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Examining the potential of rehearsal interjections to support the teaching of mathematical practice: the case of mathematical defining

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Abstract

This study investigates the potential of rehearsal interjections to provide opportunities for novice teachers and teacher educators to discuss topics related to teaching the practice of mathematical defining. Through analysis of video recordings of seven elementary novice teachers' rehearsals about geometric definitions, we identified the problems of practice that initiated rehearsal interjections, the topics discussed during rehearsal interjections, and relations between initiating problems of practice and topics discussed. We found that initiating problems of practice focused overwhelmingly on pedagogical issues, with most related to aspects specific to the teaching of mathematical defining. Likewise, discussions during interjections tended to focus on definitional pedagogical topics. Although epistemic topics were mentioned, they were only conveyed implicitly and at times in conflicting manners. Moreover, few opportunities arose for novices to make sense of student thinking about definitions and the mathematics of shape. Our results illustrate ways in which the goal of improving pedagogy, although important, can overshadow learning of other aspects for teaching mathematical practice.

Keywords Mathematical definitions · Mathematical defining · Mathematical practice · Geometry · Rehearsals · Instructional triangle · Pre-service teacher education

Introduction

Recent educational policies have placed increasing emphasis on the role of disciplinary practices in mathematics learning (e.g., National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). These policies encourage student activity aligned with professional practices of mathematics, such as conjecturing, defining, generalizing, and proving. Such classroom spaces position students as authors of mathematics and thus shift the mathematical authority away from the teacher and textbook (Ball & Bass, 2000; Lampert, 1990; Lehrer et al., 2013).

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Teaching in this way is challenging. Teachers must learn to provide space for students to invent mathematical ideas, guide students in using those ideas to collectively construct mathematical systems, and facilitate discussions with competing mathematical arguments (Ball, 1993; Lampert, 1990).

In this paper, we investigate the potential of *rehearsals of teaching* (Lampert et al., 2013) as a site for elementary school pre-service novice teachers (NTs) to learn to engage students in mathematical practice. Rehearsals are simulations in which one NT teaches a lesson with other NTs acting as school-age students. The teaching is *interjected* (paused) by the teacher educator (TE) or NT to address problems of practice that arise (Ghousseini et al., 2015). During these interjections, the TE provides guidance and feedback related to problems raised. Research from teacher learning communities has shown that discussions of problems of practice can create opportunities for teachers to make sense of components of the instructional triangle: students, teaching, and content (Horn & Little, 2010; Lin, 2016). Yet, at the same time, these opportunities are not guaranteed: Even when a problem of practice is raised, interactions within a community can instead direct the discussion away from topics about components of the instructional triangle (Horn & Little, 2010; Lin, 2016). Despite the importance of problems of practice for teachers' learning, we know little about the kinds of problems that arise when teachers are learning to engage students in particular mathematical practices and how the subsequent discussion focuses on the instructional triangle.

We focus our inquiry on the disciplinary practice of *mathematical defining*. Defining can serve as an accessible starting point for students of all ages to engage in mathematical practice (e.g., de Villiers, 1998; Keiser, 2000). By engaging in defining, students can improve their mathematical understanding, agency, and mathematical communication (e.g., Ambrose & Kenehan, 2009; Borasi, 1992; Lehrer et al., 1999; Leikin & Winicki-Landman, 2001; Zandieh & Rasmussen, 2010; Zaslavsky & Shir, 2005). Yet, little research has investigated how to support NTs to engage students in defining. We focused on the elementary level because of its emphasis on classification and definitions of geometric shapes. We asked:

1. What problems of practice initiated rehearsal interjections?
2. What topics about the components of the instructional triangle were discussed during rehearsal interjections?
3. How were these topics related to the initiating problems of practice?

Theoretical perspectives

In this section, we first situate our work within two theoretical perspectives—situative perspective and deliberate practice—and conceptualize teacher learning within rehearsals in relation to the instructional triangle and problems of practice. We present an expanded version of the instructional triangle that includes the epistemic assumptions that guide the enactment of mathematical practice. We then describe each aspect of the expanded instructional triangle in relation to the literature on the teaching and learning of defining: epistemic assumptions of mathematical defining, students' learning of defining, and teachers' roles in supporting defining.

Perspectives on teacher learning within rehearsals

We draw on perspectives that suggest that learning professional practice is situative (Greeno, 2005; Putnam & Borko, 2000); that is, learning is interactional in nature, and the context in which the learning occurs plays a key role in shaping learners' understandings. Context is not limited to replicating a physical space. Rather, context includes the forms of thinking and interactions that constitute a community of practice (Putnam & Borko, 2000). From this perspective, teachers must have opportunities to engage in interactions and forms of thinking resembling those from the classroom (e.g., Ball & Cohen, 1999; Grossman et al., 2009). Rehearsals are a promising method to address this need, as they allow for collective *deliberate practice*—repeated opportunities to engage in aspects of practice with ongoing feedback and reflection (Ericsson et al., 1993). Feedback and reflection play a key role in making practice "deliberate" so that teachers learn to enact practices with professional judgment and not as an automatic ritual (Ghousseini et al., 2015).

To assist novices in this reflection, rehearsals simplify the complexity of teaching in two ways. First, rehearsals are conducted using *instructional activities* (IA) that provide a routine lesson structure that NTs can adapt for multiple concepts (Lampert & Graziani, 2009). This adaptability allows novices to focus on practicing components of their teaching in a familiar structure. Second, novices can be encouraged to focus on a small set of core teaching practices (e.g., orienting students' thinking) (Kazemi et al., 2009).

Teaching using such practices is relational work (Grossman et al., 2009), involving interactions between teachers, students, and content (Lampert, 2010). Within rehearsals, *interjections* are key to providing time and space for NTs to identify, reflect on, and address problems of practice that arise about these relations (Ghousseini, 2017; Ghousseini et al., 2015). *Problems of practice* include any issues related to the teaching profession (Horn & Little, 2010). From a situative perspective, they are contextual and are shaped by existing classroom norms and structures and the rehearsing NT's (R-NT's) mathematical knowledge and professional commitments (Ghousseini, 2015; Ghousseini et al., 2015). When learning to teach mathematics in ways that are responsive to students, NTs may experience problems such as order and wording of teacher questions, how to respond in ways that are neutral, and managing materials and representations to be accessible (Ghousseini, 2015; Ghousseini et al., 2015).

Given our focus on understanding how NTs learn to teach mathematical practice, we focus on problems of practice about interactions when enacting a lesson. These interactions between teachers, students, and content have been represented as an instructional triangle (Cohen et al., 2003). In our study, we expand the instructional triangle to include *epistemic assumptions*—assumptions about what it means to engage in mathematical practice (Fig. 1). From a situative perspective, the epistemic assumptions form part of the context that shapes the interactions between content, teachers, and students. We thus argue that when supporting disciplinary practice, teachers must also understand the epistemic assumptions of the mathematics and how those assumptions translate into interactions of teaching and learning. In this manuscript, we refer to epistemic assumptions, teachers, students, and content as the *components of the expanded instructional triangle (EIT)*. NTs can make sense of and discuss different *topics* about each of these components or the relations among them. For example, the specific ways students might define "triangle" could be a topic of discussion related to the component *students*.

Rehearsals approximate the interactions within the EIT and thus provide a space to explore the EIT. When NTs or TEs interject the rehearsal to raise problems of practice,

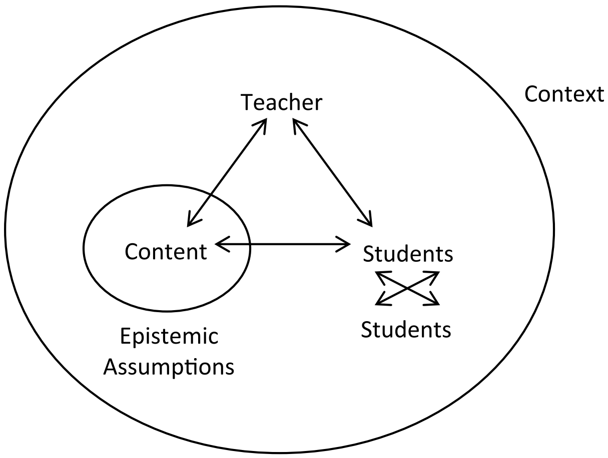

Fig. 1 Expanded instructional triangle for engaging students in mathematical practice. *Note.* Adapted from Cohen et al. (2003)

they can create opportunities to discuss and make sense of topics about components of the EIT (Ghousseini et al., 2015; Horn & Little, 2010; Lin, 2016). For example, Horn and Little (2010) describe how one teacher learning community created learning opportunities by normalizing a problem through shared experience, specifying details of the problem, unpacking its causes, and generalizing to broader principles. However, problems do not always lead to learning opportunities. Interactions within a community can instead direct the conversation away from topics about components of the EIT and/or limit opportunities for sense-making (Horn & Little, 2010; Lin, 2016). For example, Horn and Little (2010) found that a teacher learning community limited opportunities for sense-making when a problem was treated as an issue particular to one teacher instead of a general phenomenon to collectively make sense of. It is thus important for TEs to understand the kinds of problems of practice that may arise, how those problems may direct discussion toward certain topics, and how discussion may be directed away from topics about components of the EIT. In the following sections, we unpack each component of the EIT specific to mathematical defining.

Epistemic assumptions about defining

Mathematical definitions play a key role in the discipline of mathematics. We define a *mathematical definition* to be a description of the properties of a mathematical object (e.g., square, function, odd number) and the relations among those properties (Lehrer & Curtis, 2000; Polya, 1957). Mathematical definitions have several distinct features. Because definitions are created in a shared community, they must be *unambiguous* (always interpreted in the same way) (Zaslavsky & Shir, 2005) and only include *precise mathematical terminology* (Borasi, 1992; Levenson, 2012). Moreover, *alternate definitions*, those that are different yet equivalent, may exist for the same object (de Villiers, 1998). This feature is related to the *arbitrary* nature of definitions—that is, they are human constructions (Linchevsky et al., 1992). Each definition is part of a larger constructed system of definitions that are related to one another (Van Dormolen & Zaslavsky, 2003). Alternate definitions vary in form (e.g., textual vs. symbolic) or *minimality* (Linchevski et al., 1992; Van Dormolen &

Zaslavsky, 2003; Zandieh & Rasmussen, 2010; Zaslavsky & Shir, 2005). Minimal definitions only include descriptions that are necessary for guaranteeing identification of the object. Minimal definitions are often *hierarchical*; that is, they include definitions already established by the community (Van Dormolen & Zaslavsky, 2003; Zaslavsky & Shir, 2005).

Definitions are constructed through the disciplinary practice of mathematical defining. From a situative perspective, we consider *disciplinary practice* to include socially constructed forms of activity that are grounded in historical and material representations within a community (Ford & Forman, 2006). We thus consider *mathematical defining* to involve communication about definitions among members of a mathematical community (Borasi, 1992). Communication resulting in disagreement may lead to definitional arguments (Lakatos, 1976). Definitional arguments take on different forms, including: (a) whether or not to include a case as an example of a particular object, (b) whether to dismiss or keep a proposed counterexample to a proof, (c) verifying the validity of an object by appealing to a definition, or (d) justifying the equivalence or non-equivalence of two definitions (Van Dormolen & Zaslavsky, 2003). Definitional arguments can aid the refinement of proof (Lakatos, 1976; Linchevsky et al., 1992; Parameswaran, 2010). As Lakatos described in his historical analysis, mathematicians may introduce counterexamples to new proofs, leading them to contest the definitions within the proof. Sometimes new definitions are proposed in order to dismiss the counterexample while still salvaging the proof and the conjecture (a process Lakatos referred to as "monster-barring"); other times the definition remains and the proof is altered. Definitional arguments can also lead to the development of other related definitions (Borasi, 1992; Zaslavsky & Shir, 2005). For instance, in the case of the Euler Characteristic, defining "polyhedron" led to a counterexample that spurred discussions about the definitions of "polygon" and "edge" (Lakatos, 1976). Thus, defining plays an important role in introducing new objects.

Students' learning of defining

Mathematics educators have developed a variety of approaches for students to engage in the forms of disciplinary practice just described. Examples include: (a) *sorting tasks* in which students classify examples and non-examples into categories (e.g., Ambrose & Kenehan, 2009; Lehrer et al., 1989; Ouvrier-Buffet, 2006; Tabach & Nachlieli, 2015); (b) students *evaluating alternate definitions* (and sometimes non-definitions) (e.g., Borasi, 1992; Leikin & Winicki-Landman, 2001; Zaslavsky & Shir, 2005); or (c) students *authoring definitions*, either as an isolated exercise, or arising from students' need to define an object embedded in a problem or proof (e.g., Mariotti & Fischbein, 1997; Zandieh & Rasmussen, 2010). Lessons can also blend these different tasks. As we elaborate in the Methods, the NTs in our study learned to teach defining through a sorting task in which students authored definitions.

In their literature review, Kobiela and Lehrer (2015) found that by participating in these types of activities, students are able to engage in different aspects of defining practice. These aspects include: (a) proposing different potential definitions for an object; (b) further developing their definitions by constructing or evaluating examples and non-examples; (c) describing properties or relations among properties; (d) constructing arguments about examples, non-examples, or definitions; (e) revising definitions based on arguments; (f) reasoning about systematic relations between objects, properties, definitions, conjectures and/or proofs; and (g) engaging in epistemic conversations about criteria for judging acceptability of definitions. As within the larger mathematical community, students collectively engage in these aspects of defining practice through social interactions.

This research demonstrates that these forms of activity enable students to develop more robust understandings of definitions, examples, and non-examples. When learning definitions, students develop concept images—images evoked when one hears or thinks of an object (Vinner, 1991). Concept images often guide students' classifications. Students' concept images in geometry tend to be prototypes of objects that they see in everyday life (Rosch, 1973). Shapes are prototypes based on dimensions and orientation. For example, the equilateral triangle is a prototypical triangle (Tsamir et al., 2008). Children may believe that a rotated equilateral triangle is not a triangle because it is not sitting on its "bottom" (Lehrer et al., 1999). Objects that look like equilateral triangles but are not (e.g., with slightly curved sides) may be misunderstood by children to be triangles.

In addition, children often initially use visual reasoning to classify geometric objects instead of attending to properties (Lehrer et al., 1999; Tsamir et al., 2008). Visual reasoning can take several forms (Tsamir et al., 2008), including: (a) appealing to known appearance ("It doesn't look like a triangle," p. 88); (b) holistic descriptions relating to everyday objects (e.g., "it looks like a pyramid"); or (c) holistic descriptions relating to mathematical objects (e.g., "it's a square, not a triangle"). Students may also attend to non-critical attributes before focusing on defining properties (e.g., "It goes up too high to be a triangle.") (Lehrer et al., 1999; Tsamir et al., 2008).

Teachers' roles in teaching defining

Teachers play a key role in cultivating classroom environments that engage students in defining and help them develop concept images aligned with conventionally accepted definitions. A first step is planning the defining task. For a definitional sort, teachers must decide which examples and non-examples to include in the sort and what order to present these objects during the discussion (Kobiela et al., 2018). Doing so requires understanding students' initial prototypes and reasoning and how those ideas can be built upon to help them attend to and make sense of properties (Lehrer et al., 1999).

Teachers also play an important role in leading discussions with students about definitions. Existing research highlights different moves teachers may use to orchestrate these discussions. First, in order to communicate the epistemic assumptions of defining, teachers need to *articulate expectations specific to participation in defining* and the purpose of definitions (Kobiela, 2012). For example, teachers may highlight that definitions can help class members collectively decide what counts as an example of an object (Kobiela & Lehrer, 2015). Teachers can position students as authors in the defining process by *requesting that they engage in aspects of defining* (Kobiela, 2012; Kobiela & Lehrer, 2015) (e.g., "What is a triangle?"). Teachers can also ask students to *attend to precise language* (Kobiela, 2012; Lehrer et al., 1999) to move away from everyday language and visual reasoning. For example, in the younger grades, if students are struggling to describe properties, the teacher may ask them to use gestures and then suggest language to use (Lehrer et al., 1999). As students develop shared ways of articulating a definition, teachers may orient students to their developing definition by *keeping the definition at the forefront* (Kobiela, 2012). For example, teachers can encourage students to keep track of their definition, both in their personal math journals (Kobiela & Lehrer, 2015) and as a public record (Lehrer et al., 1999; Wongkamalasai, 2019), and later build upon these records (Steele et al., 2013). The teacher can also use the class's working definition when posing questions or presenting counterexamples (Kobiela & Lehrer, 2015). Finally, when students are struggling to articulate new properties for their definition, the teacher may *problematize definitions* by proposing a counterexample based on the class's definition (similar to "monster-barring" from Lakatos, 1976). A *counterexample* is a non-example that is used to make an argument about a definition. For example, Kobiela and Lehrer (2015) describe how the teacher problematized the class's definition of polygon ("sides and angles") by drawing three connected sides (forming a zigzag) and claiming that the figure was a polygon based on their definition. This drawing prompted the students to add the property of "closed" to their definition.

As this literature illustrates, the work of supporting defining is complex. For some teachers, teaching may be further complicated by their limited understanding about definitions and defining. Within geometry, teachers sometimes struggle to provide accurate definitions of objects (Ulusoy, 2021; Zazkis & Leikin, 2008) or may use informal language when providing definitions (Miller, 2018; Tsamir et al., 2015; Ulusoy, 2021). Teachers may not use their definitions to correctly classify or construct examples and non-examples (Ward, 2004) and may instead rely on prototypical concept images (Gutiérrez & Jaime, 1999; Ulusoy, 2021; Ward, 2004). Teachers may also struggle to identify contradictory elements of a definition (Yahya et al., 2019), to identify alternate definitions (Haj-Yahya, 2019; Salinas et al., 2014), or to make sense of hierarchical relations among definitions (Linchevsky et al., 1992; Susanto, 2019).

Teachers may also lack epistemic understandings of the purpose and features of definitions. In particular, teachers' epistemic understandings may be shaped by pedagogical goals, needs, or understandings—what we term a *pedagogical orientation.* For example, teachers may have a preference for non-minimal definitions because they provide more description for students (Haj-Yahya, 2019; Linchevsky et al., 1992; Sánchez & Garcia, 2014; Zazkis & Leikin, 2008). They may also dislike hierarchical definitions, feeling that they can be confusing to students (Mosvold & Fauskanger, 2013; Zazkis & Leikin, 2008). Linchevsky and colleagues also found that some pre-service teachers in their study believed in or preferred only one definition, suggesting a lack of understanding of the notion of arbitrariness and the existence of alternate definitions.

Moreover, teachers sometimes possess views of the learning and teaching of definition that may limit the opportunities they provide students for defining. For example, Mosvold and Fauskanger (2013) found that some teachers felt that students in the lower grades do not need to focus on definitions, and so teachers at this level do not need to understand definitions. Such views may result from teachers' own experiences as learners, as some have limited first-hand experience with definition construction (Moore-Russo, 2008).

The above literature shows the need to strengthen teachers' mathematical and epistemic understandings of definitions of geometric objects and their connections to teaching and learning. A few studies have shown the potential to support teachers' learning of definitions and defining through definition authoring and sorting tasks (Gutiérrez & Jaime, 1999; Moore-Russo, 2008; Steele et al., 2013; Tabach & Nachlieli, 2015). These studies highlight that engaging in defining can help teachers develop their mathematical and epistemic understandings. However, we know less of how to help teachers connect these understandings to teaching and learning.

Methods

The data reported here come from the first year of an ethics-approved multi-year design study (Cobb et al., 2003) aiming to support elementary NTs to learn to engage students in the practice of mathematical defining. Design research involves the design, implementation, and study of educational interventions to produce theories of teaching and learning (Cobb et al., 2003). Relevant to our study, a key assumption of this approach is the situated nature of learning and teaching and the importance of studying learning interactions within the complex settings in which they occur (Schoenfeld, 2012).

Rehearsal setting

The study took place in a mathematics methods course for elementary pre-service teachers in a Canadian undergraduate teaching certification program. The first author was a course instructor for two of the sections during the year of data collection. The other authors had at one time been associated with the course, either as a teaching assistant or a student.

The course was the second of two required mathematics methods courses. It focused on measurement, statistics, probability, and geometry and aimed to support NTs' content knowledge, knowledge of student thinking, and pedagogy. To target the third goal, the course focused on a set of core practices (Ball et al., 2009; Lampert et al., 2013), such as orienting students to each others' thinking and creating a public record of student thinking. The purpose of these practices was grounded within a set of principles about teaching and learning, such as "treating students as sense-makers" (Lampert et al., 2013, p. 228).

During the course, students each participated in a rehearsal for one IA. One of these IAs, a *definitions sort* (Baldinger et al., 2016; Kobiela et al., 2018), aimed to support NTs to engage students in defining. In this activity, students build upon their existing concept images to construct a definition. The IA is divided into three parts (Kobiela et al., 2018): (1) *Opening*: A teacher elicits students' initial understandings of a mathematical object; (2) *Exploration:* Individually or in groups, students sort objects into examples and non-examples; (3) *Follow-up Discussion:* The teacher orchestrates a whole class discussion in which students share their classifications and collectively build a set of definition rules. To do so, the teacher orders the objects in the discussion to target specific properties. When discussion prompts students to introduce new properties, the teacher asks them if they should create or add a rule.

Because NTs each planned, rehearsed, and enacted one IA, only a portion of NTs taught the definition sort. NTs completed the sort in groups of three. One NT taught the Opening phase (Phase 1), the second NT taught the first half of the Follow-up Discussion (Phase 2), and the third NT taught the second half of the Follow-up Discussion (Phase 3). During the rehearsal, each NT had 7–10 min to practice part of their phase (~ 30 min for the group).

NTs prepared for the definition sort rehearsals during two classes prior to the rehearsals. Activities included: reflection and discussion of previous experiences with definitions (documented in a definitions journal), engaging as students in a definitions sort, planning for a definitions sort, evaluating whether four definitions of triangles were "good" definitions,[1] and examining how to support procedural definitions (Zaslavsky & Shir, 2005).

[1] This activity was adapted from Zaslavsky & Shir (2005) and the Mathematics Teaching and Learning to Teach project materials.

These activities collectively aimed to support NTs' epistemic understandings that: (a) definitions help to introduce new objects, describe properties and relations, and facilitate communication; (b) definitions must be unambiguous and include precise terminology; (c) alternative definitions can exist; (d) mathematical defining involves arguments around potential examples and non-examples in relation to the definition and revision of definitions; and (e) "good" definitions include all the needed properties to exclude non-examples and include examples.

Data collection and participants

Data were collected from one section of the course that had the largest number of NTs rehearsing definition sorts (not the first author's section). For this paper, we drew upon video recordings of rehearsals to answer our research questions. Video recordings allowed us to examine how topics were brought up as interactions unfolded during interjections. In addition, we inductively coded the NTs' definitions journals to provide background on their experiences and views coming into the rehearsals.

Journals revealed that most NTs had learnt definitions in teacher-driven ways (e.g., memorization, copying definitions), yet felt that definitions should be taught using student-centered methods (e.g., using manipulatives, "hands on"). Nine NTs (of 27) rehearsed a definitions sort and were invited to participate in the study. Of those, seven participated. These NTs had experiences and views that reflected those of the class as a whole.[2] The journals revealed that at the start of the unit, the participants believed that mathematical definitions are descriptions, processes, or rules for a concept; a couple also noted that definitions are "fact(s)" or "truths." After engaging in the definitional sort, five reflected on definitions as constructed and/or revisable.

Three of the participating NTs rehearsed a definitions sort for defining triangle in kindergarten, two rehearsed a triangle sort for grade 3, and two a polygon sort for grades 5–6. To help them plan, the NTs were provided with instructional goals (including definitions for polygon and triangle), a lesson plan protocol to fill out, and examples of potential student contributions. In these resources, polygon was defined as a closed shape that has at least three straight sides and triangle as a closed shape that has three straight sides.

The instructor (Eugene) facilitated rehearsals in one room (for the polygon sort) and a teaching assistant (Nelson) facilitated rehearsals in the other room (with the other NTs). Both were graduate students who had received training in coaching rehearsals.[3]

Analysis

Identifying and transcribing episodes

Because of our focus on rehearsal interjections, we first identified *TE/NT exchange episodes* within the rehearsals. Episodes began with the move that initiated the interjection and ended with the turn of talk immediately before the R-NT resumed teaching.

[2] One of the seven NTs did not consent to allowing researchers to view his journal and is not included in this assessment.

[3] We do not include more information about the TEs to protect their identities.

We began with a preliminary set of rules for developing TE/NT exchanges, based on Lampert et al. (2013). We then individually created episodes, collectively reviewed episodes to determine consensus, refined episode rules, and continued this iterative process until no more changes were needed. This process resulted in 36 episodes in total. However, given our focus on discussions of the EIT initiated by problems of practice, we eliminated any episodes that occurred before the start of the teaching or that did not include any feedback or questions from the TE or NTs. We did not, however, limit episodes to those in which the discussion focused on defining because we wanted to understand the range of problems that arose. We were left with 26 episodes, with 2–6 episodes per R-NT. We then transcribed all 26 episodes to capture talk and gesture.

Table 1 shows an example of an episode and the teaching that preceded it. The start of the episode was prompted when R-NT Tricia turned to the TE and asked a question, signaling that she was stepping out of her role as teacher. The episode ended when Tricia resumed her role as teacher.

Coding initiating problems of practice

To address research question 1, we developed a coding scheme for *initiating problems of practice*—the problems that prompted the start of an interjection. Our unit of analysis was the first turn of talk within each episode that contained a question or point of feedback related to aspects or interactions within the EIT. In most cases, the unit was the very first turn of talk for the interjection. We chose to only focus on the first turn that contained a question or point of feedback because we were interested in what problem prompted the interjection before additional issues or ideas were raised. However, in order to interpret the first turn of talk, we considered the teaching that immediately preceded, as it provided context. For example, in the episode in Table 1, the first turn of talk for the episode was from Tricia: "Should I go there?" To interpret what she meant by "there," we considered the question she had asked immediately before: "So do you think we could make a rule?" We thus characterized this problem as *when and how to address the class's collective rules*.

One researcher developed an initial coding scheme by comparing the first turns of talk. We then collectively refined the scheme through an iterative process of individually coding samples of the rehearsals, comparing codes, and using disagreements to revise the coding scheme. After receiving feedback on the first iteration of this manuscript, the first author further refined the coding scheme and re-coded all data.

Although the development of codes started with the data, we drew upon language from our theoretical framework that included epistemic assumptions, students' ways of thinking and practice, types of definitional teaching moves, and aspects of teaching definitional sorts. Although these frameworks do not describe problems of practice, they provided language for our codes. For example, the code of when and how to address the class's collective rules was related to issues about enacting the teaching move of keeping definition at the forefront. We thus described the code in relation to the teaching move but added "when and how" to capture the issues that arose. In addition, because our focus was on understanding problems of practice specific to defining, we created broader codes for problems not related to defining (e.g., "general teaching").

Table 1 Example of TE/NT exchange episode

	Speaker	Transcript
Teaching prior to episode	Tricia	So which one do you think looks more like a triangle to you?
	Student	That one.
	Tricia	To this one (points to the equilateral triangle). So do you think we could make a rule?
TE/NT exchange episode	Tricia	(turns to TE) Should I go there?
	Nelson	I think that would be good.
	Tricia	Am I going the right way?
	Nelson	No, I think it's a good idea to talk about the rules because we want to try to create the public record of the rules to always refer back to. So, she said it has to be straight.
	Tricia	Okay, I-uh-hold on.

Coding topics

We then coded for topics about components of the EIT (question 2). Our unit of analysis was the entire episode, including the first turn of talk that was used to identify the problem of practice. We included the entire episode because sometimes R-NTs or TEs introduced topics when raising the problem in the first turn of talk.

We followed an iterative process to develop four coding schemes to capture topics about components of the EIT: definitional pedagogy, epistemic nature of definitions/defining, mathematics of shape, and students' thinking about definitions. For each coding scheme, we developed codes starting from the data, but drew upon existing research in each of these categories to guide what we saw in the data. For pedagogical topics, we looked for comments in which NTs and TEs mentioned anything about teaching connected to definitions and/or the sort. We drew upon the teaching moves described in our literature review to describe the types of teaching moves discussed during interjections or to identify when discussions were about the teaching moves (e.g., about the order of moves). In addition, the literature on the structure of definitional sorts allowed us to identify when discussion included topics related to issues of how to implement the sort, such as launching the sort.

For the epistemic nature of definitions and defining, we looked for comments that related to the purpose and nature of definitions or defining. We noted commonalities to the ideas described in our literature review about epistemology of definitions and defining (e.g., defining as involving communication) and teachers' understandings of the epistemology of definitions (e.g., pedagogical orientations) and used those ideas to help form our codes.

For mathematics of shape, we looked for any mention of properties, relations between properties or objects, classifications or descriptions of examples or non-examples, or specific definitions. For students' thinking about definitions, we identified comments about students' contributions about definitions, properties, examples, or non-examples.

One researcher coded the entire data set. Because multiple topics were discussed in each episode, she assigned multiple codes when needed. Finally, because we had a large number of codes, we then grouped our codes thematically (Braun & Clarke, 2006). To do so, we compared across the codes within each coding scheme, looking for commonalities. For example, within our definitional pedagogy coding scheme, we had seven different codes that each specified a definitional teaching move (e.g., keeping definition at the forefront, attending to precise language). We grouped these codes under the theme, *how to enact a definition-specific teacher move*.

To illustrate, the episode in Table 1 was coded with the definitional pedagogy topic themes of *how to enact a definition-specific teacher move* (since they discuss whether it is "good" to ask about the rules) and *how to sequence definitional moves* (since they discuss whether Tricia's question should be stated then or later). The episode was coded with the epistemic theme of *social nature of definitions* since Nelson's comment implied that the public record of the rules should serve as a reference point for the community. The episode also included the mathematics theme of *mathematical properties* and the student thinking theme of *students' contributions of a particular example* since he noted that a student had said that the object "has to be straight" —pointing to the student contribution and a property of triangles.

Table 2 Initiating problems of practice during definition sort rehearsals

Pedagogical problems specific to defining	When and how to address the class's collective rules
	When and how to engage students in highlighting properties (e.g., having them come up to show the property)
	How to keep discussion on a property when a student brings in a different idea
	Structure and logistics of the definitional sort lesson (e.g., sequencing of aspects of the lesson)
Pedagogical problems not specific to defining	General teaching issues (e.g., staying neutral)
	Providing access to help students who do not understand
Problem related to student thinking of definitions	Authenticity of a student contribution

Characterizing relations between topics and initiating problems of practice

Finally, we created tables to show the relations between topics and initiating problems of practice (question 3). To do so, for each problem of practice, we documented how many corresponding episodes were also coded with a topic theme from one of the four coding schemes: definitional pedagogy, epistemic nature of definitions/defining, mathematics of shape, and students' thinking about definitions. For example, nine episodes had been coded with an initiating problem of practice of *when and how to address the class's collective rules*. In our table, we documented that 9/9 of the episodes had been coded with a topic theme from the definitional pedagogy coding scheme, 9/9 with a theme from epistemic nature of definitions/defining, 5/9 with a theme from mathematics of shape, and 6/9 with a theme from students' thinking about definitions.

In a few episodes, we noticed that an additional problem of practice arose partway through the interjection. To ensure that the relations we identified were due to the initiating problem, in our table, we did not count topics occurring after a new problem arose.

Results

In what follows, we present our results, focusing first on the problems of practice that initiated rehearsal interjections. We then describe topics about components of the EIT that were discussed during the interjections and how those topics related to the initiating problems of practice.

Initiating problems of practice

We identified seven different problems of practice that initiated the rehearsal interjections (Table 2). Six problems were related to issues of teaching but differed in how they connected to the teaching of defining. The seventh problem related to student thinking of definitions. Two of the 26 interjections were not coded as any of the seven problems because one was unclear and the other was not initiated by a problem.

TEs and R-NTs initiated similar numbers of interjections. Twelve of the interjections were initiated by the TE, 13 interjections were initiated by the R-NT, and one

Table 3 Topics discussed during definition sort rehearsal interjections

Definitional pedagogy	How to carry out the sort (e.g., how to select and sequence objects and properties for the sort)
	How to enact a definition-specific teacher move (e.g., keeping definition at the forefront)
	How to sequence definitional teaching moves
Epistemic nature of definitions/defining	Social nature of definitions (e.g., definitions are communal)
	Role and nature of examples and non-examples (e.g., importance of having definition rules emerge from evaluation of examples and non-examples)
	Features of definitions (e.g., alternate definitions can exist)
	Pedagogical orientations (e.g., definitions serve a purpose of conveying what individuals know)
Mathematics of shape	Mathematical properties (including in relation to a particular example/non-example or definition)
	Classifications of examples and non-examples
	Visualizing non-examples
Students' thinking about definitions	Students' contributions of a particular example
	Students' contributions about proposed definitions (e.g., "they're (the students are) going to be like a rectangle has four sides.")
	Precursors to defining (i.e., what students would know or would need to know to engage in defining)

interjection was initiated by another NT.[4] Although both TEs and R-NTs initiated most types of problems of practice, TEs more often initiated: when and how to address the class's collective rules (6 of 9 episodes), general teaching (2 of 3 episodes), and when and how to engage students in highlighting properties (2 of 3 episodes). R-NTs more often initiated problems related to the structure and logistics of the definitional sort (4 of 5 episodes) and how to keep discussion on a property (1 of 1 episode). Thus, R-NTs were more concerned than TEs with how to carry out the logistics of the sorting lesson.

Topics discussed during interjections and relations to initiating problems of practice

Discussions during rehearsal interjections included a wide range of topics, spanning all components of the EIT (Table 3). Yet, the depth and frequency of the topics varied. In what follows, we describe three key findings. First, in most interjections, topics related to definitional pedagogy were discussed, even when the initiating problem of practice was related to general teaching. Second, topics related to the epistemic nature of definitions and defining were implicitly conveyed and in a few interjections, provided conflicting messages about the nature of defining. Third, in most cases, topics about the mathematics of shape and student thinking were mentioned without further analysis or sense-making. As we

[4] The norm in these rehearsals was that only the TE and R-NT could pause the rehearsal. This was an occasion when that norm was broken.

elaborate in the Discussion, these findings point to how the goal of improving pedagogy, although important, can overshadow learning of the other components of the EIT.

Abundance of topics about definitional pedagogy

Topics related to definitional pedagogy were most prevalent, occurring in almost all interjections (24 of 26). Most such comments were suggestions for enacting a specific teaching move (n = 22 interjections). Whereas these discussions focused on *how to enact* the teaching moves, seven interjections included comments about the *sequence of definitional teaching moves*. The remaining topics were specific to teaching the sort (e.g., selecting and sequencing objects and properties for the sort).

Discussions about the pedagogy of teaching defining were present following all initiating problems of practice, except for authenticity of student contribution (n = 1 interjection) (Table 4). In many cases, discussions that included definitional pedagogical topics closely related to the problem of practice, suggesting that these problems helped prompt discussion of definition-specific pedagogy. For example, of the nine interjections initiated by the problem of practice of when and how to address the class's collective rules, seven included discussions about the teaching move of keeping definition at the forefront—a move that speaks to how to address the rules. The other two interjections focused on the sequence of teaching moves—speaking to when to address the rules. Moreover, the two interjections in which topics of definitional pedagogy were not raised were prompted by problems that were not specific to defining.

To illustrate how definition-specific pedagogical problems prompted discussion of definition-specific pedagogy, we provide an example from Cassandra's rehearsal of Phase 2 of the grade 6 polygon sort. Prior to her phase, students had developed an initial list of ideas of what a polygon is and then engaged in the sort. Cassandra had started her phase by pointing to an object on the board and asked them, "Why did we think this was not a polygon?" A student explained that the sides were not connected. After asking her what she meant, Cassandra revoiced this idea, "They're not connected? So um." At this point, Eugene paused to raise the problem of when and how to address the class's collective rules.

Eugene:	I think you're forgetting uh- a step. is to- is to write down on the other side RULES that ma-
Cassandra	AGAIN[5] right?
Eugene	Yes.
Cassandra	I was confused about that if I had to write it again or not?
Eugene	YES.
Cassandra	Cause it just seems repetitive because they already got (points to the student's initial list of ideas of polygon).
Eugene	Yeah THESE (points towards the list on the board) yeah THESE are initial ideas that they have but we have NOT agreed on those ideas yet.

In this example, Eugene's initial suggestion provided an opportunity for Cassandra to articulate the problem further—highlighting that, from her perspective, the lists seemed repetitive. Cassandra's concern prompted Eugene to justify this pedagogical decision.

[5] All caps in transcript is used to indicate words emphasized through stress. The use of … signifies transcript removed.

Table 4 Relations between initiating problems of practice and topics

Problems of practice	Definitional pedagogy	Epistemic nature of definitions/ defining	Mathematics of shape	Students' thinking about definitions
Pedagogical problems specific to defining (n = 18)				
When and how to address the class's collective rules	9/9	9/9	5/9	6/9
When and how to engage students in highlighting properties	3/3	3/3	2/3	2/3
How to keep discussion on a property when a student brings in a different idea	1/1	1/1	1/1	1/1
Structure and logistics of the definitional sort lesson	5/5	2/5	2/5	3/5
Pedagogical problems not specific to defining (n = 5)				
General teaching issues	3/3	1/3	2/3	1/3
Providing access to help students who do not understand	1/2	1/2	1/2	1/2
Problem related to student thinking of definitions (n = 1)				
Authenticity of a student contribution	0/1	0/1	0/1	1/1
Unclear or "other" problems (n = 2)				
2/2	1/2	2/2	0/2	

Note. The denominator of the values indicates the total number of interjections that were initiated by the problem of practice indicated in the row. The numerator indicates how many of those interjections were coded with a topic code from the component specified in the column. For example, nine interjections were initiated with the problem of when and how to address the class's collective rules. All nine of those interjections were coded with a topic code related to definitional pedagogy.

In addition to such examples, in four of the five interjections in which the initiating problem was about general pedagogy, TEs and NTs still mentioned topics related to definitional pedagogy. In these four interjections, the TE provided a definition-specific teaching move as a solution to the general problem raised—thus shifting the topic to definition-specific teaching. For example, in one rehearsal, the R-NT had elicited students' initial ideas about triangles. A student had noted that a triangle has "points." After asking the student to explain, the R-NT paused and asked, "From here? Where do I go?" The R-NT framed the problem generally, asking what the next step in the teaching should be. In response, the TE suggested that the R-NT enact a definition-specific teaching move (requesting that the students construct an example): "So this would be a good example to have the student come up and maybe draw."

Despite the prevalence of topics about definitional pedagogy, discussions sometimes contained a mix of definition and non-definition pedagogical topics. This mixture occurred regardless of the initiating problem. During non-definition discussions of pedagogy, TEs and NTs drew upon general frameworks of pedagogy, such as the core practices. We noticed subtle differences in how TEs linked these general frameworks to definition-specific pedagogy. For example, Nelson provided feedback to an R-NT to "ask someone 'oh if we wanted to make another rule, what would it be?' So you're like orienting different ideas. Okay?" Here, he used the linking word "so" to connect the specific example of a definition teaching move (keeping definition at the forefront) to the general practice of orienting students to each other's thinking. In contrast, in another rehearsal, Nelson commented to the R-NT, "You did a really good job like orienting to each other's ideas, and I really like how you kept going back to the rules." Here, Nelson communicated the same two ideas. Yet, he framed them as two separate points of feedback, indexed linguistically by two separate independent clauses. Although subtle, linguistic markers may signal whether and how general frameworks are connected to more specific moves for supporting mathematical practice.

Implicit and conflicting messages about the epistemic nature of defining and definitions

Although topics about epistemic understandings of mathematical definitions and defining were prevalent (occurring in 18 of the 26 interjections), they were never communicated explicitly. Instead, epistemic topics were conveyed as implicit messages when TEs or NTs mentioned aspects of teaching defining. For example, Nelson provided a suggestion to an R-NT for a definition-specific teaching move that contained an implicit message that definitions are communal:

> Because sometimes what I like to do depending on the grade level is like play dumb and be like, "no no, I think it's a triangle. Look at our rules *we* all *agree* with. There's three sides, they're straight. I think it's a triangle." Someone has to *disagree* with me. And then from there we can get to the idea that no clearly that's not a triangle. And be like, "Well what's missing from our rules then? Cause according to *our* rules, this is a triangle. So *we* need to make rules to make sure that we *all agree*" (italics added for emphasis).

In this example, although Nelson communicated an epistemic message, it was framed as a suggestion for what the R-NT could say to her students—foregrounding pedagogical ideas.

In addition, in a few interjections, TEs or NTs communicated conflicting epistemic messages. These conflicts arose when an epistemic message conveyed a *pedagogical orientation* to what it means to engage in defining. Such messages included: (a) definitions are for the purpose of helping the teacher understand, (b) we evaluate definitions as a way to practice applying our definition rules, and (c) definitions serve a purpose of describing what students know about an object. These messages differed from other epistemic messages that aligned with the discipline. For example, Nelson suggested that an R-NT frame the sorting task by saying, “Well today, I want to sort it and I want to know which ones are triangles or not.” By using the first person “I,” Nelson conveyed that definitions are constructed for the purpose of helping the teacher understand. This message was counter to a message he had communicated in another rehearsal (illustrated in the prior example) —about the social nature of definitions as creating common understanding. Although rare, pedagogically oriented epistemic topics provided contradictory messages about the nature and purpose of defining and definitions.

The implicit and conflicting nature of epistemic messages both illustrate a tendency of TEs and NTs to direct discussion to pedagogical topics. This focus on pedagogy may be due, in part, to the types of problems of practice that initiated the interjections. As illustrated in Table 2, none of the initiating problems related to epistemic assumptions. Instead, interjections that included epistemic topics occurred most frequently in relation to problems about definitional teaching moves. For example, all interjections following the problems of when and how to address rules, when and how to engage students in highlighting properties, and how to keep discussion on a property included discussion of an epistemic topic. In contrast, only some of the interjections that were initiated with problems about authenticity of student thinking, general teaching, providing access, and structure and logistics of the definitional sort were followed by an epistemic topic ($n=0$ of 1 for authenticity, 1 of 3 for general teaching, 1 of 2 for providing access, 2 of the 5 for structure and logistics).

We illustrate the implicit and conflicting nature of the epistemic topics with an example from Fred’s rehearsal. Fred taught Phase 3 of the grade 6 polygon sort. The first interjection occurred after Fred oriented students to one object in the sort. In doing so, he pointed to the list of students’ initial ideas for polygon and noted, “The idea of a polygon is right here, correct?” At this moment, Eugene paused the rehearsal and asked Fred whether he was staying neutral—a personal goal that Fred had identified. After Fred reflected on the ways his question was not neutral and why, Eugene suggested to Fred that he first re-draw the object and then ask the students to debate the question, “Would this be a polygon?” to “let it [the ideas] come out from the students.” Despite agreeing, Fred instead oriented students to the rules: “What rule does this object follow?” Eugene interjected again, emphasizing his earlier suggestion: “Is it a polygon? First.” Eugene’s suggestion, prompted by the problem of when and how to address the class’s collective rules, contained an implicit epistemic message: that definitional rules should emerge from evaluation.

However, Fred likely did not identify this implicit epistemic message, or if he did, did not understand its importance. Although Fred repeated the suggested question (“Is it a polygon?”), he immediately paused the rehearsal to bring attention to the same problem: “I mean? If you ask them what RULE does it follow then they can start associating (points to the table of sorted shapes) them. THEN you can ask ‘oh is this a polygon?’” Fred’s suggestion held an implicit epistemic message that showed a conflicting pedagogical orientation: that the purpose of evaluation is to apply rules. Eugene responded by reframing the purpose of the rules—as helping students justify:

> That's part of the WHY. So you start… with seeing what they think now because you know with the discussion that's been done before students might change their minds at this point… You can even- "okay who thinks this …this object might be a polygon?" And you can have a "thumbs up, thumbs down, Unsure?" And then okay "THOSE that say that it's a polygon, WHY do you think so?" And THEN relate to those goals.

Fred's question illustrated that his teaching was guided by a different and conflicting epistemic assumption to Eugene's. Although Eugene provided some justification about the purpose of the rules, his justification was focused on prompting students' justifications, as illustrated by the question he suggested that Fred pose ("WHY do you think so?"). This conflict could have led to a discussion about their differing assumptions, but the discussion instead focused on the pedagogical aspects of supporting students' defining. Moreover, because of the implicit nature of the epistemic messages, NTs may not have realized the epistemic assumptions underlying the rationales for each teaching move.

Lack of sense-making of mathematics and student thinking

Unlike epistemic topics, comments about mathematics of shape and student thinking were explicit (each occurring in 16 of the 26 interjections[6]). However, most comments did not involve analysis or sense-making of these ideas. None of the interjections included sense-making of student thinking and only two interjections included sense-making of mathematics. Instead, topics about mathematics or students' thinking were typically mentioned when discussing what ideas had been brought up by the rehearsing students, when highlighting ideas that students might bring up, or when stating the goals of the lesson. This lack of analysis and sense-making of mathematics and student thinking may be because NTs and TEs did not feel a need to make sense of either, as evidenced by a lack of problems of practice focused on mathematics or student thinking. Only one interjection was initiated by a problem focused on student thinking. Instead, mathematics and student thinking topics were mostly prompted by pedagogical problems.

To illustrate this lack of analysis and sense-making, we provide an example from Cassandra's rehearsal. At the end of the last interjection, Eugene provided Cassandra with a suggestion about selecting and sequencing of objects for the discussion. In making his suggestion, he connected it to how students might make sense of particular examples:

Eugene	Sometimes I wonder if doing this one [object] might be more accessible first before this one… Because this one might be more complicated for the students compared to this one.
Cassandra	Really? I thought it might be the opposite.

[6] These numbers differ from those in Table 4. This is because Table 4 excludes any topics that occurred after a new problem of practice arose in the interjection (see Sect. 3.3.4).

Eugene	It could be, but it could be the opposite as well. See how the students are working on this. So if you feel that this one seems to have it more easily than this one, then reverse the order. Okay, but if you see this one, oh okay, they seem to get this, then- It's always like starting with what is more accessible to students.

In this exchange, Eugene framed his feedback in terms of a pedagogical suggestion: to have the students discuss one object from the sort before the other. He provided a justification for this suggestion, noting that one of the objects might be "more complicated" for students. However, Cassandra's question ("Really? I thought it might be the opposite.") disputed this justification, illustrating her uncertainty with the reasoning behind the pedagogical suggestion.

Although Cassandra's question may have directed the discussion toward an analysis of *why* students might think one object is more complicated, Eugene instead responded that she could observe the students and then adapt her pedagogy accordingly—re-directing the conversation toward suggestions about pedagogy.

To further illustrate our point, we provide a contrasting example from the two sole interjections in which NTs spent time making sense of mathematics. These two episodes both came from Meghan's rehearsal and showed the NTs making sense of the mathematical ideas of shape by *visualizing a non-example*. Meghan taught Phase 2 of the kindergarten triangle sort. Her rehearsal had three interjections. In the first interjection, she paused the rehearsal to address the problem of keeping the discussion on a property. Meghan was concerned that pursuing a student's idea that the "bottom" is different from the "sides" of a triangle would deviate from her goal that a triangle has three sides. Nelson suggested that she present an example to problematize the student's definition and keep focus on the number of sides: "Maybe you can show an example of another shape that has two sides and that's not a triangle and say, 'What about this one?'" This suggestion created an opportunity for Meghan to make sense of the mathematics. In trying to take up Nelson's suggestion, she asked how such an example would look like. Nelson responded, "It doesn't have to be a conventional shape. It could be like instead of hav[ing] a bottom." Nelson's suggestion prompted Meghan to draw a horizontal line connected to a vertical line at a right angle.

This same problem arose again in the third interjection. Another NT, Samantha, raised the issue partway through the interjection. Samantha's contribution led to a communal effort to visualize a new counterexample that would better attend to the student's idea of a triangle needing a bottom.

Samantha	Well with that what I would have done instead is just drawn that one [the equilateral triangle] without the bottom part instead because this is like showing.
Meghan	Yeah (points to her counterexample) I think- cause this is showing the wrong thing so it would be better (erases the counterexample) if I just did this like (draws a line connected to a horizontal line at a 60° angle)… Mmhmm. So it'd be better (points to her drawing with two connected sides and no third side) if I drew it like THIS? (looks at Nelson) And then after (gestures along the missing side) connect it?
Nelson	Yeah.
Meghan	M-kay.
Samantha	Meghan.

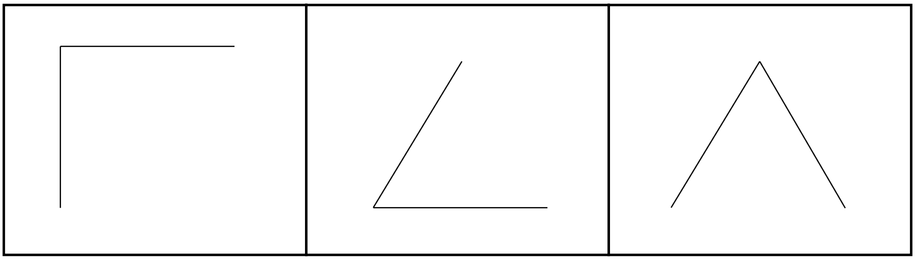

Fig. 2 The Evolution of Meghan's Counterexamples. *Note.* The first image shows the counterexample that Meghan drew during the first interjection, after being prompted by the TE to draw a counterexample. The second image shows the counterexample that Meghan drew after Samantha suggested to draw the equilateral triangle without the bottom. The third image is the final counterexample that Meghan drew after Samantha prompted her with "Don't draw the bottom."

Meghan	Yeah?
Samantha	Don't draw the bottom.
Meghan	OH (turns to face the board) okay.
Tricia	Yeah yeah leave the BOttom.
Meghan	(erases the bottom side)
Sydney	Just the two sides.
Meghan	So just like uh (draws in another side).
Tricia	Like uh- like an upside-down ice cream cone?
Samantha	Just so that it looks exactly like the one [the equilateral triangle] we were talking about earlier.

In this exchange, Samantha encouraged Meghan to re-consider her counterexample to be one that looked like the students' prototype but without a "bottom." Her suggestion demonstrated a focus on developing counterexamples that target particular ideas from students. In the process of making sense of Samantha's suggestion, Meghan and others had to visualize what the counterexample would look like without a bottom. Tricia aided in this visualization by suggesting an analogy of "an upside-down ice cream cone." Fig. 2 shows the evolution of Meghan's counterexamples. In this exchange, the NTs made sense of relations within the EIT: a student's contribution about an example, the visualization of a counterexample, and the pedagogical move of problematizing a student contribution.

In both Cassandra and Meghan's rehearsals, the TE had provided a pedagogical suggestion related to the teaching of definitions, and in both cases, the NTs expressed uncertainty. Both TEs provided support to the NTs, but the ways they did so altered the topics of discussion. In Cassandra's case, Eugene's suggestion re-directed the discussion to the pedagogical aspects of the issue she raised. In Meghan's case, Nelson's suggestion was also pedagogical, but left enough ambiguity to require Meghan to make sense of the mathematics needed to enact the pedagogical move. These contrasting examples thus illustrate how subtle differences in responses to NTs can re-direct opportunities to make sense of topics about components of the EIT.

Discussion

We sought to understand the potential of rehearsal interjections for promoting discussion to support NT teaching of one disciplinary practice: mathematical defining. Our results contribute to a small body of literature on problems of practice that NTs experience during approximations of mathematics teaching (Ghousseini, 2015; Ghousseini et al., 2015)—helping to expand understanding of the potential problem space in these contexts. Overall, the problems of practice in our study focused on aspects specific to the teaching of mathematical defining. This focus shows the potential for definition-specific problems to emerge within rehearsal contexts focused on defining. Similar to Ghousseini et al. (2015), we found that several problems centered on the wording and order of questions. However, our framework specifies what these issues look like when teaching defining.

Moreover, our study provides insight into NTs' perceptions of problems when teaching a definition sort. We found that R-NTs paused for all types of problems of practice, showing the range of the types of issues and questions they brought up. However, they paused more than TEs for problems related to structure and logistics of the definitional sort. Such issues were brought up most often in Phase 1, suggesting that, at least for NTs in our sample, these issues were more relevant for them in the first part of the lesson. Our framework of problems of practice may be used as a tool by TEs to anticipate the types of problems that NTs might raise when rehearsing definitional sorts. As Ghousseini (2015) highlights, it is important to articulate problems of practice faced by NTs in order to determine supports for NTs to address these problems. In addition, TEs can create opportunities for NTs to make sense of problems that they may be less likely to raise.

Definition-specific problems of practice helped to promote definition-specific discussions of pedagogy. This alignment is noteworthy. Research done within in-service teacher education contexts has shown that problems that arose during teacher conversations were taken up in different ways—altering the opportunities for teachers' learning (Horn & Little, 2010; Lin, 2016). As in these in-service contexts, we found that the TE played a key role in steering the discussion toward aspects of teaching specific to definitional discussions. Thus, even in the five interjections initiated by general problems, in all but one, suggestions were given about definition-specific pedagogical moves. In addition, other NTs helped to direct conversation toward teaching defining. One reason may be "shared frames of reference," that is, "shared concepts, principles, and terminology" (Horn & Little, 2010, p. 209). Engaging in conversations about teaching defining prior to the rehearsals may have provided NTs shared frames of reference. Ghousseini et al. (2015) similarly found that having shared frames of reference around a set of core principles and practices prompted NTs to raise problems during rehearsals and to collectively construct solutions to those problems.

At the same time, we found differences in how TEs connected general shared frames of reference to definition-specific frames. General frames, such as core practices, provide an accessible entry point for NTs to learn how to engage in the relational work of teaching (Ghousseini et al., 2015; Kavanagh et al., 2020). However, these general frames may not be enough to help NTs make sense of the varied complexities of teaching (Ghousseini, 2015). TEs may thus need to help NTs build from general frameworks to develop more specific frameworks, such as those for engaging students in mathematical practice.

Moreover, since epistemic messages were implicit, they may have been less apparent to NTs. For example, Fred's question showed that he may not have understood the epistemic assumptions of certain moves—despite previous efforts to develop those understandings in class activities. This result aligns with previous research that shows that teachers often have pedagogical orientations when making sense of definitions (e.g., Haj-Yahya, 2019; Linchevsky et al., 1992; Sánchez & Garcia, 2014; Zazkis & Leikin, 2008). A lack of epistemic understanding may have longer term effects. Singer-Gabella and colleagues (2016) followed NTs into their early years of teaching and found that epistemic "pressure points" mediated the way that NTs took up pedagogical practices they had learned in their university teacher education courses. TEs may thus need to help NTs explicitly connect between epistemic assumptions and teaching, learning, and content when teaching.

One explanation for the lack of discussion on exploring mathematics in the interjections could be the low difficulty level. Yet, Meghan's rehearsal shows that even with very basic mathematics—targeted at kindergarten—NTs can still have opportunities to engage in mathematical reasoning in relation to student thinking and teaching moves. These opportunities are important given that responsive example construction is a key aspect of mathematical teaching (Zaslavsky, 2019) and can be challenging for teachers (e.g., Zaslavsky & Peled, 1996). As noted earlier, teachers may rely on prototypical concept images (Gutiérrez & Jaime, 1999; Ward, 2004), and thus may struggle to visualize other potential examples and non-examples. Our results provide an example of the potential for NTs to visualize counterexamples within a rehearsal context.

One limitation of our work is that we do not have access to the enactments that NTs conducted after the rehearsals in order to see how topics discussed during rehearsals influence NTs' teaching in classrooms. At most, our analysis can highlight potential learning opportunities available through the topics discussed. More research is needed to see how these discussions relate to how NTs enact their teaching.

A second limitation is in what we can infer from the frequency and types of problems and topics. Our findings may be related to the context, including the nature of supports provided (Kavanagh et al., 2020). For example, when TEs interject by posing questions, this can encourage reflection about teaching (Averill et al., 2016). Research is thus needed to understand how certain contexts shape problems and topics related to teaching mathematical practice.

As more focus is placed on how to support mathematical practices, TEs must be attuned to how to help teachers cultivate such classroom environments. Supporting student engagement in mathematical practices is complex and requires attention to aspects and relations within the EIT. Our analysis of the rehearsal context highlights complexities in creating such opportunities—providing one step in this direction.

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