DUALITIES FOR ACCESSIBLE CATEGORIES

by

Hongde Hu

A thesis submitted to the Faculty of Graduate Studies and Research

in partial fulfilment of the requirements for the degree of

Doctor of Philosophy

Supervisor: M. Makkai

Department of Mathematics and Statistics

McGill University

Montreal, Canada

December 1992

ISBN 0-315-87588-3

ACKNOWLEDGEMENTS

I wish to express my sincere thanks to my supervisor, Professor Micheal Makkai, for his kind encouragement and inspiring motivation; without whom I could not realize this dream.

I would like to thank Professor Michael Barr for his constant kindness and support.

I would like to thank Professor Marta Bunge, Professor Jim Lambek, and Professor James Loveys for their support. I would also like to thank Professor Robert Paré for helpful discussions.

Last, but not least, I feel indebted to my wife and my mother, for their love and encouragement.

ABSTRACT

In this work we prove dualities for Diers categories, Barr categories and small Barr-exact categories. The latter duality solves a problem of M. Makkai. Further, we prove a stronger version of the strong completeness theorem on κ-Barr-exact categories. Finally we prove that an accessibly embedded subcategory of a locally presentable category satisfies the solution-set condition iff it is accessible. This improves work of J. Adámek and J. Rosický on injectivity in locally presentable categories.

RESUME

Dans ce travail, nous prouvons les dualités des catégories de Diers, des catégories de Barr et des petites catégories exactes de Barr. La dualité ultérieure permet la résolution du Probléme de M. Makkai. En outre, nous effectuons des énoncés de complétude comceptuelle pour la 2-catégorie des catégories de Diers, la 2-catégorie des catégories de Barr et la 2-catégorie des catégories localement présentables. Finalement, nous prouvons qu'une sous-catégorie accessiblement plongée d'une catégorie localement présentables satisfait la condition de l'ensemble-solution si et seulement si elle est accessible. Ceci contribute à améliorer le travail de J. Adámek, de J. Rosický sur l'injectivité des catégories localement présentables.

CONTENTS

CHAPTER 1

INTRODUCTION

Before the explicit introduction of accessible categories by M. Makkai and R. Paré in [24], various subclasses of accessible categories had been investigated for many years, beginning with Gabriel and Ulmer in 1971 and Artin, Grothendieck and Verdier in 1972, and continuing through the work of Makkai and Reyes in 1977, Diers in 1980, and Guitart and Lair in 1980-1981, and so on. The $\aleph_0$-accessible categories are the categories that are determined, in a quite precise way (free completion by filtered colimits), by some small subcategory. The idea of looking at objects which are filtered colimits of finitely presentable objects has been used in algebra. An early example is Lazare's theorem for flat modules, which says that a flat module is a filtered colimit of finitely generated free modules. The category of Hilbert spaces is another important example for an accessible category, in which all objects are $\aleph_1$-filtered colimits of $\aleph_1$-presentable ones.

Locally presentable categories, introduced by Gabriel and Ulmer, is a subclass of the class of accessible categories. As proved in Makkai and Paré's book, a category is locally presentable if and only if it is accessible and complete, or equivalently cocomplete. The classic work of Gabriel and Ulmer gives a characterization of locally

κ-presentable categories which, up to equivalence, can be written as $L_\kappa(\mathbf{C},\mathbf{S})$, the category of the functors preserving κ-limits; here $\mathbf{C}$ is a small category with κ-limits, $\mathbf{S}$ denotes $\mathbf{Set}$, the category of small sets. The result was rephrased as a duality theorem by Makkai and Pitts in 1987. The duality theorem for accessible categories, as given by Makkai and Paré, contains the statement that κ-accessible categories are exactly the categories of the form $L_\kappa Cocls((\mathbf{C},\mathbf{S}),\mathbf{S})$, the category of the functors preserving κ-limits and colimits; here $\mathbf{C}$ is a small category.

We continue the study of dualities for accessible categories: (κ-)Diers categories ((κ)-accessible with connected limits, introduced by Diers) and (κ-)Barr categories ((κ-)accessible with products). A category $\mathbf{C}$ is said to be coproduct-accessible, if it has small coproducts, and it has a small subcategory $\mathbf{B}$ consisting of coproduct-presentable objects (the functors representable by objects in $\mathbf{B}$ preserve small coproducts) such that every object of $\mathbf{C}$ is a small coproduct of objects in $\mathbf{B}$. We have proved a duality for (κ-)Diers categories. The duality theorem for (κ-)Diers categories has the following consequences:

(i) κ-Diers categories are exactly the categories of the form $\coprod L_\kappa(\mathbf{C},\mathbf{S})$, the category of the functors preserving coproducts and κ-limits, where $\mathbf{C}$ is a coproduct-accessible category with κ-limits.

(ii) The coproduct-accessible categories with κ-limits are exactly the categories of the form $CoF_\kappa(\mathbf{A},\mathbf{S})$, the category of the functors preserving κ-filtered colimits and connected limits, with $\mathbf{A}$ a κ-Diers category.

A Barr-exact category $\mathbf{C}$ is said to be Barr-exact accessible if it has a small subcategory $\mathbf{B}$ consisting of regular epi projective objects such that for every object C of $\mathbf{C}$ there is a regular epimorphism from B into C with B in $\mathbf{B}$. We have proved a duality for (κ-)Barr categories. The duality theorem has the following consequences:
(i) κ-Barr categories are exactly the categories of the form $\kappa - Reg(\mathbf{C}, \mathbf{S})$, the category of the κ-regular functors, where $\mathbf{C}$ is a κ-Barr-exact accessible category.
(ii) The κ-Barr exact accessible categories are exactly the categories of the form $\prod F_\kappa(\mathbf{A}, \mathbf{S})$, the category of the functors preserving κ-filtered colimits and products, with $\mathbf{A}$ a κ-Barr category.

Both duality theorems are Stone-type dualities with $\mathbf{S}$, the category of small sets, as the dualizing object. It is interesting to note that a Barr-exact accessible category is necessarily an essentially small category.

The term 'strong conceptual completeness' is used in [20] and [18] in two different senses, both of which are important in this thesis (the two meanings refer to two distinct strengthenings of the Conceptual Completeness Theorem of [22]). To avoid any misunderstanding, we will call 'sharp conceptual completeness' what is called 'strong conceptual completeness' in [20], and continue to use 'strong conceptual completeness' in the sense of [18]. The abbreviation 'SCC' used in [20] is, therefore, the same as 'sharp conceptual completeness'. A sharp conceptual completeness (SCC) statement for a logical doctrine means an assertion that any theory of the doctrine can be recovered from an appropriate structure formed by the models of the theory. In classical propositional logic, sharp conceptual completeness takes the form of the Stone duality theorem. We give SCC statements for various doctrines: the 2-category of locally presentable categories, the 2-category of Diers categories and the 2-category of Barr categories. Similar results for other accessible doctrines were obtained by Makkai in [20].

The notion of an exact category was introduced by M. Barr in [4]; we call it Barr-exact category. The definition of Barr-exact category consists of a combination of finite completeness and exactness conditions. More precisely, it has finite limits and stable quotients of equivalence relations. A functor between Barr-exact categories is regular if it preserves finite limits and quotients of equivalence relations. An important result on this subject is Barr's theorem on full embeddings of exact categories, saying that a small exact category has a full regular embedding into a set-valued functor category.

The notions of κ-Barr-exact category and κ-regular functor are given by M. Makkai in [21], which are an infinitary generalization of Barr-exact category and regular functor, for κ any infinite regular cardinal. Let $\mathbf{C}$ and $\mathbf{D}$ be arbitrary κ-Barr-exact categories, and denoted by $\kappa - Reg(\mathbf{C}, \mathbf{D})$ the full subcategory of $(\mathbf{C}, \mathbf{D})$ consisting of κ-regular functors from $\mathbf{C}$ into $\mathbf{D}$. To say that a κ-regular functor $F : \mathbf{C} \to \mathbf{D}$ is a quotient is to say two things, first, that for any κ-Barr-exact category $\mathbf{T}$, the induced functor of $F^* : \kappa - Reg(\mathbf{D},\mathbf{T}) \to \kappa - Reg(\mathbf{C},\mathbf{T})$ is full and faithful, and second, that F^* is essentially surjective onto those regular functors $\mathbf{C} \to \mathbf{T}$ that invert all morphisms inverted by F (see [18]). The strong conceptual completeness theorem for small κ-Barr-exact categories says that for small κ-Barr-exact categories $\mathbf{C}$ and $\mathbf{D}$, it suffices to require the first thing only, and that only for $\mathbf{T} = \mathbf{S}$ (see [18] and [8]). We prove a stronger version of the strong conceptual completeness on κ-Barr-exact categories. The main theorem we prove can be stated as follows. Given a small κ-Barr-exact category $\mathbf{C}$, let $\mathbf{A}$ be an accessible full subcategory of $\kappa - Reg(\mathbf{C},\mathbf{S})$ which is closed under κ-filtered colimits and products. If the evaluation functor $e_{\mathbf{A}} : \mathbf{A} \to (\prod F_\kappa(\mathbf{A},\mathbf{S}),\mathbf{S})$ is full and faithful, then the functor

$$F : \prod F_\kappa(\kappa - Reg(\mathbf{C},\mathbf{S}),\mathbf{S}) \to \prod F_\kappa(\mathbf{A},\mathbf{S})$$

induced by the inclusion satisfies the following property: for every functor M in $\prod F_\kappa(\mathbf{A},\mathbf{S})$, there are a functor N in $\prod F_\kappa(\kappa - Reg(\mathbf{C},\mathbf{S}),\mathbf{S})$ and a regular epi $F(N) \to M$. The stronger version of the strong conceptual completeness is that, for the above $\mathbf{C}$ and $\mathbf{A}$, the composite of $e_{\mathbf{C}}$ and F, denoted by $e : \mathbf{C} \to \prod F_\kappa(\mathbf{A},\mathbf{S})$, is a quotient. By taking $\mathbf{A}$ to be the category $\kappa - Reg(\mathbf{D},\mathbf{S})$, with $\mathbf{D}$ small κ-Barr exact, the strong conceputal completeness theorem as stated above follows from the result just stated together with the duality theorem of [21]: the canonical evaluation functor

$e_{\mathbf{D}} : \mathbf{C} \to \prod F_\kappa(\kappa - Reg(\mathbf{D}, \mathbf{S}), \mathbf{S})$ is an equivalence of categories. As pointed in [23], the strong conceptual completeness fails for **Lex**, the 2-category of small categories with finite limits, although ordinary conceptual completeness for **Lex** holds by using Gabriel-Ulmer duality: for a functor $F : \mathbf{C} \to \mathbf{D}$ preserves finite limits in **Lex**, if the induced functor $F^* : Lex(\mathbf{D}, \mathbf{S}) \to Lex(\mathbf{C}, \mathbf{S})$ is an equivalence of categories, then F is an equivalence as well.

Let $\kappa - \mathbf{Barr} - \mathbf{exact}$ be the 2-category of all κ-Barr-exact categories as objects and κ-regular functors as 1-arrows, and all natural transformations between the latter as 2-arrows, and let $\prod\mathcal{F}_\kappa$ be the 2-category of all categories with κ-filtered colimits and products and functors preserving those operations. $\mathbf{S}$ is an object living in both $\kappa - \mathbf{Barr} - \mathbf{exact}$ and $\prod\mathcal{F}_\kappa$ such that the $\prod\mathcal{F}_\kappa$ and the $\kappa - \mathbf{Barr} - \mathbf{exact}$ structures on $\mathbf{S}$ commute each other (see [21]). Such a state of affairs gives rise to a 2-adjunction

$$\kappa - \mathbf{Barr} - \mathbf{exact}^{op} \underset{G}{\overset{F}{\rightleftarrows}} \prod\mathcal{F}_\kappa$$

here $F = \prod F_\kappa(-, \mathbf{S})$, and $G = \kappa - Reg(-, \mathbf{S})$. The component $e_{\mathbf{C}} : \mathbf{C} \to F(G(\mathbf{C}))$ of the counit is the evaluation functor on $\mathbf{C}$. Therefore, Makkai's theorem on κ-Barr-exact categories gives a 'one-sided' duality for small κ-Barr-exact categories. To complete the perfect duality for small κ-Barr-exact categories, we characterize the categories of the form $\kappa - Reg(\mathbf{C}, \mathbf{S})$ with $\mathbf{C}$ a small κ-Barr-exact category. This answers a problem posed by Makkai. The characterization theorem is made difficult by the fact that $\kappa - Reg(\mathbf{C}, \mathbf{S})$ is not necessarily κ-accessible. In general, $\kappa - Reg(\mathbf{C}, \mathbf{S})$ is λ-accessible for some regular cardinal $\lambda \geq \kappa$, and the duality theorem for κ-Barr categories shows that $\kappa - Reg(\mathbf{C}, \mathbf{S})$ is κ-accessible if and only if $\mathbf{C}$ is κ-Barr-exact accessible.

Recall from [2] that a full subcategory of a locally presentable category is said to be accessibly embedded if it is closed under the κ-filtered colimits , for some infinite regular cardinal κ. Also recall from [1] that Vopěnka's principle is the following statement: the category *Gra* of graphs (sets with a binary relation) does not have a large discrete full subcategory. It has been shown in [1] that Vopěnka's principle is equivalent to the following statement: every accessibly embedded subcategory of a locally presentable category is accessible. The absolute result we prove is the following: an accessibly embedded subcategory of a locally presentable category satisfies the solution-set condition if and only if it is accessible, or equivalently it is a small cone-injectivity classes. This result improves work of J. Adámek and J. Rosický on injectivity in locally presentable categories. Adámek and Rosický have shown that the small injectivity classes of locally presentable categories are exactly the Barr categories (see [2]).

The thesis proceeds as follows.

In Chapter 2 we will summarize a certain amount of material on accessible categories and κ-Barr-exact categories we use later. We make a detailed study of coproduct-accessible categories in Chapter 3. We have a duality theorem for coproduct-accessible categories. This duality is analogous to the duality for accessible categories (see Proposition 4.2.1 in [24]). In Chapter 4 we first establish the relation between categories with κ-multicolimits and coproduct-accessible categories with κ-limits, then we show the duality for Diers categories. Chapter 5 contains the duality for Barr categories. A crucial lemma says that, with $\mathbf{A}$ a Barr category and $\prod Acc(\mathbf{A}, \mathbf{S})$ denoting the category of all accessible functors preserving products, for each $F \in \prod Acc(\mathbf{A}, \mathbf{S})$, there is a regular epi $\mathbf{A}(A, -) \to F$ with some $A \in \mathbf{A}$. As corollaries of Gabriel-Ulmer duality and dualities for Diers categories and Barr categories, we give sharp conceptual completeness statements in Section 6.1, then we introduce the concept of Barr-exact weak-accessible category. One example of this concept is the opposite category of the category of R-modules, for R an associative ring. This concept is motivated by the characterization of the categories of the form $\kappa - Reg(\mathbf{C}, \mathbf{S})$ with $\mathbf{C}$ small κ-Barr-exact. Chapter 7 contains the result mentioned above on quotient morphisms between small κ-Barr-exact categories and a perfect duality for small κ-Barr-exact categories. Chapter 8 gives a treatment of a cone-reflectivity subcategory of a locally presentable category. The proof of the main result in Chapter 8 uses some techniques developed by Adámek and Rosický in [2].

CHAPTER 2

PRELIMINARIES

Let κ be an infinite regular cardinal. Recall that a category $\mathbf{A}$ is κ-filtered if for any graph $\mathbf{G}$ of cardinality less than κ, any diagram $D : \mathbf{G} \to \mathbf{A}$ has a cocone. $\mathbf{A}$ has κ-filtered colimits, if $\mathbf{A}$ has colimits of all diagrams whose domain is a κ-filtered category. Another concept is κ-limit; it refers to the limit of a diagram whose domain category is of size less than κ. An object A of $\mathbf{A}$ is said to be κ-presentable if the representable functor $\mathbf{A}(A, -) : \mathbf{A} \to \mathbf{SET}$ preserves κ-filtered colimits existing in $\mathbf{A}$. Here $\mathbf{SET}$ is the category of sets. When $\mathbf{A}$ is locally small, $\mathbf{SET}$ may be repleed by $\mathbf{S}$, the category of small sets. The full subcategory of $\mathbf{A}$ whose objects are the κ-presentable ones is denoted by $\mathbf{A}_\kappa$. The following definition is given in [24].

Definition 2.1 *A category* $\mathbf{A}$ *is* κ*-accessible if:*

(i) $\mathbf{A}$ *has* κ*-filtered colimits;*

(ii) There is a small full subcategory $\mathbf{B}$ *of* $\mathbf{A}_\kappa$ *so that every object of* $\mathbf{A}$ *is a* κ*-filtered colimit of a diagram of objects in* $\mathbf{B}$.

A category is accessible if it is κ*-accessible for some infinite regular cardinal* κ.

Let $\mathbf{C}$ be a small category. A functor $F : \mathbf{C} \to \mathbf{S}$ is said to be κ-flat if F is a κ-filtered colimit of representable functors. The category of all κ-flat functors from $\mathbf{C}$ to

S, a full subcategory of $(\mathbf{C}, \mathbf{S})$, is denoted as $\kappa - Flat(\mathbf{C})$. The category $\kappa - Flat(\mathbf{C})$ has the universal property of being the free completion of $\mathbf{C}^{op}$ with κ-filtered colimits (Proposition 1.2.4 (ii) in [24]): κ-Flat(C) has κ-filtered colimits, and for any category $\mathbf{B}$ with κ-filtered colimits, the functor

$$Z^* : F_\kappa(\kappa - Flat(\mathbf{C}), \mathbf{B}) \to (\mathbf{C}^{op}, \mathbf{B})$$

is an equivalence of categories. Here Z^* is defined by composition with the canonical functor $Z : \mathbf{C}^{op} \to \kappa - Flat(\mathbf{C})$. As proved in [24], a category $\mathbf{A}$ is κ-accessible if and only if it is equivalent to $\kappa - Flat(\mathbf{C})$ for some small category $\mathbf{C}$. Let $\kappa - \mathbf{Acc}$ be the 2-category of all κ-accessible categories as objects with κ-accessible functors (i.e. the functors preserving κ-filtered colimits) as 1-arrows, and all natural transformations as 2-arrows. $\mathbf{P}_\kappa$ denotes the 2-category of all categories which are equivalent to one of the form $(\mathbf{C}, \mathbf{S})$ for a small category $\mathbf{C}$, whose 1-arrows are the functors preserving κ-limits and colimits, and whose 2-arrows are all natural transformations between the latter. The following duality theorem is given in [24] (For the notations, see the list at the end of this section).

Theorem 2.2 (*i*) *For each* $\mathbf{A}$ *in* $\kappa - \mathbf{Acc}$, *the evaluation functor*

$$\epsilon_{\mathbf{A}} : \mathbf{A} \to L_\kappa Cocts(F_\kappa(\mathbf{A}, \mathbf{S}), \mathbf{S})$$

is an equivalence of categories;

(*ii*) *For each* $\mathbf{B}$ *in* $\mathbf{P}_\kappa$, *the evaluation functor*

$$\eta_{\mathbf{B}} : \mathbf{B} \to F_\kappa(L_\kappa Cocts(\mathbf{B}, \mathbf{S}), \mathbf{S})$$

is an equivalence of categories.

The class of accessible categories has a number of subclasses, determined by additional structure on the categories.

Let $\mathbf{A}$ be a category. A set $\mathbf{C}$ of objects of $\mathbf{A}$ is a strong generator of $\mathbf{A}$ (see [12]) if the family of functors represented by the objects in $\mathbf{C}$ are jointly conservative (jointly reflect isomorphisms): for any $f : A \to B$ in $\mathbf{A}$, f is an isomorphism if and only if for all $C \in \mathbf{C}$, $\mathbf{A}(C, f) : \mathbf{A}(C, A) \to \mathbf{A}(C, B)$ is a bijection. Also, recall from [11] that A diagram $D : \mathbf{G} \to \mathbf{A}$ has a multicolimit if the functor

$$D - cocone : \mathbf{A} \to \mathbf{S}$$

assigning to every object A the set of cocones on D, with A as vertex, is isomorphic to a small coproduct of representable functors. The multicolimit of D is then the family of objects of $\mathbf{A}$ representing the functor in the coproduct. A category $\mathbf{A}$ is κ-multicocomplete, if it has the multicolimits of diagrams of size$< \kappa$. The following definition was given in [11].

Definition 2.3 *Let* $\mathbf{A}$ *be a locally small category.* $\mathbf{A}$ *is called locally* κ*-multipresentable if*

(i) **A** *has* κ*-filtered colimits;*

(ii) **A** *is* κ*-multicocomplete;*

(iii) **A** *has a small strong generator consisting of* κ*-presentable objects.*

A category **A** is connected if it is nonempty and for any pair of objects A and B in **A**, there is finite sequence of morphisms

$$A \rightarrow C_1 \leftarrow C_2 \rightarrow \cdots \rightarrow C_{2n-1} \leftarrow B$$

joining A to B. A category has connected limits, if it has limits of all diagrams whose domain is a connected category. We have

Theorem 2.4 *Let* **A** *be a category. The following are equivalent.*

(i) **A** *is* κ*-accessible with connected limits;*

(ii) **A** *is locally* κ*-multipresentable;*

(iii) **A** *is* κ*-accessible and multicocomplete.*

The above theorem can be found in [11] and [24]. **A** as in Theorem 2.4 is called κ-Diers.

Recall from [10] that a functor $F : \mathbf{A} \rightarrow \mathbf{B}$ has a left multiadjoint if for each object $B \in \mathbf{B}$ there is a small family of morphisms $< g_i : B \rightarrow F(A_i) >_{i \in I}$ such that every morphism $g : B \rightarrow F(A)$ with $A \in \mathbf{A}$ can be uniquely written as a composite $g = F(f) \circ g_i$ for some g_i and some morphism $f : A_i \rightarrow A$. As proved in [10], if a functor $F : \mathbf{A} \to \mathbf{B}$ has a left multiadjoint, then F preserves small connected limits existing in $\mathbf{A}$. The following multiadjoint functor theorem is given by Y. Diers (see Theorem 3.6.1 in [10]).

Theorem 2.5 *Let $\mathbf{A}$ be a category with small connected limits, and assume that $F : \mathbf{A} \to \mathbf{B}$ preserves all small connected limits. Assume that F satisfies the solution-set condition, i.e. for each object $B \in \mathbf{B}$ there is a small family of morphisms $< g_i : B \to F(A_i) >_{i \in I}$ such that every morphism $g : B \to F(A)$ with $A \in \mathbf{A}$ can be written as a composite $g = F(f) \circ g_i$ for some g_i and some morphism $f : A_i \to A$. Then F has a left multiadjoint.*

Remark 2.6 *Let $\mathbf{A}$ be a κ-Diers category, $F : \mathbf{A} \to \mathbf{S}$ a κ-accessible functor preserving small connected limits. By Proposition 6.1.2 in [24], any accessible functor satisfies the solution-set condition. Therefore F has a left multiadjoint. Similarly to the Representability Theorem (see Theorem 5.6.3 in [17]), we have that F is multi-representable, i.e. it is a sum of representable functors $\mathbf{A}(A_i, -), i \in I$. Since F preserves κ-filtered colimits, it is easily seen that each $\mathbf{A}(A_i, -)$ preserves κ-filtered colimits, i.e. $A_i \in \mathbf{A}_\kappa$.*

A category $\mathbf{A}$ is called locally κ-presentable if it is locally small, cocomplete, and has a small strong generator consisting of κ-presentable objects (see [12]). A nice theorem (see Theorem 6.1.4. in [24]) says that an accessible category is complete if and only if it is cocomplete. Gabriel and Ulmer have shown in [12] that a category $\mathbf{A}$ is locally κ-presentable if and only if it is equivalent to the category of the form $L_\kappa(\mathbf{C}, \mathbf{S})$; here $\mathbf{C}$ is a small category with κ-limits. Since $L_\kappa(\mathbf{C}, \mathbf{S})$ is κ-accessible (see Corollary 2.1.9. in [24]), it follows that $\mathbf{A}$ is locally κ-presentable if and only if it is complete κ-accessible.

Let $\kappa - \mathbf{LEX}$ be the 2-category of categories having κ-limits, whose 1-arrows are functors preserving κ-limits, the 2-arrows are all natural transformations between such functors. $\mathbf{LF}_\kappa$ is the 2-category of categories having limits and κ-filtered colimits, functors preserving limits and κ-filtered colimits, and all natural transformations. $\mathbf{S}$ is an object of both $\kappa - \mathbf{LEX}$ and $\mathbf{LF}_\kappa$. The fact that κ-limits commute with limits and κ-filtered colimits in $\mathbf{S}$, gives rise to a 2-adjunction

$$\kappa - \mathbf{LEX}^{op} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{LF}_\kappa$$

$$F = LF_\kappa(-, \mathbf{S}), G = L_\kappa(-, \mathbf{S})$$

The unit and counit at any object of the respective kind are defined by the evaluation functors. Let $\mathbf{L}_\kappa$ be the full sub-2-category of $\kappa - \mathbf{LEX}$ with objects that are essentially small, and $\mathbf{LP}_\kappa$ be the full sub-2-category of $\mathbf{LF}_\kappa$ whose objects are locally κ-presentable. We have a Stone adjunction based on $\mathbf{S}$ (see [23])

$$\mathbf{L}_\kappa^{op} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{LP}_\kappa.$$

Gabriel-Ulmer duality says that the above 2-adjunction is a biequivalence (see Theorem 1.2 in [23], for the case $\kappa = \aleph_0$), i.e. we have

Theorem 2.7 (*i*) *If* $\mathbf{C}$ *in* $\mathbf{L}_\kappa$, *then* $L_\kappa(\mathbf{C},\mathbf{S}) \in \mathbf{LP}_\kappa$, *and the evaluation functor*

$$\epsilon_{\mathbf{C}} : \mathbf{C} \to LF_\kappa(L_\kappa(\mathbf{C},\mathbf{S}),\mathbf{S})$$

is an equivalence of categories;

(*ii*) *if* $\mathbf{A}$ *in* $\mathbf{LP}_\kappa$, *then* $LF_\kappa(\mathbf{A},\mathbf{S}) \in \mathbf{L}_\kappa$, *and the evaluation functor*

$$\eta_{\mathbf{A}} : \mathbf{A} \to L_\kappa(LF_\kappa(\mathbf{A},\mathbf{S}),\mathbf{S})$$

is an equivalence.

Recall from [2] that for any class M of morphisms of a category $\mathbf{B}$, $M-inj$ denotes the collection of all objects A of $\mathbf{B}$ which are M-injective, i.e. for each $m : B \to C$ in M and each morphism $f : B \to A$, f factors through m

$$\begin{tikzcd} B \arrow[r, "m"] \arrow[d, "f"'] & C \arrow[dl, "f'"] \\ A & \end{tikzcd}$$

here f' is some morphism from C into A. An injectivity class of $\mathbf{B}$ is a class of objects of $\mathbf{B}$ of the form $M - inj$, for some collection M of morphisms of $\mathbf{B}$. When M is small, we call a class of the form $M - inj$ a small-injectivity class of $\mathbf{B}$.

Let $\mathbf{A}$ be a small injectivity class $M - inj$ of a locally presentable category $\mathbf{B}$. Given $A_i \in \mathbf{A}$ $(i \in I)$, then $\prod_{i \in I} A_i \in \mathbf{A}$: for each $m : B \to A$ in M, and each $f : B \to \prod A_i$, we have $f_i' : A \to A_i$ with $p_i \circ f = f_i' \circ m$ for each $i \in I$; here p_i are the product projections. Thus the morphism $f' : A \to \prod A_i$ with components f_i' fulfils $f = f' \circ m$. That is, $\mathbf{A}$ is closed under products in $\mathbf{B}$. Also, we have that $\mathbf{A}$ is closed under κ-filtered colimits in $\mathbf{B}$, for some infinite regular cardinal κ. Indeed, we can take κ being a regular cardinal larger or equal to the presentability of all domains of morphisms in M. As proved in [2], the small injectivity classes of locally presentable categories are exactly the classes of accessible categories with products. We call them Barr categories. A category is called κ-Barr category if it is κ-accessible with products.

Let us recall the notion of regular and Barr-exact categories (see [4] , [5]). A morphism in a category is said to be a regular epimorphism if it is a coequalizer of some pair of morphisms. A category is regular if it has finite limits, coequalizers of kernel-pairs, and in which any pullback of a regular epimorphism is again a regular epi. A functor $F : \mathbf{C} \to \mathbf{D}$ between regular categories is regular if it preserves finite limits and regular epis. $Reg(\mathbf{C}, \mathbf{D})$ denotes the category of regular functors from $\mathbf{C}$ into $\mathbf{D}$; it is a full subcategory of the functor category $(\mathbf{C}, \mathbf{D})$.

A diagram

$$A \overset{f}{\underset{g}{\rightrightarrows}} B$$

in **S** is an equivalence relation if the map $a \mapsto < f(a), g(a) > : A \to B \times B$ is one-to-one and its image is an equivalence relation on B in the ordinary sense. Let **C** be a category. A diagram as above in **C** is an equivalence relation if for each $C \in \mathbf{C}$, the induced diagram

$$\mathbf{C}(C, A) \overset{\mathbf{C}(C,f)}{\underset{\mathbf{C}(C,g)}{\rightrightarrows}} \mathbf{C}(C, B)$$

is an equivalence relation in **S**. A Barr-exact category is a regular category in which every equivalence relation is a kernel-pair.

The notions of κ-regular category and κ-regular functor are introduced in [21], for κ any infinite regular cardinal. They are a natural generalization of the notions of regular category and regular functor.

Definition 2.8 *A category* **C** *is κ-regular if it is regular, has κ-limits, and satisfies the principle of $<\kappa$ dependent choices (DC_κ): let α be an ordinal less than κ, and let $\Gamma =< A_\beta, f_{\beta,\gamma} : A_\beta \to A_\gamma >_{\gamma \leq \beta < \alpha}$ be an inverse diagram of type α in* **C** *such that*

(i) $f_{\beta+1,\beta}$ is a regular epi, for every β with $\beta + 1 < \alpha$; and

(ii) the restriction $\Gamma \mid \leq \beta$ of Γ to the domain consisting of all ordinals $\gamma \leq \beta$ is a limit diagram: C_β is a limit of $\Gamma \mid < \beta$ (Γ restricted to ordinals $< \beta$) with limit projections $f_{\beta,\gamma} : A_\beta \to A_\gamma (\gamma < \beta)$, for every limit ordinal $\beta < \alpha$.

Then every $f_{\beta,\gamma}$ is a regular epi, for all $\gamma \leq \beta < \alpha$.

A κ-Barr-exact category is a κ-regular category which is a Barr-exact. A functor is κ-regular if it is between κ-regular categories, and it preserves all regular epis and all κ-limits.

The following theorem can be found in [3] or [12] (Lemma 1.4.9.).

Theorem 2.9 *Suppose a regular functor $F : \mathbf{C} \to \mathbf{D}$ is full, and conservative, i.e. it reflects isomorphisms, and* $\mathbf{C}$ *is Barr-exact. Suppose that for every object D in* $\mathbf{D}$ *there is an object $C \in \mathbf{C}$ and a regular epi $e : F(C) \to D$. Then F is an equivalence of categories.*

The relationship between κ-exact categories and κ-regular theories were established in [21]. A language L which is suitable for the logic $L_{\kappa\kappa}$ consisting of a set of sorts, a set of operation symbols and a set of relation symbols. Terms are built up from sorted variables and operation symbols in the usual way. Atomic formulas are either of the form $t = u$, with t and u terms of the same sort, or $R < t_i >_{i \in I}$, with R a relation symbol, with arity assignment $R \subset \prod_{i \in I} S_i$, subject to the condition that t_i is of sort S_i, for each $i \in I$. The positive primitive (pp) formulas of $L_{\kappa\kappa}$ are the formulas which are obtained from the atomic formulas by the operations of $< \kappa$-conjunction and existential quantification over $< \kappa$ variables. A κ-regular theory in $L_{\kappa\kappa}$ is a collection of sentences of the form $\forall x(\phi \to \psi)$ (regular sentences), with ϕ, ψ pp formulas; subject to the conditions that x is exactly equal to $\mathrm{Var}(\phi)$, and $\mathrm{Var}(\psi) \subset x$; x may be the empty set. A κ-regular category $\mathbf{C}$ is the Lindenbaum-

Tarski category of the internal theory $T_{\mathbf{C}}$ of $\mathbf{C}$ in the canonical language associated with $\mathbf{C}$. The category of models of $T_{\mathbf{C}}$ is the same as the category $\kappa - Reg(\mathbf{C}, \mathbf{S})$. Also, any κ-regular category can be extended to a κ-Barr-exact category, without changing the category of κ-regular functors to $\mathbf{S}$.

The following completeness theorem for small κ-regular categories can be found in [21] (Theorem 2.3.).

Theorem 2.10 *For any small κ-regular category $\mathbf{C}$, there is a small set I, and a conservative κ-regular functor $F : \mathbf{C} \to (I, \mathbf{S})$.*

We collect here some notations used (before and) later:

κ: an infinite regular cardinal;

Each of the following categories is a full subcategory of the functor category $(\mathbf{A}, \mathbf{B})$.

$F_\kappa(\mathbf{A}, \mathbf{B})$: the category of functors preserving κ-filtered colimits;

$LF_\kappa(\mathbf{A}, \mathbf{B})$: the category of functors preserving limits and κ-filtered colimits;

$\prod F_\kappa(\mathbf{A}, \mathbf{B})$: the category of functors preserving products and κ-filtered colimits;

$CoF_\kappa(\mathbf{A}, \mathbf{B})$: the category of functors preserving connected limits and κ-filtered colimits;

$CoCocts(\mathbf{A}, \mathbf{B})$: the category of functors preserving connected limits and colimits;

$L_\kappa(\mathbf{A}, \mathbf{B})$: the category of functors preserving κ-limits;

$L_\kappa Cocts(\mathbf{A}, \mathbf{B})$: the category of functors preserving κ-limits and colimits;

$\coprod(\mathbf{A},\mathbf{B})$: the category of functors preserving coproducts;

$L_\kappa \coprod(\mathbf{A},\mathbf{B})$: the category of functors preserving κ-limits and coproducts;

$\kappa - Reg(\mathbf{A},\mathbf{B})$: the category of κ-regular functors;

$LAcc(\mathbf{A},\mathbf{B})$: the category of accessible functors preserving limits;

$CoAcc(\mathbf{A},\mathbf{B})$: the category of accessible functors preserving connected limits;

$\prod Acc(\mathbf{A},\mathbf{B})$: the category of accessible functors preserving products;

$L(\mathbf{A},\mathbf{B})$: the category of functors preserving limits;

$L\coprod(\mathbf{A},\mathbf{B})$: the category of functors preserving limits and coproducts;

$LR(\mathbf{A},\mathbf{B})$: the category of regular functors preserving limits.

CHAPTER 3

COPRODUCT-ACCESSIBLE CATEGORIES

3.1 Coproduct-flat functors

Definition 3.1 *Let* $\mathbf{C}$ *be a small category, a functor* $F : \mathbf{C} \to \mathbf{S}$ *is called coproduct-flat if* F *is a small coproduct of representable functors in* $(\mathbf{C},\mathbf{S})$*.*

Let $\mathbf{A}$ be an arbitrary category. By the full closure under coproducts of a subcategory $\mathbf{B}$ of $\mathbf{A}$ we mean the smallest full subcategory $\mathbf{B}'$ of $\mathbf{A}$ containing $\mathbf{B}$ and closed under coproducts taken in $\mathbf{A}$. Obviously, F is coproduct-flat if and only if F belongs to the full closure under small coproducts of the representable functors in $(\mathbf{C}, \mathbf{S})$.

Let $F_!$ be the left Kan extension of F along the Yoneda embedding, $Y : \mathbf{C} \to (\mathbf{C}^{op}, \mathbf{S})$, and assume that F is coproduct-flat. Note that small connected limits commute with coproducts in $\mathbf{S}$, and the left Kan extension $()_!$ is cocontinuous; also for $F = \mathbf{C}(C, -)$, $F_!$ is representable (represented by $\mathbf{C}(-, C)$); it follows that $F_!$ preserves small connected limits.

Theorem 3.2 *Let* $\mathbf{C}$ *be a small category with split idempotents. A functor* $F : \mathbf{C} \to \mathbf{S}$ *is coproduct-flat if and only if* $F_!$ *preserves small connected limits.*

Proof: We only need to show that if $F_!$ preserves small connected limits, then F is coproduct-flat. By Proposition 1.2.4(i) in [24], we have that $F_! \in CoCocts((\mathbf{C}^{op}, \mathbf{S}), \mathbf{S})$, and hence $F_! \in CoF_{\aleph_0}((\mathbf{C}^{op}, \mathbf{S}), \mathbf{S})$. Since the functor category $(\mathbf{C}^{op}, \mathbf{S})$ is an $\aleph_0$-Diers category (it is locally $\aleph_0$-presentable), Remark 2.6 gives that $F_!$ is multipresentable, i.e. $F_! = \coprod_{i \in I} Nat(M_i, -)$, with $M_i \in (\mathbf{C}^{op}, \mathbf{S})$. It follows that $Nat(M_i, -)$ is cocontinuous from $F_!$ cocontinuous, i.e. M_i is 0-presentable in $(\mathbf{C}^{op}, \mathbf{S})$. Let $M_i = colim\mathbf{C}(-, {C_k}^i)$ be the canonical colimit, then the isomorphism $M_i \to colim\mathbf{C}(-, {C_k}^i)$ factors through a colimit projection:

$$M_i \to \mathbf{C}(-, {C_k}^i) \to colim\mathbf{C}(-, {C_k}^i)$$

so M_i is a retract of $\mathbf{C}(-, {C_k}^i)$, but $\mathbf{C}$ has split idempotents, thus M_i is representable. We conclude that F is coproduct-flat.

The following example shows that the condition on $\mathbf{C}$ in Theorem 3.2 cannot be deleted.

Example 3.3 *Let* $\mathbf{C}$ *be the category of one object* C *and having only one non-trivial morphism* e *which is an idempotent.* $M : \mathbf{C} \to \mathbf{S}$ *is a subfunctor of* $\mathbf{C}(C, -)$ *which is defined by* $M(C) = \{e\}$ *and* $M(e)(-) = (-) \circ e$. *Then* M *is a retract of* $\mathbf{C}(C, -)$. *We have that* $M_!$ *preserves small connected limits. Clearly,* M *isn't coproduct-flat.*

The full subcategory of all coproduct-flat functors of (C,S), is denoted as c-

Flat(**C**). Note that the opposite Yoneda embedding Y' factors through $c - Flat(\mathbf{C})$, giving rise to the functor $Z : \mathbf{C}^{op} \to c - Flat(\mathbf{C})$.

Proposition 3.4 *Let* **C** *be a small category. Then we have*

(i) The functor $Z : \mathbf{C}^{op} \to c - Flat(\mathbf{C})$ *has the universal property of being the free completion of* $\mathbf{C}^{op}$ *with small coproducts, i.e. c-Flat(***C***) has small coproducts, and for any category* **A** *with small coproducts, the functor*

$$Z^* : \coprod(c - Flat(\mathbf{C}), \mathbf{A}) \to (\mathbf{C}^{op}, \mathbf{A})$$

is an equivalence of categories. Here Z^* *is defined by composition with* Z.

(ii) The quasi inverse of of the equivalence Z^* *in (i) takes any functor* $F : \mathbf{C}^{op} \to \mathbf{A}$ *to its left Kan extension* $F_!$ *along* Z.

Proof: By Theorem 5.35 in [15].

Corollary 3.5 *For any small category* **C** *with split idempotents, we have an equivalence of categories*

$$Y^* : CoCocts((\mathbf{C}^{op}, \mathbf{S}), \mathbf{S}) \to c - Flat(\mathbf{C})$$

defined by composition with the Yoneda embedding Y.

Proof: By Proposition 1.2.4(i) in [24] and Theorem 3.2.

3.2 Coproduct-accessible categories

Let $\mathbf{A}$ be an arbitrary category. An object A of $\mathbf{A}$ is *coproduct-presentable*, if the representable functor $\mathbf{A}(A, -) : \mathbf{A} \to \mathbf{SET}$ preserves small coproducts existing in $\mathbf{A}$. As we know, in $\mathbf{SET}$, coproducts are disjoint unions, hence A is a coproduct presentable object if every morphism $A \to \coprod A_i$ into a coproduct $\coprod A_i$ factors uniquely through a coproduct coprojection.

The full subcategory of $\mathbf{A}$ whose objects are the coproduct presentable ones is denoted by $\mathbf{A}_c$.

Proposition 3.6 *In any category, a colimit of a small connected diagram in which the objects are all coproduct presentable is coproduct presentable itself.*

The above proposition is a consequence of the fact that small coproducts commute with small connected limits in $\mathbf{SET}$.

Definition 3.7 *A category* $\mathbf{A}$ *is called coproduct-accessible if*

(*i*) $\mathbf{A}$ *has small coproducts, and*

(*ii*) *there is a small full subcategory* $\mathbf{B}$ *of* $\mathbf{A}_c$ *so that every object of* $\mathbf{A}$ *is a small coproduct of objects in* $\mathbf{B}$.

Proposition 3.8 *For every small category* $\mathbf{C}$, *c-Flat(*$\mathbf{C}$*) is coproduct-accessible.*

Proof: That small coproducts in $c - Flat(\mathbf{C})$ are computed pointwise follows from the facts that the inclusion functor $c - Flat(\mathbf{C}) \rightarrow (\mathbf{C}, \mathbf{S})$ preserves small coproducts, hence representable functors are coproduct-presentable objects in c-Flat($\mathbf{C}$).

Recall that a functor $F : I \rightarrow J$ is final if for each $j \in J$ the comma category j/F is non-empty and connected. We have

Proposition 3.9 *Let* $\mathbf{A}$ *be a coproduct-presentable category. For every object* A *of* $\mathbf{A}$*, let* $A = \coprod_{i \in I} A_i$ *with* A_i *in* $\mathbf{A}_c$*, and the functor*

$$F : I \rightarrow \mathbf{A}_c/A$$

be defined by $F(i) = p_i$*; here* p_i *is the coprojection. Then* F *is final.*

Proof: If $(f : B \rightarrow A) \in \mathbf{A}_c/A$, then, since B is coproduct presentable, there is $i \in I$ with $f = p_i \circ f'$; here $f' : B \rightarrow A_i$ is some morphism, and p_i is the coprojection. This shows that f/F is non-empty. Let $g_i : f \rightarrow p_i$ and $g_j : f \rightarrow p_j$ be two morphisms in f/F, that is, $f = p_i \circ g_i$ and $f = p_j \circ g_j$. By the coproduct-presentability of B, we have $i = j$ and $g_i = g_j$. Thus f/F is connected.

Corollary 3.10 *Let* $\mathbf{A}$ *be a coproduct-accessible category, then*

(i) $\mathbf{A}_c$ *is essentially small, and an object of* $\mathbf{A}$ *is coproduct presentable if and only if it is a retract of some object in* $\mathbf{B}$*, the category* $\mathbf{B}$ *referred to in Definition 3.7.*

(ii) $\mathbf{A}_c$ *is dense in* $\mathbf{A}$, *i.e. for every object* A *of* $\mathbf{A}$, *the canonical cocone*

$$\iota_A : (\mathbf{A}_c/A)^+ \to \mathbf{A}$$

with vertex A *is colimiting.*

Proof: The proof of (i) is essentially same as Proposition 2.1.5(i) in [24]; (ii) follows from Theorem IX.3.1 in [17] and Proposition 3.9.

Proposition 3.11 *Let* $\mathbf{A}$ *be a coproduct-accessible category,* $\mathbf{C}^{op} = \mathbf{A}_c$, *and let* $i : \mathbf{C}^{op} \to \mathbf{A}$ *be the inclusion. Then the functor*

$$\sum : \mathbf{A} \to (\mathbf{C}, \mathbf{S})$$

$$A \longmapsto \mathbf{A}(i(-), A)$$

is full and faithful, and its essential image consists of the coproduct-flat functors from $\mathbf{C}$ *to* $\mathbf{S}$, *i.e. we have an equivalence*

$$\mathbf{A} \simeq c - Flat(\mathbf{C}).$$

Also, the diagram

$$\begin{tikzcd} \mathbf{C}^{op} \arrow[r, "i"] \arrow[d, "Z"'] & \mathbf{A} \arrow[dl, "\Sigma"] \\ c - Flat(\mathbf{C}) & \end{tikzcd}$$

commutes (for Z, see Proposition 3.4).

Proof: For any C in $\mathbf{C}$, by the coproduct presentability of C, the functor $\Sigma(-)(C)$: $\mathbf{A} \to \mathbf{S}$ preserves coproducts. That Σ preserves coproducts follows from the fact that colimits are computed pointwise in $(\mathbf{C}, \mathbf{S})$. The density of $\mathbf{A}_c$ in $\mathbf{A}$ is equivalent to saying that Σ is full and faithful. Note that $\Sigma(C) \cong \mathbf{C}(C, -)$ for every C in $\mathbf{C}$, and c-Flat($\mathbf{C}$) consists of those objects which are coproducts of objects of the form $\Sigma(C)$. Since $\mathbf{A}$ has small coproducts and Σ is a full and faithful functor preserving them, we have that the essential image of Σ is c-Flat($\mathbf{C}$).

The last assertion of the proposition is immediate from the definitions.

Corollary 3.12 *A category is coproduct-accessible if and only if it is equivalent to c-Flat($\mathbf{C}$) for some small category $\mathbf{C}$.*

Corollary 3.13 *In a coproduct-accessible category, we have*

(*i*) *Small coproducts are stable under pullback, and*

(*ii*) *Small coproducts commute with small connected limits to the extent that the latter exist.*

Proof: Note that c-Flat($\mathbf{C}$) is closed under the small coproducts in the functor category ($\mathbf{C}$,$\mathbf{S}$), and in the latter, small coproducts are stable under pullback. As for (ii), the full and faithful functor $\sum : \mathbf{A} \to (\mathbf{C}, \mathbf{S})$ of Proposition 3.11 preserves small coproducts, as well as all limits existing in $\mathbf{A}$, the reason being that representable functors preserve all existing limits. Also note that small coproducts commute with small connected limits in the functor category.

3.3 Duality of coproduct accessible categories

Let $\mathcal{A}$ denote the 2-category of all categories with small connected limits and small colimits, with functors preserving such limits and small colimits as 1-arrows, and all natural transformations as 2-arrows. Let $\mathcal{B}$ be the 2-category of categories with small coproducts and functors that preserve small coproducts, whose 2-arrows are all natural transformations between them. **S**, the category of small sets, is an object living in both $\mathcal{A}$ and $\mathcal{B}$ such that the two structures on **S** commute with each other, i.e. small coproducts commute with small connected limits and all colimits in **S**. Such a state of affairs gives rise to a pair of adjoint functors. We obtain the 2-adjunction

$$\mathcal{A}^{op} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{B}$$

$$\eta : id_{\mathcal{B}} \longrightarrow G \circ F (unit)$$

$$\varepsilon : F \circ G \longrightarrow id_{\mathcal{A}^{op}} (counit)$$

given by

$$G = CoCocts(-, \mathbf{S}), F = \coprod(-, \mathbf{S});$$

both unit and counit are defined as the following evaluation functors.

$$\eta_{\mathbf{B}} : \mathbf{B} \longrightarrow CoCocts(\coprod(\mathbf{B}, \mathbf{S}), \mathbf{S})$$

$$\varepsilon_{\mathbf{A}} : \mathbf{A} \longrightarrow \coprod(CoCocts(\mathbf{A}, \mathbf{S}), \mathbf{S})$$

We consider the full sub-2-category $\mathcal{C}$ of $\mathcal{A}$ on the objects that are categories equivalent to one of the form (**C**, **S**) for a small category **C** with split idempotents, and the full sub-2-category $\mathcal{D}$ of $\mathcal{B}$ whose objects are coproduct-accessible categories **A** so that $\mathbf{A}_c$ has split idempotents. By Proposition 3.4, Corollary 3.5, and Corollary 3.12, we have the 2-adjunction

$$\mathcal{C}^{op} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$$

with the restricted unit and counit

$$\eta : id_{\mathcal{D}} \longrightarrow G \circ F, \varepsilon : F \circ G \longrightarrow id_{\mathcal{C}^{op}}.$$

Theorem 3.14 *The 2-adjunction $(F, G, \eta, \varepsilon)$ between $\mathcal{C}$ and $\mathcal{D}$ is a biequivalence, in other words,*

(i) if $\mathbf{C}$ is in $\mathcal{C}$, then $CoCocts(\mathbf{C}, \mathbf{S}) \in \mathcal{D}$, and the evaluation functor

$$\varepsilon_{\mathbf{C}} : \mathbf{C} \longrightarrow \coprod(CoCocts(\mathbf{C}, \mathbf{S}), \mathbf{S})$$

is an equivalence;

(ii) if $\mathbf{A}$ is in $\mathcal{D}$, then $\coprod(\mathbf{A}, \mathbf{S}) \in \mathcal{C}$, and the evaluation functor

$$\eta_{\mathbf{A}} : \mathbf{A} \longrightarrow CoCocts(\coprod(\mathbf{A}, \mathbf{S}), \mathbf{S})$$

is an equivalence.

Proof (i) follows from Proposition 3.4 and Corollary 3.5. (ii) can be obtained from Corollary 3.4 and Corollary 3.12.

CHAPTER 4

DUALITY FOR DIERS CATEGORIES

4.1 Coproduct-accessible categories with κ-limits

Proposition 4.1 *Let* $\mathbf{C}$ *be a coproduct-accessible category. For every object* A *of* $\mathbf{C}$*, the slice category* $\mathbf{C} \downarrow A$ *is coproduct-accessible.*

Proof: Consider the forgetful functor $U : \mathbf{C} \downarrow A \to \mathbf{C}$. U creates colimits, hence $\mathbf{C} \downarrow A$ has small coproducts.

If C is a coproduct presentable object in $\mathbf{C}$, then any morphism $k : C \to A$ is a coproduct presentable object in $\mathbf{C} \downarrow A$. In fact, given a set of arrows $f_i : C_i \to A$, consider the coproduct $\coprod C_i$ with coprojections q_i. By the universal property of coproduct, there is a unique morphism $f : \coprod C_i \to A$ such that $f_i = f \circ q_i$ for all i. Such a morphism f is the coproduct of f_i in $\mathbf{C} \downarrow A$. If $h : k \to f$, then, since $c \in \mathbf{C}_c$, we have a unique i and a unique $l : C \to C_i$ such that

$$\begin{tikzcd}
C_i \arrow[r, "f_i"] & A \\
C \arrow[u, "l"] \arrow[ur, "k"] \arrow[r, "h"'] & \coprod C_i \arrow[u, "f"']
\end{tikzcd}$$

commutes; this means $h : k \to \coprod f_i$ factors uniquely through a unique f_i, showing that k is coproduct presentable.

For any object $g : B \to A$ of $\mathbf{C} \downarrow A$, let $B = \coprod C_i$ with coprojections $q_i : C_i \to B$, we already have that $g = \coprod g \circ q_i$, and $g \circ q_i$ are coproduct presentable objects in $\mathbf{C} \downarrow A$.

Remark 4.2 *Note that if* $\mathbf{C}$ *has* κ*-limits, then the slice category* $\mathbf{C} \downarrow A$ *has* κ*-limits too.*

Let $\mathbf{X}$ be any category. Recall ([24], p.115) that the category of families in $\mathbf{X}$, Fam($\mathbf{X}$), has as objects pairs (I, X) where I is a set and X is an I-indexed family of objects X_i of $\mathbf{X}$. A morphism $(I, X) \to (J, Y)$ in Fam($\mathbf{X}$) is a pair (f, x), here $f : I \to J$ is a function and x is a family of morphisms $x_i : X_i \to Y_{f(i)}$ indexed by I. Composition of morphisms is given by $(g, y) \circ (f, x) = (g \circ f, f^*y \cdot x)$ where $(f^*y \cdot x)_i = y_{f(i)} \circ x_i$.

Proposition 4.3 *Let* $\mathbf{C}$ *be an arbitrary coproduct-accessible category, and* $i : \mathbf{C}_c \to \mathbf{C}$ *be the inclusion functor. Then the functor*

$$\Sigma : Fam(\mathbf{C}_c) \to \mathbf{C}$$

$$(I, < C_i >) \longmapsto \coprod C_i$$

is an equivalence of categories.

Proof: The functoriality of Σ is directly verified by the universal property of coproduct. To show that Σ is full and faithful, let $(I, < C_i >)$ and $(J, < D_j >)$ be any two objects in $Fam(\mathbf{C}_c)$, and $f : \coprod C_i \to \coprod D_j$ is a morphism. Let $p_i : C_i \to \coprod C_i$ be the coprojections of $\coprod C_i$, $q_j : D_j \to \coprod D_j$ are the coprojections of $\coprod D_j$, for each $i \in I$, by the coproduct presentability of C_i, we have $f \circ p_i = q_{i'} \circ f_i$ for a unique morphism $f_i : C_i \to D_{i'}$. Thus we define a morphism $(k, < f_i >)$ between $(I, < C_i >)$ and $(J, < D_j >)$ as follows.

$$k : I \to J \,(i \mapsto i')$$

Then $\Sigma((k, < f_i >)) = f$. This proves the fullness of Σ, the faithfulness of Σ follows from the uniqueness of f_i. Finally , Σ is surjective on objects, since every object of $\mathbf{C}$ is a coproduct of coproduct presentable objects.

Remark 4.4 *As shown in [24], if category $\mathbf{X}$ is accessible, then $Fam(\mathbf{X})$ is an accessible category. Given a coproduct-accessible category $\mathbf{C}$, $\mathbf{C}_c$ is a small category. If $\mathbf{C}_c$ has split idempotents, by Theorem 2.2.2 in [24], $\mathbf{C}_c$ is accessible, so in that case , $\mathbf{C}$ is accessible.*

Recall that a category $\mathbf{A}$ is κ-multicocomplete, if it has the multicolimits of diagrams of size$< \kappa$. The notion of κ-multicomplete is dual to that of κ-multicocomplete.

Proposition 4.5 *Let $\mathbf{C}$ be a coproduct-accessible category with κ-limits. We have*

(*i*) *$\mathbf{C}_c$ is κ-multicomplete; and*

(*ii*) *$\mathbf{C}_c$ has split idempotents.*

Proof: (i) Let $\mathbf{I}$ be a graph of size$< \kappa$. For any diagram $G : \mathbf{I} \to \mathbf{C}_c$, let $C = limG$ in $\mathbf{C}$, with limit projections $< p_i : C \to C_i >$. For any cone $< f_i : D \to C_i >$ in $\mathbf{C}_c$, there is a unique morphism $f : D \to C$ such that $f_i = p_i \circ f$. Let $C = \coprod A_j$ with all A_j in $\mathbf{C}_c$. The coproduct presentability of D gives that f factors uniquely through a coproduct coprojection q_j, i.e. there is a unique morphism $g : D \to A_j$ so that $f = q_j \circ g$. We have

$$f_i = p_i \circ (q_j \circ g) = (p_i \circ q_j) \circ g.$$

This shows that every cone on $\mathbf{I}$ factors uniquely through a unique cone $< p_i \circ q_j : A_j \to C_i >$.

(ii) Given an idempotent $e : A \to A$ in $\mathbf{C}_c$, we need to show that e factors as $e = f \circ g$ with $g \circ f = id$ in $\mathbf{C}_c$. Consider the equalizer of (id_A, e), say $f : D \to A$. Since $e = e \circ e$, there is an unique morphism $h : A \to D$ such that $e = f \circ h$. Let $D = \coprod D_i$ with the coprojections $g_i : D_i \to D$, where all D_i in $\mathbf{C}_c$. The coproduct presentability of A gives that h factors uniquely through a coprojection g_i, thus $e = f \circ (g_i \circ h') = (f \circ g_i) \circ h'$. But $f = e \circ f$, hence $f \circ g_i = e \circ (f \circ g_i) = (f \circ g_i) \circ h' \circ (f \circ g_i)$. Note that the coproduct coprojections are monomorphisms in coproduct-accessible category, also the equalizer f is monomorphism , therefore $h' \circ (f \circ g_i) = id$.

Theorem 4.6 *Every coproduct-accessible category with κ-limits is accessible.*

Proof: By Proposition 2.2.2 in [24], every small category with split idempotents is accessible. Given a coproduct-accessible category $\mathbf{C}$ with κ-limits, $\mathbf{C}_c$ is accessible by proposition 4.5 (ii). We conclude that $\mathbf{C}$ is accessible from Remark 4.4.

Proposition 4.7 *Let $\mathbf{A}$ be an arbitrary κ-Diers category. Then $CoF_\kappa(\mathbf{A}, \mathbf{S})$ is a coproduct-accessible category having κ-limits, and the full subcategory of coproduct presentable objects of $CoF_\kappa(\mathbf{A}, \mathbf{S})$ is equivalent to $\mathbf{A}_\kappa^{op}$, the opposite category of the full subcategory of κ-presentable objects of $\mathbf{A}$.*

Proof: Note that κ-limits and coproducts commute with connected limits and κ-filtered colimits in $\mathbf{S}$, hence the category $CoF_\kappa(\mathbf{A}, \mathbf{S})$ is closed under the κ-limits and coproducts in $(\mathbf{A}, \mathbf{S})$. For any κ-presentable object A in $\mathbf{A}$, the functor $\mathbf{A}(A, \text{-})$ is in $CoF_\kappa(\mathbf{A}, \mathbf{S})$. Using the Yoneda Lemma and the fact that small colimits in $(\mathbf{A}, \mathbf{S})$ are computed pointwise, it is easy to see that $\mathbf{A}(A, \text{-})$ is a coproduct presentable object in $CoF_\kappa(\mathbf{A}, \mathbf{S})$. By Remark 2.6, every functor F in $CoF_\kappa(\mathbf{A}, \mathbf{S})$ is multi-representable, i.e. it is a coproduct of representable functors $\mathbf{A}(A_i, -)$; here A_i is κ-presentable. Thus $CoF_\kappa(\mathbf{A}, \mathbf{S})$ is a coproduct-accessible category with κ-limits.

Corollary 4.8 *Let $\mathbf{A}$ be an arbitrary κ-Diers category. Then $\mathbf{A}_\kappa$ is κ-multicocomplete.*

Proof: By proposition 4.5(i) and Proposition 4.7.

Corollary 4.9 *A category is coproduct-accessible with κ-limits if and only if it is equivalent to the category of the form $c - Flat(\mathbf{C})$; here $\mathbf{C}$ is a small category with κ-multicolimits.*

Proof: By Corollary 3.13, a category is coproduct-accessible if and only if it is equivalent to $c - Flat(\mathbf{C})$ with $\mathbf{C}$ small. If $\mathbf{C}$ is a small category with κ-multicolimits, by Theorem 3.0 in [11], then $\kappa - Flat(\mathbf{C}^{op})$ is κ-Diers. Note that $c - Flat(\mathbf{C})$ is equivalent to the category $CoF_\kappa(\kappa - Flat(\mathbf{C}^{op}), \mathbf{S})$ in Proposition 4.7, hence $c - Flat(\mathbf{C})$ has κ-limits. Conversely, if $c - Flat(\mathbf{C})$ has κ-limits, by Proposition 4.5 (i), then $\mathbf{C}$ is κ-multicocomplete.

4.2 Duality for Diers categories

Proposition 4.10 *Let $\mathbf{C}$ be a coproduct-accessible category with κ-limits, and $i : \mathbf{C}_c \to \mathbf{C}$ be the inclusion functor. Then the functor*

$$\sum : L_\kappa \coprod(\mathbf{C}, \mathbf{S}) \to (\mathbf{C}_c, \mathbf{S})$$

$$F \longmapsto F \circ i$$

is full and faithful, and its essential image consists of the κ-flat functors from $\mathbf{C}_c$ to $\mathbf{S}$, i.e. we have an equivalence

$$L_\kappa \coprod(\mathbf{C}, \mathbf{S}) \simeq \kappa - Flat(\mathbf{C}_c).$$

Proof: First of all, for every functor M in $L_\kappa \coprod(\mathbf{C}, \mathbf{S})$, we have that $M \circ i \in \kappa - Flat(\mathbf{C}_c)$. Indeed, by Theorem 1.2.2 in [24], this is equivalent to saying that the category $el(M \circ i)$, the category of elements of $M \circ i$, is κ-filtered. Let I be a graph of size less than κ, and a diagram

$$D : I \to el(M \circ i)$$

$$k \longmapsto x_k \in M \circ i(C_k)$$

Since $\mathbf{C}$ has κ-limits, let $(\pi_k : C \to C_k)_{k \in I}$ be the limit cone in $\mathbf{C}$. So $(M(\pi_k) : M(C) \to M(C_k))_{k \in I}$ is a limit cone in $\mathbf{S}$. Assume that $C = \coprod C_j$ with $C_j \in \mathbf{C}_c$, then $M(C)$ is the disjoint union of $M(C_j)$. It follows that there is some C_j and $x \in M(C_j)$ so that $M(\pi_k)(x) = x_k$ for each $k \in I$.

The fact that $\sum$ is full and faithful follows from Proposition 3.4. Also, note that $\kappa - Flat(\mathbf{C}_c)$ consists exactly of those objects which are κ-filtered colimits of the form $\sum(\mathbf{C}(C, -))$ with $C \in \mathbf{C}_c$. We conclude that the essential image of $\sum$ is $\kappa - Flat(\mathbf{C}_c)$.

Corollary 4.11 *For any coproduct-accessible category* $\mathbf{C}$ *with* κ*-limits, the category* $L_\kappa\coprod(\mathbf{C},\mathbf{S})$ *is a* κ*-Diers category; moreover, the full subcategory of* κ*-presentable objects in* $L_\kappa\coprod(\mathbf{C},\mathbf{S})$ *is equivalent to the opposite category of* $\mathbf{C}_c$.

Proof: That the category $L_\kappa\coprod(\mathbf{C},\mathbf{S})$ has connected limits and κ-filtered colimits follows from the fact that connected limits and κ-filtered colimits commute with κ-limits and coproducts in $\mathbf{S}$. So, by Proposition 4.10 and Theorem 2.4, we conclude that $L_\kappa\coprod(\mathbf{C},\mathbf{S})$ is a κ-Diers category.

Let $\mathbf{A}$ be a κ-Diers category. We have the evaluation functor

$$e_{\mathbf{A}} : \mathbf{A} \to L_\kappa\coprod(CoF_\kappa(\mathbf{A},\mathbf{S}),\mathbf{S})$$

defined by

$$A \longmapsto [M \longmapsto M(A)]$$

$$A \longmapsto [h \longmapsto h_A]$$

$$f \longmapsto [M \longmapsto M(f)].$$

By Proposition 4.7, $CoF_\kappa(\mathbf{A},\mathbf{S})$ is a coproduct-accessible category having κ-limits, and the full subcategory of coproduct presentable objects in $CoF_\kappa(\mathbf{A},\mathbf{S})$ is equivalent to the opposite category of $\mathbf{A}_\kappa$. Proposition 4.10 gives an equivalence

$$\sum : L_\kappa \coprod (CoF_\kappa(\mathbf{A}, \mathbf{S}), \mathbf{S}) \to \kappa - Flat(\mathbf{A}_\kappa^{op}).$$

Let E be the functor

$$E : \mathbf{A} \to \kappa - Flat(\mathbf{A}_\kappa^{op})$$

$$A \longmapsto \mathbf{A}(i(-), A)$$

By Proposition 2.1.8 in [24], E is an equivalence. Clearly , $E = \sum \circ e_{\mathbf{A}}$. Thus we have

Theorem 4.12 *For any κ-Diers category* **A**, *the canonical functor $e_{\mathbf{A}}$ is an equivalence of categories.*

Let $\mathcal{A}$ be the 2-category of all categories with κ-limits and coproducts as objects and functors preserving κ-limits and coproducts as arrows, and all natural transformations between them as 2-arrows. $\mathcal{B}$ the 2-category of all categories with connected limits and κ-filtered colimits and functors preserving κ-filtered colimits and connected limits. **S** is an object living in both $\mathcal{A}$ and $\mathcal{B}$ such that the two structures on **S** commute with each other . We obtain the 2-adjunction

$$\mathcal{A}^{op} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$$

Here $F = CoF_\kappa(-,\mathbf{S}), G = L_\kappa\coprod(-,\mathbf{S})$; both unit and counit are defined, at any object of the respective kind, as the evaluation functor. We consider the full sub-2-category $L_\kappa\coprod - \mathbf{ACC}$ of $\mathcal{A}$ whose objects are coproduct-accessible with κ-limits, and the full sub-2-category $\kappa - \mathbf{Diers}$ of $\mathcal{B}$ whose objects are κ-Diers categories. By Proposition 4.7 , 4.11, we have the 2-adjunction

$$L_\kappa\coprod - \mathbf{ACC}^{op} \underset{G}{\overset{F}{\rightleftarrows}} \kappa - \mathbf{Diers}$$

with restricted unit and counit, i.e. the component $e_{\mathbf{A}}$ at $\mathbf{A}$ in $\kappa - \mathbf{Diers}$ is the functor in Theorem 4.12, the counit ε at $\mathbf{C}$ in $L_\kappa\coprod - \mathbf{ACC}$ is defined as the evaluation functor

$$\varepsilon_{\mathbf{C}} : \mathbf{C} \longrightarrow CoF_\kappa(L_\kappa\coprod(\mathbf{C},\mathbf{S}),\mathbf{S}).$$

Theorem 4.13 *The pair of adjoint 2-functors obtained is a biequivalence. in other words,*

(*i*) *if* $\mathbf{A}$ *in* $\kappa - \mathbf{Diers}$, *then* $CoF_\kappa(\mathbf{A},\mathbf{S})$ *in* $L_\kappa\coprod - \mathbf{ACC}$, *and* e_A *is an equivalence;*

(*ii*) *if* $\mathbf{C}$ *in* $L_\kappa\coprod - \mathbf{ACC}$, *then* $L_\kappa\coprod(\mathbf{C},\mathbf{S})$ *in* $\kappa - \mathbf{Diers}$, *and* $\varepsilon_{\mathbf{C}}$ *is an equivalence.*

Proof: By Proposition 4.7, 4.11, and Theorem 4.12, we only need to show that $\varepsilon_{\mathbf{C}}$ is an equivalence of categories, for any coproduct-accessible category $\mathbf{C}$ with κ-limits. First of all, we show that $\varepsilon_{\mathbf{C}}$ is a full and faithful functor. Proposition 1.2.4 (ii) in [24] and Proposition 4.10 give an equivalence $Y^* : (\mathbf{C}_c{}^{op},\mathbf{S}) \to F_\kappa(L_\kappa\coprod(\mathbf{C},\mathbf{S}),\mathbf{S})$. By observation, the following diagram

$$\begin{tikzcd} \mathbf{C} \arrow[r, "\varepsilon_{\mathbf{C}}"] \arrow[d, "\Sigma"'] & F_\kappa(L_\kappa\coprod(\mathbf{C},\mathbf{S}),\mathbf{S}) \\ (\mathbf{C}_c{}^{op},\mathbf{S}) \arrow[ur, "\gamma^*"'] & \end{tikzcd}$$

commutes. We obtain that $\varepsilon_{\mathbf{C}}$ is full and faithful from Proposition 3.12.

Secondly, $\varepsilon_{\mathbf{C}}$ in Theorem 4.13 (ii) is essentially surjective on objects. Indeed, for any functor M in $CoF_\kappa(L_\kappa\coprod(\mathbf{C},\mathbf{S}),\mathbf{S})$, by Proposition 4.7 and Proposition 4.11, M is a pointwise coproduct of $Nat(\mathbf{C}(C_i,-),-)$, i.e. $\varepsilon_{\mathbf{C}}(C_i)$; here all C_i in $\mathbf{C}_c$. But the evaluation functor $\varepsilon_{\mathbf{C}}$ preserves coproducts, we have $M = \varepsilon_{\mathbf{C}}(\coprod C_i)$.

The following conceptual completeness theorem is an immediate consequence of Theorem 4.13.

Proposition 4.14 (*i*) *If $F : \mathbf{C} \to \mathbf{D}$ is a functor between coproduct-accessible categories having κ-limits that preserves coproducts and κ-limits, if the induced functor*

$$F^* : L_\kappa\coprod(\mathbf{D},\mathbf{S}) \to L_\kappa\coprod(\mathbf{C},\mathbf{S})$$

is an equivalence of categories, then F itself is an equivalence.

(*ii*) *If $M : \mathbf{A} \to \mathbf{B}$ is a functor between κ-Diers categories preserving connected limits and κ-filtered colimits, and the induced functor*

$$M^* : CoF_\kappa(\mathbf{B}, \mathbf{S}) \to CoF_\kappa(\mathbf{A}, \mathbf{S})$$

is an equivalence of categories, then M is an equivalence .

Let $\mathbf{C}$ be a κ-multicocomplete category. Recall from [11] that a functor $F : \mathbf{C} \to \mathbf{S}$ is multicontinuous, means that, for any diagram $D : \mathbf{G} \to \mathbf{C}$ of size less than κ, if the multicolimit of D is $< C_i >_{i \in I}$, then $colim F \circ D = \coprod_{i \in I} F(C_i)$. $Mul_\kappa(\mathbf{C}, \mathbf{S})$ denotes the full subcategory of $(\mathbf{C}, \mathbf{S})$ consisting of the multicontinuous functors from $\mathbf{C}$ into $\mathbf{S}$. The following result also proved in [11](Corollary 6.2).

Corollary 4.15 *If $\mathbf{C}$ and $\mathbf{D}$ are small κ-multicocomplete categories, and the categories $Mul_\kappa(\mathbf{C}^{op}, \mathbf{S})$ and $Mul_\kappa(\mathbf{D}^{op}, \mathbf{S})$ are equivalent, then $\mathbf{C}$ and $\mathbf{D}$ are equivalent.*

Proof: For any small κ-multicocomplete category $\mathbf{C}$, as shown in [11], $Mul_\kappa(\mathbf{C}^{op}, \mathbf{S})$ is a κ-Diers category.

Here is a set of conditions ensuring that an accessible category is a κ-Diers category (also see Proposition 6.1.8 in [24]).

Proposition 4.16 *Suppose $\mathbf{B}$ is a κ-Diers category, $\mathbf{A}$ is an accessible category with κ-filtered colimits and small connected limits, $F : \mathbf{A} \to \mathbf{B}$ preserves them, and F is conservative, i.e. it reflects isomorphisms. Then $\mathbf{A}$ is κ-Diers.*

Proof: For any $B \in \mathbf{B}_\kappa$, the composite of F and the representable functor $\mathbf{B}(B, -)$ preserves small connected limits and κ-filtered colimits; hence we have a small jointly conservative family consisting of $\mathbf{B}(B, -) \circ F$ with $B \in \mathbf{B}_\kappa$. By Proposition 6.1.8 in [24], we obtain that $\mathbf{A}$ is κ-Diers.

Remark 4.17 *In case of κ-Barr category, Proposition 4.16 fails. In fact, take a small κ-Barr-exact category* $\mathbf{C}$*, the inclusion functor $\kappa - Reg(\mathbf{C}, \mathbf{S}) \to L_\kappa(\mathbf{C}, \mathbf{S})$ preserves κ-filtered colimits and products. But $\kappa - Reg(\mathbf{C}, \mathbf{S})$ is not necessarily κ-Barr, unless* $\mathbf{C}$ *is κ-Barr-exact accessible (see next chapter).*

CHAPTER 5

DUALITY FOR BARR CATEGORIES

5.1 Barr-exact accessible categories

Let $\mathbf{C}$ be an arbitrary category. An object C of $\mathbf{C}$ is called projective presentable (regular epi projective) if the representable functor $\mathbf{C}(C,-) : \mathbf{C} \to \mathbf{SET}$ takes regular epimorphisms into surjective morphisms.

C is regular epi projective means that C is a projective object with respect to the class of regular epimorphisms of $\mathbf{C}$, i.e. if $e : A \to B$ is a regular epimorphism in $\mathbf{C}$, then every morphism from C into B factors through e.

The full subcategory of $\mathbf{C}$ whose objects are the regular epi projective ones is denoted by $\mathbf{C}_p$.

Definition 5.1 *A category* $\mathbf{C}$ *is called projective-accessible if*

(*i*) $\mathbf{C}$ *has kernel-pairs of arbitrary arrows and has coequalizers of kernel-pairs;*

(*ii*) *there is a small full subcategory* $\mathbf{B}$ *of* $\mathbf{C}$ *consisting of projective presentable objects such that for any object* C *of* $\mathbf{C}$ *there is* B *in* $\mathbf{B}$, *and a regular epimorphism from* B *into* C.

Proposition 5.2 *Let* $\mathbf{C}$ *be a projective-accessible category. Then,* $\mathbf{C}$ *is essentially small.*

Proof: In fact, given C in $\mathbf{C}$, let $\eta : B \to C$ be a regular epi morphism with $B \in \mathbf{B}$, and $< f, g : D \to B >$ be the kernel-pair of η. Take a regular epi $e : B' \to D$ with B' in $\mathbf{B}$, then we have a coequalizer diagram

$$B' \overset{f'}{\underset{g'}{\rightrightarrows}} B \xrightarrow{\eta} C$$

Here $f' = f \circ e$ and $g' = g \circ e$. It follows that $\mathbf{C}$ is essentially small from the smallness of $\mathbf{B}$.

Definition 5.3 *Let* $\mathbf{C}$ *be a projective-accessible category.* $\mathbf{C}$ *is called* κ*-Barr-exact accessible if it is a Barr-exact category with* κ*-limits.* $\mathbf{C}$ *is Barr-exact accessible if it is* κ*-Barr-exact accessible for some* κ.

Later, we will show that any κ-Barr-exact category $\mathbf{C}$ is κ-Barr-exact, i.e. $\mathbf{C}$ satisfies the principle of $< \kappa$ dependent choices (DC_κ).

Let $\mathbf{C}$ be an arbitrary Barr-exact category, and $\mathbf{D}$ a full subcategory of $\mathbf{C}$. We give a condition as follows ensuring that $\mathbf{D}$ is dense in $\mathbf{C}$, i.e. the 'restricted' Yoneda embedding $Y : \mathbf{C} \to (\mathbf{D}^{op}, \mathbf{S})$ is full and faithful. A similar argument was given by M. Barr (see Theorem 14. in [5]).

Proposition 5.4 *Let* $\mathbf{C}$ *be a Barr-exact category, and* $\mathbf{D}$ *a full subcategory of* $\mathbf{C}$. *If for every object* C *of* $\mathbf{C}$ *there is a regular epimorphism from* D *into* C *with* D *in* $\mathbf{D}$, *then* $\mathbf{D}$ *is dense in* $\mathbf{C}$.

Proof: We have to show that for any $C \in \mathbf{C}$, the canonical cocone $y_C : (\mathbf{D}/C)^+ \longrightarrow \mathbf{C}$ with vertex C is colimiting. Let C be a fixed object of $\mathbf{C}$, and take an arbitrary cocone

$$(f : D \to C) \longmapsto (f^* : D \to C')$$

on $y_C|(\mathbf{D}/C)$; for any morphism g in $\mathbf{D}$, if g and f are composable, we have $(f \circ g)^* = f^* \circ g$. We will prove that there is a unique morphism $h : C \to C'$ such that $f^* = h \circ f$ for every $f : D \to C$.

Take a regular epimorphism $e : A \to C$ with $A \in \mathbf{D}$. Let e be the coequalizer of the kernel pair $(u, v : A' \to A)$, and $a : B \to A'$ be a regular epimorphism with $B \in \mathbf{D}$. Then e is coequalizer of the pair of $(u \circ a, v \circ a)$. Writing $u' = u \circ a$, $v' = v \circ a$, since A and B are in $\mathbf{D}$. We have

$$(e \circ u')^* = e^* \circ u', (e \circ v')^* = e^* \circ v',$$

hence $e^* \circ u' = e^* \circ v'$. By the universal property of coequalizer , there is a unique morphism $h : C \to C'$ such that $e^* = h \circ e$.

It remains to show that for any $f : D \to C$, we have $f^* = h \circ f$. Consider the following diagram

$$\begin{tikzcd}
P \arrow[r, "k"] & K \arrow[r, "f'"] \arrow[d, "e'"'] & A \arrow[d, "e"] & \\
& D \arrow[r, "f"'] & C \arrow[r, "h"'] & C'
\end{tikzcd}$$

Here K is the pullback of morphisms (e, f), and k is a regular epimorphism with $P \in \mathbf{D}$. Since $\mathbf{C}$ is a Barr-exact category, e' is a regular epimorphism too. We have

$$h \circ e \circ f' \circ k = e^* \circ f' \circ k = (e \circ f' \circ k)^* = (f \circ e' \circ k)^* = f^* \circ e' \circ k.$$

Thus $h \circ f \circ e' \circ k = f^* \circ e' \circ k$. Note that a composite of two regular epimorphisms is a regular epimorphism , thus $e' \circ k$ is a regular epimorphism. It follows that $f^* = h \circ f$ as required.

The uniqueness of h is assured by the equality $e^* = h \circ e$ as we noted above.

Corollary 5.5 *Let* $\mathbf{C}$ *be a Barr-exact accessible category, and* $i : \mathbf{C}_p \to \mathbf{C}$ *is the inclusion. Then* $\mathbf{C}_p$ *is dense in* $\mathbf{C}$, *and the functor*

$$\sum : \mathbf{C} \longrightarrow (\mathbf{C}_p{}^{op}, \mathbf{S})$$

$$C \longmapsto \mathbf{C}(i(-), C)$$

is a full and faithful regular functor. If $\mathbf{C}$ *is* κ*-Barr-exact accessible, then* $\mathbf{C}$ *is* κ*-Barr-exact.*

Proof: For any C in $\mathbf{C}_p$, by the regular epi projective presentability of C, the functor $\sum(-)(C) : \mathbf{C} \to \mathbf{S}$ is regular. That $\sum$ is a regular functor follows from the fact that limits and colimits are computed pointwise in $(\mathbf{C}_p{}^{op}, \mathbf{S})$. By Proposition 5.3, $\mathbf{C}_p$ is dense in $\mathbf{C}$. The density of $\mathbf{C}_p$ in $\mathbf{C}$ is equivalent to saying that $\sum$ is full and faithful.

Note that the functor category $(\mathbf{C}_p{}^{op}, \mathbf{S})$ satisfies the principle of $< \kappa$ dependent choices (DC_κ), so does $\mathbf{C}$.

Let $\mathbf{A}$ be an arbitrary locally small category, it has products and κ-filtered colimits, and there is a small full subcategory $\mathbf{B}$ of $\mathbf{A}$ so that every object in $\mathbf{A}$ is a κ-filtered colimit of a diagram of objects in $\mathbf{B}$. Makkai has proved that the category $\prod F_\kappa(\mathbf{A}, \mathbf{S})$ is an essentially small κ-Barr-exact category (see Proposition 6.9. in [21]). In particular, if $\mathbf{A}$ is κ-accessible with products, these conditions are satisfied. Following Makkai's proof of the above result, we establish the following proposition.

Proposition 5.6 *Let* $\mathbf{A}$ *be a* κ*-accessible category with small products. Then for any functor* F *in* $\prod F_\kappa(\mathbf{A}, \mathbf{S})$*, there is a* κ*-presentable object* A *in* $\mathbf{A}$*, and a regular epimorphism* $\eta : \mathbf{A}(A, -) \to F$ *. Moreover, every* $F \in \prod F_\kappa(\mathbf{A}, \mathbf{S})$ *is the codomain object of a coequalizer of a pair of morphisms between the representable functors represented by* κ*-presentable objects.*

Proof: Let $\mathbf{B}$ be a small full subcategory of $\mathbf{A}$ consisting of κ-presentable objects so that every object of $\mathbf{A}$ is a κ-filtered colimit of a diagram of objects in $\mathbf{B}$. Given a functor $F \in \prod F_\kappa(\mathbf{A}, \mathbf{S})$, for every $B \in \mathbf{B}$, let us enumerate all elements of $F(B)$:$< a_i^B >_{i \in J_B}$; here J_B is an ordinal number. Consider the small product $\prod_{B \in \mathbf{B}} B^{J_B}$ in $\mathbf{A}$. The product is the colimit of a κ-filtered diagram $(< B_s >_{s \in S}, < a_{st} : B_s \to B_t >_{s \leq t})$ with the colimit coprojections $< e_s : B_s \to \prod_{B \in \mathbf{B}} B^{J_B} >_{s \in S}$, where each $B_s \in \mathbf{B}$. Let K be the join of all J_B, and by $< a_k >_{k \in K}$ denoting the set of elements of $F(B)$ with all $B \in \mathbf{B}$, $\pi_k : \prod_{B \in \mathbf{B}} B^{J_B} \to B$ be the product projections. Since F preserves products, then $F(\prod_{B \in \mathbf{B}} B^{J_B}) = \prod_{B \in \mathbf{B}} F(B)^{J_B}$, and $F(\pi_k)$ are the product projections in $\mathbf{S}$. We have that there is $a \in F(\prod_{B \in \mathbf{B}} B^{J_B})$ such that $F(\pi_k)(a) = a_k$, for all $k \in K$. Also note that F preserves κ-filtered colimits, the morphisms $F(e_s)$: $F(B_s) \to F(\prod_{B \in \mathbf{B}} B^{J_B})$ make $F(\prod_{B \in \mathbf{B}} B^{J_B})$ a κ-filtered colimit of the diagram $(< F(B_s) >_{s \in S}, < F(a_{st}) >_{s \leq t})$ in $\mathbf{S}$. Thus there is $s \in S$, and some $c \in F(B_s)$ such that $F(e_s)(c) = a$. It follows that

$$F(\pi_k \circ e_s)(c) = a_k, k \in K$$

We use A for B_s. The Yoneda Lemma gives a natural transformation $\eta : \mathbf{A}(A, -) \to F$, defined by $\eta(id_A) = c$. For every $B \in \mathbf{B}$, we have that η_B is surjective in $\mathbf{S}$. Note that every object of $\mathbf{A}$ is a κ-filtered colimit of objects in $\mathbf{B}$ and F preserves κ-filtered colimits, it is easy to see that $\eta_{A'}$ is surjective , for all $A' \in \mathbf{A}$. $\prod F_\kappa(\mathbf{A}, \mathbf{S})$ is Barr-exact and the inclusion $\prod F_\kappa(\mathbf{A},\mathbf{S}) \to (\mathbf{A},\mathbf{S})$ is regular; in the latter , η being a regular epimorphism means that η_A is surjective for all $A \in \mathbf{A}$. We conclude that η is a regular epimorphism .

We have established that every $F \in \prod F_\kappa(\mathbf{A},\mathbf{S})$ has a regular epimorphism from a representable functor $\mathbf{A}(A,-)$ into F with $A \in \mathbf{B}$. Thus we have a coequalizer diagram

$$G \underset{g}{\overset{f}{\rightrightarrows}} \mathbf{A}(A,-) \xrightarrow{\eta} F$$

Using again the previous conclusion, there is a regular epimorphism $e : \mathbf{A}(B,-) \to G$ with $B \in \mathbf{B}$. Then η is a coequalizer of the morphisms $(f \circ e, g \circ e)$ between the representable functors $\mathbf{A}(A,-)$ and $\mathbf{A}(B,-)$.

Proposition 5.7 *For any κ-accessible category $\mathbf{A}$ having small products, $\prod F_\kappa(\mathbf{A},\mathbf{S})$, the category of the functors preserving κ-filtered colimits and small products, is a small κ-Barr exact accessible category, and the full subcategory of projective presentable objects of $\prod F_\kappa(\mathbf{A},\mathbf{S})$ is equivalent to $\mathbf{A}_\kappa$.*

Proof: It follows from Proposition 5.6 that $\prod F_\kappa(\mathbf{A},\mathbf{S})$ is κ-Barr-exact accessibie. The smallness follows from Proposition 5.2. Note that, for any A in $\mathbf{A}_\kappa$, the representable functor $\mathbf{A}(A,-)$ is in $\prod F_\kappa(\mathbf{A},\mathbf{S})$. Clearly, if $A \in \mathbf{A}$, then $\mathbf{A}(A,-)$ is projective presentable.

Definition 5.8 *A κ-accessible category having small products is called a κ-Barr category. A category is a Barr category if it is a κ-Barr category for some infinite regular cardinal number κ.*

Proposition 5.9 *Let $\mathbf{C}$ be a small κ-Barr exact accessible category, and $i : \mathbf{C}_p \to \mathbf{C}$ be the inclusion functor. Then the functor*

$$Z : \kappa - Reg(\mathbf{C}, \mathbf{S}) \longrightarrow (\mathbf{C}_p, \mathbf{S})$$

$$M \longmapsto M \circ i$$

is full and faithful, and its essential image consists of the κ-flat functors from $\mathbf{C}_p$ into $\mathbf{S}$, i.e. we have an equivalence

$$\kappa - Reg(\mathbf{C}, \mathbf{S}) \simeq \kappa - Flat(\mathbf{C}_p).$$

Proof: First of all , for every functor M of $\kappa - Reg(\mathbf{C}, \mathbf{S})$, we verify that $M \circ i \in \kappa - Flat(\mathbf{C}_p)$. As we know , it is sufficient to show that the category $el(M \circ i)$ of elements of the functor $M \circ i$ is κ-filtered . Let I be a graph of size less than κ, and a diagram

$$G : I \longrightarrow el(M \circ i)$$

$$k \longmapsto x_k \in M \circ i(C_k)$$

Since $\mathbf{C}$ has κ-limits, we have the limit cone $(\pi_k : C \to C_k)_{k \in I}$ in $\mathbf{C}$. By Definition 5.2, there is a regular epimorphism $e : D \to C$ with D in $\mathbf{C}_p$. Note that M is a κ-regular functor; thus $(M(\pi_k) : M(C) \to M(C_k))_{k \in I}$ is a limit cone in $\mathbf{S}$, and $M(e) : M(D) \to M(C)$ is surjective. There is an $x \in M(C)$ such that $M(\pi_k)(x) = x_k$ for all $k \in I$. Choosing some $d \in M(D)$ with $M(e)(d) = x$, then $(x_k \in M \circ i(C_k) \to d \in M \circ i(D))_{k \in I}$ is a cocone on G.

Secondly, we are going to show that the functor Z is full and faithful . Let $M, N \in \kappa - Reg(\mathbf{C}, \mathbf{S})$, and $t : M \circ i \to N \circ i$ a natural transformation. We want to find $\eta : M \to N$ such that $Z(\eta) = t$. For any C in $\mathbf{C}$, take a regular epimorphism $e : A \to C$ with A in $\mathbf{C}_p$, and let

$$D \underset{g}{\overset{f}{\rightrightarrows}} A \xrightarrow{\ e\ } C$$

be the kernel pair of e; e is the coequalizer of (f, g). Let $d : B \to D$ be a regular epimorphism, then e is a coequalizer of the morphisms $(f \circ d, g \circ d)$. Let $f' = f \circ d$, $g' = g \circ d$. By M and N being κ-regular functors, we have that $M(e)$ is a coequalizer of the morphisms $(M(f'), M(g'))$, and $N(e)$ is a coequalizer of the morphisms $(N(f'), N(g'))$. Naturality of t gives rise to the commutative diagrams as follows.

$$
\begin{tikzcd}
M(B) \arrow[r, "t_B"] \arrow[d, "M(f')"'] & N(B) \arrow[d, "N(f')"] \\
M(A) \arrow[r, "t_A"'] & N(A)
\end{tikzcd}
$$

$$
\begin{tikzcd}
M(B) \arrow[r, "t_B"] \arrow[d, "M(g')"'] & N(B) \arrow[d, "N(g')"] \\
M(A) \arrow[r, "t_A"'] & N(A)
\end{tikzcd}
$$

Since $N(e) \circ N(f) = N(e) \circ N(g)$, we obtain that

$$N(e) \circ t_A \circ M(f) \circ M(d) = N(e) \circ t_A \circ M(g) \circ M(d).$$

But as $M(d)$ is surjective, it follows that

$$(N(e) \circ t_A) \circ M(f) = (N(e) \circ t_A) \circ M(g).$$

By the universal property of the coequalizer $M(e)$, there is a unique morphism $t_C :$ $M(C) \to N(C)$ such that the following diagram

$$
\begin{tikzcd}
M(A) \arrow[r, "t_A"] \arrow[d, "M(e)"'] & N(A) \arrow[d, "N(e)"] \\
M(C) \arrow[r, "t_C"'] & N(C)
\end{tikzcd}
$$

commutes.

Claim: Let C and D be arbitrary objects in $\mathbf{C}$, $h : D \to C$ is any morphism. We have the following commutative diagram

$$\begin{tikzcd}
M(D) \arrow[r, "t_D"] \arrow[d, "M(h)"'] & N(D) \arrow[d, "N(h)"] \\
M(C) \arrow[r, "t_C"'] & N(C)
\end{tikzcd}$$

Proof of claim: First of all, consider a morphism $g : B \to C$ with B in $\mathbf{C}_p$. Let $e : A \to C$ be a regular epimorphism with A in $\mathbf{C}_p$. We take the pullback of (e, g), say Q with $e' : Q \to B$ and $g' : Q \to A$. For such a Q , there is a regular epimorphism $a : P \to Q$ with $P \in \mathbf{C}_p$. Note that since e' is a regular epimorphism, so is $e' \circ a$. Let $e'' = e' \circ a$ and $g'' = g' \circ a$. Consider the following commutative diagram

$$\begin{tikzcd}
P \arrow[r, "e''"] \arrow[d, "g''"'] & B \arrow[d, "g"] \\
A \arrow[r, "e"'] & C
\end{tikzcd}$$

Applying M and N on above the commutative diagram, we have a diagram as follows.

$$\begin{tikzcd}
& N_P \arrow[rr, "N_{e''}"] \arrow[dd, "N_{g''}"' near start] & & N_B \arrow[dd, "N_g"] \\
M_P \arrow[ur, "t_P"] \arrow[rr, "M_{e''}" near start] \arrow[dd, "M_{g''}"'] & & M_B \arrow[ur, "t_B"] \arrow[dd, "M_g" near start] & \\
& N_A \arrow[rr, "N_e" near start] & & N_C \\
M_A \arrow[ur, "t_A"] \arrow[rr, "M_e"'] & & M_C \arrow[ur, "t_C"'] &
\end{tikzcd}$$

Here M_S, N_S, M_h and N_h denote $M(S)$, $N(S)$, $M(h)$ and $N(h)$, respectively, for any object S and any morphism h in $\mathbf{C}$. We have

$$t_C \circ M(g) \circ M(e' \circ a) = N(g) \circ t_B \circ M(e' \circ a).$$

It follows that $t_C \circ M(g) = N(g) \circ t_B$ from $M(e' \circ a)$ being surjective.

Consider now an arbitrary $h : D \to C$, and let $b : P \to D$ be a regular epimorphism with $P \in \mathbf{C}_p$, then

$$t_C \circ M(h \circ b) = N(h \circ b) \circ t_P = N(h) \circ (t_D \circ M(b)).$$

We obtain that $t_C \circ M(h) = N(h) \circ t_D$, as $M(b)$ is surjective.

The claim shows that there is a natural transformation $\eta : M \to N$ such that $Z(\eta) = t$. This proves the fullness of Z; the faithfulness of Z is clear (by t_C uniqueness).

Note that the category $\kappa - Reg(\mathbf{C}, \mathbf{S})$ is closed under the κ-filtered colimits in $(\mathbf{C}, \mathbf{S})$, and colimits are computed pointwise in the latter, thus Z preserves κ-filtered colimits. For any C in $\mathbf{C}_p$, $\mathbf{C}(C,-) \in \kappa - Reg(\mathbf{C},\mathbf{S})$, $\kappa - Flat(\mathbf{C}_p)$ consists exactly of those objects of the codomain of Z that are κ-filtered colimits of objects of the form $Z(\mathbf{C}(C,-))$. We obtain that the essential image of Z is $\kappa - Flat(\mathbf{C}_p)$.

Corollary 5.10 *For any κ-Barr-exact accessible category* $\mathbf{C}$*, the category κ-Reg(*$\mathbf{C}$*,* $\mathbf{S}$*) is a κ-Barr category.*

Proof: By Proposition 5.9 and Proposition 2.1.4 in [24].

5.2 Duality for Barr categories

Let $\kappa - \mathbf{BARR} - \mathbf{EX}$ be the 2-category of all κ-Barr-exact categories as objects and κ-regular functors as 1-arrows, and all natural transformations between the latter as 2-arrows. $\prod\mathcal{F}_\kappa$ is the 2-category of all categories with κ-filtered colimits and products as objects , all functors preserving κ-filtered colimits and small products as 1-arrows, and all natural transformations between the latter as 2-arrows. $\mathbf{S}$, the category of small sets, is an object in both $\kappa - \mathbf{BARR} - \mathbf{EX}$ and $\prod\mathcal{F}_\kappa$, and each of the $\kappa - \mathbf{BARR} - \mathbf{EX}$-operations commute with each of the $\prod\mathcal{F}_\kappa$-operations on $\mathbf{S}$(see [16]). Such a statement gives rise to a pair of adjoint 2-functors

$$\kappa - \mathbf{BARR} - \mathbf{EX}^{op} \underset{G}{\overset{F}{\rightleftarrows}} \prod\mathcal{F}_\kappa$$

$$F = \prod F_\kappa(-, \mathbf{S}), G = \kappa - Reg(-, \mathbf{S});$$

both unit and counit are defined , at any object of the respective kind, as the evaluation functor. Consider the full sub-2-category $\kappa - \mathbf{PAcc}$ of $\kappa - \mathbf{BARR} - \mathbf{EX}$ whose objects are κ-Barr exact accessible categories. $\kappa - \mathbf{Barr}$ is the full sub-2-category of $\prod \mathcal{F}_\kappa$ with objects κ-Barr categories. By Proposition 5.7 and Corollary 5.10, we have the 2-adjunction

$$\kappa - \mathbf{PAcc}^{op} \underset{G}{\overset{F}{\rightleftarrows}} \kappa - \mathbf{Barr}$$

with restricted unit and counit.

Theorem 5.11 *The pair of adjoint 2-functors as mentioned is biequivalence. In other words,*

(*i*) *If* $\mathbf{C} \in \kappa - \mathbf{PAcc}$, *then* $\kappa - Reg(\mathbf{C}, \mathbf{S}) \in \kappa - \mathbf{Barr}$, *and the evaluation functor*

$$\varepsilon_{\mathbf{C}} : \mathbf{C} \to \prod F_\kappa(\kappa - Reg(\mathbf{C}, \mathbf{S}), \mathbf{S})$$

is equivalence of categories; and

(*ii*) *if* $\mathbf{A} \in \kappa - \mathbf{Barr}$, *then* $\prod F_\kappa(\mathbf{A}, \mathbf{S}) \in \kappa - \mathbf{PAcc}$, *and the evaluation functor*

$$\eta_{\mathbf{A}} : \mathbf{A} \to \kappa - Reg(\prod F_\kappa(\mathbf{A}, \mathbf{S}), \mathbf{S})$$

is an equivalence of categories.

Proof: (i) For any $\mathbf{C} \in \mathbf{PAcc}$, by Proposition 5.5, the canonical functor $\sum : \mathbf{C} \to (\mathbf{C}_p{}^{op}, \mathbf{S})$ is full and faithful. Using Proposition 1.2.4(ii) in [24] and Proposition 5.9, we have an equivalence

$$Z : (\mathbf{C}_p{}^{op}, \mathbf{S}) \to F_\kappa((\kappa - Reg(\mathbf{C}, \mathbf{S}), \mathbf{S}).$$

Also, the diagram

$$\begin{tikzcd} \mathbf{C} \arrow[r, "\varepsilon_{\mathbf{C}}"] \arrow[d, "\Sigma"'] & F_\kappa(\kappa - Reg(\mathbf{C}, \mathbf{S}), \mathbf{S}) \\ (\mathbf{C}_p{}^{op}, \mathbf{S}) \arrow[ur, "Z"'] & \end{tikzcd}$$

commutes. Thus $\varepsilon_{\mathbf{C}}$ is full and faithful functor, using the fact that small limits and colimits are computed pointwise, we easily see that $\varepsilon_{\mathbf{C}}$ is a κ-regular functor. Proposition 5.6 gives that every object in $\prod F_\kappa(\kappa - Reg(\mathbf{C}, \mathbf{S}), \mathbf{S})$ is a codomain of a regular epimorphism , whose domain is a functor of the form $\varepsilon_{\mathbf{C}}(C)$ with $C \in \mathbf{C}_p$. By Theorem 2.9, we obtain that $\varepsilon_{\mathbf{C}}$ in Theorem 5.11(i) is an equivalence. For (ii), since $\prod F_\kappa(\mathbf{A}, \mathbf{S})$ is a κ-Barr-exact accessible category, we can apply Proposition 5.9. Let

$$Y : \mathbf{A}_\kappa^{op} \to \prod F_\kappa(\mathbf{A}, \mathbf{S})$$

$$A \mapsto \mathbf{A}(A, -)$$

be the canonical functor, we have that the induced functor

$$Y^{*} : \kappa - Reg(\prod F_{\kappa}(\mathbf{A}, \mathbf{S}), \mathbf{S}) \longrightarrow \kappa - Flat(\mathbf{A}_{\kappa}^{op})$$

$$M \longmapsto M \circ Y$$

is an equivalence of categories. By Proposition 2.1.8. in [24], the functor

$$\sum : \mathbf{A} \longrightarrow \kappa - Flat(\mathbf{A}_{\kappa}^{op})$$

$$A \longmapsto \mathbf{A}(i(-), A)$$

is an equivalence, where $i : \mathbf{A}_{\kappa} \to \mathbf{A}$ is the inclusion functor. It is easy to see that $\sum = \eta_{\mathbf{A}} \circ Y^{*}$. We conclude that $\eta_{\mathbf{A}}$ is an equivalence of categories.

Remark 5.12 *J. Adámek and R. Rosický have recently shown in [2] that accessible categories with products are exactly the small injectivity classes of locally presentable categories. By Theorem 5.11, a category is a κ-Barr iff it is equivalent to $\kappa - Reg(\mathbf{C}, \mathbf{S})$ for some κ-Barr exact accessible category $\mathbf{C}$, such a category $\mathbf{C}$ is essentially small. As we know, the category $\kappa - Reg(\mathbf{C}, \mathbf{S})$ is the injectivity class for the small set of regular monomorphisms between representable functors in the category $L_{\kappa}(\mathbf{C}, \mathbf{S})$.*

The following conceptual completeness theorem is an immediate consequence of Theorem 5.11.

Corollary 5.13 *(i) If $F : \mathbf{C} \to \mathbf{D}$ be a κ-regular functor between κ-Barr-exact accessible such that the induced functor*

$$F^{*} : \kappa - Reg(\mathbf{D}, \mathbf{S}) \to \kappa - Reg(\mathbf{C}, \mathbf{S})$$

is an equivalence of categories, then F is an equivalence.

(ii) Let $\mathbf{A}$ and $\mathbf{B}$ be any two κ-Barr categories. If $G : \mathbf{A} \to \mathbf{B}$ is a κ-accessible functor preserving small products such that the induced functor

$$G^{*} : \prod F_{\kappa}(\mathbf{B}, \mathbf{S}) \longrightarrow \prod F_{\kappa}(\mathbf{A}, \mathbf{S})$$

is an equivalence of categories, then G is an equivalence as well.

Corollary 5.14 *A small κ-Barr exact category has a full regular embedding into a Barr-exact accessible category.*

Proof: Let $\mathbf{C}$ be small κ-Barr-exact, then the evaluation functor

$$e_{\mathbf{C}} : \mathbf{C} \longrightarrow \prod F_{\kappa}(\kappa - Reg(\mathbf{C}, \mathbf{S}), \mathbf{S})$$

is an equivalence (see [21]). Let $\kappa - Reg(\mathbf{C}, \mathbf{S})$ be λ-accessible, for some regular cardinal $\lambda > \kappa$. Thus we have a full κ-regular functor from $\mathbf{C}$ into $\prod F_{\lambda}(\kappa - Reg(\mathbf{C}, \mathbf{S}), \mathbf{S})$, and the latter is Barr-exact accessible from Theorem 5.11.

Let $\mathbf{D}$ be a small category with finite limits. Recall that $\mathbf{C}$ is Barr-exact hull of $\mathbf{D}$, if and only if $\mathbf{C}$ is Barr-exact , and we have a $F \in Lex(\mathbf{D}, \mathbf{C})$ such that for any Barr-exact category $\mathbf{E}$, the induced functor

$$F^{*} : Reg(\mathbf{C}, \mathbf{E}) \longrightarrow Lex(\mathbf{D}, \mathbf{E})$$

$$M \mapsto M \circ F$$

is an equivalence of categories(see [21]). We have

Proposition 5.15 *Let $\mathbf{D}$ be a small category with finite limits. Then the Barr-exact hull of $\mathbf{D}$ is a Barr-exact accessible category. More precisely, $\mathbf{C}$ is equivalent to the category $\prod F_{\aleph_0}(Lex(\mathbf{D}, \mathbf{S}), \mathbf{S})$.*

Proof: Note that $Lex(\mathbf{D}, \mathbf{S})$ is an $\aleph_0$-Barr category, so, by Theorem 5.11, we have an equivalence

$$e_{\mathbf{D}}{}^{*} : Reg(\prod F_{\aleph_0}(Lex(\mathbf{D}, \mathbf{S}), \mathbf{S}), \mathbf{S}) \longrightarrow Lex(\mathbf{D}, \mathbf{S})$$

$$M \longmapsto M \circ e_{\mathbf{D}}$$

Where $e_{\mathbf{D}} : \mathbf{D} \rightarrow \prod F_{\aleph_0}(Lex(\mathbf{D}, \mathbf{S}), \mathbf{S})$ is the evaluation functor. We conclude that the Barr-exact hull of $\mathbf{D}$ is equivalent to $\prod F_{\aleph_0}(Lex(\mathbf{D}, \mathbf{S}), \mathbf{S})$ from Corollary 6.5. in [16]. Using Theorem 5.11 again, the latter is a Barr-exact accessible category.

Let $\mathbf{L}_\kappa$ be the 2-category of small categories with κ-limits, whose 1-arrows are functors preserving κ-limits, and 2-arrows are all natural transformations between such functors. κ-**Barr-ex** be the 2-category of small κ-Barr-exact categories, whose 1-arrows are κ-regular functors, and 2-arrows are all natural transformations. For case $\kappa = \aleph_0$, Carboni and Magno in [9] have described an 2-adjunction between $\mathbf{L}_{\aleph_0}$ and $\aleph_0$-**Barr-ex**:

$$\aleph_0 - \mathbf{Barr} - \mathbf{ex} \underset{G}{\overset{F}{\rightleftarrows}} L_{\aleph_0}$$

here F is the forgetful functor, and G is the Barr-exact hull (Barr-exact completion). Such a 2-adjunction can be generalized to any infinite regular cardinal κ:

$$\kappa - \mathbf{Barr} - \mathbf{ex} \underset{G}{\overset{F}{\rightleftarrows}} L_\kappa$$

F is the forgetful functor, and G is the κ-Barr-exact hull. By duality for (κ)-Barr categories, for each small category $\mathbf{C}$ with κ-limits, its κ-Barr-exact hull $\mathbf{C}_{ex}$ is equivalent to the category $\prod F_\kappa(L_\kappa(\mathbf{C},\mathbf{S}),\mathbf{S})$; such a category is κ-Barr-exact accessible, and the unit of the above 2-adjunction at any $\mathbf{C}$ is defined by the evaluation functor

$$\eta_{\mathbf{C}} : \mathbf{C} \to \prod F_\kappa(L_\kappa(\mathbf{C},\mathbf{S}),\mathbf{S})$$

Clearly, $\eta_{\mathbf{C}}$ is full and faithful. By the properties of Barr-exact accessible categories, we have for any $C \in \mathbf{C}$, $\eta_{\mathbf{C}}(C)$ is regular projective in $\mathbf{C}_{ex}$; for any $D \in \mathbf{C}_{ex}$, there is a $C \in \mathbf{C}$ and a regular epi $\eta_{\mathbf{C}}(C) \to D$ in $\mathbf{C}_{ex}$; for any f in $\mathbf{C}$, if $\eta_{\mathbf{C}}(f)$ is a regular epi, then f is a split epi.

CHAPTER 6

SOME RECONSTRUCTION RESULTS

6.1 SSC on doctrines: LP, Diers and Barr

In this section, we will give sharp conceptual completeness results for the doctrines: **LP**, **Diers** and **Barr**.

LP is the 2-category of all locally presentable categories, whose 1-arrows are accessible functors preserving small limits, and whose 2-arrows are all natural transformations between the latter.

Diers is the 2-category of Diers categories, its 1-arrows are accessible functors preserving small connected limits, and its 2-arrows are all natural transformations between them. **Barr** is the 2-category of Barr categories, its 1-arrows are accessible functors preserving products, and its 2-arrows are all natural transformations.

Theorem 6.1 (*i*) *For each* $\mathbf{A}$ *in* **LP**, *the evaluation functor*

$$\epsilon_{\mathbf{A}} : \mathbf{A} \to L(LAcc(\mathbf{A}, \mathbf{S}), \mathbf{S})$$

is an equivalence of categories.

(*ii*) *For each* $\mathbf{A}$ *in* **Diers**, *the evalution functor*

$$\epsilon_{\mathbf{A}} : \mathbf{A} \to L\coprod(CoAcc(\mathbf{A}, \mathbf{S}), \mathbf{S})$$

is an equivalence of categories.

(*iii*) *For each* $\mathbf{A}$ *in* **Barr**, *the evaluation functor*

$$\epsilon_{\mathbf{A}} : \mathbf{A} \to LR(\prod Acc(\mathbf{A}, \mathbf{S}), \mathbf{S})$$

is an equivalence of categories.

Proof: The above results follow from Gabriel-Ulmer duality and duality theorems for (κ)-Diers and (κ)-Barrs categories, respectively. Since the proofs are essentially same in each case, we only give the proof of (ii).

Let $\mathbf{A}$ be any Diers category. We assume that $\mathbf{A}$ is a κ-Diers category, Note that for any $A \in \mathbf{A}$, the representable functor $\mathbf{A}(A, -)$ is in $CoAcc(\mathbf{A}, \mathbf{S})$. Let $Y : \mathbf{A} \to CoAcc(\mathbf{A}, \mathbf{S})^{op}$ be the induced functor of the Yoneda embedding, and also $Z : CoAcc(\mathbf{A}, \mathbf{S})^{op} \to (CoAcc(\mathbf{A}, \mathbf{S}), \mathbf{S})$ be induced by the Yoneda embedding. Let $ev_{\mathbf{A}}$ be the evaluation functor from $\mathbf{A}$ into the category $(CoAcc(\mathbf{A}, \mathbf{S}), \mathbf{S})$. By observation, we have that the diagram

$$\begin{tikzcd}
\mathbf{A} \arrow[r, "ev_{\mathbf{A}}"] \arrow[d, "Y"'] & (CoAcc(\mathbf{A},\mathbf{S}),\mathbf{S}) \\
CoAcc(\mathbf{C},\mathbf{S})^{op} \arrow[ur, "Z"'] &
\end{tikzcd}$$

commutes. It follows that $ev_{\mathbf{A}}$ is full and faithful from the fact that both of Y and Z are full and faithful. Therefore, $\epsilon_{\mathbf{A}}$ is full and faithful.

To show that $\epsilon_{\mathbf{A}}$ is essentially surjective on objects, let $M \in L\coprod(CoAcc(\mathbf{A},\mathbf{S}),\mathbf{S})$, and let $l_\kappa : CoF_\kappa(\mathbf{A},\mathbf{S}) \to CoAcc(\mathbf{A},\mathbf{S})$ be the inclusion functor. We have a functor induced by l_κ, denoted by G_κ,

$$G_{\kappa 1} : L\coprod(CoAcc(\mathbf{A},\mathbf{S}),\mathbf{S}) \to L_\kappa\coprod(CoF_\kappa(\mathbf{A},\mathbf{S}),\mathbf{S})$$

$$M \mapsto M \circ l_\kappa$$

Let e_κ be the evaluation functor in Theorem 4.12. For any regular cardinal κ' with $\kappa' \trianglerighteq \kappa$, by Theorem 4.12, $e_{\kappa'}$ is an equivalence. Thus, there is an object $A_{\kappa'}$ in $\mathbf{A}$ so that $M \circ l_{\kappa'} = e_{\kappa'}(A_{\kappa'})$. To show $M = \epsilon_{\mathbf{A}}(A)$ for some $A \in \mathbf{A}$, we only need to show that $A_{\kappa'} = A_\kappa$ for all κ' with $\kappa' \trianglerighteq \kappa$. In fact, let $i : CoF_\kappa(\mathbf{A},\mathbf{S}) \to CoF_{\kappa'}(\mathbf{A},\mathbf{S})$ be the inclusion. We have a induced functor of i

$$I : L_{\kappa'}\coprod(CoF_{\kappa'}(\mathbf{A},\mathbf{S}),\mathbf{S}) \to L_\kappa\coprod(CoF_\kappa(\mathbf{A},\mathbf{S}),\mathbf{S})$$

$$E \mapsto E \circ i$$

and $e_\kappa = I \circ e_{\kappa'}$. Note that $M \circ l_{\kappa'} \circ i = M \circ l_\kappa$; hence $e_\kappa(A_\kappa) = e_\kappa(A_{\kappa'})$. But e_κ is full and faithful, so we conclude that $A_\kappa = A_{\kappa'}$.

Remark 6.2 *For any functor $F \in LAcc(\mathbf{A}, \mathbf{S})$, F has a left adjoint, hence F is a representable functor. We have that if $\mathbf{A}$ is locally presentable, then $\mathbf{A}$ is equivalent to the category $L(\mathbf{A}^{op}, \mathbf{S})$; if $\mathbf{B}$ is locally copresentable (i.e. $\mathbf{B}^{op}$ is locally presentable), then $\mathbf{B}$ is equivalent to the category $LAcc(\mathbf{B}^{op}, \mathbf{S})$.*

6.2 Barr-exact weak-accessible categories

Let $\mathbf{A}$ be a Barr category. The category $\prod Acc(\mathbf{A}, \mathbf{S})$ is a complete Barr-exact category. We call it a Barr-exact weak-accessible category. The meaning of 'weak' is that the full subcategory of projective presentable objects of it is not necessary small, but its opposite category is accessible.

Definition 6.3 *A category $\mathbf{B}$ is called Barr-exact weak-accessible, if*

(i) $\mathbf{B}$ is a complete Barr-exact category; and

(ii) The opposite category of $\mathbf{B}_p$ is a Barr category, and for every object $C \in \mathbf{B}$ there is a regular epimorphism from B into C with B in $\mathbf{B}_p$.

Proposition 6.4 *Let* $\mathbf{B}$ *be a Barr-exact weak-accessible category. Then*

(*i*) $\mathbf{B}$ *is locally small;*

(*ii*) $\mathbf{B}_p$ *is dense in* $\mathbf{B}$.

Proof: Since $\mathbf{B}_p$ is locally small, the proof of (i) is same as that of Proposition 2.1.5(i) in [24]. (ii) follows from Proposition 5.4.

Proposition 6.5 *For any Barr category* $\mathbf{A}$, $\prod Acc(\mathbf{A}, \mathbf{S})$ *is a Barr-exact weak-accessible category, and the full subcategory of projective presentable objects of* $\prod Acc(\mathbf{A}, \mathbf{S})$ *is equivalent to the category* $\mathbf{A}$.

Proof: For every object A of $\mathbf{A}$, note that the representable functor $\mathbf{A}(A, -)$ is projective presentable in $\prod Acc(\mathbf{A}, \mathbf{S})$. Given $F \in \prod Acc(\mathbf{A}, \mathbf{S})$, say that $F \in \prod F_\kappa(\mathbf{A}, \mathbf{S})$, by Propositon 5.6, we have a regular epimorphism from a representable functor into F. The last assertion obviously follows.

Proposition 6.6 *Let* $\mathbf{B}$ *be a Barr-exact weak-accessible category,* $i : \mathbf{B}_p \to \mathbf{B}$ *be the inclusion functor. Then the functor*

$$\sum : \mathbf{B} \longrightarrow (\mathbf{B}_p^{op}, \mathbf{S})$$

$$B \longmapsto \mathbf{B}(i(-), B)$$

is full and faithful, and its essential image consists of the accessible functors from $\mathbf{B}_p^{op}$ to $\mathbf{S}$ that preserve products, i.e. we have an equivalence

$$\mathbf{B} \simeq \prod Acc(\mathbf{B}_p^{op}, \mathbf{S}).$$

Proof: Note that , by the definition of projective presentability, for any $B \in \mathbf{B}_p$, the functor $\sum(-)(B) : \mathbf{B} \to \mathbf{S}$ is a regular functor, since colimits are computed pointwise in $(\mathbf{B}_p^{op}, \mathbf{S})$, so $\sum$ is regular.

That $\sum$ is full and faithful follows from the density of $\mathbf{B}_p$. By Proposition 5.6, Given $F \in \prod Acc(\mathbf{B}_p^{op}, \mathbf{S})$, there is a regular epimorphism $\mathbf{B}_p^{op}(B, -) \to F$ with $B \in \mathbf{B}_p^{op}$. Note that for $B \in \mathbf{B}_p$, $\sum(B) \cong \mathbf{B}_p^{op}(B, -)$. By Theorem 2.9, we obtain that the essential image of $\sum$ is $\prod Acc(\mathbf{B}_p^{op}, \mathbf{S})$.

Proposition 6.7 *Let $\mathbf{B}$ be a Barr-exact weak-accessible category. For every $B \in \mathbf{B}_p$, we have $\mathbf{B}(B, -) \in LR(\mathbf{B}, \mathbf{S})$, and the canonical functor*

$$y_{\mathbf{B}} : \mathbf{B}_p^{op} \longrightarrow LR(\mathbf{B}, \mathbf{S})$$

$$B \mapsto \mathbf{B}(B, -)$$

is an equivalence of categories. Thus $LR(\mathbf{B}, \mathbf{S})$ is a Barr category.

Proof: That $y_{\mathbf{B}}$ is full and faithful follows from the Yoneda lemma. By Proposition 6.6, we have

$$\sum^{*} : LR(\prod Acc(\mathbf{B}_p^{op}, \mathbf{S}), \mathbf{S}) \longrightarrow LR(\mathbf{B}, \mathbf{S})$$

is an equivalence of categories. By Theorem 6.1(iii), we conclude that $y_{\mathbf{B}}$ is an equivalence.

Let **LR** be the 2-category of all Barr-exact weak-accessible categories as objects and regular functors preserving limits as 1-arrows, and all natural transformations between the latter as 2-arrows. **Barr** is the 2-category of all Barr categories, whose 1-arrows are accessible functors preserving small products, and whose 2-arrows are all natural transformations between the latter. We have the following duality result.

Theorem 6.8 *(i) If* $\mathbf{B} \in \mathbf{LR}$*, then* $LR(\mathbf{B}, \mathbf{S}) \in \mathbf{Barr}$*, and the evaluation functor*

$$\varepsilon_{\mathbf{B}} : \mathbf{B} \longrightarrow \prod Acc(LR(\mathbf{B}, \mathbf{S}), \mathbf{S})$$

is an equivalence of categories.

(ii) If $\mathbf{A} \in \mathbf{Barr}$*, then* $\prod Acc(\mathbf{A}, \mathbf{S}) \in \mathbf{LR}$*, and the evaluation functor*

$$\eta_{\mathbf{A}} : \mathbf{A} \longrightarrow LR(\prod Acc(\mathbf{A}, \mathbf{S}), \mathbf{S})$$

is an equivalence of categories.

Proof: (i) By Proposition 6.7, the functor induced by $y_{\mathbf{B}}$

$$Y_{\mathbf{B}} : \prod Acc(LR(\mathbf{B},\mathbf{S}),\mathbf{S}) \longrightarrow \prod Acc(\mathbf{B}_p^{op},\mathbf{S})$$

$$M \longmapsto M \circ y_{\mathbf{B}}$$

is an equivalence. Clearly , we have $\sum = Y_{\mathbf{B}} \circ \varepsilon_{\mathbf{B}}$, where $\sum$ is given in Proposition 6.6. Thus $\varepsilon_{\mathbf{B}}$ is an equivalence of categories.

(ii) is given by Theorem 6.1(iii).

Example 6.9 *Let R be an associative ring with unit. Mod_R denotes the category of right R-modules. Both of Mod_R and $Mod_R{}^{op}$ are Barr-exact categories. Note that Mod_R is locally κ-presentable category, for any infinite regular cardinal κ (see [10]). Thus the category Mod_R^{op} is a complete Barr-exact category. By the Gabriel-Ulmer duality, we have that $Mod_R \simeq L_\kappa((Mod_R)_\kappa^{op},\mathbf{S})$. Let Inj_R denote the subcategory of Mod_R consisting of the injective modules (projective presentable objects of $Mod_R{}^{op}$). As shown in [16], $Inj_R \simeq \kappa - Reg((Mod_R)_\kappa^{op},\mathbf{S})$ for some regular cardinal. Each functor in $L_\kappa((Mod_R)_\kappa^{op},\mathbf{S})$ is the domain of some regular monomorphism whose codomain is a functor in $\kappa - Reg((Mod_R)_\kappa^{op},\mathbf{S})$ (see [5], [21]). So the category Mod_R^{op} is a Barr-exact weak-accessible category. It follows that $Mod_R^{op} \simeq \prod Acc(Inj_R,\mathbf{S})$ from Theorem 6.8. Using Theorem 6.8 again, we have*

Proposition 6.10 *Given two rings R_1 and R_2, if they have equivalent categories of injective modules (left or right), then $Mod_{R_1} \simeq Mod_{R_2}$, i.e. they are Morita equivalent.*

This result is also shown in [21].

CHAPTER 7

DUALITY FOR κ-BARR-EXACT CATEGORIES

7.1 Some basic facts on κ-Barr-exact categories

Let $\mathbf{C}$ be a small κ-Barr-exact category. The category $\kappa - Reg(\mathbf{C}, \mathbf{S})$ is closed under κ-filtered colimits in the functor category $(\mathbf{C}, \mathbf{S})$. Let $< M_i >_{i \in I}$ be a small family of κ-regular functors from $\mathbf{C}$ to $\mathbf{S}$. Then the induced functor $< M_i >_{i \in I}$: $\mathbf{C} \to (I, \mathbf{S})$ is κ-regular as well, and every epi splits (has a right inverse) in $(I, \mathbf{S})$, hence the composite of

$$\mathbf{C} \xrightarrow{< M_i >} (I, \mathbf{S}) \xrightarrow{\prod_I} \mathbf{S}$$

is also κ-regular; here $\prod_I$ takes $< M_i >_{i \in I}$ to $\prod_{i \in I} M_i$. Since this composite is the same as the product $\prod_{i \in I} M_i$ in the category ($\mathbf{C}$, $\mathbf{S}$), we conclude that $\kappa - Reg(\mathbf{C}, \mathbf{S})$ is closed under (small) products in ($\mathbf{C}$, $\mathbf{S}$). As shown in [21], $\kappa - Reg(\mathbf{C}, \mathbf{S})$ is an accessible category, thus it is a Barr category with κ-filtered colimits.

The notion of regular monomorphism is the dual of that of regular epimorphism: a morphism $m : \mathbf{C} \to \mathbf{D}$ is a regular mono in a category $\mathbf{A}$ if and only if the same morphism in $\mathbf{A}^{op}$ is a regular epi. For any category $\mathbf{A}$, let $\mathbf{B}$ be a full subcategory of $\mathbf{A}$. Recall from [6] that an object A of $\mathbf{A}$ is $\mathbf{B}$-injective if whenever $f : C \to D$ is a regular monic in $\mathbf{A}$ between objects of $\mathbf{B}$, then $\mathbf{A}(f, A) : \mathbf{A}(D, A) \to \mathbf{A}(C, A)$ is surjective. The $\mathbf{B}$-projective notion is the dual to that of a $\mathbf{B}$-injective.

The following result is given by M. Barr (see Theorem 1 in [5]) for the case $\kappa = \aleph_0$.

Proposition 7.1 *Let $\mathbf{C}$ be a small κ-Barr-exact category. Then $(L_\kappa(\mathbf{C}, \mathbf{S}))^{op}$, the opposite category of the category of the functors preserving κ-limits from $\mathbf{C}$ to $\mathbf{S}$, is a κ-Barr-exact category, and the functor $Y : \mathbf{C} \to (L_\kappa(\mathbf{C}, \mathbf{S}))^{op}$ induced by the Yoneda embedding is a κ-regular functor.*

Proof: The proof of the proposition is essentially the same as that of Theorem 1 in [5]. The fact that $L_\kappa(\mathbf{C}, \mathbf{S})^{op}$ satisfies $< \kappa$ dependent choices (DC_κ) follows from that a κ-filtered colimit of regular monos is a regular mono in $L_\kappa(\mathbf{C}, \mathbf{S})$.

The Yoneda lemma gives the following fact: for $M \in L_\kappa(\mathbf{C}, \mathbf{S})$ and an object C of $\mathbf{C}$, we have the bijection $h_C : M(C) \to Nat(\mathbf{C}(C, -), M)$; moreover for $f : C \to D$ in $\mathbf{C}$, and $Y(f) : \mathbf{C}(C, -) \to \mathbf{C}(D, -)$, the following diagram

$$\begin{tikzcd} M(C) \arrow[r, "h_C"] \arrow[d, "M(f)"'] & Nat(\mathbf{C}(C,-), M) \arrow[d, "Nat(Y(f), M)"] \\ M(D) \arrow[r, "h_D"'] & Nat(\mathbf{D}(D,-), M) \end{tikzcd}$$

commutes. In other words, the aspect of M being a functor on $\mathbf{C}$ can be fully recovered inside the category $L_\kappa(\mathbf{C}, \mathbf{S})$. We have

Proposition 7.2 *Let M be a functor in $L_\kappa(\mathbf{C}, \mathbf{S})$. M is a κ-regular functor from $\mathbf{C}$ into $\mathbf{S}$ if and only if the following holds: for any morphism $f : \mathbf{C}(C, -) \to M$ and regular mono $m : \mathbf{C}(C, -) \to \mathbf{C}(D, -)$, there is $g : \mathbf{C}(D, -) \to M$ such that*

$$\begin{tikzcd} \mathbf{C}(C,-) \arrow[r, "m"] \arrow[d, "f"'] & \mathbf{C}(D,-) \arrow[dl, "g"] \\ M & \end{tikzcd}$$

commutes, i.e. M is an R-injective object in $L_\kappa(\mathbf{C}, \mathbf{S})$. Here R is the subcategory of $L_\kappa(\mathbf{C}, \mathbf{S})$ whose objects are isomorphic to the representable functors.

As pointed out in [21], $L_\kappa(\mathbf{C}, \mathbf{S})$ can be generated by $\kappa - Reg(\mathbf{C}, \mathbf{S})$ using limits, i.e. every functor in $L_\kappa(\mathbf{C}, \mathbf{S})$ is a limit of a diagram of functors in $\kappa - Reg(\mathbf{C}, \mathbf{S})$. This result is due to M.Barr (see [5]) for the case $\kappa = \aleph_0$. The following result is proved in [21] (Proposition 6.3.).

Proposition 7.3 *Let $\mathbf{C}$ be a small κ-Barr-exact category. Then for every $M \in L_\kappa(\mathbf{C}, \mathbf{S})$ there are $N \in \kappa - Reg(\mathbf{C}, \mathbf{S})$, and a regular monomorphism $m : M \to N$ in $L_\kappa(\mathbf{C}, \mathbf{S})$. Therefore, every $M \in L_\kappa(\mathbf{C}, \mathbf{S})$ is the domain object of an equalizer of a pair of morphisms in $\kappa - Reg(\mathbf{C}, \mathbf{S})$.*

Since $L_\kappa(\mathbf{C},\mathbf{S})^{op}$ is Barr-exact, it follows that $\kappa - Reg(\mathbf{C},\mathbf{S})$ is codense in $L_\kappa(\mathbf{C},\mathbf{S})$ from the above proposition and Proposition 5.3.

Given any small κ-Barr-exact category $\mathbf{C}$, we have the evaluation functor

$$ev_{\mathbf{C}} : \mathbf{C} \dashrightarrow (\kappa - Reg(\mathbf{C},\mathbf{S}),\mathbf{S})$$

$$C \mapsto [h \mapsto h_C]$$

$$f \mapsto [M \mapsto M(f)]$$

Using the fact that small limits and colimits are computed pointwise in the functor category $(\kappa - Reg(\mathbf{C},\mathbf{S}),\mathbf{S})$, it is easy to see that $ev_{\mathbf{C}}$ is a κ-regular functor. For any $C \in \mathbf{C}$, since $\kappa - Reg(\mathbf{C},\mathbf{S})$ is closed under κ-filtered colimits and products in $(\mathbf{C}, \mathbf{S})$, then $ev_{\mathbf{C}}(C)$ preserves κ-filtered colimits and products. Thus the evaluation functor $ev_{\mathbf{C}}$ induces a functor, denoted by $e_{\mathbf{C}}$,

$$e_{\mathbf{C}} : \mathbf{C} \to \prod F_\kappa(\kappa - Reg(\mathbf{C},\mathbf{S}),\mathbf{S})$$

The full subcategory $\prod F_\kappa(\kappa - Reg(\mathbf{C},\mathbf{S}),\mathbf{S})$ of $(\kappa - Reg(\mathbf{C},\mathbf{S}),\mathbf{S})$ is a κ-Barr-exact category.

Let I be a nonempty set. Recall that a filter F over I is defined to be a subset of the powerset of I such that

(i) $I \in F$;

(ii) if $X, Y \in F$, then $X \cap Y \in F$;

(iii) if $X \in F$ and $X \subseteq Y \subseteq I$, then $Y \in F$.

A filter F is proper, if $\phi \in F$. F is said to be κ-complete, if it is closed under $< \kappa$ intersections, i.e., for $E \subseteq F$ with the cardinal of E less than κ, then $\bigcap E \in F$. Given a κ-complete filter over I, and a family of sets A_i, let $\prod A_i$ be the cartesian product. Define an equivalence relation $\sim$ on the set of all vectors $< a_i; i \in P >$ such that $P \in F$ and $a_i \in A_i$ for $i \in P$ as

$$< a_i; i \in P > \sim < b_i; i \in P' > \text{ iff } \{i \in P \cap P'; a_i = b_i\} \in F$$

Denoted by $< a_i > /F$ the equivalence class of $< a_i >$, the κ-reduced product of A_i over F is the set of all equivalence classes.

Let F be a κ-complete filter over a set I, $< A_i >_{i \in I}$ a family of sets. The κ-reduced product $\prod_F A_i$ is the κ-filtered colimit of the diagram whose vertices are the products $\prod_{j \in P} A_j$ $(P \in F)$, and whose edges are the projections $\prod_{j \in P} A_j \to \prod_{j \in Q} A_j$ $(Q \subset P)$. When each A_i is the same set A, then the reduced product $\prod_F A$ is called a reduced power of A. Denote it by A^F.

Let $\mathbf{C}$ be a small κ-Barr-exact category, and let $\mathbf{A}$ be a full subcategory of $\kappa - Reg(\mathbf{C}, \mathbf{S})$ which is closed under κ-filtered colimits and products. Then $\mathbf{A}$ is closed under κ-reduced product in $\kappa - Reg(\mathbf{A}, \mathbf{S})$. We now assume that $\mathbf{A}$ is accessible, and let $i : \mathbf{A} \to \kappa - Reg(\mathbf{C}, \mathbf{S})$ be the inclusion. We have the functor

$$e : \mathbf{C} \to \prod F_\kappa(\mathbf{A}, \mathbf{S})$$

defined as

$$C \mapsto e_{\mathbf{C}}(C) \circ i : \mathbf{A} \to \mathbf{S}$$

here $e_{\mathbf{C}}$ is the evaluation functor . Clearly, the functor e is κ-regular, and by Proposition 5.7, $\prod F_\kappa(\mathbf{A}, \mathbf{S})$ is small κ-Barr-exact. We write $\mathbf{A}^*$ for $\prod F_\kappa(\mathbf{A}, \mathbf{S})$. The following is due to W. Boshuck (see [8]).

Proposition 7.4 *Let* $\mathbf{A}$ *be an accessible full subcategory of* $\kappa - Reg(\mathbf{C}, \mathbf{S})$ *which is closed under* κ*-filtered colimits and products. Then the above* e *is full on subobjects, i.e., for every* $C \in \mathbf{C}$*, the poset morphism induced by* e

$$e^C : Sub_{\mathbf{C}}(C) \to Sub_{\mathbf{A}^*}(e(C))$$

is surjective. Here $Sub_{\mathbf{C}}(C)$ *is the poset of subobjects of* C*; to the subobject determined by the monomorphism* $m : D \to C$*,* e^C *assigns the subobject determined by* $e(m) : e(D) \to e(C)$.

Proof: To show the proposition, we first need the following lemma which is similar to Lemma 4.3. in [19].

lemma 7.5 *Let* $M, N \in \mathbf{A}^*$*,* $C \in \mathbf{C}$*,* $a \in M(C)$ *and* $b \in N(C)$*. Soppose that for all* $\Sigma \in Sub_{\mathbf{C}}(C)$*,* $b \in N(\Sigma)$ *implies that* $a \in M(\Sigma)$*. Then there is a* κ*-complete proper* *filter F and a homomorphism $h : N \to M^F$ such that $\delta_{\mathbf{C}}(a) = h_C(b)$; here δ is the canonical embedding of M into the reduced power M^F.*

The condition in Lemma 7.5 is equivalent to saying that every pp formula (in the canonical language associated with **C**) satisfied by b in N is satisfied by a in M. In fact, it is a variant of Tarski's theorem on substructures-extensions. We refer to this as Theorem 7.1.4' in [22].

Next we turn to the proof of the proposition. Let $X \to e(C)$ be a subobject of $e(C)$ in $\mathbf{A}^*$, and let S be the set of all subobjects $D \to C$ such that $X(M) \subset M(D)$ for all $M \in \mathbf{A}^*$. We are going to prove the claim as follows. For all $M \in \mathbf{A}^*$,

$$X(M) = \bigcap_{D \in S} M(D).$$

Obviously, the left hand side is contained in the right. To show the converse, let a belong to $M(D)$ for all $D \in S$. Let $J = \{D \in Sub(C) : a \notin M(D)\}$. Then $S \cap J = \emptyset$. For each $D \in J$, there are $N_D \in \mathbf{A}^*$ and $b_D \in (X(N_D) - N_D(C))$. Let $N = \prod_{D \in J} N_D$ and $b =< b_D >_{D \in J}$. Note that if $D \in Sub(C)$ and $b \in N(D)$, then $D \notin J$. So $a \in M(D)$. By Lemma 7.5, there is a homomorphism $h : N \to M^F$ from N into a reduced power of M such that $\delta_C(a) = h_C(b)$. Since $b \in X(N)$, we have

$$< a > /F = \delta_C(a) = h_C(b) \in X(M^F) = (X(M))^F.$$

Therefore $a \in X(M)$ for F-almost all i. Since F is proper, hence for at least one i.

We will see that there is $D \in S$ such that $X(M) = M(D)$ for all $M \in \mathbf{A}^{*}$. Suppose not, i.e., for all $D \in S$ there are $N_D \in \mathbf{A}^{*}$ and $b_D \in (N_D(D) - X(N_D))$. Let N be the product of N_D for $D \in S$, and $b = < b_D >$. This is contrary to the above claim.

7.2 A stronger version of the strong conceptual completeness

Recall from Chapter 5 that, $\kappa - \mathbf{BARR} - \mathbf{EX}$ denotes the 2-category of all κ-Barr-exact categories as objects and κ-regular functors as 1-arrows, and all natural transformations between the latter as 2-arrows. $\prod\mathcal{F}_\kappa$ is the 2-category of all categories with κ-filtered colimits and products as objects , all functors preserving κ-filtered colimits and small products as 1-arrows, and all natural transformations between the latter as 2-arrows. We have 2-adjunction

$$\kappa - \mathbf{BARR} - \mathbf{EX}^{op} \underset{G}{\overset{F}{\rightleftarrows}} \prod\mathcal{F}_\kappa$$

$$F = \prod F_\kappa(-, \mathbf{S}), G = \kappa - Reg(-, \mathbf{S});$$

both unit and counit are defined, at any object of the respective kind, as the evaluation functor.

Consider a small κ-Barr-exact category $\mathbf{C}$, and $\mathbf{A} \rightarrow \kappa - Reg(\mathbf{C}, \mathbf{S})$ a full and faithful arrow in $\prod\mathcal{F}_\kappa$. We prove that if $\mathbf{A}$ is accessible, and the evaluation functor $e_{\mathbf{A}}$:

$\mathbf{A} \to \kappa - Reg(\prod F_\kappa(\mathbf{A},\mathbf{S}),\mathbf{S})$ is full and faithful, then its transpose $\mathbf{C} \to \prod F_\kappa(\mathbf{A},\mathbf{S})$ is a quotient.

Given $\mathbf{A}$ in $\prod\mathcal{F}_\kappa$, we denote $\prod F_\kappa(\mathbf{A},\mathbf{S})$ by $\mathbf{A}^*$, and $\mathbf{A}^{**}$ denotes the category $\kappa - Reg(\mathbf{A}^*,\mathbf{S})$. If $\mathbf{A}$ is accessible, we will show that the evaluation functor $e_\mathbf{A}$ is full and faithful if and only if for every $A \in \mathbf{A}$, the canonical cocone

$$\phi_{\mathbf{A}(A,-)} : (\mathbf{A}(A,-)/\mathbf{A}^*)^- \to (\mathbf{A},\mathbf{S})$$

with vertex $\mathbf{A}(A,-)$ is limiting; under these conditions, we consider that $\mathbf{A}$ is a full subcategory of $\kappa - Reg(\mathbf{C},\mathbf{S})$ which is closed under κ-filtered colimits and products. Then, we prove that the functor

$$F : \prod F_\kappa(\kappa - Reg(\mathbf{C},\mathbf{S}),\mathbf{S}) \to \prod F_\kappa(\mathbf{A},\mathbf{S})$$

induced by the inclusion satisfies the following property: for every functor $M \in \mathbf{A}^*$, there are a functor N in $\prod F_\kappa(\kappa - Reg(\mathbf{C},\mathbf{S}),\mathbf{S})$ and a regular epi $F(N) \to M$. The proof proceeds in three stages. Firstly, for each $A \in \mathbf{A}$, we build up a morphism $f : \mathbf{A}^{**}(e_\mathbf{A}(A),-) \to M'$ in the category $(\mathbf{A}^{**},\mathbf{S})$ from a regular epi $\eta : \mathbf{A}(A,-) \to M$ (such a regular epi is given by Proposition 5.6) in $(\mathbf{A}, \mathbf{S})$; secondly, we show that f has certain features by using the codensity of $\mathbf{A}^{**}$ in $L_\kappa(\mathbf{A}^*,\mathbf{S})$. Finally, we obtain a regular epi $F(N) \to M$ from the features of f.

For $\mathbf{C}$ small κ-Barr-exact, we use the notation $\mathbf{C}^*$ for $\kappa - Reg(\mathbf{C}, \mathbf{S})$, and $\mathbf{C}^{**}$ for $\prod F_\kappa(\mathbf{C}^*, \mathbf{S})$.

Definition 7.6 *Let* $\mathbf{A}$ *be an arbitrary category, and* $\mathbf{B}$ *a full subcategory of* $\mathbf{A}$. *Let* $i : \mathbf{B} \to \mathbf{A}$ *be the inclusion. An object* A *of* $\mathbf{A}$ *is called* $\mathbf{B}$-κ-*presentable if the functor* $\mathbf{A}(A, -) \circ i$ *preserves* κ-*filtered colimits existing in* $\mathbf{B}$. *Thus an object is* κ-*presentable if it is* $\mathbf{A}$-κ-*presentable.*

The $\mathbf{B}$-κ-copresentability of an object A in $\mathbf{A}$ is the dual to the $\mathbf{B}$-κ-presentablity. In elementary terms, A is $\mathbf{B} - \kappa$-copresentable if every morphism f from a κ-cofiltered limit $lim_I B_i$ in $\mathbf{B}$ into A factors through a limit projection p_i

$$\begin{tikzcd} lim_I B_i \arrow[r, "f"] \arrow[d, "p_i"'] & A \\ B_i \arrow[ur, "f'"'] & \end{tikzcd}$$

and any two different factorizations

$$\begin{tikzcd} lim_I B_i \arrow[r, "p_i"] \arrow[d, "p_j"'] & B_i \arrow[d, "f"] \\ B_j \arrow[r, "g"'] & A \end{tikzcd}$$

of the same morphism have a common "refinement"

$$\begin{tikzcd} B_k \arrow[r, "u"] \arrow[d, "v"'] & B_i \arrow[d, "f"] \\ B_j \arrow[r, "g"'] & A \end{tikzcd}$$

Here u and v are some morphisms in the limit cone.

Let $\mathbf{C}$ be a small κ-Barr-exact category, and let $i : \kappa - Reg(\mathbf{C}, \mathbf{S}) \to L_\kappa(\mathbf{C}, \mathbf{S})$ be the inclusion. The codensity of $\kappa - Reg(\mathbf{C}, \mathbf{S})$ in $L_\kappa(\mathbf{C}, \mathbf{S})$ is an equivalent way of saying that the functor

$$\sum : L_\kappa(\mathbf{C}, \mathbf{S})^{op} \to (\kappa - Reg(\mathbf{C}, \mathbf{S}), \mathbf{S})$$

$$M \mapsto Nat(M, i(-))$$

induced by i is full and faithful (see Theorem X.6.2 in [17]). By observation , we have following commutative diagram (this diagram will be appeared several times later without explanation)

$$\begin{tikzcd} \mathbf{C} \arrow[r, "ev_{\mathbf{C}}"] \arrow[d, "Y"'] & (\kappa - Reg(\mathbf{C}, \mathbf{S}), \mathbf{S}) \\ L_\kappa(\mathbf{C}, \mathbf{S})^{op} \arrow[ur, "\sum"'] & \end{tikzcd}$$

here Y is induced by the Yoneda embedding. Therefore, $ev_{\mathbf{C}}$ is full and faithful. Note that $\sum$ makes colimit diagrams in $L_\kappa(\mathbf{C}, \mathbf{S})$ into the corresponding limits diagram in

$(\kappa - Reg(\mathbf{C},\mathbf{S}),\mathbf{S})$. Also for any object C in $\mathbf{C}$, $\mathbf{C}(C,-)$ is κ-presentable in $L_\kappa(\mathbf{C},\mathbf{S})$

, we have

Proposition 7.7 *Let $\mathbf{C}$ be a small κ-Barr-exact category. Then*

(*i*) *The evaluation functor $ev_{\mathbf{C}}$ is full and faithful.*

(*ii*) *For any object C in $\mathbf{C}$, then $ev_{\mathbf{C}}(C)$ is $\mathbf{P}$-κ-copresentable in $(\mathbf{C}^*,\mathbf{S})$; here $\mathbf{P}$ consists of those objects which are κ-cofiltered limits of diagrams of objects of the form $ev_{\mathbf{C}}(D)$ with $D \in \mathbf{C}$.*

We now assume that $\mathbf{A}$ is an accessible with κ-filtered colimits and products. Then we have the evaluation functor

$$e_{\mathbf{A}} : \mathbf{A} \to \kappa - Reg(\textstyle\prod F_\kappa(\mathbf{A},\mathbf{S}),\mathbf{S})$$

defined by

$$A \longmapsto [M \longmapsto M(A)]$$

$$f \longmapsto [M \longmapsto M(f)].$$

The functor $e_{\mathbf{A}}$ preserves κ-filtered colimits and products. The following is a set of condition ensuring that $e_{\mathbf{A}}$ is full and faithful.

Proposition 7.8 *For an accessible category $\mathbf{A}$ with κ-filtered colimits and products, the following conditions are equivalent:*

(*i*) *The evaluation functor $e_{\mathbf{A}}$ is full and faithful.*

(*ii*) *For any $A \in \mathbf{A}$, the canonical cone*

$$\phi_{\mathbf{A}(A,-)} : (\mathbf{A}(A,-)/\mathbf{A}^{*})^{-} \to (\mathbf{A},\mathbf{S})$$

with vertex $\mathbf{A}(A,-)$ is limiting.

Proof: Let $e_{\mathbf{A}}$ be full and faithful. Consider the following induced functor of $e_{\mathbf{A}}$,

$$G : \prod Acc(\mathbf{A}^{**},\mathbf{S}) \to \prod Acc(\mathbf{A},\mathbf{S})$$

$$F \to F \circ e_{\mathbf{A}}$$

Then G preserves limits. For small κ-Barr-exact category $\mathbf{A}^{*}$, the diagram

$$\begin{tikzcd} \mathbf{A}^{*} \arrow[r, "ev_{\mathbf{A}^{*}}"] \arrow[d, "Y"'] & (\mathbf{A}^{**},\mathbf{S}) \\ L_{\kappa}(\mathbf{A}^{*},\mathbf{S})^{op} \arrow[ur, "\sum"'] & \end{tikzcd}$$

commutes; Let $ev_{\mathbf{A}^{*}}(\mathbf{A}^{*})$ be the image of $ev_{\mathbf{A}^{*}}$. For each $A \in \mathbf{A}$, then the canonical cone

$$L_{\sum(e_{\mathbf{A}}(A))} : (\sum(e_{\mathbf{A}}(A))/ev_{\mathbf{A}^{*}}(\mathbf{A}^{*}))^{-} \to (\mathbf{A}^{**},\mathbf{S})$$

is limiting. It is easy to see that $G(ev_{\mathbf{A}^*}(M)) \cong M$ for any $M \in \mathbf{A}^*$, and $G(\sum(e_{\mathbf{A}}(A))) \cong \mathbf{A}(A,-)$ for any $A \in \mathbf{A}$. Also, $\sum$ is full and faithful, so (ii) follows from the limiting diagram mentioned above.

Assuming (ii), let $\mathbf{A}$ be λ-accessible for some $\lambda \geq \kappa$, and let $i : \prod F_\kappa(\mathbf{A},\mathbf{S}) \to \prod Acc(\mathbf{A},\mathbf{S})$ be the inclusion; here $\prod Acc(\mathbf{A},\mathbf{S})$ is the full subcategory of $(\mathbf{A},\mathbf{S})$ whose objects are the accessible functors preserving products. Let Z be the functor induced by i

$$Z : LR(\prod Acc(\mathbf{A},\mathbf{S}),\mathbf{S}) \to \kappa - Reg(\prod F_\kappa(\mathbf{A},\mathbf{S}),\mathbf{S})$$

$$M \mapsto M \circ i$$

then Z preserves κ-filtered colimits and products.

By Theorem 6.1.(iii), the evaluation functor $\eta_{\mathbf{A}} : \mathbf{A} \to LR(\prod Acc(\mathbf{A},\mathbf{S}),\mathbf{S})$ is an equivalence of categories. Also, we have that the diagram

$$\begin{tikzcd}
LR(\prod Acc(\mathbf{A},\mathbf{S}),\mathbf{S}) \arrow[d, "Z"'] & \mathbf{A} \arrow[l, "\eta_{\mathbf{A}}"'] \arrow[dl, "e_{\mathbf{A}}"] \\
\kappa - Reg(\prod F_\kappa(\mathbf{A},\mathbf{S}),\mathbf{S}) &
\end{tikzcd}$$

commutes. So, to show that $e_{\mathbf{A}}$ is full and faithful, it suffices to show that the functor Z is full and faithful.

Let $M, N \in LR(\prod Acc(\mathbf{A}, \mathbf{S}), \mathbf{S})$, and $t : M \circ i \to N \circ i$ a natural transformation. We are going to construct a natural transformation $\eta : M \to N$ with $Z(\eta) = t$. For any $A \in \mathbf{A}$, we use A' to denote the representable functor $\mathbf{A}(A, -)$. Let $A' = lim_{i \in I} F_i$ be the canonical limit with all $F_i \in \mathbf{A}^*$. Since M and N preserve limits, thus we have $M(A') = lim_{i \in I} M(F_i)$ and $N(A') = lim_{i \in I} N(F_i)$. By the naturality of t, the diagram

$$\begin{tikzcd} M(F_j) \arrow[r, "t_{F_j}"] \arrow[d, "M(k)"'] & N(F_j) \arrow[d, "N(k)"] \\ M(F_i) \arrow[r, "t_{F_i}"'] & N(F_i) \end{tikzcd}$$

commutes for any $k : F_j \to F_i$; hence $M(A')$ is a cone of diagram $N(\phi_{A'})$. But $N(A')$ is the limit of that diagram, so we have a unique morphism $t_{A'} : M(A') \to N(A')$ such that the diagram

$$\begin{tikzcd} M(A') \arrow[r, "t_{A'}"] \arrow[d, "M(f)"'] & N(A') \arrow[d, "N(f)"] \\ M(F_i) \arrow[r, "t_{F_i}"'] & N(F_i) \end{tikzcd}$$

commutes for each limit projection f. Since A' is the canonical limit of the diagram whose objects are in $\prod F_\kappa(\mathbf{A}, \mathbf{S})$, then, for any $F \in \prod F_\kappa(\mathbf{A}, \mathbf{S})$, and any morphism $g : A' \to F$, the diagram

$$\begin{tikzcd}
M(A') \arrow[r, "t_{A'}"] \arrow[d, "M(g)"'] & N(A') \arrow[d, "N(g)"] \\
M(F) \arrow[r, "t_F"'] & N(F)
\end{tikzcd}$$

commutes.

Claim: For any two objects $A, B \in \mathbf{A}$, and any morphism $h : A' \to B'$, we have that the diagram

$$\begin{tikzcd}
M(A') \arrow[r, "t_{A'}"] \arrow[d, "M(h)"'] & N(A') \arrow[d, "N(h)"] \\
M(B') \arrow[r, "t_{B'}"'] & N(B')
\end{tikzcd}$$

commutes.

Proof of claim: Let $B' = lim_{j \in J} F_j$ be the canonical limit with projections $a : B' \to F_j$. Let $k = a \circ h$, then the diagram

$$\begin{tikzcd}
M(A') \arrow[r, "t_{A'}"] \arrow[d, "M(k)"'] & N(A') \arrow[d, "N(k)"] \\
M(F_j) \arrow[r, "t_{F_j}"'] & N(F_j)
\end{tikzcd}$$

commutes. Therefore we have that

$$N(a) \circ N(h) \circ t_{A'} = t_{F_j} \circ M(a) \circ M(h);$$

but $t_{F_j} \circ M(a) = N(a) \circ t_{B'}$, hence

$$N(a) \circ N(h) \circ t_{A'} = N(a) \circ t_{B'} \circ M(h)$$

holds for each projection $N(a)$. It follows that $N(h) \circ t_{A'} = t_{B'} \circ M(h)$.

Let M' and N' be M and N restricted to the subcategory of $\prod Acc(\mathbf{A}, \mathbf{S})$ whose objects are isomorphic to the representable functors, the above claim defines a natural transformation $\eta : M' \to N'$.

For any $F \in \prod Acc(\mathbf{A}, \mathbf{S})$, by Proposition 5.6, there are $A \subset \mathbf{A}$ and a regular epimorphism: $A' \to F$. By a proof similar to that of Proposition 5.9, we can see that there is a unique natural transformation η between M and N with that $Z(\eta) = t$. This proves the fullness and faithfulness of Z; hence $e_{\mathbf{A}}$ is full and faithful.

Theorem 7.9 *Let* $\mathbf{C}$ *be a small* κ*-Barr-exact category, and let* $\mathbf{A}$ *be an accessible subcategory of* $\kappa - Reg(\mathbf{C}, \mathbf{S})$ *which is closed under* κ*-filtered colimits and products. Let* $i : \mathbf{A} \to \kappa - Reg(\mathbf{C}, \mathbf{S})$ *be the inclusion. If* $e_{\mathbf{A}}$ *is full and faithful, then the functor* F *induced by* i

$$F : \prod F_\kappa(\kappa - Reg(\mathbf{C}, \mathbf{S}), \mathbf{S}) \to \prod F_\kappa(\mathbf{A}, \mathbf{S})$$

$$M \mapsto M \circ i$$

satisfies the following property: for every functor M in $\prod F_\kappa(\mathbf{A}, \mathbf{S})$, there are a functor N in $\prod F_\kappa(\kappa - Reg(\mathbf{C}, \mathbf{S}), \mathbf{S})$ and a regular epi $F(N) \to M$.

Proof: Note that both $\mathbf{C}^*$ and $\mathbf{A}$ are accessible categories with κ-filtered colimits and products, hence $\mathbf{C}^{**}$ and $\mathbf{A}^*$ are small κ-Barr-exact. Therefore, the evaluation functors

$$ev_{\mathbf{C}^{**}} : \mathbf{C}^{**} \to (\mathbf{C}^{***}, \mathbf{S})$$

$$ev_{\mathbf{A}^*} : \mathbf{A}^* \to (\mathbf{A}^{**}, \mathbf{S})$$

are full and faithful. By assumption, $e_{\mathbf{A}} : \mathbf{A} \to \mathbf{A}^{**}$ is a full and faithful functor. Let $e^{\#}{}_{\mathbf{A}}$ be the functor

$$e^{\#}{}_{\mathbf{A}} : (\mathbf{A}^{**}, \mathbf{S}) \to (\mathbf{A}, \mathbf{S})$$

$$M \mapsto M \circ e_{\mathbf{A}}$$

induced by $e_{\mathbf{A}}$. For any small κ-Barr-exact $\mathbf{C}$, the diagram

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "ev_{\mathbf{C}}"] \arrow[d, "Y_{\mathbf{C}}"'] & (\mathbf{C}^{*}, \mathbf{S}) \\
L_{\kappa}(\mathbf{C}, \mathbf{S})^{op} \arrow[ur, "\sum_{\mathbf{C}}"'] &
\end{tikzcd}$$

commutes; here $Y_{\mathbf{C}}$ is induced by the Yoneda embedding. Both $ev_{\mathbf{C}}$ and $\sum_{\mathbf{C}}$ are full and faithful. Consider the diagram

$$\begin{tikzcd}
\mathbf{A}^{op} \arrow[r, "e'_{\mathbf{A}}"] \arrow[d, "Y"'] & L_{\kappa}(\mathbf{A}^{*}, \mathbf{S})^{op} \arrow[d, "\sum_{\mathbf{A}^{*}}"] \\
(\mathbf{A}, \mathbf{S}) & (\mathbf{A}^{**}, \mathbf{S}) \arrow[l, "e^{\#}{}_{\mathbf{A}}"]
\end{tikzcd}$$

and the diagram

$$\begin{tikzcd}
\mathbf{A}^{*} \arrow[r, "Y_{\mathbf{A}^{*}}"] \arrow[d, "l"'] & L_{\kappa}(\mathbf{A}^{*}, \mathbf{S})^{op} \arrow[d, "\sum_{\mathbf{A}^{*}}"] \\
(\mathbf{A}, \mathbf{S}) & (\mathbf{A}^{**}, \mathbf{S}) \arrow[l, "e^{\#}{}_{\mathbf{A}}"]
\end{tikzcd}$$

Here $e'_{\mathbf{A}}$ denotes the functor induced by the composite of $e_{\mathbf{A}} : \mathbf{A} \to \mathbf{A}^{**}$ and the inclusion $i_{\mathbf{A}^{**}} : \mathbf{A}^{**} \to L_{\kappa}(\mathbf{A}^{*}, \mathbf{S})^{op}$, Y is the Yoneda embedding, and l is the inclusion. Without difficulty, we can see that $Y \cong e^{\#}{}_{\mathbf{A}} \circ \sum_{\mathbf{A}^{*}} \circ e'_{\mathbf{A}}$ and $l \cong e^{\#}{}_{\mathbf{A}} \circ \sum_{\mathbf{A}^{*}} \circ Y_{\mathbf{A}^{*}}$. The fact that $e_{\mathbf{A}}$ is full and faithful implies that $\sum_{\mathbf{A}^{*}}(e'_{\mathbf{A}}(A)) \cong \mathbf{A}^{**}(e_{\mathbf{A}}(A), -)$. Thus, $\mathbf{A}(A, -) \cong e^{\#}{}_{\mathbf{A}}(\mathbf{A}^{**}(e_{\mathbf{A}}(A), -))$ and $M \cong e^{\#}{}_{\mathbf{A}}(e_{\mathbf{A}^{*}}(M))$. By Proposition 5.6, for

$M \in \mathbf{A}^{*}$, there is an object A in $\mathbf{A}$ and a regular epi $\eta : \mathbf{A}(A, -) \to M$. Using the Yoneda lemma, η is uniquely determined by an element $a \in M(A)$; also, note that $\mathbf{A}^{**}(\mathbf{A}^{**}(e_{\mathbf{A}}(A), -), e_{\mathbf{A}^{*}}(M)) \cong M(A)$, that is, $a \in \mathbf{A}^{**}(\mathbf{A}^{**}(e_{\mathbf{A}}(A), -), e_{\mathbf{A}^{*}}(M))$. By using the Yoneda lemma again, we obtain a morphism $f : \mathbf{A}^{**}(e_{\mathbf{A}}(A), -) \to e_{\mathbf{A}^{*}}(M)$ which is dertermined by a. We obtain that $e^{\#}{}_{\mathbf{A}}(f) \cong \eta$.

Next, let F^{*} be the functor

$$F^{*} : \mathbf{A}^{**} \to \mathbf{C}^{***}$$

$$X \mapsto X \circ F$$

induced by F, and let $F^{*\#}$ be the functor

$$F^{*\#} : (\mathbf{C}^{***}, \mathbf{S}) \to (\mathbf{A}^{**}, \mathbf{S})$$

$$E \mapsto E \circ F^{*}$$

induced by F^{*}. Restricting $F^{*\#}$ on the category $\prod Acc(\mathbf{C}^{***}, \mathbf{S})$ gives a functor

$$F^{**} : \prod Acc(\mathbf{C}^{***}, \mathbf{S}) \to \prod Acc(\mathbf{A}^{**}, \mathbf{S})$$

Note that F^{**} preserves limits. Let $i_{\mathbf{C}^{**}} : \mathbf{C}^{***} \to L_{\kappa}(\mathbf{C}^{**}, \mathbf{S})$ be the inclusion. By Remark 7.9, $e_{\mathbf{C}^{*}} : \mathbf{C}^{*} \to \mathbf{C}^{***}$ is full and faithful, so we have that the composite

$$i_{\mathbf{C}^{**}} \circ e_{\mathbf{C}^{*}} \circ i : \mathbf{A} \to L_\kappa(\mathbf{C}^{**}, \mathbf{S})$$

is a full and faithful functor which preserves κ-filtered colimits and products. Denote $i_{\mathbf{C}^{**}} \circ e_{\mathbf{C}^{*}} \circ i$ by G, and look at the following commutative diagram

$$\begin{tikzcd}
\mathbf{C}^{**} \arrow[r, "e_{\mathbf{C}^{**}}"] \arrow[d, "Y_{\mathbf{C}^{**}}"'] & (\mathbf{C}^{***}, \mathbf{S}) \\
L_\kappa(\mathbf{C}^{**}, \mathbf{S})^{op} \arrow[ur, "\sum_{\mathbf{C}^{**}}"'] &
\end{tikzcd}$$

For any A in $\mathbf{A}$, $G(A)$ can be written as a κ-filtered colimit in $L_\kappa(\mathbf{C}^{**}, \mathbf{S})$, say $G(A) = colim_{j\in J}\mathbf{C}^{**}(N_j, -)$; here J is κ- filtered category, and all N_j are in $\mathbf{C}^{**}$. Since $\sum_{\mathbf{C}^{**}}$ makes colimit diagrams in $L_\kappa(\mathbf{C}^{**}, \mathbf{S})$ into the corresponding limit diagrams in $(\mathbf{C}^{***}, \mathbf{S})$, we have that $\sum_{\mathbf{C}^{**}}(G(A)) = lim_{j\in J}\sum_{\mathbf{C}^{**}}(\mathbf{C}^{**}(N_j, -))$, i.e., we have that

$$\sum\nolimits_{\mathbf{C}^{**}}(G(A)) = lim_{j\in J} e_{\mathbf{C}^{**}}(N_j)$$

is the κ-cofiltered limit in $(\mathbf{C}^{***}, \mathbf{S})$. Note that $\prod Acc(\mathbf{C}^{***}, \mathbf{S})$ is closed under limits in $(\mathbf{C}^{***}, \mathbf{S})$. We conclude that

$$\sum\nolimits_{\mathbf{C}^{**}}(G(A)) = lim_{j\in J} e_{\mathbf{C}^{**}}(N_j)$$

is the κ-cofiltered limit in $\prod Acc(\mathbf{C}^{***}, \mathbf{S})$. Therefore, we have that

$$F^{**}(\sum_{\mathbf{C}^{**}}(G(A))) = lim_{j\in J}F^{**}(e_{\mathbf{C}^{**}}(N_j))$$

is the κ-cofiltered limit in $\prod Acc(\mathbf{A}^{**}, \mathbf{S})$.

Consider the diagram

$$\begin{tikzcd} \mathbf{A}^{op} \arrow[r, "G"] \arrow[d, "H"'] & L_\kappa(\mathbf{C}^{**}, \mathbf{S})^{op} \arrow[d, "\sum_{\mathbf{C}^{**}}"] \\ (\mathbf{A}^{**}, \mathbf{S}) & (\mathbf{C}^{***}, \mathbf{S}) \arrow[l, "F^{*\#}"] \end{tikzcd}$$

Here H is induced by $e_{\mathbf{A}}$ and the Yoneda embedding $Y_{\mathbf{A}^{**}} : (\mathbf{A}^{**})^{op} \to (\mathbf{A}^{**}, \mathbf{S})$. By the fullness and faithfulness of G, we have that $H \cong F^{*\#} \circ \sum_{\mathbf{C}^{*}} \circ G$; hence

$$\mathbf{A}^{**}(e_{\mathbf{A}}(A), -) \cong lim_{j\in J}F^{**}(e_{\mathbf{C}^{**}}(N_j)).$$

Also, by the definitions of $e_{\mathbf{C}^{**}}$ and F^{**}, we have that $F^{**}(e_{\mathbf{C}^{**}}(N_j)) \cong e_{\mathbf{A}^{*}}(F(N_j))$, for all $N_j \in \mathbf{C}^{**}$. Therefore, we have the κ-cofiltered limit

$$\mathbf{A}^{**}(e_{\mathbf{A}}(A), -) \cong lim_{j\in J}e_{\mathbf{A}^{*}}(F(N_j))$$

in $(\mathbf{A}^{**}, \mathbf{S})$; here all $F(N_j)$ are in $\mathbf{A}^{*}$.

For any $M \in \mathbf{A}^{*}$, there is $f : \mathbf{A}^{**}(e_{\mathbf{A}}(A), -) \to e_{\mathbf{A}^{*}}(M)$ (see before). By Proposition 7.7(ii), the morphism f can be factored through a limit projection p_j

$$\begin{tikzcd} \mathbf{A}^{**}(e_{\mathbf{A}}(A), -) \arrow[r, "f"] \arrow[d, "p_j"'] & e_{\mathbf{A}^*}(M) \\ e_{\mathbf{A}^*}(F(N_j)) \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

Applying $e^{\#}{}_{\mathbf{A}^*}$ on the above diagram, we have the following commutative diagram

$$\begin{tikzcd} \mathbf{A}(A, -) \arrow[r, "\eta"] \arrow[d, "e^{\#}{}_{\mathbf{A}^*}(p_j)"'] & M \\ F(N_j) \arrow[ur, "e^{\#}{}_{\mathbf{A}^*}(h)"'] & \end{tikzcd}$$

It follows that $e^{\#}{}_{\mathbf{A}^*}(h) : F(N_j) \to M$ is a regular epi from that η is regular epi.

Let us note a connection of the last theorem with the duality theorem, Theorem 5.1 in [21], for κ-Barr exact categories. With our notation, the latter says that for every small κ-Barr exact category $\mathbf{C}$, $e_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}^{**}$ is an equivalence of categories. First, note that from this it follows that for $\mathbf{A} = \mathbf{C}^*$, $e_{\mathbf{A}} : \mathbf{A} \to \mathbf{A}^{**}$ is an equivalence. Conversely, if we only assume that, for $\mathbf{A} = \mathbf{C}^*$, $e_{\mathbf{A}}$ is full and faithful, then, by Theorem 7.13, it follows that $e_{\mathbf{C}}$ is a quotient morphism; since $e_{\mathbf{C}}$ is conservative (even full and faithful), it follows that $e_{\mathbf{C}}$ is an equivalence, which is the assertion of the duality theorem. This argument does not constitute a new proof of the duality theorem as long as we do not have an independent proof for $e^*_{\mathbf{C}}$ being full and faithful.

The auther is presently working on an attempt to find such an independent proof, specifically one that avoids the use of 'principal prime' models, a characteristic feature of Makkai's proof.

Let $F : \mathbf{C} \to \mathbf{D}$ be a κ-regular functor. Denote by $Inv(F)$ the collection of all those morphisms f in $\mathbf{C}$ for which $F(f)$ is an isomorphism in $\mathbf{D}$. Let Σ be a collection of morphisms in $\mathbf{C}$. Recall from [18] that F is said to be obtained by inverting the morphisms in Σ if we have the following universal property: for any κ-Barr-exact category $\mathbf{B}$, the functor induced by F

$$F^* : \kappa - Reg(\mathbf{D}, \mathbf{B}) \to \kappa - Reg(\mathbf{C}, \mathbf{B})$$

$$M \mapsto M \circ F$$

induces an equivalence of $\kappa - Reg(\mathbf{D}, \mathbf{B})$ onto the full subcategory of $\kappa - Reg(\mathbf{C}, \mathbf{B})$ consisting of those $G : \mathbf{C} \to \mathbf{B}$ for which $\Sigma \subset Inv(G)$. F is a quotient morphism (a quotient) if it is obtained by inverting the morphisms in $Inv(F)$.

As pointed out in [18], the definition of a quotient morphism has a general character. It can be repeated in other, similar, situations. Such a situation is given by a concrete 2-category. E.g., **Lex**, the 2-category of small categories with finite limits, whose 1-arrows are functors preserving finite limits, and all natural transformations as 2-arrows; **Pretop**, the 2-category of small pretoposes with 1-arrows pretoposes morphisms, and 2-arrows are all natural transformations between them (see [18] and

[22]). We discuss quotient morphisms in the 2-category $\kappa - \mathbf{Barr} - \mathbf{ex}$, i.e. the 2-category of small κ-exact categories with 1-arrows κ-regular functors, whose 2-arrows are natural transformations.

The following proposition gives a characterization of the quotient morphism between Barr-exact categories (see [19] and [22]).

Proposition 7.10 *A regular functor $F : \mathbf{C} \to \mathbf{D}$ is a quotient if and only if F satisfies the following conditions.*

(i) F is full on subobjects;

(ii) for any object $D \in \mathbf{D}$, there is a regular epi $e : F(C) \to D$ with $C \in \mathbf{C}$.

Remark 7.11 *Let $F : \mathbf{C} \to \mathbf{D}$ be a regular functor. Suppose that F is a quotient and conservative functor, then F is an equivalence (see [19]).*

For a quotient F, by the above definition, F^* is full and faithful, for any κ-Barr-exact category $\mathbf{B}$. The following strong conceptual completeness says that, suppose that F is a κ-regular functor between small κ-Barr-exact categories, to show that F is a quotient, it suffices to require the full and faithful condition for F^* on $\mathbf{S}$.

Proposition 7.12 *For a κ-regular functor $F : \mathbf{C} \to \mathbf{D}$ between two small κ-Barr-exact categories to be a quotient, it is sufficient that the induced functor*

$$F' : \kappa - Reg(\mathbf{D}, \mathbf{S}) \to \kappa - Reg(\mathbf{C}, \mathbf{S})$$

is full and faithful.

The proposition is a special case of the next Theorem, by taking $\mathbf{A}$ the category $\kappa - Reg(\mathbf{D}, \mathbf{S})$.

Theorem 7.13 *Let* $\mathbf{A}$ *be an accessible full subcategory of the category* $\kappa - Reg(\mathbf{C}, \mathbf{S})$ *which is closed under* κ*-filtered colimits and products, and let* $i : \mathbf{A} \to \kappa - Reg(\mathbf{C}, \mathbf{S})$ *be the inclusion. Then the the composite of* $e_{\mathbf{C}}$ *and* F *in Theorem 7.9, denoted by* e

$$e : \mathbf{C} \to \prod F_{\kappa}(\mathbf{A}, \mathbf{S})$$

is a quotient if and only if $e_{\mathbf{A}}$ *is full and faithful.*

Proof: Note that $e_{\mathbf{C}}$ is an equivalence of categories, if $e_{\mathbf{A}}$ is full and faithful, By Theorem 7.9, for each M in $\prod F_{\kappa}(\mathbf{A}, \mathbf{S})$, there are $C \in \mathbf{C}$ and a regular epi $e(C) \to M$. Proposition 7.4 says that e is full on subobjects, by using proposition 7.12, we obtain that e is a quotient.

Assume that e is a quotient. By the definition of a quotient, the induced functor of e

$$e^{*} : \mathbf{A}^{**} \to \mathbf{C}^{*}$$

is full and faithful. Consider the diagram

$$\begin{tikzcd}
\mathbf{A}^{**} \arrow[r, "e^{*}"] \arrow[d, "e_{\mathbf{A}}"'] & \mathbf{C}^{*} \arrow[dl, "i"] \\
\mathbf{A} &
\end{tikzcd}$$

We conclude that $e_{\mathbf{A}}$ is full and faithful from the fullness and faithfulness of i and e^{*}.

Proof of Proposition 7.12: Let $F : \mathbf{D} \to \mathbf{C}$ be a κ-regular functor. Then F' preserves κ-filtered colimits and products. Suppose additional that, F' is full and faithful, we obtain that

$$F'^{*} : \mathbf{C}^{**} \to \mathbf{D}^{**}$$

is a quotient from Theorem 7.13. But $\mathbf{C}$ and $\mathbf{D}$ are equivalent to $\mathbf{C}^{**}$ and $\mathbf{D}^{**}$, resepectively; hence F is a quotient.

Remark 7.14 *Assuming Vopěka's principle, the accessibility of* $\mathbf{A}$ *in this section can be removed. As shown in [2] (see Corollary IV.7 in [2]), assuming Vopěnka's principle, each full subcategory of a locally presentable category which is closed under* κ*-filtered colimits and products is small injectivity class, hence it is a Barr category; here* κ *is some infinite regular cardinal.*

Remark 7.15 *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be any two small categories with finite limits, and let* $F : \mathbf{C} \to \mathbf{D}$ *be a functor preserving finite limits. If* F *is a quotient, then the induced* *functor F' : $Lex(\mathbf{D},\mathbf{S}) \to Lex(\mathbf{C},\mathbf{S})$ is full and faithful. The converse, however, is not true (see Example 2.5 in [23]). We note that for the above F, suppose that F' is an equivalence of categories, by Gabriel-Ulmer duality, then F is an equivalence of categories as well.*

7.3 Duality for κ-Barr-exact categories

In this section we will characterize the categories of the form $\mathbf{C}^*$, i.e., $\kappa - Reg(\mathbf{C},\mathbf{S})$, for $\mathbf{C}$ small κ-Barr-exact.

Let $\mathbf{C}$ be a small κ-Barr-exact category. The category $\kappa - Reg(\mathbf{C},\mathbf{S})$ is a Barr category with κ-filtered colimits. In general, $\kappa - Reg(\mathbf{C},\mathbf{S})$ is not κ-accessible. In fact, the duality theorem for κ-Barr categories implies that $\kappa - Reg(\mathbf{C},\mathbf{S})$ is κ-accessible if and only if $\mathbf{C}$ is κ-Barr-exact accessible.

Given two infinite cardinals κ and λ, recall from [24] that $\kappa \trianglelefteq \lambda$ if $\kappa \leq \lambda$ and for every set X of cardinality less than λ, $P_\kappa(X)$, the partially ordered set of subsets of X of cardinality less than κ, has a cofinal subset of cardinality less than λ. As proved in [24](Theorem 2.3.10), if $\mathbf{A}$ is κ-accessible and $\kappa \trianglelefteq \lambda$, then $\mathbf{A}$ is λ-accessible.

Let $\mathbf{A}$ be a λ-Barr category. We write $\mathbf{A}^*$ for the category $\prod F_\lambda(\mathbf{A},\mathbf{S})$, and write $\mathbf{A}^{**}$ for the category $\lambda - Reg(\mathbf{A}^*,\mathbf{S})$. By the duality theorem for λ-Barr categories, the evaluation functor $\eta_{\mathbf{A}} : \mathbf{A} \to \mathbf{A}^{**}$ is an equivalence of categories. Let $i : \mathbf{A}^{**} \to$

$L_\lambda(\mathbf{A}^\star, \mathbf{S})$ be the inclusion. By the codensity of i, the induced functor of the composite of $\eta_{\mathbf{A}}$ and i, denoted by Σ_λ,

$$\Sigma_\lambda : L_\lambda(\mathbf{A}^\star, \mathbf{S})^{op} \to (\mathbf{A}, \mathbf{S})$$

$$M \mapsto \mathbf{A}(M, i \circ \eta_{\mathbf{A}}(-))$$

is full and faithful, and Σ_λ makes colimit diagrams in $L_\lambda(\mathbf{A}^\star, \mathbf{S})$ into the corresponding limit diagrams in $(\mathbf{A}, \mathbf{S})$. We have

Proposition 7.16 *Let* $\mathbf{A}$ *be a* λ*-Barr category, and let* $N \in (\mathbf{A}, \mathbf{S})$ *be a functor of the form* $\Sigma_\lambda(M)$*, for some* $M \in L_\lambda(\mathbf{A}^\star, \mathbf{S})$*. If* N *is* $\mathbf{A}^\star$*-projective in* $(\mathbf{A}, \mathbf{S})$*, then* N *is isomorphic to a representable functor* $\mathbf{A}(A, -)$ *for some* $A \in \mathbf{A}$*.*

Proof: Let $Y : \mathbf{A}^\star \to L_\lambda(\mathbf{A}^\star, \mathbf{S})^{op}$ be the induced functor of the Yoneda embedding. Note that the composite of Y and Σ_λ is the evaluation functor, so it is a full and faithful λ- regular functor. For each regular monomorphism $m : Y(M') \to Y(M'')$ in $L_\lambda(\mathbf{A}^\star, \mathbf{S})$ with $M', M'' \in \mathbf{A}^\star$, $\Sigma_\lambda(m)$ is regular epi in $(\mathbf{A}, \mathbf{S})$. Write m' for $\Sigma_\lambda(m)$. Given a morphism $f : Y(M') \to M$, by the assumption on N, we have that $\Sigma_\lambda(f)$ factors as

$$\begin{tikzcd}
\Sigma_\lambda(Y(M')) & \Sigma_\lambda(Y(M'')) \arrow[l, "m'"'] \\
\Sigma_\lambda(M) \arrow[u, "\Sigma_\lambda(f)"] \arrow[ur, dotted, "h"'] &
\end{tikzcd}$$

for some $h : \Sigma_\lambda(M) \to \Sigma_\lambda(Y(M''))$. Since Σ_λ is full and faithful, we let $h = \Sigma_\lambda(g)$ with some $g : Y(M'') \to M$. Therefore, we have that f factors as

$$\begin{tikzcd}
Y(M') \arrow[r, "m"] \arrow[d, "f"'] & Y(M'') \arrow[dl, "g"] \\
M &
\end{tikzcd}$$

We conclude that M is in $\mathbf{A}^{**}$ from Proposition 7.2. But $\eta_{\mathbf{A}}$ is an equivalence, so $\Sigma_\lambda(M)$ is isomorphic to a representable functor.

Consider a small κ-Barr-exact category $\mathbf{C}$, and we let $\kappa-\mathit{Reg}(\mathbf{C},\mathbf{S})$ be λ-accessible; here λ is an infinite cardinal with $\kappa \trianglelefteq \lambda$.

Let $i : \kappa - \mathit{Reg}(\mathbf{C},\mathbf{S}) \to L_\kappa(\mathbf{C},\mathbf{S})$ be the inclusion. Then we have a full and faithful functor induced by i, denoted by Z_κ,

$$Z_\kappa : L_\kappa(\mathbf{C},\mathbf{S})^{op} \to (\mathbf{C}^*,\mathbf{S})$$

$$M \mapsto \mathbf{C}^*(M, i(-))$$

makes colimit diagrams in $L_\kappa(\mathbf{C},\mathbf{S})$ into the corresponding limit diagrams in $(\mathbf{C}^*,\mathbf{S})$.

Proposition 7.17 *Let* $\mathbf{C}$ *be a small* κ*-Barr-exact category. For any object* $M \in L_\kappa(\mathbf{C},\mathbf{S})$*, if* $Z_\kappa(M)$ *is* $e_{\mathbf{C}}(\mathbf{C})$*-projective in* $(\mathbf{C}^{**},\mathbf{S})$*, then* $M \in \mathbf{C}^*$.

The proof of Proposition 7.17 is essentially the same as that of Proposition 7.16.

Since $\mathbf{C}^*$ is a λ-Barr category, the duality theorem for λ-Barr categories gives an equivalence of categories (the evaluation functor):

$$\eta_{\mathbf{C}^*} : \mathbf{C}^* \to \lambda - Reg(\prod F_\lambda(\mathbf{C}^*,\mathbf{S}),\mathbf{S}).$$

The category $\prod F_\lambda(\mathbf{C}^*,\mathbf{S})$ is λ-Barr-exact accessible, of course, it is small λ-Barr-exact. Also, the induced functor of the evaluation functor $e_{\mathbf{C}}$, denoted by the same $e_{\mathbf{C}}$,

$$e_{\mathbf{C}} : \mathbf{C} \to \prod F_\lambda(\mathbf{C}^*,\mathbf{S})$$

is a κ-regular functor. Let i_λ be the composite of $\eta_{\mathbf{C}^*}$ and the inclusion $\lambda - Reg(\prod F_\lambda$ $(\mathbf{C}^*,\mathbf{S}),\mathbf{S}) \to L_\lambda(\prod F_\lambda(\mathbf{C}^*,\mathbf{S}),\mathbf{S})$. By the codensity of i_λ, the functor defined as

$$Z_\lambda : L_\lambda(\prod F_\lambda(\mathbf{C}^*,\mathbf{S}),\mathbf{S})^{op} \to (\mathbf{C}^*,\mathbf{S})$$

$$M \mapsto \mathbf{C}^*(M, i_\lambda(-))$$

is full and faithful, and Z_λ takes colimit diagrams in the domain category into the limit diagrams in the codomain category. Let $Y_{\mathbf{C}}$ be the composite of $e_{\mathbf{C}}$ and the functor $Y : \prod F_\lambda(\mathbf{C}^*, \mathbf{S}) \to L_\lambda(\prod F_\lambda(\mathbf{C}^*, \mathbf{S}), \mathbf{S})^{op}$; here Y is induced by the Yoneda emdedding. Note that Y is λ-regular, hence $Y_{\mathbf{C}}$ is κ-regular, and the diagram

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "ev_{\mathbf{C}}"] \arrow[d, "Y_{\mathbf{C}}"'] & (\mathbf{C}^*, \mathbf{S}) \\
L_\lambda(\prod F_\lambda(\mathbf{C}^*, \mathbf{S}), \mathbf{S})^{op} \arrow[ur, "Z_\lambda"'] &
\end{tikzcd}$$

commutes. We have

Proposition 7.18 *Let $\mathbf{C}$ be a small κ-Barr-exact category. Then, we have*

(i) for every $M \in \mathbf{C}^*$, the canonical cocone

$$\psi_{\eta_{\mathbf{C}^*}(M)} : (Y_{\mathbf{C}}(\mathbf{C})/\eta_{\mathbf{C}^*}(M))^+ \to L_\lambda(\prod F_\lambda(\mathbf{C}^*, \mathbf{S}), \mathbf{S})$$

with vertex $\eta_{\mathbf{C}^*}(M)$ is colimiting;

(ii) for $N \in L_\lambda(\prod F_\lambda(\mathbf{C}^*, \mathbf{S}), \mathbf{S})$, suppose that N is a κ-filtered colimit of a diagram of objects of the form $Y_{\mathbf{C}}(C)$ with $C \in \mathbf{C}$, and it is $Y_{\mathbf{C}}(\mathbf{C})$-injective in $L_\lambda(\prod F_\lambda(\mathbf{C}^*, \mathbf{S}), \mathbf{S})$. Then N is $Y(\prod F_\lambda(\mathbf{C}^*, \mathbf{S}))$-injective in $L_\lambda(\prod F_\lambda(\mathbf{C}^*, \mathbf{S}), \mathbf{S})$;

(iii) for any $M \in L_\kappa(\mathbf{C}^{**}, \mathbf{S})$, there are an object N in $\mathbf{C}^*$ and a regular monomorphism $M \to e_{\mathbf{C}^*}(N)$ in $L_\kappa(\mathbf{C}^{**}, \mathbf{S})$.

Proof: Note that for every $M \in \mathbf{C}^*$, the representable functor $\mathbf{C}^*(M, -)$ is isomorphic to $Z_\lambda(\eta_{\mathbf{C}^*}(M))$. Since Z_λ is full and faithful, and it makes the colimit diagrams in $L_\lambda(\prod F_\lambda(\mathbf{C}^*, \mathbf{S}), \mathbf{S})$ into the corresponding limit diagrams in $(\mathbf{C}^*, \mathbf{S})$, we can see that (i) is follows from the fact that the canonical cone

$$\phi_M : (M/(e_{\mathbf{C}}(\mathbf{C}))^- \to (\mathbf{C}^*, \mathbf{S})$$

with vertex M is limiting.

For (ii), since N is a κ-filtered colimit of a diagram of objects of the form $Y_{\mathbf{C}}(C)$ with $C \in \mathbf{C}$, so, $Z_\lambda(N)$ is a κ-cofiltered limit of a diagram of objects $e_{\mathbf{C}}(C)$ with $C \in \mathbf{C}$. It follows that there is $M \in L_\kappa(\mathbf{C}^*, \mathbf{S})$ so that $Z_\lambda(N)$ is isomorphic to $Z_\kappa(M)$. For any regular epi $p : e_{\mathbf{C}}(C) \to e_{\mathbf{C}}(D)$ in $(\mathbf{C}^*, \mathbf{S})$ with $C, D \in \mathbf{C}$, we can see that there is a regular epi $q : C \to D$ so that $p = e_{\mathbf{C}}(q)$. Thus, $Y_{\mathbf{C}}(q) : Y_{\mathbf{C}}(D) \to Y_{\mathbf{C}}(C)$ is a regular mono in $L_\lambda(\prod F_\lambda(\mathbf{C}^*, \mathbf{S}), \mathbf{S})$. Given a morphism $f : Z_\lambda(N) \to e_{\mathbf{C}}(D)$, i.e., $f : Z_\lambda(N) \to Z_\lambda(Y_{\mathbf{C}}(D))$, by the fullness of Z_λ, there is $g : Y_{\mathbf{C}}(D) \to N$ so that $f = Z_\lambda(g)$. By the assumption on N, g factors as

$$\begin{tikzcd}
Y_{\mathbf{C}}(D) \arrow[r, "Y_{\mathbf{C}}(q)"] \arrow[d, "g"'] & Y_{\mathbf{C}}(C) \arrow[dl, "h"] \\
N &
\end{tikzcd}$$

for some $h : Y_{\mathbf{C}}(C) \to N$. Applying Z_λ on the above diagram, since $e_{\mathbf{C}} = Z_\lambda \circ Y_{\mathbf{C}}$, we have that f factors as

$$\begin{tikzcd}
e_{\mathbf{C}}(D) & e_{\mathbf{C}}(C) \arrow[l, "p"'] \\
Z_\lambda(N) \arrow[u, "f"] \arrow[ur, "Z_\lambda(h)"'] &
\end{tikzcd}$$

Thus $Z_\lambda(N)$ is $e_{\mathbf{C}}(\mathbf{C})$-projective in $(\mathbf{C}^{\bullet}, \mathbf{S})$, i.e., $Z_\kappa(M)$ is $e_{\mathbf{C}}(\mathbf{C})$-projective in $(\mathbf{C}^{\bullet}, \mathbf{S})$. By Proposition 7.19, we have that $M \in \mathbf{C}^{\bullet}$. We conclude that $Z_\lambda(N)$ is isomorphic to the representable functor $\mathbf{C}^{\bullet}(M, -)$. Note that $\mathbf{C}^{\bullet}(M, -)$ is projective in $(\mathbf{C}^{\bullet}, \mathbf{S})$, a fortiori, it is $\prod F_\lambda(\mathbf{C}^{\bullet}, \mathbf{S})$-projective. For $S, T \in \prod F_\lambda(\mathbf{C}^{\bullet}, \mathbf{S})$, and $r : Y(S) \to Y(T)$, as we know, r is a regular mono in $L_\lambda(\prod F_\lambda(\mathbf{C}^{\bullet}, \mathbf{S}), \mathbf{S})$ if and only if $Z_\lambda(r)$ is a regular epi in $(\mathbf{C}^{\bullet}, \mathbf{S})$. Therefore, N is $Y(\prod F_\lambda(\mathbf{C}^{\bullet}, \mathbf{S}))$-injective in $L_\lambda(\prod F_\lambda(\mathbf{C}^{\bullet}, \mathbf{S}), \mathbf{S})$.

(iii) follows from that Proposition 7.3 and $e_{\mathbf{C}^{\bullet}}$ is an equivalence of categories.

Let $\mathbf{A}$ be a λ-Barr category with κ-filtered colimits. Denoted by $\mathbf{A}^{\bullet}$ the category $\prod F_\kappa(\mathbf{A}, \mathbf{S})$, and denoted by $\mathbf{A}^{\star}$ the category $\prod F_\lambda(\mathbf{A}, \mathbf{S})$, and by $\mathbf{A}^{\star\star}$ the category $\kappa - Reg(\mathbf{A}^{\star}, \mathbf{S})$. Let $\eta_{\mathbf{A}} : \mathbf{A} \to \mathbf{A}^{\star\star}$ be the evaluation functor, and let $Y_{\mathbf{A}^{\star}} : \mathbf{A}^{\star} \to L_\lambda(\mathbf{A}^{\star}, \mathbf{S})^{op}$ be the induced functor of the Yoneda embedding. The following proposition shows that the properties in Proposition 7.18 on $\mathbf{A}$ give a sufficient condition so that $\mathbf{A}$ is the category of the form $\mathbf{C}^{\bullet}$. Therefore, we have a characterization of the categories of the form $\kappa - Reg(\mathbf{C}, \mathbf{S})$ with $\mathbf{C}$ small κ-Barr-exact.

Proposition 7.19 *Let $\mathbf{A}$ be an arbitrary category with κ-filtered colimits and products. Suppose that there is an infinite cardinal λ with $\kappa \trianglelefteq \lambda$ so that $\mathbf{A}$ is λ-accessible, and $\mathbf{A}$ satisfying the following conditions.*

(i) For every object A of $\mathbf{A}$, the canonical cocone

$$\psi_{\eta_{\mathbf{A}}(A)} : (Y_{\mathbf{A}^*}(\mathbf{A}^*)/\eta_{\mathbf{A}}(A))^+ \to L_\lambda(\mathbf{A}^*, \mathbf{S})$$

with vertex $\eta_{\mathbf{A}}(A)$ is colimiting;

(ii) for $M \in L_\lambda(\mathbf{A}^*, \mathbf{S})$, suppose that M is a κ-filtered colimit of a diagram of objects of the form $Y_{\mathbf{A}^*}(P)$ with $P \in \mathbf{A}^*$, and is $Y_{\mathbf{A}^*}(\mathbf{A}^*)$-injective in $L_\lambda(\mathbf{A}^*, \mathbf{S})$. Then M is $Y_{\mathbf{A}^*}(\mathbf{A}^*)$- injective in $L_\lambda(\mathbf{A}^*, \mathbf{S})$;

(iii) for any $M \in L_\kappa(\mathbf{A}^*, \mathbf{S})$, there are an object A in $\mathbf{A}$ and a regular monomorphism $M \to e_{\mathbf{A}}(A)$ in $L_\kappa(\mathbf{A}^*, \mathbf{S})$; here $e_{\mathbf{A}}$ is the evaluation functor

$$e_{\mathbf{A}} : \mathbf{A} \to \kappa - Reg(\mathbf{A}^*, \mathbf{S})$$

Then $\mathbf{A}$ is equivalent to the category of models of the small κ-Barr-exact category $\prod F_\kappa(\mathbf{A}, \mathbf{S})$. Therefore, $e_{\mathbf{A}}$ is an equivalence of categories.

Proof: Since $\mathbf{A}$ is a λ-Barr category, the duality theorem for λ-Barr categories gives that the evaluation functor $\eta_{\mathbf{A}}$ is an equivalence of categories. Let G_λ be the composite of the $\eta_{\mathbf{A}}$ and the inclusion $\mathbf{A}^{**} \to L_\lambda(\mathbf{A}^*, \mathbf{S})$. By Proposition 7.3, G_λ is codense; hence the functor

$$\Sigma_\lambda : L_\lambda(\mathbf{A}^*, \mathbf{S})^{op} \to (\mathbf{A}, \mathbf{S})$$

$$M \mapsto Nat(M, G_\lambda(-)) : \mathbf{A} \to \mathbf{S}$$

is full and faithful, and Σ_λ makes colimit diagrams in $L_\lambda(\mathbf{A}^*, \mathbf{S})$ into the limiting diagrams in $(\mathbf{A}, \mathbf{S})$. Let $\eta^{\#}_{\mathbf{A}} : (\mathbf{A}^{**}, \mathbf{S}) \to (\mathbf{A}, \mathbf{S})$ be the induced functor of $\eta_{\mathbf{A}}$, and let $ev_{\mathbf{A}^*} : \mathbf{A}^* \to (\mathbf{A}^{**}, \mathbf{S})$ be the evaluation functor. Denote by F_λ the composite of $ev_{\mathbf{A}^*}$ and $\eta^{\#}_{\mathbf{A}}$. Then the diagram

$$\begin{tikzcd} \mathbf{A}^* \arrow[r, "F_\lambda"] \arrow[d, "Y_{\mathbf{A}^*}"'] & (\mathbf{A}, \mathbf{S}) \\ L_\lambda(\mathbf{A}^*, \mathbf{S})^{op} \arrow[ur, "\Sigma_\lambda"'] & \end{tikzcd}$$

commutes; here $Y_{\mathbf{A}^*}$ is the induced functor of the Yoneda embedding. It follows from the condition (i) that for every object A of $\mathbf{A}$, the representable functor $\mathbf{A}(A, -)$ is the limit of the canonical diagram

$$L_{\mathbf{A}(A,-)} : \mathbf{A}(A, -)/\mathbf{A}^* \to (\mathbf{A}, \mathbf{S}).$$

By Proposition 7.8, $e_{\mathbf{A}}$ is full and faithful.

Let G be the composite of the inclusion $i : \mathbf{A}^{**} \to L_\kappa(\mathbf{A}^*, \mathbf{S})$ and $e_{\mathbf{A}}$. Then G is full and faithful. By Proposition 7.3, it follows that G is codense from (iii). Thus, The induced functor of G

$$\Sigma_\kappa : L_\kappa(\mathbf{A}^*, \mathbf{S})^{op} \to (\mathbf{A}, \mathbf{S})$$

$$M \mapsto \mathbf{A}(M, G(-))$$

is full and faithful, and Σ_κ makes colimit diagrams in $L_\kappa(\mathbf{A}^*, \mathbf{S})$ into the corresponding limit diagrams in $(\mathbf{A}, \mathbf{S})$. Let $e^{\#}_{\mathbf{A}} : (\mathbf{A}^{**}, \mathbf{S}) \to (\mathbf{A}, \mathbf{S})$ be the induced functor of $e_{\mathbf{A}}$, and let $ev_{\mathbf{A}^*} : \mathbf{A}^* \to (\mathbf{A}^{**}, \mathbf{S})$ be the evaluation functor. Denote by F the composite of $ev_{\mathbf{A}^*}$ and $e^{\#}_{\mathbf{A}}$. We have that the diagram

$$\begin{tikzcd}
\mathbf{A}^* \arrow[r, "F"] \arrow[d, "Y_{\mathbf{A}^*}"'] & (\mathbf{A}, \mathbf{S}) \\
L_\kappa(\mathbf{A}^*, \mathbf{S})^{op} \arrow[ur, "\Sigma_\kappa"'] &
\end{tikzcd}$$

commutes; here $Y_{\mathbf{A}^*}$ is induced by the Yoneda embedding. For any $M \in \mathbf{A}^{**}$, then $\Sigma_\kappa(M)$ is a κ-cofiltered limit of a diagram of objects of $\mathbf{A}^*$. Note that for any N in ($\mathbf{A}$, $\mathbf{S}$), if N is a limit of a diagram of objects of $\mathbf{A}^*$, then N is isomorphic to $\Sigma_\lambda(M')$ for some $M' \in L_\lambda(\mathbf{A}^*, \mathbf{S})$; hence, $\Sigma_\kappa(M)$ is isomorphic to $\Sigma_\lambda(M')$ with some $M' \in L_\lambda(\mathbf{A}^*, \mathbf{S})$. Also, $\Sigma_\kappa(M)$ is $\mathbf{A}^*$-injective in $(\mathbf{A}, \mathbf{S})$, i.e., $\Sigma_\lambda(M')$ is $\mathbf{A}^*$-injective in (**A**, **S**). Note that the condition (ii) is equivalent to saying that for $N \in (\mathbf{A}, \mathbf{S})$, if N is a κ-cofiltered limit of a diagram of objects of $\mathbf{A}^*$, and N is $\mathbf{A}^*$-injective in (**A**, **S**), then N is $\mathbf{A}^*$-injective in (**A**, **S**). Therefore, $\Sigma_\lambda(M')$ is $\mathbf{A}^*$-injective in (**A**, **S**). By Proposition 7.16, $\Sigma_\lambda(M')$ is isomorphic to a representable functor $\mathbf{A}(A, -)$, for some $A \in \mathbf{A}$. We have that $\Sigma_\kappa(M)$ is isomorphic to $\mathbf{A}(A, -)$. Since Σ_κ is full and faithful, we conclude that **A** is equivalent to $\mathbf{A}^{**}$. That $e_{\mathbf{A}}$ is an equivalence of categories is clear.

Let $\kappa - \mathbf{BARR} - \mathbf{EX}$ be the 2-category of all κ-Barr-exact categories as objects and κ-regular functors as 1-arrows, and all natural transformations between the latter as 2-arrows. $\prod\mathcal{F}_\kappa$ is the 2-category of all categories with κ-filtered colimits and products as objects , all functors preserving κ-filtered colimits and small products as 1-arrows, and all natural transformations between the latter as 2-arrows. We have a 2-adjunction

$$\kappa - \mathbf{BARR} - \mathbf{EX}^{op} \underset{G}{\overset{F}{\rightleftarrows}} \prod\mathcal{F}_\kappa$$

$$F = \prod F_\kappa(-, \mathbf{S}), G = \kappa - Reg(-, \mathbf{S});$$

both unit and counit are defined, at any object of the respective kind, as the evaluation functor. Consider the full sub-2-category $\kappa - \mathbf{Barr} - \mathbf{ex}$ of $\kappa - \mathbf{BARR} - \mathbf{EX}$ whose objects are small κ-Barr exact categories. $\prod\mathbf{F}_\kappa$ is the full sub-2-category of $\prod\mathcal{F}_\kappa$ with objects in Proposition 7.19. By Theorem 5.1 in [21] and Proposition 7.19, we have the 2-adjunction

$$\kappa - \mathbf{Barr} - \mathbf{ex} \underset{G}{\overset{F}{\rightleftarrows}} \prod\mathbf{F}_\kappa$$

The following duality theorem for small κ-Barr exact categories solves a problem posed by M. Makkai in [21].

Theorem 7.20 *The pair of adjoint 2-functors restricting to* $\kappa - \mathbf{Barr} - \mathbf{ex}$ *and* $\prod\mathbf{F}_\kappa$ *is a biequivalence. In other words,*

(i) If $\mathbf{C}$ *is a small* κ*-Barr-exact category, then* $\kappa - Reg(\mathbf{C}, \mathbf{S}) \in \prod\mathbf{F}_\kappa$*, and the evaluation functor*

$$\varepsilon_{\mathbf{C}} : \mathbf{C} \to \prod F_\kappa(\kappa - Reg(\mathbf{C}, \mathbf{S}), \mathbf{S})$$

is an equivalence of categories; and

(ii) if $\mathbf{A} \in \prod\mathbf{F}_\kappa$*, then* $\prod F_\kappa(\mathbf{A}, \mathbf{S}) \in \kappa - \mathbf{Barr} - \mathbf{ex}$*, and the evaluation functor*

$$\eta_{\mathbf{A}} : \mathbf{A} \to \kappa - Reg(\prod F_\kappa(\mathbf{A}, \mathbf{S}), \mathbf{S})$$

is an equivalence of categories.

Proof: (i) is given by Theorem 5.1 in [21]. (ii) is given by Proposition 7.19.

The following conceptual completeness is a consequence of Theorem 7.20. Such a statement strengthens Corollary 5.13.

Proposition 7.21 *(i) If $F : \mathbf{C} \to \mathbf{D}$ is a κ-regular functor between small κ-Barr-exact categories such that the induced functor*

$$F^* : \kappa - Reg(\mathbf{D}, \mathbf{S}) \to \kappa - Reg(\mathbf{C}, \mathbf{S})$$

is an equivalence of categories, then F is an equivlance.

(ii) Let $\mathbf{A}$ and $\mathbf{B}$ be any two categories in $\prod \mathbf{F}_\kappa$. If a functor $G : \mathbf{A} \to \mathbf{B}$ preserves κ-filtered colimits and products such that the induced functor

$$G^* : \prod F_\kappa(\mathbf{B}, \mathbf{S}) \to \prod F_\kappa(\mathbf{A}, \mathbf{S})$$

is an equivalence of categories, then F is an equivalence as well.

CHAPTER 8

CONE-REFLECTIVITY CLASSES

IN LOCALLY PRESENTABLE CATEGORIES

Definition 8.1 *Let* $\mathbf{B}$ *be a locally presentable category, and* $\mathbf{A}$ *a full subcategory of* $\mathbf{B}$. $\mathbf{A}$ *is said to be accessibly emdedded if it is closed under* κ*-filtered colimits in* $\mathbf{B}$, *for some regular cardinal* κ.

Recall from [2] that a full subcategory $\mathbf{A}$ of a category $\mathbf{B}$ is said to be cone-reflective if the inclusion functor $\mathbf{A} \to \mathbf{B}$ satisfies the solution-set condition, i.e. for each object B of $\mathbf{B}$ there exists a small cone $< r_i : B \to A_i >_{i \in I}$ with $A_i \in \mathbf{A}$ such that for any $A \in \mathbf{A}$, every morphism $B \to A$ factors through some r_i. As proved in [2], assuming Vopěnka's principle, every subcategory of a locally presentable category is cone-reflective. The main result in this Chapter is that if $\mathbf{A}$ is a cone-reflective accessibly embedded subcategory of a locally presentable category, then it is an accessible category.

The following lemma can be found in [24] (Lemma 1.1.2).

lemma 8.2 *Suppose that* $\mathbf{J}$ *is* κ*-filtered and the functor* $F : \mathbf{I} \to \mathbf{J}$ *satisfies that for every* $J \in \mathbf{J}$, *there exists* I *in* $\mathbf{I}$ *and a morphism* $J \to F(I)$. *If* F *is full and faithful,* *then* $\mathbf{I}$ *is* κ*-filtered and* F *is final, i.e. for any diagram* $\sum : \mathbf{J} \to \mathbf{A}$, *colim* $\sum$ *exists in and only if colim* $\sum \circ F$ *exists and the canonical morphism colim* $\sum(F) \to$ *colim* $\sum$ *is an isomorphism.*

In what follows, κ, λ and subscripted variants of them always denote infinite regular cardinals.

Let $\mathbf{A}$ be a full subcategory of $\mathbf{B}$, $B \in \mathbf{B}$ and $\mathbf{D}$ a set of objects of $\mathbf{A}$. Let us say that $\mathbf{D}$ weakly reflects B (in $\mathbf{A}$) if for every $A \in \mathbf{A}$ and $f : B \to A$ there is $D \in \mathbf{D}$ and a factorization

$$\begin{tikzcd} B \arrow[r, "m"] \arrow[d, "f"'] & D \arrow[dl, "f'"] \\ A & \end{tikzcd}$$

where m and f' are some morphisms. Note that to say that $\mathbf{A}$ is cone-reflective in $\mathbf{B}$ is to say that for every $B \in \mathbf{B}$, there is a small set $\mathbf{D} \subset \mathbf{A}$ weakly reflecting B. If $\mathbf{B}$ is accessible, then this is equivalent to saying that for every $B \in \mathbf{B}$ there is κ such that $\mathbf{D}_\kappa = \mathbf{A} \bigcap \mathbf{B}_\kappa$ weakly reflects B. Note that, of course, if $\kappa < \kappa'$ and $\mathbf{D}_\kappa$ weakly reflects B, so does $\mathbf{D}_{\kappa'}$.

Proposition 8.3 *Let* $\mathbf{B}$ *be an* κ*-accessible category,* $\mathbf{A}$ *a full subcategory of* $\mathbf{B}$ *closed under* κ*-filtered colimits in* $\mathbf{B}$*. If every* $B \in \mathbf{B}_\kappa$ *is weakly reflected in* $\mathbf{A}$ *by* $\mathbf{D} = \mathbf{A} \bigcap \mathbf{B}_\kappa$*, then* $\mathbf{A}$ *is* κ*-accessible.*

Proof: $\mathbf{A}$ has κ-filtered colimits, by assumption. For any $A \in \mathbf{A}$, we have a canonical diagram $G : \mathbf{B}_\kappa/A \to \mathbf{B}$, and $A = colimG$. This colimit is κ-filtered. Let $\mathbf{D} = \mathbf{A} \bigcap \mathbf{B}_\kappa$. Since $\mathbf{A}$ is closed under κ-filtered colimits in $\mathbf{B}$, all objects in $\mathbf{D}$ are κ-presentable in $\mathbf{A}$. We have a full and faithful functor $F : \mathbf{D}/A \to \mathbf{B}_\kappa/A$. Let $G' : \mathbf{D}/A \to \mathbf{B}$ be the canonical diagram. Given an object $f : B \to A$ in $\mathbf{B}_\kappa/A$, by assumption, there is a factorization $f = f' \circ m$, with $f' : D \to A$ in $\mathbf{D}/\mathrm{A}$. That A is the κ-filtered colimit $colimG'$ in $\mathbf{B}$, and as a consequence, also in $\mathbf{A}$. This completes the proof.

The proof of the following theorem uses some techniques in the proof of Theorem IV.3 in [2].

Theorem 8.4 *Let* $\mathbf{B}$ *be a locally presentable category, and* $\mathbf{A}$ *an accessibly embedded subcategory of* $\mathbf{B}$. *If* $\mathbf{A}$ *is cone-reflective, then it is accessible.*

Proof: We may assume that $\mathbf{B} = (\mathbf{C}, \mathbf{S})$ with $\mathbf{C}$ small. The reason is that every locally presentable category is a reflective subcategory of a functor category $(\mathbf{C}, \mathbf{S})$ for some small category $\mathbf{C}$, and the inculsion functor is accessibly embedded. If λ is a regular cardinal bigger than the cardinal of $\mathbf{C}$ and $\aleph_0$, then a functor $F \in \mathbf{B}$ is λ-presentable in $\mathbf{B}$ if and only if the cardinal of $\coprod_{C \in \mathbf{C}} F(C)$ is less than λ. It easily follows that if $\mu = sup_{i<\nu}\kappa_i$ with $\kappa_i \leq \kappa_j$ for $i < j < \nu$, and $B \in \mathbf{B}_{\mu^+}$, then we can write B as a colimit of a ν-chain, $B = colim_{i<\nu}B_i$, $b_{i,j} : B_i \to B_j$, with $B_i \in \mathbf{B}_{\kappa_i}$.

Let κ be a regular cardinal such that $\mathbf{A}$ is closed under κ-filtered colimits in $\mathbf{B}$.

Let us define κ_i for $i < \kappa$ by induction. Let $\kappa_0 = \kappa$. With $0 < i < \kappa$, having defined κ_j for all $j < i$, for i limit, let κ_i be a regular cardinal bigger than κ_j for all $j < i$, and for $i = j + 1 < \kappa$, let κ_{j+1} be a regular cardinal $\geq \kappa_j$ such that all objects in $\mathbf{B}_{\kappa_j}$ are weakly reflected by $\mathbf{D}_{\kappa_{j+1}}$; since each $\mathbf{B}_{\kappa'}$ is small, and since every $B \in \mathbf{B}$ is weakly reflected by $\mathbf{D}_{\kappa'}$, for some κ', such κ_{j+1} clearly exists.

Let $\mu = sup_{i<\kappa}\kappa_i$, and $\lambda = \mu^+$. We claim that every $B \in \mathbf{B}_\lambda$ is weakly reflected in $\mathbf{A}$ by $\mathbf{D}_\lambda$. Since $\mathbf{B}$ is clearly κ'-accessible for all $\kappa' \geq \aleph_0$, in particular, for $\lambda = \kappa'$, and $\lambda > \kappa$, by Proposition 8.3, this will suffice for the proof of the theorem.

Let $B \in \mathbf{B}_\lambda$. According to what was said above, let us represent B as the colimit of a κ-chain $(b_{i,j} : B_i \to B_j)_{i<j<\kappa}$, with $B_i \in \mathbf{B}_{\kappa_i}$. Let $\phi_i : B_i \to B$ be the colimit coprojection. Let $A \in \mathbf{A}$ and $f : B \to A$ be arbitrary; we want to find $A^* \in \mathbf{D}_\lambda$ with a factorization

$$\begin{tikzcd} B \arrow[r, "f^*"] \arrow[d, "f"'] & A^* \arrow[dl, "a^*"] \\ A & \end{tikzcd}$$

By induction on $i < \kappa$, we will define objects $A_i \in \mathbf{D}_{\kappa_{i+1}}$, morphisms $a_{i,j} : A_i \to A_j$, $f_i : B_i \to A_i$, $\psi_i : A_i \to A$ such that the the following diagrams

$$\begin{tikzcd}
B_i \arrow[r, "b_{i,j}"] \arrow[d, "f_i"'] & B_j \arrow[d, "f_j"] \\
A_i \arrow[r, "a_{i,j}"'] & A_j
\end{tikzcd}$$

$$\begin{tikzcd}
B_i \arrow[r, "\phi_i"] \arrow[d, "f_i"'] & B \arrow[d, "f"] \\
A_i \arrow[r, "\psi_i"'] & A
\end{tikzcd}$$

$$\begin{tikzcd}
A_i \arrow[r, "a_{i,j}"] \arrow[d, "\psi_i"'] & A_j \arrow[dl, "\psi_j"] \\
A &
\end{tikzcd}$$

$$\begin{tikzcd}
A_i \arrow[r, "a_{i,k}"] \arrow[d, "a_{i,j}"'] & A_k \arrow[dl, "a_{j,k}"] \\
C &
\end{tikzcd}$$

commute for all $i < j < k < \kappa$.

For $i = 0$, we let $A_0 \in \mathbf{D}_{\kappa_1}$, $f_0 : B_0 \to A_0$ and $\psi_0 : A_0 \to A$ such that

$$\begin{tikzcd}
B_0 \arrow[r, "\phi_0"] \arrow[d, "f_0"'] & B \arrow[d, "f"] \\
A_0 \arrow[r, "\psi_0"'] & A
\end{tikzcd}$$

commutes; these items are obtained from a suitable factorization of the morphism $f \circ \phi_0 : B_0 \to A$, possible by the choice of κ_1 and $B_0 \in \mathbf{D}_{\kappa_0}$.

Fix k, $0 < k < \kappa$, and assume that all items with indices $< k$ have been defined. Let $C = colim(a_{i,j} : A_i \to A_j)_{i<j<k}$ with coprojections $a_i : A_i \to C$, and $B^* = colim(b_{i,j} : B_i \to B_j)_{i<j<k}$ with coprojections $b_i^* : B_i \to B^*$.

Since $A_i \in \mathbf{B}_{\kappa_i} \subset \mathbf{B}_{\kappa_k}$, and $\mathbf{B}_{\kappa_k}$ is closed under $< \kappa \leq \kappa_k$-sized colimits, $C \in \mathbf{B}_{\kappa_k}$. Similarly, $B_i \in \mathbf{B}_{\kappa_{i+1}} \subset \mathbf{B}_{\kappa_k}$, and so $B^* \in \mathbf{B}_{\kappa_k}$.

By the univeral property of B^* , we have $b^* : B^* \to B_k$ such that

$$\begin{tikzcd} B_i \arrow[r, "b_i^*"] \arrow[d, "b_{i,k}"'] & B^* \arrow[dl, "b^*"] \\ B_k & \end{tikzcd}$$

commute, and $c : B^* \to C$ such that

$$\begin{tikzcd} B_i \arrow[r, "b_i^*"] \arrow[d, "f_i"'] & B^* \arrow[d, "c"] \\ A_i \arrow[r, "a_i"'] & C \end{tikzcd}$$

commute, for all $i < k$.

By the universal property of C, we have $a : C \to A$ such that

$$\begin{tikzcd} A_i \arrow[r, "a_i"] \arrow[d, "\psi_i"'] & C \arrow[dl, "a"] \\ A & \end{tikzcd}$$

commute for all $i < k$. We form the pushout of $< c, b^* >$:

$$\begin{tikzcd} B^* \arrow[r, "b^*"] \arrow[d, "c"'] & B_k \arrow[d, "h"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

Since $B^*, B_k, C \in \mathbf{B}_{\kappa_k}$, we have $D \in \mathbf{B}_{\kappa_k}$. For morphisms $< a : C \to A, f \circ \phi_k : B_k \to A >$, we have that

$$\begin{aligned} a \circ c \circ b_i^* &= a \circ a_i \circ f_i \\ &= \psi_i \circ f_i \\ &= f \circ \phi_i \\ &= f \circ \phi_k \circ b^* \circ b_i^* \end{aligned}$$

holds for all b_i^* with $i < k$. We obtain $a \circ c = (f \circ \phi_k) \circ b^*$ from b_i^* coprojections. By using the universl property of pushout D, we have a unique morphism $l : D \to A$ such that $a = l \circ g$ and $f \circ \phi_k = l \circ h$. Since $D \in \mathbf{B}_{\kappa_k}$, and every object in $\mathbf{B}_{\kappa_k}$ is weakly reflected by $\mathbf{D}_{\kappa_{k+1}}$, there is $A_k \in \mathbf{D}_{\kappa_{k+1}}$ with $\psi_k : A_k \to A$ such that the diagram

$$\begin{tikzcd}
D \arrow[r, "m"] \arrow[d, "l"'] & A_k \arrow[dl, "\psi_k"] \\
A &
\end{tikzcd}$$

commutes. We have defined the items A_k and ψ_k.

Next, we define $f_k = m \circ h : B_k \to A_k$ and $a_{i,k} = m \circ g \circ a_i : A_i \to A_k$. Note that the diagrams

$$\begin{tikzcd}
B_i \arrow[r, "b_i^*"] \arrow[d, "f_i"'] & B^* \arrow[r, "b^*"] \arrow[d, "c"] & B_k \arrow[d, "h"] \\
A_i \arrow[r, "a_i"'] & C \arrow[r, "g"'] & D
\end{tikzcd}$$

commute for all $i < k$; and $b_{i,k} = b^* \circ b_i^*$. Then the diagrams

$$\begin{tikzcd}
B_i \arrow[r, "b_{i,k}"] \arrow[d, "f_i"'] & B_k \arrow[d, "f_k"] \\
A_i \arrow[r, "a_{i,k}"'] & A_k
\end{tikzcd}$$

commute for all $i < k$, and the diagram

$$\begin{tikzcd}
B_k \arrow[r, "\phi_k"] \arrow[d, "f_k"'] & B \arrow[d, "f"] \\
A_k \arrow[r, "\psi_k"'] & A
\end{tikzcd}$$

commutes. It is clear that $\psi_i = a_{i,k} \circ \psi_k$ and $a_{i,k} = a_{i,j} \circ a_{j,k}$ hold for all $i < j < k$. This completes the construction.

Put $A^* = colim(a_{i,j} : A_i \to A_j)_{i<j<\kappa}$ with coprojections $p_i : A_i \to A^*$. Since $\mathbf{A}$ is closed under κ-filtered colimits in $\mathbf{B}$, $A^* \in \mathbf{A}$. Also, since $A_i \in \mathbf{B}_{\kappa_{i+1}} \subset \mathbf{B}_\lambda$ and $\kappa < \lambda$, we have that $A^* \in \mathbf{B}_\lambda$; that is, $A^* \in \mathbf{D}_\lambda$. By the construction above, we have $f^* : B \to A^*$ such that the diagrams

$$\begin{tikzcd} B_i \arrow[r, "\phi_i"] \arrow[d, "f_i"'] & B \arrow[d, "f^*"] \\ A_i \arrow[r, "p_i"'] & A^* \end{tikzcd}$$

commute for all $i < \kappa$; also, we have $a^* : A^* \to A$ such that the diagrams

$$\begin{tikzcd} A_i \arrow[r, "p_i"] \arrow[d, "\psi_i"'] & A^* \arrow[dl, "a^*"] \\ A & \end{tikzcd}$$

commute for all $i < \kappa$; hence we have that

$$f \circ \phi_i = \psi_i \circ f_i = a^* \circ p_i \circ f_i = a^* \circ f^* \circ \phi_i$$

for all $i < \kappa$. Since $< \phi_i >_{i<\kappa}$ is a colimit cocone, we conclude that the diagram

$$\begin{tikzcd} B \arrow[r, "f^*"] \arrow[d, "f"'] & A^* \arrow[dl, "a^*"] \\ A & \end{tikzcd}$$

commutes. We complete the proof.

Remark 8.5 *The above theorem gives the characterization of accessible categories. Indeed, Let* $\mathbf{A}$ *be an* κ*-accessible category, we have a full and faithful* κ*-accessible functor* $\mathbf{A} \rightarrow (\mathbf{A}_\kappa{}^{op}, \mathbf{S})$*, by Proposition 6.1.2 in [24], such a functor satisfies the solution-set condition.*

Remark 8.6 *Let* $\mathbf{A}$ *be an accessible full subcategory of an accessible category* $\mathbf{B}$*. Suppose that the inclusion functor from* $\mathbf{A}$ *to* $\mathbf{B}$ *satisfies the solution-set condition, J. Rosický and W. Tholen have recently proved that the inclusion functor is accessibly embedded (see Theorem 3.10 in [26]). Also, they have proved in [26] that Vopěnka's principle is equivalent to the the following statement: every functor between accessible categories is accessibly embedded if and only if it satisfies the solution-set condition.*

Remark 8.7 *Recall from [2] that A subcategory* $\mathbf{A}$ *of* $\mathbf{B}$ *is called weakly reflective if for each* $B \in \mathbf{B}$*, there exists a morphism* $r : B \rightarrow B^*$ *with* $B^* \in \mathbf{A}$ *such that*

(*i*) *for each* $f : B \rightarrow A$ *with* $A \in \mathbf{A}$*, there exists a morphism* $f' : B^* \rightarrow A$ *so that* $f = f' \circ r$*;*

(ii) **A** *is closed under retracts.*

Adámek and Rosický proved in [2] that a weakly reflective accessibly embedded subcategory of a locally presentable category is a Barr category. Observe that in a complete category, each weakly reflective subcategory is closed under products. Thus the above theorem improves that result. Also the condition (ii) is not necessary, since if **A** *is an accessible subcategory of* **B**, *and it is accessibly embedded, then* **A** *is closed under the retracts.*

The following concept generalizes the concept of injectivity class (see [13] and [16]).

Definition 8.8 *For each class M of small cones in a category* **B**, *$M - inj$ denotes the collection of objects A in* **B** *which are M-injective, i.e. for each cone $< m_i : B \to B_i >_{i \in I}$ in M, and any morphism $f : B \to A$, there exists some i such that $f = f' \circ m_i$ for some morphism $f' : B_i \to A$. A small cone-injectivity class is a class of objects of the form $M - inj$ for some small class M of small cones.*

Corollary 8.9 *A subcategory of a locally presentable category is a small cone-injectivity class if and only if it is a cone-reflective accessibly embedded subcategory.*

Proof: As proved in [13] and [16], such a subcategory is accessible and accessibly embedded.

References

[1] J.Adámek , J.Rosický and V.Trnková , Are all limit-closed subcategories of locally presentable categories reflective?, Lecture Notes in Math., no.1348, Springer-Verlag,1988, 1-18

[2] J.Adámek and J.Rosický, On injectivity in locally presentable categories, to appear in Trans. Amer. Math. Soc.

[3] M.Artin, A.Grothendieck, and J.L.Verdier, Theorie des Topos et Cohomologie Etale des Schemas, Lecture Notes in Math., no. 269, 270, Springer-Verlag, 1972.

[4] M.Barr, Exact categories, Lecture Notes in Math., no.236, Springer-Verlag, 1971, 1-120

[5] M.Barr, Representations of categories, J. Pure Appl. Algebra 41 (1986), 113-137

[6] M.Barr, Models of Horn theories, Contemporary Math.92(1989),1-7

[7] M.Barr, Accessible categories and models of linear logic, J. Pure Appl. Algebra 69 (1990), 219-232

[8] W.H.Boshuck, Reduced products and regular logic, M.S. Thesis, McGill University, 1992

[9] A.Carboni and R.C.Magno, The free exact category on a left exact one, J. Austral. Math. Soc. (Series A) vol 33 (1982), 295-301

[10] Y.Diers, Familles universelles de morphismes, Ann. Soc. Bruxelles, Tome 93, no.3, 1979

[11] Y.Diers, Catégories localement multiprésentables, Arch. Math. 34 (1980), 344-356

[12] P.Gabriel and F.Ulmer, Lokal präsentierbare kategorrien, Lecture Notes in Math., no.221, Springer-Verlag, 1971

[13] R. Guitart and C.Lair, Calcul syntaxique des modèles et calcul des formulas internes, Diagrammes 4(1980)

[14] H.Hu, Dualities for accessible categories, to appear in the proceedings of the conference of Category Theory 1991, AMS.

[15] G.M. Kelly, Basic concepts of enriched category theory, London Math. Lecture Note Ser., no.64, 1982

[16] C. Lair, Catégories modelables et catégories esquissables, Diagrammes 6(1981)

[17] S.Maclane, Categories for the Working Mathematician, Springer-Verlag, 1971

[18] M.Makkai, Ultraproducts and categorical logic, Lecture Notes in Math., no.1130, Springer-Verlag 1985, 222-309

[19] M.Makkai, Stone duality for first order logic, Adv. in Math. 65 (1987), 97-170

[20] M.Makkai, Strong conceptual completeness for first order logic, Ann. Pure Appl. Logic 40 (1988), 167-215

[21] M.Makkai, A theorem on Barr-exact categories, with an infinitary generalization, Ann. Pure Appl. Logic 47(1990), 225-268

[22] M.Makkai and G.R.Reyes, First order categorical logic, Lecture Notes in Math., no.611, Springer-Verlag, 1977

[23] M.Makkai and A.M.Pitts, Some results on locally finitely presentable categories, Trans. Amer. Math. Soc. 229 (1987), 473-496

[24] M.Makkai and R.Paré, Accessible categories: the foundations of categorical model theory, Contemporary Math. 104 (1990)

[25] R.Paré, Some application of categorical model theory, Contemporary Math. 92(1989), 325-339

[26] J.Rosický and W.Tholen, Accessibility and the solution set condition, preprint