SOLUTIONS TO PROBLEMS IN PLANE ELASTICITY EXPRESSED BY MEANS OF SINGULAR INTEGRAL EQUATIONS

by

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ACKNOWLEDGEMENTS

The results given in this paper have been taken from the book by N.I. Muskhelishvili: Some Basic Problems of the Mathematical Theory of Elasticity, from which it has been this writer's intention to extract the "thread of reasoning" behind the highly successful method which the Russians have devised for the solution of problems in Plane Elasticity. Many more results and far greater consideration of the underlying concepts are given in the above work than has been possible to present here: indeed, it is hoped that this paper will serve as an introduction to the method, and that the reader will become enticed into taking up Muskhelishvili's book himself. Muskhelishvili's beautiful style and lucid explainations (He presupposes only a knowledge of the elements of complex variables and advanced calculus!) make reading this book a most rewarding experience.

My special thanks to Professor Charles Fox, of McGill University, for having introduced me to this book, and for assisting me with this paper.

Montreal H. Miller

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1. INTRODUCTION: In recent years the Russian mathematicians have obtained brilliant solutions to problems in Plane Elasticity by representing them in terms of complex functions. This new technique is fully explained, along with many significant solutions, in the book Some Basic Problems of the Mathematical Theory of Elasticity by N.I. Muskhelishvili, who is one of the foremost proponents of this method. This work has been translated into English by J.R.M. Radok from the Third Edition (1949) in the original Russian and was published by P. Nordhoff of Holland in 1953. For the interested reader, a condensed treatment of this method may also be found in the Second Edition of I.S. Sokolnikoff's Mathematical Theory of Elasticity (McGraw-Hill, 1956).

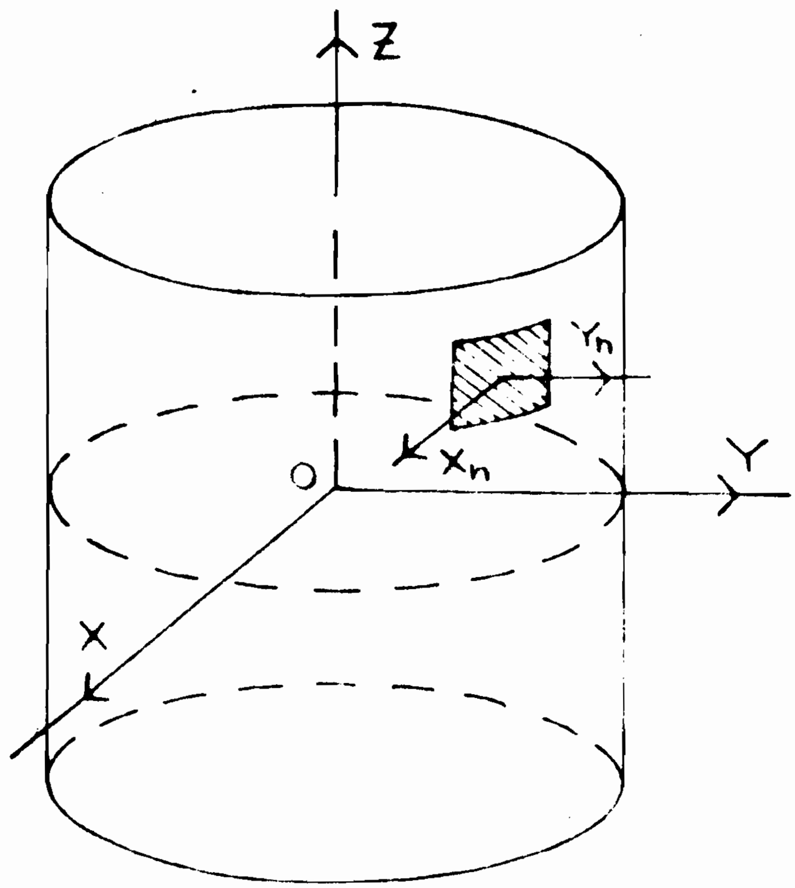
The basic problems in Theory of Elasticity are to determine the displacements and strains within the body from those on the boundary, since it is here that these quantities can be measured. In Plane Elasticity, the problems reduce to boundary-value problems in two-dimensions: hence it is natural to attempt to utilize the powerful results of complex variable theory. The success of this method rests with the Plemelj Formulae and the Cauchy Integral Formula, which together establish necessary and sufficent conditions for a (holomorphic) function to exist throughout a given region which will attain prescribed values at the boundary of the region. These lead to solution expressed as Singular Integral Equations, the solutions of which are given in Muskhelishvili's book for a large number of important cases.

Of the many problems that may be solved in this way, only one of these---the most fundamental---will be given here: In this case, the cross-section of the body under plane strain is simply-connected, i.e., it has the property that every closed curve in the region may be contracted down to a point while remaining all the while within the region. This case has the advantage of illustrating the new method without being unduly complicated. It should be pointed out that the complex-variable method is also highly successful for multi-connected regions, and also for regions which are semi-infinite, these latter being quite important in application. Muskhelishvili's book also treats the important topic of the Theory of Compound Bars (e.g., a steel-reinforced concrete beam) by this method in the later chapters of his book.

It should be pointed out that we will assume that the bodies we deal with are elastically isotropic and homogeneous. By isotropic it is meant that, at any point, the body behaves the same in any direction (it may then be shown that the elastic behavior is completely determined by two real constants, λ, μ, called the Lamé constants), and by homogeneous it is meant that these constants do not vary at different points in the body.

In what follows we shall assume that functions are integrable or differentable whenever we perform such operations, even though we have not specifically mentioned the conditions enabling us to perform these operations.

At the end of this paper an Appendix has been added to facilitate reference to standard equations and the notation used in this paper.



PLANE STRAIN

2. THE PHYSICAL PROBLEMS: Two physical problems considered in the Mathematical theory of Elasticity constitute what is known as PLANE ELASTICITY: these are the case of plane strain and the case of generalized plane strain, and both give rise to equations of the same mathematical formulation.

Plane Strain: A body will be said to be in the state of plane strain, parallel to the xy-plane, if the following conditions hold for the displacements u,v,w (in the direction of the x-, y-, z-axis, respectively) at every point (x, y, z) of the body:

w = o, and

u and v are functions of x, y only and not of z.

In addition, it will be assumed that body is cylindrical, with generators parallel to the z-axis, and with flat ends parallel to the xy-plane. It will further be assumed that the external forces are applied only on the sides of the cylinder, and that these act parallel to the xy-plane. (see figure)

Under these conditions the stress-strain relations become

$$X_x = \lambda\Theta + 2\mu\frac{\partial u}{\partial x}, \quad Y_y = \lambda\Theta + 2\mu\frac{\partial v}{\partial y}, \quad X_y = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{1}$$

$$Z_z = \lambda\Theta, \quad X_z = Y_z = 0,$$

<u>where</u> $$\Theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} ,$$

while the equations of equilibrium become

$$(2) \quad \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + X = 0 \quad , \frac{\partial X_y}{\partial x} + \frac{\partial Y_y}{\partial y} + Y = 0 \, , \quad Z = 0 .$$

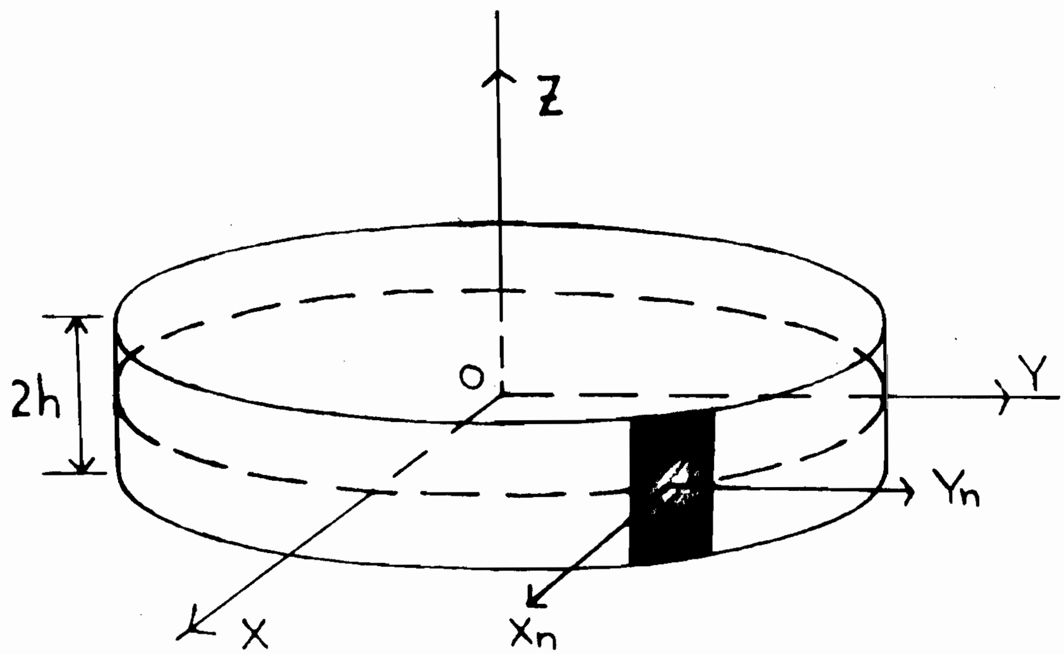
(see Appendix for these equations in the general three- dimensional case).

The equation $Z = 0$ indicates that the body force (which is assumed known in these problems) acts parallel to the xy-plane, while the equations (1) indicate that the stress components X_x, Y_y, X_y are independent of z, since u, v (and hence Θ) are functions of x, y only. Also note Z_z is a quantity dependent on X_x and Y_y, since (1) may be solved to give

$$Z_z = \lambda\Theta = \frac{\lambda}{2(\lambda+\mu)} (X_x + Y_y) .$$

The condition $w = o$ for all points of the body assures one that points in any cross-section (cut parallel to the xy-plane) will remain in this same plane after deformation: this is the basis whereby the problem may be looked upon as one in two-dimensions. To do this, the external forces, which before acted on an element of <u>area</u>, will now be considered as acting on an element of <u>arc</u>, instead.

<u>Generalized Plane Strain</u>: Here the body is understood to be



GENERALIZED

PLANE STRAIN

a cylinder of very small height 2h (i.e., a thin plate), the middle surface of which is taken as the xy-plane. It will be assumed that the flat surfaces of the plate are free from external forces, and that the forces acting on the rim of the cylinder are parallel to the middle surface, and are symmetrically distributed with respect to the middle surface. This same assumption will be made for the body forces.

Under these conditions the points lying in the middle plane will remain in it after deformation, and the displacement component w will be very small for points lying off the middle surface, and therefore may effectively be taken as zero. Also, the variation of the components u and v across the thickness of the plate will be insignificant, and hence the problem may be satisfactorily approximated by replacing u and v by their mean values

$$u^*(x,y) = \frac{1}{2h}\int_{-h}^{+h} u(x,y,z)\,dz \;;\; v^*(x,y) = \frac{1}{2h}\int_{-h}^{+h} v(x,y,z)\,dz \;.$$

It may be shown that one may take $Z_z = 0$ with good approximation in this problem. (see Muskhelishvili, p. 93)

By taking the mean values of the equilibrium equations, (see Appendix) over the plate, we obtain

$$(3) \qquad \frac{\partial X_x^*}{\partial x} + \frac{\partial X_y^*}{\partial y} + X^* = 0 \;;\; \frac{\partial X_y^*}{\partial x} + \frac{\partial Y_y^*}{\partial y} + Y^* = 0 \;,$$

(where the asterisk denotes the mean value) making use of

$$\frac{\partial X_z^*}{\partial z} = \frac{1}{2h}\int_{-h}^{+h} \frac{\partial X_z}{\partial z}\,dz = \frac{X_z}{2h}\Big|_{-h}^{+h} = 0\ ;\ \frac{\partial Y_z^*}{\partial z} = \frac{1}{2h}\int_{-h}^{+h} \frac{\partial Y_z}{\partial z}\,dz = \frac{Y_z}{2h}\Big|_{-h}^{+h} = 0$$

(since at $z = \pm h$ by assumption $X_z = Y_z = 0$).

Furthermore, the stress-strain relations (see Appendix) become, upon solving the third of these for $\frac{\partial w}{\partial z}$ (since $Z_z = 0$) and substituting this expression in the remaining two equations,

$$X_x = \left(\frac{2\lambda\mu}{\lambda+2\mu}\right)\Theta + 2\mu\frac{\partial u}{\partial x}\ ;\ Y_y = \left(\frac{2\lambda\mu}{\lambda+2\mu}\right)\Theta + 2\mu\frac{\partial v}{\partial y}\ .$$

By taking the mean values of these expressions and the relation

$$X_y = Y_x = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

we obtain

$$(4)\quad X_x^* = \lambda^*\Theta^* + 2\mu\frac{\partial u^*}{\partial x}\ ;\ Y_y^* = \lambda^*\Theta^* + 2\mu\frac{\partial v^*}{\partial y}\ ;\ X_y^* = \mu\left(\frac{\partial v^*}{\partial x} + \frac{\partial u^*}{\partial y}\right)$$

where

$$\lambda^* = \frac{2\lambda\mu}{\lambda+2\mu}\ ,\quad \Theta^* = \frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y}\ .$$

3. THE BASIC EQUATIONS, AND REDUCTION TO THE CASE OF ZERO BODY FORCES: As we have seen above, both physical problems, plane strain and generalized plane strain, are governed by equations of the same form, i.e.,

$$(5)\quad \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + X = 0\ , \quad \frac{\partial X_y}{\partial x} + \frac{\partial Y_y}{\partial y} + Y = 0$$

$$(6)\quad X_x = \lambda\Theta + 2\mu\frac{\partial u}{\partial x}\ , \quad Y_y = \lambda\Theta + 2\mu\frac{\partial v}{\partial y}\ , \quad X_y = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

where it is understood that for the case of generalized plane strain all quantities are to be replaced by their mean values over the thickness of the plate, and λ is to be replaced by

$$\lambda^* = \frac{2\lambda\mu}{\lambda + 2\mu}\ .$$

All the quantities X_x, Y_y, X_y, u, v in these equations are functions of x, y only: that is, they are defined in the cross-section of the body cut by the xy-plane, which has been assumed to be simply-connected. Hereafter we shall designate the interior of this cross-section as the region S, bounded by the closed curve L. (The positive direction of L will be taken so that S lies on the left of L).

It is known from the general theory of Elasticity the solution X_x, Y_y, X_y, u, v of the equation (5) and (6) is unique for prescribed values of either X_x, Y_y , X_y or u, v on the boundary (except for arbitrary displacements expressing rigid-body translations or rotations).

It may also be shown that the equations (5) along with

$$(7) \quad \Delta\left(X_x + Y_y\right) = \frac{-2(\lambda+\mu)}{\lambda+2\mu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right),$$

where the operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, (For derivation, see Muskhelishvili, p. 96-97) serve to completely determine the stresses X_x, Y_y, X_y from their known values at the boundary: the displacements may then be found from the relations (6).

The solution of the equations is considerably simplified in the absence of body forces, i.e., when $X = Y = 0$. It is possible, however, to reduce the general problem to this case: for suppose __any__ particular solution $X_x^{(0)}$, $Y_y^{(0)}$, $X_y^{(0)}$, $u^{(0)}$, $v^{(0)}$ has been found; put

$$X_x = X_x^{(1)} + X_x^{(0)}, \qquad \text{etc.,}$$

then $X_x^{(1)}$, etc., will satisfy the same equations, but with $X = Y = 0$. As an example of such a particular solution, consider the case of body forces due to gravity $X = 0$, $Y = -gp$, where g is the gravitational acceleration and p is the density.

Equations (5) and (7) become

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} = 0, \quad \frac{\partial X_y}{\partial x} + \frac{\partial Y_y}{\partial y} = g\rho, \quad \Delta\left(X_x + Y_y\right) = 0,$$

which have the particular solution

$$X_x = X_y = 0 \ , \quad Y_y = g\rho y \ .$$

(A particular solution of (5) and (7) for the case of centrifugal body forces is given by Muskhelishvili on p. 101).

Hereafter, we will consider that the problem has been reduced to one with zero body forces: i.e., the equations (5) and (7), are now

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} = 0 \quad , \quad \frac{\partial X_y}{\partial x} + \frac{\partial Y_y}{\partial y} = 0$$

and

$$\Delta \left(X_x + Y_y \right) = 0 \ .$$

4. THE AIRY STRESS FUNCTION: In the absence of body forces, equations (5) become

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} = 0 \quad , \quad \frac{\partial X_y}{\partial x} + \frac{\partial Y_y}{\partial y} = 0$$

which are the necessary and sufficient conditions that the expressions

$$X_y dx - X_x dy \quad , \quad Y_y dx - X_y dy$$

are the exact differentials of functions $-B(x, y)$ and $A(x, y)$ with

$$\frac{\partial B}{\partial x} = -X_y \quad , \quad \frac{\partial B}{\partial y} = X_x \quad ;$$

$$\frac{\partial A}{\partial x} = Y_y \quad , \quad \frac{\partial A}{\partial y} = -X_y \quad .$$

The first and last of these yield the further relation

$$\frac{\partial B}{\partial x} = \frac{\partial A}{\partial y}$$

which is again the necessary and sufficient condition that $Adx + Bdy$ is the exact differential of a function $U(x, y)$ with

$$\frac{\partial U}{\partial x} = A \quad , \quad \frac{\partial U}{\partial y} = B .$$

The function $U(x, y)$ is the Airy Stress Function, and it is easily deduced from the above that the stress are related to $U(x, y)$ by

$$(8) \quad \frac{\partial^2 U}{\partial x^2} = Y_y \quad , \quad \frac{\partial^2 U}{\partial y^2} = X_x \quad , \quad \frac{\partial^2 U}{\partial x \partial y} = -X_y \quad .$$

Furthermore, in the absence of body forces equation (7) becomes

$$\Delta\left(X_x + Y_y\right) = 0 \quad .$$

Noting that $X_x + Y_y = \Delta U$ from the proceeding equations, we have that

$$\Delta\Delta U = 0 \quad ,$$

or that

$$\frac{\partial^4 U}{\partial x^4} + 2\frac{\partial^4 U}{\partial x^2 \partial y^2} + \frac{\partial^4 U}{\partial y^4} = 0$$

which is called the Biharmonic Equation.

If, in a given problem, the stress function $U(x, y)$ can be found, then the problem is completely solved since both the stresses and displacements at every point of the region can be given in terms of U. We assume that $U(x, y)$ has continuous partial derivatives up to and including the fourth order and that, in order to allow for the possibility of rigid translations and rotations, the derivatives are single valued starting from the second order.

5. THE DISPLACEMENTS IN TERMS OF THE STRESS FUNCTION:

We have seen that the stresses X_x, Y_y, X_y may be determined from the Airy Stress Function by the relations (8). We now seek to also express the displacements u, v in terms of U(x, y), and in so doing shall unearth a relationship which leads to the representation of U(x, y) by complex functions.

The equations (6) may be written as

$$\frac{\partial^2 U}{\partial y^2} = (\lambda+2\mu)\frac{\partial u}{\partial x} + \lambda\frac{\partial v}{\partial y} \quad , \quad \frac{\partial^2 U}{\partial x^2} = \lambda\frac{\partial u}{\partial x} + (\lambda+2\mu)\frac{\partial v}{\partial y}$$

(since $\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$).

By substituting into the first of these the expression obtained by solving for $\frac{\partial u}{\partial x}$ in the second, one obtains

$$(9) \quad 2\mu\frac{\partial v}{\partial y} = -\frac{\partial^2 U}{\partial y^2} + \frac{\lambda+2\mu}{2(\lambda+\mu)}\Delta U \quad .$$

Similarly,

$$(10) \quad 2\mu\frac{\partial u}{\partial x} = -\frac{\partial^2 U}{\partial x^2} + \frac{\lambda+2\mu}{2(\lambda+\mu)}\Delta U \quad .$$

Since $\Delta\Delta U = 0$, $P(x, y) = \Delta U(x, y)$ is a harmonic function. Let $Q(x, y)$ be the conjugate harmonic of $P(x, y)$ then Q is uniquely determined from P except for an arbitrary constant. Then the function defined by

$$f(z) = P(x, y) + i\,Q(x, y)$$

where $z = x + iy$

is holomorphic (i.e., analytic and single-valued) in the region S, since we have assumed S to be simply-connected.

Next, define $\varphi(z)$ by

$$\varphi(z) = p + iq = \frac{1}{4}\int_a^z f(\zeta)\,d\zeta \tag{11}$$

for arbitrary a and any $z = x + iy$ in S. (The $\frac{1}{4}$ has been introduced for convenience.) Note that if the lower limit of intregration is chosen as 0 (0 is assumed to lie in the region), then $\varphi(o) = o$. Since $f(z)$ is holomorphic in S, the integration indicated above is independent of the path, provided this path lies wholly in S. Furthermore, it may be shown that $\varphi(z)$ is also holomorphic in S, and that

$$\varphi'(z) = \frac{\partial p}{\partial x} + i\frac{\partial q}{\partial x} = \frac{1}{4}(P + iQ)$$

which by the aid of the Cauchy-Riemann equations give

$$P = 4\frac{\partial p}{\partial x} = 4\frac{\partial q}{\partial y} \quad . \tag{12a}$$

These last relations, when substituted in (9) and (10) yield

$$2\mu\frac{\partial v}{\partial y} = -\frac{\partial^2 U}{\partial y^2} + \frac{2(\lambda+2\mu)}{\lambda+\mu}\frac{\partial q}{\partial y}$$

and

$$2\mu\frac{\partial u}{\partial x} = -\frac{\partial^2 U}{\partial x^2} + \frac{2(\lambda+2\mu)}{\lambda+\mu}\frac{\partial p}{\partial x}$$

which may now be integrated with respect to y and x respectively, giving

$$2\mu v = -\frac{\partial U}{\partial y} + \frac{2(\lambda+2\mu)}{\lambda+\mu}q + f_2(x)$$

$$2\mu u = -\frac{\partial U}{\partial x} + \frac{2(\lambda+2\mu)}{\lambda+\mu}p + f_1(y),$$

By the last of the relations (6), i.e.,

$$X_y = \frac{-\partial^2 U}{\partial x \partial y} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

and the Cauchy-Riemann relation $\frac{\partial p}{\partial y} + \frac{\partial q}{\partial x} = 0$, one may determine that $\frac{\partial}{\partial x}f_2(x) + \frac{\partial}{\partial y}f_1(y) = 0$ and hence that $\frac{\partial}{\partial x}f_2(x) = -\frac{\partial}{\partial y}f_1(y)$ = constant = a. From this we have

$$f_2(x) = ax + b, \quad f_1(y) = -ay + c$$

(where a, b, c are arbitrary constants): since these represent rigid-body displacement and as such do not affect the stress distribution in the body, they may be omitted. Thus we have obtained

$$\text{(12b)} \quad 2\mu v = -\frac{\partial U}{\partial y} + \frac{2(\lambda+2\mu)}{\lambda+\mu}q, \quad 2\mu u = -\frac{\partial U}{\partial x} + \frac{2(\lambda+2\mu)}{\lambda+\mu}p,$$

which are the desired relationships giving the displacements u, v in terms of the stress function U.

To summarize, then, if the Airy Stress Function $U(x, y)$ is known, then the stress components may be determined from the relation (8), and the displacement components found from the relation (12b), where p and q are determined from U by the definition (11).

6. COMPLEX REPRESENTATION OF THE AIRY STRESS FUNCTION:

The representation of $U(x, y)$ in terms of two complex functions, holomorphic in S, may now be readily obtained. In fact, we have already found one of these complex functions: it is $\varphi(z)$, defined by (11).

To determine the second complex function, note that since from (12a) $\Delta(px + qy) = P$ (remembering that $\varphi(z) = p + iq$), it follows that

$$U - (px + qy) = p_1$$

is a harmonic function. Let q_1 be the conjugate harmonic determined from p_1. Then

$$\chi(z) = p_1 + i q_1$$

will be holomorphic in S.

But now observe that $U(x, y)$ is given by

$$U(x, y) = \mathcal{R}[\bar{z}\varphi(z) + \chi(z)]$$

(where $z = x + iy$ and $\mathcal{R}$ designates the "real part" of the complex expression). This is the desired representation of $U(x, y)$ in terms of two complex functions, which may also be written

$$2U = \bar{z}\varphi(z) + z\overline{\varphi(z)} + \chi(z) + \overline{\chi(z)} \tag{13}$$

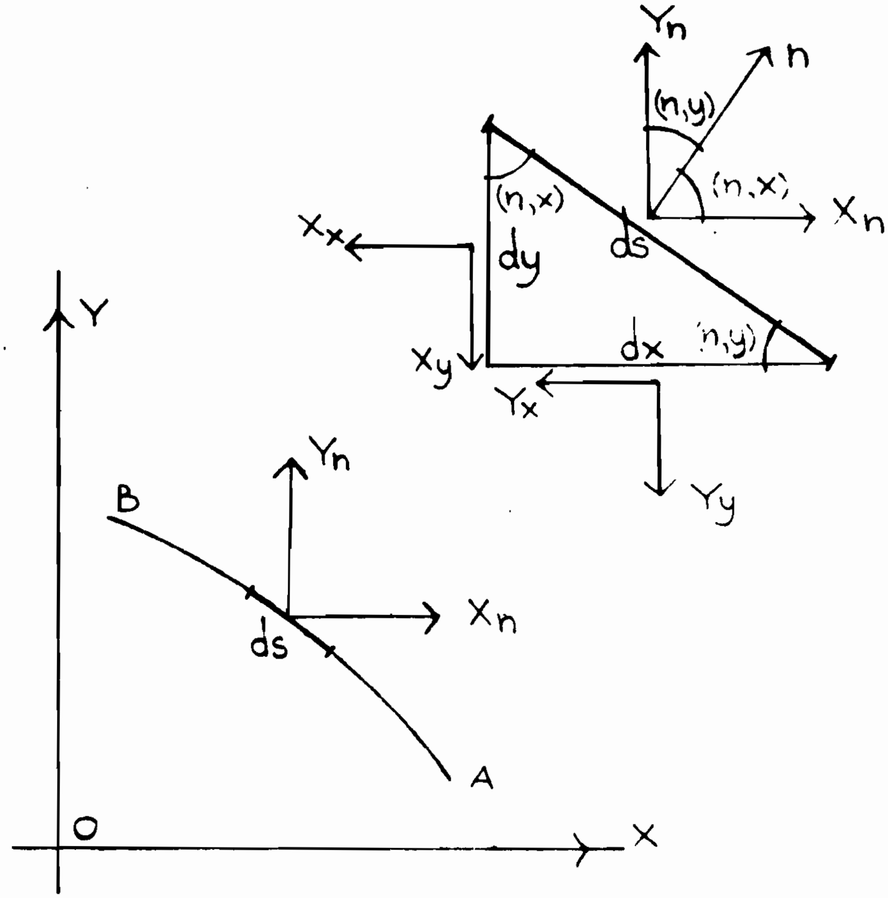
(because $2\mathcal{R}(z) = \bar{z} + z$, etc.,) The "bar" designates the "complex conjugate" value, i.e., if $\gamma = a + ib$, then $\bar{\gamma} = a - ib$.

As a consequence of this representation, it follows that $U(x, y)$ is also holomorphic in S: that is, $U(x, y)$ has derivatives of all orders in S (since $\varphi(z)$ and $\chi(z)$ have this property).

As would be imagined, it is possible to give relations for the stresses X_x, Y_y, X_y and also the displacements u, v in terms of the complex functions $\varphi(z)$ and $\chi(z)$: This will be done in the next sections. These representations will then become the basis by which we will approach the fundamental boundary value problems. Since we will be using the first derivative of U, it will be convenient to introduce

$$\psi(z) = \frac{d\chi}{dz} .$$

In the sequel, $\varphi(z)$ and $\psi(z)$ will not only be understood to be holomorphic in S, but also continuous on $S + L$ (L being the boundary of S). This last condition is a natural one, since the stresses and displacements are continuous on $S + L$. (Otherwise the body would be ruptured.)



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7. REPRESENTATION OF THE STRESS IN TERMS OF $\varphi(z)$ AND $\psi(z)$, AND THE FIRST FUNDAMENTAL PROBLEM. Consider any arc AB lying in the region S. (The direction given by a point moving along AB from A to B will be taken as positive.) Let ds designate a small element of this arc, and let X_n and Y_n designate the components of the stress acting on ds: then X_n and Y_n are related to components X_x, Y_y, X_y by

$$X_n = X_x \cos(x,n) + X_y \cos(y,n)$$

$$Y_n = X_y \cos(x,n) + Y_y \cos(y,n)$$

(keeping in mind that $X_y = Y_x$: see Appendix) where n is the normal to ds (taken to the right of the tangent pointing in the positive direction of AB), and (x, n) and (y, n) denote the angles this normal makes with the positive x-and y-axis, respectively.

But $\cos(x, n) = \frac{dy}{ds}$, and $\cos(y, n) = \frac{-dx}{ds}$, while also

$$X_x = \frac{\partial^2 U}{\partial y^2}, \quad Y_y = \frac{\partial^2 U}{\partial x^2}, \quad -X_y = \frac{\partial^2 U}{\partial x \partial y}$$

from (8), so that the above relations become

$$X_n = \frac{\partial^2 U}{\partial y^2}\frac{dy}{ds} + \frac{\partial^2 U}{\partial x \partial y}\frac{dx}{ds} = \frac{d}{ds}\left(\frac{\partial U}{\partial y}\right)$$

$$Y_n = \frac{-\partial^2 U}{\partial x \partial y}\frac{dy}{ds} - \frac{\partial^2 U}{\partial x^2}\frac{dx}{ds} = \frac{-d}{ds}\left(\frac{\partial U}{\partial x}\right)$$

These may be combined in complex form, giving

$$\text{(14a)}\quad X_n + iY_n = \frac{d}{ds}\left(\frac{\partial U}{\partial y} - i\frac{\partial U}{\partial x}\right) = -i\frac{d}{ds}\left(\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right)$$

where it is easily deduced from (13) that

$$\text{(14b)}\quad \frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} \quad .$$

The above expressions allow us to formulate the

FIRST FUNDAMENTAL BOUNDARY-VALUE PROBLEM: Given the values of the stress components X_n, Y_n acting along the boundary L of the region S, determine the components of stress X_x, Y_y, X_y and the components of displacement u, v at all points *within* the region.

This problem may be formulated, using (14a) and (14b), in the following way: firstly, since the stresses must be continuous throughout the region and also at its boundary, the boundary L may be taken for the arc AB used in deriving the relations (14a) and (14b) above. Along L, the values X_n, Y_n are assumed known, and we shall suppose that they are given as functions of the arc-lengths. For convenience, denote the expression $\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}$ as

$$\text{(15a)}\quad f = f_1 + if_2 = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}$$

along the boundary L. From (14a) and (14b), (15a) can be written

$$(15b) \quad f = f_1 + i f_2 = i \int_0^s (X_n + i Y_n)\, ds + \text{const.}$$

Since X_n and Y_n are known along L, f may be determined, and then (15a) enables us to solve for the functions $\varphi(z)$ and $\psi(z)$, by employing Cauchy-type integrals, which will be done in Section 11. Of course, once $\varphi(z)$ and $\psi(z)$ have been found, the problem is solved since the stress function U may now be determined, and the stresses and displacements are given in terms of U.

For the problem in which the boundary displacements are known (the Second Fundamental Problem) the analysis leads to an equation of the type (15a) again. This will be discussed in detail in the next section.

8. REPRESENTATION OF THE DISPLACEMENTS IN TERMS OF $\varphi(z)$ AND $\psi(z)$, AND THE SECOND FUNDAMENTAL PROBLEM: Here our purpose is to formulate the

SECOND FUNDAMENTAL BOUNDARY-VALUE PROBLEM: Given the values of the displacements g_1 g_2 along the boundary L of the region S, determine the components of stress X_x, Y_y, X_y and the components of displacement u, v at all points within the region.

This formulation is readily obtained, since the equations (12b) may be written in complex form to give

$$2\mu(u+iv) = -\left(\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right) + \frac{2(\lambda+2\mu)}{\lambda+\mu}(p+iq)$$

which becomes, utilizing the relations (11), and (14b)

$$2\mu(u+iv) = k\varphi(z) + z\overline{\varphi'(z)} - \overline{\psi(z)}$$

where

$$k = \frac{\lambda+3\mu}{\lambda+\mu} \quad .$$

Along the boundary L, the left-hand side of this relation assumes known values, i.e., $u + iv = g_1 + ig_2$ where g_1 and g_2 are given displacements. We now have, along L,

$$(16) \quad g = 2\mu(g_1+ig_2) = k\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}$$

from which the functions $\varphi(z)$ and $\psi(z)$ throughout the region may be found by the method we shall see in Section 11.

Note that the formulation (16) of the Second Fundamental Problem is almost entirely analogous to the formulation (15) of the First Fundamental Problem, since κ is a constant. For this reason we will restrict our attention to the solution of the First Fundamental Problem, since the Second is solved in an identical manner.

It should be pointed out that there also exists the Mixed Fundamental Boundary-Value Problem, in which the stresses X_n, Y_n are known on only a portion of the boundary L, while the displacements g_1, g_2 are known on the remaining portion of L. This problem will not be considered here.

As would be expected, there exists a certain amount of arbitrariness in the determination of the functions $\varphi(z)$ and $\psi(z)$: in fact, it may be shown that the state of stress within the body remains unaltered when $\varphi(z)$ and $\psi(z)$ are replace by

$$\varphi(z) + iCz + \gamma$$

$$\psi(z) + \gamma'$$

where C is a real constant and γ, γ' are complex constants. This is the extent of the arbitrariness for a given state of stress. To remove this arbitrariness, in the case of the

First Fundamental Problem, one may impose that

$$\varphi(0) = 0 \quad ; \quad \mathfrak{I}\left(\varphi'(0)\right) = 0$$

(where $\mathfrak{I}$ designates the "imaginary part"): in this way the constants γ and C will be fixed values. The constant will be determined by the choice of the arbitrary constant appearing in the expression (15b).

In the case when the displacements are given, as in the Second Fundamental Problem, it may be shown that the extent of the arbitrariness is reduced in that

$$C = 0 \quad , \quad K\gamma - \gamma' = 0$$

so that only one of the constants γ , γ' may be chosen at will. Thus, in this case, the arbitrariness may be removed by setting

$$\varphi(0) = 0 \ .$$

(These points are considered in detail in Muskhelishvili in Section 34 and also on page 146.)

9. RESULTS FROM COMPLEX VARIABLE THEORY: Several results from the theory of functions of a complex variable will be of use to us, so we pause now to recollect these. Since the problems we are considering in this paper are only those cases in which the region is bounded by a simple closed contour, we will list here only theorems which are pertainent to such regions (see Muskhelishvili, Chapters 12 and 13 for similar theorems for multiconnected regions and the half-plane).

Let L be a simple closed contour: then L divides the plane into three parts: the finite region S^+, enclosed by L; the line L (the positive direction of L is taken so that S^+ lies to the left of L); and the region S^-, which is understood to include the point at infinity.

1°. Let $f(z)$ be a function, holomorphic in S^+ and continuous in $S^+ + L$. Then

$$\frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-z} = f(z) \quad \text{for } z \text{ in } S^+$$

$$\frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-z} = 0 \quad \text{for } z \text{ in } S^-,$$

The first of these is Cauchy's Integral Formula, while the second follows from Cauchy's theorem.

2°: Let $f(z)$ be a function, holomorphic in S^- (including the point at infinity) and continuous in $S^- + L$. (It will be remembered that this means for sufficiently large $|z|$

$$f(z) = c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots$$

so that $f(\infty) = c_0$.) Then

$$\frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-z} = -f(z) + f(\infty) \quad \text{for } z \text{ in } S^-$$

$$\frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-z} = f(\infty) \quad \text{for } z \text{ in } S^+.$$

On the other hand, if $f(t)$ is a continuous function defined on L, then the function defined by

$$F(z) = \frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-z}$$

is <u>holomorphic</u> for all z not on L. It is important to know the limits of $F(z)$ as z approaches a point t_0 of L: these are given by the

<u>Plemelj Formulae</u>: Denoting by $F^+(t_0)$ and $F^-(t_0)$ the limit-values of the function $F(z)$ defined above as $z \to t_0$ from S^+ and S^- respectively, then

$$F^+(t_0) = \frac{1}{2} f(t_0) + \frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-t_0}$$

$$F^+(t_0) = -\frac{1}{2} f(t_0) + \frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-t_0},$$

where the integrals appearing are to be taken as <u>Cauchy Principal Values</u>, i.e.,

$$\frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-t_0} = \lim_{r\to 0}\int_{L-l}\frac{f(t)\,dt}{t-t_0}$$

where r is the radius of a small circle with center t_0 and $L-l$ represents the parts of L lying outside this circle. These integrals may be shown to exist provided $f(t)$ satisfies a Holder condition on L in a neighbourhood of t_0; that is, for all t_1, t_2 lying on L in some neighbourhood of t_0, the following condition holds

$$|f(t_1) - f(t_2)| \leqslant A|t_1 - t_2|^{\alpha}$$

where A, α are real constants and $0 < \alpha \leqslant 1$.

The results above combine to give the very important criteria:

I. The necessary and sufficient condition for a continuous function $f(t)$, given on L, to be the boundary value of some function, holomorphic in S^+, is that

$$\frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-z} = 0 \qquad \text{for all } z \text{ in } S^-.$$

II. The necessary and sufficient condition for a continuous function $f(t)$, given on L, to be the boundary value of some function, holomorphic in S^- (including the point at infinity), is that

$$\frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-z} = a$$

for all z in S^+,

where **a** is some constant (which is equal to the value of the above mentioned holomorphic function at infinity).

These criteria take on a very convenient form when L is the circle of unit radius with center at the origin. Let γ designate this circle, and designate the points of γ by σ, so that

$$\sigma = e^{i\theta}, \quad 0 \leq \theta < 2\pi .$$

Denote by Σ^+ and Σ^- the interior and the exterior of the circle, and choose the positive direction on γ so that Σ^+ lies to the left of γ. Let $F(z)$ be a function defined on Σ^+ (or Σ^-), and define $\bar{F}(1/z)$ by the relation

$$\bar{F}\left(\frac{1}{z}\right) = \overline{F\left(\frac{1}{\bar{z}}\right)} .$$

(The bar denoting the conjugate complex value.) Then, if $F(z)$ is holomorphic in Σ^+ (or Σ^-), then $\bar{F}(1/z)$ will be holomorphic in Σ^- (or Σ^+). Furthermore, if $F(z)$ has the boundary-value $f(\sigma)$ on γ, then $\bar{F}(1/z)$ will take the boundary-value $\overline{f(\sigma)}$, since as $z \to \sigma$, $1/\bar{z}$ also tends to σ.

The criteria may now be rewritten as

I': A necessary and sufficient condition for the function $f(\sigma)$, continuous on the circle γ, to be the boundary-value of some function, holomorphic inside γ, is

$$\frac{1}{2\pi i}\int_{\gamma}\frac{\overline{f(\sigma)}\,d\sigma}{\sigma - z} = \bar{a} \qquad \text{for all } z \text{ inside } \gamma,$$

where $\bar{a}$ is a constant (which is equal to the value of the above mentioned function at $z = 0$).

II': A necessary and sufficient condition for the function $f(\sigma)$, continuous on the circle γ, to be the boundary-value of some function, holomorphic outside γ, is

$$\frac{1}{2\pi i}\int_{\gamma}\frac{\overline{f(\sigma)}\,d\sigma}{\sigma - z} = 0 \qquad \text{for all } z \text{ outside } \gamma,$$

Also, the following "principal parts" formulae are of use in many problems. It should be recalled that if a function $f(z)$ can be expanded, in the neighbourhood of a point $\underline{a}$, in the form

$$f(z) = G(z) + f_0(z)$$

where $f_0(z)$ is holomorphic at $\underline{a}$, and $G(z)$ has the form

$$G(z) = \frac{A_1}{z-a} + \frac{A_2}{(z-a)^2} + \cdots + \frac{A_l}{(z-a)^l}$$

(A_1, A_2, ..., A_l being constants), it is said that $f(z)$ has a pole of order l at $\underline{a}$ with the principal part $G(z)$. If the expansion is at the point at infinity, the principal part must be taken as

$$G(z) = A_0 + A_1 z + \cdots + A_l z^l \qquad ,$$

where the constant term A_0 has been included.

3^o: Let $f(z)$ be holomorphic in S^+ and continuous in $S^+ + L$ with the possible exception of the points a_1, a_2, ..., a_n of S^+, where it may have poles with the principal parts $G_1(z)$, $G_2(z)$, ... , $G_n(z)$. Then

$$\frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-z} = f(z) - G_1(z) - G_2(z) - \cdots - G_n(z)$$

for z in S^+

and

$$\frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-z} = -G_1(z) - G_2(z) - \cdots - G_n(z)$$

for z in S^-.

4°: Let $f(z)$ be holomorphic in S^{-} and continuous in $S^{-} + L$ with the possible exclusion of the finite points $a_1, a_2, \ldots, a_n$ of S^{-} and also the point $z = \infty$, where it may have poles with the principal parts $G_1(z), \ldots, G_n(z)$. $G_{\infty}(z)$. Then

$$\frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-z} = -f(z) + G_1(z) + \cdots + G_M(z) + G_{\infty}(z)$$

for z in S^{-}

and

$$\frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-z} = G_1(z_1) + \cdots + G_M(z) + G_{\infty}(z)$$

for z in S^{+}.

These formulae have been given without proof: all the results listed in this section are fully explained in Chapters 12 and 13 of Muskhelishvili's book, as are the notions of conformal mapping considered next.

10. **CONFORMAL** MAPPING WITH RESPECT TO THE UNIT CIRCLE: Because of the exceptional form of the Criteria I' and II', in many problems it is advantageous to transform the fundamental problems, by a change of variable, into ones defined on the unit circle. This is accomplished by replacing z in the formulations (15) and (16) by

$$z = \omega(\zeta)$$

where the above represents a conformal mapping of the unit circle onto the region S where S is the infinite or finite part of the plane, bounded by a single contour. (By conformal it is meant that $\omega(\zeta)$ is a holomorphic function of ζ, with $\omega'(\zeta) \neq 0$ for any ζ lying inside of the unit circle γ. Under these conditions it is known that the inverse mapping $\zeta = \omega^{-1}(z)$ is single-valued, and that the mapping is angle-preserving.)

To restrict our attention to the First Fundamental Problem----the Second would be treated analogously---the formulation of the boundary condition for the region S has been seen to be (15)

$$f(z) = \varphi(z) + z\,\overline{\varphi'(z)} + \overline{\psi(z)}$$

where $f(z)$ is known for z lying on the boundary L and is given by

$$f(z) = i\int_0^s (X_M + iY_M)\, ds \quad + \text{const.}$$

along the boundary L.

Then, substituting $z = \omega(\zeta)$, we obtain

$$(17) \quad f_1(\zeta) = \varphi_1(\zeta) + \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}} \overline{\varphi_1'(\zeta)} + \overline{\psi_1(\zeta)}$$

where we denote

$$f_1(\zeta) = f(\omega(\zeta)),\ \varphi_1(\zeta) = \varphi(\omega(\zeta)),\ \psi_1(\zeta) = \psi(\omega(\zeta))$$

which represents the same problem, except now defined on the unit circle $|\zeta| \leq 1$ instead of on S, and with f_1, φ_1, ψ_1 holomorphic on $|\zeta| < 1$ and continuous on $|\zeta| \leq 1$. As proposed earlier, we may set

$$\varphi_1(0) = 0\ ;\ \mathfrak{J}(\varphi_1'(0)) = \mathfrak{J}\frac{\varphi'(0)}{\omega'(0)} = 0$$

and fix the constant in the expression for f at the top of the page in some definite manner, to eliminate an arbitrariness in the functions φ_1 and ψ_1. Because of the form of (17) no confusion will result if we drop the subscripts and write simply

$$(18) \quad f(\zeta) = \varphi(\zeta) + \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}} \overline{\varphi'(z)} + \overline{\psi(z)}\ .$$

In a similar manner, the formulation (16) of the Second Fundamental Boundary-Value Problem takes the form

$$(19)\quad g = 2\mu(g_1 + ig_2) = k\,\varphi(z) - \frac{\omega(z)}{\overline{\omega'(z)}}\,\overline{\varphi'(z)} - \overline{\psi(z)}$$

defined on the unit circle. To avoid arbitrariness of the functions φ and ψ in this case, we set

$$\varphi(0) = 0 \ .$$

It is useful also to have

$$\omega(0) = 0 \ ;$$

this is always possible, since if $\omega^{-1}(\alpha) = 0$, where $\alpha \neq 0$, then the coordinates imposed on the region S could be translated so that the point α becomes the new origin, i.e., the mapping defined by

$$\omega^{*^{-1}}(z-\alpha) = \omega^{-1}(z)$$

maps S onto the unit circle, and has $\omega^*(0) = 0$.

11. GENERAL SOLUTIONS OF THE FUNDAMENTAL PROBLEMS: We are now in a position to show how the solutions of the fundamental boundary-value problems may be obtained from the boundary conditions (18) and (19). These solutions will be expressed in terms of singular integral equations for each of the functions $\varphi(\zeta)$ and $\psi(\zeta)$, i.e., equations involving Cauchy-type integrals.

Continuing to work with the First Fundamental Problem (the Second Fundamental Problem would be treated analogously), the boundary condition takes the form (18), i.e.,

$$(20)\qquad \varphi(\sigma) + \frac{\omega(\sigma)}{\omega'(\sigma)}\overline{\varphi'(\sigma)} + \overline{\psi(\sigma)} = f_1 + i f_2 = f$$

where $\sigma = e^{i\theta}$ is an arbitrary point of γ, and $\varphi(\sigma)$, $\varphi'(\sigma)$, $\psi(\sigma)$ must be interpreted as boundary values for $\zeta \to \sigma$ from inside γ. This condition may also be written in conjugate complex form

$$(21)\qquad \overline{\varphi(\sigma)} + \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\varphi'(\sigma) + \psi(\sigma) = f_1 + i f_2 = \overline{f}\ .$$

The quantity $f = f_1 + if_2$ is defined on L by

$$f = f_1 + i f_2 = i\int_0^s (X_n + iY_n)\,ds + \text{const.}$$

where s is the arc length of L and the constant may be fixed arbitrarily. This expression may be written as a function of σ (or of θ) since s may be given as a function of σ.

Note that once the function $\varphi(\zeta)$ is known inside and on γ, $\psi(\zeta)$ may be determined quite readily: in fact, the values of $\psi(\sigma)$ along the boundary γ are given by

$$\psi(\sigma) = \bar{f} - \overline{\varphi(\sigma)} - \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\,\varphi'(\sigma)$$

(obtained by solving (21) for $\psi(\sigma)$) and hence $\psi(\zeta)$ is given inside the circle by Cauchy's Integral Formula

$$\psi(\zeta) = \frac{1}{2\pi i}\int_{\gamma}\frac{\psi(\sigma)\,d\sigma}{\sigma - \zeta} \tag{22}$$

Thus the problem reduces to that of determining $\varphi(\zeta)$ from the boundary condition (20).

Writing (20) as

$$\overline{\psi(\sigma)} = f - \varphi(\sigma) - \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\overline{\varphi'(\sigma)}$$

and denoting the right-hand side as $\overline{F(\sigma)}$, it is seen that $F(\sigma)$ must be the boundary value of some function $\psi(\zeta)$, holomorphic inside γ. By the Criteria I' (page 29), the necessary and sufficient condition for this to be so is

$$\frac{1}{2\pi i}\int_\gamma \frac{\overline{F(\sigma)}\,d\sigma}{\sigma - \zeta} = \bar{a} \qquad \text{FOR ALL } \zeta \text{ INSIDE } S,$$

where $\bar{a}$ is a constant. (Here $a = \psi(0)$.)

This gives (for ζ inside γ)

$$\bar{a} = \frac{1}{2\pi i}\int_\gamma \frac{f\,d\sigma}{\sigma-\zeta} - \phi(\zeta) - \frac{1}{2\pi i}\int_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\frac{\overline{\phi'(\sigma)}\,d\sigma}{\sigma-\zeta}$$

or

$$\phi(\zeta) + \frac{1}{2\pi i}\int_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\frac{\overline{\phi'(\sigma)}\,d\sigma}{\sigma-\zeta} + \bar{a} = \frac{1}{2\pi i}\int_\gamma \frac{f\,d\sigma}{\sigma-\zeta} \tag{23}$$

where we have used

$$\phi(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{\phi(\sigma)\,d\sigma}{\sigma-\zeta}$$

since $\phi(\zeta)$ must be holomorphic inside γ. (23) is the promised solution for the function $\phi(\zeta)$, expressed in terms of a singular integral equation. It must be understood that at the moment the constant a is as yet unknown: however, if a solution $\phi^*(\zeta)$ may be found for the equation (23) using an arbitrary value a^* for a, where $\phi^*(0) = a_0$, then

$$\phi(\zeta) = \phi^*(\zeta) - a_0$$

is the desired solution of (23), since $\phi(0) = 0$, and the correct value for $\bar{a}$ is given by

$$\bar{a} = \bar{a}^* - a_0 \quad ,$$

After $\varphi(\zeta)$ has been determined from the singular integral equation (23), the function $\psi(\zeta)$ is determined from the singular integral equation

$$(24) \quad \psi(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{\bar{f}\,d\sigma}{\sigma-\zeta} - \frac{1}{2\pi i}\int_\gamma \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\frac{\varphi'(\sigma)}{\sigma-\zeta}\,d\sigma$$

which is obtained from the relation (22) by substituting the relation for $\psi(\sigma)$ given by (21). In this manipulation, use is made of

$$\frac{1}{2\pi i}\int_\gamma \frac{\overline{\varphi(\sigma)}}{\sigma-\zeta}\,d\sigma = \overline{\varphi(0)} = 0$$

by virtue of Criteria I' (page 29). Thus both $\varphi(\zeta)$ and $\psi(\zeta)$ may be expressed as solutions to certain integral equations.

12. EXAMPLE: THE SOLUTION OF THE FIRST FUNDAMENTAL PROBLEM FOR THE INFINITE PLANE WITH AN ELLEPTICAL HOLE. To illustrate the method of solution proposed above, we may consider the case of an infinite plate with an elliptical hole, subjected to plane strains. In actual application, of course, such a region does not exist: however, it is approximated when the size of the plate is large compared with that of the hole. In this problem, the applied forces, acting on the rim of the elleptical hole, shall be assumed to have zero resultant. Under the additional assumptions that the stresses vanish at the point at infinity and that there is no rotation at infinity, the functions $\varphi(\zeta)$ and $\psi(\zeta)$ are holomorphic throughout the infinite region, including the point at infinity (see Muskhelishvili, Section 34 and also page 146). To avoid arbitrariness, we may set

$$\varphi(\infty) = 0 , \quad \mathfrak{I}\left[\varphi'(\infty)\right] = 0 .$$

Choosing the origin to coincide with the center of the ellipse, so that the elliptical boundary is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

we may map the finite region of the unit disk $|\zeta| \leq 1$ onto the infinite plane with the elliptical hole by the mapping

$$z = \omega(\zeta) = R\left(\frac{1}{\zeta} + m\zeta\right)$$

(where R and m are real constants, with $R > 0$, $0 \leq m < 1$, determined so that $a = R(1 + m)$, $b = R(1 - m)$), which is conformal because this mapping is holomorphic and $\omega'(\zeta)\neq 0$ for all points other than $\zeta = \frac{\pm 1}{\sqrt{m}}$, which lie outside the unit circle.

Using the notation $\varphi_1(z) = \varphi(\zeta)$, etc., we have that

$$\varphi_1(0) = \varphi(\infty) = 0\,, \quad \mathcal{J}[\varphi_1'(0)] = \mathcal{J}[\varphi'(\infty)] = 0$$

since $\omega(0) = \infty$.

In this problem

$$\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} = \frac{1}{\sigma}\frac{(1+m\sigma^2)}{(\sigma^2+m)}\,, \quad \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)} = \frac{-\sigma(\sigma^2+m)}{(1-m\sigma^2)}$$

so that the singular integral equation (23) becomes

$$(25)\quad \varphi(\zeta) + \frac{1}{2\pi i}\int_\gamma \frac{1}{-\sigma}\frac{(1+m\sigma^2)}{(\sigma^2-m)}\frac{\overline{\varphi'(\sigma)}}{\sigma-\zeta}\,d\sigma + \overline{a} = \frac{1}{2\pi i}\int_\gamma \frac{f\,d\sigma}{\sigma-\zeta}$$

where the integral on the right-hand side is assumed known.

Writing

$$\overline{g(\sigma)} = \frac{1}{-\sigma}\frac{(1+m\sigma^2)}{(\sigma^2-m)}\overline{\varphi'(\sigma)}$$

or

$$g(\sigma) = \frac{-\sigma(\sigma^2+m)}{(1-m\sigma^2)}\varphi'(\sigma)$$

we have that $g(\sigma)$ is the boundary value on γ of the function

$$g(\zeta) = \frac{-\zeta(\zeta^2+m)}{(1-m\zeta^2)}\,\varphi'(\zeta)$$

which is holomorphic inside γ. By virtue of I' on page 29,

$$\frac{1}{2\pi i}\int_\gamma \frac{\overline{g(\sigma)}\,d\sigma}{\sigma-\zeta} = \overline{g(0)} = 0$$

so that the integral on the left-hand side of (25) vanishes, yielding

$$(26)\quad \varphi(\zeta)+\bar{a} = \frac{1}{2\pi i}\int_\gamma \frac{f\,d\sigma}{\sigma-\zeta}$$

which gives us the solution for the function $\varphi(\zeta)$ inside γ. ($\bar{a}$ is determined in the above by the condition $\varphi(0) = 0$.)

$\psi(\zeta)$ is determined by the equation (24), which in this case becomes

$$\psi(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{\bar{f}\,d\sigma}{\sigma-\zeta} - \frac{1}{2\pi i}\int_\gamma \frac{-\sigma(\sigma^2+m)}{(1-m\sigma^2)}\frac{\varphi'(\sigma)}{\sigma-\zeta}\,d\sigma .$$

In order to evaluate the second integral appearing in this expression, note that

$$\frac{-\sigma(\sigma^2+m)}{(1-m\sigma^2)}\,\varphi'(\sigma)$$

is holomorphic within γ, and hence by Cauchy's Integral Formula

$$(27)\quad \psi(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{\bar{f}\,d\sigma}{\sigma-\zeta} + \frac{\zeta(\zeta^2+m)}{(1-m\zeta^2)}\,\varphi'(\zeta)$$

Thus the solutions $\Phi(\zeta)$, $\Psi(\zeta)$ for this problem are given by (26) and (27). Specific solutions for various case of applied forces (thus determining the function f in the above expressions) may be found in Muskhelishvili.

13. CONCLUDING REMARKS: In the foregoing, we have illustrated the basis of the new Russian method for obtaining solution to the fundamental problems of Plane Elasticity by means of singular integral equations, where, for simplicity, we have restricted our attention to those cases having cross-sections which are simply-connected regions. The method may be generalized to regions which are multiply-connected, such as an infinite plate with a finite number of holes. And as one would guess in view of the treatment of the example given in this paper (i.e., the infinite plate with an elliptical hole), it is possible to develope the method for infinite regions in a manner analogous to that for finite regions. These topics are fully discussed in Muskhelishvili's book.

Several other important problems treated by Muskhelishvili are worthwhile mentioning here. Since the mapping function $\omega(\zeta)$ is conformal within the unit circle, it may be expanded as an infinite power series about the origin. Muskhelishvili gives the general solutions to the fundamental problems when the mapping $\omega(\zeta)$ is a polynomial (in this case, a partial sum of the infinite series), since the polynomials thus obtained map the unit disk onto regions approximating the given region to any degree of accuracy. Also important are those problems when the mapping function $\omega(\zeta)$ is a rational function, i.e., a fraction in which both the numerator and denominator are polynomials, since the Schwarz-Christoffel transformation of the unit disk on the region may take this form.

Another important problem is the case of the semi-infinite plane subjected to the pressure of a rigid-stamp acting on its boundary: the treatment of this problem also appears in Muskhelishvili's other book, <u>Singular Integral Equations</u>, which has been translated into English by J.R.M. Radok and is published by P. Nordhoff in Holland.

Upon reading the bibliography of Muskhelishvili's work, one is given the impression that this field is one of great activity, especially with the Russians, and that many more solutions, of a particular nature, are yet to be discovered. Surely tremendous strides have been taken in the last few decades. And more is yet to come: we'll see...

APPENDIX

GENERAL NOTATION AND EQUATIONS FOR A THREE-DIMENSIONAL ELASTIC BODY

ASSUMPTIONS:

The elastic body is assumed to be isotropic and homogeneous (see definitions, pages 2 and 3), and also perfectly elastic: that is, the body will fully recover its original shape when the forces causing its distortion are removed. Furthermore, when distorted, the body is assumed to be in a state of elastic equilibrium, i.e., the forces acting on the body, and every sub-body of the body, satisfy the conditions of static equilibrium (and also the conditions of dynamic equilibrium if the body is in motion through space). The conditions of static equilibrium are:

I. The vector resultant of all forces acting on the body is zero;

II. The total moment about any point (of the body) caused by the forces acting on the body is zero.

Two types of forces act on the body: surface forces (given per unit area), or stresses; and volumetric (given per unit volume), or body forces.

NOTATION:

In the definitions below, it should be remembered that these quantities are functions of the point (x,y,z) of the body, where they are taken.

X_x, X_y, Y_z, Z_n, etc.: The Components of Stress.

These are to be interpreted as follows: Let T be a plane through the point (x,y,z) having n as its normal vector: then the (vector) components of the stress acting or T at the point (x,y,z) are denoted by X_n, Y_n, Z_n , taken in the directions of the positive x-, y-, and z- axes respectively. Thus, Y_x designates the component of the stress in the direction of the y-axis acting on a plane having normal parallel to the x-axis (at some prescribed point (x,y,z)), and analogously for the other symbols.

Under the conditions of elastic equilibrium, these components are related by

$$
\begin{aligned}
X_x \cos(n,x) + X_y \cos(n,y) + X_z \cos(n,z) &= X_n \\
Y_x \cos(n,x) + Y_y \cos(n,y) + Y_z \cos(n,z) &= Y_n \\
Z_x \cos(n,x) + Z_y \cos(n,y) + Z_z \cos(n,z) &= Z_n
\end{aligned}
$$

(at some prescribed point (x,y,z)), where $\cos(n,x)$, $\cos(n,y)$, $\cos(n,z)$ are the direction-cosines of the normal n with the x-, y-, and z-axes respectively.

X, Y, Z: The Components of the Body Force.

These designate the vector components of the body force, per unit volume, taken in the directions of the x-, y-, and z-axes, respectively, at some prescribed point (x,y,z).

u, v, w: The Components of Displacement.

These are defined as follows: Suppose that when the body is in an un-strained state, the co-ordinates of a given point of the body are (x_0, y_0, z_0), and that after displacement, due to the applied forces, the co-ordinates of this point (with respect to the previous system of co-ordinate axes) are (x, y, z). Then

$$u = x - x_0 \;;\quad v = y - y_0 \;;\quad w = z - z_0$$

are the (vectorial) components of the displacement of the given point, in the directions of the x-, y-, and z-axes, respectively.

The Components of the Strain Tensor, which are not used in this paper, but occure in the literature, may now be given as

$$e_{xx} = \frac{\partial u}{\partial x} \;;\quad e_{yy} = \frac{\partial v}{\partial y} \;;\quad e_{zz} = \frac{\partial w}{\partial z} \;;$$

$$e_{xy} = e_{yx} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \;;$$

$$e_{xz} = e_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \;;$$

$$e_{yz} = e_{zy} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \;.$$

Δ : The Laplacian Operator.

If U(x,y,z) is a function of x, y, and z, then Δ is defined by

$$\Delta U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \;.$$

In two-dimensions this becomes

$$\Delta U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \;.$$

EQUATIONS:

The Equations of Equilibrium:

I. From the first condition of elastic equilibrium, it follows that

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} + X = 0$$

$$\frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} + Y = 0$$

$$\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} + Z = 0 .$$

II. From the second condition of elastic equilibrium, it follows that

$$X_y = Y_x \; ; \quad X_z = Z_x \; ; \quad Y_z = Z_y \; .$$

(i.e., the Stress Tensor is symmetric.)

The Stress-Strain Relations:

The Stress components are related to the Strain components in the following way:

$$X_x = \lambda\,(e_{xx} + e_{yy} + e_{zz}) + 2\mu\, e_{xx}$$

$$Y_y = \lambda\,(e_{xx} + e_{yy} + e_{zz}) + 2\mu\, e_{yy}$$

$$Z_z = \lambda\,(e_{xx} + e_{yy} + e_{zz}) + 2\mu\, e_{zz}$$

$$Y_z = \mu\, e_{yz} \; ; \quad Z_x = \mu\, e_{zx} \; ; \quad X_y = \mu\, e_{xy} \; .$$

These relations may be re-written in terms of the displacement components (using the relations given on page iii) yielding

$$X_x = \lambda\theta + \mu\frac{\partial u}{\partial x} \quad ;$$

$$Y_y = \lambda\theta + \mu\frac{\partial v}{\partial y} \quad ;$$

$$Z_z = \lambda\theta + \mu\frac{\partial w}{\partial z} \quad ;$$

$$X_y = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) ;$$

$$X_z = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) ;$$

$$Y_z = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) .$$

where $\theta = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}$

and where λ and μ are the Lamé constants (see page 2).

For further reference, it may be helpful to consult the article entitled "Elasticity" in the Encyclopedia Britannica (Eleventh Edition).

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