Singular G-Monopoles on $S^1 \times \Sigma$

Benjamin H. Smith

Doctor of Philosophy

Department of Mathematics and Statistics
Faculty of Science

McGill University

Montréal, Quebec

December 15, 2014

A thesis submitted to McGill University
in partial fulfillment of the requirements for the degree of Doctor of Philosophy

ACKNOWLEDGEMENTS

I would like to begin by thanking NSERC, Jacques Hurtubise, McGill University and ISM for providing financial support throughout the duration of this project. This has allowed me the space and freedom to pursue mathematics with fewer external pressures. A second thanks goes out to my trusty supervisor Jacques Hurtubise, who guided me through the infinite hazy-maze of abstraction and sketched ideas just enough for me to discover them on my own. Without his expertise and intuition, this wouldn't have been possible.

I would also like to thank a short list of instructors at McGill University for introducing me to algebraic groups, driving my enthusiasm for illustrative mathematics and having a great collection of rare books they were willing to lend.

To some colleagues (both physical and virtual) Stewart Jarvis, Tom Baird and Michael Wong who, respectively, volunteered time and consideration in reply to several emails regarding clarification of their work, spent several hours one-on-one filling gaps in my understanding of complex analysis and line bundles and donated well-written expanded notes on the topic of principal bundles. Thanks also to fellow students Andrew Fiori, Katherine Daignault and Kael Dixon who donated countless hours listening to my "almost there but not quite" lectures and, on occasion, making key remarks to push me further.

Special thanks to Leta, Nico and Juan for encouraging my good health and fitness. To Second Cup and BDP for a spacious, friendly working environments and finally to my family and many friends for encouragement and parties!

ABSTRACT

This thesis provides a self-contained account of the functorial correspondence between irreducible singular G-monopoles on $S^1 \times \Sigma$ and $\vec{t}$-stable meromorphic bundles on Σ. The main theorem from the work of B. Charbonneau and J. Hurtubise is thus generalized here from unitary to arbitrary compact, connected gauge groups. The required distinctions and similarities for unitary versus arbitrary gauge are clearly outlined and many parallels are drawn for easy transition. Some basic theory involving induced connections on associated bundles is developed for the purpose of describing covariantly constant G-equivariant maps between principal bundles. Several calculations from the references are expanded throughout in order to demystify any uncertainties.

Once the main correspondence theorem is complete, the spectral data of our monopoles is provided and the groundwork for a monopole theory on Sasakian manifolds, along with their analogous abelianization is discussed.

ABRÉGÉ

Cette thèse décrit une correspondance fonctorielle entre des G-monopoles irréductibles avec singularités sur $S^1 \times \Sigma$ et les G^C-fibrés avec automorphismes méromorphes $\vec{t}$-stables sur Σ. Ainsi nous généralisons le théorème principal d'un travail de Charbonneau et Hurtubise à tous les groupes de jauge compacts. Les différences et les resemblances entre le cas unitaire et le cas général sont clairement soulignés et beaucoup de parallèles sont tracées pour faciliter la transition. Un peu de théorie élémentaire à propos des connexions sur les fibrés associés est développée pour décrire leurs sections G-equivariantes et parallèles. Plusieurs résultats provenant des références sont développées pour plus de clarté. Une fois la correspondance principale établie, nous présentons les données spectrales de nos monopoles , pour le cas étudié, et aussi dans un contexte plus général, sur des fibrés en cercle non triviaux sur une surface de Riemann, ce qui fait intervenir la géométrie de Sasaki.

TABLE OF CONTENTS

CHAPTER 1
Introduction

The Bogomolony equation, often seen on vector bundles [5, 28] can be equivalently stated and solved on the more general class of principal G-bundles. This has been done by Jarvis [25, 26] in the non-compact case where the base is $\mathbb{R}^3$. He provides a bijective correspondence between the space of based rational maps $f : \mathbb{C}P^1 \to G^c/P$ and the framed moduli space of monopoles whose Higgs field takes values in the orbit G/P on the sphere at infinity.

The main goal of this dissertation is to provide a complete, self-contained proof of the bijective Kobayashi-Hitchin type correspondence between the moduli space of *singular G-monopoles* over $S^1 \times \Sigma$ and the space of *$\vec{t}$-polystable holomorphic pairs* $(\mathscr{P}, \rho)$. Since working with complex vector bundles is equivalent to working with principal $\mathrm{GL}_n(\mathbb{C})$-bundles, the results of [5] form a model for the constructions and results found here. Hence, the constructions here will be parallel with the exception of the fact that Charboneau and Hurtubise had advantage of working in the Lie algebra of skew-hermitian matrices, which form an inductive system. Careful considerations will be made about the properties of the more general Lie algebras involved. For this reason, G^c will be a complex reductive Lie group which can be realized as the complexification of a compact, connected real reductive Lie group G.

The main theorem, stated in full generality, is as follows;

Theorem 1.0.1. *There is a bijective correspondence between the moduli space*

$$\mathscr{M}_{k_0}^{irr}(G, S^1 \times \Sigma, \{(p_i, \mu_i)\}_{i=1}^N)$$

of irreducible principal G-monopoles over $S^1 \times \Sigma$ *with singularities at* $p_i \in S^1 \times \Sigma$ *of* μ*-Dirac type, degree* k_0 *over* $\{0\} \times \Sigma$ *and the moduli space*

$$\mathscr{M}_s(\Sigma, \boldsymbol{K}, \vec{t})$$

of $\vec{t}$*-stable holomorphic pairs* (P, ψ)*, where* P *is a holomorphic principal G-bundle of degree* k_0 *over* Σ *and* ψ *is a meromorphic section of* $\mathrm{Aut}_G(P)$ *taking the form*

$$F_i(z)\mu_i(z - z_i)G_i(z)$$

when expressed locally near z_i *with* F_i, G_i *holomorphic-invertible and* μ_i *is a cocharacter of the complexified gauge group* G^c*.*

In less cryptic terminology, this theorem states that one can parameterize the moduli space of G-monopoles [1] over $S^1 \times \Sigma$ having singularities of Dirac-type by the more tractable complex algebraic moduli space of $\vec{t}$-stable meromorphic pairs. There is a family of these moduli spaces, parameterized by the location of singularities on Σ and indexed by the combinatorial data given by the degree, and charge of the bundle at the singularities.

The overall picture portrayed here is that of an intricate correspondence between a moduli space of singular monopoles, which are solutions to a non-linear

[1] i.e. a space of solutions to a partial differential equation

partial differential equation involving the curvature tensor and a much simpler, algebraic, moduli space of holomorphic principal G-bundles on Σ equipped with a meromorphic section of its automorphism bundle. The bijective correspondence between these spaces is given by a forgetful map which only keeps track of the monopole's information along $\{0\} \times \Sigma$ and the monodromy of a scattering map around the circle. Thus, the surjectivity of this forgetful map is one of the main technical and philosophical issues behind the result. That is to say, the real work lies in verifying that a $\vec{t}$-stable meromorphic G-Higgs bundle (P, ρ) is the skeletal information required to uniquely construct a solution to the monopole equation.

The method used to reconstruct a monopole from its singular data is an interesting application of heat flow on the space of positive hermitian metrics which, in the course of doing so, makes use of the celebrated Hopf-fibration. Heuristically, one wishes to, holomorphically, patch together a G-bundle on $S^1 \times \Sigma$ having the correct prescribed 'twisting', so to be in the correct topological isomorphism class. This is done by patching together a metric (using a partition of unity) that will be a parametrix of the solution having the correct singular points. Once this metric is defined, the heat flow is employed to evenly distribute the curvature, induced by the metric, towards a solution to the monopole equation.

In history, there have been several results involving classifications of these types and the general picture is known as the *Kobayashi-Hitchin correspondence*. There are three foundational works in this area; namely the papers of Donaldson [9, 10] and Uhlenbeck-Yau [44, 45] in establishing the Kobayashi-Hitchin correspondence for holomorphic vector bundles on compact Kähler manifolds [32]. The progression of these results is the work of many mathematicians starting with Narasimhan-Seshardi [35] for Riemann surfaces, Donaldson [9, 10, 11] again for Riemann surfaces and also algebraic surfaces, and Uhlenbeck-Yau [44, 45] for compact Kähler manifolds. A careful analysis of heat flow in these settings, and more generally in situations with singularities, is due to Simpson [42]. A good reference for the completed Kobayashi-Hitchin correspondence was presented by Lúbke and Teleman [32] in great detail and generality.

In our situation, the solutions to the Bogomolny (monopole) equation are required to have singularities. In 1988, Simpson [42] provided a short list of assumptions sufficient to guarantee the required long term existence of the heat equation in these cases. Our domains and initial conditions fit Simpson's profile (as first employed in [5]) and so we have the existence of our solutions with the exception of singular neighbourhoods that must be considered with separately.

It was M. Pauly [37], following unpublished work of Kronheimer [31] who first dealt with Dirac-type singular monopoles on 3-balls. He displayed, via a radially extended version of the Hopf fibration, a correspondence between Dirac-type monopoles on $B^3\setminus\{0\}$ and smooth S^1-invariant anti self-dual connections on $B^4\setminus\{0\}$. This was used to solve the problem of classifying singular Hermitian-Einstein ($G = \mathrm{U}(n)$) monopoles on $S^1 \times \Sigma$ which was worked out by B. Charbonneau and J. Hurtubise [5].

1.1 Chapter Breakdown

The first part consists mainly of the literature review and examples of required topics and results to be employed throughout the main subject matter. It begins with the core facts and ideas behind principal G-bundles, connections and curvature. This includes some reoccurring examples (e.g. the Hopf fibration) which is extensively used in the proof of our correspondence theorem. Another important example found here is the realization of equivariant bundle maps $P \to P'$ as sections of an interesting associated bundle (cf. Lemma 2.4.1 and Section 3.5). This example is required for the proof of injectivity (Proposition 7.1.11) in the correspondence theorem and drove what led to a small contribution (Proposition **??**) to the general theory of induced connections on associated bundles. There are many available resources for this type of material but the content found here (in chapters 2 and 3) are heavily influenced by [4, 13, 15, 23, 28, 29, 30, 33, 43] along with some notes by M.L. Wong.

The next important bit of core material is that of stability theory for complex vector and principal bundles on Riemann surfaces taken mainly from [4, 16, 17, 32, 34] for vector bundles and [24, 38] for principal. A proof of a result I like to refer to as *reduced curvature* for Hermitian-Einstein bundles (found in both [17] - chapter 0 and [32] - chapter 2) is translated into the language of principal bundles and proven in our notation (Theorem 4.2.2). This result allows us to integrate more easily on our 3-manifolds assists in removing the use of induction in the proof of G-monopole stability.

The heart of this document lies in part II which provides a self-contained account of the correspondence theorem between the differential geometric family of *irreducible singular G-monopoles over $S^1 \times \Sigma$* and the equivalent algebraic family of *$\vec{t}$-stable meromorphic pairs over Σ*. This theorem and much of the groundwork behind it was inspired mostly by [5] which proved the analogous theorem for complex vector bundles (i.e. the case $G = \mathrm{U}_n$). More substantially, chapters 5 and 6 are geared toward the development and understanding of the geometric objects of interest (namely, monopoles and bundle pairs). Some important constructions here are the μ-Dirac monopole (Section 5.2) which generalize the standard Dirac monopole found in [5] to the principal bundle setting. Explicit definitions and calculations are made regarding the stability of these objects while keeping the reader informed about the choices made and ties back to more familiar notions. One should notice the lack of mathematical induction required in the proof of Proposition 6.3.5 which, contrary to [5] (Proposition 3.6), allows us to discuss the stability of monopoles for reductive groups that do not carry an inductive system (e.g. G_2).

Chapter 7 is dedicated entirely to the proof, via Propositions 7.1.2 and 7.1.11, of our main Theorem 7.1.1. The proof technique is quite standard for correspondence theorems of this type. It involves defining a forgetful map from the differential/monopole side to the algebraic (recording only the skeletal/algebraic data involved in the solution to the differential equation) and proceeds by showing bijectivity. Once all of the correct notions (induced connections on associated bundles and isomorphism between monopoles and bundle pairs) are in place, the proof of injectivity is rather quick and not much different than found in [5]. The surjectivity, shown by demonstrating that any stable bundle pair gives rise to a monopole, is where M. Pauly's "Hopf-trick" [37] and Simpson's heat-flow [42] are employed. Essentially, this process is described as follows;

(i) one takes the algebraic data of a stable meromorphic pair,

(ii) patches this information together to define a G-bundle on the $S^1 \times \Sigma$ with appropriate singularity data,

(iii) applies Simpson's heat flow on the space of hermitian metrics (with initial metric specified by the alignment of our fibres from parti(ii)) to obtain a hermitian metric whose associated curvature tensor is evenly distributed[2] and

(iv) finally checks to make sure that the resulting bundle has not been affected to drastically near the singularities (i.e. still has the required Dirac monopole data at the singularities).

Once, this correspondence theorem is proven, there are several directions in which this project can be taken further. Indeed, this correspondence between the monopole moduli space and the space of stable holomorphic pairs is only two pieces in a circle of three corresponding families of objects, the third being the spectral data of the monopole. The topic of this dissertation is the bijection between monopoles and stable holomorphic pairs. This circle of equivalent moduli spaces and all the technical analysis involved has not yet been investigated for the case of principal bundles with reductive structure groups and careful study of the relations between these objects could lead to many beautiful results. When Σ has genus one, there is in addition a Nahm transformation, taking us to instantons over $\Sigma^* \times \mathbb{R} \times S^1$. Alternatively, with the level of comfort developed from this document, one may wish

[2] in these stages, we are making use of the well-known equivalence ([4]) between topological isomorphism classes of bundles with connected components of the space of hermitian metrics and the and their uniquely induced metric-compatible connections

to consider the moduli space of singular G-monopoles on a non-trivial circle bundle over Σ.

Chapter 8 and 9 are dedicated to the general layout for the problem of spectral data associated to meromorphic pairs and the notion of singular G-monopoles over Sasakian manifolds. The spectral curves and line bundles associated to meromorphic pairs for complex vector bundles are well-understood and discussed near the end of [5]. A brief review is covered here for the purpose of abstracting to arbitrary reductive groups where we take a closer look at the orbit theory and generalize these notions to what are called cameral covers. These cameral covers, introduced by Donagi and Scognamillo [7, 39] and investigated in several similar contexts by [8, 22, 20, 21, 40], have been of more recent mathematical interest in describing the Hitchin map on the moduli space of Higgs bundles. Here, they are adjusted to our particular system and the lift of our meromorphic data reduces to a maximal torus bundle over the cameral cover.

Finally, chapter 9 deals with the idea of constructing singular G-monopoles over a circle bundle of positive degree which can be shown to admit the structure of a compact Sasakian 3-manifold. These differ from $S^1 \times \Sigma$ in the sense that the domain is now a non-trivial circle bundle over a Riemann surface. The appropriate metrics, forms and mechanics of Sasakian geometry is provided and the monopole equations here are quickly achieved. The classical example of S^3, realized as admitting non-trivial S^1-fibration over S^2, is examined as a great candidate and calculations are provided to match the theory. The holomorphic and meromorphic structures appearing in this context are slightly different and give rise to a correspondence of singular monopoles on Sasaki manifolds with what we call twisted holomorphic bundles over Σ. The generalized notion of twisted spectral data is then addressed with some ideas left for future work.

Part I

Core Material

CHAPTER 2
Principal Bundles

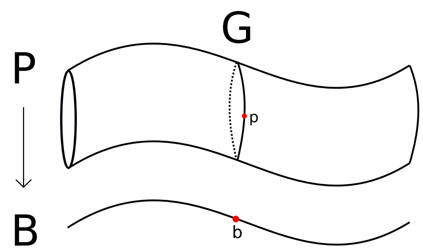
2.1 Definition and basic facts

In order to generalize the theory of vector bundles to bundles having arbitrary structure group (i.e. not necessarily GL_n) we must introduce a more general notion of bundle. These objects will be constructed from the perspective of their progeny, the vector bundle. This is followed all the way to the theory of connections and curvature.

Namely, a *principal G-bundle P* over a base manifold B is a smooth left G-space P and a smooth projection $\pi : P \to B$ satisfying that

- The action of G is free and π descends to the quotient P/G inducing a diffeomorphism $[\pi] : P/G \to B$, and
- P is locally trivializable[1]

[1] meaning, for each $b \in B$ (the base manifold) there is an open neighbourhood $U \ni b$ and a G-equivariant diffeomorphism $\varphi_U : P_{|_U} \to U \times G$ mapping p to $(\pi(p), \sigma_U(\pi(p)))$ for some map $\sigma : U \to G$.



If the base B has dimension n and G has dimension r then $\dim P = n + r$ (based on the local description). Denoting by $\rho : G \times P \to P$ the group action, $\rho(g, p) = g \cdot p$, of G on P we get two types of useful maps $\rho_g : P \to P; p \mapsto \rho(g, p)$ and $\rho^p : G \hookrightarrow P; g \mapsto \rho(g, p)$. If L_g and R_g denote the action of G on itself by left and right multiplication then the following list of identities hold between these maps

1. $\pi \circ \rho_g = \pi$, since G is a fibre-preserving action on P,
2. $\rho_e = \mathbf{1}_P$,
3. $\rho_{gh} = \rho_g \circ \rho_h$,
4. $\rho^p \circ R_g = \rho^{g \cdot p}$, and
5. $\rho_g \circ \rho^p = \rho^p \circ L_g$.

A *section* of a principal bundle P is a map $\sigma : B \to P$ such that $\pi \circ \sigma = \mathbf{1}_B$. The trivial bundle $P = B \times G$ admits a section $\sigma(b) = (b, e)$. Conversely, if P admits some section $\sigma \in \Gamma(P)$, we can define an isomorphism of principal bundles $B \times G \to P$ by $(b, g) \mapsto g \cdot \sigma(b)$. Hence, unlike vector bundles, we find

Lemma 2.1.1. *A principal bundle is trivial if and only it admits a section.*

Also, as holds for any fibered space, we have

Lemma 2.1.2. *A principal bundle over a contractible base is topologically trivial.*

2.2 Examples

We are already familiar with the trivial principal bundle $P = B \times G$ which is really just the local model motivating the definition. The next natural motivatinal example, coming from vector bundles is:

1. The *Frame bundle* $\mathscr{F}M$ of a manifold M. Fibre-wise $\mathscr{F}_x M$ is defined as the space of frames (i.e. bases) for the tangent space $T_x M$. Local triviality is inherited from the corresponding structure on TM. This is a GL_n-bundle over M where GL_n acts by conjugation/change of basis. Given a metric, one has[2] the *orthogonal frame bundle* $\mathscr{O}\mathscr{F}M$ by restricting our attention to the space of orthogonal frames. This is a *subbundle* of $\mathscr{F}M$ and is a principal O_n-bundle on M. This notion of subbundle and *reduction of structure group* will be elaborated in more detail later.

Another method for creating principal bundles and extracting useful topological information comes from the theory of *homogeneous spaces*.

2. Let G be a Lie group and $H \leq G$ a closed Lie subgroup so that the base B is given as the homogeneous space G/H. Then the sequence $H \hookrightarrow G \xrightarrow{\pi} B$ defines a principal H-bundle over B. There is, furthermore, a right action, R_g mapping fibres to fibres

Notice that the total space in this particular instance of principal bundles is always the group G. Let us put this construction to work and see what we can find.

a) Let $G = \mathrm{SL}_2(\mathbb{C})$ which acts naturally on the left of $\mathbb{C}^2$ and define $H \leq G$ as the stabilizer of the first standard basis vector $e_1 = (1, 0) \in \mathbb{C}^2$. Then ,

[2] via an application of the Grahm-Schmidt process

$H = \mathrm{stab}_G(e_1) \cong (\mathbb{C}, +)$ via the identification

$$\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \longleftrightarrow z.$$

As per the theory of group actions and homogeneous spaces we find the base manifold $B \sim G/H$ as the orbit

$$\mathcal{O}_{e_1} = G \cdot e_1 = \mathbb{C}^2 \backslash \{\vec{0}\}.$$

Now, the principal H-bundle $q : G \to G/H; g \mapsto [g] = gH$ is trivial since

$$\sigma : B = \mathbb{C}^2 \backslash \{\vec{0}\} \to G; (z, w) \to \begin{pmatrix} z & -\bar{w}/k \\ w & \bar{z}/k \end{pmatrix} \text{ for } k = |z|^2 + |w|^2$$

is a global section. Hence, we have a diffeomorphism

$$\mathrm{SL}_2(\mathbb{C}) \cong G/H \times H = (\mathbb{C}^2 \backslash \vec{0}) \times \mathbb{C} = (S^3 \times \mathbb{R}) \times \mathbb{R}^2 = S^3 \times \mathbb{R}^3.$$

b) Let G be the *Lorentz group* $L_+^\uparrow = \{A \in \mathrm{SO}(1,3) : A_{00} \geq 1\}$ which acts naturally on *Minkowski space* $E^{1,3}$. Define H as the stabilizing subgroup of the first standard basis vector $e_0 = (1, 0, 0, 0) \in E^{1,3}$. A simple computation shows that $H \cong \mathrm{SO}(3)$ via the identification

$$\begin{pmatrix} 1 & 0\ 0\ 0 \\ \begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix} & A \end{pmatrix} \longleftrightarrow A.$$

The base space $G/H = \mathcal{O}_{e_0}$ is found to be

$$\{x \in E^{1,3} : \eta(x, x) = 1, x_0 > 0\} \cong \{(t, x, y, z) \in \mathbb{R}^4 : t > 0, t^2 - x^2 - y^2 - z^2 = 1\} \cong \mathbb{R}^3$$

where $\eta = \left(\begin{smallmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right)$ is the Lorentzian "metric". Hence, the principal SO(3)-bundle $L_+^\uparrow \to \mathbb{R}^3$ is trivial since the base is contractible and we have the diffeomorphism

$$L_+^\uparrow \cong \mathbb{R}^3 \times \mathrm{SO}(3)$$

which will soon reduce to $\mathbb{R}^3 \times S^3/(\mathbb{Z}/2)$ upon further investigation of SO(3).

2.2.1 The Hopf Fibration

In order to establish notation, as the mechanics found here will be of importance later on, recall some

Facts about SU(2)

The compact, connected, 3-dimensional, real, linear algebraic group SU(2) is diffeomorphic to S^3 via the embedding of the unit-length quaternions into $M_2(\mathbb{C})$. There is an obvious canonical action of SU(2) on $\mathbb{C}^2$, but also it acts naturally on $\mathbb{R}^3$ via the adjoint representation on its Lie algebra. Indeed,

$$\mathrm{Lie}(\mathrm{SU}(2)) = \mathfrak{su}_2 = \mathrm{span}_{\mathbb{R}}\{i\sigma_1, i\sigma_2, i\sigma_3\}$$

where

$$\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right), \sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$$

are known as the *Pauli-spin matrices*. Let us adjust this basis to $e_j := -\frac{i}{2}\sigma_j$.

In this basis, any non-zero $X \in \mathfrak{su}_2$ can be expressed uniquely as $X = -\frac{i}{2}\alpha\vec{n}\cdot\vec{\sigma}$ where $\alpha \in \mathbb{R}_+$ and $\vec{n} \in S^2$ and $\vec{n}\cdot\vec{\sigma} = \sum_{i=1}^3 n_i\sigma_i$. The one-parameter subgroups of SU(2) are now given as $\eta(t) = \exp(tX)$.

Note that $(\vec{n}\cdot\vec{\sigma})^2 = I_2$ for all choices of $\vec{n}$. So, in the power series expansion of the one-parameter subgroups we find that

$$\eta(t) = \sum_{k=0}^{\infty} \frac{i\alpha(\vec{n}\cdot\vec{\sigma})^k}{k!} = I_2\cos(t\alpha) + i\vec{n}\cdot\vec{\sigma}\sin(t\alpha).$$

Furthermore, with this notation, the exponential map is easily seen to be surjective[3] .

Lemma 2.2.1. *The adjoint representation of* SU(2) *naturally defines a surjective 2:1 group homomorphism* $f : \mathrm{SU}(2) \to \mathrm{SO}(3)$.

Proof. The matrix $f(A) \in M_3(\mathbb{R})$ is obtained through the action of SU(2) on the basis elements of $\mathfrak{su}_2$ by conjugation. That is, column j of $f(A)$ is obtained as the real coefficients from

$$A\sigma_j A^* = \sum_i f(A)_{ij}\sigma_i.$$

[3] For an arbitrary element of SU(2) can be expressed as $A = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix}$ with $|z|^2 + |w|^2 = 1$, in real coordinates, writing $z = a_0 + ia_3$ and $w = a_1 + ia_2$ then decomposing A as $a_0 I_2 + i(\vec{a}\cdot\vec{\sigma})$. This suggests that if we choose α such that $\cos(\alpha) = a_0$ and $\vec{n} = \vec{a}/|\vec{a}|$, then $A = \exp(i\alpha(\vec{n}\cdot\vec{\sigma}))$.

One can easily verify that the conjugations $A\sigma_i A^*$ where $A = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \in \mathrm{SU}(2)$ is expressed in complex coordinates give rise to

$$f(A) = \begin{pmatrix} \Re(z^2 - \bar{w}^2) & \Im(z^2 + \bar{w}^2) & 2\Re(z\bar{w}) \\ -\Im(z^2 - \bar{w}^2) & \Re(z^2 + \bar{w}^2) & -2\Im(z\bar{w}) \\ -2\Re(zw) & -2\Im(zw) & |z|^2 - |w|^2 \end{pmatrix}.$$

Notice that $\mathbb{Z}/2 \subseteq \ker(f)$ since $f(A) = f(-A)$ we would like to claim that this is an equality. To check this, note that we seek to find all A for which $A\sigma_i A^* = \sigma_i$. This is equivalent to solving $[A, \sigma_i] = 0$ for all $i = 1, 2, 3$. The only matrices which commute with all three of our Pauli matrices must be scalar. That is, we have boiled the kernel down to $A = \lambda I_2$. However, the only scalar matrices in SU(2) are $\pm I_2$.

Thus, we have a short exact sequence of Lie groups

$$\mathbb{Z}/2 \hookrightarrow \mathrm{SU}(2) \twoheadrightarrow \mathrm{SO}(3).$$

$\square$

With this out of the way, we see that SU(2) acts naturally on both $\mathbb{C}^2$ (by matrix multiplication), but also on $\mathbb{R}^3$ via the adjoint representation. The *Hopf map* is a quadratic map defined by

$$\pi : \mathbb{C}^2 \to \mathbb{R}^3$$

$$\chi = (z, w) \mapsto \vec{r} := \chi^* \vec{\sigma} \chi = (z\bar{w} + w\bar{z}, i(z\bar{w} - w\bar{z}), |z|^2 - |w|^2)$$

and satisfies the following list of interesting properties.

Proposition 2.2.2. *The Hopf map satisfies the following;*

(i) For arbitrary $\chi \in \mathbb{C}^2$, the Hermitian matrix constructed by $\chi\chi^*$ can be expressed in terms of $\vec{r} = \pi(\chi)$ as

$$\chi\chi^* = \frac{1}{2}(rI_2 + \vec{r}\cdot\vec{\sigma})$$

where $r := \chi^*\chi = |\chi|^2$.

(ii) restriction of π to the 3-sphere of radius $\sqrt{r}$ in $\mathbb{C}^2$ has the 2-sphere of radius r as its image in $\mathbb{R}^3$.

(iii) π is an $\mathrm{SU}(2)$ equivariant map meaning that the following diagram commutes

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, "A"] \arrow[d, "\pi"'] & \mathbb{C}^2 \arrow[d, "\pi"] \\ \mathbb{R}^3 \arrow[r, "f(A)"'] & \mathbb{R}^3. \end{tikzcd}$$

(iv) π is surjective, which is simply a result of the transitivity of the action $\mathrm{SU}(2)$ on S^3 and the equivariance of π.

Now, expressing $\pi : S^3 \to S^2$ in terms of angular coordinates[4] then $\pi(\theta,\varphi,\psi) = (\theta,\varphi)$ is just projection on the first two factors. Recall the conversion $z = \cos(\theta/2)e^{-\frac{i}{2}(\psi+\varphi)}, w = \sin(\theta/2)e^{-\frac{i}{2}(\psi-\varphi)}$ so that

$$\begin{aligned}
\pi(\theta,\varphi,\psi) &= \pi(\cos(\tfrac{\theta}{2})e^{-\frac{i}{2}(\psi+\varphi)}, \sin(\tfrac{\theta}{2})e^{-\frac{i}{2}(\psi-\varphi)}) \\
&= \begin{pmatrix} \cos(\frac{\theta}{2})\sin(\frac{\theta}{2})e^{-i\varphi} + \cos(\frac{\theta}{2})\sin(\frac{\theta}{2})e^{i\varphi} \\ i\left[\cos(\frac{\theta}{2})\sin(\frac{\theta}{2})e^{-i\varphi} - \cos(\frac{\theta}{2})\sin(\frac{\theta}{2})e^{i\varphi}\right] \\ \cos(\frac{\theta}{2})^2 - \sin(\frac{\theta}{2})^2 \end{pmatrix} \\
&= \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\cos\varphi \\ \cos\theta \end{pmatrix} = (\theta,\varphi)
\end{aligned}$$

With this, notice that $\zeta \in S^3$ is a (unit length) solution to the equation[5] $(\vec{n}\cdot\vec{\sigma})\zeta = \zeta$. Recall the identity, $\zeta\zeta^* = \frac{1}{2}(I_2 + \vec{n}\cdot\vec{\sigma})$ so then

$$(\vec{n}\cdot\vec{\sigma})\zeta = (2\zeta\zeta^* - I_2)\zeta = 2\zeta - \zeta = \zeta.$$

Furthermore, the set of all solutions is parameterized by $e^{-i\alpha}\zeta$ as a similar computation will provide. Computing the fibres of the projection $\pi : S^3 \to S^2$ is shown by the

---

[4] that is, *Euler angles* (θ,φ,ψ) for S^3 and *spherical coordinates* (θ,φ) for S^2

[5] where $\vec{n} = (x,y,z) \in S^2, \vec{\sigma} = (\sigma_x,\sigma_y,\sigma_z)$ and the product $\vec{n}\cdot\vec{\sigma} = x\sigma_x, +y\sigma_y + z\sigma_z$ denotes an arbitrary unit element in the Lie algebra of SU(2)

transitivity of SU(2) on S^3. Indeed,

$$\pi^{-1}(\vec{n}) = \{B\zeta \in S^3 : f(B) \in \mathrm{stab}_{\mathrm{SO}(3)}(\vec{n})\}.$$

Recall that the stabilizer of $\vec{n} \in \mathbb{R}^3$ is the set of rotations about the axis defined by $\vec{n}$. This is then a subgroup of SO(3) isomorphic to SO(2) $\cong S^1$. One can compute this fibre explicitly by realizing that $f(B)$ comes from some 1-parameter subgroup of $B(t)$ of SU(2). More particularly, this subgroup is

$$B(t)\zeta = e^{-\frac{i}{2}t\vec{n}\cdot\vec{\sigma}}\zeta = \left(\cos\tfrac{t}{2}I_2 - i\sin\tfrac{t}{2}\vec{n}\cdot\vec{\sigma}\right)\zeta = \left(\cos\tfrac{t}{2} - i\sin\tfrac{t}{2}\right)\zeta = e^{-it/2}\zeta = e^{i\alpha}\zeta.$$

In conclusion, we have realized a principal S^1-bundle structure on S^3 providing an action of SU(2) on $\mathbb{R}^3$ with the orbit of any fixed element $\vec{n} \in S^2$ is S^2 and stabilizer is a closed Lie subgroup $H = S^1 \leq S^3$. Hence, we find the principal S^1 fibration of S^3 over S^2. As already discussed, through cohomology classes of the corresponding spaces, this bundle is not trivial.

2.2.2 Pullback bundles

Definition 2.2.3. *Given a smooth map $f : A \to B$ and a principal G-bundle P over B, then the* pullback bundle

$$f^*P := \{(x,p) \in A \times P : f(x) = \pi(p)\}$$

is a principal G-bundle on A.

2.2.3 Torus bundles on 2-spheres

We cover the sphere S^2 by two unit disks U_+, U_- intersecting on $S^1 \times I$, for some interval I, which is homotopy equivalent to S^1. Hence, the transition functions for a T-bundle on S^2 are given by the homotopy class of a single map $\mu : S^1 \to T$ (For example, a cocharacter of T).

2.3 The vertical tangent bundle

If we set $\mathfrak{g}$ to denote the Lie algebra of G, then each vector $\xi \in \mathfrak{g}$ determines a 1-parameter subgroup of G, and also a vector field in any manifold on which G acts. That is, the line, $t\xi$, through the origin in $\mathfrak{g}$ defines a morphism from $\mathbb{R}$ into the diffeomorphisms of P by $t \mapsto \exp(t\xi)$ and the action of $\mathbb{R}$ on a point $p \in P$ is given by $\exp(t\xi) \cdot p$ and the *fundamental vector field v_ξ on G* is generated as

$$v_\xi(p) = \left.\frac{\partial}{\partial t}\right|_{t=0} \exp(t\xi) \cdot p = d\rho^p(\xi)$$

Lemma 2.3.1. *The fundamental vector field on a principal bundle enjoys the following properties.*

(i) *For any $\xi \neq 0$, v_ξ is nowhere zero*

(ii) *If $\xi_1, \dots, \xi_k \in \mathfrak{g}$ are linearly independent then the v_{ξ_i} are independent vector fields.*

(iv) *The map $\xi \mapsto v_\xi$ is G equivariant, meaning that*

$$v_\xi(g \cdot p) = v_{\mathrm{Ad}_g \xi}(p)$$

for all $g \in G$ and $\xi \in \mathfrak{g}$.

We define the *vertical tangent bundle* $VP \leq TP$ of the tangent bundle for P as the kernel, $\ker(d\pi)$, of the differential of the projection $\pi : P \to B$.

Our 'properties' lemma above ensures that a basis $\xi_1, \ldots, \xi_n$ for $\mathfrak{g}$ will generate a basis of sections $\{v_{\xi_i}\}$ for the VP. This is understood by a simple dimension count (i.e. since $\dim \mathfrak{g} = \dim G$). Furthermore, this also holds a proof of the fact that VP is a trivial bundle over P having constructed a global frame. That is, we have the following tautological result:

Lemma 2.3.2. *The vertical tangent bundle VP is isomorphic to the trivial vector bundle $P \times \mathfrak{g}$ via $v_\xi(p) \longleftrightarrow (p, \xi)$.*

2.4 Associated Bundles

Given any linear representation $\rho : G \to \mathrm{GL}(V)$ of G on some n-dimensional complex vector space V one can form the *associated vector bundle*

$$E_\rho := P \times_\rho V$$

which is the space of equivalence classes $P \times V / \sim$ with $(g \cdot p, v) \sim (p, \rho(g)v)$ for all $g \in G$.

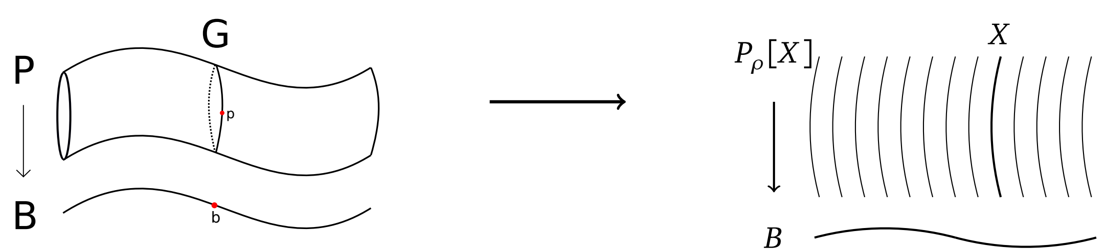
Example 1.

1. The most common example of associated vector bundles is the *adjoint bundle* $\mathrm{ad}_P := P \times_{\mathrm{ad}} \mathfrak{g}$ associated to the adjoint representation of G on its Lie algebra. That is,
$$\mathrm{ad} : G \to \mathrm{End}(\mathfrak{g});\, g \mapsto \mathrm{ad}_g := (v \mapsto gvg^{-1})$$
2. The associated line bundles are classified by the group characters, $X(G) = \{\rho : G \to \mathbb{C}^*\}$, of G. In particular, for matrix groups, the *determinant* is a canonical

choice of character and the associated vector bundle is the familiar *determinant bundle* constructed as the top exterior power.

More generally, these ideas can be extended. In fact, let X be any smooth G-space and P some principal G bundle over B. The fact that there is a group action $\varphi : G \times X \to X; (g, x) \mapsto g \cdot x$ gives a map $\rho : G \to \mathrm{Aut}(X)$ by $\rho(g)(x) = \varphi(g, x) = g \cdot x$. Then, in the same way associated vector bundles are constructed, we get $P_{\varphi}[X] = P \times_{\varphi} X$.

Pictorially, associated bundles are illustrated by the following diagram:



It will be important to notice that, unlike the case of associated vector bundles, these associated fibre bundles do not necessarily admit any special algebraic structure. That is to say, an associated fibre bundle may not be anywhere near principal. A great example of this is the G-fibre bundle $\mathrm{Hom}_G(P, P')$ of G-equivariant maps $\tau : P \to P'$ between principal bundles. We will be able to realize this as a G-bundle associated to the principal $G \times G$ *fibre product bundle* $P \times_B P'$. However, it is immediately apparent that this space does not admit a free action of G unless G is abelian.

Example 1. *The G-fibration* $\pi : \mathrm{Hom}_G(P, P') \to B$ *can be defined as a set as the union of fibres*

$$\mathrm{Hom}_G(P, P') = \bigsqcup_{b \in B} \mathrm{Hom}_G(P, P')_b = \bigsqcup_{b \in B} \mathrm{Hom}_G(P_b, P'_b).$$

These fibres are equivalent to the space of G-equivariant bijections from G to G. However, a more appropriate realization is found as the associated G-bundle to the fibre product $G \times G$*-bundle* $P \times_B P'$ *via the group action*

$$\varphi : (G \times G) \times G \to G; (g, h, x) \mapsto g^{-1} x h.$$

Stated more precisely

Lemma 2.4.1. *There is an isomorphism*

$$\mathrm{Hom}_G(P, P') \longleftrightarrow (P \times_B P') \times_{G \times G} G$$

given by the fiberwise correspondence between G-equivariant bijections from G to G and the topological quotient $(G \times G \times G)/(G \times G)$ *defined by the equivalence relation* φ *above.*

Proof. Indeed, given an element of the equivalence class $(a, b, g^{-1}xh) \sim (ga, hb, x)$, define a G-equivariant bijection τ which sends a to $g^{-1}xhb$. Then certainly $\tau(ga) = g\tau(a) = xhb$ which would have come from the same element (ga, hb, x). Conversely, given some G-equivariant bijection, fix $a, b \in G$ and let $g, h \in G$ be arbitrary (also fixed). Then the unique element $x := g\tau(a)b^{-1}h^{-1}$ would satisfy that $\tau(a) = g^{-1}xhb$ so also $\tau(ga) = g\tau(a) = xhb$, corresponding to the fact that this bijection maps uniquely to an element in the equivalence class $(a, b, g^{-1}xh) \sim (ga, hb, x)$ for fixed $a, b, x \in G$ and any $g, h \in G$. That is, we have verified the following;

$$\mathrm{Hom}_G(P, P') \longleftrightarrow (P \times_B P') \times_{G \times G} G.$$

$\square$

Notice that, although this is a fibration (with fibres isomorphic to G), we do not have a well-defined, let alone free, action of G on the total space. The G-action can only be well-defined for the central elements, $Z(G)$, of G. Observe, for $g \in G$, and $f_b \in \mathrm{Hom}_G(P, P')$ if we wanted to have, say $g \cdot f_b(x) = f_b^g(x) := f_b(g \cdot x)$, then the G equivariance would impose that for all $h \in G$

$$ghf_b(x) = f_b(ghx) = f_b^g(hx) = hf_b^g(x) = hf_b(gx) = hgf_b(x)$$

which holds if and only if $gh = hg$ (i.e. $g \in Z(G)$).

This example will be important to keep in mind throughout the rest of this document as it will reappear several times and will be the object of future definitions (e.g. induced connections on associated bundles)

2.4.1 Sections of associated bundles

Recall that a section of $\pi : P \to B$ is a map $\sigma : B \to P$ such that $\pi \circ \sigma = \mathbf{1}_B$. Now, if $P_\rho[X]$ is any associated X-bundle then the sections $\Gamma(P_\rho[X])$, defined as maps $\sigma : B \to P_\rho[X]$, are in correspondence with the space of G-equivariant maps $f : P \to X$. That is

Lemma 2.4.2.

$$\Gamma(P_\rho[X]) = C^\infty(P, X)^G$$

whose elements satisfy $f(g \cdot p) = \rho(g) \cdot f(p)$.

Proof. A section $\sigma : B \to P_\rho[X]$ admits a natural lift to a G-equivariant map $\tilde{\sigma} : P \to P \times X$ which, because $\pi \circ \sigma = \mathrm{Id}_B$, factors as $\mathrm{Id}_P \times f$ (i.e. $\tilde{\sigma}(p) = (p, f(p))$). The G-equivariance of $\tilde{\sigma}$ is expressed as $(g \cdot p, \rho(g) \cdot p) = \sigma(g \cdot p) = \sigma(p) = (p, f(p))$ which naturally implies the equivariance of f.

Conversely, any G-equivariant $f : P \to X$ when extended by the identity to $F = \mathrm{Id}_P \times f : P \to P \times X$ remains equivariant and thus descends to a map $\bar{F} : B = P/G \to P \times_G X = P_\rho[X]$. □

2.4.2 The gauge group of P

Another very important example is the associated bundle $\mathrm{Ad}_P := P \times_{\mathrm{Ad}} G$ with the adjoint action, $g \cdot x := gxg^{-1}$, of G on itself[6] (by conjugation). This is not a principal bundle, but merely a bundle of groups. The sections $\Gamma(\mathrm{Ad}_P)$ act on the bundle P and this is the well-known *gauge group*, $\mathscr{G}_P$ of P. In fact this gauge group of P is in correspondence with the group of automorphisms of P which act trivially on the base. That is

Lemma 2.4.3.

$$\mathscr{G}_P = \mathrm{Aut}(P)$$

Proof. Given $f \in \mathscr{G}_P = \Gamma(\mathrm{Ad}_P)$ in the form $f : P \to G$ such that

$$f(g \cdot p) = g f(p) g^{-1}$$

[6] notice the distinction here between our little ad which was the action of G on its Lie algebra. In fact $\mathrm{ad} = d\,\mathrm{Ad}$ is the differential.

construct $\psi : P \to P$ by $\psi(p) := f(p) \cdot p$. Now since

$$\psi(g \cdot p) = f(g \cdot p) \cdot g \cdot p = gf(p)g^{-1}gp = gf(p) \cdot p = g \cdot \psi(p),$$

this is base preserving and G-equivariant meaning $\psi \in \mathrm{Aut}(P)$. Conversely, any $\psi \in \mathrm{Aut}(P)$ uniquely defines a map $f : P \to G$ by the condition $\psi(p) = f(p) \cdot p$ which satisfies (2.4.2) because of the equivariance of ψ. $\square$

CHAPTER 3
Connections and Curvature

Here, we will briefly recall some ideas about connections on vector bundles in order to motivate definitions and draw parallels. For more information and a longer list of results see any standard text on differential, Riemannian or complex geometry. A short list of references is [15, 17, 22, 28]. See also [43] for a similar notation and train of thought.

3.1 On vector bundles

A *connection* on a complex vector bundle $E \to M$ is a linear map

$$\nabla : \Gamma(E) \to \Omega^1(E)$$

satisfying a *Leibniz rule*

$$\nabla(f \cdot \sigma) = df \cdot \sigma + f \cdot \nabla(\sigma), \quad \text{for } f \in \mathscr{C}^\infty(M, \mathbb{C}), \sigma \in \Gamma(E).$$

A connection is the vector bundle analogue of the differential of a function:

Example. Consider some multi-variable function $f : \mathbb{R}^n \to \mathbb{R}$, then the differential of f (aka, its gradient) is given as

$$\nabla f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i = \sum_{i=1}^{n} f_i dx^i.$$

Notice that the image of the differential is a 1-form and has been set up exactly so that the directional derivative $D_{\vec{u}}f = \langle \nabla f, \hat{u} \rangle$ where the standard inner product here is, more generally, the non-degenerate pairing between a vector space and its dual.

Thus, given a connection ∇ we define the *covariant derivative* of a section $\sigma \in \Gamma(E)$ along a tangent vector field $X \in \Gamma(TM)$

$$\nabla_X \sigma := \nabla \sigma(X) \in \Gamma(E).$$

3.1.1 The connection 1-form

Given an atlas $\{(U_\alpha, \varphi_\alpha)\}$ of trivializing charts for E the connection can be expressed locally on each U_α (with respect to a choice of trivializing frame) as

$$\nabla_\alpha = \nabla_{|U_\alpha} = d + \omega_\alpha$$

where ω_α is a matrix valued 1-form on U_α (i.e. $\omega_\alpha \in \Omega^1(U_\alpha, \operatorname{End} E)$). Here we find that the connection form transforms under on overlaps via $g : U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{C})$ as

$$\omega_\beta = g_{\alpha\beta}^{-1} \omega_\alpha g_{\alpha\beta} - d g_{\alpha\beta} g_{\alpha\beta}^{-1}$$

and one can verify that any locally defined matrix-valued 1-form transforming in this fashion gives rise to a well defined connection on E. That is

Proposition 3.1.1. *A connection ∇ on E is equivalent to a family $\{(U_\alpha, \varphi_\alpha, \omega_\alpha)\}$ satisfying*

$$\omega_\beta = g_{\alpha\beta}^{-1} \omega_\alpha g_{\alpha\beta} - d g_{\alpha\beta} g_{\alpha\beta}^{-1}$$

for each $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{C})$, where $\{(U_\alpha, \varphi_\alpha)\}$ is an atlas of trivializing charts for E and $\omega_\alpha \in \Omega^1(U_\alpha, \mathrm{End}(E))$ is an $\mathrm{End}(E)$-valued 1-form on each U_α.

3.1.2 Parallel transport

Along with the notion of covariant differentiation comes a way of determining if a vector field is covariantly constant. In fact, this particular realization of connection is most directly related to the name connection for (as we will see) it provides a tangible means of comparing the relative alignment between different fibres.

A section $\sigma \in \Gamma(E)$ is said to be *covariantly constant* along a vector field X if $\nabla_X \sigma \equiv 0$.

Example 2. *A path $\gamma : [0,1] \to B$ in the base is said to be* geodesic *if $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$ (where we note that here, ∇ is a connection on the tangent bundle of M).*

Suppose instead, we are given a curve $\gamma : [0,1] \to B$ and a point $e_0 \in E_{\gamma(0)}$ rather than a section. The *parallel transport* of e_0 along γ is the point $p_1 \in P_{\gamma(1)}$ determined by $\sigma(\gamma(1))$ where σ is the unique section of E (along γ) satisfying that

$$\nabla_{\dot{\gamma}} \sigma = 0$$

with the initial condition $\sigma(\gamma(0)) = e_0$. Notice that solving for such a σ here is a local problem so the parallel condition is expressed on a trivialization of E as

$$\cot \gamma(\sigma) + A(\dot{\gamma})\sigma = 0$$

which is simply a first order ODE so the initial condition guarantees uniqueness.

3.2 On principal bundles

Here, I will give a brief account from several sources. We follow published articles [13, 15] and [22] along with notes on the subject by K. Klonoff and M. L. Wong.

Definition 3.2.1. *Let $\pi : P \to B$ be a principal G bundle and select $p \in P$ so that $\pi(p) = b \in B$. A* connection *on P can be defined as any of the following equivalent objects:*

1. *A G-invariant horizontal distribution on P. That is, a G-invariant subbundle $HP \leq TP$ which is transverse to the vertical tangent bundle VP.*
2. *A differential form $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ satisfying that*
   - *(i)* $\omega_{g \cdot p}(d\rho_g(v)) = \mathrm{Ad}_g(\omega_p(v))$ *for all $p \in P, g \in G$; and*
   - *(ii)* $\omega(v_\xi) = \xi$ *for all $\xi \in \mathfrak{g}$.*
3. *A G-equivariant splitting of the short exact sequence of vector bundles*

$$0 \to VP \to TP \to \pi^* TB \to 0.$$

The first and second are the most common among definitions and the second being the version most used in practice since it can be handled locally.

3.2.1 The Maurer-Cartan form

An equivalent formulation of the second definition is given using the Maurer-Cartan form, the canonical left invariant 1-form on G, defined by

$$\omega_G : T_g G \to \mathfrak{g}; v \mapsto dL_{g^{-1}}(v).$$

An alternate condition (ii) for the connection 1-form can be expressed in terms of the Maurer-Cartan form as

$$(\rho^p)^* \omega = \omega_G$$

which says that ω evaluates as the canonical left invariant 1-form on G when restricted to any vertical subspace of P.

Lemma 3.2.2 (Parallel transport)**.** *With a connection ∇ on a principal G-bundle $\pi : P \to B$, if $\gamma : [0,1] \to B$ is a smooth curve on B, then for any fixed $p \in P_{\gamma(0)} = \pi^{-1}(\gamma(0))$ there is a unique lift $\tilde{\gamma} : [0,1] \to P$ satisfying that $\tilde{\gamma}(0) = p$, $\pi \circ \tilde{\gamma} = \gamma$, and $\tilde{\gamma}'(t) \in H_{\gamma(t)}P$ for all $t \in [0,1]$.*

Let us point out that this is the key motivation for connection. Indeed, such a unique lift provides a G-equivariant diffeomorphism (i.e. it connects) between the fibres over the endpoints of γ. In brief, define a map

$$\mathscr{P}_\gamma : P_{\gamma(0)} \to P_{\gamma(1)}$$

by $\mathscr{P}_\gamma(p) = \tilde{\gamma}(1) \in P_{\gamma(1)}$ where $\tilde{\gamma}$ is the unique horizontal lift from the above lemma. Ideas of this sort will come up later when we discuss the *scattering map* for singular monopoles.

Lemma 3.2.3 (Local description of the connection form)**.** *If $\{(U_\alpha, \psi_\alpha)\}$ is locally trivializing atlas for P giving rise to sections $\sigma_\alpha : U_\alpha \to P$ and $g_{\alpha\beta} : U_{\alpha\beta} \to G$ represents a transition function (so that $s_\beta = g_{\alpha\beta} s_\alpha$) then, for any connection form $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ we get local connection forms*

$$\omega_\alpha := s_\alpha^* \omega \in \Omega^1(U_\alpha) \otimes \mathfrak{g}$$

which are related on overlapping neighbourhoods as

$$\omega_\beta = g_{\alpha\beta} \omega_\alpha g_{\alpha\beta}^{-1} - d g_{\alpha\beta} g_{\alpha\beta}^{-1}. \tag{3.1}$$

Conversely, any family of local $\mathfrak{g}$-valued 1-forms related by this type of transition uniquely determine a connection on P.

Summary

To briefly summarize the ideas so far would be to quickly review the equivalent objects to a connection on P. That is, beginning with a connection form $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ we can construct a G-invariant horizontal distribution on P via $HP = \ker(\omega)$. This horizontal distribution comes equipped with a natural projection $\pi_V : TP \to VP$ and happens to be the G-equivariant splitting of the sequence $0 \to VP \to TP \to \pi^*TB \to 0$. Finally, a connection form ω is the form obtained from the G-equivariant splitting ω' as the triple composition $\omega := \mathrm{pr}_{\mathfrak{g}} \circ \varphi \circ \omega'$.

3.3 On associated bundles

The purpose of this section will not be immediately apparent. Our aim here is to provide formalism for overcomming the lack of tensor and other algebraic product operations in the category of groups. The motivating example for this section comes from the comparison between $\mathrm{Hom}(E, E)$ and $\mathrm{Hom}_G(P, P)$ which are both bundles associated to the vector bundle E and principal bundle P. For vector bundles one simply finds that $\mathrm{Hom}(E, E)$ is canonically equivalent to the tensor product bundle $E \otimes E^*$ where, as we have shown, $\mathrm{Hom}_G(P, P)$ isn't necessarily a principal bundle. In the case of tensor product bundles, the theory of connections is a simple an application of linear algebra and requires almost no further investigation. However, for $\mathrm{Hom}(P, P)$ we must first develop some means of realizing such a bundle associated to P followed by a well-defined notion of induced connection.

Brief recollection of terms, notation and objects at hand

We will make use of the following objects

- B - A complex manifold or Riemann surface

- G - A (typically compact or reductive) Lie group
- $\pi : P \to B$ a principal G-bundle over B with action map $\rho : G \times P \to P$.
- The bundle isomorphism $\zeta : P \times \mathfrak{g} \xrightarrow{\sim} VP$ on the *vertical tangent bundle* $VP \hookrightarrow TP$.
- X an arbitrary G-space with action map $\varphi : G \times X \to X$.
- The associated X bundle $P_\varphi = P_\varphi[X] = P \times_\varphi X$ over B with projection denoted by π_φ.
- A *connection* ∇ viewed as a linear surjective bundle map $\Pi : TP \to VP$ such that[1] $\Pi^2 = \Pi$ and satisfying equivariance resulting from the action of G.
- The *connection 1-form* $\omega \in \Omega^1(TP) \otimes \mathfrak{g}$ which is, essentially, related to ∇ by the expression $\nabla = \zeta \circ \omega_p$.

3.3.1 Definition and Examples

Heuristically, a connection on P_φ should be a linear projection of TP_φ onto the (associated) vertical VP_φ. This will involve determining exactly what is meant by the associated vertical and also some theory on the push-forward of our action maps.

3.3.2 The associated vertical and differential of the action map

Definition 3.3.1. *The* vertical tangent bundle VP_φ *to* P_φ *is defined as the kernel* $d\pi : TP_\varphi \to TB$.

Proposition 3.3.2. *[Interpreted from [27]]*

$$VP_\varphi := \ker(d\pi_\varphi) \cong P \times_\varphi TX.$$

[1] So to be a projection on each fibre and hence also, since the image is VP, restricting to the identity on $VP \hookrightarrow TP$.

Proof. The composition

$$\Phi : P \times X \xrightarrow{\Pi_P} P \xrightarrow{\pi} B$$

is constant on equivalence classes defined by the diagonal action of G, so descends to our projection $\pi_\varphi : P \times_\varphi X \to B$. If we denote by

$$\psi : G \times P \times X \to P \times X; \quad (g, p, x) \mapsto (\rho(g, p), \varphi(g, x)),$$

the diagonal action of G on $P \times X$, then its differential

$$d\psi : TG \times TP \times TX \to TP \times TX$$

induces an action of TG on $TP \times TX$ which is elaborated upon immediately after this proof.

Now, $d\pi_\varphi$ is the map descended to from the TG-equivariant $d\Phi : TP \times TX \to TB$ whose kernel is clearly

$$\ker(d\Phi) = VP \times TX$$

so that

$$\ker(d\pi_\varphi) = \ker(d\Phi)/TG \cong (P \times \mathfrak{g}) \times_{TG} TX \cong (P \times \mathfrak{g} \times X)/(G \times \mathfrak{g}) \cong P \times_G TX$$

□

Definition 3.3.3. *A connection on the X-bundle $P_\varphi[X]$ associated to P by $\varphi : G \times X \to X$ is realized as an equivariant surjective vector bundle map*

$$\Pi : TP_\varphi[X] \to VP_\varphi$$

such that $\Pi^2 = \Pi$. *Equivalently, the connection 1-form is expressed* $\omega \in \Omega^1(P_\varphi) \otimes VP_\varphi$.

Besides the immediately apparent resemblance to a connection in the principal setting, the following short list of examples should also provide concrete evidence for such a definition.

Example.

1. $X = G$ and $\varphi = \rho$ so that $P_\varphi[X] = P_\rho[G] = P \times_\rho G \cong P$. Here, the associated bundle is the original principal bundle and the definition above is exactly that for a principal bundle.
2. $X = \{x\}$ is a single point so that $P_\varphi[X] = P \times_\varphi \{x\} \cong B$. Here, every section is necessarily constant, the vertical bundle will be zero and the connection is the zero map.
3. $X = V^n$ an n-dimensional complex vector space, so that φ can be interpreted as a linear representation and $P_\varphi = P \times_\varphi V^n = E_\varphi$ an associated vector bundle. Then

$$\omega \in \Omega^1(E_\varphi) \otimes \pi^*(E_\varphi)$$

which resembles the old fashioned connection 1-form on a vector bundle.

The associated bundle and induced connection of great concern for our purposes will be that of Example 2 in Chapter 3. Recall that the bundle $\hom_G(P, P')$ of G-equivariant maps between two principal bundles is realized as the associated bundle $P_\varphi[G] = (P \times_B P') \times_\varphi G$ where $\varphi : G \times G \times X \to X; (g, h, x) \mapsto g^{-1}xh$.

3.3.3 TG and its induced action on $TP \times TX$

Determining exactly how this induced action of TG on $TP \times TX$ looks will play a role in properly defining induced connections and also serves a purpose right here for completeness of the previous proposition.

Remark 1. *Before we begin, it is important to realize that $TG \cong G \times \mathfrak{g}$, as a group, is commonly referred to as the group of* 1-jets *of G.*

Given

$$\psi : G \times P \times X \to P \times X$$

the differential[2] $d\psi$ is expressed, point-wise, by

$$d_{(g,p,x)}\psi = d_g\psi^{(p,x)} + d_{(p,x)}\psi^g$$

where

$$d_g\psi^{(p,x)} : TG \to TP \times TX$$

is the G-differential of $\psi^{(p,x)} : G \to P \times X; g \mapsto (\rho(g,p), \varphi(g,x))$ (defined by holding (p,x) fixed) and similarly for $d_{(p,x)}\psi^g$. Since this is just a combination of the two actions ρ and φ, we can write

$$d_{(p,x)}\psi^g = d\rho_g \oplus d\varphi_g$$

---

[2] inspired by simply looking at a map $f : X \times Y \to Z$ expressed as $z = f(x,y)$ so that, abusing notation $dz = df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$.

acting separately on $TP \times TX$. From this we see that

$$d_{(g,p,x)}\psi(v_g, v_p, v_x) = d_g\psi^{(p,x)}(v_g) + (d\rho_g(v_p), d\varphi_g(v_x)) \in T_{g\cdot p}P \times T_{g\cdot x}X \tag{3.2}$$

and, provided maps are given explicitly, could be easily calculated in local charts using standard techniques from multi-variable calculus. To see that this is indeed an action of the group $TG \cong G \times \mathfrak{g}$ (multiplicative in the G-factor and additive in $\mathfrak{g}$) observe that

- the identity $(e, 0) \in TG$ acts trivially;

$$d_{(e,p,x)}\psi(0, v_p, v_x) = d_e\psi^{(p,x)}(0) + (d\rho_e(v_p), d\varphi_e(v_x)) = (v_p, v_x) \in T_pP \times T_xX$$

- Given two elements $v_g \in T_gG, v_h \in T_hG$, the group operation[3] here is given by

$$v_g \boxplus v_h := dR_h(v_g) + dL_g(v_h) \in T_{gh}G.$$

  So, we wish to see that

$$d\psi(v_g, d\psi(v_h, v_x)) = d\psi(v_g \boxplus v_h, v_x).$$

  Note: For notational convenience and without loss of generality we are assuming ψ is simply an action on X (an arbitrary G-space) which encompasses the particular action we are dealing with.

[3] Note that $TG \cong G \times \mathfrak{g}$ is, in general, non-abelian (unless G is) so that $v_g \boxplus v_h$ is different from $v_h \boxplus v_g$ where there are the same resulting vector, but one lies over the point gh and the other at hg.

By formal expansion and making use of the fact that ψ satisfies $\psi(g,\psi(h,x)) = \psi(gh,x)$ (being a group action), we see

$$\begin{aligned}
d\psi(v_g, d\psi(v_h, v_x)) &= d\psi^{\psi(h,x)}(v_g) + d\psi_g\left(d\psi^x(v_h) + d\psi_h(v_x)\right) \\
&= d\psi^{\psi(h,x)}(v_g) + d\psi_g \circ d\psi^x(v_h) + d\psi_g \circ d\psi_h(v_x) \\
&= d\psi^{\psi(h,x)}(v_g) + d\psi_g \circ d\psi^x(v_h) + d\psi_{gh}(v_x) \\
&= d(\psi^x \circ R_h)(v_g) + d(\psi^x \circ L_g)(v_h) + d\psi_{gh}(v_x) \\
&= d\psi^x(dR_h(v_g) + dL_g(v_h)) + d\psi_{gh}(v_x) \\
&= d\psi(v_g \boxplus v_h, v_x)
\end{aligned}$$

where we have used that, $\psi^{\psi(h,x)} = \psi^x \circ R_h$ and $\psi_g \circ \psi^x = \psi^x \circ L_g$.

3.4 Curvature

The basic idea of curvature is to measure, for a given connection, ∇ (equivalently a horizontal distribution $HP \le TP$), how far the horizontal distribution, HP, is from being an *integrable distribution*[4] . We already know that $VP \cong P \times \mathfrak{g}$ is vacuously integrable but what about HP?

More formally, given a principal bundle with connection (P, ∇) along with a corresponding connection 1-form $\omega \in \Omega^1(P;\mathfrak{g})$, the *curvature* F_∇ is defined in several equivalent ways:

---

[4] Here we resort to the Frobenius integrability saying that HP is integrable if for $X, Y \in \Gamma(HP)$ then $[X,Y] \in \Gamma(HP)$.

1. For tangent vector fields $X, Y \in \Gamma(TM)$, $F_\nabla(X,Y) := d\omega((1-\nabla)(X),(1-\nabla)(Y))$;

   Notice here that $1-\nabla$ represents the *horizontal projection* of TP onto HP, hence this map really acts like

$$\begin{aligned} F(X,Y) &= d\omega((1-\nabla)(X),(1-\nabla)(Y)) \\ &= d\omega(X_H, Y_H) \\ &= X_H \cancel{\omega(Y_H)}^0 - Y_H \cancel{\omega(X_H)}^0 - \omega([X_H, Y_H]) = -\omega([X_H, Y_H]) \end{aligned}$$

   and is measuring how much of $[X_H, Y_H]$ lies in the vertical direction.

   Note furthermore, that this definition of curvature is equivalent to to the pullback by the horizontal projection $F_\nabla = (1-\nabla)^* d\omega$.

2. The curvature is viewed equivalently, as a $\mathfrak{g}$-valued 2-form on P as

$$\Omega := d\omega + \frac{1}{2}[\omega,\omega] \in \Omega^2(P;\mathfrak{g}), \tag{3.3}$$

   where,

$$\frac{1}{2}[\omega,\omega](X,Y) := [\omega(X),\omega(Y)].$$

To verify these are equivalent it suffices to show that

$$d\omega(X_H, Y_H) = d\omega(X,Y) + [\omega(X),\omega(Y)].$$

Indeed, since everything here is bilinear, it suffices to verify for the cases

(i) X, Y are horizontal;

Here, $X = X_H, Y = Y_H$ and we are blessed with the fact that $(1-\nabla)^*\omega = 0$, so this is trivial.

(ii) X, Y are vertical;

Say $X = v_\mu$ and $Y = v_\nu$ are fundamental vector fields generated by $\mu, \nu \in \mathfrak{g}$, then the left hand side is

$$d\omega(X_H, Y_H) = d\omega(0,0) = 0$$

and evaluating the right hand side is

$$\begin{aligned} d\omega(X,Y) + [\omega(X), \omega(Y)] &= X\omega(Y) - Y\omega(X) - \omega([X,Y]) + [\omega(X), \omega(Y)] \\ &= X(\nu) - Y(\mu) - \omega(v_{[\mu,\nu]}) + [\mu, \nu] \\ &= 0 - 0 - [\mu, \nu] + [\mu, \nu] = 0 \end{aligned}$$

(iii) X is horizontal and Y is vertical;

So $X = X_H$ and $Y = v_\eta$, then the left hand side is

$$d\omega(X, 0) = 0$$

and the right is

$$\begin{aligned} d\omega(X_H, v_\eta) + [\omega(X_H), \omega(v_\eta)] &= X_H\omega(v_\eta) - v_\eta\omega(X_H) - \omega([X_H, v_\eta]) + [0, \eta] \\ &= 0 - 0 - \omega([X_H, v_\eta]) + 0 \end{aligned}$$

Now, we rely on the following lemma from [29])

Lemma 3.4.1. *If $X \in \Gamma(HP)$ is a horizontal vector field and $Y = v_\eta \in \Gamma(VP)$ is a vertical vector field generated by $\eta \in \mathfrak{g}$, then $[X,Y]$ is horizontal.*

We have now shown a few interesting details about our curvature tensor.

Lemma 3.4.2. *The curvature form Ω for F defined in equation* (3.3) *(i) vanishes on $VP \times VP$*

(ii) vanishes if and only if HP is an integrable distribution.

(iii) is G-equivariant

Having clearly described the induced action of TG on $TP \times TX$, lets get to our main point of...

3.5 Induced Connections

For the purpose of a single example, namely $\mathrm{Hom}_G(P, P')$, required for the proof of injectivity in the monopole correspondence (Proposition 7.1.11), we must develop a notion of induced connections on associated bundles. Unlike the vector bundle case, where associated bundles such as the dual, determinant, direct sum or tensor product of vector bundles (where induced connections are very naturally described via linear algebraic techniques), our task will be slightly more subtle. Here, I will recall some relevant linear algebraic constructions directly related to the problem at hand. I will also outline some of the thought process involved in the analogous definition for principal bundles.

3.5.1 On vector bundles

Given two vector bundles (with connections) $(E, \nabla^E), (F, \nabla^F)$ of ranks n, m respectively. Then we have the following three important constructions:

1. The *Tensor product bundle $E \otimes F$* with induced connection

$$\nabla := \nabla^E \otimes 1_F + 1_E \otimes \nabla^F.$$

Indeed, this is linear and satisfies the Leibniz rule. Observe, for sections s_E, s_F of E and F respectively, then the induced connection is defined to act as

$$\nabla(s_E \otimes s_F) = \nabla^E s_e \otimes s_F + s_E \otimes \nabla^F s_F.$$

so to say that the connection 1-form is

$$A = A^E \otimes 1_F + 1_E \otimes A^F.$$

2. The *Dual bundle* E^* has induced connection defined as follows: Let $s \in \Gamma(E^*)$ and $t \in \Gamma(E)$ be sections of E^* and E respectively. The application of s on t is then a smooth function $s(t) : B \to \mathbb{C}$ and hence has the usual exterior derivative for a connection. Then, as in the tensor product case, we should have

$$d(s(t)) = (\nabla^* s)(t) + s(\nabla(t)).$$

Locally, then, we have that $A^* = -\bar{A}^T$.

3. The *bundle-morphism bundle* $\mathrm{Hom}(E, F) \cong F \otimes E^*$ is just the combination of our previous two. So, as we have seen

$$\nabla = \nabla^F \otimes 1_{E^*} + 1_F \otimes \nabla^{E^*}$$

and locally

$$A = A^F \otimes 1_{E^*} - 1_F \otimes \overline{A^E}^T.$$

One then has the induced connection ∇ expressed in terms of ∇^E, ∇^F as

$$\boxed{\nabla^F(\tau(s)) = (\nabla\tau)(s) + \tau(\nabla^E s)}$$

where, locally, τ is expressed as $f \otimes e^*$ where f and e^* are sections of F and E respectively. This interpretation for a connection on on $\mathrm{Hom}(E, F)$ will lead to the correct formalism in developing the analogous object for $\mathrm{Hom}_G(P, P')$. However, let us first set aside this specific example and develop some general theory.

3.5.2 On associated bundles

Given connection $\nabla : TP \to VP$ on P, the induced connection ∇_φ on the associated bundle $P_\varphi[X]$ is defined via the following quotient diagram;

$$\begin{tikzcd}
TP \times TX \arrow[r, "\nabla\times\mathbf{1}"] \arrow[d, "d\psi"'] & VP \times TX \arrow[d, "d\psi"] \\
TP \times_{TG} TX \arrow[r, "\nabla_\varphi"] & VP \times_{TG} TX
\end{tikzcd}$$

where Proposition 3.3.2 implies that $VP \times_{TG} TX \cong P \times_\varphi TX = VP_\varphi$, so that $\nabla_\varphi : TP_\varphi \to VP_\varphi$.

Lemma 3.5.1. *∇_φ is a well-defined connection on P_φ.*

Proof. We must first verify that $\nabla \times \mathbf{1}$ is TG-equivariant so that ∇_φ is a properly defined map. For this, consider $v_g \in TG$ and $(v_p, v_x) \in TP \times TX$ and apply,

$$\begin{aligned}
(\nabla \times \mathbf{1})(d\psi(v_g, v_p, v_x)) &= (\nabla \times \mathbf{1})\left(d\psi^{(p,x)}(v_g) + (d\rho_g(v_p), d\varphi_g(v_x)\right) \\
&= (\nabla \times \mathbf{1})d\psi^{(p,x)}(v_g) + (\nabla d\rho_g(v_p), d\varphi_g(v_x)) \\
&= d\psi^{(p,x)}(v_g) + (\nabla d\rho_g(v_p), d\varphi_g(v_x)) \\
&= d\psi(v_g, (\nabla \times \mathbf{1})(v_p, v_x))
\end{aligned}$$

where from the second to third line, we have made the assumption that $d\psi^{(p,x)}(v_g) \in VP \times TX$, so to say that the image TG are vertical where ∇ acts as the identity. Hence. $\nabla \times \mathbf{1}$ is TG-equivariant.

It remains to demonstrate that $\nabla_\varphi^2 = \nabla_\varphi$ and is a surjective vector bundle map. This is essentially trivial since $(\nabla \times \mathbf{1})^2 = \nabla^2 \times \mathbf{1}^2 = \nabla \times \mathbf{1}$, all maps involved in the diagram above are linear and $\nabla : TP \to VP$ is surjective. So ∇_φ is a well-defined connection on P_φ. □

3.5.3 Local form of induced connections in associated fibre bundles

Recall that the local connection form for ∇ on P is given, by pullback with respect to a choice of gauge. That is, for a local section $\sigma : B \to P$ we have

$$\omega = \sigma^* \nabla \in \Omega^1(B) \otimes \mathfrak{g}.$$

Now, this form takes on values in $\mathfrak{g}$ which can be pushed forward to TX via our action map. That is,

$$\omega_\varphi(b, x) := d\varphi^x(\omega(b)).$$

Furthermore, in the case that X is a Lie group, one can push forward by $dL_{x^{-1}}$ (so to land in the tangent space at the identity).

Example 3. *For our purposes, our original principal bundle is the $G \times G$ (fibre product) bundle $P \times_B P'$ and $X = G$, with the action map $\varphi : G \times G \times G \to G$ defined by $\varphi(g, h, x) = g^{-1}xh$. The connection here is the direct product connection $\nabla \oplus \nabla'$ with local expression given most conveniently by the block diagonal matrix, say (ξ, η) where ξ is the connection form for ∇ on P and η for ∇' on P'.*

So, the picture here will look something like

$$\Omega^1(P \times_B P') \otimes \mathfrak{g} \oplus \mathfrak{g} \xrightarrow{\sigma^* \times \sigma'^*} \Omega^1(B) \otimes \mathfrak{g} \oplus \mathfrak{g} \xrightarrow{d\varphi^x} \Omega^1(B) \otimes T_x G$$

$$\nabla \times \nabla' \mapsto (\xi, \eta) \mapsto d\varphi^x(\xi, \eta)$$

Expanding,

$$\begin{aligned} d\varphi^x(\xi, \eta) &= \frac{d}{dt}_{|t=0} \left(e^{-t\xi} \cdot x \cdot e^{t\eta} \right) \\ &= \left[-\xi e^{-t\xi} \cdot x \cdot e^{t\eta} + e^{-t\xi} \cdot x \cdot \eta e^{t\eta} \right]_{t=0} \\ &= -\xi x + x\eta \end{aligned}$$

by abuse of notation. Now, recognize that viewing $\mathfrak{g}$ as the tangent space of G at the identity, we have defined this map point wise and landed in the wrong fibre of TG. That is, we have not recovered a lie algebra valued element.

Indeed,

$$d\varphi^x : T_g G \oplus T_h G \to T_{\varphi^x(g,h)} G$$

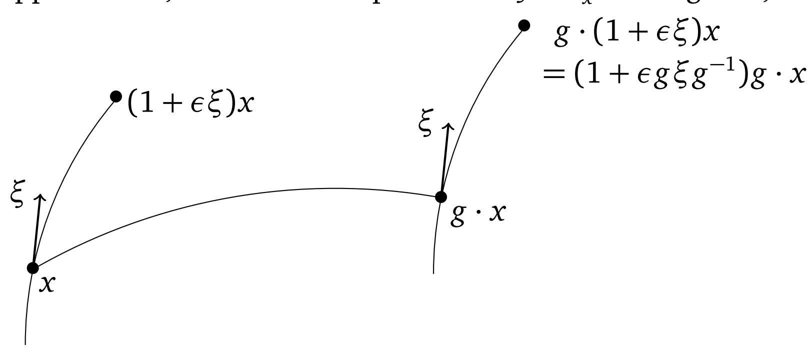
or in our case

$$d\varphi^x : T_e G \oplus T_e G \to T_{\varphi^x(e,e)} G = T_x G.$$

Finally, applying $dL_{x^{-1}}$,

$$
\begin{aligned}
dL_{x^{-1}} \circ d\varphi^x(\xi, \eta) &= dL_{x^{-1}}(-\xi x + x\eta) \\
&= \frac{d}{dt}_{|t=0} \left(x^{-1} \cdot e^{t(-\xi x + x\eta)}\right) \\
&= \left[x^{-1}(-\xi x + x\eta) \cdot e^{t(-\xi x + x\eta)}\right]_{t=0} \\
&= x^{-1}(-\xi x + x\eta) \\
&= \eta - \mathrm{ad}_{x^{-1}}(\xi)
\end{aligned}
$$

The following illustration demonstrates how adjoint actions continue to make unexpected appearances; Let X be a G-space with $\xi \in T_xX$ and $g \in G$, then



CHAPTER 4
Stability Theory on Riemann Surfaces

Stability, in algebraic geometry, is a term used when one is faced with the action of a reductive group on a projective algebraic variety and seeks to construct a 'well-behaved' quotient space. Taking the naive quotient by such an action often results in topological spaces which are no longer algebraic, or even Hausdorff! Fortunately, it is possible to achieve a quotient which falls within the realm of our model by simply disregarding a small family of points (those which are unstable), taking special consideration to the equivalence classes of those which are semi-stable followed by taking the standard topological quotient of this remaining subspace. These issues are of great importance to us here as moduli spaces of bundles are, first and foremost, realized as a space of isomorphism classes generated by the action of our gauge group. The development of this theory is due to D. Mumford [34], where he describes the moduli space of vector bundles explicitly as an example.

4.1 Vector bundle stability

The notion of a stable vector bundle has been widely investigated (cf. [4, 16, 41] and many more) and goes back as far as Mumford's book, [34], on Geometric Invariant Theory (GIT) where vector bundle stability appears as a example towards the end. The common working definition of vector bundle stability is given as a slope comparison between a bundle and its subbundles.

We, thus, adopt the following definition:

Definition 4.1.1. *A complex vector bundle $E \to X$ over a Riemann surface is* stable (semi-stable) *if for all subbundles $V \subseteq E$ we have*

$$\mu(V) < (\leq)\mu(E)$$

where μ is the slope *of E defined in terms of the degree (δ) and rank (rk) as*

$$\mu(E) := \frac{\delta(E)}{rk(E)}.$$

At first glance, this definition might seem unmotivated and strange. However, a geometric explanation of this appears in the work of [9] and [4]. They describe the semi-stable bundles as those flowing to the minima under the Yang-Mills energy functional on the moduli space of bundles. In particular, but not exclusively, the Yang-Mills functional on the moduli space of holomorphic structures for a fixed smooth isomorphism class of complex vector bundle yields the stable elements as minima. Atiyah and Bott have provided a careful account of this statement in [4]. Along with the *Harder-Narisimhan filtration theorem*, the stability theory of vector bundles over a Riemann surface boils down to examining the degree of special subbundles of $\mathrm{Hom}(E, E)$. With this in mind, we can now formulate an equivalent definition of stability that will easily adapt to principal bundles.

A choice of subbundle V of E allows one to write the exact sequence of bundles

$$0 \to V \hookrightarrow E \to E/V \to 0$$

which splits if and only if $E = V \oplus E/V$ and in the smooth category, provides a decomposition of $\mathrm{End}(E)$ as follows:

$$\mathrm{End}(E,E)=E^*\otimes E=(V^*\otimes V)\oplus((E/V)^*\otimes V)\oplus(V^*\otimes E/V)\oplus((E/V)^*\otimes E/V).$$

This can be pictured as the matrix decomposition

$$\left(\begin{array}{c|c} V^*\otimes V & (E/V)^*\otimes V \\ \hline V^*\otimes E/V & (E/V)^*\otimes E/V \end{array}\right).$$

More generally, for a flag of subbundles $0 = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_r = E$ with $D_i := V_i/V_{i-1}$, we can again decompose the endomorphisms in the smooth category to obtain,

$$\mathrm{End}(E)=\bigoplus_{i<j}\mathrm{End}(D_i,D_j)=\bigoplus_{i<j}D_i^*\otimes D_j.$$

The following technical lemma allows us to form and equivalent definition for vector bundle stability and will prove helpful in bridging the gap between vector and principal bundle stability.

Lemma 4.1.2. *If $V \hookrightarrow E$ is a vector subbundle, then*

$$\delta((E/V)^*\otimes V)=\mathrm{rk}(E)\delta(V)-\mathrm{rk}(V)\delta(E). \tag{4.1}$$

Proof. It suffices, by the splitting principle, to verify this claim under the assumption that E is a direct sum of line bundles. With $E = \bigoplus_{i=1}^n L_i$, any proper subbundle V corresponds to a proper subset $S \subset \{1,2,\ldots,n\}$ in the sense that $V = \bigoplus_{j\in S} L_j$ having rank $\mathrm{rk}(V) = |S|$. Computing the Chern character of each V and E/V^* separately gives

$$ch(V)=\sum_{j\in S}(1+l_i) \text{ and } ch((E/V)^*)=\sum_{j\notin S}(1-l_j)$$

where $l_i := c_1(L_i)$ is the degree of the corresponding line bundle in E. The benefit of using the Chern character is to take advantage of the fact that it is a ring homomorphism, so that

$$
\begin{aligned}
ch((E/V)^* \otimes V) = {} & ch((E/V)^*) \cdot ch(V) \\
& = \sum_{j \notin S}(1-l_j) \cdot \sum_{j \in S}(1+l_i) \\
& = (\mathrm{rk}(E) - \mathrm{rk}(V) - \sum_{j \notin S} l_j)(\mathrm{rk}(V) + \sum_{j \in S} l_j) \\
& = (\mathrm{rk}(E) - \mathrm{rk}(V) - \delta(E/V)) \cdot (\mathrm{rk}(V) - \delta(V)) \\
& = (\mathrm{rk}(E) - \mathrm{rk}(V)) \cdot \mathrm{rk}(V) + \mathrm{rk}(E)\delta(V) - \mathrm{rk}(V)(\delta(E/V) + \delta(V)) - \delta(V)\delta(E/V) \\
& = (\mathrm{rk}(E) - \mathrm{rk}(V)) \cdot \mathrm{rk}(V) + \mathrm{rk}(E)\delta(V) - \mathrm{rk}(V)\delta(E) - \delta(V)\delta(E/V)
\end{aligned}
$$

and we have identified the Chern character, via Newton's identities, as being expressed in terms of Chern classes by

$$ch(F) = \mathrm{rk}(F) + c_1(F) + \frac{1}{2}\left(c_1(F)^2 - 2c_2(F)\right) + \cdots$$

implying that

$$\mathrm{rk}((E/V)^* \otimes V) = (\mathrm{rk}(E) - \mathrm{rk}(V)) \cdot \mathrm{rk}(V)$$

and

$$\delta((E/V)^* \otimes V) = \mathrm{rk}(E)\delta(V) - \mathrm{rk}(V)\delta(E)$$

□

Equation (4.1) allows us to immediately see that the stability of E is equivalently stated as

Definition 4.1.3. *A complex vector bundle $E \to X$ over a Riemann surface is* stable (semi-stable) *if for all subbundles $V \subseteq E$ we have*

$$\delta((E/V)^* \otimes V) < (\leq) 0.$$

Example 4. *For the sake of simplicity, suppose that $\delta(E) = 0$ so that a subbundle $V \leq E$ is destabilizing precisely when V has positive degree. That is $\delta(V) > 0$ implies $\mu(V) > 0 = \mu(E)$. Equation* (4.1) *now implies $\delta((E/V)^* \otimes V) = \mathrm{rk}(E) \cdot \delta(V)$ which has the same signature as $\delta(V)$.*

That is to say, the stability of a bundle (at least in the degree zero setting) implies the non-existence of holomorphic sections of $\det(\mathrm{Hom}(E/V, V))$. *It is in this way that a working definition for stable principal bundles is constructed.*

4.2 Principal bundle stability

For a principal G^c bundle $P \to X$, it is important to keep in mind the analogous vector bundle scenario for consistent definitions. The following construction of stable principal bundle should be compared with the standard notion of vector bundle stability outlined above in 4.1.

Remark 2. *Note that viewing G^c as a subgroup of* $\mathrm{GL}_n(\mathbb{C})$ *for some n which is always possible*[1] *since G^c is the complexification of a compact connected real Lie group (which*

[1] since compact groups admit faithful unitary representations so naturally the complexification can be represented faithfully in $\mathrm{GL}_n = \mathrm{U}_n^{\mathbb{C}}$

always admit faithful unitary representations). In this way, the embeddings extend to parabolic subgroups, their Levi subgroups and their unipotent radicals. That is, at least in the case of maximal parabolic subgroups H where only two blocks are required,

$$L \leq \begin{pmatrix} \mathrm{GL}_k & 0 \\ 0 & \mathrm{GL}_{n-k} \end{pmatrix} \text{ and } U \leq \begin{pmatrix} I_k & M_{k,n-k} \\ 0 & I_{n-k} \end{pmatrix}$$

whose Lie algebras are correspondingly of the form

$$\mathfrak{l} = \begin{pmatrix} \mathfrak{l}_1 & 0 \\ 0 & \mathfrak{l}_2 \end{pmatrix} \leq \begin{pmatrix} M_k & 0 \\ 0 & M_{n-k} \end{pmatrix} \text{ and } \mathfrak{u} = \begin{pmatrix} 0 & \mathfrak{u} \\ 0 & 0 \end{pmatrix} \leq \begin{pmatrix} 0 & M_{k,n-k} \\ 0 & 0 \end{pmatrix}.$$

These results, serving only for illustrative purposes, follow directly from a correspondence involving the simple roots of G^c. The important lemma here (found in [14] pp. 147) being about a correspondence between parabolic subgroups (containing a fixed Borel) and subsets of positive simple roots. Overall, the idea is that given any faithful representation $G^c \hookrightarrow \mathrm{GL}_n$ then the image of a parabolic subgroup in G^c will be a subgroup of a parabolic subgroup in GL_n. This winds up being due to the fact that a parabolic subgroup H of G^c can be viewed as the exponential image of its Lie algebra $\mathfrak{h}$ which decomposes into the direct sum $\mathfrak{h} = \mathfrak{b} \oplus (\bigoplus_{\alpha \in S} \mathfrak{h}_\alpha)$ where $S \subset \Delta$ is a collection of positive roots and each $\mathfrak{h}_\alpha$ is the root subspace *of $\mathfrak{g}$ corresponding to α. With these ideas in mind, it suffices to acknowledge that root systems are preserved under faithful representation.*

The *Levi-decomposition* of a maximal parabolic subgroup $H \leq G^c$ is a semi-direct product $H = L \ltimes U$.

An important representation to consider comes from the adjoint action of H on the unipotent Lie algebra $\mathfrak{u}$. Using the 2-block case for illustration, observe that an element $h = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$ of H acts on u as

$$\mathrm{Ad} : H \to \mathrm{GL}(\mathfrak{u})$$

$$\mathrm{Ad}_h(u) = huh^{-1} = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix} \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A^{-1} & -A^{-1}XB \\ 0 & B^{-1} \end{pmatrix} = \begin{pmatrix} 0 & AUB^{-1} \\ 0 & 0 \end{pmatrix}.$$

Perhaps more importantly, its derivative

$$\mathrm{ad} := d\,\mathrm{Ad}\,|_H : \mathfrak{h} \to \mathrm{End}(\mathfrak{u})$$

Notice immediately (even before differentiation) that the U component of H acts trivially on $\mathfrak{u}$, so to say that $U \cong \ker \mathrm{Ad}_H^{\mathfrak{u}}$ and the associated vector bundle of interest $P_H(\mathfrak{u}) := P_H \times_{\mathrm{Ad}_H} \mathfrak{u}$ is isomorphic to just $P_L(\mathfrak{u})$ (in the case of 2-blocks).

Recall, the *adjoint representation* of G on its Lie algebra is a canonical choice of faithful representation and gives rise to the following associated vector bundle

$$\mathrm{Ad}(P) := P \times_G \mathfrak{g}.$$

This vector bundle, the *adjoint bundle* of P, represents precisely the endomorphism bundle of some complex vector bundle, E with structure group $\mathrm{Ad}(G) = \mathrm{GL}(\mathfrak{g})$.

From this perspective, a consistent notion of degree (first Chern class) and stability for principal bundles arises. Indeed, note that a fixed vector subbundle V of E corresponds, in P, to a reduction of structure group $P_H \hookrightarrow P$ whose fibres, H, are a maximal parabolic subgroup of G defined as the point-wise stabilizer, in G, of the subspace. This reduction of structure group forms the bundle whose sections are those of Ad(P) leaving invariant the corresponding sub(vector)bundle.

Remark 3. *More generally, a flag of subbundles* $0 = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_r = E$ *corresponds to a reduction of structure group where the parabolic subgroup* H *is no longer required to be maximal.*

The Lie algebra $\mathfrak{h}$ of H splits into the direct sum of its unipotent part, $\mathfrak{u}$, along with its *Levi subalgebra* $\mathfrak{l} = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ to form a decomposition

$$\mathfrak{u} \hookrightarrow \mathfrak{h} \to \mathfrak{l}.$$

More importantly, this gives rise to the exact sequence of Lie algebras

$$0 \to \mathfrak{u} \hookrightarrow \mathfrak{h} \hookrightarrow \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0.$$

Now, as in the vector bundle scenario, we may view the decomposition of $\mathfrak{g}$ as

$$\mathfrak{g} = \left(\begin{array}{c|c} \mathfrak{l}_1 & \mathfrak{u} \\ \hline \mathfrak{g}/\mathfrak{h} & \mathfrak{l}_2 \end{array}\right).$$

where it should be noted that $\mathfrak{g}/\mathfrak{h} \cong \mathfrak{u}^*$.

With this decomposition and the definition following Lemma 4.1.2, we may define a notion of stability for a principal G-bundle which is consistent with the definition provided by Ramanathan in [38].

Definition 4.2.1. *If* G *is a reductive Lie group, then a holomorphic principal* G^c*-bundle* P *over a Riemann surface is* stable (semi-stable) *if for every reduction to a maximal parabolic subgroup* H *in* G^c *with unipotent lie algebra* $\mathfrak{u}$*, the associated vector bundle*

$P_H \times_H \mathfrak{u}$ has negative (non-positive) degree. That is,

$$\delta(P_H \times_H \mathfrak{u}) < (\leq)\, 0.$$

Remark 4. *1. Overall, from the perspective of stability for vector bundles, one should be thinking of $P_H \times_H \mathfrak{u}$ as being isomorphic to the tensor product of vector bundles $(E/V)^* \otimes V$ where $V \hookrightarrow E$ is the vector sub bundle of E stabilized by the maximal parabolic subgroup H of G.*

2. Equivalently, (P, ∇) is defined to be stable *if for the character, $\chi = \det \circ \mathrm{Ad}^H_{\mathfrak{u}}$ of H, the degree of the induced S^1-bundle $P_H(\chi)$ is negative. That is,*

$$\delta^{\chi}(P, \nabla) := \delta(P_H(\chi)) < 0. \tag{4.2}$$

The degree of a circle bundle here is the integer corresponding to the Euler-class (known to be the same as the first Chern-class natural associated complex line bundle).

4.2.1 The Stability of Hermitian-Einstein bundles

The following result has been adapted from [32] and re-expressed in the language of principal bundles.

Theorem 4.2.2 (Lübke & Teleman)**.** *Hermitian-Einstein G-bundles over Σ are polystable.*

Proof. Suppose that a Hermitian-Einstein G-bundle (P, ∇) admits a holomorphic reduction $P_H \subset P$ to a maximal parabolic subgroup $H \leq G$. The decomposition of $\mathfrak{g}$ induced by $\mathfrak{h}$ allows us to decompose the connection form ω (in a unitary gauge) of ∇ into

$$\omega = \omega_1 + \omega_2 + \mathscr{F}^* + \mathscr{F} \in \mathfrak{g} \otimes \Omega^1(\Sigma)$$

where $\mathfrak{g} = \mathfrak{l}_1 \oplus \mathfrak{l}_2 \oplus \mathfrak{u} \oplus \mathfrak{g}/\mathfrak{h}$

$$\mathscr{F} := \nabla|_{TP_H} - \nabla_H \in \mathscr{A}^{1,0}(\mathfrak{g}/\mathfrak{h})$$

is referred to as the *second fundamental form* of ∇ and visualized matrically as

$$\mathscr{F} = \begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix}.$$

Having this expression for the connection form, the curvature is then decomposed similarly according to $\mathfrak{g} = \mathfrak{l}_1 \oplus \mathfrak{l}_2 \oplus \mathfrak{u} \oplus \mathfrak{g}/\mathfrak{h}$ as

$$\Omega_P = d\omega_P + \omega_P \wedge \omega_P = \underbrace{\Omega_{L_1} + \Omega_{L_2} + \mathscr{F} \wedge \mathscr{F}^*}_{\in(\mathfrak{l}_1 \oplus \mathfrak{l}_2)\otimes\Omega^2(\Sigma)} + \bigstar$$

where $\bigstar$ denotes all terms in $\mathfrak{u} \oplus \mathfrak{g}/\mathfrak{h}$ will be neglected since characters are evaluated on maximal tori. Thus, upon projection to $\mathfrak{l} = \mathfrak{l}_1 \oplus \mathfrak{l}_2$, this is simply expressed

$$\pi_L \circ \Omega_P = \Omega_L + \mathscr{F} \wedge \mathscr{F}^*$$

which globally reads as

$$F_{\pi_L(\nabla)} = \pi_L \circ F_\nabla - \mathscr{F} \wedge \mathscr{F}^*.$$

The Hermitian-Einstein condition on F_∇ allows us to write $F_\nabla = iC \cdot \omega_\Sigma$ and evaluation of the character $\chi = \mathrm{Ad}^H_{\mathfrak{u}}$ on H and will be denoted accordingly as

$$\mathrm{tr}^\chi := d\chi : \mathfrak{t} \to \mathbb{C}.$$

The Chern-form, $c_1^{\chi}(F_{\pi_L(\nabla)})$, associated to χ is defined by the map

$$\begin{aligned} c_1^{\chi} : \mathfrak{l} \otimes \Omega^2(\Sigma) &\to \mathbb{C} \otimes \Omega^2(\Sigma) \\ F_{\pi_L(\nabla)} &\mapsto \tfrac{i}{2\pi}\operatorname{tr}^{\chi}(F_{\pi_L(\nabla)}) \end{aligned}$$

and we find that

$$\begin{aligned} \tfrac{i}{2\pi}\operatorname{tr}^{\chi}(F_{\pi_L(\nabla)}) &= \tfrac{i}{2\pi}\operatorname{tr}^{\chi}(F_{\nabla} - \mathscr{F} \wedge \mathscr{F}^*) \\ &= \tfrac{i}{2\pi}\operatorname{tr}^{\chi}(iC)\omega_{\Sigma} - \tfrac{i}{2\pi}\operatorname{tr}^{\chi}(\mathscr{F} \wedge \mathscr{F}^*) \\ &= -\operatorname{rk}(G)||\mathscr{F}||_{\chi}^2 \cdot \omega_{\Sigma} \end{aligned}$$

Note that $\operatorname{tr}^{\chi}(iC) = 0$ since the centre of the Lie algebra is contained in the kernel of the adjoint representation.

So then

$$\delta(P_L(\chi)) := \int_{\Sigma} c_1^{\chi}(F_L) = -\operatorname{rk}(G)||\mathscr{F}||_{\chi}^2 \int_{\Sigma} \omega_{\Sigma} = -\operatorname{rk}(G)||\mathscr{F}||_{\chi}^2 \cdot \mathrm{Vol}_{\Sigma} \leq 0$$

and equality holds if and only if $\mathscr{F} = 0$ which, furthermore, implies the existence of a reduction to the Levi subgroup of H. $\square$

Part II

Application to Monopoles

CHAPTER 5
Background and basic objects

Throughout this section, we will be working with a complex reductive Lie group G^c of rank n, its maximal compact subgroup G, a Riemann surface Σ with Hermitian metric, a circle S^1 of circumference τ with standard metric and we shall impose the product metric on the manifold $S^1 \times \Sigma$.

5.1 Bogomolny equations and generalizations

Let P be a principal G-bundle on

$$Y := S^1 \times \Sigma \backslash \{p_1, \ldots, p_N\}$$

where each p_i has coordinates $(t_i, z_i) \in S^1 \times \Sigma$ and, for the sake of convenience, the t_i's and z_i's are assumed to be distinct. The restriction of P to sufficiently small spheres about each p_i comes with a reduction to the torus. The bundle on this sphere is defined by some cocharacter $\mu_i : S^1 \to T$ of the maximal real torus $T \subset G$. Suppose that P admits a reduction to G with a G-connection ∇ and a section of the $\mathfrak{g}$-adjoint bundle

$$\Phi \in H^0(Y, \mathrm{ad}(P)),$$

called a *Higgs field*. The triple (P, ∇, Φ) satisfies the *Bogomolny equation* if

$$F_\nabla = *d_\nabla \Phi. \tag{5.1}$$

This is a first order partial differential equation. In fact, this equation is a reduction of the anti self-dual (ASD) equations[1] over $S^1 \times Y$. This reduction and many other simple facts in this preliminary section can be found in [5]. As the purpose here is to extend the main result of [5] from the vector bundle setting to principal bundles with reductive structure group G, the reader is referred to our main source for many technicalities.

Unfortunately equation 5.1 imposes unnecessarily strong constraints on the first Chern classes (i.e. that they average to zero in a suitable sense) so the following, slightly weaker, form will be considered here to allow for solutions with arbitrary degree. That is to say, the triple (P, ∇, Φ) is said to satisfy the *Hermitian-Einstein-Bogomolny (HEB) equation* if

$$F_\nabla - iC \cdot \omega_\Sigma = *d_\nabla \Phi \tag{5.2}$$

where C is in the center, $\mathscr{Z}(\mathfrak{g})$, of $\mathfrak{g}$, $\omega_\Sigma \in \Omega^2(\Sigma)$ represents the Kähler form of our Riemann surface and the only difference here is an extra term which allows for non-zero global central curvature.

Remark 5. *Note that central elements of* $\mathfrak{g}$ *are invariant under conjugation and thus may be equivalently viewed as sections of* $\mathrm{ad}(P)$*.*

Now, since the base is essentially a product manifold, equation (5.2) splits naturally into components and this is the content of the following lemma.

---

[1] A connection A on a bundle over a 4-manifold is *anti-self-dual* if $*F_A = -F_A$

Lemma 5.1.1. *The HEB-equation* (5.2) *can be re-expressed as the following three equations;*

$$F_\Sigma - \nabla_t \Phi = iC, \tag{5.3}$$

$$[\nabla_\Sigma^{0,1}, \nabla_t - i\Phi] = 0 \tag{5.4}$$

and

$$[\nabla_\Sigma^{1,0}, \nabla_t + i\Phi] = 0 \tag{5.5}$$

where F_Σ is the surface component of the curvature tensor (i.e. $F = F_\Sigma \omega + \cdots$) and $\nabla = \nabla_\Sigma^{0,1} d\bar{z} + \nabla_\Sigma^{1,0} dz + \nabla_t dt$.. Note that the third equation is merely the dual of the second.

Proof. This is shown by breaking equation (5.2) into components and remembering that it is "unitary" (in the G-sense). There is the surface component, $\Sigma = \langle x, y\rangle$, and the time component, $\langle t\rangle$. Extracting the surface component of (5.2) gives

$$F_\Sigma - iC = \nabla_t \Phi$$

where we note that the Hodge-star on the right hand side of (5.2) takes surface components to time components and vice-versa. Equation (5.3) is known as the *variational constraint*.

For the equation (5.4), extract components $\langle x, t\rangle$, $\langle y, t\rangle$ and combine them. On the left hand side, the $\langle x, t\rangle$ component of curvature is realized as the commutator $[\nabla_x, \nabla_t]$ which gives the equation

$$[\nabla_x, \nabla_t] - 0 = -\nabla_y \Phi = -[\nabla_y, \Phi]$$

where the negative is recognized as coming from the Hodge-star operator on the ordered basis $\{x, y, t\}$. Similarly, the $\langle y, t\rangle$ component gives

$$[\nabla_y, \nabla_t] = \nabla_x \Phi = [\nabla_x, \Phi].$$

Multiplying the second equation by i and adding these two together gives

$$[\nabla_x + i\nabla_y, \nabla_t] = [-\nabla_y + i\nabla_x, \Phi] = [\nabla_x + i\nabla_y, i\Phi]$$

and simplification of this is precisely equation (5.4). $\square$

5.2 The μ-Dirac monopole

This section is based on standard knowledge about complex line bundles on S^2. The Dirac monopole shall be referred to frequently and built upon within this document, so we provide the necessary formulae here for easy reference. Throughout this section and the remainder of this article, let $\mu \in X_*(T^c)$ be a cocharacter of a fixed complexified maximal torus $T^c \subset G^c$.

Definition 5.2.1. *For any real compact torus T, a μ-*Dirac monopole *is a principal T-bundle over $\mathbb{R}^3\backslash\{0\}$ of degree μ, equipped with a connection ∇ and Higgs field ϕ satisfying the Hermitian-Einstein-Bogomolny equation* (5.2)*, defined as follows:*

On $\mathbb{R}^3$, one has spherical coordinates related to Euclidean by

$$(t, x, y) = (R\cos\theta, R\cos\psi\sin\theta, R\sin\psi\sin\theta)$$

and volume form

$$dV = R^2 \sin\theta dR d\theta d\psi = -r^2 dr d(\cos\theta d\psi).$$

For any $\mu \in X_*(T)$, the cocharacters $\mathrm{Hom}(S^1, T)$, define the principal T-bundle L_μ over $\mathbb{R}^3\backslash\{0\}$ by the transition function $g_\pm = \mu$ from the neighbourhood where $U_+ = \mathbb{R}^3\backslash\{t \geq 0\}$ to $U_- = \mathbb{R}^3\backslash\{t \leq 0\}$. Any section on this bundle may be expressed by maps

$$\sigma_+ : \left\{\mathbb{R}^3\backslash\{t \geq 0\}\right\} \to T$$

$$\sigma_- : \left\{\mathbb{R}^3\backslash\{t \leq 0\}\right\} \to T$$

satisfying $\sigma_- = g_\pm \sigma_+$.

Now, consider a connection defined locally by the Lie-algebra-valued 1-forms

$$A_+ = \frac{i\mu_*}{2}(1+\cos\theta)d\psi, A_- = \frac{i\mu_*}{2}(-1+\cos\theta)d\psi$$

where $\mu_* \in \mathrm{Lie}(T)$ is the differential of μ evaluated at 0 and the Higgs field $\phi = \frac{i\mu_*}{2R}$. It is clear that

$$\nabla\phi = d\phi + [A,\phi] = d\phi = -\frac{i\mu_*}{2R^2}dR = *\left(\frac{i\mu_*}{2}d(\cos\theta d\psi)\right) = *F_\nabla,$$

so that the pair (∇, ϕ) satisfies the Bogomolny equation (5.1).

If $U_\pm$ represents the open cover of $\mathbb{R}^3\backslash\{0\}$ obtained by removing the positive/negative z-axes, the overlap $U_+ \cap U_-$ is homotopy-equivalent to a circle and so the transition functions defining such a bundle can be given, up to homotopy, by a cocharacter $\mu \in X_*(T)$ and sections σ are uniquely expressed as maps $\sigma_\pm : U_\pm \to T$ satisfying $\sigma_+ = \chi \cdot \sigma_-$.

Following this idea one has

Lemma 5.2.2. *The μ-Dirac monopoles are all induced from the standard S^1-Dirac monopole by the cocharacter $\mu \in X_*(T)$.*

Proof. First note that, as for any bundle over a sphere, the smooth isomorphism class of any torus bundle is determined by the homotopy classes of maps $[S^1, T]$ for which we may choose a cocharacter $\mu \in X_*(T)$ as a representative. Thus this torus bundle is isomorphic to the T-bundle induced by μ from the line bundle, L_1 of charge 1 over $\mathbb{R}^3 \backslash \{0\}$. That is, we may consider bundles of the form

$$L_1(\mu) := L_1 \times_{S^1} T$$

where the diagonal action on L_1 is as usual and via μ on T.

Having that any T-bundle on $\mathbb{R}^3 \backslash \{0\}$ realized as $L_1(\mu)$ for some cocharacter $\mu \in X_*(T)$, it is natural to choose the necessary connection and Higgs field to be obtained through μ as well. Indeed, with connection form defined locally on the open cover $U_{\pm} := \mathbb{R}^3 \backslash \{\mp z \geq 0\}$ as

$$\omega_{\pm} = \mu_*(A_{\pm})$$

and Higgs field

$$\Phi := \mu_*(\phi)$$

where A and ϕ are the connection and Higgs field for the model Dirac monopole of charge 1, defined in [5]. It is then tautological to verify that $(L_1(\chi), \omega, \Phi)$ satisfies the monopole equation. $\square$

With this identification, there is no need to pursue the structure of the μ-Dirac monopole any further. Calculations for the change between holomorphic and unitary gauges are the same as for vector bundles (see [5]) and provided here as follows;

Some calculations for the $U(1)$ Dirac monopole

We compute the $\bar{\partial}$ operator $\nabla^{0,1} = \frac{1}{2}(\nabla_x + i\nabla_y)$ and the change of gauge from unitary to holomorphic gauge. That is, a non-unitary trivialization by a section σ such that $\nabla^{0,1}\sigma = 0$. Using the fact that $\cos\theta = t/R$ and $d\psi = \frac{xdy-ydx}{r^2}$ (where $R^2 = t^2+r^2$) between spherical and Euclidean and also that $z = x+iy$ and $\bar{z} = x-iy$ between real and complex coordinates; the connection may be expressed as

$$A_+ = \frac{ik}{2}(1+t/R)\frac{xdy-ydx}{r^2} = \frac{k(1+t/R)}{4r^2}(\bar{z}dz - zd\bar{z})$$

which has $(0,1)$ component

$$A_+^{0,1} = -\frac{k(1+t/R)}{4r^2}zd\bar{z} = -\frac{k(t+\sqrt{t^2+z\bar{z}})}{4z\bar{z}\sqrt{t^2+z\bar{z}}}(zd\bar{z}).$$

Now consider the radial (in r) form

$$-\frac{k(t+\sqrt{t^2+r^2})}{4r^2\sqrt{t^2+r^2}}(2rdr)$$

which has the same $(0,1)$-term (provided one recognizes $r = \sqrt{z\bar{z}}$ so that $2rdr = zd\bar{z}+\bar{z}dz$).

One can change the trivialization to eliminate the $(0,1)$ part by applying a change g_0 which solves

$$\frac{\partial}{\partial r}\ln(g_+) = -\frac{k}{2}\frac{(t+\sqrt{t^2+r^2})}{r\sqrt{t^2+r^2}}.$$

This equation is solved by

$$g_+ = (R-t)^{-\frac{k}{2}}$$

and this transforms the connection, via $A \mapsto A - (dg)g^{-1}$ into

$$-(k/2R)dt + (k(R+t)/2r^2R)\bar{z}dz.$$

Now, in this trivialization, we have $\nabla^{0,1} = \partial^{0,1}$ and $\nabla_t - i\phi = \partial_t$. In this holomorphic gauge, the metric is given by

$$(g_+^* g_+)^{-1} = (R-t)^k.$$

Similarly, for the second chart where $\theta \neq \pi$, the change of trivialization is given by $g_- = (R+t)^{k/2}$, the connection transforms to

$$-(k/2R)dt - (k(R-t)/2r^2R)\bar{z}dz$$

and again we have $\nabla^{0,1} = \partial^{0,1}, \nabla_t - i\phi = \partial_t$ with holomorphic metric given by $(g_-^* g_-)^{-1} = (R+t)^{-k}$. These new trivializations are related by[2]

$$g_- g_\mp g_+^{-1} = z^k.$$

[2] This is subtle but can be verified; Hint: $g_- g_\mp g_+^{-1} = (R+t)^{k/2} e^{ik\psi}(R-t)^{k/2}$, draw a picture and recall that $R^2 = t^2 + x^2 + y^2$.

CHAPTER 6
Singular G-monopoles, holomorphic structures and meromorphic pairs

This chapter introduces and elaborates on all of the essential analytic and topological details involving both singular G-monopoles on $S^1 \times \Sigma$ and their eventual algebraic equivalent, meromorphic bundle pairs. The stability of both is discussed in depth including motivation and consistency arguments from the standard theory.

We then define the map, $\mathscr{H}$, from monopoles to bundle pairs and provide justification that its image lies in the space of stable pairs. Similar proofs of this flavour are found in [17, 32, 28] and heavily rely on the fact that, loosely stated, the curvature in holomorphic subbundles decreases. This is the essential idea used in the proof of the Kobayashi-Hitchin correspondence, but here we will have to adapt the argument for meromorphic Chern forms. On compact complex manifolds, where integration and Chern classes are well-defined, one can further use this to show stability of irreducible Hermitian-Einstein bundles.

The general phenomenon of reduced curvature happens to carry over to a wider class of holomorphic fibrations and can be applied here in the setting of principal bundles. A proof of this reduced curvature in holomorphic subbundles was given for singular Hermitian-Einstein (unitary gauge) monopoles in [5]. However, their proof relies on an inductive argument on the rank of the group and does not carry over to arbitrary reductive gauge since, for example, the exceptional Lie group G_2 does not admit an inductive system. Because of this, it will be of priority to provide proof of stability without the use of induction.

6.1 Definitions

Singular G-monopoles

For a point p in a three manifold Y, let R represent the geodesic distance to p and use a normal coordinate system (t, x, y) centered at p for which the metric in these coordinates is represented by $I + \mathcal{O}(R)$ as $R \to 0$. Let (θ, ψ) represent angular coordinates, as above, for the μ-Dirac monopole on the sphere of constant radius $R = c$ and denote the open ball defined by $R < c$ by B^3.

Definition 6.1.1. *A solution* (P, ∇, Φ) *to the HEB equation* (5.2) *on* $Y \setminus \{p\}$ *has a* singularity of μ-Dirac type *at* p *if:*

- *locally, on* $B^3 \setminus \{p\}$*,* P *admits a reduction of structure group to* T *which is* G*-isomorphic (replacing* unitarily isomorphic*) to the bundle of a* μ*-Dirac monopole* T_μ*, and*
- *under this isomorphism, in the two open sets,* $U_\pm$*, trivializing* P *on* B^3 *induced by standard trivializations of the* T_μ *(so that the* P*-trivializations have transition function given by* μ*), one has, in both trivializations, that*[1]

$$\Phi = \frac{d\mu}{2R} + \mathcal{O}(1) \textit{ and } \nabla(R\Phi) = \mathcal{O}(1)$$

[1] Note here that $\chi_*(0) = \frac{d\chi}{d\psi}|_{\psi=0}$ is intended to mimic the formulation in GL_n which reads

$$i\,\mathrm{diag}(k_1, \ldots, k_n) = \frac{d}{d\psi}|_{\psi=0}\,\mathrm{diag}(e^{ik_1\psi}, \ldots, e^{ik_n\psi})$$

Furthermore, a solution to equation (5.2) *with singularities* $\{p_j\}_{j=1}^N$ *of* μ_j*-Dirac type a is called a* singular G-monopole *(of Dirac-type).*

Remark 6. *The first part of this definition says that a solution with singularity of Dirac type is locally (in a neighbourhood of a singular point) comparable to a* μ*-Dirac monopole. From the perspective of bundle construction via sheaf cohomology, any section* $\sigma \in \Gamma(P)$ *locally takes values in the maximal torus* T *of* G*.*

The second part of the definition ensures, first that the Higgs field respects the local decomposition of P *into Dirac monopoles and the second constraint, via equation* (5.2)*, implies that the curvature is* $\mathcal{O}(R^{-2})$ *and hence integrable in neighbourhoods of singularities. Indeed,*

$$\mathcal{O}(1) = \nabla(R\Phi) = dR \wedge \Phi + R \cdot d_\nabla \Phi = dR \wedge \Phi + R \cdot (*F_\nabla - *iCI_n \cdot \omega_\Sigma)$$

implying

$$*F_\nabla = \frac{1}{R}(\mathcal{O}(1) - dR \wedge \Phi) + *iC \cdot \omega_\Sigma = \frac{\mathcal{O}(1) + \mathcal{O}(R^{-1})}{\mathcal{O}(R)} + \mathcal{O}(1) = \mathcal{O}(R^{-2})$$

as $R \to \infty$*.*

Definition 6.1.2. *The* moduli space of irreducible singular G-monopoles[2] *on* $S^1 \times \Sigma$ *having Dirac singularities of type* μ_j *at* $p_j = (t_j, z_j)$ *for* $j = 1, \dots, N$ *will, from now on, be denoted as*

$$\mathcal{M}_{k_0}^{irr}(G, S^1 \times \Sigma, \{(p_j, \mu_j)\}_{j=1}^N) \tag{6.1}$$

[2] defined here simply as a set

and any triple denoted by (P, ∇, Φ) will be an element of this space.

Holomorphic structures and scattering

A *holomorphic structure* on Y, will be an intermediary object, realized through complexification of P, when passing from monopoles to meromorphic pairs. However, holomorphic structures on Y can be defined independently from the information provided above.

Definition 6.1.3. *A* holomorphic structure *on a G^c-bundle P^c over Y is defined by two commuting, covariant differential operators*

$$\nabla_{\Sigma}^{0,1} : \Gamma(P) \to \Gamma(P) \otimes (T\Sigma^{0,1})^* \text{ and } \nabla_t^c : \Gamma(P) \to \Gamma(P)$$

expressed locally as

$$(\bar{\partial}_z + A_{\Sigma}^{0,1})d\bar{z} \text{ and } \partial_t - i\varphi$$

such that near singularities there exists a reduction to G and ∇_t^c has the asymptotics of a Dirac-singularity.

This definition allows us to understand the meaning of a holomorphic section over our odd-dimensional base manifold.

Definition 6.1.4. *A section $\sigma \in \Gamma(P^c)$ is* holomorphic *if it is parallel with respect to both $\nabla_{\Sigma}^{0,1}$ and ∇_t^c. That is, σ is holomorphic in the usual sense when restricted to any complex slice Σ_t, and satisfies $\nabla_t^c \sigma = 0$ (i.e. respecting the commutative nature of the operators).*

One sees, via Equation (5.4) in Lemma 5.1.1, that the complexification of a monopole (P, ∇, Φ) admits a holomorphic structure. To state things more clearly, that is

Proposition 6.1.5. *There exists a forgetful map from monopoles to holomorphic structures on Y given by*

$$(P, \nabla, \Phi) \mapsto (P^c, \nabla^{0,1}_{\Sigma}, \nabla^c)$$

where $\nabla^{0,1}_{\Sigma} = \nabla^{0,1}_{|\{0\}\times\Sigma}$ *and* $\nabla^c = \nabla_t - i\Phi$.

To holomorphic structures, we may apply the following scattering technique.

The *scattering operator* is the second differential operator, ∇^c, of a holomorphic structure (also, found as the second term in the commutator from equation (5.4)). This is a first order (linear) differential equation in the S^1-direction of $S^1 \times \Sigma$ and amounts to a complex parallel transport[3] operator. That is, setting $P^c := P \times_G G^c$ (i.e. the complexification of P) let parallel sections $\sigma \in \Gamma(P^c)$ satisfy

$$\nabla^c \sigma = 0.$$

As usual, this provides a smooth, fibre-wise isomorphism (at least whenever the curve $[t, t'] \times \{z\}$ contain no singularities)

$$\rho_{t,t'} : P^c_{(t,z)} \to P^c_{(t',z)}$$

defined more precisely as follows:

For each $p \in P^c_{(t,z)}$, let γ be the unique solution to $\nabla^c \gamma = 0$ with $\gamma(t) = p$. Then $\rho_{t,t'}(g) = \gamma(t')$.

[3] Indeed, when the Higgs field is zero, this is exactly the parallel transport in the t-direction.

For intervals $[t, t']$ containing no singularities, integration of the scattering operator defines an isomorphism between $P_{\{t\}\times\Sigma}$ and $P_{\{t'\}\times\Sigma'}$. When there is a singularity at some time $t_i \in (t, t')$ consider, for simplicity, the singularity at the origin of a chart for Σ with time considerations as $-1 < 0 < 1$. The result ([5] Proposition 2.5) is that

Proposition 6.1.6. *In holomorphic trivializations at $t = \pm 1$ the scattering map $\rho_{-1,1}$ is locally expressed in the form*

$$h(z)\mu(z)g(z)$$

with $h, g : U \subset \mathbb{C} \to G$ holomorphic and $\mu : \mathbb{C}^* \to T^c$ is a map into a maximal torus of G. Note that the coordinate z has been chosen so that the singularity is at 0.

We say that a map $\rho : U \to G$ admitting this type of local decomposition is encoded by μ at z.

To see the result in the principal bundle setting, note that by [5], it holds in any representation of G.

Meromorphic pairs

For us, a meromorphic bundle is a pair $(\mathscr{P}, \rho)$ where $\mathscr{P}$ is a holomorphic principal G-bundle over a Riemann surface Σ and $\rho \in \mathscr{M}(\mathrm{Aut}(P))$ is a section of $\mathrm{Aut}(P)$ which is meromorphic over Σ. More concretely,

Definition 6.1.7. *A* meromorphic pair *of type $(\vec{\mu}, \vec{z}) = \{(\mu_1, z_1), \ldots, (\mu_N, z_N)\}$ is a pair (P, ρ) where P is a holomorphic principal G-bundle on Σ and $\rho \in \mathscr{M}(\mathrm{Aut}(P))$ is a meromorphic automorphism of P whose singular data is encoded by the cocharacter μ_j at $z_j \in \Sigma$. So, in fact, $\rho : P \to P$ is an automorphism of P on the Zariski-open $\Sigma \backslash \{z_1, \ldots, z_N\}$.*

An example of such objects is achieved when considering the forgetful map which takes the holomorphic structure of a singular G-monopole (P, ∇, Φ) to $(P^c_t, \rho_{t,t+\tau})$ where $P^c_t := P^c_{|\{t\}\times\Sigma}$ is the restriction of the complexified bundle P^c on $S^1 \times \Sigma$ to time $t \in S^1$ and $\rho_{t,t+\tau}$ the monodromy obtained from scattering along S^1 with $\nabla^c = \nabla_t - i\Phi$. In fact, it was for this particular example that the definition has been formed. More concretely, let us state that

Proposition 6.1.8. *Every holomorphic structure $(P^c, \nabla^{0,1}_\Sigma, \nabla^c)$ on Y gives rise to a meromorphic pair $(\mathscr{P}, \rho)$ by restriction of P^c to any non-singular slice $\{t\} \times \Sigma$ and the monodromy obtained by integrating the scattering operator ∇^c around the circle.*

With this, it will be convenient to denote the space of meromorphic pairs as follows,

Definition 6.1.9. *The* moduli space of meromorphic bundle pairs *over Σ of singular type $\boldsymbol{K} = \{(\mu_j, z_j)\}_{j=1}^N$ will be denoted by*

$$\mathscr{M}(\Sigma, \boldsymbol{K}) \tag{6.2}$$

From monopole to stable pairs

In summary, now that the objects of importance are well-defined and familiar, we define the forgetful map as the composition of maps from monopoles to holomorphic structures and finally to meromorphic pairs

$$\mathscr{H} : \mathscr{M}^{irr}_{k_0}(G, S^1 \times \Sigma, \{(p_i, \vec{k}_i)\}_{i=1}^N) \to \mathscr{M}(\Sigma, \mathbf{K})$$

as

$$\mathscr{H}(P, \nabla, \Phi) := (P^c_{|\{0\}\times\Sigma}, \rho_{0,\tau})$$

where $P^c_{|\{0\}\times\Sigma}$ is the restriction of the complexification $P^c \to Y$ to the slice $\{0\}\times\Sigma$ (note $t = 0$ is assumed to be a non-singular time) and $\rho_{0,\tau}$ is the meromorphic automorphism of P_0 resulting from the monodromy by scattering all the way around the circumference S^1.

Remark 7. *The restriction to any non-singular time $t \in [0,\tau]$ would provide an equivalent correspondence. For this purpose, one may wish to denote the function more precisely as $\mathscr{H}_t(P,\nabla,\Phi) := (P_t, \rho_{t,t+\tau})$. However, we will stick with denoting $\mathscr{H}_0$ by $\mathscr{H}$.*

We first note that the $P^c_{|\{0\}\times\Sigma}$ component in the image of $\mathscr{H}$ is a *holomorphic principal G-bundle* over Σ. This follows because the slice $\{0\}\times\Sigma$ of $S^1\times\Sigma$ has been conveniently chosen not to contain any singular points. Since $P^c_{|\{0\}\times\Sigma}$ is the restriction of a monopole, it is furthermore already equipped with the holomorphic differential $\nabla^{0,1}_{\Sigma}$ (seen first in 5.4).

6.2 The topology and degree of a G-bundle on Y

Recall the topological classification for principal G-bundles over a fixed base manifold Y is given by homotopy classes of maps $[Y;BG]$ where BG is the classifying space of G. In our case, the base manifold $Y = (S^1\times\Sigma)\backslash\{p_i\}_{i=1}^N$ is the complement of a finite collection of points in a compact 3-manifold. Thus Y deformation retracts (i.e. is homotopic) to some 2-dimensional CW-complex with $(N+1)$ cells in dimension 2. Namely $Y \simeq Y^1 \cup Y^2$ is the skeletal decomposition where $Y^2 = \Sigma \cup \left(\bigcup_{i=1}^N S^2_i\right)$. In fact, since there are N punctures in Y, the integer second homology is $H_2(Y;\mathbb{Z}) \cong \mathbb{Z}^{N+1}$.

With G, a compact, connected real algebraic group one finds

$$0 = \pi_0(G) = \pi_1(BG)$$

implying that

$$\pi_1(G) = \pi_2(BG) \cong H_2(BG)$$

where the last equivalence is due to Hurewicz Theorem since $\pi_1(BG) = 1$. Thus, classification of G-bundles on Y amounts to the classification of the bundles on a bouquet of $(N+1)$ 2-spheres since the 1-skeleton contracts to a point after mapping to BG.

Let us consider the characteristic classes obtained by pullback from $H^2(BG)$. We have (by the Universal coefficient theorem and Hurewicz theorem respectively) that

$$H^2(BG, \mathbb{R}) \cong H_2(BG; \mathbb{R})^* \cong H_2(BG; \mathbb{Z}) \otimes \mathbb{R} \cong \pi_1(G) \otimes \mathbb{R}.$$

Following some results involving the theory of Lie groups found in [14] (Chapter 3) we have the exact sequence

$$\mathscr{Z}(G) \hookrightarrow G \twoheadrightarrow \mathrm{Ad}(G)$$

after applying the fundamental group functor implies that

$$\pi_1(\mathscr{Z}(G)) \to \pi_1(G) \twoheadrightarrow \pi_1 \mathrm{Ad}(G).$$

Now, $\pi_1(\mathrm{Ad}(G))$ is finite which implies that, after removing torsion,

$$\pi_1(G) \otimes \mathbb{R} \cong \pi_1(\mathscr{Z}(G)) \otimes \mathbb{R}.$$

We will construct *characteristic classes* for our bundles from the curvature tensor $F_\nabla \in \mathfrak{g} \otimes \Omega^2(Y)$ through a contraction by a character $\chi : G \to S^1$. Notice that characters of G factor through the *commutator subgroup*

$$[G,G] = \{aba^{-1}b^{-1} \in G : a,b \in G\}$$

(since S^1 is abelian) and, as a result, is actually well-defined on the quotient $G/[G,G]$. This quotient group is discretely equivalent to the center, $\mathscr{Z}(G)$, of G in the sense that the right side of the following diagram is a finite covering;

$$\mathscr{Z}(G) \hookrightarrow G \twoheadrightarrow G/[G,G].$$

However, on the level of Lie algebras, this induces an exact sequence

$$\mathscr{Z}(\mathfrak{g}) \hookrightarrow \mathfrak{g} \to \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$$

and hence an isomorphism $\mathscr{Z}(\mathfrak{g}) \cong \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$. Thus, then the derivative of a character $d\chi : \mathfrak{g} \to i\mathbb{R}$ corresponds to a well-defined map $d\underline{\chi} : \mathscr{Z}(\mathfrak{g}) \to i\mathbb{R}$. Also, including exponential maps to the diagram, we see that

$$\begin{tikzcd} & \mathscr{Z}(G) \arrow[r, "\underline{\chi}"] & S^1 \\ \exp^{-1}(1) \arrow[r, hook] & \mathscr{Z}(\mathfrak{g}) \arrow[u, "\exp"] \arrow[r, "d\underline{\chi}"] & i\mathbb{R} \arrow[u, "\exp"'] \end{tikzcd}$$

where $\exp^{-1}(1)$ is canonically isomorphic to $\pi_1(\mathscr{Z}(G))$.

In short, to measure the 'degree' of a monopole (at least, modulo torsion) is to integrate a specific form along surfaces in $S^1 \times \Sigma$. These forms are analogous to the first Chern class from complex geometry.

The first most natural associated circle bundles to a principal G-bundle P arise from extending a group homomorphism $\chi : G \to \mathbb{C}^*$ across the fibers of P. That is, given any character $\chi \in X(G)$ there is an S^1 bundle $P(\chi) := P \times_\chi \mathbb{C}$ to which we have a valid notion of measuring topological degree. The characters of G, however, do not provide us with a means of comparing degrees of P with any of its sub-objects. This will be considered later in the discussion on stability.

Given a singular G monopole $(P\,\nabla, \Phi)$ on Y, i.e. a bundle-connection-Higgs field solution to

$$F_\nabla = iC \cdot \omega_\Sigma + *d_\nabla \Phi$$

we seek to develop

6.2.1 The Chern-form of a monopole

One has a well-defined curvature tensor F_∇ given as a section of $\Omega^2(\mathrm{ad}(P)) = \mathrm{ad}(P) \otimes \bigwedge^2 T^*Y$. In order to obtain a first Chern form (i.e. an element of $H^2(Y, \mathbb{C})$), one must 'trace-out' the Lie algebra portion of this curvature to obtain a gauge-invariant section in $\Omega^2(Y)$. The degree is then measured as an integral of this form over Y. More concretely, to a basis $\{e_i\}_{i=1}^k$ of characters for G, we get Chern forms $\{\omega_i\}$ and so degree maps $\delta_i : H_2(Y) \to \mathbb{R}$ which can be adjusted to take integer values as usual.

Let us take a quick peek at some of the group theory involved with the choice of representation alluded to above.

Groups, representations and characters of importance

Since the Chern form, alluded to above, is (at least locally) a Lie algebra valued function whose integral will be of geometric relevance in two somewhat distinct cases, it will be of importance to first study the representation and character theory associated to reductive groups and their parabolic subgroups.

In brief, and in analogy with vector subbundles we will be concerned with a maximal parabolic subgroup $H \leq G$ along with a corresponding Lie algebra decomposition

$$\mathfrak{g} = \begin{pmatrix} \mathfrak{l}_1 & \mathfrak{u} \\ \mathfrak{g}/\mathfrak{h} & \mathfrak{l}_2 \end{pmatrix}$$

where $\mathfrak{h} = \mathfrak{l}_1 \oplus \mathfrak{l}_2 \oplus u$ is according to the Levi decomposition

$$L \hookrightarrow H \to U.$$

Remark 8. *One important property about this character that is worth mentioning is that the center $\mathscr{Z}(G)$ of G lies in the kernel of this adjoint representation so that the constant scalar portion of our curvature tensor does not affect the 2-form we aim to construct.*

In summary, these characters will serve geometrically relevant in two cases:

1. When $\chi \in X^*(G)$ is any character of G. This is used to determine the *degree* of a monopole and is analogous to complex vector bundles when $\chi = \det$ (the only non-trivial character of GL_n whose derivative at the identity is the usual $\mathrm{tr} : M_n \to \mathbb{C}$)
2. When $\chi = |\mathrm{Ad}_L^{\mathfrak{u}}| \in X^*(L)$ is the unique character of L (the Levi-subgroup of a maximal parabolic subgroup H of G) determined as the top exterior power of the adjoint representation of L on $\mathfrak{u}$ (the corresponding unipotent sub Lie algebra of $\mathfrak{h}$). This will be used to measure the stability of a monopole.

Finally,

Definition 6.2.1. *The* Chern-form *associated to a character $\chi \in X^*(G)$ of a G bundle P will be defined as*

$$c_1^\chi(P, \nabla, \Phi) := \frac{i}{2\pi} \operatorname{tr}^\chi(F_\nabla) \in \Omega^2(Y)$$

where $\operatorname{tr}^\chi = d\chi(0)$ sometimes denoted χ_* is the derivative of the character $\chi : G \to \mathbb{C}^*$ at the identity.

With this, then given our monopole with singularities at $\vec{t}$

Definition 6.2.2. *For any character $\chi \in X_*(G)$, the $(\chi, \vec{t})$-*degree*, $\delta^\chi : H_2(Y) \to \mathbb{Z}$, of a monopole (P, ∇, Φ) (eventually to be referred to as the $(\chi, \vec{t})$-degree of a bundle pair) is the integral of the Chern-form*

$$\delta^\chi(P, \nabla, \Phi) := \frac{1}{\tau} \int_Y c_1^\chi(P, \nabla, \Phi) \wedge dt.$$

Note: *Geometrically, this represents the average (along S^1) of the usual χ-degrees along each holomorphic slice $P_{\{t\}\times\Sigma}$. We note that the degree of a bundle can be evaluated on any two cycle of Y (i.e. $H_2(S^1 \times \Sigma \backslash \{p_i\}_{i=1}^N)$ is large), but that a particular choice has been made here (namely, a weighted sum over all 2-cells in the deformation retraction of Y as a 2-complex).*

Let us now examine the actual integration of the curvature tensor over Y.

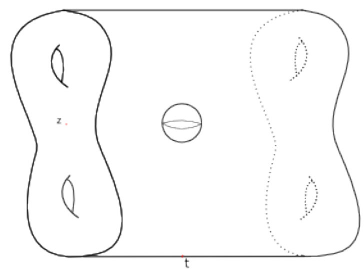
Integration on $S^1 \times \Sigma$

Now, for the purpose of integration, write $Y_\epsilon := Y \backslash \bigcup_{j=1}^N D_\epsilon(p_j)$ to denote a closed subspace of Y. This Y_ϵ limits topologically to Y as a nested family of closed subspaces so that integration on Y is the limit (as ϵ tends to 0) of integration on Y_ϵ.

Stokes' theorem[4] will be of use as

$$\partial\left([t-\epsilon, t+\epsilon]\times\Sigma\backslash D_{\epsilon/2}(p_j)\right) = \Sigma_+ - \Sigma_- - S^2_{\epsilon/2}(p_j) \tag{6.3}$$

corresponding to the following diagram



where $\Sigma_\pm$ denotes the surface $\{t\pm\epsilon\}\times\Sigma$ upon restriction to times $t\pm\epsilon$. Also, even more handy will be the fact that

$$\partial(S^1\times\overline{D_\epsilon(z_j)}\backslash B_{\epsilon/2}(p_j)) = S^1\times\partial\overline{D_\epsilon(z_j)} - S^2_{\epsilon/2}$$

corresponding to a cylindrical neighbourhood of radius ϵ about z_j in the illustration above along all of S^1 (rather than being restricted to the subinterval $[t,t']$).

To measure the *degree* of a bundle, one usually resorts to a well-chosen associated line bundle (the determinant or top exterior power bundle). Given a character $\chi\in X^*(G)$ of G, define the real valued function $f^\chi : S^1\backslash\{t_1,\dots,t_N\}\to\mathbb{R}$ as the integral of the χ-Chern form $c^\chi(P,\nabla,\Phi)$ upon restriction of P to $\{t\}\times\Sigma$ for each

[4] The Fundamental Theorem of Calculus $\int_{\partial M}\omega = \int_M d\omega$

$t \in S^1 \backslash \{t_1, \ldots, t_N\}$. More concretely, $f^{\chi}(t)$ is expressed as

$$f^{\chi}(t) = \frac{i}{2\pi} \int_{\{t\} \times \Sigma} c_1^{\chi}(P, \nabla, \Phi).$$

It is clear (from standard theory of Chern classes) that f^{χ} is, in fact, an integer valued function. Furthermore, the following lemma describes all of its important properties:

Lemma 6.2.3. *The function f_t^{χ} defined above is an integer-valued, piecewise constant function on $S^1 \backslash \{t_i\}_{i=1}^N$ satisfying that for all sufficiently small $\epsilon > 0$ and singular time $t = t_j$ (for some j)*

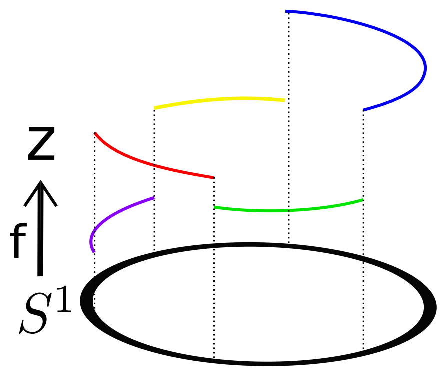
$$f_{t+\epsilon}^{\chi}(P, \nabla) = f_{t-\epsilon}^{\chi}(P, \nabla) + (\chi \circ \mu_j)_*.$$

Note that if no singular time occurs on the interval $[t, t']$, then the boundary $\partial\,([t, t'] \times \Sigma) = \Sigma_{t'} - \Sigma_t$ and

$$f_{t'}^{\chi}(P, \nabla) = f_t^{\chi}(P, \nabla)$$

which says that f_t^{χ} is a piecewise constant function on S^1 whose discontinuities are achieved at the singular times.

Pictorially we have the graph of f^{χ} given as



Proof. That this is integer valued follows directly from the fact that the Chern-form, upon restriction to $\{t\} \times \Sigma$, is an integer cohomology class. Piecewise constancy follows from the fact that the scattering map $\rho_{t,t'}$ for times $t_i < t < t' < t_{i+1}$ between singularities defines an isomorphism $P_t \cong P_{t'}$. Thus, $c_1^{\chi}(P_t, \nabla, \Phi) = c_1^{\chi}(P_{t'}, \nabla, \Phi)$ and certainly then $f_t^{\chi} = f_{t'}^{\chi}$.

Now, on the level of homology in $Y = (S^1 \times \Sigma) \backslash \{p_1, \ldots, p_N\}$ where for any non-singular time t,

$$\Sigma_t := \{t\} \times \Sigma \in H_2(Y)$$

represents the fundamental homology class for the subcurve $\{t\} \times \Sigma \subset Y$. We thus have, with respect to the orientation's prescribed signature in Equation (6.3) and Stokes' theorem

$$f_{t+\epsilon}^{\chi}(\xi) := \int_{\Sigma_{t+\epsilon}} \xi = f_{t-\epsilon}^{\chi}(\xi) + \int_{S_{\epsilon}^2} \xi + \int_Y d\xi$$

for any $\xi \in H^2(Y)$. Luckily here, $\xi = c_1^\chi(F_\nabla) = \mathrm{tr}^\chi F_\nabla$ so

$$d\xi = d \circ \mathrm{tr}^\chi F_\nabla = \mathrm{tr}^\chi \circ dF_\nabla = \mathrm{tr}^\chi(d_\nabla F_\nabla - [\nabla, F_\nabla]) = -\mathrm{tr}^\chi[\nabla, F_\nabla] = 0$$

where we have made use of the Bianchi identity (that $d_\nabla F_\nabla = 0$) and that $[\mathfrak{g}, \mathfrak{g}] \leq \ker \mathrm{tr}^\chi$. So, we have thus far demonstrated

$$f^\chi_{t+\epsilon}(P, \nabla) = f^\chi_{t-\epsilon}(P, \nabla) + \int_{S^2_{\epsilon/2}} \mathrm{tr}^\chi(F_\nabla).$$

It remains to evaluate the integral $\frac{1}{2\pi}\int_{S^2_\epsilon} \mathrm{tr}^\chi(F_\nabla)$. This easily evaluates as

$$\frac{1}{2\pi}\int_{S^2_\epsilon} \mathrm{tr}^\chi(F_\nabla) = (\chi \circ \mu_j)_*.$$

since χ defines an associated line bundle for the T-bundle given by μ so the computation follows from the asymptotic form of the curvature tensor about p_j. □

Lemma 6.2.3 allows us to define and breakdown the *χ-degree* of a monopole $\delta^\chi(P, \nabla, \Phi)$ into the integral of this piecewise constant function f^χ_t as

Corollary 6.2.4. *This integration reduces to discrete inputs*[5] *, and evaluates as*

$$\delta^\chi(P, \nabla, \phi) = \chi_* \circ C \cdot \mathrm{Vol}_\Sigma + \frac{1}{\tau}\sum_{j=1}^{N}(\tau - t_j)(\chi \circ \mu_j)_*$$

for characters $\chi \in X^*(G)$.

---

[5] according the the singularity data of the Higgs-field, volume of domain, rank of fibre and constant scalar factor of curvature

Proof.

$$\begin{aligned}\delta^\chi(P,\nabla,\Phi) = \int_Y c_1^\chi(P,\nabla,\Phi)\wedge dt &= \int_{S^1\setminus\{t_i\}_{i=1}^N} f_t^\chi dt \\ &= \sum_{i=0}^N (t_{i+1}-t_i) f_{t_i^*}^\chi \\ &= \sum_{i=0}^N (t_{i+1}-t_i)\left(f_0^\chi + \sum_{j=1}^i \mathrm{tr}^\chi(\mu_j)\right) \\ &= \chi_*\circ C\cdot\tau\cdot \mathrm{Vol}_\Sigma + \sum_{i=0}^N (t_{i+1}-t_i)\sum_{j=1}^i \mathrm{tr}^\chi(\mu_j) \\ &= \chi_*\circ C\cdot\tau\cdot \mathrm{Vol}_\Sigma + \sum_{j=0}^N (\tau - t_j)\,\mathrm{tr}^\chi(\mu_j)\end{aligned}$$

where $t_i^* \in (t_i, t_{i+1})$ represents any point between the i^{th} singular interval and the first line makes use of the fact that an integral over a codimension 0 submanifold of Y will have the same value.

□

This now looks very much like the formula provided for the average and $\vec{t}$-degree given in [5] and will be the definition of stability for bundle pairs $(\mathscr{P},\rho)$ of the next section.

6.3 Stability theory of monopoles and pairs

Definition 6.3.1. *A holomorphic structure* $(P^c, \nabla^{0,1}, \nabla^c)$ *is* stable *if for every maximal parabolic subgroup* $H \leq G^c$ *such that there is an* H*-invariant holomorphic reduction to* $P_H \subset P^c$*,*

$$\delta^\chi(P_L) < 0$$

where $\chi = \det \circ \mathrm{Ad}_L^{\mathfrak{u}}$ *is the unique character of* L *(coming from the Levi-decomposition* $H = L \ltimes U$*) whose derivative is the sum of the roots of* U*.*

Proposition 6.3.2. *Let* (P, ∇, Φ) *be a singular* G *monopole, then the holomorphic structure* $(P^c, \nabla_\Sigma^{0,1}, \nabla^c)$ *obtained from the monopole satisfies that if* H *is a maximal parabolic subgroup of* G^c *such that* $P^c = P \times_G G^c$ *admits a holomorphic reduction to* P_H*, then*

$$\delta^\chi(P_H, \nabla, \phi) = -\int_Y ||\mathscr{F}||_\chi^2 \cdot \omega_\Sigma \wedge dt, \quad \chi = \det \circ \mathrm{Ad}_L^{\mathfrak{u}} \in X^*(L).$$

Hence, the holomorphic structure is stable.

Proof. Since $P^c = P \times_G G^c$ admits a holomorphic reduction to P_H, it then projects to an L-bundle $P_L = \pi_L \circ P_H$. On the level of adjoint bundles, with the Levi-subalgebra $\mathfrak{l} \leq \mathfrak{h}$ according to the proof in Section 4.2.1, its curvature satisfies the following relation with the total curvature and its *second fundamental form* $\mathscr{F}$

$$F_{\pi_L(\nabla)} = \pi_L \circ F_\nabla - \mathscr{F} \wedge \mathscr{F}^*.$$

So, for the character $\chi = |\mathrm{Ad}_L^{\mathfrak{u}}|$ of L, by definition, we have

$$\delta^\chi(P_L) = \lim_{\epsilon \to 0} \int_{Y_\epsilon} c_1^\chi(F_{\pi_L(\nabla)}) \wedge dt.$$

Upon substituting the HEB equation (5.2) for F_∇, this evaluates as

$$\begin{aligned}
\lim_{\epsilon\to 0}\frac{i}{2\pi}\int_{Y_{\epsilon/2}} & \operatorname{tr}^\chi(iC\cdot\omega_\Sigma + *d_\nabla\Phi - \mathscr{F}\wedge\mathscr{F}^*)\wedge dt \\
&= -\int_Y ||\mathscr{F}||_\chi^2\cdot\omega_\Sigma\wedge dt + \lim_{\epsilon\to 0}\frac{i}{2\pi}\int_{Y_\epsilon}\partial_t\Phi^\chi dt\wedge\omega_\Sigma \\
&< \lim_{\epsilon\to 0}\frac{i}{2\pi}\int_{Y_\epsilon}\partial_t\Phi^\chi dt\wedge\omega_\Sigma
\end{aligned}$$

since $\mathscr{Z}(\mathfrak{g}_\mathbb{C})\subset\ker\chi$ (implying that $\operatorname{tr}^\chi(C)=0$) and, although non-constant, $||\mathscr{F}||_\chi^2$ is strictly negative (when our monopole is irreducible). We now want to demonstrate that the remaining term to vanishes.

Notice immediately that the remaining term is reduced to

$$\lim_{\epsilon\to 0}\frac{1}{2\pi}\sum_{j=1}^N\int_{S^1\times\overline{D_\epsilon(z_j)}\setminus B_{\epsilon/2}(p_j)}\partial_t i\Phi^\chi\cdot\omega_\Sigma\wedge dt$$

because away from any nonsingular circle ($S^1\times\{z_j\}$) this amounts to

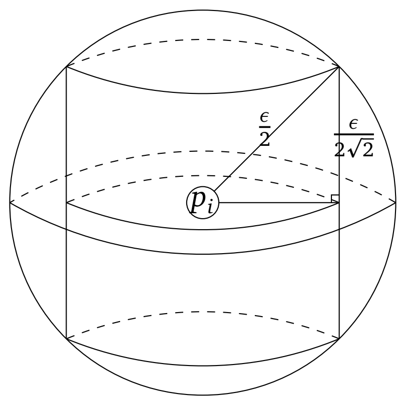
$$\int_{S^1}\partial_t i\Phi^\chi dt = 0$$

being the integral of the derivative over a closed interval.

Now, writing $\partial_t i\Phi^\chi dt\wedge\omega_\Sigma = d(i\Phi^\chi\omega_\Sigma)$ as an exact form and by Stokes' theorem, each

$$\int_{S^1\times\overline{D_\epsilon(z_j)}\setminus B_{\epsilon/2}(p_j)}\partial_t i\Phi^\chi\cdot\omega_\Sigma\wedge dt = \int_{S^1\times S^1_\epsilon} i\Phi^\chi\cdot\omega_\Sigma - \int_{S^2_{\epsilon/2}} i\Phi^\chi\cdot\omega_\Sigma$$

The first term here vanishes in the limit as $\epsilon \to 0$ and the second term is reinterpreted in a different coordinate system. Currently, there are two local coordinate systems under consideration. Namely, the connection and Higgs field have been expressed in terms of the spherical coordinates $\{dR, d\theta, d\psi\}$ whereas the form of integration is in terms of 'holomorphic-Euclidean' coordinates $\{dz, d\bar{z}, dt\}$. A happy medium for choice of coordinates here will be to choose a cylinder inscribed in the $\epsilon/2$-ball whose dimensions are chosen to be radius $\epsilon/2\sqrt{2}$ and height $\epsilon/\sqrt{2}$ (These are homotopy equivalent in Y and hence have the same values upon integration).



Upon recognizing changing the domain of integration to the cylinder, the second term is then realized as being bounded above by $\sup_{C_\epsilon}(i\Phi^\chi) \cdot 2 \cdot \mathrm{Vol}_{D_\epsilon}$ which is of order $\mathcal{O}(\epsilon^2)$ according to the volume of the caps on the cylinder and thus limits to zero. That is to say,

$$\lim_{\epsilon \to 0} \frac{1}{2\pi} \sum_{j=1}^{N} \int_{S^1 \times \overline{D_\epsilon(z_j)} \setminus B_{\epsilon/2}(p_j)} \partial_t i\Phi^\chi \cdot \omega_\Sigma \wedge dt = 0$$

as required.

$\square$

Consistency with vector bundles

Our above discussion verifies consistency of the stability definition with both monopoles and vector spaces. Indeed, examining this in the language of associated vector bundles amounts to the following;

$$
\begin{aligned}
&\delta_{\vec{t}}(\mathscr{P}_H \times_H \mathfrak{u}) := \delta_{\vec{t}}((E/V)^* \otimes V) \\
&\quad = \sum_{i=0}^{N} (t_{i+1} - t_i) \left(c_1((E/V)^* \otimes V)) + \sum_{l \leq i} \left[n \sum_{j \in S} k_j^l - m \sum_{j=1}^{n} k_j^l \right] \right) \\
&\quad = \tau(nc_1(V) - mc_1(E)) + \sum_{i=1}^{N} (\tau - t_i) \left(n \sum_{j \in S} k_j^i - m \sum_{j=1}^{n} k_j^i \right) \\
&\quad = \tau(nc_1(V) - mc_1(E)) + \tau \left(n \cdot \underbrace{\sum_{i=1}^{N} \sum_{j \in S} k_j^i}_{=0} - m \cdot \underbrace{\sum_{i=1}^{N} \sum_{j=1}^{n} k_j^i}_{=0} \right) - \sum_{i=1}^{N} \left(n \sum_{j \in S} k_j^i - m \sum_{j=1}^{n} k_j^i \right) t_i \\
&\quad = \tau(nc_1(V) - mc_1(E)) - \sum_{i=1}^{N} \left(n \sum_{j \in S} k_j^i - m \sum_{j=1}^{n} k_j^i \right) t_i
\end{aligned}
$$

which is ≤ 0 if and only if

$$
\frac{1}{m} \left(\tau c_1(V) - \sum_{i=1}^{N} \sum_{j \in S} k_j^i t_i \right) \leq \frac{1}{n} \left(\tau c_1(E) - \sum_{j=1}^{n} k_j^i t_i \right).
$$

That is to say, V was initially chosen as a $\vec{t}$-stable subbundle of E in the sense of [5]. Note that, here V is meant to represent the vector subbundle of the associated vector bundle E to $\mathscr{P}$ (via some faithful complex representation) corresponding to the choice of parabolic subgroup.

The $\vec{t}$-degree and stability of a bundle pair

The approach taken in [5] to define the proper notion of stability for the data contained in a bundle pair $(\mathscr{P}, \rho)$ is to examine the average degree of a monopole (P, ∇, Φ) defined over $Y = (S^1 \times \Sigma) \setminus \{p_i\}_{i=1}^N$ and 'massage' this definition until it can be computed using only the information contained in the image $(\mathscr{P}, \rho) = \mathscr{H}(P, \nabla, \Phi)$. That is to say, taking what was initially the integral $\int_Y c_1^\chi(P, \nabla, \Phi) \wedge dt$ of the first Chern form over the 3-manifold Y and discretizing it as a finite sum of only the information contained in the holomorphic bundle $\mathscr{P}$ and the singularity data contained in ρ. We have already developed some useful tools for integrating the Chern form of a singular G-monopole so now we may provide definitions without further verification.

We are now able to state two definitions.

Definition 6.3.3. *Let $(\mathscr{P}, \rho)$ be a meromorphic pair*

1. The $(\chi, \vec{t})$-degree *of a bundle pair $(\mathscr{P}, \rho)$ is defined as*

$$\delta_{\vec{t}}^{\chi}(\mathscr{P}, \rho) = \sum_{i=0}^{N} (t_{i+1} - t_i) \left(\delta^{\chi}(\mathscr{P}) + \sum_{j=1}^{i} (\chi \circ \mu_j)_*(0) \right)$$

where $\delta^{\chi}(\mathscr{P})$ is previously defined in Equation (4.2).

2. A bundle pair $(\mathscr{P}, \rho)$ is $\vec{t}$-stable *if for every ρ-invariant holomorphic reduction to $\mathscr{P}_H \subset \mathscr{P}$ where $H \leq G^c$ is a maximal parabolic subgroup of G one has*

$$\delta^{|\mathrm{Ad}_L^{\mathfrak{u}}|}(\mathscr{P}_H, \rho) < 0.$$

Note that $|\mathrm{Ad}_L^{\mathfrak{u}}|$ (as a character of H) is the determinant of the adjoint representation of L on $\mathfrak{u}$, where $H = L \ltimes U$ is its Levi-decomposition and $\mathfrak{u} = \mathrm{Lie}(U)$.

Now, adopting notation from the previous defined space of meromorphic pairs,

Definition 6.3.4. *The* moduli space of $\vec{t}$-stable meromorphic bundle pairs *over* Σ *of singular type* $\boldsymbol{K} = \{(\mu_j, z_j)\}_{j=1}^N$ *will be denoted by*

$$\mathscr{M}_{\vec{t}s}(\Sigma, \boldsymbol{K}) \tag{6.4}$$

and the lack of subscript $\vec{t}s$ *here will denote the larger space of just meromorphic pairs discussed earlier.*

We thus define that a holomorphic structure is $\vec{t}$-stable if its associated pair is and we have shown (through discretizing the integration - Lemma 6.2.3) that the holomorphic structure associated to an irreducible singular monopole is $\vec{t}$-stable. In more appropriate terminology, that is the following statement.

Proposition 6.3.5. *If* $(P, \nabla, \Phi) \in \mathscr{M}_{k_0}^{irr}(G, S^1 \times \Sigma, \{(p_i, \vec{k}_i)\}_{i=1}^N)$ *then its image* $(P_0, \rho_{0,\tau})$ *under* $\mathscr{H}$ *is* $\vec{t}$*-stable.*

Proof. Everything for this proof has already been set up and only requires a small argument. Suppose $(\mathscr{P}, \rho) = \mathscr{H}(P, \nabla, \Phi)$ and let H be a maximal parabolic subgroup of G^c corresponding to a holomorphic, ρ-invariant reduction $\mathscr{P}_H$ of $\mathscr{P}$. We need that $\delta_{\vec{t}}^{|\mathrm{Ad}_L^{\mathfrak{u}}|}(\mathscr{P}_L)$ is negative, which has already been verified and is the result of Proposition 6.3.2. $\square$

CHAPTER 7
Correspondence Theorem

Now that the objects of interest are well-defined and the stability theory has been taken care of, this chapter is focused solely on the proof of the bijective correspondence theorem stated below. The surjectivity of $\mathscr{H}$ (defined in the previous chapter) is quite analytic and heavily relies on the proof found in [5]. The injectivity of $\mathscr{H}$ also, somewhat, follows their techniques but depends more on the theory of induced connections on associated principal bundles (developed at the end of Chapter 3).

7.1 Equivalence between stable pairs and monopoles

In this section, the bijective equivalence is stated slightly differently from the main Theorem 1.0.1 but is still equivalent. The theorem is merely restated with respect to the notation and language developed thus far.

Theorem 7.1.1. *If $\{p_i\}_{i=1}^N$ is a finite subset of $S^1 \times \Sigma$ which projects to N different points on Σ then the map*

$$\mathscr{H} : \mathscr{M}_{k_0}^{irr}(S^1 \times \Sigma, \{p_i, \mu_i\}_{i=1}^N) \to \mathscr{M}_{\vec{t}s}(\Sigma, k_0, \mathbf{K})$$

$$(P, \nabla, \Phi) \to (P_0, \rho_{0,\tau})$$

is a bijection. [1]

The proof demonstrated throughout the following two propositions 7.1.2 for surjectivity and 7.1.11 for injectivity. To get to the heart of the proof, let us first recall the objects at hand. The moduli space $\mathscr{M}^{irr}_{k_0}(S^1 \times \Sigma, \{p_i, \mu_i\}_{i=1}^N)$ (on the left) represents equivalences classes of triples (P, ∇, Φ) satisfying Equation (5.2) and having μ-Dirac singularities of type μ_i at each $p_i \in S^1 \times \Sigma$ and topological type k_0 along $\{t_0\} \times \Sigma$. For notational simplicity alone, assume that within the set $\{p_i = (t_i, z_i)\}_{i=1}^N$, none of the t_i's or z_i's coincide. Each triplet (P, ∇, Φ) consists of a principal G-bundle on $S^1 \times \Sigma \backslash \{p_1, \ldots, p_N\}$, a connection $\nabla \in \Omega^1(\mathrm{ad}(P))$ and a meromorphic Higgs field $\Phi \in \mathscr{M}(\mathrm{ad}(P))$.

The moduli space $\mathscr{M}_{\vec{t}s}(\Sigma, k_0, \mathbf{K})$ (on the right) consists of equivalence classes of $\vec{t}$-stable pairs (P, ρ) where P is a holomorphic principal G-bundle on Σ of topological type k_0 and $\rho \in \mathscr{M}(\mathrm{Aut}(P))$ is a meromorphic automorphism with singularities prescribed by the $(\mu_i, z_i) \in \mathbf{K}$.

Surjectivity

Proposition 7.1.2. *For any $\vec{t}$-stable pair $(\mathscr{P}, \rho)$ on Σ of type $\mathbf{K} = ((\mu_1, z_1), \ldots, (\mu_N, z_N))$ with singular time data $0 < t_1 \leq t_2 \leq \cdots \leq t_n < \tau$, there is a singular G-monopole on $S^1 \times \Sigma$ with Dirac singularities of weight μ_j at $p_j = (t_j, z_j)$ for which $\mathscr{H}(P, \nabla, \phi) = (\mathscr{P}, \rho)$.*

[1] The statement when irreducibility is removed is between poly-stable pairs.

Proof. This proof is structured almost exactly as in our main source of reference [5] which makes use of Pauly's application, [37], of the Hopf fibration and Simpson's heat flow on the space of metrics [42] . Some images are provided here to aid in understanding and the necessary generalizations to adapt the proof for general G are elaborated upon. The reader is encouraged to compare the proofs. Let us briefly summarize the four main steps of this proof;

- ρ is used to extend $\mathscr{P}$ to a bundle P on $Y := (S^1 \times \Sigma)\backslash\{p_1, \dots, p_N\}$ having the correct twisting around spheres about the p_j's and a holomorphic structure. Thus it will be holomorphic on all Σ_t and will lift to a holomorphic bundle $\bar{P}$ on the (open) complex manifold $X = S^1 \times Y$ (subset of $\overline{X} = S^1 \times S^1 \times \Sigma$). Furthermore, $\bar{P}$ is invariant under the action of S^1 on the left factor.
- Since $\bar{P}$ has a holomorphic structure, for any Hermitian metric (i.e. any reduction of P to G, i.e. a section of $P(G^c/G)$), there is a unique metric connection[2] which is compatible with the holomorphic structure. Choose a Hermitian metric on $\bar{P}$ whose induced connection around the j^{th} singularity is that of a μ-Dirac monopole of weight μ_j.
- This metric serves as an initial metric for the heat flow of Simpson's paper [42]. Taking the limit as time tends to infinity produces a principal-HE connection on $\bar{P}$ which is invariant under the S^1 action and so, descends to a bundle over Y. This will be our singular G-monopole, however one further analytic technicality remains.

[2] this is the Chern connection in the case of a $U_n \subset \mathrm{GL}_n(\mathbb{C})$ gauge.

- Simpson's theorem does not immediately provide the necessary regularity at the singular points. To see they are indeed of Dirac type (in the limit), the proof is finished by lifting locally on 3-balls using the Hopf map $B^4 \to B^3$.

Step 1: The bundle P on Y will be the top left most portion of the following diagram of bundles:

$$\begin{tikzcd}
\widetilde{\pi^*(\mathscr{P})} \arrow[d] & \pi^*(\mathscr{P}) \arrow[l, "\tilde{q}"'] \arrow[r] \arrow[d] & \mathscr{P} \arrow[d] \\
Y & \tilde{Y} \arrow[l, "q"'] \arrow[r, "\pi"] & \Sigma
\end{tikzcd}$$

where $\pi : \tilde{Y} \to \Sigma$ is the natural projection of a 'doubled-up' version[3]

$$\tilde{Y} = ((-\tau, \tau) \times \Sigma) \setminus \cup_j \big((-\tau, t_j - \tau) \cup (t_j, \tau)\big) \times \{z_j\}$$

of Y (shown in Figure 1)

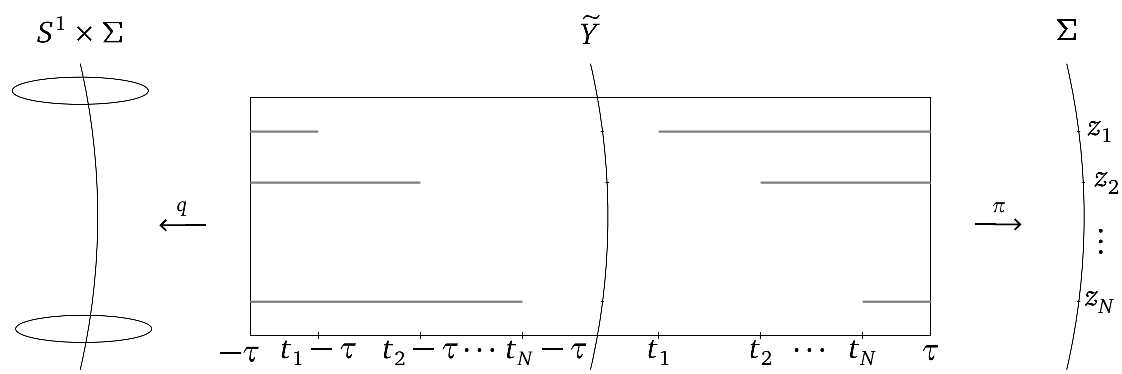


Figure 1

[3] recall that τ here refers to the circumference of S^1 in our domain

and $q : \tilde{Y} \to Y$ is generically a double cover defined by the identification[4] $(t,z) \sim (t+\tau,z)$ (i.e. so that in the pre-image $q^{-1}(t,z) = \{(t,z),(t+\tau,z)\}$). Essentially, q wraps $\tilde{Y}$ around itself so, after gluing, to produce the S^1 part of Y from intervals.

The pull-back bundle $\pi^*(\mathscr{P})$ is always well-defined and the fact that ρ is locally represented (cf. Proposition 6.1.6) around the singularities z_j as $h_j(z)\cdot\mu_j(z)\cdot g_j(z)$ will indeed restrict to the Dirac monopole bundle T_{μ_j} in a punctured neighbourhood of p_j.

The quotient of this pull-back bundle $P := \widetilde{\pi^*(\mathscr{P})}$ is given by the equivalence relation $(t,z,v) \sim (t+\tau,z,\rho(z)v)$ with $t \in (-\tau,0), z \in \Sigma$ and $v \in \mathscr{P}_z$ (the fibre of $\mathscr{P}$ at z). Finally, P is trivially lifted via the pull-back projection $\pi_2 : X \to Y$ to an S^1-invariant holomorphic principal G-bundle on the open complex 4-manifold $X = S^1 \times Y$. In terms of the diagram of bundles, this is appended to the left hand side as follows:

$$\begin{tikzcd}
\bar{P} := \pi_2^*(P) \arrow[r] \arrow[d] & P \arrow[d] & \pi^*(\mathscr{P}) \arrow[l, "\tilde{q}"'] \arrow[r] \arrow[d] & \mathscr{P} \arrow[d] \\
X \arrow[r, "\pi_2"] & Y & \tilde{Y} \arrow[l, "q"'] \arrow[r, "\pi"] & \Sigma
\end{tikzcd}$$

This completes the first step.

Step 2: As briefly described above, this involves choosing an appropriate Hermitian metric (i.e. a reduction to G - recall that G^c/G is contractible) so that the induced

[4] Note the implied use of additive modular arithmetic on the interval $[-\tau,\tau]$ here.

”Chern” connection has the proper singular data to represent the Dirac singularities in neighbourhoods of the points p_i. The metric is constructed by choosing a finite open cover of Y (by $2N+2$ open sets), specifying the metric on each piece and pasting it all together via a partition of unity. Alternatively, or equivalently the bundle can be constructed by specifying transition functions on the overlaps of this special open cover from which a natural choice of Hermitian metric will arise. In either case, it will be necessary to specify the open cover first and then describe how this works from both perspectives.

To define the cover, choose sufficiently small (disjoint) open disks $D_1, \ldots, D_N$ in Σ about each of the z_i's as well as an extra, entirely disjoint, disk D_0. The last disk will be where the initial curvature is concentrated so to accommodate for the first Chern class of bundle $\mathscr{P}$ (of degree k_0) initially chosen for our stable pair. One should recognize from the choice of cover that we are going to prescribe that the metric is altered when we move forward in time and reach singular points. At these points, a Dirac metric is glued in accordingly to each disk about the singularities. Thus, let C_i for $i = 1, \ldots, N$ be another family of disks about each z_i which are properly contained in the D_i's. Furthermore, let ϵ be chosen such that $4\epsilon < \min(t_1, t_2 - t_1, \ldots, t_N - t_{N-1}, \tau - t_N)$. Our cover is then defined as

$$
\begin{aligned}
U_0 &= ((-2\epsilon, t_N + 2\epsilon) \times \Sigma) \setminus \left(\cup_j (t_j - \epsilon, t_N + 2\epsilon) \times C_j\right) \\
U_{N+1} &= (t_N + \epsilon, \tau - \epsilon) \times \Sigma \\
U_{j-} &= \left((t_j - 2\epsilon, t_j + 2\epsilon) \times D_j\right) \setminus \left((t_j, t_j + 2\epsilon) \times \{z_j\}\right) \\
U_{j+} &= \left((t_j - 2\epsilon, t_j + 2\epsilon) \times D_j\right) \setminus \left((t_j + 2\epsilon, t_j) \times \{z_j\}\right)
\end{aligned}
$$

for $j = 1, \ldots, N$ and an image of this open cover is provided in Figure 2.

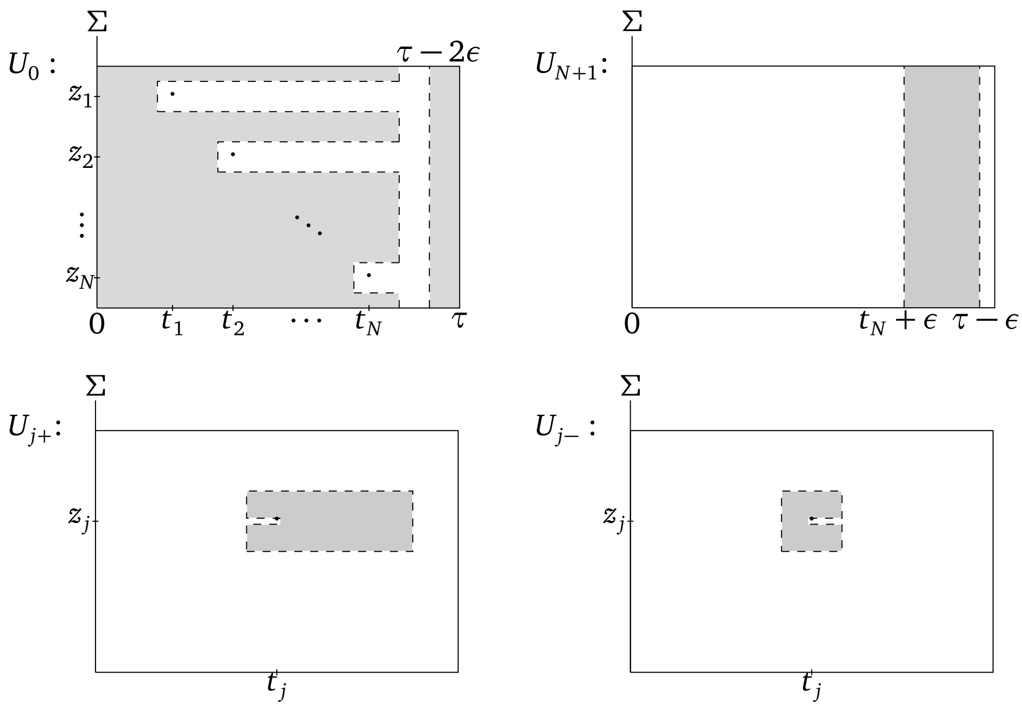


Figure 2.

On this cover, transition functions are specified as

$$
\varphi_{0,j-} = g_j,\ \varphi_{j-,j+} = \mu_j,\ \varphi_{0,j+} = g_j \cdot \mu_j,\ \varphi_{j+,N+1} = h_j
$$

and

$$\varphi_{0,N+1} = \begin{cases} \rho^{-1},\ t \in (t_N + \epsilon, t_N + 2\epsilon) \\ \\ 1,\ t \in (\tau - 2\epsilon, \tau - \epsilon) \end{cases} .$$

Now, the bundle and its transition functions reflect those of μ-Dirac monopoles on $U_{j\pm}$. Choose the hermitian metrics $\mu_j(R-t)$ on U_{j-} and $\mu_j(R+t)$ on U_{j+}. These are compatible with each other under change of basis and are patched together, along with the metric lifted from P on U_0, U_{N+1}, via a partition of unity

This metric k can be lifted to a metric $\bar{k}$ on $\overline{P}$ subject to the following properties:

Lemma 7.1.3. *The pair $(\overline{P}, \bar{k})$ above satisfies*

- *$\bar{P}$ is invariant under the S^1 action on the left factor of X*
- *$\bar{k}$ is S^1 invariant.*
- *In neighbourhoods of inverse image of the p_j's the pair $(\bar{P}, \bar{k})$ corresponds to an S^1-invariant instanton of charge specified by μ_j .*
- *$(\bar{P}, \bar{k})$ satisfies a bound $|\Lambda F_{\bar{k}}| \leq c < \infty$*

This completes the second step.

Step 3: Taking the metric $\bar{k}$ in G^c/G constructed in step 2 as the starting point for Simpson's heat flow

$$\begin{aligned} H^{-1}\frac{dH}{du} &= -i\Lambda F_H^{\perp} \\ H_0 &= K \end{aligned} \tag{7.1}$$

Let us remark that this equation remains valid in G^c/G as the left and right hand side both take values in $i \cdot \mathfrak{g}$.

Now, the asymptotic behaviour of this heat flow is governed by the following result of C.T. Simpson;

Theorem 7.1.4 (Simpson [42], Theorem 1)**.** *Let* (X, ω) *satisfy conditions in Lemma 7.1.5 and suppose* E *is an* S^1*-invariant bundle on* X *with* S^1*-invariant metric* K *satisfying that* $\sup|\Lambda F_K| < c$. *If* E *is stable in the sense that it arises from a stable pair on* Σ*, then there is an* S^1*-invariant metric* H *with* $\det(H) = \det(K)$*,* H *and* K *mutually bounded* $\bar{\partial}(K^{-1}H) \in L^2$ *and such that* $\Lambda F_H^{\perp} = 0$. *Additionally, [5], if* R *is the geodesic distance to one of the singularities,* $R \cdot d(K^{-1}H)$ *is bounded by a constant.*

Note that, as demonstrated in [5], our notion of stability coincides with Simpson's notion of stability here although they are presented in a slightly different fashion.

Lemma 7.1.5. *Our manifold* $X = S^1 \times ((S^1 \times \Sigma)\backslash\{p_1, \ldots, p_N\})$ *satisfies the three necessary conditions for Simpson's Theorem*

1. X *is Kähler and of finite volume;*
2. *There exists a* ≥ 0 *exhaustion function with bounded Laplacian on* X*;*
3. *There is an increasing* $a : [0, \infty) \to [0, \infty)$ *such that* $a(0) = 0$ *and* $a(x) = x$ *for all* $x > 1$ *so that if* f *is a bounded positive function on* X *with* $\Delta(f) \leq B$*, then*

$$\sup_X |f| \leq C(B)a\left(\int_X |f|\right)$$

and furthermore, if $\Delta(f) \leq 0$ *then* $\Delta(f) = 0$.

Proof. See [5] and [37]. □

Step 4:

Remark 9. *Let us first heuristically describe the following process of determining the regularity of our solutions about the singular points. We restrict our initial metric, H_0, to a neighbourhood of p (diffeomorphic to $B^3\backslash\{p\}$ and desingularize by extending the pullback of our Hopf map $\pi^* : \Omega(B^3 \times S^1) \to \Omega(B^4)$. This process of desingularization was brought to light by Kronheimer [31] and studied in more depth by Pauly [37]. After all differential forms of interest are pulled back and appropriately scaled, we apply Simpson's heat flow (with Dirichlet boundary conditions) to the HEB equation which is known to correspond to an S^1-invariant instanton equation. A new Hermitian metric is achieved in the limit of the heat flow and then pushed back down to a metric describing a Dirac monopole and, because of the Dirichlet boundary condition, glues right back into the global picture. Finally, due to uniqueness of the solutions to Simpson's heat flow, this "alternate" solution is found to coincide with the previous and thus the previous metric satisfies the required regularity at the singularities.*

Please note that all proofs of technical lemmas found in the step which do not contribute to the development of notation are left to the end of the chapter.

If the metric K is chosen as constructed above in this proof and limits the H_∞ as in Theorem 7.1.4, this gives a solution to HE monopole equations on $Y = S^1 \times X$. It remains to see why these solutions have desired μ-Dirac monopole-type singularities. For this, we resort to the local construction, described earlier, involving the Hopf fibration used by both Kronheimer [31] and Pauly [37].

Recall, the Hopf map from section 2.2. defines a lift. Adding the variable s to B^3 to expand to $S^1 \times B^3$ allows us to write the HEB equations (5.2) for (∇, ϕ) as the HE equations (5.1) for $\tilde{\nabla} = \nabla + \phi ds$. Defining that

$$\pi^* ds = \xi = \frac{1}{i}(w_1 d\bar{w}_1 - \bar{w}_1 dw_1 - w_2 d\bar{w}_2 + \bar{w}_2 dw_2)$$

makes the pull-back of π into a map $\pi^* : \Omega(S^1 \times B^3) \to \Omega(B^4)$, although π is merely a map from B^4 to B^3 and surely does not extend to have range $S^1 \times B^3$ (since the Hopf bundle is non-trivial). The process of Kronheimer and Pauly to smooth out the Dirac singularity then associates to (∇, ϕ) the unwound connection

$$\hat{\nabla} = \pi^* \tilde{\nabla} = \pi^* \nabla + \pi^* \phi \xi.$$

Lemma 7.1.6. *The curvatures here are related by*

$$F_{\hat{\nabla}} = \pi^* F_{\tilde{\nabla}} + \pi^* \phi d\xi.$$

For a singular G-monopole on B^3 there is an equation giving by declaring that after its lift to $S^1 \times B^3$ the projection onto the self-dual 2-forms (with kernel the anti-self-dual forms) takes the specific value $C \cdot \omega_\Sigma$.

Examining the lifts of the 1-forms under $\pi^* : \Omega^1(S^1 \times B^3) \to \Omega^1(B^4)$ one sees

Lemma 7.1.7. *In complex coordinates, the bi-type of each differential form is preserved.*

Proof. Indeed, with $dz = dx + idy$ and $dt - ids$ as our basis for the $(1,0)$-forms on $B^3 \times S^1$, these lifts are given by

$$dz = 2(w_1 dw_2 + w_2 dw_1) \qquad dt - ids = 2(\bar{w}_1 dw_1 - \bar{w}_2 dw_2)$$

$$d\bar{z} = 2(\bar{w}_1 d\bar{w}_2 + \bar{w}_2 d\bar{w}_1) \qquad dt + ids = 2(w_1 d\bar{w}_1 + w_2 d\bar{w}_2)$$

Now, examining the pullbacks of (1,1)-forms; the Kähler form

$$\begin{aligned}\pi^*(\Omega) &= \frac{i\alpha}{2} dz \wedge d\bar{z} - dt \wedge \xi \\ &= 2i\left[(\alpha|w_1|^2 + |w_2|^2) dw_1 \wedge d\bar{w}_1 + (\alpha|w_2|^2 + |w_1|^2) dw_2 \wedge d\bar{w}_2 \right. \\ &\qquad \left. +(\alpha - 1)(w_2\bar{w}_1 dw_1 \wedge d\bar{w}_2 + w_1\bar{w}_2 dw_2 \wedge d\bar{w}_1)\right]\end{aligned}$$

and the other 3, anti-self-dual, (1,1)-forms

$$\begin{aligned}\epsilon_1 &= \frac{1}{4}(dz \wedge d\bar{z} - \alpha(dt - i\xi) \wedge (dt + i\xi)) \\ &= (|w_2|^2 - |w_1|^2)(dw_1 \wedge d\bar{w}_1 - dw_2 \wedge d\bar{w}_2) \\ &\quad + (1-\alpha)(|w_1|^2 dw_1 \wedge d\bar{w}_1 + |w_2|^2 dw_2 \wedge d\bar{w}_2) \\ &\quad + (1+\alpha)(w_2\bar{w}_1 dw_1 \wedge d\bar{w}_2 + w_1\bar{w}_2 dw_2 \wedge d\bar{w}_1), \\ \epsilon_2 &= \frac{1}{4}(dz \wedge (dt + i\xi)) \\ &= w_1 w_2 (dw_1 \wedge d\bar{w}_1 - dw_2 \wedge d\bar{w}_2) - w_2^2 dw_1 \wedge d\bar{w}_2 + w_1^2 dw_2 \wedge d\bar{w}_1 \\ \epsilon_3 &= \frac{1}{4}(d\bar{z} \wedge (dt - i\xi)) \\ &= \bar{w}_1 \bar{w}_2 (dw_1 \wedge d\bar{w}_1 - dw_2 \wedge d\bar{w}_2) - w_1^2 dw_1 \wedge d\bar{w}_2 + w_2^2 dw_2 \wedge d\bar{w}_1\end{aligned}$$

□

Remark 10. *Note that the Kähler form on $S^1 \times B^3$ has been chosen with an opposite orientation than usual. That is, the volume form (expressed in real coordinates) is* *obtained by*

$$\begin{aligned}
dV &= \Omega \wedge \Omega \\
&= \left(\frac{\alpha i}{2} dz \wedge d\bar{z} - dt \wedge ds\right) \wedge \left(\frac{\alpha i}{2} dz \wedge d\bar{z} - dt \wedge ds\right) \\
&= -\alpha(dxdydtds + dtdsdxdy) \\
&= -2\alpha dxdydtds \\
&= 2\alpha dxdy \underbrace{dsdt}
\end{aligned}$$

where the roles of ds and dt have been reversed. It is this subtle choice which correspondingly reverses the roles of self-dual and anti-self-dual 2-forms.

Divide the lift of the Kähler form by $4(|w_1|^2 + |w_2|^2)$ and get

$$\begin{aligned}
\tilde{\Omega} =& \frac{i}{2}(dw_1 \wedge d\bar{w}_1 + dw_2 \wedge d\bar{w}_2) \\
&+ \frac{i(\alpha-1)}{2(|w_1|^2 + |w_2|^2)}(|w_1|^2 dw_2 \wedge d\bar{w}_2 + |w_2|^2 dw_1 \wedge d\bar{w}_1 + w_2\bar{w}_1 dw_1 \wedge d\bar{w}_2 + w_1\bar{w}_2 dw_2 \wedge d\bar{w}_1) \\
&= \omega + (\alpha-1)(Q/R^2)
\end{aligned}$$

where ω is the standard Kähler form in the Euclidean metric on B^4, $R^2 = |w_1|^2 + |w_2|^2$ is the radial distance from the singularity and Q is some quadratic expression in $w_i, \bar{w}_i$.

Lemma 7.1.8. *These lifted (1,1) forms $\{\tilde{\Omega}, \epsilon_1, \epsilon_2, \epsilon_3\}$ miraculously form an orthogonal basis of $\Lambda^{1,1}(B^4)$ with respect to the usual Euclidean inner product and for that, we find $|\tilde{\Omega}|^2 = \alpha^2 + 1$.*

Hence, with these calculations in mind, the orthogonal projection onto the linear subspace spanned by $\tilde{\Omega}$ is given by

$$P_{\tilde{\Omega}}(F) = \frac{\langle F, \tilde{\Omega}\rangle}{\langle \tilde{\Omega}, \tilde{\Omega}\rangle}\tilde{\Omega}.$$

One checks that

$$P_{\tilde{\Omega}}(F) = \frac{\langle F, \omega\rangle}{2}\omega + \frac{(\alpha-1)}{(\alpha^2+1)}\left(\frac{\langle F, Q\rangle\omega + \langle F, \omega\rangle Q}{R^2} + (\alpha-1)\frac{\langle F, Q\rangle}{R^4}Q - \frac{(\alpha+1)}{2}\langle F, \omega\rangle\omega\right)$$

Let p_k represent a homogeneous polynomial of degree k and p_∞ a smooth function (in the variables $w_i, \bar{w}_i$). In this notation, the above then reads more simply as

$$P_{\tilde{\Omega}}(F) = p_\infty + p_\infty\frac{p_6}{R^2} + p_\infty\frac{p_{12}}{R^4}$$

so the coefficients of the projectors are in C^3.

So the equation for the HE connection on $S^1 \times B^3$ (below) is

$$P_\Omega(F_{\check{\nabla}}) = C \cdot \Omega$$

and lifting this becomes

$$P_{\tilde{\Omega}}(F_{\hat{\nabla}} - \pi^*\phi\, d\xi) = \pi^*(C \cdot \Omega).$$

If $\alpha \equiv 1$ (Euclidean case) then $\Lambda(d\xi) = P_\omega(d\xi) = 0$; but for $\alpha = 1 + w\bar{w}f(w, \bar{w})$, $\Lambda(d\xi) = P_{\tilde{\Omega}}(d\xi)$ is a bounded quartic near the origin.

We are interested in the Hermitian connection obtained from the limit of the metric solved for by the heat equation. Note that since the coefficients of the projectors are C^3 the initial condition space for the connection will be C^3. That is to say $\tilde{\nabla}$ coming from $\tilde{\nabla}_0$ are both Chern connections. This is done by keeping the same $(0,1)$ part and modifying the $(1,0)$ part so that $\tilde{\nabla}$ is the Chern connection for the new metric $H_\infty = H_0 h$. According to Simpson ([42], Lemma 3.1), the connection changes by

$$A^{0,1} = A_0^{0,1}, A^{1,0} = A_0^{1,0} + h^{-1}\nabla_0^{1,0} h.$$

Hence,

Lemma 7.1.9. *Given the change in metric as $H_\infty = H_0 h$, the Higgs field then changes as*

$$\phi = \phi_0 - \frac{i}{2} h^{-1}(\nabla_{0,t} + i\phi_0)h$$

and the curvature of the Hermitian connection $\tilde{\nabla}$ transforms as

$$F_{\tilde{\nabla}} = F_{\tilde{\nabla}_0} + \tilde{\nabla}_0(h^{-1}\tilde{\nabla}_0^{1,0}h) + \frac{1}{2}[h^{-1}\tilde{\nabla}_0^{1,0}h, h^{-1}\tilde{\nabla}_0^{1,0}h].$$

From this, we can read the $(1,1)$ part of the curvature as

$$F_{\tilde{\nabla}}^{1,1} = F_{\tilde{\nabla}_0}^{1,1} + \nabla_0^{0,1}(h^{-1}\nabla_0^{1,0}h).$$

Lemma 7.1.10. *Upon lifting to B^4, the curvature becomes*

$$F_{\hat{\nabla}} - \pi^*\phi d\xi = F_{\hat{\nabla}_0} - \pi^*\phi_0 d\xi + \hat{\nabla}_0^{0,1}(\pi^* h^{-1}\hat{\nabla}_0^{1,0}\pi^* h) + \pi^* h^{-1}[\pi^*\phi\partial\xi, \pi^* h].$$

Proof. See appendix on various tedious formulas in gauge theory. □

The original connection $\hat{\nabla}_0$ has zero curvature about the monopole (being a direct sum of flat connections) so locally this equation reduces to

$$F_{\hat{\nabla}} - \pi^*\phi d\xi = -\pi^*\phi_0 d\xi + \hat{\nabla}_0^{0,1}(\pi^* h^{-1}\hat{\nabla}_0^{1,0}\pi^* h) + \pi^* h^{-1}[\pi^*\phi\partial\xi, \pi^* h]$$

If α is uniformly 1, the forms $d\xi$ and $\bar{\partial}\xi$ are anti-self-dual and these equations reduce to

$$\hat{D}(h) \equiv \Delta h - 2i\Lambda\hat{\nabla}_0^{0,1} h h^{-1} \hat{\nabla}_0^{1,0} h = 0.$$

When α is not uniformly 1, this is an elliptic equation $\hat{D}(h) = 0$, a deformation of the one above whose coefficients are C^1 (taking into account the poles of ϕ_0 and the behaviour of $\Lambda(-\pi^*\phi_0 d\xi)$).

Now, h was obtained as the limit to infinity of a heat flow h_u and is smooth away from singularities. Upstairs (on B^4) h_u solves the heat equation $\partial_u h_u = \hat{D}(h_u)$. Take a four-ball about the singular point (which maps to a 3-ball downstairs) with initial conditions for the heat flow (given by our solution at the previous step) $\partial_u \hat{h}_u = \hat{D}(\hat{h}_u)$ $\hat{h}_0 = h_0 = 1$ and as a boundary for the Dirichlet condition $\hat{h} = h$ (hence, same boundary and same initial conditions will imply same result). Applying the work of Donaldson, or simply again the result of Simpson, we get a C^1 solution $\hat{h}_u$ which is S^1 invariant as the initial boundary conditions are and which satisfies the same boundary conditions and initial conditions as $h(t)$. Again, one has a limit $\hat{h} = \hat{h}_\infty$ solution to $\hat{D}(\hat{h}) = 0$.

Both h and $\hat{h}$ descend to the 3-ball and solve the HEB equations there. One can refer to the lemma of Simpson saying that one has uniqueness if solutions are bounded, which they are. Thus $\hat{h} = h$ which means the global h given by Simpson's result has the required smoothness at the singular points (since $\hat{h}$ does) to ensure the Higgs field and its covariant derivative have the correct Dirac type singularities. □

Injectvity

Please note, the change in notation for τ below as we are no longer in need of this to denote circumference.

Proposition 7.1.11. *If two singular G-monopoles (P, ∇, Φ) and (P', ∇', Φ') yield isomorphic holomorphic data, then they are isomorphic (i.e. $\mathscr{H}$ is injective).*

In order to justify this claim, we must first provide a proper formal description of what it means for monopoles to be isomorphic. These notions will, as usual, heavily rely on the analogous vector bundle scenario so I will discuss both in parallel to solidify understanding and justify our constructions.

Let us first examine the statement and proof of this result from [5] for the vector bundle analogue. The statement is more or less word for word, with the principal bundles replaced by vector bundles and reads something like

"Two singular HE-monopoles with isomorphic holomorphic data are isomorphic."

Remark 11. *The main source of technical difficulty lies with the lack of a tensor product operation for principal bundles. If this were not an issue, one could imagine a natural way of inducing a monopole structure on some algebraically combined version of the two monopoles (i.e. one whose sections are the bundle maps $P \to P'$) and proceed in showing that the sections here as well as the induced Higgs field are covariantly constant with respect to the induced connection.*

Now, with the proof for vector bundles in mind (c.f. [5]), we have already developed the required tools and examples to proceed. We recall that $\mathrm{Hom}_G(P, P')$ is realized as the associated G-fibre bundle $(P \times_B P') \times_\varphi G$ where φ is the action of $G \times G$ on G defined as $(g, h) \cdot x := g^{-1}xh = L_{g^{-1}} \circ R_h(x)$.

So now,

Proof. If (P, ∇, ϕ) and (P', ∇', ϕ') are singular G monopoles such that $\mathscr{H}(P, \nabla, \phi) = (\mathscr{P}, \rho) \cong (\mathscr{P}', \rho') = \mathscr{H}(P', \nabla', \phi')$, then $\mathscr{P} \cong \mathscr{P}'$ are isomorphic as holomorphic principal bundles via some G-equivariant bundle map $\tau : \mathscr{P} \to \mathscr{P}'$ which furthermore satisfies $\tau \circ \rho = \rho' \circ \tau$. This holds more generally for each $\mathscr{P}_t$ and $\mathscr{P}_t'$ (as a result of scattering and intertwining with meromorphic data) meaning that τ aligns the invariant fibres of ρ and ρ' and so extends to an isomorphism $\hat{\tau}$ between P and P' over $S^1 \times \Sigma$. This isomorphism $\hat{\tau}$ is viewed as a section of the G-fibre bundle $\mathrm{Hom}_G(P, P')$ which is equipped with the induced connection $\hat{\nabla} = (\nabla \times \nabla') \times_\varphi \mathbf{1}$, a Higgs field $\hat{\phi} = \phi' \otimes \mathbf{I} - \mathbf{I} \otimes \phi$ and furthermore $\hat{\tau}_* \in \ker(\hat{\nabla}_\Sigma^{0,1}) \cap \ker(\hat{\nabla}_t - i\hat{\phi})$. Using the identities

$$\begin{aligned} \hat{\nabla}_\Sigma^{1,0}\hat{\nabla}_\Sigma^{0,1} &= (\Delta_\Sigma + i\hat{F}_\Sigma)\omega \\ (\hat{\nabla}_t + i\hat{\phi})(\hat{\nabla}_t - i\hat{\phi}) &= \hat{\nabla}_t^2 + \hat{\phi}^2 - i\hat{\nabla}_t\hat{\phi}, \end{aligned} \tag{7.2}$$

(doing the integration by parts performed in a representation of G) one finds

$$\begin{aligned} 0 &= -\int_{S^1\times\Sigma} \langle \hat{\tau}_*, (\hat{\nabla}_t + i\hat{\phi})(\hat{\nabla}_t - i\hat{\phi})\hat{\tau} + \omega^{-1}\hat{\nabla}_\Sigma^{1,0}\hat{\nabla}_\Sigma^{0,1}\hat{\tau}_*\rangle d\nu \\ &= \int_{S^1\times\Sigma} \langle \hat{\tau}_*, (-\hat{\phi}^2 - \hat{\nabla}_t^2 - \hat{\Delta}_\Sigma)\hat{\tau}_*\rangle d\nu \\ &= \int_{S^1\times\Sigma} |\hat{\phi}\hat{\tau}_*|^2 + |\hat{\nabla}_t\hat{\tau}_*|^2 + |\hat{\nabla}_\Sigma\hat{\tau}_*|^2 d\nu. \end{aligned}$$

Hence, $\hat{\tau}$ is covaritantly constant and as a map $E \to E'$ it intertwines the two Higgs fields (That is, $\hat{\phi} \circ \hat{\tau} = 0$ is equivalent to $\phi' \circ \hat{\tau} - \hat{\tau} \circ \phi = 0$). Therefore, the two monopoles are isomorphic.

□

7.1.1 Remaining proofs of technical Lemmas from Proposition 7.1.2

7.1.6 *Proof.* At first glance, this might appear to be a typo. However, the extra term is coming from the fact that $\xi = \pi^* ds$ is no longer a closed form. Indeed,

$$\begin{aligned}
F_{\hat{\nabla}} &= d\hat{A} + \hat{A} \wedge \hat{A} \\
&= d(\pi^* A + \pi^* \phi \xi) + (\pi^* A + \pi^* \phi \xi) \wedge (\pi^* A + \pi^* \phi \xi) \\
&= d(\pi^* A) + d\pi^* \phi \wedge \xi + \pi^* \phi d\xi + (\pi^* A + \pi^* \phi \xi) \wedge (\pi^* A + \pi^* \phi \xi) \\
&= \pi^*[d(A + \phi ds) + (A + \phi ds) \wedge (A + \phi ds)] + \pi^* \phi d\xi \\
&= \pi^* F_{\tilde{\nabla}} + \pi^* \phi d\xi
\end{aligned}$$

□

7.1.8 *Proof.* Note here that these basis vectors have been expressed in terms of the basis $\beta = \{dw_i \wedge d\bar{w}_j : i, j \in \{1, 2\}\}$ which is different from the real Euclidean metric under consideration here. This is first translated into real coordinates as;

$$dw_i = dx_i + i dy_i \qquad\qquad d\bar{w}_i = dx_i - i dy_i$$

So then the norms of these are all 2. The inner product inherited on the exterior algebra remains orthonormal on any basis induced from an orthonormal basis, so expressing our forms in complex coordinates reveals

$$dw_1 \wedge d\bar{w}_1 = (dx_1 + idy_1) \wedge (dx_1 - idy_1) = -2idx_1 \wedge dy_1 = \begin{bmatrix} -2i \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$dw_1 \wedge d\bar{w}_2 = dx_1 \wedge dx_2 - idx_1 \wedge dy_2 + idy_1 \wedge dx_2 + dy_1 \wedge dy_2 = \begin{bmatrix} 0 \\ 1 \\ -i \\ i \\ 1 \\ 0 \end{bmatrix}$$

$$dw_2 \wedge d\bar{w}_1 = -dx_1 \wedge dx_2 - idx_1 \wedge dy_2 + idy_1 \wedge dx_2 - dy_1 \wedge dy_2 = \begin{bmatrix} 0 \\ -1 \\ -i \\ i \\ -1 \\ 0 \end{bmatrix}$$

$$dw_2 \wedge d\bar{w}_2 = -2idx_2 \wedge dy_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -2i \end{bmatrix}$$

and now it is clear that this basis is orthogonal but all vectors have norm 2. For example, observe that (in the standard complex basis β for $(1,1)$-forms) we can express $\epsilon_2 = (w_1w_2, -w_2^2, -w_1^2, w_1w_2)$ and $\epsilon_3 = (-\bar{w}_1\bar{w}_2, -\bar{w}_1^2, \bar{w}_2^2, \bar{w}_1\bar{w}_2)$ so that

$$\langle \epsilon_2, \epsilon_3 \rangle = 4(-w_1^2w_2^2 - w_1^2w_2^2 + w_1^2w_2^2 + w_1^2w_2^2) = 0$$

and

$$
\begin{aligned}
||\tilde{\Omega}||^2 &= \frac{1}{4}\left\langle \omega + \frac{\alpha-1}{R^2}Q, \omega + \frac{\alpha-1}{R^2} \right\rangle \\
&= \frac{1}{4}\left(||\omega||^2 + \frac{\alpha-1}{R^2}(\langle\omega, Q\rangle + \langle Q, \omega\rangle) + \left(\frac{\alpha-1}{R^2}\right)^2 ||Q||^2 \right) \\
&= \frac{1}{4}\left(4+4+\frac{\alpha-1}{R^2}(4R^2+4R^2) + \left(\frac{\alpha-1}{2}\right)^2 4R^4 \right) \\
&= 2+2(\alpha-1)+(\alpha-1)^2 \\
&= \alpha^2+1
\end{aligned}
$$

□

7.1.9 *Proof.* First, for ϕ, recall that this is the ds-term in the connection $\tilde{A}$,

$$
\begin{aligned}
\tilde{A} &= \tilde{A}_z dz + \tilde{A}_{\bar{z}} d\bar{z} + \tilde{A}_u du + \tilde{A}_{\bar{u}} d\bar{u} \\
&= \tilde{A}_z dz + \tilde{A}_{\bar{z}} d\bar{z} + \tilde{A}_u (dt - ids) + \tilde{A}_{\bar{u}}(dt + ids) \\
&= \tilde{A}_z dz + \tilde{A}_{\bar{z}} d\bar{z} + (\tilde{A}_u + \tilde{A}_{\bar{u}})dt + i(\tilde{A}_{\bar{u}} - \tilde{A}_u)ds)
\end{aligned}
$$

while

$$
\tilde{A} = \tilde{A}_0 + h^{-1}\tilde{\nabla}_0^{1,0} h
$$

Extracting the ds-terms from both then reveals our claim.

Now, for curvature,

$$
\begin{aligned}
F_{\tilde{\nabla}} &= d\tilde{A} + \frac{1}{2}[\tilde{A},\tilde{A}] \\
&= d(\tilde{A}_0 + h^{-1}\tilde{\nabla}_0^{1,0}h) + \frac{1}{2}[\tilde{A}_0 + h^{-1}\tilde{\nabla}_0^{1,0}h, \tilde{A}_0 + h^{-1}\tilde{\nabla}_0^{1,0}h] \\
&= (d\tilde{A}_0 + \frac{1}{2}[\tilde{A}_0,\tilde{A}_0]) + d(h^{-1}\tilde{\nabla}_0^{1,0}h) + [\tilde{A}_0, h^{-1}\tilde{\nabla}_0^{1,0}h] + \frac{1}{2}[h^{-1}\tilde{\nabla}_0^{1,0}h, h^{-1}\tilde{\nabla}_0^{1,0}h] \\
&= F_{\tilde{\nabla}_0} + \tilde{\nabla}_0(h^{-1}\tilde{\nabla}_0^{1,0}h) + \frac{1}{2}[h^{-1}\tilde{\nabla}_0^{1,0}h, h^{-1}\tilde{\nabla}_0^{1,0}h].
\end{aligned}
$$

□

7.1.10 Recall, what we are dealing with is a pair (∇, Φ) on B^3 (satisfying the Hermitian-Einstein-Bogomolny equations) which, by construction, corresponds to a Hermitian-Einstein connection $\tilde{\nabla} = \nabla + \Phi ds$ on $B^3 \times S^1$. This is further lifted via the Hopf map $\pi : B^4 \to B^3$ by simply declaring that $\pi^* ds = \xi$, so that

$$\hat{\nabla} = \pi^*\tilde{\nabla} = \pi^*\nabla + \pi^*\Phi\xi.$$

One relevant formula is an expression for the curvature tensor upstairs in terms of the lifted data from downstairs. This can be done in two ways (on $\mathbb{R}^3$ or $\mathbb{R}^3 \times S^1$), however it will be most cleanly expressed on $\mathbb{R}^3 \times S^1$ in terms of $\tilde{\nabla}$. So we shall proceed with these computations first.

Formulas upstairs expressed on $\mathbb{R}^3 \times S^1$ Indeed,

$$
\begin{aligned}
\hat{F} &= d\hat{A} + \hat{A} \wedge \hat{A} \\
&= d(\pi^* A + \pi^* \phi \xi) + (\pi^* A + \pi^* \phi \xi) \wedge (\pi^* A + \pi^* \phi \xi) \\
&= d(\pi^* A) + d\pi^* \phi \wedge \xi + \pi^* \phi d\xi + (\pi^* A + \pi^* \phi \xi) \wedge (\pi^* A + \pi^* \phi \xi) \\
&= \pi^*[d(A + \phi ds) + (A + \phi ds) \wedge (A + \phi ds)] + \pi^* \phi d\xi \\
&= \pi^* \tilde{F} + \pi^* \phi d\xi.
\end{aligned}
$$

That is,

$$\hat{F} = \pi^* \tilde{F} + \pi^* \Phi d\xi. \tag{7.3}$$

and this holds in any gauge.

To change the metric (via the heat flow equation) while remaining within a holomorphic gauge, it is a result of Simpson [42], Lemma 3.1 that when the new metric H_∞ is expressed as the product $H_0 h$ where h is positive definite and self-adjoint (generalized to $h \in \Gamma(G^c/G)$) then locally, the connection matrices change as follows

$$\tilde{A}^{0,1} = \tilde{A}_0^{0,1}, \qquad \tilde{A}^{1,0} = \tilde{A}_0^{1,0} + h^{-1} \tilde{\nabla}_0^{1,0} h$$

or equivalently,

$$\tilde{A} = \tilde{A}_0 + h^{-1} \tilde{\nabla}_0^{1,0} h.$$

The change in curvature is computed locally as

$$\begin{aligned}
\tilde{F} &= d\tilde{A} + \tfrac{1}{2}[\tilde{A},\tilde{A}] \\
&= d(\tilde{A}_0 + h^{-1}\tilde{\nabla}_0^{1,0}h) + \tfrac{1}{2}[\tilde{A}_0 + h^{-1}\tilde{\nabla}_0^{1,0}h, \tilde{A}_0 + h^{-1}\tilde{\nabla}_0^{1,0}h] \\
&= d\tilde{A}_0 + \tfrac{1}{2}[\tilde{A}_0,\tilde{A}_0] + dh^{-1}\tilde{\nabla}_0^{1,0}h + [\tilde{A}_0, h^{-1}\tilde{\nabla}_0^{1,0}h] + \tfrac{1}{2}[h^{-1}\tilde{\nabla}_0^{1,0}h, h^{-1}\tilde{\nabla}_0^{1,0}h] \\
&= \tilde{F}_0 + \tilde{\nabla}_0(h^{-1}\tilde{\nabla}_0^{1,0}h) + \tfrac{1}{2}[h^{-1}\tilde{\nabla}_0^{1,0}h, h^{-1}\tilde{\nabla}_0^{1,0}h]
\end{aligned}$$

and thus, the (1,1) part is simply

$$\tilde{F}^{1,1} = \tilde{F}_0^{1,1} + \tilde{\nabla}_0^{0,1}(h^{-1}\tilde{\nabla}_0^{1,0}h) \tag{7.4}$$

Lifting, equation (7.4) and recalling the relation provided in equation (7.3) gives says that

$$\hat{F}^{1,1} - \pi^*\Phi d\xi = \hat{F}_0^{1,1} - \pi^*\Phi_0 d\xi + \pi^*\tilde{\nabla}_0^{0,1}(h^{-1}\tilde{\nabla}_0^{1,0}h)$$

where it remains to compute the lift $\pi^*\tilde{\nabla}_0^{0,1}(h^{-1}\tilde{\nabla}_0^{1,0}h)$.

Note that although we have that $\pi^*\tilde{\nabla} = \hat{\nabla}$, it *does not* follow that $\pi^*(\tilde{\nabla} \circ \tilde{\nabla})$ is equal to $\hat{\nabla} \circ \hat{\nabla}$. That is to say that the pull-back π^* does not simply distribute itself over all terms. In fact, we have already encountered this in equation (7.3). In essence, we are dealing with the fact that d and π^* do not commute as usual and this is due to the fact that $\pi^* ds = \xi$ which is no longer closed.

Lemma 7.1.12. *[Technical lemma] For $\alpha, \beta \in \Gamma(G^c/G)$ we have,*

$$\pi^*\tilde{\nabla}(\alpha\tilde{\nabla}\beta) = \hat{\nabla}(\pi^*\alpha\hat{\nabla}\pi^*\beta) + \pi^*\alpha[\pi^*\Phi d\xi, \pi^*\beta].$$

Furthermore, also

$$\pi^*\tilde{\nabla}^{0,1}(\alpha\tilde{\nabla}\beta) = \hat{\nabla}^{0,1}(\pi^*\alpha\hat{\nabla}\pi^*\beta) + \pi^*\alpha[\pi^*\Phi\bar{\partial}\xi, \pi^*\beta]$$

and for our purposes

$$\pi^*\tilde{\nabla}^{0,1}(\alpha\tilde{\nabla}^{1,0}\beta) = \hat{\nabla}^{0,1}(\pi^*\alpha\hat{\nabla}^{1,0}\pi^*\beta) + \pi^*\alpha[\pi^*\Phi\bar{\partial}\xi, \pi^*\beta].$$

Proof.

$$\begin{aligned}
\pi^*\tilde{\nabla}(\alpha\tilde{\nabla}\beta) &= \pi^*\left(\tilde{\nabla}(\alpha)\wedge\tilde{\nabla}(\beta) + \alpha\tilde{\nabla}(\tilde{\nabla}(\beta))\right) \\
&= \hat{\nabla}(\pi^*\alpha)\wedge\hat{\nabla}(\pi^*\beta) + \pi^*\alpha\pi^*\tilde{F}(\beta) \\
&= \hat{\nabla}(\pi^*\alpha)\wedge\hat{\nabla}(\pi^*\beta) + \left(\pi^*\alpha(\hat{F} - \pi^*\Phi d\xi)(\pi^*\beta)\right) \\
&= \hat{\nabla}(\pi^*\alpha)\wedge\hat{\nabla}(\pi^*\beta) + \pi^*\alpha\hat{\nabla}(\hat{\nabla}(\pi^*\beta)) - [\pi^*\Phi d\xi, \pi^*\beta] \\
&= \hat{\nabla}(\pi^*\alpha\hat{\nabla}\pi^*\beta) - [\pi^*\Phi d\xi, \pi^*\beta]
\end{aligned}$$

The other two identities (involving holomorphic and anti-holomorphic projections of $\tilde{\nabla}$) follow essentially the same calculation, but use the fact that

$$\pi^*\bar{\partial}\tilde{A} = \bar{\partial}\pi^*\tilde{A} - \pi^*\Phi\bar{\partial}\xi$$

rather than just

$$\pi^*d\tilde{A} = d\pi^*\tilde{A} - \pi^*\Phi d\xi$$

which is the core identity behind these formulas and is recognized as coming from equation (7.3). Notice also, that the last term remains unchanged between the second and third equations provided in this lemma. This is due to the fact that it is only the second application of covariant differentiation that introduces the extra term and has nothing to do with the first. □

Substituting the third formula from Lemma 7.1.12 into the relation following equation (7.4) gives a relation between the curvature tensors upstairs upon changing metrics. That is,

$$\hat{F}^{1,1} - \pi^*\Phi d\xi = \hat{F}_0^{1,1} - \pi^*\Phi_0 d\xi + \hat{\nabla}_0^{0,1}(\pi^*h^{-1}\hat{\nabla}_0^{1,0}\pi^*h) + \pi^*h^{-1}[\pi^*\Phi\bar{\partial}\xi, \pi^*h] \quad (7.5)$$

CHAPTER 8
Spectral curves and abelianization of meromorphic pairs

In this section and the next, we briefly consider two venues for furthur investigation. This first is the abelianization of our stable meromorphic pairs, which gives rise to another corresponding family of geometric objects.

In the case of vector bundles our meromorphic pair $(\mathscr{E},\rho)$ can be transformed into an n-sheeted ramified cover S_ρ of Σ recording the spectrum of the automorphism ρ and a sheaf $\mathscr{L}$ which is (generically) a line bundle on the spectral cover S_ρ, corresponding to the eigenvectors of ρ. More generally, for reductive G-fibrations, a similar process will yield pairs $(S_\rho,\mathscr{Q})$ where $S_\rho \to \Sigma$ is a $|W(G,T)|$-sheeted ramified cover of Σ (called a *cameral cover*) and $\mathscr{Q}$ is a T-bundle over S_ρ.

An inverse for these constructions are provided in several places throughout the literature with varying levels of abstraction and difficulty (cf. [7, 8, 22, 20, 21, 39, 40] for lots of information regarding these ideas)

8.1 Spectral data associated to a bundle pair

Consider the bundle pair $(\mathscr{E},\rho)$ where $\mathscr{E}$ is a holomorphic vector bundle over a Riemann surface Σ and ρ is a meromorphic automorphism of $\mathscr{E}$. To elaborate a bit further, we may express this as an automorphism away from $\{z_1,z_2,\ldots,z_n\}$ with ρ expressed locally as $\rho(z)=g(z)\operatorname{diag}\big((z-z_j)^{k_1},\ldots,(z-z_j)^{k_n}\big)h(z)$ near z_j (i.e. it is meromorphic in the sense that it has poles and zeros at some points). This bundle pair actually arises as the monodromy information for a singular $U(n)$-monopole, but this will not be relevant for our construction.

Analogously, a principal bundle pair $(\mathscr{P}, \rho)$ will be a principal G-bundle over Σ and $\rho \in \mathscr{M}(\mathrm{Ad}_P)$ a meromorphic section of $\mathrm{Ad}_P = P \otimes_G G$ (where G acts by conjugation). The procedure developed here is referred to as the *abelianization* of the bundle pair.

8.1.1 Vector bundles

Given a vector bundle pair $(\mathscr{E}, \rho)$, we can define a *spectral curve* S_ρ as the compactification of

$$S^0_\rho = \{(z, \lambda) \in \Sigma \times \mathbb{C}^* : \det(\rho(z) - \lambda \cdot I_n) = 0\}.$$

That is, S_ρ is a projective subvariety of $\Sigma \times \mathbb{C}P^1$. This is generically an n-fold branched cover, which we denote by π, of Σ whose fiber at $z \in \Sigma$ consists of the eigenvalues of $\rho(z)$. Note also that the intersection of S_ρ with $\Sigma \times \{0, \infty\}$ encodes the singular information of ρ at the $\{z_j\}'s$.

Considering the pullback

$$\begin{tikzcd} \pi^*\mathscr{E} \arrow[r] \arrow[d] & \mathscr{E} \arrow[d] \\ S_\rho \arrow[r, "\pi", "n:1"'] & \Sigma \end{tikzcd}$$

allows us to define the co-kernel of

$$\begin{tikzcd} \pi^*\mathscr{E} \otimes \mathscr{O}(-1) \arrow[rr, "\rho(z) - \lambda \cdot I_n"] \arrow[dr] & & \pi^*\mathscr{E} \arrow[r] \arrow[dl] & \mathscr{L} \\ & S_\rho & & \end{tikzcd}$$

which is generically a line bundle $\mathscr{L}$ over S_ρ whose fiber at (z,λ) is isomorphic to the eigenspace

$$E_\lambda(\rho(z)) = \ker(\rho(z) - \lambda \cdot I_n).$$

The pair $(S_\rho, \mathscr{L})$ is called the *spectral data* of our bundle pair and it can be shown that this information encodes the bundle pair $(\mathscr{E}, \rho)$: the holomorphic bundle $\mathscr{E}$ can be reconstructed from $(S_\rho \xrightarrow{\pi} \Sigma, \mathscr{L})$ as the push-forward $\pi_*(\mathscr{L})$; similarly, $\rho \in \mathscr{M}(\mathrm{Aut}(\mathscr{E}))$ is reconstructed as the push-forward $\pi_*(\mu_\lambda)$ of multiplication by λ.

8.1.2 Principal bundles

Given a meromorphic principal pair $(\mathscr{P}, \rho)$, we wish to construct a pair $(S_\rho, \mathscr{Q})$ that analogously encodes the "eigen data".

The spectral information (Cameral cover)

From ([24] section 6.2), given the data $(\mathscr{P}, \rho)$, the meromorphic endomorphism $\rho \in \mathscr{M}(\mathrm{Aut}(P))$ has a notion of spectrum given by examining its orbits under conjugation by G as follows:

Fix a maximal torus T (analogous to diagonal matrices) and to each $z \in \Sigma$, associate to $\rho|_{P_z}$, the Weyl group orbit in T of the closure of the G-orbit (under conjugation) of the second coordinate in the equivalence class $\rho(p_z) = [p_z, \psi(z)] = \{(g \cdot p_z, g\psi(z)g^{-1}) : g \in G\}$. That is,

$$S_\rho^0 := \{(z,\alpha) \in \Sigma \times T : \alpha \in \overline{\mathscr{O}_G(\psi(z))} \cap T\} \tag{8.1}$$

where $\mathscr{O}_G(\psi(z)) = \{g\psi(z)g^{-1} : g \in G\}$ is the conjugacy class of $\psi(z)$ in G.

Now, at first glance, since our torus here will be $T \cong (\mathbb{C}^*)^n$ where $n = \operatorname{rank} G$, a first natural assumption might be that a compactification should be simply given by including the points $\{0, \infty\}$ for each copy of $\mathbb{C}^*$. However, in most cases, this naive approach will not yield the desired *Weyl-invariant compactification*. Assuming (to be discussed below) for a second that such an invariant compactification was at our fingertips, then $S_\rho := \overline{S^0_\rho}^W$ defines a (generically) $|W(G,T)|$-fold branched cover of the Riemann surface, denoted by $q : S_\rho \to \Sigma$. This S_ρ is a projective subvariety of $\Sigma \times \overline{T}^W$.

Remark 12. *Let us elaborate, for a minute, the case $G = \mathrm{GL}_n$ and relate the spectral curve for vector bundles (an n-fold cover over the base) to this generalization which is n!-fold. This map q is related to the case above via a projection map onto the first coordinate of diagonal matrices. That is, $q = pr_1 \circ \pi$ from above. Recall, π is an n-fold cover and projection on to the first coordinate of such a cover is $(n-1)$!-sheeted. Pictorially, for vector bundles, we have*

$$\begin{tikzcd} \tilde{S}_\rho \arrow[r, "pr_1", "(n-1)!"'] \arrow[rr, bend left, "q", "n!"'] & S_\rho \arrow[r, "\pi", "n"'] & \Sigma \end{tikzcd}$$

as simply

$$(\lambda_1(z), \lambda_2(z), \dots, \lambda_n(z)) \mapsto \lambda_1(z) \mapsto z$$

and this $\tilde{S}_\rho$ here represents the generalization for G-bundles.

A maximal torus bundle on the cameral cover

Next, returning to the case of a general group G, with the spectral information in hand and as above, we pullback $\mathscr{P}$ via q:

$$\begin{tikzcd}
q^*\mathscr{P} \arrow[r] \arrow[d] & \mathscr{P} \arrow[d] \\
S_\rho \arrow[r, "q", "|W|:1"'] & \Sigma
\end{tikzcd}$$

Fixing some Borel subgroup $B \leq G$ containing T (the analogue for G of the upper triangular matrices in $\mathrm{GL}_n(\mathbb{C})$), it is known (by the Lie-Kolchin Theorem) that any group element may be conjugated into B. However, there is no canonical choice for doing so. Having now separated the different possible semi-simple components in the G-orbit of ρ, this lifted bundle q^*P should now admit a canonical reduction to a principal B-bundle over S_ρ.

Indeed, writing B as the semi-direct product $T \ltimes U$, define

$$P_B = \{p_{(z,\alpha)} \in q^*P : q^*\rho(p) = [p, \alpha \cdot u], \text{ for some } u \in U\} \subset q^*P.$$

That is to say P_B is the family for frames for which P is of the form $\alpha \cdot u$. Then, appealing to the fact that Borel subgroups are self-normalizing (i.e. $N_G(B) = B$), one find that the condition

$$(p, \alpha \cdot u) \sim (h \cdot p, h\alpha \cdot uh^{-1}) = (h \cdot p, \alpha \cdot u')$$

for some $u' \in U$ holds if and only if $h \in B$. Hence P_B is a reduction of the pullback q^*P over S_ρ to B. Furthermore, the lifted map $q^*\rho$ is naturally found as a section of the associated reduction $\mathrm{Aut}(P_B) = \mathrm{Ad}_{P_B}$.

Now, through the isomorphism $B \cong T \ltimes U$, which gives the exact sequence

$$U \hookrightarrow B \overset{\pi}{\twoheadrightarrow} T,$$

the reduced B-bundle P_B as an element of the non-abelian sheaf cohomology group $H^1(S_\rho; B)$ naturally also defines an element $\pi \circ P_B \in H^1(S_\rho; T)$ which we denote by $\mathscr{Q}$. This $\mathscr{Q}$ is the desired T-bundle over S_ρ alluded to above for which we would like to consider the pair $(S_\rho, \mathscr{Q})$ as the *abelianization* of $(\mathscr{P}, \rho)$.

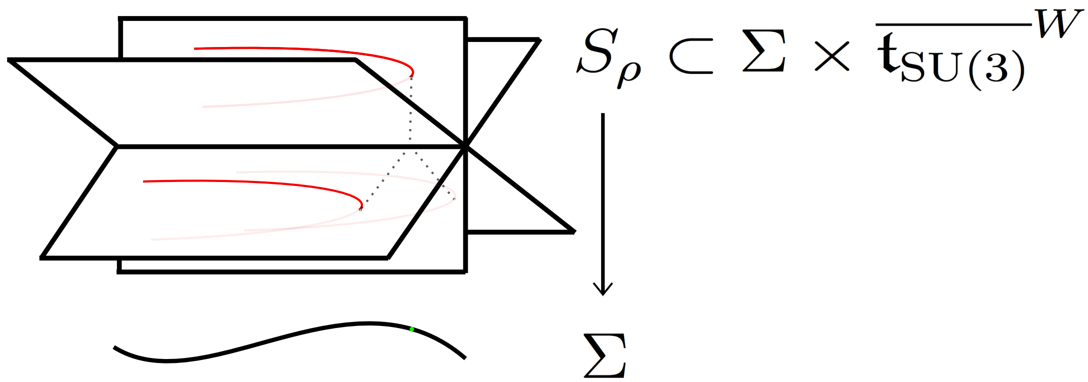
Note that, furthermore the unipotent information $U_{\mathscr{Q}}$ is realized as the pre-image $\pi^{-1}(\mathscr{Q}) \in H^1(S_\rho, U_{\mathscr{Q}})$ where, say at $(z, \alpha) \in \S_\rho$, $U_{(z,\alpha)} = \pi^{-1}(\alpha) = \{b \in B : \exists u \in U, b = \alpha \cdot u\}$.

Remark 13. *A reversal of this procedure, at least in the generic setting, is outlined in [22] section 2. By generic, one means that the logarithm of the cameral cover (so to take values in $\mathfrak{t}$ rather than T) crosses walls of the Weyl-chamber transversally and never more than one at a time. This implies that the stabilizers at branch points are isomorphic to $\mathbb{Z}/2$ and there exists a choice of gauge for which ρ's orbit contains an element appearing, in matrix form, as*

$$\begin{pmatrix} \lambda_1 & & & 0 \\ & \ddots & & \\ & & \begin{matrix} a & 1 \\ 0 & a \end{matrix} & \\ & & & \ddots & \\ 0 & & & & \lambda_{n-2} \end{pmatrix}$$

with distinct $\lambda_1, \ldots, \lambda_{n-2}$.

For the group SU(3) *an image of such a cameral cover would look like;*



8.1.3 Weyl-invariant compactifications of maximal tori

Now, as mentioned, we should take a look at the process in which the *Weyl-invariant compactification* of maximal tori is accomplished.

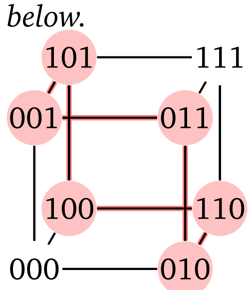
In the standard case, when $G = \mathrm{GL}_n$, one simply compactifies its maximal torus $(\mathbb{C}^*)^n$ to $(\mathbb{C}P^1)^n$ by the natural extension of the two point $(\{0, \infty\})$ compactification of $\mathbb{C}^*$. Any point here is invariant under permutation (i.e. the Weyl-group of GL_n). Notice that SL_n has the same Weyl-group as GL_n, but the maximal torus is only $(n-1)$-dimensional. Of course then, since algebraic groups admit an embedding into GL_N (for some N), we can expect to realize the compactification of their maximal tori as compact subvarieties of $(\mathbb{C}^*)^N$. In fact, given a complex reductive Lie group G of rank k, the visual procedure is as follows;

Consider maximal $T \subset G$ (isomorphic to $(\mathbb{C}^*)^k$) along with the embedding $\iota : G \hookrightarrow \mathrm{GL}_N$. Compactify the torus to $\overline{T} \cong (\mathbb{C}P^1)^k$ and find its image under ι as a k-dimensional subvariety in $\overline{T_{\mathrm{GL}_N}} \cong (\mathbb{C}P^1)^N$.

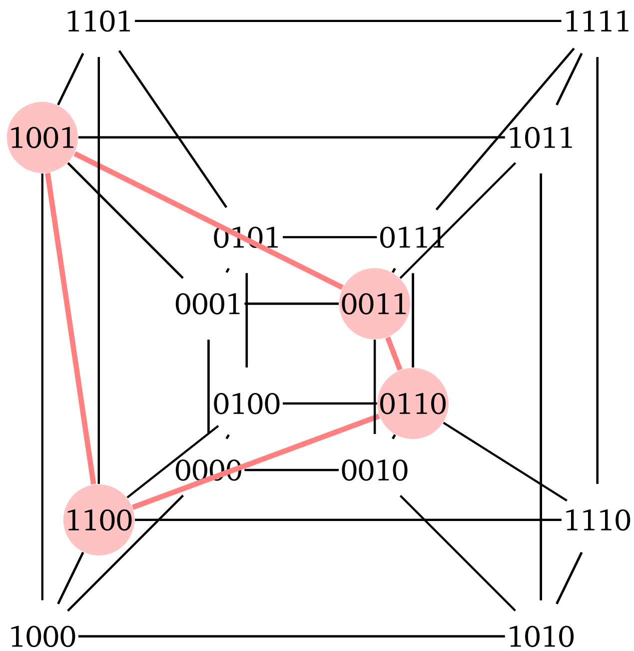
Example 5. *We can provide a sketch of some low-dimensional cases*

1. $G = \mathrm{SL}_3(\mathbb{C}) = \{A \in \mathrm{GL}_3 : \det A = 1\}$ *has rank 2. A natural choice of maximal torus is already embedded in* T_{GL_3} *subvariety* $\{(x, y, z) \in (\mathbb{C}^*)^3 : xyz = 1\}$

(this may require desingularization at ∞. Notice immediately that certain combinations of zeros and infinities in $(\mathbb{C}P^1)^3$ are not compatible with the constraint $xyz = 1$. It suffices to check the image of $(\mathbb{C}P^1)^2$ in $(\mathbb{C}P^1)^3$ under the map $(x,y) \mapsto (x,y,(xy)^{-1})$. Upon doing so, one finds the hexagon (indicated in red) as the image of $\overline{T_{\mathrm{SL}_3}}$ inside of the cube (a real sketch of $\overline{T_{\mathrm{GL}_3}}$) illustrated below.



2. $G = \mathrm{Sp}_2(\mathbb{C}) = \{A \in \mathrm{GL}_4 : AJA^* = J\}$ *has rank 2. A natural choice of maximal torus is embedded in* T_{GL_4} *as* $\{(x,y,z,w) : xz = 1, yw = 1\}$ *and verifying the image of zeros and infinities through the map* $(x,y) \mapsto (x,y,x^{-1},y^{-1})$ *reveals the following quadrilateral (in red) as a codimension 2 subvariety in the real illustration of* $(\mathbb{C}P^1)^4$ *as a 4-cube below.*


1101
1111
1001
1011
0101
0111
0001
0011
0100
0110
0000
0010
1100
1110
1000
1010

CHAPTER 9
Monopoles on Sasakian manifolds

A second avenue of future research, given the results on G-monopoles over a trivial S^1-fibration on Σ, is to consider an analogous classification theorem for some non-trivial S^1-bundles on a Riemann surface. If they are positive, they can be given a Sasakian structure which will allow us to extend some of our methods to this case.

Given the analysis involved for singular G monopoles on $S^1 \times \Sigma$, these compact Sasakian 3-manifolds are, in some ways, merely a (literal) twist away. Thus, a brief summary of this new geometry and a few appropriately adjusted definitions immediately suggests a classification theorem for monopoles of this type.

It will then be of interest to investigate the spectral data corresponding to these twisted objects.

9.1 Singular monopoles on Sasaki manifolds

A natural extension of this project, initiated in [1], would be to examine the moduli spaces over a less trivial domain. That is to say $S^1 \times \Sigma$ is the trivial S^1-bundle over Σ, but what about having non-trivial circle bundles over Σ as our monopole's domain? This type of problem was briefly alluded to at the end of [5] where they considered their domain X to be a flat S^1-bundle on Σ. These bundles correspond to 1-dimensional representations of the fundamental group of Σ. This quickly reveals the problem of monopole moduli spaces on certain flat S^1-bundles over Σ to $\pi_1(\Sigma)$-invariant objects on finite covering spaces of Σ.

Taking this slightly further, one has that any positive S^1-bundle on Σ can be given a Sasakian structure. This allows us to broaden our horizons to bundles which admit curvature. With the mechanics of Sasakian geometry under control, the theory developed thus far should translate. However, now the appropriate metric is no longer simply the product metric on $S^1 \times \Sigma$. Thus, let us first explain the basics of Sasakian geometry

9.1.1 Sasakian Manifolds

Sasakian geometry is informally referred to as "odd-dimensional Kähler" geometry. It merges the theory of Riemannian, symplectic and complex manifold theory and a Saskian 3-manifold is nested nicely between two well-known Kähler structures (one being on its positive cone and the other being the quotient of its Reeb foliation).

Definition 9.1.1. *A* Sasaki 3-manifold *is a triple* (X, α, g) *where* $\alpha \in \Gamma(T^*X)$ *is a normalized* contact form *making* X *into a contact manifold and* g *is a metric compatible with* α *in the following sense; the contact structure* α *defines a unit Reeb vector field* ξ *which is perpendicular to the contact planes and acts on* X *as a Killing vector field.*

We shall suppose the orbits of ξ are compact and take the quotient of X by S^1 action generated by the flow of ξ. Since X is a 3-manifold, this quotient reveals a projection $\pi : X \to \Sigma$ which is a surface orbifold. If the action of S^1 is, furthermore, assumed to be free, then this projection is that of a circle bundle on Σ.

A first example of such a manifold is the 3-sphere S^3 which is realized (via the Hopf map) as a non-trivial principal S^1-bundle over S^2.

A Sasakian structure on X is equivalent to the cone $C(X) := \mathbb{R}^+ \times X$ with metric $dr^2 + r^2 g$ being Kähler with form given by

$$\Omega = r^2 d\alpha - 2r\alpha \wedge dr.$$

Recalling that, [2], there is a quotient surface $\pi : X \to \Sigma$ at hand allowing us to decompose the metric $g = \pi^* h + \alpha \otimes \alpha$ where h is a Hermitian metric on Σ. If we write ω for the Kähler form on Σ which, recall, is the orthogonal complement of ξ, then $d\alpha = \pi^*\omega$ and we may rewrite the form on $C(X)$ as

$$\Omega = r^2 \pi^* \omega - 2r\alpha \wedge dr.$$

The volume form Ω^2 on X upon contraction by the radial vector is conveniently expressed as $V_X = d\alpha \wedge \alpha$ and the Kähler form on Σ pulled back to X is given as

$$\omega = d\alpha.$$

Locally, we write

$$i\mu(z,\bar{z})dz \wedge d\bar{z} = 2\mu(z,\bar{z})dx \wedge dy.$$

On the cone $C(X)$, there is a basis of vector fields (of constant norm in r) $\{\xi/r, \partial_r, v_z/r, v_{\bar{z}}/r\}$ where the first two are mutually orthonormal and orthogonal to the other two.

Thus, we can see that these Sasaki manifolds lie quite nicely between two related complex structures. The first is J on Σ and the second is the extension of J to $C(X)$ by the definition $J(\xi/r) := \partial_r$. With this, we find

$$T_{1,0}M = \text{span}\{\xi/r - i\partial_r, v_z/r\}(= i - \text{eigen space of } J)$$

$$T^*_{1,0}M = \text{span}\{r\alpha + idr, rdz\}$$

$$\Omega^{1,1}M = \text{span}\{r\alpha \wedge dr, r^2 d\alpha, \sigma_3, \sigma_4\}$$

where

$$\sigma_3 = (r\alpha + idr) \wedge d\bar{z}$$

$$\sigma_4 = (r\alpha - idr) \wedge dz$$

The wedge product is then found to be

$$\Omega \wedge \Omega = -4r^3 d\alpha \wedge \alpha \wedge dr.$$

Adjusting $\tilde{\Omega} := \frac{1}{r^2}\Omega = d\alpha - 2\alpha \wedge dt$ where $t = \log r$ moves us from the cone to the product $X \times S^1$ and here we have the following $\partial\bar{\partial}$-*Lemma*

Lemma 9.1.2.

$$\partial\bar{\partial}\left(1/r^2\right) = \frac{-i}{r^2}(d\alpha + 2\alpha \wedge dr).$$

implying that

$$\partial\bar{\partial}\left(\frac{1}{r^2}\Omega\right) = \partial\bar{\partial}(1/r^2) \wedge \Omega = 0.$$

Thus, $\tilde{\Omega}$ defines an S^1-invariant *Gauduchon* metric on $C(X)$ which then descends to a quotient manifold $N = X \times S^1$. An important pair of operators on 4-manifolds like M are given as

$$L : \wedge^{p,q}M \to \wedge^{p+1,q+1}M; \eta \mapsto \eta \wedge \tilde{\Omega}$$

and its formal adjoint $\Lambda = L^\dagger$ (under $\tilde{g}$).

For any $\eta \in \wedge^{p,q} M$ we find,

$$[L, \Lambda]\eta = (p+q-2)\eta$$

and using this in the case η is a 2-form gives

$$L^2\Lambda(\eta) = L\Lambda L\eta = 2L\eta$$

so

$$\Lambda(\eta)\tilde{\Omega} \wedge \tilde{\Omega} = 2\eta \wedge \tilde{\Omega}.$$

Hence, as will be required in the discussion of monopoles, the projection of any 2-form onto the Gauduchon (rather than Kähler) component is,

$$\Lambda(\delta d\alpha + \beta(\alpha \wedge dt) + \gamma\sigma_3 + \epsilon\sigma_4) = \delta - \beta/2.$$

With basis of 1-forms $\alpha, dz, d\bar{z}$ for X, $g(\alpha, \alpha) = 1/2, g(dz, d\bar{z}) = \mu^{-1}$, the volume form V_X as above and the Hodge dual coming from $g(a, b)V_X = a \wedge *b$ the Laplacian on functions $f \in \mathscr{C}^\infty(X; \mathbb{C})$ is computed as

$$\Delta(f) = \frac{1}{2}\xi^2 f + \mu^{-1}\left[v_{\bar{z}} \circ v_z(f) + v_z \circ v_{\bar{z}}(f)\right].$$

To summarize, our Sasakian 3-manifolds admit a non-trivial S^1-fibration over Σ so fit into the exact sequence

$$S^1 \hookrightarrow X \twoheadrightarrow \Sigma$$

and $N := S^1 \times X$ admits a Gauduchon metric and in particular admits the structure of an *elliptic fibration* over Σ.

Having all of this rich structure at hand suggests that a theory of singular monopoles and a classification theorem analogous to Theorem 1.0.1 is entirely plausible.

We now exhibit the Sasaki structure for the Hopf fibration.

9.1.2 Contact structure for Hopf-fibration

Consider the 3-sphere S^3 as embedded in $\mathbb{C}^2$ with standard Hermitian inner product as follows;

$$S^3 = \left\{\vec{z} = (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\right\} \subset \mathbb{C}^2$$

Define a complex line (real plane) at a point $\vec{z} \in S^3$ by

$$L_{\vec{z}} = \{\vec{w} \in \mathbb{C}^2 : \langle \vec{z}, \vec{w} \rangle = 0\}.$$

This $L_{\vec{z}}$ as a real subspace of $T_{\vec{z}}S^3$ is a tangent plane. We may equivalently view this tangent plane as the kernel of the exterior form

$$\alpha = z_1 d\bar{z}_1 + z_2 d\bar{z}_2 \in T^*_{\vec{z}}\mathbb{C}^2.$$

Notice that the action of S^1, given by

$$e^{i\theta} \cdot (z_1, z_2) := (e^{i\theta} z_1, e^{i\theta} z_2),$$

generates a flow (the derivative at $\theta = 0$) (iz_1, iz_2) which is tangent to S^3 and orthogonal to the hyperplane $L_{\vec{z}}$ thus verifying α as a contact structure on S^3.

After normalization the contact structure becomes

$$\hat{\alpha} = \frac{z_1 d\bar{z}_1 + z_2 d\bar{z}_2}{|z_1|^2 + |z_2|^2} = \frac{\alpha}{r^2}$$

and for future calculations it will be worthwhile to compute

$$d\hat{\alpha} = d\left(\frac{1}{r^2} \cdot \alpha\right) = -2r^{-3} dr \wedge \alpha + \frac{1}{r^2} d\alpha = \frac{1}{r^2} d\alpha + \frac{2}{r^3} \alpha \wedge dr.$$

Now, following the general construction above, the Kähler form on the cone $C(S^3)$ is given in terms of the contact form as

$$\begin{aligned}
\Omega &= r^2 d\hat{\alpha} - 2r\hat{\alpha} \wedge dr \\
&= r^2\left(\frac{1}{r^2} d\alpha + \frac{2}{r^3} \alpha \wedge dr\right) - 2r\hat{\alpha} \wedge dr \\
&= d\alpha + \frac{2}{r} \alpha \wedge dr - \frac{2r}{r^2} \alpha \wedge dr \\
&= d\alpha \\
&= dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2
\end{aligned}$$

So the $(1,1)$-form associated to the Gauduchon metric on the *Hopf surface* $\mathscr{H} = S^3 \times S^1$ (realized as the quotient of the cone) is then

$$\tilde{\Omega} = \frac{1}{r^2} \Omega = \frac{d\alpha}{r^2} = \frac{dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2}{|z_1|^2 + |z_2|^2}$$

which agrees (although unnormalized) entirely with the form provided in [3].

9.1.3 Singular G-monopoles on Sasaki manifolds

Given a G-bundle on $N := X \times S^1$, equip it with a G-connection ∇. In our notation this connection is expressed as

$$\nabla_{v_z} dz + \nabla_{v_{\bar{z}}} d\bar{z} + \nabla_{\xi}\alpha + \nabla_{\frac{\partial}{\partial t}} dt = (v_z + A_z)dz + (v_{\bar{z}} + A_{\bar{z}})d\bar{z} + (\xi + A_{\xi})\alpha + \left(\tfrac{\partial}{\partial t} + \varphi\right) dt$$

where z is the coordinate on Σ.

Lie brackets on X are $[v_z, v_{\bar{z}}] = -i\mu(z,\bar{z})\xi$ (since $\alpha([v_z, v_{\bar{z}}]) = -d\alpha(v_z, v_{\bar{z}})$) , and $[\xi, v_z] = [\xi, v_{\bar{z}}] = 0$ (since ξ is Killing), and on N $[\frac{\partial}{\partial t}, \xi] = 0$. This allows us to compute the curvature tensor as

$$\begin{aligned} F = &\left(i\mu^{-1}(z,\bar{z})[\nabla_{v_z}, \nabla_{v_{\bar{z}}}] + \nabla_{\xi}\right) d\alpha + [\nabla_{\xi}, \nabla_{\frac{\partial}{\partial t}}]\alpha \wedge dt \\ &+ [\nabla_{\sigma}, \nabla_{v_z}]\alpha \wedge dz + [\nabla_{\sigma}, \nabla_{v_{\bar{z}}}]\alpha \wedge d\bar{z} + [\nabla_{\frac{\partial}{\partial t}}, \nabla_{v_z}]dt \wedge dz + [\nabla_{\frac{\partial}{\partial t}}, \nabla_{v_{\bar{z}}}]dt \wedge d\bar{z} \\ = &F_{\Sigma} d\alpha + F_{\alpha}\alpha \wedge dt + \cdots \end{aligned}$$

Remark 14. *We note that the only peculiar term in the curvature above is the $d\alpha$-term and it may not be immediately apparent where the ∇_{ξ} has come from. Note that our basis $v_z, v_{\bar{z}}, \alpha$ consists of a form, α, which is* <u>*not closed*</u> *and so remains after the exterior differentiation $d(A_{\xi}\alpha) = A_{\xi} d\alpha$.*

In the case where the connection is t-invariant (i.e. when lifting monopoles on X to t-invariant connections on N) the commutator $[\nabla_{\xi}, \nabla_{\frac{\partial}{\partial t}}]$, hence F_{α}, is simply $\nabla_{\xi}(\varphi)$.

Now, as in the case for monopoles on $S^1 \times \Sigma$, we fix a finite collection of points $A = \{a_1, \ldots, a_N\} \subset X$ whose projection, via $\pi : X \to \Sigma$, we denote by

$B = \{b_j = \pi(a_j)\}_{j=1}^N \subset \Sigma$. We also fix a collection of cocharacters $\{\mu_j\}_{j=1}^N \subset X_*(G)$ associated to each of the a_j which encode singular information. Now,

Definition 9.1.3. *A* singular G-monopole *with constant $c \in \mathscr{Z}(\mathfrak{g})$ on our Sasaki manifold X of type $(A, \vec{\mu})$ is a G-bundle, G-connection, $\mathfrak{g}$-endomorphism triple (P, ∇, Φ) defined over $X \backslash A$ satisfying that;*

- *when lifted to N (inserting Φ for the dt-component), the resulting connection $\tilde{\nabla}$ is not only metric compatible, but compatible with the complex structure in the sense that*

$$F_{\tilde{\nabla}}^{0,2} = [\nabla_{v_{\bar{z}}}, \nabla_{\xi} - i\Phi] = 0$$

  and by conjugation

$$F_{\tilde{\nabla}}^{2,0} = [\nabla_{v_z}, \nabla_{\xi} + i\Phi] = 0$$

- *the contraction by the Gauduchon form $\tilde{\Omega}$ should be central. That is*

$$\Lambda(\tilde{F}) = iC.$$

  More explicitly, this reads

$$F_\Sigma - \frac{F_\alpha}{2} = \frac{2\left(i\mu^{-1}(z,\bar{z})[\nabla_{v_z}, \nabla_{v_{\bar{z}}}] + \nabla_\xi\right) - \nabla_\xi \Phi}{2} = iC$$

- *the singularities of Φ should be of the μ_j-Dirac type at each a_j as defined in section 6.1.*

9.1.4 Holomorphic and meromorphic structure on monopoles

Here, as before, we have holomorphic information for P over X which is prescribed by the commuting operators $\nabla_\Sigma^{0,1} = \nabla_{v_{\bar{z}}}$ and $\nabla_\xi^c = \nabla_\xi - i\Phi$. This implies that for all open subsets $U \subset \Sigma$ and sections $\psi : U \to X$ then the restriction P_ψ of the holomorphic section of P to $\psi(U)$ is then holomorphic in the usual sense.

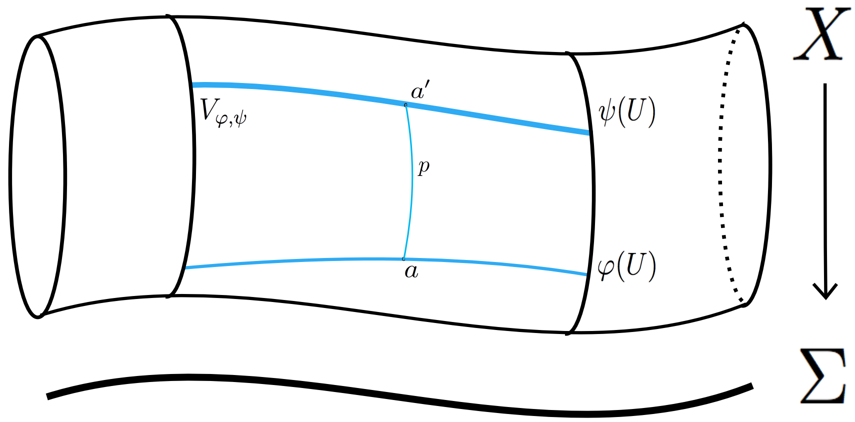
Now, given two sections $\psi, \varphi : U \to X$, ∇^c_ξ can be used to parallel transport $\rho_{\psi,\varphi}$ between P_ψ and P_φ defining a (almost everywhere) holomorphic isomorphism (cf. the scattering operator 6.1.6) except for possibly when the singular points p_j lie in between. A monodromy is locally defined in the same fashion, by scattering one full cycle around the fibres of X and these will be denoted $m_\varphi : P_\varphi \to P_\varphi$ for a section $\varphi : U \to X$. Now, with this in mind we define

Definition 9.1.4. *A section of $P \to X$ is* holomorphic *if it is parallel with respect to both $\nabla^{0,1}_\Sigma$ and $\nabla_\xi - i\Phi$ and, thus, the triple (P, ∇, Φ) on X is referred to a* holomorphic structure.

A meromorphic structure associated to a monopole (P, ∇, Φ) on X will be constructed in the same fashion as before with the exception of having a globally defined holomorphic bundle over Σ along with a meromorphic bundle automorphism. This time, due to the non-trivial nature of X as an S^1-bundle, everything must be constructed locally and patched together via the transition functions.

So, a *meromorphic structure* on $P \to X$ with poles at Z (of type μ_j) is simply a holomorphic structure $(\nabla^{0,1}, \nabla^c)$ on the restriction of P^c to the complement $X \backslash Z$. With this structure we employ the following parallel transport construction; for local sections φ, ψ of $X \to \Sigma$ defined on open $U \subset \Sigma$ having disjoint images, let $V_{\varphi,\psi}$ be defined formally and pictorially as a subset of P as follows;

$$V_{\varphi,\psi} = \{p \in P : \exists \theta < \theta' \in (0, 2\pi), a \in \varphi(U), a' \in \psi(U), s(\theta, a) = p, s(\theta', a) = a'\}$$



the transition maps $\rho_{\varphi,\psi} : P_\psi \to P_\varphi$ obtained by integrating along ∇^c_ξ are required to be holomorphic isomorphisms when $Z \cap V_{\varphi,\psi} = \phi$ occurring in the following cases;

(i) away from points in Z, and

(ii) when the paths of integration along fibers over points in Z do not contain singularities.

In all the cases when $z_i \in V_{\varphi,\psi}$, $\rho_{\varphi\psi}$ are required to be meromorphic and take the local form

$$\rho_{\varphi,\psi} = F(z)\mu_i(z - z_i)G(z)$$

with F, G holomorphic and invertible.

9.1.5 Holomorphic data on the curve Σ

Given a holomorphic structure, (P, ∇, Φ), on a Sasaki manifold $X \to \Sigma$, we cover Σ by carefully chosen charts $\{U_\alpha\}$ so that sections $\varphi_\alpha : U_\alpha \to X$ can be given to have disjoint images. That is, if p_i, p_j live in the same orbit (over $q \in \Sigma$), there is a φ_α so that $\varphi_\alpha(q)$ lies in the positive path (along S^1) from p_i to p_j. Let $Q = \{q_i\}_{i=1}^N \subset \Sigma$ be the points at which $\pi^{-1}(q_i)$ admits singularities and $X_0 := X \backslash Q$. Define bundles P_α

over U_α as the restriction of $P \to X$ to the image of φ_α in X (i.e. $P_\alpha = P|_{\varphi_\alpha(U_\alpha)}$) along with monodromies

$$m_\alpha : P_\alpha \to P_\alpha$$

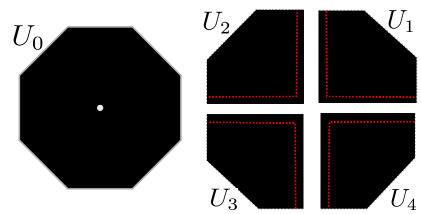
which are isomorphisms on $X_0 \cap U_\alpha$ and singular on $Q \cap U_\alpha$. If p_i is alone on S^1-orbit, then the singular type of m_α at q_i is of type μ_i (as usual). There are also transition maps $\rho_{\alpha\beta} : P_\beta \to P_\alpha$ on the overlap $U_\alpha \cap U_\beta$ providing a twisted cocycle condition

$$\rho_{\alpha\beta}\rho_{\beta\alpha} = m_\alpha$$

$$\rho_{\alpha\beta}\rho_{\beta\gamma} = \begin{cases} \rho_{\alpha\gamma} \\ \rho_{\alpha\gamma} m_\gamma \end{cases} .$$

depending on whether $U_\alpha, U_\beta, U_\gamma$ are ordered cyclically or not as one is always moving in the positive direction along the circle. The collection $\{U_\alpha, \varphi_\alpha, P_\alpha, m_\alpha, \rho_{\alpha\beta}\}$ constructed from any holomorphic structure (P, ∇, Φ) are referred to as *twisted bundles over* Σ.

As an example, the open sets, U_α can be chosen explicitly depending on the degree, say k, of X as an S^1-bundle over Σ. Indeed, it is clear that $X \to \Sigma$ is trivial over the complement of any point and is topologically classified by an integer k (in fact this is computed by integrating $d\alpha$ over Σ). Thinking of Σ by its polygon model (i.e. a $4g$-gon), U_0 is defined as the complement of a fixed point p and $U_1, \ldots, U_k$ are defined as an ϵ-neighbourhood of the angular sector $\left(2\pi\frac{s-1}{k+1}, 2\pi\frac{s}{k+1}\right)$ which look like pizza slices as illustrated below for the case $k = 4$ on a genus 2 Riemann surface;



9.2 Twisted spectral data

A singular G monopole (P, ∇, ϕ) on Sasakian $X \xrightarrow{\pi} \Sigma$ gives rise to the data, $\{U_\alpha, \varphi_\alpha, P_\alpha, m_\alpha, \rho_{\alpha\beta}\}$, of a twisted G-bundle over Σ. The monodromies (taking the place of meromorphic automorphism ρ from the $S^1 \times \Sigma$ case) are related by conjugation on overlapping neighbourhoods $U_\alpha \cap U_\beta$ as depicted in the following commutative diagram;

$$\begin{tikzcd} P_\alpha \arrow[r, "m_\alpha"] & P_\alpha \\ P_\beta \arrow[u, "\rho_{\alpha\beta}"] \arrow[r, "m_\beta"'] & P_\beta \arrow[u, "\rho_{\alpha\beta}"'] \end{tikzcd}$$

This allows us to define (for a fixed maximal torus and Borel $T \subset B \subset G$) *local cameral covers* $S_\alpha \xrightarrow{q_\alpha} U_\alpha$ in the same fashion as (8.1). A particularly nice, yet tautological, result of the monodromies $\{m_\alpha\}$ being related by conjugation is that for $z \in U_\alpha \cap U_\beta$, the conjugacy classes of $m_\alpha(z)$ and $m_\beta(z)$ coincide to define a global cover $S \xrightarrow{q} \Sigma$ as the topological quotient identifying points on overlapping neighbourhoods of the base. Again, after the proper compactification of T this cameral cover is a Weyl-invariant, projective curve in $\Sigma \times \bar{T}^W$ which is thus a $|W|$-sheeted branched cover of Σ.

Now, since we are dealing with twisted bundles over Σ, we cannot simply pull-back to the global cover via q, but must locally consider pullbacks $q_\alpha^*(P_\alpha) \to S_\alpha$.

Each of this pullbacks admit a reduction to B (denoted P_B^α) and project naturally via $\pi : B \to T$ to give a family of T-bundles over each S_α. Furthermore, as before, the lifts of the monodromies $q_\alpha^*(m_\alpha)$ are realized as a sections of the associated reduction $\mathrm{Aut}(P_B^\alpha) = \mathrm{Ad}_{P_B^\alpha}$. Patching these torus bundles together on overlaps $S_\alpha \cap S_\beta$ is achieved via the transition functions $\tau_{\alpha\beta} = \pi \circ q_\alpha^* \circ \rho_{\alpha\beta}$ and thus we have constructed a shifted T-torsor over S consisting of the following data

$$\mathscr{T} = \{\pi(P_B^\alpha), \tau_{\alpha\beta}\}.$$

That is to say, that the pair $(\mathscr{T}, S)$ is the analogous information required in the abelianization of a singular G monopole over a Sasakian 3-manifold and that this process is reversible.

CHAPTER 10
Conclusion

The originality of the results found here hinge greatly on the generalization structure group under consideration. Many generalizations of this type have evolved mainly over choice of domains (e.g. from algebraic curves to arbitrary compact complex manifolds) but often consider only the case of complex vector bundles. The greatest obstacle here lies within the deeper understanding of the structure theory of complex algebraic groups while keeping in mind their relationship with the general linear group. Four major differences encountered here and worth summarizing are:

1. The notion of degree of a G-monopole,
2. local decomposition for meromorphic maps into G^c
3. our notion of stability (inspired by Ramanathan) which, after much comparison with the usual vector-bundle analogue, has been reevaluated and brought back closer to its original orbit-theoretical roots. That is, we have backed away from the slope-condition (a comparison made between a vector bundle and its subbundles) and described a stability measurement more intrinsic to the nature of principal bundles.
4. Also, the concrete description of $\mathrm{Hom}_G(P, P')$ of G-equivariant maps between bundles as a naturally associated bundle to the fibre-product of $P \times_B P'$ which allowed for a description of induced connections so to determine relevant information such as covariant constancy of its sections.

Future investigations in this realm will undoubtedly reveal more of the subtle intricacies and structural differences between fibre-type. This is already apparent from the abelianization procedure outlined in chapter 8 (c.f. spectral curves and cameral covers). Furthermore, with all of this theory in order, we are also in place to start working with more exotic domains (e.g. Sasaki manifolds) which appear to reveal similar, yet twisted, classification results.

Part III

Appendices

Appendix A - Metrics on a principal bundle

Having worked out the details of the Cartan Decomposition for $\mathrm{GL}_n(\mathbb{C})$, it would make sense to consider, first, the related example.

Example 6 ($G = U_2$)**.** *Consider the standard metric on the group of* 2×2 *positive definite Hermitian matrices defined by the line element* $\mathrm{Tr}(H^{-1}\delta H)^2$. *This is easy to evaluate on the subgroup of positive diagonal elements. Indeed, if* $D = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ *with* $\lambda, \mu > 0$*, then our line element becomes*

$$\mathrm{Tr}(D^{-1}\delta D)^2 = \mathrm{Tr}\left[\begin{pmatrix} 1/\lambda & 0 \\ 0 & 1/\mu \end{pmatrix}\begin{pmatrix} d\lambda & 0 \\ 0 & d\mu \end{pmatrix}\right]^2 = \mathrm{Tr}\begin{pmatrix} \frac{d\lambda^2}{\lambda^2} & 0 \\ 0 & \frac{d\mu^2}{\mu^2} \end{pmatrix} = \frac{d\lambda^2}{\lambda^2} + \frac{d\mu^2}{\mu^2}.$$

With this, the Christoffel symbols *can be computed using the standard formula*

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^{n} g^{kl}\left(\frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l}\right)$$

where g_{ij} *is the* (i,j) *entry of the symmetric matrix obtained from the quadratic form* ds^2 *and* g^{ij} *denotes the* (i,j) *entry of its inverse. That is, with* $g = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ *and we find*

$$\Gamma^1 = \tfrac{1}{2}x^2\begin{pmatrix} \frac{-2}{x^3} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{-1}{x} & 0 \\ 0 & 0 \end{pmatrix} \quad \textit{and} \quad \Gamma^2 = \tfrac{1}{2}y^2\begin{pmatrix} 0 & 0 \\ 0 & \frac{-2}{y^3} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{-1}{y} \end{pmatrix}.$$

Plugging this into Euler-Lagrange/geodesic equations *(for the case* $n = 2$*)*

$$\frac{d^2x^k}{dt^2} + \sum_{i,j=1}^{n} \Gamma^k_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0,\ 1 \le i \le n \tag{10.1}$$

says that the minimal path $\gamma(t) = (x(t), y(t))$ between two points here must satisfy that

$$\frac{d^2x}{dt^2} - x = 0 \quad \textit{and} \quad \frac{d^2y}{dt^2} - y = 0.$$

This is solved with the path $x(t) = A\sinh(t) + B\cosh(t)$ and $y(t) = C\sinh(t) + D\cosh(t)$ and setting

$$\gamma(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \textit{ and } \gamma(1) = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

gives that $A = C = 1, B = \frac{\lambda - \cosh(1)}{\sinh(1)}$ and $D = \frac{\mu - \cosh(1)}{\sinh(1)}$. Then the arc length of $\gamma(t)$ is computed as

$$\mathscr{L}(\gamma) = \int_0^1 \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))}dt = \int_0^1 \sqrt{\frac{\dot{x}(t)^2}{x(t)^2} + \frac{\dot{y}(t)^2}{y(t)^2}}dt$$

and upon substitution becomes the definite integral appearing in the statement of the problem.

The left hand side is is the Euclidean norm of the logarithm of $D = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ (which makes sense here because of the following section on the Cartan decomposition). It is known that the exponential map from diagonal hermitian matrices is an isometric diffeomorphism and imposes a linearized metric on the group, so that the arc length of the geodesic between the points specified by $\gamma(t)$ above is in fact the Euclidian norm of the vector $(\ln(\lambda), \ln(\mu))$.

The general statement for U_n (found in both [12, 26]) is that if h, k are positive definite $n \times n$ Hermitian matrices with $d(h, k)$ denoting the geodesic distance from h to k with respect to the standard metric on $\mathrm{GL}_n(\mathbb{C})$ given by $\mathrm{Tr}(H^{-1}\delta H)^2$ and

$$\sigma(h,k) := \mathrm{tr}(h^{-1}k) + \mathrm{tr}(k^{-1}h) - 2n$$

then, there exist positive constants c_1, c_2 such that

$$c_1\sigma(h,k) \leq d^2(h,k) \leq c_2\sigma(h,k).$$

Here we have carefully examined the inequality on the right more concretely by spelling out these functions where as the left inequality. I previously stated that our bounds are actually tighter. This is because

$$\begin{aligned}
\sigma(1,D) &= \sum_{i=1}^{n}(\lambda_i + \lambda_i^{-1} - 2) \\
&= \sum_{i=1}^{n}(e^{\ln\lambda_i} + e^{-\ln\lambda_i} - 2) \\
&= 2\sum_{i=1}^{n}[\cosh(\ln\lambda_i) - 1] \\
&= 2\cdot\sum_{i=1}^{n}\left(\sum_{k\geq 1}\frac{(\ln\lambda_i)^{2k}}{(2k)!}\right) \\
&\geq 2\sum_{i=1}^{n}(\ln\lambda_i)^2
\end{aligned}$$

implying that the constant $c_2 = \frac{1}{2}$.

The point of having such a measurement, is that σ acts as a “semi-norm” in the sense that $\sigma(h,k) \geq 0$ with equality if and only if $h = k$. Furthermore, with $d^2(h,k) \leq C\cdot\sigma(h,k)$ then any notion of analytical convergence found using σ will imply the desired convergence in the actual metric! This allows us to determine convergence results about the Yang-Mills-Higgs flow for the monopole problem (see Simpson [42]).

The general Cartan decomposition for G^c allows us to analogously view the space of positive metrics on P as the sections of the G^c/G bundle $P^c/G :=$ $(P \times_G G^c)/G$. The function σ here used to determine convergence is defined via a faithful unitary representation ϕ of G so that for two metrics $h, k \in \Gamma(P^c/G)$,

$$\sigma_\phi(h,k) := \mathrm{tr}(\phi(h^{-1}k)) + \mathrm{tr}(\phi(k^{-1}h)) - 2\dim\phi.$$

Appendix B - The Yang-Mills-Higgs flow

The Yang-Mills-Higgs action is an energy functional defined appropriately on the space of Higgs bundles. That is, a total energy for a pair (A, Φ) which restricts to the Yang-Mills action in the case that the Higgs field Φ vanishes. This functional is denoted by YMH and defined as

$$YMH(A, \Phi) := ||F_A||_2 + ||d_A\Phi||_2.$$

First variation of YMH

Proposition 10.0.1. *The first variations for the Yang-Mills-Higgs functional are*

(i) $\frac{\partial A}{\partial \epsilon} = d_A^* F_A + [d_A\Phi, \Phi]$ *and*

(ii) $\frac{\partial \Phi}{\partial \epsilon} = d_A^* d_A \Phi.$

Proof. For arbitrary forms $\eta \in \mathfrak{g} \otimes \Omega^1(B), \mu \in \mathfrak{g} \otimes \Omega^0(M)$, we can vary A, Φ, as

$$A_\epsilon = A + \epsilon\eta \quad \text{and} \quad \Phi_\epsilon = \Phi + \epsilon\mu$$

so that

$$\begin{aligned}
F_{A_\epsilon} &= dA_\epsilon + A_\epsilon \wedge A_\epsilon \\
&= dA + \epsilon d\eta + A \wedge A + \epsilon(A \wedge \eta + \eta \wedge A) + \epsilon^2 \eta \wedge \eta \\
&= F_A + \epsilon d_A \eta + \epsilon^2 \eta \wedge \eta, \\
d_{A_\epsilon} \Phi &= d_A \Phi + \epsilon[\eta, \Phi],
\end{aligned}$$

and

$$d_A \Phi_\epsilon = d_A \Phi + \epsilon d_A \mu.$$

For the variation in the A direction

$$\begin{aligned}
||F_{A_\epsilon}||_2 &= \int_B F_{A_\epsilon} \wedge *F_{A_\epsilon} \\
&= \int_B (F_A + \epsilon d_A \eta + \epsilon^2 \eta \wedge \eta) \wedge *(F_A + \epsilon d_A \eta + \epsilon^2 \eta \wedge \eta) \\
&= ||F_A||_2 + 2\epsilon \langle F_A, d_A \eta \rangle_2 + \epsilon^2 \left(||d_A \eta||_2 + 2\langle F_A, \eta \wedge \eta \rangle_2 \right) + 2\epsilon^3 \langle d_A \eta, \eta \wedge \eta \rangle_2
\end{aligned}$$

and similarly,

$$||d_{A_\epsilon} \Phi||_2 = ||d_A \Phi||_2 + 2\epsilon \langle d_A \Phi, [\eta, \Phi] \rangle + \epsilon^2 ||[\eta, \Phi]||_2.$$

Hence,

$$\frac{\partial YMH(A_\epsilon, \Phi)}{\partial \epsilon}\Big|_{\epsilon=0} = \langle F_A, d_A \eta \rangle + \langle d_A \Phi, [\eta, \Phi] \rangle.$$

Similarly, using that

$$||d_A \Phi_\epsilon||_2 = ||d_A \Phi||_2 + 2\epsilon \langle d_A \Phi, d_A \mu \rangle + \epsilon^2 ||d_a \mu||_2$$

we find

$$\frac{\partial YMH(A,\Phi_\epsilon)}{\partial \epsilon}|_{\epsilon=0} = \langle d_A\Phi, d_A\mu\rangle.$$

This remainder of this result makes use of the technical result involving inner products in Lemma (10.0.2) below. □

Before stating the next technical result required to complete the previous Proposition, recall that for linear algebraic groups, we will use $\langle A,B\rangle := \mathrm{Tr}(AB^*)$ as the inner product on the Lie algebra, between elements $A,B \in \mathfrak{g}$. Alternatively, one might wish to use the *Killing form* $\langle a, b\rangle_{\mathfrak{g}} := \mathrm{Tr}\, R(a)R(b)$ where $R : \mathfrak{g} \to \mathrm{GL}(g)$ is the *regular representation* of $\mathfrak{g}$.

Lemma 10.0.2. *If $\alpha,\beta \in \mathfrak{g}\otimes\Lambda^1(V)$ and $\gamma \in \mathfrak{g}$ is self-adjoint, then*

$$\langle \alpha, [\beta,\gamma]\rangle = \langle [\alpha,\gamma], \beta\rangle.$$

Proof. As above, write $\alpha = \sum_i \alpha_i v_i, \beta = \sum_i \beta_i v_i$. Then

$$\begin{aligned}
\langle \alpha, [\beta, \gamma] \rangle &= \sum_{i,j} \langle \alpha_i v_i, [\beta_j, \gamma] v_j \rangle \\
&= \sum_{ij} \mathrm{Tr}(\alpha_i [\beta_j, \gamma]^*) \cdot \langle v_i, v_j \rangle \\
&= \sum_{ij} \mathrm{Tr}(\alpha_i \gamma^* \beta_j^* - \alpha_i \beta_j^* \gamma^*) \cdot \langle v_i, v_j \rangle \\
&= \sum_{ij} \mathrm{Tr}(\alpha_i \gamma \beta_j^* - \gamma \alpha_i \beta_j^*) \cdot \langle v_i, v_j \rangle \\
&= \sum_{ij} \mathrm{Tr}([\alpha_i, \gamma] \beta_j^*) \cdot \langle v_i, v_j \rangle \\
&= \sum_{ij} \langle [\alpha_i, \gamma] \beta_j^* \rangle_{\mathfrak{g}} \cdot \langle v_i, v_j \rangle \\
&= \langle [\alpha, \gamma], \beta \rangle
\end{aligned}$$

Where we note that, here, we didn't need to make use of the inner product on $\Lambda^k(V)$ and so, have not discussed it. $\square$

Here, we wish to describe the flow in terms of complex gauge transformations, $g \in \Gamma(P^c)$ on pairs (A, Φ). Parameterize a family g_s of transformations with $g_0 = I$ and write $(A_s, \Phi_s) = g_s(A_0, \Phi_0)$

Recall, the equation (dropping the subscript s on g).

$$A_s = A_0 - \bar{\partial}_z^{A_0} g \cdot g^{-1} d\bar{z} + (g^*)^{-1} \partial_z^{A_0} g^* dz + \tfrac{1}{2} \left((-\partial_t^{A_0} + i\Phi_0) g \cdot g^{-1} + (g^*)^{-1} (\partial_t^{A_0} + i\Phi_0) g^* \right) dt$$

In this notation, the first variations are expressed (according to S. Jarvis [26]) as

(i) $\frac{\partial A_s}{\partial s} = -\bar{\partial}_z^{A_0} \xi d\bar{z} + \partial_z^{A_0} \xi^* dz + \frac{1}{2} (\partial_t (\xi^* - \xi) + i[\Phi_0, \xi + \xi^*]) dt$ and

(ii) $\frac{\partial \Phi_s}{\partial s} = \frac{1}{2i} \left(\partial_t^{A_0} (\xi^* + \xi) + i[\Phi_0, \xi^* - \xi] \right)$

where $\xi = \frac{\partial g}{\partial s} g^{-1}$.

The next, unverified claim here is that if (A_0, Φ_0) satisfies two-thirds (namely equations (**??**)) part the Bogomolny equation, then so will every pair in the sequence. Luckily, this easily seen by expanding $[\partial_t^{A_s} - i\Phi_s, \bar{\partial}_z^{A_s}]$ in terms of the holomorphic gauge h and (A_0, Φ_0).

Proposition 10.0.3 (Jarvis, [26] Proposition 8)**.** *If* (A_0, Φ_0) *satisfy the integrability condition* (5.3) *and* g_s *is a smooth sequence of gauge transformations satisfying*

$$\frac{\partial g}{\partial s} g^{-1} = -iB_t$$

where $B = *F_{A_s} - d_{A_s}\Phi_s$ *and* B_t *is the* dt *component, then* (A_s, Φ_s) *will be a solution to the Yang-Mills-Higgs flow.*

Furthermore, the corresponding flow of metrics H_S *is*

$$\frac{\partial H_s}{\partial s} = -2iH_s B(H_s),$$

with $B(H_s) = B(g_s \cdot (A_0, \Phi_0))$*.*

The solution to the Yang-Mills-Higgs flow is recovered from the H_s *as follows: A sequence* g_s *such that* $H_s = H_0 g_s^* g_s$ *(where adjoint is taken with respect to* H_0*) gives a sequence of pairs* $g_s \cdot (A_0, \Phi_0)$ *that is gauge equivalent to a solution of the flow. If we demand further that*

$$g^{-1}\frac{\partial g}{\partial s} = \tfrac{1}{2} H^{-1}\frac{\partial H}{\partial s},$$

then this solution will be exact.

Recall, as stated originally in [12], $\sigma : \Gamma(P^c/P) \times \Gamma(P^c/P) \to \mathbb{Z}$ defined as $\sigma(h,k) := \mathrm{tr}(h^{-1}k) + \mathrm{tr}(k^{-1}h) - 2n$ gives a measurement on the space of metrics, P^c/P, for P which is commensurate with the geodesic norm.

Proposition 10.0.4 (Jarvis, [26] Proposition 9)**.** *If H^1, H^2 satisfy*

$$\frac{\partial H^i}{\partial t} = -2iH^i B(H^i)$$

then

$$\left(\frac{\partial}{\partial t} + \Delta\right)\sigma(H^1, H^2) \leq 0.$$

Lemma 10.0.5 (Jarvis, [26] Lemma 10)**.** *If (A_s, Φ_s) is a solution to the YMH-flow coming form a flow of metrics H_s, then*

(i)

$$\left(\frac{\partial}{\partial s} + \Delta_A + \Phi^*\Phi\right)B = 0$$

(ii)

$$\left(\frac{\partial}{\partial s} + \Delta\right)|B|^2 = -2\left(|d_A^* F_A - [d_A\Phi, \Phi]|^2 + |d_A^* d_A \Phi|^2\right)$$

(iii) when $|B| \neq 0$,

$$\left(\frac{\partial}{\partial s} + \Delta\right)|B| \leq 0$$

The proof of this makes use of the *Weitzenbock formula* for 1-forms α

$$\left(d_A^* d_A + d_A d_A^*\right)\alpha = \nabla_A^* \nabla_A \alpha - *[*F_A \wedge \alpha]. \tag{10.2}$$

and *Kato's inequality*

$$|\nabla_A B|^2 > |d|B||^2. \tag{10.3}$$

Appendix C - Some gauge theory

Throughout this paper, we are working almost exclusively with a connection $\nabla = d + A$ and a Higgs field Φ on a principal bundle $\pi : P \to B$. The base space here is the very particular 3-manifold, $S^1 \times \Sigma_g$ to be viewed locally as $\mathbb{R} \times \mathbb{C}$

Gauge transformations $g \in \mathscr{G}^c$ act on the principal bundle, or any vector or matrix representation of it, by multiplication or conjugation. A vector $v \in E_\varphi$ transforms as gx, the connection form A by $gAg^{-1} - g^{-1}dg$. As an operator, the connection ∇_A becomes $g \circ \nabla_A \circ g^{-1}$ and a metric H (represented by a self-adjoint matrix) transforms as $(g^*)^{-1}Hg^{-1}$. An endomorphism $d \in \mathrm{End}(E_\varphi)$ such as the Higgs field is transformed by gdg^{-1} and its adjoint with respect to a given metric H is the endomorphism $d^\dagger$, defined uniquely by $\langle x, dy \rangle = \langle d^\dagger x, y \rangle$. In terms of matrices, this is $d^\dagger = H^{-1}d^*H$ and applying the standard gauge transformation to endomorphisms transforms the adjoint as usual $gd^\dagger g^{-1}$.

Unitary versus holomorphic gauge

Given a "complex" chart in the base manifold (i.e. a local neighbourhood of the form $\mathbb{R} \times \mathbb{C}$), there are two useful types of trivialization we may consider: A *unitary gauge*, where the metric $H = I$ is the identity and a *holomorphic gauge* for which $A^{0,1} = 0$ or $\bar{\partial}_z^A = 0$.

Unitary gauge Here, the condition that ∇_A be unitary implies that the connection form A is skew-Hermitian (i.e. that $A = -A^*$) in such a gauge. Then also $A^{1,0} = -(A^{0,1})^*$ so that ∂_z^A is derived easily from $\bar{\partial}_z^A$.

Holomorphic gauge The metric here is a self-adjoint matrix H. There is a gauge transformation g , taking the holomorphic gauge to a unitary one; g can be taken to be any G^c-valued function satisfying that $H = g^*g$. In the holomorphic gauge the connection takes the form $\nabla_A = \partial^A + \bar{\partial}$ (since $A^{0,1} = 0$) transforms in the unitary gauge to $g \circ \nabla_A \circ g^{-1}$. Hence, $A^{0,1} = -g^{-1}\bar{\partial} g$ in the new unitary gauge.

Taking adjoints, we find that $A^{1,0} = dg^*(g^*)^{-1}$, which must hold even without any prior information about $A^{1,0}$ in the holomorphic gauge. So, in the new unitary gauge we have

$$\nabla_A = g \circ \bar{\partial} \circ g^{-1} + (g^*)^{-1} \circ \partial \circ g^*.$$

Applying the inverse gauge transformation g^{-1} to return to the holomorphic gauge gives

$$\partial^A = g^{-1}(g^*)^{-1} \circ \partial \circ g^* g = H^{-1} \circ \partial H.$$

Combining this with that fact that $A^{0,1} = 0$ in this gauge implies the connection 1-form is expressed purely in terms of the metric as

$$A = H^{-1}\partial H.$$

Multiple structures A choice of gauge allows us to compare the connection ∇ to the usual exterior derivative d whose difference is the connection 1-form A. Will be working with a convergence on the space of metrics of a measurement depending on A relative to a particular metric, so it will make sense to have some means of comparison or relationship between two connection forms relative to different choices of metric.

Identifying holomorphic structures: If A_1, A_2 are two connections which are consistent with the same holomorphic structure (so that $\bar{\partial}^{A_1} = \bar{\partial}^{A_2}$). Then, in a holomorphic gauge where $A_1^{0,1} = A_2^{0,1} = 0$ then $A_i = H_i^{-1}\partial H_i$ and the metrics are related by $H_1 = H_0 h$ for some endomorphism $h = H_0^{-1}H_1$. Computing $h^{-1}\partial_z^{A_0}h$,

$$H_1^{-1}H_0\left[\partial_z(H_0^{-1}H_1) + H_0^{-1}(\partial_z H_0)H_0^{-1}H_1\right] = H_1^{-1}\partial_z H_1.$$

So in a gauge where both connections are holomorphic,

$$d^{A_1} = d^{A_0} + h^{-1}\partial^{A_0}h. \tag{10.4}$$

Identifying unitary structures: In this common unitary gauge, connections take the form

$$d^{A_i} = g_i \circ \bar{\partial} \circ g_i^{-1} + (g_i^*)^{-1} \circ \partial \circ g_i^*.$$

Put $g = g_1 g_0^{-1}$, then

$$d^{A_1} = g \circ \bar{\partial}^{A_0} \circ g^{-1} + (g^\dagger)^{-1} \circ \partial^{A_0} \circ g^\dagger. \tag{10.5}$$

In the initial unitary gauge, $g^\dagger = g^*$, but if it is defined with respect to H_0 this holds in any gauge and $g^\dagger g = H_0^{-1}g^* H_0 g = H_0^{-1}H_1 = h$.

Equation (10.5) can be thought of as defining an action of $\mathscr{G}^c$ on the space of connections as in [13] section 6.1.1. This is not the usual action, but coincides on $\mathscr{G}$ when $g = g^*$. Also, the usual gauge action of g on $d^{g(A)} := g \circ \bar{\partial}^{A_0} \circ g^{-1} + (g^\dagger)^{-1} \circ \partial^{A_0} \circ g^\dagger$ recovers

$$g^{-1} \circ d^{g(A)} \circ g = d^{A_0} + h^{-1}\partial^{A_0}h$$

from equation (10.4).

Bogomolny equations

The equivalent of a holomorphic gauge is a choice of gauge $q \in \Gamma(P^c)$ where the operators $\bar{\partial}_z^A = \bar{\partial}_z$ and $\partial_t^A - i\Phi = \partial_t$. Such a gauge can be found by solving some ordinary differential equations provided the operators commute (part of the Bogomolny equation). We can transform this gauge into a unitary gauge with a gauge transformation g so that $q = pg$ with $p \in \Gamma(P)$. Here $\bar{\partial}_z^A = g \circ \bar{\partial}_z \circ g^{-1}$ and $\partial_t^A - i\Phi = g \circ \partial_t \circ g^{-1}$. The connection form is now unitary, so we can take adjoints to find $\partial_z^A = (g^*)^{-1} \circ \partial_z \circ g^*$ and $\partial_t^A + i\Phi = (g^*)^{-1} \circ \partial_t \circ g^*$. These will be put together below to deduce required equations from [26].

Index

REFERENCES

[1] I. Biswas, J. Hurtubise, *Singular monopoles on compact Sasakian 3-manifolds*, To appear 2015.

[2] C. Boyer and K. Galicki, *Sasakian Geometry*, Oxford Mathematics Monographs (2008).

[3] P. Braam, J. Hurtubise, *Instantons on Hopf surfaces and monopoles on solid tori*, J. reine angew. Math. **400** (1989), 146–172.

[4] M. Atiyah, R. Bott, *The Yang-Mills equations over Riemann surfaces*, (1982)

[5] B. Charbonneau, J. Hurtubise, *Singular Hermitian-Einstein Monopoles on the Product of a Circle and a Riemann Surface*, Int. Math. Research Notices, v.2011, No. 1, pp.175-216.

[6] M. Daniel, C. M. Viallet, *The geometrical setting of gauge theories of the Yang-Mills type*, Reviews of Modern Physics, Vol. 52, No. 1, January 1980.

[7] R.Y. Donagi, *Spectral Covers*, MSRI series Vol. **28**, 1995.

[8] R.Y. Donagi, D. Gaitsgory, *The gerbe of Higgs bundles*, Transform. Groups **7** (2002) 109-153,

[9] S.K. Donaldson, *A new proof of a theorem of Narasimhan and Seshadri*, J. Differential Geom. Vol. 18, Number 2 (1983), 269-277.

[10] S.K. Donaldson, *Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, Proc. London Math. Society, **50** (1985) 1–26.

[11] S.K. Donaldson, *Infinite determinants, stable bundles and curvature*, Duke Mathematical Journal, **54** (1987) 231–247.

[12] S.K. Donaldson, *Boundary value problems for Yang-Mills fields*, Journal of Geometry and Physics 8, no. 1-4 (1992):89-122.

[13] S. K. Donaldson, P. B. Kronheimer, *The Geometry of Four Manifolds*, Oxford Mathematical Monographs, Oxford: Clarendon Press, ISBN 0-19-850269-9 (1997).

[14] J. J. Duistermaat, J. A. C. Kolk, *Lie Groups*, Universitext, Springer Berlin Heidelberg ISBN: 978-3-540-15293-4 (2000).

[15] M. Fecko, *Differential Geometry and Lie groups for Physicists*, Cambridge Univ. Press (2006).

[16] D. Gieseker, *Geometric invariant theory and the moduli of bundles*, Lecture Publication Series, IAS/Park City Mathematics Series v.00, (1994).

[17] Griffith's and Harris, *Principals of Algebraic Geometry*, Wiley Interscience (1978).

[18] N. Hitchin, *On the construction of Euclidean monopoles*, Commun. Math. Phys. 89,145-190 (1983).

[19] J.C. Hurtubise, *The algebraic geometry of the Kostant-Kirillov form*, J. London Math. Soc. (2) 56 (1997) 504–518.

[20] J.C. Hurtubise and E. Markman, *Rank 2 integrable systems of Prym varieties*, Adv. Theo. Math. Phys., 2 (1998), 633–695.

[21] J.C. Hurtubise and E. Markman, *Elliptic Sklyanin integrable systems for arbitrary reductive groups*, Adv. Theor. Math. Phys., 6 (2002), 873–978.

[22] D. Huybrechts, *Complex Geometry*, Springer (2005).

[23] D. Husemöller, *Fibre Bundles*, Springer Verlag, ISBN 0-387-94087-1 (1994).

[24] D. Hyeon and D. Murphy, *Note on the stability of principal bundles*, Proc. Amer. Math. Society (2004).

[25] S. Jarvis, *Euclidean Monopoles and Rational Maps*, Proc. London Math. Soc. (3) 77 (1998) 170-192.

[26] S. Jarvis, *Construction of Euclidean Monopoles*, Proc. London Math. Soc. (3) 77 (1998) 193-214.

[27] A. Kapustin, E. Witten, *Electric-Magnetic Duality And The Geometric Langlands Program*. Commun. Number Theory Phys. **1** (2007), 1-236.

[28] S. Kobayashi, *The Differential Geometry of Complex Vector Bundles*, Princeton University Press (1987).

[29] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, Wiley Classics Library (1963).

[30] A. Kriegl, P. W. Michor, *The Convenient Setting of Global Analysis*, American Mathematical Society, (1997).

[31] P. B. Kronheimer, *Master's Thesis*, Oxford, 1986.

[32] M. Lübke, A. Teleman, *The Kobayashi-Hitchin Correspondence*, World Scientific (1995).

[33] P. W. Michor, *Topics in Differential Geometry*, Graduate studies in mathematics v. 93, American Math Soc. (2008).

[34] D. Mumford, *Geometric invariant theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Band 34. Springer-Verlag, Berlin-New York, 1965.

[35] M.S. Narasimhan, C.S. Seshadri, *Stable and unitary bundles on a compact Riemann surface*, Ann. of Math. **82** (1965), 540–564.

[36] Paul Norbury, *Magnetic Monopoles On Manifolds with Boundary*, Transactions of the Amer. Math. Soc. **3** (2011), 1287-1309.

[37] M. Pauly, *Monopole moduli spaces for compact 3-manifolds*, Mathematische Annalen 311, no. 1 (1998): 125-46.

[38] A. Ramanathan, *Stable principal bundles on a compact Riemann surface*, Math. Ann. 213 (1975), pp. 129-152.

[39] R. Scognamillo, *Prym-Tjurin varieties and the Hitchin map*, Math. Ann. **303**, 47-62 (1995).

[40] R. Scognamillo, *An elementary approach to the abelianization of the Hitchin system for arbitrary reductive groups*, Compos. Math. **110** (1998), 17-37,

[41] C. S. Seshadri, *Space of unitary vector bundles on a compact Riemann surface*, Ann. of Math. (2) 85 (1967), 303-336.

[42] C. T. Simpson, *Constructing variations of Hodge structure using Yang-Mills theory and applications of uniformization*, Journal of the AMS 1, no.4 (1988): 867-918.

[43] B. H. Smith, *The Differential Geometry of Instantons*, Masters thesis, Univ. of Waterloo (2009).

[44] K. Uhlenbeck, S.T. Yau, *On the existence of Hermitian Yang-Mills connections in stable vector bundles* Comm. Pure Appl. Math., **39** (1986) 257–293.

[45] K. Uhlenbeck, S.T. Yau, *A note on our previous paper: On the existence of Hermitian Yang-Mills connections in stable vector bundles* Comm. Pure Appl. Math., **42** (1989) 703–707.