Colourings and games on sparse graphs

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Abstract

This thesis contains five manuscripts concerning various problems in structural graph theory and graph searching. The problems all concern sparse graphs, either graphs with forbidden minors, with bounded maximum degree, or trees.

First, we consider two variants of Hadwiger's conjecture on the chromatic number. Hadwiger's conjecture states that the number of colours required to properly colour the vertices of a graph (that is, adjacent vertices receive different colours) is at most the order of the largest complete graph we can obtain by contracting edges and deleting vertices (we call the obtained graph a minor of the original graph). Our variants are obtained by modifying the condition that the forbidden minor is a complete graph. In the first case, we show that the conjecture holds if we replace the forbidden complete graph by a sparse bipartite graph. In fact, a stronger result on degeneracy is shown, which is qualitatively tight. In the second variant, instead of forbidding a single minor, we forbid all graphs with some fixed numbers of vertices and edges and bound the average degree of the original graph.

We then present results on a variant of Menger's theorem in which the obtained paths are pairwise non-adjacent, which is motivated by recent questions in coarse graph theory. Using strong edge colourings, this variant is proved for graphs with bounded maximum degree. Some stronger results for subcubic graphs are shown, in part in a computer-assisted proof, as well as for graphs with a forbidden topological minor.

We also consider graph burning, a process representing the propagation of information on networks. It is shown that the Burning Number Conjecture, the main problem on the topic, holds asymptotically, using probabilistic techniques.

Finally, the game of cops and robbers on graphs with an excluded minor is studied. New upper bounds on the cop number for these graphs are proved, generalizing and strengthening the previous work of Andreae.

Abrégé

Cette thèse contient cinq articles sur divers problèmes en théorie des graphes structurelle et en jeux de recherche sur les graphes. Les problèmes concernent tous des graphes creux, soit des graphes avec des mineurs interdits, avec degré maximal borné ou des arbres.

Nous considérons tout d'abord deux variantes de la Conjecture de Hadwiger. Cette conjecture stipule que le nombre de couleurs nécessaire pour colorier proprement les sommets d'un graphe (c'est-à-dire, les sommets adjacents reçoivent des couleurs distinctes) est au plus l'ordre du plus grand graphe complet qu'on peut obtenir en contractant des arêtes et supprimant des sommets (on appelle le graphe obtenu un mineur du graphe original). Nos variantes sont obtenues en modifiant la condition que le mineur interdit soit un graphe complet. Dans le premier cas, nous montrons la conjecture tient si on remplace le graphe complet qu'on interdit par un graphe biparti creux. En fait, un résultat plus fort sur la dégénérescence est démontré, qui est qualitativement serré. Dans la seconde variante, au lieu d'interdir un unique mineur, nous interdisons tous les graphes avec des nombres fixés de sommets et d'arêtes et bornons le degré moyen du graphe original.

Nous présentons ensuite des résultats sur une variante du Théorème de Menger, motivée par des questions récentes en théorie des graphes grossière, dans laquelle les chemins obtenus ne sont pas adjacents. En utilisant des coloriages d'arêtes forts, nous prouvons cette variante pour les graphes avec degré maximal borné. Des résultats plus forts pour les graphes sous-cubiques sont prouvés, en partie dans une preuve assistée à l'ordinateur, ainsi que pour les graphes avec un mineur topologique interdit.

Nous considérons aussi le brûlage de graphes, un processus représentant la propagation d'information sur des réseaux. Il est prouvé que la Conjecture du nombre de brûlage, le principal problème sur le sujet, tient asymptotiquement, en utilisant des méthodes probabilistes.

Finalement, le jeu de policiers-voleur sur les graphes avec un mineur interdit est étudié. De nouvelles bornes supérieures sur le nombre de policiers pour ces graphes sont prouvées, généralisant et renforcissant les résultats de Andreae.

Contribution

In this thesis, five manuscripts are presented. For all of these manuscripts, all coauthors are considered to be co-first authors, as in pure mathematics the tradition is to list the authors in alphabetical order. I participated significantly in the preparation of each manuscript.

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Part I

Introduction and literature review

1

Introduction

This thesis consists of five manuscripts on various problems in structural graph theory and graph searching games. Each topic concerns graphs which are sparse in some sense, generally graphs with a forbidden minor, but also graphs with bounded maximum degree.

Each manuscript contains an introduction specific to its precise topic. However, in this section, we will introduce each topic more broadly and present some of the relevant similar results, sometimes presenting simple proofs when possible.

We will use mostly standard graph-theoretic notation. Our graphs will generally be simple, finite graphs. If G is a graph, we write $V(G)$ and $E(G)$ for its set of vertices and edges, respectively. If $u \in V(G)$, we write $N(u)$ for its *neighbourhood*, i.e. its set of neighbours, and $N[u] := N(u) \cup \{u\}$ for its *closed neighbourhood*. Then, $d(u) := |N(u)|$ is the *degree* of u. We write $\delta(G)$ and $\Delta(G)$ for the minimum and maximum degrees of G, respectively. If S is a vertex or a set of vertices of G, we write $G - S$ for the subgraph of

G induced on $V(G) \setminus S$, and if M is an edge or a set of edges, $G - M$ for the (spanning) subgraph of G obtained by removing the edges in M. For $u, v \in V(G)$, we write $\mathsf{dist}(u, v)$ for the *distance* between u and v, that is the number of edges in the shortest path between u and v (and ∞ if no such path exists). For $s, t \in \mathbb{N}$, we will write K_t for the complete graph on t vertices, and $K_{s,t}$ for the complete bipartite graph with parts of size respectively s and t. For $r \in \mathbb{Z}^{\geq 0}$, we write $B(u, r) := \{v \in V(G) : \mathsf{dist}(u, v) \leq r\}$ for the *ball* of radius r centered at u. We will use the convention that $[n] = \{1, 2, \ldots, n\}$, and that $0 \notin \mathbb{N}$.

1.1 Graph minors

1.1.1 Motivation and definition

Many of the manuscripts in this paper concern the structure of graphs forbidding a minor. In this section, we will introduce this concept and discuss its motivation.

Let G be a graph, and $uv \in E(G)$. We define G/uv as the graph obtained by *contraction* of uv as the graph with vertex set $(V(G) \setminus \{u, v\}) \cup \{w\}$ and edge set $(E(G) \setminus \{e : u \in e \text{ or } v \in e\}) \cup \{wx : x \in V(G) \setminus \{u, v\}, \text{ and } ux \in E \text{ or } vx \in E\}$. Intuitively, we are "merging" the vertices u and v to create a new vertex w, and removing any multi-edges that are created by this operation.

We then say that a graph H is a *minor* of G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges.

One of the easiest ways of working with minors is with the concept of models. A *model* of H in G is a function μ which maps vertices of H to pairwise disjoint subsets of vertices of $V(G)$, with the conditions that for every $u \in V(G)$, the subgraph of G induced by $\mu(u)$ is connected, and that for every $uv \in H$, there exists an edge between a vertex of $\mu(u)$ and a vertex of $\mu(v)$. The following is standard and easy to prove.

Lemma 1.1.1. *There exists a model of H in G if and only if H is a minor of G.*

Intuitively, if H is a minor of G, then in the corresponding model μ of H in G, $\mu(u)$ is the set of vertices which are contracted into u.

The main motivation for studying graph minors is its connection to graph topology. We say a graph is *planar* if it can be drawn in the plane (vertices corresponding to distinct points, edges corresponding to continuous curves) with distinct edges never intersecting (other than at their endpoints). In a drawing of a planar graph G, a *face* is a connected region of $\mathbb{R}^2 \setminus G$. We write $F(G)$ for the set of faces in a planar drawing of a planar graph G (the following result will make it clear that the size of this set does not depend on the choice of the drawing). The following is one of the best known results on planar graphs.

Theorem 1.1.2 (Euler's formula)**.** *If G is a connected non-null planar graph, then $|V(G)| - |E(G)| + |F(G)| = 2$.*

The following proof is standard.

Proof. We prove the following stronger statement. Write $\mathrm{cc}(G)$ for the number of connected components of G. We claim that $|V(G)| - |E(G)| + |F(G)| = 1 + \mathrm{cc}(G)$.

We fix $|V(G)|$ and show the statement by induction on $|E(G)|$. For the base case, consider the graph with no edges. Each vertex is its own connected component, and there is only one face (the unbounded one). Then,

$$|V(G)| - |E(G)| + |F(G)| = |V(G)| + 1 = 1 + \mathrm{cc}(G).$$

For the inductive step, consider the process of adding one edge. Let $uv \in E(G)$. If u, v are in distinct components of $G - uv$, then adding the edge uv does not create any new faces (keeping the rest of the drawing unchanged) but does reduce the number of connected components by one, and so

$$\begin{aligned} |V(G)| - |E(G)| + |F(G)| &= |V(G-uv)| - (|E(G-uv)| + 1) + |F(G-uv)| \\ &= (1 + \mathrm{cc}(G-uv)) - 1 = \mathrm{cc}(G) + 1. \end{aligned}$$

Otherwise, adding uv does not create any new connected components, but does split one face into two. Hence,

$$\begin{aligned}|V(G)| - |E(G)| + |F(G)| &= |V(G - uv)| - (|E(G - uv)| + 1) + (|F(G - uv)| + 1) \\ &= \mathrm{cc}(G - uv) + 1 = \mathrm{cc}(G) + 1.\end{aligned}$$

□

The following consequence will be useful to us. We use the formulation (and proof) seen at [24]. We say the *girth* of a graph (which is not a forest) is the length of its shortest cycle. Forests are often said to have infinite girth, however we will not use this convention in the next result.

Corollary 1.1.3. *If G is a connected planar graph of girth at least $g \geq 3$, then $|E(G)| \leq (|V(G)| - 2)\,\frac{g}{g-2}$.*

Proof. In any planar drawing of G, every edge is in exactly 2 faces (possibly, the same face with multiplicity 2). On the other hand, the boundary of every face contains at least g edges. Hence, by double counting, $2|E(G)| \geq g|F(G)|$. The result following by substituting this inequality into Theorem 1.1.2. □

It is an immediate consequence of this corollary that K_5 (5 vertices, 10 edges, girth 3) and $K_{3,3}$ (6 vertices, 9 edges, girth 4) are not planar graphs.

If a graph G is planar, then G/uv is also planar. Broadly speaking, one can take a planar drawing of G and merge u and v along the curve corresponding to uv (one must be a bit careful as to not create any intersections with the other edges incident to u or v). Hence, if a graph contains either K_5 or $K_{3,3}$ as a minor, it cannot be planar. It turns out that this is actually a full characterization of planar graphs.

Theorem 1.1.4 (Wagner's theorem)**.** *[103] A graph G is planar if and only if it does not contain K_5 and $K_{3,3}$ as minors.*

One of the most important results in graph theory is the Robertson-Seymour graph minor theorem [87], which was proved in a series of 20 papers. We say a family (possibly, infinite) of graphs $\mathcal{G}$ is minor-closed if every minor of a graph in $\mathcal{G}$ is also in $\mathcal{G}$ (up to isomorphism).

Theorem 1.1.5 ([87])**.** *If $\mathcal{G}$ is a minor-closed family of $\mathcal{G}$, then there exists a a finite set of graphs* $\mathrm{Ob}(\mathcal{G})$*, called the obstruction set of $\mathcal{G}$, such that $G \in \mathcal{G}$ if and only if G does not have any minor in* $\mathrm{Ob}(\mathcal{G})$*.*

We assume the obstruction set is always chosen to be minimal (and doesn't countain isomorphic graphs). In this language, the obstruction set of planar graphs is $\{K_5, K_{3,3}\}$ by Wagner's theorem. Another example is the class of outerplanar graphs. We say a graph is *outerplanar* if it has a planar drawing such that all the vertices are on the outside face. It is easily seen that this family is minor-closed. The following is well-known.

Theorem 1.1.6. *A graph G is outerplanar if and only if it does not contain K_4 and $K_{2,3}$ as minors.*

The argument here is simple (see [51] for example).

Proof. If G is outerplanar, then $U(G)$, the graph obtained by adding a universal vertex of G (a vertex adjacent to all other vertices), is planar, as we can place this vertex in the outer face.

This first implies that $K_4, K_{2,3}$ are not outerplanar, as $K_5, K_{3,3}$ are subgraphs of $U(K_5), U(K_{2,3})$, respectively. Hence, if G contains either K_4 or $K_{2,3}$ as a minor, it is not outerplanar.

On the other hand, if G is not outerplanar, then $U(G)$ is not planar: either G is itself not planar, or otherwise in every planar drawing of G there does not exist a face on which all the vertices are present, so we cannot add our universal vertex. Hence, $U(G)$ contains either K_5 or $K_{3,3}$ as a minor, say H, by Wagner's theorem. By Lemma 1.1.1, there exists a model μ of H in G. Let v be the universal vertex of $U(G)$. Let u be the unique vertex of H such that $v \in \mu(u)$, if it exists, and otherwise let u be chosen arbitrarily. It is easily verified that μ, restricted to $V(H-u)$, is a model of $H-u$ in $U(G)-v=G$. To conclude the proof, it suffices to note that $K_5-u\simeq K_4$ and $K_{3,3}-u\simeq K_{2,3}$. $\square$

We note that planar graphs are exactly the graphs which can be embedded on the sphere (orientable surface of genus 0) without crossings. We consider bounding the cop number for higher genus graphs as well. The same argument as before shows us that the class of graphs of genus at most g is minor-closed. However, the obstructions sets for these families can be extremely large. Notably, we say a graph is *toroidal* if it can be drawn without crossings on the torus. The obstruction set for toroidal graphs is known to contain at least 17,535 graphs [73].

One can also embed graphs on non-orientable surfaces. For example, the obstruction set for graphs which can be drawn on the real projective plane without crossings contains exactly 35 graphs [48, 10].

All of these are surfaces: two-dimensional manifolds. Another way of generalizing planar graphs is thus to determine which graphs can be embedded in $\mathbb{R}^3$. However, it is easily seen that any graph can be drawn in $\mathbb{R}^3$ without crossings: there are too many degrees of freedom. A more interesting condition is the following. Note that in a planar drawing of a graph, every cycle separates the plane: there are no edges between a vertex inside and outside this cycle. In particular, two cycles cannot intertwine. Thus, we say a graph is said to be *linklessly embeddable* if it can be drawn in $\mathbb{R}^3$ such that there are no two cycles which form a *link* (imagine two rings, each of which passes through the other once). Robertson et al. [89] have shown that the obstruction set of linklessly embeddable graphs is the Petersen family (it contains 7 graphs and it includes, in particular, K_6 and the Petersen graph), see Figure 5.5.1.

Graphs of bounded treewidth are also minor-closed (see the definition of tree-decompositions in Section 4.3, the *treewidth* of a graph is the minimum over all tree-decompositions of this graph of the maximum size of a bag, minus one).

1.1.2 Topological minors

We briefly mention a related topic, the concept of *topological minors*. We say a graph is a *subdivision* of a graph H if it can be obtained by replacing some (or none) of the edges of H by paths (i.e., by repeatedly *subdividing* edges). Then, we say H is a *topological minor* of a graph G if G contains a subdivision of H as subgraph.

We first note that this concept is more restrictive than minors.

Lemma 1.1.7. *If H is a topological minor of G, then H is a minor of G .*

Proof. Let H' be the subdivision of H which is a subgraph of G. Given that H' can be obtained from H by a sequence of edge subdivisions, by contracting these edges we can obtain H from H'. Hence, H is a minor of H', which is itself a subgraph (which thus also a minor) of G. □

In general, containing a graph as a minor and as a subdivision are not equivalent. There are however some specific cases for which this turns out to be true. One of these is the case of planar graphs. The following preceded Wagner's theorem.

Theorem 1.1.8 (Kuratowski's theorem)**.** *[58] A graph G is planar if and only if it does not contain K_5 and $K_{3,3}$ as topological minors.*

More generally, the following lemma is well-known.

Lemma 1.1.9. *If H is a minor of G and $\Delta(H) \leq 3$, then H is a topological minor of G.*

Proof. By Lemma 1.1.1, there exists a model μ of H in G. Let $u \in V(H)$. As $\Delta(H) \leq 3$, there are at most 3 edges coming out of $\mu(u)$ which are used by the model. For simplicity, assume there are exactly 3. Let $a, b, c \in \mu(u)$ be the ends of these edges. As $\mu(u)$ is connected, there exists a tree in the graph induced by $\mu(u)$ and which contains a, b, c. Let T_u be such a tree, chosen to be minimal. By minimality, only a, b, c can be leaves of T_n so it can only be a subdivision of a claw, the unique tree with 3 leaves (if we allow here the paths to have length 0). A subgraph isomorphic to a subdivision of H can then be obtained by taking the union of T_u for every $u \in H$ and the edges between the bags. In this subdivision, the vertex corresponding to u is the "center" of every claw. $\square$

We further note in Chapter 4, we will use Theorem 4.3.1, an important structure theorem for topological minors,in which minors plays an important part. Given its technical nature and since it will only be used once, we postpone its statement.

1.2 Colourings

1.2.1 Vertex colourings and Hadwiger's conjecture

A *proper vertex k-colouring* of a graph G is a function $f : V(G) \to [k]$ such that adjacent vertices always receive distinct colours. We say G is *k-colourable* if a proper vertex k-colouring of G exists, and we define the *chromatic number* $\chi(G)$ as the smallest k such that G is k-colourable. Of course, this is well-defined, as any graph is trivially $|V(G)|$-colourable.

Guthrie conjectured in 1852 that any planar graph is 4-colourable. In practical terms, this would mean that any map in which regions (say, countries) are simply connected can be coloured using 4 colours in a way that adjacent regions receive different colours. For example, [46] is a world map using 4 colours. Of course, this bound cannot be improved: K_4 is a planar graph but cannot be properly coloured with 3 colours.

This conjecture was later proved by Appel and Haken [8, 9] (see also Robertson et al. [86]) in a famous, computer-assisted proof.

Theorem 1.2.1 (Four colour theorem [8, 9])**.** *If G is a planar graph, then $\chi(G) \leq 4$.*

Although the proof of this theorem is very involved, and computer-assted, proving that, say, $\chi(G) \leq 6$, is trivial. Indeed, it is a well-known consequence of Euler's formula (see Corollary 1.1.3) that any planar graph must contain a vertex with degree at most 5. We proceed by induction. Let G be a planar graph and u be a vertex of degree at most 5. Then, $G - u$ is also planar, so it can be properly coloured with 6 colours. Given that u has at most

5 neighbours, we can extend this 6-colouring to G: at least one of the 6 colours does not appear in its neighbourhood.

Recall that Wagner's theorem states that planar graphs are exactly the $\{K_5, K_{3,3}\}$-minor-free graphs. Generalizing the Four colour conjecture (now theorem), Hadwiger [49] conjectured the following.

Conjecture 1.2.2 (Hadwiger's conjecture [49])**.** *If $t \in \mathbb{N}$ and G is a K_t-minor-free graph, then $\chi(G) \leq t - 1$.*

Hadwiger [49] has shown the cases $t \leq 4$. The conjecture is trivial for $t = 1, 2$, since K_1-minor-free graphs and K_2-minor-free graphs are respectively empty and edgeless. Any cycle can be contracted into a K_3, so K_3-minor-free graphs are forests. These are easily seen to be 2-colourable: choose an arbitrary vertex u, and partition the vertices according to whether their distance to u is even or odd. Wagner [104] showed that the case $t = 5$ was equivalent to the Four colour theorem, hence it was proved by Appel and Haken [8, 9]. The case $t = 6$ was proved by Robertson et al. [88]. All other cases $t \geq 7$ remain open; in fact it has not even be proved that K_7-minor-free graphs are 7-colourable.

Hadwiger's conjecture is considered to be one of the most important conjectures in graph theory, and one of the most difficult. Given how hard this conjecture is, much of the progress has been in approaching the conjecture in another direction: if a graph is K_t-minor-free, what is the best upper bound on $\chi(G)$ we can get?

For many years, the best such bound was $\chi(G) \leq O\left(t\sqrt{\log t}\right)$, as proved by Kostochka [56, 57] and Thomason [99]. Their proofs use degeneracy: a graph G is said to be *k-degenerate* if every subgraph of G has minimum degree at most k. A k-degenerate graph necessarily has chromatic number at most $k+1$, by the same argument as our proof above of the Six colour theorem (planar graphs are 5-degenerate). However, it is known that this bound cannot be improved by degeneracy alone, due to random graph examples [99].

This bound was later improved by Norin et al. [75]. Their argument is broadly as follows. Small graphs are coloured greedily, by repeatedly extracting independent sets and assigning a colour to these. Furthermore, by the result of Kostochka and Thomason mentioned above, every graph which has high average degree contains a forbidden complete minor, so we can assume that the graph is sparse. Otherwise, they use a *density increment* argument: if the graph forbids some large complete minor, it must contain some small subgraph for which the average degree is high (for its size). Repeatedly applying the density increment, they extract many such small subgraphs. We will also use this tool in Chapter 2. Using that the graph will need to be sufficiently connected (otherwise, it is easy to colour), they use some linkage arguments (we will also use similar arguments in Chapter 2) to piece together these small pieces to form a sufficiently large complete minor.

We note that currently, the best upper bound is $O(t \log \log t)$, which was proved by Delcourt and Postle [37].

In the work we present here, we will approach Hadwiger's conjecture in another direction: instead of forbidding a complete minor, we will forbid a sparser minor and attempt to obtain the same bound on the chromatic number as Hadwiger's conjecture predicts (which depends on the number of vertices). This approach, named the H-Hadwiger conjecture, was suggested by Seymour [93, 94].

In Chapter 2, we will forbid a graph H which is bipartite, has bounded maximum degree, and is structurally sparse. In this case, we will in fact not even use colouring directly, and simply bound the degeneracy of graphs forbidding H as a minor, given the relationship between chromatic number and degeneracy mentioned above. We will also show that this degeneracy result is best possible, in multiple ways.

In Chapter 3, we will slightly modify this problem by not forbidding only one graph H as a minor, but forbid all graphs on t vertices and a given number of edges. How many edges can we forbid in t-vertex minors and still obtain the desired chromatic number? This problem was previously studied by Norin and Seymour [76] in the case of graphs with independence number 2. In general, a result of Mader [64] (see Lemma 3.2.1) directly yields the result if we forbid t-vertex minors with at most 25% of possible edges them. Here as well, we will not work directly with the chromatic number, but instead use average degree: if the average degree of any graph forbidding all of these minors cannot be more than $t-1$, the minimum degree also cannot be more than $t-1$, and thus degeneracy again shows that our graph is t-colourable. We will show that we can obtain a fraction of $\sqrt{2}-1$ of all possible edges, that it is not possible to obtain more than 75% of them only using average degree, and we will obtain some exact results for small t.

1.2.2 Strong edge colourings

There are many variants of colouring as defined in the previous subsection. The most notable is *proper edge colouring*. Here, the difference is what we wish to colour every edge, such that incident edges receive different colours. The *chromatic index* of a graph G, denoted $\chi'(G)$, is the smallest number of colours with which it is possible to properly colour the edges of G.

It is clear that $\chi'(G) \geq \Delta(G)$, given that around a vertex of maximum degree every edge must have a different colour. It is also easy to see that $\chi'(G) \leq 2\Delta(G) - 1$: any edge is incident to at most $2(\Delta(G)-1)$ other edges, and so we can colour it greedily using the extra colour.

One of the classic results in graph theory is Vizing's theorem [102], which states that $\chi'(G) \leq \Delta(G) + 1$.

Our proof method in Chapter 4 uses a variant of proper edge colouring called *strong edge colouring*. In a strong edge colouring, we require not only incident edges to be given distinct colours, but also any pair of edges for which there exists a third edge, incident to both of them. We denote $\chi'_s(G)$ the *strong chromatic index* of G, the smallest number of colours with which it is possible to properly to strongly colour the edges of G.

Consider an edge uv. There are at most $\Delta(G)-1$ other edges incident with u, and for each one of these, there are further up to $\Delta(G)-1$ edges incident with them at their other endpoint. This also holds for v. Hence, there are at most $2((\Delta(G)-1)+(\Delta(G)-1)^2) = 2\Delta(G)(\Delta(G)-1)$ edges which must be given a colour different from uv. Hence, here again by the greedy argument, one more colour suffices to strongly colour G. Although there are stronger bounds, for our purposes it is only important to know that $\chi'_s(G) \leq O(\Delta(G)^2)$.

1.3 Coarse graph theory

Coarse geometry is the study of the large-scale properties of geometric objects. It was pioneered by, in particular, Mostow and Gromov, and had particular impact in geometric group theory. By seeing graphs as metric spaces, it has some connections to structural graph theory, for instance Bonamy et al. [15] have shown that the asymptotic dimension of any minor-closed graph family is at most 2.

Motivated by these, Georgakopoulos and Papasoglu [47] have recently proposed studying *Coarse Graph Theory*, i.e. the study of the large-scale geometry of graphs.

In Section 1.1.1, we defined models in order to study graph minors. In a model μ of a graph H in a graph G, we required that if $uv \in E(H)$, then there exists an edge with one end in $\mu(u)$ and one end in $\mu(v)$. However, this requirement could be weakened to requiring there exists a path P from $\mu(u)$ to $\mu(v)$, which is internally disjoint from $\bigcup_{x\in V(H)} \mu(x)$, as well as from other such paths between other parts of the model. Indeed, to obtain the edge uv in the minor, we contract every edge of P except one. In Chapter 5, our definition of models will be closer to this version.

From this version of models, it is natural to consider minors in which, for example, all of these paths are quite long. Indeed, even if G contains H as a minor, this does not necessarily tell us much about the global structure of G, for instance if H is already a minor of a small subgraph of G. We may then define the following.

A *fat minor* of a graph H in a graph G is a generalization of a minor to large distances. Precisely, we say H is K-*fat minor* of G if there exists connected sets $B_v \subseteq V(G)$ (for $v \in V(H)$) which are pairwise at distance at least K apart, and for each $uv \in E(H)$ there exists a B_u-B_v path P_{uv} such that P_e and $P_{e'}$ are distance-K-apart for every $e \neq e' \in E(H)$, and such that P_e and B_v are distance-K-apart for every $v \in V(H)$ and $e \in E(G)$ not containing v.

As we have mentioned earlier, planar graphs are exactly the graphs not containing K_5 and $K_{3,3}$ as minors. What can we say about graphs which contain one of these as minors, but do not contain them as K-fat minors? As the obstructions to being planar are local, could we say that are these graphs are planar in some global sense? For instance, if we "zoom out", does the graph look planar? More generally, we will need the following definition.

We say two graphs G, H are K*-quasi-isometric* if there exists $f : V(G) \to V(H)$ such that $\frac{\mathsf{dist}_G(u,v)}{K} - K \leq \mathsf{dist}_H(f(u), f(v)) \leq K\mathsf{dist}_G(u,v) + K$ for every $u, v \in V(G)$.

With this definition, we can rephrase the previous question as follows. If G does not contain K_5 and $K_{3,3}$ as K-fat minors, is G K'-quasi-isometric to a planar graph?

In general, Georgakopoulos and Papasoglu [47] have conjectured the following.

Conjecture 1.3.1. *For every graph H, there exists $f_H : \mathbb{N} \to \mathbb{N}$ such that the following holds. If G is a graph and $K \in \mathbb{N}$, then G has no K-fat H minor if and only if G is $f_H(K)$-quasi-isometric to a graph with no H minor.*

This conjecture is known to hold for a few graphs H, such as K_3 [47], $K_{2,3}$ [28], K_4^- [44], K_4 [4], and stars ($K_{1,m}$) [47].

The following is a critical tools in graph theory.

Theorem 1.3.2 (Menger's theorem)**.** *[69] If $k \in \mathbb{N}$, G is a graph and $X, Y \subseteq V(G)$, then there exists either*

(1) *k pairwise disjoint X-Y-paths, or*

(2) *a set of less than k vertices which separates X and Y.*

Menger's theorem and other related connectivity tools are especially crucial in the study of graph minors, as we will see in Chapter 2 and Chapter 3. In Chapter 2 in particular, our general approach is based on constructing disjoint small parts of the desired minor and finding paths between these pieces in order to construct the entire minor. In order to prove

Conjecture 1.3.1, having a coarse version of Menger's theorem could be useful: we would like the paths that are found to be pairwise far-apart. Georgakopoulos and Papasoglu [47], and independently Albrechtsen et al. [3], have conjectured the following.

Conjecture 1.3.3 (Coarse Menger's Conjecture)**.** *For every $k \in \mathbb{N}$, there exists $c = c(k) \in \mathbb{N}$ satisfying the following. If $d \in \mathbb{N}$, G is a graph and $X, Y \subseteq V(G)$, then there exists either*

(1) *k disjoint X-Y-paths $P_1, \ldots, P_k$ such that $\mathsf{dist}(P_i, P_j) \geq d$ for all distinct i, j, or*

(2) *a set $Z \subseteq V(G)$ of size less than k such that $B(Z, cd)$ separates X and Y.*

Both groups have shown this conjecture to hold for $k = 2$: if G does not contain two far-apart X-Y paths, then X, Y can be separated by removing one ball of bounded diameter.

It turns out however that both Conjecture 1.3.1 and Conjecture 1.3.3 are false. Conjecture 1.3.3 was disproved by Nguyen et al. [74], and based on their construction Conjecture 1.3.1 was disproved by Davies et al. [36]. We note that this is not mentioned in Chapter 4, as our manuscript predates these developments. We note that Davis et al. have mentioned that their construction does not disprove Conjecture 1.3.1 for planarity, as their counter-example family contains arbitrarily fat K_5 and $K_{3,3}$ minors.

In Chapter 4, inspired by Conjecture 1.3.3, we work on a similar problem in the case $d = 2$: that is, we want the obtained paths to be induced. In our case, we will replace the condition that a ball of bounded diameter separates X and Y by the condition that it is a set of Ck vertices which separates X and Y (for some constant C depending on the maximum degree of G). This version is also quite natural, as it is closer to the Menger's theorem. A version in which the graph does not have bounded degree, but forbids a topological minor, is also deduced using a structure theorem for topological minors. We will also show improved bounds in the subcubic case, notably using a computer-assisted proof.

We note that some of the results of Chapter 4 were obtained independently and concurrently by Gartland et al. [45].

1.4 Graph searching

Graph searching is an area of research in graph theory which studies various games (or processes), such as cops and robbers, graph burning, firefighting, and their many variants, in which the players explore the graph in some sense.

1.4.1 Cops and robbers

As the name suggests, this is a pursuit-evasion game in which cops, controlled by the first player, attempt to catch a robber, controlled by the second player. The game is played on the vertices of a connected graph G, and the possible moves correspond to following an edge. On the first turn, the first player places its cops on (not necessarily distinct) vertices of the graph, after which the second player chooses a position for the robber. In subsequents turns, the first player may move any number of its cops, after which the second player may move the robber. The cops win if they capture the robber. Otherwise, the robber wins. The *cop number*, denoted $c(G)$, is the smallest number of cops for which the first player has a winning strategy on G.

We note that in our version of the game, there is full visibility, and cops may share vertices. We note however that there is a large number of variants of the game, in which, for instance, the robber must move [1], the robber is invisible [38], only one cop can move per turn [78], or the winning-condition is relaxed to being at a certain distance from the cop [17] or surrounding the robber with the cops playing on edges [34].

This game was first introduced over 40 years ago by Nowakowski and Winkler [77] and Quilliot [83], and they fully characterized graphs on which one cop may capture the robber. We say a vertex $u \in V(G)$ is a *corner* (*pitfall*, in [1]) if there exists a distinct vertex $v \in V(G)$ such that $N[u] \subseteq N[v]$. If the robber is on u and there is a cop on v, then the cops will necessarily win at the next turn, as the robber cannot escape to a vertex which is not adjacent to this cop. In the formulation of [1], we can characterize one-cop-win graphs G as follows. The proof is a simple induction argument.

Theorem 1.4.1 ([77, 83, 1])**.** *If G is a connected graph, then $c(G) = 1$ if and only if there either exists a corner $u \in V(G)$ such that $c(G - u) = 1$, or G contains a single vertex.*

Aigner and Fromme [1] introduced the version of the game with multiple cops. Their most significant result is the following.

Theorem 1.4.2 ([1])**.** *If G is a connected planar graph, then $c(G) \leq 3$.*

The proof idea is as follows. It is based on the fact that a single cop may guard a geodesic path. On a planar graph, two cops guarding two $u - v$ geodesic paths confine the robber to either stay inside or outside the region delimited by these paths. The third cop may then protect a geodesic path which goes through this region, reducing the size of the region in which the robber is confined. One of the two original cops now no longer needs to guard its path, and we repeat the strategy until the robber's territory vanishes.

The geodesic path guarding strategy has become one of the most important tools in the study of this game. As it will be crucial to our main result in Chapter 5, we reproduce its proof here (closer to the proof of Andreae [7]). The following is a slightly stronger form of the result of Aigner and Fromme, which is used by Andreae [7]. Note that this is exactly Theorem 5.3.2.

Lemma 1.4.3 ([1, 7])**.** *If G is a connected graph, $u, v \in V(G)$, P is a shortest $u - v$ path and C is a cop currently on u, then there exists a strategy for C to keep guarding u and, after a finite number of turns, also guard P.*

Here, by C *guarding* a set of vertices S, we mean that if the robber were to move to a vertex in S, it would be immediately caught by C at the next turn. The cop does not have to stay on or be adjacent to a vertex to guard it: if the robber is far away from it, it might also be acceptable for the cop to be far away as well.

Proof. Let $\phi : V(G) \to V(P)$ be defined as following. For $w \in V(G)$, let $\phi(w)$ be the unique vertex of $V(P)$ which is at distance exactly $\min(\mathsf{dist}_G(u, w), \mathsf{dist}_G(u, v))$ from u.

We will take w to be the position of the robber. We first note that at every turn, $\phi(w)$ either does not change, or moves by to a vertex on P adjacent to its previous position, given that $\mathsf{dist}_G(u, w)$ can itself only change by a value of at most 1.

We know that C is currently on u. In the first part of C's strategy, it will at every turn move towards $\phi(w)$. Given that $V(P)$ is finite, and $\phi(w)$ can only move at the same speed as cops, the cop C will eventually "catch" $\phi(w)$. Afterwards, C will simply follow $\phi(w)$ at every step.

First note that at every point in this strategy, C is at least as close to u as $\phi(w)$ is, at least after the cops' turn, since C attempts to catch $\phi(w)$, by starting at u. Hence if at any point the robber moves to u, $\phi(w) = u$ and so the cop will immediately capture the robber. Thus, at all points of the game, u is guarded.

The second phase of the strategy begins after at most $\mathsf{dist}(u, v)$ turns. During this phase, all vertices of P are guarded. Indeed, after every cops' turn, C is on $\phi(w)$. Hence, if the robber enters P, given that this vertex will necessarily become $\phi(w)$, the cop will catch the robber on the following turn. This completes the proof. $\square$

This type of argument is a standard tool in the study of the game of cops and robbers. The function ϕ is an example of a *retract*, a special type of homomorphism for which, in particular, the vertices in the image of ϕ map to themselves. In these arguments, if the robber is on w, the cops will chase $\phi(w)$, the "robber's shadow" on the image. Once the shadow has been caught, this can be used to restrict the region that the robber can safely access, which can be used in various strategies. We note that Theorem 1.4.1 can be proved using this type of argument, by setting $\phi(w) = w$ except for $\phi(u) = v$. See [13], in particular, for more details on retract-based bounds.

As we mentioned earlier, planar graphs are exactly the graphs forbidding both K_5 and $K_{3,3}$ as minors. This motivated Andreae [7] to study the cop number of graphs forbidding minors, with the main result being the following.

Theorem 1.4.4. *Let H be a graph and $h \in V(H)$ be a vertex such that $H - h$ has no isolated vertex. If G is a connected H-minor-free graph, then $c(G) \leq |E(H - h)|$.*

In Chapter 5, we will generalize and improve this result. As the statement is fairly technical, we will postpone stating it exactly. However, broadly speaking, our improvements are most significant when H is small or sparse. For instance, we will show that $c(G) \leq |E(H - h - M)|$, if M is a matching such that $H - h - M$ does not have an isolated vertex.

Andreae also showed that forbidding only one of K_5 or $K_{3,3}$ as a minor is sufficient to get an upper bound of 3 on the cop number, strengthening Theorem 1.4.2.

As discussed earlier in this chapter, the connections between graph minors and graph topology extend far beyond planar graphs. We will thus now briefly survey some of the main results on cops and robbers and graph topology.

We have noted earlier that outerplanar graphs are exactly the graphs forbidding $K_{2,3}$ and K_4 as minors. It was shown by Clarke [33] that connected outerplanar graphs have cop number at most 2. Omitting the technical details, the strategy here is to place one cop on one end of a chord, separating the graph into two sides, say with the robber on the "left". The other cop then moves to block the next chord, trapping the robber further to the left. Repeating the argument, the robber is eventually caught. This result is also implied by some more general results of Andreae using forbidden minors [7], and by some bounds on treewidth [53], given that outerplanar graphs have treewidth at most 2.

The situation of graphs with higher genus has also been of interest. Lehner [60] has shown that toroidal graphs have cop number at most 3, which had previously been asked/conjectured by Andreae [7] and Schroeder [91]. The proof is inspired by the proof of Theorem 1.4.2. The main idea tool here is to write the torus as $\mathbb{R}^2/\mathbb{Z}^2$. Tiling the plane with this drawing of the graph, one obtains an infinite planar graph on which we can play a parallel, modified game, to obtain a winning strategy on the original graph.

More generally, suppose G is a connected graph which can be embedded on a surface with orientable genus g. Quilliot [84] showed that $c(G) \leq 2g + 3$. Schroeder [91] proved that

$c(G) \leq \left\lfloor \frac{3g}{2} \right\rfloor + 3$. Bowler et al. [23] showed that $c(G) \leq \frac{4g}{3} + \frac{10}{3}$. Erde and Lehner [40] have announced a proof that $c(G) \leq \left(3 - \sqrt{3} + o(1)\right) g$. Schroeder [91] has conjecture an upper bound of $g + 3$, which would be achieved if each cop had a strategy to protect a cycle which is not homotopic to a point, as the removal of such a cycle would reduce the genus of the graph (the base case being planar graphs). Mohar [22] has conjectured that $c(G) \leq g^{\frac{1}{2}+o(1)}$, based on a lower bound which comes from certain random graphs.

There are also some bounds depending on the non-orientable genus. The first such bound is by Andreae [7], and uses Theorem 1.4.4 applied to complete minors. Clarke et al. [32] have essentially reduced the problem to orientable genus (at least asymptotically), by showing that the upper bound for non-orientable genus g is at most the upper bound for orientable genus $g - 1$. Similarly, the upper bound for non-orientable genus g is at most the upper bound for orientable genus $2g + 1$.

One of the applications of our results in Chapter 5 will be improving the upper bound on the cop number of linklessly embeddable graphs from 9 to 6 (the former is a direct application of Theorem 1.4.4 to one of the graphs in the Petersen family).

We have introduced topological minors earlier in this chapter. Joret et al. [53] have noted that it is a consequence of Andreae's work [6, 7] that the cop number of the class of graphs forbidding H as a topological minor is only bounded if H has maximum degree at most 3. We have noted in Lemma 1.1.9 that for these graphs it is equivalent to forbid H as a minor or as a topological minor.

Motivated by Andreae's work, Joret et al. [53] have characterized the graphs H for which the classes of H-induced-subgraph-free or H-subgraph-free graphs are bounded. Although we will not expand on this topic here, some relevant references are [30, 52, 62, 66, 67, 79, 95, 96, 97, 100].

We close this section by reviewing Meyniel's conjecture, which is the most famous conjecture in the field. Although it is not directly related to forbidding minors, we will mention it in Section 5.5. Here, we wish to bound the cop number of general graphs: we want a bound which depends only on the order of the graph.

Conjecture 1.4.5 (Meyniel's conjecture[42])**.** *If G is a connected graph on n vertices, then $c(G) = O(\sqrt{n})$.*

Suppose G is a connected graph on n vertices. Frankl [42] showed that $c(G) \leq (1 + o(1))\frac{n \log\log n}{\log n}$. The idea is as follows. If the graph has high maximum degree, place a cop on a vertex of maximum degree to guard its neighbourhood. Otherwise, by the Moore bound the graph necessarily has large diameter, so we can apply Lemma 1.4.3 to guard a long path. In both cases, we can reduce the number of vertices in the graph by using one cop, and we induct. Chiniforooshan [29] improved this bound to $O\left(\frac{n}{\log n}\right)$. The idea is similar, however here the tool is a variation of Lemma 1.4.3 in which it is proved that a group of 5 cops can guard a geodesic caterpillar (i.e. guard the path, as well as all vertices adjacent to it). Currently, the best known upper bound is $\frac{n}{2^{-(1+o(1))\sqrt{\log n}}}$, which was proved by Scott and Sudakov [92] and Lu and Peng [63] using probabilistic arguments.

We note that this conjecture cannot be improved, as there exists graphs with cop number $\Omega(\sqrt{n})$, such as incidence graphs of projective planes [80] and other types of graphs based on designs [16], and Cayley graphs [27].

Meyniel's conjecture is known to hold for some classes of graphs, such as random graphs [81, 82], Cayley graphs [25, 27], graphs of diameter 2 [63, 104], and of course the many classes of graphs for which the cop number is simply bounded, such as the ones mentioned above.

The problem of upper bounding the cop number by a function of the order of the graph can be reformulated as follows: what is the order of the smallest graph with cop number at least k. This question was first asked by Andreae [7]. For small graphs, it has been shown that the smallest graph with cop number 3 is the Petersen graph [7, 11], and there are no graphs with cop number 4 on 18 vertices or fewer [101] (the Robertson graph [85], on 19 vertices, has cop number 4 [7]).

1.4.2 Graph burning

Graph burning is a process which models the spread of information in networks. It was defined by Bonato et al. [19, 20, 90]. However it had also independently previously appeared in a paper of Alon [5], the question being asked at Intel by Brandenburg and Scott, in this case relating to the transmission of information between processors. The game goes as follows. Suppose we wish to burn every vertex of a graph G. At the start of the game, no vertex is burning. However, once a vertex has begun burning, it will remain burning until the end of the game. At every turn, we can choose a new vertex to burn. Furthermore, at every turn, every vertex adjacent to a vertex which was previously burning also begins burning.

We would like to know how long it takes for us to burn all vertices of G, given an optimal strategy. The *burning number* of G, denoted $b(G)$, is the smallest number of turns required to burn all vertices of G. It is easily seen that this is fundamentally a covering problem: if the game lasts k turns, the vertices which are eventually burned by the fire we started at the vertex v at the i-th turn is exactly $B(v, k-i)$. Hence, $b(G)$ can be defined as the smallest integer k such that there exists $v_1, \ldots, v_k \in V(G)$ such that $V(G) = \bigcup_{i=1}^{k} B(v_i, k-i)$.

We begin with a few simple bounds on the burning number of specific types of graphs. For instance, if G is the complete graph K_n on $n \geq 2$ vertices, then $b(G) = 2$, since $B(v, 1) = V(G)$ for any vertex v.

At the other extreme, if G is an n-vertex graph with no edges, then $b(G) = n$, given that we will have to manually burn every vertex of the graph. This last case is somewhat uninteresting, as the graph is disconnected. For this reason, most of the literature is restricted to the burning number of connected graphs. We note that there is some interesting work on disconnected graph, for instance [21], although it is mostly used to derive bounds for connected graphs, in this case spiders.

Alon [5] showed that the burning number of the d-dimensional hypercube is $\lceil \frac{d}{2} \rceil + 1$. In this case, the optimal strategy is curiously to, after burning an arbitrary vertex, burn the vertex opposite to it on the hypercube, and then do nothing (burning other vertices does not help).

In their original paper, Bonato et al. [20], show the following result for paths P_n (on n vertices).

Theorem 1.4.6 ([20])**.** *If $n \in \mathbb{Z}^{\geq 0}$, then $b(P_n) = \lceil \sqrt{n} \rceil$.*

As the proof is simple and quite informative for our treatment of the topic, we reproduce it here. The main idea is that on a paths, the optimal cover is to always place the balls to be disjoint (at least on paths which for which the order is a square number), which is not true in general.

Proof. We first show that $b(P_n) \leq \lceil \sqrt{n} \rceil$. We first note that we may suppose that $\sqrt{n}$ is an integer. Indeed, if $n' \leq n$, a cover of P_n with balls of radii $0, \ldots, k-1$ can directly be converted into a cover of $P_{n'}$, by choosing an embedding of $P_{n'}$ into P_n (say, by identifying an end vertex), and placing the balls at the same positions on $P_{n'}$ as they are on P_n, except for the positions which do exist on the shorter path, in which case the corresponding balls are placed at the end vertex.

Hence, we may now write $n = k^2$, and show the claim by induction on k. The base case $k = 0$ is trivial: it takes no balls to cover the empty graph. Now, we prove the inductive step. Given a path of length k^2, place the ball of radius $k-1$ at the k-th vertex from the end of the path. Then, the set of vertices which are not covered form a path of length $k^2 - (2(k-1)+1) = (k-1)^2$. Hence, by induction the uncovered path can be covered by balls of radii $0, \ldots, k-2$. Using these balls as well as our ball of radius $k-1$ shows that $b(P_{k^2}) \leq k$, as desired.

We now show that $b(P_n) \geq \lceil \sqrt{n} \rceil$. Write $k = b(P_n)$, and let $v_1, \ldots, v_k \in V(G)$ such that $V(G) = \bigcup_{i=1}^{k} B(v_i, k-i)$. Note that on a path, $|B(v,r)| \leq 2r+1$ for every $v \in V(G)$ and $r \in \mathbb{Z}^{\geq 0}$, as $B(v,r)$ induces a path containing v and all vertices at distance at most r in either direction. Hence,

$$\begin{aligned} n = |V(P_n)| = \left|\bigcup_{i=1}^{k} B(v_i, k-i)\right| &\le \sum_{i=1}^{k} |B(v_i, k-i)| \le \sum_{i=1}^{k} (2(k-i)+1) \\ = 2k^2 + k - 2\sum_{i=1}^{k} i &= 2k^2 + k - 2 \cdot \frac{k(k+1)}{2} = k^2, \end{aligned}$$

and thus $k \ge \sqrt{n}$. Given that k is necessarily an integer, we obtain that $k \ge \lceil \sqrt{n} \rceil$, as desired. $\square$

Bonato et al. [20] have conjectured that paths are the extremal examples for the burning number.

Conjecture 1.4.7 (Burning Number Conjecture)**.** *[20] If G is a connected graph on n vertices, then $b(G) \le \lceil \sqrt{n} \rceil$.*

This conjecture has become the most important in the area of graph burning; we will now survey the results related to this conjecture. If T is a spanning tree of G, then $b(T) \ge b(G)$ (removing edges can only hurt us), so it suffices to show this conjecture for trees.

We first note however that paths are not the only tight examples for this conjecture. Bonato et al. [20] note that this is also tight for cycles (the extra edge does not help us). Indeed, they show that the burning number of a graph is the minimum burning number over all spanning trees of G.

A spider graph is a tree which contains exactly one vertex of degree greater than two. We now show that there exists some spiders for which we cannot do better than the Burning Number Conjecture. The following proof has appeared in [98].

Theorem 1.4.8 ([98])**.** *For every $k \in \mathbb{N}$, there exists a spider T_k such that $b(T_k) \ge \left\lceil \sqrt{|V(T_k)|} \right\rceil = k$.*

Proof. Let $k \in \mathbb{N}$, and let T_k be the spider defined as follows: identify an end vertex of k copies of a path on k vertices. T_k then has $k(k-1)+1$ vertices. Noting that $\left\lceil \sqrt{k(k-1)+1} \right\rceil = k$, we must thus show that we cannot cover T_k with balls of radii $0, \ldots, k-2$.

T_k contains k leaves, which are all mutually at distance $2k-2$ of each other. However, our largest radius is $k-2$, which can cover a subgraph of diameter at most $2(k-2) = 2k-4$. In particular, none of our $k-1$ balls may cover more than one of the k leaves. Hence, however we place the balls, they will never cover all leaves. $\square$

The best general upper bound on the burning number has improved multiple times in recent years. Let G be a connected graph on n vertices. In their original paper, Bonato et al. [20] showed that $b(G) \leq 2\lceil\sqrt{n}\rceil + 1$, by relating the burning number to the distance-k-dominating number. Bessy et al. [14] showed that $b(G) \leq \sqrt{\frac{12n}{7}} + 3$ and that $b(G) \leq \sqrt{\frac{32}{19} \cdot \frac{n}{1-\varepsilon}} + \sqrt{\frac{27}{19\varepsilon}}$ for every $0 < \varepsilon < 1$. Land and Lu [59] showed that $\left\lceil \sqrt{\frac{3n}{2} + \frac{33}{16}} - \frac{3}{4} \right\rceil$. Finally, Bastide et al. [12] proved that $b(G) \leq \sqrt{\frac{4n}{3}} + 1$. Although we will not prove these results here, we will prove the following intermediary result, which gives an idea to the type of induction used.

Theorem 1.4.9 ([14])**.** *If G is a connected graph on n vertices, then $b(G) \leq \lceil\sqrt{2n}\rceil$.*

Proof. We show that $b(G) \leq k$, where k is the smallest integer such that $2n \leq k(k+1)$ (in particular, $k \leq \lceil\sqrt{2n}\rceil$. As in the proof of Theorem 1.4.6, we may further assume that $2n = k(k+1)$ for some k, otherwise add arbitrary leaf vertices to G. Burning the obtained graph directly translates to burning G.

We prove by induction on k that $b(G) \leq k$. Again, the case $k = 0$ is trivial.

As mentioned earlier, we may suppose G is a tree. Let u be an arbitrary vertex of G and let v be a vertex which is as far as possible from v. Let w be the vertex at distance $k-1$ from w on the unique path from u to v (choose w to be v if this path is not long enough). Let G' be the subtree of G obtained by taking the component of $G - w$ which contains u.

First note that G' (if it is not empty) contains at most $n - k$ vertices, as it does not contain the path from w to v. Furthermore, note that every vertex not in G' is covered by the ball $B(w, k-1)$: this is a consequence of the fact that v was chosen to be as distant from u as possible.

Given that $2(n-k) = k(k+1) - 2k = k^2 - k = (k-1)((k-1)+1)$, we may apply induction to cover G', which with $B(w, k-1)$ yields a cover of G, as desired. $\square$

In this proof, with balls of radii $0, \ldots, k-1$, we can burn approximately $\frac{k^2}{2}$ vertices. The approach was to take the biggest available radii r, burn a subtree of order roughly r, and induct. Always choosing the largest radius is the most obvious approach, as it allows a simple inductive hypothesis. However we can also do induction by maintaining a collection of radii which are not necessarily consecutive numbers. For instance, in [59], given m available radii (not necessarily $0, \ldots, m-1$), they prove that one can choose one of these radii, say r, and remove a subtree of order $r + \left\lfloor \frac{m-1}{3} \right\rfloor$ (this subtree must be chosen more carefully than in the proof above). Thus, we can burn $\sum_{m=0}^{k-1} \left\lfloor \frac{m-1}{3} \right\rfloor \approx \frac{k^2}{6}$ more vertices than in the above proof with the same radii $0, \ldots, k-1$. If $n \approx \frac{k^2}{2} + \frac{k^2}{6}$, then $n \approx \sqrt{\frac{3n}{2}}$. The proof in [12] uses a similar idea, where with the ball of radius r they can burn $r + \frac{m}{2}$ vertices.

The following is another proof of Theorem 1.4.9, which was communicated to me by Alon. Although it does not generalize to stronger results, it is quite elegant.

Proof. Consider the multigraph G' obtained from G by doubling each edge. By construction, every vertex of G' has even degree, and so G' is Eulerian: there is a circuit C of length $2n$ which covers every edge of G' exactly once. In particular, there exists a surjective homomorphism $f : P_{2n} \to G'$. We know by Theorem 1.4.6 that P_{2n} can be covered with balls of radius $0, \ldots, 2n-1$. We can obtain a cover of G' with balls of the same radii as follows: if a ball of radius r is centered on u in P_{2n}, center it at $f(u)$ in G'. Given that f is surjective, every vertex of G' will be covered by one of the balls. To complete the proof, it suffices to see that the same cover works for G: given that no additional adjacencies were added to obtain G' (only edge multiplicities are changed), the balls on G' are the same as on G. $\square$

In Chapter 6, we will improve these results by showing that the Burning Number Conjecture holds asymptotically, that is $b(G) \leq (1 + o(1))\sqrt{n}$.

We note that the Burning Number Conjecture is known to hold exactly for some graphs:

spiders [35, 21], caterpillars (trees with a dominating path) [61], graphs with minimum degree at least 23 [55], sufficiently large graphs with minimum degree at least 4 [12] and trees with no vertices of degree 2 [72]. There are also specific bounds for other classes of graphs, such as random graphs [70, 39] and graph products [70, 71].

Part II

Minors and Hadwiger's conjecture

2

Limits of degeneracy for colouring graphs with forbidden minors

Sergey Norin[1], Jérémie Turcotte[1]

Motivated by Hadwiger's conjecture, Seymour asked which graphs H have the property that every non-null graph G with no H minor has a vertex of degree at most $|V(H)|-2$. We show that for every monotone graph family $\mathcal{F}$ with strongly sublinear separators, all sufficiently large bipartite graphs $H \in \mathcal{F}$ with bounded maximum degree have this property. None of the conditions that H belongs to $\mathcal{F}$, that H is bipartite and that H has bounded maximum degree

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can be omitted.

2.1 Introduction

In 1943, Hadwiger proposed the following conjecture relating the chromatic number and complete minors.[2]

Conjecture 2.1.1 (Hadwiger's conjecture [A9])**.** *For every positive integer t every graph with no K_t minor is $(t-1)$-colourable.*

Hadwiger's conjecture is considered by many to be one of the most important open problems in graph theory. It is a notoriously difficult problem; in particular, it generalizes the Four Colour Theorem. Hadwiger's conjecture is only known to be true for $t \leq 6$. The cases $t \leq 4$ were proved by Hadwiger [A9]. The case $t = 5$ was proved by Appel and Haken as a consequence of their famous proof of the Four Colour Theorem [A1, A2]. (The equivalence between these two statements was established earlier by Wagner [A33].) The case $t = 6$ was proved by Robertson, Seymour and Thomas [A26], also using the Four Colour Theorem. See Seymour [A29] for a recent survey of results and open problems related to Hadwiger's conjecture.

In the 1980s, Kostochka [A14, A15] and Thomason [A32] proved that every graph with no K_t minor is $O(t\sqrt{\log t})$-colourable, and the order of magnitude of their upper bound remained unchanged until recently. In fact, Kostochka and Thomason established a stronger result. They have shown that every graph G with $\delta(G) = \Omega(t\sqrt{\log t})$ has a K_t minor, where we use $\delta(G)$ to denote the minimum degree of a graph G. Equivalently, every non-null graph with no K_t minor has a vertex of degree $O(t\sqrt{\log t})$. A standard "degeneracy" inductive argument implies that every graph with no K_t minor is $O(t\sqrt{\log t})$-colourable. Note that the

[2]All graphs in this paper are finite, simple and undirected. Given graph H and G, we say that H is *a minor* of G and write $H \preceq G$ if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. We denote the complete graph on t vertices by K_t. A *k-colouring* of a graph G is a map $c : V(G) \to S$ for some set S of colours with $|S| = k$ such that $c(u) \neq c(v)$ for every pair of adjacent $u, v \in V(G)$. A graph G is *k-colourable* if it admits a k-colouring.

"easy" cases of Hadwiger's conjecture ($t \leq 4$) also follow via degeneracy, i.e. for $t \leq 4$ every non-null graph G with $\delta(G) \geq t-1$ has a K_t minor.

The Kostochka-Thomason bound on the order minimum degree sufficient to force a K_t minor cannot be improved [A14, A15, A8] and the possibility that colouring graphs with no K_t minor requires $\Omega(t\sqrt{\log t})$ colours was left open, until recently, when the "degeneracy barrier" was broken by Postle, Song and the first author [A23]. Even more recently, Delcourt and Postle [A4] have shown that graphs with no K_t minor are $O(t \log\log t)$-colourable.

Given the apparent difficulty of Hadwiger's conjecture, many relaxations have been considered. We are interested in the following relaxation proposed by Seymour. Let $\mathsf{v}(H)$ denote the number of vertices of a graph H.

Conjecture 2.1.2 (Seymour [A29, A30]). *For every graph H with $\mathsf{v}(H) = t$, every graph with no H minor is $(t-1)$-colourable.*

As H is a subgraph of K_t, the validity of Conjecture 2.1.2 for H is implied by Hadwiger's conjecture for $t = \mathsf{v}(H)$. Note further that K_{t-1} has no H minor and is not $(t-2)$-colourable, and so the number of colours in Conjecture 2.1.2 is optimal for every H. Woodall [A35] and Seymour (in private communication) previously conjectured a more narrow weakening of Hadwiger's conjecture for $H = K_{s,t}$, where $K_{s,t}$ is the complete bipartite graph with parts of sizes respectively s and t.

Conjecture 2.1.2 is known to hold for some graphs H. Hendrey and Wood [A12] have proved it when H is the Petersen graph. Lafferty and Song [A20, A19] have shown that Conjecture 2.1.2 holds for some graphs H on 8 and 9 vertices; see therein for further background and references on Conjecture 2.1.2 for small H. Denote by $K^*_{s,t}$ the graph obtained from the complete bipartite graph $K_{s,t}$ by making all the vertices in the part of size s pairwise adjacent. Kostochka [A16] has shown that Conjecture 2.1.2 holds for $H = K^*_{s,t}$ when t is sufficiently large compared to s, and later proved [A13, Theorem 3] that $t = \Omega((s \log s)^3)$ suffices. It follows that Conjecture 2.1.2 holds whenever H is bipartite and one of the parts of the bipartition is sufficiently large compared to the other.

In [A30], Seymour asked, in particular, for which graphs H does Conjecture 2.1.2 follow from degeneracy.

Question 2.1.3 (Seymour [A30])**.** *For which graphs H with $\mathsf{v}(H) = t$ does every non-null graph with no H minor have a vertex of degree at most $t-2$?*

Let us say that H is *a Hadwiger-amenable graph* or *an HA graph*, for brevity, if the answer to Question 2.1.3 is positive.

As noted in [A30], every graph G contains as a subgraph every tree of order $\delta(G)+1$. It follows that trees are Hadwiger-amenable, but to the best of our knowledge no other general classes of HA graphs were previously known. Let us note that if H is an HA graph then not only is every graph G with no H minor $(t-1)$-colourable, but is also $(t-1)$-list-colourable (and even the stronger $(t-1)$-DP-colourable), while many other methods establishing bounds for colouring are harder to extend to list-colouring. Hadwiger's conjecture, in particular, is known to be false for list colouring [A3] (see also [A31]).

Our main result states that every sufficiently large bipartite graph with bounded maximum degree and good separation properties is an HA graph. We will also show that none of these conditions can be entirely dismissed, hence providing a very rough characterization of large HA graphs.

Before stating our main result more precisely, let us introduce necessary definitions and notation, some of which has been already mentioned above. We will use the notation $\mathbb{N} = \{1, 2, \dots\}$ and $[n] = \{1, \dots, n\}$ (for $n \in \mathbb{N}$). Let G be a graph. We write $\mathsf{v}(G)$ and $\mathsf{e}(G)$ for, respectively, the number of vertices and edges of G. If $S \subseteq V(G)$, then $G[S]$ will denote the subgraph of G induced by S and $G - S$ will denote the subgraph of G induced by $V(G) \setminus S$. We denote by $\delta(G)$ and $\Delta(G)$ the minimum degree and maximum degree of G, respectively.

A *(proper) separation* of a graph G is a pair of subsets (A, B) of vertices of G such that $A \cup B = V(G)$, $A \nsubseteq B$, $B \nsubseteq A$ and no edge of G has one end in $A - B$ and the other in $B - A$. The *order* of the separation (A, B) is $|A \cap B|$. A separation is said to be *balanced* if $|A|, |B| \leq \frac{2}{3}\mathsf{v}(G)$. A graph family $\mathcal{F}$ has *strongly sublinear separators* if $\mathcal{F}$ is closed under taking subgraphs and there exists $c > 0$ and $0 < \beta < 1$ such that every graph $G \in \mathcal{F}$ has a balanced separation of order at most $c\mathsf{v}(G)^{\beta}$.

We are now ready state our main result.

Theorem 2.1.4. *For every graph family $\mathcal{F}$ with strongly sublinear separators and every $\Delta \in \mathbb{N}$, there exists $M = M_{2.1.4}(\mathcal{F}, \Delta)$ satisfying the following. If $H \in \mathcal{F}$ is a bipartite graph with $\Delta(H) \leq \Delta$ and $\mathsf{v}(H) \geq M$ then H is Hadwiger-amenable. That is, if G is a non-null graph with $\delta(G) \geq \mathsf{v}(H) - 1$, then H is a minor of G.*

In Section 2.7, we show that none of the conditions of Theorem 2.1.4 can be omitted, while the rest of the paper is occupied by the proof Theorem 2.1.4. We outline the proof of Theorem 2.1.4 in Section 2.2, and derive it from a number of technical results, which are proved in Sections 2.3–2.6.

2.2 Proof outline

In this section, we present the tools used in the proof of Theorem 2.1.4 and outline the proof.

2.2.1 Tools

Models

We will often certify that a graph H is a minor of a graph G by exhibiting a model of H in G. A *model* μ of H in G assigns to every vertex of $v \in V(H)$ a set $\mu(v)$ of vertices of G such that

- $\mu(u) \cap \mu(v) = \emptyset$ for every pair of distinct $u, v \in V(H)$,
- $G[\mu(v)]$ is connected for every $v \in V(H)$, and
- for every edge $uv \in E(H)$ there exist $u' \in \mu(u)$ and $v' \in \mu(v)$ such that $u'v' \in E(G)$.

For $U \subseteq V(H)$, let $\mu(U) = \bigcup_{v \in U} \mu(v)$ for brevity. Note that the last two conditions in the above definition of a model can be replaced by the following single condition.

- $G[\mu(U)]$ is connected for every $U \subseteq V(G)$ such that $H[U]$ is connected.

The following properties of models are well-known and not difficult to verify.

Lemma 2.2.1. *If H and G are graphs, then*

(a) *$H \preceq G$ if and only if there exists a model of H in G, and*

(b) *if $F \subseteq E(H)$, μ is a model of $H - F$ in G and $\{P_f\}_{f \in F}$ is a collection of paths in G such that*

- *P_{uv} has one end in $\mu(u)$, the other in $\mu(v)$, and is otherwise disjoint from $\mu(V(H))$ for every $uv \in F$, and*
- *the paths $\{P_f\}_{f \in F}$ are pairwise internally vertex-disjoint,*

then $H \preceq G$.

Density

While Theorem 2.1.4 is concerned with the minimum degree condition necessary to guarantee existence of given graph as a minor, the average degree conditions have been much more thoroughly investigated and provide a starting point for our argument.

We define the *density* of a non-null graph G as $\mathsf{d}(G) = \frac{\mathsf{e}(G)}{\mathsf{v}(G)}$. We note that $\mathsf{d}(G)$ is half the average degree of G, and so $\frac{\delta(G)}{2} \leq \mathsf{d}(G)$. The *extremal function* of a graph H, denoted by $c(H)$, as the supremum of densities of H-minor-free graphs, i.e. $H \preceq G$ for every non-null graph G with $\mathsf{d}(G) > c(H)$.

One of our main tools is the following result obtained independently by Haslegrave, Kim and Liu [A10][3], and by Hendrey, Norin and Wood [A11].

Theorem 2.2.2 ([A10, Theorem 2.2], [A11, Theorem 1.1])**.** *For every graph family $\mathcal{F}$ with strongly sublinear separators and every $\varepsilon > 0$, there exists $M = M_{2.2.2}(\mathcal{F}, \varepsilon)$ such that*

$$c(H) \leq (1+\varepsilon)\frac{\mathsf{v}(H)}{2}$$

[3]Haslegrave, Kim and Liu [A10] state their theorem for proper minor-closed families, but their proof holds for families with strongly sublinear separators.

for every bipartite graph $H \in \mathcal{F}$ and $\mathsf{v}(H) \geq M$.

Theorem 2.2.2 implies that a slight weakening of Theorem 2.1.4 holds with the condition $\delta(G) \geq \mathsf{v}(H) - 1$ replaced by $\delta(G) \geq (1+\varepsilon)\mathsf{v}(H)$. At several points in the proof, we will be able to replace G or a subgraph of G by a denser minor and apply Theorem 2.2.2.

Using strongly sublinear separators

For $s \in \mathbb{N}$, let $\mathcal{F}_s$ be the class of all graphs with component size (that is, the number of vertices in the component) at most s. Then, it is clear that $\mathcal{F}_s$ is a family with strongly sublinear separators. We will use this to apply Theorem 2.2.2 to graphs with bounded component size.

In addition to Theorem 2.2.2 we use several other technical results from [A11]. The following result states that graphs in families with strongly sublinear separators are in fact always close to graphs with bounded component size. It is essentially [A11, Lemma 7.3], but as the statement is slightly different we include its proof in Section 2.A for completeness.

Lemma 2.2.3. *For every graph family $\mathcal{F}$ with strongly sublinear separators and every $\delta > 0$ there exists $s = s_{2.2.3}(\mathcal{F}, \delta)$ such that for any graph $H \in \mathcal{F}$ there exists a graph H' and $F \subseteq E(H')$ such that*

- $H \preceq H'$,
- $\mathsf{v}(H) \leq \mathsf{v}(H') \leq (1+\delta)\mathsf{v}(H)$,
- $\Delta(H') \leq \Delta(H) + 2$,
- $|F| \leq \delta \mathsf{v}(H)$,
- *for every component J of $H' - F$ we have $\mathsf{v}(J) \leq s$ and J is isomorphic to a subgraph of H, and*
- *no edge of F has both ends in the same component of $H' - F$.*

Note that in Lemma 2.2.3, H' is not necessarily bipartite even if H is bipartite. However, $H' - F$ is bipartite if H is bipartite. By Lemma 2.2.3, in the proof of Theorem 2.1.4 it suffices to show that $H' \preceq G$ for H' satisfying the conditions of the lemma. We do so by building a model of $H' - F$ and then, given some appropriate collection of paths, extending it to all of H' using Lemma 2.2.1(b).

Connectivity

To extend a given model we need the graph H to satisfy certain connectivity assumptions. The actual assumptions that are both usable and attainable are somewhat technical. Let us now introduce them.

Let $d, k \geq 0$, let G be a graph and let $X \subsetneq V(G)$. We say that a pair (G, X) is (d, k)-*dense* if

- $\deg_G(v) \geq d$ for every $v \in V(G) - X$,
- for every separation (A, B) of G of order at most k we have $A \setminus B \subseteq X$ or $B \setminus A \subseteq X$.

For brevity, if $d = 0$ we can omit the first condition, that is we write that (G, X) is a k-*dense* pair.

Lemma 2.2.4. *If $k > 0$ and G is a non-null graph, then there exists a subgraph G' of G and $X \subsetneq V(G')$ such that $|X| \leq 2k$ and (G', X) is $(\delta(G), k)$-dense.*

Proof. If G admits no separation of order less than $2k$ then $(G, \emptyset)$ is $(\delta(G), k)$-dense. Thus we assume that there exists a separation (A, B) of G of order less than $2k$ and choose such a separation with A minimal.

We claim that $(G[A], A \cap B)$ is $(\delta(G), k)$-dense. Clearly $\deg_{G[A]}(v) = \deg_G(v) \geq \delta(G)$ for every $v \in V(G[A]) \setminus (A \cap B)$, and so we assume for a contradiction that there exists a separation (A', B') of $G[A]$ with $|A' \cap B'| \leq k$ such that neither of $(A' \setminus B') \setminus (A \cap B) = A' \setminus (B \cup B')$ and $(B' \setminus A') \setminus (A \cap B) = B' \setminus (A' \cup B)$ is empty. As

$$|B \cap (A' \setminus B')| + |B \cap (B' \setminus A')| \leq |A \cap B| < 2k,$$

we assume without loss of generality that $|B \cap (A' \setminus B')| < k$. Then, $(A', B \cup B')$ is a separation of G of order

$$|A' \cap (B \cup B')| = |A' \cap B'| + |B \cap (A' \setminus B')| < 2k.$$

Furthermore, $A' \subsetneq A$ given that (A', B') is a separation of $G[A]$. Thus $(A', B \cup B')$ contradicts the choice of (A, B). This contradiction implies the claim and thus the lemma.

$\square$

By Lemma 2.2.4, we can replace the graph G in Theorem 2.1.4 by a $(\mathsf{v}(H) - 1, \varepsilon\mathsf{v}(H))$-dense pair (G', X) with $|X| \leq 2\varepsilon\mathsf{v}(H)$ for any $\varepsilon > 0$.

In one of the cases of the proof of our main result, the collections of paths needed to extend models as discussed in the previous subsection will be found by using the following lemma, which is a slight modification of [A11, Lemma 6.5] that involves our non-standard connectivity condition. The proof of [A11, Lemma 6.5] translates to the setting we need with trivial changes[4].

Lemma 2.2.5 ([A11, Lemma 6.5]). *For every $\varepsilon > 0$, there exists $\delta = \delta_{2.2.5}(\varepsilon) > 0$ satisfying the following. If (G, X) is an $\varepsilon\mathsf{v}(G)$-dense pair, then there exists $Z \subseteq V(G)$ with $|Z| \leq \varepsilon\mathsf{v}(G)$ such that for all $p_1, q_1, \ldots p_t, q_t \in V(G) \setminus (X \cup Z)$ with $t \leq \delta\,\mathsf{v}(G)$, there exist a collection of pairwise internally vertex disjoint paths $P_1, \ldots, P_t$ in G such that P_i has ends p_i and q_i and $V(P_i) \setminus \{p_i, q_i\} \subseteq Z$ for every $1 \leq i \leq t$.*

In another case of the proof, we will instead use the following result allowing us to build a minor from pieces, the proof of which also uses this idea of obtaining the model by finding appropriate paths between models of subgraphs. We will prove this theorem in Section 2.6.

[4]In the proof of [A11, Lemma 6.5], one uses Menger's theorem [A22] and the fact that the connectivity of the graph is at least $\varepsilon\mathsf{v}(G)$ to find at least $\varepsilon\mathsf{v}(G)$ paths between a pair of non-adjacent vertices u, v. Here, instead of using connectivity, for such $u, v \notin X$ we use the $\varepsilon\mathsf{v}(G)$-density to obtain that there is no vertex-cut separating u and v with order at most $\varepsilon\mathsf{v}(G)$, and again apply Menger's theorem. The rest of the proof is identical.

Theorem 2.2.6. *There exists $C = C_{2.2.6} \in \mathbb{N}$ satisfying the following. Let H be a graph, let $F \subseteq E(H)$ be such that $H - F$ is a disjoint union of graphs $H_1, \dots, H_k$ and such that no edge of F has both ends in the same component of $H - F$. If G is a graph and $G_1, \dots, G_k$ are pairwise vertex-disjoint subgraphs of G such that*

- $\mathsf{d}(G_i) \geq C(c(H_i) + |F|)$ *for every $i \in [k]$, and*
- $\left(G, V(G) \setminus \bigcup_{i=1}^{k} V(G_i)\right)$ *is $2|F|$-dense,*

then $H \preceq G$.

2.2.2 Proof outline

We will prove the following more technical version of Theorem 2.1.4.

Theorem 2.2.7. *For every $\Delta \in \mathbb{N}$, there exists $\alpha = \alpha_{2.2.7}(\Delta) > 0$ such that for every graph family $\mathcal{F}$ with strongly sublinear separators and every $\varepsilon > 0$ there exists $M = M_{2.2.7}(\mathcal{F}, \Delta, \varepsilon)$ such that the following holds. If $H \in \mathcal{F}$ is a bipartite graph with $\Delta(H) \leq \Delta$ and $\mathsf{v}(H) \geq M$ and (G, X) is a $(\mathsf{v}(H) - 1, \varepsilon\mathsf{v}(H))$-dense pair such that $|X| \leq \alpha\mathsf{v}(H)$, then $H \preceq G$.*

Theorem 2.1.4 follows directly from Theorem 2.2.7 by setting $M_{2.1.4}(\mathcal{F}, \Delta) = M_{2.2.7}\left(\mathcal{F}, \Delta, \frac{\alpha_{2.2.7}(\mathcal{F},\Delta)}{2}\right)$ and applying Lemma 2.2.4 to G with $k = \frac{\alpha_{2.2.7}(\mathcal{F},\Delta)}{2}\mathsf{v}(H)$ to obtain (G', X) as noted above.

The proof of Theorem 2.2.7 is separated into three cases. When $\mathsf{v}(G)$ is only slightly larger then $\mathsf{v}(H)$ we use the following lemma, which we prove in Section 2.3, to find H as a subgraph of G.

Lemma 2.2.8. *Let $\Delta \in \mathbb{N}$, H be a bipartite graph with $\Delta(H) \leq \Delta$, G be a graph and $X \subsetneq V(G)$ be such that $\deg_G(v) \geq \mathsf{v}(H) - 1$ for every $v \in V(G) \setminus X$. If*

$$\mathsf{v}(G) \leq \left(1 + \frac{1}{4\Delta(\Delta+1)}\right)\mathsf{v}(H) - 1$$

and

$$|X| \leq \frac{\mathsf{v}(H)}{(\Delta+1)(\Delta^2+1)},$$

then H is isomorphic to a subgraph of G.

When $\mathsf{v}(G)$ is somewhat larger but still within a constant factor of $\mathsf{v}(H)$ we use the following result, which we prove in Section 2.4, in combination with Lemmas 2.2.3 and 2.2.5.

Theorem 2.2.9. *For every $\Delta \in \mathbb{N}$, $\nu > 0$, there exists $\mu = \mu_{2.2.9}(\Delta, \nu) > 0$ such that for every $s \in \mathbb{N}$ there exists $M = M_{2.2.9}(\Delta, \nu, s)$ such that if H is a bipartite graph with $\Delta(H) \leq \Delta$, maximum component size s and $\mathsf{v}(H) \geq M$, and G is a graph such that $\delta(G) \geq (1-\mu)\mathsf{v}(H)$ and $\mathsf{v}(G) \geq (1+\nu)\mathsf{v}(H)$, then H is a minor of G.*

Finally, when $\mathsf{v}(G)$ is much larger that $\mathsf{v}(H)$, we use the following density increment lemma in combination with Lemma 2.2.3 and Theorem 2.2.6.

Theorem 2.2.10. *There exists $\varepsilon = \varepsilon_{2.2.10} > 0$ such that for every $K \in \mathbb{N}$ there exist $\varepsilon' = \varepsilon'_{2.2.10}(K)$ and $C = C_{2.2.10}(K) \geq 1$ such that for every $D \geq C$ and every G such that $\delta(G) \geq D$ and $\mathsf{v}(G) \geq CD$, then either*

1. *G contains vertex-disjoint subgraphs $J_1, \dots, J_K$ such that $\mathsf{v}(J_i) \leq \frac{D}{\varepsilon}$ and $\mathsf{d}(J_i) \geq \varepsilon D$ for every $i \in [K]$, or*

2. *G contains a minor H such that $\mathsf{d}(H) \geq (1+\varepsilon')\frac{D}{2}$.*

We are now ready to derive Theorem 2.2.7 from the above-mentioned results.

Proof of Theorem 2.2.7. We begin by introducing the necessary parameters. Let $\beta = \frac{\varepsilon_{2.2.10}}{4C_{2.2.6}}$ and $K = \lceil 2/\beta \rceil$.

Given Δ, let $\nu = \frac{1}{8(\Delta+1)(\Delta^2+1)}$ and $\alpha = \min\left(\frac{\nu}{3}, \frac{\mu_{2.2.9}(\Delta+2,\nu)}{3}, \frac{\varepsilon'_{2.2.10}(K)}{3(1+\varepsilon'_{2.2.10}(K))}\right)$. Let $\gamma = \frac{(1+\varepsilon'_{2.2.10}(K))(1-2\alpha)-1}{2}$; note that $\gamma > 0$ by choice of α.

Given $\mathcal{F}, \varepsilon$, let $\varepsilon' = \min\left(\frac{\varepsilon}{C_{2.2.10}(K)}, \frac{\alpha}{C_{2.2.10}(K)}\right)$, $\delta = \min\left(\frac{\nu}{3(1+\nu)}, \frac{\alpha}{2(1-3\alpha)}, \delta_{2.2.5}(\varepsilon'), \frac{\varepsilon}{2}, \beta\right)$, $s = s_{2.2.3}(\mathcal{F}, \delta)$ and finally

$$M = \max\left(\frac{1}{\frac{1}{4\Delta(\Delta+1)} - \frac{1}{4(\Delta+1)(\Delta^2+1)}}, \frac{2}{\alpha}, M_{2.2.9}(\Delta+2, \nu, s), \frac{Ks}{K\beta - 1 - \delta}, \frac{C_{2.2.10}(K)}{1-2\alpha}, M_{2.2.2}(\mathcal{F}, \gamma), K(M_{2.2.2}(\mathcal{F}$$

Given H and (G, X) as in the statement, we show that $H \preceq G$. As mentioned earlier, we divide the proof into three cases depending on the ratio between $\mathsf{v}(G)$ and $\mathsf{v}(H)$.

Case 1: $\mathsf{v}(G) \le (1+2\nu)\mathsf{v}(H)$.

Given that

$$\mathsf{v}(G) \le (1+2\nu)\mathsf{v}(H) = \left(1 + \frac{1}{4(\Delta+1)(\Delta^2+1)}\right)\mathsf{v}(H) \le \left(1 + \frac{1}{4\Delta(\Delta+1)}\right)\mathsf{v}(H) - 1,$$

where the last inequality follows by choice of M, and

$$|X| \le \alpha\mathsf{v}(H) \le \nu\mathsf{v}(H) \le \frac{\mathsf{v}(H)}{(\Delta+1)(\Delta^2+1)},$$

the conditions of Lemma 2.2.8 are satisfied. Hence, H is isomorphic to a subgraph of G, and so $H \preceq G$.

For the remaining two cases, apply Lemma 2.2.3 with the above choice of δ to H to obtain H' and $F \subseteq E(H')$ such that $H \preceq H'$, $\mathsf{v}(H) \le \mathsf{v}(H') \le (1+\delta)\mathsf{v}(H)$, $\Delta(H') \le \Delta(H) + 2 \le \Delta + 2$, $|F| \le \delta\mathsf{v}(H)$, for every component J of $H' - F$ we have $\mathsf{v}(J) \le s$ and J is isomorphic to a subgraph of H (in particular, $H' - F$ is bipartite), and no edge of F has both ends in the same component of $H' - F$.

Case 2: $(1+2\nu)\mathsf{v}(H) \le \mathsf{v}(G) \le C_{2.2.10}(K)\mathsf{v}(H)$.

Given that (G, X) is $\varepsilon\mathsf{v}(H)$-dense and $\varepsilon'\mathsf{v}(G) \le \varepsilon' C_{2.2.10}(K)\mathsf{v}(H) \le \varepsilon\mathsf{v}(H)$, it is also $\varepsilon'\mathsf{v}(G)$-dense. Apply Lemma 2.2.5 with ε' instead of ε to (G, X) to obtain Z satisfying the conditions of the lemma. In particular,

$$|Z| \le \varepsilon'\mathsf{v}(G) \le \varepsilon' C_{2.2.10}(K)\mathsf{v}(H) \le \alpha\mathsf{v}(H).$$

Let $G' = G - (X \cup Z)$. Then $|X \cup Z| \leq |X| + |Z| \leq 2\alpha \mathsf{v}(H) \leq \frac{2}{3}\nu \mathsf{v}(H)$ and so

$$\mathsf{v}(G') = \mathsf{v}(G) - |X \cup Z| \geq \left(1 + \frac{4}{3}\nu\right) \mathsf{v}(H) \geq (1+\nu)(1+\delta)\mathsf{v}(H) \geq (1+\nu)\mathsf{v}(H'),$$

and

$$\delta(G') \geq \delta(G) - |X \cup Z| \geq (1-2\alpha)\,\mathsf{v}(H) - 1 \geq \left(1 - \frac{5}{2}\alpha\right) \mathsf{v}(H) \geq (1-3\alpha)(1+\delta)\mathsf{v}(H)$$
$$\geq (1 - \mu_{2.2.9}(\Delta+2, \nu))\mathsf{v}(H'),$$

where the fourth inequality follows by choice of M. Furthermore, $\mathsf{v}(H' - F) \geq \mathsf{v}(H) \geq M \geq M_{2.2.9}(\Delta + 2, \nu, s)$.

Thus by Theorem 2.2.9 applied to the graph $H' - F$ in place of H, and G' in place of G, there exists a model μ of $H' - F$ in G'. As $|F| \leq \delta \mathsf{v}(H) \leq \delta_{2.2.5}(\varepsilon')\mathsf{v}(G)$ and Z was chosen to satisfy the conditions of Lemma 2.2.5, there exist pairwise internally vertex disjoint paths $\{P_{uv}\}_{uv \in F}$ in G, such that for every edge $uv \in F$ the path P_{uv} has one end in $\mu(u)$ the other in $\mu(v)$ and is otherwise disjoint from $V(G')$ and so from $\mu(V(H))$. Thus by Lemma 2.2.1(b) there exists a model of H' in G, and so $H \preceq H' \preceq G$, as desired.

Case 3: $C_{2.2.10}(K)\mathsf{v}(H) \leq \mathsf{v}(G)$.

Let $H_1, \ldots, H_K$ be disjoint subgraphs of $H' - F$ such that for every $i \in [K]$, H_i is a union of connected components of $H' - F$ and $\mathsf{v}(H_i) \leq \beta \mathsf{v}(H)$, and then subject to these conditions $\sum_{i=1}^{K} \mathsf{v}(H_i)$ is maximal, and subject to this condition $\max_{i,j \in [K]} |\mathsf{v}(H_i) - \mathsf{v}(H_j)|$ is minimal.

We first claim that $\mathsf{v}(H') = \sum_{i=1}^{K} \mathsf{v}(H_i)$. Suppose otherwise for a contradiction that this is not the case, that is at least one component J of $H' - F$ is not in any of the $H_1, \ldots, H_K$. Given that $\mathsf{v}(J) \geq s$, we know that for every $i \in [K]$, $\mathsf{v}(H_i) > \beta \mathsf{v}(H) - s$ (otherwise J could be added to H_i). This implies that $(1+\delta)\mathsf{v}(H) \geq \mathsf{v}(H') > \sum_{i=1}^{K} \mathsf{v}(H_i) > K(\beta \mathsf{v}(H) - s)$, from which it follows that $\mathsf{v}(H) < \frac{Ks}{K\beta - 1 - \delta}$ since $K\beta - 1 - \delta \geq 1 - \delta > 0$. This is a contradiction to the choice of M. This finishes the proof of the claim.

Let $G' = G - X$ and $D = (1-2\alpha)\mathsf{v}(H)$. Note that $D \geq C_{2.2.10}(K)$ by choice of M. We have that

$$\mathsf{v}(G') = \mathsf{v}(G) - |X| \geq C_{2.2.10}(K)\mathsf{v}(H) - \alpha\mathsf{v}(H) \geq C_{2.2.10}(K)(1-2\alpha)\mathsf{v}(H) = C_{2.2.10}(K)D$$

and

$$\delta(G') \geq \delta(G) - |X| \geq \mathsf{v}(H) - 1 - \alpha\mathsf{v}(H) \geq (1-2\alpha)\mathsf{v}(H) = D,$$

where the last inequality follows by choice of M.

Hence by Theorem 2.2.10 applied to G' either

1. G' contains vertex-disjoint subgraphs $G_1, \ldots, G_K$ such that $\mathsf{d}(G_i) \geq \varepsilon_{2.2.10}(1-2\alpha)\mathsf{v}(H)$ for every $i \in [K]$, or

2. G' contains a minor G'' such that $\mathsf{d}(G'') \geq (1+\varepsilon'_{2.2.10}(K))(1-2\alpha)\frac{\mathsf{v}(H)}{2} > (1+\gamma)\frac{\mathsf{v}(H)}{2}$.

Given that $\mathsf{v}(H) \geq M \geq M_{2.2.2}(\mathcal{F}, \gamma)$, Theorem 2.2.2 yields that $c(H) \leq (1+\gamma)\frac{\mathsf{v}(H)}{2}$. Thus if (2) holds we have $H \preceq G'' \preceq G' \preceq G$.

Suppose on the other hand note that (1) holds. By the last condition in the choice of the subgraphs, for every $i, j \in [K]$ we have $|\mathsf{v}(H_i) - \mathsf{v}(H_j)| \leq s$, since every component of $H' - F$ has order at most s (if this is not the case for some pair, transfer a component from the largest of the two subgraphs to the smallest). As the average order of the H_i is $\frac{\mathsf{v}(H')}{K} \geq \frac{\mathsf{v}(H)}{K}$, this implies that $\mathsf{v}(H_i) \geq \frac{\mathsf{v}(H)}{K} - s \geq \frac{M}{K} - s \geq M_{2.2.2}(\mathcal{F}_s, 1)$ for every $i \in [K]$. Given that $H' - F$ has bounded component size s, $H_1, \ldots, H_K \in \mathcal{F}_s$. Hence, Theorem 2.2.2 yields that $c(H_i) \leq (1+1)\frac{\mathsf{v}(H_i)}{2} = \mathsf{v}(H_i)$ for every $i \in [K]$. Then

$$\mathsf{d}(G_i) \geq \varepsilon_{2.2.10}(1-2\alpha)\mathsf{v}(H) \geq 2C_{2.2.6}\beta\mathsf{v}(H) \geq C_{2.2.6}(\mathsf{v}(H_i) + \delta\mathsf{v}(H)) \geq C_{2.2.6}(c(H_i) + |F|)$$

for every $i \in [K]$. Furthermore, (G, X) is $\varepsilon\mathsf{v}(H)$-dense and so $\left(G, V(G) \setminus \bigcup_{i=1}^K V(G_i)\right)$ is $|F|$-dense given that $2|F| \leq 2\delta\mathsf{v}(H) \leq \varepsilon\mathsf{v}(H)$ and $X \subseteq V(G) \setminus \bigcup_{i=1}^K V(G_i)$. Thus by

Theorem 2.2.6 (applied to H' instead of H), $H \preceq H' \preceq G$, finishing the proof in this case. $\square$

Let us highlight why, using our methods, we must split the proof in three cases. The obstacles are broadly as follows.

- Lemma 2.2.8 is only applicable when $\mathsf{v}(G)$ is very close to $\mathsf{v}(H)$.
- Theorem 2.2.9 is not applicable when $\mathsf{v}(G)$ is too close to $\mathsf{v}(H)$.
- Lemma 2.2.5 requires an $\varepsilon'\mathsf{v}(G)$-dense pair, however under our hypothesis we only have an $\varepsilon\mathsf{v}(H)$-dense pair, so we cannot use this lemma if $\mathsf{v}(G)$ is arbitrarily large compared to than $\mathsf{v}(H)$.
- Our use of Theorem 2.2.6 requires us to find a large number K of pieces with good density, which we obtain using Theorem 2.2.10. The latter requires that G be much larger than D; in the proof, D is only slightly larger than $\mathsf{v}(H)$.

2.3 Small case

In this section, we prove Lemma 2.2.8, which we restate for convenience.

Lemma 2.2.8. *Let $\Delta \in \mathbb{N}$, H be a bipartite graph with $\Delta(H) \leq \Delta$, G be a graph and $X \subsetneq V(G)$ be such that $\deg_G(v) \geq \mathsf{v}(H) - 1$ for every $v \in V(G) \setminus X$. If*

$$\mathsf{v}(G) \leq \left(1 + \frac{1}{4\Delta(\Delta+1)}\right)\mathsf{v}(H) - 1$$

and

$$|X| \leq \frac{\mathsf{v}(H)}{(\Delta+1)(\Delta^2+1)},$$

then H is isomorphic to a subgraph of G.

Proof. If follows from the fact that at least one vertex of G has degree at least $\mathsf{v}(H)-1$ that

$$\mathsf{v}(G) \geq \mathsf{v}(H) \geq (\Delta+1)(\Delta^2+1)|X| > (\Delta+1)|X|.$$

Let $Y = V(G) \setminus X$, and if possible let $X_0 \subseteq X$ be chosen maximal such that $|N_G(X_0) \cap Y| < \Delta|X_0|$. In particular,

$$|N_G(X_0) \cap Y| < \Delta|X_0| \leq \Delta|X| < \mathsf{v}(G) - |X| = |Y|.$$

Otherwise, let $X_0 = \emptyset$. In both cases, this implies there exists $v \in Y$ with no neighbours in X_0. and so $\deg_G(v) \leq \mathsf{v}(G - X_0) - 1$. It then follows from the lower bound on $\deg_G(v)$ from the statement that

$$\mathsf{v}(G - X_0) \geq \mathsf{v}(H).$$

Then, let $X' = X \setminus X_0$.

We construct an injective homomorphism $\phi : V(H) \to V(G \setminus X_0)$, that is an injection such that $\phi(u)\phi(v) \in E(G)$ for every $uv \in E(H)$. The existence of such an injection implies the lemma.

We first choose (A, B) a bipartition of H such that $|A| \leq |B|$ and subject to this $|B|$ is minimum. We claim that $|A| \geq \frac{\mathsf{v}(H)}{\Delta+1}$. If not, then $|B| \geq \mathsf{v}(H) - |A| > \left(1 - \frac{1}{\Delta+1}\right)\mathsf{v}(H) > |A|$ and in particular B contains no isolated vertices by our choice of the bipartition. Thus $\Delta|A| \geq \mathsf{e}(H) \geq |B| = \mathsf{v}(H) - |A|$, implying the desired inequality.

We then choose a set of vertices in A to be mapped to X' as follows. Let $A_0 \subseteq A$ be chosen so that $|A_0| = |X'|$ and no two vertices of A_0 have a common neighbour. Such a choice is possible. Indeed, otherwise then there exists $A_0 \subseteq A$ such that $|A_0| < |X'| \leq |X|$ and every vertex in $A - A_0$ shares a neighbour with a vertex in A_0. As at most $\Delta^2|A_0|$

vertices of $A \setminus A_0$ share neighbours with vertices in A_0, we have

$$\frac{\mathsf{v}(H)}{\Delta+1} - |X| \leq |A| - |X| \leq |A \setminus A_0| \leq \Delta^2|A_0| < \Delta^2|X|,$$

implying $|X| > \frac{\mathsf{v}(H)}{(\Delta+1)(\Delta^2+1)}$, a contradiction. Define $\phi|_{A_0}$ to be an arbitrary bijection $A_0 \to X'$.

By the choice of X_0 we have $|N_G(S) \cap Y| \geq \Delta|S|$ for every $S \subseteq X'$, and so by Hall's theorem (applied Δ times) there exist pairwise disjoint sets $(Y_v)_{v\in X'}$ such that $Y_v \subseteq N_G(v) \cap Y$ and $|Y_v| = \Delta$ for every $v \in X'$. Hence, we can extend the injection ϕ to B so that $\phi(N_G(v) \cap B) \subseteq Y_v \subseteq N_G(v) \cap Y$ for every $v \in A_0$.

Let $A' = A \setminus A_0$. It remains to define ϕ on A'. Let $Y' = Y \setminus \phi(B)$. For every $v \in A'$ let

$$Y_v = \bigcap_{w\in N_H(v)\cap B} N_G(\phi(w)) \cap Y'.$$

In other words, Y_v is the set of possible choices for extending ϕ to v in Y' for ϕ to still be a homomorphism. In order for ϕ to also be an injection, our goal is then find a set of distinct representatives of the set system $(Y_v)_{v\in A'}$.

By Hall's theorem such a system of representatives exists as long as

$$\left|\bigcup_{v\in S} Y_v\right| \geq |S| \tag{2.1}$$

for every $S \subseteq A'$.

We thus finish the proof by verifying that Hall's condition holds. First note that

$$|Y'| = |Y| - |\phi(B)| = (\mathsf{v}(G - X_0) - |X'|) - |B| \geq \mathsf{v}(H) - |B| - |X'| = |A| - |X'|.$$

First suppose $|S| \leq \frac{|A|}{2}$. We can rewrite $Y_v = Y' \setminus \left(\bigcup_{w\in N_G(v)\cap B}(Y' \setminus N_G(\phi(w)))\right)$ for every $u \in A'$. Thus

$$\begin{aligned}
|Y_v| &\geq |Y'| - \sum_{w \in N_H(v) \cap B} |Y' \setminus N_G(\phi(w))| \geq |A| - |X'| - \sum_{w \in N_H(v) \cap B} |V(G) \setminus N_G(\phi(w))| \\
&\geq |A| - |X| - \sum_{w \in N_H(v) \cap B} (\mathsf{v}(G) - \deg_G(\phi(w))) \geq |A| - \frac{\mathsf{v}(H)}{4(\Delta+1)} - \Delta(\mathsf{v}(G) - \mathsf{v}(H) + 1) \\
&\geq |A| - \frac{\mathsf{v}(H)}{2(\Delta+1)} \geq \frac{|A|}{2} \\
&\geq |S|,
\end{aligned}$$

as desired.

We now suppose that $|S| > \frac{|A|}{2}$. We claim that $\bigcup_{v \in S} Y_v = Y'$ and so that $\left|\bigcup_{v \in S} Y_v\right| = |Y'| \geq |A| - |X| = |A| - |A_0| = |A'| \geq |S|$, as desired. It thus only remains to establish this claim.

Suppose for a contradiction it does not hold, that is there exists $S \subseteq A'$ (with $|S| > |A|/2$) such that $u \in Y'$ but for which $u \notin Y_v$ for every $v \in S$. Let $B' \subseteq B$ be the set of $w \in B$ such that $\phi(w) \in Y \setminus N_G(u)$. Then $|B'| \leq |Y \setminus N_G(u)| \leq \mathsf{v}(G) - \deg(u) \leq \mathsf{v}(G) - \mathsf{v}(H) + 1$.

Note that every $v \in S$ has a neighbour in B' by the choice of S and u. It follows that

$$|S| \leq |N_H(B') \cap A| \leq \Delta|B'| \leq \Delta(\mathsf{v}(G) - \mathsf{v}(H) + 1) \leq \frac{\mathsf{v}(H)}{4(\Delta+1)} \leq \frac{|A|}{4},$$

a contradiction. This finishes the proof of the claim and thus of the lemma. $\square$

2.4 Minors with bounded component size

In this section, we prove Theorem 2.2.9, which allows us to construct large bipartite minors with bounded component size.

Given a graph H, we say a graph G is H-free if G does not contain a copy of H, that is a subgraph isomorphic to H. We now cite a density increment result of Krivelevich and Sudakov [A17] for $K_{s,s}$-free graphs; note that a slightly weaker result of Kühn and Osthus

[A18, Theorem 11] would be sufficient for our purposes.

Theorem 2.4.1 ([A17, Theorem 4.5])**.** *For every $s \in \mathbb{N}^{\geq 2}$, there exists $C = C_{2.4.1}(s) > 0$ such that if a graph G is $K_{s,s}$-free, then it contains a minor J such that $\mathsf{d}(J) \geq C \cdot (\mathsf{d}(G))^{1+\frac{1}{2(s-1)}}$.*

We are now ready to prove Theorem 2.2.9, which we restate for convenience.

Theorem 2.2.9. *For every $\Delta \in \mathbb{N}$, $\nu > 0$, there exists $\mu = \mu_{2.2.9}(\Delta, \nu) > 0$ such that for every $s \in \mathbb{N}$ there exists $M = M_{2.2.9}(\Delta, \nu, s)$ such that if H is a bipartite graph with $\Delta(H) \leq \Delta$, maximum component size s and $\mathsf{v}(H) \geq M$, and G is a graph such that $\delta(G) \geq (1-\mu)\mathsf{v}(H)$ and $\mathsf{v}(G) \geq (1+\nu)\mathsf{v}(H)$, then H is a minor of G.*

Proof. We begin by introducing the necessary parameters. We are given Δ, ν. Let $\varepsilon > 0$ small enough such that

(E1) $\varepsilon < \min\left(\frac{1}{10}, \nu\right)$

(E2) $\frac{1-\varepsilon^2-4\varepsilon}{1-2\varepsilon^2} \geq 1 - \frac{1}{3\Delta}$,

(E3) $1 - 6\varepsilon - \frac{12\varepsilon}{\nu} \geq \frac{3}{4}$,

(E4) $(1-2\varepsilon)(1+\nu) - (1-\varepsilon) > 0$,

(E5) $\frac{\nu}{1+\frac{\nu}{2}} > 8\varepsilon$,

(E6) $\left(1 - \frac{2\varepsilon(2+\nu)}{1-2\varepsilon^2}\right) \geq \frac{3}{4}$,

(E7) $\frac{(1+\nu+2\varepsilon^2)}{1+\frac{\nu}{2}} > (1+2\varepsilon)$.

It is easy to verify that this is possible. Choose $\mu = \varepsilon^2$.

We are given s; we assume $s \geq 2$ without loss of generality. Let $M_1 = \frac{M_{2.2.2}(\mathcal{F}_s, \varepsilon^2)}{2\varepsilon^2}$, M_2 large enough that $\frac{C_{2.4.1}(s)}{2}\left(\frac{\varepsilon^2 M_2}{2}\right)^{\frac{1}{2(s-1)}} > (1+\varepsilon^2)$, $M_3 = \frac{2s}{\varepsilon^2}$ and finally $M = \max(M_1, M_2, M_3)$.

Since $\mathsf{v}(H) \geq M_1 > M_{2.2.2}(\mathcal{F}_s, \varepsilon^2)$, by Corollary 2.2.2 we have that either G contains H as a minor or $\mathsf{d}(G) \leq c(H) \leq (1+\varepsilon^2)\frac{\mathsf{v}(H)}{2}$. We may assume the latter case since in the first case the lemma holds. On the other hand, we have that $\mathsf{d}(G) \geq \frac{\delta(G)}{2} \geq (1-\varepsilon^2)\frac{\mathsf{v}(H)}{2}$.

Let G_0 be an induced subgraph of G with $\mathsf{v}(G_0)$ maximum such that

(S1) $\mathsf{v}(G_0) \leq (1-2\varepsilon^2)\mathsf{v}(H)$,

(S2) G_0 contains a spanning subgraph isomorphic to H_0, where H_0 is a union of connected components of H, and, subject to the above,

1. if $\mathsf{v}(G) \geq (2+\nu)\mathsf{v}(H)$, select G_0 minimizing $\sum_{v \in V(G_0)} \deg_G(v)$, and

2. if $(1+\nu)\mathsf{v}(H) \leq \mathsf{v}(G) < (2+\nu)\mathsf{v}(H)$, select G_0 maximizing $\sum_{v \in V(G_0)} \deg_{G_0}(v)$.

This is always possible, given that the empty graph respects (S1) and (S2).

Denote $G' = G - V(G_0)$. Our goal is to show that G' contains $H' = H - V(H_0)$ as a minor. If follows from (S2) that $\mathsf{v}(G_0) = \mathsf{v}(H_0)$. Condition (S1) then implies that

$$\mathsf{v}(H') = \mathsf{v}(H) - \mathsf{v}(H_0) \geq 2\varepsilon^2 \mathsf{v}(H).$$

Since $\mathsf{v}(G_0) = \mathsf{v}(H_0)$, every vertex in G' has at most $\mathsf{v}(H_0)$ neighbours outside of G', hence

$$\delta(G') \geq (1-\varepsilon^2)\mathsf{v}(H) - \mathsf{v}(H_0) = (1-\varepsilon^2)\mathsf{v}(H) - (\mathsf{v}(H) - \mathsf{v}(H')) = \mathsf{v}(H') - \varepsilon^2 \mathsf{v}(H)$$

and $\mathsf{d}(G') \geq \frac{\mathsf{v}(H') - \varepsilon^2 \mathsf{v}(H)}{2}$.

First, consider the case $(1-2\varepsilon^2)\mathsf{v}(H) - s \geq \mathsf{v}(G_0)$. Then, G' does not contain subgraphs isomorphic to any of the components of H', as otherwise we could add the corresponding induced subgraph of G' to G_0 without violating either (S2) or (S1), and so contradicting the choice of G_0. Since H is bipartite and has component size at most s, its components are subgraphs of $K_{s,s}$. Therefore G' has no subgraph isomorphic to $K_{s,s}$. By Theorem 2.4.1, G'

contains a minor J such that

$$\begin{aligned}
\mathsf{d}(J) &\geq C_{2.4.1}(s) \cdot (\mathsf{d}(G'))^{1+\frac{1}{2(s-1)}} \\
&\geq C_{2.4.1}(s) \left(\frac{\mathsf{v}(H') - \varepsilon^2 \mathsf{v}(H)}{2} \right)^{1+\frac{1}{2(s-1)}} \\
&\geq C_{2.4.1}(s) \left(\frac{(2\varepsilon^2 - \varepsilon^2)\mathsf{v}(H)}{2} \right)^{\frac{1}{2(s-1)}} \left(\frac{\left(1 - \frac{\varepsilon^2}{2\varepsilon^2}\right) \mathsf{v}(H')}{2} \right) \\
&> (1+\varepsilon^2)\frac{\mathsf{v}(H')}{2}
\end{aligned}$$

by choice of M_2. Since $\mathsf{v}(H') \geq 2\varepsilon^2\mathsf{v}(H) \geq 2\varepsilon^2 M_1 = M_{2.2.2}(\mathcal{F}_s, \varepsilon^2)$, by Theorem 2.2.2 we have that J contains H' as a minor. Hence, G' contains H' as a minor, as desired.

It remains to consider the case $(1-2\varepsilon^2)\mathsf{v}(H) - s \leq \mathsf{v}(G_0) \leq (1-2\varepsilon^2)\mathsf{v}(H)$. Note that this implies that $2\varepsilon^2\mathsf{v}(H) \leq \mathsf{v}(H') \leq 2\varepsilon^2\mathsf{v}(H) + s$ and that

$$\mathsf{v}(G_0) \geq (1-2\varepsilon^2)\mathsf{v}(H) - s \geq (1-2\varepsilon^2)M_3 - s = (1-2\varepsilon^2)\frac{2s}{\varepsilon^2} - s \geq s.$$

If $\mathsf{d}(G') > (1+\varepsilon^2)\frac{\mathsf{v}(H')}{2}$, then as in the previous paragraph G' contains H' as a minor by Theorem 2.2.2. Hence we assume $\mathsf{d}(G') \leq (1+\varepsilon^2)\frac{\mathsf{v}(H')}{2}$; we show this leads to a contradiction.

Denote by A the set of vertices of G with degree smaller than $(1+2\varepsilon)\mathsf{v}(H)$. We claim that $|A| \geq (1-\varepsilon)\mathsf{v}(G)$. Indeed, as $\delta(G) \geq (1-\varepsilon^2)\mathsf{v}(H)$, otherwise we would have

$$\mathsf{d}(G) \geq \frac{|A|(1-\varepsilon^2)\mathsf{v}(H) + (\mathsf{v}(G) - |A|) \cdot (1+2\varepsilon)\mathsf{v}(H)}{2\mathsf{v}(G)} \geq (1+\varepsilon^2+\varepsilon^3)\frac{\mathsf{v}(H)}{2} > (1+\varepsilon^2)\frac{\mathsf{v}(H)}{2},$$

contradicting our earlier assumption.

Denote by B the set of vertices of $v \in V(G')$ such that $\deg_{G'}(v) < 4\varepsilon\mathsf{v}(H)$. Similarly to the above, we show that $|B| \geq (1-\varepsilon)\mathsf{v}(G')$. Indeed, otherwise we would have

$$\mathsf{d}(G') \geq \frac{(\mathsf{v}(G') - |B|) \cdot 4\varepsilon\mathsf{v}(H)}{2\mathsf{v}(G')} \geq 2\varepsilon^2\mathsf{v}(H) \geq \mathsf{v}(H') - s \geq \left(1 - \frac{s}{2\varepsilon^2\mathsf{v}(H)}\right)\mathsf{v}(H') > (1+\varepsilon^2)\frac{\mathsf{v}(H')}{2}$$

which is again a contradiction (the last inequality uses our choice of M_3). Note that each vertex $v \in B$ has at least

$$\deg_G(u) - \deg_{G'}(u) \geq (1-\varepsilon^2-4\varepsilon) \geq (1-5\varepsilon)\mathsf{v}(H) \geq \frac{(1-5\varepsilon)\mathsf{v}(H)}{(1-2\varepsilon^2)\mathsf{v}(H)}\mathsf{v}(G_0) \geq \left(1 - \frac{1}{3\Delta}\right)\mathsf{v}(G_0)$$

neighbours in G_0, using (E2).

Another consequence is that there are at least $(1-5\varepsilon)\mathsf{v}(H) \cdot (1-\varepsilon)\mathsf{v}(G') \geq (1-6\varepsilon)\mathsf{v}(H)\mathsf{v}(G')$ edges between the vertices of G' (specifically, the vertices in B) and the vertices of G_0. Let C the set of vertices of G_0 with at least $\frac{\mathsf{v}(G')}{1+\frac{\nu}{2}}$ neighbours in G'. Then the number of edges between vertices of G' and G_0 is upper bounded by $|C| \cdot \mathsf{v}(G') + (\mathsf{v}(G_0) - |C|) \cdot \frac{\mathsf{v}(G')}{1+\frac{\nu}{2}}$, implying

$$|C| + \frac{\mathsf{v}(G_0) - |C|}{1+\frac{\nu}{2}} \geq (1-6\varepsilon)\mathsf{v}(G_0),$$

and therefore

$$|C| \geq \left(1 - 6\varepsilon - \frac{12\varepsilon}{\nu}\right)\mathsf{v}(G_0) \geq \frac{3}{4}\mathsf{v}(G_0)$$

where the last inequality uses (E3).

Let $A' = A \cap V(G_0)$ if $(1+\nu)\mathsf{v}(H) \leq \mathsf{v}(G) < (2+\nu)\mathsf{v}(H)$ and $A' = V(G_0)$ otherwise. We claim that $|C \cap A'| > \frac{1}{2}\mathsf{v}(G_0)$.

If $A' = V(G_0)$, clearly $|C \cap A'| = |C| \geq \frac{3}{4}\mathsf{v}(G_0) > \frac{1}{2}\mathsf{v}(G_0)$. Otherwise, $\mathsf{v}(G) < (2+\nu)\mathsf{v}(H)$, and

$$\begin{aligned}
|A'| &\geq |A| - \mathsf{v}(G') \geq (1-\varepsilon)\mathsf{v}(G) - \mathsf{v}(G') = \mathsf{v}(G_0) - \varepsilon\mathsf{v}(G) > \mathsf{v}(G_0) - \varepsilon(2+\nu)\mathsf{v}(H) \\
&\geq \mathsf{v}(G_0) - \varepsilon(2+\nu)\frac{\mathsf{v}(G_0)+s}{1-2\varepsilon^2} \geq \left(1 - \frac{2\varepsilon(2+\nu)}{1-2\varepsilon^2}\right)\mathsf{v}(G_0) \\
&\geq \frac{3}{4}\mathsf{v}(G_0)
\end{aligned}$$

using (E6) and the bound $\mathsf{v}(G_0) \geq s$. We then have

$$|C \cap A'| = |C| + |A'| - |C \cup A'| > \frac{3}{4}\mathsf{v}(G_0) + \frac{3}{4}\mathsf{v}(G_0) - \mathsf{v}(G_0) = \frac{1}{2}\mathsf{v}(G_0),$$

and the claim also holds in this case.

As $|A| \geq (1-\varepsilon)\mathsf{v}(G)$ and $|B| \geq (1-\varepsilon)\mathsf{v}(G')$, we have

$$\begin{aligned}|A \cap B| = |(A \setminus V(G_0)) \cap B| &\geq (|A| - \mathsf{v}(G_0)) - (\mathsf{v}(G') - |B|) \\ &\geq (1-\varepsilon)\mathsf{v}(G) - \mathsf{v}(G_0) - \varepsilon\mathsf{v}(G') = (1-2\varepsilon)\mathsf{v}(G) - (1-\varepsilon)\mathsf{v}(G_0) \\ &\geq (1-2\varepsilon)(1+\nu)\mathsf{v}(H) - (1-\varepsilon)\mathsf{v}(G_0) \geq ((1-2\varepsilon)(1+\nu) - (1-\varepsilon))\mathsf{v}(G_0) \\ &> 0\end{aligned}$$

by (E4). Choose $x \in A \cap B$.

Our goal is to show that there exists $y \in C \cap A'$ such that $N_{H_0}(y) \subseteq N_G(x)$ (considering H_0 as a subgraph of G with some fixed embedding by (S2)).

Suppose this is not the case. Then, for each $y \in C \cap A'$, y is adjacent (in H) to at least one vertex of $V(G_0) \setminus N_G(x)$. Hence, the total degree (in H) over vertices of $V(G_0) \setminus N_G(x)$ is at least $|C \cap A'| > \frac{1}{2}\mathsf{v}(G_0)$. By the Pigeonhole principle, at least one vertex $z \in V(G_0) \setminus N_G(x)$ has degree at least $\frac{\frac{1}{2}\mathsf{v}(G_0)}{|V(G_0) \setminus N_G(x)|}$ in H. Being in B, x is adjacent to a proportion of at least $1 - \frac{1}{3\Delta}$ of the vertices of G_0, which means that $\deg_{H_0}(z) \geq \frac{\frac{1}{2}\mathsf{v}(G_0)}{\frac{1}{3\Delta}\mathsf{v}(G_0)} > \Delta$. This is a contradiction to the maximum degree of H. Hence, such a y exists.

Removing y of G_0 and instead adding x still yields H_0 as a subgraph of G: keep the same embedding of H_0 in G_0 but replace y by x, which is possible since $N_{H_0}(y) \subseteq N_G(x)$.

Consider the case $\mathsf{v}(G) \geq (2+\nu)\mathsf{v}(H)$. In this case, we claim that x has smaller degree in G than y, which is the contradiction we are looking for. Indeed, being in A, the degree of

x is at most $(1+2\varepsilon)\mathsf{v}(H)$. On the other hand, since $y \in C$, its degree is at least

$$\frac{\mathsf{v}(G')}{1+\frac{\nu}{2}} = \frac{\mathsf{v}(G)-\mathsf{v}(G_0)}{1+\frac{\nu}{2}} \geq \frac{(2+\nu)\mathsf{v}(H)-(1-2\varepsilon^2)\mathsf{v}(H)}{1+\frac{\nu}{2}} = \frac{(1+\nu+2\varepsilon^2)}{1+\frac{\nu}{2}}\mathsf{v}(H) > (1+2\varepsilon)\mathsf{v}(H),$$

the last inequality holding by (E7). This contradicts our initial choice of G_0.

Now, consider the case $(1+\nu)\mathsf{v}(H) \leq \mathsf{v}(G) < (2+\nu)\mathsf{v}(H)$. In this case, we claim x has greater degree in G_0 than y, which will contradict the choice of G_0. Since $y \in A' \subseteq A$, $\deg_G(y) \leq (1+2\varepsilon)\mathsf{v}(H)$, and since $y \in C$, it is adjacent to at least $\frac{1}{1+\frac{\nu}{2}}\mathsf{v}(G')$ vertices of G'. Hence, y is adjacent to at most $(1+2\varepsilon)\mathsf{v}(H) - \frac{1}{1+\frac{\nu}{2}}\mathsf{v}(G')$ vertices of G_0. On the other hand, being in B, x is adjacent to at least $(1-5\varepsilon)\mathsf{v}(H)-1$ vertices of G_0 excluding y. It then suffices to show that

$$(1-5\varepsilon)\mathsf{v}(H) - 1 > (1+2\varepsilon)\mathsf{v}(H) - \frac{1}{1+\frac{\nu}{2}}\mathsf{v}(G').$$

Using that $\mathsf{v}(G') = \mathsf{v}(G) - \mathsf{v}(G_0)$, we can rewrite the desired inequality as

$$\frac{1}{1+\frac{\nu}{2}}\mathsf{v}(G) > 1 + 7\varepsilon\mathsf{v}(H) + \frac{1}{1+\frac{\nu}{2}}\mathsf{v}(G_0).$$

Using that $\mathsf{v}(G_0) \leq (1-2\varepsilon^2)\mathsf{v}(H) < \mathsf{v}(H)$, $\mathsf{v}(G) \geq (1+\nu)\mathsf{v}(H)$ and $\mathsf{v}(H) \geq M_3 \geq \frac{2s}{\varepsilon^2} \geq \frac{1}{\varepsilon}$, it suffices to show that

$$\frac{1+\nu}{1+\frac{\nu}{2}} > 8\varepsilon + \frac{1}{1+\frac{\nu}{2}}.$$

This is a direct consequence of (E5). This is the contradiction we are looking for, since replacing y by x would give more edges inside G_0.

In all cases, the union of the H' minor in G' and the H_0 subgraph of G_0 gives us an H minor in G. □

We note that for our purposes, it would be sufficient to prove this result for graphs G such that $\mathsf{v}(G) \leq C_0\mathsf{v}(H)$, for some large C_0. In fact, the previous proof could be modified so that the cutoff point between the two cases is $C_0\mathsf{v}(H)$ instead of $(2+\nu)\mathsf{v}(H)$, and so only one case would be required in the proof. However, we believe the more general result is interesting in its own right.

2.5 Density increment

In this section, we prove Theorem 2.2.10, which states that that under certain conditions we can either increase the density of our graph or find a large number of small subgraphs of constant density. We begin with a technical lemma, which, given a graph with average degree close to the minimum degree, allows us to extract a subgraph with maximum degree close to the average degree, while only losing a small amount of density.

Lemma 2.5.1. *For every $\gamma, \alpha, \beta > 0$ with $\beta < 1$, there exists $\varepsilon = \varepsilon_{2.5.1}(\gamma, \alpha, \beta) > 0$ such that for any $D \in \mathbb{N}$, if G is a graph with $\delta(G) \geq D$ and $\mathsf{d}(G) \leq (1+\varepsilon)\frac{D}{2}$, then G contains a subgraph G' such that $\mathsf{d}(G') \geq (1-\gamma)\frac{D}{2}$, $\Delta(G') \leq (1+\alpha)D$ and $\mathsf{v}(G') \geq (1-\beta)\mathsf{v}(G)$.*

Proof. Choose $\varepsilon = \min\left(\frac{\gamma}{2\left(1+\frac{1}{\alpha}\right)}, \alpha\beta\right)$. Let D, G be as in the statement.

Let X be the set of vertices of G of degree greater than $(1+\alpha)D$, and set $G' = G - X$. Clearly $\Delta(G') \leq (1+\alpha)D$. We wish to prove that $\mathsf{d}(G') \geq (1-\gamma)\frac{D}{2}$ and $\mathsf{v}(G') \geq (1-\beta)\mathsf{v}(G)$.

We have that

$$\frac{\mathsf{e}(G)}{\mathsf{v}(G)} = \mathsf{d}(G) \leq (1+\varepsilon)\frac{D}{2}.$$

Hence, we have that

$$(1+\varepsilon)D\mathsf{v}(G) \geq 2\mathsf{e}(G) = \sum_{u \in X} \deg(u) + \sum_{u \in V(G)\setminus X} \deg(v) \geq \sum_{u \in X} \deg(u) + D(\mathsf{v}(G) - |X|)$$

or again that

$$\sum_{u \in X} \deg(u) \leq D(\varepsilon \mathsf{v}(G) + |X|).$$

By definition of X, we then have that $(1+\alpha)D|X| \leq D(\varepsilon\mathsf{v}(G) + |X|)$, hence $|X| \leq \frac{\varepsilon}{\alpha}\mathsf{v}(G)$.

This implies that $\mathsf{v}(G') = \mathsf{v}(G) - |X| \geq \left(1 - \frac{\varepsilon}{\alpha}\right)\mathsf{v}(G) \geq (1-\beta)\mathsf{v}(G)$.

We also get that

$$\sum_{u \in X} \deg(u) \leq \left(1 + \frac{1}{\alpha}\right) D\varepsilon \mathsf{v}(G)$$

and so

$$\mathsf{d}(G') = \frac{\mathsf{e}(G')}{\mathsf{v}(G')} \geq \frac{\frac{D}{2}\mathsf{v}(G) - \left(1 + \frac{1}{\alpha}\right) D\varepsilon \mathsf{v}(G)}{\mathsf{v}(G)} = \left(1 - 2\varepsilon\left(1 + \frac{1}{\alpha}\right)\right)\frac{D}{2} \geq (1-\gamma)\frac{D}{2}.$$

□

The next follows directly from the density increment result of Norin and Song [A25, Theorem 4.1] by taking $K = \frac{1}{8\varepsilon}$.

Lemma 2.5.2 ([A25]). *If $0 < \varepsilon < \frac{1}{100}$ and G is a graph with $\mathsf{d}(G) \geq \frac{2}{\varepsilon}$, then G contains either*

1. *a subgraph J such that $\mathsf{v}(J) \leq \frac{\mathsf{d}(G)}{2\varepsilon}$ and $\mathsf{d}(J) \geq \varepsilon \mathsf{d}(G)$, or*
2. *a minor H of G such that $\mathsf{d}(H) \geq (1+\varepsilon)\mathsf{d}(G)$.*

Recursively applying this lemma, we may now deduce the desired result, which we restate for convenience.

Theorem 2.2.10. *There exists $\varepsilon = \varepsilon_{2.2.10} > 0$ such that for every $K \in \mathbb{N}$ there exist $\varepsilon' = \varepsilon'_{2.2.10}(K)$ and $C = C_{2.2.10}(K) \geq 1$ such that for every $D \geq C$ and every G such that $\delta(G) \geq D$ and $\mathsf{v}(G) \geq CD$, then either*

1. *G contains vertex-disjoint subgraphs $J_1, \ldots, J_K$ such that $\mathsf{v}(J_i) \leq \frac{D}{\varepsilon}$ and $\mathsf{d}(J_i) \geq \varepsilon D$ for every $i \in [K]$, or*
2. *G contains a minor H such that $\mathsf{d}(H) \geq (1+\varepsilon')\frac{D}{2}$.*

Proof. Take $\varepsilon < \frac{1}{300}$. Let K be given. Set $0 < \gamma < 1 - \frac{1}{(1+3\varepsilon)^{\frac{1}{K+1}}}$ (note that this implies $(1-\gamma)^{K+1} \geq \frac{2}{3}$). We may now choose $C = \frac{K+1+\frac{1}{3\gamma}}{\varepsilon}$ and $\varepsilon' = \min\left(\varepsilon_{2.5.1}(\gamma, 1, \frac{1}{\varepsilon C}), (1+3\varepsilon)(1-\gamma)^K - 1\right)$. Let D, G be as in the statement.

We will show that either we can construct a minor H of G such that $\mathsf{d}(H) \geq (1+\varepsilon')\frac{D}{2}$ or that we can recursively construct a sequence of subgraphs $G_0, \ldots, G_K$ and $J_1, \ldots, J_K$ of G such that

(a) G_0 is a subgraph of G, and G_i, J_i are disjoint subgraphs of G_{i-1} for $i \in [K]$;

(b) $\mathsf{v}(J_i) \leq \frac{D}{\varepsilon}$ and $\mathsf{d}(J_i) \geq \varepsilon D$ for $i \in [K]$; and

(c) $\mathsf{v}(G_i) \geq \left(C - \frac{i+1}{\varepsilon}\right) D$, $\mathsf{d}(G_i) \geq (1-\gamma)^{i+1}\frac{D}{2}$ for $i \in \{0\} \cup [K]$.

We first consider the base case of this construction. Either $\mathsf{d}(G) > (1+\varepsilon')\frac{D}{2}$, in which case taking $H = G$ yields the result, or applying Lemma 2.5.1 yields a subgraph G_0 of G such that $\mathsf{d}(G_0) \geq (1-\gamma)\frac{D}{2}$, $\Delta(G_0) \leq (1+1)D = 2D$ and $\mathsf{v}(G_0) \geq \left(1 - \frac{1}{\varepsilon C}\right)\mathsf{v}(G) \geq \left(C - \frac{1}{\varepsilon}\right) D$.

Let $i \in [K]$. Suppose we have already constructed $G_0, \ldots, G_{i-1}, J_1, \ldots, J_{i-1}$. We have that $\mathsf{d}(G_{i-1}) \geq (1-\gamma)^i \frac{D}{2} \geq \frac{C}{3} > \frac{2}{3\varepsilon}$. Hence, we can apply Lemma 2.5.2 (to G_{i-1} with 3ε) to get that G_{i-1} contains either

1. a subgraph J_i such that $\mathsf{v}(J_i) \leq \frac{\mathsf{d}(G_{i-1})}{6\varepsilon}$ and $\mathsf{d}(J_i) \geq 3\varepsilon\mathsf{d}(G_{i-1})$, or

2. a minor H such that $\mathsf{d}(H) \geq (1+3\varepsilon)\mathsf{d}(G_{i-1})$.

If we are in case (2), then we are done since

$$\mathsf{d}(H) \geq (1+3\varepsilon)\mathsf{d}(G_{i-1}) \geq (1+3\varepsilon)(1-\gamma)^i\frac{D}{2} \geq (1+\varepsilon')\frac{D}{2}.$$

Otherwise we are in case (1). Set $G_i = G_{i-1} - V(J_i)$. Condition (a) is direct. Condition (b) is due to the fact that

$$\mathsf{v}(J_i) \leq \frac{\mathsf{d}(G_{i-1})}{6\varepsilon} \leq \frac{\Delta(G_0)}{12\varepsilon} \leq \frac{D}{6\varepsilon} < \frac{D}{\varepsilon}$$

and that

$$\mathsf{d}(J_i) \geq 3\varepsilon\mathsf{d}(G_{i-1}) \geq 3\varepsilon(1-\gamma)^i\frac{D}{2} \geq \varepsilon D.$$

For condition (c), we have that

$$\mathsf{v}(G_i) = \mathsf{v}(G_{i-1}) - \mathsf{v}(J_i) \geq \left(C - \frac{i}{\varepsilon}\right) D - \frac{D}{\varepsilon} = \left(C - \frac{i+1}{\varepsilon}\right) D$$

and

$$\begin{aligned}
\mathsf{d}(G_i) = \frac{\mathsf{e}(G_i)}{\mathsf{v}(G_i)} &\geq \frac{\mathsf{e}(G_{i-1}) - \Delta(G_0)\cdot \mathsf{v}(J_i)}{\mathsf{v}(G_i)} \geq \frac{\mathsf{v}(G_{i-1})\mathsf{d}(G_{i-1}) - 2D \cdot \frac{\mathsf{d}(G_{i-1})}{6\varepsilon}}{\mathsf{v}(G_i)} \\
&\geq \frac{\mathsf{v}(G_i) - \frac{D}{3\varepsilon}}{\mathsf{v}(G_i)}\mathsf{d}(G_{i-1}) \geq \left(1 - \frac{\frac{D}{3\varepsilon}}{\left(C - \frac{i+1}{\varepsilon}\right) D}\right)\mathsf{d}(G_{i-1}) \\
&\geq (1-\gamma)\mathsf{d}(G_{i-1})
\end{aligned}$$

from which it follows that $\mathsf{d}(G_i) \geq (1-\gamma)^{i+1}\frac{D}{2}$.

If the construction was not interrupted (yielding a satisfactory minor H), we now have the desired sequence of subgraphs $J_1, \ldots, J_K$. □

2.6 Building a minor from pieces

In this section, we prove Theorem 2.2.6, which allows us to build a minor from pieces of sufficient density, using some ideas from the proof of [A23, Theorem 2.6]. We first need the following definitions.

Given an injection $\phi : V(H) \to V(G)$ we say that a model μ of H in G is *ϕ-rooted* if $\phi(v) \in \mu(v)$ for every $v \in V(H)$. Finally, we say that G is *H-linked* if $\mathsf{v}(G) \geq \mathsf{v}(H)$ and for every injection $\phi : V(H) \to V(G)$ there exists a ϕ-rooted model of H in G. Note that every H-linked graph has an H minor, but the converse does not hold.

A *simple edge extension* of a graph H is a graph H' obtained from H by adding a new vertex joined by an edge to at most one vertex of H, or two new vertices joined by an edge. In particular, we have $H \subset H'$ and $\mathsf{e}(H') \leq \mathsf{e}(H) + 1$. A *$k$-edge extension* of H is obtained from H by a sequence of at most k simple edge extensions. If a graph G is H'-linked for every k-edge extension H' of H then we write that G is *$(H+k)$-linked* for brevity, and we say that a graph is *k-linked* if it is $(O + k)$-linked, where O is the null graph.

We will use the following version of Menger's theorem [A22].

Theorem 2.6.1 ([A22])**.** *If $k \in \mathbb{N}$, G is a graph and $U, W \subseteq V(G)$, then either there exists $A, B \subseteq V(G)$ such that $|A \cap B| \leq \ell - 1$, $U \subseteq A$ and $W \subseteq V$, or there exists ℓ pairwise vertex-disjoint paths each with one end in A and one end in B.*

The following lemma is a key element in our proof of Theorem 2.2.6.

Lemma 2.6.2. *Let H be a graph let $F \subseteq E(H)$ be such that $H - F$ is a disjoint union of graphs $H_1, \ldots, H_k$ and such that no edge of F has both ends in the same component of $H - F$. If G is a graph and $G_1, \ldots, G_k$ are pairwise vertex-disjoint subgraphs of G such that*

- *G_i is $(H_i + |F|)$-linked for every $i \in [k]$, and.*
- *$\left(G, V(G) \setminus \bigcup_{i=1}^{k} V(G_i)\right)$ is $2|F|$-dense,*

then $H \preceq G$.

Roughly speaking, we will find find paths between the G_i which correspond to the edges of F by applying Menger's theorem using the second condition of the lemma, after which the first condition will allow us to construct a rooted model of H_i in each G_i while simultaneously rerouting the paths found previously in order to avoid these models, which together yield a model of H.

Proof. For brevity, let $Z = \bigcup_{i=1}^{k} V(G_i)$.

For every $i \in [k]$, G_i is $|F|$-linked, and so contains at least $2|F|$ vertices. Hence, we can choose U a set of $2|F|$ vertices of G_1. Furthermore, it is possible to choose for each $uv \in F$ two vertices r_{uv} and s_{uv} such that $r_{uv} \in V(G_i)$ if $u \in V(H_i)$ and $s_{uv} \in V(G_i)$ if $v \in V(H_i)$ (and such that all these vertices are distinct). Let W be the set of these $2|F|$ vertices. Note that U and W might intersect.

Apply Theorem 2.6.1 to G, U, W with $\ell = 2|F|$ instead. There first possibility is there exists $A, B \subseteq V(G)$ such that $|A \cap B| \leq k - 1$, $U \subseteq A$ and $W \subseteq V$. Since $|A \cap B| \leq$

$2|F|-1 < |U|$, we have that $U \setminus B = U \setminus (A \cap B) \neq \emptyset$, and analogously $W \setminus B \neq \emptyset$. In particular, $A \setminus B, B \setminus A \neq \emptyset$, and so (A, B) is a separation. Given that $(G, V(G) \setminus Z)$ is $2|F|$-dense, $A \setminus B \subseteq V(G) \setminus Z$. In particular, $(U \setminus B) \cap V(G_1) \subseteq (A \setminus B) \cap V(G_1) = \emptyset$. This is a contradiction since $U \subseteq V(G_1)$ and $U \setminus B \neq \emptyset$.

Hence, for each $r_{uv} \in W$ (resp. $s_{uv} \in W$), there a path R_{uv} (resp. S'_{uv}) with one end in U, say r'_{uv} (resp. s'_{uv}), and the other end is r_{uv} (resp. s_{uv}), and all these paths are pairwise vertex-disjoint. By choosing these paths to be as short as possible, we may assume that no internal vertices of the paths are in G_1.

As G_1 is $|F|$-linked, there exists in G_1 pairwise vertex-disjoint paths T_{uv}, indexed by $uv \in F$, such that R_{uv} has r'_{uv} and s'_{uv} as ends. For each $uv \in F$, set P_{uv} as the path obtained by concatenating R_{uv}, T_{uv} and S_{uv}. These are pairwise vertex-disjoint paths in G such that if $u \in V(H_i)$ and $v \in V(H_j)$ then P_{uv} has ends $r_{uv} \in V(G_i)$ and $s_{uv} \in V(G_j)$. We may also assume that P_{uv} paths is shortest possible, and in particular that it has no internal vertices in $G_i \cup G_j$.

We will need to modify the paths $\{P_{uv}\}_{uv \in F}$ after we find appropriate rooted models in the graphs $G_1, \ldots, G_s$, and we prepare for this as follows.

Fix $uv \in F$. We construct an auxiliary graph J_{uv} with vertex set $[k]$ such that indices i', i'' are adjacent if there exists a subpath of P_{uv} with ends in $V(G_{i'})$ and $V(G_{i''})$ and otherwise disjoint from Z. Denote such a subpath by $P_{uv}^{i'i''}$. Suppose $u \in V(H_i)$ and $v \in V(H_j)$. Clearly, i and j belong to the same component of J_{uv}, and so there exist a sequence $I_{uv} = (i_1, i_2 \ldots, i_\ell)$ of distinct indices such that $i_1 = i$ and $i_\ell = j$, and i_t and i_{t+1} are adjacent in our auxiliary graph for each $1 \leq t \leq \ell$. For each $1 \leq t \leq \ell$, we define $s_{uv}^{i_t}$ and $r_{uv}^{i_t}$ to be the ends of $P_{uv}^{i_{t-1}i_t}$ and $P_{uv}^{i_t i_{t+1}}$, respectively, in $V(G_{i_t})$ (except for $s_{uv}^{i_1}, r_{uv}^{i_\ell}$ which are not defined). It is possible that $r_{uv}^{i_t} = s_{uv}^{i_t}$. Note that $r_{uv}^{i_1} = r_{uv}$ and $s_{uv}^{i_\ell} = s_{uv}$.

For each $i \in [k]$, let H'_i be the $|F|$-extension of H_i defined as follows. For each edge $uv \in F$ such that $u, v \notin V(H_i)$, if r_{uv}^i and s_{uv}^i are both defined and distinct, we add p_{uv}^i and q_{uv}^i to H_i joined by an edge, and if $r_{uv}^i = s_{uv}^i$ is defined, we add an isolated vertex

$p^i_{uv} = q^i_{uv}$. Furthermore, if $u \in V(H_i)$, add a vertex p^i_{uv} to H_i and an edge from u to p^i_{uv}, and if $v \in V(H_i)$, add a vertex q^i_{uv} to H_i and an edge from v to q^i_{uv}.

Let an injection $\phi_i : V(H'_i) \to V(G_i)$ be defined as follows. For $uv \in F$, let $\phi_i(p^i_{uv}) = r^i_{uv}$ and $\phi_i(q^i_{uv}) = s^i_{uv}$ whenever r^i_{uv} and s^i_{uv}, respectively, are defined. For $w \in V(H_i)$, we choose $\phi_i(w)$ arbitrarily subject to ϕ_i being injective (this is possible given than G_i is $(H_i + |F|)$-linked and so contains at least $\mathsf{v}(H_i) + 2|F|$ vertices).

As G_i is $(H_i + |F|)$-linked there exists a ϕ_i-rooted model μ_i of H'_i in G_i. Let μ' be a model of $H - F$ in G obtained by defining $\mu'(v) = \mu_i(v)$ when $v \in V(H_i)$.

To extend μ' to a model μ of H it suffices to find a set of pairwise internally vertex-disjoint paths $\mathcal{Q} = \{Q_{uv}\}_{uv \in F}$ such that for each $uv \in F$ the path Q_{uv} has one end in $\mu'(u)$ and the other in $\mu'(v)$, and otherwise disjoint from $\mu'(V(H))$. To do this, we consider the subgraph G_{uv} of G induced by the set

$$W_{uv} = (V(P_{uv}) \setminus Z) \cup \mu'(p^{i_1}_{uv}) \cup \mu'(q^{i_\ell}_{uv}) \cup \bigcup_{i \in I_{uv}\,:\, u,v \notin V(H_i)} \left(\mu'(p^i_{uv}) \cup \mu'(q^i_{uv})\right),$$

Note that the elements of $\{W_{uv}\}_{uv \in F}$ are pairwise vertex-disjoint and so it suffices to show that for each $uv \in F$ there exists a path Q_{uv} with one end in $\mu'(u)$, the other in $\mu'(v)$ and all internal vertices in W_{uv}. As W_{uv} contains vertices $x_{uv} \in \mu'(p^{i_1}_{uv})$ with a neighbour in $\mu'(u)$ and $y_{uv} \in \mu'(q^{i_\ell}_{uv})$ with a neighbour in $\mu'(v)$, it suffices to show that x_{uv} and y_{uv} belong to the same component of G_{uv}. This follows from our construction. Indeed, if $I_{uv} = (i_1, i_2 \ldots, i_\ell)$, then G_{uv} contains

- a path from x_{uv} to $r_{uv} = r^1_{uv}$ in $\mu'(p^{i_1}_{uv})$,
- a path $P^{i_1 i_2}_{uv}$ from $r^{i_1}_{uv}$ to $s^{i_2}_{uv}$,
- a path from $s^{i_2}_{uv}$ to $r^{i_2}_{uv}$ in $\mu'(q^{i_2}_{uv}) \cup \mu'(p^{i_2}_{uv})$,
- similarly paths from $r^{i_{t-1}}_{uv}$ to $s^{i_t}_{uv}$ and from $s^{i_t}_{uv}$ to $r^{i_t}_{uv}$ for $t = 3, \ldots, \ell$,
- a path from $s^\ell_{uv} = s_{uv}$ to y_{uv} in $\mu'(q^{i_\ell}_{uv})$.

This completes the proof of the lemma. $\square$

In order to prove Theorem 2.2.6 using Lemma 2.6.2, we need to find well-linked pieces of G using dense pieces of G.

We define the *connectivity* of G, denoted by $\kappa(G)$, as the minimum order of a separation of G, except when G is complete in which case $\kappa(G) = \mathsf{v}(G) - 1$. It is easy to see that the connectivity of a graph gives a lower bound on the degrees of the vertices of the graph, that is $\mathsf{d}(G) \geq \frac{\kappa(G)}{2}$. The following result of Mader [A21] allows us to find a subgraph of connectivity on the same order as the average degree.

Theorem 2.6.3 ([A21])**.** *Every graph G contains a subgraph G' such that $\kappa(G') \geq \frac{\mathsf{d}(G)}{2}$.*

We need the following result of Wollan [A34].

Theorem 2.6.4 ([A34, Theorem 1.1])**.** *If H and G are graphs such that $\kappa(G) \geq \mathsf{v}(H)$ and $\mathsf{d}(H) \geq 9c(H) + 26833\mathsf{v}(H)$, then G is H-linked.*

The following form will be easier to use.

Corollary 2.6.5. *There exists $C = C_{2.6.5} \in \mathbb{N}$ such that if H and G are graphs such that $\kappa(G) \geq Cc(H)$, then G is H-linked.*

Proof. Set $C = 6 \cdot 28633$. Let H, G be as in the statement. If $\mathsf{v}(H) \leq 2$, the statement is trivial. Hence we may suppose that $\mathsf{v}(H) \geq 3$.

First note that $c(H) > \frac{\mathsf{v}(H)}{2} - 1$ for every graph H. Indeed, the complete graph $K_{\mathsf{v}(H)-1}$ on $\mathsf{v}(H) - 1$ vertices does not contain $\mathsf{v}(H)$ as a minor, but $\mathsf{d}(K_{\mathsf{v}(H)-1}) = \frac{\binom{\mathsf{v}(H)-1}{2}}{\mathsf{v}(H)-1} = \frac{\mathsf{v}(H)}{2} - 1$. In particular, $Cc(H) > C\left(\frac{\mathsf{v}(H)}{2} - 1\right) \geq 2 \cdot 26833\mathsf{v}(H)$, using that $\mathsf{v}(H) \geq 3$.

Then,

$$\kappa(G) \geq Cc(H) \geq \mathsf{v}(H)$$

and

$$\mathsf{d}(G) \geq \frac{\delta(G)}{2} \geq \frac{\kappa(G)}{2} \geq \frac{Cc(H)}{2} \geq 9c(H) + 26833\mathsf{v}(H).$$

We may therefore apply Theorem 2.6.4 to obtain that G is H-linked, which completes the proof. $\square$

We may now upper bound the extremal function for graphs with edge extensions.

Lemma 2.6.6. *If J is a k-edge extension of a graph H, then $c(J) \leq 2C_{2.6.5}c(H) + 4k$.*

Proof. Suppose G is such that $\mathsf{d}(G) > 2C_{2.6.5}c(H) + 4k$. We wish to prove that H is a minor of G.

By Theorem 2.6.3, G contains a subgraph G' such that $\kappa(G') \geq \frac{\mathsf{d}(G)}{2} \geq C_{2.6.5}c(H) + 2k$.

As J is a k-edge extension of H, we may consider H as an induced subgraph of J. Let A be the the vertices added by simple edge extensions. In particular, $|A| \leq 2k$ and $V(J) = V(H) \cup A$. Let B be the vertices of $V(H)$ to which incident edges have been added by (the first type of) simple edge extensions. Note that by definition of simple edge extensions, $J' = J[A \cup B] - E(J[B])$ is necessarily a forest.

Given that $\delta(G') \geq \kappa(G') \geq 2k$, we may greedily construct an embedding of J' in G' : consecutively for each component (which is a tree) of J', select an arbitrary unused vertex of G' and repeatedly use the minimum degree to find new vertices in order construct the tree. Let $\phi : V(J') \to V(G')$ be the (injective) mapping of vertices in this embedding.

Given that $\kappa(G' - \phi(A)) \geq \kappa(G') - 2k \geq C_{2.6.5}c(H)$, there exists by Corollary 2.6.5 a $\phi|_B$-rooted model μ of H in $G' - \phi(A)$. We can then expand μ to a model of J in G' by setting $\mu(a) = \{\phi(a)\}$ for every $a \in A$. Hence, $J \preceq G' \preceq G$ as desired, which completes the proof. $\square$

Finally, we prove Theorem 2.2.6, which we restate for convenience.

Theorem 2.2.6. *There exists $C = C_{2.2.6} \in \mathbb{N}$ satisfying the following. Let H be a graph, let $F \subseteq E(H)$ be such that $H - F$ is a disjoint union of graphs $H_1, \ldots, H_k$ and such that no edge of F has both ends in the same component of $H - F$. If G is a graph and $G_1, \ldots, G_k$ are pairwise vertex-disjoint subgraphs of G such that*

- $\mathsf{d}(G_i) \geq C(c(H_i) + |F|)$ *for every* $i \in [k]$*, and*

- $\left(G, V(G) \setminus \bigcup_{i=1}^{k} V(G_i)\right)$ *is* $2|F|$*-dense,*

then $H \preceq G$*.*

Proof. We show that the statement holds for $C = 8C_{2.6.5}^2$.

By Theorem 2.6.3, for each $i \in [k]$ there exists a subgraph G_i' of G_i such that $\kappa(G_i') \geq \frac{\mathsf{d}(G_i)}{2}$.

We first claim that G_i' is $(H_i + |F|)$-linked for every $i \in [k]$. We thus need to show that if J is an $|F|$-edge extension of H_i, then G_i' is J-linked. By Lemma 2.6.6, we have that

$$\kappa(G_i') \geq \frac{\mathsf{d}(G_i)}{2} \geq 4C_{2.6.5}^2(c(H_i) + |F|) \geq C_{2.6.5}(2C_{2.6.5}c(H_i) + 4|F|) \geq C_{2.6.5}c(J).$$

By Corollary 2.6.5, G_i' is J-linked, which completes the proof of the claim.

Since $V(G) \setminus \bigcup_{i=1}^{k} V(G_i) \subseteq V(G) \setminus \bigcup_{i=1}^{k} V(G_i')$, we have that $\left(G, V(G) \setminus \bigcup_{i=1}^{k} V(G_i')\right)$ is also $2|F|$-dense.

Applying Lemma 2.6.2 with G_i' instead of G_i, we have that $H \preceq G$ as desired. □

2.7 Tightness

In this section, we show the necessity of the conditions imposed in Theorem 2.1.4. Indeed, for each condition we provide examples showing that the theorem does not hold when removing this condition but maintaining all others.

In fact, in all of these examples, we will even impose a lower bound on the minimum (or average, in the last case) degree of G which is larger than $\mathsf{v}(H) - 1$ as in Theorem 2.1.4.

Removing the maximum degree bound on H.

Lemma 2.7.1. *For* $s, t \in \mathbb{N}$ *such that* $s \ll t$*, there exists a graph* G *with* $\delta(G) \geq 2s + t - 2\sqrt{2s} - 2$ *and no* $K_{s,t}$ *minor.*

Proof. Let $d = \lceil\sqrt{2s}\rceil$ and let G' be a d-regular graph of girth (length of shortest cycle) greater than $(d+1)s$ with

$$\mathsf{v}(G') = 2\left(s + \left\lfloor \frac{t-\sqrt{2s}}{2} \right\rfloor\right).$$

Note that it is well known that n vertex d-regular graphs of girth at least g exist for all even n large enough compared to d and g. In particular, by a result of Sauer [A27, A28] (see [A7, Theorem 7]) they exist for even $n \geq 2(d-1)^{g-2}$ for $d, g \geq 3$. Thus the graph G' as above exists for $t \gg s$.

We show that the complement G of G' satisfies the lemma. Let $k = \mathsf{v}(G') - s - t$. As

$$\delta(G) = \mathsf{v}(G) - d - 1 = 2\left(s + \left\lfloor \frac{t-\sqrt{2s}}{2} \right\rfloor\right) - \lceil\sqrt{2s}\rceil - 1 \geq 2s + t - 2\sqrt{2s} - 2,$$

it remains to show G has no $K_{s,t}$ minor.

Suppose for a contradiction that there exists a model μ of $K_{s,t}$ in G. Let A and B denote the parts of the bipartition of $K_{s,t}$ of sizes s and t, respectively. Let A' denote the set of vertices v in A such that $|\mu(v)| = 1$ and let $A'' = \bigcup_{v \in A'} \mu(v)$. Symmetrically, let B' denote the set of vertices v in B such that $|\mu(v)| = 1$ and let $B'' = \bigcup_{v \in B'} \mu(v)$. Then

$$\mathsf{v}(G) \geq \sum_{v \in A \cup B} |\mu(v)| \geq 2(s+t) - |A''| - |B''|.$$

As $k = \mathsf{v}(G) - s - t$ and $|B''| \leq t$, it follows that

$$|A''| + |B''| \geq s + t - k \qquad \text{and} \qquad |A''| \geq s - k. \tag{2.2}$$

Let $F = G'[A'' \cup N_{G'}(A'')]$. As every vertex in A'' is adjacent in G to every vertex of B'' we have $N_{G'}(A'') \cap B'' = \emptyset$. Thus

$$\mathsf{v}(F) \leq \mathsf{v}(G) - |B''|. \tag{2.3}$$

Moreover, $\mathsf{v}(F) \leq (d+1)|A''| \leq (d+1)s$ and so F is a forest (since any cycle in G' contains more than $(d+1)s$ vertices). As $E(F)$ contains all edges of G' incident to vertices of A'', and at most $|A''|-1$ edges have both ends in A'', we have

$$\mathsf{v}(F) - 1 \geq \mathsf{e}(F) \geq d|A''| - (|A''| - 1)\,. \tag{2.4}$$

Combining (2.3) and (2.4) we have $\mathsf{v}(G) \geq (d-1)|A''| + |B''| + 2$, which by (2.2) implies $s + t + k = \mathsf{v}(G) \geq s + t - k + (d-2)(s-k) + 2$ and so $k \geq \frac{(d-2)s+2}{d}$. It follows that

$$\mathsf{v}(G') = s+t+k \geq s+t+\frac{(d-2)s+2}{d} > 2s+t-\frac{2s}{d} \geq 2s+t-\sqrt{2s} > 2\left(s + \left\lfloor \frac{t-\sqrt{2s}}{2} \right\rfloor\right),$$

which contradicts the choice of $\mathsf{v}(G')$. $\square$

Removing the condition that H is bipartite.

Lemma 2.7.2. *For every $s \in \mathbb{N}$, there exists a graph G with $\delta(G) \geq \frac{22}{3}s - 2$ and no sK_7 minor.*

Proof. Let $t = \left\lfloor \frac{11s-1}{3} \right\rfloor$ and let G be a complete 3-partite graph with each part of size t. Then $\delta(G) = 2t \geq \frac{22}{3}s - 2$ and so it only remains to show that G has no sK_7 minor. We show that for any model μ of K_7 in G we have $\sum_{v \in V(K_7)} |\mu(v)| \geq 11$. This would prove the statement, since this would imply that any model of sK_7 in G would require at least $11s > 3t = \mathsf{v}(G)$ vertices.

Suppose that μ is a model of K_7 in G with $\sum_{v \in V(K_7)} |\mu(v)| \leq 10$ and let $A = \{v \in V(K_7) : |\mu(v)| = 1\}$. Then $|A| \geq 4$, and by the pigeonhole principle there exist $v, v' \in A$ such that the unique elements of $\mu(v)$ and $\mu(v')$ belong to the same part of the tripartition of G. This is a contradiction, as $v, v' \in E(K_7)$ but there is no edge in G between the elements of $\mu(v)$ and $\mu(v')$. $\square$

Removing the condition that H has good separation properties.

We first need the following theorem, which is a direct consequence of a result of Norin, Reed, Thomason and Wood [A24, Theorem 4].

Theorem 2.7.3 ([A24])**.** *There exists $d_0 \in \mathbb{N}$ such that for every integer $d \geq d_0$ and every integer $n \geq 2d+1$, there exist a graph H with $\mathsf{v}(H) = n$ and $\mathsf{e}(H) = dn$ and a graph G with $\mathsf{d}(G) \geq \frac{1}{4}n\sqrt{\ln d}$ such that H is not a minor of G.*

We wish to use Theorem 2.7.3 to construct an example showing that Theorem 2.1.4 does not hold if we remove that H is in a family with strongly sublinear separators. We will need the following lemma in order for H to be bipartite and have bounded maximum degree.

Lemma 2.7.4. *For every $\Delta \in \mathbb{N}^{\geq 3}$ and every graph H, there exists a bipartite graph H' with $\Delta(H') \leq \Delta$ and $\mathsf{v}(H') \leq \frac{4\mathsf{e}(H)}{\Delta-2} + 2\mathsf{v}(H)$ such that H is a minor of H'.*

Proof. For each $v \in V(H)$, let $k(v) = \left\lfloor \frac{\deg_H(v)}{\Delta-2} + 1 \right\rfloor$, and let $A(v)$ and $B(v)$ be sets with $|A(v)| = |B(v)| = k(v)$ and all the sets in $\{A(v), B(v)\}_{v \in V(H)}$ are pairwise disjoint.

Let $A = \bigcup_{v \in V(H)} A(v)$, $B = \bigcup_{v \in V(H)} B(v)$. We construct H' so that (A, B) is a bipartition of it; in particular, $V(H') = A \cup B$. Note that

$$\mathsf{v}(H') = 2 \sum_{v \in V(H)} k(v) \leq 2 \sum_{v \in V(H)} \left(\frac{\deg_H(v)}{\Delta - 2} + 1 \right) = \frac{4\mathsf{e}(H)}{\Delta - 2} + 2\mathsf{v}(H),$$

as desired.

We now define the edge set of H'. For each $v \in V(H)$, let $H'[A(v) \cup B(v)]$ be a path (chosen arbitrarily) respecting the bipartition (A, B), which we denote by $P(v)$. For every edge $uv \in E(H)$, add an edge to H' with one end in $A(u)$ and another in $B(v)$. Note that by the choice of $k(v)$ it is possible to add such edges so that $\Delta(H') \leq \Delta$. Contracting each path $P(v)$ to a single vertex we obtain H as a minor of H'. $\square$

Corollary 2.7.5. *There exist $\Delta_0 \in \mathbb{N}$ such that for every integer $\Delta \geq \Delta_0$ and every integer $n \geq 2\Delta$, there exist a bipartite graph H' with $\mathsf{v}(H') \leq 6n$ and $\Delta(H') \leq \Delta$ and a graph G' with $\delta(G') \geq \frac{1}{5}n\sqrt{\ln \Delta}$ such that H' is not a minor of G'.*

Proof. Let d_0 be as in Theorem 2.7.3. We show that $\Delta_0 = \max\{5, d_0 + 2\}$ satisfies the corollary. For $\Delta \geq \Delta_0$ and $n \geq 2\Delta$, let $d = \Delta - 2$. Then $d \geq d_0$ and $n \geq 2d + 1$ so by Theorem 2.7.3 there exists a graph H with $\mathsf{v}(H) = n$ and $\mathsf{e}(H) = dn$ and a graph G with $\mathsf{d}(G) \geq \frac{1}{4}n\sqrt{\ln d}$ such that H is not a minor of G.

Let G' be a subgraph of G with $\delta(G') \geq \mathsf{d}(G)$ (it is a standard result that this is always possible by repeatedly removing vertices of smaller degree, see for instance [A5, Proposition 1.2.2]). Then

$$\delta(G') \geq \frac{1}{4}n\sqrt{\ln(\Delta - 2)} \geq \frac{1}{5}n\sqrt{\ln(\Delta)},$$

as $\sqrt{\ln(\Delta - 2)} \geq \frac{4}{5}\sqrt{\ln(\Delta)}$ for $\Delta \geq 5$. By Lemma 2.7.4 there exists bipartite graph H' with $\Delta(H') \leq \Delta$ and $\mathsf{v}(H') \leq \frac{4\mathsf{e}(H)}{\Delta - 2} + 2\mathsf{v}(H) = 6\mathsf{v}(H)$ such that H is a minor of H'.

As H is a minor of H', G' is a minor of G and H is not a minor of G, it follows that H' is not a minor of G', and so H' and G' are as desired. □

Corollary 2.7.5 provides interesting examples for our purposes when $\Delta \geq e^{900}$.

Replacing the minimum degree of G with average degree.

Lemma 2.7.6. *For all $s, t \in \mathbb{N}$ and $\varepsilon > 0$, there exists a graph G with $\mathsf{d}(G) \geq st - 1 + \frac{t-1}{2} - \varepsilon$ and no $sK_{t,t}$ minor.*

Proof. Let k be a positive integer, and let $G = G(s, t, k)$ be constructed as follows. Let $X_1, \ldots, X_k, Y$ be disjoint sets such that $|X_i| = t$ for $i = [k]$ and $|Y| = st - 1$. Let $V(G) = X_1 \cup \ldots \cup X_k \cup Y$ and let distinct $u, v \in V(G)$ be adjacent unless $u \in X_i$ and $v \in X_j$ for some $i \neq j$. In other words, X_i is a clique of order t for every $i \in [k]$ and Y is a set of $st - 1$

universal vertices. First, we have that $\mathsf{v}(G) = kt + st - 1$. Then

$$\mathsf{e}(G) = \binom{st-1}{2} + (st-1) \cdot kt + k\binom{t}{2}.$$

One easily computes that when $k \to \infty$, we have

$$\frac{\binom{st-1}{2}}{kt+st-1} \to 0, \ \frac{(st-1)\cdot kt}{kt+st-1} \to st-1 \text{ and } \frac{\binom{t}{2}}{kt+st-1} \to \frac{t-1}{2}.$$

Hence, $\mathsf{d}(G) \geq st - 1 + \frac{t-1}{2} - \varepsilon$ for sufficiently large k.

We will show that for every model μ of $K_{t,t}$ in G we have $|Y \cap (\bigcup_{v \in V(K_{t,t})} \mu(v))| \geq t$. Given that $|Y|$ has size $st-1$, this will imply that $sK_{t,t}$ is not a minor of G and thus prove the lemma.

Suppose for a contradiction that μ is a model of $K_{t,t}$ in G and $\left|Y \cap \left(\bigcup_{v \in V(K_{t,t})} \mu(v)\right)\right| \leq t-1$. Let $U = \{v \in V(K_{t,t}) : \mu(v) \cap Y = \emptyset\}$. Then $|U| \geq t+1$ and so $K_{t,t}[U]$ is connected. It follows that $G\left[\bigcup_{v \in U} \mu(v)\right]$ is connected, but $\bigcup_{v \in U} \mu(v) \subseteq V(G) - Y$ and so $\bigcup_{v \in U} \mu(v) \subseteq X_i$ for some $1 \leq i \leq k$. Hence, $t \geq \left|\bigcup_{v \in U} \mu(v)\right| \geq |U| \geq t+1$, a contradiction. □

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2.A Proof of Lemma 2.2.3

A *decomposition* of a graph H is a set $\mathcal{B} \subseteq \mathcal{P}(V(H))$ such that for every edge $uv \in E(H)$, there exists $B \in \mathcal{B}$ such that $u, v \in B$. For $C \in \mathbb{N}$, we say $\mathcal{B}$ is C-*bounded* if $|B| \leq C$ for all $B \in \mathcal{B}$. The *excess* of $\mathcal{B}$ is the quantity $\sum_{B \in \mathcal{B}} |B| - \mathsf{v}(G)$.

We need the following well-known lemma [A6], using the formulation by Hendrey, Norin and Wood [A11].

Lemma 2.A.1 ([A11, Lemma 7.1])**.** *For every graph family $\mathcal{F}$ with strongly sublinear separators and every $\varepsilon > 0$ there exists $C = C_{2.A.1}(\mathcal{F}, \varepsilon)$ such that every graph $H \in \mathcal{F}$ admits a C-bounded decomposition with excess at most $\varepsilon \mathsf{v}(G)$.*

We can then derive Lemma 2.2.3, which we restate for convenience. We recall that this is essentially [A11, Lemma 7.3].

Lemma 2.2.3. *For every graph family $\mathcal{F}$ with strongly sublinear separators and every $\delta > 0$ there exists $s = s_{2.2.3}(\mathcal{F}, \delta)$ such that for any graph $H \in \mathcal{F}$ there exists a graph H' and $F \subseteq E(H')$ such that*

- *$H \preceq H'$,*
- $\mathsf{v}(H) \le \mathsf{v}(H') \le (1+\delta)\mathsf{v}(H)$,
- $\Delta(H') \le \Delta(H) + 2$,
- $|F| \le \delta\mathsf{v}(H)$,
- *for every component J of $H' - F$ we have $\mathsf{v}(J) \le s$ and J is isomorphic to a subgraph of H, and*
- *no edge of F has both ends in the same component of $H' - F$.*

Proof. Take $s = C_{2.A.1}(\mathcal{F}, \delta)$ and let $\mathcal{B}$ be a s-bounded decomposition of H such that $\sum_{B \in \mathcal{B}} |B| \le (1+\delta)\,\mathsf{v}(H)$, which exists by Lemma 2.A.1.

We first prove the statement when H does not contain any isolated vertices.

Say $|\mathcal{B}| = k$. Then, write $H_1, \dots, H_k$ as the subgraphs of H induced respectively by the vertex sets in $\mathcal{B}$. From now on, we consider the vertex sets of $H_1, \dots, H_k$ to all be mutually disjoint. For each $x \in \mathsf{v}(H)$, let $V_x = \{y_1, \dots, y_{i_x}\}$ (for some $i_x \in \mathbb{N}$) be the set of vertices respectively in $H_1, \dots, H_k$ which are copies of x. Then, let $F_x = \{y_1y_2, \dots, y_{i_x-1}y_{i_x}\}$, $F = \bigcup_{x \in V(H)} F_x$ and finally $H' = H_1 \cup \dots \cup H_k + F$.

Contracting the edges in F shows that H is a minor of H', which is possible since the copies of $x \in V(H)$ form a path in H' and all other edges of H are contained in at least one of $H_1, \dots, H_k$ by the definition of a decomposition. Furthermore, $\mathsf{v}(H') = \sum_{i=1}^{k} \mathsf{v}(H_i) = \sum_{B \in \mathcal{B}} |B| \le (1+\delta)\mathsf{v}(H)$. Since every vertex of H' gains at most 2 edges in H' compared to the original vertex in H (the possible extra edges being those in F), $\Delta(H') \le \Delta(H) + 2$. By our construction, when a vertex $x \in V(H)$ has multiplicity i_x in $\mathcal{B}$, $i_x - 1$ edges are added to F. Since the total multiplicity of all vertices is $\mathsf{v}(H')$, then $|F| = \mathsf{v}(H') - \mathsf{v}(H) \le \delta\mathsf{v}(H)$. By construction the components of $H' - F$ are $H_1, \dots, H_k$, which are induced subgraphs of H and have order at most s. Finally, given that edges of F are between copies of a vertex in different $H_1, \dots, H_k$, no edge of F has both ends in the same component of $H' - F$.

To prove the general statement, let H_0 be the graph H with all isolated vertices removed. Let H_0' be the graph resulting from applying the statement to H_0 (which is possible given that $\mathcal{F}$ is closed under taking subgraphs), and let H' be the disjoint union of H_0' and of $\mathsf{v}(H) - \mathsf{v}(H_0)$ isolated vertices. It is easy to verify that H' respects the conditions of the statement. $\square$

Bridging text 1

The previous chapter concerned the H-Hadwiger conjecture of Seymour [93, 94], which is a weakening of Hadwiger's conjecture obtained by replacing the condition that G be K_t-minor-free by the condition that G be H-minor-free, where H is a graph on t vertices. We showed that this conjecture holds if H is bipartite, has bounded maximum degree, is chosen in a graph family with strongly sublinear separators, and is sufficiently large. Although these conditions might be stringent, this remains one of the only classes of graphs for which the conjecture is known to hold: as we mentioned, it is known to hold for sufficiently imbalanced bipartite graphs, trees and some small graphs. Notably, our proof does not use colouring directly, and instead shows that these graphs are $(t-2)$-degenerate.

In the next chapter, we consider the same problem but replace the condition that we only forbid one graph H by forbidding all H with some given number of edges. Of course, this remains a weakening of Hadwiger's conjecture: if we choose the maximum possible $\binom{t}{2}$ edges, we only forbid the complete graph K_t and thus obtain Hadwiger's conjecture.

Similarly to the previous chapter, we will also only use degeneracy to colour the graph in the next chapter. However, we will work with a weaker condition: instead of showing that that the minimum degree in the class is upper bounded by $t-2$, our arguments will only make use of the average degree (or equivalently, of the number of edges), proving that the average degree is strictly less that $t-1$.

Our method is based on a lemma of Mader [64], which shows that we can assume that every neighbourhood has minimum degree at least half of the average degree of the graph. We then, broadly speaking, randomly sample subgraphs on t vertices in these denser neighbourhoods (or unions of these neighbourhoods), as an attempt to construct minors with many edges. With a lower bound on the average degree, this method will be successful in constructing a minor with t vertices and $\left(\sqrt{2}-1-o(1)\right)\binom{t}{2}$ edges.

In the previous chapter, we showed that our result was best possible in many ways. Here, in a similar attempt to show the limits of our approach, we will present examples showing that we cannot do better than $\left(\frac{3}{4}+o(1)\right)\binom{t}{2}$ edges, if we are only working with the average degree.

Using specific arguments using extremal functions, we will show the exact maximum number of edges in t-vertex minors we can be sure to obtain when forbidding a minor with t vertices and maximum degree is less than $t-1$ for small t.

3

Finding dense minors using average degree

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Motivated by Hadwiger's conjecture, we study the problem of finding the densest possible t-vertex minor in graphs of average degree at least $t-1$. We show that if G has average degree at least $t-1$, it contains a minor on t vertices with at least $(\sqrt{2}-1-o(1))\binom{t}{2}$ edges.

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We show that this cannot be improved beyond $\left(\frac{3}{4}+o(1)\right)\binom{t}{2}$. Finally, for $t \leq 6$ we exactly determine the number of edges we are guaranteed to find in the densest t-vertex minor in graphs of average degree at least $t-1$.

3.1 Introduction

In this paper all graphs are simple and finite. We say a graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges (and removing any loops and parallel edges). A *k-colouring* of a graph G is an assignment of colours from $\{1,\dots,k\}$ to the vertices of G such that adjacent vertices are assigned distinct colours. The *chromatic number* $\chi(G)$ is the smallest integer k such that G admits a k-colouring.

Hadwiger [B3] conjectured that if $\chi(G) \geq t$, then G contains K_t, the complete graph on t vertices, as a minor. Hadwiger's conjecture is one of the most famous open problems in graph theory. The study of Hadwiger's conjecture has spawned a large number of variants, strengthenings and relaxations; see [B13] for a survey of this area.

For instance, there has been much progress in recent years in attempting to find the smallest function $f(t)$ for which $\chi(G) \geq f(t)$ implies that G contains a K_t minor. The current best bound $f(t) = \Omega(t \log\log t)$ is due to Delcourt and Postle [B1].

Another strategy is to approach Hadwiger's conjecture by relaxing the condition that the minor be complete. Seymour [B13, B14] asks for which graphs H on t vertices does $\chi(G) \geq t$ guarantee H as a minor. Kostochka [B4, B5] as well as the second and fourth authors of this paper [B10] have proved that this holds for large classes of bipartite graphs.

The second author and Seymour [B9] study the related question of maximizing the edge density of t-vertex minors of G if $\chi(G) \geq t$, that is finding a minor of G on t vertices with as many edges as possible. Unlike in the previous relaxation we allow the minor to change depending on G. It is shown in [B9] that if G has n vertices and independence number 2 (and so $\chi(G) \geq \lceil \frac{n}{2} \rceil$), then G contains a minor on $\lceil \frac{n}{2} \rceil$ vertices and at least $(0.98688 - o(1))\binom{\lceil \frac{n}{2} \rceil}{2}$

edges.

Letting both the number of vertices and edges of the minor vary, Nguyen [B8] showed that there exists $C > 0$ such that if $\varepsilon \in (0, \frac{1}{256})$ and $\chi(G) \geq Ct \log\log\left(\frac{1}{\varepsilon}\right)$, then G contains a minor on t vertices with at least $(1-\varepsilon)\binom{t}{2}$ edges. Setting $\varepsilon = \frac{1}{t^2}$ recovers the result of Delcourt and Postle mentioned above.

In this paper, we study the following strengthening of the question considered by the second author and Seymour: What is the densest t-vertex minor of G if the average degree is at least $t-1$? This is motivated by the well-known fact that $\chi(G) \geq t$ implies that G contains a subgraph G' with minimum degree at least $t-1$.

It is a direct consequence of a result of Mader [B6, 65] (see Lemma 3.2.1 below) that every graph of average degree at least $t-1$ contains a minor on t vertices with at least $\frac{1}{4}\binom{t}{2}$ edges.

Our main result is the following improvement, which we prove in Section 3.2. Let $\overline{\mathsf{d}}(G)$ denote the average degree of a graph G.

Theorem 3.1.1. *If $t \in \mathbb{N}$ and G is a graph with average degree $\overline{\mathsf{d}}(G) \geq t$, then G contains a minor on t vertices with at least $\left(\sqrt{2}-1-\frac{24}{t}\right)\binom{t}{2}$ edges.*

Note that in Theorem 3.1.1 we assume for convenience that $\overline{\mathsf{d}}(G) \geq t$ and not $\overline{\mathsf{d}}(G) \geq t-1$ as in the problem above. However, this only affects the lower order terms.

In Section 3.3, we show that the constant in the previous result cannot be improved past $\frac{3}{4}$.

Theorem 3.1.2. *For $t \in \mathbb{N}$, there exists a graph G with average degree $\overline{\mathsf{d}}(G) \geq t$ such that G does not contain a minor on t vertices with more than $\left(\frac{3}{4}+o(1)\right)\binom{t}{2}$ edges.*

In fact, we will describe a large class of graphs which satisfy the condition of Theorem 3.1.2.

Finally, in Theorem 3.1.3, proved in Section 3.4 we determine for small values of t the exact number of edges we are guaranteed to find in the densest t-vertex minor.

Theorem 3.1.3. *If $2 \leq t \leq 6$ is an integer and G is a graph with average degree $\overline{\mathsf{d}}(G) \geq t-1$, then G contains a minor on t vertices with at least*

- 1 *edge if $t = 2$,*
- 3 *edges if $t = 3$,*
- 5 *edges if $t = 4$,*
- 8 *edges if $t = 5$, and*
- 11 *edges it $t = 6$.*

Furthermore, none of these values can be improved.

3.1.1 Notation

Let G be a graph. We denote by $V(G)$ the set of vertices of G and $E(G) \subseteq \binom{V(G)}{2}$ the set of edges of G. We will write $\mathsf{v}(G) = |V(G)|$ for the number of vertices of G and $\mathsf{e}(G) = |E(G)|$ for the number of edges of G. If $u \in V(G)$, we write $N_G(u)$ for the (open) neighbourhood of u, $N_G[u] = N_G(u) \cup \{u\}$ for the closed neighbourhood of u, and $\deg_G(u) = |N_G(u)|$ for the degree of u in G. For $S \subseteq V(G)$ we denote by $N_G[S]$ the set $\bigcup\{N_G[s] : s \in S\}$. We omit subscripts when the graph is clear from context. We denote the minimum degree of G by $\delta(G) = \min_{u \in V(G)} \deg_G(u)$ and the average degree of G by $\overline{\mathsf{d}}(G) = \frac{\sum_{u \in V(G)} \deg_G(u)}{\mathsf{v}(G)}$. If $X \subseteq V(G)$, we write $G[X]$ for the subgraph of G induced by X, and $G - X = G[V(G) \setminus X]$ for the subgraph obtained by removing the vertices in X; if $X = \{u\}$ we will write $G - u$ for $G - X$. If $e \in E(G)$, we write $G - e$ for the graph obtained by removing e, and G/e for the graph obtained from G by contracting the edge e (and removing any resulting loops or duplicate edges); in particular G/e is a minor of G. If Z is a real-valued random variable, we write $\mathbb{E}[Z]$ for the expected value of Z.

3.2 Lower bound

In this section, we prove Theorem 3.1.1.

The following result of Mader [64, pages 265–266] will allow us to get a lower bound on the minimum degree in the neighbourhood of each vertex, by taking a minor of our graph. We include a short proof for the sake of completeness, since the paper [B6] is only available in German.

Lemma 3.2.1. *If G is a graph, then G contains a minor H such that $\overline{\mathsf{d}}(H) \geq \overline{\mathsf{d}}(G)$ and $\delta\left(H[N[u]]\right) > \frac{\overline{\mathsf{d}}(G)}{2}$ for every $u \in V(H)$.*

Proof. Let H be a minor of G such that $\overline{\mathsf{d}}(H) \geq \overline{\mathsf{d}}(G)$ which minimizes $\mathsf{v}(H)$. Suppose for a contradiction that H does not respect the statement, i.e. there is some $u \in V(H)$ such that $\delta\left(H[N[u]]\right) \leq \frac{\overline{\mathsf{d}}(G)}{2}$.

If $\deg_H(u) = 0$, then it is direct that $\overline{\mathsf{d}}(H-u) \geq \overline{\mathsf{d}}(H)$, which contradicts the minimality of H. Hence, we may suppose that u has at least one neighbour.

Given that the degree of u in $H[N[u]]$ is as least as large as the degree in $H[N[u]]$ of every other vertex in $N[u]$, there exists $v \in N(u)$ such that $\deg_{H[N[u]]}(v) \leq \frac{\overline{\mathsf{d}}(G)}{2} \leq \frac{\overline{\mathsf{d}}(H)}{2}$. In other words, $|N[u] \cap N(v)| \leq \frac{\overline{\mathsf{d}}(H)}{2}$. Then,

$$\overline{\mathsf{d}}(H/uv) = \frac{2\mathsf{e}(H/uv)}{\mathsf{v}(H/uv)} = \frac{2(\mathsf{e}(H) - |N[u] \cap N(v)|)}{\mathsf{v}(H) - 1} \geq \frac{2\mathsf{e}(H) - \overline{\mathsf{d}}(H)}{\mathsf{v}(H) - 1} = \frac{\mathsf{v}(H) \cdot \overline{\mathsf{d}}(H) - \overline{\mathsf{d}}(H)}{\mathsf{v}(H) - 1} = \overline{\mathsf{d}}(H)$$

which is a contradiction to the minimality of H. $\square$

The next easy lemma tells us when removing vertices does not decrease average degree.

Lemma 3.2.2. *If G is a graph and $X \subsetneq V(G)$ is such that exactly M edges of G have at least one end in X and $M \leq \frac{\overline{\mathsf{d}}(G) \cdot |X|}{2}$, then $\overline{\mathsf{d}}(G - X) \geq \overline{\mathsf{d}}(G)$.*

Proof. We may compute directly that

$$\overline{\mathsf{d}}(G - X) = \frac{2\mathsf{e}(G - X)}{\mathsf{v}(G - X)} = \frac{2\left(\mathsf{e}(G) - M\right)}{\mathsf{v}(G) - |X|} \geq \frac{2\mathsf{e}(G) - \overline{\mathsf{d}}(G) \cdot |X|}{\mathsf{v}(G) - |X|} = \frac{\mathsf{v}(G) \cdot \overline{\mathsf{d}}(G) - \overline{\mathsf{d}}(G) \cdot |X|}{\mathsf{v}(G) - |X|} = \overline{\mathsf{d}}(G).$$

$\square$

The next lemma allows us to extract a dense subgraph on t vertices; this is a standard application of the first moment method.

Lemma 3.2.3. *If $t \in \mathbb{N}$ and G is a graph with $\mathsf{v}(G) \geq t$, then G contains a subgraph on t vertices with at least $\frac{\bar{\mathsf{d}}(G)}{\mathsf{v}(G)}\binom{t}{2}$ edges.*

Proof. The statement is trivial when $t = 1$, so we may assume that $t \geq 2$. Let Z be a uniformly random subset of $V(G)$ of size t. Given $uv \in E(G)$, the probability that uv is an edge of $G[Z]$ is $\frac{\binom{\mathsf{v}(G)-2}{t-2}}{\binom{\mathsf{v}(G)}{t}} = \frac{t(t-1)}{\mathsf{v}(G)(\mathsf{v}(G)-1)}$. As $\mathsf{e}(G) = \frac{\mathsf{v}(G)\cdot\bar{\mathsf{d}}(G)}{2}$, we have

$$\mathbb{E}[\mathsf{e}(G[Z])] = \frac{\mathsf{v}(G)\cdot\bar{\mathsf{d}}(G)}{2} \cdot \frac{t(t-1)}{\mathsf{v}(G)(\mathsf{v}(G)-1)} = \frac{\bar{\mathsf{d}}(G)}{\mathsf{v}(G)-1}\binom{t}{2} \geq \frac{\bar{\mathsf{d}}(G)}{\mathsf{v}(G)}\binom{t}{2}.$$

Hence, there exists at least one choice of Z such that $\mathsf{e}(G[Z]) \geq \frac{\bar{\mathsf{d}}(G)}{\mathsf{v}(G)}\binom{t}{2}$. Hence the statement holds for $G[Z]$. $\square$

In the next lemma, we find a dense subgraph on t vertices by extending an already dense, but not large enough, set of vertices X to a set of size t by sampling the remaining vertices in another set Y. We will apply this lemma when X is a union of closed neighbourhoods and Y a closed neighbourhood (or vice versa), the conditions on the minimum degrees of the induced subgraphs on these sets will come from Lemma 3.2.1.

Lemma 3.2.4. *If $t \in \mathbb{N}$, G is a graph and $X, Y \subseteq V(G)$ are such that $|X| \leq t$, $|X \cup Y| \geq t$ and $\delta(G[X]), \delta(G[Y]) \geq \frac{t}{2}$, then there is a subgraph of G on t vertices with at least $\left(\frac{1}{2}\left(x + \frac{(1-x)^2}{y}\right) - \frac{1}{t}\right)\binom{t}{2}$ edges, where $x = \frac{|X|}{t}$ and $y = \frac{|Y|}{t}$.*

Proof. If $|X| = t$, the statement follows directly by considering the t-vertex subgraph $G[X]$, which contains at least $\frac{1}{2}\cdot|X|\cdot\delta(G[X]) \geq \frac{1}{2}\cdot t\cdot\frac{t}{2} \geq \frac{1}{2}\binom{t}{2}$ edges. Hence, we may suppose that $|X| \leq t-1$, and as a consequence that $|Y \setminus X| \geq 1$.

Let Y' be a uniformly random subset of $Y \setminus X$ of size $t - |X|$ and set $Z = X \cup Y'$, which is possible given that $|X \cup Y| \geq t$ implies $t - |X| \leq |Y \setminus X|$.

The number of edges with both ends in X is at least $\frac{1}{2} \cdot |X| \cdot \delta(G[X]) \geq \frac{t|X|}{4}$. The number of edges with both ends in $Y \setminus X$ is $\frac{1}{2}\sum_{v \in Y \setminus X} |N(v) \cap (Y \setminus X)|$ and the number of edges between X and $Y \setminus X$ is $\sum_{v \in Y \setminus X} |N(v) \cap X|$, so the number of edges with both ends in $X \cup Y$ and at least one end in $Y \setminus X$ is

$$\begin{aligned}\frac{1}{2}\sum_{v \in Y \setminus X} |N(v) \cap (Y \setminus X)| + \sum_{v \in Y \setminus X} |N(v) \cap X| &\geq \frac{1}{2}\sum_{v \in Y \setminus X} |N(v) \cap (X \cup Y)| \\ &\geq \frac{1}{2}|Y \setminus X| \cdot \delta(G[X \cup Y]) \geq \frac{t|Y \setminus X|}{4}.\end{aligned}$$

Suppose uv is an edge of $G[X \cup Y]$ with at least one end in $Y \setminus X$, say $u \in Y \setminus X$. Then, uv is an edge in $G[Z]$ if and only if $\{u, v\} \subseteq Z$. If $v \in Y \setminus X$, then the probability that uv is an edge is the probability that $\{u, v\} \subseteq Y'$, which is $\frac{\binom{|Y \setminus X|-2}{t-|X|-2}}{\binom{|Y \setminus X|}{t-|X|}} = \frac{(t-|X|)(t-|X|-1)}{|Y \setminus X|(|Y \setminus X|-1)} \geq \frac{(t-|X|)(t-|X|-1)}{|Y \setminus X|^2}$. If $v \in X$, then the probability that uv is an edge is the probability that $u \in Y'$, which is $\frac{t-|X|}{|Y \setminus X|} \geq \frac{(t-|X|)(t-|X|-1)}{|Y \setminus X|^2}$. Here we use the fact that $t - |X| \leq |Y \setminus X|$. Hence, we have

$$\begin{aligned}\mathbb{E}[\mathsf{e}(G[Z])] &\geq \frac{t|X|}{4} + \frac{t|Y \setminus X|}{4} \cdot \frac{(t-|X|)(t-|X|-1)}{|Y \setminus X|^2} \\ &\geq \frac{t|X|}{4} + \frac{t}{4} \cdot \frac{(t-|X|)(t-|X|-1)}{|Y|} \\ &\geq \frac{(t-1) \cdot tx}{4} + \frac{t-1}{4} \cdot \frac{(t-tx)(t-tx-1)}{ty} \\ &= \frac{1}{2}\left(x + \frac{(1-x)(1-x-\frac{1}{t})}{y}\right)\binom{t}{2} \\ &\geq \left(\frac{1}{2}\left(x + \frac{(1-x)^2}{y}\right) - \frac{1}{t}\right)\binom{t}{2},\end{aligned}$$

where in the last step we used that $1 - x \leq y$. Hence, there is at least one choice of Y' such that $G[Z] = G[X \cup Y']$ has t vertices and at least $\left(\frac{1}{2}\left(x + \frac{(1-x)^2}{y}\right) - \frac{1}{t}\right)\binom{t}{2}$ edges, as desired. □

The next lemma finds a dense subgraph on t vertices given a dense set X if the vertices outside of X all have sufficiently large degree. Contrary to the previous lemma, here the set

X is not extended to a set of size t, instead its properties will allow us to show that G is itself not too large, and so a good candidate to apply Lemma 3.2.3.

Lemma 3.2.5. *Let $t, c \in \mathbb{N}$, $\lambda > 1 + \frac{3}{t}$ and let G be a graph with $\overline{\mathsf{d}}(G) \geq t$ and such that $\overline{\mathsf{d}}(H) < t$ for all non-null proper subgraphs H of G. If $\emptyset \neq X \subsetneq V(G)$ is such that $\delta(G[X]) \geq \frac{t}{2}$ and $\deg_G(u) > \lambda t - 1$ for all $u \in V(G) \setminus X$, then G contains a subgraph on t vertices with at least $\frac{(\lambda - 1 - \frac{3}{t})t}{(\lambda - \frac{3}{4})|X|}\binom{t}{2}$ edges.*

Proof. First note that $\overline{\mathsf{d}}(G - X) < t \leq \overline{\mathsf{d}}(G)$. Let M be the number of edges with at least one end in X. Lemma 3.2.2 implies that $M > \frac{\overline{\mathsf{d}}(G) \cdot |X|}{2} \geq \frac{t|X|}{2}$. In particular, we then have that the sum of degrees of vertices in X is

$$\begin{aligned}
\sum_{v \in X} \deg_G(v) &= \sum_{v \in X} |N(v) \cap X| + \sum_{v \in X} |N(v) \setminus X| \\
&= \left(\frac{1}{2} \sum_{v \in X} |N(v) \cap X| + \sum_{v \in X} |N(v) \setminus X| \right) + \frac{1}{2} \sum_{v \in X} |N(v) \cap X| \\
&= M + \frac{1}{2} \sum_{v \in X} |N(v) \cap X| \\
&> \frac{t|X|}{2} + \frac{|X|\delta(G[X])}{2} \\
&\geq \frac{3t|X|}{4}.
\end{aligned}$$

Furthermore, we have

$$\sum_{v \in V(G) \setminus X} \deg_G(v) \geq (\mathsf{v}(G) - |X|)(\lambda t - 1).$$

Let u be any vertex of G. As $\overline{\mathsf{d}}(G - u) < t$,

$$\overline{\mathsf{d}}(G) = \frac{2\mathsf{e}(G)}{\mathsf{v}(G)} = \frac{2(\mathsf{e}(G-u) + \deg_G(u))}{\mathsf{v}(G)} = \frac{\mathsf{v}(G-u)\overline{\mathsf{d}}(G-u) + 2\deg_G(u)}{\mathsf{v}(G)} < \overline{\mathsf{d}}(G-u) + 2 \leq t + 2.$$

Hence,

$$\mathsf{v}(G)(t+2) > \mathsf{v}(G)\cdot\overline{\mathsf{d}}(G) = \sum_{v\in V(G)} \deg_G(v) \geq \frac{3t|X|}{4} + (\mathsf{v}(G)-|X|)(\lambda t - 1).$$

Rearranging yields that

$$\mathsf{v}(G) < \frac{(\lambda - \frac{3}{4} - \frac{1}{t})|X|}{\lambda - 1 - \frac{3}{t}} < \frac{(\lambda - \frac{3}{4})|X|}{\lambda - 1 - \frac{3}{t}}.$$

Since $\overline{\mathsf{d}}(G) \geq t$, we in particular have that $\mathsf{v}(G) \geq t$. Finally, to get the desired subgraph we apply Lemma 3.2.3 to G to get a subgraph on t vertices with at least

$$\frac{\overline{\mathsf{d}}(G)}{\mathsf{v}(G)}\binom{t}{2} \geq \frac{(\lambda - 1 - \frac{3}{t})t}{(\lambda - \frac{3}{4})|X|}\binom{t}{2}$$

edges. □

The next lemma is a core element of our proof, which we summarize here. We want to find a set X of vertices such that $G[X]$ has minimum degree at least $\frac{t}{2}$ which has order as close as possible to t. We will be taking X to be a union of closed neighbourhoods in our graph (as, by Lemma 3.2.1, we will be able to assume that closed neighbourhoods have this minimum degree); we can construct this set by sequentially adding neighbourhoods of vertices. The closer $|X|$ is to t, the denser a subgraph of G of order t we will be able to find; when $|X| > t$ we can use Lemma 3.2.3 to sample a dense subset of size exactly t and when $|X| < t$ we can use either Lemma 3.2.4 or Lemma 3.2.5 (depending on the degrees of vertices outside of X). In practice, we will introduce some tolerance and attempt to find X such that $\alpha t \leq |X| \leq \beta t$ (this is Case (1) in the following lemma). The first way of failing is when constructing X, at some point the size is smaller than αt, but the size jumps over βt when adding any other neighbourhood of size smaller than αt; this is Case (3), to which we will apply Lemma 3.2.4. The other way of failing is to run out of vertices of small degree, in which case we are in Case (2). In this case, the fact that all remaining vertices have large degree will allow us to apply Lemma 3.2.5. The parameters α, β will later be chosen to optimize the trade-offs between these cases.

Lemma 3.2.6. *If $t \in \mathbb{N}$, G is a graph such that $\delta(G[N[u]]) \geq \frac{t}{2}$ for every $u \in V(G)$, and $\frac{1}{2} \leq \alpha < \beta \in \mathbb{R}$, then either*

(1) there exists $X \subseteq V(G)$ such that $\alpha t \leq |X| \leq \beta t$ and $\delta(G[X]) \geq \frac{t}{2}$,

(2) there exists $S \subseteq V(G)$ (possibly empty) such that $|N[S]| < \alpha t$ and $\deg_G(u) > \beta t - 1$ for every $u \in V(G) \setminus N[S]$, or

(3) there exist $X, Y \subseteq V(G)$ such that $|X|, |Y| < \alpha t$, $|X \cup Y| > \beta t$ and $\delta(G[X]), \delta(G[Y]) \geq \frac{t}{2}$.

Proof. If there exists $u \in V(G)$ such that $\alpha t - 1 \leq \deg_G(u) \leq \beta t - 1$, then setting $X = N[u]$ we are in Case (1). Hence, we may assume that for every $u \in V(G)$, $\deg_G(u) < \alpha t - 1$ or $\deg_G(u) > \beta t - 1$.

Let $S = \{u \in V(G) : \deg(u) < \alpha t - 1\}$. If S is such that $|N[S]| < \alpha t$, then we are in Case (2), since $S \subseteq N[S]$ and every vertex not in S has degree at least $\alpha t - 1$, and hence greater than $\beta t - 1$ by the assumption above.

Otherwise, we can find $B = \bigcup_{i=1}^{k} N[x_i]$, for $x_1, \ldots, x_k \in V(G)$ all of degree smaller than $\alpha t - 1$, such that $|B| \geq \alpha t$. Pick such a set which minimizes k. Again, note that $\delta(G[B]) \geq \frac{t}{2}$. If $|B| \leq \beta t$, we are in Case (1) with $X = B$. Hence suppose that $|B| > \beta t$.

Note that necessarily $k \geq 2$, as if $B = N[x_1]$, we have $|B| = \deg_G(x_1) + 1 < \alpha t$. By minimality of k, $|\bigcup_{i=1}^{k-1} N[x_i]| < \alpha t$. Hence, Case (3) holds for $X = \bigcup_{i=1}^{k-1} N[x_i]$ and $Y = N[x_k]$. $\square$

Finally, we are ready to derive Theorem 3.1.1, which we restate for convenience, from the above lemmas.

Theorem 3.1.1. *If $t \in \mathbb{N}$ and G is a graph with average degree $\overline{\mathsf{d}}(G) \geq t$, then G contains a minor on t vertices with at least $\left(\sqrt{2} - 1 - \frac{24}{t}\right)\binom{t}{2}$ edges.*

Proof. The statement is trivial for $t \le 24$, so assume $t \ge 25$. We may suppose that G has no proper minor H such that $\overline{\mathsf{d}}(H) \ge t$. By Lemma 3.2.1, G has a minor G' such that $\overline{\mathsf{d}}(G') \ge \overline{\mathsf{d}}(G) \ge t$ and $\delta(G'[N[u]]) \ge \frac{t}{2}$ for every $u \in V(G)$. By assumption, G' is not a proper minor of G, so $G' = G$.

Let $\alpha = \frac{4}{5}$, $\beta = \nu = \frac{6}{5}$ and let $\gamma = \sqrt{2} - 1 - \frac{24}{t}$. We wish to prove that G contains a subgraph on t vertices with at least $\gamma\binom{t}{2}$ edges. As $1/2 < \alpha < \beta$ we can apply Lemma 3.2.6 to G and consider three cases depending on the outcome of Lemma 3.2.6 which holds.

First suppose we are in Case (1), i.e. there exists $X \subseteq V(G)$ such that $\alpha t \le |X| \le \beta t$ and $\delta(G[X]) \ge \frac{t}{2}$. There are two subcases here. The first is $t \le |X| \le \beta t$. Lemma 3.2.3 applied to $G[X]$ then ensures that G contains a subgraph on t vertices with at least

$$\frac{\overline{\mathsf{d}}(G[X])}{\mathsf{v}(G[X])}\binom{t}{2} \ge \frac{\frac{t}{2}}{\beta t}\binom{t}{2} = \frac{1}{2\beta}\binom{t}{2} = \frac{5}{12}\binom{t}{2} > \gamma\binom{t}{2}$$

edges.

The other subcase is $\alpha t \le |X| \le t$. In the following, we assume without loss of generality that X is chosen of maximum size subject to satisfying the conditions of Case (1) in Lemma 3.2.6 as well as $|X| \le t$. In particular, this implies that for every vertex $u \in V(G)\backslash X$, we have $|X \cup N[u]| > t$, for otherwise we could replace X with $X \cup N[u]$, contradicting the maximality assumption.

Let us consider two possibilities. The first is that there exists $u \in V(G) \setminus X$ such that $\deg_G(u) \le \nu t - 1$. Let $Y = N[u]$ and write $|X| = xt$ and $|Y| = yt$. By Lemma 3.2.4 there exists a subgraph of G on t vertices with at least

$$\left(\frac{1}{2}\left(x + \frac{(1-x)^2}{y}\right) - \frac{1}{t}\right)\binom{t}{2} \ge \left(\frac{1}{2}\left(x + \frac{(1-x)^2}{\nu}\right) - \frac{1}{t}\right)\binom{t}{2}$$

edges. Hence, the theorem holds in this case as

$$\gamma \le \min_{\alpha \le x \le 1} \frac{1}{2}\left(x + \frac{(1-x)^2}{\nu}\right) - \frac{1}{t} = \frac{1}{2}\left(\alpha + \frac{(1-\alpha)^2}{\nu}\right) - \frac{1}{t} = \frac{5}{12} - \frac{1}{t}.$$

Otherwise, $\deg_G(u) > \nu t - 1$ for every $u \in V(G) \setminus X$. By Lemma 3.2.5 G contains a subgraph on t vertices with at least

$$\frac{(\nu - 1 - \frac{3}{t})t}{(\nu - \frac{3}{4})|X|}\binom{t}{2} \geq \frac{(\nu - 1 - \frac{3}{t})t}{(\nu - \frac{3}{4})t}\binom{t}{2} > \left(\frac{\nu - 1}{\nu - \frac{3}{4}} - \frac{12}{t}\right)\binom{t}{2} = \left(\frac{4}{9} - \frac{12}{t}\right)\binom{t}{2} > \gamma\binom{t}{2}$$

edges, where the second inequality holds as $\nu > 1$. This finishes the proof in the case that outcome (1) of Lemma 3.2.6 holds.

Now suppose that outcome (2) of Lemma 3.2.6 holds, i.e. there exists $S \subseteq V(G)$ such that $|N[S]| < \alpha t$ and $\deg_G(u) > \beta t - 1$ for every $u \in V(G) \setminus N[S]$. Note that $S \neq \emptyset$, since otherwise deleting an arbitrary edge of G results in a proper minor H satisfying $\overline{\mathsf{d}}(H) \geq \delta(H) \geq \beta t - 2 \geq t + \frac{t}{5} - 2 \geq t + 3$, a contradiction. Additionally $\delta(G[N[S]]) \geq \frac{t}{2}$ since $\delta(G[N[s]]) \geq \frac{t}{2}$ for each $s \in S$. By Lemma 3.2.5 with $X = N[S]$, G contains a subgraph on t vertices with at least

$$\frac{(\beta - 1 - \frac{3}{t})t}{(\beta - \frac{3}{4})|X|}\binom{t}{2} > \frac{(\beta - 1 - \frac{3}{t})t}{(\beta - \frac{3}{4})t}\binom{t}{2} > \left(\frac{\beta - 1}{(\beta - \frac{3}{4})} - \frac{24}{t}\right)\binom{t}{2} = \left(\frac{4}{9} - \frac{24}{t}\right)\binom{t}{2} > \gamma\binom{t}{2}$$

edges, where the second inequality uses $\beta > 1$.

Finally, suppose that outcome (3) of Lemma 3.2.6 holds, i.e. there exist $X, Y \subseteq V(G)$ such that $|X|, |Y| < \alpha t$, $|X \cup Y| > \beta t$ and $\delta(G[X]), \delta(G[Y]) \geq \frac{t}{2}$. Without loss of generality suppose $|X| \geq |Y|$ and let $|X| = xt$ and $|Y| = yt$. By Lemma 3.2.4 there exists a subgraph of G on t vertices with at least

$$\left(\frac{1}{2}\left(x + \frac{(1-x)^2}{y}\right) - \frac{1}{t}\right)\binom{t}{2} \geq \left(\frac{1}{2}\left(x + \frac{(1-x)^2}{x}\right) - \frac{1}{t}\right)\binom{t}{2}$$

edges. Hence, it suffices to show that

$$\frac{1}{2}\left(x + \frac{(1-x)^2}{x}\right) \geq \sqrt{2} - 1.$$

This last inequality simplifies to $\left(\sqrt{2}x - 1\right)^2 \geq 0$ for $x > 0$ and hence holds for all such x, as desired. $\square$

3.3 Upper bound

In this section, we prove Theorem 3.1.2. We first need the following definitions.

For $k \in \mathbb{N}$, *k-trees* are the graph family defined in a recursive manner as follows:

- The complete graph K_{k+1} is a k-tree.
- If G is a k-tree and $C \subseteq V(G)$ is a clique in G with $|C| = k$, then the graph obtained from G by adding a new vertex with neighbourhood C is also a k-tree.

It follows readily from this definition that for any k-tree G, $\mathsf{e}(G) = \binom{k}{2} + k(\mathsf{v}(G) - k)$. Furthermore, every minor of a k-tree with at least $k+1$ vertices is also a spanning subgraph of a k-tree. Indeed, the treewidth $\mathsf{tw}(G)$ of a graph G with at least $k+1$ vertices is at most k if and only if G is a spanning subgraph of a k-tree [B12, B16], and it is well known that treewidth is minor-monotone (that is, $\mathsf{tw}(G') \leq \mathsf{tw}(G)$ for every minor G' of a graph G). Sufficiently large k-trees, with appropriately chosen parameter k, will be our first candidates for Theorem 3.1.2.

Let $S_r = K_{1,r}$ be the star graph with r leaves. We define the graph $S_{k,r,s}$ as the graph obtained from S_r by replacing every leaf by cliques $A_1, \ldots, A_r$ on s vertices and replacing the central vertex by a clique C on k vertices. In particular, every vertex of C is adjacent to every other vertex in the graph. Such vertices are said to be *universal*.

These graphs, with appropriately chosen parameters, will also be candidates for Theorem 3.1.2. First note that

$$\overline{\mathsf{d}}(S_{k,r,s}) = \frac{2\left(\binom{k}{2} + r\binom{s}{2} + krs\right)}{k + rs}.$$

Given graphs G and H, we say the collection $(B_u)_{u \in V(H)}$ of pairwise disjoint non-empty subsets of $V(G)$ is a *model* of H in G if $G[B_u]$ is connected for every $u \in V(H)$ and G

contains at least one edge between B_u and B_v if $uv \in E(H)$. It is easy to see that there exists a model of H in G if and only if H is a minor of G. It is also direct that if $|B_u| = 1$ for every $u \in V(H)$, then G contains a subgraph isomorphic to H (precisely, on vertex set $\bigcup_{u\in V(H)} B_u$).

We now show that with these graphs, we may restrict ourselves to finding a dense subgraph on t vertices, which is simpler than finding minors.

Lemma 3.3.1. *If $k, r, s \in \mathbb{N}$ and H is a minor of $S_{k,r,s}$, then $S_{k,r,s}$ has a subgraph isomorphic to H.*

Proof. Let $\mathcal{B} = (B_u)_{u\in V(H)}$ be a model of H in G which minimizes $\sum_{u\in V(H)} |B_u|$. If $|B_u| = 1$ for every $u \in V(H)$, then we are done by the above remark. Hence, assume $|B_v| \geq 2$ for some $v \in V(H)$.

If $B_v \subseteq A_i$ for some $1 \leq i \leq r$, let $x \in B_v$. If $B_v \cap C \neq \emptyset$, let $x \in C \cap B_v$. Given the structure of $S_{k,r,s}$ and that the subgraph induced by B_v is connected, these are the only two possible cases. Let $B'_v = \{x\}$ and $B'_u = B_u$ for $u \in V(H) \setminus \{v\}$.

It is easy to verify that in both of these cases, if $w \in V(G) \setminus B_v$ is adjacent to at least one vertex of B_v, then w is adjacent to x. Hence, $(B'_u)_{u\in V(H)}$ is a model of H in G which contradicts the minimality of $\mathcal{B}$. □

We now compute an upper bound on the density of t-vertex subgraphs of $S_{k,r,s}$.

Lemma 3.3.2. *If $k, r, s, t \in \mathbb{N}$ are such that $k + rs \geq t \geq k$, then $S_{k,r,s}$ does not contain a subgraph on t vertices with more than*

$$f(k, s, t) = \binom{k}{2} + k(t - k) + \left\lfloor \frac{t-k}{s} \right\rfloor \binom{s}{2} + \binom{t - k - \left\lfloor \frac{t-k}{s} \right\rfloor s}{2} \tag{3.1}$$

edges.

Proof. Let $X \subseteq V(S_{k,r,s})$ such that $|X| = t$ which maximizes the number of edges in $G[X]$ with first priority and then, subject to $\mathsf{e}(G[X])$ being maximum, maximizes $|X \cap C|$ with second priority. This is possible given that $\mathsf{v}(S_{k,r,s}) = k + rs \geq t$. Our goal is thus to upper bound $\mathsf{e}(G[X])$.

We first claim that $C \subseteq X$. Suppose to contrary that there exists $c \in C \setminus X$. Given that $|X| = t \geq k = |C| > |C \setminus \{c\}|$, this implies there exists $x \in X \setminus C$. Let $X_0 = X \setminus \{x\}$ and $X' = X_0 \cup \{c\}$. Then,

$$\mathsf{e}(G[X]) = \mathsf{e}\left(G\left[X_0\right]\right) + |N(x) \cap X_0| \leq \mathsf{e}\left(G\left[X_0\right]\right) + |N(c) \cap X_0| = \mathsf{e}(G[X']),$$

where the inequality follows from the fact that c is a universal vertex of G. Since $|X' \cap C| > |X \cap C|$, this contradicts our choice of X. Hence, we have $C \subseteq X$.

The number of edges in $G[C]$ is $\binom{k}{2}$. Given that $|X \setminus C| = t - k$ and every vertex in $X \setminus C$ is connected to all k vertices in C, the number of edges between C and $X \setminus C$ is $k(t-k)$.

For $1 \leq i \leq r$, let $a_i = |X \cap A_i|$. In particular, $\sum_{i=1}^{r} a_i = t - k$ and for every i we have $0 \leq a_i \leq s$. Then $\mathsf{e}(G[X \setminus C]) = \sum_{i=1}^{r} \binom{a_i}{2}$. Suppose there exist distinct $1 \leq i, j \leq r$ such that $0 < a_i, a_j < s$. Without loss of generality, suppose $a_i \geq a_j$. Under this assumption, it is easy to verify that $\binom{a_i}{2} + \binom{a_j}{2} < \binom{a_i+1}{2} + \binom{a_j-1}{2}$, and so by choosing one more vertex in A_i and one fewer vertex in A_j we could obtain a subgraph with more edges, contradicting the maximality of $\mathsf{e}(G[X])$. Hence $a_i \in \{0, s\}$ for all except at most one index $i \in \{1, \ldots, r\}$. It then follows from $\sum_{i=1}^{r} a_i = t - k$ that $a_i = s$ for exactly $\left\lfloor \frac{t-k}{s} \right\rfloor$ choices of i, and the possible remaining non-empty set of the form $X \cap A_i$ contains exactly $(t-k) - \left\lfloor \frac{t-k}{s} \right\rfloor s$ vertices. Hence, $G[X \setminus C]$ contains exactly $\left\lfloor \frac{t-k}{s} \right\rfloor \binom{s}{2} + \binom{t-k-\left\lfloor \frac{t-k}{s} \right\rfloor s}{2}$ edges. This concludes the proof of the lemma. $\square$

We may now prove Theorem 3.1.2, which we restate for convenience.

Theorem 3.1.2. *For $t \in \mathbb{N}$, there exists a graph G with average degree $\overline{\mathsf{d}}(G) \geq t$ such that G does not contain a minor on t vertices with more than $\left(\frac{3}{4} + o(1)\right)\binom{t}{2}$ edges.*

Proof. We prove the theorem in two ways.

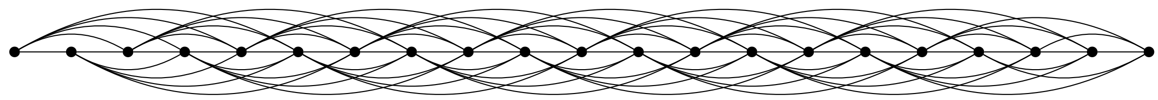
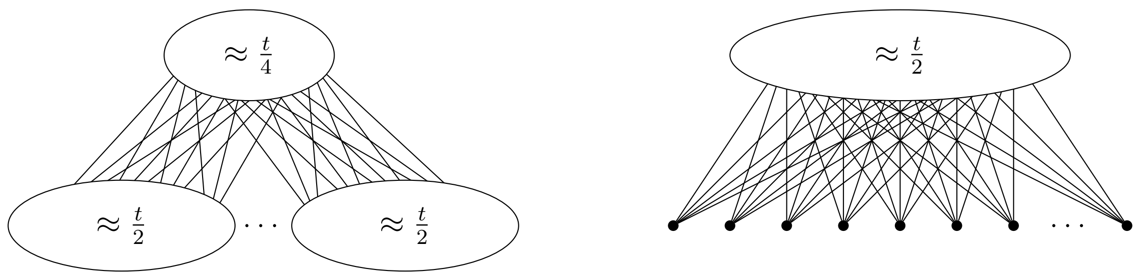


Figure 3.3.1: Example of a k-tree in the proof of Theorem 3.1.2: $\lceil \frac{t+1}{2} \rceil$-th power of a path, here illustrated for $t = 8$.



(a) $s(t) = \left(\frac{1}{2} + o(1)\right) t$ (b) $s(t) = 1$

Figure 3.3.2: Examples of graphs $S_{k(s(t),t),r,s(t)}$ in the proof of Theorem 3.1.2.

Using k-trees

In order to prove the theorem, consider any $k(t)$-tree G, where $k(t) = \left(\frac{1}{2} + o(1)\right) t > \frac{t}{2}$, and for which $\mathsf{v}(G)$ is sufficiently large (as a function of $k(t)$) such that

$$\overline{\mathsf{d}}(G) = 2\frac{\binom{k(t)}{2} + k(t)(\mathsf{v}(G) - k(t))}{\mathsf{v}(G)} = 2k(t) - \frac{(k(t)+1)k(t)}{\mathsf{v}(G)} > t.$$

Let G' be any minor of G on t vertices. As noted earlier, G' must be a spanning subgraph of some $k(t)$-tree. Hence,

$$\mathsf{e}(G') \leq \binom{k(t)}{2} + k(t)(t - k(t)) = \left(\frac{1}{2} + o(1)\right)^2 \frac{t^2}{2} + \left(\frac{1}{2} + o(1)\right)^2 t^2 = \left(\frac{3}{4} + o(1)\right) \binom{t}{2},$$

as desired.

See Figure 3.3.1 for an example of such a graph.

Using $S_{k,r,s}$

In order to prove the theorem, we consider the graphs $S_{k(s(t),t),r,s(t)}$ with $k(s(t),t) = \left\lceil \frac{t-s(t)}{2} \right\rceil + 1$ and some choice of $s(t) \geq 1$, to be specified later. Given that $\lim_{r\to\infty} \overline{\mathsf{d}}(S_{k(s(t),t),r,s(t)}) =$

$s(t)-1+2k(s(t),t) = s(t)-1+2\left(\left\lceil\frac{t-s(t)}{2}\right\rceil+1\right) \geq t+1$, we have that $\overline{\mathsf{d}}(S_{k(s(t),t),r,s(t)}) \geq t$ for sufficiently large r.

Applying Lemma 3.3.1 and Lemma 3.3.2, we only need show that $f(k(s(t),t),s(t),t) = \left(\frac{3}{4}+o(1)\right)\binom{t}{2}$. We show that this holds for various choices of $s(t)$.

One possible choice for $s(t)$ is $s(t) = \left(\frac{1}{2i}+o(1)\right)t$ for fixed $i \in \mathbb{N}$ (see Figure 3.3.2(a) for the case with $i=1$.) Then $k(s(t),t) = \left(\frac{1}{2}-\frac{1}{4i}+o(1)\right)t$. We may then compute that

$$\binom{k(s(t),t)}{2} = \left(\frac{1}{2}-\frac{1}{4i}+o(1)\right)^2\frac{t^2}{2} = \left(\frac{1}{4}-\frac{1}{4i}+\frac{1}{16i^2}+o(1)\right)\binom{t}{2}$$

and

$$k(s(t),t)(t-k(s(t),t)) = \left(\frac{1}{2}-\frac{1}{4i}+o(1)\right)\left(1-\left(\frac{1}{2}-\frac{1}{4i}+o(1)\right)\right)t^2 = \left(\frac{1}{2}-\frac{1}{8i^2}+o(1)\right)\binom{t}{2}.$$

Given that

$$\left\lfloor\frac{t-k(s(t),t)}{s(t)}\right\rfloor = \left\lfloor\frac{t-\left(\frac{1}{2}-\frac{1}{4i}+o(1)\right)t}{\left(\frac{1}{2i}+o(1)\right)t}\right\rfloor = \left\lfloor\frac{\frac{1}{2}+\frac{1}{4i}+o(1)}{\frac{1}{2i}+o(1)}\right\rfloor = i$$

for sufficiently large t, we have

$$\left\lfloor\frac{t-k(s(t),t)}{s(t)}\right\rfloor\binom{s(t)}{2} = (i+o(1))\left(\frac{1}{2i}+o(1)\right)^2\frac{t^2}{2} = \left(\frac{1}{4i}+o(1)\right)\binom{t}{2}$$

and

$$\begin{aligned}\binom{t-k(s(t),t)-\left\lfloor\frac{t-k(s(t),t)}{s(t)}\right\rfloor s(t)}{2} &= \left(1-\left(\frac{1}{2}-\frac{1}{4i}+o(1)\right)-(i+o(1))\left(\frac{1}{2i}+o(1)\right)\right)^2\frac{t^2}{2}\\ &= \left(\frac{1}{16i^2}+o(1)\right)\binom{t}{2}.\end{aligned}$$

Thus we obtain for the value of $f(k(s(t),t),s(t),t)$:

$$\left(\left(\frac{1}{4}-\frac{1}{4i}+\frac{1}{16i^2}+o(1)\right)+\left(\frac{1}{2}-\frac{1}{8i^2}+o(1)\right)+\left(\frac{1}{4i}+o(1)\right)+\left(\frac{1}{16i^2}+o(1)\right)\right)\binom{t}{2}$$
$$=\left(\frac{3}{4}+o(1)\right)\binom{t}{2},$$

as desired.

Another possible case is $s(t)=o(t)$ (see Figure 3.3.2(b) for the case $s(t)=1$). An analogous computation to above yields the result in this case (this can informally be seen by letting i tend to infinity). In fact, in this case the result also follows from the approach for k-trees discussed above. $\square$

3.4 Small graphs

In this section, we prove Theorem 3.1.3. We first need the following definitions.

A *(proper) separation* of a graph G is a pair (A,B) such that $A,B\subseteq V(G)$, $A\cup B=V(G)$, $A\setminus B, B\setminus A\neq\emptyset$ and there are no edges between vertices in $A\setminus B$ and $B\setminus A$. The *order* of (A,B) is $|A\cap B|$. We say a graph G is *k-connected* if $\mathsf{v}(G)\geq k+1$ and G does not have a separation of order strictly smaller than k. Note that complete graphs are the only graphs to not have any separation.

Given a graph H and $k\in\mathbb{N}$, we say a graph G is an *(H,k)-cockade* if G is isomorphic to H or if G can be obtained from smaller (H,k)-cockades G' and G'' by identifying a clique of size k of G' with a clique of size k of G''. A simple inductive argument can be used to show that if G is an (H,k)-cockade then $\mathsf{e}(G)=\frac{\mathsf{v}(G)-k}{\mathsf{v}(H)-k}\mathsf{e}(H)-\frac{\mathsf{v}(G)-\mathsf{v}(H)}{\mathsf{v}(H)-k}\binom{k}{2}$.

We first prove the following upper bound on the extremal function for minors in $\mathcal{K}_6^{-4}$, the class of graphs on 6 vertices and 11 edges. See, for instance, the introduction of [B11], and references therein, for a summary of similar results on the extremal functions of small graphs. By K_5^- we denote the graph obtained from K_5 by removing one edge.

Theorem 3.4.1. *If G is a graph such that $\mathsf{e}(G) \geq \frac{5}{2}\mathsf{v}(G) - \frac{7}{2}$, then G contains a minor with 6 vertices and 11 edges, unless G is isomorphic to K_1, K_5^- or K_5.*

Proof. First note that $\lceil \frac{5}{2}n - \frac{7}{2} \rceil = -1, 2, 4, 7, 9, 12$ when, respectively, $n = 1, 2, 3, 4, 5, 6$. It is then immediate that the only graphs G with $\mathsf{v}(G) \leq 5$ and at least $\frac{5}{2}\mathsf{v}(G) - \frac{7}{2}$ edges are K_1, K_5^- and K_5. If $\mathsf{v}(G) = 6$, then G contains at least 12 edges, and thus the statement also holds in this case. This shows that the theorem holds if G has at most 6 vertices, and therefore we may assume $\mathsf{v}(G) \geq 7$.

Towards a contradiction, suppose then that the statement is false, and let G be a counterexample that minimizes $\mathsf{v}(G)$ and then, subject to $\mathsf{v}(G)$ being minimum, minimizes $\mathsf{e}(G)$. The latter condition in particular implies that $\mathsf{e}(G) = \lceil \frac{5}{2}\mathsf{v}(G) - \frac{7}{2} \rceil < \frac{5}{2}\mathsf{v}(G)$.

First suppose G is 3-connected. If some $e \in E(G)$ is in fewer than two triangles, then

$$\mathsf{e}(G/e) \geq \mathsf{e}(G) - 2 \geq \frac{5}{2}\mathsf{v}(G) - \frac{11}{2} = \frac{5}{2}(\mathsf{v}(G/e) + 1) - \frac{11}{2} = \frac{5}{2}\mathsf{v}(G/e) - 3.$$

By minimality of G, G/e contains a minor on 6 vertices and 11 edges (using that $\mathsf{v}(G/e) \geq 6$ to exclude the small exceptions). As this contradicts our assumptions on G, every edge in G lies on at least two triangles.

Next suppose there exists $A \subseteq V(G)$ of size 5 such that $\mathsf{e}(G[A]) = 8$. Let $u \in V(G) \backslash A$. As G is 3-connected, Menger's theorem [B7] implies there exist internally vertex-disjoint $u - A$ paths P_1, P_2, P_3 in G (with no internal vertices in A). Then, $G[A \cup V(P_1) \cup V(P_2) \cup V(P_3)]$ contains a minor on 6 vertices and at least 11 edges, which can be obtained by contracting all but one edge in each of P_1, P_2, P_3. Hence such a set A does not exist.

Given that $\overline{\mathsf{d}}(G) = \frac{2\mathsf{e}(G)}{\mathsf{v}(G)} < 5$, there exists $u \in V(G)$ of degree at most 4 (and at least 3, by 3-connectivity).

Consider the case $\deg_G(u) = 4$. As every edge of G is in at least two triangles, $\delta(G[N_G[u]]) \geq 3$. In particular, $\mathsf{e}(G[N_G[u]]) = \frac{1}{2}\overline{\mathsf{d}}(G[N_G[u]])\mathsf{v}(G[N_G[u]]) \geq \frac{15}{2}$, and as this is an integer $\mathsf{e}(G[N_G[u]]) \geq 8$. However, we have excluded such a choice $A = N_G[u]$ earlier.

Hence, we may suppose that $\deg_G(u) = 3$, say $N_G(u) = \{x, y, z\}$. Again as every edge of G is in at least two triangles, $G[N_G[u]]$ is necessarily isomorphic to K_4. Let $v \in N_G(x) \setminus N[u]$ (such a vertex necessarily exists as otherwise $(\{u, x, y, z\}, V(G) \setminus \{u, x\})$ would form a separation of order 2 in G, contradicting 3-connectivity). As proved above, vx is in at least one triangle. If v is adjacent to y or z, then $G[\{u, x, y, z, v\}]$ contains at least 8 edges, which we have excluded earlier. Hence, there exists $w \in V(G) \setminus N_G[u]$ which is adjacent to both x and v. Again, by Menger's theorem there exist at least three internally vertex-disjoint paths between $\{v, w\}$ and $\{u, y, z\}$. At most one of these can contain x. Let P_1, P_2 be two of these paths which do not contain x, we may also suppose they do not contain any of $\{u, x, y, z, v, w\}$ as internal vertices. Then, $G[\{u, x, y, z, v, w\} \cup V(P_1) \cup V(P_2)]$ contains a minor on 6 vertices and 11 edges, which can be obtained by contracting all but one edge in each of P_1, P_2.

Hence, we may now suppose that G is not 3-connected. As $\mathsf{v}(G) \geq 7$, we may then suppose G is not a complete graph, so let (A, B) be a separation of G of minimal order. In particular, $|A \cap B| \leq 2$.

We divide the rest of the proof into cases depending on the size of $A \cap B$. First suppose $|A \cap B| = 0$. By minimality of G, if $\mathsf{e}(G[A]) \geq \frac{5}{2}|A| - 2$ (in particular, $G[A]$ cannot be isomorphic to K_1, K_5^- or K_5), then $G[A]$ contains a minor on 6 vertices and 11 edges, which is a contradiction. Hence, we may assume that $\mathsf{e}(G[A]) < \frac{5}{2}|A| - 2$, and similarly that $\mathsf{e}(G[B]) < \frac{5}{2}|A| - 2$. Then,

$$\mathsf{e}(G) = \mathsf{e}(G[A]) + \mathsf{e}(G[B]) < \frac{5}{2}(|A| + |B|) - 4 = \frac{5}{2}\mathsf{v}(G) - 4 < \frac{5}{2}\mathsf{v}(G) - \frac{7}{2},$$

which is a contradiction to our hypothesis, so this case is not possible.

Now suppose $|A \cap B| = 1$, say $A \cap B = \{x\}$. Then G is connected, and in particular $G[A]$ and $G[B]$ are connected and $|A|, |B| \geq 2$. If $G[A]$ is isomorphic to K_5, then $G[A \cup y]$ is a 6-vertex graph with at least 11 edges, where y is a neighbour of x in B. Hence, $G[A]$ is not isomorphic to K_5, and similarly for $G[B]$. Then, by minimality of G, if $\mathsf{e}(G[A]) \geq \frac{5}{2}|A| - 3$ (note that this implies that A cannot isomorphic to K_5^-), $G[A]$ contains a minor on 6 vertices and 11 edges, which is a contradiction. Hence, we may assume that $\mathsf{e}(G[A]) < \frac{5}{2}|A| - 3$, and similarly that $\mathsf{e}(G[B]) < \frac{5}{2}|B| - 3$. Then,

$$\mathsf{e}(G) = \mathsf{e}(G[A]) + \mathsf{e}(G[B]) < \frac{5}{2}(|A| + |B|) - 6 = \frac{5}{2}(\mathsf{v}(G) + 1) - 6 = \frac{5}{2}\mathsf{v}(G) - \frac{7}{2},$$

which contradicts our hypothesis, so this case is not possible.

Finally, suppose $|A \cap B| = 2$; say $A \cap B = \{x, y\}$. Since G is 2-connected, every component of $G - \{x, y\}$ has an edge to both x and y. Hence, there exists an $x - y$ path P_1 with at least two edges and with internal vertices in $A \setminus B$, and an $x - y$ path P_2 with at least two edges and with internal vertices in $B \setminus A$.

If $G[A]$ is isomorphic to K_5^- or K_5, then $G[A \cup V(P_2)]$ contains a minor on 6 vertices and at least 11 edges, which we can obtain by contracting all but two of the edges of P_2. Hence, $G[A]$, and similarly $G[B]$, are not isomorphic to K_5^- or K_5.

By minimality of G, if $\mathsf{e}(G[A]) \geq \frac{5}{2}|A| - \frac{7}{2}$ or $\mathsf{e}(G[B]) \geq \frac{5}{2}|B| - \frac{7}{2}$, then G contains a minor on 6 vertices and 11 edges, which is a contradiction. Given that the number of edges must be an integer, we have that $\mathsf{e}(G[A]) \leq \frac{5}{2}|A| - 4$ and $\mathsf{e}(G[B]) \leq \frac{5}{2}|B| - 4$.

If $xy \in E(G)$, then

$$\mathsf{e}(G) = \mathsf{e}(G[A]) + \mathsf{e}(G[B]) - 1 \leq \frac{5}{2}(|A| + |B|) - 9 = \frac{5}{2}(\mathsf{v}(G) + 2) - 9 < \frac{5}{2}\mathsf{v}(G) - \frac{7}{2},$$

which is a contradiction to our hypothesis.

Hence, $xy \notin E(G)$. Suppose first that $\mathsf{e}(G[A]) \leq \frac{5}{2}|A| - \frac{9}{2}$ and $\mathsf{e}(G[B]) \leq \frac{5}{2}|B| - \frac{9}{2}$. Then,

$$\mathsf{e}(G) = \mathsf{e}(G[A]) + \mathsf{e}(G[B]) \leq \frac{5}{2}(|A| + |B|) - 9 = \frac{5}{2}(\mathsf{v}(G) + 2) - 9 < \frac{5}{2}\mathsf{v}(G) - \frac{7}{2},$$

which would be a contradiction to our assumptions on G. Thus, at least one of the two above inequalities is invalid. Without loss of generality, we may assume that $\mathsf{e}(G[A]) > \frac{5}{2}|A| - \frac{9}{2}$ and thus $\mathsf{e}(G[A]) \geq \frac{5}{2}|A| - 4$. Now this implies $\mathsf{e}(G[A] + xy) = \mathsf{e}(G[A]) + 1 \geq \frac{5}{2}|A| - 3 > \frac{5}{2}|A| - \frac{7}{2}$. Contracting P_2 into an edge, we obtain that $G[A] + xy$ is a minor of G. Using the minimality of G, we then find that $G[A] + xy$ is isomorphic to K_5^- or K_5. The case $G[A] + xy \simeq K_5^-$ is impossible, as it would mean that $\mathsf{e}(G[A]) = 8 < \frac{5}{2}|A| - 4$. Thus, we have $G[A] + xy \simeq K_5$. But then by removing superflous vertices and edges from B and contracting all but two of the edges in P_2, we obtain a minor of G isomorphic to the graph obtained from $G[A] \simeq K_5^-$ by adding a new vertex adjacent to x and y. This is a graph on 6 vertices and 11 edges, as desired.

As we have found a contradiction to our initial assumption that G is a smallest counterexample in every possible case, this completes the proof of the theorem. $\square$

Although Theorem 3.4.1 is sufficiently strong for our purposes, it might be possible to improve it, as we are not aware of any family of graphs G with no $\mathcal{K}_6^{-4}$ minor for which $\mathsf{e}(G) \approx \frac{5}{2}\mathsf{v}(G)$. However, $(K_5^-, 1)$-cockades do not contain any $\mathcal{K}_6^{-4}$ minor and contain $\approx \frac{9}{4}\mathsf{v}(G)$ edges.

We are now ready to prove Theorem 3.1.3, which we restate for convenience.

Theorem 3.1.3. *If $2 \leq t \leq 6$ is an integer and G is a graph with average degree $\overline{\mathsf{d}}(G) \geq t-1$, then G contains a minor on t vertices with at least*

- 1 *edge if* $t = 2$,
- 3 *edges if* $t = 3$,
- 5 *edges if* $t = 4$,
- 8 *edges if* $t = 5$, *and*
- 11 *edges it* $t = 6$.

Furthermore, none of these values can be improved.

Proof.

Upper bounds

We show that the values in the statement cannot be improved. For $t = 2, 3$, these values cannot be improved as no graph on t vertices can have more than $\binom{t}{2}$ edges.

For $t = 4, 5$, consider the graphs $S_{2,r,t-3}$ as defined in Section 3.3. One easily verifies that for such t and $r \geq 4$,

$$\overline{\mathsf{d}}(S_{2,r,t-3}) = \frac{2\left(\binom{2}{2} + r\binom{t-3}{2} + 2r(t-3)\right)}{2 + r(t-3)} \geq t - 1.$$

By Lemma 3.3.1 and Lemma 3.3.2, $S_{2,r,t-3}$ does not contain any minor on t vertices with more than $f(2, t-3, t)$ vertices, which we can directly compute to be 5 and 8 for, respectively, $t = 4$ and $t = 5$.

For $t = 6$, consider any $(K_5^-, 2)$-cockade G with $\mathsf{v}(G) \geq 26$. First note that $\mathsf{e}(G) = \frac{\mathsf{v}(G)-2}{5-2} \cdot 9 - \frac{\mathsf{v}(G)-5}{5-2}\binom{2}{2} = \frac{8}{3}\mathsf{v}(G) - \frac{13}{3} \geq \frac{5}{2}\mathsf{v}(G)$ and so $\overline{\mathsf{d}}(G) \geq 5$. However, we claim that G cannot contain any minor on 6 vertices and 12 edges.

It is easy to see that every graph on 6 vertices with at least 12 edges is either 3-connected or contains a K_5 subgraph. However, as G is constructed in a tree-like fashion by identifying edges between copies of K_5^-, the only 3-connected minors of G are in fact minors of K_5^- (more generally, for $k = 0, 1, 2$, it is well known that if G is a k-sum of G_1 and G_2, and H is 3-connected, then H is a minor of G if and only if H is a minor of G_1 or of G_2). Hence, G can contain neither a 3-connected 6-vertex graph nor K_5 as a minor, proving our claim.

Lower bounds

$\boldsymbol{t = 2}$ **:** Any graph with average degree at least one contains an edge, and thus contains a minor on two vertices with one edge.

$\boldsymbol{t = 3}$ **:** It is well known that if G is a forest, $\mathsf{e}(G) \leq \mathsf{v}(G) - 1$, and in particular that $\overline{\mathsf{d}}(G) = \frac{2\mathsf{e}(G)}{\mathsf{v}(G)} < 2$ (if G is non-null). Given that $\overline{\mathsf{d}}(G) \geq 3 - 1 = 2$, G is not a forest and thus contains a cycle C and thus G contains K_3 as a minor.

$\boldsymbol{t = 4}$ **:** It is well known, and easy to see, that K_4^--minor-free graphs are the graphs for which every component is a *cactus graph*, that is a connected graph in which every block is either an edge or a cycle. It is also well known (for instance, [B15, Exercice 4.1.31]) that if G is a cactus graph, then $\mathsf{e}(G) \leq \left\lfloor \frac{3(\mathsf{v}(G)-1)}{2} \right\rfloor$ edges (the proof proceeds by induction on the number of blocks). Given that G has average degree at least 3, $\mathsf{e}(G) \geq \frac{3}{2}\mathsf{v}(G)$ and so G contains a K_4^- minor, i.e. a minor on four vertices with five edges, as claimed.

$\boldsymbol{t = 5}$ **:** Dirac [B2, Theorem 1B] proved that for any graph G such that $\mathsf{e}(G) \geq 2\mathsf{v}(G) - 2$, either G contains a minor on 5 vertices and 8 edges or G is a $(K_4, 1)$-cockade. Note that in the latter case, $\mathsf{e}(G) = \frac{\mathsf{v}(G)-1}{4-1} \cdot 6 - \frac{\mathsf{v}(G)-4}{4-1}\binom{1}{2} = 2(\mathsf{v}(G) - 1) < 2\mathsf{v}(G)$. Hence, if $\overline{\mathsf{d}}(G) \geq 4$, we have that $\mathsf{e}(G) \geq 2\mathsf{v}(G)$ and so G contains a minor on 5 vertices and 8 edges.

$\boldsymbol{t = 6}$ **:** Given that $\mathsf{d}(G) \geq \frac{5}{2}\mathsf{d}(G)$, Theorem 3.4.1 implies this result directly (noting that none of the small exceptions in Theorem 3.4.1 have average degree at least 5). □

3.5 Concluding remarks

We have considered the problem of finding the best possible α such that every graph with average degree at least t contains a minor on t vertices with at least $(\alpha - o(1))\binom{t}{2}$ edges; we have shown that $\sqrt{2} - 1 \leq \alpha \leq \frac{3}{4}$. It would be interesting to further improve these bounds.

We note that in our proof of Theorem 3.1.1, we have only used contractions to be able to consider the smallest minor of G such that $\overline{\mathsf{d}}(G) \geq t$ and apply Lemma 3.2.1. Once we have obtained that all closed neighbourhoods have minimum degree greater than $\frac{t}{2}$, we only consider subgraphs. In this setup, we cannot improve our lower bound on α beyond $\frac{1}{2}$. Indeed, consider the line graph of the complete graph K_n. It is $t := 2(n-2)$-regular, has closed neighbourhoods of minimum degree $(n-1) > \frac{t}{2}$ and it is not hard to verify that this graph contains no subgraph on t vertices with more than $(1+o(1))n^2 = (\frac{1}{2} + o(1))\binom{t}{2}$ edges.

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Part III

Menger's theorem and induced paths

Bridging text 2

In the previous two chapters, we discussed related weakenings of Hadwiger's conjecture, in which the number of edges in the forbidden minor(s) is relaxed.

In this chapter, we consider minors in which the pieces of the model are pairwise far-apart, which we have defined in the introduction as fat minors. Georgakopoulos and Papasoglu [47] have conjectured a version of Hadwiger's conjecture in the coarse context, where the colouring is replaced by Assouad-Nagata dimension and the forbidden minor condition is replaced by forbidding arbitrarily fat K_t minors.

As discussed in the introduction of this thesis, relating H-fat-minor-free graphs to the large-scale structure of the graph (specifically, quasi-isometry to an H-minor-free graph) was suggested by Georgakopoulos and Papasoglu [47] as one the first problems to study in this new field, and they suggested proving a coarse version of Menger's conjecture as one of the main tools to do so.

Although the suggested Coarse Menger's conjecture was disproved by Nguyen et al. [74], many variants and weakenings of it may still be true.

In the following chapter, we consider the induced case: we wish to find k paths which are pairwise at distance at least two between sets of vertices X and Y. In the specific variant we work on, if we cannot find the desired paths, we obtain a set of at most Ck vertices which separate X and Y, rather than a set $k-1$ of balls as in the original Coarse Menger conjecture. The former would be implied by the latter, in the case of graphs of bounded maximum degree. Using strong edge colourings, we will show that such a constant C exists, and find better and sometimes tight bounds in the case of subcubic graphs.

We also prove an analogous result for graphs with a forbidden topological minor, using a structure theorem which reduces the problem to either having bounded degree (where we will use the previously mentioned result) or having a forbidden minor (in which case there is an easy argument).

4

On an induced version of Menger's theorem

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We prove Menger-type results in which the obtained paths are pairwise non-adjacent, both for graphs of bounded maximum degree and, more generally, for graphs excluding a topological minor. More precisely, we show the existence of a constant C, depending only on the maxi-

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mum degree or on the forbidden topological minor, such that for any pair of sets of vertices X, Y and any positive integer k, there exists either k pairwise non-adjacent X-Y-paths, or a set of fewer than Ck vertices which separates X and Y. We further show better bounds in the subcubic case, and in particular obtain a tight result for two paths using a computer-assisted proof.

4.1 Introduction

Given a graph G and $X, Y \subseteq V(G)$, we say a set of vertices Z *separates* X and Y if Z intersects every X-Y-path[4]. In general, we say two paths are disjoint if they do not share any vertices. Menger's theorem is a fundamental result of graph theory, relating the existence of many disjoint paths between two sets of vertices in a graph with the absence of small separators.

Theorem 4.1.1 (Menger's theorem [C12])**.** *If $k \in \mathbb{N}$, G is a graph and $X, Y \subseteq V(G)$, then there exists either*

(1) *k pairwise disjoint X-Y-paths, or*

(2) *a set of less than k vertices which separates X and Y.*

It is a natural question to ask under which circumstances we can guarantee the paths in point (1) to be far apart from each other in the graph metric, rather than just disjoint. Georgakopoulos and Papasoglu [C6], motivated by questions in metric geometry, and Albrechtsen et al. [C1], have conjectured a "Coarse Menger's theorem". Before stating it, we need the following notation. If G is a graph, $Z \subseteq V(G)$ and $d \in \mathbb{N}$, we write $B_G(Z, d)$ for the *ball of radius d around Z*, i.e. all vertices at distance at most d from one of the vertices in Z. If $Z = \{z\}$, we may simply write $B_G(z, d)$.

Conjecture 4.1.2. *For every $k \in \mathbb{N}$, there exists $c = c(k) \in \mathbb{N}$ satisfying the following. If $d \in \mathbb{N}$, G is a graph and $X, Y \subseteq V(G)$, then there exists either*

[4] An *X-Y-path* is defined as a path that has one endpoint in X and the opposite endoint in Y. Notably, this definition also allows a single vertex in $X \cap Y$ to qualify as an X-Y-path (of length 0).

(1) *k disjoint X-Y-paths $P_1, \ldots, P_k$ such that $\mathsf{dist}_G(P_i, P_j) \geq d$ for all distinct i, j, or*

(2) *a set $Z \subseteq V(G)$ of size less than k such that $B_G(Z, cd)$ separates X and Y.*

We note that the conjecture of Albrechtsen et al. is in fact stronger, as it does not allow c to depend on k. McCarty and Seymour have shown that it is sufficient to prove the conjecture for the case $d = 3$ for the entire conjecture to hold, see [C1, Theorem 4].

Both Georgakopoulos and Papasoglu [C6] and Albrechtsen et al. [C1] have shown that Conjecture 4.1.2 holds for $k = 2$, the constant of 129 below is from the latter authors.

Theorem 4.1.3 ([C6, C1])**.** *If $d \in \mathbb{N}$, G is a graph and $X, Y \subseteq V(G)$, then there exists either*

(1) *two disjoint X-Y-paths P_1, P_2 such that $\mathsf{dist}_G(P_1, P_2) \geq d$, or*

(2) *$z \in V(G)$ such that $B_G(z, 129d)$ separates X and Y.*

A natural special case of Conjecture 4.1.2 is the case when the graph G has maximum degree bounded by a constant Δ. Indeed, in this case one can upper-bound the size of the ball $B_G(Z, cd)$ in the statement of Conjecture 4.1.2 by $|B_G(Z, cd)| \leq |Z| \cdot \sum_{i=0}^{cd} \Delta^i < \Delta^{cd+1}k$. In particular, if the strong version of Conjecture 4.1.2 proposed by Albrechtsen et al. holds (that is, with c independent of k), then also the following must be true.

Conjecture 4.1.4. *For every $d, \Delta \in \mathbb{N}$ there exists a constant $C = C(d, \Delta) > 0$ such that the following holds. If $k \in \mathbb{N}$, G is a graph with $\Delta(G) \leq \Delta$ and $X, Y \subseteq V(G)$, then there exists either*

(1) *k X-Y-paths pairwise at distance at least d in G, or*

(2) *a set of less than Ck vertices in G which separates X and Y.*

The reduction for $d \geq 3$ to $d = 3$ by McCarty and Seymour also holds for Conjecture 4.1.4, see Section 4.A.

In the first main result of this paper, namely Theorem 4.1.5 below, we prove Conjecture 4.1.4 in the first non-trivial case when $d = 2$, that is, when we look for a family of disjoint X-Y-paths that are pairwise non-adjacent[5]. For brevity, when saying that paths are pairwise non-adjacent, they will also be meant to be disjoint.

To state Theorem 4.1.5 concisely we need a bit of terminology. Let G be a graph. We say $M \subseteq E(G)$ is an *induced matching* if $\mathsf{dist}_G(e_1, e_2) \geq 2$ for every distinct $e_1, e_2 \in M$. A *strong edge colouring* of G is a partition of the edges of G into induced matchings. In other words, the edges are coloured such that no two edges of the same colour are adjacent. The *strong chromatic index* of G, denoted by $\chi'_s(G)$, is the smallest number of matchings in a strong edge colouring of G. The strong chromatic index is well studied, and there are many known bounds depending on the maximum degree $\Delta(G)$ of G, as well as for more specific classes of graphs. In general, $\chi'_s(G) \leq 2\Delta(G)(\Delta(G) - 1) + 1$, which can be seen by counting the number of edges at distance at most 2 of any edge and colouring greedily. For large enough Δ, currently the best bound, by Hurley, de Verclos and Kang [C10], is $\chi'_s(G) \leq 1.772\Delta^2$ when $\Delta(G) \leq \Delta$.

Theorem 4.1.5. *If $k \in \mathbb{N}$, G is a graph and $X, Y \subseteq V(G)$, then there exists either*

(1) *k pairwise non-adjacent X-Y-paths, or*

(2) *a set of less than $2^{\chi'_s(G)}k$ vertices which separates X and Y.*

Given the bounds on $\chi'_s(G)$ mentioned above, we may obtain the $d = 2$ case of Conjecture 4.1.4 as a direct corollary of Theorem 4.1.5.

By Menger's theorem (Theorem 4.1.1), Theorem 4.1.5 is equivalent to the following result.

Theorem 4.1.6. *If $k \in \mathbb{N}$, G is a graph, $X, Y \subseteq V(G)$ and there exists at least $2^{\chi'_s(G)}k$ pairwise disjoint X-Y-paths, then there exist k pairwise non-adjacent X-Y-paths in G.*

[5]Since for every X-Y-path P in a graph G there exists an induced X-Y-path P' such that $V(P') \subseteq V(P)$, one can see that the existence of a family of k pairwise non-adjacent X-Y-paths is equivalent to the existence of a family $\mathcal{P}$ of k different X-Y-paths, such that the union of the paths in $\mathcal{P}$ forms an induced subgraph of G. This explains the naming of the paper.

We believe this result to be interesting in its own right, as it is a quite natural analogue to Menger's theorem in an induced setting. An example of another result of this type would be Korhonen's [C11] proof of the grid minor theorem for induced minors for bounded degree graphs.

The idea behind our proof is as follows. Given a large number of disjoint X-Y-paths and a strong colouring of the edges in between the paths, we contract all edges of a colour class (say, green) and apply Menger's theorem to find many disjoint X-Y-paths in the contracted graph, which we can then lift back to the original graph. By this contraction, we will be guaranteed to not have any green edges between the paths, and the strong colouring will guarantee that there are no edges of the original paths that go between the new paths. After repeating this argument for every colour, we find a collection of pairwise non-adjacent paths.

We will prove Theorem 4.1.6 in Section 4.2. In fact, given that we do not need to colour the edges of the original paths, we can obtain an improvement over the constant $2^{\chi'_s(G)}$, which is most significant when the maximum degree is small. In particular, in Section 4.4, we will show the following two results. For brevity, if $\mathcal{P}$ is a collection of subgraphs (typically, of paths) of a graph G, then write $V(\mathcal{P}) = \bigcup_{P \in \mathcal{P}} V(P)$ and $E(\mathcal{P}) = \bigcup_{P \in \mathcal{P}} E(P)$.

Theorem 4.1.7. *If G is a graph, $X, Y \subseteq V(G)$, and there exists a collection $\mathcal{P}$ of at least $16k$ disjoint X-Y-paths such that every vertex in $V(\mathcal{P})$ is incident to at most one edge in $E(G) \setminus E(\mathcal{P})$, then there exist at least k pairwise non-adjacent X-Y-paths in G.*

This is a large improvement over the constant 2^{10} which would be obtained from Theorem 4.1.6 with the bound of $\chi'_s(G) \leq 10$ when $\Delta(G) \leq 3$ proved independently by Andersen [C2] and by Horák, Qing and Trotter [C9].

For the case $k = 2$, we will show in a computer-assisted proof the following tight result.

Theorem 4.1.8. *If G is a graph, $X, Y \subseteq V(G)$, and there exists a collection $\mathcal{P}$ of five disjoint X-Y-paths such that every vertex in $V(\mathcal{P})$ is incident to at most one edge in $E(G) \setminus E(\mathcal{P})$, then there exist two pairwise non-adjacent X-Y-paths in G.*

Furthermore, the statement does not necessarily hold if either

(a) $\mathcal{P}$ *contains four paths instead of five, or*

(b) *we replace the condition that every vertex in* $V(\mathcal{P})$ *is incident to at most one edge in* $E(G) \setminus E(\mathcal{P})$ *by the condition that the maximum degree of* G *is three.*

Both the proof methods of Theorems 4.1.7 and 4.1.8 and (b) in the latter indicate that the maximum degree of $G - E(\mathcal{P})$ is a more natural parameter to bound than the maximum degree of G.

A direct consequence of (1) in this result is that the $16k$ in Theorem 4.1.7 cannot be improved below $4k - 3$ (consider a disjoint union of $k - 1$ copies of a graph containing 4 disjoint X-Y-paths but no two non-adjacent X-Y-paths).

We say a graph H is a *topological minor* of a graph G if G contains a subdivision[6] of H as a subgraph. In Section 4.3, using the structure theorem for graphs excluding a topological K_r-minor first proved by Grohe and Marx [C7], as well as Erde and Weißauer [C4], we then generalize our induced Menger's theorem to the class of graphs excluding the complete graph K_r as a topological minor.

Theorem 4.1.9. *For every* $r \in \mathbb{N}$*, there exists* $c = c_{4.1.9}(r) > 0$ *such that the following holds.*

If G *is a graph not containing* K_r *as a topological minor and* $X, Y \subseteq V(G)$*, then there exists either*

(1) k *pairwise non-adjacent* X-Y*-paths, or*

(2) *a set of less than* ck *vertices which separates* X *and* Y*.*

We note that Theorems 4.1.5 and 4.1.9 were also proved, with slightly different constants, independently by Gartland et al. [C5].

[6] As usual, by a *subdivision* of H we here mean any graph that is isomorphic to a graph that can be obtained from H by replacing its edges by paths of positive length.

4.2 Graphs with bounded maximum degree

Theorem 4.1.6 follows directly from the following result, by taking $\mathcal{M}$ as the colour classes of edges in $E(G[V(\mathcal{P})]) \setminus E(\mathcal{P})$ (in other words, the edges not in $\mathcal{P}$ but with both ends in vertices in $\mathcal{P}$) in a strong edge colouring of G.

Theorem 4.2.1. *If $m, k \in \mathbb{N}$, G is a graph, $X, Y \subseteq V(G)$, $\mathcal{P}$ is a collection of $2^m k$ pairwise disjoint X-Y-paths and $\mathcal{M}$ is a partition of $E\left(G[V(\mathcal{P})]\right) \setminus E(\mathcal{P})$ into m induced matchings of G, then there exist k pairwise non-adjacent X-Y-paths in G.*

Proof. First note that we may without loss of generality assume that $V(G) = V(\mathcal{P})$, by restricting G, X, Y to $V(\mathcal{P})$. This implies that $E(G) = E(\mathcal{P}) \cup (\cup\mathcal{M})$, where $\cup\mathcal{M} := \bigcup_{M\in\mathcal{M}} M$.

We prove the statement by induction on m. If $m = 0$, then $\mathcal{M} = \emptyset$. In particular, $E\left(G[V(\mathcal{P})]\right) \setminus E(\mathcal{P}) = \emptyset$, and so the $2^0 k = k$ paths of $\mathcal{P}$ are pairwise non-adjacent.

We now show the inductive step. We may assume that the paths of $\mathcal{P}$ are chosen such that the sum of the lengths of paths in $\mathcal{P}$ is smallest possible among all collections of k disjoint X-Y-paths. This immediately implies that every $P \in \mathcal{P}$ is an induced path in G, and that every path $P \in \mathcal{P}$ intersects X and Y only in its endpoints.

Let $M \in \mathcal{M}$, chosen arbitrarily, and write $\mathcal{M}^* := \mathcal{M} \setminus \{M\}$. We define $G' = G - E(\mathcal{M}^*)$, and let G'' be obtained from G' by contracting the edges of M.[7] Let $f : V(G) \to V(G'')$ denote the mapping of vertices underlying the resulting contraction. The sets corresponding to X, Y in G'' are then $X' := f(X), Y' := f(Y')$.

By Menger's theorem (Theorem 4.1.1), there exists either

1. a collection $\mathcal{P}'$ of $2^{m-1}k$ disjoint X'-Y'-paths in G'', or

2. a set Z' of size less than $2^{m-1}k$ vertices separating X' and Y' in G''.

[7] An edge is *contracted* by identifying its end vertices, and removing any resulting loops and parallel edges.

First suppose we are in case (2). Let $Z := f^{-1}(Z')$. As we have contracted by a matching, the preimage of any vertex in G'' is of size at most two, and so $|Z| \leq 2|Z'| < 2^m k$. We claim Z separates X and Y in G', which would be a contradiction as there are at least $2^m k$ pairwise disjoint X-Y-paths in G' (the paths in $\mathcal{P}$ are not affected by removing M^*). If P is an X-Y-path in G', then $f(V(P))$ corresponds to the vertex set of a X'-Y' walk in G'', from which we can extract an X'-Y'-path P'. Given that Z' separates X' and Y' in G'', there exists $v' \in Z' \cap V(P') \neq \emptyset$. By construction of P', there exists $v \in V(P)$ such that $f(v) = v'$. By definition of Z, $v \in Z$. Hence, $v \in Z \cap V(P)$ as desired. Therefore, we are necessarily in case (1).

Given a path $P' \in \mathcal{P}'$, it is easily seen that $G'[f^{-1}(V(P'))]$ is connected and so there exists an X-Y path P in G' such that $f(V(P)) \subseteq V(P')$. We may further suppose that this path is induced in G'. We call such a path a *lift* of P'. Let $\mathcal{P}_2$ be the collection of lifts of paths of $\mathcal{P}'$. Given that the paths of $\mathcal{P}'$ are pairwise disjoint, so are those in $\mathcal{P}_2$.

Let $\mathcal{M}_2$ be the collection of matchings of $\mathcal{M}^*$ restricted to edges in $E\left(G[V(\mathcal{P}_2)]\right) \setminus E(\mathcal{P}_2)$.

We claim $\mathcal{M}_2$ partitions the edges of $E\left(G[V(\mathcal{P}_2)]\right) \setminus E(\mathcal{P}_2)$. The fact that the matchings in M_2 are pairwise disjoint is direct from their construction as restrictions of matchings in $\mathcal{M}^*$. Let $e = uv \in E\left(G[V(\mathcal{P}_2)]\right) \setminus E(\mathcal{P}_2)$, and suppose for a contradiction that e is not in any matching of $\mathcal{M}_2$.

First suppose that $e \in M$. We cannot have $u, v \in V(P)$ for $P \in \mathcal{P}_2$, since $e \notin E(\mathcal{P}_2)$, $M \subseteq E(G')$ and P is induced in G'. Hence, $u \in V(P_1)$ and $v \in V(P_2)$ for distinct paths $P_1, P_2 \in \mathcal{P}_2$. However, P_1, P_2 are lifts of paths in G'', say $P_1', P_2' \in \mathcal{P}'$. In particular, $f(u) \in V(P_1')$ and $f(v) \in V(P_2')$. As $uv \in M$, $f(u) = f(v)$, and so P_1' and P_2' are not disjoint, which is a contradiction to the choice of $\mathcal{P}'$. Hence, we may now suppose that $e \notin M$.

Given that $e \notin M$ and e is not in any matching of $\mathcal{M}^*$, $e \notin \cup\mathcal{M}$. By our first assumption, necessarily $e \in E(\mathcal{P})$. If we show that both u and v are incident to some edges of M, this would be a contradiction to the fact that M is a strong matching.

We will show that u is incident to some edge of M; the proof for v is analogous. Let $P \in \mathcal{P}_2$ be the path such that $u \in V(P)$. There are two cases to consider. First suppose u is not an endpoint of P, i.e. u has distinct neighbours $z_1, z_2 \in V(P)$. It is impossible that both $uz_1, uz_2 \in E(\mathcal{P})$, given that this is a collection of paths (so u cannot be incident to three edges of $E(\mathcal{P})$) and we know that $e \in E(\mathcal{P})$; without loss of generality say $uz_1 \in \cup\mathcal{M}$. Recall that $\mathcal{P}_2$ is a collection of paths in G' and so necessarily $uz_1 \in M$, as desired. Now suppose that u is an endpoint of P, hence $u \in X \cup Y$. If $u \in X \cap Y$, then e would not appear in $\mathcal{P}$, as we have assumed those paths to be as short as possible. Hence, $u \notin Y$ and so there exists some edge $uz \in E(\mathcal{P}_2)$. By our hypothesis that the paths in $\mathcal{P}$ are shortest possible, u is not an interior vertex of any path in $\mathcal{P}$, i.e. u appears in at most one edge of $\mathcal{P}$, which we already know to be e. Hence $uz \notin E(\mathcal{P})$ and so $uz \in \cup\mathcal{M}$. By the same argument as previously, $uz \in M$. This completes the proof of the claim.

Note that $|\mathcal{P}_2| = |\mathcal{P}'| = 2^{m-1}k$ and $|\mathcal{M}_2| = |\mathcal{M}^*| = |\mathcal{M}| - 1 = m - 1$. Hence, by the induction hypothesis applied to G', we obtain k pairwise non-adjacent X-Y-paths in G, as desired. □

We now briefly discuss why our proof does not work in the $d = 3$ case. The main difficulty is that there is no nice analogue of moving to an induced subgraph; the first step of the previous proof was that we can assume that $V(G) = V(\mathcal{P})$. Indeed, when we find non-adjacent paths in an induced subgraph of the original graph, they are also non-adjacent in the latter. This no longer holds when $d = 3$. We would need to consider the vertices on paths length two between any paths in $\mathcal{P}$, but after an iteration of our method, these vertices might now be part of $\mathcal{P}$. Hence, we must also consider the paths of length two between those, and so on. It is unclear how one might then manage the complications arising from this.

4.3 Excluding a topological minor

In this section, we will prove Theorem 4.1.9. We first need some definitions and notation. Let G be a graph. If $S \subseteq V(G)$, we write $G[S]$ for the subgraph of G induced by S.

A *separation* in G is to be understood as a pair (A, B) of subsets of $V(G)$ such that $A \cup B = V(G)$ and there exists no edge in G with endpoints in $A \setminus B$ and $B \setminus A$. It is slightly unusual but convenient for us to allow in this definition also degenerate cases in which $A \subseteq B$ or $B \subseteq A$. Given a separation (A, B) of G, we call $A \cap B$ its *separator* and refer to $|A \cap B|$ as the *order* of the separation (A, B).

A *tree-decomposition* of G is a pair $(T, \mathcal{V})$, where T is a tree and $\mathcal{V} = (V_t)_{t \in V(T)}$ is a collection of subsets of $V(G)$ satisfying the following properties:

- for every $v \in V(G)$, the set $\{t \in V(T) : v \in V_t\}$ induces a non-empty subtree of T, and
- for every $uv \in V(G)$, there exists at least one $t \in V(T)$ such that $u, v \in V_t$.

Given a tree decomposition $(T, \mathcal{V})$ of G, for every edge $e = t_1t_2 \in E(T)$, we denote $S(e) := V_{t_1} \cap V_{t_2}$ and call $\max_{e \in E(T)} |S(e)|$ the *adhesion* of the tree-decomposition $(T, \mathcal{V})$. Given a vertex $t \in V(T)$, the *torso at* t, denoted by $\tau(t)$, is defined as the graph obtained from $G[V_t]$ by adding, for every edge $e \in E(T)$ incident to t, an edge between any two non-adjacent vertices in $S(e)$, in other words we make $S(e)$ a clique for every incident edge e of t.

For every edge $e = t_1t_2 \in E(T)$, there exists a natural corresponding separation in G, namely

$$\left(\bigcup_{t \in (T-e)(t_1)} V_t, \bigcup_{t \in (T-e)(t_2)} V_t \right),$$

where $(T - e)(t_i)$ denotes the set of vertices of the unique component of $T - e$ that contains t_i. From the definition of a tree decomposition it is not hard to see that this indeed is a separation in G, with $V_{t_1} \cap V_{t_2} = \left(\bigcup_{t \in (T-e)(t_1)} V_t\right) \cap \left(\bigcup_{t \in (T-e)(t_2)} V_t\right)$ being the corresponding separator.

Finally, we say a graph H is a *minor* of G if a graph isomorphic to H can be obtained from G be removing vertices and edges, and contracting edges. It is direct that if G contains H as a topological minor, it also contains H as a minor.

The following structure theorem is a key element of our proof of Theorem 4.1.9. We use the exact statement of Erde and Weißauer [C4, Theorem 4], see also Grohe and Marx [C7, Theorem 4.1].

Theorem 4.3.1 ([C7, C4]). *If $r \in \mathbb{N}$ and G is a graph excluding K_r as a topological minor, then G admits a tree-decomposition of adhesion less than r^2 such that every torso either*

(1) *has fewer than r^2 vertices of degree at least $2r^4$, or*

(2) *is K_h-minor-free, for $h = 2r^2$.*

Broadly speaking, our proof of Theorem 4.1.9 will proceed as follows. Given a collection of X-Y-paths in a smallest counterexample G, we will apply Theorem 4.3.1 and find a torso of the tree decomposition which intersects every path in the collection. Then, in order to find the desired collection of paths, we will either apply our result for bounded maximum degree (Theorem 4.1.6), if we are in case (1), or use the following lemma, if we are in case (2).

Lemma 4.3.2. *If $h, k \in \mathbb{N}$, G is a K_h-minor-free graph, $X, Y \subseteq V(G)$ and there exists k pairwise disjoint X-Y-paths in G, then there exists at least $\frac{k}{2(h-1)}$ pairwise non-adjacent X-Y-paths in G.*

Proof. Let $\mathcal{P}$ be a collection of k disjoint X-Y-paths in G. Let H be the minor of G obtained from $G\left[\bigcup_{P\in\mathcal{P}} V(P)\right]$ by contracting each path $P \in \mathcal{P}$ into a single vertex. In this way, the vertices of H have a natural one-to-one correspondence with the paths in $\mathcal{P}$, and two vertices in H are adjacent if and only if the corresponding paths in $\mathcal{P}$ are adjacent. Since G is K_h-minor-free, so is H. Hence, by a classical result of Duchet and Meyniel [C3], we have that H contains an independent set of size at least $\alpha(H) \geq \frac{\mathsf{v}(H)}{2(h-1)} = \frac{k}{2(h-1)}$. The subcollection $\mathcal{P}' \subseteq \mathcal{P}$ corresponding to this independent set in H now consists of pairwise non-adjacent X-Y-paths, as desired. □

Theorem 4.1.9 follows directly from the following result, by applying Menger's theorem (Theorem 4.1.1) and choosing $c_{4.1.9}(t) \geq \frac{1}{\varepsilon_{4.3.3}(t)}$. The additive $\frac{1}{2}$ is used solely for formal reasons, as it simplifies the inductive proof.

Theorem 4.3.3. *For every $r \in \mathbb{N}$, there exists $\varepsilon = \varepsilon(r) > 0$ such that the following holds.*

If G is a graph not containing K_r as a topological minor, $X, Y \subseteq V(G)$, $k \in \mathbb{N}$ and there are k pairwise disjoint X-Y-paths in G, then there also exists a family of at least $\varepsilon k + \frac{1}{2}$ pairwise non-adjacent X-Y-paths in G.

Proof. Fix $r \in \mathbb{N}$; we prove the statement with the constant $\varepsilon(r) := 2^{-(8r^8+3)}$. Towards a contradiction, suppose the claim is not true, and consider a counterexample G with $\mathsf{v}(G)$ minimum. Hence, there exist $X, Y \subseteq V(G)$ and $k \in \mathbb{N}$ such that on the one hand, there exists a collection $\mathcal{P}$ consisting of k pairwise disjoint X-Y-paths in G, but on the other hand, every collection $\mathcal{Q}$ of pairwise non-adjacent X-Y-paths in G has size less than $\varepsilon k + \frac{1}{2}$. Note that the latter fact implies that $1 < \varepsilon k + \frac{1}{2}$, so $k > \frac{1}{2\varepsilon}$. Our next claim uses the minimality assumption on G to guarantee that for every separation (A, B) in G of sufficiently small order, one of its two sides must intersect all paths in $\mathcal{P}$.

Claim 4.3.3.1. *If (A, B) is a separation in G of order $|A \cap B| < 2^{8r^8+1}$, then $V(P) \cap A \neq \emptyset$ for every $P \in \mathcal{P}$ or $V(P) \cap B \neq \emptyset$ for every $P \in \mathcal{P}$.*

Proof of Claim 4.3.3.1. Suppose towards a contradiction that there exist two paths $P_1, P_2 \in \mathcal{P}$ such that $V(P_1) \subseteq A \setminus B$ and $V(P_2) \subseteq B \setminus A$. Let $\mathcal{P}_1 := \{P \in \mathcal{P} : V(P) \subseteq A \setminus B\}$, $\mathcal{P}_2 := \{P \in \mathcal{P} : V(P) \subseteq B \setminus A\}$, and let us denote $k_1 := |\mathcal{P}_1|$, $k_2 := |\mathcal{P}_2|$. Note that since (A, B) is a separation, every path $P \in \mathcal{P} \setminus (\mathcal{P}_1 \cup \mathcal{P}_2)$ must intersect the separator $A \cap B$. Since the paths in $\mathcal{P}$ are pairwise disjoint, this implies that $k - (k_1 + k_2) \leq |A \cap B| < 2^{8r^8+1}$. Note that $\mathcal{P}_1$ is a collection of $k_1 \geq 1$ pairwise disjoint $(X \cap (A \setminus B))$-$(Y \cap (A \setminus B))$-paths in $G[A \setminus B]$, and $\mathcal{P}_2$ is a collection of $k_2 \geq 1$ pairwise disjoint $(X \cap (B \setminus A))$-$(Y \cap (B \setminus A))$-paths in $G[B \setminus A]$. Since by our minimality assumption on G both graphs $G[A \setminus B]$ and $G[B \setminus A]$ satisfy the hypothesis of the theorem, we find that there is a collection $\mathcal{Q}_1$ of at least $\varepsilon k_1 + \frac{1}{2}$ pairwise non-adjacent $(X \cap (A \setminus B))$-$(Y \cap (A \setminus B))$-paths in $G[A \setminus B]$, and a collection $\mathcal{Q}_2$ of at least $\varepsilon k_2 + \frac{1}{2}$ pairwise non-adjacent $(X \cap (B \setminus A))$-$(Y \cap (B \setminus A))$-paths in $G[B \setminus A]$. Since there are no edges in G between $A \setminus B$ and $B \setminus A$, the collection $\mathcal{Q} := \mathcal{Q}_1 \cup \mathcal{Q}_2$ also consists of pairwise non-adjacent X-Y-paths in G. We furthermore have

$$\begin{aligned}|\mathcal{Q}| = |\mathcal{Q}_1| + |\mathcal{Q}_2| &\geq \left(\varepsilon k_1 + \frac{1}{2}\right) + \left(\varepsilon k_2 + \frac{1}{2}\right) = \varepsilon(k_1 + k_2) + 1 \\ &> \varepsilon\left(k - 2^{8r^8+1}\right) + 1 = \varepsilon k + \frac{1}{2} + \left(\frac{1}{2} - \varepsilon 2^{8r^8+1}\right) > \varepsilon k + \frac{1}{2}.\end{aligned}$$

This is a contradiction on our initial assumptions that such a collection $\mathcal{Q}$ cannot exist. Hence our assumption was false, and this concludes the proof of the claim. ■

Next, we apply Theorem 4.3.1 to G, which yields a tree-decomposition $(T, (V_t)_{t\in V(T)})$ of G of adhesion less than r^2, such that every torso $\tau(t)$ has at most r^2 vertices of degree at least $2r^4$, or is K_h-minor-free for $h := 2r^2$.

Claim 4.3.3.2. *There exists a vertex $t^* \in V(T)$ such that $V(P) \cap V_{t^*} \neq \emptyset$ for every $P \in \mathcal{P}$.*

Proof of Claim 4.3.3.2. For every edge $e = t_1t_2$ of T, we have that

$$\left(\bigcup_{t\in(T-e)(t_1)} V_t, \bigcup_{t\in(T-e)(t_2)} V_t\right)$$

forms a separation in G of order $|S(e)| < r^2 < 2^{8r^8+1}$. Hence, by Claim 4.3.3.1, every path in $\mathcal{P}$ intersects $\bigcup_{t\in(T-e)(t_1)} V_t$, or every path in $\mathcal{P}$ intersects $\bigcup_{t\in(T-e)(t_2)} V_t$. We can therefore find an orientation $\vec{T}$ of T such that for every edge $e = t_1t_2$ oriented from t_1 to t_2 in $\vec{T}$, we have $V(P) \cap \left(\bigcup_{t\in(T-e)(t_2)} V_t\right) \neq \emptyset$ for every $P \in \mathcal{P}$. Since T is a tree, there must exists a vertex $t^* \in V(T)$ that is a sink in the orientation $\vec{T}$ of T. We now claim that $V(P) \cap V_{t^*} \neq \emptyset$ for every $P \in \mathcal{P}$. Suppose otherwise towards a contradiction. Let $P \in \mathcal{P}$ be such that $V(P) \cap V_{t^*} = \emptyset$ and let $R := \{t \in V(T) : V(P) \cap V_t \neq \emptyset\}$. Since P is a connected subgraph of G, it readily follows from the definition of a tree-decomposition that R induces a subtree of T, which does not include t^*. Hence, there is an edge $e = t't^* \in E(T)$ incident to t^* such that $R \subseteq (T - e)(t')$. This, however, contradicts the fact that $V(P) \cap \left(\bigcup_{t\in(T-e)(t^*)} V_t\right) \neq \emptyset$, which follows since e is oriented from t' to t^* in $\vec{T}$. This concludes the proof of the claim.

■

Let H be the graph obtained from $G[V_{t^*}]$ by adding an edge between every pair x, y of non-adjacent vertices in $G[V_{t^*}]$ for which there exists a path in G with endpoints x, y all whose internal vertices are in $V(G) \setminus V_{t^*}$.

Claim 4.3.3.3. *For every pair of vertices $x, y \in V_{t^*}$ with $xy \notin E(G)$ for which there exists a path in G with endpoints x, y all whose internal vertices are in $V(G) \setminus V_{t^*}$, there exists an edge $f = tt^* \in E(T)$ incident with t^* such that $x, y \in S(f)$. In particular, $G[V_{t^*}] \subseteq H \subseteq \tau(t^*)$.*

Proof of Claim 4.3.3.3. Let P be an x-y-path in G such that $V(P) \cap V_{t^*} = \{x, y\}$. Let $S := \{s \in V(T) : V_s \cap (V(P) \setminus \{x, y\}) \neq \emptyset\}$. It follows readily from the definition of a tree-decomposition (and since $P - \{x, y\}$ is a connected subgraph of G) that S induces a connected subgraph of T, i.e., $T[S]$ is a subtree of T. We furthermore have $V_{t^*} \cap (V(P) \setminus \{x, y\}) = \emptyset$, and thus $t^* \notin S$. Therefore, there exists an edge $f = tt^*$ incident with t^* such that S is contained in $(T - f)(t)$. Let $x, x_1, \ldots, x_\ell, y$ be the vertex-trace of P. By definition of a tree-decomposition, there exist bags V_{t_1} and V_{t_2} such that $x, x_2 \in V_{t_1}$ and $x_{\ell-1}, y \in V_{t_2}$. This directly implies that $t_1, t_2 \in S \subseteq (T - f)(t)$. Hence, we have

$$x, y \in \left(\bigcup_{s \in (T-f)(t)} V_s \right) \cap V_{t^*} = V_t \cap V_{t^*}.$$

This proves that $x, y \in S(f)$, as desired. This concludes the proof. ■

Next, we define a family $\mathcal{P}^*$ of k disjoint paths in H as follows. For every path $P \in \mathcal{P}$, let P^* denote the path in H that has vertex-set $V(P) \cap V_{t^*}$ and visits the vertices in $V(P) \cap V_{t^*}$ in the same order as P. This indeed forms a path in H, since for every subpath $x, x_1, \ldots, x_\ell, y$ of P with $x, y \in V_{t^*}$ and $x_1, \ldots, x_\ell \notin V_{t^*}$, we have $xy \in E(H)$ by definition.

For every endpoint v of a path $P \in \mathcal{P}$, let us denote by $v^* \in V_{t^*}$ the unique vertex in $V(P) \cap V_{t^*}$ that is closest to v along the path P. Necessarily, v^* is an endpoint of P^*. Using this notation, we now define two subsets $X^*, Y^* \subseteq V_{t^*}$ as

$$X^* := \{v^* : v \in X \text{ and } v \text{ is the endpoint of some path in } \mathcal{P}\},$$

$$Y^* := \{v^* : v \in Y \text{ and } v \text{ is the endpoint of some path in } \mathcal{P}\}.$$

In particular, $\mathcal{P}^*$ is a collection of k pairwise disjoint X^*-Y^*-paths in H.

Claim 4.3.3.4. *There exists a family $\mathcal{Q}^*$ consisting of pairwise non-adjacent X^*-Y^*-paths in H such that $|\mathcal{Q}^*| \geq \varepsilon k + \frac{1}{2}$.*

Proof of Claim 4.3.3.4. By the properties of the tree-decomposition $(T, (V_t)_{t \in V(T)})$, we know that $\tau(t^*)$ either has at most r^2 vertices of degree at least $2r^4$, or is K_h-minor-free for $h = 2r^2$. In particular, the same is true for the subgraph H of $\tau(t^*)$.

Let us start with considering the first case. Let $Z \subseteq V_{t^*}$ denote the set of vertices of degree at least $2r^4$ in H. Then we know that $|Z| \leq r^2$ and $\Delta(H - Z) < 2r^4 := \Delta$. Let $\mathcal{P}'$ be the set of paths of $\mathcal{P}^*$ which do not intersect Z. Then, since the paths in $\mathcal{P}^*$ are pairwise disjoint, we have that $|\mathcal{P}^*| \geq k - |Z| \geq k - 2r^4$. Hence, Theorem 4.1.6 implies that there exists a collection $\mathcal{Q}^*$ of pairwise non-adjacent $(X^* \setminus Z)$-$(Y^* \setminus Z)$-paths in $H - Z$ such that

$$|\mathcal{Q}^*| \geq \left\lfloor \frac{1}{2^{2\Delta^2}}(k - 2r^4) \right\rfloor .$$

Clearly, $\mathcal{Q}^*$ is also a family of pairwise non-adjacent X^*-Y^*-paths in H. Using that $k > \frac{1}{2\varepsilon}$ and $\varepsilon = 2^{-(8t^8+3)}$, we can lower bound its size as follows.

$$|\mathcal{Q}^*| \geq \frac{k}{2^{2\Delta^2}} - \frac{2r^4}{2^{2\Delta^2}} - 1 = \frac{k}{2^{8r^8}} - \frac{2r^4}{2^{8r^8}} - 1 = 8\varepsilon k - \frac{2r^4}{2^{8r^8}} - 1 = \varepsilon k + \frac{1}{2} + \left(7\varepsilon k - \frac{2r^4}{2^{8r^8}} - \frac{3}{2}\right) > \varepsilon k + \frac{1}{2}.$$

This establishes the claim in the first case.

Next, consider the case that H is K_h-minor-free. Then, by Lemma 4.3.2 there exists a collection $\mathcal{Q}^*$ of pairwise non-adjacent X^*-Y^*-paths in H of size at least $\frac{k}{2(h-1)} = \frac{k}{2(2r^2-1)} >$

$2^{-(8r^8+2)}k = 2\varepsilon k > \varepsilon k + \frac{1}{2}$, as desired. This concludes the proof of the claim also in the second possible case. ■

We now finish the proof of the theorem by using $\mathcal{Q}^*$, as given by Claim 4.3.3.4, to construct a family $\mathcal{Q}$ of pairwise non-adjacent X-Y-paths in G of size $|\mathcal{Q}| = |\mathcal{Q}^*| \geq \varepsilon k + \frac{1}{2}$.

For every edge $xy \in E(H) \setminus E(G[V_{t^*}])$, pick and fix a path P_{xy} in G that has endpoints x, y and no internal vertices in V_{t^*} (such a path always exists by definition of H). Furthermore, for every edge $xy \in E(G[V_{t^*}])$, we let P_{xy} denote the path consisting of the single edge xy in G.

Now consider any path $Q^* \in \mathcal{Q}^*$ and let $x_1, x_2, \ldots, x_q$ be its sequence of vertices, such that $x_1 \in X^*$ and $x_q \in Y^*$. Then, by definition, there exist $x \in X$, $y \in Y$ such that $x^* = x_1$, $y^* = x_q$. We now define $W(Q^*)$ as the walk in G that starts at x, follows the unique path in $\mathcal{P}$ that x is an endpoint of, until it reaches $x^* = x_1$, then follows the concatenation of the paths $P_{x_i x_{i+1}}$ for $1 \leq i < q$ and then follows the unique path in $\mathcal{P}$ that y is an endpoint of, until it reaches y.

Claim 4.3.3.5. *If $Q_1^*, Q_2^* \in \mathcal{Q}^*$ are distinct, then $W(Q_1^*)$ and $W(Q_2^*)$ are non-adjacent in G.*

Proof of Claim 4.3.3.5. Suppose towards a contradiction that there exist $a \in V(W(Q_1^*)), b \in V(W(Q_2^*))$ that are at distance at most 1 in G. Let $a' \in V(Q_1^*)$ be a vertex closest to a along $W(Q_1^*)$, and let $b' \in V(Q_2^*)$ be defined similarly for b. In particular, there exist paths R_1 and R_2 that form subwalks of $V(W(Q_1^*))$ and $V(W(Q_2^*))$, respectively, such that R_1 has endpoints a, a' and $V(R_1) \cap V_{t^*} = \{a'\}$, and analogously R_2 has endpoints b, b' and $V(R_2) \cap V_{t^*} = \{b'\}$. Now, as $a = b$ or $ab \in E(G)$, then the walk W in G that starts at a', follows R_1 to a, moves to b, and follows R_2 until it reaches b', satisfies $V(W) \cap V_{t^*} = \{a', b'\}$. This implies that there exists an a'-b'-path R in G with $V(R) \subseteq V(W)$, in particular we have $V(R) \cap V_{t^*} = \{a', b'\}$. By definition of H, this implies that $a' = b'$ or $a'b' \in E(H)$, in either case a contradiction, since Q_1^* and Q_2^* are be non-adjacent in H. This concludes the proof of the claim. ■

We can now define $\mathcal{Q}$ by, for every $Q^* \in \mathcal{Q}^*$, short-cutting the walk $W(Q^*)$ into a path in G that has the same endpoints and such that $V(Q) \subseteq V(W(Q^*))$. By Claim 4.3.3.5, any two distinct paths in $\mathcal{Q}$ are non-adjacent. Clearly, $|\mathcal{Q}| = |\mathcal{Q}^*|$ by definition, and Claim 4.3.3.4 now implies that $\mathcal{Q}$ consists of at least $\varepsilon k + \frac{1}{2}$ pairwise non-adjacent X-Y-paths in G. This yields the desired contradiction, completing the proof of the theorem. $\square$

4.4 Subcubic graphs

In this section, we show our results on subcubic graphs. We begin by proving Theorem 4.1.7, which we restate for convenience.

Theorem 4.1.7. *If G is a graph, $X, Y \subseteq V(G)$, and there exists a collection $\mathcal{P}$ of at least $16k$ disjoint X-Y-paths such that every vertex in $V(\mathcal{P})$ is incident to at most one edge in $E(G) \setminus E(\mathcal{P})$, then there exist at least k pairwise non-adjacent X-Y-paths in G.*

Proof. Let $F := E(G[V(\mathcal{P})]) \setminus E(\mathcal{P})$. By Theorem 4.2.1, it suffices to partition F into four induced matchings. By hypothesis, no two edges of F share an end vertex.

A standard tool for studying strong edge colouring is to find a proper vertex colouring of the square of the line graph; we slightly vary this argument given that we only want to partition the edges in F. We construct the auxiliary graph H with vertex set F and such that $e_1, e_2 \in F$ are adjacent if $\mathsf{dist}_G(e_1, e_2) = 1$. In other words, e_1, e_2 are adjacent if and only if one of the end vertices of e_1 and one of the end vertices of e_2 are consecutive vertices on one of the paths in $\mathcal{P}$. This implies that $\Delta(H) \leq 4$, since the maximum degree in a path is two and since no two edges in F are incident.

By construction, a proper vertex-colouring of H with four colours yields the desired partition; each colour class is an induced matching. It suffices to show that H is 3-degenerate. Suppose, for a contradiction, that H has a 4-regular subgraph H'. Let $P \in \mathcal{P}$ such that some edge $e \in V(H')$ has an end $u \in V(P)$ and choose such e and u so that the distance from u to an end of P along P is minimum. Then at most one neighbour of u in P is an end

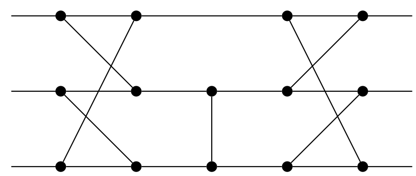

Figure 4.4.1: Example requiring four colours for any strong edge colouring of non-horizontal edges.

of an edge in $V(H')$. It follows that e has degree at most three in H', obtaining the desired contradiction to the assumption that H' is 4-regular.

$\square$

We note that the 4-colouring of the auxiliary graph H in the proof above is best possible. The configuration shown in Figure 4.4.1 (where the horizontal edges are part of the paths of $\mathcal{P}$), is an example in which we cannot partition the edges outside $\mathcal{P}$ into three induced matchings. In particular, in this case the auxiliary graph is the Moser spindle [C13], which is easily verified to not be 3-colourable.

We now prove Theorem 4.1.8, which we restate for convenience.

Theorem 4.1.8. *If G is a graph, $X, Y \subseteq V(G)$, and there exists a collection $\mathcal{P}$ of five disjoint X-Y-paths such that every vertex in $V(\mathcal{P})$ is incident to at most one edge in $E(G) \setminus E(\mathcal{P})$, then there exist two pairwise non-adjacent X-Y-paths in G.*

Furthermore, the statement does not necessarily hold if either

(a) *$\mathcal{P}$ contains four paths instead of five, or*

(b) *we replace the condition that every vertex in $V(\mathcal{P})$ is incident to at most one edge in $E(G) \setminus E(\mathcal{P})$ by the condition that the maximum degree of G is three.*

Proof. We begin by proving the first part of the statement.

We define a *path system* as a quadruple $\mathcal{H} = (H, A, B, \mathcal{Q})$ where H is a graph, $A, B \subseteq V(H)$ and $\mathcal{Q}$ is a 5-tuple of five disjoint A-B-paths such that

1. $V(H) = V(\mathcal{Q})$,

2. no vertex in A is incident in H to any edge in $E(H) \setminus E(\mathcal{Q})$,

3. every vertex of $V(H) \setminus A$ is incident in H to exactly one edge in $E(H) \setminus E(\mathcal{Q})$, and

4. there does not exist any collection $\mathcal{Q}'$ of 5 pairwise disjoint A-B-paths such that $V(\mathcal{Q}') \subsetneq V(H)$.

Note that conditions (1) and (4) imply that the paths in $\mathcal{Q}$ contain no vertices in $A \cup B$ other than their endpoints, and if $x \in A \cap B$ then one of the paths consists of exactly x. In particular $|A| = |B| = 5$.

We may suppose without loss of generality that $\mathcal{G} = (G, X, Y, \mathcal{P})$ is a path system. By restricting G, X and Y to vertices $V(\mathcal{P})$, we may suppose that (1) holds; any pair of non-adjacent paths in an induced subgraph remains non-adjacent in the original graph. We may suppose (2) holds as if a vertex $x \in X$ is incident to some edge in $E(H) \setminus E(\mathcal{Q})$, we can add a new vertex x' to G as well as the edge $x'x$, replace X by $(X \setminus \{x\}) \cup \{x'\}$, and prepend $x'x$ to the path of $\mathcal{P}$ with x as an endpoint. Any x'-Y-path in the new graph directly yields a x-Y-path with the same set of neighbours. We may suppose (3) holds for vertices not in Y given that finding a pair of non-adjacent paths in a graph directly yields such a pair in a subdivision of this graph. Furthermore, if some vertex $y \in Y \setminus X$ is not incident to some edge in $E(G) \setminus E(\mathcal{P})$, then y has a neighbour in the path of $\mathcal{P}$ of which it is an endpoint, which exists by (1) (this path cannot be a singleton as $y \notin X$); say $y'y \in E(\mathcal{P})$. We may then remove y from G and from its path of $\mathcal{P}$ and replace Y by $(Y \setminus \{y\}) \cup \{y'\}$; any X-y'-path directly extends to a X-y-path with the same neighbours. By repeating this argument, we may suppose that (3) holds for vertices in Y. Finally, we may suppose (4), as otherwise we could then replace $\mathcal{P}$ with these paths. It is easily verified that none of these reductions are in conflict.

In the following, for $k \in \mathbb{N}$ we write $[k] = \{1, \ldots, k\}$.

We now define an operation (which will have two variants) which allows us to easily construct and represent path systems. Let $\mathcal{H} = (H, A, B, \mathcal{Q} = (Q_1, Q_2, Q_3, Q_4, Q_5))$ be a

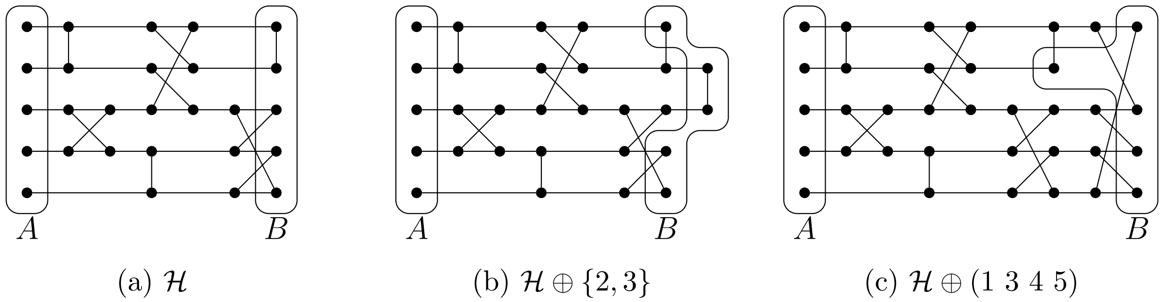

Figure 4.4.2: Example of a path system $\mathcal{H}$ and two examples for the $\oplus$ operation. The paths are labelled from 1 to 5 from top to bottom.

path system. For $i \in [5]$, write b_i for the vertex of B in Q_i.

Let $\mathcal{E} := \binom{[5]}{2}$ be the collection of (unordered) pairs of integers between 1 and 5. Let $\{i_1, i_2\} \in \mathcal{E}$. We define $\mathcal{H} \oplus \{i_1, i_2\}$ as the path system obtained by

- adding new vertices b'_{i_1}, b'_{i_2} and the edges $b_{i_1}b'_{i_1}$, $b_{i_2}b'_{i_2}$ and $b'_{i_1}b'_{i_2}$ to H,
- appending the edges $b_{i_1}b'_{i_1}$ and $b_{i_2}b'_{i_2}$, respectively, to Q_{i_1} and Q_{i_2}, and
- replacing B with $(B \setminus \{b_{i_1}, b_{i_2}\}) \cup \{b'_{i_1}, b'_{i_2}\}$

An example of this operation is provided in Figure 4.4.2(b).

Let $\mathcal{C}$ be the set of cyclic permutations of length at least two with values in $[5]$. We write such cycles as $(i_1 \dots i_k)$, for instance $(i_1\ i_2\ i_3) = (i_2\ i_3\ i_1) = (i_3\ i_1\ i_2)$. For $(i_1\ \dots\ i_k) \in \mathcal{C}$, we say $\mathcal{H} \oplus (i_1\ \dots\ i_k)$ is the path system obtained by

- adding new vertices $c_{i_1}, \dots, c_{i_k}$ and $b'_{i_1}, \dots, b'_{i_k}$ and the edges $b_{i_1}c_{i_1}, \dots, b_{i_k}c_{i_k}$, $c_{i_1}b'_{i_1}, \dots, c_{i_k}b'_{i_k}$ and $c_{i_1}b'_{i_2}, \dots, c_{i_k}b'_{i_1}$ to H,
- appending edges $b_{i_j}c_{i_j}$ and $c_{i_j}b'_{i_j}$ to Q_{i_j} for every $j \in [k]$, and
- replacing B with $(B \setminus \{b_{i_1}, \dots, b_{i_k}\}) \cup \{b'_{i_1}, \dots, b'_{i_k}\}$.

An example is provided in Figure 4.4.2(c).

Let $\mathcal{H}_0 = (H_0, A_0, B_0, \mathcal{Q}_0)$ be the path system consisting of the graph H_0 with singleton vertices $V(H_0) = A_0 = B_0 = \{v_1, v_2, v_3, v_4, v_5\}$ and $\mathcal{P}_0$ the 5 paths of length 0 in H_0.

We now show that every path system can be obtained from $\mathcal{H}_0$ using the $\oplus$ operation. We say two path systems $\mathcal{H}_1 = (H_1, A_1, B_1, \mathcal{Q}_1)$ and $\mathcal{H}_2 = (H_2, A_2, B_2, \mathcal{Q}_2)$ are *isomorphic*, which we denote by $\mathcal{H}_1 \simeq \mathcal{H}_2$, if there exists a graph isomorphism $h : V(H_1) \to (H_2)$ which maps A_1 to A_2, B_1 to B_2 and $\mathcal{Q}_1$ to $\mathcal{Q}_2$ (the ordering of the paths must be the same).

Claim 4.4.0.1. *For every path system $\mathcal{H}$, there exists some sequence $m_1, \ldots, m_k \in \mathcal{E} \cup \mathcal{C}$ such that $\mathcal{H} \simeq \mathcal{H}_0 \oplus m_1 \oplus \cdots \oplus m_k$.*

Proof of Claim 4.4.0.1. Write $\mathcal{H} = (H, A, B, \mathcal{Q})$. We prove the statement by induction on $|E(H) \setminus E(\mathcal{Q})|$.

For the base case, if $|E(H) \setminus E(\mathcal{Q})| = 0$, then (3) implies that $A = V(\mathcal{Q})$, and so necessarily $\mathcal{H} \simeq \mathcal{H}_0$.

We now show the inductive step. Let F be the set of edges in $E(H) \setminus E(\mathcal{Q})$ incident to B. As $|E(H) \setminus E(\mathcal{Q})| > 0$, condition (2) implies that $A \neq B$, and so by (3) we have that $F \neq \emptyset$.

First suppose there exists an edge of F with both ends in B, say $b'_{i_1} b'_{i_2}$, where $b'_{i_1} \in V(Q_{i_1})$ and $b'_{i_2} \in V(Q_{i_2})$. Let $\mathcal{H}'$ be the path system resulting from removing b'_{i_1}, b'_{i_2}. More precisely, if b_{i_1} and b_{i_2} are, respectively, the neighbours of b'_{i_1} in Q_{i_1} and of b'_{i_2} in Q_{i_2} (these exist since $b'_{i_1}, b'_{i_2} \notin A$), then $\mathcal{H}' = (H', A, B', \mathcal{Q}')$ where $H' = H\left[V(H) \setminus \{b'_{i_1}, b'_{i_2}\}\right]$, $B' = \left(B \setminus \{b'_{i_1}, b'_{i_2}\}\right) \cup \{b_{i_1}, b_{i_2}\}$ and $\mathcal{Q}'$ is identical to $\mathcal{Q}$ except that the edges $b_{i_1} b'_{i_1}$ and $b_{i_2} b'_{i_2}$ are removed from, respectively, Q_{i_1} and Q_{i_2}. It is direct from the definitions that $\mathcal{H} = \mathcal{H}' \oplus \{i_1, i_2\}$. In particular, condition (3) implies that $b'_{i_1} b'_{i_2}$ were not incident to any edge other than $b_{i_1} b'_{i_1}$, $b_{i_2} b'_{i_2}$ and $b'_{i_1} b'_{i_2}$. Furthermore, $E(H') \setminus E(\mathcal{Q}') = (E(H) \setminus E(\mathcal{Q})) \setminus \{b'_{i_1} b'_{i_2}\}$. By induction, there exists a sequence $m_1, \ldots, m_{k-1} \in \mathcal{E} \cup \mathcal{C}$ such that $\mathcal{H}' \simeq \mathcal{H}_0 \oplus m_1 \oplus \cdots \oplus m_{k-1}$. Hence, we obtain that $\mathcal{H} \simeq \mathcal{H}_0 \oplus m_1 \oplus \cdots \oplus m_{k-1} \oplus \{i_1, i_2\}$, as desired.

Otherwise, we construct an auxiliary digraph J with vertex set F as follows. Let $c_{i_1} b'_{i_2}, c_{i_3} b'_{i_4} \in F$, where $b'_{i_2} \in V(Q_{i_2}) \cap B$ and $b'_{i_4} \in V(Q_{i_4}) \cap B$, and $c_{i_1} \in V(Q_{i_1}) \setminus B$ and $c_{i_3} \in V(Q_{i_3}) \setminus B$. Further note that $i_1 \neq i_2$ and $i_3 \neq i_4$: these edges are not in $\mathcal{Q}$ and by (4) the paths of $\mathcal{Q}$ are necessarily induced. In our auxiliary digraph J, we put a directed edge from $c_{i_1}b_{i_2}$ to $c_{i_3}b_{i_4}$ in J if and only if $i_2 = i_3$.

Every edge in F has in-degree at least one in J (in fact, it is necessarily exactly one). Indeed, let $c_{i_1}b'_{i_2} \in F$. Let b'_{i_1} be the unique vertex of $V(Q_{i_1}) \cap B$. As $c_{i_1} \notin B$, necessarily $b'_{i_1} \notin A$. Hence, by (3), there exists there is some edge $e \in F$ incident to b'_{i_1}. In particular, there is a directed edge in J from e to $c_{i_1}b'_{i_2}$.

Hence, there exists in J a directed cycle. By definition, the sequence of vertices in this directed cycle is of the form $c_{i_1}b'_{i_2}, \ldots, c_{i_k}b'_{i_1}$, where $c_{i_j} \in V(Q_{i_j}) \setminus B$ and $b'_{i_j} \in V(Q_{i_j}) \cap B$ for every $j \in [k]$. We claim $c_{i_j}b'_{i_j} \in E(Q_{i_j})$ for every $j \in [k]$. Suppose otherwise that for some $j \in [k]$ there exists at least one vertex x between c_{i_j} and b_{i_j} on Q_{i_j}. Then, the following collection of paths would contradict (4): in $\mathcal{Q}$, replace the paths $Q_{i_1}, \ldots, Q_{i_k}$ with the paths formed by following P_{i_j} until c_{i_j} and then following the edge $c_{i_j}b'_{i_{j+1}}$ (with addition modulo k). In particular, this new set of paths does not contain x.

Let $\mathcal{H}'$ be the path system resulting from removing c_{i_j} and b'_{i_j} for every $j \in [k]$. More precisely, if we write b_{i_j} for the neighbour of c_{i_j} in $V(Q_{i_j})$ which is not b'_{i_j} (this vertex exists since $c_{i_j} \notin A$ by (2)), for every $j \in [k]$, then $\mathcal{H}' = (H', A, B', \mathcal{Q}')$ where where $H' = H\left[V(H) \setminus \{c_{i_1}, \ldots, c_{i_k}, b'_{i_1}, \ldots, b'_{i_k}\}\right]$, $B' = \left(B \setminus \{b'_{i_1}, \ldots, b'_{i_k}\}\right) \cup \{b_{i_1}, \ldots, b_{i_k}\}$ and $\mathcal{Q}'$ is identical to $\mathcal{Q}$ except that the edges $b_{i_j}c_{i_j}$ and $c_{i_j}b'_{i_j}$ are removed from Q_{i_j}, for every $j \in [k]$. Similarly to above, it is then direct from the definitions that $\mathcal{H} \simeq \mathcal{H}_0 \oplus m_1 \oplus \cdots \oplus m_{k-1} \oplus (i_1 \ \ldots \ i_k))$, and by induction, there exists a sequence $m_1, \ldots, m_{k-1} \in \mathcal{E} \cup \mathcal{C}$ such that $\mathcal{H}' \simeq \mathcal{H}_0 \oplus m_1 \oplus \cdots \oplus m_{k-1}$. Hence, we obtain that $\mathcal{H} \simeq \mathcal{H}_0 \oplus m_1 \oplus \cdots \oplus m_{k-1} \oplus (i_1 \ \ldots \ i_k)$, as desired. ∎

We define a *state* as an unordered pair $\{S_1, S_2\}$ of non-empty disjoint subsets of $[5]$. Given a path system $\mathcal{H} = (H, A, B, \mathcal{Q} = (Q_1, Q_2, Q_3, Q_4, Q_5))$, we say a state $S = \{S_1, S_2\}$ is *$\mathcal{H}$-reachable* if there exist sets $C_1, C_2 \subseteq V(H)$ such that

- C_1, C_2 are disjoint and non-adjacent in H,

- $C_1 \cap A \neq \emptyset$ and $C_2 \cap A \neq \emptyset$,
- $C_1 \cap V(Q_i) \cap B \neq \emptyset$ if and only if $i \in S_1$ and $C_2 \cap V(Q_i) \cap B \neq \emptyset$ if and only if $i \in S_2$, and
- $H[C_1], H[C_2]$ are connected.

Claim 4.4.0.2. *For every path system $\mathcal{H}$, there exists an $\mathcal{H}$-reachable state.*

Proof of Claim 4.4.0.2. Let $S = \{S_1, S_2\}$ be a state and $m \in \mathcal{E} \cup \mathcal{C}$. We construct a collection of states $f(S, m)$ by saying $S' \in f(S, m)$ if and only if S' can be written as $S' = \{S_1, S_2\}$ such that the following holds: writing, $\mathcal{H}_0 \oplus m = (H_m, A_m, B_m, \mathcal{Q}_m)$, there there exists $C_1, C_2 \subseteq V(H_m)$ such that

- C_1, C_2 are disjoint and non-adjacent in H_m,
- $C_1 \cap V(Q_i) \cap A_m$ if and only if $i \in S_1$ and $C_2 \cap V(Q_i) \cap A_m$ if and only if $i \in S_2$,
- $C_1 \cap V(Q_i) \cap B_m \neq \emptyset$ if and only if $i \in S_1'$ and $C_2 \cap V(Q_i) \cap B_m \neq \emptyset$ if and only if $i \in S_2'$,
- for every vertex of $C_1 \cap B_m$, its connected component in $H[C_1]$ contains a vertex of $C_1 \cap A_m$, and for every vertex of $C_2 \cap B_m$, its connected component in $H[C_2]$ contains a vertex of $C_2 \cap A_m$.

It is easily verified that for any path system $\mathcal{H}$, if S is $\mathcal{H}$-reachable, then every state in $f(S, m)$ is $\mathcal{H} \oplus m$-reachable. The crucial observation is that $\mathcal{H} \oplus m$ can be obtained from $\mathcal{H}$ and $\mathcal{H}_0 \oplus m$ by identifying the vertices in the B set from the former and the vertices from the A set in the latter. From this, these conditions are exactly those which allow us to extend the non-adjacent sets C_1, C_2 corresponding to states in $\mathcal{H}$ to sets in $\mathcal{H} \oplus m$.

If $\mathcal{S}$ is a collection of states, let $g(\mathcal{S}, m) = \bigcup_{S \in \mathcal{S}} f(\mathcal{S}, m)$. Hence, for any $\mathcal{H}$, if $\mathcal{S}$ is a collection of $\mathcal{H}$-reachable states, then $g(\mathcal{S}, m)$ is a collection of $\mathcal{H} \oplus m$-reachable states. We say a collection of states $\mathcal{S}'$ is a *descendant* of $\mathcal{S}$ if there exists some sequence $m_1, \dots, m_k \in \mathcal{E} \cup \mathcal{C}$ such that $\mathcal{S}' = g(\dots g(g(\mathcal{S}, m_1), m_2) \dots, m_k)$.

Let $\mathcal{S}_0 := \{\{\{x\}, \{y\}\} : x, y \in [5], x \neq y\}$. It is direct from the definition that every state in $\mathcal{S}_0$ is $\mathcal{H}_0$-reachable. By Claim 4.4.0.1, every $\mathcal{H}$ can be written as $\mathcal{H} = \mathcal{G}_0 \oplus m_1 \oplus \dots \oplus m_k$ for some sequence of $m_1, \dots, m_k \in \mathcal{E} \cup \mathcal{C}$. In particular, $g(\dots g(g(\mathcal{S}_0, m_1), m_2) \dots, m_k)$ is a collection of $\mathcal{H}$-reachable states. We want to show that this collection is non-empty. In order to prove the claim, it thus suffices to show that $\emptyset$ is not a descendant of $\mathcal{S}_0$. Our strategy is thus as follows : start with the collection $\mathcal{S}_0$, and then repeatedly apply $g(\cdot, m)$ for every $m \in \mathcal{E} \cup \mathcal{C}$ until no new collection of states are found. If the $\emptyset$ is never encountered, we are done.

Along the way, we may in fact trim some branches off of this process, in the three following ways:

(i) If $\mathcal{S}$ is a collection of $\mathcal{H}$-reachable states and σ is a permutation of [5], then $\sigma(\mathcal{S})$, which is obtained by applying σ to the elements in $\mathcal{S}$, is $\mathcal{H}'$-reachable, where $\mathcal{H}'$ is identical to $\mathcal{H}'$ except that we have permuted the order of the paths of $\mathcal{Q}$ according to σ.

    If $g(\dots g(g(\sigma(S), m_1), m_2) \dots, m_k) = \emptyset$, then $g(\dots g(g(\mathcal{S}', \sigma^{-1}(m_1)), \sigma^{-1}(m_2)) \dots, \sigma^{-1}(m_k))$ is empty, and so if $\emptyset$ is not a descendant of $\mathcal{S}$, it is not a descendant of $\sigma(\mathcal{S})$ either. In particular, we need only keep one collection of states from each equivalence class under permutation.

(ii) If $\mathcal{S} \subseteq \mathcal{S}'$ are collections of states, then $g(\mathcal{S}, m) \subseteq g(\mathcal{S}', m)$. In particular, if $\emptyset$ is not a descendant of $\mathcal{S}$, it is not a descendent of $\mathcal{S}'$ either. Hence, during our searching process, we may throw out any collection of states which is not minimal.

(iii) We do not need to use the definition of f to compute every instance of $f(S, m)$. Indeed, similarly to (i), it is easily verified from the definitions that if σ is a permutation of [5], then $f(S, m) = \sigma^{-1}(f(\sigma(S), \sigma(m)))$, so we only need to compute f for one pair (S, m) in every equivalence class.

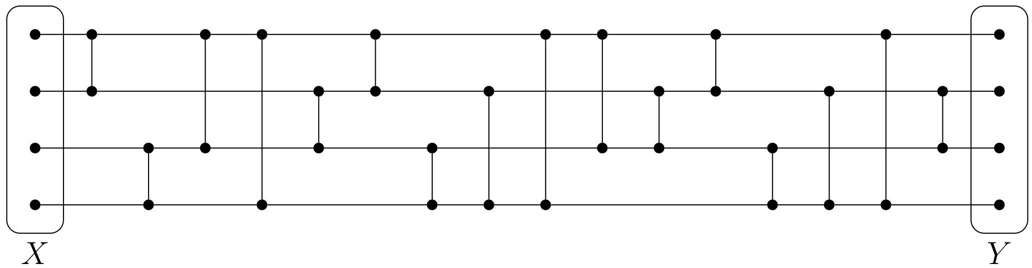

Figure 4.4.3: Counter-example (a) in Theorem 4.1.8.

We have implemented this approach in Mathematica [C14], the code is provided in [C8]. This code is fully commented, consult these for further implementation details. As a benchmark, this script can run in under 14 minutes on a 2020 MacBook Air with M1 chip and 16 GB ram running Mathematica 13.0.0.0. ■

In particular, there exists some $\mathcal{G}$-reachable state. This state is a certificate of the existence of two non-adjacent sets C_1, C_2 which induce connected graphs and both intersect X and Y. From these, we may extract two non-adjacent X-Y-paths, as desired. This concludes the proof of the first part of the statement.

We now show that this result is best possible, in two ways.

First, consider the statement when we are given four paths instead of five. By modifying the approach used above to prove the statement for five paths so as to it also taking into account paths which "backtrack", we were able to find a counter-example to this modified statement; it is shown in Figure 4.4.3. A short Mathematica script which verifies that no pair of non-adjacent X-Y-paths exists in this graph is provided in [C8].

Secondly, consider the statement if we replace the condition that every vertex in $V(\mathcal{P})$ is incident to at most one edge in $E(G) \setminus E(\mathcal{P})$ by the condition that the maximum degree is three. Of course, the difference only concerns vertices in X and in Y, given that these vertices are the only ones with fewer than two neighbours in their path. The graph formed by a adding a matching between a five-cycle (with vertices X) and the complement of a copy of this five-cycle (with vertices Y), as shown in Figure 4.4.4, is easily verified to not contain any pair of non-adjacent edges between X and Y. □

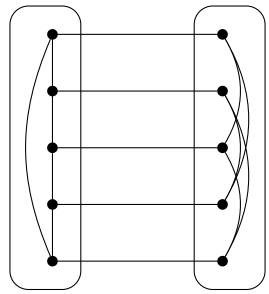

Figure 4.4.4: Counter-example (b) in Theorem 4.1.8.

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4.A Reduction of Conjecture 4.1.4 to distance 3

In this appendix, we show that reduction for $d \geq 3$ to $d = 3$ by McCarty and Seymour also holds for Conjecture 4.1.4, see Section 4.A. For this paper to be self-contained, we reproduce this proof here with only slight modifications for the change of setting.

Theorem 4.A.1. *If Conjecture 4.1.4 holds for $d = 3$, it also holds for $d \geq 3$ with constant $C(d, \Delta) = C(3, \Delta^{d+1})k$.*

Proof. Suppose G be a graph with $\Delta(G) \leq \Delta$ and $X, Y \subseteq V(G)$, and $k \in \mathbb{N}$. Let H be the d-th power of G, i.e. the graph obtained from G by adding edges to any pair of vertices at distance at most d in G. Note that the maximum degree of H is at most Δ^{d+1}. Given that Conjecture 4.1.4 holds for distance 3, there exists either

(1) k X-Y-paths pairwise at distance at least 3 in H, or

(2) a set of less than $C(3, \Delta^{d+1})k$ vertices in H which separates X and Y.

First suppose we are in case (1). Let $P'_1, \ldots, P'_k$ be X-Y paths in H, pairwise at distance at least 3. We wish to construct $P_1, \ldots, P_k$, X-Y paths in H, pairwise at distance at least d. For each $i \in [k]$, P'_i can be converted into a path in G with the same endpoints by following paths of length at most d in G between every pair of consecutive vertices in P'_i (these exist by definition of H), and then converting the resulting walk to a path by removing cycles. In particular, any vertex in P_i is at distance at most $\frac{d}{2}$ from P'_i. If $d_G(P_i, P_j) \geq d$ for every distinct $i, j \in [k]$, then we are done. Otherwise, suppose there exists distinct $i, j \in [k]$ such that $\mathsf{dist}_G(P_i, P_j) \leq d$. In particular, the distance between P'_i and P'_j is at most $\frac{d}{2} + d + \frac{d}{2} = 2d$. By definition of H, this implies that $\mathsf{dist}_H(P'_i, P'_j) \leq 2$, which is a contradiction.

Otherwise, we are in case (2). As G is a subgraph of H, the set of size less than $C(3, \Delta^{d+1})k$ which separates X and Y in H also separates X and Y in G. □

4.B Code used

```
(* ::Package:: *)

(* ::Title:: *)
(*Proving the existence of two non-adjacent paths*)


```

```
(* :: Subtitle :: *)
(*Code used in the paper "On an induced version of Menger's theorem" by Kevin Hendrey, Sergey Norin, Raphael Steiner and J\[EAcute]r\[EAcute]mie Turcotte*)
(**)
(*As part of the proof that given a (close to) subcubic graph with 5 X-Y paths, there exists a pair of non-adjacent X-Y paths, this script shows that there is a reachable state for every path system.*)


(* :: Section :: *)
(*Preparatory functions*)


(* :: Subsection :: *)
(*Basic definitions*)


(* :: Input :: Initialization :: *)
pathCount=5;
(* The number of paths *)

pathList=Range[pathCount];
(* We will always label the paths 1,..., pathCount *)


(* :: Input :: Initialization :: *)
edgeList={#,"Edge"}&/@Subsets[pathList,{2}];
(* The list of edges we can append to a path system (\mathcal E), the indices are always in growing order *)

cycleList ={#,"Cycle"}&/@DeleteDuplicatesBy[Flatten[Permutations/@Subsets[pathList,{2,Infinity}],1], Cycles[{#}]&];
(* The list of cycles we can append to a path system (\mathcal C), for each cycle one way of writing it is chosen *)
```

```
edgeAndCycleList=Join[edgeList, cycleList ];
(* For convenience, we will work with the union of these lists , each edge or cycle has a labelled
    attached to it *)


(* ::Input:: Initialization :: *)
 stateList =Select[Subsets[Subsets[pathList ,{1,4}],{2}], Intersection [#[[1]],#[[2]]]=={}&];
(* The list of states *)

convertStateToIndex[state_]:= FirstPosition [ stateList , state ][[1]]
(* Given a state , returns the index of this state in stateList *)


(* ::Input:: Initialization :: *)
 startStatesIndices =convertStateToIndex/@Select[Subsets[Subsets[pathList ,{1,1}],{2}], Intersection
    [#[[1]],#[[2]]]=={}&];
(* The list of indices of initial states (\mathcal S_0) *)


(* ::Subsection:: *)
(*Computing f*)


(* ::Subsubsection:: *)
(*Defining the graph H_m*)


(* ::Input:: Initialization :: *)
(* We define H_m, A_m and B_m from the path system \mathcal H_m =\mathcal H_0 \[CirclePlus] m

We will write the vertices which start the paths (written b_i in the definition of \[ CirclePlus ]) as {1,
    i },and those which are added and end the paths (written b_i' in the definition of \[ CirclePlus ]) as
```

```
{2,i}. In the case of cycles, the middle vertices (written c_i in the definition of \[CirclePlus]) will be denoted by {3/2,i}. Note that this notation also allows a direct embedding, every vertex having itself as coordinates.
*)

am={1,#}&/@pathList;
(* The vertices in A_m, does not depend on the operation *)

bm[m_]:=Join[{2,#}&/@m[[1]],{1,#}&/@Complement[pathList,m[[1]]]]
(* The vertices in B_m, depends on which edge or cycle m is appended *)

bmNotam[m_]:=Complement[bm[m],am]
(* The vertices in B_m\A_m, depends on which edge or cycle m is appended *)

hm[{{i1_,i2_},"Edge"}]:=Graph[
Join[am,bmNotam[{{i1,i2},"Edge"}]],
(* Vertices *)
{UndirectedEdge[{1,i1},{2, i1}], UndirectedEdge[{1,i2},{2, i2}], UndirectedEdge[{2,i1},{2, i2}]}, (* Edges *)
VertexCoordinates->Join[am,bmNotam[{{i1,i2},"Edge"}]] (* Coordinates for embedding*)
]
(* The version of H_m for appending an edge *)

hm[{cycle_,"Cycle"}]:=Graph[
Join[am,bmNotam[{cycle,"Cycle"}],Table[{3/2,i},{i, cycle }]], (* Vertices *)
Flatten[Table[{UndirectedEdge[{1,cycle [[ i ]]},{3/2, cycle [[ i ]]}], UndirectedEdge[{3/2,cycle [[ i ]]},{2, cycle [[ i ]]}], UndirectedEdge[{3/2,cycle [[ i ]]},{2, cycle [[ Mod[i+1,Length[cycle],1]]]}]},{ i ,Length[cycle ]}]], (* Edges *)
VertexCoordinates->Join[am,bmNotam[{cycle,"Cycle"}],Table[{3/2,i},{i,cycle}]] (* Coordinates for embedding*)
]
(* The version of H_m for appending a cycle *)
```

```
(* ::Subsubsection:: *)
(*Generating all possible sets C_1,C_2*)


(* ::Input:: Initialization :: *)
twoDisjointSubsets[list_]:=Select[Tuples[Subsets[list],{2}],!IntersectingQ[#[[1]],#[[2]]]&]
(* Given a list list, returns a list of all (ordered) pairs of disjoint subsets *)

possibleConfigurations[{s1_,s2_},m_]:={Join[{1,#}&/@s1,#[[1]]],Join[{1,#}&/@s2,#[[2]]]}&/@twoDisjointSubsets[Complement[VertexList[hm[m]],am]]
(* Given a state (s1,s2) and an edge or cycle m, returns all possible pairs of disjoint subsets of V(H_m) C_1,C_2 such that the second condition, regarding the intersections with A, is respected. In other words, the intersections with A is fixed, but for the remainder of the sets we impose no conditions other than disjointness. *)


(* ::Subsubsection:: *)
(*Testing non-adjacent sets*)


(* ::Input:: Initialization :: *)
jointNeighbourhood[g_,vList_]:=DeleteDuplicates[Join[vList,Flatten[AdjacencyList[g,#]&/@vList,1]]]
(* Given a graph g and a list of vertices vList, returns the list of vertices which appear either in vList or are adjacent to some vertex in vList *)

nonAdjacentSets[g_,{vList1_,vList2_}]:=!IntersectingQ[jointNeighbourhood[g,vList1],vList2]&&!IntersectingQ[jointNeighbourhood[g,vList2],vList1]
(* Given a graph g and a pair of lists of vertices vList1,vList2, returns True if and only if vList1 and vList2 are (disjoint and) non-adjacent in g *)


(* ::Subsubsection:: *)
(*Testing for connectivity condition*)
```

```
(* :: Input :: Initialization :: *)
connectedToA[m_,vList_]:=AllTrue[Intersection[vList,bm[m]],IntersectingQ [VertexComponent[Subgraph[hm[m],vList],{#}],am]&]
(* Given an edge or a cycle m and a list of vertices vList of H_m, returns True if and only if for every vertex in vList \[ Intersection ] B_m, its component in H[vlist] contains a vertex of A_m *)


(* :: Subsubsection :: *)
(*Defining f*)


(* :: Input :: Initialization :: *)
 selectedConfigurations [s_,m_]:=
Select[ possibleConfigurations [s,m] ,(* Only sets respecting the second condition are proposed *)

nonAdjacentSets[hm[m],#]&&
(* Test for the first condition *)

connectedToA[m,#[[1]]]
&&connectedToA[m,#[[2]]]&&
(* Tests for the fourth condition *)

 IntersectingQ [#[[1]], bm[m]]&&IntersectingQ[#[[2]],bm[m]]
(* In order to yield a valid state (both parts must be non−empty), both C_1 and C_2 must intersect B *)
&
]
(* Given a state s and an edge or cycle m, returns the pairs C_1,C_2 which respect the first , third and fourth conditions as specified in the proof of the claim. *)

f[stateIndex_,mIndex_]:=
```

```
Union[convertStateToIndex/@
(* This converts the states into their indices in stateList , and we sort these and remove duplicates *)
Map[Sort,
(* We sort the values inside each part of the states , as well as the order of the two parts in the state
    , to use the same form as in stateList *)
Map[Transpose[Intersection[#,bm[edgeAndCycleList[[mIndex]]]]][[2]]&,
(* To extract the possible states from the possible configurations , we first take the intersection of
    these with B_m, and then keep only the labelling of the path numbers *)

 selectedConfigurations [ stateList [[ stateIndex ]], edgeAndCycleList[[mIndex]]]
,{2}]
,{1,2}]
]
(* Given the index in stateList of a state s and the index in edgeAndCycleList of an edge or cycle m,
    returns the indices of states in f(s,m). We proceed by computing the possible configurations (sets
    C_1,C_2) as above, and the state is then exactly the indices of the paths in which these intersect
    B_m (hence the third condition holds). *)


(* ::Subsection:: *)
(*Defining g*)


(* ::Input:: Initialization :: *)
g[ collectionStateIndices_ ,mIndex_]:=Union@@(f[#,mIndex]&/@collectionStateIndices)
(* Given a collection of indices in stateList of states S and the index in edgeAndCycleList of an edge
    or cycle m, returns the indices of states in f(S,m) *)


(* ::Subsection:: *)
(*Time improvements using permutations*)
```

```
(* :: Input :: Initialization :: *)
permutationList=AssociationThread[pathList,#]&/@Permutations[pathList];
(* The list of permutations of the (indices of the) paths *)

reversePermutation[permutationIndex_]:=FirstPosition[permutationList,KeySort[Association[Reverse/@Normal[permutationList[[permutationIndex]]]]]][[1]]
(* Given the index in permutationList of a permutation, returns the index of the inverse permutation. KeySort ensures that the permutation is written in the same form as in permutationList. Permutation reversing code taken from https://mathematica.stackexchange.com/questions/284775/how-can-i-invert-the-association *)

replaceState[stateIndex_,permutationIndex_]:=replaceState[stateIndex,permutationIndex]=convertStateToIndex[Sort[Map[Sort,stateList[[stateIndex]]/. permutationList[[permutationIndex]]]]]
(* Returns the index of the state obtained by applying the permutation with index permutationIndex in permutationList to the state with index stateIndex. As earlier, we use sorting to ensure that the states are in the same format as in stateList before finding their indices. *)

canonicalCollectionStates[collectionStateIndices_]:=Sort[Table[Sort[replaceState[#,i]&/@collectionStateIndices],{i,1,Length[permutationList]}]][[1]]
(* Given a collection of indices in stateList of states S, considers all collections of states which can be obtained from S by applying a permutation, and chooses a canonical one using sorting. *)

equivalent[m1_,m2_]:=Cycles[{m1[[1]]}]==Cycles[{m2[[1]]}]&&m1[[2]]==m2[[2]]
(* Given two edges or cycles m1,m2, possibly not written in the same way as in edgeAndCycleList, returns True if and only if m1 and m2 represent the same edge or cycle *)

convertEdgeOrCycleToIndex[m_]:=FirstPosition[edgeAndCycleList, SelectFirst[edgeAndCycleList, equivalent[#,m]&]][[1]]
(* Given an edge or cycle m, possibly not written in the same way as in edgeAndCycleList, returns its index in edgeAndCycleList *)

replaceEdgeOrCycle[mIndex_,permutationIndex_]:=
convertEdgeOrCycleToIndex[edgeAndCycleList[[mIndex]]/.permutationList[[permutationIndex]]]
```

```
(* Returns the index in edgeAndCycleList of the edge or cycle obtained by applying the permutation with index permutationIndex to the edge or cycle with index mIndex *)


(* ::Subsection:: *)
(* Filtering out subsets*)


(* ::Input:: Initialization :: *)
minimalSubsets[list_]:=DeleteDuplicates[SortBy[list,Length],Intersection[#1,#2]===Sort[#1]&]
(* Given a list of lists, keeps only those which are minimal (not containing any of the other lists), this version is from https://mathematica.stackexchange.com/questions/8154/how-to-select-minimal-subsets *)

minimalSubsets[list1_,list2_]:=Select[minimalSubsets[list1],Function[u,AllTrue[list2,Intersection[u,#]!=Sort[#]&]]]
(* Given two lists of lists list1 and list2, keeps only lists in list1 which are minimal and which also don't contain lists in list2 *)


(* ::Section:: *)
(*Main computation*)


(* ::Subsection:: *)
(*Preloading f*)


(* ::Input:: Initialization :: *)
(* Given that computing f can be slow, we only wish to compute f once, which is what the below script does, as it saves the outputs of f. As noted in improvement (iii) in the paper, if \[Sigma] is a permutation of [5], then \[Sigma]^-1 f(\[Sigma]S,\[Sigma]m)=f(S,m). This can be used to greatly reduce the computation time by only running the underlying computation of f once for each
```

```
equivalence class. However, if you wish to not use this reduction, change symmetryReduction to
False. *)

symmetryReduction=True;

Do[
permIndex=SelectFirst[Range[Length[permutationList]],replaceState[stateIndex,#]<stateIndex||(
    replaceState[stateIndex,#]==stateIndex &&replaceEdgeOrCycle[mIndex,#]<mIndex)&];
(* The index of a permutation \[Sigma] such that f(\[Sigma]S,\[Sigma]m) has already been computed, using
    the order in which the Do runs. If no such permutation exists, this will be Missing["NotFound"].
    *)

If[permIndex===Missing["NotFound"]||!symmetryReduction,

f[stateIndex,mIndex]=f[stateIndex,mIndex],
(* This computes f using its definition *)

f[stateIndex,mIndex]=Sort[replaceState[#,reversePermutation[permIndex]]&/@f[replaceState[stateIndex,
    permIndex],replaceEdgeOrCycle[mIndex,permIndex]]]
(* This computes f from already computed values using a permutation*)

]
,{stateIndex,Length[stateList]},{mIndex,Length[edgeAndCycleList]}]//AbsoluteTiming

(* Runs in 274.472 seconds on a 2020 MacBook Air with M1 chip and 16GB ram running Mathematica
    13.0.0.0
We note that a less naive,but less legible implementation of the definition of f can bring this down to
    around 25 seconds, more information can be provided upon request. *)


(* ::Subsection:: *)
(*Main computation*)

```

```
(* :: Input :: Initialization :: *)
AbsoluteTiming[
    processedList = {};
    (* Will contain the collections of states to which we have already applied g to *)

    toProcess ={ startStatesIndices };
(* Will contain the collections of states to which we have not yet applied g to starts containing only \
    mathcal S_0 *)

found = False;
(* This variable will become True if the empty collection of states is ever found as the result of
    applying g (we want this to never happen, and so that found is still False at the end) *)

    While[
        toProcess != {},
                (* As long as we keep finding unseen collections
        of states , we continue the process *)

        toProcess = First[Reap[
(* The new toProcess is obtained by collecting the applications of g (see below) to the previous states
    in toProcess (using Reap and Sow to collect these efficiently ) *)

Do[
(* For every collection of states in toProcess and every edge and cycle in edgeAndCycleList, we compute
    g *)

                    newCollection = canonicalCollectionStates [g[ currentCollectionStates , mIndex]];
(* We apply g, and then take an equivalent collection of states obtained by applying some permutation,
    which implements improvement (i) *)

                    If [newCollection != currentCollectionStates ,
 (* We are only interested in collections of states which are new *)
```

```

                    Sow[newCollection];

                    If[newCollection == {},
                        found = True
                    ]
(* Changed found to true if we have found the empty collection of states *)
                ];

                ,{ currentCollectionStates , toProcess},{mIndex, 1, Length[edgeAndCycleList]}];

            processedList = minimalSubsets[Join[processedList , toProcess ]]
(* We append the now processed states to processedList , and remove non−minimal elements. This is
    compatible with ( ii ) since , when we remove elements in the new toProcess list which contain
    elements in processedList , we only need to test this for minimal elements. *)

 ][[2]],  {}];

        toProcess = minimalSubsets[toProcess, processedList ] (* We remove non−minimal subsets in
            toProcess as well as those which contain elements in processedList as described in
            improvement (ii ) in the paper *)
    ];

    found
]

(* Runs in 559.123642‘ seconds on a 2020 MacBook Air with M1 chip and 16GB ram running Mathematica
    13.0.0.0 *)
```

4.C Code used

```
(* ::Package:: *)

```

```
(* :: Title :: *)
(*Testing the existence of two non-adjacent paths*)


(* :: Subtitle :: *)
(*Code used in the paper "On an induced version of Menger's theorem" by Kevin Hendrey, Sergey Norin, Raphael Steiner and J\[EAcute]r\[EAcute]mie Turcotte*)
(**)
(*This short script verifies that the example provided in the paper of a subcubic graph with four X-Y paths does not contain a pair of non-adjacent X-Y paths.*)


(* :: Input :: Initialization :: *)
(* We first define the graph *)
vertexList ={{1,0},{1,1},{1,3},{1,4},{1,6},{1,9},{1,10},{1,12},{1,15},{1,17},{2,0},{2,1},{2,5},{2,6},
  {2,8},{2,11},{2,12},{2,14},{2,16},{2,17},{3,0},{3,2},{3,3},{3,5},{3,7},{3,10},{3,11},{3,13},{3,16},
 {3,17},{4,0},{4,2},{4,4},{4,7},{4,8},{4,9},{4,13},{4,14},{4,15},{4,17}};
edgeList={{1,0}\[UndirectedEdge]{1,1},{1,1}\[UndirectedEdge]{1,3},{1,3}\[UndirectedEdge]{1,4},{1,4}\[UndirectedEdge]{1,6},{1,6}\[UndirectedEdge]{1,9},{1,9}\[UndirectedEdge]{1,10},{1,10}\[UndirectedEdge]{1,12},{1,12}\[UndirectedEdge]{1,15},{1,15}\[UndirectedEdge]{1,17},{2,0}\[UndirectedEdge]{2,1},{2,1}\[UndirectedEdge]{2,5},{2,5}\[UndirectedEdge]{2,6},{2,6}\[UndirectedEdge]{2,8},{2,8}\[UndirectedEdge]{2,11},{2,11}\[UndirectedEdge]{2,12},{2,12}\[UndirectedEdge]{2,14},{2,14}\[UndirectedEdge]{2,16},{2,16}\[UndirectedEdge]{2,17},{3,0}\[UndirectedEdge]{3,2},{3,2}\[UndirectedEdge]{3,3},{3,3}\[UndirectedEdge]{3,5},{3,5}\[UndirectedEdge]{3,7},{3,7}\[UndirectedEdge]{3,10},{3,10}\[UndirectedEdge]{3,11},{3,11}\[UndirectedEdge]{3,13},{3,13}\[UndirectedEdge]{3,16},{3,16}\[UndirectedEdge]{3,17},{4,0}\[UndirectedEdge]{4,2},{4,2}\[UndirectedEdge]{4,4},{4,4}\[UndirectedEdge]{4,7},{4,7}\[UndirectedEdge]{4,8},{4,8}\[UndirectedEdge]{4,9},{4,9}\[UndirectedEdge]{4,13},{4,13}\[UndirectedEdge]{4,14},{4,14}\[UndirectedEdge]{4,15},{4,15}\[UndirectedEdge]{4,17},{1,1}\[UndirectedEdge]{2,1},{3,2}\[UndirectedEdge]{4,2},{1,3}\[UndirectedEdge]{3,3},{1,4}\[UndirectedEdge]{4,4},{2,5}\[UndirectedEdge]{3,5},{1,6}\[UndirectedEdge]{2,6},{3,7}\[UndirectedEdge]{4,7},{2,8}\[UndirectedEdge]{4,8},{1,9}\[UndirectedEdge]{4,9},{1,10}\[UndirectedEdge]{3,10},{2,11}\[UndirectedEdge]{3,11},{1,12}\[UndirectedEdge]{2,12},{3,13}\[UndirectedEdge]{4,13},{2,14}\[UndirectedEdge]{4,14},{1,15}\[UndirectedEdge
```

```
]{4,15},{2,16}\[UndirectedEdge]{3,16}};
g=Graph[vertexList, edgeList, VertexCoordinates->(#->Reverse[#*{-1,1}]&/@vertexList)]
x={{1,0},{2,0},{3,0},{4,0}};
y={{1,17},{2,17},{3,17},{4,17}};


(* ::Input:: Initialization :: *)
(* We compute the list of X-Y paths *)
pathList=Flatten[Table[#&/@FindPath[g,vx,vy,Infinity,All],{vx,x},{vy,y}],2]


(* ::Input:: Initialization :: *)
(* We test whether there exist two non-adjacent X-Y paths by testing for each path found above whether
   removing the vertices in this path and their neighbours from the graph leaves the remaining vertices
   of X and Y in the same connected component *)

removeVertices[g_,vList_]:=Subgraph[g,Complement[VertexList[g],vList]] (* Given a graph g and a list of
   vertices vList, returns the induced subgraph of g obtained by removing the vertices in vList *)

jointNeighbourhood[g_,vList_]:=DeleteDuplicates[Join[vList,Flatten[AdjacencyList[g,#]&/@vList,1]]]
(* Given a graph g and a list of vertices vList, returns the list of vertices which appear either in
   vList or are adjacent to some vertex in vList *)

deleteJointNeighbourhood[g_,path_]:=removeVertices[g,jointNeighbourhood[g,path]]
(* Given a graph g and a list of vertices vList, returns the induced subgraph of g obtained by removing
   the vertices in or adjacent to vList *)

connectedVertices[g_,{v1_,v2_}]:=v1==v2 ||FindPath[g,v1,v2]!={}
(* Given a graph g and vertices v1 and v2, returns True if and only if there exists at least one path
   between v1 and v2 *)

connectedSets[g_,vList1_,vList2_]:=AnyTrue[Tuples[{Intersection[vList1,VertexList[g]], Intersection[
   vList2, VertexList[g]]}], connectedVertices[g,#]&]
```

```
(* Given a graph g and lists of vertices vList1 and vList2 (not necessarily all in g, we will restrict
    to those in g), returns True if and only if there exists at least one path with one end in vList1
    and one end in vList2 *)

AnyTrue[pathList,connectedSets[deleteJointNeighbourhood[g,#],x,y]&]
(* Main computation, returns True if and only if a pair of non-adjacent X-Y paths exists in our graph *)
```

Part IV

New bounds in graph searching

Bridging text 3

In Chapters 2 and 3, we worked on constructing minors in graphs with high minimum and average degrees, respectively, in order to work towards Hadwiger's conjecture. In Chapter 4, motivated by a conjecture related to fat minors, we showed some versions of Menger's theorem in which the obtained paths are induced, one of which concerns the related concept of topological minors.

In the following chapter, we will once again work on graphs with a forbidden minor. However, we will consider a different graph invariant: rather than work with the chromatic number (or degree) as in Chapters 2 and 3, we will show bounds on the cop number of such graphs.

Andreae [7] was the first to derive such bounds. We will generalize Andreae's methods to prove stronger bounds for many graphs. One of our main motivations is to find a good bound on the cop number of linklessly embeddable graphs, which we have mentioned in the introduction.

We recall that Andreae has shown that both $K_{3,3}$-minor-free graphs and K_5-minor-free graphs have cop number at most 3, strenghthening Aigner and Fromme's [1] result for planar graphs. However, this required a separate proof from Theorem 1.4.4 (although the general framework was similar). In our case, our more general result implies the $K_{3,3}$ bound, without requiring a separate proof.

Similarly to some the proofs in the introduction and in Chapter 2, our proof will be formulated in the language of models to construct the desired minor. However, rather than using connectivity tools to connect the bags with paths, we will use use geodesic paths obtained using Lemma 1.4.3.

5

Improved bounds on the cop number when forbidding a minor

Franklin Kenter[1], Erin Meger[2], Jérémie Turcotte[3]

Andreae (1986) proved that the cop number of connected H-minor-free graphs is bounded for every graph H. In particular, the cop number is at most $|E(H-h)|$ if $H-h$ contains no isolated vertex, where $h \in V(H)$. The main result of this paper is an improvement on this bound, which is most significant when H is small or sparse, for instance when $H-h$

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can be obtained from another graph by multiple edge subdivisions. Some consequences of this result are improvements on the upper bound for the cop number of $K_{3,t}$-minor-free graphs, $K_{2,t}$-minor-free graphs and linklessly embeddable graphs.

5.1 Introduction

The game of cops and robbers is a combinatorial game played on a graph G [D11, 83, 1]. One player plays as the cops and the other plays as the robber. The objective for the cops is to capture the robber by occupying the same vertex as the robber with one or more cops; the objective for the robber is to evade capture forever. At the start of the game, the cops player places m cops on (not necessarily distinct) vertices of the graph, then the other player places the robber on a vertex. Thereafter, starting with the cops player, the players alternate moving any of their pieces (cops or robber) to an adjacent vertex. While there are many variants of the game, we focus on the classical version of the game where a player may decline to move any (or all) of their pieces on their turn. For a graph G, the *cop number*, denoted by $c(G)$, is the minimum number of cops sufficient for the cops player to have a winning strategy [D1].

This game has a rich history in topological graph theory as surveyed in [D4]. A classical result by Aigner and Fromme is that for any connected planar graph G, the cop number is at most 3 [D1]. A long series of results have also established a relationship between the genus of the graph, g, and the cop number. Quilliot's [D12] bound is that $c(G) \leq 2g+3$; this has since been improved to $c(G) \leq \lfloor \frac{3}{2}g + 3 \rfloor$ by Schröder [D16] and subsequently to $c(G) \leq \frac{4}{3}g + \frac{10}{3}$ by Bowler, Erde, Lehner and Pitz [D5]. Lehner [D10] very recently showed that for any connected toroidal graph $c(G) \leq 3$, solving a question of Andreae [D2]. Originally, Schröder [D16] conjectured that $c(G) \leq g + 3$; however, more daringly, Bonato and Mohar [D4] conjectured that, in fact, $c(G) \leq g^{1/2+o_g(1)}$.

An *edge contraction* is an operation by which we obtain from G a new graph G' by identifying the two end vertices of an edge and removing resulting loops and multiple edges.

We say H is a *minor* of G if a graph isomorphic to H can be obtained from G by removing vertices and edges and by contracting edges. Given a family of graphs $\mathcal{H} = \{H_i\}_{i\in I}$, we say G is *$\mathcal{H}$-minor-free* if H_i is not a minor of G for every $i \in I$. If $\mathcal{H} = \{H\}$, we simply write that G is H-minor-free. Many topological classes of graphs can be defined using forbidden minors. Most notably, Wagner [D19] proved that planar graphs are exactly the $\{K_5, K_{3,3}\}$-minor-free-graphs (where K_t is the complete graph on t vertices and $K_{s,t}$ is the complete bipartite graph with parts of size respectively s and t). More generally, Robertson and Seymour [D13] famously proved that for any minor-closed family of graphs $\mathcal{F}$ there exists a finite set of graphs $\mathcal{H}$ such that $\mathcal{F}$ are exactly the $\mathcal{H}$-minor-free graphs.

Shortly after Aigner and Fromme's result on the cop number of planar graphs, Andreae [D2] studied the cop number of graphs with a forbidden minor. In particular, Andreae proved the following theorem.

Theorem 5.1.1. *[D2] Let H be a graph and $h \in V(H)$ be a vertex such that $H - h$ has no isolated vertex. If G is a connected H-minor-free graph, then $c(G) \leq |E(H - h)|$.*

Andreae also proves, using similar but more specific methods, that connected $K_{3,3}$-minor-free graphs and K_5-minor-free graphs have cop number at most 3, strengthening Aigner and Fromme's result, as well as showing that connected $K_{3,3}^-$-minor-free graphs and K_5^--minor-free graphs (here H^- designates the graph H with one edge removed) have cop number at most 2, and that when forbidding the $(t + 1)$-vertex wheel graph W_t as a minor the cop number is at most $\lceil \frac{t}{3} \rceil + 1$.

Joret, Kaminsky and Theis [D9], inspired by Andreae's paper, have considered the cop number when forbidding a subgraph or an induced subgraph, and when bounding the treewidth of the graph. The question of bounding the cop number of graphs with one or multiple forbidden induced subgraphs has gained traction in recent years, see for instance [D17, 30, 67] and references therein. However, there have been no improvements to the upper bounds on the cop number of graphs with an excluded minor since Andreae's paper.

Our main result generalizes and improves the bounds in Theorem 5.1.1. As it requires some technical definitions, we postpone its statement to Section 5.4. The improved bounds are most significant when H is sparse, for instance graphs when $H - h$ is a subdivision of a much smaller graph, or is small. In particular, we show a few notable applications of our main result:

- We show that the cop number of linklessly embeddable graphs is at most 6; previously the upper bound was 9.
- We show that our main result encompasses Theorem 5.1.1, as well as many of Andreae's more specific results mentioned earlier.
- We improve the known upper bounds on the cop number of $K_{3,t}$-minor-free graphs and $K_{2,t}$-minor-free graphs by a factor of 2.
- We provide an example where our method improves the cop number by a factor of 4.

In Section 5.2, we define the notation we will be using throughout this paper. In Section 5.3, we recall the classical path guarding strategy for cops and deduce a more convenient form for our use. In Section 5.4, we will state and prove our main result. Finally, in Section 5.5, we derive the corollaries of our main result mentioned above.

5.2 Notation

We begin with some notation, which is mostly standard. For $n \in \mathbb{N}$, we write $[n] = \{1, \dots, n\}$. If A is a set, we will use the notation $\binom{A}{2} = \{\{u, v\} : u, v \in A, u \neq v\}$ for the set of unordered pairs of distinct elements of A; in general we will write uv or vu to represent the pair $\{u, v\}$. If $f : X \to Y$ and $A \subseteq X$, then $f(A) = \{f(a) : a \in A\}$ is the image of A. If $B \subseteq Y$, then $f^{-1}(B) = \{a \in A : f(a) \in B\}$ is the pre-image of B.

Let G be a graph, which we always consider to be simple and finite. We denote by $V(G)$ the set of vertices of G and by $E(G) \subseteq \binom{V(G)}{2}$ the set of edges of G. If $v \in V(G)$, we write $N(v)$ for the *neighbourhood* of v and $N[v] = N(v) \cup \{v\}$ for the *closed neighbourhood* of v.

Given a set $S \subseteq V(G)$, we write $N[S] = \bigcup_{v \in S} N[v]$ for the *closed neighbourhood* of S, and we define the *coboundary* of S as the set $N(S) = N[S] \setminus S$, i.e. the vertices adjacent to but not in S (note that $N(S)$ is not $\bigcup_{v \in S} N(v)$, as it is often defined).

If $S \subseteq V(G)$, we write $G[S]$ for the subgraph of G induced by S. We also write $G - S$ for $G[V(G) \setminus S]$. If $x \in V(G)$, we use the shorthand $G - x$ for $G - \{x\}$. If $A \subseteq \binom{V(G)}{2}$, then we write $G - A$ for the graph on vertex set $V(G)$ with edge set $E(G) \setminus A$. If $e \in E(G)$, then we write $G - e$ for $G - \{e\}$.

A *matching* in G is a set of edges which pairwise do not share vertices.

We denote by $U(G)$ the graph obtained from G by adding an universal vertex (a vertex adjacent to all other vertices). Given graphs G_1, G_2, we write $G_1 + G_2$ for the graph obtained as the disjoint union of G_1 and G_2. Analogously, mG is the graph obtained as the disjoint union of m copies of G.

We use the convention that the length of a path or a cycle is the number of edges it contains. For non-empty paths this is the number of vertices minus 1; for cycles this is exactly the number of vertices. A path may have length 0, whereas a cycle necessarily has length at least 3. When a $u - v$ path has length 1, we will often write it simply as the edge uv to simplify notation. The *end vertices* of a path are its first and last vertices (given some arbitrary orientation of the path). In general, our cycles will have a clearly defined *root* vertex; we will say the *end vertices* of the cycle to be this root. We say the *interior* of a path or a cycle is its set of vertices except for the end vertices. Given a non-empty $x - y$ path (or x-rooted cycle, if $x = y$) P and $z \in V(P)$, write $P[x, z]$ for the subpath of P with ends x and z, and $P[z, y]$ analogously. If P_1 and P_2 are internally-disjoint, respectively $x - y$ and $y - z$ paths, then we write $P_1 \oplus P_2$ for their concatenation; this is also a path except when $x = z$ in which case it is an x-rooted cycle. For convenience, we write $\emptyset$ for the empty path.

Given an $x - y$ path P, we say $\{X_i\}_{i \in I}$, a family of subsets of vertices of P, is *non-intertwined* if there do not exist distinct $i, j \in I$ and three distinct vertices $a_1, a_2 \in X_i$,

$b \in X_j$ such that these vertices appear in the order $a_1 - b - a_2$ in P (perhaps with additional vertices in between). If P is an x-rooted cycle such that the neighbours of x in the cycle are y, z, then we say $\{X_i\}_{i \in I}$, a family of subsets of vertices of P, is non-intertwined if it is is a non-intertwined family of subsets of the path $P - xy$ or of the path $P - xz$.

5.3 Guarding paths

Let G be a connected graph, and $\mathcal{C}$ be the set of cops playing the game of cops and robbers. We say a cop $C \in \mathcal{C}$ *guards* a subset $S \subseteq V(G)$ if its strategy guarantees that if the robber enters S it is immediately captured by C. For now, we require that this strategy be independent of the strategy of the other cops, that is the strategy still works even if the other cops change their strategies. Let us note that if C guards S, then it also guards any $S' \subseteq S$; we are not claiming that S is an exhaustive list of all vertices this cop is blocking the robber from entering. In general, if one vertex is guarded by multiple cops, we will select one of the cops to guard this vertex. In particular, if all the vertices guarded by C are also guarded by other cops, this cop can be given a new strategy as it is currently useless.

The following result of Aigner and Fromme, originally used to prove that planar graphs have cop number at most 3, is one of the main tools in the study of the game of cops and robbers.

Theorem 5.3.1. *[D1] If G is a connected graph, $u, v \in V(G)$ and P is a shortest $u-v$ path, then there exists a strategy for one cop to, after a finite number of turns, guard P.*

Andreae noticed that the proof of Theorem 5.3.1 gives us the following stronger result, which we have reformulated for convenience.

Theorem 5.3.2. *[D2] If G is a connected graph, $u, v \in V(G)$, P is a shortest $u - v$ path and C is a cop currently on u, then there exists a strategy for C to keep guarding u and, after a finite number of turns, also guard P.*

Theorem 5.3.2 is indeed stronger than Theorem 5.3.1, since if a cop C has no current strategy, then one can still use C to guard a new path by first sending the cop to u in a finite number of turns and then applying the strategy given by Theorem 5.3.2. The importance of this result is that sometimes a cop is already busy guarding a vertex and it is important that it keeps guarding it whilst it prepares to guard the path.

Andreae's proof of Theorem 5.1.1 uses this path guarding strategy repeatedly to incrementally reduce the *robber's territory*, i.e. the vertices the robber can reach without being caught by a cop, while gradually constructing an instance of the forbidden minor as the game is being played. The fact that the minor is forbidden implies that the game will eventually end, with the robber being caught. This is also our approach. With the objective of being as rigorous as possible, we will use the following corollary, which is implicit in [D2] (see, in particular, Section 2). This formulation will clarify the dependencies between the strategies of the cops, since Theorem 5.3.2 is usually applied not to G but to a subgraph of G (roughly speaking, to the subgraph induced by the current robber's territory). The strategy we get thus only holds as long as the robber is guaranteed to not leave this subgraph.

Corollary 5.3.3. *Let G be a connected graph, let $R \subseteq V(G)$ such that $G[R]$ is connected and the robber is in R, and let $u \in N[R]$ and $v \in N(R)$ be distinct vertices such that v has at least one neighbour in $R \setminus \{u\}$.*

If a cop C is currently on u, then there exists a $u-v$ path P of length at least two with all internal vertices of P in R and a strategy for C to keep guarding u and, after a finite number of turns, also guard P, under the conditions that the robber never moves to $N(R) \setminus \{u, v\}$, and that the robber does not move to v before C guards P.

Proof. Consider the graph $G' = G[R \cup \{u, v\}] - uv$. Let P be a shortest $u - v$ path in G'; such a path exists given that v has a least one neighbour in R which is not u. By choice of G', the only vertices of P which are possibly not in R are u, v, and so all internal vertices of P are in R. Given that $uv \notin E(G')$, P must contain at least one vertex other than u, v, and so P must have length at least two.

Apply Theorem 5.3.2 to get a strategy for C to guard u and, after a finite number of turns, also guard P, if the game is played on G'. We claim this strategy is also a valid strategy when playing on G. Of course, given that G' is a subgraph of G, every move of the cop remains valid. We need to show that the robber cannot, without being caught, make any move on G which it could not have done on G'. Suppose to the contrary that we are at the first turn where the robber makes such a move. The first type of illegal move is using the edge uv, which implies the robber was on u or v at the previous turn. The first case is impossible, given that u was guarded by C, and the latter is impossible because either the robber was supposed to not move to v (if the cop was not yet guarding P) or C was guarding v (if the cop was guarding P). The other type of illegal move is moving outside $R \cup \{u, v\}$. By hypothesis, the robber can only leave R through either u or v, which is impossible by the same argument as above. □

5.4 Main result

In this section, we state and prove the main result of this paper, which is an upper bound on the cop number for graphs G with some forbidden minor H. By and large, the proof optimizes and greatly extends the techniques used in the proof of Theorem 5.1.1, with very technical modifications. We summarize these key elements following the proof.

The statement of the result requires the two following definitions.

Definition 5.4.1. Given a graph H, we say that the tuple $\mathcal{H} = (h, W, \mathcal{P}, M, f)$ is a *decomposition* of H, where

(a) $h \in V(H)$,

(b) $\emptyset \neq W \subseteq V(H - h)$,

(c) $\mathcal{P}$ is a collection of distinct pairwise internally vertex-disjoint paths and (rooted) cycles with end vertices in W such that every edge of $H - h$ is contained in some $P \in \mathcal{P}$,

(d) $M \subseteq \mathcal{P}$ is a collection of paths of length 1 which forms a matching of $H - h$, and

(e) $f : W \to \mathcal{P} \setminus M$ is such that u is an end of $f(u)$ for every $u \in W$.

Figure 5.4.1(a) gives an example of a decomposition.

Conditions (b) and (c) can be seen as stating that the graph $H - h$ is a subdivision of a multigraph (with loops allowed) on the vertex set W. Note that in condition (d) M does not need to be a perfect matching, and may be empty. Given that a path or (rooted) cycle $P \in \mathcal{P}$ may only have at most 2 vertices, condition (e) implies that $|f^{-1}(P)| \in \{0, 1, 2\}$.

Intuitively, a decomposition of H is, after choosing some vertex $h \in H$, a way of representing $H - h$ around a "core" set of vertices W, between which there are paths (those in $\mathcal{P}$). We will use this decomposition as the blueprint when we attempt to construct H as a minor of the graph G on which the game is played.

To further motivate this definition, we broadly outline the idea behind the proof. We will progressively construct a minor of H inside G, using the properties of the game to show that we can add every vertex and edge of H to our partial minor. The robber's territory, that is the region in which the robber will be confined to, will correspond (will be contracted to) to the vertex h. The cops' territory, that is the region guarded by the cops, will consist of bags (denoted A_w, for every $w \in W$) and paths between these bags (denoted Q_P, for $P \in \mathcal{P}$), with the property that if P is a path in H between w_1 and w_2, then Q_P will be a path in G between A_{w_1} and A_{w_2}. If we can ensure that every A_w is non-empty (and has a neighbour in the robber's territory) and that every $P \in \mathcal{P}$ has a corresponding (and sufficiently long) path Q_P in G, we will have obtained a minor of H in G. Broadly speaking, the paths Q_P in the cop territory will completely contain every vertex which is outside of but adjacent to the robber's territory, and a cop will always guard every such vertex, ensuring that the robber is confined to its territory. Indeed, for every $P \in \mathcal{P}$, a group of cops $\mathcal{C}_P$ will be assigned to guard the path Q_P. If, for example, a path P between w_1 and w_2 does not yet have a corresponding path Q_P in our model, one of those cops will, using the results of the previous section, start protecting a path path between A_{w_1} and A_{w_2} going through the robber's territory, which will thus reduce it. In some specific cases, we will be able to add a path to the model without requiring any cops to protect it: these are edges of M. The ends of this path will be guarded by cops assigned to other paths of the model; this is why we require M to be a matching. Furthermore, as we have noted above, we want every bag A_w to have a least one neighbour in the robber's territory. In our proof, we will in fact be able to guarantee that only one vertex of A_w will be adjacent to the robber's territory. As this vertex only needs to be guarded by one cop, the role of the function f is to indicate the group of cops (the group assigned to $Q_{f(w)}$) which will be responsible for guarding this vertex. We will formally define this partially constructed minor in the context of the game as a *state*, an example of which is shown in Figure 5.4.1 below.

We may then define the following parameter for each path of $\mathcal{P}$. It will always be approximately be the length of the path, but takes into account these technicalities: we only need to know the length of the part of the path for which the corresponding vertices in G needs to be guarded by cops.

Definition 5.4.2. Given a decomposition $\mathcal{H} = (h, W, \mathcal{P}, M, f)$ of a graph H, we define for each path $P \in \mathcal{P}$ the following parameter:

$$\ell_P = \begin{cases} 0 & P \in M \\ \max(|E(P)| - 1 + |f^{-1}(P)|, 1) & P \notin M. \end{cases}$$

We are now ready to state our main result.

Theorem 5.4.3. *If $\mathcal{H}$ is a decomposition of a graph H and G is a connected H-minor-free graph, then*

$$c(G) \leq \mathbb{1}_\ell + \sum_{P \in \mathcal{P}} \left\lceil \frac{\ell_P}{3} \right\rceil,$$

where the indicator function $\mathbb{1}_\ell$ is equal to 1 if and only if there is some $P \in \mathcal{P}$ with $\ell_P \notin \{0, 1, 2, 4\}$.

In this theorem, we do not impose any conditions on which decomposition is picked. As can be seen in Definition 5.4.1, a graph H may have multiple possible decompositions. When using this theorem, the best bound will be obtained by choosing an optimal decomposition of H, roughly speaking by choosing a decomposition that yields the smallest possible sum of ℓ_P. Note that the minor relation is transitive, and so if H is a minor of H', then if a graph is H-minor-free, it is also H'-minor-free. Hence, one might obtain a better upper bound by applying Theorem 5.4.3 to H' instead of H. We make such an application of Theorem 5.4.3 in the proof of Corollary 5.5.5. This is also useful when H does not have any decomposition, for instance if $H - h$ contains an isolated vertex. Further discussion on applications of this result is provided in Section 5.5.

The structure of the proof is as follows:

- Set up terminology surrounding both G and the $\mathcal{H}$ decomposition, using precisely the number of cops given by the upper bound.
- Define a game state to detail the particular relationship between G and the forbidden minor H through its decomposition. This state relates paths in G to paths in H, using the terminology of *initialized* to indicate that the feature will later be used to build the minor. Furthermore, we will say that the cops are *active* when they have been assigned a particular strategy and are actively guarding vertices of G.
- Define a partial order on the game states. The robber will be captured when the robber's territory decreases to zero, which can be achieved by taking smaller and smaller game states.
- Assume that we are in some minimal game state, where the robber's territory is non-zero for the sake of contradiction. That is, we assume that our current bound is insufficient for the cops to win.
- Explore the game state to show that certain features of the decomposition must be present in G, by assuming their absence and finding a smaller game state, thus contradicting the minimality of the game state.

- Finally, we will show that with all features present, we must in fact have an H minor of G, which is forbidden, contradicting the assumption that the game is a minimal game state with non-empty robber's territory.

We now prove Theorem 5.4.3.

Proof. Let G be a connected H-minor-free graph, and $\mathcal{H} = (h, W, \mathcal{P}, M, f)$ be a decomposition of H. We will play the game of cops and robbers with $\mathbb{1}_\ell + \sum_{P \in \mathcal{P}} \left\lceil \frac{\ell_P}{3} \right\rceil$ cops. Let $\mathcal{C}$ be the set of all cops.

For each $P \in \mathcal{P}$, we define $\mathcal{C}_P$ to be a set of $\left\lceil \frac{\ell_P}{3} \right\rceil$ cops, such that for all pairs of distinct paths $P_1, P_2 \in \mathcal{P}$, $\mathcal{C}_{P_1} \cap \mathcal{C}_{P_2} = \emptyset$. Since $\ell_P = 0$ if and only if $P \in M$, it follows that $\mathcal{C}_P = \emptyset$ if and only if $P \in M$. In particular, for every vertex $w \in W$, $\mathcal{C}_{f(w)} \neq \emptyset$, since the function f maps only to non-matching paths in $\mathcal{P}$. If the indicator function $\mathbb{1}_\ell = 1$ (if some $P \in \mathcal{P}$ is such that $\ell_P \notin \{0, 1, 2, 4\}$), we then define C_ℓ be an additional, distinct cop. Note that the set $\mathcal{C}_P$ to which each cop belongs, may change throughout the proof as the cops "switch roles" but they will always do so in a way that leaves the sizes of these sets unchanged.

In order to show that the cops have a winning strategy, we next need to define the concept of a game state. We will then define a partial order between game states. We will call the process of going from one game state to another smaller game state a *transition*.

Definition. We say the game is in *state* $(\mathcal{A}, \mathcal{Q}, R, s)$ if all of the following hold.

(1) $\mathcal{A} = (A_w)_{w \in W}$ is a collection of pairwise disjoint subsets of $V(G)$ (which we will call *bags*) such that for every $w \in W$, $G[A_w]$ is either connected or contains no vertices. We say w is *initialized* if $A_w \neq \emptyset$.

(2) $\mathcal{Q} = (Q_P)_{P \in \mathcal{P}}$ is a collection of pairwise internally vertex-disjoint paths such that if P has end vertices u, v, then either Q_P is empty or Q_P has end vertices respectively in A_u and A_v, and such that internal vertices are not in any of the sets in $\mathcal{A}$. If P is not a path but a rooted cycle, the end vertices of Q_P are allowed (but not obliged) to be

the same (i.e. Q_P is allowed to be a rooted cycle). We say $P \in \mathcal{P}$ is *initialized* if Q_P is not empty.

(3) If $u, v \in W$ are initialized and $uv \in M$, then uv is initialized.

(4) R is the set of vertices of the connected component of $G - \left(\bigcup_{w \in W} A_w \cup \bigcup_{P \in \mathcal{P}} V(Q_P)\right)$ containing the robber.

(5) For each $w \in W$, A_w contains at most one vertex adjacent to R.

(6) $s : \mathcal{C} \to 2^{V(G) \setminus R}$ is a function such that $C \in \mathcal{C}$ is following a strategy to guard the vertices in $s(C)$ which is irrespective of the strategy of the other cops, but holds only as long as the robber does not leave R by moving to a vertex in $N(R) \setminus s(C)$. A cop is said to be *active* if $s(C) \neq \emptyset$. Inactive cops C may follow any strategy.

(7) Every vertex in the coboundary of R is in one of the images of s, i.e. $N(R) \subseteq \bigcup_{C \in \mathcal{C}} s(C)$.

(8) If $P \in \mathcal{P}$ is initialized, then $s(\mathcal{C}_P)$ is a non-intertwined family of subsets of $V(Q_P)$.

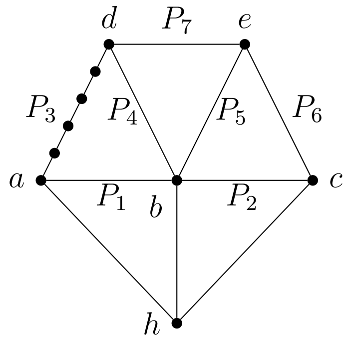
(9) If $P \in \mathcal{P}$ is uninitialized, with end vertices u, v, and $C \in \mathcal{C}_P$, then $s(C)$ is either empty or contains a unique vertex, in A_u or A_v.

(10) If A_w contains a vertex x adjacent to R, then $x \in s(C)$ for some $C \in \mathcal{C}_{f(w)}$.

(11) If $\mathbb{1}_\ell = 1$, the extra cop C_ℓ is inactive.

Furthermore, we will use the notations $|\mathcal{A}| = |\{w \in \mathcal{W} : A_w \neq \emptyset\}|$ and $|\mathcal{Q}| = |\{P \in \mathcal{P} : Q_P \neq \emptyset\}|$ to denote respectively the number of initialized vertices of W and the number of initialized paths of P. When helpful, we will call the graph $G\left[\bigcup_{w \in W} A_w \cup \bigcup_{P \in \mathcal{P}} V(Q_P)\right]$ the *model* since it is a partial construction of a graph which could be contracted into $H - h$.

A visualization of an example of a game state is provided in Figure 5.4.1.



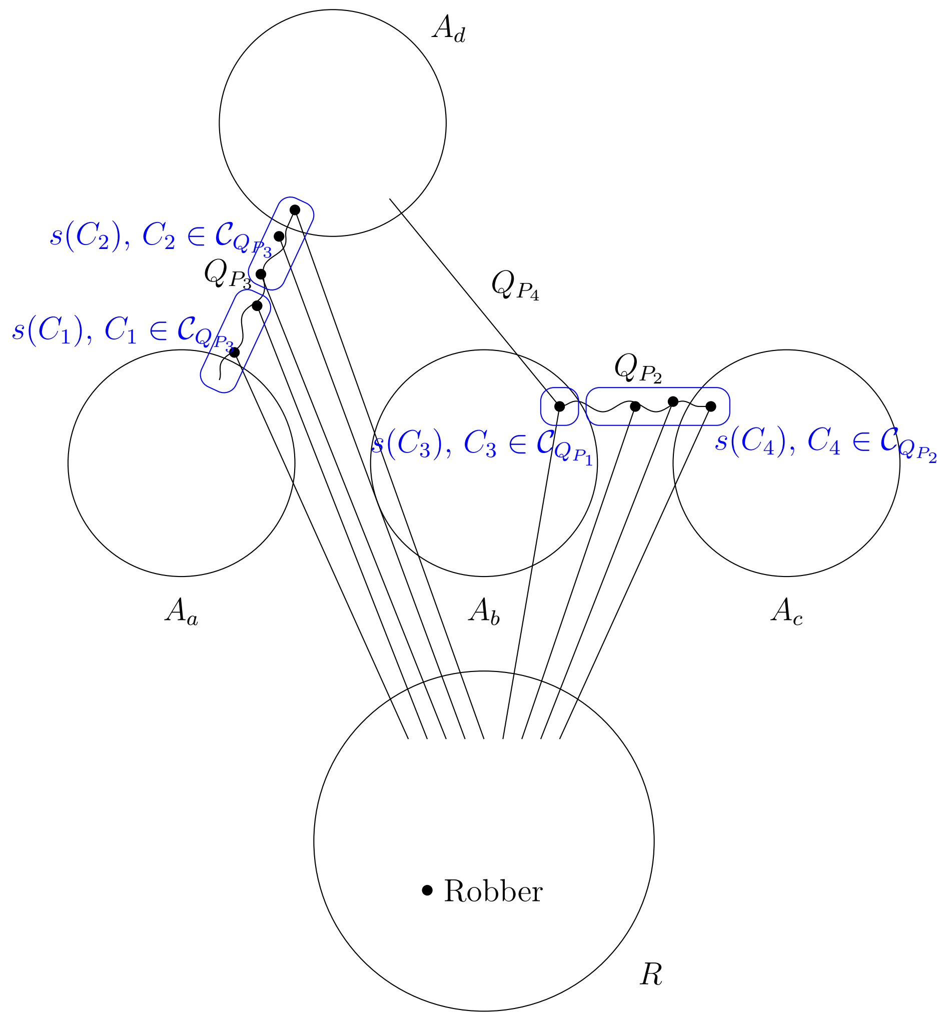
$W = \{a, b, c, d, e\}$

$\mathcal{P} = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7\}$

$M = \{P_4, P_6\}$

$f(a) = P_3, f(b) = P_1, f(c) = P_2$
$f(d) = P_3, f(e) = P_5$

(a) Decomposition $(h, W, \mathcal{P}, M, f)$ of a graph H.



(b) State $(\mathcal{A}, \mathcal{Q}, R, s)$. We notice that e is the only uninitialized vertex of W, and that the uninitialized paths are P_1, P_5, P_6 and P_7. As b and d are initialized, it was obligatory for P_4 to be initialized. Further notice that despite P_1 not being initialized, one of the cops of $\mathcal{C}_{Q_{P_1}}$ is active, and is protecting a vertex in A_b by sitting on it. Finally, note that every vertex adjacent to R that is also in one of the bags A_w is protected by a cop in $\mathcal{C}_{Q_{f(w)}}$. As $P_4 \in M$, there no cops assigned to protect it, so it cannot contain in its interior any vertices adjacent to R.

Figure 5.4.1: Example of a decomposition of a graph H and of a state of a game played on an H-minor-free graph G.

Note that condition (11) does not state that the extra cop is never used. It simply implies that when the game in a specific state, it not used. This cop will however be used when transitioning from one state to another, as we will see below.

Let us also note that once the game is in a state, it may remain in this state as long as the cops' strategies do not change. Indeed, the robber is in R by (4), and cannot leave R due to (7), as long as the cops maintain their current strategies, which is possible, as specified in (6), as long as the robber does not leave R. In general, note that every cop $C \in \mathcal{C}$ may change its actual strategy, as long as the vertices it guards are the same, since the strategies of the other cops do not depend on it.

Definition. We can define a partial order on states by setting $(\mathcal{A}', \mathcal{Q}', R', s') < (\mathcal{A}, \mathcal{Q}, R, s)$ if

(i) $R' \subsetneq R$ (the robber's territory is decreased),

(ii) $R' = R$ and $\sum_{C \in \mathcal{C}} |s(C)| > \sum_{C \in \mathcal{C}} |s'(C)|$ (the number of guarded vertices, with multiplicity, decreases),

(iii) $R' = R$, $\sum_{C \in \mathcal{C}} |s(C)| = \sum_{C \in \mathcal{C}} |s'(C)|$ and $|\mathcal{A}'| + |\mathcal{Q}'| < |\mathcal{A}| + |\mathcal{Q}|$ (the number of pieces of the model decreases), or

(iv) $R' = R$, $\sum_{C \in \mathcal{C}} |s(C)| = \sum_{C \in \mathcal{C}} |s'(C)|$, $|\mathcal{A}'| + |\mathcal{Q}'| = |\mathcal{A}| + |\mathcal{Q}|$ and $\sum_{w \in W} |A'_w| > \sum_{w \in W} |A_w|$ (the total size of the bags increases).

It is easy to see that this defines a well-founded relation on the set of states, in particular, using that these parameters have a finite number of possible values.

If the cops change strategies changes to bring the game from one state to a smaller state, we will say the *type of the transition* is the condition (either (i), (ii), (iii) or (iv)) in the definition of the partial order by virtue of which the new state is smaller.

For brevity, in general when defining a new smaller state $(\mathcal{A}', \mathcal{Q}', R', s')$, we will only define the values A'_w ($w \in W$), Q'_P ($P \in \mathcal{P}$) and $s'(C)$ ($C \in \mathcal{C}$) which are different from $(\mathcal{A}, \mathcal{Q}, R, s)$

(in particular, all cops except those mentioned will maintain their current strategies). In general, we will also not explicitly define R' as it will always be the component of $G - \bigcup_{w \in W} A'_w \cup \bigcup_{P \in \mathcal{P}} V(Q'_P)$ containing the robber. In all instances, we will indeed have $R' \subseteq R$ since only vertices not adjacent to R will ever be removed from the model.

With the technical definitions completed, we now proceed with proving that the cops have a winning strategy. Suppose for the sake of contradiction that the robber has a strategy to escape any strategy employed by these cops.

In order to find a contradiction, we first place the cops arbitrarily on G and assign them no strategy. Then,

$$((\emptyset)_{w \in W}, (\emptyset)_{P \in \mathcal{P}}, V(G), s(\cdot) = \emptyset)$$

is a valid state since all of the conditions hold trivially. Then, the cops will follow a strategy to minimize the game state. Given the defined partial order, it must be the case that after some finite amount of time the game is in some minimal state $(\mathcal{A}, \mathcal{Q}, R, s)$. We will investigate this game state and construct an H-minor if the robber territory is non-empty (i.e. if the game has not been won).

To avoid repetition, we first explain more precisely why, with the strategies we will use, (6) continues to hold for every cop C during a transition from $(\mathcal{A}, \mathcal{Q}, R, s)$ to $(\mathcal{A}', \mathcal{Q}', R', s')$. There are essentially two kinds of strategy changes we will use, which we summarize here:

Case I: The first kind to consider is that, as noted above, the cop C maintains its current strategy (and it has maintained it during the transition between the game states) and that $s'(C) \subseteq s(C)$ (of course, we will always choose $s'(C)$ in a way that $s(C) \setminus s'(C)$ will only contain vertices which are also guarded by other cops, so (7) will still hold). To show that (6) still holds, we suppose that we know that the robber is in R' but never leaves R' by moving to a vertex in $N(R') \setminus s'(C) \supseteq N(R') \setminus s(C)$, and we must show that C guards $s'(C)$. We however know that C guards $s(C) \supseteq s'(C)$ as long as the robber does not leave R by moving to a vertex in $N(R) \setminus s(C)$, so it suffices to verify this last condition. Recall that

$R' \subseteq R$. The robber cannot leave R directly from R' since it would have to be through a vertex in $N(R') \setminus s(C)$, which is forbidden. The robber also cannot leave R by first going to a vertex in $R \setminus R'$, as this would be forbidden since $R \cap N(R') \subseteq N(R') \setminus s(C)$ since $s(C)$ contains no vertex of R.

Case II: The other kind uses Corollary 5.3.3: we get a new $x - y$ path (or x-rooted cycle if $x = y$) Q with internal vertices in R such that, after a finite of moves, a cop C will be guarding Q, under the condition that the robber does not leave R by going on a vertex of $N(R) \setminus \{x, y\}$. When using this argument, generally $V(Q)$ will be the new part of the model and we will set $s'(C) = V(Q)$, although it will be clear in the proof when this is not the case. To show that (6) holds, we need to prove that if the robber does not leave R' by moving to a vertex in $N(R') \setminus s'(C)$, the new strategy that C is following still works. Note that by the definition of R' in (4), $N(R') \subseteq N(R) \cup V(Q)$. If the robber were to leave R by going to a vertex in $N(R) \setminus \{x, y\}$, it could be directly from R' if this vertex is in $N(R') \setminus V(Q)$, which is forbidden. Otherwise, the robber would need to go first through $V(Q)$ to reach vertices in $R \setminus R'$ which is impossible given that $V(Q)$ is guarded by C.

We will now prove a series of claims about the minimal state $(\mathcal{A}, \mathcal{Q}, R, s)$ the game is currently in. Most of the proofs of these claims will be by contradiction, showing that if the claim does not hold, then the cops can, after a finite number of turns, bring the game into a smaller state.

Claim 5.4.3.1. *For every $C \in \mathcal{C}$, $s(C)$ contains only vertices adjacent to R.*

Proof. If for some $C \in \mathcal{C}$, there exists $x \in s(C)$ such that x has no neighbour in R, then C does not need to explicitly guard x, as the robber cannot reach that vertex (see (7)). Let $s'(C) = s(C) \setminus \{x\}$. Then, $(\mathcal{A}, \mathcal{Q}, R, s')$ is a new valid state, with transition of type (ii), which is a contradiction to the minimality of the current game state. $\square$

Claim 5.4.3.2. *s has disjoint images, i.e. if $C_1, C_2 \in \mathcal{C}$ are distinct cops, then $s(C_1) \cap s(C_2) = \emptyset$.*

Proof. If $x \in A_w$ for some $w \in W$, is in multiple elements of $s(\mathcal{C})$, condition (10) implies that at least one of the cops guarding x is some $C \in \mathcal{C}_{f(w)}$. Defining s' such that x is only contained in $s'(C)$, and otherwise identically to s, yields a new state $(\mathcal{A}, \mathcal{Q}, R, s')$ with transition, similarly to the previous claim, of type (ii).

If a vertex x in the interior of Q_P (for some $P \in \mathcal{P}$) is in multiple elements of $s(\mathcal{C})$, choose arbitrarily which of these cops will keep guarding x and proceed as previously. Note that (8) is maintained, the only difference is that the sets of $s(\mathcal{C}_P)$ can no longer overlap.

In both of these situations, we get a contradiction to the minimality of the current game state. $\square$

Claim 5.4.3.3. *If $P \in \mathcal{P} \setminus M$ is initialized, either the interior of Q_P contains a neighbour of R, or $P = f(u) = f(v)$ for distinct $u, v \in W$ so that both ends of Q_P, say $x \in A_u$ and $y \in A_v$ have neighbours in R.*

Proof. Suppose there exists $P \in \mathcal{P} \setminus M$ for which the statement does not hold. Let $x \in A_u, y \in A_v$ be the end vertices of Q_P (note that it is possible that $x = y$ if P is a cycle).

By Claim 5.4.3.1, only vertices adjacent to R appear in the elements of $s(\mathcal{C}_P)$, and by (8) only vertices of Q_P can appear in $s(\mathcal{C}_P)$. In particular, given that Q_P contains no internal vertex adjacent to R, only x, y can appear in $s(\mathcal{C}_P)$. Note that by (10) and Claim 5.4.3.2, the ends x, y of Q_P may only appear in $s(\mathcal{C}_P)$ if respectively $P = f(u)$, $P = f(v)$. Since we are assuming that the claim does not hold for P, this holds for at most one of x, y (unless $x = y$). Hence, either $s(C) = \emptyset$ for all $C \in \mathcal{C}_P$, or only one cop of $\mathcal{C}_P$ is active and only guards one vertex in A_u (without loss of generality). These two cases will correspond to the two possible cases in (9) of the new smaller state, which we now define. Set $Q'_P = \emptyset$. Note that R is still a component of $G - \left(\bigcup_{w \in W} A_w \cup \bigcup_{P \in \mathcal{P}} V(Q'_P)\right)$, given that no neighbour of R was in the interior of the removed path. It is then easy to verify that $(\mathcal{A}, \mathcal{Q}', R, s)$ is a new valid state with transition of type (iii), which is a contradiction to the minimality of the current game state. □

Claim 5.4.3.4. *If $C \in \mathcal{C}$ is such that $s(C) = \{x\}$, we may assume that C is guarding x by sitting on it.*

Proof. Suppose $C \in \mathcal{C}$ is such that $s(C) = \{x\}$. Of course, there are possibly many strategies that C could be using to block the robber from moving to x. For instance, C could be using some larger path guarding strategy such as in Corollary 5.3.3, or sitting on a neighbour of x until the robber enters x.

Given that C is guarding x if the robber only moves in $G[R \cup \{x\}]$, the distance (at the cops' turn) between x and C in G cannot be more than one larger than the distance between x and the robber in $G[R \cup \{x\}]$. Indeed, otherwise the robber could follow a shortest path to x and not be caught by C on the way.

Let C abandon its current strategy and move towards x via the shortest path in G until the cop reaches x, after which it will sit on x to guard it. We claim that once this is done, the state of the game will be unchanged; it suffices to show that the robber could not have escaped R through x, given that all of other cops may follow their strategies as long as the robber not leave R, as specified in (6). Given the distances between x and C and the robber discussed above, the cop will either arrive at x before the robber does or capture the robber on x.

Given there are a finite number of cops and this strategy takes at most $\text{diam}(G) \leq |V(G)|$ turns, we can apply the above strategy for every cop if needed. Hence, from now on, if a cop is only guarding one vertex, we may suppose it is sitting on that vertex.

Note that once the change of strategy is complete, the state of the game is unchanged. □

Claim 5.4.3.5. *If $w \in W$ is initialized, then A_w contains a vertex adjacent to R.*

Proof. Suppose to the contrary that there is some $w \in W$ which is initialized but such that A_w contains no vertex adjacent to R. There are a few cases to consider here.

If there are no initialized paths in $\mathcal{P} \setminus M$ incident to w, then set $A'_w = \emptyset$. If $wv \in M$ for some $v \in W$, let $Q'_{wv} = \emptyset$. The cop assignment s is still well defined. Indeed, A_w contained no vertex adjacent to R, and thus by Claim 5.4.3.1 none of its elements was guarded by a cop. In the case where $wv \in M$, no internal vertex of Q_{wv} is guarded by a cop given that $\mathcal{C}_{wv} = \emptyset$. It is easy to see that $(\mathcal{A}', \mathcal{Q}', R, s)$ defines a new valid state, this time with transition of type (iii).

Suppose now $f(w)$ is initialized. Say $f(w)$ has ends w, v and $Q_{f(w)}$ has end vertices x, y, with $x \in A_w$ and $y \in A_v$ (note that if $w = v$, it is possible that $x = y$). We know that x is not adjacent to R. By Claim 5.4.3.3, $Q_{f(w)}$ must then contain a vertex in its interior which is adjacent to R. Take z such a vertex which is as close as possible to x in $Q_{f(w)}$ (when traversing it from x to y). Let $A'_w = A_w \cup V(Q_{f(w)}[x, z])$ and $Q'_{f(w)} = Q_{f(w)}[z, y]$ (note that $Q'_{f(w)}$ is necessarily a path, even if $Q_{f(w)}$ was a cycle). As A_w contained no vertex adjacent to R and by our choice of z, A'_w still respect (5). Then, $(\mathcal{A}', \mathcal{Q}', R, s)$ will define a new valid state, here the transition being of type (iv), since the only change is that A'_w absorbed some vertices of $Q_{f(w)}$. Note that it is important for (10) that we absorbed parts of $Q_{f(w)}$ and not of any incident path, so that the vertex of A'_w be guarded by one of the cops of $\mathcal{C}_{f(w)}$.

Hence, we may suppose that $f(w)$ is uninitialized, but there exists $P \in \mathcal{P} \setminus M$ containing w which is initialized. Suppose P has end vertices w, v and $f(w)$ has end vertices w, u. Say Q_P has ends x, y (where $x \in A_w$ and $y \in A_v$). Claim 5.4.3.3 again yields that there exists z in the interior of Q_P which has a neighbour in R. Again take z as close as possible to x in Q_P. Given that all active cops are guarding at least one vertex adjacent to R by Claim 5.4.3.1 and that A_w contains no neighbour of R, either all cops of $\mathcal{C}_{f(w)}$ are inactive, or one of them is guarding a vertex $a \in A_u$ and the others are inactive (in this second case, necessarily $u \neq w$ and so $f(w)$ is not a cycle). Let $C \in \mathcal{C}_{f(w)}$ be a cop which, depending on the case above, is either inactive or on a.

In the first of these two cases, we first send the inactive cop C to sit on z to guard it. Let $A'_w = A_w \cup V(Q_P[x, z])$, $Q'_P = Q_P[z, y]$, $s'(C) = \{z\}$ and $s'(C') = s(C') \setminus \{z\}$, where

$C' \in \mathcal{C}_P$ is the cop which was guarding z previously. Then, the game is now in the new state $(\mathcal{A}', \mathcal{Q}', R, s')$; this transition has type (iv).

In the other case, using Corollary 5.3.3 (note by Claim 5.4.3.4 that C is sitting on a) there exists an $a-z$ path Q with internal vertices in R (of which there is at least one), which can, after a finite number of turns (during which C still guards a), be guarded by C. Let $A'_w = A_w \cup V(Q_P[x,z])$, $Q'_P = Q_P[z,y]$, $Q'_{f(w)} = Q$, $s'(C) = V(Q)$. The game is now in the new state $(\mathcal{A}', \mathcal{Q}', R', s')$; this transition has type (i). Note that in this case we initialized $f(w)$, which was necessary as a (or the) cop of $\mathcal{C}_{f(w)}$ was already busy guarding one vertex.

In both of these cases, it is important for (10) to hold that one of the cops of $C_{f(w)}$ guards z, which is now the unique vertex of A'_w adjacent to R'.

In all cases, we can reach a strictly smaller game state, which is a contradiction to the minimality of the current game state. $\square$

Claim 5.4.3.6. *If $w \in W$ and $C \in \mathcal{C}_{f(w)}$ is such that $s(C) = \{x\}$, where $x \in A_w$, and $f(w)$ is uninitialized, then x has at least 2 neighbours in R.*

Proof. Suppose to the contrary that x has exactly 1 neighbour a in R (by Claim 5.4.3.1, x cannot have no neighbours in R). By Claim 5.4.3.4, C is sitting on x. Move C to a. Let $A'_u = A_u \cup \{a\}$ and $s'(C) = \{a\}$. The game is now in state $(\mathcal{A}', \mathcal{Q}, R', s')$, with transition of type (i), which is a contradiction to the minimality of the current game state. Note that given that x only has a as a neighbour in R, x is not a neighbour of R', and so (5) and (7) are indeed still respected. Further note that (8) is still respected since $f(w)$ is uninitialized. $\square$

Claim 5.4.3.7. *Every $w \in W$ is initialized.*

Proof. Suppose w is uninitialized. Throughout the proof, let w, v be the ends of $f(w)$. When initializing w, recall that by (10) we must take care that a cop of $\mathcal{C}_{f(w)}$ will guard the possible vertex which is adjacent to the robber's territory. We will consider two main cases.

We first consider the case in which $wu \notin M$ for every initialized $u \in W$. Under this assumption, we can initialize w without being concerned with (3) (of course, as long as w is the only vertex we are initializing). There are two subcases to consider.

The first subcase is if all cops of $\mathcal{C}_{f(w)}$ are inactive. Let $x \in R$ be arbitrary (which exists as the game is still being played). Send some $C \in \mathcal{C}_{f(w)}$ to guard x by sitting on it. Let $A'_w = \{x\}$ and $s'(C) = \{x\}$. It is easy to verify that $(\mathcal{A}', \mathcal{Q}, R', s')$ is indeed a new valid state, with transition of type (i).

The second subcase is if not all cops of $\mathcal{C}_{f(w)}$ are inactive. Given that $A_w = \emptyset$, there can be no paths incident to A_w, and so $f(w)$ is necessarily uninitialized. By (5), (9), Claim 5.4.3.1 and Claim 5.4.3.2, there is only one active cop in $\mathcal{C}_{f(w)}$, say C, which is guarding (and sitting on, by Claim 5.4.3.4) a vertex $y \in A_v$. Let $x \in R$ be any neighbour of y, which must exist by Claim 5.4.3.1. It is easy to see that C can guard the path xy, for instance by sitting on y and moving to x if the robber goes on x. Set $A'_w = \{x\}$, $Q'_{f(w)} = xy$ and $s'(C) = \{x, y\}$. Then $(\mathcal{A}', \mathcal{Q}', R', s')$ is a new valid state, with transition of type (i).

The other case to consider is if there exists some initialized $u \in W$ such that $wu \in M$. Note that necessarily $w \neq u$ in this case. Given that M is a matching, there is only one such u, and so to ensure that (3) is respected after initializing w we only need to consider this u. Recall that by definition of f, $f(w) \notin M$ and so $f(w) \neq wu$. By Claim 5.4.3.5, there exists $y \in A_u$ such that y has at least one neighbour in R. There are once again two main subcases here.

The first subcase now is that all cops of $\mathcal{C}_{f(w)}$ are inactive, let $C \in \mathcal{C}_{f(w)}$. Let $x \in R$ be a neighbour of y. Send C to guard x by sitting on it. Let $A'_w = \{x\}$, $Q'_{wu} = xy$, and $s'(C) = \{x\}$. Then $(\mathcal{A}', \mathcal{Q}', R', s')$ is indeed a new valid state, with transition of type (i).

The other subcase is that there exists $C \in \mathcal{C}_{f(w)}$ which is active. Given that A_w is uninitialized, $f(w)$ is also uninitialized. As earlier, the cop C must then be guarding (by sitting on) a vertex $x \in A_v$ adjacent to R (and so $w \neq v$), and by Claim 5.4.3.2, any other cop in $\mathcal{C}_{f(w)}$ is inactive.

First suppose $u \neq v$. Apply Corollary 5.3.3 to get a $x-y$ path Q through R of length at least 2, such that C can guard Q after a finite number of turns. Let z be the penultimate vertex of Q, i.e. z is the vertex of Q adjacent to y. Set $A'_w = \{z\}$, $Q'_{wu} = zy$, $Q'_{f(w)} = Q[x, z]$ and $s(C) = V(Q) \setminus \{y\}$. Then $(\mathcal{A}', \mathcal{Q}', R', s')$ is a new valid state, with transition of type (i).

Now suppose $u = v$. By (5) we have that $x = y$, so we know that y is already guarded by C. Given that $C \in \mathcal{C}_{f(w)}$ but is sitting on a vertex of A_u, by (10) and Claim 5.4.3.2 we have that necessarily $f(w) = f(u)$, which we recall is uninitialized. By Claim 5.4.3.6, y has at least two neighbours in R, let a be such a such a neighbour. In particular, $|f^{-1}(f(w))| = 2$. Also, given that $wu \in \mathcal{P}$ already, $f(w)$ cannot be an edge (since H is not a multigraph), and thus has length at least two. Hence, $\ell_{f(w)} \geq 3$. If $\ell_{f(w)} = 3$, then $\mathbb{1}_\ell = 1$ and the extra cop C_ℓ is present in the game, so let $C' = C_\ell$. If $\ell_{f(w)} \geq 4$, then $|\mathcal{C}_{f(w)}| \geq 2$ and so let $C' \in \mathcal{C}_{f(w)}$ which is distinct from C (in particular, C' is inactive). First move C' to a. Then, apply Corollary 5.3.3 to get an $a-y$ path Q of length at least two (in particular, not using the edge ay) with internal vertices in R which can be guarded by C' after a finite number of turns. Note that is important when applying Corollary 5.3.3 that C keeps guarding y while C' goes to guard Q, as otherwise the robber could escape R through y. Only once C' is guarding Q can C stop guarding y. Let $A'_w = \{a\}$, $Q'_{wu} = ay$, $Q'_{f(w)} = Q$, $s'(C') = V(Q)$ and $s'(C) = \emptyset$. If $C' = C_\ell$, we also need to switch the labels of C and C_ℓ, i.e. C becomes the new extra cop, and C_ℓ becomes a cop of $C_{f(w)}$. The game is now in state $(\mathcal{A}', \mathcal{Q}', R', s')$, with transition of type (i).

Note that in all of these subcases, the reason no cop is required in $\mathcal{C}_{wu}$ is because Q'_{wu} is only an edge, hence contains no internal vertex adjacent to the robber's territory. The ends of this path, if they are adjacent to the robber's territory, are protected by the cops designated by f given (10).

In all cases, we can reach a strictly smaller game state, which is a contradiction to the minimality of the current game state. $\square$

Claim 5.4.3.8. *Every $P \in \mathcal{P}$ is initialized.*

Proof. Suppose to the contrary there exists some uninitialized $P \in \mathcal{P}$. If possible, choose P such that there is $u \in W$ for which $P = f(u)$. By Claim 5.4.3.7, all vertices in W are initialized. By (3), $P \notin M$ and so $\mathcal{C}_P$ is necessarily non-empty.

Suppose P has end vertices u, v. There are two main cases to consider: when $u \neq v$ and $u = v$.

First suppose that $u \neq v$. By Claim 5.4.3.5, there exists $x \in A_u$ and $y \in A_v$ adjacent to R. As in the previous claims, (9) implies that any active cop of $\mathcal{C}_P$ is either sitting on x or y, without loss of generality say it is on x. If no cop of $\mathcal{C}_P$ is active, first send one inactive cop of $\mathcal{C}_P$ to x. In both cases, there is a cop $C \in \mathcal{C}_P$ sitting on x. Using Corollary 5.3.3, there exists at least one $x - y$ path Q with internal vertices in R, which can be guarded by C after a finite number of turns (and such that during these turns, x remains guarded by C). Define $Q'_P = Q$ and $s'(C) = V(Q)$. Once C is following this new strategy, the game is now in state $(\mathcal{A}, \mathcal{Q}', R', s')$, with transition of type (i).

We now consider the case $u = v$, so P is an u-rooted cycle. By Claim 5.4.3.5, there exists $x \in A_u$ adjacent to R. There are two subcases here based on the number of neighbours of x that are in R.

First suppose x has at least two neighbours in R, let one of them be $a \in R$. By (10), we know that x is guarded by a cop $C' \in \mathcal{C}_{f(u)}$. We first want to find a cop (distinct from C') to guard a new path used to initialize P. If $P \neq f(u)$, all cops of $\mathcal{C}_P$ are necessarily inactive by (9) and Claim 5.4.3.2 given that x is already guarded by C', so let $C \in \mathcal{C}_P$. Suppose now that $P = f(u)$. Recall that we want to find an inactive cop distinct from C'. Given that a cycle has length at least 3 we have $\ell_P \geq 3$. If $\ell_P = 3$, then $\mathbb{1}_\ell = 1$ and the extra cop C_ℓ is present in the game, so let $C = C_\ell$. If $\ell_P \geq 4$, $|\mathcal{C}_P| \geq 2$ and so let $C \in \mathcal{C}_P$ which is distinct from C'. In both cases, C is inactive and thus available to take on a new strategy. Move C to a, and apply Corollary 5.3.3 to get an $a - x$ path Q of length at least two (in particular, not using the edge ax) with internal vertices in R. This path can be guarded by C after a finite number of turns. Note that it is important when applying Corollary 5.3.3 that C' keeps guarding x while C prepares to guard Q (which is why we wanted C to be distinct from C'). Set $Q'_P = xa \oplus Q$ and $s'(C) = V(Q)$. In the case where $C = C_\ell$, also set $s'(C') = \emptyset$. This only happens if, in particular, $C' \in \mathcal{C}_{f(u)}$, and so C' was necessarily sitting on x by (9). In this case, we also need to switch the labels of C' and C_ℓ, that is C' becomes the new extra cop, and C_ℓ becomes a cop of C_P. The game is now in state $(\mathcal{A}, \mathcal{Q}', R', s')$, with transition of type (i).

Second, suppose x has exactly one neighbour a in R. If $P = f(u)$, then by (10), there necessarily exists $C \in \mathcal{C}_P$ which is guarding x. By (9), $s(C) = \{x\}$. Given that $P = f(u)$ is uninitialized, this contradicts Claim 5.4.3.6, so $P \neq f(u)$. In particular, by the same argument in the previous case, no cop of $\mathcal{C}_P$ is active, so we let $C \in \mathcal{C}_P$. By our initial choice of P, $f(u)$ must be initialized. Let w be the other end of $f(u)$, and let y be the vertex of A_w adjacent to R, which exists by Claim 5.4.3.5. We next consider whether the interior of $Q_{f(u)}$ contains vertices adjacent to R, and separate this into two subsubcases.

First suppose the interior of $Q_{f(u)}$ contains no vertex adjacent to R. Note that by Claim 5.4.3.3, $f(u)$ is not a cycle, so $u \neq w$. Apply Corollary 5.3.3 to get an $x - y$ path Q which goes through R which can be guarded by C after a finite number of turns. The cops of $\mathcal{C}_{f(u)}$ guarding (the ends of) $Q_{f(u)}$ may now be relieved. Let $Q'_{f(u)} = Q$, $s'(C) = V(Q)$ and $s'(C') = \emptyset$ for every $C' \in \mathcal{C}_{f(w)}$. Note however that this would no longer respect (8), given that C is in $\mathcal{C}_P$ but is guarding $Q'_{f(u)}$. Hence, switch the roles of C and of one cop $C' \in \mathcal{C}_{f(w)}$, that is we redefine $\mathcal{C}_{f(w)} = (\mathcal{C}_{f(w)} \setminus \{C'\}) \cup \{C\}$ and $\mathcal{C}_P = (\mathcal{C}_P \setminus \{C\}) \cup \{C'\})$. Then, the game is now in state $(\mathcal{A}, \mathcal{Q}', R', s')$, with transition of type (i).

Second, suppose now the interior of $Q_{f(u)}$ contains a vertex adjacent to R (note that here it is possible that $u = w$). Choose z to be such a vertex as close to x as possible, i.e. the interior of $Q_{f(u)}[x, z]$ contains no vertex adjacent to R. Apply Corollary 5.3.3 to get an $x - z$ path Q which goes through R which can be guarded by C after a finite number of turns.

Define $A'_u = A_u \cup V(Q_{f(u)}[x,z])$, $Q'_{f(u)} = Q_{f(u)}[z,y]$ and $Q'_P = Q$. Given that $x, z \in A'_u$, Q'_P is indeed an $A'_u - A'_u$ path. Let $s(C) = V(Q)$ and let $s'(C') = s(C') \setminus \{x\}$ for the cop $C' \in \mathcal{C}_{f(u)}$ which was previously guarding x, in order for (8) to still hold. The game is now in state $(\mathcal{A}', \mathcal{Q}', R', s')$, with transition of type (i). Note that given that x only had a as a neighbour in R, a is necessarily in Q, and so z is the only vertex of A'_u which is potentially adjacent to R', hence (5) still holds.

In all cases, we can reach a strictly smaller game state, which is a contradiction to the minimality of the current game state. □

Claim 5.4.3.9. *For every $P \in \mathcal{P}$, $\sum_{C \in \mathcal{C}_P} |s(C)| \geq \ell_P$.*

Proof. Suppose to the contrary that there exists P such that $\sum_{C \in \mathcal{C}_P} |s(C)| < \ell_P$.Since $|\mathcal{C}_P| = \left\lceil \frac{\ell_P}{3} \right\rceil$ by definition, there must be some $C \in \mathcal{C}_P$ with $|s(C)| \leq 2$. If possible, choose this C with $|s(C)| \leq 1$.

By Claim 5.4.3.8, P is initialized, so $s(\mathcal{C}_P)$ is a non-intertwined family of subsets of $V(Q_P)$. Let u, v be the end vertices of P, and x, y the end vertices of Q_P, where $x \in A_u$ and $y \in A_v$. Notice that when, $\ell_P \in \{0, 1, 2, 4\}$, we have $2\left\lceil \frac{\ell_P}{3} \right\rceil \geq \ell_P$, and so there exists $C \in \mathcal{C}_P$ such that $s(C) \leq 1$; in this case, such a C would have been chosen above. Hence, if $|s(C)| = 2$, we know it must be the case that $\ell_P \notin \{0, 1, 2, 4\}$.

If $|s(C)| = 1$, let z_1 be the vertex C is currently sitting on. If $|s(C)| = 0$, let z_1 be the vertex of Q_P which is adjacent to R (such a vertex exists by Claim 5.4.3.3) and closest to x (when traversing Q_P from x to y), and then send C to z_1.

If z_1 is the only vertex of Q_P adjacent to R, by Claim 5.4.3.3 z_1 is necessarily an internal vertex of Q_P. In this case, by Claim 5.4.3.5, A_v must contain a vertex $z_2 \neq z_1$ which is adjacent to R. Otherwise, let z_2 be the first vertex adjacent to R which appears after z_1 when traversing Q_P from x to y.

By Corollary 5.3.3, there exists a $z_1 - z_2$ path Q of length at least two with internal vertices in R such that C has a strategy to keep guarding z_1 and, after a finite number of turns, guard Q. If $z_2 \in Q_P$, let $Q'_P = Q_P[x, z_1] \oplus Q \oplus Q_P[z_2, y]$. Otherwise we chose $z_2 \in A_v$, and so let $Q'_P = Q_P[x, z_1] \oplus Q$. Note that in all cases, the parts of Q_P which are being dropped did not contain any neighbour in R, so (7) still holds, and Q'_P still has ends in A_u and A_v. Let $s(C) = V(Q)$. It is direct that this maintains (8). The game is now in state $(\mathcal{A}', \mathcal{Q}', R', s')$, with transition of type (i).

Suppose now that $|s(C)| = 2$. By the choice of C above, $\ell_P \notin \{0, 1, 2, 4\}$. In particular, $\mathbb{1}_\ell = 1$, and so the cop C_ℓ exists and is inactive. Let z_1, z_2 be the vertices of $s(C)$, suppose without loss of generality that z_1 appears before z_2 when traversing Q_P from x to y. By (8) and Claim 5.4.3.1, $Q_P[z_1, z_2]$ contains no internal vertex adjacent to R (in the very specific case where Q_P is a cycle with root $x = z_1$, given the somewhat technical definition of non-intertwined for cycles, it is possible that one might need the consider Q_P to be travelled in the opposite direction for this to hold).

By Corollary 5.3.3, there exists a $z_1 - z_2$ path, call it Q, of length at least two with internal vertices in R such that C_ℓ has a strategy to guard Q, after a finite amount of turns. Set $Q'_P = Q_P[x, z_1] \oplus Q \oplus Q_P[z_2, y]$, $s(C_\ell) = V(Q)$ and $s(C) = \emptyset$. With C now inactive, we relabel C to be C_ℓ and vice versa, in order for (8) and (11) to hold. The game is now in state $(\mathcal{A}', \mathcal{Q}', R', s')$, with transition of type (i).

In all cases, we can reach a strictly smaller game state, which is a contradiction to the minimality of the current game state. □

Claim 5.4.3.10. *H is a minor of G.*

Proof. We use the model in the current state $(\mathcal{A}, \mathcal{Q}, R, s)$ to construct a minor of H in G.

First, contract all edges in the connected component R and call the resulting vertex h'. Any vertex in $V(G) \setminus R$ adjacent to R is now adjacent to h'.

For every $w \in W$, A_w is non-empty by Claim 5.4.3.7. By the definition in (1), $G[A_w]$ is connected for every $w \in W$. Hence, we may contract every edge between vertices in A_w to obtain one vertex, which we denote w'. Since A_w contains at least one vertex adjacent to R by Claim 5.4.3.5, h' is adjacent to w' in the resulting graph.

By Claim 5.4.3.8, every $P \in \mathcal{P}$ is initialized. For every $P \in \mathcal{P}$ and every edge $uv \in Q_P$, contract the edge uv if either u or v is not adjacent to h'. Let P' be the resulting path (or cycle), which has ends u' and v'. With these contractions, every vertex of P' is adjacent to h'.

By Claim 5.4.3.2, the images of s are disjoint, which contain only vertices adjacent to R (and now h') by Claim 5.4.3.1. Let $P \in \mathcal{P}$ with ends u, v. By (10), the end vertices of Q_P, say x, y, are in one of the sets of $s(\mathcal{C}_P)$ only if $P = f(u)$ and $P = f(v)$ respectively. Thus, P' contains $\left(\sum_{C \in \mathcal{C}_P} |s(C)|\right) - |f^{-1}(P)|$ internal vertices. By Claim 5.4.3.9, P' then contains at least $\ell_P - |f^{-1}(P)| \geq |E(P)| - 1$ internal vertices. As the number of edges in a path or a rooted cycle is one more than the number of internal vertices, P' contains at least $|E(P)|$ edges. We may contract further edges of P' in order for P' to contain exactly $|E(P)|$ edges.

Mapping h to h', w to w' for every $w \in W$ and P to P' for every $P \in \mathcal{P}$, we conclude that H is isomorphic to a subgraph of our contracted graph, and so H is a minor of G. $\square$

Given that G is H-minor-free, Claim 5.4.3.10 yields the contradiction. This completes the proof of the theorem. $\square$

5.4.1 Key ideas of the proof

We now highlight a few key elements of the proof of Theorem 5.4.3.

The method introduced by Andreae in [D2] consists in, as long as the robber is not caught, gradually constructing a minor of $H - h$ by buildings bags corresponding to the vertices of $H - h$ and using path guarding strategies for cops in order to add paths between bags when the corresponding vertices are adjacent in $H - h$. These paths are taken through the robber's territory, gradually reducing its size. Once the minor of $H - h$ is completed, contracting the robber's territory then yields a minor of H. As the graph is H-minor-free, this process cannot be completed, and hence the cops must eventually capture the robber.

Our proof builds on this basic framework in multiple ways.

1. In the proof of Theorem 5.1.1, exactly one cop is used to recreate each edge of $H - h$ in the minor by guarding a path between two bags corresponding to adjacent vertices of $H - h$ (which yields the bound $c(G) \leq |E(H - h)|$). In a specific proof sketch for wheel graphs (a cycle plus a universal vertex), Andreae [7, Theorem 3] uses the fact that one cop can be used to recreate at least three vertices of the cycle: when a cop is guarding fewer than three (for simplicity, say there are two) vertices adjacent to the robber's territory, the extra cop can relieve this cop by guarding a new path between these two vertices through the robber's territory. Using a more specific assignment of cops, in which now cops are grouped together to guard paths (and rooted cycles) between pairs of "core" vertices W, the same idea can be used for general graphs.

2. To go from a model of $H - h$ to a model of H, one requirement is that each bag A_w (which will be contracted to give w in the minor) must contain at least one vertex adjacent to R (at least, when $wh \in E(H)$). Furthermore, the existence of at least one such vertex is needed when adding a new $u - v$ path to the model, as it allows us to get a new path between A_u and A_v passing through R. In Andreae's proof, when A_w no longer has a neighbour in R, it gains one by absorbing parts of one of the paths incident to it, or otherwise it is uninitialized. Using this approach directly with the previous improvement, we would get Corollary 5.5.2 below. However this is not optimal, as it requires that the group of cops of any of the paths be large enough to guard not only the required number of neighbours of R internally in the path, but in the ends of the path as well. Hence, another key idea in our proof is that it is in fact possible to designate for each w from which path to absorb vertices to acquire a vertex adjacent to R; this is the role played by f. If this is not possible, i.e. if $f(w)$ is uninitialized, a neighbour of R will be acquired from another path incident to A_w (if one such path exists), and then we use a cop of $\mathcal{C}_{f(w)}$ to guard this vertex. This is reflected by (10).

3. In the last point, what happens if instead of a long path between $u, v \in W$, there is simply an edge, and uv is not in the image of f? In some sense, to get the minor we do not need any neighbour of R to be present in the path Q_{uv} which will be contracted to uv. However, if we take Q_{uv} to be a path between A_u and A_v, a cop is still potentially needed: even though Q_{uv} is not required to contain a neighbour in R, it might contain one. However, when we first initialize A_u or A_v, we can do so in a way that Q_{uv} is only an edge and thus no cop will need to be assigned to guard it. This is the role the matching M plays. Let us note that this only works when M is a matching; this is a consequence of the fact that we cannot much control the order in which sets A_w are initialized and uninitialized.

4. When using a group of cops to recreate a path to build the minor of $H - h$, one extra cop is often required. More precisely, we expect every cop to be able to guard at least three vertices adjacent to R. However, if one cop is guarding exactly two vertices adjacent to R, there is generally no way for that cop to start guarding a new path through R between these vertices without losing control of one of these vertices, and all other cops might also be busy. We can use the extra cop to do so, after which the first cop can be relieved (the cops may then switch their roles); this is what is suggested for wheel graphs in [D2]. In some very specific cases with short paths, we can guarantee that if the cops are not on average guarding at least three neighbours of R, then one of these cops is necessarily guarding at most one neighbour of R, in which case the extra cop is not required. We will see in the applications in the next section that this difference can be very useful when H is small. We note that the extra cop is also used in very specific technical situations involving edges of the matching or cycles in Claim 5.4.3.7 and Claim 5.4.3.8.

Note that many of the technicalities of the proof concern the interplay between these various improvements. Indeed, we often need to break the proofs of the claims into various cases depending on whether, for instance, the $P \in \mathcal{P}$ in question is a path or a cycle, is in the image of f or not and is in the matching M or not. These complexities also require a more technical proof statement and system of states and state transitions. Our proof is also quite formal when it comes to path guarding strategies, hence the use of Corollary 5.3.3 and the specific formulation of condition (6).

5.5 Applications

In this section, we will see various consequences of our main result Theorem 5.4.3.

5.5.1 Simplified versions of the main result

In many cases, one might not need the full flexibility of Theorem 5.4.3, which is quite technical. In this section we present some simpler versions of this result. This will also allow us to better isolate the various improvements described in Section 5.4.1.

Firstly, we have a version of Theorem 5.4.3 in which the only difference with Theorem 5.1.1 is the addition of a matching of "free" edges.

Corollary 5.5.1. *Let H be a graph, $h \in V(H)$ and M be a matching of $H - h$ such that $H - h - M$ has no isolated vertex. If G is a connected H-minor-free-graph, then $c(G) \leq |E(H - h)| - |M|$.*

Proof. Let $W = V(H - h)$, let $\mathcal{P} = E(H - h)$ (considering every edge as a path of length 1) and let f be arbitrary; at least one such function exists since every vertex of $H - h - M$ is not isolated. Then, $(h, W, \mathcal{P}, M, f)$ is a decomposition of H.

For every $P \in \mathcal{P}$, we have that $|E(P)| = 1$ and thus

$$\begin{aligned}\ell_P &= \begin{cases} 0 & P \in M \\ \max(|E(P)| - 1 + |f^{-1}(P)|, 1) & P \notin M \end{cases} \\ &\leq \begin{cases} 0 & P \in M \\ 2 & P \notin M. \end{cases}\end{aligned}$$

This implies that $\mathbb{1}_\ell = 0$. Hence, by Theorem 5.4.3 we have that

$$c(G) \le \mathbb{1}_\ell + \sum_{P \in \mathcal{P}} \left\lceil \frac{\ell_P}{3} \right\rceil \le \sum_{e \in E(H-h)\setminus M} \left\lceil \frac{2}{3} \right\rceil + \sum_{e \in M} \left\lceil \frac{0}{3} \right\rceil = |E(H-h)| - |M|.$$

□

We might also want a version of Theorem 5.4.3 in which we use the improvements for long paths in $H - h$ but without some of the technicalities.

Corollary 5.5.2. *Let H be a graph and $h \in V(H)$ be a vertex such that $H - h$ has no isolated vertex. Let $W \subseteq V(H-h)$ be non-empty and let $\mathcal{P}$ be a collection of pairwise internally vertex-disjoint paths and cycles with end vertices in W such that every edge of $H - h$ is contained in some $P \in \mathcal{P}$.*

If G is a connected H-minor-free graph, then

$$c(G) \le 1 + \sum_{P \in \mathcal{P}} \left\lceil \frac{|V(P)|}{3} \right\rceil .$$

Proof. Let f be arbitrary; at least one valid choice exists given that no vertex of W is isolated. Then, $(h, W, \mathcal{P}, \emptyset, f)$ is a decomposition of H.

If $P \in \mathcal{P}$ is a path, $|f^{-1}(P)| \le 2$, and so $\ell_P \le |E(P)| + 1 = |V(P)|$. If $P \in \mathcal{P}$ is a cycle, then $|f^{-1}(P)| \le 1$, and so $\ell_P \le |E(P)| = |V(P)|$. Furthermore, $\mathbb{1}_\ell \le 1$. □

5.5.2 Recovering Andreae's results

Here we show that Theorem 5.4.3 is indeed a generalization of Andreae's results. Firstly, we indeed recover Theorem 5.1.1, which we restate for convenience.

Theorem 5.1.1. *[D2] Let H be a graph and $h \in V(H)$ be a vertex such that $H - h$ has no isolated vertex. If G is a connected H-minor-free graph, then $c(G) \le |E(H-h)|$.*

Proof. Apply Corollary 5.5.1 with $M = \emptyset$. □

Consider the wheel graph $W_t = U(C_t)$ (where C_t is the cycle graph on t vertices). As noted in the introduction, Andreae proved the following. As noted in Section 5.4.1, the proof of that result partially inspired Theorem 5.4.3. We can recover that result.

Theorem 5.5.3. *If G is a connected W_t-minor-free graphs ($t \geq 3$), then $c(G) \leq \lceil \frac{t}{3} \rceil + 1$.*

Proof. Apply Corollary 5.5.2 with h being the universal vertex, $W = \{u\}$ where u is some arbitrary vertex of $W_t - h$ and $\mathcal{P}$ containing only the cycle $W_t - h$ which we root at u. $\square$

Further results of Andreae, for $K_{3,3}$-minor-free graphs and $K_{2,3}$-minor-free graphs, are recovered in the next subsection. We note however that we cannot recover all of Andreae's results for small graphs, in particular the upper bound of 3 on the cop number of connected K_5-minor-free graphs. Andreae's method to prove this, although similar to the methods used to prove Theorem 5.1.1, constructs the minor more carefully, in a way which only works for very small graphs. In particular, whereas in the general framework used to prove Theorem 5.1.1 and Theorem 5.4.3 we do not have much control over what A_w (the set of vertices which are going to be contracted to obtain w) looks like, in Andreae's proof for K_5-minor-free graphs the structure of the model of H is much more rigid; some edges in the minor can be obtained as a consequence of the fact that when building a minor of a small graph, one can keep track of the presence of specific vertices and edges.

5.5.3 Complete bipartite graphs

We can improve the bound from $2t$ to t for $K_{3,t}$-minor-free graphs.

Corollary 5.5.4. *If G is a connected $K_{3,t}$-minor-free graph ($t \geq 2$), then $c(G) \leq t$.*

Proof. Let h, a, b be the vertices in the part of $K_{3,t}$ with 3 vertices. Then, $K_{3,t} - h$ consists of exactly t internally disjoint paths $P_1, \ldots, P_t$ of length 2 between a and b. Let $W = \{a, b\}$, $\mathcal{P} = \{P_1, \ldots, P_t\}$ and define f by $f(a) = P_1$, $f(b) = P_2$. Then, $(h, W, \mathcal{P}, \emptyset, f)$ is a decomposition of H.

We have that $|E(P_i)| = 2$ for $i \in [t]$, $|f^{-1}(P_1)| = |f^{-1}(P_2)| = 1$ and $|f^{-1}(P_i)| = 0$ for $i \in [t] \setminus \{1, 2\}$. Hence $\ell_{P_i} \leq 2$ for every $i \in [t]$, and in particular $\mathbb{1}_\ell = 0$. Theorem 5.4.3 then yields the result. $\square$

In particular, we recover the bound for $K_{3,3}$-minor-free graphs from [D2] without needing a separate argument.

We can also improve the upper bounds for $K_{2,t}$-minor-free from t to essentially half that.

Corollary 5.5.5. *If G is a connected $K_{2,t}$-minor-free graph $(t \geq 1)$, then $c(G) \leq \left\lceil \frac{t+1}{2} \right\rceil$.*

Proof. First note that it suffices to show the result when t is odd, since $K_{2,t-1}$-minor-free graphs are also $K_{2,t}$-minor-free, and in this case $\left\lceil \frac{(t-1)+1}{2} \right\rceil = \left\lceil \frac{t}{2} \right\rceil = \frac{t-1}{2} + 1$.

Consider the graph $H = U\left(U\left(\frac{t-1}{2} K_2 + K_1\right)\right)$. In other words, if h is one of the universal vertices, $H - h$ is a graph obtained by identifying one vertex of $\frac{t-1}{2}$ triangles and of one edge. In particular, $K_{2,t}$ is a subgraph of H, and so it suffices to prove that connected H-minor-free graphs have cop number at most $\frac{t-1}{2} + 1$. Let G be such a graph.

Let a be the universal vertex of $H - h$. We have that $H - h$ is the union of $\frac{t-1}{2}$ internally-disjoint a-rooted cycles of length 3 (write $\mathcal{P}_{aa}$ for this collection of cycles), and one other edge ab. Let $W = \{a, b\}$, let $\mathcal{P} = \mathcal{P}_{aa} \cup \{ab\}$ and define f by $f(a) = f(b) = ab$. Then, $(h, W, \mathcal{P}, \emptyset, f)$ is a decomposition of H.

For every cycle (of length 3) $P \in \mathcal{P}_{aa}$, we have $\ell_P = 2$, and $\ell_{ab} = 2$. In particular, $\mathbb{1}_\ell = 0$. Theorem 5.4.3 yields that $c(G) \leq \frac{t-1}{2} + 1$ as desired. $\square$

Note that both of these corollaries give us an bound of 2 for $K_{2,3}$-minor-free graphs, which does not follow from Theorem 5.1.1 but is a consequence of Andreae's [D2] stronger upper bound of 2 on the cop number of $K_{3,3}^-$-minor-free graphs.

5.5.4 Complete graphs

One of the consequences of Theorem 5.1.1 is that if G is a connected K_t-minor-free graph for $t \geq 3$, then $c(G) \leq \binom{t-1}{2} = \frac{(t-1)(t-2)}{2}$.

We can improve this result.

Corollary 5.5.6. *If G is a connected K_t-minor-free graph ($t \geq 4$), then $c(G) \leq \left\lfloor \frac{(t-2)^2}{2} \right\rfloor$.*

Proof. Let h be an arbitrary vertex of K_t, and let M be a maximum matching of $K_t - h \simeq K_{t-1}$, which has size $\left\lfloor \frac{t-1}{2} \right\rfloor$. Note that $K_t - h - M$ contains no isolated vertex since $t \geq 4$. Then, Corollary 5.5.1 yields that

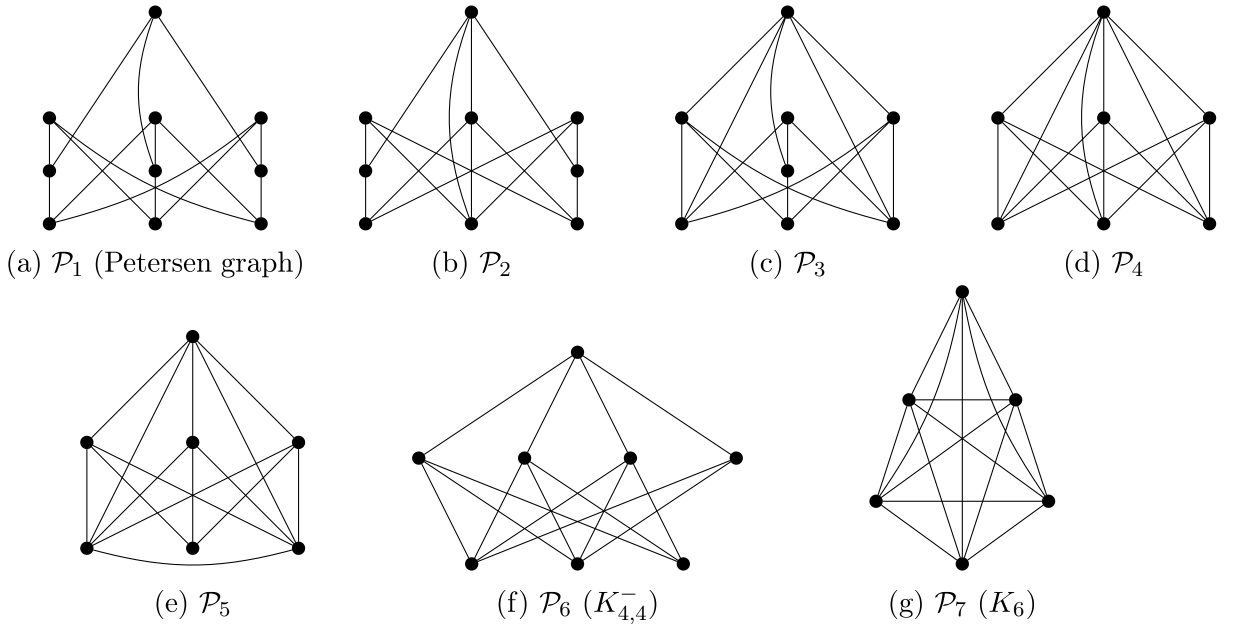
$$c(G) \leq |E(H-h)| - |M| = \binom{t-1}{2} - \left\lfloor \frac{t-1}{2} \right\rfloor = \left\lfloor \frac{(t-2)^2}{2} \right\rfloor.$$

□

The cop number of K_t-minor-free graphs in particular has received some interest. Andreae [D2] posed as an open problem to find K_t-minor-free graphs with large cop number.

Furthermore, Bollobás, Kun and Leader [D3] noted the bound on K_t-minor-free graphs is related to Meyniel's conjecture. Meyniel's conjecture [D8] is the most famous and important conjecture on the game of cops and robbers. It states that $c(G) = O(\sqrt{n})$ if G is a connected graph on n vertices. A weaker but still open conjecture is the weak or soft Meyniel conjecture, stating that $c(G) = O(n^{1-\delta})$ for some fixed $\delta > 0$. Bollobás, Kun and Leader note that if we prove that K_t-minor-free graphs have cop number at most $O(t^{2-\varepsilon})$, then weak Meyniel holds for $\delta = \frac{\varepsilon}{4-\varepsilon}$. Briefly, their argument goes as follows. Suppose we wish to bound the cop number of an arbitrary graph G on n vertices. If G has a vertex u of degree $\Omega(n^\delta)$, then place a cop on this vertex and proceed by induction on $G - N[u]$. Otherwise, G has $O(n^{\delta+1})$ edges, and so G cannot contain a complete minor on more than $O(n^{\frac{\delta+1}{2}})$ vertices. We may then apply the bound for graphs forbidding a complete minor to obtain the desired result.

We note that Bollobás, Kun and Leader's argument holds more generally. Suppose $\{G_t\}_{t\geq 1}$ is a family of graphs indexed by t such that $e(t) = |E(G_t)|$ is monotone increasing, and let f be a monotone increasing upper bound on the cop numbers of these graphs, i.e. $c(G_t) \leq f(t)$. If $f(e^{-1}(m)) = O(m^{1-\varepsilon})$, then weak Meyniel holds for $\delta = \frac{\varepsilon}{2-\varepsilon}$.



(a) $\mathcal{P}_1$ (Petersen graph) (b) $\mathcal{P}_2$ (c) $\mathcal{P}_3$ (d) $\mathcal{P}_4$

(e) $\mathcal{P}_5$ (f) $\mathcal{P}_6$ ($K_{4,4}^-$) (g) $\mathcal{P}_7$ (K_6)

Figure 5.5.1: Petersen family.[4]

In other words, if we find any class of graphs G_t (not only complete graphs) for which the order of the cop number of G_t-minor-free graphs is polynomially smaller than that of the number of edges of G_t, one gets an improvement towards Meyniel.

5.5.5 Linklessly embeddable graphs

A *linkless embedding* of a graph is an embedding of the graph into $\mathbb{R}^3$ such that every pair of two disjoint cycles forms a trivial link (i.e., they do not pass through one another). A graph that has a linkless embedding is called *linklessly embeddable*. Robertson, Seymour, and Thomas [D15] showed that the linklessly embeddable graphs are exactly the graphs excluding the *Petersen family* (see Figure 5.5.1) as minors. The Petersen family contains seven graphs that are all $\Delta - Y$ equivalent (i.e. can be obtained by replacing an induced claw by a triangle) to K_6, which notably includes $K_{4,4}^-$ and the Petersen graph.

Given the various topological results on the game of cops and robbers discussed in the introduction, one might then be interested in determining the maximum cop number of linklessly embeddable graphs. It follows from Theorem 5.1.1 that for any linklessly embeddable

[4]Drawings based on [D14].

graph, $c(G) \leq 9$: take $H = \mathcal{P}_4$ and let h be the degree 6 vertex ($H - h$ is then $K_{3,3}$).

Using our main result, we are able to improve this upper bound.

Corollary 5.5.7. *If G is a $\mathcal{P}_i$-minor-free graph ($i \in [4]$), then $c(G) \leq 6$. In particular, if G is a connected linklessly embeddable graph, then $c(G) \leq 6$.*

Proof. Let h be the top vertex of $\mathcal{P}_i$ in the drawings in Figure 5.5.1. Then, we can represent $\mathcal{P}_i - h$ as follows. Let $W = \{a_1, a_2, a_3, b_1, b_2, b_3\}$ be vertices of $\mathcal{P}_i - h$ such that $a_i b_j$ is an edge for every distinct $i, j \in [3]$, and for every $i \in [3]$ there is either an edge or a path of length 2 between a_i and b_i. Let $\mathcal{P}$ be the collection of these paths of length 1 or 2. Let $M = \{a_1b_2, a_2b_3, a_3b_1\}$. For $i \in [3]$, let $f(a_i) = f(b_{i+2}) = a_i b_{i+2}$ (with indices modulo 3). With these choices, $(h, W, \mathcal{P}, M, f)$ is a decomposition of h.

We then have that $\ell_P \leq 2$ for every $P \in \mathcal{P} \setminus M$ (of which there are 6), since either P is an edge which is the f-image of 2 vertices of W (hence $\ell_P = 2$), or P is a path with 2 edges but is not in the image of f (hence $\ell_P = 1$). In particular, $\mathbb{1}_\ell = 0$. Theorem 5.4.3 then yields that $c(G) \leq 6$. $\square$

As for lower bounds, we were unable to find any linklessly embeddable graph with cop number at least 4 (planar graphs being linklessly embeddable, the dodecahedral graph is an example with cop number 3 [D1]). It hence remains open to determine what the maximum cop number of linklessly embeddable graphs is.

One might also be interested in relating the cop number to the *Colin de Verdière* spectral graph parameter $\mu(G)$. While we omit the formal definition here, Colin de Verdière showed that $\mu(G) \leq 1$ if and only if G is disjoint union of paths, $\mu(G) \leq 2$ if and only if G is outerplanar, and $\mu(G) \leq 3$ if and only if G is planar [D7]. This pattern was extended by Van der Holst, Lovász and Schrijver who showed $\mu(G) \leq 4$ if and only if G is linklessly embeddable [D18].

For the first three of these classes, it turns out the upper bound on the cop number is the same as the upper bound on the Colin de Verdière invariant, and that the examples for which the cop number bound is tight are also tight for the Colin de Verdière bounds. It is trivial that paths have cop number 1. Furthermore, if G is a connected outerplanar graph, i.e. a graph which can be embedded in the plane without edge crossings and such that all vertices are on the outer face, then $c(G) \leq 2$. This was originally stated and proved by Clarke [D6], however it was also a consequence of Andreae's bound for $K_{2,3}$-minor-free graphs and K_4-minor-free graphs, as it is well-known that the outerplanar graphs are exactly the $\{K_{2,3}, K_4\}$-minor-free graphs. Any outerplanar graph of cop number 2 (for example, a cycle of length at least 4), is necessarily not a path and thus has Colin de Verdière number 2. Finally, as mentioned earlier, Fromme and Aigner [D1] have proved that any connected planar graph G has $c(G) \leq 3$. Any planar graph of cop number 3 (for example, the dodecahedral graph) is necessarily not outerplanar, and thus must have Colin de Verdière number 3.

Hence, one might wonder whether this pattern continues to linklessly embeddable graphs with an upper bound of 4 for the cop number in this class, and more generally whether $c(G) \leq \mu(G)$ for all connected graphs G.

5.5.6 Greater improvement factor

We have seen earlier that we can improve the bound on H-minor-free graphs by a factor of 3 (relative to the number of edges of $H - h$) when $H - h$ can be obtained by subdividing the edges of another graph many times, for instance when applying Corollary 5.5.2 to the wheel graph. In fact, in some cases we can essentially get an improvement factor of 4. Let us see an example.

Define H_t to be the graph formed by identifying the end vertices of m copies of a five vertex path, as shown in Figure 5.5.2. Recall that $U(H_t)$ is the graph H_t with an additional universal vertex. Theorem 5.1.1 shows that if G is a connected $U(H_t)$-minor-free graph, then $c(G) \leq 4m$. We can improve this result by almost a factor of 4.

Corollary 5.5.8. *If G is a connected $U(H_t)$-minor-free graph ($t \geq 1$), then $c(G) \leq t + 2$.*

Proof. Let h be the universal vertex of $U(H_t)$ and let a, b be the two end vertices of H_t (those

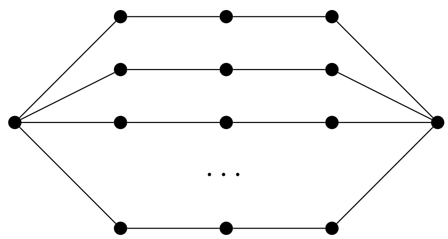


Figure 5.5.2: The graphs H_t.

obtained by identification). Let $W = \{a, b\}$ and let $\mathcal{P} = \{P_1, \ldots, P_t\}$ be the collection of t internally disjoint $a - b$ paths of length 4 of H_t. Finally, define f by $f(a) = f(b) = P_1$. Then, $(h, W, \mathcal{P}, \emptyset, f)$ is a decomposition of H.

We have that $\ell_{P_1} = 5$ and $\ell_{P_i} = 3$ for $i \in [t] \setminus \{1\}$. With these values, $\mathbb{1}_\ell = 1$. Theorem 5.4.3 yields that

$$c(G) \leq \mathbb{1}_\ell + \left\lceil \frac{\ell_{P_1}}{3} \right\rceil + \sum_{i=2}^{t} \left\lceil \frac{\ell_{P_i}}{3} \right\rceil = 1 + 2 + (t-1) = t + 2.$$

□

5.6 Future directions

In Section 5.5.6, we have seen that our results allow us to, in some cases, obtain an improvement of factor 4 over the previous results. There still appears to be a lot of work to be done further optimizing the upper bounds on the cop number when forbidding an minor, both for general classes of graphs and specific graphs.

It would be interesting to get a better upper bound on the cop number when forbidding multiple minors, especially when they are very similar (for instance, for linklessly embeddable graphs). This might in particular yield interesting results for various topological classes of graphs, where the obstruction set usually contains a large number of graphs.

Finding lower bounds, i.e. constructing graphs with some forbidden minor but relatively high cop number, also appears difficult.

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Bridging text 4

In the previous chapter, we studied the game of cops and robbers on connected graphs with a forbidden minor. In the next chapter, we will study graph burning, which can also be formulated as a game on connected graphs, although only with one player. These games are two of the most studied games in the field of graph searching.

We will show an upper bound on the burning number of a connected graph, the invariant describing the minimum length of a game, which converges to the square root of the order of the graph. The non-asymptotic version of this result is generally referred to as the Burning Number Conjecture [20].

The Burning Number Conjecture can be formulated as stating that paths have the highest possible burning number. This is a marked difference with the cop number, where paths have cop number 1. Arguably, a more appropriate comparison is with a different invariant related to the cop number: the capture time [18].

As noted in the introduction, it will suffice to work on trees. As suggested by the title of this thesis, trees are also sparse graphs (they are also exactly the connected K_3-minor-free graphs).

To tackle this problem efficiently, we will convert the problem to a continuous version, and then use a probabilistic approach to find a cover of the graph. Whereas in Chapter 3 the probabilistic methods used are exclusively a direct application of the first-moment (expectation) method, here we will need to use more intricate, measure-theoretic tools, as well as Chernoff bounds. The methods used in this chapter will motivate a variant, called fractional burning, which will be discussed in depth in the discussion section of this thesis.

6

The Burning Number Conjecture Holds Asymptotically

Sergey Norin[1], Jérémie Turcotte[1]

The burning number $b(G)$ of a graph G is the smallest number of turns required to burn all vertices of a graph if at every turn a new fire is started and existing fires spread to all adjacent vertices. The Burning Number Conjecture of Bonato et al. (2016) postulates that $b(G) \leq \lceil \sqrt{n} \rceil$ for all connected graphs G on n vertices. We prove that this conjecture holds asymptotically, that is $b(G) \leq (1 + o(1))\sqrt{n}$.

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6.1 Introduction

In contrast with graph burning, graph burning is a one player game. Consider the following process on a finite, usually connected, graph G. At the start, all vertices are said to be unburned. Then, at every step (or turn) of the process, we may choose to burn (start a new fire) at some vertex. Furthermore, at every step, all vertices adjacent to burning vertices will start burning as well (the fire spreads to neighbouring vertices at every turn). How many steps does it take before all vertices of the graph are burned?

The *burning number* of G, denoted by $b(G)$, is the smallest number of turns required to burn the entire graph G. Formally, $b(G)$ is the smallest integer k such that we can cover G with balls of radii $0, \ldots, k-1$.

This process first appeared in print in a paper of Alon [E1] and was motivated by a question of Brandenburg and Scott at Intel, and formulated as a transmission problem involving a set of processors. Alon [E1] proved that the burning number of a d-dimension hypercube is exactly $\lceil \frac{d}{2} \rceil + 1$.

This process was later independently defined by Bonato et al. [E6] (also in the related [E5, E18]), and named *graph burning.* In this case, the inspiration is the spread of information (news, memes, opinions, etc.) in social networks. Since its reintroduction only a few years ago, a large body of work on graph burning has emerged, concentrating mainly on bounding the burning number, either in general or on specific classes of graphs, and on the complexity of graph burning. For instance, it was shown by Bessy et al. that graph burning is NP-complete, even when restricted to trees of maximum degree three. Kobayashi and Otachi have shown that it is W[2]-complete when parametrized by k. Even $(\frac{5}{3} - \varepsilon)$-approximation of the burning number of NP-hard for general graphs, as shown by Martinsson [E15]. See [E4] and references therein for a fairly recent survey of results on graph burning.

Bonato et al. [E6] show that $b(P_n) = \lceil \sqrt{n} \rceil$ and conjecture that paths are in fact the connected graphs with the largest burning number.

Conjecture 6.1.1 (Burning Number Conjecture [E6]). *If G is a connected graph on n vertices, then $b(G) \leq \lceil \sqrt{n} \rceil$.*

Since the introduction of graph burning, this conjecture has been the central open problem in the field. The conjecture is known to hold on multiple classes of graphs, such as spiders [E9, E8], caterpillars [E14], some p-caterpillars [E11], sufficiently large graphs with minimum degree at least 4 [E2], and others (for instance [E16]).

The best known bound on the burning number of general graphs has been improved multiple times. Suppose G is a connected graph on n vertices.

- Bonato et al. [E6] show that $b(G) \leq 2\lceil \sqrt{n} \rceil - 1$.
- Bessy et al. [E3] show that $b(G) \leq \sqrt{\frac{32n}{19(1-\varepsilon)}} + \sqrt{\frac{27}{19\varepsilon}}$ for every $0 < \varepsilon < 1$ and $b(G) \leq \sqrt{\frac{12n}{7}} + 3$.
- Land and Lu [E13] show that $b(G) \leq \left\lceil \frac{\sqrt{24n+33}-3}{4} \right\rceil$.
- Bastide et al. [E2] show that $b(G) \leq \left\lceil \sqrt{\frac{4n}{3}} \right\rceil + 1$.

Our main result is the following, which shows that Conjecture 6.1.1 holds asymptotically.

Theorem 6.1.2. *If G is a connected graph on n vertices, then*

$$b(G) \leq (1 + o(1))\sqrt{n}.$$

Bastide et al. [E2, Theorems 2-3] prove that the Burning Number Conjecture holds for sufficiently large graphs with minimum degree at least four, and almost holds (up to adding some small constant) for graphs with minimum degree three. We note that with their argument, one can deduce from Theorem 6.1.2 that the Burning Number Conjecture also holds for sufficiently large graphs with minimum degree three. In fact, for these graphs we can get the bound $b(G) \le \left(\sqrt{\frac{3}{4}} + o(1)\right)\sqrt{n}$ for graphs with minimum degree three, and $b(G) \le \left(\sqrt{\frac{3}{5}} + o(1)\right)\sqrt{n}$ for graphs with minimum degree at least four.

The remainder of the paper is occupied by the proof of Theorem 6.1.2, which we now outline.

As noted in [E6], if T is a spanning tree of G, then $b(G) \le b(T)$; a strategy to burn T still works when edges are added. Hence, it suffices to prove Theorem 6.1.2 for trees and we focus on trees from now on.

The results on the burning number mentioned above largely use traditional graph theoretical techniques, i.e. induction.

While we borrow many ideas from the previous work, we deviate from this pattern by shifting to a continuous and probabilistic setting. First, as the burning number is defined in terms of existence of a cover of the graphs by balls with certain radii, we embrace the metric nature of the problem and work with metric, rather than discrete, trees, i.e. metric spaces obtained by replacing the edges of a tree by real intervals. This has several advantages, in particular allowing us to ignore the scaling issues. Section 6.2 is devoted to introducing the setting. In Section 6.3 we prove the existence of covers of any given metric tree by balls, which are "frugal", in a sense that the sum of the radii of balls is about as small as can be expected, and flexible, allowing us to choose from many different possible radii. In Section 6.4 we bootstrap the results of Section 6.3 to prove Theorem 6.4.1 - a fractional version of Theorem 6.1.2. Informally, Theorem 6.4.1 says that for every metric tree T and every r there exists a distribution on covers of T by balls of radii at most r which is still frugal in expectation and such that the distribution is almost uniform - uses all radii with roughly the same probability. In Section 6.5 we use Theorem 6.4.1 to prove a version of Theorem 6.1.2 for metric trees and then transfer the result back to the discrete setting. The key observation here is that if we divide the tree T into many pieces and independently choose a cover of each piece with the properties guaranteed by Theorem 6.4.1 then the radii in the resulting cover will be almost uniformly distributed, i.e. close to the desired set of radii $\{1, 2, \dots \lceil \sqrt{n} \rceil\}$.

We conclude in Section 6.6 with a few remarks, in particular briefly discussing potential generalization of the metric version of the Burning Number Conjecture that allows less restrictive sets of radii.

6.2 Metric trees

As mentioned in the introduction, we will primarily be working with metric trees and we start this section by formally defining them and introducing the necessary notation.

A *metric tree* T is obtained from a non-null (discrete) tree T_0 by replacing each edge with a close interval of positive length, so that the end points of the interval are identified with the end vertices of the edge. The *length* of T, denoted by $|T|$, is the sum of the length of the intervals comprising it. Note that for any two points u, v in a metric tree T there exists a unique path (simple curve) in T with ends u and v, which we denote by $T[u, v]$, and we denote its length by $d_T(u, v)$ or simply by $d(u, v)$. It is easy to see that (T, d_T) is a metric space. In fact, T is also compact, since it is obtained from identifying the ends of a finite number of closed (compact) intervals[2].

A *leaf* of T is a leaf of T_0, and a *branch point* of T is a vertex of T_0 with degree three or greater. Leaves and branch points of a metric tree T can be also defined intrinsically, i.e. a point $v \in T$ is a leaf if and only if $T \setminus v$ is connected, and a branch point of T is a point of T such that $T \setminus v$ has at least three connected components. Let $L(T)$ and $\mathrm{Br}(T)$ denote the sets of leaves and branch points of T, respectively. The components of $T \setminus (L(T) \cup \mathrm{Br}(T))$ are homeomorphic to open intervals. The *segments* of T are the closures of these intervals. Thus every point in $T \setminus \mathrm{Br}(T)$ belongs to a unique segment, and the ends of every segment lie in $L(T) \cup \mathrm{Br}(T)$.

The *diameter* of T, denoted by $\mathrm{diam}(T)$, is the maximum distance between two points

[2]Given an open cover of T, one can find a finite subcover by taking the union of the finite subcovers obtained from the compactness of each interval.

in T, that is

$$\operatorname{diam}(T) = \max_{u,v \in T} d_T(u,v).$$

This is well defined by the compactness of T (and given the continuity of d_T).

A metric tree T is *trivial* if $|T| = 0$, in which case T consists of a single point, and T is *non-trivial*, otherwise. A metric tree T' is a *subtree* of a metric tree T if $T' \subseteq T$. For instance, for any $u, v \in T$, $T[u,v]$ is a subtree of T with 2 leaves, unless $u = v$ in which case $T[u,v]$ is trivial. A collection $\mathcal{T} = \{T_1, \ldots, T_k\}$ of subtrees of a metric tree T is *a decomposition of* T if $T = \cup_{i \in [k]} T_i$ and $T_1, T_2, \ldots, T_k$ are pairwise internally disjoint, i.e. $|T| = \sum_{i=1}^{k} |T_i|$.

For a subtree T' of a metric tree T we denote by $\bar{T}'$ the closure of $T \setminus T'$ (in other words, we remove T' from T, except for the boundary points of T'). We denote by $\bar{\operatorname{comp}}(T')$ the set of components of $\bar{T}'$. Note that every component of $\bar{T}'$ is a subtree of T, that $\{T'\} \cup \bar{\operatorname{comp}}(T')$ is a decomposition of T, and that every component of $\bar{T}'$ shares exactly one point with T'.

A subtree T' of T is *a branch of* T if $\bar{T}'$ is connected and non-empty, i.e. $\bar{T}'$ is a subtree of T. It follows from the observations above that for every proper subtree T' of a metric tree T every component of $\bar{T}'$ is a branch of T. The *anchor* $\operatorname{anc}(T')$ of a branch T' of T is the unique point of $T' \cap \bar{T}'$. The *depth* $\operatorname{dp}(T')$ of a branch T' of T is defined as

$$\operatorname{dp}(T') = \max_{v \in T'} d_T(\operatorname{anc}(T'), v).$$

Again, this is well defined by compactness.

For some minimality arguments, we will need a stronger compactness property. For a metric tree T, let $\mathcal{S}(T)$ be the set of subtrees of T. It is easy to see that $\mathcal{S}(T)$ is the set of closed (hence compact) and connected (in this case as in $\mathbb{R}$, this is equivalent to being path-connected) subsets of T. Let $d_{\mathcal{S}(T)}$ be the Hausdorff distance [E10] on $\mathcal{S}(T)$, formally if T_1, T_2 are subtrees of T,

$$d_{\mathcal{S}(T)}(T_1, T_2) = \max\left(\max_{v_1 \in T_1} d_T(v_1, T_2), \max_{v_2 \in T_2} d_T(v_2, T_1)\right),$$

where, for $v \in T$ and a subtree T', $d_T(v, T') = \min_{v' \in T'} d_T(v, v')$. These are well defined (we can write max, min instead of the usual sup, inf) by compactness of T. In fact, $(\mathcal{S}(T), d_{\mathcal{S}(T)})$ is a compact metric space; the space of non-empty compact subsets of a compact set with this metric is compact, and $\mathcal{S}(T)$ is a closed subspace of this space given that the limit of connected sets is connected [E10] (these properties are fairly straightforward to prove).

Let us now prove some basic properties of metric trees.

Lemma 6.2.1. *If $z \geq 0$ and T' is a subtree of a metric tree T such that* $\mathrm{dp}(J) \leq z$ *for every* $J \in \bar{\mathrm{comp}}(T')$*, then*

$$\mathrm{diam}(T) \leq \mathrm{diam}(T') + 2z.$$

Proof. Let $u_1, u_2 \in T$ be such that $d_T(u_1, u_2) = \mathrm{diam}(T)$. For $i = 1, 2$, if $u_i \notin T'$, let T_i be the component of $\bar{T}'$ such that $u_i \in T_i$, and otherwise, let $T_i = \{u_i\}$ be a trivial branch of T; in particular, we always have that $\mathrm{anc}(u_i) \in T'$. Then

$$\begin{aligned}
\mathrm{diam}(T) = d_T(u_1, u_2) &\leq d_T(u_1, \mathrm{anc}(T_1)) + d_T(\mathrm{anc}(T_1), \mathrm{anc}(T_2)) + d_T(\mathrm{anc}(T_2), u_2) \\
&\leq \mathrm{dp}(T_1) + \mathrm{diam}(T') + \mathrm{dp}(T_2) \\
&\leq \mathrm{diam}(T') + 2z,
\end{aligned}$$

as desired. □

Lemma 6.2.2. *If T', T'' are branches of a metric tree T such that $T'' \subseteq T'$, then*

$$\mathrm{dp}(T') \geq \mathrm{dp}(T'') + d_T(\mathrm{anc}(T'), \mathrm{anc}(T'')).$$

Proof. Since $T'' \subseteq T'$, we have that $\bar{T}' \subseteq \bar{T}''$. Hence, we know that $\mathrm{anc}(T') \in \bar{T}' \subseteq \bar{T}''$. If $\mathrm{anc}(T') \in T''$, then $\mathrm{anc}(T') \in T'' \cap \bar{T}''$ and so $\mathrm{anc}(T') = \mathrm{anc}(T'')$.

The statement is trivial in this case. Hence we can suppose that $\mathrm{anc}(T') \notin T''$. Let $u \in T''$ such that $\mathrm{dp}(T'') = d_T(\mathrm{anc}(T''), u)$. Since $\mathrm{anc}(T') \notin T''$, any path between u and $\mathrm{anc}(T')$ must go through the unique point of $T'' \cap \bar{T}''$, which is $\mathrm{anc}(T'')$. Hence, $d_T(\mathrm{anc}(T'), u) =$

$d_T(\mathrm{anc}(T'), \mathrm{anc}(T'')) + d_T(\mathrm{anc}(T''), u)$.

We may thus conclude that

$$\begin{aligned}
\mathrm{dp}(T') &= \max_{v \in T'} d_T(\mathrm{anc}(T'), v) \\
&\geq d_T(\mathrm{anc}(T'), u) \\
&= d_T(\mathrm{anc}(T''), u) + d_T(\mathrm{anc}(T'), \mathrm{anc}(T'')) \\
&= \mathrm{dp}(T'') + d_T(\mathrm{anc}(T'), \mathrm{anc}(T'')),
\end{aligned}$$

as desired. $\square$

Given a metric tree T, the following lemma will allow us in Section 6.3 to focus the most technical part of our analysis on a fairly simple subtree T' of T which has most 3 leaves.

Lemma 6.2.3. *If T is a metric tree, then there exists $z \geq 0$ and a subtree T' of T such that*

(i) T' has at most three leaves,

(ii) $\mathrm{dp}(J) \leq z$ for every $J \in \bar{\mathrm{comp}}\, T'$,

(iii) for every $v \in \mathrm{L}(T')$ there exists $J \in \bar{\mathrm{comp}}\, T'$ such that $\mathrm{anc}(J) = v$ and $\mathrm{dp}(J) = z$,

(iv) $|T| \geq |T'| + 4z$.

Proof. Let (T', z) satisfying properties (ii)-(iv) be chosen with z maximum. This is possible given that the subspace of $\mathcal{S}(T) \times \left[0, \frac{|T|}{4}\right]$ of possible solutions is closed hence compact (for instance, using the continuity of $|\cdot|$ and of $\max_{J \in \bar{\mathrm{comp}}(\cdot)} \mathrm{dp}(J)$). Note that $(T, 0)$ satisfies these properties, and so this subspace is nonempty.

If T' has at most three leaves then the lemma holds, and so we assume for a contradiction that T' has at least four leaves.

Let $\delta > 0$ be such that the length of every segment of T' is greater than δ. For every $v \in \mathrm{L}(T')$, let $p_v \in T'$ be chosen so that p_v belongs to the unique segment of T' with end v

and $d_{T'}(v, p_v) = \delta$. Let T'' be the subtree of T' such that $p_v \in \mathrm{L}(T'')$ for every leaf $v \in L(T')$. In other words, T'' is obtained from T' by deleting for each leaf the half-open interval of length δ containing it.

We claim that $z' = z + \delta$ and T'' satisfy properties (ii)-(iv). Note that this claim contradicts the choice of z and T', and thus implies the lemma.

For every $J \in \bar{\mathrm{comp}}\, T''$ either

- $J \in \bar{\mathrm{comp}}\, T'$, or
- $\mathrm{anc}(J) = p_v$ for some $v \in \mathrm{L}(T')$ and J is the union of $T[v, p_v]$ and all the components of $\bar{T}'$ with anchors in $T[v, p_v]$.

In the first case we have $\mathrm{dp}(J) \le z < z'$ by the choice of (T', z) .

Now suppose that J is a component of $\bar{T}''$ satisfying the conditions of the second case. Then $\mathrm{dp}(J) \ge z + \delta = z'$ by Lemma 6.2.2. Moreover, for every $u \in J$ either $u \in T'$, in which case $d_T(p_v, u) \le \delta \le z'$, or there exists $J' \in \bar{\mathrm{comp}}\, T'$ such that $u \in J'$ and $\mathrm{anc}(J') \in T[v, p_v]$, in which case $d_T(p_v, u) \le d_T(p_v, \mathrm{anc}(J')) + \mathrm{dp}(J') \le \delta + z = z'$. It follows that $\mathrm{dp}(J) \le z'$, and so $\mathrm{dp}(J) = z'$. Therefore properties (ii) and (iii) hold; for the latter, note that every $p_v \in \mathrm{L}(T'')$, there is a component of $\bar{\mathrm{comp}}\, T''$ of the second form (the component containing p_v itself).

Finally, $|T''| = |T'| - \delta|\,\mathrm{L}(T')|$. As $|\,\mathrm{L}(T')| \ge 4$ by our assumption, we have $|T| \ge |T'| + 4z \ge |T''| + 4(z + \delta) = |T''| + 4z'$. Thus (iv) also holds. □

A key part of our argument involves dividing the tree T into pieces so that all but one of them have length at least a certain threshold l, and none of them are much bigger. The precise definition that works for our purposes is the following.

For a metric tree T and $l \ge 0$, we say that T is *l-minimal* if $|T| \ge l$, and there exists a decomposition $\{T', T''\}$ of T, such that $|T'| \le l$ and $|T''| \le l$. Note that this implies that $|T| \le 2l$.

Lemma 6.2.4. *If $l > 0$ and T is a metric tree such that $|T| \geq l$, then there exists a decomposition $\{T_0, T_1\}$ of T such that T_1 is l-minimal.*

Proof. Let a decomposition $\{T_0, T_1\}$ of T be chosen so that $|T_1| \geq l$ and subject to this T_1 is minimal. This is possible given that $\mathcal{S}(T)$ is compact (consider the subspace of $\mathcal{S}(T)$ defined by valid T_1, it is not too hard to see that it is closed, in particular using the continuity of $|\cdot|$). We wish to show that T_1 is l-minimal. Let v be the unique point in $T_0 \cap T_1$. Note that if $T_1 = T$, then $T_0 = \{v\}$ will simply be the trivial subtree containing one of the leaves of T.

Suppose first that v is not a leaf of T_1. Then there exist a decomposition $\{T', T''\}$ of T_1 such that T', T'' are non-trivial branches of T_1 with anchor v. If $|T'| \leq l$ and $|T''| \leq l$ then T_1 is l-minimal as desired. Thus we assume without loss of generality that $|T'| > l$. Then $\{T_0 \cup T'', T'\}$ is a decomposition of T contradicting the choice of $\{T_0, T_1\}$.

It remains to consider the case when v is a leaf of T_1. Let v' be chosen in the interior of the segment of T_1 containing v, so that $d_T(v, v') < l$. Then there exists a unique decomposition $\{T', T''\}$ of T_1 such that T', T'' are branches of T_1 with anchor v'. Without loss of generality assume that $v \in T''$. Then T'' is an interval with ends v and v' and so $|T''| < l$. If $|T'| \geq l$ then $\{T_0 \cup T'', T'\}$ again contradicts the choice of $\{T_0, T_1\}$. Otherwise, T_1 is l-minimal, as desired. $\square$

The next result immediately follows from Lemma 6.2.4 by induction on the size of the decomposition.

Lemma 6.2.5. *For every tree T and every $l > 0$ there exist a decomposition $\{T_0, T_1, \ldots, T_k\}$ of T such that $|T_0| \leq l$, and $T_1, \ldots, T_k$ are l-minimal.*

The following lemma is a variant of Lemma 6.2.5 that will allow us to break a metric tree in pieces of roughly the same size.

Lemma 6.2.6. *For every metric tree T and any $0 < l \leq |T|$ there exists a decomposition $\mathcal{T}$ of T such that $l \leq |T'| \leq 3l$ for every $T' \in \mathcal{T}$.*

Proof. Applying Lemma 6.2.5 we obtain a decomposition $\mathcal{T} = \{T_0, \dots, T_k\}$ of T such that $|T_0| \leq l$ and $T_1, \dots, T_k$ are l-minimal. By l-minimality, $l \leq |T_i| \leq 2l$ for every $i \in [k]$. Since $|T| \geq l$, if $k = 0$ then $|T_0| = l$ and $\mathcal{T}$ satisfies the lemma. If $k \geq 1$, since T is connected, T_0 has to share a point with at least one other T_i, without loss of generality say T_1. Then, $l \leq |T_0 \cup T_1| \leq 2l + l = 3l$ and so $\{T_0 \cup T_1, T_2, \dots, T_k\}$ satisfies the statement. $\square$

6.3 Covers

In this section, we analyze the possible (multi) sets of radii of balls that can be used to cover a given metric tree. These sets, which we simply call covers are formally defined as follows.

For a metric tree T, $r \geq 0$ and $v \in T$, we denote by $B_T(v, r)$ the closed ball of radius r with center v, i.e.

$$B_T(v, r) = \{u \in T | d_T(v, u) \leq r\}.$$

A sequence $(r_1, \dots, r_m)$ of non-negative reals is a *cover* of T if there exist $v_1, \dots, v_m \in T$ such that $T = \cup_{i=1}^m B_T(v_i, r_i)$. We write $\mathcal{C}(T)$ for the set of all covers of T.

The equivalent of the following lemma, which precisely determines the minimum radius of a single ball that can cover a given metric tree, is well-known for discrete trees, but we include the proof for completeness.

Lemma 6.3.1. *If T is a tree, then there exists $y \in T$ such that $T = B_T\left(y, \frac{\mathrm{diam}(T)}{2}\right)$. In particular, $(r) \in \mathcal{C}(T)$ for every $r \geq \frac{\mathrm{diam}(T)}{2}$.*

Proof. Let $u, v \in T$ such that $d_T(u, v) = \mathrm{diam}(T)$. Let y be the point on the path $T[u, v]$ at distance exactly $\frac{\mathrm{diam}(T)}{2}$ from both u, v.

Let $w \in T$. We claim that $d(w, y) \leq \frac{\mathrm{diam}(T)}{2}$. Suppose otherwise that $d(w, y) > \frac{\mathrm{diam}(T)}{2}$. Let $\mathcal{T}$ be the decomposition defined by breaking T at y, in other words $\mathcal{T}$ is the set of minimal branches which anchor y. Let $T_w \in \mathcal{T}$ be such that $w \in T_w$, and analogously for u, v. Since u, v are necessarily in different elements of $\mathcal{T}$, we have that, without loss of generality, $T_u \neq T_w$.

Hence, the path $T[u,w]$ must contain y and thus has length $d_T(u,y)+d_T(y,w) > \mathrm{diam}(T)$, which is a contradiction. $\square$

One first application of Lemma 6.3.1 is the following.

Lemma 6.3.2. *If $r_1, \ldots, r_k \in \mathbb{R}_+$ and T is a metric tree such that $|T| \leq \sum_{i=1}^{k} r_i$, then*

$$(r_1, \ldots, r_k) \in \mathcal{C}(T).$$

Proof. By induction on k. If $|T| \leq r_1$ then (r_1) is a cover of T by Lemma 6.3.1 and the lemma holds. Otherwise by Lemma 6.2.4 there exists a decomposition $\{T_0, T_1\}$ of T such that T_1 is r_1-minimal. Then $|T_1| \geq r_1$ and (r_1) is a cover of T_1 by Lemma 6.3.1 since $\mathrm{diam}(T_1) \leq |T_1| \leq 2r_1$. Meanwhile, $|T_0| \leq \sum_{i=1}^{k-1} r_i$ and so $(r_2, \ldots, r_k)$ is a cover of T_0 by the induction hypothesis. It follows that $(r_1, \ldots, r_k)$ is a cover of T, as desired. $\square$

A similar (but discrete) lemma appears in [E3] (a slightly stronger version appears in [E13]). Note that this last lemma already gives an interesting bound on the burning number of graphs. Consider the radii $\{0, \ldots, k\}$ and a tree T on n vertices (which we will consider a metric tree). Then, if $|T| \leq \sum_{i=1}^{k} i = \frac{k(k+1)}{2}$, i.e. if $k \geq \sqrt{2n} + O(1)$, then T can be burned (some of the centers of the balls might not be on vertices, so we might also need to increase each radii by 1, see the proof of Lemma 6.5.4 for more details). This approach to getting a bound of the form $b(G) \leq \sqrt{2n} + O(1)$ first appeared in [E3] (it also appears in [E13, E7]).

The following lemma is the main result of this section and is one of the key parts of our proof of Theorem 6.1.2. It shows that every metric tree T has covers of size two with the sum of the radii at most $\frac{|T|}{2}$, and a broad range of choices of radii. There is a certain technical trade-off here, where for certain trees the range is smaller but then so is the sum of radii. The precise values of parameters matter here and are just right to make the following arguments work.

Lemma 6.3.3. *If T is an l-minimal tree for some $l > 0$, then there exists $0 \leq a \leq \min\{\frac{l}{2} - \frac{|T|}{4}, \frac{|T|}{12}\}$ such that*

- $$\left(\frac{|T|}{4} - 3a - x, \frac{|T|}{4} + a + x\right)$$

  is a cover of T *for all* $0 \leq x \leq \frac{|T|}{4} - 3a$*, and*

- $$\left(\frac{|T|}{4} + a + x\right)$$

  is a cover of T *for every* $x \geq \frac{|T|}{4} - 3a$*.*

Proof. Let T' and z be obtained by applying Lemma 6.2.3 to T. Let $v_0, v_1, v_2, v_3 \in T'$ be chosen so that $L(T') \subseteq \{v_1, v_2, v_3\}$, v_0 is the unique point shared by the paths $T[v_1, v_2]$, $T[v_1, v_3]$ and $T[v_2, v_3]$.[3] As T' has at most three leaves, by condition (i) of Lemma 6.2.3, such a choice is always possible. Let $l_i = |T[v_0, v_i]|$ for $i = 1, 2, 3$. Without loss of generality, suppose $l_1 \leq l_2 \leq l_3$.

Then $|T'| = l_1 + l_2 + l_3$ and as T' satisfies condition (iv) of Lemma 6.2.3 we have

$$|T| \geq l_1 + l_2 + l_3 + 4z. \tag{6.1}$$

We have $\mathrm{diam}(T') = \max_{u,v \in L(T')} d_T(u, v) = d_T(v_2, v_3) = l_2 + l_3$. From Lemma 6.2.1 (using condition (ii) of Lemma 6.2.3) it follows that $\mathrm{diam}(T) \leq l_2 + l_3 + 2z$.

We show that

$$a = \max\left\{0, \frac{l_1 + l_2}{2} + z - \frac{|T|}{4}\right\}$$

satisfies the conditions of the lemma.

First, we need to verify that $a \leq \frac{|T|}{12}$ and $a \leq \frac{l}{2} - \frac{|T|}{4}$. The first of this inequalities follows immediately from (6.1), as

$$\frac{l_1 + l_2}{2} + z \leq \frac{l_1 + l_2 + l_3 + 3z}{3} \leq \frac{|T|}{3},$$

[3] More explicitly, if T' has exactly three leaves we choose v_0 to be the unique branch point of T' and v_1, v_2, v_3 to be the leaves. If T has at most two leaves we choose v_1 and v_2 so that $L(T') \subseteq \{v_2, v_3\}$ and choose $v_0 = v_1 \in T'$ arbitrarily.

and so $a \leq \max\left\{0, \frac{|T|}{3} - \frac{|T|}{4}\right\} = \frac{|T|}{12}$.

Showing that $a \leq \frac{l}{2} - \frac{|T|}{4}$ takes a bit more effort. As T is l-minimal there exists a decomposition $\{S_0, S_1\}$ of T such that $|S_0| \leq l$ and $|S_1| \leq l$. In particular, $|T| = |S_0| + |S_1| \leq 2l$, and so $\frac{l}{2} - \frac{|T|}{4} \geq 0$. Thus we only need to verify the case where $a = \frac{l_1+l_2}{2} + z - \frac{|T|}{4} > 0$. We thus want to prove that $l \geq 2\left(a + \frac{|T|}{4}\right) = l_1 + l_2 + 2z$. Substituting (6.1) in the above inequality $a > 0$, we get $l_1 + l_2 > l_3$. In particular, $l_1 > 0$ and so T' has exactly three leaves v_1, v_2 and v_3. By condition (iii) of Lemma 6.2.3 there exists a component T_i' of $\bar{T}'$ with $v_i = \mathrm{anc}(T_i')$ and $\mathrm{dp}(T_i') = z$ for every $i \in \{1, 2, 3\}$. Thus there exists $w_i \in T_i'$ such that $d_T(v_i, w_i) = z$.

Without loss of generality we assume that there exist distinct $i, j \in \{1, 2, 3\}$ such that $w_i, w_j \in S_0$. Then

$$l \geq |S_0| \geq d_T(w_i, w_j) = d_T(w_i, v_i) + d_T(v_i, v_j) + d_T(v_j, w_j) = z + (l_i + l_j) + z \geq l_1 + l_2 + 2z,$$

as desired.

For $x \geq 0$, let $r_1 = \frac{|T|}{4} + a + x$ and $r_2 = \frac{|T|}{4} - 3a - x$ (note that since $a, x \geq 0$, we always have $r_1 \geq r_2$). It remains to show that (r_1) or (r_1, r_2) is a cover of T.

If $r_1 \geq \frac{l_2+l_3}{2} + z \geq \mathrm{diam}(T)/2$, then (r_1) is a cover of T by Lemma 6.3.1, and so the lemma holds if $x \geq \frac{l_2+l_3}{2} + z - a - \frac{|T|}{4}$. Note that if we also have that $x \leq \frac{|T|}{4} - 3a$, then we can say that (r_1, r_2) is a cover, given that (r_1) alone is a cover. This explains the different cutoff on x between the statement of the theorem and in this proof.

Thus we assume $x \leq \frac{l_2+l_3}{2} + z - a - \frac{|T|}{4}$. In this regime we will show that (r_1, r_2) is a cover of T. First, we have

$$\begin{aligned}
r_2 &= \frac{|T|}{4} - 3a - x \\
&\geq \frac{|T|}{2} - 2a - \frac{l_2 + l_3}{2} - z \\
&= \min\left\{\frac{|T|}{2} - \frac{l_2+l_3}{2} - z, \frac{|T|}{2} - \left(l_1 + l_2 + 2z - \frac{|T|}{2}\right) - \frac{l_2+l_3}{2} - z\right\} \\
&\geq \min\left\{\frac{l_1+l_2+l_3+4z}{2} - \frac{l_2+l_3}{2} - z, l_1 + l_2 + l_3 + 4z - l_1 - l_2 - 2z - \frac{l_2+l_3}{2} - z\right\} \\
&= \min\left\{\frac{l_1}{2} + z, \frac{l_3 - l_2}{2} + z\right\} \\
&\geq z.
\end{aligned}$$

In particular, we are necessarily in the case $x \leq \frac{|T|}{4} - 3a$, given that otherwise $r_2 < 0 \leq z$.

Note that by our earlier assumption $r_2 - z \leq r_1 - z \leq \frac{l_2+l_3}{2}$. In particular, it is possible to choose $p_1, p_2 \in T[v_2, v_3]$ so that $d_T(p_1, v_2) = r_1 - z$ and $d_T(p_2, v_3) = r_2 - z$. We first wish to show that $T' \subseteq B_T(p_1, r_1 - z) \cup B_T(p_2, r_2 - z)$. Informally, we use the smaller radii to cover the end of longest branch of T' and we use the larger radii to cover the 2nd longest branch of T' (including v_0), in order to, as we will see below, maximize the "overflow" onto the shortest branch of T'.

First, we show that $T[v_2, v_3] \subseteq B_T(p_1, r_1 - z) \cup B_T(p_2, r_2 - z)$. To establish this it suffices to show that $d_T(p_1, p_2) \leq (r_1 - z) + (r_2 - z)$. By the above remark (that $r_2 - z \leq r_1 - z \leq \frac{l_2+l_3}{2}$), p_1 is closer to v_2 than p_2 is, and similarly p_2 is closer to v_3 than p_1 is (in other words, these points appear in the order $v_2 - p_1 - p_2 - v_3$ on $T[v_2, v_3]$) and so $d_T(v_2, v_3) = l_2 + l_3 - (r_2 - z) - (r_1 - z)$.

Therefore,

$$
\begin{aligned}
d_T(p_1,p_2) - ((r_1-z)+(r_2-z)) &= l_2+l_3-2(r_2-z)-2(r_1-z) \\
&= l_2+l_3-|T|+4a+4z \\
&= \max\{l_2+l_3-|T|+4z, l_2+l_3-2|T|+2l_1+2l_2+8z\} \\
&\le \max\{-l_1, l_2-l_3\} \\
&\le 0,
\end{aligned}
$$

as desired.

Next we show that $T[v_1,v_2] \subseteq B_T(p_1,r_1-z)$. As we already have seen that $v_2 \in B_T(p_1,r_1-z)$, it suffices to show that $d_T(p_1,v_1) \le r_1-z$. First note that

$$
2(r_1-z) \ge \frac{|T|}{2}+2a-2z \ge \frac{|T|}{2}+\left(l_1+l_2+2z-\frac{|T|}{2}\right)-2z = l_1+l_2.
$$

Thus,

$$
\begin{aligned}
d_T(p_1,v_1) &= d_T(p_1,v_0)+d_T(v_0,v_1) \\
&= |(r_1-z)-l_2|+l_1 \\
&= \max\{r_1-z-(l_2-l_1), l_1+l_2-(r_1-z)\} \\
&\le r_1-z.
\end{aligned}
$$

It follows that $T' = T[v_2,v_3] \cup T[v_1,v_2] \subseteq B_T(p_1,r_1-z) \cup B(p_2,r_2-z)$. Finally, we show that $T \subseteq B_T(p_1,r_1) \cup B(p_2,r_2)$, i.e. we show that for every $u \in T$ we have $d_T(u,p_i) \le r_i$ for some $i \in \{1,2\}$. We already established this for $u \in T'$, so we may assume that $u \in \bar{T}'$. Let v be the anchor of the component T'' of $\bar{T}'$ containing u. Then $v \in T'$ and so $d_T(v,p_i) \le r_i - z$ for some $i \in \{1,2\}$. Moreover, $d_T(u,v) \le \mathrm{dp}(T'') \le z$, where the last inequality holds by the choice of T' (condition (ii) of Lemma 6.2.3). Thus $d_T(u,p_i) \le d_T(u,v)+d_T(v,p_i) \le r_i$, as desired. $\square$

In [E2, E16], it is used that if there are many leaves in a tree, we can cut them off, burn (obtain a cover of) the remaining subtree and then increment all radii by 1 to burn the entire tree. Here, we have pushed this idea further by using Lemma 6.2.3 to cut off as much as we need to obtain a subtree with at most 3 remaining leaves.

6.4 Random covers of metric trees

In this section we prove a fractional version of Theorem 6.1.2 for metric trees. To state it we first need to formalize the notion of a random cover of a metric tree and define the necessary parameters of such a cover.

Let T be a metric tree. Endow $\mathcal{C}(T)$ with the topology of $\sqcup_{m\in\mathbb{N}}\mathbb{R}_+^m$. Let ν be a finite Borel measure on $\mathcal{C}(T)$. (Our main focus is the case when ν is a probability measure.)

The key parameter of interest to us is the *expectation measure* $E\nu$ of ν, which is a Borel measure on $\mathbb{R}_+$ defined as follows. For $\boldsymbol{r} = (r_1, \ldots, r_m) \in \mathcal{C}(T)$ and $B \subseteq \mathbb{R}_+$, let $\#(B, \boldsymbol{r})$ denote the number of components of $\boldsymbol{r}$ (i.e. radii) that lie in B, i.e.

$$\#(B, \boldsymbol{r}) = |\{i \mid 1 \leq i \leq m, r_i \in B\}|.$$

Then, we can define

$$E\nu(B) = \int \#(B, \boldsymbol{r}) d\nu(\boldsymbol{r})$$

for every Borel $B \subseteq \mathbb{R}_+$.[4] In particular, when ν is a probability measure, then $E\nu(B)$ is the expected number of radii in a random cover $\boldsymbol{r}$ that lie in B. Note also that when ν is concentrated on covers of size m then $E\nu$ is the sum of m marginals of ν.

Before stating the main result of this section, we need to introduce a few more technical definitions. First, to convert the random covers in to a particular uniform one in the next

[4]By rescaling we can assume that ν is a probability measure. Consider each $\boldsymbol{r} = (r_1, \ldots, r_m) \in \mathcal{C}(T)$ as a sum of discrete measures $\sum_{i=1}^{m} \delta_{r_i}$. Then ν is a *point process* on $\mathbb{R}_+$, and $E\nu$ is its *expectation or intensity measure*. It is well-known that $E\nu$ is a well-defined and is indeed a measure, see e.g. [E17, Lemma 1.1.1].

section, it will be convenient to ensure that the covers we consider are somewhat tame. The precise notion of tameness is given in the next definition. For $l \in \mathbb{R}$, we say that a cover $\boldsymbol{r} = (r_1, \ldots, r_m)$ of T is *l-good* if $\|\boldsymbol{r}\|_1 = \sum_{i=1}^m r_i \leq |T| + l$. Let $\mathcal{C}(T, l) \subseteq \mathcal{C}(T)$ denote the set of l-good covers of T.

Secondly we will want the expectation measure of our distribution on covers to be close to uniform and as a result many of the calculation involve the uniform measures on real intervals. For $b \geq a \geq 0$, let $\boldsymbol{U}[a, b]$ denote the uniform probability (Borel) measure on $[a, b]$. For future reference, the following is useful identity relating the measures $\boldsymbol{U}[a, b]$, $\boldsymbol{U}[a, c]$ and $\boldsymbol{U}[b, c]$ for $c \geq b \geq a$, $c > a$:

$$\boldsymbol{U}[a, c] = \frac{b-a}{c-a}\boldsymbol{U}[a, b] + \frac{c-b}{c-a}\boldsymbol{U}[b, c]. \tag{6.2}$$

We are finally ready to state the main result of this section.

Theorem 6.4.1. *If $\varepsilon, r > 0$ and T is a metric tree such that $|T| \geq 24\varepsilon^{-1} r$, then there exists a probability measure ν on $\mathcal{C}(T, r)$ such that*

$$E\nu \leq (1+\varepsilon)\frac{|T|}{r}\boldsymbol{U}[0, r]. \qquad {}^{5}$$

Informally, Theorem 6.4.1 implies that if T is large enough compared to r then there exists a distribution on (r-good) covers of T that uses only radii in $[0, r]$, uses all such radii approximately equally often, and moreover the expected sum of the radii in a cover is not much larger than $\frac{|T|}{2}$.[6]

The rest of the section is occupied by the proof of Theorem 6.4.1 starting with introducing additional notation.

[5]For two measures on the same measure space μ_1, μ_2 we write $\mu_1 \leq \mu_2$ if $\mu_1(B) \leq \mu_2(B)$ for every measurable B.

[6]The last property might not be obvious from the statement, but will be made clearer by subsequent calculations. Note moreover that if T is an interval then the sum of the radii in every cover of T is at least $\frac{|T|}{2}$ so this property and the coefficient $(1+\varepsilon)\frac{|T|}{r}$ in the theorem statement can not be improved, except for eliminating the ε error term.

For a Borel measure μ on $\mathbb{R}_+$, let

$$\mathsf{m}(\mu) = 2 \int x d\mu(x),$$

i.e. $\mathsf{m}(\mu)$ is the first moment of μ rescaled for convenience by a factor of two. If ν is a probability measure on covers $\mathcal{C}(T)$ then $\mathsf{m}(E\nu)$ is twice the expected value of the sum of radii in a cover chosen according to ν, i.e. the expected maximum length of an interval that can be covered by such a cover. Due to this property we use $\mathsf{m}(E\nu)$ to keep track of the "quality" of the distribution ν.

Note that m is a linear map from the space of Borel measures on $\mathbb{R}_+$ to $\mathbb{R}_+$ and that

$$\mathsf{m}(\boldsymbol{U}[a, b]) = b + a. \tag{6.3}$$

We prove of Theorem 6.4.1 iteratively for smaller and smaller ε. A single iteration hinges on us finding a probability measure on $\mathcal{C}(T)$ that can be complemented by others with expectation measures of the form $c_i\boldsymbol{U}[0, a_i]$ for $a_i < r$ to produce the desired measure with expectation roughly uniform on the interval $[0, r]$.

The following definition makes the properties we need precise. For $r, \delta > 0$, we say that a probability measure ν on $\mathcal{C}(T)$ is (r, δ)-*controlled* if there exist $\alpha_1, \dots, \alpha_k \geq 0$ and $a_1, a_2, \dots, a_k \in [0, (1-\delta)r]$ such that

$$E\nu = \sum_{i=1}^{k} \alpha_i \boldsymbol{U}[a_i, r],$$

With the main definitions in place, we collect all the basic properties of measures on $\mathcal{C}(T, l)$ that we need in the following lemma.

Lemma 6.4.2. *Let T be a metric tree.*

(a) *Let $f_1, f_2, \dots, f_m : [0, 1] \to \mathbb{R}_+$ be affine functions such that $(f_1(x), f_2(x), \dots, f_m(x))$ is a cover of T for every $x \in [0, 1]$. Let $a_i = \min\{f_i(0), f_i(1)\}$ and $b_i = \max\{f_i(0), f_i(1)\}$*

for $i = 1, \ldots, m$ *and let* $l = \max\{\sum_{i=1}^m f_i(0), \sum_{i=1}^m f_i(1)\} - |T|$. *Then there exists a probability measure on* ν *on* $\mathcal{C}(T, l)$ *such that*

$$E\nu = \sum_{i=1}^{m} \boldsymbol{U}[a_i, b_i]. \tag{6.4}$$

(b) *Let* $\{T_1, \ldots, T_k\}$ *be a decomposition of a tree* T. *Let* $r, \delta, l_1, \ldots, l_k \geq 0$ *be such that for every* $1 \leq i \leq k$ *there exists an* (r, δ)*-controlled probability measure* ν_i *on* $\mathcal{C}(T_i, l_i)$. *Then there exists an* (r, δ)*-controlled probability measure* ν *on* $\mathcal{C}(T, \sum_{i=1}^k l_i)$, *such that*

$$\mathsf{m}(E\nu) = \sum_{i=1}^{k} \mathsf{m}(E\nu_i). \tag{6.5}$$

(c) *Let* $l \geq 0$, *let* $\nu_0, \nu_1, \ldots, \nu_k$ *be probability measures on* $C(T, l)$ *and let* $p_0, \ldots, p_k \geq 0$ *be such that* $\sum_{i=0}^k p_i = 1$. *Then there exists a probability measure* ν *on* $C(T, l)$ *such that*

$$E\nu = \sum_{i=0}^{k} p_i \cdot E\nu_i. \tag{6.6}$$

Proof. **(a):** Let the map $F : [0, 1] \to \mathcal{C}(T, l)$ be defined by

$$F(x) = (f_1(x), f_2(x), \ldots, f_m(x)).$$

As F is continuous it is Borel measurable and we can define $\nu = \boldsymbol{U}[0, 1] \circ F^{-1}$ to be the image measure of the uniform probability measure on $[0, 1]$ under the map F.[7] Then ν is a probability measure on $\mathcal{C}(T, l)$ and the marginals $\nu_1, \ldots, \nu_i$ of ν satisfy $\nu_i = \boldsymbol{U}[0, 1] \circ f_i^{-1}$. As the expectation measure of ν is the sum of its marginals we have

$$E\nu = \sum_{i=1}^{m} \left(\boldsymbol{U}[0, 1] \circ f_i^{-1}\right). \tag{6.7}$$

[7] I.e. for every Borel $B \subseteq \mathbb{R}^m$ we have $\nu(B) = (\boldsymbol{U}[0, 1])(F^{-1}(B))$

As f_i is an affine bijection from $[0,1]$ to $[a_i, b_i]$ we have $\boldsymbol{U}[0,1] \circ f_i^{-1} = \boldsymbol{U}[a_i, b_i]$, and so (6.7) implies (6.4), as desired.

(b): Let $\boldsymbol{r}^i = (r_1^i, \ldots, r_{m_i}^i) \in \mathcal{C}(T_i, l_i)$ for $i = 1, \ldots, k$. Define

$$\begin{aligned}\boldsymbol{R} &= \boldsymbol{R}(\boldsymbol{r}^1, \ldots, \boldsymbol{r}^k) \\ &= (r_1^1, \ldots, r_{m_1}^1, r_1^2, \ldots, r_{m_2}^2, \ldots, r_1^k, \ldots, r_{m_k}^k)\end{aligned}$$

to be the concatenation of these covers. Then $\boldsymbol{R}$ is a cover of T as there exist balls of radii $r_1^i, \ldots, r_{m_i}^i$ whose union includes T_i for every i, and so the union of such balls over all i is T. Note also that

$$\|\boldsymbol{R}\|_1 = \sum_{i=1}^{k} \|\boldsymbol{r}^i\|_1 \leq \sum_{i=1}^{k} (|T_i| + l_i) = |T| + \sum_{i=1}^{k} l_i,$$

and so $\boldsymbol{R}$ is $(\sum_{i=1}^k l_i)$-good.

Thus $\boldsymbol{R}$ is a Borel measurable map from $\prod_{i=1}^k \mathcal{C}(T_i, l_i)$ to $\mathcal{C}(T, \sum_{i=1}^k l_i)$, and we define $\nu = (\otimes_{i=1}^k \nu_i) \circ \boldsymbol{R}^{-1}$. That is, ν is the probability measure on $(\sum_{i=1}^k l_i)$-good covers of T obtained by taking the union (more formally, a concatenation) of l_i-good covers of T_i chosen for $i = 1, \ldots, k$ independently at random according to the probability measure ν_i. Then $E\nu = \sum_{i=1}^k E\nu_i$, implying that ν is (r, δ)-controlled as each ν_i is, and implying (6.5) by linearity of $\mathsf{m}(\cdot)$.

(c): Let $\nu = \sum_{i=0}^k p_i \nu_i$. That is ν the probability measure on l-good covers of T obtained by randomly choosing an index $\{0, \ldots, k\}$ with i chosen with probability p_i and then choosing a cover of T according to the probability measure ν_i. The identity (6.5) holds as the expectation measure is linear. $\square$

Our next two lemmas establishes existence of a measure that is needed to perform a single iteration in the proof of Theorem 6.4.1 as outlined above. The first lemma finds a measure on covers of a single part of an appropriate decomposition of T, and the second combines the measures for each part of the decomposition.

Lemma 6.4.3. *If $0 < r$, $0 < \delta \leq \frac{1}{2}$ are real and T is a $2(1-\delta)r$-minimal metric tree, then there exists an (r, δ)-controlled probability measure ν on $\mathcal{C}(T, 0)$ such that*

$$\mathsf{m}(E\nu) \leq |T| + \delta r.$$

Proof. Let $l = 2(1-\delta)r$. Let $0 \leq a \leq \frac{l}{2} - \frac{|T|}{4}$ be as in Lemma 6.3.3. Then $|T| \geq l \geq \frac{|T|}{2} + 2a$.

Suppose first that $\frac{|T|}{2} - 2a \leq (1-\delta)r$. By Lemma 6.3.3 we have that

$$\left(\frac{|T|}{4} + a + x\right)$$

is a cover of T for every $x \geq \frac{|T|}{4} - 3a$. In other words, (x') is a cover of T for every $x' \geq \frac{|T|}{4} + a + \left(\frac{|T|}{4} - 3a\right) = \frac{|T|}{2} - 2a$. In particular, this holds if $x' \in [(1-\delta)r, r]$ by our previous assumption. Hence, by Lemma 6.4.2(a) there exists a probability measure ν on $\mathcal{C}(T, r - |T|)$ such that

$$E\nu = \boldsymbol{U}[(1-\delta)r, r].$$

Given that $|T| \geq 2(1-\delta)r \geq r$ we have that $\mathcal{C}(T, r - |T|) \subseteq \mathcal{C}(T, 0)$ and so ν is a probability distribution on $\mathcal{C}(T, 0)$.

It is direct from the definition that ν is (r, δ)-controlled (take $k = 1, \alpha_1 = 1, a_1 = (1-\delta)r$). Using (6.3), we have

$$\mathsf{m}(E\nu) = (2-\delta)r = l + \delta r \leq |T| + \delta r.$$

It follows that ν satisfies the conditions of the lemma.

Thus we assume $\frac{|T|}{2} - 2a \geq (1-\delta)r$. For $y \in [0, 1]$,

$$\left(\max\left\{0, \frac{|T|}{4} - 3a - y\left(r - \frac{|T|}{4} - a\right)\right\}, \frac{|T|}{4} + a + y\left(r - \frac{|T|}{4} - a\right)\right) \tag{6.8}$$

is a cover of T by Lemma 6.3.3 applied with $x = y\left(r - \frac{|T|}{4} - a\right)$. Note that as $r \geq \frac{l}{2} \geq \frac{|T|}{4} + a$, we indeed have $x \geq 0$.

Suppose further that $\frac{|T|}{2} - 2a \geq r$. By increasing the first component , we further deduce that

$$\left(\frac{|T|}{4} + a - y\left(r - \frac{|T|}{4} + 3a\right), \frac{|T|}{4} + a + y\left(r - \frac{|T|}{4} - a\right)\right)$$

is a cover of T for every such y. That this is actually an increase of the first radii is a consequence of the last supposition (to show that the new first radii is non-negative) and of the fact that $y \leq 1$ (for showing the inequality in the second case).

By Lemma 6.4.2(a) there exists a probability distribution ν on $\mathcal{C}\left(T, \frac{|T|}{2} + 2a - |T|\right)$ such that

$$E\nu = \boldsymbol{U}\left[\frac{|T|}{2} - r - 2a, \frac{|T|}{4} + a\right] + \boldsymbol{U}\left[\frac{|T|}{4} + a, r\right]. \tag{6.9}$$

As $\frac{|T|}{2} + 2a \leq l \leq |T|$, $\mathcal{C}\left(T, \frac{|T|}{2} + 2a - |T|\right) \subseteq \mathcal{C}(T, 0)$ and so ν is a probability distribution on $\mathcal{C}(T, 0)$. Using (6.2) we have

$$\boldsymbol{U}\left[\frac{|T|}{2} - r - 2a, r\right] = \frac{r - \frac{|T|}{4} + 3a}{2r - \frac{|T|}{2} + 2a}\boldsymbol{U}\left[\frac{|T|}{2} - r - 2a, \frac{|T|}{4} + a\right] + \frac{r - \frac{|T|}{4} - a}{2r - \frac{|T|}{2} + 2a}\boldsymbol{U}\left[\frac{|T|}{4} + a, r\right],$$

so we can rewrite (6.9) as

$$E\nu = \frac{2r - \frac{|T|}{2} + 2a}{r - \frac{|T|}{4} + 3a}\boldsymbol{U}\left[\frac{|T|}{2} - r - 2a, r\right] + \frac{4a}{r - \frac{|T|}{4} + 3a}\boldsymbol{U}\left[\frac{|T|}{4} + a, r\right].$$

As $0 \leq \frac{|T|}{2} - r - 2a \leq \frac{|T|}{4} + a \leq \frac{l}{2} = (1-\delta)r$, it follows that ν is (r, δ)-controlled. Using (6.3) and (6.9), we have

$$\mathsf{m}(E\nu) = \left(\left(\frac{|T|}{2} - r - 2a\right) + \left(\frac{|T|}{4} + a\right)\right) + \left(\left(\frac{|T|}{4} + a\right) + r\right) = |T|.$$

Thus ν satisfies the lemma in this case.

It remains to consider the case $(1-\delta)r \geq \frac{|T|}{2} - 2a \leq r$. By increasing the first component in the cover (6.8) in a different way, we see that

$$\left(\frac{|T|}{4}+a-y\left(\frac{|T|}{4}+a\right), \frac{|T|}{4}+a+y\left(r-\frac{|T|}{4}-a\right)\right)$$

is a cover of T for every $y \in [0,1]$. Verifying that the new first radii is non-negative is direct, and furthermore this new radii is indeed an increase in the other case as a consequence of $\frac{|T|}{2}-2a \leq r$. As

$$\max_{y\in\{0,1\}}\left(\frac{|T|}{4}+a-y\left(\frac{|T|}{4}+a\right)+\frac{|T|}{4}+a+y\left(r-\frac{|T|}{4}-a\right)\right)=\max\left\{r, \frac{|T|}{2}+2a\right\}\leq l\leq |T|,$$

by Lemma 6.4.2(a) there exists a probability distribution ν on $\mathcal{C}(T,0)$ such that

$$E\nu = \boldsymbol{U}\left[0, \frac{|T|}{4}+a\right]+\boldsymbol{U}\left[\frac{|T|}{4}+a, r\right]. \tag{6.10}$$

As in the previous case, we can use (6.2) to get

$$\boldsymbol{U}\left[0,r\right]=\frac{\frac{|T|}{4}+a}{r}\boldsymbol{U}\left[0, \frac{|T|}{4}+a\right]+\frac{r-\frac{|T|}{4}-a}{r}\boldsymbol{U}\left[\frac{|T|}{4}+a, r\right],$$

we rewrite (6.10) as

$$E\nu = \frac{r}{\frac{|T|}{4}+a}\boldsymbol{U}[0,r]+\frac{\frac{|T|}{2}+2a-r}{\frac{|T|}{4}+a}\boldsymbol{U}\left[\frac{|T|}{4}+a, r\right].$$

As observed earlier $\frac{|T|}{4}+a \leq (1-\delta)r$ and so ν is (r,δ)-controlled. Finally, using (6.3) and (6.10), we have

$$\mathsf{m}(E\nu) = r+\frac{|T|}{2}+2a \leq |T|+\delta r$$

and ν satisfies the lemma in this last case. $\square$

Lemma 6.4.4. *For every $0<\delta\leq 1/2$, every $r>0$ and every metric tree T there exists an*

(r,δ)-controlled probability measure ν on $\mathcal{C}(T,r)$ such that

$$\mathsf{m}(E\nu) \le (1+\delta)|T| + 2r.$$

Proof. Let $l = 2(1-\delta)r \ge r$. By Lemma 6.2.5 there exists a decomposition $\{T_0, T_1, \dots, T_k\}$ of T such that $|T_0| \le l$, and $T_1, \dots, T_k$ are l-minimal. Note that

$$|T| \ge \sum_{i=1}^{k} |T_i| \ge kl \ge kr.$$

By Lemma 6.4.3, for every $i \in [k]$ there exists an (r,δ)-controlled distribution ν_i on $\mathcal{C}(T_i, 0)$ such that $\mathsf{m}(E\nu_i) \le |T_i| + \delta r$.

Meanwhile, by Lemma 6.3.1, $\operatorname{diam}(T_0) \le \frac{|T_0|}{2} \le (1-\delta)r$ and so (x) is a cover of T_0 for every $x \ge (1-\delta)r$, and in particular for $x \in [(1-\delta)r, r]$. Applying Lemma 6.4.2(a), there exists a probability measure ν_0 on $\mathcal{C}(T_0, r - |T_0|) \subseteq \mathcal{C}(T_0, r)$ such that $E\nu_0 = \boldsymbol{U}[(1-\delta)r, r]$, and so ν_0 is (r,δ)-controlled and $\mathsf{m}(E\nu_0) \le (2-\delta)r$.

Then, by Lemma 6.4.2(b) there exists an (r,δ)-controlled probability measure ν on $\mathcal{C}(T,r)$, such that

$$\mathsf{m}(E\nu) \le \sum_{i=1}^{k} |T_i| + k\delta r + (2-\delta)r \le |T| + k\delta r + 2r \le (1+\delta)|T| + 2r,$$

as desired. □

With all the ingredients in place, we start the proof of the main result of this section, which we restate for convenience.

Theorem 6.4.1. *If $\varepsilon, r > 0$ and T is a metric tree such that $|T| \ge 24\varepsilon^{-1}r$, then there exists a probability measure ν on $\mathcal{C}(T,r)$ such that*

$$E\nu \le (1+\varepsilon)\frac{|T|}{r}\boldsymbol{U}[0,r]. \quad [8]$$

[8] For two measures on the same measure space μ_1, μ_2 we write $\mu_1 \le \mu_2$ if $\mu_1(B) \le \mu_2(B)$ for every

Proof. First we show that the theorem holds for $\varepsilon \geq 12/11$. Let $k = \lceil \frac{|T|}{r} \rceil \leq \frac{|T|}{r} + 1$. By Lemma 6.3.2

$$\Bigg(\underbrace{x, \dots, x}_{k \text{ times}}, \underbrace{r-x, \dots, r-x}_{k \text{ times}} \Bigg) \in \mathcal{C}(T)$$

for every $x \in [0, r]$, since the sum of these radii is $kr \geq |T|$. By Lemma 6.4.2 there exists a probabilistic distribution ν on $\mathcal{C}(T, kr - |T|) \subseteq \mathcal{C}(T, r)$ such that

$$E\nu = 2k \cdot \boldsymbol{U}[0, r] \leq 2\left(\frac{|T|}{r} + 1\right) \boldsymbol{U}[0, r] \leq 2\left(1 + \frac{\varepsilon}{24}\right) \frac{|T|}{r} \boldsymbol{U}[0, r] \leq (1+\varepsilon) \frac{|T|}{r} \boldsymbol{U}[0, r],$$

as desired, where the last inequality holds by the choice of ε to be relatively large.

We now prove that the theorem holds for ε such that $\varepsilon \geq \frac{15}{n}$ for some integer n by induction on n, which implies that the theorem holds all $\varepsilon > 0$. The base case for $n \leq 13$ was established above.

Suppose now $\frac{15}{n-1} > \varepsilon \geq \frac{15}{n}$ for $n \geq 14$. Let $\varepsilon' = \varepsilon + \frac{\varepsilon^2}{13}, \delta = \frac{\varepsilon}{8}$. Then

$$\varepsilon' \geq \frac{15}{n}\left(1 + \frac{15}{13n}\right) \geq \frac{15}{n-1}$$

by our lower bound on n. Thus the theorem holds for ε' by the induction hypothesis.

By Lemma 6.4.4 there exists a probability distribution ν_0 on $\mathcal{C}(T, r)$, as well as reals $\alpha_1, \dots, \alpha_k \geq 0$ and $0 \leq a_1, a_2, \dots, a_k \leq (1-\delta)r$, such that

$$E\nu_0 = \sum_{i=1}^{k} \alpha_i \boldsymbol{U}[a_i, r],$$

and

$$\mathsf{m}(E\nu_0) \leq (1+\delta)|T| + 2r \leq \left(1 + \frac{\varepsilon}{8}\right)|T| + \frac{\varepsilon}{12}|T| \leq \left(1 + \frac{\varepsilon}{4}\right)|T|, \tag{6.11}$$

where the second to last inequality uses the choice of δ and the condition $r \leq \frac{\varepsilon|T|}{24}$ in the theorem statement.

measurable B.

By (6.3) and the linearity of $\mathsf{m}(\cdot)$ we have $\mathsf{m}(E\nu_0) = \sum_{i=1}^{k} \alpha_i(r+a_i)$, and so (6.11) implies

$$\sum_{i=1}^{k} \alpha_i(r+a_i) \leq \left(1+\frac{\varepsilon}{4}\right)|T|. \tag{6.12}$$

Note that $|T| \geq \frac{24r}{\varepsilon} \geq \frac{24a_i}{\varepsilon'}$ for every $i \in [k]$. Then by the induction hypothesis, for each $i \in [k]$ there exists a probabilistic distribution ν_i on $\mathcal{C}(T, a_i) \subseteq \mathcal{C}(T, r)$ such that $E\nu_i \leq (1+\varepsilon')\frac{|T|}{a_i}\boldsymbol{U}[0, a_i]$. Let

$$q = (1+\varepsilon')|T| + \sum_{i=1}^{k} \frac{\alpha_i a_i^2}{r-a_i}, \qquad p_0 = \frac{(1+\varepsilon')|T|}{q}$$

and let

$$p_i = \frac{\alpha_i a_i^2}{(r-a_i)q}$$

for $i \in [k]$. Of course, $\sum_{i=0}^{k} p_i = 1$. By Lemma 6.4.2(c) here exists is a probability measure ν on $\mathcal{C}(T, r)$ such that

$$\begin{aligned}
E\nu &\leq p_0 \cdot E\nu_0 + \sum_{i=1}^{k} (p_i \cdot E\nu_i) \\
&= p_0 \sum_{i=1}^{k} \left(\frac{p_i}{p_0} \cdot E\nu_i + \alpha_i \boldsymbol{U}[a_i, r]\right) \\
&\leq p_0 \sum_{i=1}^{k} \left(\frac{\alpha_i a_i^2}{(1+\varepsilon')|T|(r-a_i)} \cdot (1+\varepsilon')\frac{|T|}{a_i}\boldsymbol{U}[0, a_i] + \alpha_i \boldsymbol{U}[a_i, r]\right) \\
&= p_0 \sum_{i=1}^{k} \left(\frac{\alpha_i r}{(r-a_i)}\left(\frac{a_i}{r}\boldsymbol{U}[0, a_i] + \frac{r-a_i}{r}\boldsymbol{U}[a_i, r]\right)\right) \\
&= p_0 \left(\sum_{i=1}^{k} \frac{\alpha_i r}{(r-a_i)}\right)\boldsymbol{U}[0, r],
\end{aligned}$$

using in particular (6.2).

Thus it suffices to show that

$$\frac{(1+\varepsilon)|T|}{r} \geq p_0 \left(\sum_{i=1}^{k} \frac{\alpha_i r}{(r-a_i)}\right). \tag{6.13}$$

Substituting the value of p_0, expanding q and rearranging, we obtain that inequality (6.13) is equivalent to

$$\sum_{i=1}^{k} \frac{\alpha_i r^2}{r - a_i} \leq (1+\varepsilon)|T| + \frac{1+\varepsilon}{1+\varepsilon'} \sum_{i=1}^{k} \frac{\alpha_i a_i^2}{r - a_i},$$

which we can also rewrite as

$$\sum_{i=1}^{k} \alpha_i (r + a_i) + \frac{\varepsilon' - \varepsilon}{1+\varepsilon'} \sum_{i=1}^{k} \frac{\alpha_i a_i^2}{r - a_i} \leq (1+\varepsilon)|T|. \tag{6.14}$$

As $a_i \leq (1-\delta)r$ for every $i \in [k]$ we have

$$\frac{a_i^2}{r - a_i} \leq \frac{r(r + a_i)}{2(r - a_i)} \leq \frac{(r + a_i)}{2\delta},$$

and therefore the second term of the left-hand side of (6.14) is upper bounded by

$$\frac{\varepsilon' - \varepsilon}{2\delta} \sum_{i=1}^{k} \alpha_i (r + a_i) = \frac{4\varepsilon}{13} \sum_{i=1}^{k} \alpha_i (r + a_i),$$

where the equality holds by the choice of δ and ε'. Thus (6.14) is implied by

$$\left(1 + \frac{4\varepsilon}{13}\right) \sum_{i=1}^{k} \alpha_i (r + a_i) \leq (1+\varepsilon)|T|. \tag{6.15}$$

By (6.12), the inequality (6.15) is further implied by

$$\left(1 + \frac{4\varepsilon}{13}\right) \left(1 + \frac{\varepsilon}{4}\right) |T| \leq (1+\varepsilon)|T|.$$

This last inequality can finally easily seen to hold for $\varepsilon \leq 2$. □

6.5 Proof of Theorem 6.1.2

In this section, we will deduce our main result from Theorem 6.4.1. We will need some classical results in probability theory.

Theorem 6.5.1 (Markov's inequality)**.** *If $X \geq 0$ is a random variable and $a > 0$, then*

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}.$$

Theorem 6.5.2 (Hoeffding's inequality)**.** *[E12] If $X_1, \ldots, X_m$ are independent random variables with values in $[a, b]$, $X = \sum_{i=1}^{m} X_i$ and $t > 0$, then*

$$\mathbb{P}\left(X \geq \mathbb{E}[X] + t\right) \leq \exp\left(-\frac{2t^2}{m(b-a)^2}\right).$$

We now convert the random covers given by Theorem 6.4.1 into uniform covers, proving a metric (but not fractional) equivalent of Theorem 6.1.2.

Theorem 6.5.3. *For every $\varepsilon > 0$, there exists $K = K_{6.5.3}(\varepsilon)$ such that if $K \leq k \in \mathbb{N}$ and T is a metric tree such that $|T| \leq (1-\varepsilon)k^2$, then $(1, 2, \ldots, k)$ is a cover of T.*

Proof. We may of course assume that $\varepsilon < 1$. Choose $N \in \mathbb{N}$ large enough so that $\lambda := 1 - \frac{\left(1+\frac{1}{N}\right)(1-\varepsilon)}{\left(1-\frac{1}{N}\right)^2} > 0$. Choose $P \geq 1$ large enough so that $N \exp\left(-\frac{\lambda^2}{24N^4(1-\varepsilon)^2}P\right) < \lambda$. Let $D = 24N$ and $K = \frac{PD}{1-\varepsilon}$. We show that K satisfies the theorem.

We assume without loss of generality $|T| = (1-\varepsilon)k^2 \geq PDk$ by extending T if needed.

By Lemma 6.2.6, there exist a decomposition $\{T_1, \ldots, T_m\}$ of T such that $Dk \leq |T_i| \leq 3Dk$ for every $i \in \{1, \ldots, m\}$. Note that this implies that $mDk \leq |T| \leq 3mDk$ and in particular $m \geq \frac{P}{3}$.

Setting $r = \left(1 - \frac{1}{N}\right)k$, apply Theorem 6.4.1 to T_i for each $i \in \{1, \ldots, m\}$ with the parameter $\varepsilon = \frac{24r}{|T_i|}$to obtain a probability measure ν^i on $\mathcal{C}(T_i, r)$ such that

$$E\nu^i \leq \left(1 + \frac{24r}{|T_i|}\right) \frac{|T_i|}{r} \boldsymbol{U}[0,r] \leq \left(1 + \frac{1}{N}\right) \frac{|T_i|}{r} \boldsymbol{U}[0,r].$$

Let $\boldsymbol{r}^i$ be a random cover of T_i following the law ν^i. For $0 \leq \alpha \leq 1 - \frac{2}{N}$, let

$$Z_i(\alpha) = \#\left(\left[\alpha k, \left(\alpha + \frac{1}{N}\right) k\right], \boldsymbol{r}^i\right)$$

be the random variable equal to the number of radii in $\boldsymbol{r}_i$ that belong to the interval $\left[\alpha k, (\alpha + \frac{1}{N})k\right]$. Then $\mathbb{E}\left[Z_i(\alpha)\right] = E\nu^i\left(\left[\alpha k, (\alpha + \frac{1}{N})k\right]\right)$ by definition of $E\nu^i$ and so

$$\mathbb{E}\left[Z_i(\alpha)\right] \leq \left(1 + \frac{1}{N}\right) \frac{|T_i|}{r} \cdot (\boldsymbol{U}[0,r]) \left(\left[\alpha k, \left(\alpha + \frac{1}{N}\right) k\right]\right) = \frac{\left(1 + \frac{1}{N}\right)|T_i|}{N\left(1 - \frac{1}{N}\right)^2 k}$$

for every $1 \leq i \leq m$. Hence,

$$\mathbb{E}\left[\sum_{i=1}^m Z_i(\alpha)\right] \leq \frac{\left(1 + \frac{1}{N}\right)|T|}{N\left(1 - \frac{1}{N}\right)^2 k} = \frac{\left(1 + \frac{1}{N}\right)(1-\varepsilon)k^2}{N\left(1 - \frac{1}{N}\right)^2 k} = \frac{(1-\lambda)k}{N}.$$

Firstly, using Markov's inequality (Theorem 6.5.1),

$$\mathbb{P}\left(\sum_{i=1}^m Z_i(0) \geq \frac{k}{N}\right) \leq \frac{\mathbb{E}\left[\sum_{i=1}^m Z_i(0)\right]}{\frac{k}{N}} \leq 1 - \lambda.$$

This above inequality holds for any α, but we will need a stronger result in general. When $\alpha \geq \frac{1}{N}$, we will use a Chernoff-type bound.

Since $\boldsymbol{r}^i$ is r-good, for every $i \in \{1, \ldots, m\}$ we have

$$Z_i(\alpha) \cdot \alpha k \leq \|\boldsymbol{r}^i\|_1 \leq |T_i| + r \leq (3D+1)k \leq 4Dk$$

and thus

$$0 \leq Z_i(\alpha) \leq \frac{4D}{\alpha} \leq \frac{4N|T|}{mk} = \frac{4N(1-\varepsilon)k}{m}.$$

using the conditions $|T| \geq mDk$ and $|T| = (1-\varepsilon)k^2$.

Then, applying Hoeffding's inequality (Theorem 6.5.2), we get

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^{m} Z_i(\alpha) \geq \frac{k}{N}\right) &\leq \mathbb{P}\left(\sum_{i=1}^{m} Z_i(\alpha) \geq \mathbb{E}\left[\sum_{i=1}^{m} Z_i(\alpha)\right] + \left(\frac{k}{N} - \frac{(1-\lambda)k}{N}\right)\right) \\
&\leq \exp\left(-\frac{2\left(\frac{\lambda k}{N}\right)^2}{m \cdot \left(\frac{4N(1-\varepsilon)k}{m}\right)^2}\right) \\
&= \exp\left(-\frac{\lambda^2}{8N^4(1-\varepsilon)^2} m\right) \\
&\leq \exp\left(-\frac{\lambda^2}{24N^4(1-\varepsilon)^2} P\right).
\end{aligned}$$

Combining these, by the union bound we have

$$\begin{aligned}
\mathbb{P}\left(\exists j \in \{0,\ldots,N-2\}, \sum_{i=1}^{m} Z_i\left(\frac{j}{N}\right) \geq \frac{k}{N}\right) &\leq \sum_{j=0}^{N-2} \mathbb{P}\left(\sum_{i=1}^{m} Z_i\left(\frac{j}{N}\right) \geq \frac{k}{N}\right) \\
&< (1-\lambda) + N\exp\left(-\frac{\lambda^2}{24N^4(1-\varepsilon)^2} P\right) \\
&< 1
\end{aligned}$$

by our choice of P. Hence, there exist covers $\boldsymbol{r}^1, \ldots, \boldsymbol{r}^m$ of $T_1, \ldots, T_m$, respectively using radii in $[0, (1-\frac{1}{N})k] = [0, r]$ such that the total number of radii in $\left[\frac{j}{N}k, \frac{j+1}{N}k\right]$ that are used is smaller than $\frac{k}{N}$, and so at most $\left\lfloor \frac{k}{N} \right\rfloor$.

The concatenation of these covers is a cover $\boldsymbol{r}$ of T, from which we now construct a cover using radii $\{0, \ldots, k\}$. Let $j \in \{1, \ldots, N-2\}$. The interval $\left(\frac{j+1}{N}k, \frac{j+2}{N}k\right]$ contains at least $\left\lfloor \frac{k}{N} \right\rfloor$ integers. Thus we can increase the radii in $\boldsymbol{r}$ that lie in the interval $\left[\frac{j}{N}k, \frac{j+1}{N}k\right]$ replacing them by distinct integers in $\left(\frac{j+1}{N}k, \frac{j+2}{N}k\right]$. The radii in the resulting modified cover are distinct integers in the interval $[\frac{1}{N}k, k]$, implying that $(1, 2, \ldots, k)$ is a cover of T, as desired. $\square$

We now come back to the original discrete setting. Let T be a (discrete) tree. As in the metric setting, for $r \geq 0$ and $v \in V(T)$, we denote by $B_T(v,r)$ the closed ball of radius r with center v, i.e.

$$B_T(v,r) = \{u \in V(T) | d_T(v,u) \leq r\},$$

where $d_T(\cdot,\cdot)$ is the usual graph metric, i.e. $d_T(v,u)$ is the number of edges in the unique path with ends u and v. As before, $(r_1,\dots,r_m)$ is a *cover* of T if there exist $v_1,\dots,v_m \in V(T)$ such that $V(T) = \cup_{i=1}^m B_T(v_i,r_i)$. We denote by T^M a metric tree obtained from T by replacing each edge by an interval of length one. Note that $d_{T^M}(u,v) = d_T(u,v)$ for any $u,v \in V(T)$.

Lemma 6.5.4. *If T is a discrete tree and $(r_1,\dots,r_m)$ is a cover of the corresponding metric tree T^M, then $(r_1+1,\dots,r_m+1)$ is a cover of T.*

Proof. Let $v_1,\dots,v_m \in T^M$ be such that $T^M = \cup_{i=1}^m B_{T^M}(v_i,r_i)$.

For each $i \in [m]$, let $u_i \in V(T)$ be an end point of the segment of T^M containing v_i. In particular $d_{T^M}(u_i,v_i) \leq 1$, and so $B_{T^M}(v_i,r_i) \subseteq B_{T^M}(u_i,r_i+1)$, implying $T^M = \cup_{i=1}^m B_{T^M}(u_i,r_i+1)$. As $V(T) \subseteq T^M$ and the distances between vertices of T are preserved in T^M, we have $V(T) = \cup_{i=1}^m B_T(u_i,r_i+1)$, implying that $(r_1+1,\dots,r_m+1)$ is a cover of T. □

Our main result, which we restate for convenience, readily follows from Theorem 6.5.3 and Lemma 6.5.4.

Theorem 6.1.2. *If G is a connected graph on n vertices, then*

$$b(G) \leq (1+o(1))\sqrt{n}.$$

Proof. We need to show that for every $0 < \varepsilon < 1$, there exists N such that if G is a connected graph on $n \geq N$ vertices then $b(G) \leq (1+\varepsilon)\sqrt{n}$.

Let $\varepsilon' = \frac{\varepsilon}{2}$, we show that

$$N = \max\left\{ \left(K_{6.5.3}(\varepsilon')\right)^2, \frac{6}{\varepsilon} \right\}$$

has the above property.

Let G be a connected graph on $n \geq N$ vertices. As noted in the introduction, it suffices to consider any spanning tree T of G; burning T will also burn G, hence $b(G) \leq b(T)$. Let T^M be the metric tree corresponding to T.

Set $k = \left\lceil \sqrt{\frac{n}{1-\varepsilon'}} \right\rceil$. We have that $k \geq \sqrt{\frac{n}{1-\varepsilon'}} \geq \sqrt{\frac{1}{1-\varepsilon'} N} \geq K_{6.5.3}(\varepsilon')$ and $|T^M| = n - 1 \leq (1-\varepsilon') \left\lceil \sqrt{\frac{n}{1-\varepsilon'}} \right\rceil^2 = (1-\varepsilon')k^2$.

Hence, by Theorem 6.5.3, $(1, \dots, k)$ is a cover of T^M. By Lemma 6.5.4 it follows that $(2, \dots, k+1)$ is a cover of T. Thus

$$b(T) \leq (k+1) + 1 = \left\lceil \sqrt{\frac{n}{1-\frac{\varepsilon}{2}}} \right\rceil + 2 \leq \left(1 + \frac{\varepsilon}{2}\right)\sqrt{n} + 3 \leq (1+\varepsilon)\sqrt{n},$$

as desired, where the last inequality uses the condition $n \geq N \geq \frac{6}{\varepsilon}$. □

6.6 Concluding remarks

6.6.1 Eliminating the error

We have shown that the Burning Number Conjecture holds asymptotically, i.e. $b(G) \leq (1 + o(1))\sqrt{n}$ for every connected n vertex graph G. A natural next direction would be to attempt to eliminate the error term, proving the conjecture in full.

Unfortunately, our method does not seem to give much insight in the behaviour of burning number of small graphs. It might be conceivable that our argument can be used as a starting point for the proof of the Burning Number Conjecture for sufficiently large n, but such an extension is likely to be quite difficult and require additional ideas.

On the other hand, eliminating the error term in Theorem 6.4.1 is more likely to be within reach. The conclusion of Theorem 6.4.1 does not hold with $\varepsilon = 0$ if T is an interval with $|T| < 2r$, as the radii of length $\frac{|T|}{2}$ can not be utilized without waste, but we conjecture that this is the only obstruction.

Conjecture 6.6.1. *If $r > 0$ and T is a metric tree such that $|T| \geq 2r$, then there exists a probability measure ν on $\mathcal{C}(T)$ such that*

$$E\nu \leq \frac{|T|}{r}\boldsymbol{U}[0, r].$$

It suffices to prove Conjecture 6.6.1 for metric trees T that do not admit a decomposition $\{T_1, T_2\}$ such that $|T_1|, |T_2| \geq 2r$, which might be possible by extending Lemma 6.3.3 from the class of $2r$-minimal trees to this larger class. We were able to do implement this strategy to show that Conjecture 6.6.1 holds for metric trees with at most three leaves, using by a more detailed case analysis of the possible ratios between the length of the three branches in Lemma 6.3.3.

6.6.2 General radii

Given a sequence of radii $(r_1, \ldots, r_k)$ what is the maximum D such that $(r_1, \ldots, r_k)$ is a cover of every metric tree T with $|T| \leq D$. By considering intervals (i.e. paths) we see that $D \leq 2\sum_{i=1}^{k} r_i$. A metric analogue of the Burning Number Conjecture suggests that the equality holds for the sequence $(1, \ldots, k)$, but it is unclear which properties of sequence make the conjecture plausible, motivating the following question.

Question 6.6.2. *Which sequences $(r_1, \ldots, r_k)$ of positive reals have the property that $(r_1, \ldots, r_k)$ a cover of every metric tree with $|T| \leq 2\sum_{i=1}^{k} r_i$?*

We believe that the following large and natural class of sequences of radii, which includes the sequences $(1, \ldots, k)$, respects this property. We say a sequence $(r_1, \ldots, r_k)$ of non-negative reals is *convex* if

$$r_1 \leq r_2 - r_1 \leq r_3 - r_2 \leq \ldots \leq r_k - r_{k-1}.$$

Conjecture 6.6.3. *If $(r_1, \ldots, r_k)$ is a convex sequence of non-negative reals and T is a metric tree such that $|T| \leq 2\sum_{i=1}^{k} r_i$, then $(r_1, \ldots, r_k)$ is a cover of T.*

Conjecture 6.6.3 is motivated by the fact that we convinced ourselves that a (rather technical) fractional variant of this conjecture holds asymptotically.[9]

The methods of Section 6.3 can be used to show that Conjecture 6.6.3 holds for $k = 2$.

Theorem 6.6.4. *If $0 \leq r_1 \leq \frac{r_2}{2}$, and T is a metric tree such that $|T| \leq 2(r_1 + r_2)$, then (r_1, r_2) is a cover of T.*

Proof. We first claim that T is $\frac{2|T|}{3}$-minimal. Trivially, $|T| \geq \frac{2|T|}{3}$, so it suffices to prove that there exists a decomposition $\{T', T''\}$ of T such that $|T'|, |T''| \leq \frac{2|T|}{3}$. Let $\{T', T''\}$ be a decomposition chosen to minimize $\max\{|T'|, |T''|\}$. (Such a choice is possible by compactness and continuity of the length function.) Suppose for a contradiction that this value is greater than $\frac{2|T|}{3}$. Without loss of generality, we have that $|T'| > \frac{2|T|}{3}$. Let v be the unique point of intersection of T', T''. Let $\{T_1, \ldots, T_k\}$ be the unique decomposition of T' such that each T_i is a non-trivial branch of T with anchor v and v is a leaf of T_i. If $k = 1$, we can slightly nudge v into the segment of T_1 incident to v, hence slightly decreasing $|T'|$ and slightly increasing $|T''|$. Otherwise $k > 1$, and so we can assume without loss of generality that $|T_1| \leq \frac{|T'|}{2}$. Then, consider the decomposition $\{T' \setminus (T_1 \setminus \{v\}), T'' \cup T_1\}$ of T. Then $|T' \setminus (T_1 \setminus \{v\})| < |T'|$ and

$$|T'' \cup T_1| = |T''| + |T_1| \leq (|T| - |T'|) + \frac{|T'|}{2} = |T| - \frac{|T'|}{2} < |T| - \frac{\frac{2|T|}{3}}{2} = \frac{2|T|}{3} < |T'|,$$

yielding the desired contradiction, and finishing the proof of the claim.

Let a be obtained by applying Lemma 6.3.3 to T with $l = \frac{2|T|}{3}$. In particular, $a \leq \frac{\frac{2|T|}{3}}{2} - \frac{|T|}{4} = \frac{|T|}{12}$ and then $\frac{|T|}{4} + a \leq \frac{|T|}{3} \leq \frac{2(r_1+r_2)}{3} \leq r_2$. Hence, there exists $x \geq 0$ such that $\frac{|T|}{4} + a + x = r_2$. If $x \geq \frac{|T|}{4} - 3a$ then (r_2) is a cover of T by Lemma 6.3.3 and hence so is

[9] More precisely the measure $\frac{|T|}{r}\boldsymbol{U}[0, r]$ in Theorem 6.4.1 can be replaced by any measure μ on $[0, r]$ such that $\mathsf{m}(\mu) \geq |T|$ and μ has non-increasing density with respect to the Lebesgue measure, i.e. there exists a non-increasing function $f : [0, r] \to \mathbb{R}_+$ such that $\mu([a, b]) = \int_a^b f(x)dx$ for all $0 \leq a < b \leq r$.

(r_1, r_2). Thus we assume $x \leq \frac{|T|}{4} - 3a$ and

$$\left(\frac{|T|}{4} - 3a - x, \frac{|T|}{4} + a + x\right) = \left(\frac{|T|}{4} - 3a - x, r_2\right)$$

is a cover of T by Lemma 6.3.3. As $|T| \leq 2(r_1 + r_2) = 2r_1 + \frac{|T|}{2} + 2a + 2x$, it follows that $r_1 \geq \frac{|T|}{4} - a - x \geq \frac{|T|}{4} - 3a - x$, and so (r_1, r_2) is a cover of T, as desired. $\square$

In fact, the answer to Question 6.6.2 for sequences of length two is exactly the set of convex sequences, i.e. convexity is not only sufficient, but necessary, for (r_1, r_2) to be the cover of every metric tree T with $|T| \leq 2(r_1 + r_2)$. Consider radii $0 \leq r_1 \leq r_2$ such that $r_1 > \frac{r_2}{2}$ and let T be metric tree with three leaves, where all three segments have length $\frac{2(r_1+r_2)}{3}$. Firstly, $|T| = 2(r_1 + r_2)$. We claim that (r_1, r_2) is not a cover of T. Suppose otherwise, that $T = B_T(r_1, p_1) \cup B_T(r_2, p_2)$ for some $p_1, p_2 \in T$. If v is a leaf of T, then v must be at distance at most r_2 from p_1 or p_2. However, the segment of which v is a leaf has length $\frac{2(r_1+r_2)}{3} > \frac{r_2+2r_2}{3} = r_2$. Hence, either p_1 or p_2 must be strictly contained in this segment, i.e. distinct from v. Since T has three branches, it is impossible for every branch to strictly contain p_1 or p_2, yielding the desired contradiction.

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Part V

Discussion and Conclusion

7

Discussion

In this thesis, we have studied the structure of sparse graphs through various aspects. In the first section of this chapter, we summarize the themes studied in this thesis and discuss which of and how the manuscripts relate to each theme. In the remainder of the chapter, we will discuss three topics which naturally extend some of the work presented in this thesis.

7.1 Survey of themes

Graph minors

Chapters 2, 4 and 5 all concern graph minors directly. Indeed, in all three of these, we derive some properties for graphs forbidding one or multiple graphs as minors. In Chapter 2, our main result upper bounds the minimum degree when a sparse bipartite graph H is forbidden as a minor. In Chapter 3, we bound the average degree when we forbid all minors with a fixed number of vertices and a fixed number of edges. In Chapter 5, we upper bound the cop number when a minor H is forbidden, roughly speaking as a function of the number of edges in H, but also taking into account its structure.

The connection of Chapter 4 to graph minor theory is twofold. Firstly, the main motivation for this problem is the (now disproven) coarse Menger's conjecture, which was suggested as a tool to prove a (now disproven) conjecture on fat minors. Furthermore, after proving our upper bound for graphs with bounded maximum degree, we use it to prove an analogous results for graphs with a forbidden topological minor, a close variant of graphs minors.

Extremal functions, i.e. upper bounds on the average degree of graphs with a forbidden minor, is a tool we used multiple times. In Chapter 2, we applied Theorem 2.2.2, a bound on the extremal functions for sufficiently large bipartite graphs taken from families with strongly sublinear separators, multiple times. We also used Theorem 2.6.4, a result which shows that when a graph is sufficiently connected as a function of the extremal function of a graph H, we can construct the graph H as a rooted minor, which is a stronger concept. In Chapter 3, for the proof of Theorem 3.1.3, our result for small t, we used multiple extremal functions as well. Although some of these had already been known, Theorem 3.4.1 is a new bound on the extremal function for the class of graphs on 6 vertices and 11 edges. We will also briefly discuss a connection between extremal functions and our work on cops and robbers in Section 7.3.

Colourings

Our main motivation in Chapters 2 and 3 was to work towards Hadwiger's conjecture, which states that K_t-minor-free graphs are $(t-1)$-colourable. As we have discussed previously, our methods do not interact directly with colouring: only degeneracy is used to greedily colour the vertices of the graph. Interestingly, this allows our results to be useful to other types of colourings, such as list colouring and DP-colouring, for which the degeneracy argument also works directly.

We also note that in Chapter 4, strong edge colouring was the main tool we used. We will discuss the limits of this tool below, in Section 7.2.

Connectivity

Connectivity is one of the most important topics in graph theory. We have used and studied it at multiple places in this thesis.

In Chapter 2, we have used what is often referred to as a "connectivity toolbox". The main results used are presented in Section 2.2.1. We have in particular used results relating to linkages and rooted minors, with the objective of building minors from smaller pieces.

In Chapter 3, connectivity is used in the proof of Theorem 3.4.1. The proof is split into cases about whether the graph is 0-, 1-, 2-, or 3-connected. Indeed, as is standard with these types of proofs on extremal functions, we take a separation of the graph and induct on each side of this separation, and attempt to construct one of the forbidden minors.

Of course, Chapter 4 is our main study of connectivity. As exhibited by its use in Chapters 2 and 4, connectivity is a very useful tool in building graph minors. It is then natural that Georgakopoulos and Papasoglu [47] suggested proving a coarse version of Menger's theorem in order to study fat minors.

We proved some versions of Menger's theorem when the condition that the paths be pairwise disjoint be replaced by the condition that the paths be pairwise non-adjacent. However, not only do our results concern connectivity, but some the tools used are standard connectivity tools. Most notably, in our proof of Theorem 4.1.5, one of the main ideas is to apply Menger's theorem on a contracted graph and then lift the paths back up to the original graph. Connectivity is also used in the proof of Theorem 4.1.9 when forbidding a topological minor (see Claim 4.3.3.1, in particular).

Games on graphs

Both Chapters 5 and 6 concern games on graphs, cops and robbers for the former and graph burning for the latter. These two games or processes are some of the best-known and most studied in the field of graph searching. Later in this chapter, in Section 7.3, we will discuss the cop number of linklessly embeddable graphs in more details. In Section 7.4, we will discuss *fractional graph burning*, a variant of graph burning which is naturally suggested by the methods used in Chapter 6.

Metric problems

As discussed previously, one of the main motivations for our work in Chapter 4 was the coarse graph theory suggested by Georgakopoulos and Papasoglu [47]. This is fundamentally a metric problem, where we are concerned about the distances between objects, in our case between the pieces of the model (hence the definition of fat minors). In fact, in their original paper, Georgakopoulos and Papasoglu work in a more general setting than graphs, rather they work on length spaces, metric spaces which, broadly speaking, "look like" graphs.

Although the original formulation of graph burning as a game does not suggest it to be a metric problem, we have seen that it is fundamentally a covering problem. In Chapter 6, we have taken this viewpoint one step further and considered metric trees. This tool was particularly useful as it allowed us to decompose the tree into small pieces by cutting it into smaller trees at any point (not only vertices), and by covering them with balls of fractional radii.

7.2 The limits of strong colouring for coarse Menger

In Chapter 4, we used strong colouring in order to prove an induced version of Menger's theorem. The strategy was broadly as follows.

For some constant C depending only on the maximum degree of G, we wish to show there exists either a set of fewer than Ck vertices which separates X and Y, or k pairwise non-adjacent X-Y paths (Theorem 4.1.5). By Menger's theorem, we may suppose that we begin with Ck pairwise disjoint paths (Theorem 4.1.6). Our strategy is to strongly colour the edges which go between these paths (i.e., partition these edges into strong, or distance 2, matchings). Then, we contract one colour class and apply Menger's theorem to find fewer, but still many, disjoint paths in this new contracted graphs. As the contracted colour class formed a matching, we only lose half the number of paths. We then lift them back to paths in the original graph, which due to the contraction, do not contain any edges of the contracted colour between them. Edges from the original paths may however be used in the new paths. It is because the matching is strong that the edges from the original paths cannot go between the new paths. We do this for every colour class in order to obtain the result.

It is natural to attempt to generalize this strategy to the distance $d = 3$ case: suppose we want the paths not only to be non-adjacent, but to not have any common neighbours. I have, unsuccessfully, attempted to prove such a result, with my coauthors in Chapter 4, Hendrey, Norin and Steiner, as well as with Chudnovsky and Seymour. The following is a summary of discussions and email exchanges I had with them.

One version of this approach would be as follows. Write $\mathcal{P}$ for the original path system, and write $\mathcal{Q}$ for the paths of length at most two with endpoints in $\mathcal{Q}$. Instead of colouring only the edges which go between the paths in $\mathcal{P}$, one might attempt colouring each path in $\mathcal{Q}$ in a way that if the endpoints of these paths are at distance at most two in $\mathcal{P}$, they obtain distinct colours. At first glance, this would appear to work. Indeed, after the first iteration of the procedure, we obtain a new path family $\mathcal{P}'$ between which there are no edges or paths of the contracted colour. Furthermore, the strong colouring condition again ensures that there are no edges or paths of length at most two between paths in $\mathcal{P}'$ which were part of $\mathcal{P}$. However, we might find that between two paths of $\mathcal{P}$, there is an edge or path of length two which was neither in $\mathcal{P}$ nor in $\mathcal{Q}$: it could have went directly between two paths of $\mathcal{Q}$.

Perhaps, we might choose to modify the condition on the colouring in a way that paths in $\mathcal{Q}$ receive distinct colours not only if their endpoints in $\mathcal{P}$ are close, but if they are close in the entire graph. This works, but only for one iteration. Indeed, to preserve the induction, after one iteration one must necessarily consider all paths of length at most two between the new paths $\mathcal{P}'$, call these $\mathcal{Q}'$. Now, some these paths had not been previously coloured, which poses an issue. So, in the original colouring, not only should we colour paths in $\mathcal{Q}$, but also paths between paths in $\mathcal{Q}$, and so on.

It is unclear how to do this. For one, if we consider all of these levels, one edge may be in multiple paths of length two. In some sense, the strength of our method in the $d = 2$ case is that we can "move to an induced subgraph", as stated by Steiner in personal communication. For higher distances, everything else gets "pulled in".

We recall that Gartland et al. [45] independently proved a statement similar to Theorem 4.1.5, with a slightly weaker bound. In their proof, they strongly colour the vertices instead of the edges. We have also attempted generalizing their proof to $d = 3$, and also had issues arise in every variation of it.

7.3 Cops and robbers and linklessly embeddable graphs

In Chapter 5, we generalized the methods of Andreae [7] in order to improve the upper bounds on the cop number when forbidding a graph. In Andreae's bound, one cop is needed to reconstruct each edge of $H - h$ in the H-minor-free graph on which the graph is played. Our improvement is, broadly speaking, twofold:

1. If the graph $H - h$ contains a path such that every vertex on this path has degree exactly two, we only need approximately one third of the cops that Andreae's main result requires (note however that this idea was based on Andreae's [7] argument for wheel graphs). We have further refined this idea by assigning to each bag of the model a cop (or group of cops) whose role it is to protect the vertex of this bag which is adjacent to the robber's territory (this is the function f in the decomposition of H), which in some cases allows us to use fewer cops.

2. We can also choose a matching in $H - h$ for which no cops need to be assigned.

We have elaborated on these improvements in more detail in Section 5.4.1.

We have claimed that our result is most significant when H is sparse or small. Indeed, if a graph is sparse or small, removing a matching from it reduces its number of edges by a significant proportion. Furthermore, graphs which can be obtained from another graph with many edge subdivisions, and thus contain many of the types of paths discussed above, are necessarily sparse.

As we have discussed in the introduction, one of the most studied aspects of the game of cops and robbers has been its connection to graph topology. Hence, one of the most important consequences of our main result, Theorem 5.4.3, is that (connected) linklessly embeddable graphs have cop number at most 6 (Corollary 5.5.7). In fact, four of the graphs in the Petersen family have the property that excluding them alone as a minor yields this upper bound.

One notices that many of the graphs in the Petersen family are structurally similar (which is natural, given that they are obtained from one another by $\Delta - Y$ replacement). The drawings in Figure 5.5.1 of the graphs $\mathcal{P}_i$, for $i \in [4]$ in particular (as well as $i = 5$, to a lesser extent), show this quite clearly. Hence, it might be possible to improve the upper bound when they are simultaneously forbidden, in a proof which leverages this similarity. The intuition here would be that at some point in the proof, we might not be sure where an edge, vertex or path is located exactly, but each possibility gives one of the forbidden minors.

We recall that Andreae [7], in addition to the main result Theorem 5.1.1, had shown in a more detailled proof that $K_{3,3}$-minor-free graphs and K_5-minor-free graphs have cop number at most 3 (strenghtening Theorem 1.4.2). We have put significant effort, without success, in generalizing this proof in order to prove a stronger bound for linklessly embeddable graphs.

In fact, it is unclear what the optimal bound should be. As discussed in Section 5.5.5, a cop number of 4 is suggested for this class by the Colin de Verdière invariant (of course, this is based on a very small number of data points). While preparing the manuscript presented Chapter 5, we have attempted, unsuccesfully, to construct linklessly embeddably graphs with cop number at least 4.

The following result was shown by Aigner and Fromme [1] in order to lower bound the cop number of some graphs.

Theorem 7.3.1 ([1]). *If G is a connected graph with girth at least 5, then $c(G) \geq \delta(G)$.*

Proof. Suppose at some point in the game the robber has not yet been caught, and it is the robber's turn. Let u be the vertex on which the robber is located. No cop can be on or adjacent to more than one vertex in $N(u)$: this would necessarily form a cycle of length 3 or 4 which includes this vertex and u. Hence, if there are fewer than $\delta(G)$ cops playing, then there is necessarily at least one neighbour of u on which there is no cop and to which no cop is adjacent, allowing the robber to not be caught at this turn. By repeating this argument, the robber has a strategy to never be caught.

To finish the proof, it thus suffices to show that the robber has an initial position such that it is not immediately captured. Suppose the cops have chosen their initial positions. Let u be an arbitrary vertex on which there is no cop, and let v be the vertex to which the robber could escape to if it were on u, which exists by the previous argument. The robber can begin the game by starting on v. $\square$

See [42, 43, 26], for instance, for similar and more general arguments of the type.

Aigner and Fromme have, for example, used this result to show that the bound on the cop number of planar graphs is tight (consider the dodecahedral graph, which has girth 5 and minimum degree 3). The following question then arises naturally.

Question 7.3.2. *Does there exist a linklessly embeddable graph with girth at least 5 and minimum degree at least 4?*

We say a graph G is *apex* if it contains a vertex u such that $G-u$ is planar. Apex graphs are important in the study of graph minors. Notably, Robertson et al. [88] showed Hadwiger's conjecture for $t = 6$ by showing that any minimal counter-example would necessarily be apex (however, these graphs are also 5-colourable, as can be shown by applying the Four-colour theorem to $G-u$ and using the extra colour for u). Apex graphs are also linklessly embeddable [88]. We first show that, under mild conditions, these natural candidates do not answer Question 7.3.2.

Lemma 7.3.3. *If G is a graph such that $G-u$ is a connected planar graph, then G cannot have girth at least 5 and minimum degree at least 4.*

Proof. As G has girth at least 5, $G - u$ must either have (finite) girth at least 5, or be a tree. The latter case is impossible, as all trees contain vertices of degree 1, and so G would contain vertices of degree 2.

Consider the former case. By Corollary 1.1.3, $|E(G-u)| \leq \frac{5}{3}|V(G-u)|$. In particular, G has average degree at most $\frac{10}{3}$. As $G-u$ has minimum degree at least 3, this implies that a proportion of at most $\frac{1}{3}$ of the vertices of $G-u$ may have degree larger than 3. As all vertices have degree at least 4 in G, this means that $d(u) \geq \frac{2}{3}|V(G-u)|$.

This implies that u is necessarily adjacent to some pair of vertices which are, in $G-u$, at distance at most two, forming a triangle or a 4-cycle, which would be a contradiction. Indeed, it is easily seen that the domination number of $G-u$ is at most $\frac{|V(G-u)|}{2}$ when $|V(G-u)| \geq 2$ (consider a rooted spanning tree of G, pick the neighbour of the deepest vertex, remove its downward neighbourhood and induct). Hence, by the pigeonhole principle there necessarily exists two vertices in $N(u)$ which are adjacent or equal to the same vertex in this dominating set. $\square$

Note that we could exclude more apex graphs by considering the version of Euler's formula which takes into account the number of components.

Question 7.3.2 is in fact related to many results and open problems on extremal functions for graph minors (which we have discussed and used in Chapters 2 and 3 and earlier in this chapter): there are many very similar bounds and open problems. Recall that both K_6 and the Petersen graph are in the Petersen family, the obstruction set for linklessly embeddable graphs. One approach to showing that no graphs fulfills the conditions of Question 7.3.2 would be to upper bound the average degree at under 4. However, Hendrey and Wood [50] have shown that n-vertex Petersen-minor free graphs have at most $5n-8$ edges, and that this bound is best possible. Mader [65] showed that for K_6, the extremal function is $4n-10$, and is also best possible. Aigner-Horev and Krakovski [2] showed that 6-connected graphs with girth at least 6 and no K_6-minor have at most $3n-8$ edges (Jorgensen [54] has conjecture the same, without the condition on the girth), and a similar bound with coefficient

$\frac{16}{5}$ when the girth is relaxed to 5. Chudnovsky et al. [31] proved that bipartite graphs with minimum degree at least 6 contains a K_6-minor. McCarty and Thomas [68] have shown that bipartite linklessly embeddable graphs have at most $3n - 10$ edges. However, to my knowledge, Question 7.3.2 remains open.

7.4 Fractional graph burning

The Burning Number Conjecture is motivated by the bound for paths, Theorem 1.4.6. In its proof, every ball of radius r is used to cover (burn) exactly $2r + 1$ vertices. Although we cannot require this to hold for every ball when burning general trees (or graphs), we want it to hold on average. Our proof of Theorem 3.1.1 shows that this is close to being true: on average a ball of radius r covers close to $2r$ vertices.

One of our main ideas was to break up the tree into small pieces, on which we are able to find many covers of size two (see Lemma 6.3.3) which are almost optimal (in the sense that on average every ball covers almost what it would be expected to on a path). In order to prove Theorem 3.1.1, these were then used by randomly choosing one of these covers for every small subtree. We then piece these together on the original tree in such a way that these small covers don't use balls of the same radii (perhaps, by shifting them slightly).

The Burning Number Conjecture states that if we have radii $0, \ldots, k - 1$, we can cover every vertex of a tree on n vertices at least once if $k = \lceil \sqrt{n} \rceil$. Our result for small (metric) trees says that if make k larger (or equivalently, make n smaller), we should be able to cover the tree many times. Conjecture 6.6.1 states that, at least in the metric setting, we should be able to do this exactly: on average a ball of radius r should cover a length of $2r$, whatever the size of the metric tree (as long as it is sufficiently large, in order for the largest radius to not be wasted).

If we translate this back to the original setting, we would like to show that if T is a tree on n vertices, and we have radii $0, \ldots, k - 1$, we could partition these radii into $\frac{k^2}{n}$ covers of T, as long as $n \geq 2k - 1$. Of course, this would be best possible, as on a path no radius

r can contribute to burning more than $2r+1$ vertices, and so we cannot burn more than $\sum_{r=0}^{k-1} 2r+1 = k^2$ vertices, with multiplicity. However, this is impossible due to the discrete nature of the problem: as soon as $\frac{k^2}{n}$ is not an integer, there is some necessarily some waste (which can also be seen in the Burning Number Conjecture, given the ceiling that is used).

Norin has suggested in personal communication to relax the problem by making it fractional, however in a different way than we used in Chapter 6.

We say a *k-cover* of a graph G is a collection $\mathcal{B}$ of pairs (B, w), where B is a ball in G with radius $r \in \{0, \dots, k-1\}$ and $w \in \mathbb{R}^+$ (we call w the *weight* of this ball).

We define, for every $r \in \{0, \dots, k-1\}$, $\eta_{\mathcal{B}}(r) := \sum_{(B(v,r'),w)\in\mathcal{B}: r'=r} w$. Note that $\eta_{\mathcal{B}}$ can be seen as a vector in $\mathbb{R}^k$, indexed by coordinates $0, \dots, k-1$. We say $\mathcal{B}$ is *frugal* if $\eta_{\mathcal{B}}(r) \leq 1$ for every r. In other words, we allow there to be multiple balls with the same radius in the cover, as long as the total weight of these balls is at most 1.

Let $\nu_{\mathcal{B}}(u)$ be the total weight of balls covering $u \in V(G)$ according to $\mathcal{B}$. Formally, $\nu_{\mathcal{B}}(u) := \sum_{(B,w)\in\mathcal{B}: u \in B} w$. We say a k-cover is *thick* if $\nu_{\mathcal{B}}(u) \geq \frac{k^2}{n}$ for every $u \in V(G)$.

Badiambile, Langevin and Norin have suggested the following conjecture in personal communication, based on testing using linear programs.

Conjecture 7.4.1. *Let $n, k \in \mathbb{N}$. If $n \geq 2k$ and n is even, or if $n \geq 3k$ and n is odd, then there exists a thick frugal k-cover of P_n.*

However, as later noted by Norin, also in personal communication, the analogous conjecture does not hold for general trees, even by changing the lower bound on n (unless one takes $n = k^2$, in which case the statement is simply a weakening of the Burning Number Conjecture). Here is an example which shows this.

Proof. As suggested by Norin, consider the spider graph G with $q \geq 2$ branches each containing two edges. In particular, G contains $2q+1$ vertices, and take $k = 2$. We show there does not exist any thick frugal 2-cover of G. Suppose for a contradiction that such a cover, say $\mathcal{B}$, exists.

Only balls of radius 0 and 1 are present in $\mathcal{B}$. Without loss of generality, we may assume that $\mathcal{B}$ does not contains two weighted balls with the same radius and centered at the same vertex. For $i \in [r]$, write a_i for the weight of the ball of radius 0 on the leaf of the i-th branch of G (if it exists, and 0 otherwise), b_i for the weight of the ball of radius 0 on the middle vertex of the i-th branch of G, x_i for the total weight of the balls of radius 1 which are centered at either the leaf or the central vertex of the i-th branch. Write $\sum_{i=1}^{q} a_i = a$, $\sum_{i=1}^{q} b_i = b$ and $\sum_{i=1}^{q} x_i = x$. Finally, let y be the weight of the ball of radius 1 centered at the vertex of degree q. Of course, $a, b, x, y \geq 0$.

It is then direct that $\nu_{\mathcal{B}}(u)$ is $a_i + x_i$ if u is the leaf of the i-th branch, and is $b_i + x_i + y$ if u is the middle vertex of the i-th branch. As $\nu_{\mathcal{B}}(u) \geq \frac{2^2}{|V(G)|} = \frac{4}{2q+1}$, we know that $a + x = \sum_{i=1}^{q}(a_i + x_i) \geq \frac{4q}{2q+1}$ and that $b + x + qy = \sum_{i=1}^{q}(b_i + x_i + y) \geq \frac{4q}{2q+1}$.

Furthermore, we have that that $a + b = \sum_{i=1}^{q} a_i + \sum_{i=1}^{q} b_i \leq \eta_{\mathcal{B}}(0) \leq 1$, and that $x + y = \sum_{i=1}^{q} x_i + y = \eta_{\mathcal{B}}(1) \leq 1$.

It is easily verified (either by hand or with an inequality solver such as Mathematica [105]) that this system of inequalities, with variables a, b, x, y, has no solutions when $q \geq 2$. $\square$

In the remainder of the section, we will approach Conjecture 7.4.1. To simplify the proofs, and to gain greater understanding of fractional burning. we will use some linear-algebraic notation. For a vector w, we write $\mathsf{w}(i)$ for the i-th component of w, with indices starting at 0.

For $n, k \in \mathbb{N}$, let

$$W_{n,k} := \left\{ \mathsf{w} \in \left(\mathbb{Z}^{\geq 0}\right)^k : \sum_{r=0}^{k-1}(2r+1)\mathsf{w}(r) = \mathsf{w} \cdot \begin{pmatrix} 1 & 3 & 5 & \cdots & 2k-1 \end{pmatrix} = n \right\}.$$

Further define

$$V_{n,k} := \operatorname{cone}(W_{n,k}) = \left\{ \sum_{i=1}^{\ell} \alpha_i \mathsf{w}_i : \ell \in \mathbb{N}, \mathsf{w}_i \in W_{n,k}, \alpha_i \in \mathbb{R}^{\geq 0} \right\},$$

the conical hull of $W_{n,k}$. We first show that we can reformulate fractional burning in this language. Define $\mathbf{1}_k := (1 \quad 1 \quad \cdots \quad 1)$, the constant 1 vector in k dimensions.

Theorem 7.4.2. *Let $n, k \in \mathbb{N}$. There exists a thick frugal k-cover of P_n if and only if $\mathbf{1}_k \in V_{n,k}$.*

Proof. We begin by proving the if direction. As $\mathbf{1}_k \in V_{n,k}$, there exists $\ell \in \mathbb{N}$, and $\mathsf{w}_i \in W_{n,k}$ and $\alpha_i \in \mathbb{R}^{\geq 0}$ (for $1 \leq i \leq \ell$), such that $\sum_{i=1}^{\ell} \alpha_i \mathsf{w}_i(r) = 1$ for every $r \in \{0, \ldots, k-1\}$.

Notice that every $\mathsf{w}_i \in W_{n,k}$ represents a (discrete) cover of P_n with balls with radii in $0, \ldots, k-1$, possibly using the same radii multiple times. Indeed, $\mathsf{w}_i(r)$ gives the number of times the radius r is used, and the condition $\sum_{r=0}^{k-1}(2r+1)\mathsf{w}_i(r) = n$ ensures that the sum of the diameters of these balls is exactly n: these covers have no waste. As seen in the proof of Theorem 1.4.6, on a path it suffices for the sum of diameters of the balls to be n to be able to obtain a cover: place the balls one after the other in an arbitrary order. Write $\mathcal{B}_i$ for the collection of balls implicit in w_i, giving each one weight α_i. In particular, $\nu_{\mathcal{B}_i}(u) = \alpha_i$ for every $u \in V(P_n)$. Let $\mathcal{B} := \cup_{i=1}^{\ell} \mathcal{B}_i$. We verify that $\mathcal{B}$ is a thick frugal k-cover of P_n.

For every $r \in \{0, \ldots, k-1\}$,

$$\eta_{\mathcal{B}}(r) = \sum_{(B(v,r'),w) \in \mathcal{B} : r' = r} w = \sum_{i=1}^{\ell} \sum_{(B(v,r'),w) \in \mathcal{B}_i : r' = r} w = \sum_{i=1}^{\ell} \alpha_i \mathsf{w}_i(r) = 1,$$

as desired. Hence, $\mathcal{B}$ is a frugal k-cover of G.

Furthermore, we know that

$$k^2 = \sum_{r=0}^{k-1}(2r+1) = \sum_{r=0}^{k-1}(2r+1)\left(\sum_{i=1}^{\ell} \alpha_i \mathsf{w}_i(r)\right) = \sum_{i=1}^{\ell} \alpha_i \sum_{r=0}^{k-1}((2r+1)\mathsf{w}_i(r)) = \sum_{i=1}^{\ell} \alpha_i n.$$

Hence, $\nu_{\mathcal{B}}(u) = \sum_{i=1}^{\ell} \nu_{\mathcal{B}_i}(u) = \sum_{i=1}^{\ell} \alpha_i = \frac{k^2}{n}$, so $\mathcal{B}$ is thick, as desired.

We now prove the only if direction. The intuition here is that we want to "decompose" the cover of P_n into "layers", in which every ball has the same weight. We need thus introduce the following additional definition. We say a cover $\mathcal{B}$ of a graph G is *uniform* if $w_{\mathcal{B}}(u)$ has the same value for every $u \in V(G)$. We show inductively that we can do this decomposition.

Claim 7.4.2.1. *If P_n has a uniform k-cover $\mathcal{B}$, then $\eta_{\mathcal{B}} \in V_{n,k}$.*

Proof of claim. We prove the statement by induction on $|\mathcal{B}|$. The base case $\mathcal{B} = \emptyset$ is trivial, since $\eta_{\mathcal{B}}$ is the constant zero vector. Hence, suppose $|\mathcal{B}| > 0$.

We first claim there exists a subset $\mathcal{C} \subseteq \mathcal{B}$ such that every vertex of P_n is in exactly one (weighted) ball in $\mathcal{C}$.

We may construct it as follows. As $\mathcal{B}$ is non-empty and uniform, every vertex is in at least one (weighted) ball of $\mathcal{B}$. Let $\mathcal{C} \subseteq \mathcal{B}$ be a family of disjoint balls which covers completely the longest possible subpath P' of P_n, and is otherwise minimal (i.e., the covered vertices form a path exactly). If $P' = P_n$, we are done. Otherwise, there exists a vertex $u \in V(P_n) \setminus V(P')$, which is adjacent to a vertex, say v, of P'. Let $(B, w) \in \mathcal{C}$ be the ball containing v. If every ball in $\mathcal{B}$ containing u also contains v, then $\nu_{\mathcal{B}}(v) \geq \nu_{\mathcal{B}}(u) + w > \nu_{\mathcal{B}}(u)$, and so $\mathcal{B}$ is not uniform. Hence, there must exist $(B', w') \in \mathcal{C}$ containing u but not v, contradicting the maximality of $\mathcal{C}$.

Define $w_{\min} := \min_{(B,w) \in \mathcal{C}} w > 0$ the minimum weight over balls in $\mathcal{C}$, and let $\mathcal{C}_1 := \{(B, w_{\min}) : (B, w) \in \mathcal{C}\}$ and $\mathcal{C}_2 := \{(B, w - w_{\min}) : (B, w) \in \mathcal{C}\}$ (omitting the balls with weight 0). We then define $\mathcal{B}' := (\mathcal{B} \setminus \mathcal{C}) \cup \mathcal{C}_2$. By choice of $w_{\min}$, the value $w - w_{\min}$ will be 0 for at least one $(B, w) \in \mathcal{C}$, so $|\mathcal{B}'| < |\mathcal{B}|$.

As the balls of $\mathcal{C}$ cover every vertex of P_n exactly once, and by choice of the new weights, $\mathcal{B}'$ is a uniform k-cover of P_n (every vertex has weight decreased by exactly $w_{\min}$). Hence, by induction, we have that $\eta_{\mathcal{B}'} \in V_{n,k}$.

Let $\mathsf{w} \in \left(\mathbb{Z}^{\geq 0}\right)^k$ be defined by $w(r)$ being the number of balls of radius r in the weighed cover $\mathcal{C}$. As $\mathcal{C}$ covers every vertex exactly once, $\mathsf{w} \in W_{n,k}$. Furthermore, $\eta_{\mathcal{C}_1} = w_{\min}\mathsf{w}$, and in particular, $\eta_{\mathcal{C}_1} \in V_{n,k}$.

Noting that $\eta_{\mathcal{B}} = \eta_{\mathcal{B}'} + \eta_{\mathcal{C}_1}$, we obtain that $\eta_{\mathcal{B}} \in V_{n,k}$, as desired. $\square$

We know that there exists a thick frugal k-cover of P_n, say $\mathcal{B}$. We first show that $\mathcal{B}$ is uniform. (Intuitively, in order for a cover to be both thick and frugal on a path, there can be no waste whatsoever.) We know that

$$\begin{aligned}\sum_{u\in V(P_n)} \nu_{\mathcal{B}}(u) &= \sum_{u\in V(P_n)} \sum_{(B,w)\in\mathcal{B}:u\in B} w \le \sum_{(B(r,v),w)\in\mathcal{B}} w\cdot(2r+1)\\ &= \sum_{r=0}^{k-1} \sum_{(B(v,r'),w)\in\mathcal{B}:r'=r} (2r+1)w = \sum_{r=0}^{k-1}(2r+1)\eta_{\mathcal{B}}(r) \le \sum_{r=0}^{k-1}(2r+1) = k^2.\end{aligned}$$

Due to this inequality, as $\nu_{\mathcal{B}}(u) \ge \frac{k^2}{n}$ for every $u \in P_n$ ($\mathcal{B}$ is thick), it is necessary that $\nu_{\mathcal{B}}(u) = \frac{k^2}{n}$ for every $u \in P_n$, as desired.

This furthermore implies that the inequalities in the equation above is necessarily an equality. We used that $\eta_{\mathcal{B}}(r) \le 1$ for every r, since $\mathcal{B}$ is frugal. Hence, necessarily $\eta_{\mathcal{B}} = \mathbf{1}_k$.

Applying the claim now directly yields the desired result. □

Using this equivalence result, we first show that the lower bounds in Conjecture 7.4.1 cannot be improved. We will use the following well-known lemma.

Lemma 7.4.3 (Farkas' lemma [41])**.** *If* $\mathsf{A} \in \mathbb{R}^{m\times n}$ *and* $\mathsf{b} \in \mathbb{R}^m$*, then exactly one of the following holds.*

1. *There exists* $\mathsf{x} \in \mathbb{R}^n$ *such that* $\mathsf{Ax} = \mathsf{b}$ *and* $\mathsf{x} \ge \mathsf{0}$*.*

2. *There exists* $\mathsf{y} \in \mathbb{R}^m$ *such that* $\mathsf{A}^\top\mathsf{y} \ge \mathsf{0}$ *and* $\mathsf{b}^\top\mathsf{y} < 0$*.*

It is easy to show that if 2 holds, then 1 cannot hold. We will only be using this direction.

Theorem 7.4.4. *Let* $n, k \in \mathbb{N}$ *such that* $k \ge 3$*. If* n *is even and* $n < 2k$*, or if* n *is odd and* $n < 3k$*, then* $\mathbf{1}_k \notin V_{n,k}$*.*

We note that the case of n even can be seen directly by noticing that if $n \le 2k-2$, the largest ball, of radius $2(k-1)+1 = 2k-1$ will have some waste, and so necessarily a thick frugal k-cover cannot exist. However, the case of odd n is more complex, so we will prove this more formally.

Proof. Consider the matrix A in which every column is a vector of $W_{n,k}$. By definition of $V_{n,k}$, $\mathbf{1}_k \in V_{n,k}$ if and only if there exists $\mathsf{x} \geq \mathsf{0}$, a vector with only non-negative weights, such that $\mathsf{A}\mathsf{x} = \mathbf{1}_k$.

Hence, to show the theorem, it suffices by Lemma 7.4.3 to find $\mathsf{y} \in \mathbb{R}^k$ such that $\mathsf{A}^\top \mathsf{y} \geq 0$ and $\mathbf{1}_k^\top \mathsf{y} < 0$. The first condition can be reformulated as $\mathsf{w} \cdot \mathsf{y} \geq 0$ for every $\mathsf{w} \in W_{n,k}$.

Let $d = 2$ if n is even, and $d = 3$ if n is odd. Define y by $\mathsf{y}(r) = n - d(2r+1)$ for $r \in \{0, \ldots, k-1\}$, except if $r = \frac{n-1}{2}$ in which case we define $\mathsf{y}(r) = 0$. (Note that this definition of y was chosen by looking at solutions to linear programs.)

If $\frac{n-1}{2} \notin \{0, \ldots, k-1\}$,

$$\mathbf{1}_k^\top \mathsf{y} = \sum_{r=0}^{k-1} \mathsf{y}(i) = \sum_{r=0}^{k-1} (n - d(2r+1)) = kn - dk^2.$$

In this case, $\mathbf{1}_k^\top \mathsf{y} < 0$ if and only if $n < dk$, which is exactly the conditions stated in the theorem. If $\frac{n-1}{2} \in \{0, \ldots, k-1\}$, the sum is the same except for the term $\mathsf{y}\left(\frac{n-1}{2}\right)$, for which the value $n - dn$ is replaced by 0. In this case (necessarily, n is odd and so $d = 3$), we get

$$\begin{aligned}
\mathbf{1}_k^\top \mathsf{y} &= \sum_{r=0}^{k-1} \mathsf{y}(i) = (kn - dk^2) - (n - dn) = (k - 1 + d)n - dk^2 \\
&= (k+2)\left(2 \cdot \frac{n-1}{2} + 1\right) - 3k^2 \leq (k+2)(2k-1) - 3k^2 < 0,
\end{aligned}$$

as desired, where the last inequality holds for $k > 2$.

Let $\mathsf{w} \in W_{n,k}$. We know that $\sum_{r=0}^{k-1} (2r+1)\mathsf{w}(r) = n$. The terms $(2r+1)$ on the left hand side are always odd. Hence, if n is even ($d = 2$ case), then $\sum_{r=0}^{k-1} \mathsf{w}(r)$ must also be even, and so at least 2. Otherwise, if n is odd ($d = 3$), $\sum_{r=0}^{k-1} \mathsf{w}(r)$ must also be odd. Note that by definition of $W_{n,k}$, this value can only be one if w is the vector which has only value 1 for coordinate $r = \frac{n-1}{2}$, and otherwise zero. Otherwise, $\sum_{r=0}^{k-1} \mathsf{w}(r) \geq 3$. Hence, $\sum_{r=0}^{k-1} \mathsf{w}(r) \geq d$, except in one specific case.

First suppose this last inequality holds. Then,

$$\mathsf{w} \cdot \mathsf{y} = \sum_{r=0}^{k-1} \mathsf{w}(r) \cdot \mathsf{y}(r) = n \sum_{r=0}^{k-1} \mathsf{w}(r) - d \sum_{r=0}^{k-1} (2r+1)\mathsf{w}(r) \geq dn - dn = 0.$$

In the special case where $\sum_{r=0}^{k-1} \mathsf{w}(r) = 1$, we noted that the vector w has only one non-zero value in position $r = \frac{n-1}{2}$. In this case, $\mathsf{y}(r) = 0$, and so $\mathsf{w} \cdot \mathsf{y} = 0$.

This completes the proof of the theorem. □

We now proceed to proving some cases of Conjecture 7.4.1. We begin by showing the easiest case $n = 2k$. Write $\mathsf{e}_{i,k}$ for the vector of length k which has value 1 in position i (with indices starting at zero), and 0 in all other positions. Define

$$W_{n,k}^{(m)} := \left\{ \mathsf{w} \in W_{n,k} : \sum_{r=0}^{k-1} \mathsf{w}(r) = m \right\},$$

as well as

$$V_{n,k} := \operatorname{cone}(W_{n,k}).$$

Lemma 7.4.5. *If $n, k \in \mathbb{N}$ are such that $n = 2k$, then $\mathbf{1}_k \in V_{n,k}^{(2)}$.*

Proof. Let $\mathsf{w}_i = \mathsf{e}_{i,k} + \mathsf{e}_{k-i-1,k}$, for $i \in \{0, \ldots, k-1\}$. It is direct that $\sum_{r=0}^{k-1} \mathsf{w}_i(r) = 2$ for every i. Furthermore,

$$\sum_{r=0}^{k-1} (2r+1)\mathsf{w}_i(r) = (2i+1) + (2(k-i-1)+1) = 2k = n,$$

so $\mathsf{w}_i \in W_{n,k}^{(2)}$.

Finally, the i-th coordinate appears only in the vectors w_i (due to $\mathsf{e}_{i,k}$) and w_{k-i-1} (due to $\mathsf{e}_{k-(k-i-1)-1,k} = \mathsf{e}_{i,k}$). This implies that $\left(\sum_{i=0}^{k-1} \mathsf{w}_i\right)(r) = 2\mathsf{e}_{r,k}(r) = 2$ for every $r \in \{0, \ldots, k-1\}$. Hence, $(2 \quad 2 \quad \cdots \quad 2) \in V_{n,k}^{(2)}$, and so $\mathbf{1}_k \in V_{n,k}^{(2)}$ as desired. □

The following lemma is also useful in an inductive proof, as is allows us to "shift up" some radii. For any $k \in \mathbb{N}$, we define f_i as the function from $\mathbb{R}^k \to \mathbb{R}^{k+i}$ for which $f_i(\mathsf{w})$ is obtained by prepending i zeroes to w.

Lemma 7.4.6. *Let $n, k, m \in \mathbb{N}$, $i \in \mathbb{Z}^{\geq 0}$. If $\mathbf{1}_k \in V^{(m)}_{n,k}$, then $f_i(\mathbf{1}_k) \in V^{(m)}_{n+2mi,k+i}$.*

Proof. Let $\mathsf{w} \in W^{(m)}_{n,k}$. Given that $\sum_{r=0}^{k-1} \mathsf{w}(r) = m$, it is trivial that $\sum_{r=0}^{k+i-1} f_i(\mathsf{w})(r) = m$.

Furthermore, given that $\sum_{r=0}^{k-1}(2r+1)\mathsf{w}(r) = n$, then

$$\begin{aligned}\sum_{r=0}^{k+i-1}(2r+1)f(\mathsf{w})(r) &= \sum_{r=i}^{k+i-1}(2r+1)f(\mathsf{w})(r) = \sum_{r=0}^{k-1}(2(r+i)+1)\mathsf{w}(r)\\ &= \sum_{r=0}^{k-1}(2r+1)\mathsf{w}(r) + 2i\sum_{r=0}^{k-1}\mathsf{w}(r) = n+2mi,\end{aligned}$$

and so $f(\mathsf{w}) \in W^{(m)}_{n+2mi,k+i}$.

As $V^{(m)}_{n+2mi,k+i}$ is the conical hull of $W^{(m)}_{n+2mi,k+i}$ and $\mathbf{1}_k \in V^{(m)}_{n,k}$, this directly implies that $f(\mathbf{1}_k) \in V^{(m)}_{n+2mi,k+i}$. □

We may now prove most of Conjecture 7.4.1.

Theorem 7.4.7. *Let $n, k \in \mathbb{N}$. If n is even and $n \geq 2k$, or if n is odd and $n \geq 4k-3$, then $\mathbf{1}_k \in V_{n,k}$.*

Proof. We first show the statement for even n. We prove the statement by induction on n, then on k. For $n = 2$, then necessarily $k = 1$, and so the base case is implied by Lemma 7.4.5. We may now suppose that $n \geq 4$.

First suppose $n \geq 4k$. As $n \geq 4$, there exists even $n_1, n_2 \in \mathbb{N}$ such that $n_1 + n_2 = n$. Here, it is simpler to work covers (which, we recall, are equivalent by Theorem 7.4.2). By induction, there exists a thick frugal k-cover $\mathcal{B}_1$ of P_{n_1}, and a thick frugal k-cover $\mathcal{B}_2$ of P_{n_2}. We can see P_n as a concatenation of P_{n_1} and P_{n_2}. Hence, we may define

$$\mathcal{B} := \left\{\left(B, w\cdot\frac{n_1}{n_1+n_2}\right) : (B,w) \in \mathcal{B}_1\right\} \cup \left\{\left(B, w\cdot\frac{n_2}{n_1+n_2}\right) : (B,w) \in \mathcal{B}_2\right\}.$$

It is easily verified that $\mathcal{B}$ is a thick frugal k-cover of P_n.

Hence, we may suppose $n < 4k$. As n is even, this implies that $2k - \frac{n}{2} \geq 1$ and is an integer. Also, $\frac{n}{2} - k \geq 0$ and is an integer. By Lemma 7.4.5, $\mathbf{1}_{2k-\frac{n}{2}} \in V_{4k-n,2k-\frac{n}{2}}$. By Lemma 7.4.6, we then have that $f_{\frac{n}{2}-k}\left(\mathbf{1}_{2k-\frac{n}{2}}\right) \in V_{(4k-n)+2\cdot 2\cdot\left(\frac{n}{2}-k\right),\left(2k-\frac{n}{2}\right)+\left(\frac{n}{2}-k\right)} = V_{n,k}$.

Given that $n < 4k$, we have $\frac{n}{2} - k < k$, so by induction, we have that $\mathbf{1}_{\frac{n}{2}-k} \in V_{n,\frac{n}{2}-k}$. By appending zeroes to it (and overloading notation), we have $\mathbf{1}_{\frac{n}{2}-k} \in V_{n,k}$. By definition, $\mathbf{1}_k = f_{\frac{n}{2}-k}\left(\mathbf{1}_{2k-\frac{n}{2}}\right) + \mathbf{1}_{\frac{n}{2}-k} \in V_{n,k}$, as desired.

We now prove the case when n is odd. We know that $n-(2k-1) \geq 2(k-1)$ and is even, so $\mathbf{1}_{k-1} \in V_{n-(2k-1),k-1}$. Furthermore, note that if $\mathsf{w} \in W_{n-(2k-1),k-1}$, then $(\mathsf{w} \quad 1) \in W_{n,k}$. As $V_{n-(2k-1),k-1}$ is the conical hull of $W_{n-(2k-1),k-1}$, this implies that $(\mathbf{1}_{k-1} \quad 1) = \mathbf{1}_k \in V_{n,k}$, as desired. $\square$

Thus, only the cases where n is odd and $3k \leq n \leq 4k-4$ are left. We will only prove the cases for n of the form $3k+6i$. Given that any ball with integer radius covers an odd number of vertices, if n is odd, we will not be able to cover it with two balls. We first prove the base case $n = 3k$.

Lemma 7.4.8. *If $n, k \in \mathbb{N}$ are such that $n = 3k$ is odd, then $\mathbf{1}_k \in V^{(3)}_{n,k}$.*

Proof. We prove this statement by induction on (odd) k. The case $k = 1$ is trivial as $(3) \in W^{(3)}_{3,1}$, and so $\frac{1}{3}(3) = (1) = \mathbf{1}_1 \in V^{(3)}_{3,1}$. The case $k = 3$ is also easy, as $\mathbf{1}_3 \in W^{(3)}_{9,3}$.

We now proceed with the inductive step, so we may suppose $k \geq 5$ (i.e., $n \geq 15$). Note that $n - 6 = 3(k-2)$. Hence, by induction, we know that $\mathbf{1}_{k-2} \in V^{(3)}_{n-6,k-2}$ (this is well defined for $n > 6$). By Lemma 7.4.6, $f_1(\mathbf{1}_{k-2}) \in V^{(3)}_{(n-6)+2\cdot 3\cdot 1,(k-2)+1} = V^{(3)}_{n,k-1}$. By appending a zero to $f_1(\mathbf{1}_{k-2})$ (and overloading notation), we may write that $f_1(\mathbf{1}_{k-2}) \in V^{(3)}_{n,k-1}$.

Let $\mathsf{w}_i := \mathsf{e}_{i,k} + e_{\frac{k-1}{2}-i,k} + e_{k-1,k}$ for $i \in \{0, \dots, \frac{k-1}{2}\}$, and let $\mathsf{v}_i := e_{0,k} + e_{\frac{k-1}{2}+i,k} + e_{k-1-i,k}$ for $i \in \{1, \dots, \frac{k-1}{2} - 1\}$ (note that this set is only non-empty as $k \geq 5$). One easily verifies that $\mathsf{w}_i, \mathsf{v}_i \in W^{(3)}_{n,k}$ for every i.

We claim that

$$\sum_{i=0}^{\frac{k-1}{2}} \mathsf{w}_i + \sum_{i=1}^{\frac{k-1}{2}-1} \mathsf{v}_i = \begin{pmatrix} \frac{k+1}{2} & 2 & 2 & \cdots & 2 & \frac{k+1}{2} \end{pmatrix} =: \mathsf{y}.$$

Indeed, $\mathsf{e}_{0,k}$ appears in exactly $\frac{k+1}{2}$ vectors : every v_i (of which there are $\frac{k-1}{2} - 1$), as well as w_0 and $\mathsf{w}_{\frac{k-1}{2}}$. Similarly, $\mathsf{e}_{k-1,k}$ appears in exactly $\frac{k+1}{2}$ vectors : every w_i. Every other coordinate appears exactly twice (possibly, in the same vector). If $0 < j \leq \frac{k-1}{2}$, $\mathsf{e}_{j,k}$ is a term only in w_j and $\mathsf{w}_{\frac{k-1}{2}-j}$. If $\frac{k-1}{2} < j < k-1$, $\mathsf{e}_{j,k}$ appears only in $\mathsf{v}_{j-\frac{k-1}{2}}$ and v_{k-1-j}.

Hence, we deduce that $\mathsf{y} \in V_{n,k}$. Thus, to show that $\mathbf{1}_k \in V_{n,k}$, it suffices to note that $\frac{k-3}{2} \cdot f_1(\mathbf{1}_{k-2}) + \mathsf{y} = \frac{k+1}{2} \cdot \mathbf{1}_k$ (note that $\frac{k-3}{2} \geq 0$). □

We now proceed with the inductive step.

Lemma 7.4.9. *Let $n, k, i \in \mathbb{N}$. If n is odd and $n = 3k + 6i$, then $\mathbf{1}_k \in V_{n,k}$.*

Proof. The proof is similar to part of the proof for the even case.

If $3k + 6i \geq 4k - 3$, then the result follows from Theorem 7.4.7. Hence, we may assume that $3k + 6i \leq 4k - 3$, and so $i \leq \frac{k}{6}$. In particular, $k - 2i > 0$.

By Lemma 7.4.8, $\mathbf{1}_{k-2i} \in V^{(3)}_{3(k-2i),k-2i}$. By Lemma 7.4.6, we then have that $f_{2i}(\mathbf{1}_{k-2i}) \in V_{3(k-2i)+2\cdot 3\cdot 2i,(k-2i)+2i} = V_{n,k}$.

As $n = 3k + 6i \geq 24i \geq 4 \cdot 2i$, we have by Theorem 7.4.7 that $\mathbf{1}_{2i} \in V_{n,2i}$. By appending zeroes to it, we have (with overloaded notation) that $\mathbf{1}_{2i} \in V_{n,k}$.

Thus, $\mathbf{1}_k = f_{2i}(\mathbf{1}_{k-2i}) + \mathbf{1}_{2i} \in V_{n,k}$, as desired. □

Here, we do not prove the cases for the other modulo classes. For these cases, the situation gets more complex : we need to be able to consider covers with at least 5 radii. This can be proved similarly to Theorem 7.4.4. Here, we take A such that every column is a vector of $W^{(3)}_{n,k}$, and we set $\mathsf{y} \in \mathbb{R}^k$ defined by $\mathsf{y}(r) = 3(2r+1) - n$ for $r \in \{0, \ldots, k-1\}$ (this is the negative of the choice in the proof of Theorem 7.4.4, except for the one special value). We obtain

$$\mathbf{1}_k^\top \mathsf{y} = \sum_{r=0}^{k-1} (3(2r+1) - n) = 3k^2 - kn < 0$$

if $n > 3k$, and

$$\mathsf{w} \cdot \mathsf{y} = \sum_{r=0}^{k-1} \mathsf{w}(r) \cdot \mathsf{y}(r) = 3\sum_{r=0}^{k-1} (2r+1)\mathsf{w}(r) - n\sum_{r=0}^{k-1} \mathsf{w}(r) = 3n - 3n = 0.$$

8

Conclusion

In this thesis, we have presented five manuscripts on problems in structural graph theory and graph searching. Four of these either directly concern, or are inspired by, problems on the structure of graphs when forbidding a minor.

We have worked on two related relaxations of Hadwiger's conjecture. In the first, Chapter 2, we showed that graphs excluding a sparse bipartite graph on t vertices have a vertex of degree at most $|V(H)| - 2$, and in particular are $(|V(H)| - 1)$-colourable. This is the bound suggested by Hadwiger's conjecture, when $H = K_t$. This is one of the only classes of graphs H for which this relaxation, the H-Hadwiger conjecture, is known to be true. As our method only uses degeneracy, it further extends to list colouring and DP-colouring, other frequently studied, and more difficult, types of vertex colourings.

In the second, Chapter 3, we have studied a similar problem, in which we forbid all graphs H on t vertices and a fixed number of edges. Precisely, we have shown that if a graph have average degree at least $t-1$, then we can find a minor on t vertices and a proportion of $\sqrt{2}-1-o(1)$ of possible edges. We have further showed that this cannot be improved beyond $\frac{3}{4}+o(1)$.

Hadwiger's conjecture is considered one of the most important, and difficult, open problems in graph theory. Although we have shown that our methods on their own have some limits, the methods used in Chapters 2 and 3 may be helpful in approaching Hadwiger's conjecture.

In Chapter 4, we proved for graphs of bounded maximum degree (and more generally, graphs forbidding a topological minor) a modification of Menger's theorem in which the obtained paths are not only pairwise disjoint, but non-adjacent. This work was originally motivated by recent (now disproved) conjectures in coarse graph theory, where the objective is to study the large-scale structure of graphs. The result we proved is also on its own a quite natural variant of Menger's theorem. It might also be useful in problems related to induced minors.

We then studied two games on graphs, cops and robbers in Chapter 5, and graph burning in Chapter 6.

In Chapter 5, we showed new upper bounds on the cop number when excluding a minor. Our bounds often improve, sometimes significantly (up to a factor of 4), the previous bounds, which had been proved by Andreae. For example, we have improved bounds on the cop number of $K_{2,t}$-minor-free graphs and $K_{3,t}$-minor-free graphs by a factor of 2. Notably, motivated by a large literature on the cop number in relation to graph topology, we proved an upper bound of 6 on the cop number of linklessly embeddable graphs. The most promising further direction in this area of research is to improve the method by forbidding multiple graphs as minors, especially when they are very similar (which is often the case for obstruction sets, such as in the case of the Petersen family for linklessly embeddable graphs).

Finally, in Chapter 6, we studied the Burning Number Conjecture, which is considered the most important problem in the area of graph burning. We proved that this conjecture holds asymptotically. Our probabilistic methods are quite different than the previously used, most inductive, methods. This new approach might be also useful in, for example, proving stronger bounds on the burning number for specific classes of graphs, showing that the Burning Number Conjecture holds for sufficiently large graphs, and in tackling other covering and metric problems. In the discussion section, we have also studied fractional burning, a variant that our approach naturally suggests.

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