.034	0.051	0.054	0.049	0.049	0.041	0.034	0.018
$\Sigma_{\beta 1}$	0.102	0.047	0.044	0.053	0.059	0.059	0.06	0.067	0.072	0.077	0.134
$\Sigma_{\beta 2}$	0.065	0.04	0.042	0.055	0.063	0.062	0.064	0.065	0.066	0.06	0.092
$\Sigma_{\hat{\beta}W1}$	0.09	0.044	0.044	0.053	0.06	0.058	0.061	0.066	0.072	0.073	0.118
$\Sigma_{\hat{\beta}W2}$	0.05	0.035	0.042	0.056	0.063	0.063	0.065	0.066	0.066	0.059	0.082
$\Sigma_{\hat{\beta}CO1}$	0.08	0.041	0.043	0.054	0.054	0.062	0.06	0.07	0.071	0.065	0.103
$\Sigma_{\hat{\beta}CO2}$	0.041	0.032	0.043	0.061	0.061	0.068	0.071	0.071	0.067	0.054	0.073
$\Sigma_{\hat{\beta}PW1}$	0.076	0.042	0.044	0.055	0.061	0.058	0.061	0.068	0.071	0.071	0.107
$\Sigma_{\hat{\beta}PW2}$	0.037	0.032	0.042	0.058	0.066	0.066	0.065	0.069	0.065	0.058	0.076

n=30

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
Σ_{ρ}	0.002	0.01	0.021	0.033	0.035	0.049	0.044	0.039	0.03	0.027	0.004
$\Sigma_{\beta 1}$	0.053	0.029	0.03	0.04	0.042	0.061	0.054	0.054	0.054	0.052	0.076
$\Sigma_{\beta 2}$	0.027	0.022	0.026	0.038	0.048	0.061	0.058	0.055	0.05	0.045	0.04
$\Sigma_{\hat{\beta}W1}$	0.044	0.026	0.03	0.039	0.043	0.062	0.054	0.054	0.054	0.05	0.066
$\Sigma_{\hat{\beta}W2}$	0.02	0.021	0.028	0.039	0.05	0.063	0.059	0.056	0.05	0.044	0.035
$\Sigma_{\hat{\beta}CO1}$	0.037	0.024	0.032	0.037	0.041	0.069	0.055	0.051	0.045	0.049	0.055
$\Sigma_{\hat{\beta}CO2}$	0.016	0.021	0.028	0.038	0.046	0.07	0.058	0.052	0.042	0.041	0.029
$\Sigma_{\hat{\beta}PW1}$	0.034	0.025	0.03	0.039	0.045	0.063	0.056	0.054	0.053	0.048	0.06
$\Sigma_{\hat{\beta}PW2}$	0.015	0.02	0.029	0.039	0.052	0.065	0.06	0.058	0.049	0.041	0.028

n=50

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
Σ_{ρ}	0.001	0.008	0.021	0.045	0.056	0.042	0.056	0.043	0.02	0.008	0.003
$\Sigma_{\beta 1}$	0.021	0.021	0.025	0.054	0.058	0.055	0.063	0.05	0.032	0.023	0.042
$\Sigma_{\beta 2}$	0.007	0.016	0.025	0.054	0.058	0.056	0.063	0.049	0.031	0.014	0.021
$\Sigma_{\hat{\beta}W1}$	0.018	0.02	0.024	0.054	0.058	0.056	0.063	0.05	0.033	0.021	0.035
$\Sigma_{\hat{\beta}W2}$	0.005	0.015	0.026	0.055	0.058	0.056	0.063	0.049	0.031	0.011	0.016
$\Sigma_{\hat{\beta}CO1}$	0.02	0.016	0.026	0.052	0.059	0.058	0.063	0.05	0.029	0.02	0.031
$\Sigma_{\hat{\beta}CO2}$	0.005	0.011	0.027	0.052	0.059	0.059	0.064	0.049	0.029	0.013	0.015
$\Sigma_{\hat{\beta}PW1}$	0.017	0.017	0.023	0.055	0.058	0.056	0.063	0.05	0.033	0.019	0.03
$\Sigma_{\hat{\beta}PW2}$	0.005	0.014	0.026	0.056	0.058	0.056	0.063	0.05	0.03	0.011	0.016

Chapter 4

Is the Classical t-Test of the Slope Really Invalid in Linear Regression Models with Autocorrelated Errors?

ABSTRACT

A classical requirement for the t-test of individual slopes in linear regression analysis is that the random errors be independently distributed. In a Monte Carlo study, we show that although the errors are autocorrelated, the classical t-test of the slope is valid or close to validity, like most of the other testing procedures, when the explanatory variable is made of purely random $N(0,1)$ entries. These results are discussed in terms of the circularity condition used in repeated measures ANOVA and of the effective sample size in correlation analysis with autocorrelated sample data. In conclusion, we recommend that the autocorrelation of random explanatory variables be analyzed first in linear regression with time series or spatial data, before neglecting the classical t-test of individual slopes.

nificance of the population mean (Cliff and Ord 1975). The authors added "... in the case of other statistical models which assume independence, little is known about their robustness to departures from the assumption of independence." This was before the rise of the repeated measures ANOVA techniques (Crowder and Hand 1990). At the beginning of the 1990s, Cressie (1993) provided the state of the art concerning statistical methods appropriate for spatial data. Krämer and Donninger (1987, cited by Cressie) showed that OLS can be more efficient than estimated GLS in the case of weak autocorrelation among errors. We have reproduced and somewhat refined Krämer and Donninger's numerical results in one of our previous studies (Alpargu and Dutilleul 2001).

In the correlation analysis between two spatially autocorrelated processes, the t-test built on Pearson's product-moment correlation coefficient suffers from an inflated Type I error risk when the number of degrees of freedom is calculated from the classical sample size (Clifford and Richardson 1985; Clifford *et al.* 1989). A similar result holds for Spearman's rank-based correlation coefficient (Haining 1990, pp. 322-323). To adjust the t-test for the autocorrelation of the two spatial processes, the classical sample size should be replaced by an effective sample size appropriately computed to obtain the number of df (Dutilleul 1993). However, Jenkins and Watts (1968, pp. 338-339) demonstrated by an example that the cross-correlations between two time series are not biased, provided at least one of the two series is not autocorrelated. In regression analysis, Cook and Pocock (1983) pointed out that t-tests based on OLS estimates of the slopes divided by the corresponding standard errors overstate the significance of regression coefficients in the presence of positive spatial autocorrelation among the errors. In their landmark book, Upton and Fingleton (1985, p. 283) wrote "the conventional t and F tests are invalidated by the dependence among the errors", without specifying the nature of the regressor.

Relating Jenkins and Watts's demonstration to Upton and Fingleton's statement, we were curious to know whether the classical t-test of the slope (i.e., built as the OLS estimator divided by the corresponding standard error) is really invalid in linear regression models with autocorrelated errors when the explanatory variable is purely random. While addressing this question, we have also assessed the validity of 30 other testing procedures. The 31 procedures that we have considered for testing the significance of an individual slope parameter in a linear regression model with temporally autocorrelated errors are based on one of the estimation methods of OLS, GLS, estimated GLS, maximum likelihood (ML), restricted maximum likelihood (REML), first differences (FD) or first-difference ratios (FDR).

In Section 2, we review a number of procedures available in the literature for estimating the slopes in linear regression models with autocorrelated errors. Some of these procedures do not require the estimation of the covariance matrix of the errors (Subsection 2.1), whereas the others do (Subsection 2.2). In Section 3, we define the testing procedures by focusing on modified t-tests of individual slopes with different adjustments of the number of degrees of freedom. We present our Monte Carlo study in Section 4. The results of it are summarized in Section 5 and discussed in Section 6. Conclusions are drawn in Section 7.

2. ESTIMATION PROCEDURES

Consider a linear regression model with temporal AR(1) errors

$$y = X\beta + \varepsilon, \quad \text{with} \quad \varepsilon_t = \rho\varepsilon_{t-1} + u_t \quad (t = 1, 2, \ldots, n), \tag{4.1}$$

where y is an $n \times 1$ observable random vector; β is a $q \times 1$ unknown vector to be estimated; X is an $n \times q$ matrix of rank $q < n$, whose first column is a column of ones and the $q-1$ others are filled with purely random $N(0, \sigma^2_{x_j})$ entries ($j = 2, \ldots, q$); ε is an $n \times 1$ unobservable vector of random errors with zero expected value; $-1 < \rho < 1$; and $u \sim N_n(0, \sigma^2_u I)$, with I the $n \times n$ identity matrix and σ^2_u an unknown positive constant. Furthermore, the x_js and ε are uncorrelated. Let the covariance matrix of ε, $\mathrm{Cov}(\varepsilon)$, be denoted by Σ.

2.1 Without estimation of the covariance matrix of the errors

The OLS estimator of β in (4.1) is $\hat{\beta}_{\mathrm{OLS}} = (X'X)^{-1}X'y$, with covariance matrix $\mathrm{Cov}(\hat{\beta}_{\mathrm{OLS}})_1 = \sigma^2(X'X)^{-1}$ if $\rho = 0$. If $\rho \neq 0$, then the covariance matrix of $\hat{\beta}_{\mathrm{OLS}}$ is $\mathrm{Cov}(\hat{\beta}_{\mathrm{OLS}})_2 = (X'X)^{-1}X'\Sigma X\ (X'X)^{-1}$.

If ρ is known, which is not the case in practice, then the Best Linear Unbiased Estimator (BLUE) of β is the GLS estimator or Aitken estimator, $\hat{\beta}_{\mathrm{GLS}} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y$, with covariance matrix $\mathrm{Cov}(\hat{\beta}_{\mathrm{GLS}}) = (X'\Sigma^{-1}X)^{-1}$.

In the FD method, the transformation defined by $(I-W)$ is applied to model (4.1) under the assumption that ρ is equal to 1, so that the dependency among the errors is removed prior to fitting a model without intercept (Martin 1974). In the particular case of simple linear regression with equally spaced observations in time, the ratios of first differences $y_t - y_{t-1}$ and $x_t - x_{t-1}$ have an expected value equal to the slope parameter under mild conditions. This led us to consider an FDR procedure in which the slope of simple linear regression is estimated by the mean of the ratios of first differences of the dependent and explanatory variables.

2.2 With estimation of the covariance matrix of the errors

In (4.1 The covariance matrix of ε is

$$\Sigma = \sigma_\varepsilon^2 V = \sigma_\varepsilon^2 \begin{pmatrix} 1 & \rho & \rho^2 & \rho^3 & \cdots & \rho^{n-2} & \rho^{n-1} \\ \rho & 1 & \rho & \rho^2 & \cdots & \rho^{n-3} & \rho^{n-2} \\ \vdots & & & & & & \vdots \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \rho^{n-4} & \cdots & \rho & 1 \end{pmatrix}, \tag{4.2}$$

where $-1 < \rho < 1$ and $\sigma_\varepsilon^2 = \sigma_u^2/(1-\rho^2)$. The GLS estimation of β requires that ρ be known. Otherwise, ρ can be estimated by the sample autocorrelation coefficient at lag $1, r(1)$, or some other estimator (Beach and Mackinnon 1978), assuming the errors follow an AR(1) process. In the following estimated GLS procedures, an estimator of Σ is used in $\hat{\beta}_{\text{GLS}}$ and $\text{Cov}(\hat{\beta}_{\text{GLS}})$, whereas σ_ε^2 is estimated by the error mean square.

If the structure of the covariance matrix of errors is unknown, then the sample autocorrelation coefficients at lag k, $r(k) = \sum_{i=1}^{n-k} e_i e_{i+k} / \sum_{i=1}^{n} e_i^2$, (where e_i $(i = 1, 2, \ldots, n)$) are the OLS residuals, are natural candidates for estimating the true autocorrelation parameters $\rho(k)$ under the general assumption of weak stationarity. In general, the recommended time series length n is 50 or more, to obtain reliable estimates of $\rho(k)$ (Box *et al.* 1994). Usually, the first $\text{INT}(n/4)$ sample autocorrelation coefficients, where INT() denotes the integer part of the number in parentheses, are usually calculated and the remaining ones are set at zero. Therefore, the more general estimated form of Σ is

$$\hat{\Sigma} = \hat{\sigma}_\varepsilon^2 \begin{pmatrix} 1 & r(1) & & \cdots & r(m) & 0 & \cdots & 0 \\ r(1) & 1 & r(1) & & \cdots & r(m) & \cdots & 0 \\ & \ddots & \ddots & \ddots & & & & \vdots \\ & & \ddots & \ddots & \ddots & & \ddots & 0 \\ & & & \ddots & \ddots & \ddots & & r(m) \\ \vdots & & & & & & & \vdots \\ 0 & \cdots & & & & \cdots & 1 & r(1) \\ 0 & \cdots & 0 & r(m) & & \cdots & r(1) & 1 \end{pmatrix}, \tag{4.3}$$

where $m = \text{INT}(n/4)$. Furthermore, if the true autocorrelation parameters are suspected to be zero beyond a certain lag, then the significance of $\rho(k)$ $(k = 1, 2, \ldots, m = \text{INT}(n/4))$ can be assessed by an approximate z-test. Namely, if the approximate z-test lies between -2 and 2, then the hypothesis $\rho(k) = \rho(k+1) = \cdots = \rho(m) = 0$ is not rejected at the approximate 5% significance level. When $\hat{\Sigma}$ is not positive definite, the problem can be circumvented by replacing $\hat{\Sigma}$ with $\hat{\Sigma} + \lambda I$ (Graybill 1983, pp. 408-409), with λ a positive scalar appropriately chosen.

If the family of distribution of the errors is known, then the ML and REML methods can be applied, conditional on X, to estimate β, σ_u^2, and ρ if it is unknown. The ML estimators of the parameters of model (4.1) are: $\hat{\beta}_{\mathrm{ML}} = (X'A'AX)^{-1}X'A'Ay$ and $\hat{\sigma}^2_{\mathrm{ML}} = (Ay)'P(Ay)/n$, where $A = (I - \hat{\rho}W)$ and $P = I-(AX)\{(AX)'(AX)\}^{-1}(AX)'$; $\hat{\rho}$ minimizes $M^* = \log(Ay)'P(Ay)-(2/n)\log|A|$ (Upton and Fingleton 1985). $W = (w_{ij})$ is defined as $w_{ij} = 1$ if $j = i-1$, and 0 otherwise.

The REML procedure is a simplification of the ML procedure (Patterson and Thompson 1971). In (4.1), the REML estimators maximize

$$L^* = -\frac{1}{2}\log|\Sigma| - \frac{1}{2}\log|X'\Sigma^{-1}X| - \frac{1}{2}(y - X\hat{\beta})'\Sigma^{-1}(y - X\hat{\beta}),$$

whereas the ML estimators maximize

$$L = -\frac{1}{2}\log|\Sigma| - \frac{1}{2}(y - X\hat{\beta})'\Sigma^{-1}(y - X\hat{\beta}).$$

If the sample size increases for a fixed number of columns of X, the ML and REML provide similar estimators; otherwise, the REML estimators are to be preferred (Diggle et al. 1996).

3. TESTING PROCEDURES

In the previous sections, the emphasis has been on the estimation of the slope parameter. In all but two of the cases, the test statistic is built as the ratio of the slope estimator divided by its standard error. The exceptions are provided by the likelihood-ratio χ^2-test in the ML procedure, and the F-test for fixed effects in the REML procedure. In most cases (i.e., when the estimation method is OLS, GLS, estimated GLS, FD, and FDR), the test statistic is assumed to follow or it actually follows a t-distribution with $n-q$ degrees of freedom (df). Depending on how it is built, the test statistic derived from the ML estimator follows a standard normal distribution or a chi-square distribution with 1 df. As for the REML procedure, the F-test statistic is nothing but the square of a t-ratio. The underlying idea in the GLS, estimated GLS, ML, REML, FD, and FDR estimation procedures is to take the dependency among the errors into account by incorporating it in the estimation procedure or by removing it from the data as much as possible. In the rest of this section, we consider an alternative approach based on the OLS estimation of the slope parameter from the raw data y, combined with a modified t-test with a number of df adjusted for the level of autocorrelation in the errors. In other words, the dependency among the errors is taken into account in the test instead of the estimator. Note that the adjusted number of df in the first modified t-test considered is restricted to be at most equal to the classical one. In the others, the adjusted number of df can be greater or smaller than the classical $n-q$ df, depending on the sign of autocorrelation of the explanatory variable and the error.

First, let the classical number of df of the t-test (i.e., $n-2$ in simple linear regression) be multiplied by a constant inspired from Box's 'epsilon' (Box 1954a, b) in the modified F-test of the repeated measures ANOVA. The multiplicative constant to be used in linear regression analysis in general would be $\varepsilon^{\star}_{\mathrm{AN}} = (\mathrm{tr}\ C'\Sigma C)^2/[(n-q)\mathrm{tr}(C'\Sigma C)^2]$, where $C = I - X(X'X)^{-1}X'$. The same estimators as those used in estimated GLS can be used to estimate Σ in $\varepsilon^{\star}_{\mathrm{AN}}$.

Secondly, we consider a modified t-test with $\hat{n}-2$ df in simple linear regression, where $\hat{n}$ is provided by the effective sample size proposed by Clifford *et al.* (1989) in simple linear correlation analysis with autocorrelated sample data. Their effective sample size is given by $\hat{\sigma}^{-2}_{\mathrm{CL}}+1$ with $\hat{\sigma}^2_{\mathrm{CL}} = \mathrm{tr}(\hat{\Sigma}_x\hat{\Sigma}_y)/[\mathrm{tr}(\hat{\Sigma}_x)\mathrm{tr}(\hat{\Sigma}_y)]$, where the estimated autocovariance matrices $\hat{\Sigma}_x$ and $\hat{\Sigma}_y$ are constructed as in (4.3), but the raw data for y (i.e., the variable to explain) and x (i.e., the regressor) are used in the calculation of sample autocorrelation coefficients.

Thirdly, we consider a modified t-test with $\hat{n}-2$ df in simple linear regression, where $\hat{n}$ is now provided by the effective sample size proposed by Dutilleul (1993) in simple linear correlation analysis with autocorrelated sample data. His effective sample size is given by $\hat{\sigma}_{\text{DU}}^{-2}+1$ with $\hat{\sigma}_{\text{DU}}^2 = \text{tr}(B\hat{\Sigma}_x B\hat{\Sigma}_y)/[\text{tr}(B\hat{\Sigma}_x)\text{tr}(B\hat{\Sigma}_y)]$, where $B = I-(1/n)J$ with J the $n\times n$ matrix of ones, and the estimated autocovariance matrices $\hat{\Sigma}_x$ and $\hat{\Sigma}_y$ are as above.

Fourthly, a hybrid procedure is considered, in which the effective sample size is estimated from $\hat{\sigma}_{\text{HY}}^2 = \text{tr}(\hat{\Sigma}_x\hat{\Sigma}_y^\star)/[\text{tr}(\hat{\Sigma}_x)\text{tr}(\hat{\Sigma}_y^\star)]$, where $\hat{\Sigma}_x$ is as above and $\hat{\Sigma}_y^\star = \hat{\Sigma}_{\varepsilon_y}$ is built by using the $r(k)$s calculated from the OLS residuals of the regression of y on x.

Finally, a combination of the effective sample sizes of Clifford *et al.* (1989) and Dutilleul (1993) is proposed by using $\hat{\sigma}_{\text{C}}^2 = \text{tr}(\hat{\Sigma}_x^\star\hat{\Sigma}_y^\star)/[\text{tr}(\hat{\Sigma}_x^\star)\text{tr}(\hat{\Sigma}_y^\star)]$, where $\hat{\Sigma}_y^\star = \hat{\Sigma}_{\varepsilon_y}$ and $\hat{\Sigma}_x^\star = \hat{\Sigma}_{\varepsilon_x}$ are built by using the $r(k)$s calculated from the OLS residuals of the regressions of y on x and of x on y, respectively.

4. MONTE CARLO STUDY

The model used for simulation was

$$y_t = a + bx_t + \varepsilon_t \quad \text{with} \quad \varepsilon_t = \rho\varepsilon_{t-1} + u_t \ \ (t = 1, 2, \ldots, n),$$

where a and b were fixed at 1 and 0, the u_ts were i.i.d. $N(0,1)$, and the value of ρ ranged from -0.9 to 0.9 by steps of 0.2, in addition to $\rho = 0$. The generation of autocorrelated errors followed a procedure similar to that of Dutilleul and Legendre (1992). The matrix X was $[1, x]$, where 1 was a column vector of ones and the entries of x were i.i.d. $N(0,1)$ observations independent of the errors ε_ts. The empirical significance levels were evaluated from 1000 simulation runs for sample sizes $n = 10, 20, 30, 50$, and 100 for each value of ρ; only the results for $n = 10, 20$, and 50 will be presented. Each empirical significance level was calculated as 0.001 times the number of rejections of the null hypothesis of a zero value for the slope b in 1000 t-, χ^2- or z-test, depending on the procedure, performed at a theoretical significance level of 5%.

The positive square-root of the (2, 2)-entry of $\text{Cov}(\hat{\beta}_{\text{OLS}})_1$, $\text{Cov}(\hat{\beta}_{\text{OLS}})_2$ or $\text{Cov}(\hat{\beta}_{\text{GLS}})$, with Σ or an estimate of it, was used to calculate the standard error of the b-estimate, depending on the procedure. For comparison purposes, we iterated the estimated GLS procedures. Iterations were stopped when two successive estimates of $\hat{b}$ differed by 0.001 or less. In ML procedure, $A = I - \hat{\rho}W$ was a lower triangular matrix with 1 on the diagonal and $-\hat{\rho}$ on the subdiagonal, the other entries being equal to zero. Following Beach and Mackinnon (1978) and Spitzer (1979), the $(1,1)$-entry of A was changed to $\sqrt{1-\hat{\rho}^2}$, where the estimate of ρ was evaluated to the nearest 0.001. In the ML procedure, we considered the χ^2- and z-tests for purposes of comparison on the basis of the sample size n. In the REML procedure, we used the F-test for fixed effects available in PROC MIXED of SAS (SAS Institute Inc. 1997). The FD and FDR procedures used the classical formula of the sample variance, except that the divisor was $n-2$ instead of $n-1$.

The following notations were used in Table 4.1. Basically, these notations refer to the different error covariance matrices used in the estimation procedures, along with whether or not the GLS estimation of β was iterative and the reference to the author that proposed a given adjustment of the number of df of the t-test. A t-test, modified or not, was performed in procedures 1-11 and 15-31.

1: Σ_0, Σ was assumed to be $\sigma^2 I$; OLS;

2: Σ_ρ, (4.2) with ρ known was used in $\hat{\beta}_{\mathrm{GLS}}$ and $\mathrm{Cov}(\hat{\beta}_{\mathrm{GLS}})$; GLS;

3: $\Sigma_{\hat{\rho}1}$, same as procedure 2, except that ρ was replaced by $r(1)$ in (4.2), and no iteration was performed in the calculation of $\hat{\beta}$; estimated GLS (as procedures 4-8);

4: $\Sigma_{\hat{\rho}2}$, same as procedure 3, that except the calculation of $\hat{\beta}$ was iterative;

5: $\hat{\Sigma}_{13}$, (4.3) was used to estimate Σ in $\hat{\beta}_{\mathrm{GLS}}$ and $\mathrm{Cov}(\hat{\beta}_{\mathrm{GLS}})$, no iteration on $\hat{\beta}$, and no test of significance of the $r(k)$s;

6: $\hat{\Sigma}_{14}$, same as procedure 5, except that the significance of $r(k)$ ($k = 1, 2, \ldots, m = \mathrm{INT}(n/4)$) was assessed, and only the $r(k)$s that were declared significantly different from 0 were used;

7: $\hat{\Sigma}_{23}$, same as procedure 5, except that the calculation of $\hat{\beta}$ was iterative;

8: $\hat{\Sigma}_{24}$, same as procedure 6, except that the calculation of $\hat{\beta}$ was iterative;

9: $\Sigma_{o\rho}$, $\hat{\beta}_{\mathrm{OLS}}$ was the estimator of β and the error covariance matrix in procedure 2 was used in $\mathrm{Cov}(\hat{\beta}_{\mathrm{OLS}})_2$ to evaluate the variance of $\hat{\beta}_{\mathrm{OLS}}$;

10: $\Sigma_{o\hat{\rho}1}$, same as procedure 9, except that the error covariance matrix in procedure 3 was used;

11: $\hat{\Sigma}_{o14}$, same as procedure 9, except that the error covariance matrix in procedure 6 was used.

12: ML_{χ^2}, β was estimated by maximum likelihood and a likelihood-ratio χ^2-test with 1 df was performed;

13: ML_{Z}, same as procedure 12, except that an asymptotic z-test was performed;

14: REML, β was estimated by restricted maximum likelihood and the significance of the slope was assessed by the F-test for fixed effects in PROC MIXED of SAS.

15: FD, first-difference procedure;

16: FDR, method of first-difference ratios;

17: $\Sigma_{\rho\mathrm{M}}$, $\hat{\beta}_{\mathrm{OLS}}$ and $\mathrm{Cov}(\hat{\beta}_{\mathrm{OLS}})_1$ were used to evaluate the t-test statistic, but the number of df was adjusted using $\varepsilon^{\star}_{\mathrm{AN}}$, which was calculated using the error covariance matrix of procedure 2;

18: $\Sigma_{\hat{\rho}1\mathrm{M}}$, same as procedure 17, except that the error covariance matrix of procedure 3 was used;

19: $\Sigma_{\hat{\rho}2\mathrm{M}}$, same as procedure 17, except that the error covariance matrix of procedure 4 was used;

20: $\hat{\Sigma}_{13\mathrm{M}}$, same as procedure 17, except that the error covariance matrix of procedure 5 was used;

21: $\hat{\Sigma}_{14\mathrm{M}}$, same as procedure 17, except that the error covariance matrix of procedure 6 was used;

22: $\hat{\Sigma}_{23\mathrm{M}}$, same as procedure 17, except that the error covariance matrix of procedure 7 was used;

23: $\hat{\Sigma}_{24M}$, same as procedure 17, except that the error covariance matrix of procedure 8 was used;

24: $\hat{\Sigma}_{CL3}$, $\hat{\beta}_{OLS}$ and $\text{Cov}(\hat{\beta}_{OLS})_1$ were used to evaluate the t-test statistic, but the number of df was adjusted using $\hat{\sigma}^2_{CL}$, no test of significance was performed on the sample autocorrelation coefficients of x and y;

25: $\hat{\Sigma}_{CL4}$, same as procedure 24, except that a test of significance was performed on the sample autocorrelation coefficients of x and y;

26: $\hat{\Sigma}_{DU3}$, same as procedure 24, except that the number of df was adjusted using $\hat{\sigma}^2_{DU}$;

27: $\hat{\Sigma}_{DU4}$, same as procedure 26, except that a test of significance was performed on the sample autocorrelation coefficients of x and y;

28: $\hat{\Sigma}_{HY3}$, same as procedures 24 and 26, except that the number of df was adjusted using $\hat{\sigma}^2_{HY}$;

29: $\hat{\Sigma}_{HY4}$, same as procedure 28, except that a test of significance was performed on the sample autocorrelation coefficients of x and y;

30: $\hat{\Sigma}_{C3}$, same as procedures 24, 26 and 28, except that the number of df was adjusted using $\hat{\sigma}^2_{C}$;

31: $\hat{\Sigma}_{C4}$, same as procedure 30, except that a test of significance was performed on the sample autocorrelation coefficients of x and y.

We used our own computer programs written in SAS/IML language and PROC MIXED of SAS (SAS Institute Inc. 1997) to implement the testing procedures. The generation of i.i.d. $N(0,1)$ observations was carried out with the random number function RANNOR of SAS (SAS Institute Inc. 1997).

5. RESULTS

The results of our Monte Carlo study for $n = 10, 20$, and 50 are reported in Table 4.1. Strictly speaking, a testing procedure is said to be valid at level α if the probability that it rejects the null hypothesis, when in fact the null hypothesis is true, is less than or equal to α. The actual significance level of each testing procedure considered here is estimated by the empirical significance level of p evaluated from 1000 simulation runs. Under the binomial distribution model, the standard deviation of p is given by $\sigma_p = \sqrt{p(1-p)/1000}$. An approximate 95% confidence interval for the actual significance level of a testing procedure is provided by $p \pm 2\sigma_p$. The largest value of p such that $p \pm 2\sigma_p$ contains the theoretical significance level of 0.05 is 0.065. Our interpretation of the results reported in Tables 4.1-4.3 is based on the strict definition of validity, combined with the variability associated with the empirical significance levels. Thus, we have used $p \leq 0.065$ as the validity condition.

Over the 31 testing procedures, the validity condition is not satisfied 55 times for 330 when $n = 10$, 51 times for 341 when $n = 20$, and 47 times for 341 when $n = 50$. (Due to the too frequent lack of convergence of the REML algorithm when $n = 10$, we do not report results for the REML procedure for that sample size.) The majority of the violations of the validity condition come from five testing procedures: procedures 5 and 7 (which are based on an estimated GLS estimator of the slope), the two ML procedures (χ^2-test and z-test), and FD. The overall rate of validity is about 85%, which is far beyond our expectations. There are only 18 cases of lack of validity if $|\rho| > 0.5$ (i.e., the autocorrelation of errors is strong) when $n = 10$ against 13 when $n = 20$ and 6 when $n = 50$. The two highest empirical significance levels are 0.268 and 0.207. They are observed, respectively, for procedure 9 when $\rho = 0.9$ and $n = 10$ and for the ML procedure (z-test) when $\rho = -0.3$ and $n = 10$. Besides some sample size effect (especially on the two ML procedures), these results indicate that most of the testing procedures satisfy the validity condition for the values of ρ and n considered.

Specifically, we have the good surprise to observe that the classical t-test of the slope (denoted Σ_o in table 4.1) satisfies the validity condition 11 times out of 11 when $n = 10, 20$, and 50. The highest empirical significance level for this test (i.e., 0.06) is observed when $\rho = -0.3$ and $n = 10$. By comparison, the t-test with $n-2$ df based on the GLS estimator of the slope (denoted Σ_ρ in Table 4.1) performs equally well, although it assumes the complete knowledge of Σ. Its highest empirical significance level is 0.056. The F-test built on the REML estimator of the slope also satisfies the validity condition 11 times out of 11 when $n = 20$ and 50. By comparison, the two tests based on the ML estimator of the slope are valid 0 time in 11 when $n = 10$ against 4 times (χ^2-test) and 1 time (z-test) when $n = 20$, and 8 times (χ^2-test) and 7 times (z-test) when $n = 50$. The FD t-test, which is based on the linear regression without intercept of the first differences of y on the first differences of x, is valid 2 times out of 11 at all sample sizes. On several occasions, the latter three tests showed an empirical significance level of 0.10 and even 0.15. For its part, the FDR procedure, which consists in a t-test for the mean performed on the ratios of a first difference of y to the corresponding first difference of x, is valid 11 times for 11 at any sample size. Its empirical significance levels range between 0.014 and 0.036.

Among the testing procedures based on an estimated GLS estimator of the slope, those that assume a stationary AR(1) covariance structure of the errors (i.e., procedures 3 and 4) perform better than the others (i.e., procedures 5-8), with one exception, when $n = 10$ (i.e., procedure 6: test of significance of the $r(k)$s and no iteration in the estimated GLS estimation of β). The iterative evaluation of the estimated GLS estimator of the slope increases the number of invalidity cases when combined with the test of significance of the $r(k)$s for $n = 10$, and has no effect when $n = 20$ and 50. The performance of the classical t-test of the slope compares well with the best of procedures 3-8. Relative to other procedures based on the OLS estimator of the slope but using different variances of it, procedure 1 and procedure 10 (i.e., specification of a stationary AR(1) covariance structure of the errors with ρ estimated by $r(1)$) and procedure 11 (i.e., no specification of the covariance structure of the errors combined with the test of significance of the sample autocorrelation coefficients) perform equally well. Only slight differences are observed among the procedures based on a modified t-test, those using $\varepsilon^{\star}_{\mathrm{AN}}$ showing the lowest empirical significance levels.

In summary, the classical t-test of the slope belongs to a group of testing procedures that have never violated the validity condition $p \leq 0.065$ for the combinations of n and ρ values considered here, with the t-test based on the GLS estimator of the slope, the REML F-test and the FDR t-test. Only the FDR t-test showed strict validity (i.e., $p \leq 0.05$). Recall that the empirical significance level of the classical t-test has always been smaller than 0.05 if $|\rho| > 0.5$. In the next section, we try to interpret these results, which are — one must honestly concede — better than expected.

6. DISCUSSION

In an attempt to find an explanation for the validity of the classical t-test of the slope, we have looked at the circularity condition that allows unmodified F-tests in the presence of heteroscedasticity and autocorrelation of some form in the repeated measures ANOVA (Huynh and Feldt 1970; Rouanet and Lépine 1970). Therefore, we have computed Box's epsilon (Box 1954a, b) for the variable to explain $y_t = 1 + x_t + \varepsilon_t$ and for the error ε_t, to evaluate how closer to circularity the covariance structure of the y_ts gets by the addition of a purely random x_t to each ε_t. Recall that (1) the intra-class correlation structure in the random one-way ANOVA with purely random errors satisfies the circularity condition and (2) the errors ε_t here follow an AR(1) process whose discrepancies of the covariance structure from circularity are well known. Box's epsilon values computed for $\sigma^2_x = \sigma^2_u = 1$ when $n = 10, 20$, and 50 and $\rho = 0, \pm 0.1, \pm 0.3, \pm 0.5, \pm 0.7$, and ± 0.9 are reported in Figure 4.1.

In view of Figure 4.1, the following observations can be made. First, Box's epsilon values for y are all greater than those for ε, which confirms that the covariance structure of the y_ts is closer to circularity than the AR(1) structure of the ε_ts. Secondly, the circularity condition is almost met by y when $\rho = 0.1$ and 0.3 for $n = 10, 20$, and 50. Thirdly, the discrepancies from circularity increase with n and ρ, with Box's epsilon values of 0.20 and 0.13 for y and ε, respectively, when $\rho = 0.9$ and $n = 50$. Note that the correlation structure of y is given by $\mathrm{Cor}(y_t, y_{t'}) = \sigma_\varepsilon^2 \rho^{|t-t'|}/(b^2\sigma_x^2 + \sigma_\varepsilon^2) \quad (t \neq t')$.

In an attempt to find a better explanation for our results, we have looked at the effective sample sizes used in correlation analysis with autocorrelated sample data (Clifford *et al.* 1989; Dutilleul 1993) and their variants introduced here in regression analysis. Using Dutilleul's (1993) expression, the theoretical value of the effective sample size is equal to the classical sample size when x_t is purely random in $y_t = a + bx_t + \varepsilon_t$, where ε_t follows an AR(1) process. Thus, the criterion of effective sample size applies to simple linear regression models in the sense of Graybill (1976, p. 143), in that no adjustment of the number of df is required in such a model when x_t or ε_t is not autocorrelated.

7. CONCLUSIONS

In this study, no evidence has been found against the validity of the classical t-test of the slope in a simple linear regression model with AR(1) errors when the explanatory variable is purely random. We have related this result to the effective sample size used in modified t-tests in correlation analysis with autocorrelated sample data. In this context, classical sample size and effective sample size are equal if at least one of the two variables analyzed for correlation is purely random. In the context of simple linear regression, the condition becomes the regressor or the error is purely random. From two ongoing studies, we may already announce that the validity of the classical t-test of the slope extends to the case of multiple linear regression when all explanatory variables are purely random, but the story is quite different when the regressors represent a trend or are, themselves, autocorrelated. This emphasizes the importance of the nature, purely random, fixed or autocorrelated, of the regressor, and restricts the warning of Upton and Fingleton (1985, p. 283) to the latter two cases.

Accordingly, we recommend that the users of regression with time series or spatial data investigate the autocorrelation of regressors first, before neglecting the classical t-test of the slope in favor of another testing procedure. Power analysis results are necessary, though, before this recommendation can be total and definitive. The assessment of autocorrelation can be undertaken through autocorrelogram, periodogram or variogram analysis (Jenkins and Watts 1968; Cressie 1993). In simple linear regression with AR(1) errors and purely random x, the t-test with $n-2$ df based on the ratios of first differences of y and x has shown strict validity. The χ^2- and z-tests based on the ML estimator of the slope, conditional on X, were frequently shown to violate the validity condition up to sample sizes of 50, as the t-tests with $n-2$ df based on estimated GLS estimators of the slope for small to moderate autocorrelation of the errors when the GLS estimation of the slope was iterative. The F-test for fixed effects used in the REML procedure is superior to the χ^2-test and z-test based on the ML estimator of the slope for sufficiently large sample sizes (i.e., $n \geq 20$).

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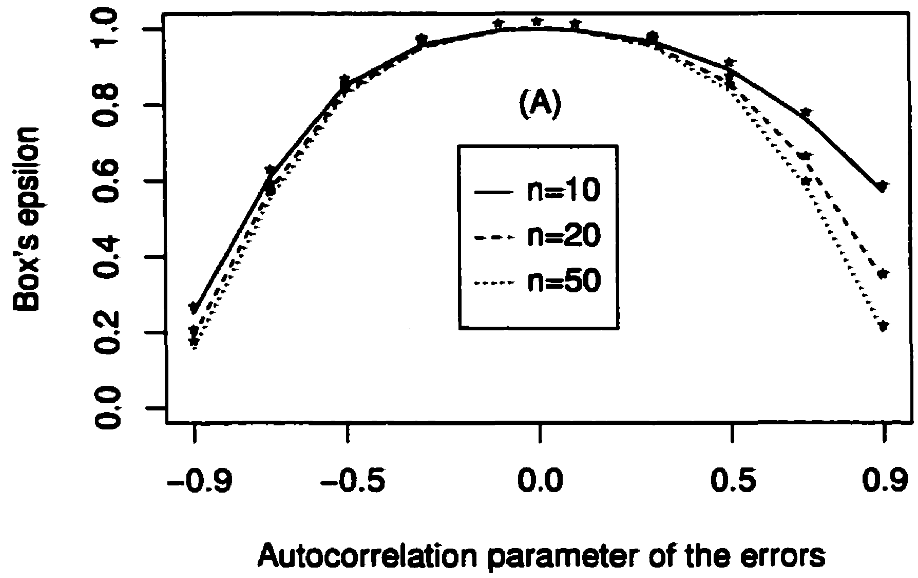

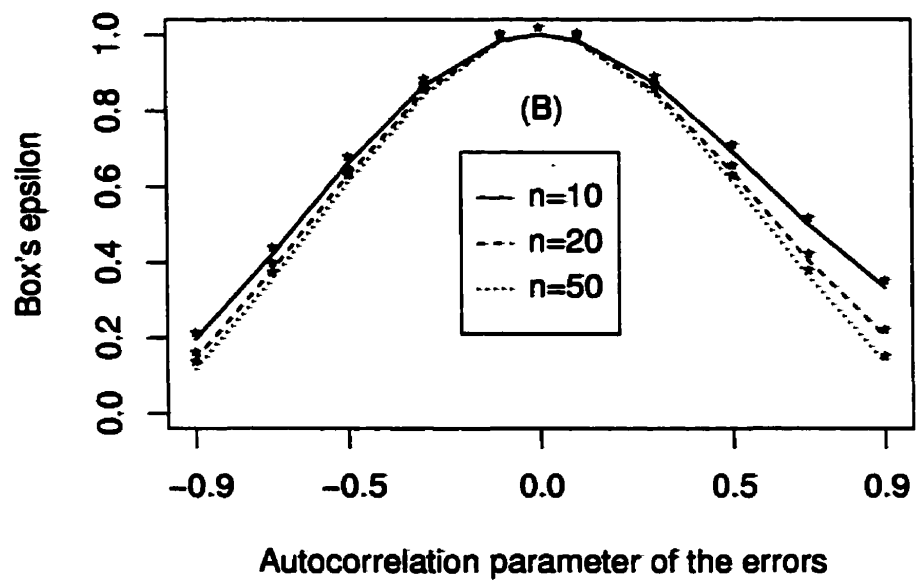

Figure 4.1: Box's epsilon for (A) the variable to explain y and (B) the errors ε in a simple linear regression model with purely random x and AR(1) errors, as a function of the sample size n and the autocorrelation parameter of the errors ρ.

Table 4.1: (first page) Empirical significance level of the 31 testing procedures for a theoretical significance level of 5% when x is purely random, as a function of the sample size n and the error autocorrelation parameter ρ. The number of times each testing procedure does not satisfy the validity condition for a given sample size is reported under the inval-column; in the inval-row is reported the number of times the validity condition is not satisfied for a given value of ρ. The empirical significance levels reported were evaluated from 1000 simulation runs. See the text for other notations.

n=10

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9	inval
Σ_o	0.034	0.049	0.041	0.06	0.052	0.046	0.053	0.038	0.048	0.048	0.04	0
Σ_ρ	0.003	0.013	0.024	0.046	0.051	0.046	0.056	0.042	0.051	0.041	0.019	0
$\Sigma_{\beta 1}$	0.013	0.029	0.035	0.054	0.067	0.055	0.06	0.051	0.046	0.048	0.041	1
$\Sigma_{\beta 2}$	0.011	0.024	0.035	0.059	0.07	0.059	0.065	0.054	0.048	0.047	0.042	1
$\hat{\Sigma}_{13}$	0.035	0.07	0.068	0.086	0.084	0.069	0.08	0.067	0.085	0.084	0.072	10
$\hat{\Sigma}_{14}$	0.033	0.05	0.041	0.061	0.052	0.046	0.053	0.038	0.048	0.048	0.04	0
$\hat{\Sigma}_{23}$	0.035	0.07	0.073	0.1	0.107	0.093	0.096	0.083	0.107	0.092	0.079	10
$\hat{\Sigma}_{24}$	0.055	0.057	0.046	0.06	0.055	0.046	0.051	0.043	0.052	0.051	0.041	0
$\Sigma_{o\rho}$	0.034	0.033	0.041	0.055	0.054	0.046	0.055	0.044	0.062	0.09	0.268	2
$\Sigma_{o\beta 1}$	0.024	0.044	0.046	0.061	0.059	0.055	0.056	0.046	0.053	0.049	0.049	0
$\hat{\Sigma}_{o14}$	0.034	0.049	0.042	0.06	0.052	0.046	0.053	0.038	0.048	0.048	0.04	0
ML_{χ^2}	0.097	0.109	0.106	0.129	0.13	0.111	0.124	0.103	0.119	0.089	0.092	11
ML_Z	0.14	0.156	0.169	0.207	0.198	0.188	0.2	0.177	0.189	0.139	0.139	11
FD	0.119	0.132	0.111	0.125	0.111	0.109	0.094	0.074	0.088	0.057	0.058	9
FDR	0.03	0.034	0.036	0.036	0.029	0.028	0.03	0.03	0.022	0.022	0.015	0
$\Sigma_{\rho M}$	0.003	0.027	0.039	0.053	0.055	0.046	0.051	0.04	0.035	0.025	0.016	0
$\Sigma_{\beta 1M}$	0.043	0.047	0.042	0.054	0.052	0.043	0.044	0.04	0.047	0.047	0.039	0
$\Sigma_{\beta 2M}$	0.024	0.038	0.037	0.052	0.051	0.042	0.043	0.038	0.043	0.041	0.036	0
$\hat{\Sigma}_{13M}$	0.029	0.045	0.037	0.057	0.052	0.043	0.048	0.035	0.041	0.044	0.037	0
$\hat{\Sigma}_{14M}$	0.034	0.049	0.041	0.06	0.052	0.046	0.053	0.038	0.048	0.048	0.04	0
$\hat{\Sigma}_{23M}$	0.048	0.05	0.04	0.053	0.054	0.045	0.044	0.037	0.046	0.044	0.037	0
$\hat{\Sigma}_{24M}$	0.056	0.058	0.046	0.059	0.055	0.046	0.051	0.043	0.052	0.051	0.041	0
$\hat{\Sigma}_{CL3}$	0.042	0.053	0.046	0.055	0.056	0.047	0.052	0.041	0.053	0.05	0.046	0
$\hat{\Sigma}_{CL4}$	0.054	0.06	0.053	0.062	0.059	0.048	0.053	0.045	0.057	0.054	0.048	0
$\hat{\Sigma}_{DU3}$	0.04	0.048	0.041	0.055	0.053	0.045	0.048	0.039	0.05	0.047	0.042	0
$\hat{\Sigma}_{DU4}$	0.053	0.057	0.046	0.059	0.055	0.046	0.051	0.043	0.052	0.051	0.041	0
$\hat{\Sigma}_{HY3}$	0.05	0.061	0.055	0.061	0.059	0.051	0.054	0.045	0.055	0.052	0.048	0
$\hat{\Sigma}_{HY4}$	0.057	0.06	0.054	0.063	0.059	0.048	0.053	0.045	0.057	0.054	0.048	0
$\hat{\Sigma}_{C3}$	0.06	0.062	0.055	0.059	0.055	0.047	0.05	0.045	0.051	0.053	0.044	0
$\hat{\Sigma}_{C4}$	0.058	0.06	0.054	0.063	0.059	0.048	0.053	0.045	0.057	0.054	0.048	0
inval	3	5	5	5	7	5	5	5	5	5	5	55

Table 4.1 (continued).

n=20

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9	inval
Σ_o	0.036	0.045	0.049	0.052	0.057	0.036	0.044	0.049	0.046	0.05	0.049	0
Σ_ρ	0.007	0.013	0.029	0.039	0.049	0.036	0.041	0.051	0.03	0.03	0.018	0
$\Sigma_{\beta 1}$	0.004	0.012	0.03	0.05	0.057	0.049	0.051	0.056	0.038	0.024	0.014	0
$\Sigma_{\beta 2}$	0.004	0.011	0.028	0.051	0.059	0.054	0.053	0.056	0.039	0.023	0.012	0
$\hat{\Sigma}_{13}$	0.025	0.058	0.104	0.106	0.111	0.099	0.111	0.106	0.098	0.088	0.07	9
$\hat{\Sigma}_{14}$	0.014	0.035	0.057	0.071	0.07	0.053	0.054	0.061	0.059	0.072	0.049	3
$\hat{\Sigma}_{23}$	0.023	0.063	0.122	0.118	0.141	0.134	0.134	0.136	0.119	0.089	0.073	9
$\hat{\Sigma}_{24}$	0.015	0.029	0.054	0.067	0.066	0.053	0.061	0.053	0.055	0.057	0.047	2
$\Sigma_{o\rho}$	0.034	0.042	0.043	0.047	0.056	0.036	0.045	0.058	0.054	0.075	0.157	2
$\Sigma_{o\beta 1}$	0.027	0.038	0.047	0.053	0.055	0.038	0.044	0.055	0.056	0.055	0.051	0
$\hat{\Sigma}_{o14}$	0.028	0.043	0.047	0.053	0.057	0.037	0.046	0.049	0.046	0.049	0.047	0
ML_{χ^2}	0.063	0.079	0.08	0.081	0.069	0.071	0.07	0.075	0.063	0.046	0.058	7
ML_Z	0.072	0.094	0.107	0.109	0.103	0.108	0.103	0.107	0.094	0.058	0.076	10
REML	0.001	0.007	0.023	0.039	0.044	0.053	0.051	0.028	0.017	0.02	0.061	0
FD	0.14	0.133	0.13	0.138	0.115	0.084	0.089	0.084	0.073	0.057	0.056	9
FDR	0.034	0.027	0.025	0.022	0.023	0.02	0.019	0.021	0.022	0.023	0.019	0
$\Sigma_{\rho M}$	0.004	0.024	0.038	0.052	0.052	0.04	0.048	0.047	0.039	0.029	0.016	0
$\Sigma_{\beta 1M}$	0.027	0.036	0.04	0.05	0.052	0.036	0.045	0.047	0.042	0.041	0.052	0
$\Sigma_{\beta 2M}$	0.013	0.027	0.037	0.048	0.052	0.036	0.045	0.047	0.04	0.036	0.039	0
$\hat{\Sigma}_{13M}$	0.03	0.038	0.038	0.051	0.057	0.033	0.039	0.048	0.042	0.044	0.045	0
$\hat{\Sigma}_{14M}$	0.033	0.043	0.048	0.052	0.057	0.035	0.044	0.049	0.045	0.05	0.049	0
$\hat{\Sigma}_{23M}$	0.037	0.035	0.039	0.051	0.052	0.037	0.041	0.044	0.044	0.045	0.048	0
$\hat{\Sigma}_{24M}$	0.036	0.046	0.05	0.054	0.052	0.039	0.049	0.047	0.046	0.047	0.055	0
$\hat{\Sigma}_{CL3}$	0.032	0.042	0.041	0.053	0.054	0.038	0.05	0.047	0.044	0.048	0.053	0
$\hat{\Sigma}_{CL4}$	0.04	0.047	0.048	0.054	0.053	0.04	0.05	0.048	0.047	0.047	0.056	0
$\hat{\Sigma}_{DU3}$	0.032	0.042	0.041	0.053	0.053	0.037	0.047	0.047	0.043	0.047	0.053	0
$\hat{\Sigma}_{DU4}$	0.039	0.046	0.048	0.054	0.052	0.04	0.049	0.047	0.046	0.047	0.054	0
$\hat{\Sigma}_{HY3}$	0.033	0.045	0.047	0.054	0.053	0.039	0.05	0.047	0.047	0.049	0.055	0
$\hat{\Sigma}_{HY4}$	0.04	0.047	0.049	0.054	0.053	0.04	0.05	0.048	0.047	0.047	0.057	0
$\hat{\Sigma}_{C3}$	0.042	0.048	0.047	0.053	0.052	0.038	0.049	0.046	0.047	0.05	0.058	0
$\hat{\Sigma}_{C4}$	0.04	0.048	0.05	0.054	0.053	0.04	0.05	0.048	0.047	0.047	0.057	0
inval	2	3	5	7	7	5	5	5	4	4	4	51

Table 4.1 (last page).

n=50

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9	inval
Σ_o	0.049	0.049	0.05	0.046	0.051	0.049	0.054	0.051	0.056	0.04	0.032	0
Σ_ρ	0.001	0.003	0.021	0.039	0.046	0.049	0.052	0.036	0.035	0.012	0.002	0
$\Sigma_{\beta 1}$	0	0.002	0.027	0.036	0.059	0.06	0.067	0.04	0.036	0.01	0.001	1
$\Sigma_{\beta 2}$	0	0.002	0.025	0.039	0.059	0.061	0.069	0.038	0.035	0.01	0.001	1
$\hat{\Sigma}_{13}$	0.032	0.053	0.098	0.093	0.106	0.109	0.115	0.109	0.097	0.069	0.026	8
$\hat{\Sigma}_{14}$	0.013	0.03	0.079	0.077	0.067	0.059	0.07	0.065	0.107	0.045	0.007	5
$\hat{\Sigma}_{23}$	0.028	0.058	0.112	0.113	0.136	0.132	0.141	0.133	0.105	0.073	0.019	8
$\hat{\Sigma}_{24}$	0.01	0.023	0.08	0.086	0.076	0.069	0.077	0.072	0.1	0.043	0.008	7
$\Sigma_{o\rho}$	0.043	0.042	0.042	0.048	0.051	0.049	0.055	0.053	0.063	0.054	0.089	1
$\Sigma_{o\beta 1}$	0.041	0.042	0.044	0.048	0.051	0.05	0.059	0.054	0.058	0.053	0.04	0
$\hat{\Sigma}_{o14}$	0.047	0.042	0.044	0.048	0.05	0.049	0.055	0.055	0.055	0.039	0.032	0
ML_{χ^2}	0.06	0.036	0.053	0.049	0.072	0.072	0.075	0.05	0.062	0.061	0.049	3
ML_Z	0.065	0.044	0.056	0.058	0.08	0.082	0.08	0.055	0.076	0.065	0.052	4
REML	0	0.002	0.023	0.042	0.051	0.052	0.06	0.049	0.032	0.002	0.009	0
FD	0.158	0.132	0.131	0.117	0.12	0.119	0.101	0.065	0.1	0.077	0.053	9
FDR	0.021	0.016	0.022	0.033	0.016	0.022	0.022	0.028	0.032	0.014	0.017	0
$\Sigma_{\rho M}$	0.019	0.037	0.041	0.043	0.056	0.057	0.058	0.047	0.059	0.036	0.014	0
$\Sigma_{\beta 1M}$	0.032	0.04	0.041	0.043	0.056	0.057	0.058	0.047	0.061	0.041	0.026	0
$\Sigma_{\beta 2M}$	0.024	0.037	0.04	0.043	0.056	0.057	0.058	0.047	0.061	0.04	0.022	0
$\hat{\Sigma}_{13M}$	0.046	0.045	0.046	0.046	0.05	0.049	0.054	0.051	0.056	0.038	0.029	0
$\hat{\Sigma}_{14M}$	0.048	0.047	0.047	0.045	0.05	0.049	0.054	0.051	0.056	0.04	0.031	0
$\hat{\Sigma}_{23M}$	0.044	0.048	0.04	0.044	0.055	0.057	0.059	0.049	0.059	0.041	0.033	0
$\hat{\Sigma}_{24M}$	0.046	0.053	0.042	0.044	0.055	0.057	0.058	0.047	0.062	0.046	0.035	0
$\hat{\Sigma}_{CL3}$	0.043	0.05	0.044	0.044	0.056	0.057	0.058	0.047	0.063	0.045	0.035	0
$\hat{\Sigma}_{CL4}$	0.048	0.054	0.044	0.044	0.056	0.057	0.058	0.047	0.063	0.046	0.036	0
$\hat{\Sigma}_{DU3}$	0.043	0.05	0.044	0.044	0.056	0.057	0.058	0.047	0.063	0.045	0.035	0
$\hat{\Sigma}_{DU4}$	0.047	0.054	0.044	0.044	0.056	0.057	0.058	0.047	0.063	0.046	0.035	0
$\hat{\Sigma}_{HY3}$	0.043	0.052	0.044	0.045	0.056	0.057	0.058	0.047	0.063	0.045	0.035	0
$\hat{\Sigma}_{HY4}$	0.047	0.054	0.044	0.044	0.056	0.057	0.058	0.047	0.063	0.046	0.036	0
$\hat{\Sigma}_{C3}$	0.046	0.054	0.044	0.044	0.055	0.057	0.058	0.047	0.063	0.045	0.035	0
$\hat{\Sigma}_{C4}$	0.047	0.055	0.044	0.044	0.056	0.057	0.058	0.047	0.063	0.046	0.036	0
inval	1	1	5	5	7	6	9	3	6	3	1	47

estimator of the slope in the mixed-model approach, two t-tests with $n - 2$ df based on first differences (FD) and first-difference ratios (FDR), and 15 modified t-test tests with a number of degrees of freedom adjusted in various ways. The REML procedure (in which the model of covariance structure of the errors is assumed to be known) and the FDR procedure (in which a t-test for the mean is performed on the ratios of first differences of the variable to explain and the regressor) are more valid than the other testing procedures, with a few exceptions. The classical t-test of the slope is valid when the regressor is trended and the error follows an AR(1) process with a negative autocorrelation parameter and when the regressor and the error both follow an AR(1) process with moderate, negative or positive, autocorrelation. We discuss our results graphically and in terms of the circularity condition used in repeated measures ANOVA and of the effective sample size in correlation analysis with autocorrelated sample data. A numerical example is presented.

Keywords: AR(1) errors, First differences, Fixed and trended vs. random and autocorrelated regressor, Least squares, Maximum likelihood, Quantitative linear models, Restricted maximum likelihood

1. Introduction

In a quantitative linear model with autocorrelated errors, the ordinary least-squares (OLS) estimator of the slope is known to be inefficient, except when the autocorrelation of errors is of the intra-class correlation type (McElroy 1967). In general, the generalized least-squares (GLS) estimator, which assumes the complete knowledge of the covariance matrix of the errors, is the best linear unbiased estimator when the errors are autocorrelated (Searle 1971). Therefore, one may expect the t-test based on the GLS estimator of the slope to be superior to that based on the OLS estimator. However, highly inefficient estimators of the parameter of a model have been shown to provide excellent tests of significance

(Fisher 1950, Sundrum 1954). From the estimation perspective, the efficiency of the estimator of a multi-parameter function is not necessarily improved when the true values of some parameters replace the estimators (Stuart 1955). On the other hand, the estimation of autocorrelation parameters may be responsible for some loss in efficiency of the estimated GLS estimator of the slope when the errors are moderately autocorrelated (Krämer and Donninger 1987).

When the sample data are positively autocorrelated in space, the classical t-test overstates the significance of the population mean (Cliff and Ord 1975) and that of individual slopes in linear regression models (Cook and Pocock 1983). Before the repeated measures ANOVA techniques (Crowder and Hand 1990), little was known about the robustness of statistical models that assume the independence of errors against the departure from this assumption. For instance, Krämer and Donninger (1987) had noticed that OLS can be more efficient than estimated GLS when the autocorrelation of errors is weak. Alpargu and Dutilleul (2001) refine Krämer and Donninger's numerical results.

In correlation with time-series data, Jenkins and Watts (1968) showed that sample cross-correlations are not biased provided at least one of the two time series is not autocorrelated. On the other hand, Upton and Fingleton (1985) claimed that the classical t- and F-tests are invalid in linear regression with spatially autocorrelated sample data, without specifying the nature of the regressors. In a previous study (Alpargu and Dutilleul, unpublished manuscript), we have provided evidence for the validity of the classical t-test of the slope when the regressor is purely random in simple linear regression with AR(1) errors. In that study, the results reported support the validity of a good number of the 31 testing procedures considered, including t-tests with $n-2$ df based on various estimators of the slope, modified t-tests with an adjusted number of degrees of freedom and the F-test for fixed effects in the mixed-model approach. The study reported here is a follow-up to Alpargu and Dutilleul (unpublished manuscript) when the regressor is fixed and trended or when it is random and follows a first-order autoregressive process like the error. Results for multiple quantitative linear models are available but will not be presented here.

The computation of slope estimators is explained in Section 2. In Section 3, we present the corresponding testing procedures, with emphasis on the computation of the adjusted number of df in the modified t-tests. Our Monte Carlo study is presented in Section 4. Our results are summarized in Section 5 and discussed in Section 6. A numerical example is presented in Section 7. Concluding remarks are made in Section 8.

2. Estimation procedures

Among the 31 testing procedures that we have considered for assessing the significance of an individual slope parameter in a linear regression model with temporally autocorrelated errors are a number that are based on the estimation methods of OLS, GLS, estimated GLS, maximum likelihood (ML), restricted maximum likelihood (REML) as well as the first-difference (FD) method and a variant of it that uses first-difference ratios (FDR). Consider a linear regression model with temporal AR(1) errors

$$y = X\beta + \varepsilon, \quad \text{with} \quad \varepsilon_t = \rho\varepsilon_{t-1} + u_t \quad (t = 1, 2, \ldots, n), \tag{5.1}$$

where y is an $n \times 1$ observable random vector; β is a $q \times 1$ unknown vector to be estimated; X is an $n \times q$ matrix of rank $q < n$; ε is an $n \times 1$ unobservable random vector of errors with mean zero and variance $\sigma_\varepsilon * 2$; $-1 < \rho < 1$; and $u \sim N_n(0, \sigma_u^2 I)$, with I the $n \times n$ identity matrix and σ_u^2 an unknown positive constant. Let Σ denote the covariance matrix of ε, $\mathrm{Cov}(\varepsilon)$.

The OLS estimator of β in (5.1) is $\hat{\beta}_{\mathrm{OLS}} = (X'X)^{-1}X'y$, with covariance matrix $\mathrm{Cov}(\hat{\beta}_{\mathrm{OLS}})_1 = \sigma_\varepsilon^2(X'X)^{-1}$ if $\rho = 0$. If $\rho \neq 0$, then the covariance matrix of $\hat{\beta}_{\mathrm{OLS}}$ is $\mathrm{Cov}(\hat{\beta}_{\mathrm{OLS}})_2 = (X'X)^{-1}X'\Sigma X\ (X'X)^{-1}$.

If ρ is known, which is not generally the case in practice, then the Best Linear

Unbiased Estimator (BLUE) of β is the GLS estimator or Aitken estimator, $\hat{\beta}_{\text{GLS}} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y$, with covariance matrix $\text{Cov}(\hat{\beta}_{\text{GLS}}) = (X'\Sigma^{-1}X)^{-1}$.

The covariance matrix of ε in (5.1) is

$$\Sigma = \sigma_\varepsilon^2 V = \sigma_\varepsilon^2 \begin{pmatrix} 1 & \rho & \rho^2 & \rho^3 & \cdots & \rho^{n-2} & \rho^{n-1} \\ \rho & 1 & \rho & \rho^2 & \cdots & \rho^{n-3} & \rho^{n-2} \\ \vdots & & & & & & \vdots \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \rho^{n-4} & \cdots & \rho & 1 \end{pmatrix}, \tag{5.2}$$

where $-1 < \rho < 1$ and $\sigma_\varepsilon^2 = \sigma_u^2/(1-\rho^2)$. The GLS estimator requires ρ to be known in (5.2). Otherwise, ρ can be estimated by the sample autocorrelation coefficient at lag 1, $r(1)$ (Alpargu and Dutilleul, unpublished manuscript), or some other estimator (Beach and Mackinnon 1978), assuming the errors follow an AR(1) process. In the estimated GLS procedures, an estimator of Σ is used in $\hat{\beta}_{\text{GLS}}$ and $\text{Cov}(\hat{\beta}_{\text{GLS}})$, whereas σ_ε^2 is estimated by the error mean square.

If the structure of the covariance matrix of errors is unknown, then the sample autocorrelation coefficients at lag k, $r(k) = \sum_{i=1}^{n-k} e_i e_{i+k} / \sum_{i=1}^{n} e_i^2$, where $e_i s$ $(i = 1, 2, \ldots, n)$ are the OLS residuals, are natural candidates for estimating the true autocorrelation parameters $\rho(k)$ under the general assumption of weak stationarity. To obtain reliable estimates of $\rho(k)$, the recommended time series length is 50 or more (Box *et al.* 1994). In practice, the first INT$(n/4)$ sample autocorrelation coefficients (where INT() denotes the integer part of the number in parentheses) are usually calculated and the remaining ones are set at zero.

REML procedures provide similar estimators. Otherwise, the REML estimator is to be preferred (Diggle et al. 1996).

In the FD procedure, the transformation defined by $(I - W)$ is applied to model (5.1) under the assumption that ρ is equal to 1, so that the dependency among the errors is removed prior to fitting a model without intercept (Martin 1974). In the particular case of simple linear regression with equally spaced observations in time, the ratios of first differences $y_t - y_{t-1}$ and $x_t - x_{t-1}$ have an expected value equal to the slope parameter under mild conditions. This led us to consider an FDR procedure in which the sample mean of first-difference ratios is used as the estimator.

3. Testing procedures

With two exceptions, the test statistics that we have considered are built as the ratio of a slope estimator divided by a standard error. The exceptions are provided by the likelihood-ratio χ^2-test in the ML procedure, and the F-test for fixed effects in the mixed-model approach of the REML procedure. In all other cases (i.e., when the estimation method is OLS, GLS, estimated GLS, FD or FDR), the test statistic is assumed to follow or it actually follows a t-distribution with $n - 2$ df. Depending on how it is built, the test statistic derived from the ML estimator follows a standard normal distribution or a χ^2 distribution with 1 df. The underlying idea in the GLS, estimated GLS, ML, REML, FD, and FDR estimation procedures is to take the dependency among the errors into account by incorporating it into the estimation procedure or by removing it from the data as much as possible.The alternative approach developed below is based on the OLS estimation of the slope parameter from the raw data y, combined with a modified t-test with a number of df adjusted for the level of autocorrelation in the errors. In other words, the dependency among the errors is taken into account in the test instead of the estimator.

First, let the classical number of df of the t-test (i.e., $n-2$ in simple linear regression) be multiplied by a constant inspired from Box's 'epsilon' (Box 1954a, b) in the modified F-test of the repeated measures ANOVA. The multiplicative constant to be used in linear regression analysis in general would be $\varepsilon^{\star}_{\mathrm{AN}} = (\mathrm{tr}\ C'\Sigma C)^2/[(n-q)\mathrm{tr}(C'\Sigma C)^2]$, where $C = I - X(X'X)^{-1}X'$. The same estimators as those used in estimated GLS can be used to estimate Σ in $\varepsilon^{\star}_{\mathrm{AN}}$.

Secondly, we consider a modified t-test with $\hat{n}-2$ df in simple linear regression, where $\hat{n}$ is provided by the effective sample size proposed by Clifford *et al.* (1989) in simple linear correlation analysis with autocorrelated sample data. Their effective sample size is given by $\hat{\sigma}_{\mathrm{CL}}^{-2}+1$ with $\hat{\sigma}^2_{\mathrm{CL}} = \mathrm{tr}(\hat{\Sigma}_x\hat{\Sigma}_y)/[\mathrm{tr}(\hat{\Sigma}_x)\mathrm{tr}(\hat{\Sigma}_y)]$, where the estimated autocovariance matrices $\hat{\Sigma}_x$ and $\hat{\Sigma}_y$ are constructed as in (5.3), but the raw data for y (i.e., the variable to explain) and x (i.e., the regressor) are used in the calculation of sample autocorrelation coefficients.

Thirdly, we consider a modified t-test with $\hat{n}-2$ df in simple linear regression, where $\hat{n}$ is now provided by the effective sample size proposed by Dutilleul (1993) in simple linear correlation analysis with autocorrelated sample data. His effective sample size is given by $\hat{\sigma}_{\mathrm{DU}}^{-2}+1$ with $\hat{\sigma}^2_{\mathrm{DU}} = \mathrm{tr}(B\hat{\Sigma}_xB\hat{\Sigma}_y)/[\mathrm{tr}(B\hat{\Sigma}_x)\mathrm{tr}(B\hat{\Sigma}_y)]$, where $B = I-(1/n)J$ with J the $n\times n$ matrix of ones, and the estimated autocovariance matrices $\hat{\Sigma}_x$ and $\hat{\Sigma}_y$ are as above.

Fourthly, a hybrid procedure is considered, in which the effective sample size is estimated from $\hat{\sigma}^2_{\mathrm{HY}} = \mathrm{tr}(\hat{\Sigma}_x\hat{\Sigma}^{\star}_y)/[\mathrm{tr}(\hat{\Sigma}_x)\mathrm{tr}(\hat{\Sigma}^{\star}_y)]$, where $\hat{\Sigma}_x$ is as above and $\hat{\Sigma}^{\star}_y = \hat{\Sigma}_{\varepsilon_y}$ is built by using the $r(k)$s calculated from the OLS residuals of the regression of y on x.

Finally, a combination of the effective sample sizes of Clifford *et al.* (1989) and Dutilleul (1993) is proposed by using $\hat{\sigma}^2_{\mathrm{C}} = \mathrm{tr}(\hat{\Sigma}^{\star}_x\hat{\Sigma}^{\star}_y)/[\mathrm{tr}(\hat{\Sigma}^{\star}_x)\mathrm{tr}(\hat{\Sigma}^{\star}_y)]$, where $\hat{\Sigma}^{\star}_y = \hat{\Sigma}_{\varepsilon_y}$ and $\hat{\Sigma}^{\star}_x = \hat{\Sigma}_{\varepsilon_x}$ are built by using the $r(k)$s calculated from the OLS residuals of the regressions of y on x and of x on y, respectively.

4. Monte Carlo study

The model used for simulation was

$$y_t = a + bx_t + \varepsilon_t \quad \text{with} \quad \varepsilon_t = \rho\varepsilon_{t-1} + u_t \quad (t = 1, 2, \ldots, n),$$

where a and b were fixed at 1 and 0, the u_ts were i.i.d. $N(0, 1)$, and the value of ρ ranged from -0.9 to 0.9 by steps of 0.2, in addition to $\rho = 0$. The generation of autocorrelated errors followed a procedure similar to that of Dutilleul and Legendre (1992). Two situations were considered for the matrix X:

Case 1: $X = [1, x]$, where $x = (1, 2, \ldots, n)'$.

Case 2: $X = [1, x]$, where the elements of x originated from an AR(1) process in time

$$x_t = \gamma x_{t-1} + v_t \quad (t = 1, 2, \ldots, n), \tag{5.4}$$

where the v_ts were i.i.d. $N(0, 1)$ and hence, $\sigma_x^2 = 1/(1 - \gamma^2)$.

In both cases, 1 was a column vector of ones. In Case 2, the autocorrelation parameters ρ and γ were fixed at the same value, and x and ε were independently distributed. The empirical significance levels were evaluated from 1000 simulation runs for sample sizes $n = 10, 20, 30, 50$, and 100 for each value of ρ; only the results for $n = 10, 20$, and 50 will be presented. Each empirical significance level was calculated as 0.001 times the number of rejections of the null hypothesis of a zero value for the slope b in 1000 t-, χ^2-, z- or F-tests, depending on the procedure, performed at a theoretical significance level of 5%.

The positive square-root of the (2, 2)-entry of $\text{Cov}(\hat{\beta}_{\text{OLS}})_1$, $\text{Cov}(\hat{\beta}_{\text{OLS}})_2$ or $\text{Cov}(\hat{\beta}_{\text{GLS}})$, with Σ or an estimate of it, was used to calculate the standard error of the b-estimate, depending on the procedure. For comparison purposes, we iterated the estimated GLS procedures. Iterations were stopped when two successive b-estimates differed by 0.001 or less. In the ML procedure, $W = (w_{ij})$ was defined as $w_{ij} = 1$ if $j = i - 1$, and 0 otherwise. Therefore, $A = I - \hat{\rho}W$ was a lower triangular matrix with 1 on the diagonal and $-\hat{\rho}$ on the subdiagonal, the other entries being equal to zero. Following Beach and Mackinnon (1978) and Spitzer (1979), the $(1,1)$-entry of A was changed to $\sqrt{1-\hat{\rho}^2}$, where the ρ-estimate was evaluated to the nearest 0.001. In the ML procedure, we considered the χ^2- and z-tests for purposes of comparison on the basis of the sample size n. The FD and FDR procedures used the classical formula of the sample variance, except that the divisor was $n-2$ instead of $n-1$.

The following notations were used in Tables 5.1-5.2. Basically, these notations refer to different error covariance matrices used in the estimation procedures, along with whether or not the GLS estimation of β was iterative and the reference to the author that proposed a given adjustment of the number of df of the t-test. A t-test, modified or not, was performed in procedures 1-11 and 15-31.

1: Σ_o, Σ was assumed to be $\sigma^2 I$; OLS;

2: Σ_ρ, (5.2) with ρ known was used in $\hat{\beta}_{\text{GLS}}$ and $\text{Cov}(\hat{\beta}_{\text{GLS}})$; GLS;

3: $\Sigma_{\hat{\rho}1}$, same as procedure 2, except that ρ was replaced by $r(1)$ in (5.2), and no iteration was performed in the calculation of $\hat{\beta}$; estimated GLS (as procedures 4-8);

4: $\Sigma_{\hat{\rho}2}$, same as procedure 3, except that the calculation of $\hat{\beta}$ was iterative;

5: $\hat{\Sigma}_{13}$, (5.3) was used to estimate Σ in $\hat{\beta}_{\text{GLS}}$ and $\text{Cov}(\hat{\beta}_{\text{GLS}})$, no iteration on $\hat{\beta}$, and no test of significance of the $r(k)$s;

6: $\hat{\Sigma}_{14}$, same as procedure 5, except that the significance of $r(k)$ $(k=1,2,\ldots,m=\text{INT}(n/4))$ was assessed, and only the $r(k)$s declared significantly different from 0 were used;

7: $\hat{\Sigma}_{23}$, same as procedure 5, except that the calculation of $\hat{\beta}$ was iterative;

8: $\hat{\Sigma}_{24}$, same as procedure 6, except that the calculation of $\hat{\beta}$ was iterative;

9: $\Sigma_{o\rho}$, $\hat{\beta}_{\text{OLS}}$ was the estimator of β and the error covariance matrix in procedure 2 was used in $\text{Cov}(\hat{\beta}_{\text{OLS}})_2$ to evaluate the variance of $\hat{\beta}_{\text{OLS}}$;

10: $\Sigma_{o\hat{\rho}1}$, same as procedure 9, except that the error covariance matrix in procedure 3 was used;

11: $\hat{\Sigma}_{o14}$, same as procedure 9, except that the error covariance matrix in procedure 6 was used.

12: ML_{χ^2}, β was estimated by maximum likelihood and a likelihood-ratio χ^2-test with 1 df was performed;

13: ML_Z, same as procedure 12, except that an asymptotic z-test was performed;

14: REML, β was estimated by restricted maximum likelihood and the significance of the slope was assessed by the F-test for fixed effects in PROC MIXED of SAS.

15: FD, first-difference procedure;

16: FDR, method of first-difference ratios;

17: $\Sigma_{\rho M}$, $\hat{\beta}_{OLS}$ and $Cov(\hat{\beta}_{OLS})_1$ were used to evaluate the t-test statistic, but the number of df was adjusted using ε^*_{AN}, which was calculated using the error covariance matrix of procedure 2;

18: $\Sigma_{\hat{\rho}1M}$, same as procedure 17, except that the error covariance matrix of procedure 3 was used;

19: $\Sigma_{\hat{\rho}2M}$, same as procedure 17, except that the error covariance matrix of procedure 4 was used;

20: $\hat{\Sigma}_{13M}$, same as procedure 17, except that the error covariance matrix of procedure 5 was used;

21: $\hat{\Sigma}_{14M}$, same as procedure 17, except that the error covariance matrix of procedure 6 was used;

22: $\hat{\Sigma}_{23M}$, same as procedure 17, except that the error covariance matrix of procedure 7 was used;

23: $\hat{\Sigma}_{24M}$, same as procedure 17, except that the error covariance matrix of procedure 8 was used;

24: $\hat{\Sigma}_{CL3}$, $\hat{\beta}_{OLS}$ and $Cov(\hat{\beta}_{OLS})_1$ were used to evaluate the t-test statistic, but the number of df was adjusted using $\hat{\sigma}^2_{CL}$, no test of significance was performed on the sample autocorrelation coefficients of x and y;

25: $\hat{\Sigma}_{\mathrm{CL4}}$, same as procedure 24, except that a test of significance was performed on the sample autocorrelation coefficients of x and y;

26: $\hat{\Sigma}_{\mathrm{DU3}}$, same as procedure 24, except that the number of df was adjusted using $\hat{\sigma}^2_{\mathrm{DU}}$;

27: $\hat{\Sigma}_{\mathrm{DU4}}$, same as procedure 26, except that a test of significance was performed on the sample autocorrelation coefficients of x and y;

28: $\hat{\Sigma}_{\mathrm{HY3}}$, same as procedures 24 and 26, except that the number of df was adjusted using $\hat{\sigma}^2_{\mathrm{HY}}$;

29: $\hat{\Sigma}_{\mathrm{HY4}}$, same as procedure 28, except that a test of significance was performed on the sample autocorrelation coefficients of x and y;

30: $\hat{\Sigma}_{\mathrm{C3}}$, same as procedures 24, 26 and 28, except that the number of df was adjusted using $\hat{\sigma}^2_{\mathrm{C}}$;

31: $\hat{\Sigma}_{\mathrm{C4}}$, same as procedure 30, except that a test of significance was performed on the sample autocorrelation coefficients of x and y.

All 31 testing procedures were included in our Monte Carlo study when x follows an AR(1) process (Case 2). As for Case 1, only 24 testing procedures were included, since FD and FDR are the same and procedures 24-29 are not applicable when x is fixed. We used our own computer programs written in SAS/IML language and PROC MIXED of SAS (SAS Institute Inc. 1997) to implement the testing procedures. The generation of i.i.d. $N(0,1)$ observations was carried out with the random number function RANNOR of SAS (SAS Institute Inc. 1997).

5. Results

The results of our Monte Carlo study for $n = 10, 20$, and 50 are reported in Tables 5.1 and 5.2. Strictly speaking a testing procedure is said to be valid at level α if the probability that it rejects the null hypothesis, when in fact the null hypothesis is true, is less than or equal to α. The actual significance level of each testing procedure considered here is estimated by the empirical significance level of p evaluated from 1000 simulation runs. Under the binomial distribution model, the standard deviation of p is given by $\sigma_p = \sqrt{p(1-p)/1000}$. An approximate 95% confidence interval for the actual significance level of a testing procedure is provided by $p \pm 2\sigma_p$. The largest value of p such that $p \pm 2\sigma_p$ contains the theoretical significance level of 0.05 is 0.065. Our interpretation of the results reported in Table 5.1 and Table 5.2 is based on the strict definition of validity, combined with the variability associated with the empirical significance levels. Thus, we have used $p \leq 0.065$ as the validity condition.

Whereas validity tends to be the rule in hypothesis testing for the slope when x is purely random (Alpargu and Dutilleul, unpublished manuscript), this is not the case when x is fixed and trended (Case 1 here) and when x is random and follows an AR(1) process (Case 2), if the errors, themselves, follow an AR(1) process in the quantitative linear model. Important differences between Case 1 and Case 2 are observed. Results for the REML procedure when $n = 10$ are not reported due to the too frequent lack of convergence of the REML algorithm at that sample size.

Case 1: Only three testing procedures show some signs of validity in the presence of positive autocorrelation of the errors (see $\rho > 0$ in Table 5.1). All three are based on a t-test with $n-2$ df. They are: FD, in which the first differences of the y_ts are computed and the null hypothesis of zero mean is tested on these first differences, and procedures 2 and 9, which assume the complete knowledge of the covariance matrix of the errors in both the computation of the GLS estimator and that of its variance or only in the computation of the variance of the OLS estimator of the slope. The FD procedure is strictly valid for all positive and negative values of ρ at all sample sizes considered in Case 1. This procedure is robust, since no assumption is made regarding the covariance structure of the errors in it. It reflects some robustness, since no particular assumption is made regarding the variance-covariance structure of the errors in this procedure. However, the zero rate of rejection of the null hypothesis in most cases may reflect a lack of power under the alternative hypothesis, and it needs further investigation. By comparison, procedure 2 satisfies the validity condition for all but one positive value of ρ, and procedures 9, 18 and 19 for only one positive value of ρ (i.e., 0.1) when $n = 10$. With procedure 9, procedure 14 (REML) gains in validity as the sample size increases, these two procedures satisfying the validity condition up to $\rho = 0.5$ when $n = 50$.

With the exceptions of the two ML procedures, all testing procedures, including the classical t-test (i.e., procedure 1), are valid or close to validity in the absence of autocorrelation or in the presence of negative autocorrelation of the errors (see $\rho \leq 0$ in Table 5.1 when $n = 10$ and 20). Among them, procedures 5 and 7 are less valid than procedures 6 and 8, which indicates the importance of assessing the significance of the $r(k)$s prior to including them in (5.3) for estimated GLS estimation of the slope when x is fixed. On the other hand, iteration in the computation of the estimated GLS estimate of the slope has no real effect on the empirical significance level of the relevant testing procedure.

Strictly speaking, the two ML procedures were never valid. Nevertheless, they gained in validity with increasing n, as could be expected on a theoretical basis. Procedure 12 started to satisfy the validity condition for some negative values of ρ when $n = 20$, whereas procedure 13 satisfied it for one negative value of ρ when $n = 50$. When $\rho > 0$ and $n = 50$, the validity of the ML χ^2-test (i.e., procedure 12) compares well with that of procedures 3 and 4 that are based on the estimated GLS estimator of the slope under the assumption that the errors follow an AR(1) process.

Case 2: A sample size effect is observed in Table 5.2. When $n = 10$, only one testing procedure satisfies the validity condition for all values of ρ. It is the t-test with $n - 2$ df of procedure 2. When $n = 50$, the set of valid testing procedures includes procedures 3 and 4 (t-test) and procedure 14 (F-test), all three procedures assuming the knowledge of the model of covariance structure of the errors. Procedures 9, 10, 12 and 16 are close to validity for all values of ρ when $n = 50$. Procedure 16 (FDR), which is equivalent to FD in Case 1 and consists in performing a t-test for the mean on the ratios of first differences of y_ts and x_ts, is strictly valid for all positive values of ρ at all sample sizes. The performance of the classical t-test of the slope (procedure 1) is close to that of the modified t-tests (procedures 17-31). The validity of these procedures is limited to values of ρ up to 0.3 in absolute value. In the absence of autocorrelation in x and ε (i.e., $\gamma = \rho = 0$), most of the testing procedures are valid when $n = 50$, with the exceptions of procedures 5, 7 and 8 (based on estimated GLS estimators of the slope), procedures 12 and 13 (based on the ML estimator of the slope) and FD. In the presence of negative autocorrelation in x and ε (i.e., $\gamma = \varepsilon < 0$), procedures 2-4, 9-10, 12-14 and 16 are the most valid, with an advantage overall for procedures 2 and 9 and procedure 14 (REML) and 16 (FDR).

Contrary to Case 1, the test of significance of the $r(k)$s prior to including them in (5.3) does not provide noticeable gains in validity to procedures 6 and 8 compared to procedures 5 and 7. As in Case 1, iteration in the computation of the estimated GLS estimate of the slope does not improve consistently and substantially the validity of procedures 7 and 8 compared to procedures 5 and 6. These results extend to procedures 17-31.

An important difference with Case 1 is that the autocorrelation effect on the validity of the testing procedures appears to be symmetrical in Case 2, with the observation of empirical significance levels for positive autocorrelation that are close to those for negative autocorrelation of the same magnitude. This symmetry is greater for $n = 50$ than for $n = 10$.

6. Discussion

When x is purely random in a simple linear regression model with AR(1) errors, Alpargu and Dutilleul (unpublished manuscript) computed Box's epsilon (Box 1954a, b) to evaluate the discrepancies of the covariance structures of the observations and the errors from the circularity condition (Huynh and Feldt 1970, Rouanet and Lépine 1970). In so doing, the authors showed that the addition of a purely random x to the error ε reduces the discrepancy from circularity without filling it completely. Thereafter, they considered the effective sample size used in correlation analysis with autocorrelated sample data, and explained the validity of the classical t-test of the slope when x is purely random by the fact that the classical sample size and effective sample size are equal in this case.

In the study reported here, Box's epsilon is the same for y and ε when x is fixed. In this case, the value of Box's epsilon for y is that of an AR(1) process with autocorrelation parameter ρ. Moreover, the value of Box's epsilon for an AR(1) process with $\rho = 0.9$ is approximately equal to that of an AR(1) process with $\rho = -0.9$ (Alpargu and Dutilleul, unpublished manuscript). It follows that Box's epsilon cannot be used to explain the drastic change in validity observed in Tables 5.1 and 5.2, depending on whether the autocorrelation of errors is negative or positive. Furthermore, the effective sample sizes of Clifford *et al.* (1989) and Dutilleul (1993) cannot be computed when x is fixed.

When x follows an AR(1) process, the effective sample sizes can be computed, but the corresponding modified t-tests as well as the other modified t-tests considered here were not very successful in maintaining the empirical significance level below the 5% threshold. In this case, Box's epsilon could be used to explain the symmetry in invalidity displayed in Table 5.2, with increasing positive and negative autocorrelation. In fact, if $b = 1$, $\gamma = \rho$ and $\sigma_v^2 = \sigma_u^2$ in Case 2, then $\mathrm{Cor}(y_t, y_{t'}) = \rho^{|t-t'|}$, so the argument of similar Box's epsilon values for AR(1) processes with opposite autocorrelation parameter values can be used to explain the results reported in Table 5.2. Note that in general (i.e., $b \neq 1$ and $\gamma \neq \rho$), $\text{Cor}(y_t, y_{t'}) = (b^2\gamma^{|t-t'|}\sigma_x^2 + \rho^{|t-t'|}\sigma_\varepsilon^2)/(b^2\sigma_x^2 + \sigma_\varepsilon^2) \quad (t \neq t')$. Also, when $\gamma = -\rho$, Box's epsilon for y is much closer to 1 than when $\gamma = \rho$. Accordingly, the classical t-test is valid or close to validity in Case 2 when $\gamma = -\rho$ and $-0.5 \leq \rho \leq 0.5$ (the results are not reported here).

In order to provide an explanation for the results we have obtained when x is trended and to complement our explanation for the case when x follows an AR(1) process, we have looked for a graphical interpretation of our results. The OLS estimator of the slope is known to be inefficient but unbiased (Searle 1971), but what does the OLS fitting of a straight line, $\hat{a} + \hat{b}t$, to a partial realization of an AR(1) process with negative or positive ρ actually mean? Similarly, what does the OLS fitting of a partial realization of an AR(1) process to a partial realization of another AR(1) process x_t with same autocorrelation parameter value mean? An illustrative example for autocorrelation parameter values of -0.9 and 0.9 is presented in Figure 5.1. Clearly, the alternating pattern over time of the AR(1) realization for $\rho = -0.9$ [Fig. 5.1 (A)] explains the validity of most testing procedures, including the classical t-test of the slope in Case 1. On the other hand [Fig. 5.1 (B)], the smooth pattern of the AR(1) realization for $\rho = 0.9$, which can be decreasing as well as increasing, explains, at least in part, the excessive empirical significance level (i.e., 0.455) of the classical t-test of the slope when $n = 10$ in Case 1. With regard to Case 2, fitting an alternating pattern to another alternating pattern and fitting a smooth pattern to another smooth pattern in time are more likely to provide a significant slope than fitting an alternating or smooth pattern to a purely random pattern [Fig. 5.1 (C) and (D)].

7. Numerical example

The data used here for illustration were collected at Gault Nature Reserve (Mont-Saint-Hilaire, Quebec, Canada) in 1994, on two transect lines denoted "11A" and "Cliff". The variable to explain is soil pH, whereas position on the transect (11A) and altitude at the sampling site (Cliff) are used as regressor in two simple linear regressions. Data were collected every 20 meters over 1 kilometre (i.e., $n = 50$), so the position on the transect can be considered fixed and trended (Case 1) whereas altitude at the sampling site varies smoothly and its 1-D pattern resembles an AR(1) process in time (Case 2). The bivariate relationships with soil pH are shown in Figure 5.3. Numerical results are reported in Table 5.3 for: the classical t-test of the slope (procedure 1); a testing procedure based on an estimated GLS estimator of the slope, in which the sample autocorrelation coefficients $r(k)$ are replaced by Moran's I correlogram ordinates (Cliff and Ord 1975)–this procedure is similar to procedure 5 in the Monte Carlo study and is denoted $\hat{\Sigma}$ here; and the ML (χ^2-test), REML (in which a spherical variogram model is used), FD and FDR procedures. In both cases (i.e., fixed and trended regressor for Transect Line 11A, and random and AR(1) regressor for Transect Line Cliff), the first three procedures provide similar slope estimates and slightly different variance estimates, resulting in lower probabilities of significance for $\hat{\Sigma}$ and ML compared to the classical t-test of the slope. All three procedures declared the slope significantly different, though. By comparison, REML, FD and FDR provide very different slope estimates in magnitude and much larger variance estimates (i.e., this is especially true for FD and FDR), resulting in probabilities of significance above the 5% threshold. For these three procedures, the slope is not declared to be significantly different from zero and this is the conclusion that one would draw on the basis of our Monte Carlo results for $n = 50$ and a moderate positive autocorrelation. The lack of power of FD for Transect Line 11A is noticeable.

8. Concluding remarks

With the exception of strong negative autocorrelation for the regressor and the errors in Case 2, FDR, which is equivalent to FD when x is trended, is the most valid among the testing procedures that do not require the knowledge of the covariance structure of the errors, whether the regressor is trended or autocorrelated. Among the procedures that do require this knowledge, REML is the most valid in Case 1 and Case 2, provided the sample size is sufficiently large (i.e., $n \geq 20$). When x is purely random, the FDR procedure had already shown strict validity (Alpargu and Dutilleul, unpublished manuscript). Thus, this procedure, which requires no a priori assumption, is robust in several respects, despite some inefficiency in estimation (Alpargu and Dutilleul 2001). The challenge now is to extend the use of ratios of first differences of the variable to explain and the regressor to multiple linear regression models; this should be possible through partial regression coefficients. A power analysis of the procedures is also recommended. In addition to $\rho = 0$, the classical t-test of the slope was shown to be valid when the regressor is trended and the errors are negatively autocorrelated ($\rho < 0$), and when the regressor and the error follow an AR(1) process with moderate autocorrelation ($0 < \rho < 0.3$).

Returning to the warning of Upton and Fingleton (1985), our study has shown that the invalidity of the classical t-test of individual slopes in quantitative linear models with autocorrelated errors is limited to the cases when x is trended and the errors are positively autocorrelated and when the regressor and the errors are autocorrelated, especially if their autocorrelation is of the same sign. Concerning Jenkins and Watts (1968), our results in Case 2 do not contradict their demonstration.

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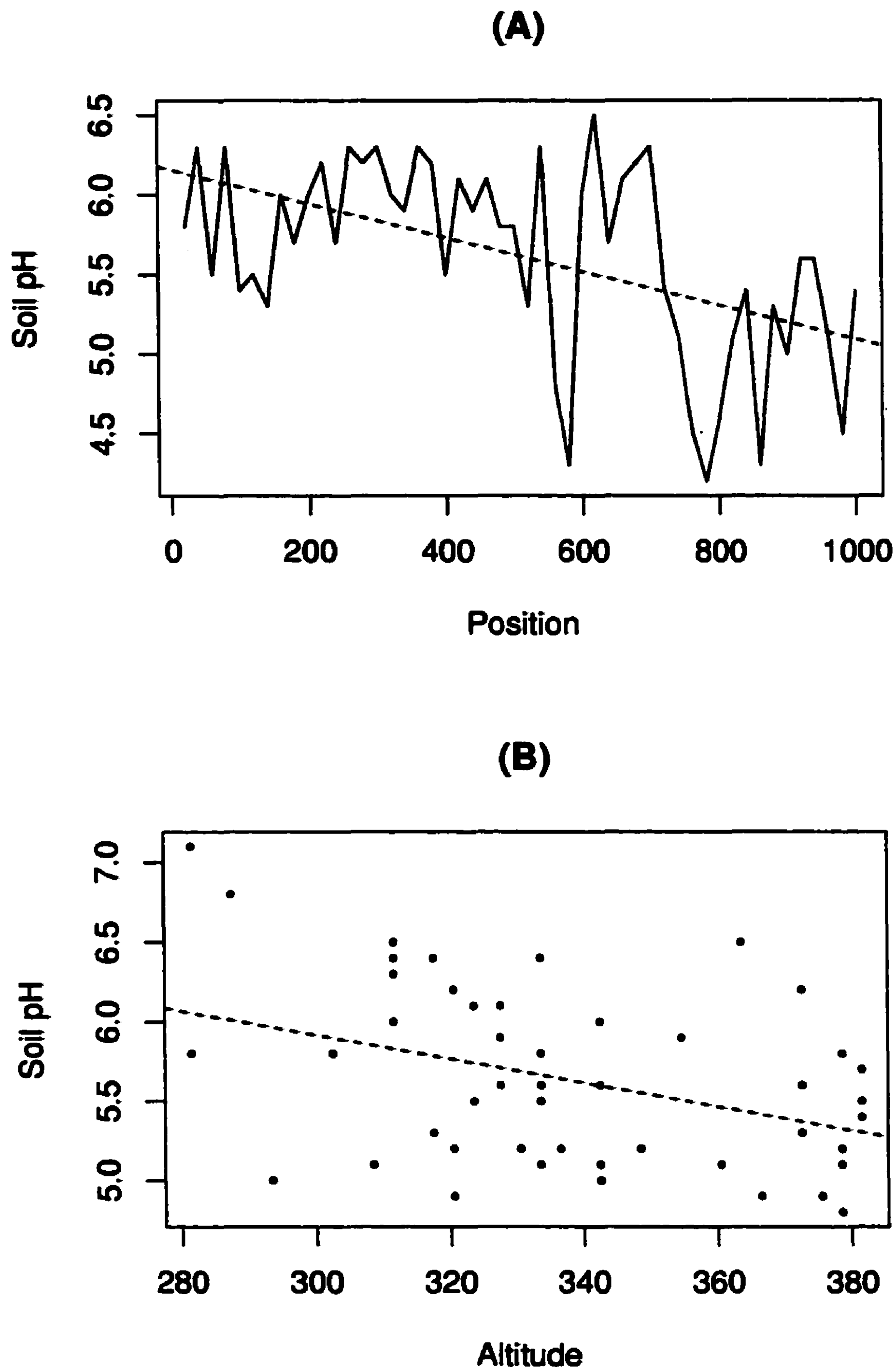

Figure 5.2: Line chart of soil pH vs. position on the transect (A, Transect Line 11A) and scatter plot of soil pH vs. altitude at the sampling site (B, Transect Line Cliff) in the Mont-Saint-Hilaire example.

Table 5.1: (first page) Empirical significance level of the 24 testing procedures available when x is fixed for a theoretical significance level of 5%, as a function of the sample size, n, and the autocorrelation parameter of the errors ρ. Empirical significance levels were computed from 1000 simulation runs. See the text for other notations.

n=10

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
Σ_o	0.001	0.004	0.012	0.019	0.028	0.053	0.072	0.132	0.188	0.312	0.455
Σ_ρ	0.004	0.015	0.031	0.045	0.037	0.053	0.056	0.068	0.056	0.062	0.049
$\Sigma_{\beta 1}$	0.002	0.007	0.025	0.05	0.065	0.096	0.085	0.148	0.192	0.293	0.387
$\Sigma_{\beta 2}$	0.002	0.007	0.025	0.05	0.065	0.096	0.086	0.151	0.186	0.286	0.372
$\hat{\Sigma}_{13}$	0.016	0.036	0.056	0.085	0.106	0.119	0.134	0.192	0.25	0.353	0.453
$\hat{\Sigma}_{14}$	0.001	0.004	0.012	0.023	0.03	0.055	0.073	0.135	0.191	0.314	0.455
$\hat{\Sigma}_{23}$	0.011	0.032	0.058	0.084	0.111	0.121	0.136	0.193	0.242	0.352	0.448
$\hat{\Sigma}_{24}$	0.001	0.004	0.012	0.024	0.03	0.056	0.074	0.137	0.191	0.319	0.452
$\Sigma_{o\rho}$	0.032	0.037	0.046	0.05	0.038	0.053	0.059	0.072	0.086	0.167	0.384
$\Sigma_{o\beta 1}$	0.014	0.019	0.036	0.065	0.079	0.098	0.092	0.151	0.191	0.306	0.412
$\hat{\Sigma}_{o14}$	0.001	0.004	0.012	0.025	0.029	0.055	0.072	0.135	0.191	0.314	0.455
ML_{χ^2}	0.098	0.097	0.107	0.125	0.131	0.133	0.136	0.186	0.2	0.289	0.388
ML_Z	0.144	0.134	0.162	0.18	0.192	0.203	0.225	0.282	0.304	0.402	0.493
FD	0	0	0	0	0	0	0	0	0	0.004	0.024
$\Sigma_{\rho M}$	0	0.002	0.007	0.018	0.027	0.054	0.072	0.131	0.171	0.27	0.386
$\Sigma_{\beta 1M}$	0.001	0.003	0.01	0.018	0.023	0.049	0.065	0.127	0.183	0.309	0.441
$\Sigma_{\beta 2M}$	0	0.003	0.01	0.018	0.023	0.049	0.065	0.126	0.183	0.305	0.44
$\hat{\Sigma}_{13M}$	0	0.003	0.011	0.017	0.026	0.05	0.066	0.126	0.182	0.305	0.449
$\hat{\Sigma}_{14M}$	0.001	0.004	0.012	0.019	0.028	0.052	0.072	0.132	0.188	0.312	0.455
$\hat{\Sigma}_{23M}$	0	0.003	0.011	0.018	0.025	0.051	0.067	0.127	0.181	0.309	0.443
$\hat{\Sigma}_{24M}$	0.001	0.004	0.012	0.02	0.028	0.053	0.073	0.134	0.188	0.317	0.452
$\hat{\Sigma}_{C3}$	0.001	0.004	0.013	0.025	0.033	0.063	0.081	0.141	0.189	0.32	0.448
$\hat{\Sigma}_{C4}$	0.002	0.005	0.012	0.02	0.032	0.061	0.079	0.139	0.198	0.328	0.459

n=20

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
Σ_o	0	0.001	0.003	0.008	0.031	0.041	0.061	0.127	0.26	0.37	0.592
Σ_ρ	0.004	0.014	0.025	0.031	0.041	0.041	0.043	0.037	0.047	0.031	0.023
$\Sigma_{\beta 1}$	0.002	0.008	0.023	0.031	0.057	0.074	0.074	0.097	0.158	0.215	0.334
$\Sigma_{\beta 2}$	0.002	0.008	0.023	0.031	0.057	0.074	0.075	0.097	0.154	0.201	0.304
$\hat{\Sigma}_{13}$	0.009	0.032	0.075	0.091	0.132	0.149	0.15	0.194	0.287	0.356	0.564
$\hat{\Sigma}_{14}$	0.004	0.019	0.037	0.05	0.062	0.071	0.091	0.144	0.27	0.351	0.569
$\hat{\Sigma}_{23}$	0.007	0.028	0.068	0.093	0.134	0.158	0.151	0.191	0.288	0.358	0.553
$\hat{\Sigma}_{24}$	0.003	0.021	0.037	0.05	0.063	0.07	0.092	0.145	0.277	0.35	0.571
$\Sigma_{o\rho}$	0.051	0.046	0.049	0.037	0.042	0.041	0.045	0.043	0.083	0.109	0.299
$\Sigma_{o\beta 1}$	0.02	0.027	0.048	0.047	0.063	0.086	0.078	0.098	0.175	0.243	0.407
$\hat{\Sigma}_{o14}$	0.004	0.013	0.026	0.035	0.047	0.065	0.081	0.138	0.262	0.349	0.569
ML_{χ^2}	0.058	0.051	0.072	0.062	0.072	0.086	0.086	0.096	0.14	0.168	0.317
ML_Z	0.067	0.069	0.094	0.09	0.1	0.117	0.125	0.152	0.218	0.262	0.432
REML	0	0	0	0.005	0.008	0.011	0.009	0.038	0.029	0.068	0.108
FD	0	0	0	0	0	0	0	0	0	0	0.004
$\Sigma_{\rho M}$	0	0	0.003	0.008	0.032	0.041	0.061	0.12	0.247	0.335	0.536
$\Sigma_{\beta 1M}$	0	0.001	0.003	0.007	0.031	0.04	0.059	0.127	0.257	0.358	0.587
$\Sigma_{\beta 2M}$	0	0.001	0.003	0.007	0.031	0.04	0.059	0.126	0.257	0.356	0.582
$\hat{\Sigma}_{13M}$	0	0.001	0.003	0.007	0.029	0.039	0.06	0.123	0.255	0.366	0.584
$\hat{\Sigma}_{14M}$	0	0.001	0.003	0.008	0.031	0.041	0.059	0.124	0.256	0.367	0.59
$\hat{\Sigma}_{23M}$	0	0.001	0.003	0.007	0.03	0.039	0.06	0.125	0.259	0.365	0.587
$\hat{\Sigma}_{24M}$	0	0.001	0.003	0.008	0.032	0.041	0.059	0.124	0.261	0.368	0.593
$\hat{\Sigma}_{C3}$	0	0.001	0.003	0.01	0.036	0.047	0.065	0.136	0.263	0.357	0.588
$\hat{\Sigma}_{C4}$	0	0.002	0.004	0.01	0.032	0.041	0.063	0.131	0.263	0.361	0.586

Table 5.2: (first page) Empirical significance level of the 31 testing procedures when x follows an AR(1) process for a theoretical significance level of 5%, as a function of the sample size n and the common value of the autocorrelation parameters. Empirical significance levels were computed from 1000 simulation runs. The autocorrelation parameter of x, γ, was fixed at the same value as that of the errors, ρ. See the text for other notations.

n=10

$\gamma=\rho$	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
Σ_o	0.365	0.168	0.092	0.055	0.056	0.053	0.06	0.069	0.088	0.14	0.213
Σ_ρ	0.013	0.023	0.032	0.032	0.047	0.053	0.062	0.049	0.056	0.043	0.029
$\Sigma_{\beta 1}$	0.199	0.09	0.07	0.045	0.058	0.069	0.074	0.079	0.11	0.13	0.181
$\Sigma_{\beta 2}$	0.168	0.08	0.068	0.048	0.065	0.073	0.076	0.084	0.111	0.132	0.171
$\hat{\Sigma}_{13}$	0.288	0.163	0.099	0.085	0.084	0.094	0.104	0.101	0.13	0.161	0.224
$\hat{\Sigma}_{14}$	0.347	0.163	0.09	0.053	0.056	0.053	0.062	0.069	0.088	0.14	0.213
$\hat{\Sigma}_{23}$	0.296	0.161	0.111	0.094	0.105	0.11	0.124	0.115	0.143	0.174	0.23
$\hat{\Sigma}_{24}$	0.391	0.175	0.1	0.058	0.055	0.053	0.063	0.07	0.086	0.149	0.221
$\Sigma_{o\rho}$	0.101	0.054	0.038	0.04	0.053	0.053	0.062	0.063	0.07	0.116	0.299
$\Sigma_{o\beta 1}$	0.238	0.11	0.072	0.051	0.058	0.063	0.066	0.075	0.099	0.137	0.2
$\hat{\Sigma}_{o14}$	0.347	0.164	0.089	0.055	0.056	0.053	0.061	0.069	0.088	0.14	0.213
ML_{χ^2}	0.143	0.121	0.113	0.111	0.102	0.132	0.155	0.131	0.144	0.145	0.157
ML_Z	0.249	0.204	0.191	0.175	0.191	0.203	0.225	0.21	0.214	0.241	0.262
FD	0.436	0.253	0.195	0.132	0.116	0.105	0.108	0.074	0.078	0.052	0.041
FDR	0.295	0.116	0.079	0.041	0.033	0.022	0.028	0.023	0.037	0.02	0.022
$\Sigma_{\rho M}$	0.235	0.117	0.078	0.053	0.051	0.053	0.061	0.067	0.07	0.102	0.135
$\Sigma_{\beta 1M}$	0.371	0.17	0.088	0.05	0.049	0.05	0.059	0.066	0.083	0.143	0.211
$\Sigma_{\beta 2M}$	0.32	0.141	0.075	0.047	0.045	0.048	0.058	0.062	0.079	0.134	0.201
$\hat{\Sigma}_{13M}$	0.344	0.165	0.086	0.05	0.051	0.052	0.056	0.064	0.083	0.13	0.204
$\hat{\Sigma}_{14M}$	0.365	0.168	0.091	0.055	0.056	0.053	0.06	0.069	0.088	0.14	0.213
$\hat{\Sigma}_{23M}$	0.391	0.171	0.089	0.053	0.045	0.053	0.058	0.064	0.08	0.132	0.209
$\hat{\Sigma}_{24M}$	0.408	0.178	0.101	0.06	0.055	0.053	0.061	0.07	0.086	0.149	0.221
$\hat{\Sigma}_{CL3}$	0.268	0.119	0.077	0.051	0.052	0.054	0.061	0.071	0.085	0.139	0.2
$\hat{\Sigma}_{CL4}$	0.355	0.164	0.109	0.064	0.059	0.059	0.067	0.073	0.09	0.154	0.228
$\hat{\Sigma}_{DU3}$	0.219	0.108	0.068	0.048	0.048	0.052	0.06	0.064	0.079	0.124	0.177
$\hat{\Sigma}_{DU4}$	0.367	0.168	0.1	0.06	0.055	0.053	0.061	0.07	0.086	0.148	0.218
$\hat{\Sigma}_{HY3}$	0.354	0.169	0.098	0.059	0.061	0.058	0.067	0.079	0.093	0.156	0.218
$\hat{\Sigma}_{HY4}$	0.393	0.18	0.111	0.065	0.059	0.059	0.067	0.073	0.09	0.154	0.23
$\hat{\Sigma}_{C3}$	0.376	0.167	0.096	0.056	0.053	0.055	0.062	0.071	0.086	0.149	0.219
$\hat{\Sigma}_{C4}$	0.406	0.181	0.111	0.065	0.059	0.059	0.067	0.073	0.09	0.154	0.23

Table 5.2 (continued).

n=20

$\gamma=\rho$	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
Σ_o	0.422	0.203	0.115	0.079	0.048	0.057	0.059	0.066	0.116	0.21	0.31
Σ_ρ	0.008	0.016	0.023	0.049	0.044	0.057	0.051	0.056	0.042	0.031	0.017
$\Sigma_{\beta 1}$	0.081	0.046	0.047	0.061	0.048	0.061	0.061	0.07	0.082	0.101	0.11
$\Sigma_{\beta 2}$	0.057	0.039	0.044	0.063	0.05	0.064	0.069	0.07	0.078	0.083	0.091
$\hat{\Sigma}_{13}$	0.287	0.154	0.118	0.128	0.098	0.112	0.107	0.118	0.139	0.197	0.251
$\hat{\Sigma}_{14}$	0.343	0.158	0.113	0.093	0.059	0.068	0.064	0.078	0.128	0.2	0.258
$\hat{\Sigma}_{23}$	0.3	0.138	0.119	0.143	0.123	0.144	0.141	0.161	0.146	0.183	0.228
$\hat{\Sigma}_{24}$	0.383	0.156	0.111	0.095	0.064	0.073	0.064	0.082	0.117	0.203	0.298
$\Sigma_{o\rho}$	0.079	0.05	0.036	0.06	0.047	0.057	0.055	0.056	0.076	0.098	0.18
$\Sigma_{o\beta 1}$	0.182	0.088	0.066	0.067	0.048	0.055	0.062	0.067	0.099	0.129	0.191
$\hat{\Sigma}_{o14}$	0.37	0.174	0.101	0.077	0.05	0.058	0.059	0.064	0.113	0.198	0.282
ML_{χ^2}	0.093	0.089	0.062	0.092	0.076	0.079	0.081	0.086	0.086	0.109	0.094
ML_Z	0.142	0.128	0.096	0.134	0.104	0.107	0.123	0.12	0.118	0.159	0.155
REML	0.025	0.033	0.041	0.046	0.061	0.058	0.051	0.054	0.04	0.032	0.087
FD	0.472	0.276	0.203	0.157	0.111	0.105	0.082	0.071	0.06	0.074	0.054
FDR	0.182	0.073	0.045	0.039	0.026	0.024	0.027	0.013	0.02	0.019	0.02
$\Sigma_{\rho M}$	0.277	0.162	0.094	0.083	0.054	0.059	0.06	0.063	0.101	0.175	0.24
$\Sigma_{\beta 1M}$	0.403	0.183	0.098	0.08	0.053	0.057	0.06	0.062	0.109	0.203	0.328
$\Sigma_{\beta 2M}$	0.353	0.163	0.091	0.079	0.053	0.057	0.06	0.062	0.105	0.195	0.311
$\hat{\Sigma}_{13M}$	0.415	0.195	0.102	0.077	0.045	0.054	0.057	0.065	0.111	0.197	0.301
$\hat{\Sigma}_{14M}$	0.42	0.203	0.114	0.079	0.048	0.057	0.059	0.065	0.116	0.208	0.309
$\hat{\Sigma}_{23M}$	0.449	0.198	0.1	0.081	0.048	0.056	0.054	0.065	0.104	0.205	0.33
$\hat{\Sigma}_{24M}$	0.462	0.218	0.115	0.086	0.054	0.059	0.06	0.063	0.113	0.217	0.345
$\hat{\Sigma}_{CL3}$	0.359	0.161	0.092	0.082	0.053	0.058	0.06	0.063	0.103	0.2	0.319
$\hat{\Sigma}_{CL4}$	0.417	0.192	0.105	0.085	0.054	0.06	0.061	0.064	0.11	0.21	0.329
$\hat{\Sigma}_{DU3}$	0.361	0.159	0.091	0.081	0.052	0.057	0.06	0.06	0.101	0.194	0.311
$\hat{\Sigma}_{DU4}$	0.418	0.192	0.105	0.085	0.054	0.059	0.06	0.064	0.109	0.206	0.327
$\hat{\Sigma}_{HY3}$	0.391	0.19	0.1	0.084	0.054	0.06	0.061	0.062	0.11	0.207	0.329
$\hat{\Sigma}_{HY4}$	0.429	0.201	0.11	0.085	0.054	0.06	0.061	0.064	0.113	0.216	0.336
$\hat{\Sigma}_{C3}$	0.409	0.192	0.101	0.084	0.053	0.059	0.061	0.062	0.11	0.208	0.33
$\hat{\Sigma}_{C4}$	0.433	0.202	0.112	0.085	0.054	0.06	0.061	0.064	0.113	0.217	0.338

Table 5.2 (last page).

n=50

$\gamma=\rho$	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
Σ_o	0.479	0.212	0.112	0.064	0.051	0.044	0.045	0.054	0.103	0.195	0.454
Σ_ρ	0	0.005	0.028	0:04	0.053	0.044	0.047	0.03	0.033	0.01	0.002
$\Sigma_{\beta 1}$	0.017	0.016	0.031	0.047	0.062	0.057	0.048	0.04	0.035	0.021	0.02
$\Sigma_{\beta 2}$	0.005	0.014	0.031	0.048	0.064	0.058	0.048	0.039	0.034	0.017	0.012
$\hat{\Sigma}_{13}$	0.338	0.149	0.113	0.126	0.114	0.114	0.123	0.109	0.125	0.157	0.347
$\hat{\Sigma}_{14}$	0.356	0.126	0.093	0.083	0.073	0.056	0.068	0.073	0.097	0.136	0.36
$\hat{\Sigma}_{23}$	0.339	0.15	0.113	0.135	0.142	0.139	0.153	0.13	0.137	0.155	0.336
$\hat{\Sigma}_{24}$	0.385	0.127	0.084	0.092	0.078	0.071	0.071	0.079	0.104	0.138	0.373
$\Sigma_{o\rho}$	0.061	0.039	0.044	0.052	0.049	0.044	0.044	0.035	0.041	0.053	0.134
$\Sigma_{o\beta 1}$	0.124	0.055	0.053	0.053	0.049	0.046	0.045	0.037	0.055	0.072	0.176
$\hat{\Sigma}_{o14}$	0.401	0.152	0.081	0.06	0.049	0.046	0.047	0.046	0.071	0.149	0.399
ML_{χ^2}	0.066	0.052	0.055	0.061	0.066	0.067	0.055	0.053	0.052	0.059	0.072
ML_Z	0.085	0.06	0.066	0.074	0.07	0.079	0.065	0.062	0.061	0.069	0.091
REML	0.008	0.018	0.04	0.047	0.046	0.047	0.044	0.048	0.031	0.027	0.016
FD	0.517	0.271	0.197	0.167	0.12	0.116	0.097	0.091	0.068	0.059	0.06
FDR	0.114	0.037	0.03	0.028	0.02	0.025	0.018	0.019	0.017	0.017	0.023
$\Sigma_{\rho M}$	0.417	0.2	0.111	0.064	0.056	0.047	0.05	0.054	0.094	0.188	0.409
$\Sigma_{\beta 1M}$	0.449	0.211	0.111	0.064	0.055	0.047	0.05	0.053	0.097	0.2	0.442
$\Sigma_{\beta 2M}$	0.431	0.203	0.107	0.064	0.055	0.047	0.05	0.053	0.097	0.193	0.434
$\hat{\Sigma}_{13M}$	0.478	0.208	0.107	0.062	0.049	0.043	0.044	0.054	0.099	0.193	0.452
$\hat{\Sigma}_{14M}$	0.479	0.212	0.107	0.063	0.05	0.044	0.045	0.052	0.103	0.194	0.454
$\hat{\Sigma}_{23M}$	0.497	0.218	0.109	0.063	0.053	0.046	0.049	0.055	0.094	0.196	0.46
$\hat{\Sigma}_{24M}$	0.501	0.224	0.113	0.064	0.055	0.047	0.05	0.053	0.101	0.205	0.467
$\hat{\Sigma}_{CL3}$	0.444	0.203	0.108	0.064	0.057	0.047	0.05	0.054	0.098	0.196	0.44
$\hat{\Sigma}_{CL4}$	0.467	0.212	0.113	0.064	0.056	0.047	0.05	0.054	0.098	0.201	0.451
$\hat{\Sigma}_{DU3}$	0.444	0.203	0.108	0.064	0.056	0.047	0.05	0.053	0.097	0.195	0.438
$\hat{\Sigma}_{DU4}$	0.466	0.213	0.112	0.064	0.056	0.047	0.05	0.054	0.098	0.2	0.449
$\hat{\Sigma}_{HY3}$	0.449	0.209	0.111	0.064	0.056	0.048	0.05	0.055	0.098	0.2	0.443
$\hat{\Sigma}_{HY4}$	0.47	0.215	0.113	0.064	0.056	0.047	0.05	0.054	0.098	0.201	0.452
$\hat{\Sigma}_{C3}$	0.451	0.213	0.111	0.064	0.055	0.047	0.05	0.055	0.098	0.201	0.451
$\hat{\Sigma}_{C4}$	0.472	0.215	0.113	0.065	0.055	0.047	0.05	0.054	0.098	0.201	0.452

Table 5.3: Simple linear regression of response variable, soil pH, on explanatory variable, position on the transect (Transect Line 11A), and altitude at the sampling site (Transect Line Cliff), in the Mont-Saint-Hilaire example.

	Transect Line 11A			Transect Line Cliff		
	$\hat{\beta}$	$\hat{\sigma}(\hat{\beta})$	p-val	$\hat{\beta}$	$\hat{\sigma}(\hat{\beta})$	p-val
Σ_o	-0.00106	0.00026	0.0002	-0.00751	0.00262	0.0062
$\hat{\Sigma}$	-0.00107	0.00025	0.0001	-0.00734	0.00326	0.029
ML	-0.00103	0.00033	0.0087	-0.00745	0.00302	0.023
REML	-0.00090	0.00076	0.2416	-0.00903	0.00764	0.2430
FD	-0.00041	0.00445	0.9274	-0.02296	0.01766	0.1999
FDR				-0.32866	0.20646	0.118
$\hat{\rho}_{ML}$	0.282		$\hat{\rho}_{ML}$	0.175		

Chapter 6

Efficiency and Validity Analyses in Mixed Multiple Quantitative Linear Models with Autocorrelated Errors

ABSTRACT

Many estimation procedures for multiple quantitative linear models with autocorrelated errors have been proposed in the literature. The reported studies focused on the parametric modeling of the errors and the efficiency of the procedures for different sample sizes. In a Monte Carlo study, we have studied the efficiency of the Estimated Generalized Least Squares, Maximum Likelihood, Restricted Maximum Likelihood, First Differences, and First-Difference Ratios procedures relative to Ordinary Least Squares. We have also studied the validity of testing procedures derived from the estimation procedures for assessing the significance of the slope when an explanatory variable x_2 is adding to the simple linear regression model, and the validity of testing the overall model with two explanatory variables x_1 and x_2. Efficiency and validity were analyzed in relation to the nature (i.e., fixed, purely random, or autocorrelated) of the explanatory variables, the sample size, and the magnitude and sign of the error autocorrelation parameter in mixed multiple quantitative linear models with AR(1) errors. The performance of the estimation and the testing procedures is illustrated in an example with environmental data collected at the Gault Nature Reserve (Mont-Saint-Hilaire, Quebec, Canada). In conclusion, we recommend the users of regression analysis with time series or spatial data to take the nature of explanatory variables into account and investigate the autocorrelation of the random explanatory variables and the errors, before making their choice of an estimation procedure and a testing procedure.

Key Words: AR(1) errors; Estimated Generalized Least Squares; First Differences; First-Difference Ratios; Maximum Likelihood; Ordinary Least Squares; Random versus fixed explanatory variable; Restricted Maximum Likelihood.

1. INTRODUCTION

Consider a situation in which one wants to explain a response variable such as soil pH measured at equally spaced sampling points on a transect. The position of the sampling points and the altitude at the sampling points are available for explaining the variability of soil pH along the transect. In addition to the simple linear regressions of soil pH on position and of soil pH on altitude, the two potential explanatory variables can be included in a multiple quantitative linear model, either sequentially by including one explanatory variable while the other is already in the model, or overall by including them both at once in the model. In multiple quantitative linear models (Graybill 1983), the explanatory variables may be fixed, such as position on the transect in the example, purely random, or autocorrelated, as altitude at the sampling site in the example. Therefore, we have considered mixed multiple quantitative linear models with two explanatory variables of different types. We have worked in the time domain instead of 1-D space, but our results for time series are readily applicable to 1-D spatial data.

Consider the multiple quantitative linear model with temporal AR(1) errors

$$y = X\beta + \varepsilon, \quad \varepsilon_t = \rho\varepsilon_{t-1} + u_t \quad (t = 1, 2, \ldots, n), \tag{6.1}$$

where y is an $n \times 1$ observable random vector; β is a $q \times 1$ unknown vector to be estimated; X is an $n \times q$ matrix of rank $q < n$; ε is an $n \times 1$ unobservable random vector of errors with zero expected value; $-1 < \rho < 1$; and $u \sim N_n(0, \sigma_u^2 I)$, with I the $n \times n$ identity matrix and σ_u^2 an unknown positive constant. Let the covariance matrix of ε, $\text{Cov}(\varepsilon)$, be denoted by Σ.

The Ordinary Least Squares (OLS) estimator of β in (6.1) is $\hat{\beta}_{\text{OLS}} = (X'X)^{-1}X'y$. Its covariance matrix is $\text{Cov}(\hat{\beta}_{\text{OLS}}) = \sigma^2(X'X)^{-1}$ if $\rho = 0$.

The covariance matrix of ε in (6.1) is

$$\Sigma = \sigma_\varepsilon^2 V = \sigma_\varepsilon^2 \begin{pmatrix} 1 & \rho & \rho^2 & \rho^3 & \cdots & \rho^{n-2} & \rho^{n-1} \\ \rho & 1 & \rho & \rho^2 & \cdots & \rho^{n-3} & \rho^{n-2} \\ \vdots & & & & & & \vdots \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \rho^{n-4} & \cdots & \rho & 1 \end{pmatrix}, \tag{6.2}$$

where $\sigma_\varepsilon^2 = \sigma_u^2/(1-\rho^2)$. The Generalized Least Squares (GLS) estimator requires that ρ be known in (6.2), which is not the case generally in practice. In the GLS procedure, $\hat{\beta}_{\text{GLS}} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y$ and $\text{Cov}(\hat{\beta}_{\text{GLS}}) = (X'\Sigma^{-1}X)^{-1}$, whereas σ_ε^2 is estimated by the error mean square. When unknown, ρ can be estimated by the sample autocorrelation coefficient at lag $1, r(1)$ (Alpargu and Dutilleul, unpublished manuscript) or some other estimator (Beach and Mackinnon 1978, Park and Mitchell 1980), assuming the errors follow an AR(1) process.

If the family of distribution of the errors is known, then the Maximum Likelihood (ML) and Restricted Maximum Likelihood (REML) methods can be applied, conditional on the regressors if random, to estimate to estimate β, σ_u^2, and ρ if it is unknown. The ML estimators of the parameters of model (6.1) are:

$\hat{\beta}_{\text{ML}} = (X'A'AX)^{-1}X'A'Ay$ and $\hat{\sigma}^2_{\text{ML}} = (Ay)'P(Ay)/n$, where $A = (I - \hat{\rho}W)$, $W = (w_{ij})$ is defined as $w_{ij} = 1$ if $j = i - 1$, and 0 otherwise, and $P = I-(AX)\{(AX)'(AX)\}^{-1}(AX)'$; $\hat{\rho}$ minimizes $M^* = \log(Ay)'P(Ay) - (2/n)\log|A|$ (Upton and Fingleton 1985).

Another likelihood-based method, defined as ML performed on linearly transformed data $y^* = By$, such that the distribution of y^* does not depend on β, is called Restricted Maximum Likelihood (REML). It was introduced by Patterson and Thompson (1971) to estimate variance components in the analysis of field experimental data. Tunnicliffe-Wilson (1989) showed that REML copes better than ML when the covariance matrix Σ is close to singularity. The ML and REML methods produce asymptotically similar estimators. When they differ, the REML estimators are superior to the ML estimators (Diggle *et al.* 1996).

In the First Differences (FD) method, the transformation defined by $(I - W)$ is applied to model (6.1) under the assumption that ρ is equal to 1. Thus, the dependency among errors is removed in part or in total, before a model without intercept is fitted to the first differences (Martin 1974). The last estimation method that we have considered is based on first-difference ratios (FDR).

Our objective in this study was threefold: first, to compare the estimation procedures using different efficiency formulas, to show that the conclusions drawn may depend heavily on the measure of efficiency chosen; second, to re-address the question of whether or not inefficient estimators can provide valid test statistics (Fisher 1950, Sundrum 1954); and third, to illustrate the use and performance of the estimation and testing procedures with the environmental data that motivated our study. These data were collected at the Gault Nature Reserve (Mont-Saint-Hilaire, Quebec, Canada).

The definition of relative efficiency is discussed in Section 2, whereas several tests of regression coefficients are reviewed in Section 3. To the best of our knowledge, the FD and FDR procedures had never been used in multiple quantitative linear models before. Therefore, we present them in detail in Section 4. In Section 5, we define our Monte Carlo study. The results are reported and discussed in Section 6. The Mont-Saint-Hilaire example is presented in Section 7. Conclusions are drawn in Section 8.

2. RELATIVE EFFICIENCY

Many estimation procedures have been studied for their validity in relation to the sample size, the error autocorrelation parameter value and the nature, fixed or random, of the explanatory variable in simple quantitative linear models with autocorrelated errors (Rao and Griliches 1969; Martin 1974; Maeshiro 1976; Park and Mitchell 1980; Alpargu and Dutilleul 2001). In simple linear regression, the efficiency of the slope estimators has usually been assessed by using the ratio of mean squared errors calculated from the slope estimates, whereas in multiple linear regression, the ratio of the determinants of the covariance matrices (i.e., generalized variances), the ratio of the traces of the covariance matrices, or the ratio of the mean squared errors, as in simple linear regression, was used.

The importance of the nature of the explanatory variables have not been stressed in multiple linear regression as it has been in simple linear regression. Nevertheless, Maeshiro (1976) briefly mentioned the lack of efficiency of the Cochrane-Orcutt (CO) estimator with respect to OLS in multiple linear regression with two fixed explanatory variables and random errors following an AR(1) process. Furthermore, the author added that results in multiple linear regression were parallel to those in simple linear regression.

Krämer (1980) argued that the OLS estimator of the vector β is almost as efficient as the Prais-Winsten (PW) estimator in simple linear regression when the disturbances are highly correlated. He used the ratio of the traces of the covariance matrices of the PW and OLS estimators a relative efficiency criterion. However, Dielman and Pfaffenberger (1989) commented that if the estimator of the intercept has a large variance compared to that of the slope coefficient estimator in a simple linear regression model with AR(1) errors, the poor performance of the OLS estimator of the slope coefficient may be masked in Krämer's relative efficiency. In their Monte Carlo study, they used the ratio of the variances of the slope coefficient estimators rather than Krämer's relative efficiency, to illustrate the advantage of using the PW estimator over the OLS estimator. In the spatial context, Richardson et al. (1992) studied spatial regression problems for irregularly spaced data points. They used Krämer's relative efficiency formula and the ratio of the variances of the slope coefficient estimators in their example, and reported discrepancies between the two measures of relative efficiency. Those articles dealt with simple linear regression. Hereinafter, we address similar questions by using three measures of relative efficiency to compare estimation procedures in multiple linear regression.

3. TESTS CONCERNING SLOPE COEFFICIENTS

Upton and Fingleton (1985) stated that the classical t- and F-tests are invalid in linear regression models when the errors are autocorrelated, without specifying the nature of the explanatory variables. In two previous studies (Alpargu and Dutilleul, unpublished manuscripts), we provided evidence for the validity of the classical t-test of the slope when the explanatory variable is purely random in a simple linear regression model with AR(1) errors and its lack of validity when the explanatory variable is fixed or follows itself an AR(1) process. We extend this validity analysis here in multiple linear regression.

3.1 TEST WHETHER $\beta_k = 0$ for a given $k = 1$ or 2

Consider the linear time-series regression model

$$y_t = \beta_0 + \beta_1 x_{1t} + \beta_2 x_{2t} + \varepsilon_t \quad (t = 1, 2, \ldots, n), \tag{6.3}$$

where ε_t follows an AR(1) process. One may want to test whether only one explanatory variable may be dropped from the multiple linear regression model (6.3), that is, to test

$$H_0 : \beta_k = 0 \quad \text{versus} \quad H_a : \beta_k \neq 0 \quad \text{for a given } k = 1 \text{ or } 2. \tag{6.4}$$

Equivalently, $H_0 : \beta_1 = 0$ versus $H_a : \beta_1 \neq 0$ can be written as

$$H_0 : y_t = \beta_0 + \beta_2 x_{2t} + \varepsilon_t \quad \text{versus} \quad H_a : y_t = \beta_0 + \beta_1 x_{1t} + \beta_2 x_{2t} + \varepsilon_t. \tag{6.5}$$

A general test statistic is provided by F^*, which is built as the ratio of the regression mean square to the error mean square. This test is known as "partial F-test". If H_0 holds, then F^* follows an F distribution with 1 and $n-3$ degrees of freedom (df). At a theoretical significance level α, the decision will be in favor of H_a if F^* exceeds the $(1-\alpha)$-quantile of the F distribution. A test statistic equivalent to F^* is $t^* = \hat{\beta}_k / s(\hat{\beta}_k)$, where $s(\hat{\beta}_k)$ is the standard error of $\hat{\beta}_k$. If H_0 holds, then t^* follows a t distribution with $n-3$ df. At a theoretical significance level α, the decision will be in favor of H_a if the absolute value of t^* exceeds the $(1-\alpha)$-quantile of the t distribution. Note that $F^* = (t^*)^2$. To test (6.4), other statistics can be used, namely the likelihood-ratio χ^2-statistic and the asymptotic z-statistic in the ML procedure.

3.2 TEST WHETHER $\beta_k = 0$ for $k = 1$ and 2

Sometimes, one is interested in testing whether all the explanatory variables may be dropped from model (6.3), which can be written as

$$H_0 : \beta_1 = \beta_2 = 0 \quad \text{versus} \quad H_a : \text{at least one of the } \beta_k' \text{s (k} = 1, 2) \text{ is not zero} \tag{6.6}$$

The test statistic generally used for this is F^*, except that the number of df of the numerator is 2 instead of 1. This test is known as the "overall F-test" in multiple linear regression. Note that to test (6.6), other statistics can be used, namely the likelihood-ratio χ^2-statistic in the ML procedure.

4. FD and FDR METHODS

We describe in detail below how to define first differences to test (6.4) and (6.6), and how to calculate ratios of first differences to test (6.4) . Let b_{y1} denote the slope estimate of the simple linear regression of y on x_1, b_{y2} that of y on x_2, b_{12} that of x_1 on x_2, and b_{21} that of x_2 on x_1; b_{y1}, b_{y2}, b_{12} and b_{21} are called zero-order regression coefficients. Let $b_{y1.2} = b_1$ be the slope estimate for x_1 and $b_{y2.1} = b_2$ the slope estimate for x_2 in (6.3); $b_{y1.2}$ and $b_{y2.1}$ are called first-order regression coefficients. The slope estimates b_1 and b_2 of the multiple linear regression can be derived from the slope estimates b_{y1}, b_{y2}, b_{12} and b_{21} of the simple linear regressions as follows:

$$b_1 = (b_{y1} - b_{y2}b_{21})/(1 - b_{12}b_{21}) \quad \text{and} \quad b_2 = (b_{y2} - b_{y1}b_{12})/(1 - b_{21}b_{12}).$$

An equivalent way of obtaining b_1 involves the residuals $y_t^\star - b_{y2}x_{2t}^\star$ (i.e., the linear effect of $x_2^\star$ is removed from $y^\star$) and $x_{1t}^\star - b_{12}x_{2t}^\star$ (i.e., the linear effect of $x_2^\star$ is removed from $x_1^\star$), where $y_t^\star = y_t - \bar{y}$, $x_{1t}^\star = x_{1t} - \bar{x}_1$ and $x_{2t}^\star = x_{2t} - \bar{x}_2$ with $\bar{y}$, $\bar{x}_1$ and $\bar{x}_2$ the sample mean of the raw data for y, x_1 and x_2, respectively. More precisely,

$$b_1 = \sum_{t=1}^{n}(y_t^\star - b_{y2}x_{2t}^\star)(x_{1t}^\star - b_{12}x_{2t}^\star) / \sum_{t=1}^{n}(x_{1t}^\star - b_{12}x_{2t}^\star)^2. \tag{6.7}$$

In other words, b_1 is obtained by OLS regression of $y_t^\star - b_{y2}x_{2t}^\star$ on $x_{1t}^\star - b_{12}x_{2t}^\star$. Similarly, b_2 is defined by interchanging the subscripts 1 and 2 in (6.7).

As for FDR, the ratios of $y_t^\star - b_{y2}x_{2t}^\star$ and $x_{1t}^\star - b_{12}x_{2t}^\star$ are calculated and the departure of the mean of these ratios from 0 is assessed by a statistic $t^\star$ with $n-3$ df, to test whether x_1 should be added to the linear regression of y on x_2.

For the overall F-test, the first differences of (6.3) are

$$y_t - y_{t-1} = \hat{\beta}_1(x_{1t} - x_{1(t-1)}) + \hat{\beta}_2(x_{2t} - x_{2(t-1)}) + (\varepsilon_t - \varepsilon_{t-1}) \quad (t = 2, \ldots, n),$$

$\hat{\beta}_1$ and $\hat{\beta}_2$ are estimated by OLS. The statistics $F^\star$ with 2 and $n-3$ df follows.

5. MONTE CARLO STUDY

The model used for simulation was

$$y_t = a + bx_{1t} + cx_{2t} + \varepsilon_t \text{ with } \varepsilon_t = \rho\varepsilon_{t-1} + u_t \quad (t = 1, 2, \ldots, n), \qquad (6.8)$$

where a was fixed at 1; b and c were fixed at 0 under the null hypothesis; the u_ts were i.i.d. $N(0,1)$; and the value of ρ ranged from -0.9 to 0.9 by steps of 0.2, in addition to $\rho = 0$. The generation of autocorrelated errors followed a procedure similar to that of Dutilleul and Legendre (1992). The slope estimates were evaluated for 1000 simulation runs for sample sizes $n = 10, 20, 30, 50$, and 100 for each value of ρ. Three types of explanatory variable were considered: fixed, purely random and AR(1). Only the results for the mixed combinations fixed-purely random, fixed-AR(1) and purely random-AR(1) for x_1 and x_2 and vice versa are reported here.

To test (6.4), the following cases were considered for matrix $X = [1, x_1, x_2]$, where 1 is a column vector of ones:

Case 1.1: x_1 is fixed, that is, $x_1 = (1, 2, \ldots, n)$, and the elements of x_2 are i.i.d. $N(0,1)$ observations.

Case 1.2: Same as Case 1.1, except that the role of x_1 and x_2 is reversed.

Case 2.1: x_1 is fixed and x_2 follows an AR(1) process with same autocorrelation parameter value as the error, that is, $x_{2t} = \rho x_{2(t-1)} + v_t$ $(t = 1, 2, \ldots, n)$, and ε and x_2 are independently distributed.

Case 2.2: Same as Case 2.1, except that the role of x_1 and x_2 is reversed.

Case 3.1: The elements of x_1 are i.i.d. $N(0,1)$ observations, and x_2 follows an AR(1) process that is independent of ε but has same autocorrelation parameter ρ.

Case 3.2: Same as Case 3.1, except that the role of x_1 and x_2 is reversed.

Following Park and Mitchell (1980), the mean squared error (MSE) of a slope estimator was calculated as

$$\text{MSE}(\hat{\beta}) = (1/1000) \sum_{i=1}^{1000} \hat{\beta}_i^2, \tag{6.9}$$

since the theoretical value of the slope parameters was zero in our Monte Carlo study. The efficiency of the estimation procedures relative to OLS was based on the mean squared errors. For example, the relative efficiency of ML procedure was calculated as Eff(ML) = MSE(ML)/MSE(OLS), and ML is said to be more (less) efficient than OLS if Eff(ML) is smaller (greater) than 1 and more (less) efficient than procedure REML, for instance, if Eff(ML) is smaller (greater) than Eff(REML).

Each empirical significance level was evaluated as 0.001 times the number of rejections of the null hypothesis of a zero value for the relevant slope at a theoretical significance level of 5% in 1000 t-tests with $n-3$ df, F-tests with 1 and $n-3$ df, asymptotic z-tests and chi-square tests with 1 df. Strictly speaking, a testing procedure is said to be valid at level α if the probability that it rejects the null hypothesis, when in fact the null hypothesis is true, is less than or equal to α.

To test (6.6), we considered three cases for matrix X:

Case 1: x_1 is fixed and the elements of x_2 are i.i.d. $N(0,1)$ observations.

Case 2: x_1 is fixed and x_2 follows an AR(1) process that is independent of ε but has same autocorrelation parameter value.

Case 3: The elements of x_1 are i.i.d. $N(0,1)$ observations and x_2 follows an AR(1) process that is independent of ε but has same autocorrelation parameter value.

The efficiency of the estimation procedures was evaluated relative to OLS, using the following measures:

1. For individual slope coefficients, the ratio of mean squared errors;
2. For the full model, the ratio of error mean squares (EMS), with EMS$(\cdot)$ = $0.001 \sum_{t=1}^{1000} (y_t - \hat{y}_t)^2$ where $\hat{y}_t$ is the estimate of the response variable at time t,

and .

3. Krämer's efficiency: $[\mathrm{MSE}(\hat{\beta}_{1.}) + \mathrm{MSE}(\hat{\beta}_{2.})]/[\mathrm{MSE}(\hat{\beta}_{1\mathrm{OLS}}) + \mathrm{MSE}(\hat{\beta}_{2\mathrm{OLS}})]$.

The procedure of calculation of the empirical significance levels in testing (6.6) was similar to that used in testing (6.4) (see above). We used our own computer programs written in SAS/IML language (SAS Institute Inc. 1997) to implement the OLS, estimated GLS, ML, FD and FDR procedures. We used the SAS procedure MIXED (SAS Institute Inc. 1997) for REML. The generation of i.i.d. $N(0,1)$ observations was carried out with the random number function RANNOR of SAS (SAS Institute Inc. 1997).

6. RESULTS AND DISCUSSION

The notation used in Tables 6.1-6.6 is self-explanatory. We need simply to mention that the efficiency of the FDR slope estimators was not reported in these tables because FDR was highly inefficient relative to OLS. The following notations were used in Tables 6.7-6.15 where empirical significance levels were reported. 1: Σ_0; β was estimated by OLS, and the classical t-test with $n-3$ df was performed on individual slopes (partial tests) and the classical F-test with 2 and $n-3$ df was used for the full model (overall test). 2: ML_{χ^2}; β was estimated by maximum likelihood, and a likelihood-ratio χ^2-test with 1 df (partial tests) or 2 df (overall test) was performed – the notation ML was used in Tables 6.13-6.15 for the overall test. 3: ML_Z; β was estimated by maximum likelihood, and an asymptotic z-test was performed on individual slopes - this test is restricted to individual slopes. 4: REML; β was estimated by restricted maximum likelihood, and the classical t-test and the likelihood-ratio χ^2-test with 2 df were used in the partial tests and the overall test, respectively. 5: FD; β was estimated on the first differences, and t-tests with $n-3$ df and an F-test with 2 and $n-3$ df were performed. 6: FDR; first-difference ratios were used in estimation, and a t-test with $n-3$ df were performed on individual slopes - this test is restricted to individual slopes.

6.1 PARTIAL TESTS

General comments

FDR is the most inefficient procedure relative to OLS, but it provides the most valid testing procedure overall. A similar result had been observed in simple linear regression (Alpargu and Dutilleul 2001; see also Alpargu and Dutilleul unpublished manuscript).

OLS is more efficient than the other estimation procedures when $-0.1 \leq \rho \leq 0.1$, that is, when the autocorrelation among errors is weak. The lack of efficiency of OLS increases when the autocorrelation among errors increases in magnitude.

When ρ in (6.2) is approximately 1, then the covariance of the errors becomes singular. As announced by Tunnicliffe-Wilson (1989), REML is then much more efficient than ML. The ML and REML procedures tend to have similar efficiencies at large sample sizes and provide the greatest efficiency overall, with some exceptions detailed below.

Specific comments

We considered six cases for the pair of explanatory variables x_1 and x_2, to see whether reversing the order of their entrance in the model, depending on their nature, has an effect on the performance of the estimation and testing procedures. In fact, such an effect is observed. For example, including x_2, fixed, in the model when x_1, purely random or AR(1), is already in the model results in a loss of efficiency for the ML estimator, compared to including x_1, purely random or AR(1), in the model when x_2, fixed, is already in the model (see the results for $\hat{\beta}_{1ML}$ and $\hat{\beta}_{2ML}$). The sample size also has a clear effect on the efficiency of estimation procedures and the validity of testing procedures.

Case 1.1 and Case 1.2: The efficiency of ML, REML and FD relative to OLS is greater in Case 1.1 than in Case 1.2 (Table 6.1). In particular, FD is inefficient for all n in Case 1.2, except when the autocorrelation among errors is very strong; on the other hand, it is efficient for all n except $n = 10$ when $\rho \geq 0.3$ in Case 1.1. When the sample size increases, the efficiency of the procedures relative to OLS increases in Case 1.1, but decreases in Case 1.2. The efficiencies of ML and REML become more similar in both cases when the sample size increases, and are almost equal when $n = 100$.

In Case 1.1 and Case 1.2, the FDR t-test is valid for all combinations of n and ρ (Tables 6.7 and 6.8). The ML χ^2-test starts to be valid when $n = 50$ in Case 1.1 and when $n = 100$ in Case 1.2. On the other hand, the ML asymptotic z-test starts to be valid when $n = 100$ in both cases. The FD t-test satisfies the criterion of strict validity (i.e., empirical significance level ≤ 0.05) when $n = 10, 20$ and 30 for $\rho = 0.9$ in Case 1.1, and for all n when $\rho \leq 0.5$ in Case 1.2. When $\rho < 0$, the classical OLS t-test is more often valid in Case 1.2 than in Case 1.1. When $\rho > 0$, this test is never valid in Case 1.2, and is valid for some combinations of n and ρ in Case 1.1. REML provides the second most valid testing procedure after FDR in Case 1.1, and the third one in Case 1.2 after FDR and FD in this order. REML is thus superior to ML_{χ^2} and ML_Z.

Case 2.1 and Case 2.2: Efficiency results in these cases are similar to those obtained in Case 1.1 and Case 1.2 (Table 6.1). In particular, the differences between the efficiencies in Case 1.1 and Case 2.1 are not large. When x_1, fixed, is in the model, the type of random explanatory variable x_2 [i.e., purely random or AR(1)] to be added to the model does not affect the efficiency of the procedures.

The ML χ^2-test is valid only two times in Case 2.1 and never in Case 2.2 (Tables 6.9 and 6.10). Increasing the sample size does not improve the validity of the REML t-test in Case 2.1. For instance, it is valid for $|\rho| \geq 0.5$ and $\rho = 0.3$ when $n = 30$, but only for $|\rho \geq 0.7$ when $n = 100$. The classical OLS t-test is valid only four times over all values of n and ρ in Case 2.1, whereas it satisfies the condition of strict validity for all $\rho < 0$ in Case 2.2. The FD t-test is valid only two times when $n = 10$ in Case 2.1, but is valid for all n when $\rho \leq 0.3$ and for $n = 10, 20$ and 30 when $\rho = 0.5$ in Case 2.2. FDR is the most valid procedure in both cases. It is valid for $\rho \geq -0.3$ when $n = 10$, for $\rho \geq -0.5$ when $n = 20$ and 30, and for $\rho \geq -0.7$ when $n = 50$ and 100. It is always valid in Case 2.2.

Case 3.1 and Case 3.2: FD is efficient for the same range of error autocorrelation values in both cases (Table 6.5). Case 3.2 provides smaller relative efficiency values than Cases 1.2 and 2.2. The lack of efficiency of OLS relative to ML increases with increasing sample size for all values of ρ in Cases 3.1 and 3.2. For all sample sizes (i.e., $n \geq 20$), the lack of efficiency of OLS relative to REML is important in both cases when the autocorrelation in the errors and the explanatory variable is very strong, whether negative or positive.

The classical OLS t-test is less often valid in Case 3.1 (Table 6.11) than in Case 3.2, where increasing the sample size does not increase the number of valid tests (Table 6.12). FD is valid only once in both cases. On the other hand, the FDR t-test is always valid in Case 3.2, whereas it is almost valid for all values of ρ, except the most negative ones, in Case 3.1. Although the ML and REML provide asymptotically similar estimates, the REML t-test is superior to the ML χ^2-test and the ML asymptotic z-test in both cases.

6.2 OVERALL TEST

Three cases that correspond to mixed combinations of the explanatory variables x_1 and x_2 (Cases 1-3) were considered. Relative efficiencies were calculated using the Error Mean Squares (EMS) and Krämer's formula (Tables 6.2, 6.4 and 6.6). Empirical significance levels are presented in Tables 6.13-6.15.

In all cases, FD is never efficient relative to OLS for any values of n when $\rho \leq 0$, whatever the relative efficiency formula may be. REML is highly inefficient for $\rho = 0.9$ in all cases, particularly in Case 3, when the EMS are compared.

When $\rho \leq 0$, the REML algorithm converged without a problem for all sample sizes in all cases. However, when ρ was moderately to strongly positive, REML always failed to converge at least once in 1000 simulation runs. Case 1 was the worst in this regard. That is a warning message to practitioners who use PROC MIXED of SAS to analyze their data.

Using the EMS to measure efficiency, ML is always efficient relative to OLS. Moreover, the EMS-based relative efficiencies are generally greater than Krämer's efficiencies.

Overall, we observed that different efficiency formulas may lead to different conclusions. For example, REML is very inefficient relative to OLS for $\rho > 0$ in all cases if the EMS are used, but not for Krämer's efficiency. From the comparison of the relative efficiencies of Tables 6.1, 6.3 and 6.5 with those of Tables 6.2, 6.4 and 6.6, it follows that the ratios of MSE of individual slope estimators have more to tell us than the EMS-based and Kramer's measures of relative efficiency, although Krämer's efficiency also involves the MSE. This is in agreement with results reported by Dielman and Pfaffenberger (1989) for the ratios of MSE and Krämer's efficiency. Note that we did not include the MSE of the intercept estimator in the numerator and denominator of Krämer's efficiency in our study.

There is a striking difference between the ML and REML likelihood-ratio χ^2-tests. This difference favors the ML testing procedure that is much more reliable than REML for all values of ρ in Cases 1-3. FD is never valid in Case 3, but provides the most valid test in Cases 1 and 2. In Case 3, ML generally provides an empirical significance level between 0.05 and 0.10 over the range of values of ρ for $n \geq 20$.

6.3 THE MONT-SAINT-HILAIRE EXAMPLE

The data used here for illustration were collected at the Gault Nature Reserve (Mont-Saint-Hilaire, Quebec, Canada) in 1994, on three transect lines denoted 11A, 11C and Cliff. The variable to explain is soil pH, whereas the position

7. CLOSING REMARKS

From our simulation results, it is not possible to draw conclusions that hold for the six cases considered in the partial tests and the three cases in the overall test. In fact, we have shown that in multiple linear regression with AR(1) errors, the efficiency of the estimation procedures and the validity of the derived testing procedures heavily depend on the nature of the explanatory variables, in addition to the sample size and the autocorrelation parameter of the error AR(1) process. We have also shown that the use of different efficiency measures may lead to different conclusions. The FDR procedure which requires no a priori assumption provides a t-test that is generally valid whatever the type of the explanatory variables may be, although the FDR slope estimators are highly inefficient. This result is in agreement with the results of Alpargu and Dutilleul (unpublished manuscript) in simple linear regression. We recommend that the users of regression analysis with time series or spatial data take the nature of explanatory variables into account and investigate the autocorrelation of the random explanatory variables and the errors, before making their choice of an estimation procedure and a testing procedure. The Mont-Saint-Hilaire example provides an illustration with real data. In closing, we note that our study sheds light on new aspects of the problem of efficient estimation and valid testing in multiple linear regression with autocorrelated errors, and we hope the reported results will be useful in future studies of this problem.

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Table 6.1: Efficiency of the three estimation procedures for individual slopes relative to Ordinary Least Squares (OLS) when x_1, fixed, is added to the model of simple linear regression of y on x_2, purely random (see the results for $\hat{\beta}_1$) and when x_2, purely random, is added to the model of simple linear regression of y on x_1, fixed (see the results for $\hat{\beta}_2$), as a function of the sample size n and the error autocorrelation parameter ρ. Note: No result is reported for the Restricted Maximum Likelihood procedure when $n = 10$ because of the too frequent lack of convergence of the maximization algorithm at that sample size.

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
						n=10					
$\hat{\beta}_{1ML}$	0.2584	0.6482	0.8201	1.0153	1.1283	1.0484	1.0672	1.0617	1.0448	0.9932	0.9774
$\hat{\beta}_{2ML}$	0.1807	0.4928	0.7	1.0374	1.3416	1.2881	1.2724	1.2586	1.269	1.2549	1.1353
$\hat{\beta}_{1FD}$	6.8506	4.8504	3.7505	3.0051	2.2304	1.891	1.7798	1.3175	1.0838	0.9203	0.9341
$\hat{\beta}_{2FD}$	2.3996	1.9973	2.0175	1.7176	1.4121	1.3152	1.1793	1.0136	0.8876	0.721	0.5864
						n=20					
$\hat{\beta}_{1ML}$	0.3318	0.7143	0.8808	0.9829	1.0185	1.0291	1.02	1.001	0.9524	0.8831	0.8675
$\hat{\beta}_{2ML}$	0.1165	0.3793	0.6671	0.9555	1.057	1.1531	1.1026	1.031	0.8052	0.551	0.4126
$\hat{\beta}_{1REML}$	0.3261	0.6877	0.8384	1.0153	1.0466	0.829	1.1011	0.5356	0.1783	0.1497	0.0565
$\hat{\beta}_{2REML}$	0.1156	0.4165	0.6506	0.9588	1.0986	0.9382	1.136	0.5213	0.1323	0.0813	0.0254
$\hat{\beta}_{1FD}$	21.3708	12.7891	7.8013	6.0653	4.2105	3.3846	3.2098	2.2953	1.5803	0.9918	0.7985
$\hat{\beta}_{2FD}$	2.9142	2.6368	2.1376	1.9244	1.6368	1.3549	1.2544	0.9962	0.7367	0.5276	0.3596
						n=30					
$\hat{\beta}_{1ML}$	0.3805	0.8333	0.9269	0.9888	1.0068	0.9996	1.0171	1.0072	0.9655	0.9013	0.8452
$\hat{\beta}_{2ML}$	0.1112	0.395	0.6907	0.8509	1.0393	1.0486	1.0169	0.9466	0.7322	0.4926	0.2602
$\hat{\beta}_{1REML}$	0.3612	0.7376	0.7897	0.9268	0.8626	0.9071	0.9174	1.0325	0.4529	0.4221	0.0683
$\hat{\beta}_{2REML}$	0.1088	0.38	0.7077	0.9957	1.0488	0.947	1.1493	0.8839	0.3566	0.1888	0.0194
$\hat{\beta}_{1FD}$	37.1632	20.9608	12.794	8.7169	5.8961	5.4945	4.2897	3.2996	2.1206	1.2943	0.8309
$\hat{\beta}_{2FD}$	3.0689	2.6445	2.3497	2.0725	1.5156	1.3963	1.2838	0.9824	0.7721	0.4766	0.2546
						n=50					
$\hat{\beta}_{1ML}$	0.573	0.8586	0.9233	0.9884	0.9986	1.0006	1	0.9885	0.9521	0.9532	0.8334
$\hat{\beta}_{2ML}$	0.1071	0.3115	0.5993	0.8843	1.0171	1.0155	1.0208	0.8686	0.6395	0.4205	0.2242
$\hat{\beta}_{1REML}$	0.5538	0.8349	0.912	0.9313	0.9771	0.9554	1.0174	1.0572	0.9336	0.8736	0.2996
$\hat{\beta}_{2REML}$	0.1033	0.3477	0.6286	0.802	1.0807	0.995	0.9244	0.8432	0.6768	0.4072	0.0764
$\hat{\beta}_{1FD}$	83.4824	38.7934	23.0585	13.6026	10.3437	8.8209	7.4455	4.7252	3.0485	1.8567	0.8835
$\hat{\beta}_{2FD}$	3.3216	2.9946	2.5805	2.0063	1.7019	1.4878	1.2434	0.9283	0.6483	0.4243	0.2241
						n=100					
$\hat{\beta}_{1ML}$	0.6996	0.8953	0.9722	1.0098	0.9975	1.0031	0.9954	0.988	0.9764	0.9434	0.8844
$\hat{\beta}_{2ML}$	0.1017	0.3215	0.5978	0.8548	0.9974	1.0136	0.9905	0.8444	0.6652	0.3616	0.1521
$\hat{\beta}_{1REML}$	0.6974	0.8655	0.9181	1.0346	0.9949	1.0818	1.0357	0.94	0.9234	0.9664	0.8563
$\hat{\beta}_{2REML}$	0.1037	0.3043	0.6227	0.8187	0.9862	1.0202	1.0165	0.8115	0.6725	0.3701	0.1488
$\hat{\beta}_{1FD}$	263.8322	98.5756	49.419	35.7795	20.3898	18.5191	12.7243	7.9012	5.416	3.3308	1.3691
$\hat{\beta}_{2FD}$	3.5515	2.9552	2.6702	2.0515	1.7029	1.4588	1.2886	0.9313	0.6999	0.363	0.1521

Table 6.3: Same as Table 6.1, except that x_1 is fixed and x_2 follows an AR(1) process.

$\gamma=\rho$	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
						n=10					
$\hat{\beta}_{1ML}$	0.3387	0.7786	0.9196	1.0118	1.093	1.0743	1.0504	1.0402	1.049	1.0217	1.0009
$\hat{\beta}_{2ML}$	0.2656	0.6315	0.8516	1.0546	1.2072	1.2178	1.2539	1.2338	1.257	1.088	1.0215
$\hat{\beta}_{1FD}$	7.5821	5.2214	3.8788	2.8815	2.2165	2.0666	1.598	1.3108	1.0437	0.8268	0.8122
$\hat{\beta}_{2FD}$	1.4071	1.4015	1.5595	1.4723	1.3462	1.2419	1.2432	1.0647	0.81	0.7099	0.5974
						n=20					
$\hat{\beta}_{1ML}$	0.4766	0.7925	0.9092	0.9845	1.0015	1.0342	1.0348	0.9944	0.9769	0.8871	0.8426
$\hat{\beta}_{2ML}$	0.1755	0.4447	0.7606	1.0114	1.0203	1.1237	1.0928	0.999	0.8558	0.5775	0.4674
$\hat{\beta}_{1REML}$	0.4259	0.6844	0.838	0.908	0.691	1.0574	0.8104	0.8155	0.8449	0.407	0.2992
$\hat{\beta}_{2REML}$	0.1857	0.4517	0.7431	1.0381	0.6786	1.0341	0.7917	0.6942	0.5679	0.251	0.1329
$\hat{\beta}_{1FD}$	24.6492	14.7504	9.6215	6.2827	4.3202	3.5792	2.9565	2.243	1.5765	1.0375	0.7935
$\hat{\beta}_{2FD}$	1.2492	1.4116	1.527	1.595	1.5049	1.395	1.2672	0.9519	0.7981	0.4557	0.3414
						n=30					
$\hat{\beta}_{1ML}$	0.4866	0.8241	0.9315	0.9863	1.0074	1.0101	1.0157	0.9921	0.9221	0.8552	0.8161
$\hat{\beta}_{2ML}$	0.1397	0.3865	0.6662	0.8948	1.0902	1.0993	1.08	0.9579	0.688	0.5142	0.2841
$\hat{\beta}_{1REML}$	0.4657	0.8096	0.9133	0.9893	0.9811	0.9487	1.0558	1.0169	0.6507	0.4586	0.3712
$\hat{\beta}_{2REML}$	0.1395	0.398	0.7336	0.9787	1.1742	1.0174	1.0801	0.9486	0.4122	0.2195	0.1132
$\hat{\beta}_{1FD}$	40.9438	21.8049	13.9951	8.796	6.7783	5.0709	4.3296	3.0227	1.9377	1.1703	0.7844
$\hat{\beta}_{2FD}$	1.2474	1.3772	1.6048	1.6479	1.5884	1.3716	1.371	1.0929	0.673	0.4553	0.2344
						n=50					
$\hat{\beta}_{1ML}$	0.6206	0.8429	0.9796	0.9852	0.9987	1.0009	1.0043	0.9746	0.968	0.905	0.801
$\hat{\beta}_{2ML}$	0.1329	0.3717	0.6477	0.8767	1.0337	1.0458	1.0361	0.8915	0.6935	0.4223	0.1924
$\hat{\beta}_{1REML}$	0.6196	0.8825	0.9956	1.0121	0.9639	0.9735	1.0061	0.898	0.9261	1.0055	0.4355
$\hat{\beta}_{2REML}$	0.1348	0.41	0.6396	0.8657	0.9894	1.0668	0.9925	0.8689	0.7254	0.4042	0.1111
$\hat{\beta}_{1FD}$	91.337	43.8366	23.9196	16.4221	10.1213	8.0695	6.7819	4.3892	2.9124	1.6967	0.8831
$\hat{\beta}_{2FD}$	1.1853	1.3567	1.5184	1.6036	1.5353	1.5095	1.2973	1.074	0.7539	0.4272	0.1768
						n=100					
$\hat{\beta}_{1ML}$	0.7911	0.9241	0.9773	0.9988	0.9982	1.0012	0.9993	0.9843	0.9904	0.9386	0.8502
$\hat{\beta}_{2ML}$	0.1271	0.3371	0.5941	0.8609	0.9726	1.0333	1.0123	0.8482	0.6496	0.3731	0.1349
$\hat{\beta}_{1REML}$	0.6333	0.7883	1.0054	0.9477	0.9159	0.8462	0.9	0.9083	0.9111	0.7902	0.7768
$\hat{\beta}_{2REML}$	0.1295	0.3682	0.6755	0.9497	0.9881	1.045	1.0742	0.9288	0.7201	0.3711	0.1129
$\hat{\beta}_{1FD}$	236.335	84.9156	53.0141	29.4193	20.3206	17.0368	12.9598	9.1106	5.8981	3.0175	1.2562
$\hat{\beta}_{2FD}$	1.1305	1.3441	1.5285	1.5775	1.5675	1.5011	1.423	1.0349	0.7587	0.381	0.1291

Table 6.4: Same as Table 6.2, except that x_1 is fixed and x_2 follows an AR(1) process.

$\gamma = \rho$	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
						n=10					
ML	0.2197	0.4428	0.6033	0.7175	0.8024	0.8346	0.8652	0.8728	0.87	0.8712	0.8502
FD	3.0944	2.9204	2.6917	2.4797	2.259	2.1261	2.0406	1.8329	1.6397	1.5154	1.3996
ML	0.267	0.6357	0.8538	1.0525	1.1996	1.2077	1.2335	1.2137	1.2266	1.0745	1.0154
FD	1.5288	1.5123	1.6333	1.5409	1.4044	1.3003	1.2788	1.0904	0.8442	0.7336	0.661
						n=20					
ML	0.2244	0.4737	0.6786	0.8198	0.9129	0.9367	0.9463	0.9242	0.8646	0.7621	0.6472
REML	1.2793	0.9834	0.9525	0.9968	1.0007	1.0046	16.7454	26.5663	134.8557	447.5199	1096.3476
FD	3.4306	3.1594	2.8506	2.5233	2.2009	2.034	1.9021	1.5896	1.3065	1.0388	0.8137
ML	0.1763	0.4465	0.7619	1.0111	1.02	1.1215	1.0912	0.9988	0.862	0.6022	0.5212
REML	0.1863	0.4529	0.7439	1.0362	0.6789	1.0347	0.7922	0.6988	0.5822	0.2634	0.1568
FD	1.3103	1.4782	1.5965	1.6621	1.5589	1.449	1.315	1.0002	0.8381	0.5021	0.4062
						n=30					
ML	0.2231	0.4931	0.7006	0.8609	0.9445	0.9629	0.9628	0.9272	0.8336	0.6929	0.5109
REML	1.1914	0.9955	0.9553	0.9867	0.9986	0.9874	1.0309	1.0887	30.2469	170.001	694.3247
FD	3.5399	3.2328	2.8937	2.5358	2.1775	2.0433	1.8864	1.5242	1.1981	0.8959	0.5971
ML	0.1401	0.3874	0.6673	0.8954	1.0893	1.0982	1.0791	0.9586	0.694	0.5284	0.3316
REML	0.1398	0.3988	0.7343	0.9787	1.172	1.0166	1.0798	0.9499	0.4184	0.2295	0.1362
FD	1.2856	1.4186	1.6567	1.6958	1.6455	1.4151	1.4112	1.1317	0.7057	0.4852	0.2834
						n=50					
ML	0.2103	0.5055	0.719	0.8781	0.9605	0.9784	0.9775	0.9222	0.8033	0.6214	0.3857
REML	1.1195	1.0114	0.9852	0.9807	0.9855	1.0043	1.0221	1.0469	1.0951	12.201	446.9904
FD	3.651	3.2883	2.9437	2.5675	2.2184	2.0256	1.8551	1.4676	1.1166	0.7687	0.4286
ML	0.1331	0.3721	0.6481	0.877	1.0336	1.0456	1.0359	0.8921	0.6967	0.4303	0.2187
REML	0.1349	0.4103	0.6401	0.8661	0.9893	1.0664	0.9925	0.8692	0.7277	0.4143	0.1251
FD	1.2092	1.3875	1.5473	1.637	1.5668	1.5403	1.3254	1.1	0.7785	0.4484	0.2073
						n=100					
ML	0.2001	0.5073	0.7377	0.8951	0.9777	0.9893	0.9837	0.9167	0.7754	0.5704	0.2845
REML	1.0841	1.0147	1.0084	1.014	1.0007	1.0137	1.0095	1.0203	1.0492	1.1455	70.8423
FD	3.725	3.3454	2.9642	2.5817	2.1967	2.003	1.8166	1.4325	1.0535	0.6867	0.307
ML	0.1272	0.3372	0.5942	0.861	0.9726	1.0333	1.0123	0.8485	0.6505	0.3755	0.1432
REML	0.1296	0.3682	0.6756	0.9496	0.988	1.0448	1.0739	0.9288	0.7207	0.3728	0.1206
FD	1.1426	1.3563	1.5444	1.5941	1.5841	1.5188	1.439	1.0506	0.7728	0.3919	0.1421

Table 6.5: Same as Table 6.1, except that x_1 is purely random and x_2 follows an AR(1) process.

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
						n=10					
$\hat{\beta}_{1ML}$	0.266	0.6141	0.8724	1.0617	1.3313	1.3155	1.2504	1.2449	1.1406	0.9155	0.7579
$\hat{\beta}_{2ML}$	0.2468	0.6004	0.9623	1.0978	1.3027	1.2484	1.2379	1.2083	0.9617	0.792	0.6307
$\hat{\beta}_{1FD}$	2.7378	2.365	2.343	1.9177	1.7169	1.3123	1.5688	1.0852	0.9449	0.6455	0.4775
$\hat{\beta}_{2FD}$	1.4076	1.5144	1.6274	1.7742	1.666	1.5435	1.5028	1.1111	0.8144	0.6109	0.3822
						n=20					
$\hat{\beta}_{1ML}$	0.1685	0.4351	0.684	0.883	1.0922	1.1352	1.0868	0.9131	0.6948	0.5006	0.3123
$\hat{\beta}_{2ML}$	0.176	0.4187	0.7405	0.9159	1.1373	1.0873	1.0899	0.9542	0.7672	0.527	0.2766
$\hat{\beta}_{1REML}$	0.1558	0.4471	0.6622	0.8715	1.1131	1.2002	1.0511	0.9078	0.6715	0.176	0.1055
$\hat{\beta}_{2REML}$	0.1788	0.395	0.7537	1.0019	1.2116	1.1649	1.2052	0.9136	0.6958	0.2151	0.1016
$\hat{\beta}_{1FD}$	3.323	2.8035	2.5244	2.1429	1.6825	1.4889	1.3034	0.9476	0.7065	0.4519	0.2906
$\hat{\beta}_{2FD}$	1.2986	1.4569	1.597	1.7323	1.6386	1.4677	1.521	1.1341	0.8431	0.5153	0.2314
						n=30					
$\hat{\beta}_{1ML}$	0.1419	0.378	0.7018	0.9357	1.0538	1.0674	1.0592	0.9279	0.6773	0.4201	0.242
$\hat{\beta}_{2ML}$	0.1429	0.37	0.6637	0.8845	1.0377	1.0434	1.0129	0.9353	0.7195	0.454	0.2049
$\hat{\beta}_{1REML}$	0.1379	0.4111	0.751	0.9185	1.0945	1.0193	0.9653	0.9287	0.736	0.3875	0.1544
$\hat{\beta}_{2REML}$	0.1612	0.3657	0.6412	0.7928	1.2236	1.0734	0.9504	0.9452	0.7591	0.4256	0.1216
$\hat{\beta}_{1FD}$	3.4783	3.1187	2.4649	2.1296	1.7289	1.4775	1.3156	1.0286	0.7244	0.41	0.2336
$\hat{\beta}_{2FD}$	1.2355	1.421	1.59	1.6954	1.5977	1.4602	1.2546	1.1016	0.8173	0.4495	0.185
						n=50					
$\hat{\beta}_{1ML}$	0.1189	0.376	0.6726	0.8492	1.0312	1.0413	1.0254	0.8594	0.6787	0.3832	0.1859
$\hat{\beta}_{2ML}$	0.1378	0.3437	0.6157	0.8903	1.0003	1.0391	1.0294	0.928	0.6546	0.3958	0.1546
$\hat{\beta}_{1REML}$	0.1246	0.3404	0.6905	0.7403	0.9947	0.8918	0.9243	0.7857	0.5907	0.3602	0.1321
$\hat{\beta}_{2REML}$	0.1297	0.3346	0.6262	0.8397	0.9446	0.9921	0.9608	0.8944	0.6885	0.4177	0.1212
$\hat{\beta}_{1FD}$	3.4483	3.0067	2.5867	2.0189	1.6976	1.4236	1.314	0.9421	0.692	0.3798	0.1841
$\hat{\beta}_{2FD}$	1.1784	1.4206	1.6017	1.5553	1.6021	1.5323	1.3585	1.1441	0.7755	0.4226	0.1459
						n=100					
$\hat{\beta}_{1ML}$	0.1023	0.3591	0.6174	0.846	1.0081	1.0138	0.9882	0.8714	0.6439	0.3697	0.1302
$\hat{\beta}_{2ML}$	0.115	0.364	0.6112	0.8649	1.0015	1.0203	0.997	0.888	0.6012	0.3599	0.113
$\hat{\beta}_{1REML}$	0.1034	0.3829	0.6482	0.8058	0.994	1.0058	0.9789	0.8468	0.6994	0.3551	0.1303
$\hat{\beta}_{2REML}$	0.1227	0.3779	0.6914	0.9051	1.1837	1.1004	1.0025	1.0017	0.5878	0.3405	0.1214
$\hat{\beta}_{1FD}$	3.6061	3.1845	2.7006	2.0903	1.7379	1.4825	1.2302	0.9609	0.6913	0.3762	0.1297
$\hat{\beta}_{2FD}$	1.1418	1.3462	1.5572	1.6037	1.6425	1.5349	1.3882	1.1451	0.6791	0.3847	0.1123

Table 6.6: Same as Table 6.2, except that x_1 is purely random and x_2 follows an AR(1) process.

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
						n=10					
ML	0.2374	0.4745	0.6303	0.7526	0.8203	0.8417	0.8501	0.8435	0.7929	0.7154	0.6579
FD	2.7529	2.5637	2.366	2.124	1.8887	1.7965	1.7354	1.4513	1.1776	0.976	0.8278
ML	0.2547	0.6066	0.9166	1.0789	1.3174	1.2836	1.2438	1.2261	1.0313	0.8417	0.675
FD	1.9579	1.9001	1.9913	1.8492	1.6922	1.4224	1.5341	1.0985	0.8652	0.6248	0.4154
						n=20					
ML	0.2361	0.5015	0.6966	0.8533	0.9248	0.9426	0.9402	0.9015	0.8024	0.6371	0.4567
REML	1.2494	1.0178	0.9898	0.9817	1.0279	0.9884	1.0458	18651.487	5433.67	513224.57	3345879.4
FD	3.2527	2.9718	2.6751	2.3362	2.0114	1.909	1.7522	1.4143	1.1094	0.7872	0.5114
ML	0.173	0.4251	0.7132	0.8988	1.1139	1.1117	1.0883	0.9342	0.734	0.5159	0.2889
REML	0.1697	0.4153	0.7095	0.9342	1.1605	1.1829	1.1273	0.9108	0.6847	0.1986	0.1029
FD	2.1025	1.9809	2.0454	1.9455	1.6614	1.4785	1.4109	1.0432	0.7805	0.4887	0.2519
						n=30					
ML	0.2144	0.4966	0.7194	0.8645	0.9482	0.9616	0.9586	0.8983	0.7862	0.6049	0.3642
REML	1.1922	1.0351	0.9891	0.9826	1.0125	1.0195	1.0237	1.0438	1.1314	75163.995	1886917.1
FD	3.4347	3.1256	2.7891	2.4541	2.115	1.9448	1.7421	1.4056	1.0733	0.7323	0.3995
ML	0.1425	0.3732	0.6804	0.908	1.0457	1.0559	1.036	0.9317	0.7008	0.4403	0.2161
REML	0.152	0.3838	0.6894	0.8503	1.1594	1.0451	0.9578	0.9373	0.7488	0.4103	0.1315
FD	2.1168	2.0978	1.9741	1.8942	1.6629	1.4692	1.285	1.0665	0.7761	0.4336	0.1997
						n=50					
ML	0.209	0.501	0.7311	0.8859	0.9672	0.9794	0.9724	0.9089	0.7758	0.5616	0.2947
REML	1.1656	1.0104	1.0077	0.9925	1.0002	1.0055	1.0019	1.0288	1.0809	1.1529	301756.29
FD	3.5811	3.2375	2.8686	2.5089	2.1326	1.9657	1.8014	1.4101	1.0476	0.6709	0.3171
ML	0.1306	0.3571	0.6401	0.8694	1.0149	1.0402	1.0275	0.8946	0.6657	0.3907	0.1643
REML	0.1277	0.337	0.6538	0.7893	0.9683	0.941	0.9431	0.8415	0.6436	0.3945	0.1246
FD	2.0478	2.0765	2.0247	1.7905	1.6471	1.477	1.3368	1.0458	0.7372	0.4054	0.1578
						n=100					
ML	0.2032	0.5108	0.7415	0.8992	0.9754	0.9899	0.9825	0.9086	0.7622	0.5374	0.2395
REML	1.0852	1.0138	1.0032	1.0012	1.0014	1.0059	1.012	1.0201	1.0256	1.0506	26898.094
FD	3.6844	3.3113	2.933	2.551	2.1827	1.9824	1.7902	1.4029	1.021	0.6368	0.2544
ML	0.1102	0.3621	0.6139	0.8557	1.0049	1.017	0.9928	0.8797	0.6187	0.3636	0.1185
REML	0.1154	0.3798	0.6725	0.8569	1.0853	1.0531	0.9912	0.9243	0.6337	0.346	0.1243
FD	2.0691	2.0408	2.0583	1.8397	1.692	1.5087	1.3121	1.053	0.6841	0.3815	0.1179

Table 6.8: Same as Table 6.7, except that the roles of x_1 and x_2 are reversed.

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
						n=10					
Σ_o	0.007	0.008	0.017	0.014	0.028	0.055	0.055	0.111	0.153	0.29	0.399
ML_{χ^2}	0.128	0.122	0.137	0.142	0.157	0.169	0.171	0.214	0.233	0.319	0.395
ML_Z	0.164	0.169	0.187	0.208	0.23	0.26	0.252	0.303	0.329	0.44	0.52
FD	0.009	0.002	0.001	0.003	0.002	0.004	0.005	0.011	0.016	0.055	0.099
FDR	0.005	0.008	0.003	0.001	0.005	0.003	0.006	0.009	0.007	0.01	0.02
						n=20					
Σ_o	0	0.002	0.005	0.011	0.038	0.058	0.07	0.13	0.206	0.395	0.598
ML_{χ^2}	0.076	0.082	0.091	0.088	0.102	0.103	0.102	0.115	0.115	0.205	0.329
ML_Z	0.095	0.101	0.111	0.116	0.142	0.139	0.136	0.169	0.171	0.282	0.452
REML	0	0	0.001	0.004	0.011	0.015	0.019	0.02	0.028	0.078	0.011
FD	0.001	0.003	0	0.001	0.003	0.004	0.001	0.011	0.016	0.061	0.177
FDR	0.008	0.012	0.009	0.006	0.005	0.009	0.006	0.011	0.008	0.01	0.034
						n=30					
Σ_o	0	0	0.005	0.013	0.037	0.048	0.068	0.132	0.246	0.388	0.588
ML_{χ^2}	0.059	0.072	0.073	0.074	0.088	0.076	0.087	0.079	0.106	0.132	0.239
ML_Z	0.074	0.085	0.092	0.087	0.114	0.098	0.106	0.112	0.158	0.191	0.342
REML	0	0	0.006	0.013	0.021	0.028	0.031	0.039	0.048	0.077	0.018
FD	0.001	0.001	0.001	0	0	0.003	0.005	0.01	0.026	0.064	0.237
FDR	0.008	0.008	0.013	0.01	0.005	0.013	0.005	0.009	0.016	0.011	0.027
						n=50					
Σ_o	0	0	0.003	0.009	0.035	0.05	0.067	0.166	0.264	0.449	0.63
ML_{χ^2}	0.051	0.054	0.064	0.075	0.073	0.075	0.053	0.084	0.089	0.127	0.173
ML_Z	0.062	0.063	0.073	0.087	0.081	0.09	0.07	0.101	0.114	0.181	0.28
REML	0	0.002	0.014	0.027	0.036	0.042	0.041	0.048	0.052	0.074	0.139
FD	0.002	0.004	0.001	0	0.002	0.002	0.003	0.008	0.032	0.114	0.271
FDR	0.011	0.017	0.015	0.016	0.026	0.018	0.016	0.013	0.014	0.016	0.029
						n=100					
Σ_o	0	0	0.001	0.003	0.029	0.037	0.068	0.152	0.249	0.404	0.659
ML_{χ^2}	0.041	0.051	0.054	0.045	0.063	0.044	0.065	0.069	0.073	0.074	0.132
ML_Z	0.042	0.054	0.055	0.049	0.068	0.051	0.071	0.08	0.082	0.102	0.188
REML	0	0.003	0.029	0.046	0.05	0.053	0.055	0.047	0.055	0.069	0.113
FD	0.002	0.002	0.002	0	0.004	0.002	0.006	0.012	0.04	0.141	0.372
FDR	0.019	0.012	0.015	0.01	0.018	0.016	0.019	0.018	0.017	0.02	0.031

Table 6.9: Same as Table 6.7, except that x_1 is fixed and x_2 follows an AR(1) process.

$\gamma=\rho$	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
						n=10					
Σ_o	0.393	0.16	0.104	0.057	0.05	0.054	0.053	0.066	0.084	0.1	0.116
ML_{χ^2}	0.16	0.154	0.158	0.144	0.152	0.172	0.168	0.175	0.185	0.221	0.214
ML_Z	0.273	0.234	0.241	0.219	0.226	0.248	0.245	0.276	0.273	0.316	0.292
FD	0.451	0.238	0.181	0.125	0.088	0.089	0.075	0.085	0.049	0.06	0.042
FDR	0.291	0.1	0.06	0.034	0.028	0.022	0.02	0.022	0.016	0.016	0.011
						n=20					
Σ_o	0.458	0.209	0.115	0.066	0.057	0.055	0.058	0.074	0.1	0.185	0.271
ML_{χ^2}	0.104	0.095	0.094	0.079	0.096	0.09	0.091	0.098	0.104	0.118	0.15
ML_Z	0.152	0.124	0.126	0.121	0.131	0.12	0.118	0.134	0.158	0.16	0.187
REML	0.022	0.039	0.041	0.052	0.055	0.038	0.042	0.042	0.043	0.037	0.032
FD	0.507	0.293	0.199	0.154	0.116	0.094	0.08	0.06	0.066	0.056	0.056
FDR	0.203	0.073	0.033	0.043	0.03	0.026	0.02	0.02	0.018	0.023	0.021
						n=30					
Σ_o	0.507	0.229	0.098	0.058	0.042	0.051	0.043	0.062	0.109	0.206	0.332
ML_{χ^2}	0.077	0.068	0.062	0.056	0.062	0.083	0.076	0.07	0.079	0.089	0.094
ML_Z	0.103	0.087	0.083	0.077	0.083	0.102	0.098	0.094	0.109	0.119	0.119
REML	0.018	0.025	0.041	0.054	0.06	0.064	0.064	0.042	0.03	0.026	0.018
FD	0.552	0.304	0.184	0.13	0.105	0.102	0.083	0.076	0.052	0.053	0.054
FDR	0.176	0.052	0.035	0.032	0.026	0.016	0.018	0.024	0.017	0.012	0.011
						n=50					
Σ_o	0.492	0.235	0.116	0.076	0.058	0.044	0.057	0.061	0.121	0.215	0.422
ML_{χ^2}	0.064	0.049	0.062	0.063	0.066	0.067	0.069	0.054	0.065	0.067	0.073
ML_Z	0.075	0.063	0.068	0.076	0.081	0.069	0.078	0.064	0.078	0.078	0.087
REML	0.009	0.021	0.039	0.057	0.052	0.052	0.05	0.048	0.045	0.019	0.013
FD	0.527	0.3	0.22	0.16	0.114	0.111	0.096	0.078	0.062	0.057	0.055
FDR	0.107	0.038	0.035	0.027	0.021	0.021	0.016	0.021	0.017	0.023	0.023
						n=100					
Σ_o	0.548	0.249	0.124	0.082	0.058	0.058	0.053	0.082	0.114	0.267	0.474
ML_{χ^2}	0.063	0.052	0.055	0.054	0.057	0.065	0.062	0.061	0.05	0.07	0.055
ML_Z	0.065	0.061	0.06	0.057	0.064	0.072	0.069	0.066	0.055	0.075	0.062
REML	0.002	0.023	0.056	0.06	0.062	0.063	0.062	0.061	0.051	0.019	0
FD	0.569	0.316	0.219	0.169	0.131	0.11	0.11	0.081	0.064	0.072	0.049
FDR	0.063	0.044	0.036	0.027	0.023	0.034	0.021	0.023	0.019	0.025	0.025

Table 6.10: Same as Table 6.9, except that the roles of x_1 and x_2 are reversed.

$\gamma=\rho$	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
						n=10					
Σ_o	0.004	0.009	0.013	0.026	0.03	0.045	0.075	0.127	0.166	0.281	0.374
ML_{χ^2}	0.11	0.137	0.123	0.144	0.167	0.167	0.204	0.223	0.265	0.352	0.415
ML_Z	0.148	0.183	0.175	0.201	0.243	0.235	0.28	0.314	0.357	0.465	0.507
FD	0	0	0	0.001	0.001	0.004	0.01	0.02	0.026	0.067	0.148
FDR	0	0.001	0.001	0.002	0.003	0.002	0.001	0.004	0.012	0.013	0.035
						n=20					
Σ_o	0	0.002	0.006	0.016	0.03	0.035	0.069	0.125	0.223	0.397	0.537
ML_{χ^2}	0.075	0.074	0.069	0.07	0.102	0.091	0.096	0.116	0.142	0.233	0.321
ML_Z	0.089	0.095	0.096	0.098	0.133	0.125	0.138	0.167	0.21	0.332	0.419
REML	0	01	0.001	0.004	0.01	0.016	0.014	0.023	0.019	0.059	0.108
FD	0	0	0	0	0.001	0.007	0.003	0.014	0.029	0.104	0.249
FDR	0.006	0.002	0.002	0.002	0.005	0.008	0.01	0.012	0.007	0.025	0.033
						n=30					
Σ_o	0	0	0.002	0.006	0.03	0.049	0.062	0.138	0.217	0.374	0.578
ML_{χ^2}	0.055	0.068	0.064	0.064	0.073	0.075	0.076	0.094	0.088	0.14	0.259
ML_Z	0.066	0.08	0.085	0.08	0.094	0.097	0.112	0.13	0.142	0.202	0.361
REML	0	0.001	0.004	0.012	0.024	0.026	0.031	0.045	0.069	0.087	0.132
FD	0	0	0	0	0.003	0.001	0.004	0.013	0.042	0.119	0.309
FDR	0.003	0.004	0.009	0.009	0.013	0.007	0.007	0.011	0.016	0.017	0.038
						n=50					
Σ_o	0	0	0.002	0.006	0.032	0.05	0.066	0.152	0.256	0.405	0.632
ML_{χ^2}	0.058	0.06	0.058	0.051	0.065	0.072	0.07	0.074	0.083	0.104	0.209
ML_Z	0.061	0.061	0.067	0.065	0.082	0.088	0.089	0.093	0.121	0.144	0.296
REML	0	0.002	0.014	0.029	0.038	0.041	0.043	0.045	0.053	0.078	0.134
FD	0	0	0	0.002	0	0.004	0.004	0.013	0.053	0.131	0.373
FDR	0.007	0.01	0.012	0.015	0.015	0.014	0.013	0.02	0.016	0.016	0.033
						n=100					
Σ_o	0	0	0	0.006	0.031	0.061	0.079	0.148	0.237	0.408	0.657
ML_{χ^2}	0.064	0.057	0.05	0.067	0.065	0.069	0.062	0.066	0.064	0.088	0.15
ML_Z	0.066	0.061	0.056	0.072	0.068	0.077	0.067	0.077	0.079	0.115	0.212
REML	0	0.002	0.022	0.037	0.04	0.04	0.041	0.042	0.046	0.061	0.103
FD	0	0	0	0	0.001	0.001	0.003	0.025	0.079	0.194	0.428
FDR	0.02	0.006	0.023	0.022	0.012	0.015	0.013	0.015	0.021	0.028	0.026

Table 6.12: Same as Table 6.11, except that the roles of x_1 and x_2 are reversed.

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
						n=10					
Σ_o	0.054	0.049	0.046	0.049	0.041	0.05	0.058	0.055	0.046	0.043	0.056
ML_{χ^2}	0.135	0.142	0.163	0.144	0.172	0.145	0.168	0.166	0.135	0.127	0.124
ML_Z	0.192	0.207	0.248	0.229	0.271	0.23	0.274	0.257	0.221	0.192	0.183
FD	0.151	0.133	0.12	0.117	0.101	0.091	0.114	0.084	0.055	0.052	0.053
FDR	0.031	0.035	0.033	0.027	0.021	0.017	0.019	0.016	0.021	0.019	0.015
						n=20					
Σ_o	0.047	0.05	0.055	0.039	0.044	0.042	0.055	0.046	0.041	0.044	0.047
ML_{χ^2}	0.078	0.08	0.094	0.082	0.079	0.083	0.098	0.079	0.081	0.07	0.06
ML_Z	0.094	0.106	0.126	0.113	0.112	0.122	0.126	0.117	0.1	0.085	0.082
REML	0.004	0.007	0.022	0.038	0.051	0.055	0.041	0.037	0.021	0.02	0.069
FD	0.156	0.152	0.144	0.122	0.099	0.093	0.108	0.099	0.07	0.061	0.048
FDR	0.034	0.035	0.026	0.022	0.017	0.024	0.024	0.024	0.023	0.018	0.018
						n=30					
Σ_o	0.064	0.049	0.032	0.056	0.041	0.045	0.039	0.045	0.046	0.056	0.059
ML_{χ^2}	0.065	0.064	0.065	0.083	0.058	0.063	0.064	0.072	0.07	0.063	0.066
ML_Z	0.078	0.081	0.076	0.1	0.084	0.088	0.094	0.085	0.09	0.077	0.079
REML	0.001	0.001	0.024	0.034	0.057	0.066	0.062	0.03	0.024	0.012	0.036
FD	0.16	0.149	0.126	0.137	0.105	0.106	0.095	0.077	0.082	0.067	0.06
FDR	0.018	0.03	0.017	0.032	0.021	0.02	0.022	0.012	0.022	0.015	0.017
						n=50					
Σ_o	0.047	0.047	0.053	0.061	0.052	0.051	0.05	0.047	0.044	0.062	0.042
ML_{χ^2}	0.063	0.053	0.057	0.066	0.062	0.061	0.069	0.067	0.054	0.058	0.051
ML_Z	0.074	0.058	0.065	0.077	0.076	0.081	0.072	0.077	0.061	0.064	0.057
REML	0	0.006	0.035	0.059	0.059	0.055	0.053	0.037	0.013	0.002	0.004
FD	0.141	0.137	0.153	0.119	0.133	0.108	0.086	0.095	0.083	0.061	0.055
FDR	0.026	0.026	0.013	0.028	0.026	0.021	0.02	0.016	0.022	0.015	0.013
						n=100					
Σ_o	0.05	0.05	0.043	0.049	0.051	0.05	0.062	0.046	0.047	0.052	0.044
ML_{χ^2}	0.06	0.06	0.046	0.062	0.06	0.055	0.066	0.05	0.053	0.061	0.056
ML_Z	0.062	0.062	0.049	0.065	0.064	0.065	0.07	0.056	0.058	0.063	0.061
REML	0	0.004	0.037	0.051	0.055	0.049	0.051	0.054	0.034	0.007	0.002
FD	0.163	0.15	0.127	0.122	0.106	0.114	0.128	0.083	0.084	0.069	0.059
FDR	0.021	0.016	0.024	0.024	0.019	0.025	0.025	0.023	0.017	0.013	0.02

Table 6.13: Empirical significance level of the testing procedures for a theoretical significance level of 5% in the multiple linear regression of y on x_1, fixed, and x_2, purely random, as a function of the sample size n and the error autocorrelation parameter ρ. Empirical significance levels were computed from 1000 simulation runs. Note: No result is reported for the likelihood-ratio χ^2-test of the Restricted Maximum Likelihood procedure when $n = 10$ because of the too frequent lack of convergence of the maximization algorithm at that sample size.

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
n=10											
Σ_o	0.027	0.031	0.024	0.041	0.038	0.046	0.054	0.093	0.125	0.246	0.353
ML	0.138	0.139	0.158	0.173	0.183	0.178	0.185	0.202	0.222	0.29	0.358
FD	0.075	0.056	0.039	0.048	0.035	0.037	0.029	0.023	0.027	0.028	0.012
n=20											
Σ_o	0.011	0.017	0.022	0.024	0.044	0.051	0.067	0.103	0.179	0.338	0.542
ML	0.076	0.079	0.111	0.103	0.102	0.101	0.109	0.109	0.113	0.148	0.264
REML	0.97	0.8	0.465	0.182	0.049	0.034	0.067	0.185	0.377	0.64	0.072
FD	0.059	0.058	0.058	0.061	0.062	0.038	0.039	0.028	0.033	0.029	0.019
n=30											
Σ_o	0.018	0.015	0.018	0.014	0.04	0.055	0.065	0.107	0.206	0.339	0.556
ML	0.071	0.078	0.072	0.07	0.086	0.094	0.084	0.089	0.101	0.108	0.191
REML	1	0.966	0.723	0.306	0.075	0.044	0.062	0.284	0.639	0.887	0.097
FD	0.082	0.061	0.042	0.042	0.035	0.052	0.035	0.044	0.035	0.018	0.018
n=50											
Σ_o	0.019	0.016	0.02	0.032	0.035	0.049	0.062	0.132	0.231	0.389	0.609
ML	0.058	0.053	0.051	0.08	0.072	0.064	0.061	0.064	0.072	0.107	0.157
REML	1	0.994	0.911	0.506	0.096	0.047	0.111	0.5	0.895	0.988	1
FD	0.075	0.075	0.072	0.057	0.039	0.052	0.042	0.029	0.024	0.021	0.021
n=100											
Σ_o	0.015	0.02	0.015	0.014	0.036	0.044	0.063	0.124	0.198	0.347	0.623
ML	0.049	0.053	0.052	0.045	0.063	0.049	0.055	0.064	0.074	0.064	0.116
REML	1	1	1	0.848	0.166	0.041	0.15	0.822	0.997	1	1
FD	0.076	0.086	0.053	0.05	0.047	0.039	0.036	0.037	0.028	0.019	0.018

Table 6.14: Same as Table 6.13, except that x_1 is fixed and x_2 follows an AR(1) process.

$\gamma = \rho$	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
						n=10					
Σ_o	0.293	0.086	0.052	0.041	0.035	0.052	0.073	0.11	0.185	0.323	0.443
ML	0.156	0.154	0.161	0.18	0.181	0.194	0.197	0.25	0.27	0.385	0.452
FD	0.366	0.143	0.096	0.06	0.034	0.033	0.031	0.03	0.016	0.019	0.017
						n=20					
Σ_o	0.364	0.122	0.051	0.038	0.045	0.049	0.069	0.121	0.223	0.432	0.658
ML	0.092	0.099	0.078	0.099	0.108	0.098	0.106	0.123	0.136	0.214	0.324
REML	0.95	0.796	0.482	0.203	0.065	0.045	0.072	0.144	0.408	0.603	0.724
FD	0.401	0.183	0.119	0.069	0.062	0.045	0.031	0.025	0.027	0.016	0.017
						n=30					
Σ_o	0.397	0.144	0.053	0.025	0.034	0.051	0.052	0.139	0.234	0.459	0.717
ML	0.07	0.064	0.065	0.07	0.065	0.088	0.077	0.085	0.095	0.14	0.231
REML	0.99	0.933	0.707	0.294	0.066	0.038	0.056	0.271	0.643	0.866	0.96
FD	0.44	0.203	0.106	0.069	0.049	0.044	0.04	0.03	0.016	0.019	0.018
						n=50					
Σ_o	0.396	0.143	0.053	0.039	0.042	0.052	0.08	0.135	0.263	0.458	0.773
ML	0.067	0.056	0.061	0.057	0.063	0.069	0.076	0.076	0.084	0.099	0.186
REML	1	0.995	0.919	0.529	0.092	0.047	0.109	0.503	0.892	0.986	0.999
FD	0.428	0.202	0.118	0.08	0.049	0.046	0.039	0.027	0.018	0.018	0.015
						n=100					
Σ_o	0.428	0.166	0.053	0.042	0.044	0.059	0.071	0.136	0.24	0.498	0.793
ML	0.068	0.058	0.046	0.063	0.065	0.072	0.061	0.058	0.06	0.083	0.125
REML	1	1	0.999	0.833	0.166	0.036	0.174	0.818	0.999	1	1
FD	0.453	0.221	0.113	0.078	0.052	0.052	0.042	0.026	0.016	0.021	0.015

Table 6.15: Same as Table 13, except that x_1 is purely random and x_2 follows an AR(1) process.

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
						n=10					
Σ_o	0.336	0.157	0.069	0.059	0.045	0.062	0.055	0.063	0.078	0.131	0.159
ML	0.171	0.179	0.174	0.164	0.17	0.174	0.177	0.195	0.175	0.194	0.182
FD	0.483	0.303	0.22	0.15	0.128	0.136	0.119	0.11	0.088	0.067	0.059
						n=20					
Σ_o	0.397	0.187	0.095	0.068	0.036	0.052	0.05	0.062	0.083	0.134	0.293
ML	0.103	0.094	0.1	0.092	0.078	0.097	0.096	0.094	0.08	0.077	0.075
REML	0.925	0.767	0.511	0.195	0.065	0.043	0.069	0.18	0.431	0.714	0.799
FD	0.53	0.343	0.231	0.2	0.132	0.134	0.115	0.113	0.071	0.074	0.058
						n=30					
Σ_o	0.394	0.178	0.092	0.065	0.043	0.048	0.056	0.051	0.084	0.178	0.372
ML	0.065	0.078	0.066	0.08	0.071	0.08	0.086	0.063	0.067	0.076	0.093
REML	0.995	0.946	0.694	0.291	0.069	0.051	0.077	0.282	0.672	0.891	0.944
FD	0.505	0.323	0.218	0.199	0.127	0.12	0.136	0.085	0.09	0.061	0.081
						n=50					
Σ_o	0.439	0.204	0.086	0.07	0.047	0.051	0.056	0.063	0.091	0.19	0.4
ML	0.063	0.065	0.064	0.066	0.06	0.063	0.072	0.057	0.066	0.059	0.059
REML	1	0.988	0.938	0.506	0.084	0.043	0.096	0.509	0.912	0.994	0.996
FD	0.53	0.335	0.233	0.186	0.142	0.133	0.122	0.107	0.087	0.07	0.057
						n=100					
Σ_o	0.469	0.209	0.092	0.061	0.046	0.053	0.062	0.068	0.105	0.204	0.474
ML	0.052	0.052	0.057	0.06	0.05	0.061	0.066	0.068	0.062	0.05	0.064
REML	1	1	0.999	0.82	0.156	0.044	0.172	0.822	0.997	1	0.998
FD	0.551	0.332	0.234	0.187	0.148	0.138	0.121	0.106	0.086	0.068	0.069

Table 6.16: Stepwise and multiple linear regressions in the Mont-Saint-Hilaire example. In Regression A, x_2 (altitude) is added to the model of simple linear regression of y (soil pH) on x_1 (position on the transect). The roles of x_1 and x_2 are reversed in Regression B. In Regression C, there is no explanatory variable under the null hypothesis against position and altitude are in the model under the alternative hypothesis. Note: ML denotes the ML asymptotic z-test. See the text for other notations.

	Regression A			Regression B			Regression C
Transect Line 11A							
	$\hat{\beta}_2$	$\hat{\sigma}(\hat{\beta}_2)$	p-val	$\hat{\beta}_1$	$\hat{\sigma}(\hat{\beta}_1)$	p-val	p-val
Σ_o	0.00224	0.00609	0.7141	-0.00125	0.00057	0.0341	0.0009
ML	0.00089	0.00749	0.9053	-0.0011	0.00071	0.1194	0.0318
REML	-0.00695	0.01549	0.6592	-0.00032	0.00149	0.8295	1
FD	-0.01889	0.02219	0.3989	0.001	0.00233	0.6711	0.6983
FDR	0.06952	0.08854	0.4363	-0.00619	0.00869	0.4799	
$\hat{\rho}_{ML}$	0.28	$\hat{\rho}_{REML}$	0.39				
Transect Line 11C							
Σ_o	-0.00856	0.00848	0.3182	0.00211	0.00102	0.0442	0.0026
ML	-0.00716	0.01011	0.4791	0.00196	0.00122	0.108	0.0303
REML	-0.00427	0.01026	0.6819	0.00162	0.00124	0.2068	0.0357
FD	0.00374	0.02895	0.8976	0.00055	0.00367	0.8819	0.9916
FDR	0.22988	0.28582	0.4253	0.02252	0.05661	0.6925	
$\hat{\rho}_{ML}$	0.24	$\hat{\rho}_{REML}$	0.33				
Transect Line Cliff							
Σ_o	-0.00948	0.00255	0.0005	-0.00069	0.00025	0.007	0.0006
ML	-0.00947	0.00266	0.0004	-0.0007	0.00026	0.0064	0.0032
REML	-0.01151	0.00792	0.1529	-0.00089	0.00078	0.2548	1
FD	-0.02356	0.01795	0.1957	-0.00282	0.00395	0.4785	0.4288
FDR	2.8405	2.06857	0.1762	0.00437	0.01426	0.7607	
$\hat{\rho}_{ML}$	0.077	$\hat{\rho}_{REML}$	0.16				

Chapter 7

Conclusions

The efficiency of estimation procedures and the validity of testing procedures in simple and multiple quantitative linear models with autocorrelated errors have been studied in this thesis. The efficiency results were discussed in terms of the mean square error of the individual slope estimators or the error mean square of the full model or Krämer's formula, relative to OLS or GLS, depending on the context. In the Monte Carlo studies, it was assumed that the random errors ε followed an AR(1) process. The importance of the nature of the explanatory variable x was stressed by considering three situations: x is fixed and trended; x is purely random; and x follows an AR(1) process with an autocorrelation parameter of the same value as that of the error process. We have also provided advice on the use of PROC MIXED of SAS.

The reported results have clearly shown that the more efficient of two estimators does not necessarily provide a more valid test of significance of the parameter of interest. In fact, FDR is highly inefficient relative to OLS, but it generally provides a valid testing procedure for most combinations of sample size n and error autocorrelation parameter ρ, whatever the type of explanatory variable(s) in simple and stepwise linear regressions may be.

In simple linear regression, GLS was the most efficient for all values of n

and ρ considered and all three types of x, but this estimation procedure is not useful in practice because it requires the complete knowledge of the covariance matrix of the errors. When n was sufficiently large (i.e., $n \geq 50$), the ML and REML procedures provided the second most efficient slope estimator after GLS. In general, the increase of the sample size does help the estimation procedures to improve their efficiency relative to OLS when x is purely random or follows an AR(1) process, but not when x is fixed and trended. Six estimated GLS procedures were considered. The first two assumed that the stationary AR(1) autocovariance structure of the errors was known but ρ had to be estimated, and the other four did not make any a priori assumption about the covariance matrix of the errors. In general, the efficiency of the first two estimated GLS procedures is close to that of the ML procedure. When x is fixed and trended, the relative efficiencies of the six estimated GLS procedures are very close.

The efficiency of two-stage estimation procedures and the validity of the derived testing procedures were studied in Chapter 3, for small to large sample sizes, negative and positive autocorrelation of the errors and the three types of explanatory variable. A proof of Anderson (1971) led us to consider the sample autocorrelation coefficient at lag 1, r(1), in two original two-stage estimation procedures. These were shown to be efficient for small to moderate values of ρ, for any sample size n and all three types of x. The corresponding testing procedures are valid or close to validity, except when x is fixed and trended, the sample size is small and the autocorrelation among errors is positive.

In Chapter 4, the validity of the classical t-test of the slope and 31 other testing procedures was studied when x is purely random and the errors follow an AR(1) process. Most of the testing procedures were shown to be valid or close to validity for most combinations of n and ρ. Box's epsilon of $y_t = 1 + x_t + \varepsilon_t$ was closer to 1 than Box's epsilon of the errors, but not close enough to satisfy the circularity condition. On the other hand, classical sample size and effective sample size were equal in this case, and this equality was retained as being the explanation for our validity results. Following this study, we strongly recommend that the users of simple linear regression with time series or spatial data investigate the autocorrelation of random explanatory variables first, before neglecting the classical t-test of the slope.

Chapter 5 was a follow-up of Chapter 4, with x fixed and trended versus x random and following an AR(1) process. Contrary to the purely random x case, invalidity tends to be the rule here, especially when x is trended and $\rho > 0$ and when x follows an AR(1) process for most values of ρ. We discussed our results in terms of Box's epsilon and effective sample size, and completed our discussion with graphics. For either type of x, the FDR t-test with $n - 2$ df was shown to be the most valid, before the REML t-test.

In Chapter 6, the efficiency of the OLS, ML, REML, FD and FDR estimation procedures and the validity of the derived testing procedures were studied in a quantitative linear model with two explanatory variables and AR(1) errors. The importance of the nature of regressors for the performance of the procedures was stressed again. In stepwise linear regression with two explanatory variables, the FDR t-test was found to be the most valid. In the example of application with environmental data, the slope estimates and their significance changed, sometimes drastically, with the procedure. The results of the FD t-test in this application confirmed some lack of power of that test with fixed explanatory variables and should motivate further investigation.

I hope that the results of this thesis will be helpful to the users of quantitative linear models with autocorrelated errors and will inspire future studies (e.g., power analysis) on the subject.

Appendix: Extra Tables

Extra Table for Chapter 4

Table 1: Empirical significance level of the testing procedures derived from 31 estimation procedures for a theoretical significance level of 5% when x is purely random, as a function of the sample size n and the autocorrelation parameter ρ. The empirical significance levels reported were obtained from 1000 simulation runs. See the text for other notations.

						n=30					
ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
Σ_o	0.05	0.037	0.029	0.039	0.052	0.062	0.045	0.05	0.049	0.044	0.037
Σ_ρ	0	0.007	0.018	0.033	0.051	0.062	0.044	0.049	0.033	0.023	0.006
$\Sigma_{\beta 1}$	0	0.008	0.018	0.026	0.057	0.073	0.051	0.05	0.031	0.018	0.004
$\Sigma_{\beta 2}$	0	0.007	0.018	0.028	0.058	0.079	0.052	0.054	0.029	0.017	0.004
$\hat{\Sigma}_{13}$	0.031	0.058	0.083	0.087	0.107	0.127	0.113	0.125	0.096	0.087	0.054
$\hat{\Sigma}_{14}$	0.016	0.023	0.044	0.065	0.067	0.082	0.066	0.079	0.073	0.047	0.019
$\hat{\Sigma}_{23}$	0.024	0.069	0.097	0.108	0.13	0.161	0.127	0.143	0.122	0.089	0.041
$\hat{\Sigma}_{24}$	0.015	0.025	0.044	0.068	0.062	0.08	0.068	0.085	0.071	0.055	0.02
$\Sigma_{o\rho}$	0.041	0.036	0.028	0.039	0.049	0.062	0.045	0.051	0.052	0.06	0.121
$\Sigma_{o\beta 1}$	0.035	0.037	0.029	0.036	0.05	0.059	0.045	0.053	0.048	0.052	0.05
$\hat{\Sigma}_{o14}$	0.048	0.035	0.022	0.038	0.053	0.061	0.047	0.05	0.051	0.045	0.035
ML_{χ^2}	0.062	0.063	0.054	0.062	0.057	0.09	0.055	0.079	0.059	0.067	0.049
ML_Z	0.077	0.078	0.069	0.075	0.074	0.108	0.081	0.098	0.078	0.077	0.061
REML	0	0.001	0.021	0.041	0.056	0.063	0.052	0.047	0.024	0.009	0.03
FD	0.145	0.13	0.126	0.114	0.112	0.112	0.099	0.095	0.076	0.082	0.051
FDR	0.039	0.033	0.022	0.029	0.02	0.031	0.019	0.018	0.02	0.022	0.023
$\Sigma_{\rho M}$	0.01	0.036	0.029	0.041	0.044	0.059	0.047	0.055	0.041	0.039	0.015
$\Sigma_{\beta 1M}$	0.04	0.039	0.029	0.04	0.043	0.057	0.047	0.056	0.043	0.048	0.036
$\Sigma_{\beta 2M}$	0.017	0.036	0.027	0.04	0.043	0.057	0.047	0.056	0.042	0.047	0.03
$\hat{\Sigma}_{13M}$	0.049	0.032	0.026	0.037	0.05	0.058	0.043	0.048	0.045	0.039	0.032
$\hat{\Sigma}_{14M}$	0.05	0.036	0.029	0.039	0.052	0.062	0.044	0.049	0.049	0.042	0.034
$\hat{\Sigma}_{23M}$	0.053	0.038	0.03	0.039	0.042	0.054	0.045	0.055	0.041	0.043	0.037
$\hat{\Sigma}_{24M}$	0.056	0.046	0.031	0.041	0.044	0.059	0.046	0.056	0.045	0.054	0.042
$\hat{\Sigma}_{CL3}$	0.053	0.046	0.032	0.041	0.044	0.058	0.047	0.057	0.044	0.054	0.045
$\hat{\Sigma}_{CL4}$	0.056	0.048	0.032	0.042	0.044	0.059	0.047	0.056	0.045	0.055	0.044
$\hat{\Sigma}_{DU3}$	0.052	0.046	0.032	0.041	0.044	0.058	0.047	0.057	0.043	0.054	0.042
$\hat{\Sigma}_{DU4}$	0.056	0.047	0.031	0.041	0.044	0.059	0.047	0.056	0.045	0.055	0.044
$\hat{\Sigma}_{HY3}$	0.053	0.048	0.032	0.041	0.044	0.058	0.047	0.057	0.046	0.054	0.047
$\hat{\Sigma}_{HY4}$	0.056	0.048	0.032	0.042	0.044	0.059	0.047	0.057	0.045	0.055	0.044
$\hat{\Sigma}_{C3}$	0.059	0.05	0.033	0.041	0.043	0.057	0.047	0.057	0.044	0.055	0.049
$\hat{\Sigma}_{C4}$	0.056	0.048	0.032	0.042	0.044	0.059	0.047	0.056	0.045	0.055	0.044

Extra Tables for Chapter 5

Table 2: Empirical significance level of the testing procedures derived from 31 estimation procedures for a theoretical significance level of 5% when x is fix, as a function of the sample size n and the autocorrelation parameter ρ. The empirical significance levels reported were obtained from 1000 simulation runs. See the text for other notations.

						n=30					
ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
Σ_o	0	0.001	0	0.01	0.035	0.049	0.073	0.127	0.242	0.392	0.613
Σ_ρ	0	0.014	0.026	0.033	0.047	0.049	0.049	0.048	0.035	0.024	0.011
$\Sigma_{\hat\beta 1}$	0	0.01	0.023	0.037	0.061	0.071	0.076	0.089	0.105	0.138	0.238
$\Sigma_{\hat\beta 2}$	0	0.01	0.023	0.037	0.062	0.071	0.076	0.089	0.103	0.133	0.212
$\hat\Sigma_{13}$	0.011	0.05	0.072	0.102	0.137	0.164	0.186	0.206	0.254	0.355	0.551
$\hat\Sigma_{14}$	0.003	0.028	0.07	0.094	0.077	0.073	0.111	0.149	0.222	0.337	0.562
$\hat\Sigma_{23}$	0.012	0.05	0.07	0.103	0.127	0.157	0.188	0.208	0.252	0.35	0.548
$\hat\Sigma_{24}$	0.003	0.024	0.066	0.098	0.076	0.076	0.11	0.15	0.221	0.329	0.561
$\Sigma_{o\rho}$	0.086	0.053	0.044	0.04	0.047	0.049	0.05	0.059	0.063	0.089	0.237
$\Sigma_{o\hat\beta 1}$	0.032	0.039	0.046	0.055	0.069	0.076	0.08	0.095	0.126	0.185	0.375
$\hat\Sigma_{o14}$	0.004	0.015	0.049	0.071	0.064	0.068	0.104	0.145	0.22	0.337	0.565
ML_{χ^2}	0.059	0.055	0.059	0.063	0.071	0.077	0.072	0.085	0.102	0.127	0.234
ML_Z	0.065	0.064	0.072	0.079	0.089	0.097	0.103	0.115	0.149	0.192	0.357
REML	0	0.002	0.005	0.014	0.024	0.029	0.031	0.044	0.048	0.091	0.139
FD	0	0	0	0	0	0	0	0	0	0	0
$\Sigma_{\rho M}$	0	0	0	0.009	0.035	0.051	0.072	0.123	0.231	0.361	0.557
$\Sigma_{\hat\beta 1M}$	0	0	0	0.01	0.035	0.049	0.073	0.125	0.235	0.376	0.591
$\Sigma_{\hat\beta 2M}$	0	0	0	0.01	0.035	0.049	0.073	0.125	0.235	0.376	0.59
$\hat\Sigma_{13M}$	0	0	0	0.01	0.035	0.046	0.07	0.121	0.237	0.387	0.608
$\hat\Sigma_{14M}$	0	0	0	0.01	0.035	0.049	0.073	0.126	0.238	0.39	0.612
$\hat\Sigma_{23M}$	0	0	0	0.01	0.035	0.048	0.07	0.119	0.236	0.387	0.603
$\hat\Sigma_{24M}$	0	0	0	0.01	0.035	0.051	0.073	0.124	0.239	0.388	0.611
$\hat\Sigma_{C3}$	0	0.001	0.002	0.012	0.038	0.06	0.082	0.128	0.239	0.376	0.586
$\hat\Sigma_{C4}$	0	0.002	0.001	0.011	0.037	0.053	0.072	0.125	0.234	0.373	0.593

Table 3: Table 2 (continued).

						n=100					
Σ_o	0	0	0.001	0.01	0.026	0.048	0.065	0.157	0.243	0.435	0.672
Σ_ρ	0	0.005	0.027	0.06	0.051	0.048	0.045	0.044	0.037	0.01	0.002
$\Sigma_{\beta 1}$	0	0.004	0.025	0.056	0.055	0.053	0.048	0.057	0.059	0.05	0.062
$\Sigma_{\beta 2}$	0	0.004	0.025	0.056	0.055	0.053	0.048	0.057	0.058	0.05	0.059
$\hat{\Sigma}_{13}$	0.009	0.034	0.059	0.087	0.102	0.116	0.136	0.184	0.236	0.384	0.616
$\hat{\Sigma}_{14}$	0	0.013	0.07	0.107	0.064	0.075	0.089	0.122	0.163	0.328	0.597
$\hat{\Sigma}_{23}$	0.009	0.033	0.061	0.086	0.1	0.116	0.133	0.184	0.235	0.385	0.618
$\hat{\Sigma}_{24}$	0	0.013	0.072	0.107	0.064	0.076	0.092	0.121	0.163	0.331	0.596
$\Sigma_{o\rho}$	0.075	0.043	0.06	0.063	0.051	0.048	0.046	0.055	0.067	0.073	0.114
$\Sigma_{o\beta 1}$	0.036	0.041	0.055	0.062	0.054	0.053	0.05	0.069	0.087	0.111	0.211
$\hat{\Sigma}_{o14}$	0	0.005	0.066	0.096	0.056	0.072	0.082	0.126	0.166	0.329	0.603
ML_{χ^2}	0.055	0.051	0.064	0.065	0.056	0.053	0.048	0.062	0.076	0.088	0.13
ML_Z	0.057	0.053	0.068	0.067	0.062	0.065	0.06	0.07	0.091	0.115	0.187
REML	0	0.001	0.028	0.041	0.047	0.046	0.044	0.046	0.058	0.064	0.11
FD	0	0	0	0	0	0	0	0	0	0	0
$\Sigma_{\rho M}$	0	0	0.001	0.01	0.025	0.05	0.066	0.156	0.241	0.428	0.643
$\Sigma_{\beta 1M}$	0	0	0.001	0.01	0.025	0.05	0.066	0.156	0.241	0.43	0.652
$\Sigma_{\beta 2M}$	0	0	0.001	0.01	0.025	0.05	0.066	0.156	0.241	0.43	0.651
$\hat{\Sigma}_{13M}$	0	0	0.001	0.01	0.025	0.047	0.065	0.157	0.241	0.431	0.669
$\hat{\Sigma}_{14M}$	0	0	0.001	0.01	0.025	0.048	0.064	0.157	0.241	0.434	0.671
$\hat{\Sigma}_{23M}$	0	0	0.001	0.01	0.024	0.048	0.066	0.157	0.24	0.434	0.669
$\hat{\Sigma}_{24M}$	0	0	0.001	0.01	0.024	0.05	0.065	0.157	0.241	0.438	0.67
$\hat{\Sigma}_{C3}$	0	0	0.001	0.013	0.025	0.052	0.068	0.156	0.24	0.427	0.652
$\hat{\Sigma}_{C4}$	0	0	0.001	0.014	0.025	0.05	0.067	0.154	0.24	0.426	0.65

Table 4: Empirical significance levels of the 31 testing procedures when x follows an AR(1) process for a theoretical significance level of 5%, as a function of the sample size, n, and the common value of the autocorrelation parameters. The autocorrelation parameter of x (i.e., γ) was fixed at the same value as that of the errors (i.e., ρ). Empirical significance levels were computed from 1000 simulation runs. See the text for other notations.

n=30

$\gamma=\rho$	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
Σ_o	0.477	0.195	0.092	0.057	0.047	0.049	0.046	0.081	0.113	0.208	0.392
Σ_ρ	0.003	0.012	0.017	0.033	0.042	0.049	0.044	0.049	0.035	0.017	0.006
$\Sigma_{\beta 1}$	0.041	0.024	0.025	0.04	0.058	0.058	0.051	0.065	0.055	0.049	0.058
$\Sigma_{\beta 2}$	0.029	0.022	0.02	0.04	0.062	0.058	0.052	0.065	0.052	0.041	0.042
$\hat{\Sigma}_{13}$	0.323	0.153	0.098	0.089	0.103	0.106	0.102	0.127	0.141	0.172	0.301
$\hat{\Sigma}_{14}$	0.365	0.14	0.105	0.072	0.072	0.059	0.061	0.098	0.128	0.156	0.308
$\hat{\Sigma}_{23}$	0.334	0.147	0.107	0.126	0.137	0.143	0.136	0.159	0.146	0.169	0.29
$\hat{\Sigma}_{24}$	0.394	0.15	0.099	0.075	0.074	0.067	0.063	0.108	0.109	0.154	0.326
$\Sigma_{o\rho}$	0.073	0.044	0.042	0.037	0.045	0.049	0.046	0.062	0.06	0.062	0.156
$\Sigma_{o\beta 1}$	0.161	0.075	0.05	0.04	0.045	0.05	0.048	0.067	0.08	0.101	0.183
$\hat{\Sigma}_{o14}$	0.392	0.147	0.076	0.059	0.051	0.049	0.049	0.077	0.102	0.165	0.328
ML_{χ^2}	0.069	0.057	0.05	0.048	0.066	0.071	0.071	0.078	0.067	0.075	0.076
ML_Z	0.107	0.074	0.066	0.066	0.089	0.097	0.084	0.097	0.085	0.1	0.106
REML	0.018	0.029	0.04	0.051	0.059	0.062	0.069	0.058	0.036	0.014	0.041
FD	0.52	0.291	0.159	0.131	0.111	0.111	0.085	0.076	0.057	0.066	0.051
FDR	0.161	0.045	0.033	0.025	0.03	0.019	0.029	0.017	0.017	0.021	0.01
$\Sigma_{\rho M}$	0.351	0.183	0.097	0.056	0.044	0.049	0.047	0.084	0.11	0.194	0.318
$\Sigma_{\beta 1M}$	0.439	0.191	0.1	0.055	0.044	0.047	0.047	0.084	0.111	0.211	0.394
$\Sigma_{\beta 2M}$	0.4	0.183	0.099	0.055	0.044	0.047	0.047	0.084	0.11	0.207	0.383
$\hat{\Sigma}_{13M}$	0.475	0.19	0.09	0.057	0.046	0.045	0.045	0.076	0.11	0.201	0.382
$\hat{\Sigma}_{14M}$	0.477	0.193	0.09	0.056	0.047	0.049	0.046	0.081	0.112	0.207	0.39
$\hat{\Sigma}_{23M}$	0.497	0.205	0.096	0.055	0.044	0.045	0.044	0.079	0.101	0.206	0.401
$\hat{\Sigma}_{24M}$	0.504	0.212	0.104	0.056	0.044	0.049	0.047	0.085	0.112	0.229	0.428
$\hat{\Sigma}_{CL3}$	0.409	0.182	0.1	0.055	0.043	0.047	0.049	0.081	0.111	0.207	0.383
$\hat{\Sigma}_{CL4}$	0.451	0.197	0.104	0.056	0.044	0.049	0.047	0.085	0.111	0.213	0.404
$\hat{\Sigma}_{DU3}$	0.409	0.18	0.1	0.055	0.043	0.047	0.048	0.078	0.11	0.206	0.377
$\hat{\Sigma}_{DU4}$	0.451	0.197	0.104	0.056	0.044	0.048	0.047	0.085	0.11	0.21	0.398
$\hat{\Sigma}_{HY3}$	0.425	0.191	0.103	0.055	0.044	0.05	0.048	0.085	0.111	0.212	0.397
$\hat{\Sigma}_{HY4}$	0.461	0.201	0.104	0.056	0.044	0.05	0.047	0.085	0.112	0.215	0.411
$\hat{\Sigma}_{C3}$	0.437	0.199	0.103	0.056	0.043	0.049	0.048	0.082	0.112	0.214	0.398
$\hat{\Sigma}_{C4}$	0.465	0.202	0.104	0.056	0.044	0.05	0.047	0.085	0.113	0.218	0.414

Table 5: Table 4 (continued).

	n=100										
$\gamma = \rho$	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
Σ_o	0.544	0.234	0.112	0.087	0.06	0.054	0.057	0.061	0.125	0.225	0.508
Σ_ρ	0	0.006	0.021	0.047	0.055	0.054	0.057	0.042	0.036	0.01	0
$\Sigma_{\hat{\beta}1}$	0.004	0.007	0.022	0.051	0.058	0.061	0.058	0.042	0.038	0.014	0.004
$\Sigma_{\hat{\beta}2}$	0.001	0.007	0.025	0.052	0.058	0.063	0.059	0.041	0.037	0.012	0.002
$\hat{\Sigma}_{13}$	0.397	0.178	0.112	0.113	0.113	0.114	0.102	0.102	0.138	0.171	0.39
$\hat{\Sigma}_{14}$	0.388	0.141	0.096	0.089	0.083	0.084	0.07	0.066	0.109	0.143	0.374
$\hat{\Sigma}_{23}$	0.396	0.176	0.108	0.125	0.123	0.129	0.126	0.128	0.14	0.171	0.394
$\hat{\Sigma}_{24}$	0.39	0.143	0.109	0.093	0.083	0.085	0.073	0.068	0.114	0.142	0.381
$\Sigma_{o\rho}$	0.062	0.043	0.042	0.063	0.058	0.054	0.055	0.048	0.048	0.048	0.081
$\Sigma_{o\hat{\beta}1}$	0.097	0.055	0.045	0.066	0.058	0.055	0.054	0.048	0.055	0.062	0.142
$\hat{\Sigma}_{o14}$	0.432	0.179	0.072	0.07	0.06	0.055	0.056	0.052	0.081	0.175	0.428
ML_{χ^2}	0.057	0.058	0.053	0.058	0.058	0.064	0.052	0.052	0.057	0.048	0.049
ML_Z	0.063	0.058	0.059	0.062	0.063	0.071	0.061	0.061	0.063	0.058	0.06
REML	0.001	0.02	0.056	0.051	0.063	0.063	0.051	0.058	0.043	0.021	0.001
FD	0.57	0.308	0.217	0.181	0.129	0.113	0.104	0.082	0.079	0.061	0.046
FDR	0.066	0.04	0.036	0.021	0.02	0.031	0.028	0.029	0.021	0.02	0.028
$\Sigma_{\rho M}$	0.487	0.243	0.125	0.089	0.055	0.056	0.054	0.063	0.12	0.239	0.468
$\Sigma_{\hat{\beta}1M}$	0.497	0.242	0.124	0.089	0.055	0.056	0.054	0.063	0.121	0.241	0.488
$\Sigma_{\hat{\beta}2M}$	0.49	0.242	0.124	0.089	0.055	0.056	0.054	0.063	0.121	0.239	0.484
$\hat{\Sigma}_{13M}$	0.543	0.234	0.108	0.087	0.06	0.054	0.057	0.061	0.125	0.224	0.508
$\hat{\Sigma}_{14M}$	0.544	0.234	0.11	0.087	0.06	0.054	0.057	0.061	0.124	0.225	0.508
$\hat{\Sigma}_{23M}$	0.551	0.24	0.116	0.089	0.056	0.056	0.054	0.064	0.116	0.236	0.514
$\hat{\Sigma}_{24M}$	0.553	0.252	0.126	0.089	0.055	0.056	0.054	0.063	0.122	0.25	0.515
$\hat{\Sigma}_{CL3}$	0.504	0.24	0.123	0.089	0.055	0.056	0.054	0.063	0.12	0.239	0.491
$\hat{\Sigma}_{CL4}$	0.507	0.242	0.124	0.089	0.055	0.056	0.054	0.063	0.121	0.241	0.496
$\hat{\Sigma}_{DU3}$	0.504	0.24	0.123	0.088	0.055	0.056	0.054	0.063	0.119	0.239	0.489
$\hat{\Sigma}_{DU4}$	0.508	0.242	0.124	0.089	0.055	0.056	0.054	0.063	0.121	0.241	0.495
$\hat{\Sigma}_{HY3}$	0.506	0.241	0.123	0.089	0.055	0.056	0.054	0.063	0.119	0.24	0.493
$\hat{\Sigma}_{HY4}$	0.509	0.243	0.125	0.089	0.055	0.056	0.054	0.063	0.121	0.241	0.496
$\hat{\Sigma}_{C3}$	0.506	0.241	0.123	0.089	0.055	0.056	0.054	0.063	0.12	0.241	0.493
$\hat{\Sigma}_{C4}$	0.511	0.244	0.126	0.089	0.055	0.056	0.054	0.063	0.121	0.242	0.496

Table 7: Same as Table 6, except that the efficiency of the slope estimators relative to OLS is computed using (1) the error mean squares $\hat{\sigma}^2$ (the first two lines) and (2) Krämer's efficiency (the last two lines).

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
						n=10					
ML	0.1318	0.3396	0.5325	0.6589	0.7657	0.8157	0.8393	0.8841	0.904	0.9095	0.9068
FD	3.5272	3.292	3.0124	2.8041	2.5691	2.4151	2.3073	2.1446	1.9626	1.8651	1.7922
ML	0.413	0.7426	0.9237	0.9993	1.0498	1.0605	1.0366	1.057	1.03	1.0211	1.0135
FD	4.3984	3.3723	2.7595	2.0397	1.6534	1.4817	1.3886	1.1307	0.987	0.9414	0.8911
						n=20					
ML	0.1624	0.4202	0.6319	0.7946	0.9014	0.9319	0.9453	0.9467	0.9019	0.8217	0.7456
FD	3.652	3.3375	3.0168	2.6889	2.3615	2.2023	2.0533	1.7539	1.49	1.2069	1.0303
ML	0.4441	0.7953	0.8979	0.9628	1.0091	1.0094	1.0186	1	0.9688	0.9149	0.9056
FD	13.5747	8.5641	5.9364	4.1015	2.7958	2.3937	2.0811	1.6269	1.1203	0.9094	0.8362
						n=30					
ML	0.1733	0.452	0.6707	0.839	0.9305	0.9613	0.9675	0.9399	0.8571	0.7222	0.59
FD	3.6983	3.3555	3.0199	2.6488	2.3153	2.1225	1.9755	1.6118	1.2908	0.9731	0.7282
ML	0.5122	0.8162	0.9489	0.9786	1.0027	1.007	1.0197	0.9938	0.9641	0.9134	0.8462
FD	25.4041	14.5885	7.5838	5.5144	3.836	3.4419	3.1522	2.0402	1.5135	1.029	0.7734
						n=50					
ML	0.1816	0.4744	0.704	0.8616	0.9554	0.9758	0.9808	0.9297	0.8181	0.6556	0.4444
FD	3.7331	3.3713	3.0028	2.6481	2.2781	2.0893	1.9187	1.5279	1.1668	0.8332	0.5119
ML	0.6108	0.8625	0.9341	0.9881	1.0067	0.9968	1.0051	0.9921	0.9692	0.9013	0.8341
FD	51.4486	23.8335	15.4792	9.3325	6.1621	5.0531	4.0345	3.0909	2.1702	1.3122	0.8137
						n=100					
ML	0.1853	0.4895	0.7271	0.8876	0.9722	0.9895	0.9849	0.9239	0.7863	0.5829	0.3143
FD	3.7651	3.3872	3.002	2.6187	2.2458	2.0341	1.8361	1.4674	1.0819	0.711	0.3452
ML	0.7287	0.9106	0.9693	0.9971	1.001	0.9979	1.0021	0.9975	0.9825	0.9405	0.8636
FD	142.3968	57.4838	28.4576	17.6619	10.3696	10.7409	8.8133	5.6729	3.6271	1.8984	0.9749

Table 8: Same as Table 6, except that both x_1 and x_2 are purely random.

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
						n=10					
$\hat{\beta}_{1ML}$	0.1729	0.4771	0.8757	1.1306	1.1571	1.3388	1.2378	1.3655	1.0602	0.7478	0.6475
$\hat{\beta}_{2ML}$	0.2163	0.4993	0.8327	1.049	1.209	1.2804	1.2161	1.1834	1.0294	0.8772	0.7038
$\hat{\beta}_{1FD}$	2.7065	2.6517	2.1887	1.9843	1.6792	1.5376	1.2897	1.1535	0.8845	0.5591	0.4134
$\hat{\beta}_{2FD}$	3.0253	2.5559	2.2963	1.9789	1.5829	1.5356	1.4194	1.0502	0.8017	0.6132	0.3702
						n=20					
$\hat{\beta}_{1ML}$	0.1367	0.3956	0.6581	0.9751	1.0713	1.1178	1.0765	0.9494	0.7435	0.5192	0.2362
$\hat{\beta}_{2ML}$	0.1154	0.3966	0.6525	0.9249	1.0662	1.1424	1.1104	0.9713	0.7866	0.5263	0.2552
$\hat{\beta}_{1FD}$	3.3255	2.5409	2.443	1.9282	1.701	1.3996	1.3011	1.0229	0.7531	0.5031	0.231
$\hat{\beta}_{2FD}$	3.1141	2.8306	2.6447	2.0319	1.657	1.4661	1.3204	1.0065	0.7985	0.521	0.2439
						n=30					
$\hat{\beta}_{1ML}$	0.1193	0.3588	0.6689	0.9428	1.0846	1.0562	1.0649	0.9372	0.6714	0.4699	0.2015
$\hat{\beta}_{2ML}$	0.1207	0.3949	0.6654	0.8975	1.0496	1.0481	1.0453	0.8595	0.6933	0.4486	0.2116
$\hat{\beta}_{1FD}$	3.1619	2.9853	2.5501	1.9371	1.562	1.4421	1.3509	1.0062	0.6911	0.4718	0.1975
$\hat{\beta}_{2FD}$	3.4588	2.9457	2.5176	2.1485	1.6784	1.4709	1.2895	0.904	0.7081	0.4241	0.2084
						n=50					
$\hat{\beta}_{1ML}$	0.1235	0.3262	0.639	0.8551	1.0353	1.0555	1.0175	0.891	0.6333	0.3998	0.158
$\hat{\beta}_{2ML}$	0.1047	0.3539	0.5924	0.8651	1.0071	1.0276	1.0037	0.881	0.6654	0.4364	0.1508
$\hat{\beta}_{1FD}$	3.458	3.0335	2.4173	2.1176	1.6813	1.4614	1.3332	0.9729	0.6451	0.3994	0.1573
$\hat{\beta}_{2FD}$	3.4197	3.1088	2.5129	2.0489	1.6472	1.5114	1.3052	0.9654	0.6832	0.437	0.1509
						n=100					
$\hat{\beta}_{1ML}$	0.1076	0.3668	0.6104	0.8509	1.0047	1.0209	1.0308	0.8369	0.5824	0.413	0.1364
$\hat{\beta}_{2ML}$	0.1088	0.3572	0.6074	0.8601	0.9745	1.0406	1.0062	0.8415	0.6398	0.3748	0.1101
$\hat{\beta}_{1FD}$	3.5548	3.1421	2.529	2.057	1.6436	1.4177	1.368	0.9086	0.5804	0.4268	0.1373
$\hat{\beta}_{2FD}$	3.5302	3.108	2.6819	2.0906	1.722	1.5205	1.3419	0.9458	0.6693	0.3758	0.1094

Table 9: Same as Table 7, except that both x_1 and x_2 are purely random.

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
						n=10					
ML	0.158	0.4005	0.5919	0.7475	0.8162	0.8455	0.8616	0.8338	0.7718	0.686	0.5483
FD	2.8636	2.6206	2.413	2.1378	1.9293	1.7692	1.6861	1.4047	1.1169	0.903	0.6413
ML	0.1929	0.4884	0.8558	1.0899	1.1837	1.3091	1.2277	1.2734	1.0444	0.8096	0.6757
FD	2.8538	2.6027	2.2385	1.9816	1.6298	1.5366	1.3499	1.1012	0.842	0.585	0.3918
						n=20					
ML	0.1766	0.4599	0.6825	0.822	0.9215	0.9402	0.9389	0.8921	0.7797	0.6156	0.3836
FD	3.3064	3.0113	2.6832	2.4058	2.0568	1.9109	1.7359	1.3959	1.0715	0.7543	0.4234
ML	0.1254	0.3961	0.6554	0.9499	1.0688	1.1294	1.0933	0.9598	0.766	0.5229	0.2458
FD	3.2135	2.6754	2.5406	1.9804	1.6789	1.431	1.3106	1.0151	0.7767	0.5123	0.2375
						n=30					
ML	0.1813	0.4789	0.7119	0.8581	0.9447	0.9632	0.96	0.9069	0.7809	0.5776	0.3164
FD	3.4596	3.1182	2.7767	2.463	2.1187	1.9193	1.7721	1.4175	1.0657	0.6981	0.3448
ML	0.12	0.376	0.6672	0.9206	1.0674	1.0521	1.0551	0.8963	0.6823	0.4591	0.2064
FD	3.3033	2.9664	2.5343	2.0406	1.6191	1.4567	1.3203	0.9524	0.6996	0.4476	0.2029
						n=50					
ML	0.1862	0.4881	0.7204	0.8869	0.9656	0.9788	0.9742	0.9054	0.7628	0.5536	0.263
FD	3.5897	3.2341	2.884	2.4993	2.1477	1.948	1.7765	1.3996	1.0263	0.6642	0.2825
ML	0.114	0.3396	0.6162	0.8601	1.0211	1.0417	1.0107	0.886	0.6502	0.4174	0.1543
FD	3.4385	3.0701	2.4642	2.083	1.6642	1.4861	1.3194	0.9691	0.6651	0.4174	0.154
						n=100					
ML	0.1872	0.5008	0.7373	0.8977	0.9793	0.9892	0.981	0.9076	0.7576	0.5327	0.2247
FD	3.6901	3.3134	2.9353	2.5534	2.1676	1.9733	1.7831	1.3977	1.0155	0.632	0.2387
ML	0.1082	0.3618	0.609	0.8554	0.9892	1.0312	1.0184	0.8391	0.6107	0.3921	0.1224
FD	3.5426	3.1243	2.6037	2.0735	1.684	1.4713	1.3549	0.9264	0.6242	0.3989	0.1224

Table 10: Same as Table 6, except that both x_1 and x_2 follow AR(1) processes.

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
						n=10					
$\hat{\beta}_{1ML}$	0.3709	0.6339	0.9426	1.0718	1.2045	1.2949	1.4169	1.1616	1.0094	0.8777	0.7999
$\hat{\beta}_{2ML}$	0.3776	0.6328	0.8985	1.0991	1.2279	1.3124	1.3457	1.1427	1.135	0.7942	0.755
$\hat{\beta}_{1FD}$	1.4987	1.7253	1.7837	1.6863	1.5208	1.4413	1.2634	1.1094	0.9106	0.6415	0.4688
$\hat{\beta}_{2FD}$	1.5266	1.6788	1.7201	1.7758	1.6544	1.4031	1.3092	1.1546	0.9355	0.6163	0.4805
						n=20					
$\hat{\beta}_{1ML}$	0.2261	0.4402	0.7167	0.9637	1.1193	1.1169	1.0793	0.9811	0.7482	0.5303	0.3715
$\hat{\beta}_{2ML}$	0.2149	0.4347	0.7962	0.9652	1.0617	1.1446	1.1111	1.0051	0.8209	0.5268	0.3304
$\hat{\beta}_{1FD}$	1.3314	1.5509	1.6917	1.6872	1.5628	1.4603	1.3936	1.0791	0.7854	0.4924	0.2885
$\hat{\beta}_{2FD}$	1.3281	1.5498	1.6481	1.6604	1.5879	1.4544	1.3406	1.0969	0.8362	0.4604	0.2618
						n=30					
$\hat{\beta}_{1ML}$	0.1588	0.4403	0.6794	0.9306	1.0306	1.1157	1.0616	0.9388	0.7325	0.4418	0.2265
$\hat{\beta}_{2ML}$	0.1619	0.4462	0.6442	0.9068	1.0006	1.0741	1.0525	0.958	0.7113	0.4292	0.2353
$\hat{\beta}_{1FD}$	1.2838	1.4747	1.5748	1.6827	1.5542	1.5262	1.4132	1.0989	0.7471	0.4378	0.1862
$\hat{\beta}_{2FD}$	1.274	1.4451	1.6721	1.6291	1.6106	1.4989	1.2952	1.1607	0.7279	0.4221	0.1998
						n=50					
$\hat{\beta}_{1ML}$	0.1204	0.369	0.6223	0.8795	1.0343	1.05	1.0072	0.8846	0.6682	0.403	0.1513
$\hat{\beta}_{2ML}$	0.1368	0.3908	0.6918	0.8798	1.0075	1.0388	1.0465	0.8768	0.6776	0.384	0.144
$\hat{\beta}_{1FD}$	1.1919	1.3997	1.5721	1.6692	1.5654	1.6067	1.306	1.0301	0.7639	0.4259	0.1445
$\hat{\beta}_{2FD}$	1.2066	1.3938	1.5312	1.63	1.597	1.5972	1.5383	1.0656	0.7724	0.4006	0.1377
						n=100					
$\hat{\beta}_{1ML}$	0.1385	0.3507	0.593	0.8193	0.9977	1.0272	1.0013	0.8586	0.64	0.373	0.1325
$\hat{\beta}_{2ML}$	0.1324	0.357	0.6354	0.8632	0.9915	1.0183	1.0112	0.8691	0.641	0.3576	0.1146
$\hat{\beta}_{1FD}$	1.1494	1.3515	1.5566	1.7021	1.5757	1.4525	1.3017	1.0983	0.7456	0.392	0.1313
$\hat{\beta}_{2FD}$	1.1453	1.3566	1.5192	1.6087	1.506	1.4619	1.3478	1.0804	0.7494	0.3689	0.1119

Table 11: Same as Table 7, except that both x_1 and x_2 follow AR(1) processes.

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
						n=10					
ML	0.3589	0.5246	0.672	0.7487	0.8113	0.8386	0.8563	0.8459	0.8193	0.76	0.7078
FD	2.5503	2.4874	2.2836	2.1125	1.9285	1.7947	1.6449	1.48	1.2743	1.0795	0.954
ML	0.3744	0.6333	0.9209	1.085	1.2162	1.3038	1.3828	1.1527	1.0697	0.8369	0.7779
FD	1.5131	1.7	1.7524	1.7296	1.5878	1.4219	1.2853	1.1308	0.9225	0.6292	0.4745
						n=20					
ML	0.3069	0.5359	0.724	0.8514	0.9186	0.9362	0.9401	0.9044	0.8272	0.668	0.5467
FD	3.1609	2.9273	2.6471	2.3537	2.0822	1.8843	1.7467	1.4388	1.1597	0.8282	0.6296
ML	0.2205	0.4375	0.7577	0.9645	1.0903	1.1301	1.0948	0.9934	0.7846	0.5285	0.351
FD	1.3297	1.5504	1.6693	1.6738	1.5754	1.4574	1.3678	1.0882	0.8108	0.4761	0.2752
						n=30					
ML	0.2653	0.5365	0.7393	0.8722	0.9504	0.9613	0.9583	0.909	0.8127	0.6324	0.4239
FD	3.398	3.0914	2.7597	2.4304	2.1032	1.9449	1.755	1.4207	1.1224	0.769	0.4682
ML	0.1604	0.4433	0.6618	0.9191	1.0147	1.095	1.0571	0.9481	0.7226	0.4353	0.231
FD	1.279	1.4596	1.6235	1.6567	1.5842	1.5126	1.355	1.1288	0.7381	0.4298	0.1931
						n=50					
ML	0.2471	0.5264	0.7386	0.8884	0.9671	0.9788	0.9737	0.9129	0.78	0.5924	0.3326
FD	3.5446	3.2157	2.8667	2.5115	2.136	1.9521	1.7806	1.4224	1.0541	0.71	0.3586
ML	0.1285	0.3795	0.6559	0.8796	1.0214	1.0442	1.0265	0.8808	0.6729	0.3934	0.1477
FD	1.1991	1.3969	1.5523	1.65	1.5806	1.6018	1.4198	1.0475	0.7681	0.4131	0.1412
						n=100					
ML	0.2193	0.5163	0.7457	0.901	0.978	0.9894	0.9821	0.9138	0.7659	0.541	0.258
FD	3.6814	3.3128	2.9345	2.5458	2.1763	1.9764	1.7923	1.4161	1.0269	0.6408	0.2741
ML	0.1353	0.3539	0.6136	0.8411	0.9945	1.0227	1.0064	0.8639	0.6405	0.3654	0.1232
FD	1.1473	1.3541	1.5384	1.6558	1.5396	1.4573	1.3257	1.0893	0.7475	0.3806	0.1213

Table 12: Empirical significance level of the testing procedures for a theoretical significance level of 5% in the stepwise linear regression where x_2, fixed, is added to the model of simple linear regression of y on x_1, fixed, as a function of the sample size n and the error autocorrelation parameter ρ. Empirical significance levels were computed from 1000 simulation runs.

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
						n=10					
Σ_o	0	0.003	0.014	0.026	0.034	0.052	0.07	0.11	0.18	0.234	0.302
ML_{χ^2}	0.127	0.139	0.139	0.173	0.163	0.203	0.222	0.239	0.293	0.355	0.397
ML_Z	0.165	0.2	0.191	0.246	0.227	0.279	0.29	0.323	0.376	0.441	0.502
FD	0	0	0.001	0.002	0.002	0.002	0.001	0.008	0.007	0.027	0.038
FDR	0	0	0	0	0.001	0.002	0.002	0.002	0.004	0.01	0.014
						n=20					
Σ_o	0	0.003	0.003	0.011	0.041	0.057	0.059	0.107	0.217	0.342	0.481
ML_{χ^2}	0.078	0.081	0.067	0.08	0.102	0.114	0.109	0.12	0.173	0.226	0.334
ML_Z	0.089	0.102	0.089	0.101	0.134	0.159	0.151	0.158	0.236	0.306	0.398
FD	0	0	0	0	0	0	0	0	0.001	0.006	0.036
FDR	0	0	0	0	0	0	0	0	0	0.001	0.001
						n=30					
Σ_o	0	0	0.002	0.014	0.037	0.045	0.064	0.134	0.218	0.378	0.556
ML_{χ^2}	0.052	0.06	0.072	0.075	0.1	0.076	0.078	0.107	0.122	0.191	0.29
ML_Z	0.059	0.065	0.088	0.093	0.128	0.096	0.108	0.134	0.162	0.247	0.359
FD	0	0	0	0	0	0	0	0	0	0	0.021
FDR	0	0	0	0	0	0	0	0	0	0.003	0.013
						n=50					
Σ_o	0	0	0.001	0.009	0.037	0.061	0.085	0.162	0.231	0.391	0.603
ML_{χ^2}	0.064	0.069	0.06	0.049	0.081	0.081	0.09	0.101	0.091	0.14	0.243
ML_Z	0.073	0.07	0.067	0.058	0.097	0.09	0.102	0.12	0.124	0.182	0.311
FD	0	0	0	0	0	0	0	0	0	0	0.004
FDR	0	0	0	0	0	0	0	0	0	0.002	0.012
						n=100					
Σ_o	0	0	0	0.012	0.03	0.05	0.071	0.138	0.273	0.423	0.636
ML_{χ^2}	0.055	0.047	0.054	0.066	0.069	0.063	0.055	0.061	0.07	0.095	0.166
ML_Z	0.057	0.05	0.057	0.069	0.072	0.073	0.059	0.072	0.082	0.124	0.233
FD	0	0	0	0	0	0	0	0	0	0	0.001
FDR	0	0	0	0	0	0	0	0	0	0	0.01

Table 13: Same as Table 12, except that both x_1 and x_2 are purely random.

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
						n=10					
Σ_o	0.039	0.054	0.048	0.055	0.065	0.057	0.047	0.056	0.048	0.055	0.052
ML_{χ^2}	0.136	0.135	0.158	0.148	0.169	0.15	0.144	0.152	0.129	0.144	0.117
ML_Z	0.188	0.183	0.222	0.226	0.245	0.237	0.234	0.233	0.21	0.218	0.195
FD	0.133	0.127	0.109	0.112	0.114	0.094	0.09	0.082	0.06	0.07	0.05
FDR	0.033	0.039	0.027	0.023	0.025	0.019	0.013	0.014	0.017	0.022	0.014
						n=20					
Σ_o	0.038	0.044	0.044	0.053	0.046	0.04	0.045	0.047	0.056	0.05	0.056
ML_{χ^2}	0.071	0.064	0.09	0.095	0.085	0.073	0.088	0.086	0.096	0.073	0.062
ML_Z	0.087	0.078	0.124	0.128	0.118	0.1	0.131	0.118	0.12	0.094	0.083
FD	0.14	0.142	0.119	0.12	0.113	0.085	0.096	0.083	0.096	0.067	0.05
FDR	0.02	0.021	0.029	0.022	0.017	0.014	0.025	0.029	0.017	0.018	0.018
						n=30					
Σ_o	0.053	0.037	0.044	0.048	0.046	0.059	0.049	0.062	0.038	0.055	0.047
ML_{χ^2}	0.07	0.064	0.062	0.072	0.068	0.088	0.072	0.069	0.057	0.061	0.054
ML_Z	0.082	0.074	0.078	0.081	0.083	0.109	0.097	0.092	0.076	0.076	0.062
FD	0.15	0.132	0.124	0.123	0.102	0.121	0.1	0.076	0.068	0.055	0.046
FDR	0.026	0.025	0.027	0.022	0.024	0.024	0.03	0.02	0.02	0.019	0.019
						n=50					
Σ_o	0.053	0.047	0.052	0.051	0.048	0.057	0.053	0.055	0.059	0.042	0.051
ML_{χ^2}	0.057	0.046	0.058	0.057	0.057	0.068	0.061	0.069	0.074	0.063	0.052
ML_Z	0.062	0.051	0.066	0.067	0.068	0.076	0.072	0.078	0.082	0.071	0.059
FD	0.157	0.139	0.134	0.119	0.106	0.119	0.096	0.09	0.102	0.067	0.047
FDR	0.017	0.026	0.024	0.02	0.021	0.027	0.024	0.022	0.027	0.015	0.022
						n=100					
Σ_o	0.049	0.045	0.043	0.052	0.06	0.058	0.056	0.052	0.042	0.057	0.061
ML_{χ^2}	0.062	0.058	0.051	0.051	0.062	0.065	0.061	0.054	0.047	0.063	0.044
ML_Z	0.065	0.061	0.057	0.054	0.068	0.07	0.069	0.057	0.048	0.069	0.046
FD	0.164	0.154	0.137	0.137	0.129	0.11	0.113	0.083	0.073	0.079	0.051
FDR	0.027	0.018	0.036	0.02	0.028	0.016	0.019	0.023	0.018	0.022	0.013

Table 14: Same as Table 12, except that both x_1 and x_2 follow AR(1) processes.

ρ	-0.9	-0.7	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.7	0.9
						n=10					
Σ_o	0.333	0.159	0.093	0.048	0.052	0.049	0.061	0.062	0.072	0.121	0.178
ML_{χ^2}	0.216	0.18	0.143	0.138	0.154	0.169	0.156	0.162	0.163	0.193	0.22
ML_Z	0.326	0.26	0.239	0.232	0.224	0.277	0.246	0.276	0.266	0.298	0.313
FD	0.431	0.29	0.191	0.139	0.107	0.103	0.09	0.081	0.045	0.064	0.061
FDR	0.187	0.094	0.051	0.038	0.023	0.025	0.023	0.014	0.01	0.014	0.018
						n=20					
Σ_o	0.417	0.204	0.109	0.067	0.052	0.046	0.053	0.082	0.1	0.186	0.289
ML_{χ^2}	0.101	0.087	0.097	0.087	0.102	0.083	0.092	0.108	0.105	0.112	0.108
ML_Z	0.155	0.127	0.145	0.13	0.133	0.136	0.13	0.15	0.158	0.155	0.157
FD	0.492	0.316	0.211	0.167	0.114	0.115	0.093	0.081	0.074	0.059	0.085
FDR	0.162	0.059	0.031	0.032	0.014	0.024	0.015	0.02	0.016	0.018	0.026
						n=30					
Σ_o	0.446	0.193	0.121	0.062	0.06	0.052	0.05	0.063	0.118	0.233	0.387
ML_{χ^2}	0.075	0.079	0.056	0.073	0.084	0.076	0.076	0.077	0.073	0.071	0.096
ML_Z	0.104	0.099	0.073	0.091	0.103	0.101	0.091	0.094	0.097	0.099	0.135
FD	0.488	0.271	0.216	0.143	0.136	0.109	0.093	0.08	0.057	0.065	0.119
FDR	0.121	0.058	0.036	0.015	0.027	0.026	0.014	0.021	0.023	0.015	0.025
						n=50					
Σ_o	0.488	0.232	0.123	0.058	0.042	0.04	0.055	0.075	0.106	0.239	0.439
ML_{χ^2}	0.065	0.06	0.07	0.059	0.058	0.06	0.07	0.061	0.051	0.073	0.056
ML_Z	0.083	0.068	0.08	0.067	0.068	0.067	0.077	0.074	0.062	0.086	0.066
FD	0.539	0.323	0.22	0.138	0.109	0.118	0.109	0.084	0.065	0.074	0.093
FDR	0.096	0.045	0.031	0.026	0.027	0.023	0.014	0.025	0.023	0.023	0.023
						n=100					
Σ_o	0.497	0.239	0.128	0.062	0.061	0.057	0.063	0.07	0.118	0.257	0.491
ML_{χ^2}	0.062	0.05	0.053	0.054	0.062	0.063	0.057	0.063	0.052	0.065	0.053
ML_Z	0.072	0.055	0.062	0.057	0.067	0.068	0.065	0.066	0.055	0.069	0.058
FD	0.531	0.319	0.216	0.15	0.126	0.11	0.105	0.082	0.07	0.071	0.101
FDR	0.054	0.026	0.029	0.024	0.022	0.027	0.027	0.022	0.016	0.019	0.025

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