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Tensor completion for a certain class of groups

by

Ekaterina Lyutikova

A thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree
of Master of Science

Departement of Mathematics and Statistic

McGill University

Montréal

March 1995



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Abstract

In this thesis we construct the tensor A -completion over an arbitrary torsion-free ring A for groups whose maximal abelian subgroups are either conjugate separated or A -modules. As a corollary, this result gives us an explicit description of the structure of tensor completion for groups of the form F/N' , in particular, for free solvable groups. Moreover, it is proved that the class of A -faithful groups is closed under free products. We also discuss some general aspects of the theory of exponential groups and tensor completions.

La complétion tensorielle d'une certaine classe de groupes

par

Ékaterina Lyutikova

Résumé

Dans cette thèse on construit la complétion A -tensorielle des groupes dont les sousgroupes maximaux abéliens sont soit conjugués-séparés, soit des A -modules. On obtient, en conséquence, une description explicite de la structure des complétions tensorielles des groupes du genre F/N' , en particulier des groupes résolubles libres. De plus, on démontre que la classe des groupes A -fidèles est stable relativement aux produits libres. Nous discutons de certains aspects généraux de la théorie des groupes exponentiels et des complétions tensorielles.

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1 Introduction

The axiomatic notion of an exponential group was introduced by R.C. Lyndon in [17] as a tool in his description of the set of all solutions of a given equation in a finitely generated free group. This study was motivated by the famous problem of A. Tarski: is the elementary theory of a free group decidable? In other words, is there an algorithm determining whether a given first-order formula is true in a free group? Although Lyndon's methods led to only partial results towards the solution of Tarski's problem, which is still open, the concept of an exponential group turned out to be of great independent interest. An A -group after Lyndon is a group that admits an action of the ring A satisfying a certain set of axioms. In particular, if A is the ring of integers, the concept of an A -group reduces to that of an ordinary group. For $A = \mathbf{Z}_m$, the integers modulo m , the A -groups are groups in which all elements have order dividing m . For A the field of rationals, the A -groups are groups in which "extraction of roots" is always possible and unique; questions related to this were studied by B.H. Neumann [27], A.I. Mal'cev [19], and A. Kontorovich [14]. G. Baumslag [5] considered, more generally, the case that A consists of those rationals that can be written with denominator a product of primes belonging to some prescribed set. Taking a different approach, P. Hall [11] and also M. Lazard [16] considered groups admitting exponents from a ring more general than the ring of rational numbers.

In [22] A. Myasnikov developed a new notion of an exponential group, adding one more axiom to Lyndon's definition. This improved notion is more convenient because it coincides exactly with the notion of a module over a ring in the abelian case, whereas abelian exponential groups after Lyndon constitute a far wider class. These two definitions coincide in the case of free groups.

The continuing interest in the theory of exponential groups is motivated by

many factors. First, many natural classes of groups are A -groups; for example, unipotent groups over a field k of characteristic zero are k -groups, pro- p -groups are exponential groups over the ring of p -adic integers, etc. (see section 2 for examples.) Second, the notion of an A -group is a natural generalization of the notion of a module over a ring to the category of non-commutative groups. Third, one can consider for an arbitrary group G its largest ring of scalars $A(G)$, over which G is an $A(G)$ -group. This notion is an analog of the notion of a centroid for rings and algebras (see [22] for details) and plays an important role in the theory of exponential groups. Finally, as shown in [21], A -groups are very helpful in the investigation of model-theoretic problems for noncommutative groups.

In [23] it was demonstrated that the key role in the study of exponential groups is played by the construction of an A -tensor completion G^A of a group G , where A is an arbitrary associative ring with identity. It is uniquely determined by a natural universal property. Tensor completions have turned out to be an extremely effective tool in the study of free A -groups (see [24]), groups acting freely on Λ -trees, where Λ is any ordered abelian group [26], and in producing interesting new examples of hyperbolic groups [13].

In the general case, it is difficult to give a constructive description for the tensor completion of an arbitrary group. However, there have been successful attempts to give such a description in some particular cases.

The first step in this direction was made by G. Baumslag [5], who introduced the $\mathbb{Q}$ -completion $F^{\mathbb{Q}}$ of a free group F , and, what is most important, described $F^{\mathbb{Q}}$ explicitly as the union of an ascending sequence of groups, each of which is a free product with a single amalgamation. This allowed him to establish some properties of the group $F^{\mathbb{Q}}$. In [24] A. Myasnikov and V. Remeslennikov developed this approach, applying it to an arbitrary ring A of characteristic 0 and to a wide class of groups, which they called CSA-groups.

By definition, G is a CSA-group if every its maximal abelian subgroup M is conjugate separated, meaning that for every $x \in G \setminus M$ $M^x \cap M = 1$. The class of CSA-groups is quite wide: it contains abelian, free, hyperbolic torsion-free groups, groups acting freely on Λ -trees, and universally free groups. Moreover, this class is universally axiomatizable, closed under free products, direct limits and extensions of centralizers (basic facts concerning these constructions are given in section 4.) In [10] it was proved that a separated HNN-extension of an arbitrary CSA*-group (CSA, without elements of order two) with associated conjugate separated subgroups is again CSA*; similar results were also obtained for amalgamated products of CSA*-groups and for the fundamental groups of certain types of graphs of CSA*-groups.

In [24] for an arbitrary CSA*-group G the construction of tensor completion was obtained as an iterated tree extension of centralizers of the group G (see section 5.) This result gives us a nice constructive description of the tensor completion for a very wide class of groups; however, it does not apply to groups containing normal abelian subgroups (such as solvable groups). In section 6 for a given ring A we introduce a new class $\mathcal{K}_A$, which consists of groups whose maximal abelian subgroups are either conjugate separated or A -modules (it contains, in particular, the class of CSA-groups.) The main result of the thesis is an explicit description of the construction of tensor completion for $\mathcal{K}_A$ -groups over an arbitrary ring of characteristic 0, which is similar to the corresponding construction for CSA-groups. As a corollary, we obtain a description of tensor completion for groups of the form F/N' , where F is a free group, N a normal subgroup of F and $N' = [N, N]$, under certain natural restrictions on the group F/N . In particular, this result applies to free solvable groups. An interesting corollary is that the tensor completion of a free nonabelian solvable group is, in general, not solvable.

In the study of tensor completions the case where a group G is a subgroup

of its A -completion (i.e. G is A -faithful) is of particular interest. In [24] it was proved that the class $\mathcal{F}$ of all A -faithful groups is universally axiomatizable and closed under subgroups, direct products and direct limits, and, in the case where A is an integral domain of characteristic 0, contains all residually torsion-free nilpotent groups. We summarize these results in subsection 3.5. In section 6 we prove that the class of torsion-free faithful A -groups over an arbitrary ring A of characteristic 0 is closed under free products. Moreover, the construction of the tensor completion for free solvable groups implies that every free solvable group is faithful over any torsion-free ring A .

The results of section 6 are new results obtained by the author.

2 Exponential groups

In this section, following [23], we describe in more detail connections and distinctions between the two definitions of an exponential group, and give a lot of natural examples. We discuss the notions of A -faithfulness, A -torsion, A -subgroups and A -ideals, and we also introduce different types of mappings from one exponential group to another. In subsection 2.5 it is shown that the class of exponential groups (in the sense of one definition or the other) is closed under direct and Cartesian products, as well as under direct and inverse limits. It is also pointed out that these operations possess certain universal properties that are analogous to the properties of the corresponding operations in the category of abelian groups.

2.1 Basic notions of the theory of exponential groups

Up to the end of this section, let us fix an arbitrary associative ring with identity A , as well as a group G .

Given an action of A on G , i. e. a mapping $G \times A \rightarrow G$, we will write the

result of the action of $\alpha \in A$ on $g \in G$ as g^α . Consider the following axioms:

1. $g^1 = g$, $g^0 = 1$, $1^\alpha = 1$;
2. $g^{\alpha+\beta} = g^\alpha \cdot g^\beta$, $g^{\alpha\beta} = (g^\alpha)^\beta$;
3. $(h^{-1}gh)^\alpha = h^{-1}g^\alpha h$;
4. $[g, h] = 1 \implies (gh)^\alpha = g^\alpha h^\alpha$.

Definition 2.1 *The group G is said to be an A -exponential group (or an A -group) after Lyndon, if an action of the ring A on G satisfying axioms 1)-3) is defined.*

Definition 2.2 *The group G is called an A -exponential group (or an A -group), if an action of the ring A on G satisfying axioms 1)-4) is defined. In this case A is called a ring of scalars of the group G .*

Observe that every abelian A -group is an A -module, and vice versa.

Let $\mathcal{L}_A$ and $\mathcal{M}_A$ be the classes of A -exponential groups after Lyndon and all A -exponential groups, respectively. Clearly, $\mathcal{L}_A \supseteq \mathcal{M}_A$. However, the class $\mathcal{L}_A$ contains abelian A -groups which are not A -modules.

Example 2.1 *Let θ be a non-identity automorphism of the ring A , and M a free A -module with base $\langle x, y \rangle$. We can define a new action $*$ of the ring A on M as follows:*

$$z * \alpha = \begin{cases} z \cdot \theta(\alpha), & z \in xA \cup yA, \\ z \cdot \alpha, & z \notin xA \cup yA. \end{cases}$$

Then the action $*$ of A on M satisfies axioms 1)-3), but if $\alpha_0 \neq \theta(\alpha_0)$, then

$$(x + y) * \alpha_0 = (x + y)\alpha_0 \neq (x + y)\theta(\alpha_0) = x * \alpha_0 + y * \alpha_0,$$

hence axiom 4) doesn't hold.

Corollary 2.1 $\mathcal{L}_A \supset \mathcal{M}_A$

Proposition 2.1 *If $G \in \mathcal{L}_A$ then for every $\alpha, \beta \in A$, $x, y \in G$*

$$[x, y] = 1 \Rightarrow [x^\alpha, y^\beta] = 1.$$

Proof. Indeed,

$$yx^\alpha = yx^\alpha y^{-1}y = (yxy^{-1})^\alpha y = x^\alpha y,$$

therefore

$$x^\alpha y^\beta = x^\alpha y^\beta y^{-\alpha} y^\alpha = (x^\alpha y x^{-\alpha})^\beta x^\alpha = y^\beta x^\alpha. \square$$

Most natural examples of exponential groups lie in the class $\mathcal{M}_A$:

1. An arbitrary group is a $\mathbf{Z}$ -group.
2. A divisible group from $\mathcal{L}_{\mathbf{Q}}$ is a $\mathbf{Q}$ -group.
3. A group of period m is a $\mathbf{Z}/m\mathbf{Z}$ -group.
4. An arbitrary A -operator group from $\mathcal{L}_A$ with the ring of operators A is an A -group.
5. A module over a ring A is an abelian A -group.
6. Free Lyndon A -groups [17] are A -groups.
7. Exponential nilpotent A -groups over a binomial ring A , introduced by P. Hall [11], are A -groups.
8. An arbitrary pro- p -group is a $\mathbf{Z}_p$ -group over the ring of p -adic integers $\mathbf{Z}_p$.
9. An arbitrary profinite group is a $\hat{\mathbf{Z}}$ -group, where $\hat{\mathbf{Z}}$ is the total completion of $\mathbf{Z}$ in the profinite topology.
10. Complex (real) nilpotent Lie groups are $\mathbf{C}$ - ($\mathbf{R}$ -) groups respectively.

Every A -group G from $\mathcal{L}_A$ is a multi-operator group in the signature $\sigma_A = \langle \cdot, ^{-1}, f_\alpha \mid \alpha \in A \rangle$, where f_α is a unary operation of raising to the power α , i.e. $f_\alpha(g) = g^\alpha$.

Since axioms 1)-3) are identities, and axiom 4) is a quasiidentity, the definition of exponential groups immediately implies the following proposition.

Proposition 2.2 1. The class $\mathcal{L}_A$ is a variety in the signature σ_A .

2. The class $\mathcal{M}_A$ is a quasivariety in the signature σ_A .

2.2 Faithfulness and torsion

For an A -group G , let us introduce the following notations:

$$x^A = \{x^\alpha \mid \alpha \in A\}, \quad x \in G,$$

$$X^A = \bigcup_{x \in X} x^A, \quad X \subseteq G.$$

Definition 2.3 An element $0 \neq \alpha \in A$ acts faithfully on G , if $G^\alpha \neq 1$. The ring of scalars A acts faithfully on G (A is a faithful ring of scalars for G), if every nonzero element of A acts faithfully on G .

$$\text{Let } \text{Ann}_A G = \{\alpha \in A \mid G^\alpha = 1\}.$$

Proposition 2.3 For every A -group G $\text{Ann}_A G$ is a two-sided ideal in A . Moreover, the action of A on G induces an action of the quotient ring $\bar{A} = A/\text{Ann}_A G$ on G such that G becomes a faithful $\bar{A}$ -group.

Proposition 2.3 allows us to obtain from an arbitrary action of A on G a faithful action of the quotient ring $\bar{A}$ on G .

Definition 2.4 An element $g \in G$ is called periodic, if $g^\alpha = 1$ for some $\alpha \in A$, $\alpha \neq 0$. In this case the right ideal $\mathcal{O}(g) = \{\alpha \in A \mid g^\alpha = 1\}$ is called the ordinal ideal of g . A group G without nontrivial periodic elements is called an A -torsion-free group.

Proposition 2.4 *An A -torsion-free group is a faithful A -group.*

Proposition 2.5 *Let $1 \neq g \in G$. If α is an invertible element of A , then $g^\alpha \neq 1$.*

Proof. If $g^\alpha = 1$, then $g = (g^\alpha)^{\alpha^{-1}} = 1$. □

Propositions 2.4 and 2.5 imply the following.

Proposition 2.6 *Let A be a division ring, then every A -group is an A -faithful A -torsion-free group.*

Finally, let us give an example of a faithful A -group in which all the elements are periodic.

Example 2.2 *Let $G = \bigoplus_{i=1}^{\infty} C(n)$, where $C(n)$ is a cyclic group of order n . Then G is a periodic faithful $\mathbb{Z}$ -group.*

2.3 A -subgroups and ideals

Let G be an A -group.

Definition 2.5 *A subgroup $H \leq G$ is called an A -subgroup, if $H^A = H$. A subgroup H is A -generated by a set $X \subseteq G$, if H is the minimal A -subgroup of G containing X , in this case we write $H = \langle X \rangle_A$.*

In this way, A -subgroups of G are subgroups of G considered as a multiplier group.

Proposition 2.7 *For $X \subseteq G$, let $X_0 = \langle X \rangle$, $X_{n+1} = \langle X_n^A \rangle$, where $\langle X \rangle$ is a subgroup (not an A -subgroup!), generated by the set X . Then*

$$\langle X \rangle_A = \bigcup_{n=0}^{\infty} X_n.$$

Proposition 2.8 *Let H be a normal subgroup in G , then $\langle H \rangle_A$ is a normal subgroup in G .*

Proof. By proposition 2.7 $\langle H \rangle_A = \bigcup_{n=0}^{\infty} H_n$, where $H_{n+1} = \langle H_n^A \rangle$. By axiom 3) for every $x \in G$ we have

$$x^{-1}H_{n+1}x = \langle (x^{-1}H_nx)^A \rangle = \langle H_n^A \rangle,$$

where the last equality is obtained by induction on n . Consequently,

$$x^{-1}\langle H \rangle_A x = \bigcup_{n=0}^{\infty} x^{-1}H_nx = \bigcup_{n=0}^{\infty} H_n = \langle H \rangle_A,$$

as required. $\square$

Observe that not for every normal A -subgroup H of the group G the quotient group G/H has a natural structure of an A -group. Below we introduce the notion of an A -ideal H of the group G supplying a subgroup H with some conditions that allow us to induce the structure of an A -group on G/H . Naturally, the notion of an A -ideal in the category $\mathcal{L}_A$ is different from that in $\mathcal{M}_A$.

Definition 2.6 *Let $G \in \mathcal{L}_A$. Then a normal A -subgroup H of the group G is called an $\mathcal{L}_A$ -ideal, if for every $g \in G$, $h \in H$, $\alpha \in A$*

$$(gh)^\alpha \in g^\alpha H.$$

Definition 2.7 *A homomorphism of A -groups $\varphi : G \rightarrow H$ is called an A -homomorphism, if*

$$(g^\alpha)^\varphi = (g^\varphi)^\alpha, \quad g \in G, \quad \alpha \in A.$$

Proposition 2.9 *For groups $G, H \in \mathcal{L}_A$ the following is true:*

1. *if $\varphi : G \rightarrow H$ is an A -homomorphism, then $\ker \varphi$ is an $\mathcal{L}_A$ -ideal in G ;*

2. if H is an $\mathcal{L}_A$ -ideal in G , then the action of A on G induces an action of A on G/H according to the rule:

$$(gH)^\alpha = g^\alpha H, \quad g \in G,$$

with respect to which G/H is an $\mathcal{L}_A$ -group.

Before we give a definition of an $\mathcal{M}_A$ -ideal, we need some preliminary notions.

Definition 2.8 For $g, h \in G$, $\alpha \in A$ the element

$$(g, h)_\alpha = h^{-\alpha} g^{-\alpha} (gh)^\alpha$$

is called an α -commutator of the elements g and h .

It is clear that:

$$(gh)^\alpha = g^\alpha h^\alpha (g, h)_\alpha,$$

$$G \in \mathcal{M}_A \iff ([g, h] = 1 \longrightarrow \forall \alpha (g, h)_\alpha = 1).$$

The last equivalence leads us to the definition of an $\mathcal{M}_A$ -ideal.

Definition 2.9 Let $G \in \mathcal{L}_A$ be a group. Then a normal A -subgroup $H \leq G$ is called an $\mathcal{M}_A$ -ideal, if for every $g, h \in G$, $\alpha \in A$ we have

$$[g, h] \in H \implies (g, h)_\alpha \in H.$$

Proposition 2.10 Let $G \in \mathcal{L}_A$. Then:

1. if H is an $\mathcal{M}_A$ -ideal in G , then H is an $\mathcal{L}_A$ -ideal in G ;
2. if $\varphi : G \rightarrow H$ is an A -homomorphism of groups from $\mathcal{M}_A$, then $\ker \varphi$ is an $\mathcal{M}_A$ -ideal in G ;
3. if H is an $\mathcal{M}_A$ -ideal in G , then $G/H \in \mathcal{M}_A$.

Proof. 1) Let H be an $\mathcal{M}_A$ -ideal in G . For $g \in G$, $h \in H$, $\alpha \in A$ we have

$$(gh)^\alpha = g^\alpha h^\alpha (g, h)_\alpha.$$

Since H is a normal subgroup of G , it follows that $[g, h] \in H$, and hence $(g, h)_\alpha \in H$. Consequently,

$$h^\alpha (g, h)_\alpha \in H \text{ and } (gh)^\alpha \in g^\alpha H,$$

i.e. H is an $\mathcal{L}_A$ -ideal in G .

2) Let $\varphi : G \rightarrow H$ be an A -homomorphism of groups from $\mathcal{M}_A$. Then $C = \ker \varphi$ is a normal A -subgroup in G . If $[g, h] \in C$, then

$$[g^\varphi, h^\varphi] = [g, h]^\varphi = 1,$$

hence in the group H we have the equality $(g^\varphi, h^\varphi)_\alpha = 1$. Consequently,

$$(g, h)_\alpha^\varphi = (g^\varphi, h^\varphi)_\alpha = 1,$$

i.e. $(g, h)_\alpha \in C$, and therefore C is an $\mathcal{M}_A$ -ideal.

3) If H is an $\mathcal{M}_A$ -ideal in G , then by item 1) H is an $\mathcal{L}_A$ -ideal in G , and hence $G/H \in \mathcal{L}_A$. From the definition of an $\mathcal{M}_A$ -ideal it follows that

$$[g, h] \in H \implies (g, h)_\alpha \in H \quad \forall \alpha \in A,$$

and this is equivalent to axiom 4) for the group G/H . □

2.4 Morphisms

In this subsection we introduce different types of mappings from one exponential group to another.

Definition 2.10 Let $G, H \in \mathcal{L}_A$. A homomorphism $\varphi : G \rightarrow H$ is said to be:

1. an A -homomorphism (or linear), if $(g^\alpha)^\varphi = (g^\varphi)^\alpha$, $g \in G$, $\alpha \in A$;

2. *semilinear*, if there exists an endomorphism $\theta : A \rightarrow A$ such that $(g^\alpha)^\varphi = (g^\varphi)^{\theta(\alpha)}$, $g \in G$, $\alpha \in A$;
3. *geometric*, if $(g^A)^\varphi \subseteq (g^\varphi)^A$.

One can define in a natural way the notions of A -monomorphisms, A -epimorphisms, A -isomorphisms and A -automorphisms.

Since a composition of A -automorphisms is again an A -automorphism, the set $\text{Aut}_A G$ of all A -automorphisms is a group. Unfortunately, the group $\text{Aut}_A G$ is not necessarily an A -group.

Example 2.3 *Let G be a vector space of dimension n over the field $\mathbb{Q}$. Then $G \in \mathcal{M}_{\mathbb{Q}}$, and $\text{Aut}_{\mathbb{Q}} G \simeq GL_n(\mathbb{Q})$. But the group $GL_n(\mathbb{Q})$ is not divisible; for example, square root can not be extracted from the matrix*

$$\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \in GL_2(\mathbb{Q}).$$

2.5 Operations in the category of exponential groups

In this subsection it will be shown that the classes $\mathcal{L}_A$ and $\mathcal{M}_A$ are closed under direct and Cartesian products, as well as under direct and inverse limits.

Let $G_i \in \mathcal{L}_A$, $i \in I$. By $\prod G_i$ and $\coprod G_i$ we denote the Cartesian and direct products of the groups G_i respectively. Let $g \in \prod G_i$, $g = (\dots, g_i, \dots)$, $\alpha \in A$. Denote an action of A on G coordinate-wise:

$$g^\alpha = (\dots, g_i^\alpha, \dots).$$

Proposition 2.2 immediately implies the following fact.

Proposition 2.11 *The classes $\mathcal{L}_A$ and $\mathcal{M}_A$ are closed under direct and Cartesian products.*

If in the standard definitions of a direct spectrum and an inverse spectrum one considers only A -homomorphisms, then it is not difficult to prove the following

Proposition 2.12 *The classes $\mathcal{L}_A$ and $\mathcal{M}_A$ are closed under direct and inverse limits.*

In the book [8] it was proved that in the category of abelian groups the operations of taking direct products of groups, as well as direct and inverse limits, possess a certain universal property. The corresponding operations in the category of exponential groups have analogous properties, which are formulated below.

Proposition 2.13 (The universal property of direct products) *Let $\phi_i : G_i \longrightarrow H$ be an A -homomorphism, $i \in I$. Then there exists a unique A -homomorphism ψ such that the diagram*

$$\begin{array}{ccc} G_i & \xrightarrow{s_i} & \Pi_i G_i \\ \phi_i \downarrow & \searrow \psi & \\ H & & \end{array} \quad (i \in I)$$

where s_i is an embedding and $[\phi_i(G_i), \phi_j(G_j)] = 1$, $i \neq j$, is commutative for all i .

Denote by $G_* = \lim_{\rightarrow} G_i$ the limit group of the direct spectrum

$$\mathcal{G} = \{G_i, (i \in I); \pi_i^j\}.$$

Proposition 2.14 (The universal property of direct limits) *Let $\sigma_i : G_i \longrightarrow H$ be A -homomorphisms such that the diagrams*

$$\begin{array}{ccc}
 G_i & \xrightarrow{\pi_i^j} & G_j \\
 & \searrow \sigma_i & \downarrow \sigma_j \\
 & & H
 \end{array} \quad (i \leq j)$$

are commutative. Then there exists a unique homomorphism $\sigma : G_* \longrightarrow H$ such that all the diagrams

$$\begin{array}{ccc}
 G_i & & \\
 \pi_i \downarrow & \searrow \sigma_i & \\
 G_* & \xrightarrow{\sigma} & H
 \end{array} \quad (i \in I, \pi_i \text{ is the projection of } G_i \text{ into } G_*)$$

are commutative.

Denote by

$$G^* = \varprojlim G_i$$

the limit group of inverse spectrum

$$\mathcal{G} = \{G_i, (i \in I); \pi_i^j\}.$$

Proposition 2.15 (The universal property of inverse limits) *Let H be an A -group and $\sigma_i : H \longrightarrow G_i$ A -homomorphisms such that the following diagrams*

$$\begin{array}{ccc}
 H & & \\
 \sigma_j \downarrow & \searrow \sigma_i & \\
 G_j & \xrightarrow{\pi_i^j} & G_i
 \end{array} \quad (i \leq j)$$

are commutative. Then there exists a unique homomorphism $\sigma : H \longrightarrow G^*$ such that all the diagrams

$$\begin{array}{ccc}
H & \xrightarrow{\sigma} & G^{\alpha} \\
& \searrow \sigma_i & \downarrow \pi_i \\
& & G_i
\end{array}
\quad (i \in I; \pi_i \text{ is the projection of } G^{\alpha} \text{ onto } G_i)$$

are commutative.

3 Tensor completions

In this section we study the basic operation in the class of exponential groups: the operation of tensor completion. It is a natural generalization of the notion of an extension of the ring of scalars of a module to the non-commutative case. Some ideas of this generalization for the class of nilpotent groups were given in [25]. Later in this section we will use tensor completion to define free constructions in the category of exponential groups, including the notion of an A -free group. In our discussion here we follow [23].

3.1 Existence and uniqueness theorems

Definition 3.1 *Let G be an A -group, and $\mu : A \longrightarrow B$ a homomorphism of rings. Then the B -group $G^{B,\mu}$ is called a tensor B -completion of the A -group G , if $G^{B,\mu}$ satisfies the following universal property:*

1. *there exists an A -homomorphism $\lambda : G \longrightarrow G^{B,\mu}$ such that $\lambda(G)$ B -generates $G^{B,\mu}$, i.e. $\langle \lambda(G) \rangle_B = G^{B,\mu}$;*
2. *for every B -group H and every A -homomorphism $\phi : G \longrightarrow H$ that agrees with μ (i.e., $(g^{\alpha})^{\phi} = (g^{\phi})^{\mu(\alpha)}$) there exists a B -homomorphism $\psi : G^{B,\mu} \longrightarrow H$ such that the following diagram is commutative:*

$$\begin{array}{ccc}
G & \xrightarrow{\lambda} & G^{B,\mu} \\
\phi \downarrow & \searrow \psi & \\
H & &
\end{array}$$

If G is an abelian A -group, then by definition 3.1 and propositions 2.1 and 2.7 the group $G^{B,\mu}$ is also abelian, i.e. it is a B -module; moreover, $G^{B,\mu}$ satisfies the same universal property as the tensor product $G \otimes_A B$ of the A -module G and the ring B . Consequently, $G^{B,\mu} \simeq G \otimes_A B$.

This observation suggests that the notion of tensor completion is a natural generalization of the notion of extension of scalars for modules. It turns out that the new notion preserves most attributes of the old one.

In what follows the ring homomorphism $\mu : A \longrightarrow B$ will be usually fixed, so we will write G^B instead of $G^{B,\mu}$.

Theorem 3.1 (Existence theorem) *Let G be an A -group, and $\mu : A \longrightarrow B$ a homomorphism of rings. Then a tensor B -completion of G exists.*

Proof. Let us introduce an equivalence relation on the class of all A -homomorphisms $\phi : G \longrightarrow H$ that agree with μ , where H is a B -group B -generated by the set $\phi(G)$. If ϕ_1 and ϕ_2 are two such homomorphisms, then $\phi_1 \sim \phi_2$ if and only if $\ker(\phi_1) = \ker(\phi_2)$ and there exists a B -isomorphism ψ between H_1 and H_2 such that $\psi\phi_1(g) = \phi_2(g)$ for every $g \in G$. The collection I of all classes of equivalent homomorphism is clearly a nonempty set. In every equivalence class let us fix a representative $\phi_i : G \longrightarrow H_i$, $i \in I$.

Let $G_0 = \prod_{i \in I} H_i$ be the Cartesian product of the groups H_i . Then G_0 is a B -group, and the mapping $\lambda : g \mapsto (\dots, \phi_i(g), \dots)$ is an A -homomorphism of G into G_0 . Let $G^B = \langle \lambda(G) \rangle_B$ be the B -subgroup generated in G_0 by the set $\lambda(G)$. Let us show that G^B is the required tensor B -completion of G . Since

the first condition from the definition of the universal property holds by the construction of G^B , it only remains to verify the second condition.

Let $\phi : G \longrightarrow H$ be an arbitrary A -homomorphism of the group G to a group H that agrees with μ , and $H_0 = \langle \lambda(G) \rangle_B \leq H$. Then $\phi \sim \phi_i$ for some $i \in I$, and there is an isomorphism ψ between H_i and H_0 defining the equivalence of ϕ_i and ϕ . Consider the diagram

$$\begin{array}{ccccc} G & \xrightarrow{\lambda} & G^B & \xrightarrow{\epsilon} & G_0 \\ \phi \downarrow & & \downarrow \pi & & \downarrow \pi_i \\ H & \xleftarrow{\epsilon} & H_0 & \xleftarrow{\psi} & H_i \end{array}$$

where ϵ and ϵ are embeddings, π_i are canonical projections and $\pi = \pi_i|_{G^B}$. Then $\pi\epsilon$ is the required B -homomorphism which agrees with μ and makes the diagram commutative. $\square$

Theorem 3.2 (Uniqueness theorem) *Let G be an A -group, and $\mu : A \longrightarrow B$ a homomorphism of rings. Then the tensor B -completion of G is unique up to a B -isomorphism.*

Proof. Let G_1 and G_2 be two arbitrary B -completions of G with respect to μ . Then by definition there exist B -homomorphisms ψ_1 and ψ_2 such that the following diagrams are commutative:

$$\begin{array}{ccc} G & \xrightarrow{\lambda_1} & G_1 \\ \lambda_2 \downarrow & \psi_1 \nearrow & \uparrow \psi_2 \\ & G_2 & \end{array}$$

Let $f_1 = \psi_1\psi_2$ and $f_2 = \psi_2\psi_1$. Since f_i is identity on $\lambda_i(G)$, and $\lambda_i(G)$ B -generate G_i , it follows that $f_i = \text{id}$, i.e. ψ_1 and ψ_2 are mutually inverse B -homomorphisms. $\square$

In applications μ is usually an embedding of rings, but even in this case the homomorphism $\lambda : G \longrightarrow G^{B,\mu}$ is not necessarily an embedding. Since in the abelian case G^B is obtained as the tensor product of the A -module G and the ring B , the corresponding examples can be found in many papers on commutative algebra and homology theory. The following proposition describes the situation when λ is an embedding.

Definition 3.2 *An A -group G is said to be a residually B -group with respect to the homomorphism μ , if for every $1 \neq g \in G$ there exists an A -homomorphism ϕ_g of the group G into a B -group H such that ϕ_g agrees with μ and $\phi_g(g) \neq 1$.*

Proposition 3.1 *Let an A -group G be a residually B -group with respect to the homomorphism μ . Then the homomorphism $\lambda : G \longrightarrow G^B$ is injective.*

Proof. Let $g \neq 1$ be an element of G and $\phi_g : G \longrightarrow H$ an A -homomorphism such that ϕ_g agrees with μ and $\phi_g(g) \neq 1$. Then there exists a homomorphism $\psi : G^B \longrightarrow H$ such that $\phi_g = \psi\lambda$. Hence $\lambda(g) \neq 1$. $\square$

3.2 The category of exponential groups

Here we discuss some categorical properties of tensor completion.

Proposition 3.2 *The class $\mathcal{M}_A$ of all A -exponential groups is a category, with morphisms being A -homomorphisms of groups.*

The category of A -exponential groups possesses virtually all attributes of the category of groups.

In the language of categories, the operation of tensor completion described above appears in the role of a functor of tensor completion. Let $\mu : A \longrightarrow B$ be a homomorphism of rings. On the basis of this homomorphism let us build a

functor $\theta^{B,\mu}$ (also denoted by θ^B) which will connect the category of A -groups $\mathcal{M}_A$ with the category of B -groups $\mathcal{M}_B$. The mapping $\theta^B : \mathcal{M}_A \rightarrow \mathcal{M}_B$ is defined on objects by the formula $\theta^B(G) = G^B$, where G is an A -group, and G^B is the tensor B -completion of G with respect to μ .

Let us define the mapping θ^B on the morphisms of the category $\mathcal{M}_A$. Let $\phi : G \rightarrow H$ be an A -homomorphism and $H, G \in \mathcal{M}_A$. Let $\lambda_G : G \rightarrow G^B$, $\lambda_H : H \rightarrow H^B$ be the canonical homomorphisms from subsection 3.1. Since the composite $\phi \circ \lambda_H$ is an A -homomorphism from G to H^B and agrees with μ , there exist B -homomorphisms ψ and ψ^B which make the following diagram commutative:

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ \lambda_G \downarrow & \searrow \psi & \downarrow \lambda_H \\ G^B & \xrightarrow{\phi^B} & H^B \end{array}$$

Now it suffices to put $\theta^B(\phi) = \phi^B$.

Theorem 3.3 *Let $\mu : A \rightarrow B$ be a ring homomorphism. Then θ^B is a covariant functor from the category $\mathcal{M}_A$ to the category $\mathcal{M}_B$.*

Proof. We will prove one axiom from the definition of a functor. Assume that $H = G$ and $\phi = 1_G$. Let us show that in this case $(1_G)^B = 1_{G^B}$. Indeed, let $\lambda : G \rightarrow G^B$ be the canonical homomorphism. then the restriction of $(1_G)^B$ on $\lambda(G)$ is identity. Since $G^B = \langle \lambda(G) \rangle_B$, it follows that $(1_G)^B$ is the identical mapping on the entire group G^B .

The other axioms from the definition of a functor can be verified similarly.

□

The following proposition can be obtained in a standard way from the universal property of tensor completion.

Proposition 3.3 *The functor of tensor completion is exact from the right and, in general, not exact from the left for short exact sequences.*

Let $A \xrightarrow{\mu_1} B \xrightarrow{\mu_2} C$ be a sequence of homomorphisms of rings A, B, C . Denote $\mu = \mu_1\mu_2 : A \longrightarrow C$. These three homomorphisms define the triple of functors:

$$\Phi^{B,\mu_1} : \mathcal{M}_A \longrightarrow \mathcal{M}_B, \Phi^{C,\mu_2} : \mathcal{M}_B \longrightarrow \mathcal{M}_C, \Phi^{C,\mu} : \mathcal{M}_A \longrightarrow \mathcal{M}_C.$$

Proposition 3.4 *In the above notations $\Phi^{B,\mu_1} \circ \Phi^{C,\mu_2} = \Phi^{C,\mu}$*

Let $\mu : A \longrightarrow B$ be an arbitrary ring homomorphism and $\text{Im}(\mu) = B_0$. Then the homomorphism μ can be canonically decomposed into the product of an epimorphism $\mu_1 : A \longrightarrow B_0$ and a monomorphism $\mu_2 : B_0 \longrightarrow B$. By proposition 3.4 we have $\Phi^{B,\mu_1} \circ \Phi^{C,\mu_2} = \Phi^{C,\mu}$. So we can talk about the canonical decomposition of the functor of the tensor completion. Therefore the proofs of theorems connected with the construction of tensor completion are naturally reduced to the two cases:

- a) μ is an epimorphism of rings;
- b) μ is an embedding of rings.

3.3 Permutability of the functor of tensor completion with basic group operations

Here we show that the operation of tensor completion is permutable with direct products and direct limits, and, generally speaking, is not permutable with Cartesian products and inverse limits. These results were proved by M. Amaglobeli in [1].

The permutability of the operation of tensor completion with direct limits allows one to reduce many questions about completions to the case of a finitely generated group. Indeed, let $\{G_i, i \in I; \pi_i^j\}$ be a direct spectrum of the

group G composed from finitely generated groups G_i . Then $G = \lim_{\leftarrow} G_i$ and $G^B \simeq \lim_{\leftarrow} G_i^B$.

Theorem 3.4 *The functor of tensor completion is permutable with direct products. In other words, if $G = \prod_j G_j$, then $G^B = \prod_j G_j^B$*

For the proof of the theorem we need the following lemma.

Lemma 3.1 *Assume that for every j there exists an A -homomorphism $\phi_j : G_j \longrightarrow H$, where H is an A -group, and for all pairs j, k with $j \neq k$ we have*

$$[\phi_j(G_j), \phi_k(G_k)] = 1 \quad (1)$$

Then there exists an A -homomorphism $\psi : \prod_j G_j \longrightarrow H$ which extends ϕ_j for every j .

Proof. Denote by $H_0 = \langle \phi_j(G_j), j \in J \rangle_A$. Let us introduce an arbitrary order on the index set J and prove that every element $h \in H_0$ can be represented in the form

$$g_{j_1} g_{j_2} \cdots g_{j_s}, \text{ where } g_{j_k} \in \phi_{j_k}(G_{j_k}) \quad (2)$$

where $j_1 < j_2 < \cdots < j_s$.

Clearly, elements of the form (2) constitute a subgroup due to condition (1). For every $\alpha \in A$ axiom 4) from the definition of an exponential group guarantees that $(g_{j_1} g_{j_2} \cdots g_{j_s})^\alpha = g_{j_1}^\alpha g_{j_2}^\alpha \cdots g_{j_s}^\alpha$, therefore the elements of the form (2) constitute an A -subgroup. Now the required A -homomorphism ψ is constructed in the following way.

Let an element $g \in \prod_j G_j$ be written as $g = (\dots, g_{j_1}, \dots, g_{j_s}, \dots)$, where dots stand for the identity element. Then define $\psi(g) = g_{j_1} g_{j_2} \cdots g_{j_s}$. It can be directly verified that ψ is a homomorphism. $\square$

To complete the proof of the theorem, let us construct a pair of mutually inverse A -homomorphisms between the groups G^B and $\prod_j G_j^B$. Consider the

extension of the canonical homomorphism $i_j : G_j \longrightarrow G_j^B$ to the group $\prod_j G_j^B$. It is clear that for $j \neq k$ $[i_j(G_j), i_k(G_k)] = 1$. By the universal property of the direct product, there exists an A -homomorphism $\phi : G \longrightarrow \prod_j G_j^B$. Then by the definition of tensor completion there is a B -homomorphism ψ_1 making the following diagram commutative:

$$\begin{array}{ccc} G & \xrightarrow{i} & G^B \\ \phi \downarrow & \nearrow \psi_1 & \\ \prod_j G_j^B & & \end{array}$$

Observe that $[\langle i(G_j) \rangle_B, \langle i(G_k) \rangle_B] = 1$, since the generators of the first subgroup commute with the generators of the second one. By the definition of the tensor completion there is a B -homomorphism β_j such that the following diagram is commutative for every $j \in J$.

$$\begin{array}{ccc} G_j & \xrightarrow{i_j} & G_j^B \\ \downarrow & \nearrow \beta_j & \\ \langle i(G_j) \rangle_B & & \end{array}$$

Then by lemma 3.1 there exists a B -homomorphism $\psi_2 : \prod_j G_j^B \longrightarrow G^B$. It can be verified directly that these homomorphisms are mutually inverse. $\square$

However, for Cartesian products the corresponding statement is false. Consider the mapping $i : \prod_j G_j \longrightarrow \prod_j G_j^B$. Then by the universal property of tensor completion we have the B -homomorphism $i_B : (\prod_j G_j)^B \longrightarrow \prod_j G_j^B$, which is, in general, not an isomorphism. A counter example exists even in the category of abelian groups.

Example 3.1 Take the field of rational numbers $\mathbb{Q}$ as the ring A , and the cyclic group of order n as G_n , $n \in \mathbb{N}$. Then $G_n^{\mathbb{Q}} = G_n \otimes \mathbb{Q} = 0$, therefore

$\overline{\prod}_n G_n^B = 0$. At the same time, in the group $\overline{\prod}_n G_n$ there are elements of infinite order, so the group $(\overline{\prod}_n G_n)^Q = \overline{\prod}_n (G_n \otimes Q)$ is nonzero.

Theorem 3.5 *The operation of tensor completion is permutable with direct limits.*

Proof. Denote by $G_* = \varinjlim G_i$ the limit group of the direct spectrum $\mathcal{G} = \{G_i, (i \in I); \pi_i^j\}$. We will construct a pair of mutually inverse mappings between the groups G_*^B and $H = \varinjlim G_j^B$. Let $\phi_j : G_j^B \rightarrow H$. Then by the universal property of direct limits there exists a B -homomorphism $\psi_1 : H \rightarrow G_*^B$ making the diagram

$$\begin{array}{ccc} G_j^B & & \\ \pi_j^B \downarrow & \searrow \phi_j & \\ G_*^B & \xleftarrow{\psi_1} & H \end{array}$$

commutative. Here $\pi_j : G_j \rightarrow G_*$ is the projection, and $\pi_j^B : G_j^B \rightarrow G_*^B$ is the corresponding homomorphism of the tensor completions.

By the universal property of tensor completion, for every index j we can construct a mapping $(\psi_2)_j$ such that the following diagram is commutative:

$$\begin{array}{ccc} G_j^B & & \\ \downarrow & \searrow \phi_j & \\ G_*^B & \xrightarrow{(\psi_2)_j} & H \end{array}$$

Since the subgroups $\pi_j(G_j)$ cover the group G_* , and since the homomorphisms $(\psi_2)_j$ and $(\psi_2)_k$ agree on the common elements, the homomorphism $\psi_2 : G_* \rightarrow H$ is well-defined. The B -homomorphism ψ_2^B will be the required inverse homomorphism for ψ_1 . $\square$

Finally, let us prove that the operation of tensor completion is not permutable with inverse limits. Denote by $G^* = \lim_{\leftarrow} G_j$ the limit group of inverse spectrum $\mathcal{G} = \{G_j, (j \in J); \pi_j^k\}$. We will construct a B -homomorphism $\sigma : (G^*)^B \rightarrow \lim_{\leftarrow} G_j^B$ as follows. Denote by $\pi_j : G^* \rightarrow G_j$ the projection of the limit group on the j -th component. Then $\pi_j^B : (G^*)^B \rightarrow G_j^B$ is the corresponding homomorphism of the tensor completions. Let $\mu_j : \lim_{\leftarrow} G_j^B \rightarrow G_j^B$ be the natural projection. Then by the universal property of inverse limits there is a homomorphism $\sigma : (G^*)^B \rightarrow \lim_{\leftarrow} G_j^B$ making the diagram

$$\begin{array}{ccc} (G^*)^B & \xrightarrow{\sigma} & \lim_{\leftarrow} G_j^B \\ & \searrow \pi_j^B & \downarrow \mu_j \\ & & G_j^B \end{array}$$

commutative. The following example shows that this homomorphism is, in general, not an isomorphism.

Example 3.2 Consider $G_k = \langle a_k \rangle$, $k \in \mathbb{N}$, where a_k is an element of order p^k for a prime number p . Then it is known that $\lim_{\leftarrow} G_k$ is isomorphic to the additive group of p -adic integers $\mathbb{Z}_p$. But $\mathbb{Z}_p^{\mathbb{Q}} = \mathbb{Z}_p \otimes \mathbb{Q}$ is a vector space over $\mathbb{Q}$ of cardinality continuum; at the same time $\lim_{\leftarrow} G_k^{\mathbb{Q}} = \lim_{\leftarrow} (G_k \otimes \mathbb{Q}) = \lim_{\leftarrow} 0 = 0$.

3.4 The category of partial exponential groups

It is usually convenient to construct the tensor completion of a group step by step, on each stage defining the action of the ring on some collection of elements. Intermediate groups that arise in this process are partial A -groups. Some operations in the class of A -groups also lead us to the notion of a partial A -group. Let A be a ring and G a group.

Definition 3.3 A group G is called a partial A -group, if g^α is defined for some pairs (g, α) , $g \in G$, $\alpha \in A$, but not necessarily for all such pairs, and axioms 1)-4) from the definition of an exponential group hold whenever the action is defined. We will denote the class of all partial A -groups by $\mathcal{P}_A$.

Example 3.3 Let A be a subring of a ring B . Then every A -group is a partial B -group.

Definition 3.4 Let $H, G \in \mathcal{P}_A$. A group homomorphism $\phi : G \rightarrow H$ is called a partial A -homomorphism, if $(g^\alpha)^\phi = (g^\phi)^\alpha$ for all pairs (g, α) for which the element g^α is defined.

Proposition 3.5 In the category $\mathcal{P}_A$ there exist direct, Cartesian and free products, as well as direct and inverse limits.

Proof. Let us consider, for example, the case of free products. Let $G = G_1 * G_2$ and $G_1, G_2 \in \mathcal{P}_A$. If H is a subgroup of G which is conjugate to one of the factors, we define a partial action of A on H in such a way that axiom (3) from the definition of an exponential group is satisfied. If g is not in a conjugate of one of the factors and $\alpha \notin \mathbb{Z}$, we assume that for this pair the element g^α is not defined. This definition of an action of A on G makes the free product a partial A -group. $\square$

Abelian partial A -group from the category $\mathcal{P}_A$ are partial A -modules. Many notions of general nature defined for A -modules can be transferred to the category of partial A -modules.

For our purposes it is important that the notion of tensor product of A -modules can be generalized to the notion of tensor product of partial A -modules. Defining this concept, we will follow [2].

Proposition 3.6 Let M and N be partial A -modules. Then there exists a pair (T, g) consisting of a partial A -module T and a partial A -bilinear mapping

$g : M \times N \longrightarrow T$ with the following property: for every partial A -module P and every partial A -bilinear mapping $f : M \times N \longrightarrow P$ there exists a unique partial A -linear mapping $f' : T \longrightarrow P$ such that $f = gf'$.

If (T, g) and (T', g') are two pairs with this property, then there exists an A -isomorphism $j : T \longrightarrow T'$ such that $gj = g'$.

Proof. The proof of uniqueness is standard. Let us prove existence. Consider the free A -module C . Elements of C are formal linear combinations of elements of $M \times N$ with coefficients from A , i.e. expressions of the form

$$\sum_{i=1}^n \alpha_i(x_i, y_i), \text{ where } \alpha_i \in A, x_i \in M, y_i \in N.$$

Let D be the submodule of C generated by all elements of the form

$$(x + x', y) - (x, y) - (x', y), (x, y + y') - (x, y) - (x, y'),$$

$$(\alpha x, y) - \alpha(x, y), (x, \alpha y) - (\alpha x, y)$$

provided that αx is defined in M , and αy in N .

Put $T = C/D$. For every basis element $(x, y) \in C$ denote by $x \otimes y$ its image in T . The module T is generated by the elements of the form $x \otimes y$, and it is clear from the definition that they satisfy the relations

$$(x + x') \otimes y = x \otimes y + x' \otimes y, x \otimes (y + y') = x \otimes y + x \otimes y',$$

$$(\alpha x) \otimes y = \alpha(x \otimes y) = x \otimes (\alpha y).$$

In other words, the mapping $g : M \times N \longrightarrow T$ defined by the formula $g(x, y) = x \otimes y$ is A -bilinear.

Every mapping f of the product $M \times N$ into the partial A -module P can be extended by linearity to a homomorphism of A -modules $\bar{f} : C \longrightarrow P$. If f , in addition, is partial A -bilinear, then $\bar{f}$ turns to 0 on the generators of D , and hence on all of D . Therefore $\bar{f}$ induces a uniquely defined partial A -homomorphism f' of the module $T = C/D$ into P for which $f'(x \otimes y) = f(x, y)$. Consequently, the pair (T, g) possesses the required property. $\square$

Remark 3.1 *The module T constructed above is called the tensor product of the modules M and N . It is denoted by $M \otimes_A N$, or simply $M \otimes N$ if it is clear what ring A is meant. Note that the tensor product $M \otimes N$ of partial A -modules is an A -module.*

3.5 Faithful A -completion

We can look at tensor completion from a slightly different point of view, considering it to be an operation in the class of partial A -groups.

Definition 3.5 *Let G be a partial A -group. Then an A -group G^A is called a tensor completion of G , if*

1. *there exists a partial A -homomorphism $\lambda : G \rightarrow G^A$ such that $\lambda(G)$ generates G^A as an A -group;*
2. *for every A -group H and every partial A -homomorphism $\phi : G \rightarrow H$ there exists an A -homomorphism $\psi : G^A \rightarrow H$ such that $\phi = \psi \circ \lambda$:*

$$\begin{array}{ccc} G & \xrightarrow{\lambda} & G^A \\ \phi \downarrow & \searrow \psi & \\ & H & \end{array}$$

Note that if A_0 is a subring of A , then any A_0 -group is a partial A -group, so that definition 3.1, in the case where the homomorphism of rings μ is an embedding, becomes a particular case of definition 3.5.

Remark 3.2 *If M is a partial abelian A -group, then the A -module $M \otimes A$, constructed in the previous subsection, is the tensor completion of M over A .*

Definition 3.6 *A partial A -group G is called faithful (over A), if the canonical homomorphism of partial A -groups $\lambda : G \rightarrow G^A$ is injective.*

Remark 3.3 *A partial A -group G is faithful if and only if G is A -embeddable into some A -group H .*

As shown in subsection 3.3, the operation of tensor completion is permutable with direct products and direct limits:

$$(\oplus G_i)^A \simeq_A \oplus G_i^A$$

$$(\varinjlim G_i)^A \simeq_A \varinjlim G_i^A$$

Using these statements and the remark above, it is not difficult to prove the following proposition:

Proposition 3.7 *Let $\mathcal{F}_A$ be a class of all A -faithful partial A -groups. Then:*

1. $\mathcal{F}_A$ is closed under direct products;
2. $\mathcal{F}_A$ is closed under direct limits.

Proof. Let $\lambda_i : G_i \rightarrow G_i^A$ be canonical homomorphisms. By the properties above, $(\oplus G_i)^A \simeq \oplus G_i^A$, and if $\lambda = \oplus \lambda_i$ is the direct sum of λ_i , then the canonical map $\lambda : \oplus G_i \rightarrow \oplus G_i^A$ is injective if and only if each λ_i is injective. This implies 1).

Let $G = \varinjlim G_i$. Then, as mentioned above, $G^A = \varinjlim G_i^A$. If $\lambda_i : G_i \rightarrow G_i^A$ is the canonical embedding, then there exists a canonical embedding

$$\lambda : \varinjlim G_i \rightarrow \varinjlim G_i^A = G^A$$

So $G = \varinjlim G_i$ is A -faithful. □

The following proposition shows that the property of being faithful is a local property.

Proposition 3.8 *Let G be a partial A -group. Then G is faithful if and only if every finitely generated subgroup of G is faithful.*

Proof. Let $\{G_i, i \in I\}$ be the set of all finitely generated partial A -subgroups of G . Then $G = \varinjlim G_i$, and the statement follows from proposition 3.7. $\square$

Proposition 3.9 *Let $\mathcal{F}_A$ be the class of all faithful partial A -groups. Then:*

1. $\mathcal{F}_A$ is closed under the operation of taking subgroups;
2. $\mathcal{F}_A$ is closed under ultraproducts;
3. $\mathcal{F}_A$ is universally axiomatizable.

Proof. Item 1) is evident by the remark above. To prove 2), consider instead of the group G the multi-sorted structure $\mathcal{A}(G, A) = \langle G, G^A, A \rangle$ with the group operations on G and G^A , ring operations on A , operations of action of A on G (partial) and on G^A , and the canonical map $\lambda : G \rightarrow G^A$. In $\mathcal{A}(G, A)$ the notion of faithfulness is evidently axiomatizable by one sentence: $g \neq 1 \implies \lambda(g) \neq 1$. Instead of the ultraproduct of groups $\prod G_i/D$ over an ultrafilter D one can consider the corresponding ultraproduct $\prod \mathcal{A}(G_i, A)/D$. By the Scolem Theorem, $\prod G_i/D$ is a faithful group over the ultrapower $\prod A/D$ of the ring A over D . But there is the diagonal embedding of A into $\prod A/D$. So $\prod G_i/D$ is a subgroup of the A -group $\prod G_i^A/D$.

Item 3) is a corollary of 1) and 2). $\square$

Remark 3.4 *Using the universal property of tensor completion, it is easy to verify that an epimorphism $G \rightarrow H$ of partial A -groups induces an epimorphism of their completions $G^A \rightarrow H^A$ (exactness from the right).*

Proposition 3.10 *Let $\lambda : G \rightarrow G^A$ be the canonical map, $N \triangleleft G$. If*

- a) $\lambda : N \rightarrow G^A$ is injective (in particular, N is a faithful group);
- b) G/N is a faithful group,

then λ is injective.

Proof. By proposition 3.3, an epimorphism $G \rightarrow H$ of partial A -groups induces an epimorphism of their completions $G^A \rightarrow H^A$ (exactness from the right). The proof now follows from the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & G/N & \longrightarrow & 1 \\ & & \downarrow \lambda_N & & \downarrow \lambda_G & & \downarrow \lambda_{G/N} & & \\ 1 & \longrightarrow & N^A & \longrightarrow & G^A & \longrightarrow & (G/N)^A & \longrightarrow & 1 \end{array}$$

□

For quite a wide class of rings A we can claim that torsion-free nilpotent groups are faithful over A .

Proposition 3.11 *Let G be a torsion-free nilpotent group, and A an integral domain of characteristic 0. Then G is faithful over A .*

Proof. Let $A^* = A \otimes \mathbb{Q}$ be the tensor completion of the ring A over $\mathbb{Q}$. Then A^* is a binomial domain (i.e. contains all binomial coefficients $\binom{a}{n}$, $a \in A$, $n \in \mathbb{N}$). It is known [11] that every torsion-free nilpotent group G has an A^* -completion in the sense of P. Hall. So G is embeddable into an A^* -group, and hence into an A -group. By remark 3.3 G is faithful over A . □

Proposition 3.12 *If a group G is residually faithful over A , then G is faithful over A .*

Proof. Let G be a residually A -faithful group, i.e. for any $1 \neq g \in G$ there exists a homomorphism of partial A -groups $\phi_g : G \rightarrow H_g$ such that $\phi_g(g) \neq 1$. It follows from the universal property of G^A that $\lambda(g) \neq 1$ if $g \neq 1$. □

Theorem 3.6 *Let G be a residually torsion-free nilpotent group. Then G is a faithful group over any integral domain A of characteristic 0.*

Proof. The statement follows from propositions 3.11 and 3.12. □

Corollary 3.1 *The following groups are faithful over any integral domain of characteristic 0:*

1. *free groups;*
2. *free solvable groups;*
3. *$F/[N, N]$ under the condition that F/N is residually torsion-free nilpotent.*

Proof. All these groups are residually torsion-free nilpotent: for free groups see [18], for 2) and 3) see [12]. $\square$

Finally, let us consider some examples of non-faithful groups.

Example 3.4 *Let G be a simple group containing an element of finite order. Then G is not faithful over the field of rational numbers $\mathbb{Q}$. Moreover, $G^{\mathbb{Q}} = 1$. Indeed, any $\mathbb{Q}$ -group has no elements of finite order, so G is not a subgroup of $G^{\mathbb{Q}}$. But in this case the homomorphism $\lambda : G \longrightarrow G^{\mathbb{Q}}$ has a nontrivial kernel. So the simplicity of G implies that $\lambda(G) = 1$, but $G^{\mathbb{Q}}$ is generated by $\lambda(G)$. Therefore, $G^{\mathbb{Q}} = 1$.*

Example 3.5 *Let G be a torsion-free group with non-unique extraction of roots. Then G is not faithful over $\mathbb{Q}$, since $G^{\mathbb{Q}}$ has unique extraction of roots.*

3.6 Free constructions

Let us formulate the notion of a free A -group. Let A be an associative ring with identity, and X an arbitrary set.

Definition 3.7 *An A -group $F_A(X)$ with the set of A -generators X is said to be a free A -group with base X , if for every A -group G an arbitrary mapping $\phi_0 : X \longrightarrow G$ can be extended to an A -homomorphism $\phi : F_A(X) \longrightarrow G$. The set X is called the set of free A -generators of $F_A(X)$. The cardinality of X is called the rank of the group $F_A(X)$.*

Theorem 3.7 *For every X and A there exists a free A -group $F_A(X)$; moreover it is unique up to an A -isomorphism and $F_A(X)$ is the tensor A -completion $F(X)^A$ of the group $F(X)$.*

Proof. Let $F(X)$ be a free group in the ordinary sense and let $\phi_0 : X \rightarrow G$ be an arbitrary mapping from X to an A -group G .

$$\begin{array}{ccc}
 X & \xrightarrow{\text{id}} & F(X) \\
 \phi_0 \downarrow & \nearrow \phi_1 & \downarrow \lambda \\
 G & \xleftarrow{\phi} & F(X)^A
 \end{array}$$

Then ϕ_0 induces a homomorphism $\phi_1 : F(X) \rightarrow G$ by a property of a free group. Moreover the latter mapping induces an A -homomorphism $\phi : (F(X))^A \rightarrow G$. Consequently, $(F(X))^A$ is a free A -group with base X . Uniqueness follows from the uniqueness of the tensor completion. $\square$

Let us now introduce the construction of free product in the category of A -groups.

Definition 3.8 *Let G_i , $i \in I$ be A -groups. An A -group $*_A G_i$ is called the free product in the category $\mathcal{M}_A$, if A -homomorphisms $\phi_i : G_i \rightarrow *_A G_i$ are such that for every A -homomorphism $\psi_i : G_i \rightarrow H$, where H is an arbitrary A -group, there exists an A -homomorphism $\psi : *_A G_i \rightarrow H$ such that the following diagrams are commutative:*

$$\begin{array}{ccc}
 X & \xrightarrow{\quad} & *_A G_i \\
 \phi_i \downarrow & \nearrow \psi & \\
 H & &
 \end{array} \quad (i \in I)$$

and $*_A G_i$ is A -generated by the set $\phi_i(g_i)$, $g_i \in G_i$, $i \in I$.

For categorical reasons it follows that the group $*_A G_i$ is defined uniquely up to an A -isomorphisms.

Theorem 3.8 *Let A be a ring containing the ring of integers $\mathbb{Z}$ as a subring, and $\{G_i, i \in I\}$ a set of A -groups. Then $*_A G_i \simeq (*G_i)^A$.*

Proof. Let $\phi_i^0 : G_i \rightarrow *_A G_i$ be canonical mappings. Since by proposition 3.5 a free product is endowed with the structure of a partial A -group, the construction of tensor completion can be applied to it.

Let $\lambda : *_A G_i \rightarrow (*G_i)^A$ be the canonical mapping from the definition of tensor completion. Put $\lambda \circ \phi_i^0 = \phi_i$. Then $\phi_i : G_i \rightarrow (*G_i)^A$ is a set of A -homomorphisms. Let $\psi_i : G_i \rightarrow H$ be arbitrary A -homomorphisms. In order to prove that the group $(*_A G_i)^A$ is the free product in the category $\mathcal{M}_A$, we should find a mapping that would make the diagram

$$\begin{array}{ccc} G_i & \xrightarrow{\quad} & *_A G_i \\ \psi_i \downarrow & \nearrow \phi & \downarrow \lambda \\ H & \xleftarrow{\psi} & (*G_i)^A \end{array}$$

commutative.

By the definition of the free product in the category $\mathcal{M}_A$, there exists a partial A -homomorphism $\phi : *_A G_i \rightarrow H$. Applying the universal property of tensor completion, we can find an A -homomorphism ψ extending ϕ , as required. The property that $(*_A G_i)^A$ by the images $\phi_i(G_i)$ also holds, and therefore $(*_A G_i)^A$ is a free product in $\mathcal{M}_A$. $\square$

It is not difficult to give a categorical definition of the construction of free product with amalgamation for A -groups. However, on this product the structure of a partial A -group can not always be introduced, and hence the construction of tensor completion can not be applied. In the case where the

structure of a partial A -group can be defined, an explicit description of the construction of free product with amalgamation is the same as for the case of free products.

4 Extensions of centralizers

In our description of tensor completions for various classes of groups, we will use the construction of extension of centralizers of a given group. Let us define this construction.

Let G be a group. The centralizer of an element $v \in G$ is denoted by $C_G(v)$. If $G = \langle X \mid R \rangle$ is a presentation of G , Y is a set of words in X and t is a new letter (not in X), then by $\langle G, t \mid [Y, t] = 1 \rangle$ we will denote the group with the representation $\langle X, t \mid R, [y, t] \rangle_{y \in Y}$.

The following definition describes, perhaps, the simplest case of the construction of extension of centralizers.

Definition 4.1 *The group $G(v, t) = \langle G, t \mid [C_G(v), t] = 1 \rangle$ is called the direct extension of rank 1 of $C(v)$.*

It is easy to see that $G(v, t)$ can be obtained an HNN-extension of G with respect to the identity isomorphism $C_G(v) \rightarrow C_G(v)$:

$$G(v, t) = \langle G, t \mid t^{-1}at = a, a \in C_G(v) \rangle,$$

or as a free product with amalgamation:

$$\langle G * (C_G(v) \times \langle t \rangle) \mid C_G(v) = C_G(v)^\phi \rangle$$

where ϕ is the canonical monomorphism $\phi : C_G(v) \rightarrow C_G(v) \times \langle t \rangle$.

In a similar way we define a general construction of direct extensions of centralizer.

Definition 4.2 Let G, H be groups, $C_G(v) = C$ the centralizer of an element v from G , then the group

$$G(v, H) = \langle G, H \mid [C, H] = 1 \rangle$$

is obtained from G by direct extension of the centralizer of v by the group H .

The group $G(v, H)$ has the following decomposition as a free product with amalgamation:

$$G(v, H) = \langle G * (C_G(v) \times H) \mid C_G(v) = C_G(v)^\phi \rangle$$

We will consider also a much more general construction of free centralizer extension.

Definition 4.3 Let G be a group, $C_G(v) = C$ the centralizer of an element v from G , and $\phi : C \longrightarrow H$ a monomorphism of groups such that $\phi(v) \in Z(H)$. Then the group

$$G(v, H) = \langle G * H \mid C = C^\phi \rangle$$

is called an extension of the centralizer $C_G(v)$ by the group H with respect to ϕ .

Some particular cases of this construction are of special interest. We will classify them with respect to the type of the embedding $\phi : C \rightarrow H$. So the extension is *central* iff $C^\phi \leq Z(H)$; *direct* iff $H = C^\phi \times B$; *abelian* iff H is an abelian group.

Proposition 4.1 The following statements are true:

1. there exist the canonical embeddings $\lambda_G : G \longrightarrow G(v, H)$ and $\lambda_H : H \longrightarrow G(v, H)$;
2. $C_{G(v, H)}(v) \geq H$;

3. the group $G(v, H)$ has the following universal property: for any group homomorphisms $\psi_G : G \rightarrow N$ and $\psi_H : H \rightarrow N$ compatible on C and C^ϕ there exists a unique homomorphism $\psi : G(v, H) \rightarrow N$ such that the following diagram is commutative:

$$\begin{array}{ccccc} G & \xrightarrow{\lambda_G} & G(v, H) & \xleftarrow{\lambda_H} & H \\ & \searrow \psi_G & \downarrow \psi & \swarrow \psi_H & \\ & & N & & \end{array}$$

The next construction allows us to extend a set of centralizers at once.

Definition 4.4 Let $\mathcal{C} = \{C_i = C_G(v_i) \mid i \in I\}$ be a set of centralizers in the group G . Suppose $\phi_i : C_i \rightarrow H_i$ is an embedding of C_i into H_i such that $\phi_i(v) \in Z(H_i)$, $i \in I$. We can form a graph of groups

$$\begin{array}{ccc} & & H_i \\ & \nearrow C_i & \vdots \\ G & & \\ & \searrow C_j & H_j \\ & & \vdots \end{array}$$

The fundamental group of this graph is called a tree extension of centralizers from $\mathcal{C}$ and is denoted by $G(\mathcal{C}, \mathcal{H}, \Phi)$, where $\mathcal{H} = \{H_i \mid i \in I\}$, $\Phi = \{\phi_i \mid i \in I\}$.

Again, we can consider central, direct and abelian extensions corresponding to the type of monomorphisms from Φ .

By definition, the group $G(\mathcal{C}, \mathcal{H}, \Phi)$ is the union (direct limit) of the chain of groups

$$G = G_0 \hookrightarrow G_1 \hookrightarrow \dots \hookrightarrow G_\alpha \xrightarrow{\phi_\alpha} G_{\alpha+1} \hookrightarrow \dots$$

where the set of centralizers $\mathcal{C} = \{C_\alpha \mid \alpha < \lambda\}$ is well ordered, $G = G_0$, $G_{\alpha+1} = G_\alpha(v_{\alpha+1}, H_{\alpha+1})$, $\phi_\alpha : G_\alpha \longrightarrow G_{\alpha+1}$ is the canonical embedding from proposition 4.1, and $G_\gamma = \lim_{\longrightarrow} G_\alpha$ (here $\alpha < \gamma$) for a limit ordinal γ .

The properties of tree extensions of centralizers are similar to those of ordinary extensions of centralizers.

Proposition 4.2 *Let $G(\mathcal{C}, \mathcal{H}, \Phi)$ be a tree extension of centralizers with respect to the set of embeddings $\phi_i : C(v_i) \longrightarrow H_i$, $i \in I$. Then:*

1. *There exist the canonical embeddings*

$$\lambda_G : G \longrightarrow G(\mathcal{C}, \mathcal{H}, \Phi), \quad \lambda_i : H_i \longrightarrow G(\mathcal{C}, \mathcal{H}, \Phi).$$

2. *The centralizer of v_i in $G(\mathcal{C}, \mathcal{H}, \Phi)$ contains the group H_i .*
3. *The group $G(\mathcal{C}, \mathcal{H}, \Phi)$ has the following universal property: for any set of group homomorphisms $\psi_G : G \longrightarrow N$, $\psi_i : H_i \longrightarrow N$ compatible with respect to ϕ_i there exists a unique homomorphism $\psi : G(\mathcal{C}, \mathcal{H}, \Phi) \longrightarrow N$ such that the following diagram is commutative for any $i \in I$:*

$$\begin{array}{ccccc} G & \xrightarrow{\lambda_G} & G(\mathcal{C}, \mathcal{H}, \Phi) & \xleftarrow{\lambda_i} & H_i \\ & \searrow \psi_G & \downarrow \psi & \swarrow \psi_i & \\ & & N & & \end{array}$$

4. *The groups $G(\mathcal{C}, \mathcal{H}, \Phi)$ corresponding to different well-orderings of the set I are isomorphic.*

Proof. 1) and 2) follow by transfinite induction from the corresponding properties of ordinary free products with amalgamation.

Let us prove 3). Following the construction of the group $G(\mathcal{C}, \mathcal{H}, \Phi)$, we need to prove this universal property for each term of the chain $G = G_0 \subset$

$G_1 \subset \dots \subset G_\alpha \subset \dots$. Induction on ordinals α . For a non-limit ordinal α we use the universal property from proposition 4.1. If α is a limit ordinal, then we use the corresponding universal property of the direct limit.

By 3) the groups $G(\mathcal{C}, \mathcal{H}, \Phi)$ for different well-orderings have the same universal property, so they are isomorphic. $\square$

Finally, we describe one more construction: the *iterated tree extension of centralizers*.

Let G be a group, $\mathcal{C} = \{C_G(v_i) \mid i \in I\}$ a set of centralizers in G , and $\Phi = \{\phi_i : C_G(v_i) \rightarrow H_i\}$ a set of embeddings of these centralizers. Then one can construct the group $G_1 = G(\mathcal{C}, \mathcal{H}, \Phi)$ and consider again the set of centralizers $\mathcal{C}_1$ in G_1 , the set of embeddings Φ_1 , and the corresponding set of groups $\mathcal{H}_1$. As above, the group $G_2 = G_1(\mathcal{C}_1, \mathcal{H}_1, \Phi_1)$ can be constructed. We can repeat this process using induction on ordinals: if a group G_α has been constructed for some ordinal α and the set of centralizers $\mathcal{C}_\alpha$ in G_α , the set of groups $\mathcal{H}_\alpha$ and the corresponding set of embeddings Φ_α have been chosen, then

$$G_{\alpha+1} = G_\alpha(\mathcal{C}_\alpha, \mathcal{H}_\alpha, \Phi_\alpha),$$

$$G_\gamma = \varinjlim_{\alpha < \gamma} G_\alpha,$$

for a limit ordinal γ . Thus, for any ordinal δ , starting from a given group G , we can obtain the group G_δ by iterating the process above.

Definition 4.5 *The group G_δ is called an iterated tree extension of centralizers up to level δ .*

Remark 4.1 *Any iterated tree extension of centralizers can be obtained by transfinite induction using operations of extension of centralizers and direct limits.*

It is easy to verify that the group G is embeddable in G_δ , and G_δ has a universal property similar to that in proposition 4.1.

5 CSA-groups

5.1 Groups with conjugate separated maximal abelian subgroups

In this section we summarize basic results about the class of CSA-groups, which plays a fundamental role in the investigation of A -free groups, tensor completions and groups with length functions. This class is quite wide; for example, it contains all torsion-free hyperbolic groups, all groups acting freely on Λ -trees, and all universally free groups. Our interest in CSA-groups is also motivated by the fact that the tensor completion for this class of groups admits a nice constructive description.

Definition 5.1 *A subgroup H of a group G is called conjugate separated (or malnormal) if $H \cap H^x = 1$ for any $x \in G \setminus H$.*

Definition 5.2 *A group G is called a CSA-group if all its maximal abelian subgroups are conjugate separated.*

Remark 5.1 *Every abelian group is a CSA-group.*

The following proposition gives us a handy tool for the work with CSA-groups.

Proposition 5.1 *Let G be a CSA-group. Then the following statements are true:*

1. *(SA-property) if M_1 and M_2 are distinct maximal abelian subgroups of G , then $M_1 \cap M_2 = 1$;*
2. *any maximal abelian subgroup of G coincides with its normalizer.*

Proof. 1) If $1 \neq x \in M_1 \cap M_2$ and $y \in M_2 \setminus M_1$, then $x \in M_1^y \cap M_1$, so $M_1^y \cap M_1 \neq 1$. Item 2) is clear. $\square$

Definition 5.3 A group G is called commutative transitive if the relation "a commutes with b" is transitive on the set $G \setminus 1$.

Remark 5.2 The following conditions for a group G are equivalent:

1. G has the SA-property;
2. G is commutative transitive;
3. the centralizer of every nontrivial element in G is abelian.

Remark 5.3 The class CSA is a proper subclass of the class of groups with transitive commutation, as can be seen on the following example. Let G be a free metabelian group. Then all proper centralizers of G are commutative. But G contains a normal abelian subgroup, so it is not a CSA-group.

In fact, the transitivity of commutation is a very strong property and implies some other interesting properties. We collect them in the following

Proposition 5.2 Let G be a group such that commutation is an equivalence relation on $G \setminus \{1\}$. Then

1. the centralizer of any nontrivial element of G is a maximal abelian subgroup of G ; conversely, any maximal abelian subgroup of $G \neq 1$ is a centralizer of some nontrivial element;
2. if G is torsion-free, then for $x, y \in G$ $x^n = y^m \implies [x, y] = 1$;
 $x^n = y^n \implies x = y$;
3. if G is nonabelian, then its center is trivial;
4. if G is nonabelian, then it is indecomposable into a nontrivial direct product.

Proof. 1), 3) and 4) are quite evident. The second implication in 2) is a corollary of the first one. Let us prove the first claim of 2).

Suppose $x^n = y^m$. Then the consecutive terms in the sequence x, x^m, y^n, y are nontrivial and commute in pairs; by transitivity of commutation we obtain $[x, y] = 1$. $\square$

Remark 5.4 *G is a CSA-group if and only if all centralizers of its nontrivial elements are abelian and conjugate separated (see statement 1 of proposition 5.2).*

Thus, CSA is a property of centralizers. Let us consider a typical situation of conjugate unseparability. A cyclic subgroup $\langle y \rangle = M < G$ is not conjugate separated if and only if there is some $x \in G \setminus M$ such that $M^x \cap M \neq 1$, or, equivalently, x and y satisfy the identity $x^{-1}y^r x = y^s$ for some integers r and s . One-relator groups with this relation have a special name and play a crucial role in what follows. Groups of the form

$$G_{r,s} = \langle a, b \mid a^{-1}b^r a = b^s \rangle, \quad r, s \in \mathbf{Z}, r > 0,$$

are called *Baumslag – Solitar groups* [6]. We will need the following properties of these groups:

- if $r = s = 1$, then $G_{r,s} \simeq \mathbf{Z} \times \mathbf{Z}$ is an abelian group;
- if $r = 1$ and $s \neq 1$, then $G_{r,s}$ is nonabelian solvable and for $s \neq -1$

$$G_{1,s} = \langle a, b \mid a^{-1}ba = b^s \rangle \simeq \mathbf{Z}[\frac{1}{s}] \rtimes \mathbf{Z}$$

where $\mathbf{Z} \simeq \langle a \rangle$, $\mathbf{Z}[\frac{1}{s}] \simeq \text{n.cl}(b)$ - the normal closure of $\langle b \rangle$;

for $s = -1$

$$G_{1,s} = \langle a, b \mid a^{-1}ba = b^{-1} \rangle \simeq \langle b \rangle \rtimes \langle a \rangle.$$

- if $r > 1$, $|s| > 1$, then $G_{r,s}$ is a nonsolvable group and $[a^{-1}ba, b] \neq 1$ (see, for example, [18, p.281]). The case $r > 1, s = \pm 1$ coincides with the second one after renaming the generators.

In the following proposition we collect some properties of CSA-groups.

Proposition 5.3 *Let G be a nonabelian CSA-group. Then*

1. *centralizers of nontrivial elements of G are abelian and coincide with maximal abelian subgroups of G (in particular, commutation is transitive);*
2. *G has no nontrivial normal abelian subgroups;*
3. *G has a trivial center;*
4. *G is directly indecomposable;*
5. *G has no non-abelian solvable subgroups;*
6. *G has no nonabelian Baumslag – Solitar subgroups;*
7. *if G is torsion-free, then G is a D -group, i.e. the operations of extracting roots in G are defined uniquely.*

Proof. Statement 1) follows from propositions 5.1 (3) and 5.2 (1). Statement 2) is clear, and 3) is a corollary of 2). To obtain 4), apply 1) and proposition 5.2 (4). In proposition 5.4 below it will be proved that a subgroup of a CSA-group is again a CSA-group. So to prove 5) one only needs to prove that a non-abelian solvable group is not a CSA-group. But the last statement follows from 2). To prove 6), it suffices to prove that a non-abelian Baumslag–Solitar group $G_{r,s}$ is not a CSA-group. If $r = 1$ and $s \neq 1$, then $G_{r,s}$ is nonabelian solvable (see remarks before the proposition), and hence by 5) it is not a CSA-group. If $r > 1$ and $|s| > 1$, then $[a^{-1}ba, b] \neq 1$. Suppose now that $G_{r,s}$ is a

CSA-group, then commutation on nontrivial elements of G must be transitive. In the sequence $a^{-1}ba$, $a^{-1}b^ra$, b^s , b all neighbours commute, but the first and the last elements do not, which contradicts the transitivity of commutation. Statement 7) follows from item 3) of proposition 5.1 and item 2) of proposition 5.2. $\square$

Proposition 5.4 1. *The class of groups CSA is finitely universally axiomatizable.*

2. *The class CSA is closed under ultraproducts.*

3. *The class CSA is closed under taking subgroups.*

Proof. To prove 1), it suffices to write down the conditions of commutativity of centralizers and their conjugate separation by universal formulas. This is straightforward. Statement 2) follows from the fact that the CSA-class is axiomatizable, and 3) from the universal axiomatization of this class. $\square$

Free groups are principal examples of nonabelian CSA-groups. The following results show that the class CSA contains almost all groups which are in some sense "close" to free groups.

Proposition 5.5 *Let G be a group such that*

a) *G is torsion-free;*

b) *any centralizer of a nontrivial element is cyclic.*

Then G is a CSA-group.

Proof. Let G satisfy conditions a) and b). Then all proper centralizers in G are abelian (in particular, commutation is transitive on $G \setminus \{1\}$), so we only need to prove that these centralizers are conjugate separated. Suppose that for some maximal abelian subgroup M in G and for some $x \in G \setminus M$ we have

$M \cap M^x \neq 1$. Then the idea is to find inside G a Baumslag-Solitar subgroup $G_{1,p}$. This will lead us to a contradiction since $G_{1,p}$ contains a noncyclic abelian subgroup $Z[\frac{1}{p}]$; see [24] for details. $\square$

Proposition 5.6 1. Any torsion-free hyperbolic group is a CSA-group.

2. Any group acting freely on a Λ -tree is a CSA-group.

3. Any universally free group is a CSA-group.

Proof. It is known (see, for example, [9]) that the centralizer of any element of infinite order in a hyperbolic group G has a cyclic subgroup of finite index (i.e. the stabilizer is virtually cyclic). So we only need to prove that any torsion-free virtually cyclic group H is infinite cyclic. It is clear that H is finitely generated and all subgroups of H are also virtually cyclic. By induction on the number of generators of H , we may assume that H is 2-generated, $H = \langle a, b \rangle$ and moreover $\langle b \rangle \triangleleft H$, $a^m \in \langle b \rangle$, $a^{-1}ba = b^n$, for some n, m . This implies $b = a^{-m}ba^m = b^{n^m}$. Therefore $1 = n^m$. Hence $n = \pm 1$ and $[a, b] = 1$ or $[a^2, b] = 1$. In the first case H is abelian torsion-free and virtually cyclic, therefore H is a cyclic group. In the second case a^2 is a central element of H , and so is the element $a^{2m} = b^k$ for some integer k . But $a^{-1}b^ka = b^{-k}$, and hence $b^k = b^{-k}$, which contradicts to the torsion-free condition on G .

Statement 2) is proved in [4, corollary 1.9].

In [28] it was proved that every universally free group acting freely on some Λ -tree. Therefore 2) implies 3). $\square$

5.2 Tensor completion for CSA-groups

The theorem below, whose proof can be found in [24], plays an important part in the construction of tensor completion for CSA-groups. Denote by CSA^* the class of all CSA-groups without elements of order 2. Results that can be

obtained under this restriction still apply to a very wide class of groups, since torsion-free hyperbolic groups, groups acting freely on Λ -trees and universally free groups are CSA*-groups.

Theorem 5.1 *The class CSA* is closed under:*

1. *free products;*
2. *extensions of centralizers by abelian groups without elements of order 2.*

Theorem 5.2 *The class CSA is closed under direct limits.*

Proof. Let $\mathcal{G} = \{G_i, \phi_j^i, I\}$ be an inductive system of CSA-groups G_i , $i \in I$, and $G^* = \varinjlim G_i$. First of all, observe that commutation is transitive on $G^* \setminus \{1\}$. Indeed, any three elements $x, y, z \in G^*$ are in some G_i , which is CSA by assumption.

The essential step of the proof is to describe centralizers of nontrivial elements in G^* . Let us prove that if M is a maximal abelian subgroup of G^* , then for any $i \in I$ $M_i = G_i \cap M$ is either maximal abelian in G_i or trivial; moreover, $M = \varinjlim M_i$.

Let M_i be nontrivial. If M_i is not maximal in G_i , then $[M_i, x] = 1$ for some $x \notin M_i$. Let us prove that $x \in M$. By transitivity of commutation, M is a centralizer of some element $z \in G$. There exists k such that $i < k$ and $x, z \in G_k$. From transitivity of commutation we deduce that $[x, z] = 1$, and therefore $x \in M$, which contradicts to the condition $x \notin M_i$. The equality $M = \varinjlim M_i$ is evident.

Now suppose that M is a maximal abelian subgroup of G^* , and for some $f \in G^*$ $M \cap M^f \neq 1$. Then $x = y^f$ for some $x, y \in M$. We can choose $k \in I$ such that $x, y, f \in G_k$ and therefore $M_k \cap M_k^f \neq 1$ in G_k . Hence $f \in M_k < M$. This proves that G^* is a CSA-group. $\square$

The following result is based on theorems 5.1 and 5.2, and can be easily proved by transfinite induction on ordinals.

Theorem 5.3 *The class CSA^* is closed under tree abelian extensions of centralizers and iterated tree abelian extensions of centralizers by abelian groups without elements of order 2.*

In section 3, using general category theory, we proved that for any group G and any ring A the tensor completion G^A exists, but this proof was not constructive. Therefore an explicit description of the tensor completion of a group remains in question. An important feature of CSA-groups that makes them an object of great interest in the study of tensor completions is the structure of centralizers in CSA-groups. Namely, all these centralizers are abelian; the intersection of two distinct centralizers is always trivial; and the centralizer of every nontrivial element is conjugate separated. Recall that the tensor completion of an abelian group was described in section 3, so we already know the tensor A -completion for each centralizer of a CSA-group G . To make the action of the ring well-defined on the entire group G , we can construct the tensor completion of G step by step, on each stage extending a centralizer M by the A -module $M \otimes A$ under the canonical monomorphism $i : M \rightarrow M \otimes A$. Then, using the fact that M is conjugate separated in G , we can extend the action of A on $M \otimes A$ to all conjugates of M (in order to satisfy axiom 3 from the definition of an A -group, according to which the action of A on G should commute with all conjugations). This idea was realized in [24] using an iterated tree extension of centralizers of a special type. Here we will give a brief description of this construction, and in the next section a similar approach will be used to construct the tensor completion for a wider class of groups, which contains CSA-groups as a subclass.

The construction of G^A is conducted under the assumption that the ring $A \geq \mathbb{Z}$ is without elements of order 2, and G is a CSA^* -group such that all maximal abelian subgroups of G are faithful over A .

Let $\mathcal{C} = \{C_i, i \in I\}$ be the set of all centralizers of G which satisfies the following conditions (S):

- any $C \in \mathcal{C}$ is not an A -module (as a partial abelian A -subgroup in G);
- any two centralizers from $\mathcal{C}$ are not conjugate in G ;
- any centralizer in G which is not an A -module is conjugate to an element of $\mathcal{C}$.

Then we can form the set $\mathcal{H}_A = \{C_i \otimes A \mid i \in I\}$ and the set of canonical embeddings $\Phi_A = \{\phi_i : C_i \hookrightarrow C_i \otimes A \mid i \in I\}$. Consider the tree extension of centralizers of this special type:

$$G^* = G(\mathcal{C}, \mathcal{H}_A, \Phi_A) = G(\mathcal{C}, A)$$

Now we can iterate this construction up to level ω :

$$G = G^{(0)} < G^{(1)} < G^{(2)} < \dots < G^{(n)} < \dots \quad (3)$$

where $G^{(n+1)} = G^{(n)}(C_n, A)$, and the set C_n of centralizers in the group $G^{(n)}$ satisfies the condition (S).

Theorem 5.4 *Under the assumptions above, the union $\bigcup_{n \in \omega} G^{(n)}$ of the chain (3) is the tensor completion of G over A .*

6 Construction of tensor completion for some classes of groups

6.1 Tensor completion for the class $\mathcal{K}_A$.

Let us introduce a new class of groups which will be the focus of our attention throughout this section.

Definition 6.1 A group G belongs to the class $\mathcal{K}_A$, if every its maximal abelian subgroup is either conjugate separated or an A -module.

In the following proposition we establish some basic properties of $\mathcal{K}_A$ -groups.

Proposition 6.1 Let $G \in \mathcal{K}_A$.

1. Suppose that $M_1 \neq M_2$ are maximal abelian subgroups of G , and at least one of M_i is not an A -module. Then $M_1 \cap M_2 = 1$.
2. If $M \leq G$ is a maximal abelian subgroup of G which is not an A -module, then M coincides with the centralizer of any its nontrivial element.

Proof.

1. Assume that M_1 is not an A -module. Suppose $1 \neq x \in M_1 \cap M_2$. Take $y \in M_2 \setminus M_1$. Then $x = y^{-1}xy$ is both in M_1 and M_1^y , but M_1 is conjugate separated. Therefore $M_2 \subset M_1$, and since both are maximal abelian, $M_1 = M_2$ — a contradiction.
2. Let $1 \neq x \in M$, suppose that $[x, y] = 1$ and $y \notin M$. Then we have $x = y^{-1}xy$, so that $M \cap M^y \neq 1$ — a contradiction. $\square$

Before stating the main theorem, let us describe the construction of *complete tensor extension of centralizers* of a group G by a ring A (this terminology was introduced in [24]). Here we will assume that G is a nonabelian group from $\mathcal{K}_A$, and all maximal abelian subgroups of G are faithful over the ring A .

Let $\mathcal{M} = \{M_i, i \in I\}$ be the set of all maximal abelian subgroups of G such that:

1. for any $M \in \mathcal{M}$ M is not an A -module;
2. $\mathcal{M}$ contains one and only one representative of every conjugacy class of maximal abelian subgroups of G that are not A -modules.

Consider the tree extension of centralizers $G^* = G(\mathcal{M}, \mathcal{H}, \Phi)$, where $\mathcal{H} = \{M_i \otimes A \mid i \in I\}$, and Φ is the set of canonical embeddings $\phi_i : M_i \hookrightarrow M_i \otimes A$. Each ϕ_i is an embedding since M_i are faithful over A by assumption.

Iterating this construction up to level ω , we obtain the chain of groups $G = G^{(0)} < G^{(1)} < \dots < G^{(n)} < \dots$, where $G^{(n+1)} = (G^{(n)})^*$.

Definition 6.2 *The group $G \otimes A = \bigcup_{n < \omega} G^{(n)}$ is called the complete tensor extension of centralizers of G .*

Main Theorem. *Let A be a ring of characteristic 0, and G a torsion-free group from the class $\mathcal{K}_A$. If all maximal abelian subgroups of G are faithful over A , then the tensor completion G^A is the complete tensor extension of centralizers of G by the ring A .*

For the proof of the Main Theorem, we will need the following results about $\mathcal{K}_A$ -groups. Denote by $\mathcal{K}_A^*$ the class of all $\mathcal{K}_A$ -groups without elements of order 2.

Theorem 6.1 *The class $\mathcal{K}_A^*$ is closed under extensions of centralizers by abelian groups without elements of order 2.*

Proof. Let $G \in \mathcal{K}_A^*$, M a conjugate separated maximal abelian subgroup of G , and B an abelian subgroup without elements of order 2 such that there exists an embedding $\phi : M \hookrightarrow B$. The extension of M by B is defined by $G(M, B) = G *_M B$. Observe that $G(M, B)$ has no elements of order 2 (see [18, cor.4.4.5]).

The following lemma describes maximal abelian subgroups in $G(M, B)$.

Lemma 6.1 *If N is a maximal abelian subgroup in $G(M, B)$, then one of the following is true:*

1. $N \leq G^h$ for some $h \in G(M, B)$ (more precisely, $N = N_1^h$ for some maximal abelian $N_1 < G$);

2. $N = B^h$ for some $h \in G(M, B)$;

3. $N = \langle z \rangle$ with $\|z\| \geq 2$, where $\|z\|$ stands for the cyclically reduced length of z .

Proof. We will use the following description of commuting elements in free products with amalgamation (see [18]): $[x, y] = 1$ iff one of the following conditions holds:

- x or y belongs to some conjugate of the amalgamated subgroup M ;
- neither x nor y is in a conjugate of M , but x is in a conjugate of a factor (G or B), then y is in the same conjugate of the factor;
- neither x nor y is in a conjugate of a factor, then $x = g^{-1}cm_1z^n$, $y = g^{-1}m_2gz^m$, where $m_1, m_2 \in M$, and $g^{-1}m_1g$, $g^{-1}m_2g$ and z commute in pairs.

Let us consider these three possible cases.

Let $x \in N$; suppose that $y \in N$, so that $[x, y] = 1$.

1. $x \in M^h$. Conjugating by h if necessary, we may assume that $x \in M$. Let $y = mg_1b_1 \dots g_nb_n$ be the canonical form of y , where $m \in M$, g_i are right representatives of G modulo M , b_i are right representatives of B modulo M , and $g_n \neq 1$. Then $xy = yx$ implies

$$xmg_1b_1 \dots g_nb_n = mg_1b_1 \dots g_nb_nx = mg_1b_1 \dots g_nxb_n,$$

so that g_n and g_nx are in the same conjugacy class of G by M , and we have: $g_nx = x_1g_n$ for some $x_1 \in M$. Hence $M^{g_n} \cap M \neq 1$, but M is conjugate separated — a contradiction. It follows that $y = mb_1 \in B$, which is case 2 of the lemma.

2. $x \in G^h$ (resp. B^h), then by the description of commuting elements in a free product with amalgamation, y is in G^h (resp. in B^h), and we obtain case 1 or 2 of the lemma.

3. Finally, suppose that $\|x\| \geq 2$ and $[x, y] = 1$. Then $x = g^{-1}m_1gz^n$, $y = g^{-1}m_2gz^m$, $m_i \in M$, $\|z\| \geq 2$, and z , $g^{-1}m_1g$ and $g^{-1}m_2g$ commute in pairs. Conjugating by g^{-1} , we reduce the problem to the case where $x = m_1z_1^n$, $y = m_2z_1^m$, $z_1 = gzg^{-1}$, and z_1 commutes with m_1 and m_2 . If $m_1 \neq 1$ or $m_2 \neq 1$, then $z_1 \in B$ according to the proof above, but this contradicts the condition $\|z\| \geq 2$. So $m_1 = 1 = m_2$, and we have $x = z_1^n$, $y = z_1^m$.

Now let z be another element of N . The same argument shows that $x = z_2^k$, $z = z_2^l$ for some z_2 with $\|z_2\| \geq 2$. Then all elements of the subgroup $H = \langle z_1, z_2 \rangle$ have length ≥ 2 . Indeed, let $u \in H$ and $\|u\| = 1$. Observe that $x = z_1^n = z_2^k$ is in the center of H . So $[u, z_1^n] = 1$, and the description of commuting elements in a free product with amalgamation implies that z_1^n is in the same conjugate to one of the factors as u . This is impossible, since $\|z_1^n\| \geq 2$. This allows us to apply the following description of subgroups of a free product with amalgamation $G_1 *_M G_2$: if $\forall x H \cap G_i^x = 1$, then H is a free group (see [18, p.243, cor.4.9.2]). It easily follows that H is cyclic, since it has a nontrivial center. We may assume that z_1 and z_2 are not proper powers, therefore $z_1 = z_2$. Hence N is cyclic, and we have case 3 of the lemma. $\square$

Let us continue the proof of the theorem.

Let $N = N_1^h$ be a maximal abelian subgroup of $G(M, B)$ of type 1. We need to prove that either $N^x \cap N \neq 1$ implies $x \in N$, or N is an A -module. Conjugating by h^{-1} if necessary, we may assume that $N = N_1 < G$. Suppose that there exist nontrivial elements $f, g \in N$ such that $f^x = g$. Then, according to [24, lemma 2], $f^y = g$ for some $y \in G$. But then $N \cap N^y \neq 1$. Since N is maximal abelian in G , we have that either N is an A -module or N is conjugate separated. In the latter case, $y \in N$, so $f = g$ and $[x, f] = 1$. More-

over, according to the description of commuting elements in free products with amalgamation, $x \in G$ (observe that x is not in the amalgamated subgroup M ; indeed, M is not maximal abelian in $G(M, A)$, so $M \neq N$, and $M \cap N = 1$). Since N is the centralizer in G of any its element, in particular f , we obtain that $x \in N$.

Now let $N = B^g$, $g \in G(M, B)$. Again, it suffices to consider the case $N = B$. The result immediately follows if B is an A -module, which is in fact the only case we will be interested in. In general, it is not difficult to prove that B is conjugate separated in $G(M, B)$. Suppose that $B \cap B^x \neq 1$, $x \in G(M, B)$. Then $f^x = h$ for some $f, h \in B$, and by [24, lemma 2], f and h are conjugate in B . But B is abelian, so $f = h$. Then $f^x = h$ implies $[f, x] = 1$, and since f is in B , so is x .

The proof for the case that N is infinite cyclic is covered by the following two lemmas.

Lemma 6.2 *In a free product of two groups with amalgamation, the centralizer of any element z such that $\|z\| \geq 2$ is cyclic.*

Proof. Let $x \in C(z)$, so that $[z, x] = 1$. From the description of commuting elements in a free product with amalgamation we obtain that $z = m_1^q u^n$, $x = m_2^q u^m$, where m_1^q, m_2^q and u commute pairwise. As before, it easily follows that $m_1 = m_2 = 1$, so that $z = u^n$ and $x = u^m$. Moreover, $\|u\| \geq 2$. It remains to show that every element of $C(z)$ is a power of u . Let $y \in C(z)$, then $[y, z] = 1$, and the previous argument shows that $z = v^k, y = v^l$ for some v with $\|v\| \geq 2$. Consider the group $H = \langle u, v \rangle$. Using the fact that $u^n = v^k \in Z(H)$ and the same argument as in the proof of lemma 1 (part 3), we obtain that H is cyclic. We may assume that u and v are not proper powers, so $u = v$. $\square$

Lemma 6.3 *Any maximal abelian subgroup of $G *_M B$ of the form $\langle z \rangle$, $\|z\| \geq 2$, is conjugate separated.*

Proof. Let $x^{-1}z^m x = z^n$ for some m and n . Then $\|z^m\| = \|z^n\|$, but since $\|z^n\| = |n| \cdot \|z\|$, it follows that either $m = n$ or $m = -n$. If $x^{-1}z^m x = z^m$, then $[x, z^m] = 1$, and hence x belongs to the centralizer of z^m . By the previous lemma, $C(z^m)$ is cyclic; it obviously contains z , so $[x, z] = 1$, but $\langle z \rangle$ is maximal abelian. Therefore $x \in \langle z \rangle$.

If $x^{-1}z^m x = z^{-m}$, then $[x^2, z^m] = 1$. It is crucial that $G *_M B$ has no elements of order 2, so that x^2 is nontrivial. We obtain that $x^2 \in C(z^m)$, which is cyclic, and hence $[x^2, z] = 1$. Furthermore, x^2 is not in a conjugate of a factor (otherwise z would be in the same conjugate, which is impossible since $\|z\| \geq 2$). Therefore $\|x^2\| \geq 2$, and its centralizer is cyclic. Since $C(x^2)$ contains both x and z , $[x, z] = 1$, but $\langle z \rangle$ is maximal abelian, so $x \in \langle z \rangle$. $\square$

This completes the proof of theorem 6.1.

Theorem 6.2 *The class $\mathcal{K}_A$ is closed under direct limits, in the case where the canonical mappings are embeddings.*

Proof. Let $\tilde{G} = \lim_{\leftarrow} G_i$, M a maximal abelian subgroup in $\tilde{G}$ which is not an A -module. Observe that $M = \lim_{\leftarrow} M_i$, where $M_i = M \cap G_i$.

Claim. M_i is a maximal abelian subgroup in G_i .

Proof. Suppose that M_i is not maximal abelian in G_i , then there exists a maximal abelian M'_i in G_i such that $M_i < M'_i$. Starting from M'_i , we construct a maximal abelian subgroup M' of $\tilde{G}$ in the following way: embed M'_i into G_{i+1} , denote by M'_{i+1} the maximal abelian subgroup of G_{i+1} containing M'_i , and so on. On limit steps take unions. In $\tilde{G}$ $M' \neq M$ (since $M \cap G_i = M_i \neq M'_i = M' \cap G_i$), and both are maximal abelian, so they are not contained in each other. There are $m' \in M'$ and $m \in M$ such that $[m', m] \neq 1$. Observe also that since M is not an A -module, there is a number j such that for all $k > j$ M_k is not an A -module. So taking k sufficiently large, we may assume that $m', m \in G_k$ and M_k is not an A -module. By construction, $M' \cap G_k = M'_k$

is a maximal abelian subgroup in G_k . The group M_k is contained in some maximal abelian $N_k < G_k$. Observe that $M'_k \neq N_k$ (m is in N_k but not in M'_k , and m' is in M'_k but not in N_k), and $M'_k \cap N_k \neq 1$ (both contain the image of M_i). We may assume that k is large enough so that N_k is not an A -module. (Indeed, if there is a j such that for all $k > j$ N_k is an A -module, it follows that $\lim_{\leftarrow} N_k = N$ is an A -module. Since N obviously contains M , and M is maximal abelian in $\hat{G}$, we have $M = N$. But M is not an A -module by assumption.) Since G_k is in class $\mathcal{K}_A$, $M'_k \cap N_k \neq 1$ means that both are A -modules — a contradiction.

To complete the proof of the theorem, suppose that $M < \hat{G}$ is not an A -module and $M^x \cap M \neq 1$. Then $m_1^x = m_2$ for some $m_1, m_2 \in M$, $x \in \hat{G}$. There exists a k such that $m_1, m_2, x \in G_k$, so $M_k^x \cap M_k \neq 1$. We assume that k is large enough so that M_k is not an A -module, and since $G_k \in \mathcal{K}_A$, M_k is conjugate separated — a contradiction. $\square$

Theorem 6.3 *The class $\mathcal{K}_A^*$ is closed under tree abelian extensions of centralizers and iterated tree abelian extensions of centralizers by abelian groups without elements of order 2.*

Proof. By definition, a tree (iterated tree) extension of centralizers of a group G is constructed by transfinite induction on ordinals. Following this construction, it suffices to prove by induction that every term of the chain $G = G_0 < G_1 < \dots < G_\alpha < \dots$ belongs to the class $\mathcal{K}_A^*$ if the initial group G is in $\mathcal{K}_A^*$. For a nonlimit ordinal this was proved in theorem 1, and for a limit ordinal in theorem 2. $\square$

Corollary 6.1 *If $G \in \mathcal{K}_A^*$, and the ring A has no additive elements of order 2, then the tree extension of centralizers G^* is a $\mathcal{K}_A^*$ -group.*

Proof. As proved in [24, proposition 13], none of the groups $M_i \otimes A$ from the construction of G^* contains elements of order 2. The rest of the proof follows

from theorem 6.3. □

Lemma 6.4 *The tree extension of centralizers G^* is a partial A -group, and the action of A on G is defined in G^* .*

Proof. Let $g \in G$, then there is a maximal abelian subgroup M of G containing g . If M is an A -module, then the action of A on g is defined in G , moreover in G^* . Suppose that M is not an A -module, then $M = M_i^x$ for some $M_i \in \mathcal{M}$, and $g = g_i^x$ for some $g_i \in M_i$. In G^* $M_i \hookrightarrow M_i \otimes A$, so the action of A is defined on g_i . For $\alpha \in A$, we let $g^\alpha = (g_i^\alpha)^x$. Since G is a $\mathcal{K}_A$ -group, M is conjugate separated, and we can extend this action correctly to all conjugates of M to satisfy axiom 3 from the definition of an A -group. In this way, the action of A is well-defined in G^* on all elements of G , since any two maximal abelian subgroups of G that are not A -modules intersect trivially. □

In what follows it is essential that G is torsion-free, and A is of characteristic 0.

Lemma 6.5 *All maximal abelian subgroups of G^* are faithful over A .*

Proof. Each maximal abelian subgroup in G^* is either an A -module, or infinite cyclic. In both cases, it is faithful over A . In fact, for an infinite cyclic group $\langle z \rangle$ we have:

$$\langle z \rangle \simeq \mathbb{Z} \hookrightarrow A \simeq \mathbb{Z} \otimes A \simeq \langle z \rangle^A$$

Lemma 6.6 *G^* has the following universal property with respect to the canonical embedding $i \hookrightarrow G^*$: for any A -group H and for any partial A -homomorphism $f : G \longrightarrow H$ there exists a partial A -homomorphism $f^* : G^* \rightarrow H$ such that $f = f^* \circ i$.*

Proof. Let M be a maximal abelian subgroup of G belonging to $\mathcal{M}$. Then $f(M)$ generates an abelian A -subgroup N in H . Using the universal property

of the tensor completion for abelian groups, we can find a homomorphism $\psi_M : M \otimes A \rightarrow N$ such that $f = \psi_M \circ i_M$, where $i_M : M \hookrightarrow M \otimes A$ is the canonical embedding. By the universal property of G^* , there exists $f^* : G^* \rightarrow H$ such that $f = f^* \circ i$. $\square$

Now we are ready to complete the proof of Main Theorem.

Let $G_0 = G$, $G^{(n+1)} = (G^{(n)})^*$, where G^* is defined as above. Then $G \otimes A = \bigcup_{n < \omega} G^{(n)}$ is the tensor A -completion of G . Indeed, $G \otimes A$ is an A -group: if $x \in G \otimes A$, then $x \in G^{(n)}$ for some n , and by lemma 6.4, the action of A on x is defined in $G^{(n+1)}$.

It remains to check the universal property of the tensor completion. Let H be an A -group and $f : G \rightarrow H$ a homomorphism. For any n , f extends to a homomorphism $f_n^* : G^{(n)} \rightarrow H$ of partial A -groups. Then $f^* = \bigcup_{n < \omega} f_n^*$ is an A -homomorphism such that $f = f^* \circ i$, where i is the canonical embedding $i : G \rightarrow G \otimes A$. $\square$

Corollary 6.2 *Let G be a torsion-free $\mathcal{K}_A$ -group, and A a ring of characteristic 0. Then G is faithful over A if and only if every abelian subgroup of it is faithful over A .*

Theorem 6.4 *Let A be a ring of characteristic 0. Then the class $\mathcal{F}_A$ of faithful torsion-free A -groups is closed under free products.*

For the proof of the theorem we will need the following

Lemma 6.7 *Let G_1 and G_2 be groups without elements of order 2. Then in the free product $G_1 * G_2$, any maximal abelian subgroup of the form $\langle z \rangle$, where $\|z\| \geq 2$, is conjugate separated.*

Proof. First let us prove that the centralizer of z in $G_1 * G_2$ is exactly $\langle z \rangle$. Let $x \in C(z)$, then $[x, z] = 1$. Consider the group $H = \langle x, z \rangle$. Clearly, H is

abelian. At the same time, H contains $\langle z \rangle$, and since $\langle z \rangle$ is maximal abelian, we have $H = \langle z \rangle$. Therefore $C(z) < \langle z \rangle$. The inverse inclusion is trivial.

To prove that $\langle z \rangle$ is conjugate separated, suppose that $\langle z \rangle^x \cap \langle z \rangle \neq 1$, then $x^{-1}z^m x = z^n$ for some m, n . Using the length argument as before, we obtain that $|m| = |n|$. There are the following two cases:

a) If $m = n$, we have $x^{-1}z^m x = z^m$, which is equivalent to $[x, z^m] = 1$. The group $H = \langle x, z \rangle$ has a nontrivial center (at least, it contains z^m). As a subgroup of a free product, H is of the form $H = F * (*_{\alpha} H_{\alpha})$, but it has a nontrivial center, and therefore coincides with one of the factors. Since $z \in H$ and $\|z\| \geq 2$, $H \neq H_{\alpha}$. Hence $H = F$, so it has to be cyclic. Therefore $[x, z] = 1$, which implies $x \in C(z) = \langle z \rangle$.

b) If $m = -n$, then $x^{-2}z^m x^2 = z^m$, and by the same argument as above, $x^2 \in \langle z \rangle$. By [18, cor. 4.4.5], $G_1 * G_2$ contains no elements of order 2, so $x^2 \neq 1$. Consider the group $\langle z, x \rangle$. Its center contains x^2 and therefore is nontrivial. In the same way as in a) we obtain that $\langle z, x \rangle$ is cyclic. Consequently, in this case $x \in \langle z \rangle$ too. $\square$

Proof of Theorem 6.4. Let $G_1, G_2 \in \mathcal{F}_A$. Then $G_1 \hookrightarrow G_1^A$ and $G_2 \hookrightarrow G_2^A$, so $G_1 * G_2 \hookrightarrow G_1^A * G_2^A$.

Claim 1. $G_1^A * G_2^A \in \mathcal{K}_A$.

Proof. Let M be a maximal abelian subgroup of $G_1^A * G_2^A$. By the Kurosh subgroup theorem ([18]), we have the following two cases:

1. $M = \langle z \rangle$ with $\|z\| \geq 2$;
2. $M < (G_1^A)^x$ or $M < (G_2^A)^y$, $x, y \in G_1^A * G_2^A$.

In the first case M is conjugate separated by lemma 6.7, and in the second case it is an A -module (indeed, $M < M^A < G_i^A$, but since M^A is an abelian group and M is maximal in G_i^A , $M = M^A$).

Claim 2. Any maximal abelian subgroup M of $G_1^A * G_2^A$ is faithful over A .

Proof. M is either an A -module or infinite cyclic; in both cases it is faithful over A .

By the Main Theorem, $G_1^A * G_2^A$ is faithful over A , and hence it is embeddable into some A -group K :

$$G_1 * G_2 \hookrightarrow G_1^A * G_2^A \hookrightarrow K$$

This proves that $G_1 * G_2$ is faithful over A . □

6.2 Tensor completion for groups of the form F/N' .

Let F be a free group, N a normal subgroup of F , and $N' = [N, N]$. In what follows we will extensively use the theorem quoted below.

Theorem 6.5 (Mal'cev, 1960) *Let F/N be torsion-free. Then two elements of F/N' commute if and only if they either belong to N/N' or belong to the same cyclic subgroup of F/N' .*

Proof. See [20]. An alternative proof using the magnus embedding can be found in [7]. □

Now consider the following conditions on a group G :

1. G is torsion-free;
2. G is an R -group (terminology due to Kontorovich, see [15]): for $x, y \in G$ $x^m = y^m$ implies $x = y$; in other words, G is a group with unique roots;
3. for every element g of G there is a primitive element f such that every root of g is a power of f . (Recall that f is called primitive if it is not a proper power.)

Lemma 6.8 (Mal'cev, 1960) *If F/N is an R-group, then so is F/N' .*

Proof. The proof is based on the following result of Auslander and Lyndon [3]: if F/N is infinite, then the center of F/N' is trivial. Suppose that $x, y \in F$, $x^m = y^m \pmod{N'}$. Then $x^m = y^m \pmod{N}$, so that we have, by assumption, $x = y \pmod{N}$, i.e. $y = xn$ for some $n \in N$ and $(xn)^m = x \pmod{N'}$. From $(xn)^m = x^m$ it follows that $n^{x^{m-1}} \cdots n^x n = 1$, where n^x stands for $x^{-1}nx$, and conjugating by x , we obtain $n^{x^m} \cdots n^{x^2} n^x = 1$. Hence $n^{x^m} = n$, which implies $x^m n = n x^m$ in F/N' . Therefore n lies in the center of the group $\langle x^m, N \rangle / [N, N]$. The group $\langle x^m, N \rangle$ is free, and $\langle x^m, N \rangle / N$ is torsion-free. By the theorem of Auslander and Lyndon we have $n \in N'$, i.e. $y = x$ in F/N' . $\square$

Lemma 6.9 *If F/N satisfies condition 3, then so does F/N' .*

Proof. Suppose for contradiction that for some $gN' \in F/N'$ there is no primitive element with the required property. Then we can build an infinite chain of the following form:

$$g = g_1^{n_1} \pmod{N'}, g_1 = g_2^{n_2} \pmod{N'}, \dots, g_{k-1} = g_k^{n_k} \pmod{N'}, \dots$$

where $n_k \neq 1$ for all k . Since $N' \leq N$, the same equalities hold modulo N , which contradicts to the fact that F/N satisfies condition 3. $\square$

Corollary 6.3 *Conditions 1–3 hold for free solvable groups.*

Proof. Condition 1 is trivial. Conditions 2 and 3 obviously hold for free abelian (one-step solvable) groups. Let us assume, by induction, that they are true for free n -step solvable groups. If F is a free group, then $F^{(n)}$ is a normal subgroup of F whose quotient group $F/F^{(n)}$ satisfies conditions 2 and 3 by the induction hypothesis. By lemmas 6.8 and 6.9 we conclude that the free $(n+1)$ -step solvable group $F/N^{(n+1)}$ satisfies these conditions. $\square$

Lemma 6.10 *If F/N satisfies conditions 1 and 2, then commutation in F/N' is transitive.*

Proof. Let $x, y, z \in F/N'$ such that $[x, y] = 1$ and $[x, z] = 1$. If $x \in N/N'$, then by theorem 6.5 $y, z \in N/N'$, which is abelian. Therefore $[y, z] = 1$. If $x \notin N/N'$, then by the same theorem there are $u, v \in F/N'$ such that $x = u^m, y = u^n, x = v^k, z = v^l$ for some positive integers m, n, k, l . Since F/N is an R -group by assumption, so is F/N' . By [15, p. 244] it follows that two elements of F/N' commute if and only if their powers commute. But both u and v are roots of x , so $[u, v] = 1$, and hence $[y, z] = 1$, as required. $\square$

In the rest of this section we will always assume that F/N satisfies conditions 1–3.

Lemma 6.11 *If $x \in F/N'$, but $x \notin N/N'$, then the centralizer of x in F/N' is cyclic.*

Proof. Take an arbitrary element $x \in (F/N') \setminus (N/N')$. By lemma 6.9, F/N' satisfies condition 3, therefore x can be represented in the form $x = y^k$, where y is a primitive element of F/N' . Let us prove that the centralizer of x in F/N' is the cyclic subgroup generated by y . Indeed, suppose that $[x, z] = 1$, then due to the transitivity of commutation in F/N' $[y, z] = 1$. By theorem 6.5, there exists an u such that $y = u^m, z = u^n$ for some m, n . But y is not a proper power, so $y = u$, and $z = y^n \in \langle y \rangle$. $\square$

Corollary 6.4 *The centralizer of any nontrivial element of F/N' is a maximal abelian subgroup of F/N' . Conversely, any maximal abelian subgroup of F/N' is a centralizer of some nontrivial element.*

Corollary 6.5 *If M is a maximal abelian subgroup of F/N' , then either $M = N/N'$, or M is cyclic.*

Lemma 6.12 *In F/N' every maximal abelian subgroup of the form $M = \langle y \rangle$, $y \notin N/N'$, is conjugate separated.*

Proof. Suppose for contradiction that for some $x \notin M$ $M^x \cap M \neq 1$, i.e. there are $m, n \in \mathbb{Z}$ such that $x^{-1}y^mx = y^n$. Denote $x^{-1}yx$ by z . Then $z^m = y^n$, therefore $[z^m, y] = 1$. Since $[z, z^m] = 1$ and commutation in F/N' is transitive, it follows that $[z, y] = 1$. This implies that $z \in M$, so $z = y^p$ for some p .

The generators of the subgroup $H = \langle x, y \rangle$ satisfy the relation $x^{-1}yx = y^p$, so H is a homomorphic image of the Baumslag-Solitar group $G_{1,p} = \langle a, b \mid a^{-1}ba = b^p \rangle$ under the epimorphism $\varphi : a \mapsto x, b \mapsto y$. Let us prove that φ is injective. It is well-known that $G_{1,p} \simeq \text{n.cl.}\langle b \rangle \rtimes \langle a \rangle$, where $\text{n.cl.}\langle b \rangle \simeq \mathbb{Z}[\frac{1}{p}]$, $\langle a \rangle \simeq \mathbb{Z}$. Suppose that $b_1 a^s \in \ker \varphi$ for $b_1 \in \text{n.cl.}\langle b \rangle$; then $y_1 x^s = 1$ for some $y_1 \in \text{n.cl.}\langle y \rangle$. But in the sequence y_1, y, x^s, x every two consecutive terms commute. By assumption $x \notin M$, so $[x, y] \neq 1$, and by transitivity of commutation we obtain that both y_1 and x^s are trivial. This proves that $\langle x, y \rangle \simeq G_{1,p}$. But $\mathbb{Z}[\frac{1}{p}] \simeq \text{n.cl.}\langle y \rangle$ is an abelian subgroup of $\langle x, y \rangle$, so either $\text{n.cl.}\langle y \rangle < N/N'$ (which is impossible since $y \notin N/N'$), or $\mathbb{Z}[\frac{1}{p}]$ is cyclic, which is also false — a contradiction. $\square$

Now we are ready to construct the tensor completion of F/N' . Observe that F/N' is not a CSA-group, since the maximal abelian subgroup N/N' is normal and hence not conjugate separated. But we can embed F/N' into a $\mathcal{K}_A$ -group in the following way. Consider the group $G = F/N' *_{N/N'} (N/N' \otimes A)$.

Theorem 6.6 *G belongs to the class $\mathcal{K}_A$.*

Proof. Let M be a maximal abelian subgroup in G , then M is of one of the following three types:

1. $M < (F/N')^x$;
2. $M < (N/N' \otimes A)^x$;

3. $M = \langle z \rangle$ with $\|z\| \geq 2$.

In case 2) M is an A -module, and in case 3) M is conjugate separated by lemma 6.3. Therefore it remains to consider case 1). Conjugating by x^{-1} if necessary, we may assume that $M < F/N'$. If $M = N/N'$, M is an A -module, so we can restrict ourselves to the case $M = \langle y \rangle$, $y \in F/N'$, $y \notin N/N'$.

Suppose that $x^{-1}y^mx = y^n$ for some $x \in G$. Write x in reduced form: $x = n_1f_1 \cdots n_sf_s$, where $n_i \in (N/N' \oplus A)$, $f_i \in F/N'$. Then we have the equality: $f_s^{-1}n_s^{-1} \cdots f_1^{-1}n_1^{-1}y^mn_1f_1 \cdots n_sf_s = y^n$. Cancellation in the left-hand side can occur only if $n_1^{-1}y^mn_1$ in the amalgamated subgroup N/N' , which by the normality of N/N' implies that $y^m \in N/N'$. Then in F/N $y^m = 1$, and $y \neq 1$, but F/N is torsion-free by assumption. Therefore $n_1 = 1$. Proceeding by induction, we obtain that all $n_i = 1$, therefore $x \in F/N'$. But in F/N' the subgroup $\langle y \rangle$ is conjugate separated by lemma 6.12. We are done.

Observe also that in any of the three cases M is faithful over A . $\square$

Summing up, we have embeddings $F/N' \hookrightarrow G \hookrightarrow G^A$, and since $G \in \mathcal{K}_A$ and all maximal abelian subgroups of G are faithful over A , the tensor completion of G is the complete tensor extension of centralizers of G by the ring A .

Theorem 6.7 $(F/N')^A = G^A$

Proof. Let $i_1 : F/N' \hookrightarrow G$, $i_2 : G \hookrightarrow G^A$. Take an arbitrary A -group H and a partial A -homomorphism $\varphi : F/N' \rightarrow H$. By the universal property of a free product with amalgamation, there exists a partial A -homomorphism $\varphi_1 : G \rightarrow H$ such that $\varphi_1 \circ i_1 = \varphi$. Then by the definition of G^A there exists an A -homomorphism $\psi : G^A \rightarrow H$ such that $\varphi_1 = \psi \circ i_2$, and $\varphi = \psi \circ i_2 \circ i_1$. Therefore G^A satisfies the universal property of the tensor completion of F/N' .

It remains to show that F/N' A -generates G^A . But it is clear that F/N' generates G as a partial A -group, and G A -generates G^A by definition. Hence

$$G^A = \langle F/N' \rangle_A.$$

□

Corollary 6.6 *A free solvable group $M_n = F/N'$, where $N = F^{(n-1)}$, is faithful over any torsion-free ring A , and the tensor completion of M_n can be constructed as the complete tensor extension of centralizers of a certain partial A -group containing M_n .*

Corollary 6.7 *The tensor completion of a nonabelian free solvable group is not solvable.*

Proof. Let $N = F^{(n-1)}$. The tensor completion of F/N' contains the subgroup $G = F/N' *_{N/N'} (N/N')^A$, which arises on the first step of our construction. If F/N' is nonabelian, then G , in its turn, contains a noncyclic free subgroup and thus is not solvable. □

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