Three Essays in Contest Theory

Jean-François Mercier

Department of Economics
McGill University, Montréal

August 2016

A thesis submitted to McGill University
in partial fulfillment of the requirements of
the degree of
Doctor of Philosophy

Acknowledgements

There are many people I wish to thank. Over the past six years, I have met many great persons who either in a direct or in an indirect way, helped me shape this thesis. I have learned a great deal from them and I wish to express my most heartfelt gratitude.

First and foremost, I wish to thank my supervisor, Licun Xue, for offering me constant advice and support since the beginning. Licun helped me and cheered me up when times were hard, when I thought I would not be able to go through it. This means a lot to me and I will be forever grateful.

I want to thank Rohan Dutta and Ngo Van Long for spending time discussing my research with me and for supporting me throughout my job market year. I also thank Vikram Manjunath who helped me a great deal at an earlier stage of my research.

Friends and colleagues played a big role during my years as a PhD student. From helping each other out when studying for our many exams, to reviewing our research papers, we helped each other in all sorts of ways and I am glad we had each other's back when we needed it. I thank Irakli Japaridze and Nagham Sayour for being part of my favorite reference group. I also wish to thank Md. Nazmul Ahsan, Guillaume Lord, Marco Serena, Álex Vázquez-Sedano, Lei Xu, Douglas Barthold, Chinmay Sharma and Vinh Nguyen.

I am particularly thankful to Ksenia Kordonsky for sharing my life during the earlier part of my PhD. Ksenia has been supportive, caring and loving and I thank her for putting up with my mood swings and for being there for me at the most difficult part of my life.

I thank the McGill economics department and its faculty members such as Hassan Benchekroun, John Galbraith, Jian Li, Fabian Lange, Theodore Papageorgiou, Francisco Alvarez-Cuadrado, Markus Poschke, Christopher Green, Chris Ragan, Paul Dickinson, Daniel Barczyk and Maxim Sinitsyn. Outside of McGill, I thank Catherine Gendron-Saulnier, Emre Ergeman, Sebastian Panthöfer, Luis Corchón, Deniz Dizdar, Philipp Denter, Ching-Jen Sun, Peter von Allmen, Derek Lougee, Peter Eccles, Diego Moreno and Takashi Kunimoto.

Et finalement, je souhaite remercier, du fond du coeur, ma soeur Julie Mercier, ma mère Reine Samson et mon père Denis Mercier. Je me suis toujours considéré chanceux d'avoir grandi en ayant comme modèles ma grande soeur et mes parents. J'ai eu la chance d'être supporté et encouragé à toutes les étapes de ma vie. Les dernières années ont été particulièrement difficiles et stressantes mais j'ai toujours eu ma famille à mes côtés.

Abstract

In the first chapter, I analyze a model of rent-seeking contest where groups compete non-cooperatively for a group-specific public good. Individuals have private information about how much they value the public good and face a free-riding problem in choosing effort levels. The probability that a group wins depends on the aggregate effort of its members relative to the aggregate effort of all contestants. For tractability, I restrict effort choices to be binary. I show that, in equilibrium, all contestants can exert positive effort ex post, despite the presence of free-riding incentives. This is in contrast to earlier results for contests with perfect information whereby only one contestant in a group exerts effort. I use simulation to show that when moving a player from a group to a group of equal or greater size, average expected effort in equilibrium decreases. Moreover, Olson's paradox, which asserts that groups of large size are less effective at winning a contest than small groups, may or may not hold. Olson's paradox can hold even though the good is purely public within the winning group. Members of the larger group expect other members to draw large valuations, which explains acute free-riding.

In the second chapter, I take the perspective of a contest designer who derives profits from aggregate effort exerted by the contestants. I develop a revelation mechanism that enables the contest designer to select a subset of contestants from a pool of candidates in a way that maximizes her profits, even though she is uninformed about the candidates' valuations for the contest prize. I prove

the existence of an incentive compatible and individually rational mechanism. I solve the designer's problem by using a 3-stage game where at Stage 0, the designer designs a mechanism, at Stage 1 the contestants participate in the mechanism then contestants are selected and at Stage 2, information is revealed and the selected contestants participate to a contest. I find that contests tend to be larger when candidates to the contest have valuations close to each other. Also, depending on the marginal cost that a contestant imposes on the designer, some candidates with low valuation for the contested good may never be selected to the contest.

In the last chapter, I extend a simple model of contest to study the question of league formation in sports economics. I model a professional sports league as a duopoly. I suggest a way to model a competitive allocation of talent into teams, by introducing a sequential game in which teams must first auction the cost of talent, and then, whichever team has made the highest bid, gets to choose first the quantity of talent to hire at the implemented market cost, and then the other team chooses a quantity from the residual pool of talent. I find that in equilibrium, leadership can be taken by the low-revenue team. Also, I find that the high-revenue team acquires more talent in equilibrium and that a revenue-sharing policy will induce the high-revenue team to acquire relatively more talent than the low-revenue team, thus producing a more uneven contest.

Abrégé

Dans cette thèse, le thème commun est la théorie des concours. Simplement, un concours est une situation dans laquelle un certain nombre de concurrents compétitionnent entre eux afin de devenir l'unique vainqueur et ainsi être récompensé d'un prix. Par exemple, si une personne voulait se départir d'un bien quelconque mais sans avoir une préférence particulière pour la personne à qui l'offrir, alors celui-ci pourrait organiser un concours dans lequel le gagnant se verrait offrir le bien. Dans ce concours, la probabilité de gagner dépend de l'effort fourni lors du concours. Comme exemples concrets, nous pouvons penser aux compétitions sportives, aux situations de litige, aux concours de recherche et autres.

Dans cette thèse, je propose trois modèles différents. Dans le premier chapitre, j'examine les concours dans lesquelles les concurrents mettent en commun leurs efforts à l'interieur du groupe auquel ils appartiennent, dans le but de gagner un bien public local, consommable uniquement par les membres du groupe vainqueur. Les concurrents sont informés de façon privée de la valeur qu'ils ont pour ce bien. J'y mets également en relief l'incitation qu'ont les concurrents à resquiller dépendamment de la taille du groupe auquel ils appartiennent et de la taille des autres groupes. Je présente aussi dans cet essai des résultats concernant la différence entre les probabilités qu'ont de gagner les grands groupes et les petits groupes. J'y conclus notamment que les grands groupes n'ont pas nécessairement l'avantage sur les plus petits.

Dans le second chapitre, j'introduis un concepteur de concours tirant un bénéfice de l'effort total des concurrents. J'y développe un mécanisme permettant au concepteur de sélectionner un sous-ensemble de concurrents, de façon à maximiser ses profits, en dépit du fait qu'il soit ignorant de l'utilité qu'ont les concurrents à participer au concours. Je démontre l'existence d'un mécanisme satisfaisant les concepts de compatibilité des incitations et de rationalité individuelle. Je résouds le problême du concepteur en utilisant un jeu en trois étapes. À l'étape 0, le concepteur choisit un mécanisme. À l'étape 1, les candidats au concours prennent part au mécanisme et un certain nombre d'entre eux sont sélectionnés pour participer en tant que concurrents à l'étape 2, dans laquelle se déroule un concours avec information parfaite. Il s'avère alors dans ce modèle que le concours final a tendance à inclure plus de concurrents lorsque les candidats valorisent le prix à gagner de façon relativement homogène. De plus, dependemment du coût marginal qu'engendre la participation d'un concurrent à l'étape 2, certains candidats valorisant faiblement le prix à gagner, risquent de ne jamais être choisis pour participer au concours.

Dans le dernier chapitre, je développe un modèle de concours simple pour étudier le problême de l'élaboration d'une ligue sportive professionnelle. Je modélise une ligue sportive professionnelle en tant que duopole. Ce faisant, je suggère une façon de modéliser l'attribution compétitive de talent dans les équipes et ce en utilisant un jeu séquentiel dans lequel les équipes doivent miser à la façon d'une enchère le coût unitaire du talent. Ensuite, l'équipe ayant offert le coût le plus élevé se voit octroyer le droit de choisir en premier la quantité de talent à acquérir au coût offert tandis que l'autre équipe doit choisir en deuxième la quantitité résiduelle voulue, payable au même coût unitaire. Dans l'équilibre de ce jeu, il est possible que ce soit l'équipe à faible revenue qui se voit octroyer le droit de choisir en premier. De plus, l'équipe à revenu élevé acquiert plus de talent à l'équilibre et je démontre qu'une politique de partage du revenu induit l'équipe à revenu élevé à acquérir relativement plus de talent, ayant alors comme conséquence d'avantager encore plus l'équire à revenu élevé.

Contents

List of Tables

List of Figures

Introduction

In this thesis, I have combined three essays, all contributing to the research on the theory of contests. What are contests? Contests are situations in which a number of contestants compete against each other by exerting costly effort towards winning a prize. A prize is a valuable abstract object such as a trophy, a lucrative contract or a favorable judgment in a court of law. It can also be a cash prize. A contest manager allocates the prize by designing a contest in which each contestant's probability of winning the prize depends on the quantity of effort exerted by all contestants. Contests differ from the usual market economy because, given the nature of the prize, contestants either are not allowed to or simply cannot secure the acquisition of the prize by engaging into a regular market trade.

The three essays contain a theoretical investigation of contests on topics such as private information, mechanism design and endogenous leadership. All three essays are single-authored and have not been published as of August 2016.

In the first chapter entitled "Group Rent-Seeking Contests with Private Information," I consider a contest in which the competing entities are groups of contestants who are each privately informed about the extent to which they value winning the contest, and participate to the contest non-cooperatively. I contribute to the literature on group contests with private information by suggesting a model in which groups of individuals can win even though the sum of their effort is not the highest of all groups and in which contestants exert either full effort or no effort. I provide a characterization of the equilibrium in such a scenario and prove its existence under fairly general conditions. I also provide intuitive results on the impact of different group structures on the behavior of contestants in equilibrium. I show that not only the size of one's group is important when deciding effort levels, but also the sizes of opposing groups. Free-riding incentives inside opposing groups are internalized by contestants and thus affect the behavior of contestants in equilibrium.

In the second chapter, entitled "Selecting Contestants for a Rent-Seeking Contest: a Mechanism Design Approach," I consider the problem of selecting the right candidates to a contest, despite having information concerning the value they have for winning the contest. In this chapter, I tackle this problem from the perspective of a contest designer who derives benefits from aggregate effort in the contest and suffers a marginal cost for each candidate invited to participate to the contest. Consequently, the contest designer must figure out a way to uncover private information held by the candidates in order to invite the subset of candidates who will, from exerting effort in the contest, maximize the the contest designer's profits. I solve the designer's problem by introducing a multi-stage mechanism, carefully designed so that candidates reveal truthfully their type, which then enables the designer to select an optimal subset of candidates to become contestants. I show existence of an incentive compatible and individually rational mechanism. I contribute to the literature by bridging contest theory and classical mechanism design theory.

In the third chapter, entitled "Endogenous Leadership in a Sports League with a Fixed Supply of Talent," I study the theory of contests using an application in the area of sports economics. In a sports league, it is commonly assumed that a team's revenue depends on its probability to win the championship, which in turn depends on the distribution of playing talent across teams. If team A has more playing talent than team B, then team A is more likely to win the League's championship and thus generates more revenues, given that the teams are ex-ante identical. I consider the problem of having to allocate a fixed supply of talent across two teams. To do so, I introduce a multi-stage game in which the first stage consists of each team submitting a bid on the cost each team is willing to pay per unit of talent. And in the second stage, teams select talent simultaneously if they submit the same first-stage bid, or select sequentially otherwise. This framework allows me to study the question of endogenous leadership. I study a scenario in which some team is able to raise more revenue than the other, for any winning probability. I call this team the rich team, or the efficient team. Moreover, my model features an exogenous revenue-sharing parameter, representing the amount of own revenue a team can keep, while the rest must be transfered to the other team. I find that contrary to usual results in the endogenous leadership literature, the rich team can arise as the follower in equilibrium. Also, I find that revenue-sharing contributes to increasing the acquisition of talent by the rich team relative to the poor team, no matter which team leads in equilibrium.

Chapter 1

Group Rent-Seeking Contests with Private Information

1.1 Introduction

In this paper, I study a type of contest in which groups of individuals compete against each other in order to obtain a group-specific public good. As opposed to *individual* contests, a *group contest* allows groups of individuals to aggregate their effort in the acquisition of a good that would benefit all members of the winning group. Many real-life situations can be modelled as a group contest. Two groups of lawyers may compete in legal proceedings; individuals with similar socio-economic status may, as a group, engage in trying to influence legislators or other public officials in favor of a specific cause; pharmaceutical firms may collaborate in order to win a patent race. In these examples, a contestant may have multiple partners. The partners' choices of effort affect the contestant's own probability of consuming the good.

An example of a contest to which my paper is closely related is when a number of different firms form joint ventures to compete for a project such as the construction of a bridge. It is common practice in the construction industry, for example, for competing firms to collaborate in designing a project and to submit a tender as one group. These firms may, in other circumstances, be competitors. Consequently, the amount of information that firms share when forming such a group may be limited or even null. Naturally, firms are usually uninformed about the extent to which their competitors value the contested good. However, it is safe to assume that in such a context, firms know who they are competing against and the way in which their competitors collaborate. That is, they know the *structure* of the opposing groups. In this example, firms must behave strategically not only against the other groups of firms who may end up producing the project instead of them, but also against their own collaborators because firms also have an incentive to free-ride within a group. It is well understood that the larger a group is, the larger are the incentives to free-ride within that group. There is, nonetheless, another source of free-riding and it is the lack of information regarding valuations. Holding constant the size of a group, if one firm sees its collaborators being more likely to derive high valuations from the project, then this firm will expect its collaborators to exert more effort on average, which in turn induces the firm to exert less effort.

The way in which such a project is awarded is not necessarily straightforward. It is often the case that the least expensive submission will be chosen. This is likely to be true for simple projects, or at least projects that must be undertaken in a very specific way. Submissions in this case are likely to be very similar and choosing among them becomes easy: choose the least expensive submission. In this context, a straightforward way of modeling this situation is using a *deterministic* contest, where the good is allocated with probability one to the group that exerts the highest aggregate effort, which is in some sense an all-pay auction. However, when it comes to complex projects that require great skills and innovation, for instance building a bridge or a skyscraper, it is most likely the case that the submissions will be different in many respects, making choosing among them more difficult. In this context, the modeling approach is the use of a *non-deterministic* contest where the highest bidder does not receive the good with probability one, but has the highest probability of receiving it. Most papers on contest theory have used the approach of non-deterministic contests, either with full information or with private information, with individualistic or with group contests. I also follow this approach.

My model is line with other papers such as Wasser (2013) and Einy *et al.* (2013) who analyse incomplete information in the case of individual contests and provide conditions for existence — and uniqueness in some families of Tullock contests — of an equilibrium. However, the functions that describe effort levels in equilibrium are not tractable. Barbieri & Malueg (2014) also develop a group contest model with private information. In contrast to my own work, Barbieri & Malueg (2014) consider a *Best-Shot* All-Pay auction, which is a deterministic contest where "each group's performance equals the best effort ("best shot") of its members, and the group with the best performance wins the contest."A group's effort supply has a closed-form solution in such a contest. A shortcoming of this approach is the inevitable necessity to use a deterministic contest, and furthermore the effort of a group is determined by only one group member as opposed to being determined by the aggregate effort of all group members. In this paper, I consider group contests in a non-deterministic framework and I assume that the effort supply of any group is the sum of efforts of all group members. The key restriction that provides for a tractable solution is to assume that, as in Dubey (2013), the effort level is a dichotomous variable.

I show that in equilibrium, agents' actions consist of choosing a threshold such that one would exert effort whenever the realization of the valuation is greater than or equal to the threshold and would not exert effort otherwise. Moreover, in equilibrium, all contestants select a positive threshold, which means that they all exert positive expected effort.

With complete information and when the cost of exerting effort is a linear function of effort, Baik (1993) showed that agents with low valuations for the public good have an incentive to let agents of their own group with high valuations exert effort so that low valuation individuals can reap the benefits without paying the cost of exerting effort. Such an extreme prediction in terms of participation is more plausibly the exception rather than the rule in many circumstances and applications. Topolyan (2014) recently noted that "there are many examples when all group members contribute to the collective cause [but the] existing theory is not fully capable of handling such situations. Whether there is an equilibrium where *all* players contribute is an interesting question." Topolyan (2014) suggests, as a solution to this paradox, a model with a continuum of equilibria where all players make contributions. It must be pointed out however that the results in Topolyan (2014) are related to deterministic contests. In this paper, the solution that I propose comes about as a novel answer to this paradox.

I also consider the impact of *partition structures* on contestants' behavior in equilibrium. In a group contest, the particular way in which contestants are grouped together is described as a partition of the contestants. I argue that not only the number of individuals within one's group matters when it comes to deciding the thresholds, but also the specific structure of the partition of competitors. For instance, one's incentive to free-ride may be diminished if contestants outside their group have greater incentive to free-ride in their own group.

Apart from Barbieri & Malueg (2014), group structures have not been taken into consideration in the literature on group contests. My results indicate that the presence of larger groups reduces the average level of effort. Thus, if a contest designer wishes to maximize the level of average effort, groups should be broken down to smaller groups whenever possible. For instance, the existence of a group of contestants should not be justified by the fact that group members find it beneficial to have the possibility to free-ride.

Lastly, I compare the implications of my model to the so-called *Olson's Paradox* (Olson (1965)), which states that "the free rider problem inside large groups is so acute that, in equilibrium, large groups exert less aggregate effort than small groups, which explains the success of the latter."[1] I show that even though free-riding exists within groups, larger groups may still exert a larger aggregate expected effort in equilibrium, which leads to a greater probability of winning the contest. However, Olson's paradox may still hold if distribution functions are concave and sufficiently skewed.

1.2 Further Review of Literature

This paper belongs to the literature on rent-seeking contests (which began with Tullock (1980)). For a complete review on the theory of contests in economics, the reader is referred to Nitzan (1994), Corchón (2007), Konrad (2009) or Long (2013). More specifically, I build upon a family of papers in which authors investigate contests with competing individuals divided into groups of contestants. The literature on group contests focuses on the existence and the characterization of the free-riding problem in games with complete information. Papers like Katz *et al.* (1990), Ursprung (1990), Baik (1993), Riaz *et al.* (1995) and Baik & Shogren (1998), although they consider various contest success functions, which are functions that map effort to a probability of winning, all establish that free-riding is an important feature of any equilibrium. For instance, Baik (1993), Baik *et al.* (2001) and Baik (2008) show that only the individuals with the unique highest valuation within each group are contributing in equilibrium. Chowdhury *et al.* (2013) modify Baik's specification by assuming that the probability of a group winning the contest is determined by the maximal individual effort within the group. This leads to an equilibrium in which free-riding takes yet another extreme form:

[1] Quote taken from Corchón (2007).

at most one player in each group exerts positive effort. However, it is not necessarily the player with the highest valuation.

In deterministic contests, Barbieri *et al.* (2013) and Topolyan (2014) show that equilibria exist where all the players contribute positively to the collective cause. It has also been shown that allowing for complementarity in the players' efforts (Kolmar & Rommeswinkel (2013)), allowing for non-linear cost of effort (Epstein & Mealem (2009)) or using a success function that depends on the minimal effort level within each group (Lee (2012)) also alleviate the severity of the free-riding problem.

One aspect of contests about which we know much less concerns what happens when the players have private information. There is a growing interest in *individual* contests with incomplete information. The literature includes Hurley & Shogren (1998a), Malueg & Yates (2004) and Sui (2009), who examine models in which the players valuations are private and distributed according to a simple discrete distribution. There are also a number of papers where only one player is affected by the information asymmetry (Harstad (1995), Hurley & Shogren (1998a), Hurley & Shogren (1998b), Schoonbeek & Winkel (2006), Pogrebna (2008), Wärneryd (2003)). Among the most recent papers, Fey (2008) analyses a contest with two players in which the cost of exerting effort is privately known, and proves the existence of an equilibrium for both discrete and continuous distributions. Wasser (2013) and Einy *et al.* (2013) generalize the analysis to more than two players, and analyse existence and uniqueness of a Bayes-Nash equilibrium in different informational settings. Finally, Ryvkin (2010) addresses the issue of player heterogeneity and how it impacts the aggregate effort level in individual contests. A common result is that the equilibrium effort levels tend to be lower when information is private than when it is public. However, Ryvkin (2010) shows that the difference in equilibrium effort levels does not generalize to contests with more than two players. The case of group contests with private information has been studied by Barbieri & Malueg (2014) and Brookins & Ryvkin (2014). The latter points out the non-tractability of the equilibrium but uses numerical techniques to depict equilibrium strategies.

There is a vast literature concerned with Olson's Paradox. The relevant stream of literature, for the purpose of the present paper, is the one related to the ways in which the paradox can be reversed. Chamberlin (1974) and McGuire (1974) suggests, without a formal demonstration, that Olson's paradox holds when the collective good, which is the good that any group aims at providing, has a sufficiently private component. If the collective group is purely public, they suggested that Olson's paradox would be reversed. More recently, counter arguments to the paradox have been proposed. Katz *et al.* (1990) and Nti (1998) argue that instead success in contests can be predicted by large valuations, small costs or contest success functions that favor certain agents. Esteban & Ray (2001) show that if the cost of exerting effort is sufficiently convex, the paradox can be reversed even if the collective good is purely private. Pecorino & Temimi (2008) extend the model of Esteban & Ray (2001) to a game of pure public good provision and show that with a fixed participation cost, large groups may fail to provide the public good. In this paper, I consider only pure (local) public goods. Nonetheless, the integration of private information and the consideration of different partition structures form a novel approach to the paradox.

1.3 The model

Let $N = \{1, ..., n\}$ be a set of individuals. Let P denote some partition[2] of N. The groups of individuals in N are grouped according to P and are participating in a rent-seeking contest where the prize is a local public good. Only one group can win the contest. Only the members of the winning group

[2]A partition P of N is defined in the following way: $\forall I, I' \in P$, $I \cap I' = \emptyset$ and $\bigcup_{I \in P} I = N$.

can consume the prize, and the others are excluded.[3] The valuation that individual $i \in N$ has for the local public good is denoted $\theta_i \in \Theta = [0, \bar{\theta}]$, where $0 < \bar{\theta} < \infty$. The list of valuations is denoted by $\theta \in \Theta^n$. $\forall i \in N$, θ_i is a random variable that is independently distributed over Θ and follows a probability distribution $f_i(\theta_i) > 0$ with cumulative distribution function F_i. The realization of θ_i is known to i and only to i. Upon the realization of θ, each contestant decides whether to exert effort ($e_i = 1$) or not ($e_i = 0$). Assuming that the effort space is dichotomous is a simplification that makes it possible to compute equilibrium effort levels in a tractable way. The cost of exerting effort is $c > 0$ and is the same for all individuals. Let $P(i)$ denote the element in P that contains player i and let $|P(i)| = p_i$. For any given partition P and any list of effort levels $e \in \{0,1\}^n$, I assume that the probability[4] that i consumes the local public good is

$$\pi_i(e, e_{-i}) = \begin{cases} \frac{\sum_{j \in P(i)} e_j}{\sum_{j \in N} e_j} & \text{if } \sum_{j \in P(i)} e_j > 0, \\ 0 & \text{if } \sum_{j \in P(i)} e_j = 0. \end{cases}$$

I follow the convention that e_{-i} denotes the list of effort levels for all individuals other than i.

The group contest can be represented as a *Bayesian game*. This game consists of a finite set N of players and a partition P of N. For all player $i \in N$, the set of possible actions is $\{0, 1\}$. Individual i is differentiated by his type θ_i. Individual i's information is the realization of his type and the

[3] Such a contest could be generalized to incorporate spillovers induced by the consumption of a local public by some $I \in P$. However, in this paper, spillovers are ruled out. Bloch & Zenginobuz (2007) consider such a scenario, although it is in different context.

[4] The functional form of π_i is slightly different from the usual form, which is known in the literature as the *Tullock contest success function*. It is a common assumption that if $\sum_{j \in N} e_j = 0$ then all contestants consume the good with equal probabilities. However, with this assumption, it is possible that a group wins the good without having a single individual in the group exerting effort. This seems rather odd when, after all, a contest should take place among individuals who signal their interest towards the good, which comes at the cost of at least signaling their interest to participate.

distribution functions of all other players. Everything else (N, P and c) is common knowledge.

In this game, a strategy for any player i is a function

$$\sigma_i : \Theta \longrightarrow \{0, 1\}.$$

Denote $\sigma_{-i} = (\sigma_j)_{j \in N \setminus i}$ and let E be the expectation operator over θ_{-i}. Note that for any given list of strategies σ_{-i} and any $e_i \in \{0, 1\}$, we have that

$$0 \leq E\left[\pi_i(e_i, \sigma_{-i})\right] \leq 1, \ \forall i \in N,$$

and in particular, $\bar{\theta} E\left[\pi_i(1, \sigma_{-i})\right] - c \leq 0$ if $c \geq \bar{\theta}$. If $c > \bar{\theta}$, i does not have an incentive to exert positive effort, regardless the realization of θ_i. From now on, I assume that $c \in (0, \bar{\theta}]$.

Given σ_{-i}, i's objective is to choose a function σ_i that maximizes i's expected utility for any realization of θ_i. Thus, an equilibrium of the game is a list σ^* such that $\forall i \in N$ and $\forall \theta_i \in \Theta$,

$$\sigma_i^*(\theta_i) \in \arg \max_{e_i \in \{0,1\}} \left\{ \theta_i E\left[\pi_i(e_i, \sigma_{-i}^*)\right] - c \cdot e_i \right\}.$$

1.3.1 Optimal strategies

I will show that the search for equilibrium strategies can be simplified by looking only at *cutoff* strategies.

Definition 1. *σ_i is a* cutoff strategy *for player i if there exists a cutoff $x_i \in \mathbb{R}$ such that*

$$\sigma_i(\theta_i) = \begin{cases} 0 & \textit{if } \theta_i < x_i \\ 1 & \textit{if } \theta_i \geq x_i. \end{cases}$$

Proposition 1. *For any σ_{-i}, a cutoff strategy is the best-response for i.*

Proof. Fix a list of strategies σ_{-i}. We have that $\forall \theta \in \Theta^n$,

$$E\left[\pi_i(1, \sigma_{-i})\right] \geq E\left[\pi_i(0, \sigma_{-i})\right], \ \forall i \in N, \tag{1.1}$$

which is true since for any σ_{-i}, i cannot decrease $P(i)$'s probability of winning by exerting effort, but it may be the case that σ_{-i} is such that $P(i)$ wins whether or not i exerts effort.

Case 1: $E\left[\pi_i(1, \sigma_{-i})\right] = E\left[\pi_i(0, \sigma_{-i})\right]$.

Then it must be that $E\left[\pi_i(0, \sigma_{-i})\right] = 1$ because the probability that $P(i)$ wins the contest strictly increases if i changes his level of effort from $e_i = 0$ to $e_i = 1$ as long as $E\left[\pi_i(0, \sigma_{-i})\right] < 1$. It is then clear that i should not exert effort for *any* realization of θ_i, which is a cutoff strategy with a cutoff of $\bar{\theta}$.

Case 2: $E\left[\pi_i(1, \sigma_{-i})\right] > E\left[\pi_i(0, \sigma_{-i})\right]$.

Then there exists $\hat{x}_i \in \mathbb{R}_+$ such that

$$\hat{x}_i E\left[\pi_i(1, \sigma_{-i})\right] - c = \hat{x}_i E\left[\pi_i(0, \sigma_{-i})\right].$$

Re-write $\hat{x}_i$ as follows:

$$\hat{x}_i = \frac{c}{E\left[\pi_i(1, \sigma_{-i})\right] - E\left[\pi_i(0, \sigma_{-i})\right].} \tag{1.2}$$

$\forall \theta_i < \hat{x}_i$, we have that

$$\theta_i E\left[\pi_i(1, \sigma_{-i})\right] - c < \theta_i E\left[\pi_i(0, \sigma_{-i})\right]$$

and it is optimal for i *not* to exert effort. Otherwise, $\forall \theta_i \geq \hat{x}_i$,

$$\theta_i E\left[\pi_i(1, \sigma_{-i})\right] - c \geq \theta_i E\left[\pi_i(0, \sigma_{-i})\right]$$

and it is optimal for i to exert effort. This is a cutoff strategy with a cutoff of $\hat{x}_i$. $\square$

Given Proposition 1, it suffices to consider only cutoff strategies. A profile of cutoff strategies implies a specific form of $E\left[\pi_i(e_i, \sigma_{-i})\right]$. Let $L, M \subseteq N$ with $|L| = l$ and $|M| = m$, and fix the cutoff strategies of all individuals different from i to x_{-i}. For $j \neq i$, since θ_j follows a cumulative distribution function F_j over Θ, i expects j to exert effort with probability $(1 - F_j(x_j))$ and to not exert effort with probability $F_j(x_j)$. Therefore, the probability that l individuals in $N \setminus P(i)$ and m individuals in $P(i) \setminus i$ exert positive effort is equal to

$$\begin{aligned}\Pi_i(l, m; x_{-i}) := \quad & \sum_{L \subset N \setminus P(i)} \left(\prod_{k \in L} (1 - F_k(x_k)) \prod_{k \in N \setminus (P(i) \cup L)} F_k(x_k) \right) \\ & \cdot \sum_{M \subset P(i) \setminus \{i\}} \left(\prod_{k \in M} (1 - F_k(x_k)) \prod_{k \in P(i) \setminus (M \cup \{i\})} F_k(x_k) \right).\end{aligned}$$

Thus,

$$E\left[\pi_i(e_i, x_{-i})\right] = \sum_{l=0}^{n-p_i} \sum_{m=0}^{p_i-1} \Pi_i(l, m; x_{-i}) \epsilon_i(e_i, l, m) \tag{1.3}$$

where

$$\epsilon_i(e_i, l, m) = \begin{cases} \frac{e_i+m}{e_i+m+l} & \text{if } e_i + m > 0 \\ 0 & \text{if } e_i + m = 0 \end{cases}$$

Individual i maximizes his expected utility by choosing x_i. This threshold choice directly translates into i's choice of expected effort which corresponds to $1 - F_i(x_i)$.

Proposition 2. *If σ is a list of cutoff strategies such that $\exists i \in N$ for which $E\left[\pi_i(1, \sigma_{-i})\right] = E\left[\pi_i(0, \sigma_{-i})\right]$, then σ cannot be an equilibrium.*

Proof. We already know that in this case, $E\left[\pi_i(0, \sigma_{-i})\right] = 1$. Then it must be the case that $|P(i)| > 1$ and among the members of $P(i)$ different from i, there must be some j whom $\forall \theta_j \in \Theta$, $\sigma_j(\theta_j) = 1$. If not, then there exists some realizations of θ for which no member of $P(i)$ different from i would exert effort and this would imply in that case that $E\left[\pi_i(0, \sigma_{-i})\right] < 1$. Now since $c > 0$, σ_j results in a negative expected utility for any realization of θ_j below c. Then there exists a profitable deviation for j. For example, the strategy σ'_j such that $\sigma'_j = \sigma_j$ for all $\theta_j \geq c$ and $\sigma'_j = 0$ otherwise makes j strictly better off. Therefore, σ cannot be an equilibrium. □

We are then left with a unique possibility: if σ is an equilibrium, then for any $i \in N$, it must be the case that $E\left[\pi_i(1, \sigma_{-i})\right] > E\left[\pi_i(0, \sigma_{-i})\right]$. In this case, Proposition 1 implies that $\forall i \in N$, i's best response function is given by equation (1.2). Using (1.3) and (1.2), we have that a list of equilibrium thresholds x^* solves the following system of equations:

$$x_i^* \sum_{l=0}^{n-p_i} \sum_{m=0}^{p_i-1} \Pi_i(l, m; x_{-i}^*) \left(\epsilon_i(1, l, m) - \epsilon_i(0, l, m)\right) = c, \ \forall i \in N \tag{1.4}$$

where

$$\epsilon_i(1, l, m) - \epsilon_i(0, l, m) = \begin{cases} \frac{l}{(1+m+l)(m+l)} & \text{if } m > 0, \\ \frac{1}{1+l} & \text{if } m = 0. \end{cases}$$

Using (1.4), we can derive the reaction of i with respect to a change in any other $j \in N$. Fix all the cutoffs other than i and j. The simplest case is a change in x_j when $j \in P(i)$. If x_j increases (decreases), then j exerts effort with a smaller (greater) probability. Consequently, the probability of m being small is increased (reduced). This change in x_j has the effect of shifting the probability weights towards low (high) values of m and consequently towards high (low) values of $\frac{l}{(1+m+l)(m+l)}$. Then, i's best response is to decrease (increase) x_i. We then have that within a group, contestants' cutoffs are strategic substitutes. If $j \notin P(i)$, it is unclear whether i and j are complements or substitutes. It is clear, that if $P(i)$ contains only i, it means that m can only take the value of 0, and so an increase (decrease) in x_j shifts the probability weights towards high (low) values of $\frac{l}{(1+l)}$ and it has inevitably a negative (positive) impact on x_i. A contestant that has no teammate always reacts negatively to the variations of opponents' effort. When $|P(i)| > 1$, a change in x_j may have either a negative impact or a positive impact on x_i. To see this, note that $\frac{l}{(1+m+l)(m+l)}$ does not necessarily increase with l. We have that

$$\frac{l}{(1+m+l)(m+l)} - \frac{l+1}{(2+m+l)(m+l+1)} = \frac{l-m}{(1+m+l)(m+l)(2+m+l)}$$

So if $l - m > 0$, then $\frac{l}{(1+m+l)(m+l)}$ decreases with l. However, whether x_i decreases or increases with x_j depends on whether $l - m$ is more or less *likely* to be positive, which is determined by the list of cutoffs. x_i and x_j are negatively correlated if the list of cutoffs is such that the probability weights on the positive values of $l - m$ is high enough so that the expected value of $\frac{l}{(1+m+l)(m+l)}$ is negatively correlated with l. This is expected to be the case when m can take sufficiently small values compared to l. If this is the case, then when j slacks off, i takes advantage of this by exerting even more effort. And when x_j decreases, x_i increases because of a discouragement effect.

However, if $|P(i)|$ is large, then $l - m$ is more likely to be negative. Consequently, for $|P(i)|$ sufficiently large, an increase in x_j can lead to an increased likelihood of having negative values of $l - m$, in turn shifting the probability weights towards low values of $\frac{l}{(1+m+l)(m+l)}$. In this case, x_i will react positively to a change in x_j. This can be explained by the fact that, on the one hand, the size of $P(i)$ is large enough so that i feels confident to slack off effort when j slacks off. On the other hand, the size of $P(i)$ is large enough so that i reacts competitively to the changes in x_j. This is analogous to how a relatively big firm would not let smaller firms increase their market power without retaliation.

1.3.2 Existence of an equilibrium

If an equilibrium x^* exists, then this equilibrium is characterized by a list x^* of cutoffs that solves (1.4), which can be rewritten as

$$x_i^* = \frac{c}{E\left[\pi_i(1, x_{-i}^*)\right] - E\left[\pi_i(0, x_{-i}^*)\right]}, \ \forall i \in N.$$

Proposition 3. *If F_i is continuous $\forall i \in N$, there exists a Nash equilibrium of the game.*

Proof. Define the function $\Phi : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+^n$ by

$$\Phi(x) = \Big(\Phi_i(x)\Big)_{i \in N}$$

where

$$\Phi_i(x) = \begin{cases} \frac{c}{E[\pi_i(1,x_{-i})] - E[\pi_i(0,x_{-i})]} & \text{if } E\left[\pi_i(1, x_{-i})\right] - E\left[\pi_i(0, x_{-i})\right] \geq c/\bar{\theta}, \\ \bar{\theta} & \text{otherwise.} \end{cases}$$

If this function possesses a fixed point, then an equilibrium exists. Consider the restricted game in which every player i chooses a threshold in $X = [0, \bar{\theta}]$. Any equilibrium of this restricted game is also an equilibrium of the non-restricted game since choosing a threshold larger than $\bar{\theta}$ is equivalent to choosing a threshold of $\bar{\theta}$ since in both cases no effort is exerted. We have that X^n is compact. This function is also continuous on X^n. To show this, note that $E\left[\pi_i(1, x_{-i})\right] - E\left[\pi_i(0, x_{-i})\right]$ is obviously continuous on X^n. The only way in which Φ_i may fail to be continuous is if $E\left[\pi_i(1, x_{-i})\right] - E\left[\pi_i(0, x_{-i})\right] = 0$. But by construction, Φ_i is identically equal to $\bar{\theta}$ for any x such that $E\left[\pi_i(1, x_{-i})\right] - E\left[\pi_i(0, x_{-i})\right] < c/\bar{\theta}$. Take any x such that $E\left[\pi_i(1, x_{-i})\right] - E\left[\pi_i(0, x_{-i})\right] = 0$. There always exists a neighborhood around x such that Φ_i is identically equal to $\bar{\theta}$. Thus, for any sequence $(x_n)_{n=0}^{\infty}$ that converges to x, it must be the case that the sequence $(\Phi_i(x_n))_{n=0}^{\infty}$ converges to $\Phi_i(x)$. Since $\forall i \in N$, Φ_i is continuous on X^n, then so is Φ. From Brouwer's fixed point theorem, Φ has a fixed point and thus an equilibrium exists. Note that this result holds for any list of continuous c.d.f functions F. $\square$

On the symmetry and multiplicity of equilibria

Proposition 3 states that an equilibrium exists but is silent about the specific structure of the equilibrium. I will focus on a symmetric equilibrium in which players that face the same strategic situation play the same strategies. Two contestants, i and j, are faced with the same strategic situation if $F_j = F_i$ and if one of the two following situations arises:

1. $j \in P(i)$;
2. $j \notin P(i)$, $|P(i)| = |P(j)|$ and $F_{P(j)\setminus j} = F_{P(i)\setminus i}$

where F_A denotes the list of c.d.f.'s for the members of $A \subseteq N$. I will use Example 1 to show that there may exist equilibria that are not symmetric even though condition 2 is satisfied.

Example 1. *Consider the partition $\{1\}, \{2\}$ with both c.d.f.'s being equal to F, and assume, without loss of generality, that $\bar{\theta} = 1$. The two relevant equations are given by equation (1.4):*

$$\begin{aligned} x_1 \left(F(x_2) + (1 - F(x_2))\frac{1}{2} \right) &= c, \\ x_2 \left(F(x_1) + (1 - F(x_1))\frac{1}{2} \right) &= c. \end{aligned}$$

- *If $F(\theta_i) = \theta_i$.*

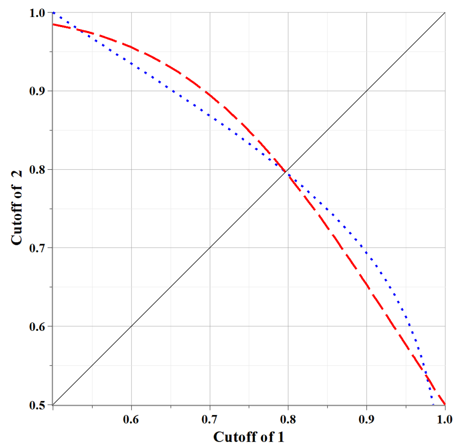
This system implies that $x_1 = x_2$ and the unique solution is given by

$$x_1 = x_2 = \frac{1}{2}(\sqrt{1+8c} - 1)$$

- *If $F(\theta_i) = \theta_i^6$.*

The system has three solutions. The solutions are plotted in Figure 1.

Figure 1.1: Best response functions for $c = 0.5$



Multiplicity of equilibria cannot be ruled out. However, it is easy to see that a symmetric equilibrium must always exist. If i and j face the same strategic situation then the i^{th} equation in (1.4) is identical to the j^{th} if x_i is interchanged with x_j. Thus it must be the case that, at least, $x_i = x_j$ is a solution. Since Proposition 3 insures the existence of at least one equilibrium, a symmetric equilibrium always exists whenever a symmetric situation arises.

A sharper result can be derived for identical players who belong to the same group. If $j \in P(i)$ and F_i and F_j are uniform distributions, then it must be the case that i and j behave identically in equilibrium.

Proposition 4. *Let $i, j \in N$ and $i \neq j$. And let $F_i = F_j$ be the uniform c.d.f. If $j \in P(i)$ then, in equilibrium, $x_i = x_j$.*

Proof. Define $M' \subset N$ with $|M'| = m - 1$ with $|M| = m$. Without loss of generality, assume that $\bar{\theta} = 1$. We have that $\Pi_i(l, m; x_{-i})$ can be re-written as follows

$$\begin{aligned}
\Pi_i(l,m;x_{-i}) := \quad & F_j(x_j) \sum_{L\subset N\setminus P(i)} \left(\prod_{k\in L}(1-F_k(x_k)) \prod_{k\in N\setminus P(i)\cup L} F_k(x_k)\right) \\
\cdot \quad & \sum_{M\subset P(i)\setminus\{i,j\}} \left(\prod_{k\in M}(1-F_k(x_k)) \prod_{k\in P(i)\setminus M\cup\{i,j\}} F_k(x_k)\right) \\
+ \quad & (1-F_j(x_j)) \sum_{L\subset N\setminus P(i)} \left(\prod_{k\in L}(1-F_k(x_k)) \prod_{k\in N\setminus P(i)\cup L} F_k(x_k)\right) \\
\cdot \quad & \sum_{M'\subset P(i)\setminus\{i,j\}} \left(\prod_{k\in M'}(1-F_k(x_k)) \prod_{k\in P(i)\setminus M'\cup\{i,j\}} F_k(x_k)\right) \\
= \quad & F_j(x_j)S(i) + (1-F_j(x_j))S'(i)
\end{aligned}$$

With this formulation, $F_j(x_j) = x_j$ and $F_i(x_i) = x_i$, the system of equations that represents the equilibrium is

$$\begin{aligned}
& x_i\Bigg(x_j \sum_{l=0}^{n-p_i}\sum_{m=0}^{p_i-1} S(i)\left(\epsilon_i(1,l,m)-\epsilon_i(0,l,m)\right) \\
+ \quad & (1-x_j)\sum_{l=0}^{n-p_i}\sum_{m=0}^{p_i-1} S'(i)\left(\epsilon_i(1,l,m)-\epsilon_i(0,l,m)\right)\Bigg) = c
\end{aligned}$$

and

$$\begin{aligned}
& x_j\Bigg(x_i \sum_{l=0}^{n-p_i}\sum_{m=0}^{p_i-1} S(j)\left(\epsilon_i(1,l,m)-\epsilon_i(0,l,m)\right) \\
+ \quad & (1-x_i)\sum_{l=0}^{n-p_i}\sum_{m=0}^{p_i-1} S'(j)\left(\epsilon_i(1,l,m)-\epsilon_i(0,l,m)\right)\Bigg) = c
\end{aligned}$$

but since the terms x_i and x_j do not appear in $S(i), S(j), S'(i), S'(j)$, we have that $S(i) = S(j) = S'(i) = S'(j)$. These two equations reduce to

$$x_i(1-x_j)\sum_{l=0}^{n-p_i}\sum_{m=0}^{p_i-1} S'(i)\left(\epsilon_i(1,l,m) - \epsilon_i(0,l,m)\right) = c$$

and

$$x_j(1-x_i)\sum_{l=0}^{n-p_i}\sum_{m=0}^{p_i-1} S'(j)\left(\epsilon_i(1,l,m) - \epsilon_i(0,l,m)\right) = c,$$

and thus

$$x_i(1-x_j) = x_j(1-x_i).$$

Therefore, it must be the case that $x_i = x_j$. □

Example 2. *Consider the partition $\{1,2\},\{3\}$ with all c.d.f.'s being the uniform distribution. From (1.4), we have three equations to solve simultaneously:*

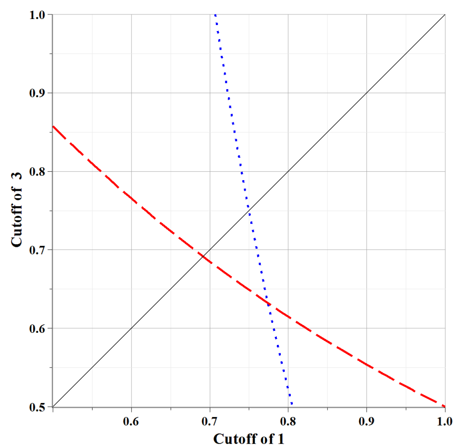
$$\begin{aligned}
x_1\left(x_2x_3 + x_2(1-x_3)\frac{1}{2} + (1-x_2)(1-x_3)\frac{1}{6}\right) &= c\\
x_2\left(x_1x_3 + x_1(1-x_3)\frac{1}{2} + (1-x_1)(1-x_3)\frac{1}{6}\right) &= c\\
x_3\left(x_1x_2 + (x_1(1-x_2) + (1-x_1)x_2)\frac{1}{2} + (1-x_1)(1-x_2)\frac{1}{3}\right) &= c
\end{aligned}$$

We can see that the two first equations imply that $x_1 = x_2$. This system is thus reduced to

$$\begin{aligned}
x_1\left(x_1x_3 + x_1(1-x_3)\frac{1}{2} + (1-x_1)(1-x_3)\frac{1}{6}\right) &= c\\
x_3\left(x_1^2 + x_1(1-x_1) + (1-x_1)^2\frac{1}{3}\right) &= c
\end{aligned}$$

The solution to this systeme is depicted in Figure 2.

Figure 1.2: Best response functions for $c = 0.5$



1.4 The symmetric equilibrium

For simplicity, I will assume that all agents share the same c.d.f. and I will only consider the symmetric equilibrium. This assumption allows me to simplify the problem since in a symmetric equilibrium, all agents who belong to a group of the same size, whether they belong to the same group or not, choose the same cutoff in equilibrium.

1.4.1 Olson's Paradox

Olson's paradox (Olson (1965)) suggests that groups of greater size may be less effective than groups of smaller size at providing a local public good. The incapacity to be as efficient as small groups comes from the fact that there are stronger free-riding incentives in large groups. I will show that the curvature of the c.d.f. affects the incentive to free-ride in large groups and can even reverse the so-called *common wisdom* (Esteban & Ray (2001)). If the good has no private consumption component, then the *common wisdom* is to agree that groups of larger size will be more efficient at providing the public good.

Contrary to this result, I provide a counter-example with convex distribution functions that actually goes in line with the Olson's paradox and is contrary to the common wisdom.

Let x^* be a list of equilibrium thresholds and σ^* as the list of equilibrium threshold strategies. For any $i \in N$, we can use (1.3) to express the expected winning probability of group $P(i)$ by computing the expected winning probability of i,

$$F_i(x_i^*)E\left[\pi_i(0, x_{-i}^*)\right] + (1 - F_i(x_i^*))E\left[\pi_i(1, x_{-i}^*)\right]. \tag{1.5}$$

Consider the partition $\{\{1,2\},\{3\}\}$ and for simplicity, let $\forall i \in \{1,2,3\}$, $F_i = F$. Denote by x_1 the equilibrium threshold of $\{1,2\}$ and by x_3 the equilibrium threshold of $\{3\}$. The expected winning probability of $\{1,2\}$ is

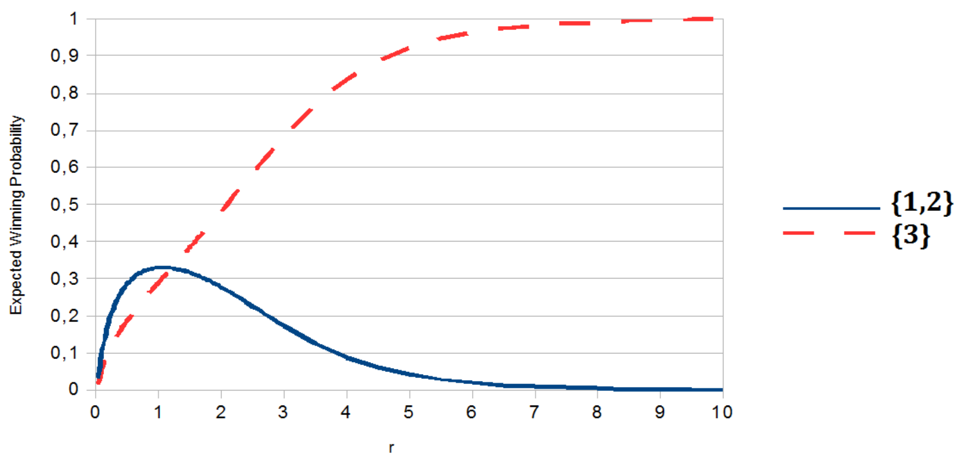
$$2(1-F(x_1))F(x_1)(1-F(x_3))\frac{1}{2}+(1-F(x_1))^2(1-F(x_3))\frac{2}{3}+(1-F(x_1)^2)F(x_3)$$

and the expected winning probability of $\{3\}$ is

$$(1 - F(x_3))\left(2(1 - F(x_1))F(x_1)\frac{1}{2} + (1 - F(x_1))^2\frac{1}{3} + F(x_1)^2\right).$$

If $F(\theta) = \theta^2$ and the cost of effort is 0.5, then at the symmetric equilibrium, the threshold chosen by group $\{1,2\}$ is 0.8833 and the one chosen by individual 3 is 0.6279. This gives $\{1,2\}$ and $\{3\}$ an expected probability of winning of 0.28 and 0.48, respectively. These probabilities do not sum to 1 because the complement is the probability that no group wins. If, instead, the distribution functions were $F(\theta) = \theta$, then the thresholds in equilibrium are, respectively, 0.7744 and 0.6318, and the winning probabilities are, respectively, 0.33 and 0.29.

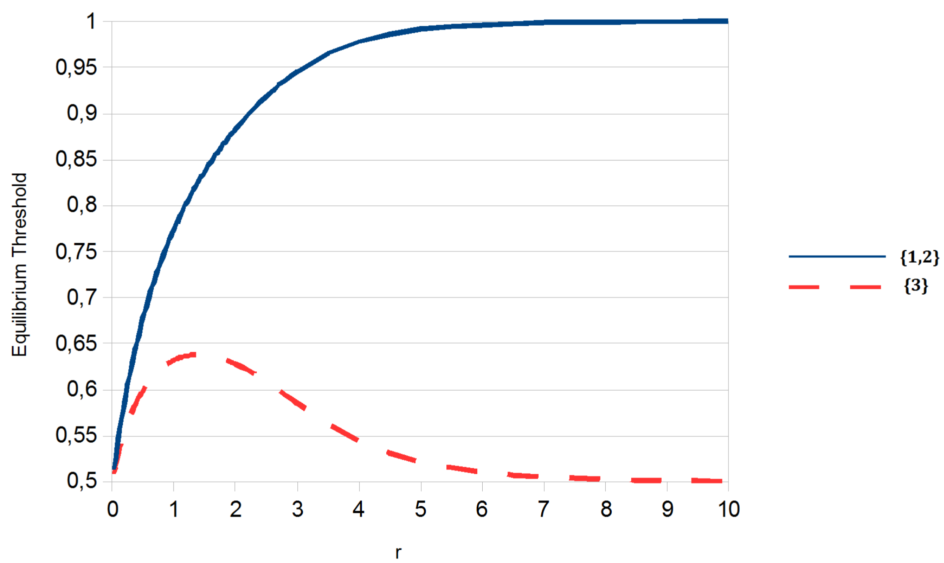
Figure 1.3: Winning probabilities for $P = \{\{1,2\},\{3\}\}$ with $c = 0.5$



In Figure 1.3, we can see the winning probabilities for the case of $F(\theta) = \theta^r$, as r goes from slightly above 0 to 10. For low values of r, Olson's paradox does not hold. The large group has a higher expected winning probability than the small group. For r large, Olson's paradox holds. We can see that as r becomes high, the winning probability of the large group converges to 0. The free-riding problem inside the larger group becomes acute for large values or r. When r is large, the probability of high realizations of the valuation is high. Thus members of the larger group expect the other member to draw a large valuation which incentivizes them to set a high threshold in equilibrium. The contestant who competes alone can internalize this acute free-riding problem and thus is willing to exert effort unless the realization of her valuation is below c, which happens with probability close to 0 when r is large.

Figure 1.4 shows the thresholds in the symmetric equilibrium. We can see that the threshold in the large group increases as r becomes larger. In the small group, the threshold increases until it reaches a maximum and then decreases towards c.

Figure 1.4: Thresholds in the symmetric equilibrium for $P = \{\{1,2\},\{3\}\}$ with $c = 0.5$



For the remainder of the paper, I will make the following simplifying assumption.

Assumption 1. *$\forall i \in N$, θ_i is independently and uniformly distributed over the unit interval $[0,1]$ and $c \in (0,1]$.*

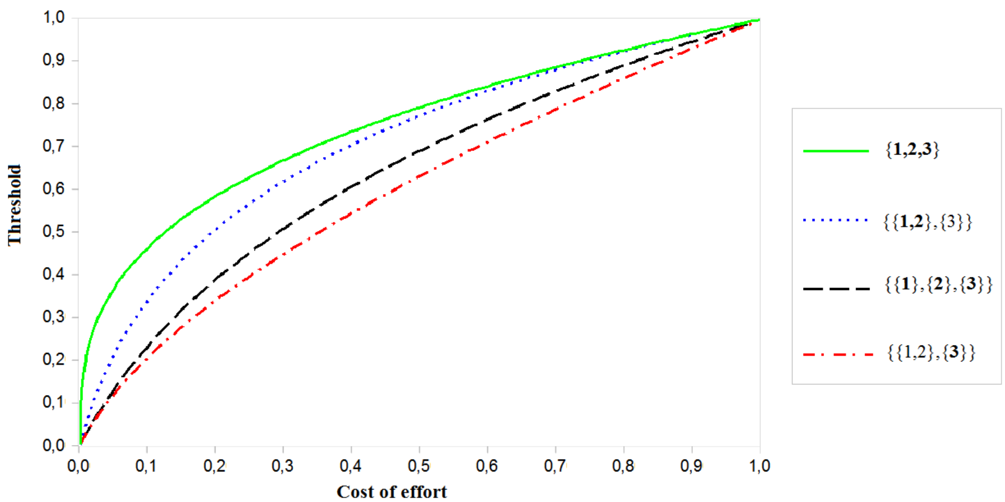
1.4.2 Expected effort as a function of c

In this section, I depict equilibrium thresholds for various values of $c \in (0,1]$. For the case of $N = \{1,2,3\}$, it is still possible to compute the thresholds in equilibrium as a function of the effort cost, c. However, for $|N| \geq 4$, the equilibrium thresholds do not have closed-form solutions. With the model that I have developed, I am unable to make a general statement concerning equilibrium thresholds for general N and P. However, depicting numerically the thresholds for small increments of c is somewhat easy. For $|N| \leq 5$, any symmetric equilibrium has at most two different thresholds as it is impossible to form more than two groups of different sizes. In what follows, I will discuss the case of $|N| = 3$ and $|N| = 4$. The similar patterns suggest that for $|N| \geq 5$, no qualitative differences should arise.

$N = \{1,2,3\}$

In Figure 5, we see that the incentive to free ride is the strongest when $P = \{1,2,3\}$. Individuals push their threshold up as mush as possible expecting that at least one individual among the two others will exert effort. The opposite occurs when $P = \{\{1\},\{2\},\{3\}\}$. In this case, individuals have no teammates to rely on, and consequently exert higher expected effort. If we compare $P = \{\{1\},\{2\},\{3\}\}$ with $P = \{\{1,2\},\{3\}\}$, we can see that individual 3 exerts more expected effort in the latter. Note that, 3 faces two opponents in both cases. However, from the threshold values in equilibrium, individual 3 is able to internalize the fact that when 1 and 2 form a group,

Figure 1.5: Equilibrium thresholds for $N = \{1, 2, 3\}$



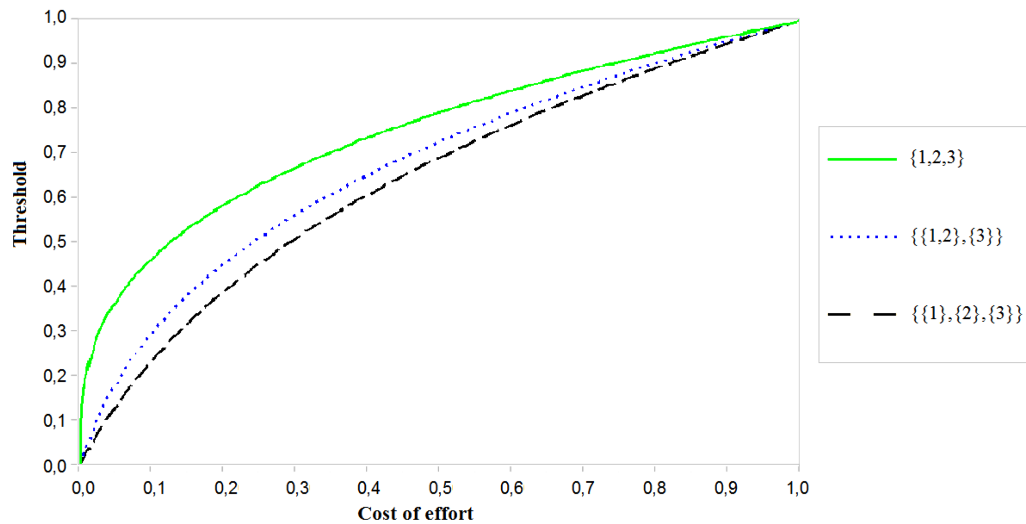
they have an incentive to free ride each other, leading to a reduction in their level of expected effort. This in turn gives 3 an incentive to take advantage of this by increasing his level of expected effort.

In Figure 6, we see that the average expected effort is the greatest when the partition is broken down to an individual contest. We can see this since the average threshold is always smaller in the individual contest.

$N = \{1, 2, 3, 4\}$

In Figure 7, I provide the result of simulations for small increments of the cost parameter c. The simulation shows that for most values of c, the highest equilibrium threshold is when the partition is $\{1, 2, 3, 4\}$. We can see a clear pattern: on the one hand, individual i's incentive to free-ride is greater as the size of $P(i)$ increases. On the other hand, individuals for whom the group size stays the same will exert higher expected effort in response to the increased free-riding incentives outside of their group.

Figure 1.6: Average equilibrium thresholds for $N = \{1, 2, 3\}$



In Figure 8, we also see that the highest average expected effort is when all contestants compete alone. Moreover, we can see that the presence of larger groups diminishes the average effort. This comes as no surprise as it has been shown by several authors that the presence of larger groups entails a larger incentive to free-ride.

1.4.3 Adding an extra player to the contest

In this section, I analyse the impact of adding an extra individual into the contest that originally has three contestants. From the perspective of 1, the extra individual, namely individual 4, enters the contest either as a teammate or as an opponent. If 4 enters as an opponent, the impact on 1's effort depends on whether 4 joins an existing group or enters the contest alone. In what follows, I compare a specific partition of $\{1, 2, 3\}$ with all of its possible extensions from the inclusion of individual 4.

Figure 1.7: Equilibrium thresholds for $N = \{1, 2, 3, 4\}$

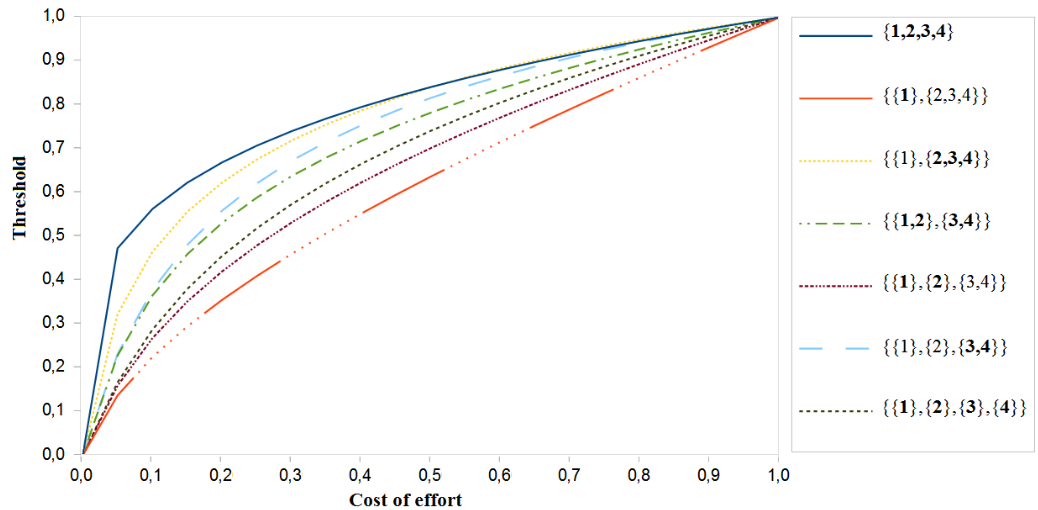
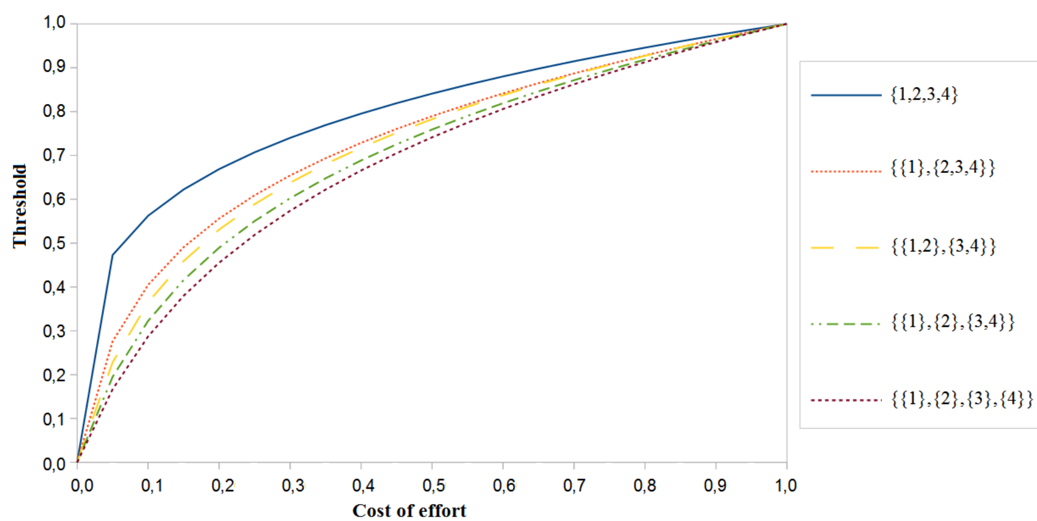


Figure 1.8: Average equilibrium thresholds for $N = \{1, 2, 3, 4\}$



There are mainly two elements to consider when choosing the right threshold: the marginal benefit of free riding and the marginal benefit of exerting effort. When an extra player is added to one's group, then the marginal benefit of free-riding and the marginal benefit of exerting effort both increase for that one player. Naturally, these have opposite effects on the expected effort. It may be the case that the incentive to free ride dominates the incentive to exert effort, in which case individuals would decrease their threshold, or vice versa. When a player is added as one's opponent, intuitively, that one player may have a lower marginal benefit of exerting effort as well as a lower marginal benefit of not exerting effort. This is so because it is expected that the winning probability of one group, in equilibrium, should decrease when there are more opponents.

$P = \{1, 2, 3\}$

Table 1.1: Equilibrium thresholds ($c = 0.5$)

P	x_1^*	x_2^*	x_3^*	x_4^*
$\{1, 2, 3\}$	0.7937	0.7937	0.7937	-
$\{1, 2, 3, 4\}$	0.8409	0.8409	0.8409	0.8409
$\{\{1, 2, 3\}, \{4\}\}$	0.8403	0.8403	0.8403	0.6470

When 4 joins the group of three, the incentive to free ride is greater than if 4 enters the contest alone. The difference in the threshold of 1 in $P' = \{\{1, 2, 3\}, \{4\}\}$ compared to $P' = \{1, 2, 3, 4\}$ is nearly infinitesimal. Whether 1 sees individual 4 as a teammate or as an opponent does not change his behavior. The size of $P(1)$ is large enough so that the equilibrium strategy of its members is almost invariant to the inclusion of a single opponent into the contest. However, the decrease in expected effort when the partition goes from $\{1, 2, 3\}$ to $\{\{1, 2, 3\}, \{4\}\}$ is puzzling. This goes against my intuition as one would imagine that with the inclusion of individual 4, 1 must exert more effort in order to secure a high probability of winning. In this case it seems that, from the perspective of 1, the increased probability that effort will be wasted makes 1 more prudent, hence the reduced expected effort in equilibrium.

$P = \{\{1, 2\}, \{3\}\}$

Table 1.2: Equilibrium thresholds ($c = 0.5$)

P	x_1^*	x_2^*	x_3^*	x_4^*
$\{\{1, 2\}, \{3\}\}$	0.7941	0.7941	0.6186	-
$\{\{1, 2, 4\}, \{3\}\}$	0.8403	0.8403	0.6370	0.8403
$\{\{1, 2\}, \{3, 4\}\}$	0.7826	0.7826	0.7826	0.7826
$\{\{1, 2\}, \{3\}, \{4\}\}$	0.8162	0.8162	0.7018	0.7018

When 4 joins $P(1)$, 1 and his teammates all increase their threshold since they now have an extra player on whom to free ride. Individual 3 can then increase slightly his threshold in response to this increased free riding incentive in the opposing group. In the case of 4 joining $P(3)$, 1 and 2 increase their expected effort by lowering their threshold. This is what one should intuitively expect as the opponent's group is now "stronger". Also, 3 now has a teammate on whom to free ride on, and so reduces his level of expected effort.

The last case is reminiscent of the puzzling case in Table 1. Individual 4 enters the contest as a single contestant. One could think that the level of competition that 1 faces is even higher because the new contestant does not have the possibility to free ride. Intuitively, it makes sense to think of the subpartition $\{\{3\}, \{4\}\}$ as being more competitive than $\{3, 4\}$. Then one should expect members of $P(1)$ to be more aggressive and decrease their threshold more than when 4 joins $P(3)$. However, it is actually the opposite that happens. This could be explained by the fact that when 4 enters the contest alone, 3 still has no marginal benefit of free riding and a decreased marginal benefit of exerting effort, which can only make him reduce his level of expected effort. Compared to when 4 joins $P(3)$, not only 3 sees his marginal benefit of exerting effort increase but 3 also sees his marginal benefit of free riding increase from null to a positive value. This may have a mixed effect on the behavior of individuals in $P(1)$. Clearly, in this case, the incentive to exert effort dominates. Now, when 4 enters the contest alone, the effect on the individuals in $P(1)$ is clearer: their opponents only have a reduced incentive to exert effort. This, in turn, reduces the marginal benefit of exerting effort since, ceteris paribus, it is more probable that they will win.

$P = \{\{1\}, \{2\}, \{3\}\}$

Table 1.3: Equilibrium thresholds ($c = 0.5$)

P	x_1^*	x_2^*	x_3^*	x_4^*
$\{\{1\}, \{2\}, \{3\}\}$	0.6914	0.6914	0.6914	-
$\{\{1\}, \{2\}, \{3, 4\}\}$	0.7018	0.7018	0.8162	0.8161
$\{\{1\}, \{2\}, \{3\}, \{4\}\}$	0.7413	0.7413	0.7413	0.7413

When the contest is originally an individual contest, we may expect that an extra player entering the contest alone would induce individuals to exert more effort due to the increased competition. However, all players see their marginal benefit of exerting effort decrease and their marginal benefit of free riding is still null. Thus they can only decrease their threshold when 4 enters the contest alone. When 4 joins $P(3)$, the incentive to free ride is evident for 3 and 4. Although 1 and 2 increase their threshold, this change in the behavior is small.

1.5 Discussion

So far, little has been said concerning the goal or the desires of the contest designer. If the contest designer is assumed to be benevolent, it may be irrelevant to discuss his interest in the contest. Moreover, if a contest designer has no value for the contested good, then the designer should not care whether or not the good is allocated, nor should he care about *who* gets the good. However, the assumption made on the functional form of the contest win probabilities, which says that if no effort is exerted then the good is not allocated, reveals that effort is implicitly valued by the designer. The reason is that not allocating the good is socially inefficient. Thus, by deciding not to allocate the good, it is implicitly assumed that the designer prefers to bear the cost of unrealized utility rather than to allocate a good to individuals who have not exerted effort. With this in mind, it is reasonable to assume that the designer may prefer contests in which individuals exert high expected effort.

It was shown that groups of greater size can expect to win the good with a higher probability in the case of a distribution function such as $F(x) = x^r$, with r small. Although in this paper I do not consider the process of coalition formation, we can still argue that individuals may prefer to belong to groups of greater size if it offers a greater expected probability of winning. Belonging to a large group may let team members diminish their expected effort since they can rely on a greater number of teammates to exert effort instead. From the perspective of the contest designer, this may not be desired. What a contest designer would want to avoid is a situation in which contestants team up for the "wrong" reasons. That is to form a group mainly because of the two properties stated above: high expected probability of winning and free-riding. Free-riding drives down the expected efforts, which is presumably bad for the contest designer. A *good* reason to form a group would be the *necessity* of forming a group. For instance, a group of architectural firms may belong to the same bidding group for a governmental contract. If they were to win the contest, the firms may produce the project jointly in a way that all firms are complementary to each other. There can be a firm specializing in drawing the plans for a building and another firm responsible of structural engineering and both of them are necessary to accomplish the project. Without collaboration, the two firms could not participate in the contest. However, in the case of two firms that are essentially identical, and are grouped together simply to lower the probability of exerting effort, it seems justifiable, from the designer's point of view, that these two firms be competitors instead of partners.

1.6 Concluding Remarks

In this paper, I have developed a model of rent-seeking group contests with private information. I have introduced a novel way to tackle the problem of non-tractability of the equilibria in group contests: using dichotomous effort levels. I show that an equilibrium exists under fairly general assumptions. This model is such that in equilibrium, all contestants exert positive expected effort. I have found that the average expected effort is maximized in individual contests. This result suggests that competing groups should be broken down into smaller groups whenever possible. Lastly, Olson's Paradox may or may not be invalidated. An interesting extension would be to allow individuals within a coalition to share information. One could then verify if free-riding incentives are less important in such a scenario.

In the previous chapter, the focus was put on the mechanics of a group contest in the presence of private information. In the next chapter, I consider instead the problem of selecting contestants. Additionally, the problem of private information also affect the contest manager, who is responsible of inviting the right candidates to the contest.

Chapter 2

Selecting Contestants for a Rent-Seeking Contest: a Mechanism Design Approach

2.1 Introduction

A contest can serve not only as a mean of allocating a good to some individual among many, but it can also be designed to induce productive effort from contestants. The promise of rewarding at least one individual with some valuable prize is what drives individuals to exert costly effort in the contest. Examples in the real-life range from sports competitions, research contests, job promotion and more. In these examples, both the contest designer and the contestants derive benefits from the contest. In professional sports like hockey, for instance, the league draws revenues from TV broadcasters and sponsorships while team owners get a share of the league's revenues, draw revenues from ticket sales and in addition, they get substantial extra revenues if their team makes it to the playoffs. The way in which professional teams sports are modeled as contests is by assuming that teams hire talent as a productive input.[1] Talent is a measure of the team strength and plays the very same role that effort plays in contests. The probability of winning the championship (the contest prize) depends positively on the team's stock of talent, and negatively on other teams' stock of talent.

However, the quality of the competition may influence greatly league revenues, hence the number of competing teams must be kept relatively small, not to dilute overall talent. Yet, every few years, discussions concerning the inclusion of new teams take place between the league managers and prospective team owners. For instance, in the National Hockey League (NHL), a group of investors based in Quebec City recently built a new amphitheater worth around \$370 CAD motivated by the desire to bring back the Quebec Nordiques franchise. Although, this construction is not sufficient to re-integrate the franchise back into NHL, it was a common belief that the construction of an up-to-date amphitheater was necessary. Not only is it necessary in the sense that NHL teams, nowadays, must operate in standard high-quality facilities, it is necessary because the investors must signal the lucrativeness of the market in which they wish to bring a new NHL team. They must show to what extent can the investors benefit from a NHL team and through this revelation, the NHL can evaluate how lucrative the project is for the league. In fact, the investors' desire is, in some sense, to be accepted to participate in this contest that is of running a NHL team. And the league evaluates whether it should let a new team, such as the Quebec Nordiques, participate in the contest.

In theory, the value to a contestant of participating in a contest depends mainly on three elements: 1) the extent to which the contestant values the contest reward, 2) the extent to which the *other* candidates value the reward and 3) the number of other contestants. The higher one's valuation is,

[1] For an extensive review of contests in the context of professional sports, the reader is referred to Szymanski (2003).

the more effort should be exerted in order to win the contest as the higher value of the reward can compensate for the cost of exerting effort. However, all contestants are rivals and one's increased effort decreases the other's probability of winning. And, obviously, if more contestants participate then everyone's effort gets diluted.

I consider a situation where individuals are candidates to a contest prior to being actual contestants. Candidates are potential contestants and may or may not be selected by the contest designer to become contestants. The work of the candidates is to convince the contest designer that they should be allowed to participate to the contest.

It has become a common assumption in the literature that the contest designer derives utility from the aggregate effort exerted during the contest. The contest designer is often viewed as someone whose objective is to design a contest in which the aggregate effort exerted is maximized. Aggregate effort is considered in this case as a productive input for the contest designer. I take on this assumption and use it in a situation where the designer is uninformed about the contestants' valuations for the prize. I assume that the prize offered is costly to the designer and that, in addition, each participant induces a fixed marginal cost. The task of the designer is simply to choose who among n individuals should participate to a contest in order to maximize the designer's profit. Put differently, the manager wants to avoid choosing contestants who would not exert enough effort or, would induce other participants to decrease their effort too much. However, the task of choosing contestants is not straightforward. The designer has no way to differentiate contestants but to ask them to reveal privately known information about their valuation, *i.e.* their types. An important issue that arises in this case is whether the information transmitted to the designer is truthful. The designer must incentivize players to reveal their type truthfully. If it is possible to do so, then the designer can choose the profit-maximizing subset of contestants.

In this paper, I solve the designer's problem by using a 3-stage game that goes as follows. At Stage 0, the designer offers a mechanism where players must reveal their type. A revelation m_i by contestant i is associated to a fixed cost $e(m_i)$. This fixed cost is put in place in the model so that candidates become accountable for their revelation. The mechanism also stipulates the probability that any subset of candidates is accepted to Stage 2, the contest stage. At Stage 1, players select a revelation to be sent to the designer, pay the fixed cost and then contestants are selected according to what had been stipulated at Stage 0. In the final stage, information about candidates's valuation is revealed. A contest with perfect information is conducted among the chosen candidates. The designer benefits from the aggregate effort exerted in Stage 2, pays a fixed cost per contestant and pays $P \geq 0$ for the contest prize.

This paper contributes to the literature by suggesting a novel way to consider optimal contests. There is a standard result in the literature that suggests that the optimal number of contestants should be two (Fullerton & McAfee (1999), for instance). This is not the case in this paper. The optimal number of contestants depends on the the cost of the prize to the designer, on the marginal cost that contestants induce and on the profile of candidates' types. I show the existence of an incentive compatible (IC) and individually rational (IR) mechanism. In an IC mechanism, the cost of revelation is increasing with respect to the revelation and may be kinked at the bottom, where some valuations may cost nothing to reveal but are associated with a zero-probability of participating to the contest.

2.2 Review of Literature

This paper fits under the umbrella of contest design. The structure or the architecture of a contest can take different forms. The most basic structure is a one-stage contest with a fixed number of contestants.[2] In my paper, I consider multiple stages where contestants are called to act in more than one stage. When contests are discriminatory enough, Gradstein & Konrad (2001) found that multistage contests, *i.e.* contests where the competitive efforts of agents take place over multiple rounds, generate more overall effort. Fu & Lu (2012) study multistage contests and focus on the maximization of overall effort when some contestants can be eliminated after each round. They find that it is optimal to give the entire prize to the winner of the final stage regardless of the number of stages and that each additional stage increases the total effort supplied by the contestants. With private information about contestants' ability, revealing information through playing in the first round may lead to inefficient outcomes in the final stage (Zhang & Wang (2009)) and has mitigated effect on total effort exertion (Lai & Matros (2007)). Moldovanu & Sela (2006) found that, depending whether the designer's objective is to maximize expected total effort or to maximize the expected highest effort, the optimal contest architecture is to, respectively, design one grand contest or split contestants into two divisions and then have the two divisions' winner compete in a final stage. Many authors allow the contest designer to restrict contestants' entry either through shortlisting (Baye *et al.* (1993), Amegashie (2000)), through entry fees (Taylor (1995), Fu & Lu (2010), Kaplan & Sela (2010), Fu *et al.* (2015)) or through entry auction (Fullerton & McAfee (1999), Giebe (2014)).

Che & Gale (2003) design a mechanism such that in the first stage, the designer selects a subset of players and offers each one of them a menu of prizes and then have them compete in the subsequent round. The designer completes the game by selecting a winner in the last stage. They find that having two contestants and imposing a ceiling on the highest possible bid of the best contestant is optimal for the contest designer. The optimality of

[2] For an extensive review concerning one-stage contests, the reader is referred to surveys such as Nitzan (1994), Corchón (2007) and Long (2013).

having only two contestants is also found in Taylor (1995) and in Fullerton & McAfee (1999). The way in which my model is different from Che & Gale (2003) is that the identity of the contestants selected to compete depend on the information that they transmit to the designer. In my model, the designer plays only in the first stage while in Che & Gale (2003), the designer plays at the first and at the last stage.

In this paper, I also attempt to bridge contest theory with mechanism design. To the best of my knowledge, Polishchuk & Tonis (2013) is the only study using the tools of mechanism design theory in a contest environment.

2.3 A comment on Fullerton & McAfee (1999)

In Fullerton & McAfee (1999), the authors are concerned with a similar problem, which is to select contestants despite having no information about the extent to which they value the contest prize.[3] However, they view the efficiency of a contest in a slightly different way than in the present paper. For them, "the entry mechanism is efficient [...] if it selects the lowest-cost contestants." The number of contestants identified as low-cost is m, which is exogenously given. There is no particular way in which m is chosen. They then construct an all pay auction in which contestants can internalize m, and find that the bidding strategy in a Bayes-Nash equilibrium is strictly monotonic if each contestant who gets entry to the contest also receives a strictly positive interim prize for becoming "finalist" in the contest.

[3]To be precise, in Fullerton & McAfee (1999), all contestants value the contest prize in the same way. However, the marginal cost of effort is different for all contestants. This marginal cost of effort is the private-information element of their model. Having private information over the value of the contest prize has a one to one relationship with having private information over the marginal cost since $\theta_i = 1/c_i$ where c_i would be the marginal cost of effort for i.

In contrast, I view an entry mechanism as being efficient if the contestants who enter the contest are a — perhaps *the* — subset of candidates who generate aggregate effort in a way that maximizes the contest designer's profits from running the contest. I do not assume that there are m high-value (low-cost) contestants, nor do I assume that an optimal contest size should be of m candidates. In fact, I even allow for the contest to be canceled, which is not a feature of Fullerton & McAfee (1999). I do not assume that contestants who enter the contest receive an interim prize.

In the model of Fullerton & McAfee (1999), it could be possible to determine the value of m that maximizes the designer's profits. However, it requires a recursive procedure where one would start at $m = 2$ and verify for all values of $m > 2$ which value of m maximizes the designer's profits. The mechanism that I propose effectively selects the optimal size without having to use such a recursive procedure. The mechanism only has to be played once to reach the desired outcome.

2.4 The model

Let $N = \{1, ..., n\}$ be a set of candidates. There is one contest designer who is responsible of selecting a subset of N to compete in a contest where one indivisible prize is offered to the winner. A prize, for player i, is worth $\theta_i \in \Theta := [\underline{\theta}, \bar{\theta}]$, where $0 \leq \underline{\theta} < \bar{\theta}$ and costs $P \geq 0$ to the designer. For all $i \in N$, θ_i is independently distributed according to a continuous cumulative distribution function F, defined over Θ. The realized value of θ_i is privately known to i.

Following the notation of Milgrom (2004),

Definition 2. *An environment is a triple* $(N, \Omega, \Theta^{|N|})$*, where*

- N *is a set of players.*

- Ω *is the set of possible outcomes over which the participants and the contest designer have preferences.*
- $\Theta^{|N|}$ *is a set of type profiles that include a type for each candidate. Here, a type is a valuation for the contest prize.*

An outcome $\omega \in \Omega$ specifies a probability distribution over the subsets of N of players who become contestants in the contest. It also specifies a fixed cost for every $i \in N$. The contest designer and the players have a utility function defined over $\Omega \times \Theta^{|N|}$ in a way that will become clear later on. The game goes as follows:

Stage 0: The contest designer selects a mechanism to be implemented in Stage 1. A mechanism is a list $((M_i)_{i \in N}, g)$ where, $\forall i \in N$, M_i is a set of possible messages that i can send to the designer and g, referred to as the *outcome function*, is defined as

$$g : M_1 \times ... \times M^{|N|} \longrightarrow \Delta(2^N) \times \mathbb{R}_{|N|}$$

associating a profile of messages m to a probability distribution $(\phi_S(m))_{S \subseteq N}$ over all subsets of N, and to a list of fixed costs $(e_i(m_i))_{i \in N}$, determining the cost to i for revealing to be of type m_i. $\phi_S(m)$ is the probability that subset S is brought to Stage 2 when the profile of messages is m.

This is a mechanism for which the designer selects a fixed cost for each candidate that must be paid in order for the candidate to be eligible for the contest. This fixed cost is not an entry fee as it is not sufficient to grant access to the contest. But in order to be considered for possible participation in the contest, the fixed cost must be paid. For example, building a new amphitheater is in some sense a fixed cost that must be paid in order to be eligible to acquire a NHL team.

The function e_i is similar in its impact on the player to the standard taxation principle in classic mechanism design. In the standard mechanism design literature, a signal m_i sent to the designer is associated with a monetary transfer — a tax — from the designer to the player. In this model, $e_i(m_i)$ is not a tax *per se* as it has no impact on the designer's revenue. However, it affects the player by imposing a cost in the same way as a (positive) tax would affect a consumer who derives utility from wealth.

Stage 1: Given the mechanism, each player selects a message $m_i \in M_i$ to be sent to the designer and pays $e_i(m_i)$. Then, according to $(\phi_S(m))_{S\subseteq N}$, a set S is realized and all players outside of S leave the game. The cost to the designer of bringing S to Stage 2 is $|S|c + P$ if $|S| > 0$ where $|S|$ is the size of S and $c > 0$. This cost is paid before the start of Stage 2. If S is empty, then Stage 2 is canceled and the designer spends nothing.

Stage 2: $(\theta_i)_{i\in S}$ becomes common knowledge. In this stage, a single contest among S takes place, which is denoted $C(S)$. One indivisible prize is attributed to the unique winner of the contest. Player i's effort is denoted x_i, and let $X_{-i} = \sum_{j\in S\setminus\{i\}} x_j$. The function that maps a profile of effort levels to the probability that contestant i wins the contest is called *the contest success function* (CSF), and is defined $\forall i \in S$ and $\forall S \subseteq N$,

$$\pi_{i,S}(x_i, X_{-i}) = \begin{cases} \frac{x_i}{x_i+X_{-i}} & \text{if } x_i + X_{-i} > 0, \\ \frac{1}{|S|} & \text{if } x_i + X_{-i} = 0. \end{cases} \tag{2.1}$$

Players are risk-neutral expected-utility maximizers and they derive negative utility from exerting effort. The marginal cost of effort is 1 and is the same for all players. There is no discounting between periods. If i leaves the game after Stage 1, her payoff is $-e_i(m_i)$. If i moves to Stage 2, her objective function, in Stage 2, is

$$\max_{x_i\geq 0} \{\theta_i\pi_{i,S}(x_i, X_{-i}) - x_i\} - e_i(m_i).$$

At the end of the game, the designer's payofft is

$$R\left(\sum_{i\in S} x_i\right) - |S|c - P$$

where $R : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is strictly concave, continuous and strictly increasing.

2.5 Solving the Game

The game is solved as follows. In Stage 2, a subset S of N has been selected to the contest. If S is empty, then Stage 2 is not played. Otherwise, a contest with perfect information is conducted. Payoffs are determined by the Nash equilibrium outcome of this stage. Going back to Stage 1, all candidates form expectations over their Stage-2 payoff conditional on the message they send to the designer. Candidates select the message that maximize their expected payoff and pay the associated fixed cost. The crucial step at this stage is to construct a cost function e, which is the same for all candidates, such that truthful revelation is incentive compatible and individually rational. Then in Stage 0, given that in Stage 1 candidates truthfully reveal their type, a simple probability distribution over the subsets of N is constructed.

2.5.1 Stage 2: the contest stage

Consider $i \in S$, and assume that S is the subset that has been selected for the contest stage.[4] Define

$$u_{i,S}(x_i, X_{-i}) := \theta_i \pi_{i,S}(x_i, X_{-i}) - x_i - e(m_i).$$

[4]In this section, some of the derivations can also be found in Fullerton & McAfee (1999).

Consider $C(S)$ and its unique equilibrium[5] x^*. Due to the concavity of $u_{i,S}$, $x_i^* > 0$ only if

$$\left.\frac{du_{i,S}}{dx_i}\right|_{x=x^*} = 0,$$

and $x_i = 0$ only if

$$\left.\frac{du_{i,S}}{dx_i}\right|_{x=x^*} \leq 0.$$

It is easy to verify that $x = 0$ is not an equilibrium.[6] Morover, in an equilibrium of $C(S)$, there must be at least two players who exert strictly positive effort.[7] Players who exert strictly positive effort in equilibrium will be referred to as *active* and others will be referred to as *passive*. The set of active players in S will be denoted by A. Without loss of generality, the following labeling will be assumed throughout the paper unless stated otherwise.

$$\theta_1 \geq \theta_2 \geq ... \geq \theta_n.$$

Consider $i \in A$. We have that i's first-order condition is

$$\theta_i \frac{X_{-i}^*}{(x_i^* + X_{-i}^*)^2} - 1 = 0. \tag{2.2}$$

Proposition 5. *If $j < i$, then $i \in A \;\Rightarrow\; j \in A$.*

Proof. Consider an equilibrium x^* of $C(S)$. Assume the contrary: $i \in A$ and $j \notin A$. Since j is passive, we have

$$0 \geq \theta_j \frac{X_{-j}^*}{(x_j^* + X_{-j}^*)^2} - 1 = \frac{\theta_j}{X_{-j}^*} - 1.$$

[5]This model of contest satisfies the assumptions detailed in Corchón (2007) for existence and uniqueness of a Nash equilibrium.

[6]Let $x_i + X_{-i} = 0$. i can profitably deviate by exerting strictly positive effort and win the prize with probability 1.

[7]If i is the only individual in S who exerts strictly positive effort, i can profitably deviate by diminishing x_i by $\epsilon > 0$ and still win the prize with probability 1.

Case 1: $\theta_j > X^*_{-j}$. This contradicts x^* being an equilibrium of $C(S)$.

Case 2: $\theta_j \leq X^*_{-j}$. Because $j \leq i$,

$$\theta_j \frac{X^*_{-j}}{(x^*_j + X^*_{-j})^2} - 1 \geq \theta_i \frac{X^*_{-j}}{(x^*_j + X^*_{-j})^2} - 1,$$

and because $i \in A$ and $j \notin A$,

$$\theta_i \frac{X^*_{-j}}{(x^*_j + X^*_{-j})^2} - 1 > \theta_i \frac{X^*_{-i}}{(x^*_j + X^*_{-j})^2} - 1.$$

Note that $x^*_j + X^*_{-j} = x^*_i + X^*_{-i}$ and $i \in A$ implies that

$$\theta_i \frac{X^*_{-i}}{(x^*_j + X^*_{-j})^2} - 1 = 0.$$

This also contradicts x^* being an equilibrium of $C(S)$. □

Define $x^*_i + X^*_{-i} = X^*$ and sum (2.2) over A. We have

$$(|A| - 1)X^* = (X^*)^2 \sum_{i \in A} \frac{1}{\theta_i},$$

from which we get

$$X^* = \frac{|A| - 1}{\sum_{i \in A} \frac{1}{\theta_i}}. \tag{2.3}$$

Then, we can retrieve the equilibrium strategy of $i \in A$

$$x^*_i = \begin{cases} \frac{|A|-1}{\sum_{i \in A} \frac{1}{\theta_i}} \left(1 - \frac{1}{\theta_i} \frac{|A|-1}{\sum_{i \in A} \frac{1}{\theta_i}}\right) & \text{if } i \in A, \\ 0 & \text{if } i \notin A. \end{cases} \tag{2.4}$$

Lemma 1. $\forall i, j \in N,\ i \leq j \ \Rightarrow \ x^*_i \geq x^*_j$.

Proof. This is a direct result from (2.4). □

Lemma 1 establishes the fact that if i values winning the contest at least as much as j, then i, in equilibrium, exerts at least as much effort as does j.

Lemma 2. *If $i \in A$ and θ_i is replaced by $\theta_i' > \theta_i$ and if the set of active contestants remains the same, then*

$$X^*\Big|_{\theta=(\theta_1,\ldots,\theta_i',\ldots,\theta_{|S|})} > X^*\Big|_{\theta=(\theta_1,\ldots,\theta_i,\ldots,\theta_{|S|})}$$

Proof. This is a direct result from (2.3). □

Lemma 2 establishes that although a higher type may discourage active players to exert effort, if all active players remain active, total effort must increase strictly.

Define $S^k = \{1, 2, 3, ..., k\}$ for $k = 3, ..., n$. We can deduce from (2.4) that l players are active in S^k if and only if

$$\theta_l > \frac{l-1}{\sum_{i=1}^{l} \frac{1}{\theta_i}}, \text{ and } \theta_{l+1} \leq \frac{l-1}{\sum_{i=1}^{l} \frac{1}{\theta_i}}.$$

Proposition 6. *Take any $S \subsetneq N$. If $j \notin S$ is active in $C(S \cup \{j\})$ then the total effort in equilibrium is greater in $C(S \cup \{j\})$ than in $C(S)$.*

Proof. Define X^{*j} as the total equilibrium effort in $C(S \cup \{j\})$ and X^* as the total equilibrium effort in $C(S)$ and A as the set of active players in $C(S)$.

Case 1: All players active in $C(S)$ are also active in $C(S \cup \{j\})$.

We have that

$$
\begin{aligned}
X^{*j} - X^* &= \frac{|A|}{\sum_{i \in A} \frac{1}{\theta_i} + \frac{1}{\theta_j}} - \frac{|A| - 1}{\sum_{i \in A} \frac{1}{\theta_i}} \\
&= \frac{\sum_{i \in A} \frac{1}{\theta_i} + \frac{1}{\theta_j} - \frac{|A|-1}{\theta_j}}{(\sum_{i \in A} \frac{1}{\theta_i} + \frac{1}{\theta_j})(\sum_{i \in A} \frac{1}{\theta_i})}
\end{aligned}
$$

which is strictly positive if

$$
\theta_j > \frac{|A| - 1}{\sum_{i \in A} \frac{1}{\theta_i} + \frac{1}{\theta_j}}.
$$

But since j is active in $S \cup \{j\}$, we have that

$$
\theta_j > \frac{|A|}{\sum_{i \in A} \frac{1}{\theta_i} + \frac{1}{\theta_j}}
$$

and thus $X^{*j} - X^* > 0$.

Case 2: Some of the players who are active in $C(S)$ are not active in $C(S \cup \{j\})$.

From Case 1 above, we can deduce that if an active player is removed from the contest, the total effort must decrease. And thus we have that the lowest total effort is achieved when only 2 players are active. Hence, the limit case is when only j and player $i' = \min_i \{i \in S\}$ are active in $C(S \cup \{j\})$. We then have that $\forall i \in S \setminus \{i'\}$,

$$
\max_{i \in S \setminus \{i'\}} \theta_i \leq \frac{1}{\frac{1}{\theta_{i'}} + \frac{1}{\theta_j}}
$$

For given θ_j and $\theta_{i'}$, Lemma 1 implies that the maximum total effort exerted in S, conditional on having only i' and j active in $C(S \cup \{j\})$, is when, $\forall i \in A \setminus \{i'\}$,

$$
\theta_i = \tilde{\theta} = \frac{1}{\frac{1}{\theta_{i'}} + \frac{1}{\theta_j}}.
$$

And, also implied by Lemma 1, the minimum possible total effort in $C(S\cup\{j\})$ is reached only if $\theta_j = \theta_{i'}$. Hence, in this limit case, we have that

$$\tilde{\theta} = \frac{\theta_{i'}}{2}.$$

We thus have that

$$\begin{aligned}
X^{*j} - X^{*} &= \frac{\theta_{i'}}{2} - \frac{|A|-1}{(|A|-1)\frac{1}{\tilde{\theta}} + \frac{1}{\theta_{i'}}} \\
&= \frac{\theta_{i'}}{2} - \frac{(|A|-1)\theta_{i'}\tilde{\theta}}{(|A|-1)\theta_{i'} + \tilde{\theta}} \\
&= \frac{\theta_{i'}}{2} - \frac{(|A|-1)\frac{\theta_{i'}^2}{2}}{(|A|-1)\theta_{i'} + \tilde{\theta}} \\
&> \frac{\theta_{i'}}{2} - \frac{(|A|-1)\frac{\theta_{i'}^2}{2}}{(|A|-1)\theta_{i'}} = 0.
\end{aligned}$$

□

2.5.2 Stage 1: the application stage

Since information is revealed in Stage 2, and its Nash equilibrium is uniquely determined by S and θ, any i can compute her Nash equilibrium payoff for any possible realization of (S,θ). Denote by $v_i(S,\theta_{-i};\theta_i)$, i's stage-2 equilibrium payoff in $C(S)$ when the profile of valuations is θ. An equilibrium (Bayes-Nash) of Stage 1 is a list of functions $(\sigma_i)_{i\in N}$ where

$$\sigma_i : \Theta \longrightarrow M_i$$

such that, $\forall i \in N$ and $\forall \theta_i \in \Theta$,

$$\sigma_i(\theta_i) \in \arg\max_{m_i \in M_i} \left\{ \mathbb{E}_{\theta_{-i}} \left[\sum_{\substack{S \subseteq N \\ i \in S}} \phi_S \left(m_i, \sigma_{-i}(\theta_{-i}) \right) v_i(S, \theta_{-i}; \theta_i) \right] - e_i(m_i) \right\}. \tag{2.5}$$

2.5.3 Stage 0: selecting the optimal mechanism

Define by $X_S(\theta)$ the sum of equilibrium effort levels of $C(S)$ for the profile θ. The designer's objective is to maximize

$$\mathbb{E}_\theta \left[\sum_{S \subseteq N} \phi_S(\sigma_1(\theta_1), ..., \sigma_n(\theta_n)) \left(R(X_S(\theta)) - |S|c - P \right) \right] \tag{2.6}$$

by choosing $(\phi_S)_{S \subseteq N}$. To do so, the designer must have a well-defined belief about the realization of θ given a profile of messages. To simplify the problem, we only consider mechanisms where $\forall i \in N$, $M_i = \Theta$. [8] Since the designer cannot differentiate players, the mechanisms will be restricted to those such that $e_i = e$, $\forall i \in N$.

[8]This restriction is reminiscent of the *revelation principle*. An intuitive definition of the revelation principle is that if an outcome can be reached as the Bayes-Nash equilibrium of some mechanism, then there exists a direct revelation mechanism that can reach the same outcome as a Bayes-Nash equilibrium. For a detailed explanation of the revelation principle, the reader is referred to Borgers *et al.* (2015). Note that this does not mean that I am considering a direct mechanism. Since the designer selects what are the allowed messages and the fixed costs, this does not qualify as being direct, *per se*. A direct mechanism only requires the message space to be the type space. In this model, the mechanism is somewhat more sophisticated as it additionally requires a participant to pay the fixed cost.

Definition 3. *The mechanism* $((\Theta)_{i\in N}, g)$ *is incentive compatible (IC) if* $\forall i \in N$ *and* $\forall \theta_i \in \Theta$*,*

$$\theta_i \in \arg\max_{m_i\in\Theta} \left\{ \mathbb{E}_{\theta_{-i}} \left[\sum_{\substack{S\subseteq N \\ i\in S}} \phi_S\left(m_i, \theta_{-i}\right) v_i(S, \theta_{-i}; \theta_i) \right] - e(m_i) \right\}.$$

Definition 4. *The mechanism* $((\Theta)_{i\in N}, g)$ *is individually rational (IR) if* $\forall i \in N$ *and* $\forall \theta_i \in \Theta$*,*

$$\mathbb{E}_{\theta_{-i}} \left[\sum_{\substack{S\subseteq N \\ i\in S}} \phi_S\left(\theta_i, \theta_{-i}\right) v_i(S, \theta_{-i}; \theta_i) \right] - e(\theta_i) \geq 0.$$

If the mechanism is IC and IR, then $\sigma_i(\theta_i) = \theta_i$, $\forall i \in N$, is an equilibrium of the mechanism. As it is often the case in the mechanism design literature, I will only consider this equilibrium from now on. Assume for the moment that there exists a function e such that the mechanism is IC and IR.

Upon receiving true signals, the designer's objective function (2.6) is solved by letting S that maximizes $X_S(\theta) - |S|c - P$ being selected with probability 1. Consequently, for any profile of types, the list of probabilities, $(\phi_S)_{S\subseteq N}$, are given by

$$\phi_S(m) = \begin{cases} & \text{if } R(X_S(m)) \geq |S|c + P, \\ 1 & \forall i \in S,\ R(X_S(m)) - R(X_{S\setminus i}(m)) \geq c \text{ and,} \\ & \forall i \in N \setminus S,\ R(X_{S\cup\{i\}}(m)) - R(X_S(m)) < c, \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

If S maximizes the designer's profits, then any $i \in S$ provides a marginal revenue at least as high as c and if any $j \notin S$ were to join S, then j would provide a marginal revenue strictly smaller than c. The continuity and the strict concavity of R ensures uniqueness of the designer's profit-maximizing S.

Example 3. *Let $R(x) = x$ and $N = \{i, j, k\}$ and use the standard labeling as previously stated. Assume the mechanism is IC and IR. From Proposition 5, if $|S| < 3$ then S is either empty or composed of 1 and 2. Let S be the profit-maximizing set of contestants.*

S is empty if

$$\frac{1}{\frac{1}{\theta_1} + \frac{1}{\theta_2}} < 2c + P \quad \textit{and} \quad \frac{2}{\frac{1}{\theta_1} + \frac{1}{\theta_2} + \frac{1}{\theta_3}} < 3c + P$$

$S = \{1, 2\}$ if

$$\frac{1}{\frac{1}{\theta_1} + \frac{1}{\theta_2}} \geq 2c + P \quad \textit{and} \quad \frac{2}{\frac{1}{\theta_1} + \frac{1}{\theta_2} + \frac{1}{\theta_3}} - \frac{1}{\frac{1}{\theta_1} + \frac{1}{\theta_2}} < c$$

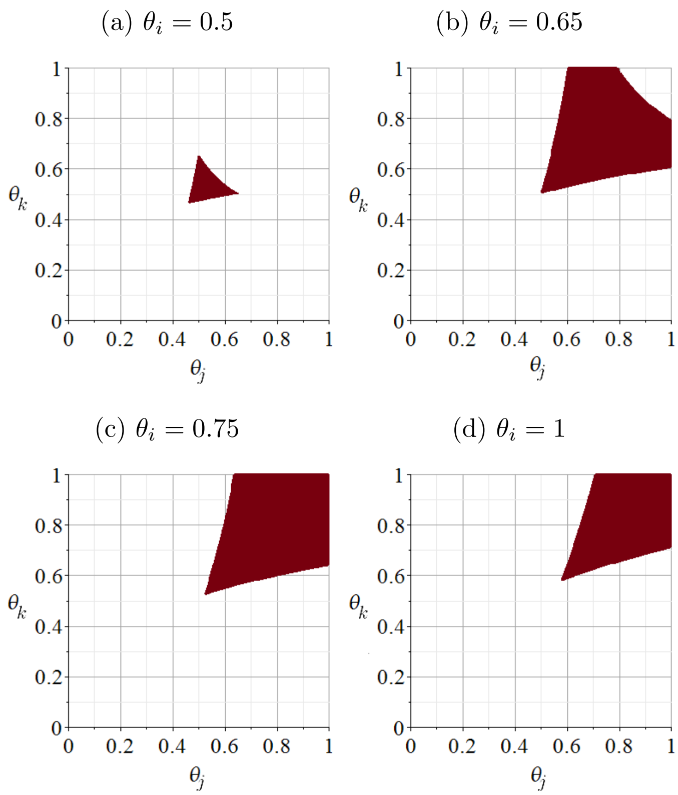
and $S = \{1, 2, 3\}$ if

$$\frac{2}{\frac{1}{\theta_1} + \frac{1}{\theta_2} + \frac{1}{\theta_3}} \geq 3c + P \quad \textit{and} \quad \frac{2}{\frac{1}{\theta_1} + \frac{1}{\theta_2} + \frac{1}{\theta_3}} - \frac{1}{\frac{1}{\theta_1} + \frac{1}{\theta_2}} \geq c.$$

Note that if θ_3 is low enough compared to θ_1 and θ_2 then player 3 is passive in $C(N)$ and consequently, the marginal effort that player 3 would bring to the contest is 0. Obviously, a passive player is never selected to Stage 2.

In Figure 1, I fix different values of θ_i and depict by a shaded area all the profiles (θ_j, θ_k) such that, $\phi_N(\theta) = 1$.

Figure 2.1: Profiles such that $\phi_N(\theta) = 1$, with $c = 0.1$ and $P = 0$.



It is natural that a candidate to the contest forms expectations over the subsets that can be selected to the contest for any signal m_i that i sends to the designer. For any subset S that contains i, it is relevant to know for what values of m_i does S start being chosen with a positive probability. Put differently, what is the smallest signal i can send for S to have a positive probability of being chosen?

It easy to see that for $S = \{1, 2\}$ to be accepted, it is necessary that $R(X_{\{1,2\}}(\theta)) \geq 2c + P$. And because $X_{\{1,2\}}(\theta)$ depends positively on θ_1 and θ_2, then the smallest value θ_2 that can be accepted to a contest is $\underline{\theta}_2$ such that

$$R\left(\frac{1}{\frac{1}{\theta}+\frac{1}{\underline{\theta}_2}}\right)=2c+P.$$

Let $|S|=k\geq 3$. For S to be selected by the designer, the first k highest values must be high enough relative to the $n-k$ smallest values. Let the $n-k$ smallest values be low enough so that the designer does not select the lowest types.

Proposition 7. *For $|S|\geq 3$, the smallest value m_i such that $S\ni i$ is accepted to the contest stage with positive probability is $\underline{\theta}_S$ such that $\theta_1=\theta_{|S|}=\underline{\theta}_S$, $R(X_S(\underline{\theta}_S))-R(X_{S\setminus\{i\}}(\underline{\theta}_S))=c$ and $\phi_S(\underline{\theta}_S)=1$, $i\leq |S|$.*

Proof. The fact that $\theta_1=\theta_{|S|}$ directly follows from (2.3) and from the fact that R is strictly increasing. From Lemma 2, we know that if for $j\in\{1,...,|S|\}$, θ_j is replaced by $\theta'_j>\theta_j$, it can be the case that $R(X_S(m_i,\theta'_{-i}))-R(X_{S\setminus\{i\}}(m_i,\theta'_{-i}))=c$ with $m_i<\theta_i$. However, if it is the case, then I will show that $\phi_S(m_i,\theta'_{-i})=0$.

For the profile θ' where only one contributor has its value increased by a small $\epsilon>0$, the strict concavity of R implies that $R(X_{S\setminus\{i\}}(\theta'))-R(X_{S\setminus\{i\}}(\theta))>R(X_S(\theta'))-R(X_S(\theta))$, and hence we get that $R(X_S(\theta'))-R(X_{S\setminus\{i\}}(\theta'))<c$. Thus, $\phi_S(\theta')=0$. □

It is also possible to rank the subsets of N with respect to the smallest signal m_i that is required for a subset $S\ni i$, to be selected to the contest round with positive probability.

For any subset S, define $\underline{\theta}_S$ as in Proposition 7.

Proposition 8. $\underline{\theta}_2\leq\underline{\theta}_3$.

Proof. We have that

$$R\left(\frac{1}{\frac{1}{\underline{\theta}_2}+\frac{1}{\theta}}\right)=2c+P$$

and

$$R\left(\frac{2}{3}\underline{\theta}_3\right) - R\left(\frac{1}{2}\underline{\theta}_3\right) = c$$

with

$$R\left(\frac{2}{3}\underline{\theta}_3\right) \geq 3c + P.$$

Consequently, we get that

$$\begin{aligned} R\left(\frac{2}{3}\underline{\theta}_3\right) - R\left(\frac{1}{\frac{1}{\underline{\theta}_2} + \frac{1}{\bar{\theta}}}\right) &\geq c \\ &= R\left(\frac{2}{3}\underline{\theta}_3\right) - R\left(\frac{1}{2}\underline{\theta}_3\right) \end{aligned}$$

and thus

$$R\left(\frac{1}{2}\underline{\theta}_3\right) \geq R\left(\frac{1}{\frac{1}{\underline{\theta}_2} + \frac{1}{\bar{\theta}}}\right).$$

Since R is strictly increasing,

$$\frac{1}{2}\underline{\theta}_3 \geq \frac{1}{\frac{1}{\underline{\theta}_2} + \frac{1}{\bar{\theta}}}.$$

Simplifying this, we get

$$\begin{aligned} \frac{\underline{\theta}_3}{2} &\geq \frac{\bar{\theta}\underline{\theta}_2}{\bar{\theta} + \underline{\theta}_2} \\ &\geq \frac{\bar{\theta}\underline{\theta}_2}{2\bar{\theta}} = \frac{\underline{\theta}_2}{2}. \end{aligned}$$

□

Proposition 9. *For any* $k = 3, ..., n-1$, $\underline{\theta}_S \leq \underline{\theta}_{S'}$, *for* $|S| = k$ *and* $|S'| = k+1$.

Proof. We have that

$$R\left(\frac{k-1}{k}\underline{\theta}_S\right) - R\left(\frac{k-2}{k-1}\underline{\theta}_S\right) = R\left(\frac{k}{k+1}\underline{\theta}'_S\right) - R\left(\frac{k-1}{k}\underline{\theta}'_S\right) = c$$

Consider, on the left-hand side, increasing $\frac{k-1}{k}\underline{\theta}_S$ to $\frac{k}{k+1}\underline{\theta}_S$ and increasing $\frac{k-2}{k-1}\underline{\theta}_S$ to $\frac{k-1}{k}\underline{\theta}_S$. The first change involves an increase of $\frac{\underline{\theta}_S}{(k+1)k}$ and the second change, an increase of $\frac{\underline{\theta}_S}{(k-1)k}$. Note that the second change is greater, in absolute value, than the first. By the concavity of R, we then have that

$$R\left(\frac{k}{k+1}\underline{\theta}_S\right) - R\left(\frac{k-1}{k}\underline{\theta}_S\right) \leq R\left(\frac{k}{k+1}\underline{\theta}'_S\right) - R\left(\frac{k-1}{k}\underline{\theta}'_S\right).$$

Because R is increasing, it must be the case that $\underline{\theta}_S \leq \underline{\theta}'_S$. $\square$

2.6 Existence of an IC and IR Mechanism

To avoid a trivial problem where the contest would always be canceled, assume the following:

Assumption 2. *There is a strictly positive probability that Stage 2 is not canceled:* $\forall i \in N$, $\exists \theta'_i \in \Theta$, $\theta'_i < \bar{\theta}$, *such that*

$$\mathbb{E}_{\theta_{-i}}\left[\sum_{\substack{S\subseteq N\\ i\in S}} \phi_S\left(\theta'_i, \theta_{-i}\right)\right] > 0.$$

Proposition 10. *For all* $i \in N$,

$$\mathbb{E}_{\theta_{-i}}\left[\sum_{\substack{S\subseteq N\\ i\in S}} \phi_S\left(m_i, \theta_{-i}\right) v_i(S, \theta_{-i}; \theta_i)\right] \tag{2.8}$$

is continuous and differentiable with respect to m_i.

Proof. Let S be a subset of N that contains i. If S is selected for the contest round, it is because (m_i, θ_{-i}) falls in the subset of all possible types such that $\phi_S(m_i, \theta_{-i}) = 1$.

Define a set-valued function $\Delta_{S|i}$ that maps m_i to a subset of Θ^{n-1} such that if $\theta_{-i} \in \Delta_{S|i}(m_i)$ then $\phi_S(m_i, \theta_{-i}) = 1$. We have that

$$E_{\theta_{-i}}\left[\phi_S(m_i, \theta_{-i})\right] = \int_{\Delta_{S|i}(m_i)} dF_{-i}(\theta_{-i}).$$

There are n potential discontinuity points to (2.8). The potential discontinuity points are those for which subsets $\{1, 2, ..., k\}$ start getting selected to the contest round with positive probability. For $k = 2, .., n$, define m_k as

$$m_k = \sup\left\{m \mid E_{\theta_{-i}}\left[\phi_{\{1,2,...,k\}}(m, \theta_{-i})\right] = 0\right\}.$$

To prove proposition 10, we must show that for $k = 2, ..., n$, $\Delta_{\{1,2,..,k\}|i}$ is continuous on Θ. If it is the case, then the domain of integration, $\Delta_{\{1,2,..,k\}|i}(m_i)$, varies continuously with respect to m_i and thus its probability mass varies continuously as well.

Definition 5. $\Delta_{\{1,2,...,k\}|i} : \Theta \rightrightarrows \Theta^{n-1}$ *is lower hemicontinuous at m if, for every open set O in Θ^{n-1} with $\Delta_{\{1,2,...,k\}|i}(m) \cap O \neq \emptyset$, there exists a $\delta > 0$ such that $\Delta_{\{1,2,...,k\}|i}(m') \cap O \neq \emptyset$ for all m' belonging in the δ-neigborhood of m in Θ, $N_{\delta,\Theta}(m)$.*

Proposition 11. $\Delta_{\{1,2,...,k\}|i}$ *is lower hemicontinuous.*

Proof. Take any m for which $\Delta_{\{1,2,...,k\}|i}(m)$ has strictly positive mass. Because R is strictly concave, no inequality in (2.7) can be satisfied with equality (otherwise $\Delta_{\{1,2,...,k\}|i}(m)$ would have mass 0.) If we take an open set O

intersecting $\Delta_{\{1,2,\ldots,k\}|i}(m)$ by the continuity of R, we can always find an open set around m such that the inequalities in (2.7) hold true for any element belonging to the open set around m.

Consider now any m such that $\Delta_{\{1,2,\ldots,k\}|i}(m)$ that has mass 0. It must then be the case that at least one inequality in (2.7) holds with equality. Any open set intersecting $\Delta_{\{1,2,\ldots,k\}|i}(m)$ has strictly positive mass. By the continuity of R we can find an open set around m such that either $\Delta_{\{1,2,\ldots,k\}|i}(m')$ has strictly positive mass and intersects O or is empty, and by convention also belongs to O. □

Definition 6. $\Delta_{\{1,2,\ldots,k\}|i} : \Theta \rightrightarrows \Theta^{n-1}$ *is upper hemicontinuous at* m *if, for every open subset* O *of* Θ^{n-1} *with* $\Delta_{\{1,2,\ldots,k\}|i}(m) \subseteq O$*, there exists a* $\delta > 0$ *such that* $\Delta_{\{1,2,\ldots,k\}|i}(N_{\delta,\Theta}(m)) \subseteq O$*.*

Proposition 12. $\Delta_{\{1,2,\ldots,k\}|i}$ *is upper hemicontinuous.*

Proof. Because of the continuity of R we can always find a neighborhood around m such that its image is contained in O. And by convention, if some element of the neighborhood is mapped to the empty set, it still belongs to O. □

We have shown that for any $k = 2, .., n$, and $\forall m \in \Theta$, $\Delta_{\{1,2,\ldots,k\}|i}$ is lower and upper hemicontinuous at m and thus $\Delta_{\{1,2,\ldots,k\}|i}$ is continuous. This means that as m varies in Θ, the domain of integration in $\mathbb{E}_{\theta_{-i}}\left[\sum_{\substack{S \subseteq N \\ i \in S}} \phi_S\left(m_i, \theta_{-i}\right) v_i(S, \theta_{-i}; \theta_i)\right]$ varies continuously, which implies that the function is continuous and differentiable with respect to m_i..

□

A sufficient condition for incentive compatibility is that $m_i = \theta_i$ solves

$$\frac{d}{dm_i}\left(\mathbb{E}_{\theta_{-i}}\left[\sum_{\substack{S\subseteq N\\ i\in S}} \phi_S\left(m_i, \theta_{-i}\right) v_i(S, \theta_{-i}; \theta_i)\right] - e(m_i)\right) = 0, \tag{2.9}$$

with the second-order condition

$$\frac{d^2}{dm_i^2}\left(\mathbb{E}_{\theta_{-i}}\left[\sum_{\substack{S\subseteq N\\ i\in S}} \phi_S\left(m_i, \theta_{-i}\right) v_i(S, \theta_{-i}; \theta_i)\right] - e(m_i)\right) < 0 \tag{2.10}$$

where the derivatives exist.

We cannot ensure that the cost function e is differentiable everywhere on Θ. For instance, an individual submitting $m_i \in [\underline{\theta}, \underline{\theta}_2]$ will not be selected to the contest. Consequently, individual rationality implies that a truthful revelation in this region is costless. Thus e can be kinked at $\underline{\theta}_2$ and also at all $\underline{\theta}_S$ for $|S| \geq 3$. Nonetheless, as long as e is strictly increasing and continuous, incentive compatibility and individual rationality can hold where e is non-differentiable.

The construction of e is as follows. On the differentiable region, by solving a simple differential equation, we can easily find a function e that solves (2.9). Because $v_i(S, \theta_{-i}; \theta_i)$ is strictly increasing with respect to θ_i, we can ensure that e is strictly increasing.

Example 4. *Let $N = \{i, j\}$.*

Let's take the perspective of contestant i. This game is solved backwards, starting from Stage 2. Assume that i and j compete against each other in a contest. Contestant i's equilibrium payoff in this stage is

$$v_i(\theta_j; \theta_i) = \frac{\theta_i^3}{(\theta_i + \theta_j)^2}.$$

Assume that contestant j reveals truthfully her type. Let individual i reports to be of type m_i. Define $\underline{\theta}_2(m_i)$ such that

$$R\left(\frac{1}{\frac{1}{m_i}+\frac{1}{\underline{\theta}_2(m_i)}}\right)=2c+P.$$

Thus, contestant i's objective is to solve

$$\max_{m_i\in\Theta}\left\{\int_{\underline{\theta}_2(m_i)}^{\bar{\theta}} v_i(\theta_j;\theta_i)dF(\theta_j)-e(m_i)\right\}.$$

Note that m_i is negatively correlated with $\underline{\theta}_2(m_i)$. The trade-off faced by i is as follows: decreasing m_i decreases the value of the fixed cost imposed on i while decreasing the probability of being selected to the contest.

In the simple case of F being the uniform distribution over $\Theta=[1,2]$ and $R(x)=x$, we have that

$$\underline{\theta}_2(m_i)=\frac{m_i(2c-P)}{m_i-2c+P}.$$

i's objective is reduced to

$$\max_{m_i}\left\{\frac{{\theta_i}^3\left(-2\,m_ic+m_iP+2\,m_i-4\,c+2\,P\right)}{\left(\theta_im_i-2\,\theta_ic+\theta_iP+2\,m_ic-m_iP\right)\left(\theta_i+2\right)}-e\left(m_i\right)\right\}.$$

In stage 1, the designer must make sure that the solution to i's problem is solved by setting $m_i=\theta_i$. The designer can ensure that i will indeed set $m_i=\theta_i$ by selecting the right functional form of e, which is given by solving a differential equation. We differentiate i's objective function with respect to m_i, then retrieve e such that i's first order condition is satisfied at $m_i=\theta_i$. We get that

$$e(\theta_i)=(P-2c)^2ln(\theta_i)+K$$

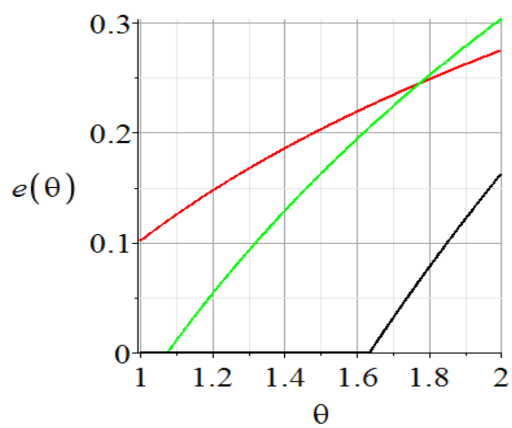
where K is the integration constant. For IR to be satisfied, no contestant should have negative payoff from revealing the true type. The critical revelation for which any smaller revelation leads to stage 2 being canceled is given by $\underline{\theta}_2$ such that

$$R\left(\frac{1}{\frac{1}{\underline{\theta}_2}+\frac{1}{\theta}}\right) = 2c + P.$$

Consequently, any $\theta_i \leq \frac{2(2c-P)}{P-2c+2}$ must pay 0. K then solves $e\left(\frac{2(2c-P)}{P-2c+2}\right) = 0$ and thus we get

$$e(\theta_i) = \begin{cases} 0 & \text{if } \theta_i \leq \max\{\frac{2(2c-P)}{P-2c+2}, 1\}, \\ (P-2c)^2 ln(\theta_i) + K & \text{otherwise.} \end{cases}$$

Figure 2.2: $e(\theta_i)$ for $c = 0.25$ (in red), $c = .35$ (in green) and $c = .45$ (in black), with $P = 0$.



2.7 Conclusion

In this paper, I developed a revelation mechanism used to select contestants so that the contest designer maximizes her profits from running a contest.

This paper offers an alternative to Fullerton & McAfee (1999) concerning the way to select contestants in a situation of private information. An interesting extension to this model is left for further research: when the contest represents a job promotion, for instance, it may very well be the case that the CEO needs to fill two, three or k positions. But since there are multiple positions to fill, and that each position requires a different set of skills or tasks to be performed, the CEO may be obligated to run k contests, each one consisting of a different set of contestants. Consequently, the CEO must, first, select the candidates that are allowed to participate in one of the k contests. Second, the CEO must decide what is an optimal way to partition the contestants into k contests. This research would achieve a generalization of my model with only one position to fill.

The previous two chapters are focused on theoretical topics in contest theory. While the next chapter also contributes to the theory of contests, it does so through an application to sports economics. In this chapter, I consider the problem of allocating a finite supply of a productive input to two contestants.

Chapter 3

Endogenous Leadership in a Sports League with a Fixed Supply of Talent

3.1 Introduction

I analyze a specific duopoly in which two professional sports teams compete against each other for the championship prize. At the start of the season, teams must acquire productive input, referred to as *talent*, in order to play in the league. However, as it is usually the case in professional sports, there is a limited supply of talent. It is generally understood that in professional sports, there is a natural threshold in terms of athletic ability, below which an athlete cannot play professionally. There is thus a natural limit to the supply of talent, from which professional teams can select from.

To obtain a higher probability of winning the championship, a team must own a larger quantity of talent than its opponent's. In this paper, I suggest a way to model a competitive allocation of talent into teams, by introducing a sequential game in which teams must first bid on the cost of talent, and then, whichever team has made the highest bid gets to choose first the quantity of talent to hire at the implemented cost, and then the other team chooses a quantity from the residual pool of talent. In the case of a tie in the first stage, teams make simultaneous demands to the market and are allocated a quantity of talent equal to their demand or equal to their demand relative to the total demand in the case that total demand exceeds total supply.

This game is in fact an endogenous leadership game, which is in some way a hybrid case of price leadership and quantity leadership. The new feature of this model is that the way in which a leader is selected is through the first stage of the game in which teams announce a unit cost for talent. Although, we can find some similarities with this model and other Stackelberg-like duopoly games, the main result of this paper is different in that, in a subgame perfect Nash equilbrium, the efficient firm, or equivalently the high-revenue firm, may stand as the follower while the low-revenue firm stands as the leader, given that the revenue advantage of the high-revenue firm is not too large.

The economic theory of professional team sports has yet to fully integrate game theory into its field. Although sports economics is a young and growing field of research, the current modeling approach tends to follow the so-called conjectural variation approach from the old industrial organization literature. With this approach, there are two equilibrium concepts in the professional sports labour market. These concepts rely on two distinct assumptions which are known as the “Walras” and the “Nash” assumptions. The former states that prior to formulating a demand to the market, team A must internalize the fact that his quantity demanded will be taken away from team B and thus affecting his revenue directly through the increase of his input *and* indirectly through the decrease of its opponent’s input. The latter assumption stipulates that the indirect effect is restricted to be null. Unfortunately, neither the “Walras” assumption nor the “Nash” is coherent with the game-theoretical perspective on competitive equilibrium. The definition of a Nash equilibrium is straightforward: For every individual among the n players, taking as given the actions of the $n-1$ other players, one cannot strictly be better off by changing his action. The Nash equilibrium concept does not allow players to assume subsequent movements in response to their own. This is precisely why the conjectural variation approach is inconsistent with the concept of competitive equilibrium. In this paper, I abstract from the standard conjectural variation approach and use the tools of game theory to forge my model.

Surprisingly, even though my approach is substantially different from the other theoretic models of sports leagues, I share the same conclusion pertaining to the impact of revenue sharing on *competitive balance*. Competitive balance is a relative measure of the inequality in talent dispersion across teams. We say that competitive balance reaches its maximal value of 1 when all teams own the same quantity of talent. It is generally accepted that a competitive balance closer to a value of 1 is better for the league as it can increase excitement from a match and, consequently, increase viewership and league revenues. But in a situation where a rich team systematically acquires more talent, it is an interesting question to ask if introducing a system of revenue sharing can affect the distribution of talent in equilibrium. I consider the simple system of revenue sharing that consists of each team sharing a given fraction of their revenues with the opposing team. I find that revenue sharing induces an even more unequal distribution of talent across teams, in equilibrium.

This paper augments the sports economics literature in two ways. First, I introduce a richer model that offers an alternative way to think about the formation of professional teams in a sports league. Second, the model proposed in this paper is used to analyze a situation where the cost of labour can be affected by the actions of the teams. I also contribute to a large literature on endogenous leadership in a duopoly by providing a specific situation in which the high-revenue firm can in fact become the follower in equilibrium.

3.2 Review of Literature

Relevant work on endogenous leadership include Hamilton & Slutsky (1990) who study a pre-play stage in which firms decide to either rule out the possibility of being a follower, or rule out the possibility of being the leader. Equilibria typically involve a sequential order of play. Van Damme & Hurkens (1999) and van Damme & Hurkens (2004) respectively study an endogenous Stackelberg leadership game and a price leadership game. In both papers, they come to the same conclusion: the efficient firm emerges as the leader, independent of whether prices or quantities are the strategic variables. Deneckere & Kovenock (1992) arrives to a similar conclusion when considering a duopoly with price setting and capacity constraints and von Stengel (2010) argues that "the seemingly natural case that both players profit from sequential play as compared to simultaneous play, but the leader more so than if he was follower, can only occur in non-symmetric games."

The use of the term "competitive equilibrium" in Szymanski (2004) refers to the so-called Walrasian fixed-supply conjecture model while the "Nash" solution to the noncooperative game of talent choice in a professional sports league is called the "Contest-Nash" solution. This equilibrium concept has been adopted in the subsequent work of Szymanski & Késenne (2004). The conjectural variation hypothesis in the field of sports economics is well-documented in Késenne (2007) and in the references therein.

My research follows the work of Madden (2011) who initiated a transition towards a more game-theoretically oriented approach to club formation. Madden suggested a new equilibrium concept where, instead of formulating demands to the market in terms of quantity of talent, teams would first decide the total budget dedicated to acquiring playing talent. Then the market decides of the cost of talent such that the whole supply of talent is distributed to teams. Szymanski (2013) stated that the work of Madden is “the most significant contribution to this literature since 2004.” However, the situation where teams may have full power over both the cost of talent and the quantity of talent has not been looked at yet. My paper focusses on this last issue.

My model shares some resemblance with the model of Jackson & Moulin (1992), but in a different context. Jackson and Moulin use a multi-stage mechanism in order to efficiently provide a public good.

3.3 The Model

A sports league consists of two teams: team A and team B. Revenues are drawn from participating to a contest against each other. The input used by teams is called *talent*, which is a positive real number. This input represents what professional sports player are assumed to be endowed with. We assume that talent is a continuous variable and that the total quantity of talent is equal to 1. Throughout the paper, the quantity of talent associated with team A and team B will be denoted $t_a \in \mathbb{R}_+$ and $t_b \in \mathbb{R}_+$, respectively.

Let the revenue functions for team A and team B be R_a and R_b, respectively, where for $i = a, b$, $R_i = R_i(t_a, t_b) \in \mathbb{R}_+$ and $i's$ profit function is

$$\pi_i(t_a, t_b; c) = R_i(t_a, t_b) - c \cdot t_i,$$

with $c > 0$ being the implemented unit cost of talent.

R_i is assumed to be continuous on $\mathbb{R}^2_+$ and strictly concave with respect to t_i. Also, R_i is assumed to be strictly increasing in t_i and strictly decreasing in t_j, $j \neq i$.

In order to distribute the total quantity of talent among the two teams, I introduce a game $\mathcal{G}$, played between team A and team B, that runs in two stages. In the first stage, both teams make a bid $c_i > 0$, $i = a, b$. The profile of bids determines the subgame played in the second stage and the implemented unit cost of talent c which is considered fixed after the first stage ends. The subgame played in the second stage is

$$G(c_a, c_b) = \begin{cases} \mathcal{G}_1 & \text{if } c_a = c_b \\ \mathcal{G}_2 & \text{if } c_a > c_b \\ \mathcal{G}_3 & \text{if } c_a < c_b \end{cases}$$

and the implemented unit cost of talent is $c = \max\{c_a, c_b\}$.

$\mathcal{G}_1$ is a normal form game, in which both teams submit a bid $t_i \geq 0$, simultaneously, that represents the quantity of talent team i is willing to acquire at the unit cost of c. If $t_a + t_b \leq 1$, then teams are allocated the quantity equal to their bid. Otherwise, both teams receive a quantity of talent proportional to their own bid relative to the sum of bids. That is, team i receives $t_i = \frac{t_i}{t_a + t_b}$. In this subgame, the profit functions are

$$\pi_i(t_a, t_b; c) = \begin{cases} R_i(t_a, t_b) - c \cdot t_i & \text{if } t_a + t_b \leq 1 \\ R_i\left(\frac{t_a}{t_a+t_b}, \frac{t_b}{t_a+t_b}\right) - c \cdot \frac{t_i}{t_a+t_b} & \text{otherwise.} \end{cases}$$

$\mathcal{G}_2$ is a sequential game in which team A acts as the leader and team B acts as the follower. Team A is allowed to choose first any quantity of talent t_a on the interval $[0, 1]$, while team B chooses second and is constrained to choose from the interval $[0, 1 - t_a]$.

$\mathcal{G}_3$ is analogous to $\mathcal{G}_2$ except that, instead, it is team B who acts as the leader and team A as the follower. In all subgames, once the quantities of talent are allocated, profits are realized and the game ends.

3.4 Solving the Game

The solution concept that will be considered is a subgame perfect Nash equilibrium in pure strategies. A (pure) strategy for A is a list $(c_a, t_{a,\mathcal{G}_1}, t_{a,\mathcal{G}_2}, t_{a,\mathcal{G}_3})$ where $c_a > 0$, $t_{a,\mathcal{G}_1} = t_{a,\mathcal{G}_1}(c)$, $t_{a,\mathcal{G}_2} = t_{a,\mathcal{G}_2}(c)$ and $t_{a,\mathcal{G}_3} = t_{a,\mathcal{G}_3}(t_b, c)$. A (pure) strategy for B is a list $(c_b, t_{b,\mathcal{G}_1}, t_{b,\mathcal{G}_2}, t_{b,\mathcal{G}_3})$ where $c_b > 0$, $t_{b,\mathcal{G}_1} = t_{b,\mathcal{G}_1}(c)$, $t_{b,\mathcal{G}_2} = t_{b,\mathcal{G}_2}(t_a, c)$ and $t_{b,\mathcal{G}_3} = t_{b,\mathcal{G}_3}(c)$. Note that only in the subgame where i is the follower that i's strategy can be made contingent on the leader's quantity of talent. In the other subgames, strategies can only be made contingent on c.

3.4.1 Subgame $\mathcal{G}_1$

Each team maximizes its profit function by choosing $t_i \in [0, 1]$. For $i = A, B$, define the Lagrange function

$$\mathcal{L}_i = \pi_i(t_a, t_b; c) - \lambda_i(t_i - 1)$$

where λ_i is the Lagrange multiplier. The Nash equilibrium in this subgame is (t_a, t_b) such that $(t_a, t_b, \lambda_a, \lambda_b)$ is the solution to the system of constrained maximization problems

$$\left(\max_{t_i} \mathcal{L}_i \;\text{ subject to } \lambda_i, t_i \geq 0,\; t_i - 1 \leq 0 \text{ and } \lambda_i(t_i - 1) = 0\right)_{i=A,B} \tag{3.1}$$

A solution to this system is a list $(t_a, t_b, \lambda_a, \lambda_b)$ satisfying the first-order conditions

$$\frac{\partial}{\partial t_a}\mathcal{L}_a = 0,\; \frac{\partial}{\partial t_b}\mathcal{L}_b = 0$$

along with

$$t_i - 1 \leq 0, \ \frac{\partial}{\partial \lambda_i} \mathcal{L}_i \lambda_i = 0, \ \text{and } \lambda_i, t_i \geq 0, \ i = a, b.$$

The Lagrange multiplier is interpreted as a measure of the extent to wich $t_i - 1 \leq 0$ is binding. The greater λ_i is, the more restrictive is the constraint. When the constraint $t_i - 1 \leq 0$ is not binding for team i, it implies that $\lambda_i = 0$.

3.4.2 Subgames $\mathcal{G}_2$ (and $\mathcal{G}_3$)

Define for $i, j = a, b$ and $i \neq j$,

$$\phi_i(t_j, c) = \arg \max_{t_i \in [0,1]} \{R_i(t_a, t_b, c) - c \cdot t_i\}$$

and

$$\psi_i(t_j, c) = \arg \max_{t_i \in [0,1-t_j]} \{R_i(t_a, t_b, c) - c \cdot t_i\}$$

Without loss of generality, consider $\mathcal{G}_2$. Taking t_a and c as given, team B chooses $\psi_b(t_a, c)$. And as the leader, A chooses $\phi_a(\psi_b(t_a, c), c_a)$. An equilibrium in this subgame is a pair (t_a, t_b) such that

$$t_a = \phi_a(\psi_b(t_a, c), c) \text{ and } t_b = \psi_b(t_a, c). \tag{3.2}$$

Analogously, an equilibrium of $\mathcal{G}_3$ is a pair (t_a, t_b) such that

$$t_a = \psi_a(t_b, c) \text{ and } t_b = \phi_b(\psi_a(t_b, c), c). \tag{3.3}$$

3.4.3 Equilibrium of $\mathcal{G}$

A subgame perfect Nash equilibrium of $\mathcal{G}$ consists of a (pure) strategy for A and a (pure) strategy for B, such that $(t_{a,\mathcal{G}_1}(c), t_{b,\mathcal{G}_1}(c))$ is part of the solution to (3.1), $(t_{a,\mathcal{G}_2}(c), t_{b,\mathcal{G}_2}(t_a, c))$ solves (3.2) and $(t_{a,\mathcal{G}_3}(t_b, c), t_{b,\mathcal{G}_3}(c))$ solves (3.3). A subgame perfect Nash equilibrium of $\mathcal{G}$ must also be such that there is no team i that can induce a subgame different from $G(c_a, c_b)$ by bidding $c_i' \neq c_i$ in the first stage and be made strictly better off, holding j's (pure) strategy constant.

3.5 Asymmetric Revenues and Revenue Sharing

In this section, I consider the commonly known Tullock Contest Success Function (CSF). Let the revenue functions be

$$R_a(t_a, t_b) = z\frac{t_a}{t_a + t_b} \quad \text{and} \quad R_b(t_a, t_b) = \frac{t_b}{t_a + t_b}$$

with $z \geq 1$.

When the Tullock CSF is used as a revenue function, it is assumed that i's revenues depend on the probability that team i wins the contest which, in turn, depends on (t_a, t_b). It is an assumption that is commonly made in the sports economics literature.

I consider the possibility that a team can generate more revenues than the other, for any (t_a, t_b). This can be the case when a team is physically located in a more lucrative market. An example on this can be found in Major League Baseball by comparing the New York Yankees and the Oakland A's; the

New York Yankees are located in a much bigger market, and thus have an advantage over the A's with respect to generating revenues.

I also consider revenue sharing, with the same modeling approach as in Szymanski & Késenne (2004). The rule for profit sharing is as follows: it is decided by the league that a team can only keep a fraction α of the revenue generated. The rest, $1-\alpha$, goes to the other team. We thus have the profit functions: for $i, j = a, b$, $i \neq j$

$$\begin{aligned}\pi_a(t_a, t_b; c, \alpha) &= \alpha R_a(t_a, t_b) + (1-\alpha) R_b(t_a, t_b) - c \cdot t_a \\ \pi_b(t_a, t_b; c, \alpha) &= \alpha R_b(t_a, t_b) + (1-\alpha) R_a(t_a, t_b) - c \cdot t_b\end{aligned}$$

with $\alpha \in (0.5, 1]$.

3.5.1 Equilibrium

For the next results to hold, z must be restricted to be in the interval $\left[1, \frac{\alpha}{1-\alpha}\right)$. The need to assume this is technical and is explained in the appendix. Intuitively, since α can take the value of 1, it prevents z from being infinitely large. By assuming this, we have that $\alpha z + \alpha - z \geq 0$.

Subgame $\mathcal{G}_1$

Proposition 13. *The list $(t_{a,\mathcal{G}_1}(c), t_{b,\mathcal{G}_1}(c))$ such that*

$$t_{a,\mathcal{G}_1}(c) = \begin{cases} \frac{(\alpha - z + \alpha z)(\alpha z - 1 + \alpha)^2}{c(z+1)^2(2\alpha - 1)^2} & \text{if } c \geq \frac{(\alpha z - 1 + \alpha)(\alpha - z + \alpha z)}{(z+1)(2\alpha - 1)} \\ 1 & \text{otherwise} \end{cases}$$

and

$$t_{b,\mathcal{G}_1}(c) = \begin{cases} \frac{(\alpha - z + \alpha z)^2(\alpha z - 1 + \alpha)}{c(z+1)^2(2\alpha - 1)^2} & \text{if } c \geq \frac{(\alpha z - 1 + \alpha)(\alpha - z + \alpha z)}{(z+1)(2\alpha - 1)} \\ 1 & \text{otherwise} \end{cases}$$

is the Nash equilibrium of $\mathcal{G}_1$.

Proof. See Appendix $\square$

It leads team A to a profit of

$$\pi_{a,\mathcal{G}_1} = \begin{cases} \frac{1}{2}(\alpha z + 1 - \alpha - c) & \text{if } c < \frac{(\alpha\, z-1+\alpha)(\alpha-z+\alpha\, z)}{(z+1)(2\,\alpha-1)} \\ \frac{(1-\alpha)\left(\alpha^2(z^3+z^2+3z+3)-\alpha(2z^2+4z+2)+2z\right)}{(z+1)(2\,\alpha-1)^2} & \text{otherwise} \end{cases}$$

and B to a profit of

$$\pi_{b,\mathcal{G}_1} = \begin{cases} \frac{1}{2}(\alpha + \alpha z - c) & \text{if } c < \frac{(\alpha\, z-1+\alpha)(\alpha-z+\alpha\, z)}{(z+1)(2\,\alpha-1)} \\ \frac{\alpha^3(z^3+5z^2+3z-1)+\alpha^2(2z^3+8z^2+6z)-\alpha(z^3+4z^2+5z)+z^2+z}{(z+1)(2\,\alpha-1)^2} & \text{otherwise} \end{cases}$$

Subgame $\mathcal{G}_2$

Proposition 14. *The list $(t_{a,\mathcal{G}_2}(c), t_{b,\mathcal{G}_2}(t_a, c))$ such that*

$$t_{a,\mathcal{G}_2}(c) = \begin{cases} 1 & \textit{if } c \leq \frac{\alpha+\alpha z-1}{2} \\ \frac{(\alpha+z\alpha-1)^2}{4c(\alpha-z+z\alpha)} & \textit{otherwise} \end{cases}$$

and

$$t_{b,\mathcal{G}_2}(t_a, c)) = \begin{cases} 0 & \textit{if } c \geq \frac{\alpha z+\alpha-z}{t_a} \\ 1 - t_a & \textit{if } c \leq t_a(\alpha z + \alpha - z) \\ \frac{-ct_a+\sqrt{ct_a\,(\alpha z+\alpha-z)}}{c} & \textit{otherwise.} \end{cases}$$

is a subgame perfect Nash equilibrium of $\mathcal{G}_2$.

Proof. See Appendix $\square$

It leads team A to a profit of

$$\pi_{a,\mathcal{G}_2} = \begin{cases} \alpha z - c & \text{if } c \leq \frac{\alpha+\alpha z-1}{2} \\ \frac{\alpha^2 z^2 - 2\,z\alpha^2 - 3\,\alpha^2 + 6\,z\alpha + 2\,\alpha - 4\,z + 1}{4(\alpha - z + z\alpha)} & \text{otherwise} \end{cases}$$

and B to a profit of

$$\pi_{b,\mathcal{G}_2} = \begin{cases} 0 & \text{if } c \leq \frac{\alpha+\alpha z-1}{2} \\ -\frac{-\alpha^2 + 2\,z\,\alpha^2 - 2\,\alpha - 2\,z\,\alpha + 3\,z^2\alpha^2 - 4\,z^2\alpha + 4\,z - 1}{4(\alpha - z + z\alpha)} & \text{otherwise} \end{cases}$$

3.5.2 Subgame $\mathcal{G}_3$

Proposition 15. *The list $(t_{a,\mathcal{G}_3}(t_b, c), t_{b,\mathcal{G}_3}(c))$ such that*

$$t_{a,\mathcal{G}_3}(c) = \begin{cases} 0 & \textit{if } c \geq \frac{\alpha z - 1 + \alpha}{t_b} \\ 1 - t_b & \textit{if } c \leq t_b(\alpha z - 1 + \alpha) \\ \frac{-ct_b + \sqrt{ct_b(\alpha z + \alpha - 1)}}{c} & \textit{otherwise} \end{cases}$$

and

$$t_{b,\mathcal{G}_3}(t_b, c) = \begin{cases} 1 & \textit{if } c \leq \frac{\alpha z + \alpha - z}{2} \\ \frac{(\alpha\,z + \alpha - z)^2}{4c(\alpha\,z - 1 + \alpha)} & \textit{otherwise} \end{cases}$$

is a subgame perfect Nash equilibrium of $\mathcal{G}_3$.

Proof. See Appendix □

It leads team A to a profit of

$$\pi_{a,\mathcal{G}_3} = \begin{cases} 0 & \text{if } c \leq \frac{\alpha z - 1 + \alpha}{t_b} \\ \frac{\alpha^2 z^2 - 2\,\alpha^2 z - 3\,\alpha^2 + 2\,\alpha\,z^2 + 2\,\alpha\,z + 4\,\alpha + z^2 - 4\,z}{4(\alpha\,z + \alpha - 1)} & \text{otherwise} \end{cases}$$

and B to a profit of

$$\pi_{b,\mathcal{G}_1} = \begin{cases} \alpha - c & \text{if } c \leq \frac{\alpha z - 1 + \alpha}{t_b} \\ \frac{-2\,\alpha^2 z + \alpha^2 + 6\,\alpha\,z + z^2 - 4\,z - 3\,\alpha^2 z^2 + 2\,\alpha\,z^2}{4(\alpha\,z - 1 + \alpha)} & \text{otherwise} \end{cases}$$

3.5.3 Equilibrium of $\mathcal{G}$

Proposition 16. *Assume that no team is allowed to submit a first-stage bid greater than* $\alpha z + \alpha - z$.

For $z = \left\{1, \frac{2\alpha - 1 + \sqrt{5}}{1 - 2\alpha + \sqrt{5}}\right\}$, (c_a, c_b) *such that* $c_a \neq c_b$ *with* $\max\{c_a, c_b\} > \frac{2\alpha - 1}{2}$, $(t_{a,\mathcal{G}_1}(c), t_{b,\mathcal{G}_1}(c))$ *as described in Proposition 13,* $(t_{a,\mathcal{G}_2}(c), t_{b,\mathcal{G}_2}(t_a, c))$ *as described in Proposition 14 and* $(t_{a,\mathcal{G}_3}(t_b, c), t_{b,\mathcal{G}_3}(c))$ *as described in Proposition 15 form a subgame perfect Nash equilibrium of* $\mathcal{G}$.

For $z \in \left(1, \frac{2\alpha - 1 + \sqrt{5}}{1 - 2\alpha + \sqrt{5}}\right)$, (c_a, c_b) *such that* $c_b > \max\left\{c_a, \frac{\alpha z + \alpha - z}{2}\right\}$ *with,* $(t_{a,\mathcal{G}_1}(c), t_{b,\mathcal{G}_1}(c))$ *as described in Proposition 13,* $(t_{a,\mathcal{G}_2}(c), t_{b,\mathcal{G}_2}(t_a, c))$ *as described in Proposition 14 and* $(t_{a,\mathcal{G}_3}(t_b, c), t_{b,\mathcal{G}_3}(c))$ *as described in Proposition 15 form a subgame perfect Nash equilibrium of* $\mathcal{G}$.

For $z \in \left(\frac{2\alpha - 1 + \sqrt{5}}{1 - 2\alpha + \sqrt{5}}, \frac{\alpha}{1 - \alpha}\right)$, (c_a, c_b) *such that* $c_a > \max\left\{c_b, \frac{\alpha z + \alpha - 1}{2}\right\}$ *with,* $(t_{a,\mathcal{G}_1}(c), t_{b,\mathcal{G}_1}(c))$ *as described in Proposition 13,* $(t_{a,\mathcal{G}_2}(c), t_{b,\mathcal{G}_2}(t_a, c))$ *as described in Proposition 14 and* $(t_{a,\mathcal{G}_3}(t_b, c), t_{b,\mathcal{G}_3}(c))$ *as described in Proposition 15 form a subgame perfect Nash equilibrium of* $\mathcal{G}$.

Assuming a cap on c simplifies the proof of proposition 16 greatly. It is possible that proposition 16 may not hold if I do not make this assumption. Nonetheless, this assumption is not only technical. One the one hand, if I considered the unrestricted case, then one would have to analyze a more complex interaction of α and z, and while it may lead to a sharper result, I do not think it would add much to the relevancy of the result. On the other hand, imposing an upper bound on c is validated by real-life professional sports leagues. In all four major professional leagues in North America (Baseball, Basketball, Football and Hockey), there exists some form of salary cap. The ultimate goal of a salary cap is to prevent domination from rich teams. As a matter of fact, in my model, we can see from prositions 14 and 15 that if one team submits a very high value of c, then c and t_a, in the case of A being the leader, for instance, can be such that B's best-response is to choose $t_b = 0$. This is an example of what can be avoided by assuming a upper bound on c.

Proof. See Appendix □

3.6 Discussion

3.6.1 Endogenous leadership

In light of the classic results pertaining to endogenous price leadership, Proposition 16 stands as an intriguing result. In this model, the firm with a revenue advantage ($z > 1$) may act as the follower in equilibrium, if z is not too high. This is perhaps a counter intuitive result. In the context of firms setting selling prices, as opposed to this model in which firms set buying prices, Deneckere & Kovenock (1992) offered an intuitive explanation as to why we should expect large firms (or high-revenue firms) to be price leaders:

> [S]mall firms, ceteris paribus, stand to lose more from being undercut than large firms. Consequently, small firms have a stronger preference than large firms for assuming a followership role in the industry, and choose not to lead or simultaneously set prices.

An intuitive explanation for Proposition 16 would be as follows. The low-revenue firm (Team B), being aware that it cannot raise as much revenue as the high-revenue firm (Team A), is willing to pay a higher unit-cost of talent, as long as it can secure a high enough quantity of talent. It is reminiscent of a team located in a small market that must rely on securing a high winning probability in order to cancel out the negative aspect of earning low profits. It was argued by Van Damme & Hurkens (1999) and in their follow-up paper van Damme & Hurkens (2004) that the firm willing to take the largest risk in waiting shall become the follower, which typically is the low-revenue firm. But we may not apply this rationale to the context of this paper, for general values of α and z. It can be argued that being the follower, in this model, involves taking the risky position of waiting to choose among the residual quantity of talent. On this point, I agree that the riskier position is the follower's position. However, what emerges from our model is that it is in fact the high-revenue firm that ends up taking the risky position in equilibrium if z is relatively small. When z is high, the more intuitive result that the high-revenue firm shall take the lead holds.

3.6.2 Competitive balance

The *competitive balance* is a relative measure of the inequality in talent dispersion across teams. We say that competitive balance reaches its maximal value when all teams own the same quantity of talent. Competitive balance *increases* when the dispersion of talent changes from a relatively unequal state to a relatively *less* unequal state. Inversely, competitive balance decreases when the dispersion of talent changes from a relatively equal state to a relatively less equal state. It is generally accepted that excitement from a professional sports match is increased when the outcome of the match is uncertain which can lead to a higher viewership and higher broadcasting revenues to the league. Although the league itself is not modeled here, we can imagine a scenario in which the league, acting as a revenue maximizer, would want to implement some policy to increase competitive balance.

From proposition 15 and 16, we have that in equilibrium, the distribution of talent is

$$t_a = \frac{(\alpha\, z + z - 2 + \alpha)\,(\alpha\, z + \alpha - z)}{4c\,(\alpha\, z - 1 + \alpha)} \text{ and } t_b = \frac{(\alpha\, z + \alpha - z)^2}{4c\,(\alpha\, z - 1 + \alpha)}$$

if team B is the leader. The competitive balance is in this case

$$\frac{t_a}{t_b} = \frac{\alpha\, z + z - 2 + \alpha}{\alpha\, z + \alpha - z}.$$

When team A is the leader, we get from proposition 14 and 16 that the distribution of talent is

$$t_a = \frac{(\alpha + \alpha z - 1)^2}{4c(\alpha + \alpha z - z)} \text{ and } t_b = \frac{(\alpha + \alpha z - 1)(\alpha + \alpha z + 1 - 2z)}{4c(\alpha + \alpha z - z)}$$

and the competitive balance is

$$\frac{t_a}{t_b} = \frac{\alpha + \alpha z - 1}{\alpha + \alpha z + 1 - 2z}.$$

We can directly see that in both cases, the competitive balance is equal to 1 when $z = 1$. When both teams have the same revenue function, no matter the sharing parameter α, in equilibrium, they acquire the same quantity of talent.

However, for $z > 1$ and for when B is the leader, we have that

$$\frac{\alpha\, z + z - 2 + \alpha}{\alpha\, z + \alpha - z} > \frac{\alpha\, z + z - 2z + \alpha}{\alpha\, z + \alpha - z} = 1.$$

And the greater z is, the greater is t_a/t_b. This satisfies the intuition that a richer team shall acquire more talent in equilibrium. But also, we have that

$$\frac{d}{d\alpha}\left(\frac{t_a}{t_b}\right) = -2\,\frac{(z^2-1)}{(\alpha\, z+\alpha-z)^2} < 0.$$

Remember that when α gets closer to 1, teams share a smaller fraction of their revenue. Consequently, introducing revenue-sharing induces the richer team to acquire relatively more talent in equilibrium, thereby reducing competitive balance.

We can easily see that when $z > 1$ and A is the leader, that $\frac{t_a}{t_b} > 1$ and that

$$\frac{d}{d\alpha}\left(\frac{t_a}{t_b}\right) = -2\,\frac{(z^2-1)}{(\alpha+\alpha z+1-2z)^2} < 0.$$

This result reinforces an earlier result from Szymanski & Késenne (2004), where revenue sharing was also shown to have a negative impact on competitive balance. Following a very different modeling approach, the two models lead to the same conclusion. Interestingly, in my model, no matter which team ends up being the leader in equilibrium, the rich team always acquires more talent than the poor team and a system of revenue-sharing is detrimental to competitive balance.

3.7 Conclusion

In this paper, a formal game-theoretical perspective on professional sports league was considered. The two main results are that 1) contrary to the classic literature on leadership in a duopoly, the efficient team can emerge as the follower in equilibrium and 2) revenue sharing has a negative impact on competitive balance. It is surprising that a modeling approach so different from the usual approach of conjectural variation share the same conclusion on competitive balance. It is also surprising that the first main result stands in clear opposition to the classic results in quantity or price leadership. Future research shall concentrate on whether there is a fundamental reason for the two results. It would be interesting to verify whether these results are robust to the functional form of the revenue functions.

Appendix

Proof of Proposition 13. Assume for the moment that c is such that the solution to both first-order conditions is interior. Thus we have that the solution to the system

$$\frac{\partial}{\partial t_a}\pi_a(t_a, t_b; c, \alpha) = \frac{\partial}{\partial t_b}\pi_b(t_a, t_b; c, \alpha) = 0$$

is

$$t_a^* = \frac{(\alpha - z + \alpha\, z)\,(\alpha\, z - 1 + \alpha)^2}{c\,(z+1)^2\,(2\,\alpha - 1)^2} \text{ and } t_b^* = \frac{(\alpha - z + \alpha\, z)^2\,(\alpha\, z - 1 + \alpha)}{c\,(z+1)^2\,(2\,\alpha - 1)^2}$$

In order for $t_a^* + t_b^* \leq 1$, it must be the case that $c \geq \frac{(\alpha\, z - 1 + \alpha)(\alpha - z + \alpha\, z)}{(z+1)(2\,\alpha - 1)}$. Otherwise, both teams are constrained. It is easy to see that the only solution in this case is that both teams ask for the total supply of talent, which gives them in return both half of the supply. The reason is that if they were not restricted, the equilibrium would be such that $t_a + t_b > 1$. Consequently any outcome of this game is such that at least one of the teams wishes to acquire more talent, or to make a higher demand. Unless both cannot make a higher demand and be strictly better, then we have reached an equilibrium. This is the case when both teams ask for the full supply. In this case, teams would want to acquire more talent but their demand are restricted to be at most one. Thus no teams can be made strictly better off by changing their demand.

This is represented by the constrained optimization problem

$$\begin{aligned} \max_{t_a} \; & \{\pi_a(t_a, t_b; c, \alpha) - \lambda_a(t_a - 1)\} \\ \max_{t_b} \; & \{\pi_b(t_a, t_b; c, \alpha) - \lambda_b(t_b - 1)\} \\ & \lambda_i(t_i - 1) = 0, \; i = a, b \\ & \lambda_i \geq 0, \; i = a, b. \end{aligned}$$

The solution to this system is

$$(t_a, t_b) = (1, 1), \; \lambda_a = \frac{1}{2}(\alpha z + \alpha - 1) - c \text{ and } \lambda_b = \frac{1}{2}(\alpha z + \alpha - z) - c.$$

Since $c < \frac{(\alpha z - 1 + \alpha)(\alpha - z + \alpha z)}{(z+1)(2\alpha - 1)}$, $\alpha > \frac{1}{2}$ and $z \geq 1$, we can verify that $\lambda_a > 0$ and that $\lambda_b > 0$ if $z < \frac{\alpha}{1-\alpha}$. If $z \geq \frac{\alpha}{1-\alpha}$, we may not have $\lambda_b > 0$, and thus an equilibrium may not exist.

□

Proof of Proposition 14. Given t_a and c, an interior solution to B's maximization problem is

$$t_b^* = \frac{-ct_a + \sqrt{ct_a\,(\alpha z + \alpha - z)}}{c}$$

which is well-defined if $\alpha z + \alpha - z \geq 0$. Constraining $t_b \in [0, 1 - t_a]$, we have that

$$\psi_b(t_a, c) = \begin{cases} 0 & \text{if } c \geq \frac{\alpha z + \alpha - z}{t_a} \\ 1 - t_a & \text{if } c \leq t_a(\alpha z + \alpha - z) \\ t_b^* & \textit{otherwise.} \end{cases}$$

Since $c < \alpha z + \alpha - z$, and because A is restricted to choose $t_a \in [0, 1]$, A can either induce B to choose $1 - t_a$ or t_b^*.

For t_a such that $c \leq t_a(\alpha z + \alpha - z)$, B chooses $1 - t_b$, which in turn gives A a profit of

$$\begin{aligned}\pi_a(t_a, 1-t_a) &= \alpha z t_a + (1-\alpha)(1-t_a) - c t_a \\ &= t_a(\alpha(z+1) - (c+1)) + (1-\alpha).\end{aligned}$$

A would then choose $t_a = 1$ as long as $\alpha(z+1) - (c+1) \geq 0$, which is indeed the case when $c < \alpha z + \alpha - z$. We then have that A's profit is equal to $\alpha z - c$.

For t_a such that $c > t_a(\alpha z + \alpha - z)$, B chooses t_b^*, which leads to a more complex objective function

$$\max_{t_a \in [0,1]} \left\{ \alpha \frac{z t_a}{t_a + t_b^*} + (1-\alpha) \frac{t_b^*}{t_a + t_b^*} - c t_a \right\}.$$

The solution is

$$t_a^* = \frac{(\alpha + z\alpha - 1)^2}{4c\,(\alpha - z + z\alpha)}$$

which implies in turn that

$$t_b^* = \frac{(\alpha + z\alpha - 1)\,(\alpha + z\alpha + 1 - 2\,z)}{4c\,(\alpha - z + z\alpha)}$$

for a profit of

$$\pi_a = \frac{\alpha^2 z^2 - 2\,z\alpha^2 - 3\,\alpha^2 + 6\,z\alpha + 2\,\alpha - 4\,z + 1}{4(\alpha - z + z\alpha)}.$$

Now, we need to check what is A's optimal action, depending on c.

Substituting $t_a = t_a^*$ in $t_a(\alpha z + \alpha - z)$ we get

$$\frac{(\alpha + z\alpha - 1)^2}{4c}.$$

This implies that if $c \leq \frac{(\alpha+z\alpha-1)^2}{4c}$ or equivalently if $c \leq \frac{\alpha+\alpha z-1}{2}$, then B will choose $t_b = 1 - t_a^*$. We know that in this case, it is better for A to choose $t_a = 1$ instead. But if $c > \frac{\alpha+\alpha z-1}{2}$, A will choose t_a^* and B will in turn choose t_b^*.

□

Proof of Proposition 15. Given t_b and c, an interior solution to A's maximization problem is

$$t_a^* = \frac{-ct_b + \sqrt{ct_b\ (\alpha z + \alpha - 1)}}{c}$$

which is well-defined if $\alpha z + \alpha - 1 \geq 0$. Constraining $t_a \in [0, 1 - t_b]$, we have that

$$\psi_a(t_b, c) = \begin{cases} 0 & \text{if } c \geq \frac{\alpha z+\alpha-1}{t_b} \\ 1 - t_b & \text{if } c \leq t_a(\alpha z + \alpha - 1) \\ t_a^* & \textit{otherwise.} \end{cases}$$

Since $c < \alpha z + \alpha - z$, and because B is restricted to choose $t_b \in [0, 1]$, B can either induce A to choose $1 - t_b$ or t_a^*.

For t_b such that $c \leq t_b(\alpha z + \alpha - 1)$, A chooses $1 - t_b$, which in turn gives B a profit of

$$\pi_b(1 - t_b, t_b) = t_b(\alpha - (1 - \alpha)z - c) + (1 - \alpha)z.$$

B would then choose $t_b = 1$ as long as $\alpha - (1-\alpha)z - c \geq 0$, which is indeed the case when $c < \alpha z + \alpha - z$. We then have that B's profit is equal to $\alpha - c$.

For t_b such that $c > t_b(\alpha z + \alpha - 1)$, A chooses t_a^*, which leads to a more complex objective function

$$\max_{t_b \in [0,1]} \left\{ \alpha \frac{t_b}{t_a^* + t_b} + (1-\alpha) \frac{z t_a^*}{t_a^* + t_b} - c t_b \right\}.$$

The solution is

$$t_b^* = \frac{(\alpha\, z + \alpha - z)^2}{4c\,(\alpha\, z - 1 + \alpha)}$$

which implies in turn that

$$t_a^* = \frac{(\alpha\, z + z - 2 + \alpha)\,(\alpha\, z + \alpha - z)}{4c\,(\alpha\, z - 1 + \alpha)}$$

for a profit of

$$\pi_b = \frac{-2\,\alpha^2 z + \alpha^2 + 6\,\alpha\, z + z^2 - 4\,z - 3\,\alpha^2 z^2 + 2\,\alpha\, z^2}{4(\alpha\, z - 1 + \alpha)}.$$

Now, we need to check what is A's optimal action, depending on c.

Substituting $t_b = t_b^*$ in $t_b(\alpha z - 1 + \alpha)$, we get

$$\frac{(\alpha\, z + \alpha - z)^2}{4c}.$$

This implies that if $c \leq \frac{(\alpha\, z + \alpha - z)^2}{4c}$ or equivalently if $c \leq \frac{\alpha z + \alpha - z}{2}$, then A will choose $t_a = 1 - t_b^*$. We know that in this case, it is better for B to choose $t_b = 1$ instead. But if $c > \frac{\alpha z + \alpha - z}{2}$, B will choose t_b^* and A will in turn choose t_a^*.

$\square$

Proof of Proposition 16. First, let's show that in equilibrium, $c_a \neq c_b$.

In the equilibrium of the subgame $\mathcal{G}_1$, A's profit is

$$\pi_{a,\mathcal{G}_1} = \frac{(1-\alpha)\,(\alpha^2(z^3+z^2+3z+3)-\alpha(2z^2+4z+2)+2z)}{(z+1)\,(2\,\alpha-1)^2}$$

while B's is

$$\pi_{b,\mathcal{G}_1} = \frac{\alpha^3(z^3+5z^2+3z-1)+\alpha^2(2z^3+8z^2+6z)-\alpha(z^3+4z^2+5z)+z^2+z}{(z+1)\,(2\,\alpha-1)^2}.$$

In an interior solution of $\mathcal{G}_2$, which happens if $c > \frac{\alpha+\alpha z-1}{2}$, A's profit is

$$\pi_{a,\mathcal{G}_2} = \frac{\alpha^2 z^2 - 2\,z\alpha^2 - 3\,\alpha^2 + 6\,z\alpha + 2\,\alpha - 4\,z + 1}{4(\alpha - z + z\alpha)}.$$

In an interior solution of $\mathcal{G}_3$, which happens if $c > \frac{\alpha+\alpha z-z}{2}$, B's profit is

$$\pi_{b,\mathcal{G}_3} = \frac{-2\,\alpha^2 z + \alpha^2 + 6\,\alpha\,z + z^2 - 4\,z - 3\,\alpha^2 z^2 + 2\,\alpha\,z^2}{4(\alpha\,z - 1 + \alpha)}.$$

We have that

$$\pi_{a,\mathcal{G}_1} - \pi_{a,\mathcal{G}_2} = -\frac{\left(1-3\,z+4\,\alpha^2 z+4\,z^2\,(-1+\alpha)^2\right)(\alpha\,z+\alpha-1)^2}{4\,(z+1)\,(2\,\alpha-1)^2\,(\alpha\,z+\alpha-z)},$$

where the roots of $\left(1-3\,z+4\,\alpha^2 z+4\,z^2\,(-1+\alpha)^2\right)$ with respect to α are

$$\frac{2z^2 \pm \sqrt{-z(z-1)^2}}{z(1+z)}.$$

For $z = 1$, the root is 1 and for $z > 1$, the roots do not belong to the real numbers. Since the coefficient before α^2 is positive and the root for $z = 1$ is unique, then it must be the case that $\left(1 - 3\,z + 4\,\alpha^2 z + 4\,z^2\,(-1+\alpha)^2\right) \geq 0$, $\forall z \geq 1$ and $\forall \alpha \in (0.5, 1]$. And thus we have that $\pi_{a,\mathcal{G}_1} - \pi_{a,\mathcal{G}_2} < 0$.

Similarly, we have that

$$\pi_{b,\mathcal{G}_1} - \pi_{b,\mathcal{G}_3} = -\frac{\left(1 + z\left(1 + 4\,\alpha^2 z + 4\,\alpha^2 - 8\,\alpha\right)\right)\left(\alpha\,z + \alpha - z\right)^2}{4\left(z+1\right)\left(2\,\alpha - 1\right)^2\left(\alpha\,z + \alpha - 1\right)}.$$

Since $z \geq 1$, we have that

$$1 + z\left(1 + 4\,\alpha^2 z + 4\,\alpha^2 - 8\,\alpha\right) \geq 2 + 8(\alpha^2 - \alpha),$$

and because $\alpha \in (0.5, 1]$, $2 + 8(\alpha^2 - \alpha) \geq 0$, and thus we get that $\pi_{b,\mathcal{G}_1} - \pi_{b,\mathcal{G}_3} < 0$.

Thus, when in $\mathcal{G}_1$, both teams have an incentive to deviate by increasing c sufficiently, to attain the subgame in which they are the leader.

Second, we have that

$$\pi_{a,\mathcal{G}_3} = \frac{\alpha^2 z^2 - 2\,\alpha^2 z - 3\,\alpha^2 + 2\,\alpha\,z^2 + 2\,\alpha\,z + 4\,\alpha + z^2 - 4\,z}{4(\alpha\,z + \alpha - 1)}.$$

and

$$\pi_{a,\mathcal{G}_3} - \pi_{a,\mathcal{G}_2} \;=\; \frac{\left(z-1\right)\left(\left(\alpha^2 - \alpha - 1\right)\left(z^2 + 2\,z + 1\right) + 5\,z\right)}{4\left(\alpha + z\,\alpha - 1\right)\left(\alpha - z + z\,\alpha\right)}.$$

We also have that

$$\pi_{b,\mathcal{G}_2} = -\frac{-\alpha^2 + 2\,z\,\alpha^2 - 2\,\alpha - 2\,z\,\alpha + 3\,z^2\alpha^2 - 4\,z^2\alpha + 4\,z - 1}{4(\alpha - z + z\alpha)}$$

and

$$\pi_{b,\mathcal{G}_2} - \pi_{b,\mathcal{G}_3} \quad = \quad -\frac{(z-1)\left(\left(\alpha^2-\alpha-1\right)\left(z^2+2\,z+1\right)+5\,z\right)}{4\left(\alpha+z\,\alpha-1\right)\left(\alpha-z+z\,\alpha\right)}.$$

Interestingly, we get that

$$\pi_{b,\mathcal{G}_2} - \pi_{b,\mathcal{G}_3} = -(\pi_{a,\mathcal{G}_3} - \pi_{a,\mathcal{G}_2}).$$

What it means is that whenever B has an incentive to deviate from subgame $\mathcal{G}_2$ to $\mathcal{G}_3$, (when $\pi_{b,\mathcal{G}_2} - \pi_{b,\mathcal{G}_3} < 0$), then it must be the case that A does not have and incentive to deviate from subgame $\mathcal{G}_3$ to $\mathcal{G}_2$ ($\pi_{a,\mathcal{G}_3} - \pi_{a,\mathcal{G}_2} > 0$) and vice versa. Then finding an equilibrium become simple: if $\pi_{b,\mathcal{G}_2} - \pi_{b,\mathcal{G}_3} < 0$, then the equilibrium will be such that $\mathcal{G}_3$ is on the equilibrium path. Otherwise, it is $\mathcal{G}_2$ that is on the equilibrium path.

In the case of $c_a < c_b$, when $c_b \leq \frac{\alpha+\alpha z-z}{2}$, A gets no profit. It is obvious that A can deviate by selecting c_a high enough to get profit from an interior solution of $\mathcal{G}_2$. If $\mathcal{G}_3$ is on the equilibrium path, then it must be the case that $c > \frac{\alpha+\alpha z-z}{2}$.

In the case of $c_a > c_b$, when $c_a \leq \frac{\alpha+\alpha z-1}{2}$, B gets no profit. It is obvious that B can deviate by selecting c_b high enough to get profit from an interior solution of $\mathcal{G}_3$. If $\mathcal{G}_2$ is on the equilibrium path, then it must be the case that $c > \frac{\alpha+\alpha z-1}{2}$.

Since, $z \geq 1$, there will be an equilibrium only if $c \geq \frac{\alpha+\alpha z-1}{2}$. Consider $\pi_{b,\mathcal{G}_2} - \pi_{b,\mathcal{G}_3}$. We have that the root of $\left(\alpha^2-\alpha-1\right)\left(z^2+2\,z+1\right)+5\,z$ with respect to z that is greater than 1 and is the largest of the two roots, is

$$\frac{2\alpha-1+\sqrt{5}}{1-2\alpha+\sqrt{5}}.$$

Then, given a certain $\alpha \in (0.5, 1]$, since the coefficient before z^2 is negative, if $z \leq \frac{2\alpha-1+\sqrt{5}}{1-2\alpha+\sqrt{5}}$ then $\pi_{b,\mathcal{G}_2} - \pi_{b,\mathcal{G}_3} \leq 0$, which means that $\mathcal{G}_3$ is on the equilibrium path. Otherwise, $\mathcal{G}_2$ is on the equilibrium path.

$\square$

Conclusion

This completes my investigation on the theory of contests. I have conducted research in the areas of group contests with private information, of mechanism design theory and of sports economics. Each paper can eventually be extended, cut, re-formulated or re-oriented for an eventual publication. For example, in the first chapter, the model studied could be included in a greater model in which contestants form groups in a first stage, and then compete in the second stage. It could also be interesting to investigate the possibility of sharing full or partial information within groups. As it stands in this thesis, I view that model as a benchmark to which many features can be added.

In the second chapter, it seems logical to consider a model in which candidates's fixed cost depend not only on their revelation, but also on all other revelations. This would bring the model closer to a more standard mechanism design model. And in the third chapter, it would appeal to a greater audience if I could generalize this model, and not restrict myself only to sports economics. In some sense, I provide a counterexample to the usual result that efficient firms shall take the leadership in a sequential game. This result is important and should be carried over to a more general area of economics, such as industrial organization. Ideally, one must find a narrative that supports my model to compel economists in general.

The body of this thesis was crafted around a simple contest model and, in the end, it took me six years of hard work to lay it down on paper. I hope that, as marginal a contribution it turns out to be, this thesis offers a mature and intelligible discussion on the theory of contests.

Bibliography

Amegashie, J Atsu. 2000. Some results on rent-seeking contests with short-listing. *Public Choice*, **105**(3-4), 245–253.

Baik, Kyung Hwan. 1993. Effort levels in contests: The public-good prize case. *Economics Letters*, **41**(4), 363–367.

Baik, Kyung Hwan. 2008. Contests with group-specific public-good prizes. *Social Choice and Welfare*, **30**(1), 103–117.

Baik, Kyung Hwan, & Shogren, Jason F. 1998. A behavioral basis for best-shot public-good contests. *Advances in Applied Microeconomics: Contests*, **7**, 169–178.

Baik, Kyung Hwan, Kim, In-Gyu, & Na, Sunghyun. 2001. Bidding for a group-specific public-good prize. *Journal of Public Economics*, **82**(3), 415–429.

Barbieri, Stefano, & Malueg, David A. 2014. Private-Information Group Contests. *Available at SSRN 2501771*.

Barbieri, Stefano, Malueg, David A, & Topolyan, Iryna. 2013. The Best-Shot All-Pay (Group) Auction with Complete Information. *Available at SSRN 2326211*.

Baye, Michael R, Kovenock, Dan, & De Vries, Casper G. 1993. Rigging the lobbying process: an application of the all-pay auction. *The american* *economic review*, 289–294.

Bloch, Francis, & Zenginobuz, Unal. 2007. The effect of spillovers on the provision of local public goods. *Review of Economic Design*, **11**(3), 199–216.

Borgers, Tilman, Strausz, Roland, & Krahmer, Daniel. 2015. *An introduction to the theory of mechanism design*. Oxford University Press, USA.

Brookins, Philip, & Ryvkin, Dmitry. 2014. *Equilibrium existence in group contests of incomplete information*. Tech. rept.

Chamberlin, John. 1974. Provision of collective goods as a function of group size. *American Political Science Review*, **68**(02), 707–716.

Che, Yeon-Koo, & Gale, Ian. 2003. Optimal design of research contests. *The American Economic Review*, **93**(3), 646–671.

Chowdhury, Subhasish M, Lee, Dongryul, & Sheremeta, Roman M. 2013. Top guns may not fire: Best-shot group contests with group-specific public good prizes. *Journal of Economic Behavior & Organization*, **92**, 94–103.

Corchón, Luis C. 2007. The theory of contests: a survey. *Review of Economic Design*, **11**(2), 69–100.

Deneckere, Raymond J, & Kovenock, Dan. 1992. Price leadership. *The Review of Economic Studies*, **59**(1), 143–162.

Dubey, Pradeep. 2013. The role of information in contests. *Economics Letters*, **120**(2), 160–163.

Einy, Ezra, Haimanko, Ori, Moreno, Diego, Sela, Aner, & Shitovitz, Benyamin. 2013. Tullock contests with asymmetric information.

Epstein, Gil S, & Mealem, Yosef. 2009. Group specific public goods, orchestration of interest groups with free riding. *Public Choice*, **139**(3-4), 357–369.

Esteban, Joan, & Ray, Debraj. 2001. Collective Action and the Group Size Paradox. *American Political Science Review*, **95**(3).

Fey, Mark. 2008. Rent-seeking contests with incomplete information. *Public Choice*, **135**(3-4), 225–236.

Fu, Qiang, & Lu, Jingfeng. 2010. Contest design and optimal endogenous entry. *Economic Inquiry*, **48**(1), 80–88.

Fu, Qiang, & Lu, Jingfeng. 2012. The optimal multi-stage contest. *Economic Theory*, **51**(2), 351–382.

Fu, Qiang, Jiao, Qian, & Lu, Jingfeng. 2015. Contests with endogenous entry. *International Journal of Game Theory*, **44**(2), 387–424.

Fullerton, Richard L, & McAfee, R Preston. 1999. Auctionin entry into tournaments. *Journal of Political Economy*, **107**(3), 573–605.

Giebe, Thomas. 2014. Innovation contests with entry auction. *Journal of Mathematical Economics*, **55**, 165–176.

Gradstein, Mark, & Konrad, Kai A. 2001. Orchestrating rent seeking contests. *Pages 361–369 of: Efficient Rent-Seeking*. Springer.

Hamilton, Jonathan H, & Slutsky, Steven M. 1990. Endogenous timing in duopoly games: Stackelberg or Cournot equilibria. *Games and Economic Behavior*, **2**(1), 29–46.

Harstad, Ronald M. 1995. Privately informed seekers of an uncertain rent. *Public Choice*, **83**(1-2), 81–93.

Hurley, Terrance M, & Shogren, Jason F. 1998a. Effort levels in a Cournot Nash contest with asymmetric information. *Journal of Public Economics*, **69**(2), 195–210.

Hurley, Terrance M, & Shogren, Jason F. 1998b. Asymmetric information contests. *European Journal of Political Economy*, **14**(4), 645–665.

Jackson, Matthew, & Moulin, Hervé. 1992. Implementing a public project and distributing its cost. *Journal of Economic Theory*, **57**(1), 125 – 140.

Kaplan, Todd R, & Sela, Aner. 2010. Effective contests. *Economics Letters*, **106**(1), 38–41.

Katz, Eliakim, Nitzan, Shmuel, & Rosenberg, Jacob. 1990. Rent-seeking for pure public goods. *Public Choice*, **65**(1), 49–60.

Késenne, Stefan. 2007. *The Economic Theory of Professional Team Sports: An Analytical Treatment*. New Horizons in the Economics of Sport Series. Edward Elgar Publishing, Incorporated.

Kolmar, Martin, & Rommeswinkel, Hendrik. 2013. Contests with group-specific public goods and complementarities in efforts. *Journal of Economic Behavior & Organization*, **89**, 9–22.

Konrad, Kai A. 2009. Strategy and dynamics in contests. *OUP Catalogue*.

Lai, Ernest K, & Matros, Alexander. 2007. *Sequential contests with ability revelation*. Tech. rept. Mimeo.

Lee, Dongryul. 2012. Weakest-link contests with group-specific public good prizes. *European Journal of Political Economy*, **28**(2), 238–248.

Long, Ngo Van. 2013. The theory of contests: a unified model and review of the literature. *European Journal of Political Economy*, **32**, 161–181.

Madden, Paul. 2011. Game Theoretic Analysis of Basic Team Sports Leagues. *Journal of Sports Economics*, **12**(4), 407–431.

Malueg, David A, & Yates, Andrew J. 2004. Sent Seeking With Private Values. *Public Choice*, **119**(1-2), 161–178.

McGuire, Martin. 1974. Group size, group homo-geneity, and the aggregate provision of a pure public good under cournot behavior. *Public Choice*, **18**(1), 107–126.

Milgrom, Paul Robert. 2004. *Putting auction theory to work*. Cambridge University Press.

Moldovanu, Benny, & Sela, Aner. 2006. Contest architecture. *Journal of Economic Theory*, **126**(1), 70–96.

Nitzan, Shmuel. 1994. Modelling rent-seeking contests. *European Journal of Political Economy*, **10**(1), 41–60.

Nti, Kofi O. 1998. Effort and performance in group contests. *European Journal of Political Economy*, **14**(4), 769–781.

Olson, Mancur. 1965. *The logic of collective action*. Harvard University Press.

Pecorino, Paul, & Temimi, Akram. 2008. The Group Size Paradox Revisited. *Journal of Public Economic Theory*, **10**(5), 785–799.

Pogrebna, Ganna. 2008. Learning the type of the opponent in imperfectly discriminating contests with asymmetric information. *Available at SSRN 1259329*.

Polishchuk, Leonid, & Tonis, Alexander. 2013. Endogenous contest success functions: a mechanism design approach. *Economic Theory*, **52**(1), 271–297.

Riaz, Khalid, Shogren, Jason F, & Johnson, Stanley R. 1995. A general model of rent seeking for public goods. *Public Choice*, **82**(3-4), 243–259.

Ryvkin, Dmitry. 2010. Contests with private costs: Beyond two players. *European Journal of Political Economy*, **26**(4), 558–567.

Schoonbeek, Lambert, & Winkel, Barbara M. 2006. Activity and inactivity in a rent-seeking contest with private information. *Public Choice*, **127**(1-2), 123–132.

Sui, Yong. 2009. Rent-seeking contests with private values and resale. *Public Choice*, **138**(3-4), 409–422.

Szymanski, Stefan. 2003. The economic design of sporting contests. *Journal of economic literature*, 1137–1187.

Szymanski, Stefan. 2004. Professional Team Sports Are Only a Game: The Walrasian Fixed-Supply Conjecture Model, Contest-Nash Equilibrium, and the Invariance Principle. *Journal of Sports Economics*, **5**(2), 111–126.

Szymanski, Stefan. 2013. Some Observations on Fort and Winfree Nash Conjectures and Talent Supply in Sports League Modeling: A Comment on Current Modeling Disagreements. *Journal of Sports Economics*, **14**(3), 321–326.

Szymanski, Stefan, & Késenne, Stefan. 2004. Competitive Balance and Gate Revenue Sharing in Team Sports. *The Journal of Industrial Economics*, **52**(1), pp. 165–177.

Taylor, Curtis R. 1995. Digging for golden carrots: an analysis of research tournaments. *The American Economic Review*, 872–890.

Topolyan, Iryna. 2014. Rent-seeking for a public good with additive contributions. *Social Choice and Welfare*, **42**(2), 465–476.

Tullock, G. 1980. Efficient rent seeking. *In:* J.M. Buchanan, R.D. Tollison, & Tullock, G. (eds), *Toward a theory of the rent-seeking society*. College Station: Texas A&M University Press.

Ursprung, Heinrich W. 1990. Public Goods, Rent Dissipation, and Candidate Competition. *Economics & Politics*, **2**(2), 115–132.

Van Damme, Eric, & Hurkens, Sjaak. 1999. Endogenous stackelberg leadership. *Games and Economic Behavior*, **28**(1), 105–129.

van Damme, Eric, & Hurkens, Sjaak. 2004. Endogenous price leadership. *Games and Economic Behavior*, **47**(2), 404–420.

von Stengel, Bernhard. 2010. Follower payoffs in symmetric duopoly games. *Games and Economic Behavior*, **69**(2), 512–516.

Wärneryd, Karl. 2003. Information in conflicts. *Journal of Economic Theory*, **110**(1), 121–136.

Wasser, Cédric. 2013. Incomplete information in rent-seeking contests. *Economic Theory*, **53**(1), 239–268.

Zhang, Jun, & Wang, Ruqu. 2009. The Role of Information Revelation in Elimination Contests*. *The Economic Journal*, **119**(536), 613–641.