On the Spectral Synthesis Property

and Its Application to PDE

by

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A thesis submitted to the faculty of

Graduate Studies and Research in partial fulfilment of

the requirements for The Degree of Doctor of Philosophy

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March, 1989

Acknowledgements

The author wishes to express his deepest gratitude to his supervisor, Professor S. W. Drury, for the help from him to understand the problem considered in this thesis and for his encouragement and advice and especially for suggesting this interesting problem. The author thanks Professor B. Marshall and Professor P. Koosis for the help from them during the preparation of the work.

Abstract

Let M be a k-dimensional manifold in R^n. Using a new idea, we extend the known result of Y. Domar on the weak spectral synthesis property of M when $k = n - 1$ by reducing the smoothness assumption upon M. For the case $k \leq n - 2$, the only known result is the curve in R^3. We discuss the situation when M is 2-dimensional quadratic manifold in R^4, and the same result as in the $(n-1)$-dimensional case is proved.

The interesting point of our method is its application to solving the uniqueness problem for some partial differential equations. Combining the Beurling-Pollard technique with our method, we can slightly improve Hörmander's result for a class of elliptic equations.

Résumé

Soit M une variété k-dimensionelle dans R^n. Utilisant une nouvelle idée, nous étendons le résultat connu de Y. Domar sur la propriété faible de synthèse spectrale de M quand $k = n-1$ en réduisant les conditions de lissage de M. Pour le cas $k \le n-2$, le seul résultat connu est la courbe dans R^3. Nous discutons du cas où M est une variété quadratique 2-dimensionelle dans R^4 et le même résultat que dans le cas d'une variété $(n-1)$-dimensionelle est prouvé.

Le point d'intérèt de notre méthode est son application à la solution du problème d'unicité de certaines équations aux dérivés partielles. En combinant la technique de Beurling-Pollard avec notre méthode, nous pouvons améliorer quelque peu le résultat d'Hörmander pour une classe d'équations elliptiques

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Chapter 1

Introduction

Let $S(R^n)$ be the space of Schwartz class functions and $S'(R^n)$ be the dual space of $S(R^n)$. Given T $\in S'(R^n)$, denote its Fourier transform by $\hat{T}$. We know that $L^p(R^n) \subset S'(R^n)$ for $1 \leq p \leq \infty$. Also as usual we denote the support of T in the distributional sense by supp(T).

Let $FL^p(R^n) = \{\hat{f}: f \in L^p(R^n), \|\hat{f}\|_{FL^p} = \|f\|_{L^p}\}, \ 1 \leq p \leq \infty$

It is well known that FL^1 is a Banach algebra and FL^q is the dual space of FL^p for $\frac{1}{p} + \frac{1}{q} = 1, 1 \leq p < \infty$. For a closed subset M in R^n, we denote

$$I(M) = \{f \in FL^1(R^n), f(M) = 0\}$$

$$J(M) = \{f \in C_0^\infty(R^n), f(M) = 0\}$$

$$K(M) = \{f \in C_0^\infty(R^n), supp(f) \cap M = \emptyset\}$$

We know that I(M) is a closed ideal in FL^1 and it is obvious to see that

$$\overline{K(M)} \subseteq \overline{J(M)} \subseteq I(M) \text{ in } FL^1 \text{ norm.}$$

We state the following two properties.

(A) : If $\overline{K(M)} = I(M)$, we say that M is of spectral synthesis

(B) : If $\overline{J(M)} = I(M)$, we say that M is of weak spectral synthesis.

For $M = \overline{B^n}$, the closed unit ball in R^n, (A) holds(see [6]).

For S^{n-1} in $R^n, n \geq 3$, L. Schwartz [20] showed that (A) does not hold.

C. Herz [12] proved that for S^1 in R^2, (A) holds Herz's idea is based on the fact that the unit circle is the orbit of the rotation group in R^2. Also the Beurling-Pollard argument (see chapter 3)used in [12] reveals that for a 1-dimensional manifold in R^n , the properties (A) and (B) are equivalent.

Adopting a method similar to Herz [12], Varopoulos [25] proved that for $S^{n-1}, n \geq 3$, (B) holds.

For M a general hypersurface, Y. Domar([3], [4]) obtained the following results:

(a) If M is a compact C^2 curve in R^2 with non-vanishing curvature, then (A) holds

(b) If M is a compact C^∞ (n-1)-dimensional manifold in $R^n (n \geq 3)$ with non-vanishing Gaussian curvature, then (B) holds.

We refer the reader to Domar's survey paper [6] for more information on the spectral synthesis property.

By a partition of unity and from the curvature assumption, we can reduce the discussion to the case where M is of the form E $= \{(x, \psi(x)), x \in U\}, U$ a small open ball in R^{n-1}. Here $\psi(x)$ is a real-valued function defined on U such that $\psi(x) \in C^k(U)$, k an integer (≥ 2) to be fixed later, and such that the inverse of the Hessian determinant $|(\frac{\partial^2 \psi}{\partial x_i \partial x_j})|$ exists and is bounded in U. Domar starts his proof by *convolving T with* *a nice function* ϕ_h *along* E to obtain a nice measure T_h on E(discussed in chapter 2)

Using a little functional analysis(see chapter 5), we can derive the property (B) from the following statement.

Given T $\in FL^\infty(R^n)$ vanishing on J(E) and T_h mentioned as above, we have

$$(i)\quad \|\hat{T}_h\|_\infty \leq C\|\hat{T}\|_\infty, \text{ C independent of h}$$

$$(ii)\quad \langle T_h, f\rangle \longrightarrow \langle T, f\rangle \text{ as h } \to 0 \text{ for all } f \in S(R^n).$$

In chapter 5, we will see that it is easy to check

$$\langle T_h, f\rangle \longrightarrow \langle T, f\rangle \text{ as h } \to 0 \text{ for f } \in S(R^n).$$

So (B) will be proved if we can show $\|\hat{T}_h\|_\infty \leq C\|\hat{T}\|_\infty$, C independent of h

In the case n=2, $E = \{(x, \psi(x)), x \in U\}, U$ a small open interval in R^1, $\psi(x) \in C^2(R^2)$. Let $(x, z) \in R \times R$. Beurling-Pollard technique implies that if $T \in FL^\infty(R^2)$ is supported in E, then $\langle T, f\rangle = 0$ for $f \in C_0^2(R^2)$, vanishing on E. More precisely for $g(x) \in C_0^2(U)$ and $f(x, z) = (e^{iz} - e^{i\psi(x)})g(x)$, Beurling Pollard technique yield $\langle T, f\rangle = 0$. This fact is crucial in Domar's proof of (a). For the case $n \geq 3$, the similar result does not hold even for S^{n-1}(see [12]), for otherwise the argument in [1] would imply that (A) holds for S^{n-1} which contradicts the Schwartz's counter example.

So for the case $n \geq 3$, the natural approach([4], [25]) is to assume suitable Beurling Pollard type conditions and obtain properties weaker than the spectral synthesis property. Let $(x, z) \in R^{n-1} \times R$. If E is a C^∞ (n-1)-dimensional manifold in R^n, then

$\psi(x) \in C^\infty(U)$ and hence for $g(x) \in C_0^\infty(U)$, the function $f(x,z) = (e^{iz} - e^{i\psi(x)})g(x)$ can be viewed as a function belonging to $C_0^\infty(R^n)$ which vanishes on E. For $T \in FL^\infty(R^n)$ supported on E, assuming the Beurling-Pollard type condition on E is amounts to having $\langle T, f\rangle = 0$ for the above f(x.z) Thus for a general compact C^∞ (n-1)-dimensional manifold M, the Beurling-Pollard type condition on M should be that we only consider those $T \in FL^\infty(R^n)$ supported on M such that $\langle T, f\rangle = 0$ for $f \in C_0^\infty(R^n)$ vanishing on M By duality this consideration derives Domar's result (b).

If $E = \{(x, \psi(x)), x \in U\}, U$ a small open ball in R^{n-1} , is a C^m hypersurface, that is, $\psi(x) \in C^m(U)$, then the function $f(x,z) = (e^{iz} - e^{i\psi(x)})g(x)$ for $g \in C_0^m(U)$ is a $C_0^m(R^n)$ function. So the Beurling-Pollard type condition on a general C^m (n-1)-dimensional manifold M is that we only consider those $T \in FL^\infty(R^n)$ supported on M such that $\langle T, f\rangle = 0$ for $f \in C_0^m(R^n)$, vanishing on M.

We let

$$J^m(M) = \{f \in C_0^m(R^n), f(M) = 0\}$$

and state the following property.

(C) : If $\overline{J^m(M)} = I(M)$ in FL^1 norm. we say that M is of m-spectral synthesis

As was pointed out by Domar([4], p.25, line 1), the method in [4] can also get the result for the manifold with the differentiability up to a certain order. Indeed, his method yields for $n \geq 3$ and $k \geq 2n+1$ that If M is a compact C^k (n-1)-dimensinal manifold in R^n with non-vanishing Gaussian curvature, then (C) holds with m=k.

The C^{2n+1} smoothness assumption in the above result is too strong if we compare it with the case n=2. The reason is that when n=2, Domar is able to prove (i) by using Carleson's inequality, Van der Corput's lemma while when $n \geq 3$ Domar bases his proof of (i) on the modified Littman's estimate([4], lemma 1) , which forces him to assume $\psi \in C^{2n+1}(U)$.

The main result in this thesis is the following theorem.

<u>Theorem 1</u> Let k be a positive integer such that $k \geq n+2$ and let E, ψ be as above. Let $T \in FL^{\infty}(R^n)$ supported on E and T_h as mentioned above. Assume $\psi(x) \in C^k(U)$ and T vanishes on $J^k(M)$. Let $(\eta, \xi) \in R^{n-1} \times R$. If we set $M_{2,\eta}(\hat{T})(\eta,\xi) = (\sup_{r>0} \frac{1}{m(B_r(\eta))} \int_{B_r(\eta)} |\hat{T}(u,\xi)|^2 du)^{\frac{1}{2}}$, then we have

$$|\hat{T}_h(\eta,\xi)| \leq C M_{2,\eta}(\hat{T})(\eta,\xi),$$

where C is independent of η, ξ, and h.

<u>Remark 1</u>: The curvature condition in theorem 1 can not be removed completely. An example is given in chapter 5.

As a corollary of theorem 1, we can extend Domar's result mentioned above.

<u>Theorem 2</u> Let $k \geq n+2$. If M is a compact C^k (n-1)-dimensional manifold in R^n with non-vanishing Gaussian curvature, then (C) holds with m=k.

<u>Remark 2</u>: As in [4], the manifold M in theorem 2 can be replaced by a compact subset E of M if E has the so called restriction cone property. The details can be found

in [4].

The following argument will show that for any positive integer m the property (B) is equivalent to the property (A) for some compact C^m (n-1)-dimensional manifold in R^n.

For a small ball U in R^{n-1} Choose $\psi(x)$ such that at any point in U, ψ is only differentiable up to a finite order ($\geq m$). Let $E = \{(x, \psi(x)), x \in U\}$ and fix a point $s_0 = (x_0, \psi(x_0)) \in E$. Using an affine transformation, we may assume $\nabla\psi(x_0) = (0, \ldots, 0)$. For $f \in C_0^\infty(R^n)$ vanishing on E, that is, $f(x, \psi(x)) = 0$ for $x \in U$, we let $H(x) = f(x, \psi(x))$. Then H is identically zero in U and hence by the chain rule we have for $1 \leq i \leq n-1$

$$\begin{aligned} 0 = H'_{x_i}(x_0) &= f'_{x_i}(x_0, \psi(x_0)) + f'_z(x_0, \psi(x_0) \cdot \psi' x_i(x_0) \\ &= f'_{x_i}(x_0, \psi(x_0)). \end{aligned}$$

So f'_{x_i} vanishes at s_0 for $i = 1, \ldots, n-1$. If $f'_z(x_0, \psi(x_0)) \neq 0$, then the implicit function theorem implies that ψ is C^∞ smooth at x_0 since f is C^∞ smooth. This contradicts our smoothness assumption of ψ at x_0 and hence all the first derivatives of f at s_0 are zero. Since $s_0 \in E$ is arbitrary, we see that all the first derivatives of f vanish on E. By a standard inductive argument we can conclude that for $f \in C_0^\infty(R^n)$, f vanishes on E implies all the derivatives of f vanish on E. From the remark 3.4 in chapter 3, we will see that for this manifold E the property (B) is equivalent to (A).

The above consideration verifies that there is no hope to prove the property (B) for a compact C^m (n-1)-dimensional manifold in $R^n (n \geq 3)$ for any finite m.

In the following L(p,q) will stands for the so called Lorentz spaces which will be discussed in chapter 3.

Let u be a solution of any partial differential equation with constant coeffients. Assume that the support of $\hat{u}$ is contained in a subset M of R^n with measure zero. If $u \in L^p, 1 \leq p \leq 2$, then from Hausdorff-Young's theorem(see chapter 2), $\hat{u}$ is a measurable function in R^n and hence is zero since M has measure zero. Thus $u = 0$ in this case.

If we consider the problem in Lorentz spaces, then by a routine interpolation argument(see chapter 3), we obtain that if the support of $\hat{u}$ is contained in M with measure zero and if $u \in L(p,q)$, $1 < p < 2$, $1 \leq q \leq \infty$, then $\hat{u} = 0$ and hence $u = 0$.

Using the Beurling-Pollard technique, we can prove the following result.

<u>Theorem 3</u> Let $T \in S'(R^n)$ and M be a k-dimensional manifold in R^n with area. If $supp(T) \subset M$ and $\hat{T} \in L(p,q)$ for $2 \leq p < \frac{2n}{k}$, $1 \leq q \leq \infty$, then T = 0.

Combining theorem 3 with theorem 1, we have the following result which slightly improves one of the Hörmander's results in [14] for the case M has non vanishing Gaussian curvature. The interested reader should compare the two different approaches.

<u>Theorem 4</u> Let M be a C^{n+2} (n-1)-dimensional manifold in R^n with non vanishing Gaussian curvature and $T \in S'(R^n)$ with $supp(T) \subset M$. If $\hat{T} \in L(p,q)$ for $2 < p < \frac{2n}{n-1}, 1 \leq q \leq \infty$; or $p = \frac{2n}{n-1}, 1 \leq q < \infty$, then T=0.

Note that in theorem 4, M need not be compact. Also from Littman's estimate in [18](see chapter 5), it is easy to see that given $\frac{2n}{n-1} < p \leq \infty, 1 \leq q \leq \infty$; or $p = \frac{2n}{n-1}, q = \infty$, we can find $T \in S'(R^n)$ with $supp(T) \subset M$ such that $\hat{T} \in L(p,q)$ and $T \not\equiv 0$. Thus the result in theorem 4 is optimal.

Remark 3: Hörmander's method in [14] cannot cover the case $p = 2, 2 < q \leq \infty$ since he assumes $\hat{T} \in L^2_{loc}(R^n)$ and uses the Plancherel's theorem. Also the approach in the proof of theorem 4 can be used to give more information about the solutions of some partial differential equations The interested reader is refered to [11] for details.

Applying theorem 4 to partial differential equations, we have the following simple example.

Example 1.1: Let u be a solution in the distributional sense of Helmholtz equation $\Delta u + u = 0$ in R^n. If $u \in L(p,q)$ for p,q in the range as mentioned in theorem 4, then u= 0.

The organization of this thesis is as follows.

In chapter 2 we recall some material which are standard in harmonic analysis, functional analysis and differential geometry.

In chapter 3 , we introduce the Lorentz spaces and the interpolation theorems. Then we put the Beurling-Pollard technique on the frame of Lorentz spaces to prove theorem 3 One of the interpolation result stated in this chapter will play a role in the proof of theorem 1.

We will prove theorem 1, 2 , and 4 altogether in chapter 4 and 5. Chapter 4 supplies some basic estimates used in chapter 5 for the proof of the theorems. Lemma 4.1 is a variation of the well known estimate of the Hardy-Littlewood maximal function while lemma 4.2 gives an estimate for the convolution operator via the maximal function defined in lemma 4.1 . Lemma 4.3 is the modification of Hormander's result ([13], theorem 1) which can be viewed as the substitution for the Littman's estimate used by Domar [4].

In chapter 6, we will come back to the spectral synthesis problem. There is no general result for M to be a k-dimensional manifold in R^n with $1 < k < n - 2$. Domar [5] proved that (A) holds for the compact C^3 curves in R^3 with non vanishing torsion. It is a little surprising to the author that the case of 2 dimensional quadratic surface in R^4 is not as difficult as expected. Using only some elementary linear algebra and the same argument as in the proof of our theorem 2, we can show that (B) holds for almost all 2-dimensional quadratic surfaces in R^4.

In the last chapter, we will discuss some interesting problems which arise logically from our results and which cannot be solved with the same method developed in this thesis. The author hopes that he can find the way to attack these problems in the future.

Chapter 2

Preliminary

In this chapter, we want to quote some basic material in functional analysis, harmonic analysis and diffrential geometry. They are standard and can be found, for example, in [26], [22], [9] and [16]. At the end, we introduce a nice construction of a family of *good measures* from a given distribution. This construction is due to Domar(see [3], or [4]) and is the starting point of our discussion.

Let $f(x)$ be a complex-valued function on R^n. The support of f is defined by $supp(f) = \overline{\{x, f(x) \neq 0\}}$. Note that $supp(f)$ is always a closed set in R^n.

Let $C^\infty(R^n)$ be the set of functions on R^n such that all the partial derivatives of $f(x)$ of all orders exist and are continuous. Denote $C_o^\infty(R^n)$ as the set of all functions $f \in C^\infty$ such that $supp(f)$ is compact.

The Schwartz class of functions, $S(R^n)$, is the class of all those C^∞ functions f such that

$$\sup_{x \in R^n} |x^\alpha(D^\beta f(x))| < \infty,$$

for all n-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ of non-negative integers. Here $D^\beta f(x) = \frac{\partial^{\beta_1+\beta_2+\cdots+\beta_n}}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \cdots \partial x_n^{\beta_n}} f(x_1, x_2, \ldots, x_n)$

For $f \in L^1(R^n)$, we define the Fourier transform of f by

$$\hat{f}(\xi) = \int_{R^n} f(x) e^{-i\xi \cdot x} dx \tag{1}$$

It is obvious to see that

$$\|\hat{f}\|_\infty \le \|f\|_1 \tag{2}$$

Also the Fourier transform operator is well-defined on $L^2(R^n)$ and the Plancherel theorem says

$$\|\hat{f}\|_2 = C\|f\|_2. \tag{3}$$

Here C is a constant only depending on the dimension n.

The corollary of the estimates (2) and (3) gives the following Hausdorff-Young theorem by using the well-known interpolation argument between L^p spaces

Hausdorff-Young Theorem If $f \in L^p(R^n), 1 \le p \le 2$, then

$$\|\hat{f}\|_q \le C\|f\|_p, \text{ for } \frac{1}{p} + \frac{1}{q} = 1, \tag{4}$$

The Fourier transform is invertible in the following sense.

The Inversion Theorem If $f \in L^2$ and $\hat{f} \in L^1$, then

$$f(x) \overset{a.e.}{=} (2\pi)^{-n} \int_{R^n} e^{ix\cdot\xi} \hat{f}(\xi) d\xi.$$

The very simple but useful property of the Fourier transform is

$$(i)^{|\beta|+|\alpha|} \xi^\beta D^\alpha \hat{f}(\xi) = \int e^{-i\xi\cdot x} D^\beta (x^\alpha f(x)) dx \tag{5}$$

From the property (5), it is easy to verify that the Fourier transform operator maps $S(R^n)$ into $S(R^n)$. We can introduce a simple topology on $S(R^n)$ (see [26],or [22]) to make it a local convex topological vector space. Denote its dual space by $S'(R^n)$. The elements of $S'(R^n)$ are called tempered distributions. In the following $\check{f}(x)$ means $f(-x)$.

For any $T \in S'(R^n)$, the Fourier transform of T is a distribution defined by

$$\langle \hat{T}, f \rangle = \langle T, \hat{f} \rangle, \text{ for all } f \in S(R^n). \tag{6}$$

We are also going to use the following generalized Parseval's identity.

$$\langle T, f \rangle = \langle \hat{T}, \check{\hat{f}} \rangle. \tag{7}$$

Let $g \in L^p(R^n), 1 \leq p \leq \infty$. and if we define T_g by

$$\langle T_g, f \rangle = \int_{R^n} g(x) f(x) dx, \text{ for } f \in S(R^n),$$

then, $T_g \in S'(R^n)$.

Thus $L^p(R^n) \subset S'(R^n)$ for $1 \leq p \leq \infty$. From Hausdorff-Young's theorem. we see that for $f \in L^p(R^n), 1 \leq p \leq 2$. $\hat{f}$ is a function almost everywhere defined on R^n. If $f \in L^p, 2 < p \leq \infty$. however. $\hat{f}$ is , in general. a distribution rather than a function. Also it is easy to see from the generalized Parseval's identity that $FL^p(R^n) \subset S'(R^n)$ for $1 \leq p \leq \infty$. If a distribution $T \in FL^p(R^n). 1 \leq p \leq \infty$. then T is a continuous linear functional on $FL^q(R^n)$ for $\frac{1}{p} + \frac{1}{q} = 1$. So $\langle T, f \rangle$ can be extended from $f \in S(R^n)$

to $f \in FL^1(R^n)$. Actually in this case we have

$$\langle T, f\rangle = \int_{R^n} \hat{T}(\xi)\hat{f}(-\xi)d\xi. \tag{8}$$

For $T \in S'(R^n)$, we say that T vanishes on U if $\langle T, f\rangle = 0$ for all $f \in S(R^n)$ with $supp(f) \subset U$. It is easy to show(see [26], chapter 6) that there is a largest open set U on which T vanishes. The support of T is the closed set in R^n defined by $supp(T) = R^n \backslash U$.

For example, if u is a solution in the distributional sense of Helmholtz equation $\Delta u + u = 0$ in R^n, then taking the Fourier transform in R^n, we have

$$|\xi|^2\hat{u}(\xi) - \hat{u}(\xi) = 0, \text{ or } (|\xi|^2 - 1)\hat{u}(\xi) = 0$$

From the last identity, it is easy to check $supp(\hat{u}) \subseteq S^{n-1}$. Here S^{n-1} is the unit sphere in R^n.

Given $f, g \in S(R^n)$, define the convolution of f and g by

$$f * g(x) = \int_{R^n} f(x-y)g(y)dy.$$

We can verify that $f*g \in S(R^n)$ and $supp(f*g) \subseteq supp(f) + supp(g)$. Here $supp(f) + supp(g)$ is the set $\{x+y; x \in supp(f), y \in supp(g)\}$. The properties of the convolution we will use later are

$$\widehat{f*g}(\xi) = \hat{f}(\xi)\hat{g}(\xi), \tag{9}$$

$$\widehat{fg}(\xi) = \hat{f} * \hat{g}(\xi). \tag{10}$$

Given $T \in S'(R^n)$ and $\phi \in S(R^n)$, we define $T * \phi \in S'(R^n)$ by

$$\langle T * \phi, f \rangle = \langle T, \check{\phi} * f \rangle.$$

The identity (9) can be generalized to

$$\widehat{T * \phi} = \hat{T} \cdot \hat{\phi}. \tag{11}$$

It is well known(see [26]) that $T * \phi$ is always a function on R^n and belongs to $C^\infty(R^n)$. As in the case of $f * g$ for $f, g \in S(R^n)$, we also have $supp(T * \phi) \subseteq supp(T) + supp(\phi)$. So, if supp(T) is compact and $\phi \in C_0^\infty(R^n)$, then $T * \phi$ is a function and belongs to $C_0^\infty(R^n)$ such that $supp(T * \phi) \subseteq supp(T) + supp(\phi)$.

Let $x = (x_1, x_2, \ldots, x_{n-1}) \in R^{n-1}, z \in R$ and let U be open in R^{n-1} with $\bar{U}$ compact. Let $\psi(x)$ be a real-valued C^{n+2} function defined on U and set

$$E = \{(x, \psi(x)), x \in U\}.$$

Needless to say, E is a C^{n+2} (n-1)-dimensional manifold in R^n.

Given $T \in FL^\infty(R^n)$ with $supp(T) \subset E$, following Domar, we now construct a family of *good measures* $\{T_h\}$ on E for a set of real h as follows.

Let

$$\pi : R^n \longrightarrow R^{n-1} \qquad \text{given by } (x, z) \longrightarrow x,$$

$$\beta : U \longrightarrow R^n \qquad \text{given by } x \longrightarrow (x, \psi(x)).$$

For $f \in S(R^n)$, we let $f_\beta(x) = f \circ \beta(x) = f(x, \psi(x))$ for $x \in U$. Suppose $supp(f_\beta)$ is compact in U, then f_β can be viewed as a $C_0^{n+2}(R^{n-1})$ function with support in

U. Since $f \in S(R^n)$ and $\psi \in C^{n+2}$ we can use the identity (5) in this chapter and Plancherel's theorem to show that

$$\int_{R^{n-1}} (1+|w|)^{2(n-1)} |\hat{f}_\beta(w)|^2 dw < \infty.$$

It follows that $f_\beta \in FL^1(R^{n-1})$ for $n \geq 2$ since we have

$$\begin{aligned} \int_{R^{n-1}} |\hat{f}_\beta| dw &\leq \left(\int_{R^{n-1}} \left(\frac{1}{1+|w|}\right)^{2(n-1)} dw\right)^{\frac{1}{2}} \cdot \left(\int_{R^{n-1}} (1+|w|)^{2(n-1)} |\hat{f}_\beta|^2 dw\right)^{\frac{1}{2}} \\ &< \infty. \end{aligned}$$

The fact that $f_\beta(x) \in L^1(R^{n-1})$ is obvious since $supp(f_\beta)$ is compact. So $|\hat{f}_\beta|$ is bounded and hence f_β belongs to $FL^p(R^{n-1})$ for $1 \leq p \leq \infty$.

We first define a distribution $\Sigma \in S'(R^{n-1})$ by

$$\langle \Sigma, g \rangle = \langle T, g \circ \pi \rangle, \text{ for } g \in S(R^{n-1}).$$

This makes sense since supp(T) is compact. From the construction of Σ, it is obvious to see that $supp(\Sigma) \subset U$. Let B^{n-1} be the open unit ball of R^{n-1} and let $\phi(x) \in C_0^\infty(R^{n-1})$ such that $supp(\phi) \subset B^{n-1}$ and $\int_{R^{n-1}} \phi(x) dx = 1$. Denote $\phi_h(x) = \frac{1}{h^{n-1}} \phi(\frac{x}{h})$

Now we define $T_h \in S'(R^n)$ for $0 < h < \frac{1}{2} dist(\partial U, supp(\Sigma))$ by

$$\langle T_h, f \rangle = \langle \Sigma * \check{\phi}_h, f \circ \beta \rangle, \text{ for } f \in S(R^n)$$

Knowing that $\Sigma * \check{\phi}_h$ has compact support, we may assume $supp(f \circ \beta)$ is compact and hence $f \circ \beta \in FL^1(R^{n-1})$. Thus T_h is well defined since $\Sigma * \check{\phi}_h \in C_0^\infty(R^{n-1})$ and hence $\in FL^\infty(R^{n-1})$.

It is easy to check that $supp(T_h) \subset E$. In fact, if $supp(f) \cap E = \emptyset$, then $f(x, \psi(x)) = 0$, for $x \in U$. This implies $f \circ \beta(x) = 0$ for all $x \in U$. So $\langle T_h, f \rangle = \langle \Sigma * \check{\phi}_h, f \circ \beta \rangle = 0$ since $\Sigma * \phi_h$ is a nice measure on U.

Furthermore T_h is a measure on E, absolutely continuous with respect to the area measure on E. To see this, we notify that $\Sigma * \check{\phi}_h$ is a $C_0^\infty(U)$ function, so

$$\langle \Sigma * \check{\phi}_h, f \circ \beta \rangle = \int_U (\Sigma * \check{\phi})(x) f(x, \psi(x)) dx$$
$$= \int_U (\Sigma * \check{\phi})(x) f(x, \psi(x)) \frac{1}{(1 + |\frac{\partial \psi}{\partial x_1}|^2 + \cdots + |\frac{\partial \psi}{\partial x_{n-1}}|^2)^{\frac{1}{2}}} \cdot (1 + |\frac{\partial \psi}{\partial x_1}|^2 + \cdots + |\frac{\partial \psi}{\partial x_{n-1}}|^2)^{\frac{1}{2}} dx$$

Since the map $(x, \psi(x)) \longrightarrow x$, from E to U, is obviously continuous, the function $\frac{(\Sigma * \check{\phi})(x)}{(1+|\frac{\partial \psi}{\partial x_1}|^2 + \cdots + |\frac{\partial \psi}{\partial x_{n-1}}|^2)^{\frac{1}{2}}}$ is continuous on E with compact support.

But $(1 + |\frac{\partial \psi}{\partial x_1}|^2 + \cdots + |\frac{\partial \psi}{\partial x_{n-1}}|^2)^{\frac{1}{2}} dx = ds$, so

$$\langle T_h, f \rangle = \langle \Sigma * \check{\phi}_h, f \circ \beta \rangle = \int_E a_h(s) f(s) ds,$$

where $a_h(s) \in C_0(E)$. This verifies our claim about T_h.

The above construction of T_h is the starting point in the proof of our theorem 1 .

Before ending this chapter, we recall(see [16],or [9]) that E has non-vanishing Gaussian curvature if and only if $|(\frac{\partial^2 \psi(x)}{\partial x_i \partial x_j})| \neq 0$ for each $x \in U$. Here $(\frac{\partial^2 \psi(x)}{\partial x_i \partial x_j})$ is the Hessian matrix of ψ defined by the following matrix,

$$\begin{pmatrix} \frac{\partial^2 \psi(x)}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 \psi(x)}{\partial x_1 \partial x_{n-1}} \\ \vdots & \cdots & \vdots \\ \frac{\partial^2 \psi(x)}{\partial x_{n-1} \partial x_1} & \cdots & \frac{\partial^2 \psi(x)}{\partial x_{n-1} \partial x_{n-1}} \end{pmatrix}.$$

Chapter 3

Lorentz Spaces And Beurling-Pollard Technique

Let $f(x)$ be a measurable complex-valued function on R^n and m be the Lebesgue measure on R^n. For $y > 0, t > 0$, denote $\lambda_f(y) = m\{x: |f(x)| > y\}$, $f^*(t) = \inf\{y, \lambda_f(y) \le t\}$. $\lambda_f(y)$ is called the distribution function of $f(x)$ while $f^*(t)$ is called the non increasing rearrangement of $f(x)$.

The Lorentz spaces $L(p,q)$ are the collection of all f such that $\|f\|_{p,q} < \infty$, where

$$\|f\|_{p,q} = \begin{cases} \left(\frac{q}{p}\int_0^\infty [t^{\frac{1}{p}} f^*(t)]^q \frac{dt}{t}\right)^{\frac{1}{q}} & 1 \le p < \infty,\ 1 \le q < \infty \\ \sup\limits_{t>0} t^{\frac{1}{p}} f^*(t) & 1 \le p \le \infty,\ q = \infty \end{cases}$$

$L(p,\infty)$ is called weak L^p spaces. Note that $L(p,q)$ is not well defined for $p = \infty, 1 \le q < \infty$.

Similar to L^p spaces, $L(p,q)$ spaces have the following properties

(1) $L(1,1)$ and $L(p,q)$ are Banach spaces for $1 < p \le \infty, 1 \le q \le \infty$ and $L(1,q)$ for $1 < q \le \infty$ are Frechet spaces

(2) $\|f\|_{p,p} = \|f\|_p = (\int_{R^n} |f(x)|^p dx)^{\frac{1}{p}}$, so $L(p,p) = L^p(R^n)$ for $1 \le p < \infty$.

(3) The dual space of $L(p,q)$ is $L(p',q')$; $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$, $1 < p < \infty, 1 < q < \infty$.

(4) The dual space of $L(p,1)$ is $L(p',\infty)$, $\frac{1}{p} + \frac{1}{p'} = 1$, $1 < p < \infty$

(5) $\|f\|_{p,\infty} \le \|f\|_{p,q_2} \le \|f\|_{p,q_1} \le \|f\|_{p,1}$, $1 \le q_1 \le q_2 \le \infty$.

(6) $\|fg\|_{p,q} \leq B\|f\|_{p_1,q_1}\|g\|_{p_2,q_2}$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$.

(7) $S(R^n)$ is dense in L(p,q).

Example 3.1 If $f(x) = (1+|x|)^{-\frac{n-1}{2}}$ in $R^n (n \geq 2)$, then $f(x) \in L(p,q)$ for and only for $\frac{2n}{n-1} < p \leq \infty, 1 \leq q \leq \infty$; or $p = \frac{2n}{n-1}, q = \infty$. In fact, $\lambda_f(y) = 0$ for $y \geq 1$ and for $y < 1$ we have

$$\begin{aligned} \lambda_f(y) &= m\{x; |f(x)| > y\} \\ &= m\{x; (1+|x|)^{-\frac{n-1}{2}} > y\} \\ &= m\{x; 1+|x| < y^{-\frac{n-1}{2}}\} \\ &= m\{x; |x| < y^{-\frac{n-1}{2}} - 1\} \\ &= C(y^{-\frac{n-1}{2}} - 1)^n. \end{aligned}$$

So,

$$\begin{aligned} f^*(t) &= \inf_y \{y; \lambda_f(y) \leq t\} \\ &= \inf_y \{y; C(y^{-\frac{n-1}{2}} - 1)^n \leq t\} \\ &= \inf_y \{y; (y^{-\frac{n-1}{2}} - 1) \leq (\frac{t}{C})^{\frac{1}{n}}\} \\ &= (1 + (\frac{t}{C})^{\frac{1}{n}})^{-\frac{n-1}{2}}, \end{aligned}$$

where C is a positive constant. The rest of the checking is trivial.

Example 3.2 If we let $f_h(x) = f(hx)$ for $h > 0$, then $\|f_h\|_{p,q} = h^{-\frac{n}{p}}\|f\|_{p,q}$.

Proof. From the definition of L(p,q) spaces, it suffices to show that $f_h^*(t) = f^*(h^n t)$.

$$\lambda_{f_h}(y) = m\{x; |f_h(x)| > y\}$$

$$
\begin{aligned}
&= m\{x; |f(hx)| > y\} \\
&= m\{\frac{x'}{h}; |f(x')| > y\} \\
&= h^{-n}\lambda_f(y), \\
f_h^*(t) &= inf\{y; \lambda_{f_h}(y) \le t\} \\
&= inf\{y; h^{-n}\lambda_f(y) \le t\} \\
&= inf\{y; \lambda_f(y) \le th^n\} \\
&= f^*(h^n t).
\end{aligned}
$$

This finishes the proof.

Example 3.3 Let $h(x) = f(-x)$ and $g(x) = (f(x))^2$. Then (i) $h^*(t) = f^*(t)$ and (ii) $g^*(t) = (f^*(t))^2$.

Proof: This can be seen from the above calculation.

We refer the reader to [15] for further information about L(p,q) spaces

An operator T which maps functions on a measure space into functions on another measure space is called *sublinear* if whenever Tf and Tg are defined and c is a constant, then $T(f+g)$ and $T(cf)$ are defined with

$$
\begin{cases}
|T(f+g)| \le |Tf| + |Tg| \\
|T(cf)| = |c| \cdot |Tf|.
\end{cases}
$$

In the following, $A \xrightarrow{T} B$ means that T is a bounded sublinear operator from A to B.

Proposition 3.1 (real interpolation, see [15])

$$If \quad L(p_1,1) \xrightarrow{T} L(q_1,\infty) \text{ and } L(p_2,1) \xrightarrow{T} L(q_2,\infty) \text{ with } q_1 < q_2, p_1 \neq p_2,$$

then

$$L(p,r) \xrightarrow{T} L(q,r),$$

where $\frac{1}{p} = \frac{1-t}{p_1} + \frac{t}{p_2}, \quad \frac{1}{q} = \frac{1-t}{q_1} + \frac{t}{q_2}, \quad 0 < t < 1, 1 \leq r \leq \infty.$

Corollary 3.1 If $f \in L(p,q)$ for $1 < p < 2, \quad 1 \leq q \leq \infty$, then we have

$$\|\hat{f}\|_{p',s} \leq C\|f\|_{p,q}, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad q \leq s.$$

Here C is a constant depending on p .

Proof: Using the two estimates $\|\hat{f}\|_\infty \leq \|f\|_1$ and $\|\hat{f}\|_2 \leq C\|f\|_2$, we see that the conclusion of the corollary follows directly form the proposition 3.1 since L(2,1) is contained in L(2,2)=L^2.

Proposition 3.2 (complex interpolation.cf[15])

$$\text{If} \quad L(p_1,q_1) \xrightarrow{T} L(r_1,s_1) \text{ and } L(p_2,q_2) \xrightarrow{T} L(r_2,s_2),$$

then

$$L(p,q) \xrightarrow{T} L(r,s).$$

Here $\frac{1}{p} = \frac{1-t}{p_1} + \frac{t}{p_2}, \quad \frac{1}{q} = \frac{1-t}{q_1} + \frac{t}{q_2}, \quad \frac{1}{r} = \frac{1-t}{r_1} + \frac{t}{r_2}, \quad \frac{1}{s} = \frac{1-t}{s_1} + \frac{t}{s_2}, \quad 0 \leq t \leq 1.$

Remark 3.1 The virtue of the real interpolation is to get the L^p-estimate from the information of weak L^p estimates. This is essential in the control of the Hilbert transform, the Hardy-Littlewood maximal function operator and the singular integral operators(see [21]).

Remark 3.2 The complex interpolation result can be easily generalized to the multilinear operators([1],p.18.exercise 13). But the generalization of the real interpolation result to the multilinear case is not trivial and the following proposition is a useful result in this direction.

Proposition 3.3 ([2], cf.[7])

Let Σ be a (n-1) simplex in $[0,1]^n$, and **C** be the complex number space. Assume that for all $(\frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_n}) \in$ set of the vertices of Σ, we have

$$L(p_1, 1) \times L(p_2, 1) \times \cdots \times L(p_n, 1) \xrightarrow{T} \mathbf{C}.$$

Then for $(\frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_n}) \in Int\Sigma$, we have

$$L(p_1, q_1) \times L(p_2, q_2) \times \cdots \times L(p_n, q_n) \xrightarrow{T} \mathbf{C}.$$

where $\quad \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_n} \geq 1.$

Remark 3.3 As a corollary of proposition 3.3 , we can give a simple proof of the *convolution theorem* ([10]).

C. Herz [12] adopted the Beurling-Pollard technique(cf. [19]) in solving the spectral synthesis problem for S^1 in R^2 and then Varopoulos extended it to S^{n-1} in R^n([18]) Domar([5], [6]) was able to prove the spectral synthesis property for a rather general curve in R^2 with the help of the Beurling-Pollard technique as we mentioned in the introduction chapter.

Let M be a k-dimensional manifold in R^n with area. Assume $T \in S'(R^n)$ supported on M. If $\hat{T} \in L(p,q)$ for $1 < p < 2$ and $1 \leq q \leq \infty$, then the corollary 3.1 yields that T is a measurable function on R^n and hence $T = 0$ since it is supported on M which has measure zero in R^n.

In the rest of this thesis, the same constant C will stand for different uniform constants.

Now we use the Beurling Pollard argument to prove the following theorem.

Theorem 3 Let M and T be as above. Then T=0 if $\hat{T} \in L(p,q)$ for $2 \leq p < \frac{2n}{k}$, $1 \leq q \leq \infty$.

Proof: From the property (5) of L(p,q), we may assume $q = \infty$. Choose $a(x) \in C_0^\infty(R^n)$ such that $supp(a) \subseteq \{x : |x| \leq 1\}$ and $\int_{R^n} a(x)dx = 1$. Given $h > 0$,denote $a_h(x) = \frac{1}{h^n} a(\frac{x}{h})$. By the definition of the Fourier transform , we have $\hat{a}(0) = 1$ and $\hat{a_h}(\xi) = \hat{a}(h\xi)$. Also it is easy to see that for each ξ $\lim_{h \to 0} \hat{a}(h\xi) = \hat{a}(0) = 1$ and for all h and ξ, $|\hat{a}(h\xi)|$ is uniformly bounded.

Denote $M_{2h} = \{x: dist(x, M) \leq 2h\}$. Let $X_{M_{2h}}(x) = \begin{cases} 1 & x \in M_{2h} \\ 0 & x \notin M_{2h} \end{cases}$, $f_h(x) = X_{M_{2h}} \cdot f(x)$ and $g_h = f - f_h$, then $supp(g_h * a_h) \cap supp(T) = \emptyset$.

To show T=0, it amounts to showing that $\langle T, f \rangle = 0$ for all $f \in S(R^n)$.

$$\begin{aligned} \langle T, f \rangle &= \langle T, f - f * a_h \rangle + \langle T, f * a_h \rangle \\ &= \langle T, f - f * a_h \rangle + \langle T, f_h * a_h \rangle + \langle T, g_h * a_h \rangle \end{aligned}$$

$$= \langle T, f - f * a_h \rangle + \langle T, f_h * a_h \rangle$$

Now $|\langle T, f - f * a_h \rangle| \le \int_{R^n} |\hat{T}(-\xi)\hat{f}(\xi)(1 - \hat{a}(h\xi)| d\xi \longrightarrow 0$ as $h \to 0$ by the Lebesgue dominated convergent theorem since $\hat{f}(\xi) \in S(R^n)$ and $\hat{T}(\xi) \in L(p,q)$ which implies $\hat{T}(-\xi)\hat{f}(\xi) \in L^1(R^n)$ from the properties (3) and (4) of L(p,q)

Thus, to show $\langle T, f \rangle = 0$, it is enough to show

$$\langle T, f_h * g_h \rangle = \int_{R^n} \hat{T}(-\xi)\hat{f}_h(\xi)\hat{a}(h\xi) d\xi \longrightarrow 0 \text{ as } h \to 0.$$

We first consider the case $p = 2$.

To begin with, we choose $2 < r < s < \infty$ such that $\frac{1}{2} = \frac{1}{r} + \frac{1}{s}$ and $(n-k)\frac{r-1}{r} - \frac{n}{s} > 0$.

Now

$$\begin{aligned}
& \int_{R^n} |\hat{T}(-\xi)\hat{f}_h(\xi)\hat{a}(h\xi)| d\xi \\
\le\ & \|\hat{T}(-\xi)\|_{2,\infty} \|\hat{f}_h(\xi)\hat{a}(h\xi)\|_{2,1} \\
\le\ & C \|\hat{f}_h(\xi)\|_{r,\infty} \|\hat{a}(h\xi)\|_{s,1}
\end{aligned}$$

Here we used the properties (4) and (6) of L(p,q). From the example 3.2, we have $\|\hat{a}(h\xi)\|_{s,1} = h^{-\frac{n}{s}} \|\hat{a}(\xi)\|_{s,1}$ and the property (5) of L(p,q) together with the Hausdorff Young theorem gives $\|\hat{f}_h\|_{r,\infty} \le \|\hat{f}_h\|_r \le \|f_h\|_{\frac{r}{r-1}}$.

But

$$|f_h\|_{\frac{r}{r-1}} = \int_{M_{2h}} |f(x) X_{M_{2h}}(x)|^{\frac{r}{r-1}} dx)^{\frac{r-1}{r}} \le C \cdot [m(M_{2h})]^{\frac{r-1}{r}} \le C h^{(n-k)\frac{r-1}{r}}$$

So,

$$\left| \int_{R^n} \hat{T}(-\xi)\hat{f}_h(\xi)\hat{a}(h\xi)d\xi \right|$$

$$\leq \; Ch^{(n-k)\frac{r-1}{r}-\frac{n}{s}} \longrightarrow 0 \text{ as } h \to 0.$$

Next we consider the case $2 < p < \frac{2n}{k}$. This time we are able to use the Plancherel theorem.

$$|\langle T, f_h * a_h\rangle| \leq \int_{R^n} |\hat{T}(-\xi)\hat{f}_h(\xi)\hat{a}_h(h\xi)|d\xi$$

$$\leq \int_{R^n} |\hat{f}_h(\xi)|^2 d\xi)^{1/2}(\int_{R^n} |\hat{a}(h\xi)\hat{T}(-\xi)|^2 d\xi)^{1/2}$$

$$= C\int_{R^n} |f_h(x)|^2 dx)^{1/2}(\int_{R^n} |\hat{a}(h\xi)\hat{T}(-\xi)|^2 d\xi)^{1/2}$$

Since $\int_{R^n} |f_h(x)|^2 dx \leq C' \int_{M_{2h}} dx \leq Ch^{n-k}$, to show $\langle T, f\rangle = 0$, it suffices to show that $I_h := (h^{n-k})^{\frac{1}{2}}(\int_{R^n} |\hat{a}(h\xi)\hat{T}(-\xi)|^2 d\xi)^{\frac{1}{2}} \longrightarrow 0$ as $h \to 0$.

We first estimate $\int_{R^n} |\hat{a}(h\xi)\hat{T}(-\xi)|^2 d\xi$.

Using the assumption $p > 2$, the properties (4) and (6) of L(p,q), we have for $\frac{2}{p} + \frac{1}{q} = 1$

$$\int_{R^n} [\hat{a}(h\xi)\hat{T}(-\xi)]^2 d\xi \leq \|[\hat{a}(h\cdot)]^2\|_{q,1}\|[\hat{T}(-\cdot)]^2\|_{\frac{p}{2},\infty}$$

$$\leq C\|\hat{a}(h\cdot)\|_{2q,\infty}\|\hat{a}(h\cdot)\|_{2q,1}(\|\hat{T}\|_{p,\infty})^2$$

$$\leq Ch^{-\frac{n}{2q}}\|\hat{a}\|_{2q,\infty} \cdot h^{-\frac{n}{2q}}\|\hat{a}\|_{2q,1}$$

$$\leq Ch^{-\frac{n}{q}}.$$

So,

$$I_h \leq C h^{\frac{n-k}{2}} \cdot h^{-\frac{n}{2q}} = C h^{\frac{n-k}{2} - \frac{n}{2q}}$$

To make $\frac{n-k}{2} - \frac{n}{2q} > 0$, it amonts to having

$$\begin{aligned} n - k &> \frac{n}{q} = (1 - \frac{2}{p})n = n - \frac{2n}{p}, \\ -k &> -\frac{2n}{p}, \\ k &< \frac{2n}{p}, \\ p &< \frac{2n}{k}. \end{aligned}$$

This is the end of the proof of theorem 3.

We will need the following lemma for the proof of theorem 4

<u>**lemma 3.1**</u> Let M be a (n-1)-dimensional manifold in R^n with area. Let $T \in S'(R^n)$ with $supp(T) \subset M$. If $\hat{T} \in L(p,q)$, then T vanishes on $J^{n+2}(R^n)$ provided

(i) $2 \leq p \leq \infty, 1 \leq q \leq \infty,$ when $n = 2$

(ii) $2 \leq p < \infty, 1 \leq q \leq \infty,$ when $n = 3$

(iii) $2 \leq p < \frac{2n}{n-3}, 1 \leq q \leq \infty,$ when $n \geq 4$.

<u>Proof</u>: From the result of theorem 3, we may assume $p \geq \frac{2n}{n-1}$. Also without loss of generality, we may assume $q = \infty$. We fix $f(x) \in J^{n+2}(R^n)$. Given a point $x_0 \in M$ and any point $x \in M_{2h}$, we have from the mean value theorem in calculus that

$$f(x) = \nabla f(c) \cdot (x - x_0),$$

where $c = (c_1, c_2, \ldots, c_n)$ is a point on the line segment between $x = (x_1, x_2, \ldots, x_n)$ and $x_0 = (x_{01}, x_{02}, \ldots, x_{0n})$. It follows that $|f_h(x)| \leq Ch$ for all $x \in M_{2h}$. Since M has area, we have

$$\int_{R^n} |f_h(x)|^2 dx \leq C \cdot h^2 \cdot h = Ch^3.$$

Following the argument in the proof of theorem 3, we see that to prove $\langle T, f \rangle = 0$, it is enough to show

$$\delta_h = \int_{R^n} \hat{T}(-\xi) \hat{f}_h(\xi) \hat{a}(h\xi) d\xi \longrightarrow 0 \text{ as } h \to 0.$$

Now

$$\begin{aligned}
|\delta_h| &\leq \int_{R^n} |\hat{T}(-\xi) \hat{f}_h(\xi) \hat{a}_h(h\xi)| d\xi \\
&\leq \int_{R^n} |\hat{f}_h(\xi)|^2 d\xi)^{1/2} (\int_{R^n} |\hat{a}(h\xi)\hat{T}(-\xi)|^2 d\xi)^{1/2} \\
&= \int_{R^n} |f_h(x)|^2 dx)^{1/2} (\int_{R^n} |\hat{a}(h\xi)\hat{T}(-\xi)|^2 d\xi)^{1/2} \\
&\leq Ch^{\frac{3}{2}} (\int_{R^n} |\hat{a}(h\xi)\hat{T}(-\xi)|^2 d\xi)^{1/2}.
\end{aligned}$$

We follow the argument in the proof of theorem 3 for the case $2 < p < \frac{2n}{k}$ to obtain

$$\begin{aligned}
|\delta_h| &\leq Ch^{\frac{3}{2}} \cdot h^{-\frac{n}{2q}} \\
&= Ch^{\frac{3}{2} - \frac{n}{2q}}.
\end{aligned}$$

To make $\frac{3}{2} - \frac{n}{2q} > 0$, it is equivalent to

$$\begin{aligned}
3 > \frac{n}{q} &= (1 - \frac{2}{p})n = n - \frac{2n}{p} \\
\text{or} \quad n - 3 &< \frac{2n}{p}. \qquad (*)
\end{aligned}$$

From $(*)$, we have

$$\begin{cases} p \leq \infty & \text{when } n = 2 \\ p < \infty & \text{when } n = 3 \\ p < \frac{2n}{n-3} & \text{when } n \geq 4 \ . \end{cases}$$

This completes the proof of lemma 3.1.

Remark 3.4: Let M be as in theorem 3. and $f \in S(R^n)$ If f and its all partial derivatives of order $\leq m$ vanish on M, then for f_h defined in the proof of theorem 3 we have $\int_{R^n} |f_h(x)|^2 dx \leq Ch^{2(m+1)} \cdot h = Ch^{2m+3}$ Let $T \in FL^2(R^n)$ supported on M. Checking the proof of theorem 3(or lemma 3.1), we see that $\langle T, f \rangle = 0$ if $2m + 3 - n > 0$, or $m > \frac{n-3}{2}$. This fact implies that for a general compact C^m $(n-1)$ dimensional manifold in $R^n (n \geq 3)$, the property (A) is equivalent to the property (B) Thus the counter-example of Schwartz convinces us that there is no hope to prove the property (B) for this manifold.

Chapter 4

Some Basic Lemmas

In this chapter we prove some basic estimates which are essential in the proof of our main theorems. We begin with a known result.

Lemma 4.1 If we set

$$M_2 f(u) = (\sup_{r>0} \frac{1}{m(B_r(u))} \int_{B_r(u)} |f(y)|^2 dy)^{\frac{1}{2}},$$

then we have $\|M_2 f\|_{2,\infty} \leq \|f\|_2$.

Proof: From the property (ii) of the example 3.3, we have $([M_2 f]^2)^*(t) = (M_2 f^*(t))^2$ which implies $\|M_2 f\|_{2,\infty} \leq (\|[M_2 f]^2\|_{1,\infty})^{\frac{1}{2}}$. It is well-known([21], chapter 1, theorem 1)that

$$\|(M_2 f)^2\|_{1,\infty} \leq C\|f^2\|_1.$$

So,

$$\|M_2 f\|_{2,\infty} \leq (\|[M_2 f]^2\|_{1,\infty})^{\frac{1}{2}} \leq C(\|f^2\|_1)^{\frac{1}{2}} = C(\int_{R^n} |f(x)|^2 dx)^{\frac{1}{2}} = C\|f\|_2.$$

The proof is complete.

Lemma 4.2 Suppose we have constant A such that

$$\{\int_{R^m} |g(y)|^2 dy\}\{\int_{R^m} |g(y)|^2 |y|^{2m} dy\} \leq A,$$

then we have $|g * f(u)| \leq C M_2 f(u)$, C independent of f.

Proof: We may assume $A \geq 1$. Also we may assume $\int_{R^m} |g(y)|^2 dy \neq 0$, for otherwise the conclusion is obvious. Let $B = (\int_{R^m} |g(y)|^2 |y|^{2m} dy)^{\frac{1}{m}}$, then $B > 0$ and we have

$$\int_{R^m} |g(y)|^2 dy \leq \frac{A}{B^m} \text{ and } \int_{R^m} |g(y)|^2 |y|^{2m} dy \leq AB^m$$

We first observe that

$$\begin{aligned} |g * f(u)| &= |\int_{R^m} g(u-y) f(y) dy| \\ &\leq \int_{|u-y|\leq B} |g(u-y) f(y) dy + \int_{|u-y|>B} |g(u-y) f(y)| dy. \end{aligned}$$

Then we control the two terms on the right hand side seperately.

$$\begin{aligned} &\int_{|u-y|\leq B} |g(u-y) f(y)| dy \\ \leq\ & (\int_{u-y|\leq B} |g(u-y)|^2 dy)^{\frac{1}{2}} (\int_{|u-y|\leq B} |f(y)|^2 dy)^{\frac{1}{2}} \\ \leq\ & (\frac{A}{B^m})^{\frac{1}{2}} (\int_{|u-y|\leq B} |f(y)|^2 dy)^{\frac{1}{2}} \\ \leq\ & A^{\frac{1}{2}} (\frac{1}{B^m} \int_{|u-y|\leq B} |f(y)|^2 dy)^{\frac{1}{2}} \\ \leq\ & C M_2 f(u), \end{aligned}$$

$$\begin{aligned} &\int_{|u-y|>B} |g(u-y) f(y)| dy \\ =\ & \int_{|u-y|>B} |g(u-y)| |u-y|^m \frac{|f(y)|}{|u-y|^m} dy \\ \leq\ & (\int_{|u-y|>B} |g(u-y)|^2 |u-y|^{2m} dy)^{\frac{1}{2}} (\int_{|u-y|>B} \frac{|f(y)|^2}{|u-y|^{2m}} dy)^{\frac{1}{2}} \\ \leq\ & (AB^m)^{\frac{1}{2}} (\sum_{n=0}^{\infty} \int_{B2^n<|u-y|\leq B2^{n+1}} \frac{|f(y)|^2}{|u-y|^{2m}} dy)^{\frac{1}{2}} \\ \leq\ & (AB^m)^{\frac{1}{2}} (\sum_{n=0}^{\infty} \frac{1}{(B2^n)^{2m}} \int_{|u-y|\leq B2^{n+1}} |f(y)|^2 dy)^{\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
&= A^{\frac{1}{2}} B^{\frac{m}{2}} \left(\sum_{n=0}^{\infty} \frac{1}{B^m 2^{(n-1)m}} \frac{1}{(B2^{n+1})^m} \int_{|u-v|\le B2^{n+1}} |f(y)|^2 dy\right)^{\frac{1}{2}} \\
&\le C\left(\sum_{n=0}^{\infty} \frac{1}{2^{(n-1)m}} (M_2 f(u))^2\right)^{\frac{1}{2}} \\
&\le C M_2 f(u).
\end{aligned}
$$

So, $|g * f(u)| \le C M_2 f(u)$, C is independent of B. The proof of Lemma 4.2 is finished.

Lemma 4.3 Let U be open in R^{2m}. Let t be a real number, k a positive integer, $2 \le k \le m+1$, $b(y,\sigma) \in C^{(k+1)}(U)$ and real-valued, $a(y,\sigma) \in C_0^k(U)$. Set $g_t(y) = \int_{R^m} e^{itb(y,\sigma)} a(y,\sigma) d\sigma$. If $|b''_{y\sigma}| \neq 0$ in $supp(a)$, then we have

$$
\int_{R^m} |g_t(y)|^2 dy \le C \frac{1}{|t|^{k-1}}, \qquad \text{for } |t| \ge 1.
$$

Here $|b''_{y\sigma}|$ denotes the determinant of the matrix

$$
\begin{pmatrix}
\frac{\partial^2 b}{\partial y_1 \partial \sigma_1} & \cdots & \frac{\partial^2 b}{\partial y_1 \partial \sigma_m} \\
\vdots & \vdots & \vdots \\
\frac{\partial^2 b}{\partial y_m \partial \sigma_1} & \cdots & \frac{\partial^2 b}{\partial y_m \partial \sigma_m}
\end{pmatrix}
$$

and C depends on $\max\{\sup_{(y,\sigma)\in supp(a)} |\frac{\partial^\beta b(y,\sigma)}{\partial y^\beta}|, 0 < |\beta| \le k+1\}$ and a, but is independent of $\sup_{(y,\sigma)\in supp(a)} |b(y,\sigma)|$.

Proof: We will follow Hörmander's argument([13], theorem 1). Since $|b''_{y\sigma}| \neq 0$ in $supp(a)$, we see that for each (y_0,σ_0) in $supp(a)$, $b''_{y\sigma}(y_0,\sigma_0)$ is invertible as an operator from R^m to R^m with l^2 norm and the norm of the inverse operator is bounded uniformly for $(y_0,\sigma_0) \in supp(a)$. Using the mean value theorem for each component of the vector function $\frac{\partial}{\partial y}(b(y,\sigma) - b(y,s))$, we have

$$|\frac{\partial}{\partial y}(b(y,\sigma) - b(y,s))| = |(b''_{y\sigma}(y,s) \cdot (\sigma - s)| + O(|\sigma - s|^2)$$

Using a smooth partition of unity if necessary, we may assume the $supp(a)$ is so small that for (y,σ) and $(y,s) \in supp(a)$, we have

$$|b''_{y\sigma}(y,\sigma)) \cdot (\sigma - s)| + O(|\sigma - s|^2) \geq C|\sigma - s|,$$

where $C > 0$ is uniformly on $supp(a)$.

Thus , we have

$$|\frac{\partial}{\partial y}(b(y,\sigma) - b(y,s))| \geq C|\sigma - s|,$$

with $C > 0$ uniformly on $supp(a)$.

Choose $f(\sigma) \in C_0^\infty(R^{2m})$ such that

$$g_t(y) = \int_{R^m} e^{itb(y,\sigma)} a(y,\sigma) f(\sigma) d\sigma.$$

This is possible since $a \in C_0^k(U)$.

Consider $a_t(\sigma, s) = \int_{R^m} e^{it(b(y,\sigma) - b(y,s))} a(y,\sigma)\overline{a(y,s)} dy$. To estimate $a_t(\sigma, s)$, we introduce the coordinates $\tilde{y}$ in R^{m-1} such that $y = (y_j, \tilde{y})$. Denote $b(y,\sigma) - b(y,s)$ as b, $a(y,\sigma)\overline{a(y,s)}$ as a and denote $\frac{\partial b}{\partial y_j}$ as b'_j, $\frac{\partial b}{\partial y} = (\frac{\partial b}{\partial y_1}, \ldots, \frac{\partial b}{\partial y_m})$ as ∇b and so on.

$$\begin{aligned} a_t(\sigma, s) &= \int_{R^m} e^{itb} a dy \\ &= \int_{R^m} \frac{e^{itb} a \sum_{j=1}^m (b'_j)^2}{|\nabla b|^2} dy \end{aligned}$$

$$
\begin{aligned}
&\doteq \sum_{j=1}^{m} \int_{R^m} \frac{e^{itb} a (b_j')^2}{|\nabla b|^2} dy \\
&= \sum_{j=1}^{m} \int_{R^m} e^{itb} itb_j' \frac{b_j' a}{it|\nabla b|^2} dy_j d\tilde{y} \\
&= \sum_{j=1}^{m} \int_{R^m} \frac{b_j' a}{it|\nabla b|^2} de^{itb} d\tilde{y} \\
&= \sum_{j=1}^{m} \int_{R^{m-1}} \int_{R} e^{itb} \frac{\partial}{\partial y_j} \left(\frac{b_j' a}{it|\nabla b|^2}\right) dy_j d\tilde{y} \\
&= \sum_{j=1}^{m} \int_{R^{m-1}} \int_{R} e^{itb} \left[\frac{(b_j' a)' it|\nabla b|^2 - 2it(\sum_{k=1}^{m} b_k' \cdot b_{kj}'') b_j' a}{(it|\nabla b|^2)^2}\right] dy_j d\tilde{y}.
\end{aligned}
$$

Note that $|b_k' \cdot b_j'| \leq |\nabla b|^2$ and $|\nabla b| \geq C|\sigma - s|$, it follows that for $t|\sigma - s| \geq 1$

$$
|a_t(\sigma, s)| \leq C \frac{1}{t|\sigma - s|}.
$$

But when $t|\sigma - s| \leq 1$, we obviously have $|a_t(\sigma, s)| \leq C$, so

$$
a_t(\sigma, s)| \leq C \frac{1}{1 + t|\sigma - s|}.
$$

Adopting the smoothness assumotion of the lemma, we can repeat the above process k times to obtain

$$
|a_t(\sigma, s)| \leq C_k \frac{1}{(1 + t|\sigma - s|)^k}.
$$

If we check the above proof, we will see that C_k depends only on $\max\{\sup_{(y,\sigma)\in supp(a)} |\frac{\partial^\beta b(y,\sigma)}{\partial y^\beta}|, 0 < |\beta| \leq k+1\}$ and a, not on $\sup_{(y,\sigma)\in supp(a)} |b(y,\sigma)|$.

Since

$$
\int_{R^m} |g_t(y)|^2 dy \;=\; \int_{R^m} \int_{R^m} a_t(\sigma, s) f(\sigma) \overline{f(s)} d\sigma ds
$$

$$\leq \int_{R^m}\int_{R^m} |a_t(\sigma,s)f(\sigma)|d\sigma|\overline{f(s)}|ds$$
$$\leq \int_{R^m}\int_{R^m} \frac{|f(\sigma)|}{(1+t|\sigma-s|)^k} d\sigma|\overline{f(s)}|ds,$$

it is enough to show

$$\int_{R^m} \frac{|f(\sigma)|}{(1+t|\sigma-s|)^k} d\sigma \leq C\frac{1}{|t|^{k-1}}, \qquad \text{for } |t| \geq 1,$$

with C independent of t and s.

For $k = m+1$, we have

$$\int_{R^m} \frac{d\sigma}{(1+t|\sigma-s|)^{m+1}} = |t|^{-m}\int_{R^m} \frac{du}{(1+|u|)^{m+1}} = C\frac{1}{|t|^m}.$$

So

$$\int_{R^m} \frac{|f(\sigma)|}{(1+t|\sigma-s|)^{m+1}} d\sigma \leq C\frac{\|f\|_\infty}{|t|^m} \leq C\frac{1}{|t|^m} = C\frac{1}{|t|^{k-1}}, \qquad |t| \geq 1.$$

For $2 \leq k < m+1$, we can find $p > 1$ such that $pk = m+1$. This implies $p(k-1) \leq m$, or $\frac{m}{p} \geq k-1$.

By Hölder's inequality,

$$\int_{R^m} \frac{|f(\sigma)|}{(1+t|\sigma-s|)^k} d\sigma$$
$$\leq \left(\int_{R^m} \frac{d\sigma}{(1+t|\sigma-s|)^{kp}}\right)^{\frac{1}{p}} \left(\int_{R^m} |f(\sigma)|^q d\sigma\right)^{\frac{1}{q}}$$
$$= |t|^{-\frac{m}{p}}\left(\int_{R^m} \frac{d\sigma}{(1+|u|)^{m+1}}\right)^{\frac{1}{p}}\|f\|_q$$
$$= C\frac{1}{|t|^{\frac{m}{p}}} \leq C\frac{1}{|t|^{k-1}}, \qquad |t| \geq 1.$$

Thus we have proved $\int_{R^m} |g_t(y)|^2 dy \leq C\frac{1}{|t|^{k-1}}$, for $|t| \geq 1$, and the constant C has the desired properties. This is the end of the proof.

Chapter 5

The Proof of Main Theorems

We are now in a position to prove the main theorems. For convenience, we restate the theorems concerned. Also we may assume $k = n+2$ without loss of generality.

Theorem 1   Let E, T, ψ, ψ_h, T_h be as in the introduction chapter and assume $\psi(x) \in C^{n+2}(U)$ and T vanishes on $J^{n+2}(E)$. Let $(\eta, \xi) \in R^{n-1} \times R$. If we set $M_{2,\eta}(\hat{T})(\eta,\xi) = (\sup_{r>0} \frac{1}{m(B_r(\eta))} \int_{B_r(\eta)} |\hat{T}(\eta,\xi)|^2 d\eta)^{\frac{1}{2}}$, then we have

$$|\hat{T}_h(\eta,\xi)| \leq C M_{2,\eta}(\hat{T})(\eta,\xi),$$

where C is independent of η, ξ, and h.

Proof: Let $X(y,\zeta) = e^{-i(\langle \eta, y\rangle + \xi\zeta)}$, we have

$$\begin{aligned}\hat{T}_h(\eta,\xi) &= \langle T_h, X\rangle = \langle \Sigma * \check{\phi}_h, X \circ \beta\rangle \\ &= \langle \Sigma, \int_{R^{n-1}} \phi_h(\sigma) e^{-i(\langle \eta, y-\sigma\rangle + \xi\psi(y-\sigma))} d\sigma\rangle.\end{aligned}$$

Since we can choose U_o open with $supp(\Sigma) \subset U_o \subset \bar{U}_o \subset U$, we can find $\tau(y) \in C_o^\infty(U)$ such that $\tau(y) = 1$ for $y \in U_o$.

Thus we can write

$$\begin{aligned}\hat{T}_h(\eta,\xi) &= \langle \Sigma, \tau(y) \int_{R^{n-1}} \phi_h(\sigma) e^{-i(\langle \eta, y-\sigma\rangle + \xi\psi(y-\sigma))} d\sigma\rangle \\ &= \langle \Sigma, \tau(y) e^{-i(\langle \eta, y\rangle + \xi\psi(y))} \int_{R^{n-1}} e^{i(\langle \eta,\sigma\rangle - \xi(\psi(y-\sigma)) - \psi(y))} \phi_h(\sigma) d\sigma\rangle\end{aligned}$$

$$= \langle \Sigma, \tau(y) e^{-i(\langle \eta, y\rangle + \xi\psi(y))} \int_{R^{n-1}} e^{i(\langle \eta, h\sigma\rangle - \xi(\psi(y-h\sigma)) - \psi(y))} \phi(\sigma) d\sigma \rangle$$

$$= \langle T, e^{-i(\langle \eta, y\rangle + \xi\zeta)} g_{\eta,\xi,h}(y) \rangle .$$

Here $g_{\eta,\xi,h}(y) = \tau(y) \int_{R^{n-1}} e^{i(\langle \eta, h\sigma\rangle - \xi(\psi(y-h\sigma)) - \psi(y))} \varphi(\sigma) d\sigma$. Note that in the last identity we used the assumption that T vanishes on $J^{n+2}(E)$ since $e^{i\xi\psi(y)} - e^{i\xi\zeta} = 0$ on E and supp(T) is compact from which we may assume the function $e^{i\xi\psi(y)} - e^{i\xi\zeta}$ has compact support.

Hence

$$\begin{aligned}
\hat{T}_h(\eta, \xi) &= \langle T, e^{-i(\langle \eta, y\rangle + \xi\zeta)} g_{\eta,\xi,h}(y) \rangle \\
&= \langle T, e^{-i(\langle \eta, y\rangle + \xi\zeta)} \frac{1}{(2\pi)^{n-1}} \int_{R^{n-1}} e^{i\langle w, y\rangle} \hat{g}_{\eta,\xi,h}(w) dw \rangle \\
&= \frac{1}{(2\pi)^{n-1}} \langle T, \int_{R^{n-1}} e^{-i(\langle \eta - w, y\rangle + \xi\zeta)} \hat{g}_{\eta,\xi,h}(w) dw \rangle \\
&= \frac{1}{(2\pi)^{n-1}} \int_{R^{n-1}} \hat{T}(\eta - w, \xi) \hat{g}_{\eta,\xi,h}(w) dw \rangle .
\end{aligned}$$

Here we used the inverse theorem of the Fourier transform. Since $T \in FL^1(R^n)$, $supp(T)$ is compact and $\hat{g}_{\eta,\xi,h} \in L^1(R^{n-1})$ for each η, ξ, h, the last identity can be verified by using a standard limit argument.

To prove the theorem, it is enough to show that

$$\left| \int_{R^{n-1}} \hat{T}(\eta - w, \xi) \hat{g}_{\eta,\xi,h}(w) dw \right| \le C M_{\lambda,\eta}(\hat{T})(\eta, \xi),$$

with C independent of η, ξ, and small h .

To this end, by lemma 4.2 , Plancherel's theorem and the inequality $|w|^{2(n-1)}$.

$C\sum_{j=1}^{n-1}|w_j|^{2(n-1)}$, we need only to show for each $\xi \neq 0$ we can find a constant B depending on $h\xi$ and a constant A independent of η, ξ and h such that for $j = 1, 2, \cdots, n-1$ we have

$$\int |g_{\eta,\xi,h}(y)|^2 dy \leq \frac{A}{B^{n-1}},$$
$$\int_{R^{n-1}} |\frac{\partial^{n-1} g_{\eta,\xi,h}(y)}{\partial y_j^{n-1}}|^2 dy \leq AB^{n-1}.$$

Let $a(y,\sigma) = \tau(y)o(\sigma), b(y,\sigma) = \frac{1}{\xi}\langle \eta, \sigma\rangle - \frac{\psi(y-h\sigma)-\psi(y)}{h}$, then we have

$$g_{\eta,\xi,h}(y) = \int_{R^{n-1}} e^{ih\xi b(y,\sigma)} a(y,\sigma) d\sigma.$$

Since $|\frac{\partial^2 b(y,\sigma)}{\partial y_j \partial \sigma}| = (-1)^{n-1}|\frac{\partial^2 \psi}{\partial y_j \partial y_k}(y - h\sigma)|$, we see that $|\frac{\partial^2 b(y,\sigma)}{\partial y \partial \sigma}| \neq 0$ in supp(a) for all small h.

Also, $b \in C^{n+1}(R^{2(n-1)})$, real-valued, $a \in C_o^\infty(R^{2(n-1)})$ and $\sup_{(y,\sigma)\in supp(a)} |\frac{\partial^\beta b(y,\sigma)}{\partial y^\beta}|$ independent of η, ξ, and h for $0 < |\beta| \leq n+1$.

So lemma 4.3 yields:

$$\int_{R^{n-1}} |g_{\eta,\xi,h}(y)|^2 dy \leq C\frac{1}{|h\xi|^{n-1}}, |h\xi| \geq 1,$$

with C independent of η, ξ, h.

To control

$$\int_{R^{n-1}} |\frac{\partial^{n-1} g_{\eta,\xi,h}(y)}{\partial y_j^{n-1}}|^2 dy$$
$$= \int_{R^{n-1}} | \int_{R^{n-1}} \frac{\partial^{n-1}}{\partial y_j^{n-1}}[e^{ih\xi b(y,\sigma)} a(y,\sigma)] d\sigma|^2 dy$$
$$= \int_{R^{n-1}} | \int_{R^{n-1}} \sum_{m=0}^{n-1} C_{n-1}^m \frac{\partial^m}{\partial y_j^m}[e^{ih\xi b(y,\sigma)}] \frac{\partial^{n-1-m}}{\partial y_j^{n-1-m}}(a(y,\sigma)) d\sigma|^2 dy,$$

it suffices to show

$$\int_{R^{n-1}} \Big| \int_{R^{n-1}} \frac{\partial^m}{\partial y_j^m}[e^{ih\xi b(y,\sigma)}] \frac{\partial^{n-1-m}}{\partial y_j^{n-1-m}}(a(y,\sigma))d\sigma\Big|^2 dy \le C|h\xi|^{n-1}, |h\xi| > 1.$$

for $m = 0, 1, 2, \dots, n-1$, C independent of η, ξ, h.

For m=0, we obviously have

$$\int_{R^{n-1}} \Big| \int_{R^{n-1}} e^{ih\xi b(y,\sigma)} \frac{\partial^{n-1}}{\partial y_j^{n-1}}(a(y,\sigma))d\sigma\Big|^2 dy \le C \le C|h\xi|^{n-1}, |h\xi| > 1,$$

since $a \in C_c^\infty(R^{2(n-1)})$.

For $1 \le m \le n-1$, denote $f(u) = e^u, u = ih\xi b(y,\sigma)$, such that $u^{(k)} = \frac{\partial^k}{\partial y_j^k}[ih\xi b(y,\sigma)] = ih\xi \frac{\partial^k}{\partial y_j^k}(b(y,\sigma))$.

We have the formula due to Faà di Bruno

$$\begin{aligned}
&\frac{\partial^m}{\partial y_j^m}(f(u(y,\sigma))) \\
&= \sum_{1 \le k \le m} \frac{m! f^{(k)}}{k_1! k_2! \dots k_l!}(u^{(1)}/1!)^{k_1}(u^{(2)}/2!)^{k_2} \dots (u^{(l)}/l!)^{k_l} \\
&\text{with } \sum_{s=1}^{l} k_s = k, k_l \ge 1 \text{ and } \sum_{s=1}^{l} s \cdot k_s = m.
\end{aligned}$$

So ,

$$\begin{aligned}
&\frac{\partial^m}{\partial y_j^m}(f(u(y,\sigma))) \\
&= C \sum_{1 \le k \le m} \frac{m!}{k_1! k_2! \dots k_l!} e^{ih\xi b(y,\sigma)}(ih\xi)^k [\frac{\partial}{\partial y_j}(b(y,\sigma))]^{k_1}[\frac{\partial^2}{\partial y_j^2}(b(y,\sigma))]^{k_2} \dots [\frac{\partial^l}{\partial y_j^l}(b(y,\sigma))]^{k_l}
\end{aligned}$$

Denote $\Phi(y,\sigma) = [\frac{\partial}{\partial y_j}(b(y,\sigma))]^{k_1}[\frac{\partial^2}{\partial y_j^2}(b(y,\sigma))]^{k_2} \dots [\frac{\partial^l}{\partial y_j^l}(b(y,\sigma))]^{k_l} \frac{\partial^{n-1-m}}{\partial y_j^{n-1-m}}(a(y,\sigma))$

then each term of $\frac{\partial^m}{\partial y_j^m}[e^{ih\xi b(y,\sigma)}] \frac{\partial^{n-1-m}}{\partial y_j^{n-1-m}}(a(y,\sigma))$ has the form $I_k = C(ih\xi)^k e^{ih\xi b(y,\sigma)} \Phi(y,\sigma)$,

C a uniform constant.

Since

$$\begin{aligned} k &= \sum_{s=1}^{l} k_s = \sum_{s=1}^{l-1} k_s + k_l \\ &\le \sum_{s=1}^{l-1} s \cdot k_s + k_l \\ &= m - lk_l + k_l = m - (l-1)k_l \\ &\le m - (l-1) \le n - l, \text{ for } \ 1 \le m \le n-1, \end{aligned}$$

and $\Phi(y,\sigma) \in C_o^{n+1-l}(R^{2(n-1)})$, lemma 4.3 yields

$$\int_{R^{n-1}} |\int_{R^{n-1}} I_k d\sigma|^2 dy \le C|h\xi|^{n-l} \le C|h\xi|^{n-1}, \ \ |h\xi| \ge 1,$$

C independent of η, ξ, h.

So, we have

$$\int_{R^{n-1}} |\frac{\partial^{n-1} g_{\eta,\xi,h}(y)}{\partial y_j^{n-1}}|^2 dy \le C|h\xi|^{n-1}, \ \ |h\xi| \ge 1, j = 1,2,\ldots,n-1,$$

C independent of η, ξ, h.

For $|h\xi| \le 1$, it is easy to see that

$$\int_{R^{n-1}} |g_{\eta,\xi,h}(y)|^2 dy \le C,$$

$$\int_{R^{n-1}} |\frac{\partial^{n-1} g_{\eta,\xi,h}(y)}{\partial y_j^{n-1}}|^2 dy \le C.$$

Hence we have the desired constants A and B such that

$$\int_{R^{n-1}} |g_{\eta,\xi,h}(y)|^2 dy \le \frac{A}{B^{n-1}},$$

$$\int_{R^{n-1}} \left|\frac{\partial^{n-1} g_{n,\xi,h}(y)}{\partial y_j^{n-1}}\right|^2 dy \leq AB^{n-1}.$$

This is the end of the proof of theorem 1.

<u>**Theorem 2**</u>: If M is a compact C^{n+2} (n-1)-dimentional manifold in R^n with non-vanishing Gaussian curvature, then (C) holds with $m = n+2$

<u>Proof</u>: The compactness of M implies that we can find $\{E_j, j = 1, \dots, m\} \subset M$ such that $M = \bigcup_{j=1}^m E_j$ and each E_j has the form as in theorem 1. Choose $\phi_j \in C_0^\infty(R^n)$ such that $supp(\phi_j) \cap M \subset E_j$ and $\sum_{j=1}^m \phi_j = 1$ in M.

Given $T \in FL^\infty(R^n)$ vanishing on $J^{n+2}(M)$ (which implies $supp(T) \subset M$) we have

$$T = (\sum_{j=1}^m \phi_j)T = \sum_{j=1}^m \phi_j T = \sum_{j=1}^m T^j.$$

Here $T^j = \phi_j T$. It is easy to see that $supp(T^j) \subset E_j$ and $\widehat{T^j} = \hat{\phi}_j * \hat{T} \in L^\infty(R^n)$ since $\hat{\phi}_j \in L^1(R^n)$. Thus $T^j \in FL^\infty(R^n)$ and hence from theorem 1 we can construct $\{T_h^j\}$ for small h such that $|\hat{T}_h(\eta, \xi)| \leq CM_{2,\eta}(\hat{T})(\eta, \xi)$ and hence

$$\|\hat{T}_h^j\|_\infty \leq C_j \|\hat{T^j}\|_\infty, \; j = 1, 2, \dots, m$$

with C_j independent of h.

By the construction of T_h^j at the end of chapter 2, we have for $f \in S(R^n)$,

$$\langle T_h^j, f\rangle = \langle \Sigma^j * \check{\phi}_h, f \circ \beta\rangle = \langle \Sigma^j, \phi_h * f \circ \beta\rangle.$$

Using the Lebesgue dominated convergent theorem, we see that

$$\|\phi_h * f \circ \beta - f \circ \beta\|_{FL^1(R^{n-1})} = \|(\hat{\phi}(h\cdot) - 1)\widehat{f \circ \beta}(\,)\|_{L^1(R^{n-1})} \to 0 \text{ as } h \to 0$$

So $\langle T_h^j, f\rangle \longrightarrow \langle \Sigma^j, f \circ \beta\rangle$ since $\Sigma^j \in FL^\infty(R^{n-1})$.

From the construction of Σ^j and the assumption $T(J^{n+2}(M)) = 0$. we have $\langle \Sigma^j, f \circ \beta\rangle = \langle T^j, f\rangle$. This yields $\langle T_h^j, f\rangle \longrightarrow \langle T^j, f\rangle$ as $h \to 0$.

Now if we let $T_h = \sum_{j=1}^{m} T_h^j$, then for $f \in S(R^n)$,

$$\langle T_h, f\rangle \longrightarrow \langle T, f\rangle \quad \text{as } h \to 0.$$

Recall that $S(R^n)$ is dense in $FL^1(R^n)$ and FL^∞ is the dual space of $FL^1(R^n)$. Now for $f \in FL^1(R^n)$, it is easy to see from the estimate $\|\hat{T}_h\|_\infty \leq C\|\hat{T}\|_\infty$ that

$$\langle T_h, f\rangle \longrightarrow \langle T, f\rangle \quad \text{as } h \to 0.$$

To prove the property (C), by Hahn-Banach theorem. it is enough to show that given $T \in FL^\infty(R^n)$ vanishing on $J^{n+2}(M)$. we have $\langle T, f\rangle = 0$ for $f \in FL^1(R^n)$ vanishing on M.

But T_h is a measure on M absolutely continuous with respect to the area measure of M, we have $a_h(s) \in C(M)$ such that

$$\langle T_h, f\rangle = \int_M a_h(s) f(s) ds = 0, \text{ for } f \in FL^1(R^n) \text{ vanishing on M}.$$

Thus for each $f \in FL^1(R^n)$ vanishing on M, we have $\langle T, f\rangle = \lim_{h\to 0}\langle T_h, f\rangle = 0$.

The proof of theorem 2 is complete.

Theorem 4 Let M be a C^{n+2} (n-1)-dimensional manifold in R^n with non-vanishing Gaussian curvature. Let $T \in S'(R^n)$ with $supp(T) \subset M$. If $\hat{T} \in L(p,q)$ for $2 \leq p < \frac{2n}{n-1}, 1 \leq q \leq \infty$; or $p = \frac{2n}{n-1}, 1 \leq q < \infty$, then T=0 .

To prove theorem 4, we need the following lemma.

<u>Lemma 5.1</u> Given T, T_h, E as in theorem 1, we have

$$\|\hat{T}_h\|_{p,q} \leq C_{p,q}\|\hat{T}\|_{p,q}$$

for

$$\begin{cases} 2 \leq p < \frac{2n}{n-3}, \quad 1 \leq q \leq \infty, \quad \text{when } n \geq 3, \\ 2 \leq p < \infty, \quad 1 \leq q \leq \infty, \quad \text{when } n = 2,3 \end{cases}$$

<u>Proof</u>: From the lemma 3.1, we see that T vanishes on $J^{n+2}(M)$ if p and q are in the range contained in the condition of this lemma. So theorem 1 yields

$$|\hat{T}_h(\eta,\xi)| \leq CM_{2,\eta}(\hat{T})(\eta,\xi)$$

Here again $M_{2,\eta}(\hat{T})(\eta,\xi) = (\sup_{r>0} \frac{1}{m(B_r(\eta))} \int_{B_r(\eta)} |\hat{T}(u,\xi)|^2 du)^{\frac{1}{2}}$

It is easy to see

$$M_{2,\eta}(\hat{T})(\eta,\xi) \leq C(\sup_{r>0} \frac{1}{m(B_r(\eta,\xi))} \int_{B_r(\eta,\xi)} |\hat{T}(y)|^2 dy)^{\frac{1}{2}} = CM_2(\hat{T})(\eta,\xi),$$

so we have $|\hat{T}_h(\eta,\xi)| \leq CM_2(\hat{T})(\eta,\xi)$ and hence $\|\hat{T}_h\|_{p,q} \leq C\|M_2(\hat{T})\|_{p,q}$ for all (p,q) such that $L(p,q)$ is well-defined.

It is easy to check that the operator M_2 is sublinear and for any $g \in L^\infty(R^n)$, we have $\|M_2(g)\|_\infty \leq \|g\|_\infty$. Thus the conclusion of this lemma follows directly from the lemma 4.1 and the proposition 3.1

Now we can prove theorem 4 easily. The case $2 \leq p < \frac{2n}{n-1}, 1 \leq q \leq \infty$, is trivial from the result in theorem 3 proved in chapter 3, so we need only consider the case $p = \frac{2n}{n-1}, 1 \leq q < \infty$.

For any open set U in R^n with $E = U \cap M$ open in M, any $\phi(x) \in C_0^\infty(U)$, we let $T_1 = \phi \cdot T$. Then we have $supp(T_1) \subset E$ and $\hat{T}_1 = \hat{\phi} * \hat{T}$. Since $\hat{\phi} \in L^1(R^n) \cap L^\infty(R^n)$, we can use the proposition 3.1 for the convolution operator with the kernel $\hat{\phi}$ to obtain that $\hat{T}_1 \in L(\frac{2n}{n-1}, q)$ since $\hat{T} \in L(\frac{2n}{n-1}, q)$.

Taking U smaller if necessary, we can choose a coordinate system in R^n such that $E = (x, \psi(x))$, ψ satisfies the properties needed for us to adopt theorem 1 since M is a C^{n+2} manifold with non-vanishing Gaussian curvature.

So for a small h, from theorem 1 we can find a good measure T_h with $supp(T_h) \subset E$ such that the corresponding density function $a_h(s) \in C^{(n+1)}(E)$ and

$$\|\hat{T}_h\|_{\frac{2n}{n-1},q} \leq C\|\hat{T}_1\|_{\frac{2n}{n-1},q}.$$

If $a_h(s)$ is not identically zero on E, Littman's asymptotic estimate in [18] yields

$$\hat{T}_h(\xi) \approx C(1 + |\xi|)^{-\frac{n-1}{2}} \text{ for large } \xi \in R^n.$$

But the function $(1 + |\xi|)^{-\frac{n-1}{2}} \notin L(\frac{2n}{n-1}, q)$ for $1 \leq q < \infty$, so we must have $a_h(s) = 0$ identically on E, that is $T_h = 0$ for all small h. By Titchmarsh's convolution theorem[cf. 17, p.135] and the definition of T_h, this implies $T_1 = 0$ and hence T=0 since U and ϕ are arbitrary. This is the end of the proof of theorem 4.

Since the function $(1 + |\xi|)^{-\frac{n-1}{2}} \in L(p, q)$ for $p = \frac{2n}{n-1}, q = \infty$; or $p > \frac{2n}{n-1}, 1 \leq q \leq \infty$, from the above Littman's estimate, we see that the result in theorem 4 is optimal.

Now we give an example to show that the curvature assumption in theorem 1 cannot be removed completely.

<u>Example 5.1</u> Let $x = (x_1, x_2) \in R^2$ and $x_0 \in R^2$ with $|x_0| = 3$. Let $U_1 = \{x; |x| < 1\}$, $U_2 = \{x; |x - x_0| < 1\}$, $U_1' = \{x; |x| < \frac{1}{2}\}$ and $U_2' = \{x; |x - x_0| < \frac{1}{2}\}$. Let $U = \{x: |x| < 5\}$ and choose $\alpha(x) \in C_0^5(U)$ such that $\alpha(x) = 1$ on $U_1' \bigcup U_2'$ and $\alpha(x) = 0$ on $U \backslash (U_1 \bigcup U_2)$. Define $\psi(x) \in C^5(U)$ by letting $\psi(x) = (2 - |x|^2)^{\frac{1}{2}} \alpha(x)$ for $x \in U_1$, $\psi(x) = x_2^2 \alpha(x)$ for $x \in U_2$, $\phi(x) = 0$ for $x \in U \backslash (U_1 \bigcup U_2)$

let $E = \{(x, \psi(x)), x \in U\}$, then E contains a sphere-piece $E_1 = \{(x, (1 - |x|^2)^{\frac{1}{2}}), x \in U_1'\}$ and a cylinder-piece $E_2 = \{(x_1, x_2; x_2^2), x \in U_2'\}$. Choose a nice measure T on E with the smooth density function contained in the piece of the sphere, then from Littman's estimate we have $\hat{T} \in L^p(R^3)$ for $p > 3$. Let $p = 4$, then lemma 3.1 yields that T vanishes on $J^5(E)$. So if theorem 1 is true for E, then we have from lemma 5.1 that

$$\|\hat{T}_h\|_4 \leq C \|\hat{T}\|_4.$$

We can make T_h for a suitable h to be a measure on E such that the C^1 density function $a_h(s)$ is not identically zero on the piece of the cylinder. Choose $\phi(x) \in C_0^\infty(R^2)$ such that ϕT_h is contained in the piece of the cylinder and non zero. Then we have since $\phi \in FL^1(R^3)$

$$\begin{aligned} \|\widehat{\phi T_h}\|_4 &= \|\hat{\phi} * \hat{T}_h\|_4 \\ &\leq C \|\hat{T}_h\|_4. \end{aligned}$$

But from Littman's estimate for the case $n = 2$, it is easy to see that $\phi T_h \notin FL^p(R^3)$

if and only if $p > 4$, for E_2 is a cylinder and the curve (x_2, x_2^2) has non-vanishing curvature. Hence for the manifold E constructed above, theorem 1 cannot hold.

Chapter 6

The Quadratic Case

Let U be open in R^2 such that the closure of U is compact. The goal of this chapter is to consider the property (B) for the 2-dimensional quadratic manifold with the form

$$M = \{(x_1, x_2; \psi_1(x_1, x_2), \psi_2(x_1, x_2)), (x_1, x_2) \in U\},$$

where $\psi_i(x_1, x_2) = a_i x_1^2 + b_i x_1 x_2 + c_i x_2^2$, $i = 1, 2$.

As usual, we denote the non-singular linear transformation group of R^n as GL_n. Let F be the subgroup of GL_4 generated by the non-singular linear transformations of (x_1, x_2) and the non-singular linear transformations of (x_3, x_4).

In the following M is called F-equivalent to M_1 if M_1 is in the orbit of M under the group F.

Let $\psi_{\xi_1,\xi_2}(x_1, x_2) = \xi_1\psi_1(x_1, x_2) + \xi_2\psi_2(x_1, x_2)$, then the Hessian determinant of ψ_{ξ_1,ξ_2}, denoted by $H\psi_{\xi_1,\xi_2}$, is only the function of (ξ_1, ξ_2). Actually $H\psi_{\xi_1,\xi_2}$ is a quadratic form of (ξ_1, ξ_2).

<u>Definition 6.1</u> Given $M = \{(x_1, x_2; \psi_1(x_1, x_2), \psi_2(x_1, x_2)), (x_1, x_2) \in U\}$ if M is not a 2-surface in any 3-dimensional subspace of R^4 and $H\psi_{\xi_1,\xi_2}$ is a degenerate quadratic form of (ξ_1, ξ_2), then M is called singular.

It is an easy exercise in linear algebra to check that the above definition is invariant under the subgroup F of GL_4. In the proof of the following theorem we will see that there are very few singular cases. Also we observe that the property (B) is invariant under the group GL_n

<u>Theorem 5</u> If M is non- singular, then (B) holds.

<u>Proof</u>: We divide the proof into several cases.

<u>Case 1</u>: One of ψ_i, $i = 1, 2$(say ψ_2) is positive definite (or equivalently negative definite).

In this case, we can find a non-singular linear transformation of (x_1, x_2) such that with the new coordinates, $\psi_1(x_1, x_2) = t_1 x_1^2 + t_2 x_2^2$, and $\psi_2(x_1, x_2) = x_1^2 + x_2^2$. So M is F-equivalent to $(x_1, x_2; t_1 x_1^2 + t_2 x_2^2, x_1^2 + x_2^2)$ If $t_1 = t_2$, using the non-singular linear transformations of (x_3, x_4), we see that M is F- equivalent to a 2-surface in R^3 with the form $(x_1, x_2; x_1^2 + x_2^2, 0)$, which has non-vanishing Gaussian curvature. So, (B) holds from theorem 2 by replacing $J^{n+2}(M)$ by $J(M)$ since M is a C^∞ manifold here.

If $t_1 \neq t_2$, by non-singular linear transformations of (x_3, x_4), we see that M is F equivalent to $M_1 = (x_1, x_2; x_1^2, x_2^2)$. For M_1, we can follow the argument in the proof of theorem 1 to obtain for $(\eta, \xi) = (\eta_1, \eta_2, \xi_1, \xi_2) \in R^4$,

$$\hat{T}_h(\eta, \xi) = \langle T, e^{-i(\langle \eta, y\rangle + \langle \xi, \zeta\rangle)} g_{\eta_1, \xi, h}(y_1) g_{\eta_2, \xi, h}(y_2)\rangle.$$

Here

$$g_{\eta_1, \xi, h}(y_1) = \tau_1(y_1) \int_R e^{i(h\eta_1\sigma_1 - \xi_1((y_1 - h\sigma_1)^2 - y_1^2))} \phi_1(\sigma_1) d\sigma_1,$$

$$g_{\eta_2, \xi, h}(y_2) = \tau_2(y_2) \int_R e^{i(h\eta_2\sigma_2 - \xi_2((y_2 - h\sigma_2)^2 - y_2^2))} \phi_2(\sigma_2) d\sigma_2.$$

Note that for $\tau(y)$and $\phi(\sigma)$ in theorem 1, here we have chosen $\tau(y) = \tau_1(y_1)\tau_2(y_2)$, $\phi(\sigma) = \phi_1(\sigma_1)\phi_2(\sigma_2)$.

Therefore as in the proof of theorem 1, we have

$$\begin{aligned} \hat{T}_h(\eta,\xi) &= C\langle T, e^{-i(\langle\eta,y\rangle+\langle\xi,\zeta\rangle)}\int_R e^{iw_1y_1}\hat{g}_{\eta_1,\xi,h}(w_1)dw_1\int_R e^{iw_2y_2}\hat{g}_{\eta_2,\xi,h}(w_2)dw_2\rangle \\ &= C\langle T, \int_{R^2} e^{-i(\langle\eta-w,y\rangle+\langle\xi,\zeta\rangle)}\hat{g}_{\eta_1,\xi,h}(w_1)dw_1\hat{g}_{\eta_2,\xi,h}(w_2)dw_2\rangle \\ &= C\int_{R^2}\hat{T}(\eta-w,\xi)\hat{g}_{\eta_1,\xi,h}(w_1)\hat{g}_{\eta_2,\xi,h}(w_2)dw_1dw_2. \end{aligned}$$

Note that the plane curve (x_1, x_1^2) has non-vanishing curvature, as in the proof of theorem 1, we can show with the help of lemma 1.3 that

$$\{\int_R|\hat{g}_{\eta_1,\xi,h}(w_1)|^2dw_1\}\{\int_R|\hat{g}_{\eta_1,\xi,h}(w_1)|^2|w_1|^2dw_1\} \le A.$$

So lemma 4.2 yields

$$|\int_R\hat{T}(\eta_1-w_1,\eta_2-w_2,\xi)\hat{g}_{\eta_1,\xi,h}(w_1)dw_1| \le M_{2,\eta_1}(\hat{T})(\eta_1,\eta_2-w_2,\xi).$$

Since we have the same control for $\hat{g}_{\eta_2,\xi,h}$ as for $\hat{g}_{\eta_1,\xi,h}$, we see that if we set

$$M_{2,\eta_1,\eta_2}(\hat{T})(\eta,\xi) = (\sup_{r_1>0,r_2>0}\frac{1}{4r_1r_2}\int_{|w_2-\eta_2|\le r_2}\int_{|w_1-\eta_1|\le r_1}|\hat{T}(w_1,w_2,\xi)|^2dw_1dw_2)^{\frac{1}{2}},$$

then we have

$$|\hat{T}_h(\eta,\xi)| \le CM_{2,\eta_1,\eta_2}(\hat{T})(\eta,\xi), \qquad C \text{ independent of } \eta,\xi,h.$$

Now the property (B) follows from the proof of theorem 2.

Case 2: None of ψ_i is positive definite but at least one of ψ_i is non-degenerate. We may assume $\psi_1(x_1,x_2) = x_1^2 - x_2^2$. The general form of ψ_2 is $ax_1^2 + bx_1x_2 + cx_2^2$

(i) $a = -c \neq 0$. It is ready to see that via the two non-singular linear trans formations of (x_3, x_4): $x_4' = \frac{1}{a}x_1$, $x_3' = x_3$ and $x_3' = x_3$, $x_4' = x_4 - x_3$, M becomes $M_1 = (x_1, x_2; x_1^2 - x_2^2, a_1x_1x_2)$. The case $a_1 = 0$ is reduced to a 2-surface in R^3 with non-vanishing Gaussian curvature since the Hessian matrix of $x_1^2 - x_2^2$ is non-singular. So we may assume $a_1 \neq 0$.

Let $\psi(x_1, x_2) = \xi_1(x_1^2 - x_2^2) + a_1\xi_2x_1x_2$, then

$$H\psi = \begin{vmatrix} 2\xi_1 & a_1\xi_2 \\ a_1\xi_2 & -2\xi_1 \end{vmatrix} = -4\xi_1^2 - a_1^2\xi_2^2 = -(4\xi_1^2 + a_1^2\xi_2^2).$$

We follow the proof of theorem 1 up to the following identity

$$\hat{T}_h(\eta,\xi) = \langle T, e^{-i(\langle\eta,y\rangle+\langle\xi,\zeta\rangle)} g_{\eta,\xi,h}(y)\rangle,$$

where $g_{\eta,\xi,h}(y) = \tau(y)\int_{R^2} e^{i(h\langle\eta,\sigma\rangle+\xi_1((y_1-h\sigma_1)^2-y_1^2))+\xi_2((y_2-h\sigma_2)^2-y_2^2))}\rho(\sigma)d\sigma$.

Now lemma 4.3 gives $\int_{R^2}|g_{\eta,\xi,h}(y)|^2dy \leq C(4\xi_1^2 + a_1^2\xi_2^2)^{-\frac{1}{2}} \leq C(\xi_1^2 + \xi_2^2)^{-\frac{1}{2}}$, with C independent of η, ξ, h. It follows that for $i = 1,2$

$$\int_{R^2} \left|\frac{\partial}{\partial y_i} g_{\eta,\xi,h}(y)\right|^2 dy \leq C(\xi_1^2 + \xi_2^2)^{\frac{1}{2}}.$$

where C is independent of η, ξ and h.

Thus the condition of lemma 4.2 is satisfied by using the Plancherel theorem So we obtain as in the proof of theorem 1 that

$$|\hat{T}_h(\eta, \xi)| \leq C M_{2,\eta}(\hat{T})(\eta, \xi), \qquad C \text{ independent of } \eta, \xi, h$$

The property (B) holds again from the proof of theorem 2.

(ii) $a \neq -c$ and a or c (say a) is not zero. We may assume $\varphi_2 = x_1^2 + a_1 x_1 x_2 + b_1 x_2^2$ with $1 + b_1 \neq 0$. In this case, M is F-equivalent to $M_1 = (x_1, x_2, x_1^2 - x_2^2, (1 + b_1) x_2^2 + a_1 x_1 x_2)$ (via the map $x_3' = x_3$, $x_4' = x_4 - x_3$) and again F-equivalent to $M_2 = (x_1, x_2, x_1^2 - x_2^2, x_1^2 + \frac{a_1}{1+b_1} x_1 x_2)$. For simplicity we denote $\frac{a_1}{1+b_1}$ as a_2.

If $|a_2| < 1$, then M_2 is F-equivalent to $M_3 = (x_1, x_2, x_1^2 - x_2^2, 2x_1^2 + 2a_2 x_1 x_2)$ and again F-equivalent to $M_4 = (x_1, x_2, x_1^2 - x_2^2, x_1^2 + a_2 x_1 x_2 + x_2^2)$.

Since

$$1 > 0, \quad \begin{vmatrix} 1 & a_2 \\ a_2 & 1 \end{vmatrix} = 1 - a_2^2 > 0,$$

we see that the quadratic form $\phi_2(x_1, x_2) = x_1^2 + 2a_2 x_1 x_2 + x_2^2$ is positive definite, which is contained in the case 1.

If $|a_2| > 1$, we consider $M_2 = (x_1, x_2, x_1^2 - x_2^2, x_1^2 + a_2 x_1 x_2)$.

Let $\varphi(x_1, x_2) = \xi_1(x_1^2 - x_2^2) + \xi_2(a_2 x_1 x_2 + x_2^2)$, then

$$H\varphi = \begin{vmatrix} 2\xi_1 & a_2\xi_2 \\ a_2\xi_2 & 2(\xi_2 - \xi_1) \end{vmatrix} = 4\xi_1(\xi_2 - \xi_1) - a_2^2\xi_2^2 = -(4\xi_1^2 - 4\xi_1\xi_2 + a_2^2\xi_2^2)$$

.

It is easy to check as above that the quadratic form $4\xi_1^2 - 4\xi_1\xi_2 + a_2^2\xi_2^2$ is positive definite and hence $\geq C(\xi_1^2 + \xi_2^2)$ with $C > 0$. The rest of the proof for property (B) is the same as in (i) of the case we are discussing now.

If $|a_2| = 1$, then M_2 and hence M is F-equivalent to $M_5 = \{x_1, x_2; x_1^2 - x_2^2, x_1^2 + x_1x_2\} = \{x_1, x_2, (x_1 - x_2)(x_1 + x_2), x_1(x_1 + x_2)\}$ and hence F-equivalent to $M_6 = \{x_1, x_2, x_1x_2, \frac{1}{2}(x_1 + x_2)x_2\}$, using the map $x_1' = x_1 - x_2$, $x_2' = x_1 + x_2$. Now it is obvious to see that M_6 is F-equivalent to $M_7 = \{x_1, x_2; x_1x_2, x_2^2\}$. Our method fails for M_7, but it is easy to check the singularity of M_7.

(iii) $\phi_1 = x_1^2 - x_2^2, \phi_2 = x_1x_2$.

Again we let $\psi(x_1, x_2) = \xi_1(x_1^2 - x_2^2) + \xi_2(x_1x_2)$ and have

$$H\psi = \begin{vmatrix} 2\xi_1 & \xi_2 \\ \xi_2 & -2\xi_1 \end{vmatrix} = -4\xi_1^2 - \xi_2^2 = -(4\xi_1^2 + \xi_2^2).$$

From this, , the property (B) follows as above.

Case 3: ϕ_1 and ϕ_2 are both degenerate.

The general form of M in this case is $\{x_1, x_2; (ax_1 + bx_2)^2, (cx_1 + dx_2)^2\}$

(i) M is F-equivalent to $M_1 = \{x_1, x_2; x_1^2, x_2^2\}$ if $ac - bd \neq 0$.

This case is contained in the case 1.

(ii) M is F-equivalent to $M_2 = \{x_1, x_2; x_1^2, 0\}$ if $ac - bd = 0$ but at least one of a, b, c, d is not zero.

This time M_2 is a cylinder in R^3. Since (x_1, x_1^2) is a curve in R^2 with non vanishing curvature, we can follow the proof of theorem 1 closely to see that the conclusion of theorem 1 is valid for M_2 here and the property (B) follows

(iii) M is F-equivalent to $M_3 = (x_1, x_2; 0, 0)$.

The proof for this case is trivial.

This is the end of the proof of theorem 5.

Chapter 7

Some Open Problems

We have already finished the proof of our results in the previous chapters. As we pointed out, the interesting point of the approach adopted in this thesis is its application to the uniqueness property of some partial differential equations. Hörmander [14] proved for a general smooth manifold M that If $T \in S'(R^n)$ with $supp(T) \subset M$ and if $\hat{T} \in L^p(R^n)$, $1 \leq p \leq \frac{2n}{n-1}$, then $T = 0$. This result is known to be optimal in L^p sense only for the case when M has non-vanishing Gaussian curvature. Our theorem 4 can be viewed as the modification of Hormander's result in the Lorentz spaces.

We may ask how about the vanishing curvature case? In this situation the 2-dimensional cone in R^3, $(\xi_1, \xi_2, (\xi_1^2 + \xi_2^2)^{\frac{1}{2}})$, is *good* since the corresponding Hessian matrix has constant rank 1(so our counter-example for theorem 1 doesn't work in this case). As the unit sphere comes from the Helmholtz equation in R^n via the Fourier transform, the above cone comes from the wave equation in R^2:

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) = 0.$$

We cannot expect the method in [14] to give the answer since the geometric property of the manifold plays no role there. If we think about the method developed in this thesis, we will see that the feature of it is to transfer the information from a general distribution supported on M to a nice measure on M. The idea behind the discussion is that we believe that If $T_1 \in S'(R^n)$ with $supp(T_1) \subset M$ and $T_2 \in S'(R^n)$ generated by a smooth measure on M(which implies $supp(T_2) \subset M$), then the L^p behaviour of $\hat{T}_1$ is at most as good as the L^p behaviour of $\hat{T}_2$

This idea is verified when M has non-vanishing Gaussian curvature. We conjecture that the same result as in our theorem 1 would hold also for the cone, namely, we could prove for $(\eta, \xi) \in R^2 \times R^1$

$$|\hat{T}_h(\eta, \xi)| \leq C M_{2,\eta}(\hat{T})(\eta, \xi).$$

Assuming this , we may have $\hat{T} \in L^p, 1 \leq p \leq 4$(rather than 3) implies $T = 0$ suggested by the result in [25].

For the case when M has co-dimension ≥ 2, the simplest case in mind is

$$M = \{(t, t^2, t^3), t \in (-1, 1)\}.$$

which is a C^∞ curve in R^3 with non-vanishing torsion

Let

$$\hat{T}_h(\xi) = \int_M e^{-i\xi\cdot s} a_h(s) ds, \quad a_h \in C_0^\infty(M)$$

Using the Van der Corput's lemma carefully, It is not difficult to [illegible] that

$$|\hat{T}_h(\xi)| \leq C_h(1 + |\xi|)^{-\frac{1}{3}}.$$

This is far from exact since we know in [8] that $\hat{T}_h \in L^p(R^3)$ for $p > 7$ and it may be true that $\hat{T}_h \in L^p$ if and only if $p > 7$ provided $a_h(s)$ is not identical zero on M

Thus theorem 1 again suggests the conjecture that if $T \in S'(R^3)$ supported on M and if $\hat{T} \in L^p(R^3)$ for $1 \leq p \leq 7$, then $T = 0$. The obstacle to proving theorem 1 in this case is that we cannot prove a result similar to lemma 4.3.

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