Averaged Conformal Field Theories and Bosonization in Quantum Gravity

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March, 2022

A thesis submitted to McGill University in partial fulfillment of the requirements of the master's degree of Science

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Abstract

We extend to fermionic conformal field theories (CFTs) a recent discovery in quantum gravity, where an ensemble average of two-dimensional free boson CFTs over the Narain moduli space gives rise to a theory of gravity in three-dimensional Anti-de Sitter (AdS) space related to Chern-Simons theory. First, we review basic features of two-dimensional CFTs, including the conformal group and algebra, the operator formalism of CFTs, and the CFT Hilbert Space. Then, we focus on CFTs on the torus in two dimensions, study their modular properties using the modular group of the torus, and compute the partition functions of three CFTs that play important roles in this project: free bosons on a circle, free bosons on a $\mathbb{Z}_2$ orbifold, and free fermions. We also review the AdS/CFT correspondence and describe how CFT plays an important role in the study of quantum gravity. In addition, we describe how the partition function of Chern-Simons gravity matches the averaged partition function of an ensemble of compact free bosonic CFTs over the Narain moduli space. To extend this result to fermionic CFTs, we apply bosonization to determine the fermionic moduli space, and build a map between moduli spaces of the c = 1 free boson and the c = 1 interacting fermionic CFTs known as the Thirring model. For c = 1 Thirring CFTs, averaging over moduli space leads to a divergence in the averaged partition function just as in the bosonic case.

Abrégé

Cette thèse vise à trouver une version fermionique d'une découverte récente en gravité quantique, soit le fait que la moyenne d'ensemble des théories conformes des champs (TCCs) bosoniques libres en deux dimensions sur l'espace de modules de Narain est liée à la gravité de Chern-Simons dans l'espace anti-de Sitter (AdS) en trois dimensions. D'abord, il sera question des TCCs en deux dimensions. De ce fait, les groupes et les algèbres conformes, le formalisme des opérateurs dans les TCCs ainsi que la représentation des états dans un espace Hilbert TCC seront développés. Les propriétés modulaires TCCs sur le tore en deux dimensions seront ensuite examinées par le biais du groupe modulaire du tore. La fonction de partition pour trois exemples de TCC, soient le TCC des bosons compacts libres sur un cercle, le TCC des bosons libres sur un orbifold $\mathbb{Z}_2$, et le TCC des fermions libres, sera calculée. Il sera aussi question de la correspondance AdS/CFT afin de démontrer l'utilité des TCCs dans l'étude de la gravité quantique et de présenter un résultat sur l'équivalence entre la fonction partition de la gravité de Chern-Simons et celle de la fonction partition moyenne d'un ensemble de TCCs bosoniques compacts libres sur l'espace des modules de Narain. Afin de dériver une version fermionique de ce résultat, l'espace des modules fermioniques sera caractérisé par le biais de la bosonization. De plus, une application entre l'espace des modules des TCCs bosoniques libres $c = 1$ et l'espace des modules des TCCs fermioniques interactives $c = 1$, nommé le modèle de Thirring, sera construite. Le fait que la fonction partition obtenue en prenant la moyenne sur l'espace des modules des TCCs Thirring $c = 1$ comprend une divergence tout comme le cas bosonique sera également démontré.

Acknowledgements

I would first like to thank my advisors Professor Sarah Harrison and Professor Alex Maloney. Their expertise in physics and mathematics and passion for knowledge were invaluable in proposing research questions and providing insightful feedback for my work. Learning theoretical physics with them was an inspiring and fruitful experience, and their kind support and counsel during the stressful pandemic helped me greatly.

I would also like to thank Professor Keshav Dasgupta for his valuable guidance throughout my undergraduate and master's studies. He helped me further deepen my understanding in physics, and I appreciate the great opportunity he offered me to assist in teaching. I am also very grateful to my peer graduate students with whom I have discussed physics during courses and meetings.

My master's study was financially supported by a scholarship and teaching assistantships offered by the physics department at McGill University, along with funding from the loans and bursaries program offered by the government of Quebec. I appreciate this help greatly since I was able to focus on my studies and research without financial hardship.

Finally, I would like to thank my parents for their company, understanding, and continual support, thank Gabriel for his help and thoughtfulness, and thank Chinyin for cheering me up with her sweetness and passion for life itself. I can never thank them enough for all they have done for me.

Contribution of Authors

Alex Maloney and Edward Witten's idea of the duality between gravity and an ensemble average gave birth to this project. The idea of extending the above relation to supersymmetric conformal field theories proposed by Alex Maloney initiated this project. The project was done in collaborations with Sarah Harrison and Alex Maloney. Nathan Benjamin and Scott Collier also contributed in the discussion of fermionizing the Narain moduli space. I re-derived and derived all the results I present in this thesis. Works of other authors are cited as much and clearly as possible.

Chapter 1

Introduction

Understanding why quantum gravity matters is one of the most important starting points in our search for a unified theory. One reason is that when we want to study, for example, the beginning of the universe where the amount of energy is huge while the size of the universe is compacted in a tiny region, we need both gravitational theory and quantum theory. In physics on very small scales, quantum field theory (QFT) is the core of elementary particle physics, which combines quantum theory, field theory, and special relativity and also provides essential tools for condensed matter physics, nuclear physics, atomic physics, and astrophysics.

A special family of QFTs in which the theories are invariant under conformal transformations is known as the set of conformal field theories (CFTs). In two dimensions, this conformal invariance (angle preserving) property of CFT gives rise to an infinite numbers of symmetries which is so powerful that finding exact solutions in the CFT context is possible. Compared to how difficult, if possible, it is to do so for a general QFT, this feature makes CFTs one of the most important fields of research in high energy physics.

In the attempt to unify quantum and gravitational theories, the duality between gravity in Anti de Sitter (AdS) space and CFTs, known as the AdS/CFT correspondence, was discovered by Maldacena in the 90s [1]. This duality suggests that certain d-dimensional strongly coupled quantum field theories and certain string theories in d+1 dimensions are two sides of the same coin, i.e. they are the same theory.

In recent studies of the AdS/CFT correspondence, disordering and averaging have played an important role. On the one hand, the idea that gravity emerges from averaging over many different configurations of the same theory has been somewhat well established. For example, when studying black holes in a quantum theory of gravity, geometrically describing an individual quantum microstate of a black hole can be very difficult in the context of general relativity. However, when we coarse-grain over many microstates, a gravitational description described by a black hole geometry emerges. On the other hand, the idea of averaging over many different theories has also been explored. For instance, in two dimensions, Jackiw-Teitelboim gravity emerges when one performs some averaging procedure over a random ensemble of quantum mechanical systems [2]. Also, for three-dimensional gravity in AdS_3 spaces, its continuous energy spectrum indicates that this theory could be considered as an ensemble average [3], and it is recently discovered that the three-dimensional Chern-Simons gravity in AdS_3 is indeed dual to an ensemble of bosonic CFTs [4, 5]. In addition, recent results manifest that the $\mathrm{AdS}_5 \times S^5$ supergravity is an ensemble average of type IIB string theories [6]. In a more recent paper, it was shown that an ensemble average of two-dimensional large-c CFTs reproduces semiclassical gravity in three dimensions [7].

The above results naturally lead to a question: whether this conjecture can be extended to fermionic theories. A step towards answering this question is to determine the moduli space for fermionic CFTs. In particular, we are interested in fermionic CFTs with a four-fermion interaction. The reason why we focus on this particular family of CFTs is that theories in this family are related to bosonic CFTs through bosonization, which is a method that has been broadly applied in theoretical high energy physics and condensed matter physics.

In quantum field theory, only specific models have exact non-perturbative results for all correlators. Among these completely solvable models, Thirring discussed the first such model describing a current-current interaction of massless fermions and constructed the eigenstates of the Hamiltonian [8]. In 1967, Klaiber gave a complete quantum solution of the Thirring model and discussed its properties [9]. Later on, Coleman proved the equivalence between the Thirring model and the quantum sine-Gordon model [10].

In two-dimensions, this phenomenon where it is possible to write fermions in terms of bosons is called bosonization, and this technique has been a powerful tool for studying particle physics and condensed matter physics.

The idea that an ensemble average of different quantum theories gives rise to a gravitational theory is closely related to black hole physics. As we know, black holes scramble information, so the idea of quantum chaos, random matrices and random CFTs is naturally related to studies of black holes. In addition, it is not unusual in condensed matter physics that a strongly coupled system rearranges itself in some way that new weakly coupled degrees of freedom emerge and the system is now better described by fields representing the emergent excitation. For example, in spin glasses such technique is used to study systems with quenched disorder. Therefore, this project also provides insights to studies of disordered systems in condensed matter physics.

This thesis will focus on studying CFTs, introducing their applications in quantum gravity, and present how gravity emerges when an ensemble of bosonic CFTs is averaged over a moduli space. Then we will study how the bosonization technique leads to the fermionic extension of this gravity as an ensemble average conjecture and we will discuss how this extension is related to the Sachdev–Ye–Kitaev (SYK) model.

The structure of this thesis is the following. In Chapter 2, we will review basic knowledge of conformal field theories in two dimensions, introduce the conformal group and algebra, the operator formalism of conformal field theory including radial quantization and the operator product expansion, and construct the CFT Hilbert space. In Chapter 3, we will take a closer look at CFTs on the torus to study their modularity properties and compute the partition functions of free bosons and free fermions. In Chapter 4, we will first give a brief introduction to the AdS/CFT correspondence, then present the recent discovery of the equivalence between gravity and averaged bosonic CFTs. Chapter 5 will focus on studying the equivalence relation between bosons and fermions by introducing the bosonization technique, then we will study how the Thirring model is related to the free boson, then map out a relation between the moduli spaces of bosonic and fermionic CFTs. We will conclude this thesis in Chapter 6 and discuss future directions in which this project points.

Chapter 2

Basics in Two Dimensional Conformal Field Theory

The most basic approach to quantum field theory is to quantize Lagrangian action for fields using canonical quantization or the path integral method. This approach requires a known action of the theory, which, unfortunately, is not always available. Luckily, such a prerequisite is unnecessary for conformal field theories. Unlike ordinary quantum field theories, a CFT can be defined through operator algebras with their corresponding representation theory and can even be solved exactly in certain cases by employing the symmetries of the theory and exploiting their consequences. In particular, we are interested in two-dimensional CFTs due to a special property of this dimension. In two dimensions, the algebra of infinitesimal conformal transformations is infinite dimensional. In this chapter, we will give a relatively detailed review of the basic properties of conformal field theories in two dimensions, including the conformal group and its algebra, the primary fields, the energy-momentum tensor, and the operator product expansion. Then we will discuss the CFT Hilbert space for free bosons and free fermions.

2.1 The Conformal Group and Algebra

A Conformal Field Theory is a field theory invariant under conformal transformations. First, we will define mathematically what a conformal transformation is, which physically preserves the angle between two lines. Let M, M' be two smooth manifolds with metric $g_{\mu\nu}$ and $g'_{\mu\nu}$ respectively, and let $U \subset M$, $V \subset M'$ be open. Take $x \in U$ and let $x' = \phi(x) \in V$, then a differentiable map $\phi : U \to V$ is a *conformal transformation* if the metric tensor $g_{\mu\nu}(x)$ transforms as

$$g'_{\mu\nu}(x') = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} g_{\alpha\beta}(x) = \Lambda(x) g_{\mu\nu}(x). \tag{2.1}$$

In other words, under a conformal transformation, the metric is unchanged up to a scale factor $\Lambda(x)$.

In the interest of this thesis, we will only consider CFTs on flat spacetimes with a constant metric $\eta_{\mu\nu} = diag(-1, \dots, +1, \dots)$. Then we can easily see that in our case, the conformal transformations

$$\eta_{\mu\nu} \longrightarrow \eta'_{\mu\nu} = \Lambda(x)\eta_{\mu\nu} \tag{2.2}$$

form a group. In fact, the Poincaré transformation is a subgroup of this group with a scale factor $\Lambda(x) = 1$.

Now let us study the conditions for conformal invariance. Let $\epsilon(x) \ll 1$ be a small parameter, and consider infinitesimal coordinate transformations $x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x)$, then we have

$$\frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} = \left(\delta^{\alpha}_{\mu} + \frac{\partial \epsilon^{\alpha}}{\partial x^{\mu}} \right) \left(\delta^{\beta}_{\nu} + \frac{\partial \epsilon^{\beta}}{\partial x^{\nu}} \right) . \tag{2.3}$$

Bringing Eq.(2.3) into Eq.(2.2) and we get

$$\eta'_{\mu\nu} = \eta_{\mu\nu} + \left(\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} \right) . \tag{2.4}$$

Then, for the transformation to be conformal, we require that $(\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu}) = K(x)\eta_{\mu\nu}$ where $K(x)$ is some function so $\eta'_{\mu\nu}$ can be written as $\Lambda(x)\eta_{\mu\nu}$ with $\Lambda(x) = 1 + K(x)$. Contracting both sides with $\eta^{\mu\nu}$ we determine the function $K(x)$ to be $2(\partial \cdot \epsilon)/d$ where d is the dimension of the flat spacetime, which leads to $\Lambda(x) = 1 + 2(\partial \cdot \epsilon)/d$. Therefore, the condition for the transformation to be conformal is

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu} . \tag{2.5}$$

The *conformal group* consists of globally defined, invertible, and finite conformal transformations, and its corresponding Lie algebra is called the *conformal algebra*. To determine the conformal group for two dimensions, first we will focus on a flat spacetime with a Euclidean metric $\eta_{\mu\nu} = diag(+1,+1)$. In two dimensions $d = 2$, the condition shown in Eq.(2.5) for the infinitesimal conformal transformation is given by the following,

$$\begin{aligned} when \;\; \mu = \nu = 0: &\qquad \partial_0\epsilon_0 = +\partial_1\epsilon_1 \,, \\ when \;\; \mu = 0 \;\; and \;\; \nu = 1: &\qquad \partial_0\epsilon_1 = -\partial_1\epsilon_0 \,, \end{aligned} \tag{2.6}$$

which are the Cauchy-Riemann equations in complex analysis, that are satisfied by the real and imaginary parts of a holomorphic function. Now let ϵ be such a holomorphic function of z, we can define the following complex variables

$$\begin{aligned} z &= x^0 + ix^1 \,, & \overline{z} &= x^0 - ix^1 \,, \\ \epsilon &= \epsilon^0 + i\epsilon^1 \,, & \overline{\epsilon} &= \epsilon^0 - i\epsilon^1 \,, \\ \partial_z &= \frac{1}{2}(\partial_0 - i\partial_1) \,, & \partial_{\overline{z}} &= \frac{1}{2}(\partial_0 + i\partial_1) \,. \end{aligned} \tag{2.7}$$

Because the complex differentiation is linear, the sums of holomorphic functions are also holomorphic, which leads to the holomorphic property of $z'(z) = z + \epsilon(z)$. We can easily show that, as an infinitesimal conformal transformation, this function gives a scale factor of $\left|\frac{\partial z'}{\partial z}\right|^2$. In fact, a meromorphic function meets our requirements as long as it has isolated singularities only outside some required open set. Therefore, we can generalize $\epsilon(z)$ to such a meromorphic function.

Now let us move on to find the generators and determine the conformal algebra. To do so, we will Laurent expand a meromorphic function $\epsilon(z)$ around $z = 0$,

$$\epsilon(z) = \sum_{n\in\mathbb{Z}} \epsilon_n(-z^{n+1}) \,, \qquad \overline{\epsilon}(\overline{z}) = \sum_{n\in\mathbb{Z}} \overline{\epsilon}_n(-\overline{z}^{n+1}) \,, \tag{2.8}$$

where $\epsilon_n, \overline{\epsilon}_n$ are constant parameters. Recall that translation $x'^\mu = x^\mu + a^\mu$ has generator $-i\partial_\mu$. Analogously, the generators corresponding to a transformation for each n are

$$l_n = -z^{n+1}\partial_z \,, \qquad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \,. \tag{2.9}$$

We see that there are infinitely many generators since $n \in \mathbb{Z}$, which will lead to an infinite dimensional algebra as mentioned at the beginning of this chapter. We will study this algebra soon, but before that, let us discuss the well-definedness of these generators.

Customarily, we treat $z, \bar{z}$ as independent variables, so we are actually considering the complex plane $\mathbb{C}^2$. However, when we take a better look at the generators $\{l_n\}$ and $\{\bar{l}_n\}$, we notice that neither of them is globally defined on $\mathbb{C}$ or on its extension, the Riemann sphere $S^2 \simeq \mathbb{C} \cup \{\infty\}$, because both $z = \bar{z} = 0$ and $z = \bar{z} = \infty$ lead to ambiguities that need to be fixed. To find generators of globally defined conformal transformations on the Riemann sphere, let us consider the following. For $z = 0$, it is easily to see that z^{n+1} diverges for any $n+1 < 0$. Therefore, $\{l_n\}$ are well defined only for $n \geq -1$, and the same for $\{\bar{l}_n\}$. For $z = \infty$, we need to perform a change of variable first. Let $z = -1/w$, then $w \to 0$ when $z = \infty$, and

$$dz = \frac{1}{w^2}dw \quad \longrightarrow \quad \partial_z = w^2\partial_w \,. \tag{2.10}$$

Bringing Eq.(2.10) back to Eq.(2.9) we get that

$$l_n = -z^{n+1}\partial_z = -\left(-\frac{1}{w}\right)^{n+1} w^2\partial_w = -\left(-\frac{1}{w}\right)^{n-1}\partial_w \,. \tag{2.11}$$

Then as $w \to 0$, $(-1/w)^{n-1}$ diverges for all $n - 1 > 0$. Thus, $\{l_n\}$ and $\{\bar{l}_n\}$ are well defined only for $n \leq +1$ for $z = \infty$. Combining with the condition at $z = 0$, we conclude that the only generators for globally defined conformal transformations on the Riemann sphere are $\{l_{-1}, l_0, l_{+1}\}$ and $\{\bar{l}_{-1}, \bar{l}_0, \bar{l}_{+1}\}$. With only these generators, we can in fact obtain all four conformal transformations. Operators

$$l_{-1} = -\partial_z \,, \qquad \bar{l}_{-1} = -\partial_{\bar{z}} \tag{2.12}$$

clearly generate translations $z \mapsto z + b$. Operators

$$l_{+1} = -z^2\partial_z = -\partial_w \,, \qquad \bar{l}_{+1} = -\bar{z}^2\partial_{\bar{z}} = -\partial_{\bar{w}} \tag{2.13}$$

generate translations $w \mapsto w - c$ in $w = -1/z$ and $\overline{w} = -1/\overline{z}$, which corresponds to the special conformal transformations $z \mapsto \frac{z}{cz+1}$. For operators l_0 and $\bar{l}_0$, we perform a change of variable where $z = re^{i\phi}$ and $\overline{z} = re^{-i\phi}$, then we find

$$l_0 + \bar{l}_0 = -r\partial_r \, , \qquad\qquad i(l_0 - \bar{l}_0) = -\partial_\phi \, , \tag{2.14}$$

which generate dilations $z \mapsto az$ and rotations $z \mapsto ze^{i\theta}$, respectively.

As we see above, in two dimensions the conformal transformations map z to $\frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{C}$. We also require that $ad - bc \neq 0$ since these transformations are invertible. Therefore, we determine the conformal group on the Riemann sphere $S^2 \simeq \mathbb{C} \cup \{\infty\}$ to be the *Möbius group* $SL(2, \mathbb{C})$.

Now let us compute the commutators of the generators to determine the corresponding algebra. Bring expressions of the generators given in Eq.(2.9) into the commutators, we obtain the following commutation relationships,

$$\begin{aligned}
&[l_m, l_n] = (m-n) l_{m+n} \, , \\
&[\bar{l}_m, \bar{l}_n] = (m-n) \bar{l}_{m+n} \, , \\
&[l_m, \bar{l}_n] = 0 \, .
\end{aligned} \tag{2.15}$$

These are two copies of the so-called *Witt algebra*, which are infinite dimensional. It is this property of the two-dimensional conformal group that makes conformal field theories in two dimensions much richer than they are in higher dimensions.

As it turns out, a quantized system has a symmetry group that is a central extension of a classical symmetry group. In our case, the Witt algebra is classical and its central extension with central charge c is the *Virasoro algebra*, denoted as Vir_c. The Virasoro algebra is characterized by the following commutation relations,

$$\begin{aligned}
&[L_m, L_n] = (m-n) L_{m+n} + c\, p(m,n) \, , \\
&[L_m, \ c\,] = [\,c, \ c\,] = 0 \, ,
\end{aligned} \tag{2.16}$$

where $\{L_n\}$ with $n \in \mathbb{Z}$ are the elements of the central extension of the Witt algebra of $\{l_n\}$

and $p(m,n)$ is a bilinear map which we will soon determine. Similarly, we have another copy of Virasoro algebra of $\{\overline{L}_n\}$ with $n \in \mathbb{Z}$ and a central charge $\bar{c}$ associated with Witt algebra of $\{\bar{l}_n\}$. In addition, just as in the Witt algebra, the generators of the Virasoro algebra are also expressed in terms of independent complex variables $z, \bar{z}$.

Now we will determine $p(m,n)$ by observing Eq.(2.16), and notice that

• We require that $p(m,n) = -p(n,m)$ since the commutator is anti-symmetric.

• Redefine $\hat{L}_n := L_n + cp(n,0)/n$ for $n \neq 0$ and $\hat{L}_0 := L_0 + cp(1,-1)/2$. We see that $[\hat{L}_n, \hat{L}_0] = n\hat{L}_n$ and $[\hat{L}_1, \hat{L}_{-1}] = 2\hat{L}_0$, which tells us that by a redefinition, we can always have $p(1,-1) = 0$ and $p(n,0) = 0$.

• By computing the Jacobi identity $[[L_m, L_n], L_0] + [[L_n, L_0], L_m] + [[L_0, L_m], L_n] = 0$ using $p(n,0) = 0$ and $p(m,n) = -p(n,m)$ as shown in Eq.(A.1) we get that $(m+n)p(n,m) = 0$, which implies that $p(n,-n)$ with $|n| \geq 2$ are the only non-zero central extensions.

• Let $m = -n+1$ and replace 0 by -1 in the above Jacobi identity, and follow the computation in Eq. (A.2) we reach the result that $p(n,-n) = \frac{1}{12}(n^3 - n)$.

Note that since $p(m,n) = 0$ for $m, n = 0, \pm 1$, the generators $\{L_{-1}, L_0, L_1\}$ generate the $SL(2,\mathbb{C})/\mathbb{Z}_2$ group just like the generators $\{l_{-1}, l_0, l_1\}$ do as we have shown. Based on our observations and calculations, we conclude that, the Virasoro algebra Vir$_c$ with central charge c has the following commutation relations,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}\,. \tag{2.17}$$

We also note that what we have done in two-dimensional Euclidean space can be carried out in two-dimensional flat spacetime with Lorentzian signature as well, simply by performing a coordinate transformation where $u = -t + x$ and $v = t + x$. We will again see that its corresponding algebra is infinite dimensional.

2.2 Primary Fields and The Energy-Momentum Tensor

Before defining a primary field, let us first remind ourselves that the generators of the Virasoro algebra are expressed in terms of two complex variables $z, \bar{z} \in \mathbb{C}$. Therefore, instead of in a two-dimensional Euclidean space $\mathbb{R}^2$, we will study fields in a complex plane $\mathbb{C}^2$.

Now let $\phi(z,\overline{z})$ be a field, it is called a *primary field of conformal dimension* $(h,\overline{h})$ if it transforms in the following way under conformal transformations $z \mapsto f(z)$,

$$\phi'(z,\overline{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \overline{f}}{\partial \overline{z}}\right)^{\overline{h}} \phi(f(z),\overline{f}(\overline{z})) \,. \tag{2.18}$$

A field is called *quasi-primary field* if Eq.(2.18) holds only for global conformal transformations, i.e. $f \in SL(2,\mathbb{C})/\mathbb{Z}_2$. If a field does not transformation as primary or quasi-primary fields, it is called a *secondary field*. It is also useful to define two quantities, the *scaling dimension* Δ and the *conformal spin* J, where

$$\Delta := h + \overline{h}\,, \qquad\qquad J := h - \overline{h}\,, \tag{2.19}$$

and we will discuss their physical meanings in the next section.

Now let us consider the map $f(z) = z + \epsilon(z)$ with $\epsilon(z) \ll 1$, and see how a primary field transforms under infinitesimal conformal transformations. Since $\epsilon(z) \ll 1$, we can ignore terms in second or higher order of $\epsilon(z)$ in our computation. As computed in the appendix (see Appendix A.2), we find that under infinitesimal conformal transformations, a primary field $\phi(z,\overline{z}) \mapsto \phi'(z,\overline{z}) = \phi(z,\overline{z}) + \delta_{\epsilon,\overline{\epsilon}}\phi(z,\overline{z})$ where the variation is given by,

$$\delta_{\epsilon,\bar{\epsilon}}\phi(z,\bar{z}) = \Big(h\partial_z\epsilon + \epsilon\partial_z + \bar{h}\partial_{\bar{z}}\bar{\epsilon} + \bar{\epsilon}\partial_{\bar{z}}\Big)\phi(z,\bar{z}) \tag{2.20}$$

For a field theory, we need its energy-momentum tensor since this tensor encodes the behavior of the theory under infinitesimal transformations in the metric. The energy-momentum tensor is usually found by varying the action of the theory under infinitesimal transformations in the metric. However, as we mentioned before, an explicit form of the action is not required in two-dimensional CFTs since the infinite dimensional algebra in two dimensions puts strong constraints on the theory. Therefore, the energy-momentum tensor can be determined using only the Noether's theorem. Recall that the Noether's theorem states that each continuous symmetry in a field theory corresponds to a conserved current j_μ, i.e. $\partial^\mu j_\mu = 0$. For a CFT with conformal symmetry $x^\mu \mapsto x^\mu + \epsilon^\mu(x)$, this conserved current can be written as

$$j_\mu = T_{\mu\nu}\epsilon^\nu \; , \tag{2.21}$$

where the symmetric tensor $T_{\mu\nu}$ is the energy-momentum tensor. Combining Eq.(2.18) with conservation law $\partial^\mu j_\mu = 0$ as computed in Appendix A.3, we get that in a CFT, the energy-momentum tensor is traceless, i.e.

$$T_\mu{}^\mu = 0 \; . \tag{2.22}$$

Now let us study what a traceless energy-momentum tensor tells us. To find expressions of $T_{\mu\nu}$ in $z, \overline{z}$ coordinates, we will change the coordinates from (x^0, x^1) to $(z, \overline{z})$ by inverting our definitions for $z, \overline{z}$ in Eq.(2.7) and use $T_{\mu\nu} = \partial_\mu x^\alpha \partial_\nu x^\beta T_{\alpha\beta}$. We find that

$$\begin{pmatrix} T_{zz} & T_{z\overline{z}} \\ T_{\overline{z}z} & T_{\overline{z}\overline{z}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(T_{00} - iT_{10}) & 0 \\ 0 & \frac{1}{2}(T_{00} + iT_{10}) \end{pmatrix} . \tag{2.23}$$

Then impose translational invariance $\partial^\mu T_{\mu\nu} = 0$, which is given by Eq.(A.4) and use the fact that $T_\mu{}^\mu = 0$ to compute $\partial_{\overline{z}} T_{zz}$ and $\partial_{\overline{z}} T_{\overline{z}\overline{z}}$, we get the following results as calculations shown in Appendix A.4,

$$\partial_{\overline{z}} T_{zz} = 0 \; , \qquad \partial_z T_{\overline{z}\overline{z}} = 0 \; . \tag{2.24}$$

Therefore, we conclude that the non-vanishing components of the energy-momentum tensor are a chiral field $T_{zz}(z, \overline{z}) = T(z)$ and an anti-chiral field $T_{\overline{z}\overline{z}}(z, \overline{z}) = \overline{T}(\overline{z})$.

2.3 The Operator Formalism for 2D CFTs

2.3.1 The Radial Quantization

To study the operator formalism for two-dimensional CFTs, we will first discuss quantization in these theories. Our setup for the CFTs will be on the two-dimensional Euclidean space, with time direction x^0 and space direction x^1, then we compactify the

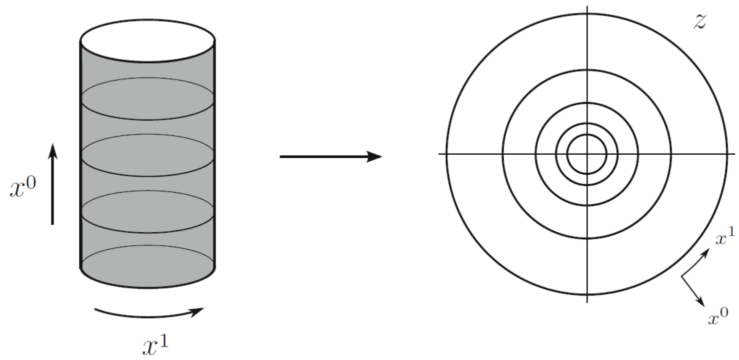

Figure 2.1: Mapping a cylinder of infinite length to the complex plane.

space direction x^1 on a unit circle to obtain a cylinder of infinite length. For the cylinder, we define the following complex coordinate $w = x^0 + ix^1$ with $w = w + 2\pi i$ and then let $z = e^w$. We see that, as shown in Figure 2.1 [11], it maps an infinitely long cylinder labelled by x^0, x^1 to the complex plane labelled by z. This mapping relates the time translations $x^0 \mapsto x^0 + a$ to complex dilation $z \mapsto e^a z$ and relates the space translations $x^1 \mapsto x^1 + b$ to rotations $z \mapsto ze^{ib}$.

Recall that in the previous section, by Eq.(2.14) and the equality $L_0 = l_0$ due to a vanishing central extension at $n = 0$, we have that $L_0 + \overline{L}_0 = -r\partial_r$ and $i(L_0 - \overline{L}_0) = -\partial_\phi$. This mapping is useful in the sense that it allows us to reach the following conclusion,

$$\begin{aligned} H &= L_0 + \overline{L}_0 \ , \\ P &= i(L_0 - \overline{L}_0) \ , \end{aligned} \tag{2.25}$$

because the Hamiltonian generates time translations, which are mapped to complex dilation, and the momentum generates the space translations, which are mapped to rotations.

To quantize primary fields $\phi(z, \overline{z})$ with conformal dimensions $(h, \overline{h})$, we perform a Laurent expansion around $z = \overline{z} = 0$,

$$\phi(z, \overline{z}) = \sum_{n, \overline{n} \in \mathbb{Z}} z^{-n-h} \overline{z}^{-\overline{n}-\overline{h}} \phi_{n,\overline{n}} \ , \tag{2.26}$$

where the Laurent modes $\phi_{n,\overline{n}}$ are promoted to operators as we do to Lorentz modes in ordinary field theories when we quantize a field, and these Laurent modes are given by

$$\phi_{n,\overline{n}} = \frac{1}{(2\pi i)^2} \oint dz d\overline{z}\; z^{n+h-1} \overline{z}^{\overline{n}+\overline{h}-1} \phi(z,\overline{z}) \,. \tag{2.27}$$

This quantization is the so-called *radial quantization* which was introduced by Fubini, Hanson, and Jackiw [12].

2.3.2 The Operator Product Expansion

Now let us study the operator product expansion and develop the operator formalism for two-dimensional CFTs. To do so, let us first revisit the energy-momentum tensor. Since the preserved conformal symmetry is associates to the current given in Eq.(2.21) the conserved charge can be expressed as $Q = \int dx^1 j_0$ at constant x^0 just as we have seen in ordinary QFTs. Now by the mapping shown in Figure 2.1, a constant x^0 is mapped to constant $|z|$, therefore, the integral over x^1 now becomes a contour integral over z and $\overline{z}$. Thus, we can write down a generalized expression for the conserved charge as the following,

$$Q = \frac{1}{2\pi i} \oint_{\mathcal{C}} dz T(z) \epsilon(z) + \frac{1}{2\pi i} \oint_{\mathcal{C}} d\overline{z} \overline{T}(\overline{z}) \overline{\epsilon}(\overline{z}) \,. \tag{2.28}$$

For an operator A, we have the relation where $\delta A = [Q, A]$ since the conserved charge generates symmetry transformations for operator A. We can determine the infinitesimal transformation of a primary field $\phi(z,\overline{z})$ generated by the conserved charge Q, which is given by

$$\delta_{\epsilon,\overline{\epsilon}} \phi(w,\overline{w}) = \frac{1}{2\pi i} \oint_{\mathcal{C}} dz [T(z)\epsilon(z), \phi(w,\overline{w})] + \frac{1}{2\pi i} \oint_{\mathcal{C}} d\overline{z} [\overline{T}(\overline{z})\overline{\epsilon}(\overline{z}), \phi(w,\overline{w})] \,. \tag{2.29}$$

where $w, \overline{w} \in \mathbb{C}$. Note that we need to consider the two situations where $w, \overline{w}$ are inside and where $w, \overline{w}$ are outside the contour $\mathcal{C}$. Taking these two situations into consideration, Eq.(2.29) becomes

$$\begin{aligned} \delta_{\epsilon,\overline{\epsilon}} \phi(w,\overline{w}) = &\frac{1}{2\pi i} \oint_{|z|>|w|} dz T(z)\epsilon(z)\phi(w,\overline{w}) - \frac{1}{2\pi i} \oint_{|z|<|w|} dz \phi(w,\overline{w}) T(z)\epsilon(z) \\ &+ \frac{1}{2\pi i} \oint_{|z|>|w|} d\overline{z} \overline{T}(\overline{z})\overline{\epsilon}(\overline{z})\phi(w,\overline{w}) - \frac{1}{2\pi i} \oint_{|z|<|w|} d\overline{z} \overline{T}(\overline{z})\overline{\epsilon}(\overline{z})\phi(w,\overline{w}) \,. \end{aligned} \tag{2.30}$$

In this case, it is useful to define the *radial ordering* of two operators as the following

$$R(A(z)B(w)) := \begin{cases} A(z)B(w) & \text{for } |z| > |w| \,, \\ B(w)A(z) & \text{for } |z| < |w| \,. \end{cases} \tag{2.31}$$

Note that for fermionic fields, there is a minus sign for the $|z| < |w|$ case due to the fermionic nature of the fields, that is to say that because the fields are Grassmann variables and follow anti-commutation relations, $R(A(z)B(w)) = -B(w)A(z)$ for $|z| < |w|$.

Then with this definition we can rewrite Eq.(2.30) as

$$\delta_{\epsilon,\bar{\epsilon}}\phi(w,\overline{w}) = \frac{1}{2\pi i}\oint_{\mathcal{C}(w)} dz\epsilon(z)R\Big(T(z)\phi(w,\overline{w})\Big) + \frac{1}{2\pi i}\oint_{\mathcal{C}(\overline{w})} d\overline{z}\overline{\epsilon}(\overline{z})R\Big(\overline{T}(\overline{z})\phi(w,\overline{w})\Big) \,. \tag{2.32}$$

where $\mathcal{C}(w)$ and $\mathcal{C}(\overline{w})$ are contours centered at w and $\overline{w}$, respectively. Recall that, in the previous section, we have found that the transformation of a primary field under infinitesimal conformal transformations takes the form of Eq.(2.20). Remind ourselves the Cauchy formula in complex analysis, which is given by

$$f^{(n-1)}(a) = \frac{(n-1)!}{2\pi i}\oint_{\mathcal{C}} dz \frac{f(a)}{(z-a)^n} \,, \tag{2.33}$$

where $f(z)$ is an infinitely differentiable function defined on the complex plane and $f^{(n)}(z)$ is its n-th derivative. Then we obtain the following identities,

$$\begin{aligned} \partial_w \epsilon(w) &= \frac{1}{2\pi i}\oint_{\mathcal{C}(w)} dz \frac{\epsilon(z)}{(z-w)^2} \,, \\ \epsilon(w) &= \frac{1}{2\pi i}\oint_{\mathcal{C}(w)} dz \frac{\epsilon(z)}{z-w} \,. \end{aligned} \tag{2.34}$$

Now we can bring these expressions back to Eq.(2.20) and compare them with Eq.(2.20), then reach the conclusion that

$$\begin{aligned} R\Big(T(z)\phi(w,\overline{w})\Big) &= \frac{h}{(z-w)^2}\phi(w,\overline{w}) + \frac{1}{z-w}\partial_w\phi(w,\overline{w}) + \text{non-singular terms} \,, \\ R\Big(\overline{T}(\overline{z})\phi(w,\overline{w})\Big) &= \frac{\overline{h}}{(\overline{z}-\overline{w})^2}\phi(w,\overline{w}) + \frac{1}{\overline{z}-\overline{w}}\partial_{\overline{w}}\phi(w,\overline{w}) + \text{non-singular terms} \,, \end{aligned} \tag{2.35}$$

The above expansion defines an algebraic product structure on the space of quantum fields, and it is called the *operator product expansion* (OPE).

Using this expression, we now give an alternative definition to primary fields, which is that *a field $\phi(z,\overline{z})$ is a primary field with conformal dimension $(h,\overline{h})$ if the OPE between the energy-momentum tensor and $\phi(z,\overline{z})$ takes the form given by Eq.(2.35)*. That is to say, a field $\phi(z,\overline{z})$ transforming under conformal transformations as Eq.(2.18) has an OPE of energy-momentum tensor with $\phi(z,\overline{z})$ which takes the form of Eq.(2.35).

With this alternative definition of a primary field, it is natural to ask if the energy-momentum tensor itself is a primary field. To answer this question, let us study the OPE of the energy-momentum tensor with itself. We claim that the OPE takes the following form,

$$R\Big(T(z)T(w)\Big) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \dots \,, \tag{2.36}$$

where c is the central charge of the theory and $|z| > |w|$. To verify this statement, we first Laurent expand $T(z)$ in the following way,

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \quad \text{where} \quad L_n = \frac{1}{2\pi i}\oint dz z^{n+1} T(z) \,. \tag{2.37}$$

Note that when we choose a conformal transformation $\epsilon(z) = -\epsilon_n z^{n+1}$ as a term in Eq.(2.8) and bring the Laurent expansion of $T(z)$ into the expression for chiral part of the conserved charge Eq.(2.28), we have the following result,

$$Q_n = \frac{1}{2\pi i}\oint dz T(z)(-\epsilon_n z^{n+1}) = -\epsilon_n \sum_{m\in\mathbb{Z}} \delta_{mn} L_m = -\epsilon_n L_n \,. \tag{2.38}$$

By identifying the Laurent modes of the energy-momentum tensor and the generators of infinitesimal conformal transformation, we can compute the commutation relation $[L_m, L_n]$ as shown in Eq.(A.7) and conclude that Eq.(2.36) indeed reproduces the Virasoro algebra Eq.(2.17) as we found before. Therefore, the statement of the OPE is verified.

We notice that in our claim, the OPE for the energy-momentum tensor with itself does not take the form of the OPE for the energy-momentum tensor with a primary field, which means that for non-vanishing central charges, the energy-momentum tensor is NOT a primary field. In fact, it transforms under conformal transformations $f(z)$ as,

$$\begin{aligned} T'(z) &= \left(\frac{\partial f}{\partial z}\right)^2 T\big(f(z)\big) + \frac{c}{12} S\big(f(z), z\big) \ , \\ S(f(z), z) &= \frac{1}{(\partial_z f)^2}\left((\partial_z f)(\partial_z^3 f) - \frac{3}{2}(\partial_z^2 f)^2\right) \ , \end{aligned} \tag{2.39}$$

where $S(w, z)$ is called the Schwarzian derivative. Let us consider infinitesimal transformations $f(z) = z + \epsilon(z)$. Then apply Eq.(2.32), the transformations in the energy-momentum tensor is given by

$$\delta_\epsilon T(z) = \frac{c}{12}\partial_z^3 \epsilon(z) + 2T(z)\partial_z \epsilon(z) + \epsilon(z)\partial_z T(z) \ . \tag{2.40}$$

Now we can compare this expression with Eq.(2.39), we get that for infinitesimal transformations $f(z) = z + \epsilon(z)$, we have

$$S(z + \epsilon(z), z) = \frac{1}{(1 + \partial_z \epsilon)^2}\left((1 + \partial_z \epsilon)(\partial_z^3 \epsilon) - \frac{3}{2}(\partial_z^2 \epsilon)^2\right) \approx \partial_z^3 \epsilon \ . \tag{2.41}$$

For a chiral primary field $\phi(z)$, we Laurent expand it similarly as in Eq.(2.26) and Eq.(2.27),

$$\phi(z) = \sum_{n \in \mathbb{Z}} z^{-n-h}\phi_n \quad \text{where} \quad \phi_n = \oint \frac{dz}{2\pi i} z^{n+h-1}\phi(z) \ . \tag{2.42}$$

Then following the computation in Eq.(A.8), we find that the holomorphic part of the OPE for a chiral field $\phi(z)$ is given by

$$[L_m, \phi_n] = [(h-1)m - n]\phi_{m+n} \tag{2.43}$$

for all $m, n \in \mathbb{Z}$. Note that for a quasi-primary field, the above holds only when m takes the value of $-1, 0, +1$.

2.3.3 The OPE of Quasi-Primary Fields

We would also like to introduce the operator algebra of quasi-primary fields and their OPE. Before we start, we will note that from now on we will write $R(A(z)B(w))$

as $A(z)B(w)$ to simplify the notation since we will always assume radial ordering for a product of fields. To start, let us consider the two-point function $\langle\phi_i(z)\phi_j(w)\rangle = g(z,w)$ as a function of z and w. By requiring invariance under translations generated by L_{-1}, we see that the function takes the form of $g(z-w)$. The invariance under dilations $z \mapsto \lambda z$ generated by L_0 leads to the relation $\lambda^{h_i+h_j} g(\lambda(z-w)) = g(z-w)$, and we find that the funtion $g(z-w) \sim (z-w)^{-(h_i+h_j)}$. The function is also invariant under transformations $z \mapsto -1/z$ generated by L_1, and it leads to the conclusion $h_i = h_j$ as one can check by setting $z^{-2h_i} w^{-2h_j} g(-1/z + 1/w)$ equal to $g(z-w)$. Therefore, we conclude that the two-point function of two quasi-primary fields is fixed by the $SL(2,\mathbb{C})/\mathbb{Z}_2$ conformal symmetry by

$$\langle\phi_i(z)\phi_j(w)\rangle = \frac{d_{ij}\delta_{h_i,h_j}}{(z-w)^{2h_i}} , \tag{2.44}$$

where d_{ij} is a structure constant. Following the same arguments as above, the three-point function of chiral quasi-primary fields is fixed by the $SL(2,\mathbb{Z})/\mathbb{Z}_2$ conformal symmertry and it is given by

$$\langle\phi_i(z_i)\phi_j(z_j)\phi_k(z_k)\rangle = \frac{C_{ijk}}{z_{ij}^{h_i+h_j-h_k} z_{jk}^{h_j+h_k-h_i} z_{ik}^{h_i+h_k-h_j}} , \tag{2.45}$$

where $z_{ij} \equiv z_i - z_j$ and C_{ijk} is again a structure constant.

Based on what we just found for the two-point and three-point functions, we can deduce a general form of the OPE of two-quasi primary fields. Let $\phi_i(z)$ be quasi-primary fields with conformal dimension h_i, where the subscript denotes different fields, we state that the OPE of two-quasi primary fields involves only other quasi-primary fields and their derivatives [13]. We will present the result here, the general form is given by

$$\begin{aligned} \phi_i(z)\phi_j(w) &= \sum_{k,n\geq 0} \frac{a^n_{ijk}}{n!} \frac{C^k_{ij}}{(z-w)^{h_i+h_j-h_k-n}} \partial^n \phi_k(w) \\ \text{with coefficients} \quad a^n_{ijk} &= \binom{2h_k+n-1}{n}^{-1} \binom{h_i-h_j+h_k+n-1}{n} \\ \text{and} \quad C_{ijk} &= C^l_{jk} d_{lk} , \end{aligned} \tag{2.46}$$

where the coeffiecients are determined using results for the two-point and three-point functions. We will also present the very useful algebra of the Laurent modes $\phi_{i,m}$ of quasi-primary fields $\phi_i(z) = \sum_m z^{-m-h_i}\phi_{i,m}$ with conformal dimension h_i,

$$
\begin{aligned}
[\phi^{(i)}_m, \phi^{(j)}_n] &= \sum_k C^k_{ij} p_{ijk}(m,n)\phi_{k,m+n} + d_{ij}\delta_{m+n,0}\binom{m+h_i-1}{2h_i-1} , \\
\text{with} \quad p_{ijk}(m,n) &= \sum_{\substack{r,s=0 \\ r+s=h_i+h_j-h_k-1}}^{\infty} C^{ijk}_{r,s}\binom{-m+h_i-1}{r}\binom{-n+h_j-1}{s} , \\
\text{where} \quad C^{ijk}_{r,s} &= (-1)^r \frac{(2h_k-1)!}{(h_i+h_j+h_k-2)!}\prod_{t=0}^{s-1}(2h_i-2-r-t)\prod_{u=0}^{r-1}(2h_j-2-s-u) .
\end{aligned}
\tag{2.47}
$$

Before moving on to the next section, we will also present an application of the OPE of two-quasi primary fields by introducing a definition of the so-called *current algebra*. This will be an essential ingredient for studying bosonic and fermionic CFTs as we will see in the next chapter. In a two-dimensional CFT, a *current* is defined as a chiral field $j(z)$ with conformal dimension $h = 1$ (or an anti-chiral field $\overline{j}(\overline{z})$ with $\overline{h} = 1$). Assume we have a CFT with N quasi-primary currents $j_i(z) = \sum_{n\in\mathbb{Z}} z^{-n-1} j_{i,m}$ where $i \in \{1,\ldots,N\}$, the algebra of the Laurent modes is given by

$$
[j^{(i)}_m, j^{(j)}_n] = \sum_k C^k_{ij} p_{111}(m,n) j_{k,m+n} + d_{ij} m \delta_{m+n,0} , \tag{2.48}
$$

where the polynomial $p_{111}(m,n) = 1$ as one can compute using the expression above and $C^k_{ij} = -C^k_{ji}$ since the commutator is anti-symmetric. We will employ this algebra when we discuss bosonic and fermionic CFTs.

2.4 The Hilbert Space of CFT

In this section, our goal is to explore the CFT Hilbert space in two-dimensions. To do so, we will first discuss the ordering prescription for two-dimensional CFTs. Similar as in ordinary quantum field theories, for operators we need an ordering prescription for products of fields at the same point in spacetime. For this purpose, we define the *normal ordering* which brings creation operators to the left. In this section, we will show how the normal ordering for the product fields arises from the regular part of an OPE. To do so, let us first determine the annihilation operators in a CFT, then find the creation operators in the theory.

Recall that on the cylinder of infinite length as shown in Figure 2.1, the infinite past labelled by $x^0 = -\infty$ is mapped to $z = \overline{z} = 0$. Then it is natural to define an *asymptotic in-state* of the form $|\phi\rangle = \lim_{z,\overline{z}\to 0} \phi(z,\overline{z})\,|0\rangle$. Recall that we have the Laurent expansion for a field $\phi(z,\overline{z})$ as given by Eq.(2.26). For this expression to be non-singular at $z = 0$, we require that $\phi_{n,\overline{n}}\,|0\rangle = 0$ for $n > -h$ and $\overline{n} > -\overline{h}$, which means that these operators can be interpreted as annihilation operators. Therefore, an asymptotic in-state can be simply written as

$$|\phi\rangle = \lim_{z,\overline{z}\to 0} \phi(z,\overline{z})\,|0\rangle = \phi_{-h,-\overline{h}}\,|0\rangle \ . \tag{2.49}$$

With similar reasoning, an *asymptotic out-state* can be written as

$$\langle\phi| = \lim_{w,\overline{w}\to\infty} w^{2h}\overline{w}^{2\overline{h}}\,\langle 0|\,\phi^{\dagger}(w,\overline{w}) = \langle 0|\,\phi_{+h,+\overline{h}}\ , \tag{2.50}$$

for a detailed derivation, see Appendix A.6. Recall that the Hamiltonian can be written as $H = L_0 + \overline{L}_0$, then let us calculate the eigenvalue of L_0, which is called the "chiral energy", for a chiral primary,

$$L_0\phi_n\,|0\rangle = (L_0\phi_n - \phi_n L_0)\,|0\rangle = [L_0, \phi_n]\,|0\rangle = -n\phi_n\,|0\rangle \ , \tag{2.51}$$

where we used $L_0\,|0\rangle = 0$ and Eq.(2.43). Since the annihilation operators $\phi_{n,\overline{n}}\,|0\rangle = 0$ for $n > -h$ and $\overline{n} > -\overline{h}$, the only values that the chiral energy can take are $n \leq -h$. Therefore, we conclude that the creation operators are operators ϕ_n with $n \leq -h$. Similarly, the anti-chiral energy is given by $\overline{L}_0\phi_{\overline{n}}\,|0\rangle = -\overline{n}\phi_{\overline{n}}\,|0\rangle$ with $\overline{n} \leq -\overline{h}$, and the creation operators are $\phi_{\overline{n}}$ with $\overline{n} \leq -\overline{h}$.

Now that we determined the creation operators in a CFT, we can define the normal ordering to be the prescription where we put all creation operators to the left, and we denote the normal ordering as $N(\chi\phi)$ or $:\chi\phi:$. In fact, the regular part of an OPE gives rise to normal ordered products naturally, which can be written as

$$\phi(z)\chi(w) = \text{singular part} + \sum_{n=0}^{\infty} \frac{(z-w)^n}{n!} N(\chi\partial^n\phi)(w) \ . \tag{2.52}$$

For $n = 0$, we have the normal ordered product for two operators. To find how the normal ordered product can be expressed in terms of the Laurent modes of fields χ and ϕ, we will do the following. First, we pick out this term by performing $\oint dz[2\pi i(z-w)]^{-1}$ on both sides of Eq. (2.52), then the contour integral vanishes for all terms with $n \neq 0$, which gives us the equality

$$\oint_{\mathcal{C}(w)} \frac{dz}{2\pi i} \frac{\phi(z)\chi(w)}{z-w} = N(\chi\phi)(w) \ . \tag{2.53}$$

Now we Laurent expand $N(\chi\phi)(w)$ as usual and get

$$\begin{aligned} N(\chi\phi)(w) &= \sum_{n\in\mathbb{Z}} w^{-n-h^\chi-h^\phi} N(\chi\phi)_n \ , \\ \text{where} \ \ N(\chi\phi)_n &= \oint_{\mathcal{C}(w)} \frac{dw}{2\pi i} w^{n+h^\chi+h^\phi-1} N(\chi\phi)(w) \ . \end{aligned} \tag{2.54}$$

Then, by bringing Eq.(2.53) back into the expression for Laurent modes $N(\chi\phi)_n$ we obtain the result

$$N(\chi\phi)_n = \sum_{k>-h^\phi} \chi_{n-k}\phi_k + \sum_{k\le -h^\phi} \phi_k\chi_{n-k} \ . \tag{2.55}$$

A detailed computation can be found in Appendix A.7. As we can see in the above expression, the annihilation operators ϕ_k with $k > -h^\phi$ are on the right in the first term, and the creation operators ϕ_k with $k \le -h^\phi$ are on the left in the second term. Thus, this expression indeed satisfies the definition for a normal ordered product.

After finding the normal ordered product of two fields, we will also introduce two useful formulas obtained using the Laurent expansion for $\partial\phi(z)$ as

$$\partial\phi(z) = \sum_{n\in\mathbb{Z}} (-n-h) z^{-n-h-1}\phi_n \ , \tag{2.56}$$

and performing similar calculations as in Appendix A.7. In addition, we can also find terms in Eq.(2.52) with higher n following similar procedure. The formulas are

$$\begin{aligned} N(\chi\partial\phi)_n &= \sum_{k>-h^\phi-1}(-h^\phi-k)\chi_{n-k}\phi_k + \sum_{k\leq -h^\phi-1}(-h^\phi-k)\phi_k\chi_{n-k} \ , \\ N(\partial\chi\phi)_n &= \sum_{k>-h^\phi}(-h^\phi-n+k)\chi_{n-k}\phi_k + \sum_{k\leq -h^\phi}(-h^\phi-n+k)\phi_k\chi_{n-k} \ , \end{aligned} \tag{2.57}$$

The normal order product can be generalized to quasi-primary fields, just like what we will do for the OPE. We will not present it in the thesis but a reference for more details can be found in [11] or [14].

Using the tools we introduced above, we now can study the CFT Hilbert space in two-dimensions. Recall that in Eq.(2.37) we have the Laurent expansion for the energy-momentum tensor, then it implies that its asymptotic in-state is given by $L_{-2}|0\rangle$ and the asymptotic in-state of $\partial T(z)$ is given by $L_{-3}|0\rangle$ by Eq.(2.49). Then we can write the normal ordered product of the energy-momentum tensor with itself as

$$\begin{aligned} N(TT)(z) &= \sum_{n\in\mathbb{Z}} z^{-n-4}N(TT)_n \ , \\ \text{where} \ \ N(TT)_n &= \sum_{k>-2} L_{n-k}L_k + \sum_{k\leq -2} L_k L_{n-k} \end{aligned} \tag{2.58}$$

as shown in Eq.(2.55). We see that in the $z \to 0$ limit, the only well-defined term has the mode with $n = -4$. In addition, the first term vanishes when we act this operator on the state $|0\rangle$ and in the second term, all $n - k > -2$ components vanish as well. Therefore, the only term that contributes to the Laurent expansion of $N(TT)$ is the Laurent mode $N(TT)_{-4} = L_{-2}L_{-2}$, and the state can be written as $L_{-2}L_{-2}|0\rangle$. Similarly, we find that the normal ordered product $N(T\partial T)$ is given by $L_{-3}L_{-2}|0\rangle$ using Eq.(2.57).

As we see in the above examples, the states are expressed in terms of creation operators — Laurent modes L_n of the energy-momentum tensor — acting on the vacuum $|0\rangle$. We define the space of all such states as a *Verma module* $\{L_{k_1}\ldots L_{k_n}|0\rangle : k_i \leq -2\}$. For a state $|\Phi\rangle$ in the Verma module, we can find a field $F \in \{T, \partial T, \ldots, N(TT), N(T\partial T), \ldots\}$ such that $\lim_{z\to 0} F|0\rangle = |\Phi\rangle$.

To define a conformal family of a primary field, we consider primary field $\phi(z)$ with conformal dimension h, and denote the state emerges from this field as $|h\rangle = \phi_{-h}|0\rangle$. Then by Eq.(2.43) and the fact that $\phi_n|0\rangle = 0$ for all $n \neq -h$ we get $L_n|h\rangle \sim \phi_{-h+n}|0\rangle = 0$ for $n > 0$. For $n = 0$, we get the chiral energy of this state, i.e. $L_0|h\rangle = h|h\rangle$.

What is more interesting are the states $L_n|h\rangle$ with $n < 0$. In fact, for each primary field $\phi(z)$, there is a corresponding infinite set of fields by taking derivatives and taking the normal ordered products with the energy-momentum tensor. We denote the *conformal family* of a primary $\phi(z)$ as

$$\begin{aligned} \Big[\phi(z)\Big] &:= \Big\{\phi, \partial\phi, \partial^2\phi, \ldots, N(T\phi), N(T\partial\phi), N(\partial T\phi), \ldots\Big\}\,, \\ \text{or}\ \Big[\phi(z)\Big] &:= \Big\{L_{k_1}\ldots L_{k_n}\phi(z) : k_i \leq -1\Big\}\,, \end{aligned} \tag{2.59}$$

where fields in the family are called *descendant fields*.

In this chapter, we introduced basic knowledge needed to study conformal field theories in two dimensions. In particular, we presented two-dimensional CFTs using the operator algebras instead of the usual Lagrangian formulation used for ordinary quantum field theories. Later on, we will make connections with the usual method and study the bosonic and fermionic conformal field theories.

Chapter 3

Conformal Field Theory on the Torus

In the previous chapter, we studied conformal field theories on the Riemann sphere $S^2 \simeq \mathbb{C} \cup \{\infty\}$, on which the chiral and anti-chiral sectors decouple and thus can be treated independently. Such theories on the Riemann sphere correspond to the tree-level contribution in the perturbation expansion of string theory, because in string theory, a tree-level amplitude diagram for string interactions is topologically a sphere augmented with states at infinity where the legs are located on the string worldsheet, as shown in Figure 3.1. The first row is the perturbative expansion of string theory for four closed string interactions as a sum of tree level, one-loop level, and higher loop level diagrams, and the second row is a topological equivalence of the first row, where crosses represent the legs, i.e. the states at infinity. In addition, loop-level contributions in the perturbative expansion of string theory are described by CFTs on higher genus Riemannian surfaces.

In this chapter, we will consider the one-loop level and discuss conformal field theories defined on a torus T^2. We will first introduce properties of the torus, in particular, the modular group. Then, we will determine the partition function of a two-dimensional CFT using these properties. Also, we will study the bosonic and the fermionic CFTs on the torus in more detail.

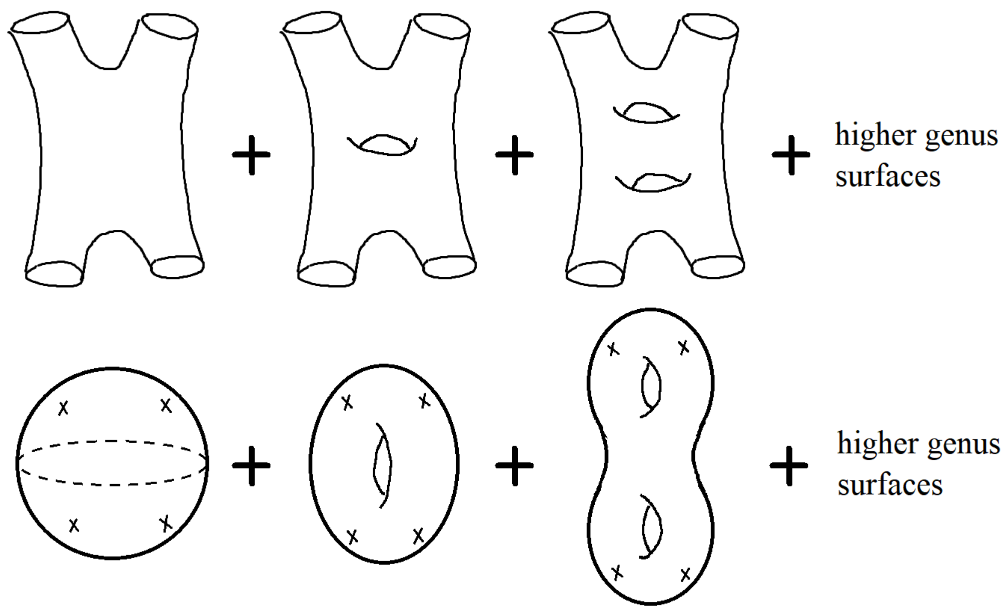

Figure 3.1: The perturbation expansion of string theory for four closed string interactions and its topological equivalence.

3.1 The Modular Group of the Torus

Recall that in the previous chapter, we discussed the mapping $z = e^w = e^{x^0 + ix^1}$, which maps the complex plane to a cylinder of infinite length. To form a torus, we can cut out a finite piece of the infinitely long cylinder and identify the boundaries of this finite piece. Another similar way to form a torus skips the intermediate cylinder mapping, instead we define a torus by identifying points $w = x^0 + ix^1$ on the complex plane $\mathbb{C}$ as $w \sim w + m\omega_1 + n\omega_2$ with a complex pair (ω_1, ω_2) and $m, n \in \mathbb{Z}$, the corresponding lattice is shown in Figure 3.2 [15]. The shaded region is the *fundamental domain* of the torus generated by (ω_1, ω_2), and the torus is formed by identifying opposite edges. We define the *complex structure* or the *modular parameter* as

$$\tau = \frac{\omega_2}{\omega_1} = \tau_1 + i\tau_2 \, , \tag{3.1}$$

and this quantity describes the shape of the torus.

It is easy to see that there are different choices of (ω_1, ω_2) giving the same lattice and the same torus, and to determine how these choices are related, let us consider the following.

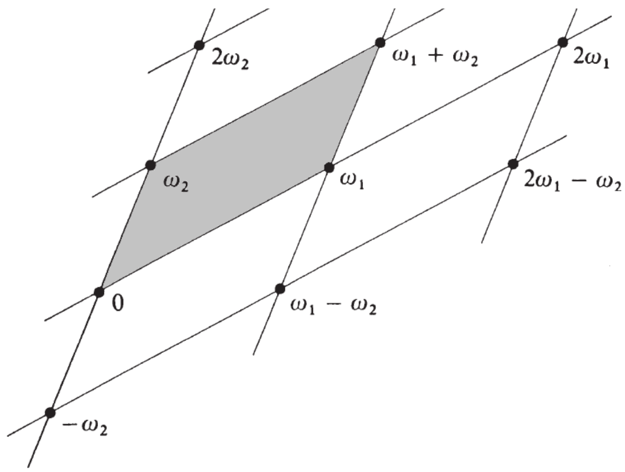

Figure 3.2: The vertices represent the periods $m\omega_1 + n\omega_2$, and the *fundamental domain* of the torus generated by (ω_1, ω_2) is the shaded region.

Assume that (ω_1, ω_2) and (ω_1', ω_2') describe the same lattice, then by Theorem 1.2 in [15], there exists a 2×2 matrix with integer entries $a, b, c, d \in \mathbb{Z}$ and determinant $ad - bc = \pm 1$ such that $\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$, where the matrices are elements of $SL(2, \mathbb{Z})$. Also, it is obvious that (ω_1, ω_2) and $(-\omega_1, -\omega_2)$ describe the same lattice as shown in Figure 3.2 and thus we can divide out the group $\mathbb{Z}_2$. Therefore, the equivalent pairs (ω_1, ω_2) and (ω_1', ω_2') are related by the $SL(2, \mathbb{Z})/\mathbb{Z}_2$ transformations and we conclude that the modular group of the torus acts on the modular parameter τ as

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} \qquad \text{with} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})/\mathbb{Z}_2 \,. \tag{3.2}$$

The original fundamental domain is shown in Figure 3.2, but now we label the lattice with modular parameter τ instead of the complex pair (ω_1, ω_2). Now, let us consider the following modular transformations.

• The modular T-transformation $T : \tau \mapsto \tau + 1$, as shown in Figure 3.3 a) [11].

• The modular U-transformation $U : \tau \mapsto \frac{\tau}{\tau+1}$, as shown in Figure 3.3 b) [11].

• The modular S-transformation $S : \tau \mapsto -\frac{1}{\tau}$. It is related to the above two transformations by $S = UT^{-1}U$ and $(ST^3) = \mathbb{I}$, and with itself it satisfies $S^2 = \mathbb{I}$.

In fact, it is sufficient to consider only the T and S transformations since they are the generators of the modular group $SL(2,\mathbb{Z})/\mathbb{Z}_2$.

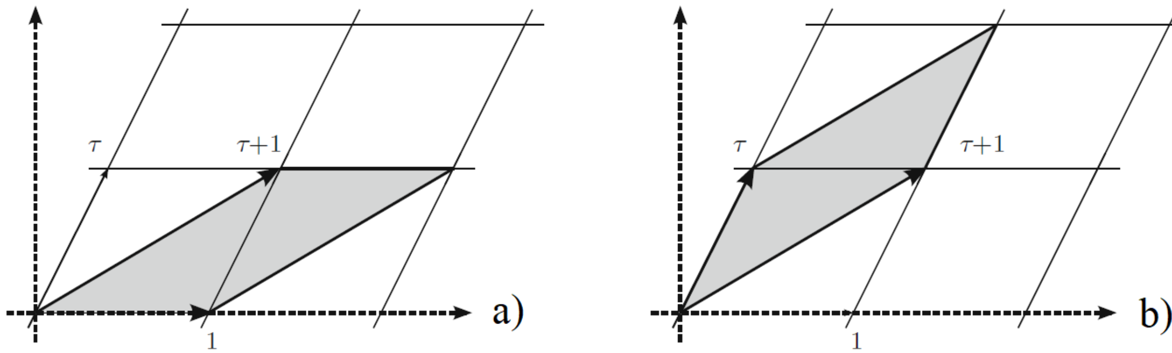

Figure 3.3: The fundamental domain transforms under a) a modular T-transformation, and b) a modular U-transformation.

3.2 The Partition Function on the Torus

Now let us consider the partition function for conformal field theories on the torus. We choose Re$\tau = \tau_1$ to be the space direction and Im$\tau = \tau_2$ to be the time direction and define the partition function in a similar way as we define the partition function in statistical mechanics,

$$\mathcal{Z}(\tau_1, \tau_2) = Tr_{\mathcal{H}}\Big(\exp(-2\pi\tau_2 H + 2\pi\tau_1 P)\Big) \tag{3.3}$$

where the trace is taken over all states in the Hilbert space $\mathcal{H}$, H is the Hamiltonian generating time translations and P is the momentum operator generating space translations.

Recall that, on the complex plane, H, P can be expressed as in Eq.(2.25), but since now we are on a torus, we need to write $L_0, \overline{L}_0$ in cylinder space as a torus can be formed by cutting out a finite piece of the infinite cylinder and identifying the boundaries. To determine $(L_{cyl})_0$, we recall that L_0 is the 0-th Laurent mode of the energy-momentum tensor, for which we know how it transforms under transformation $z = e^w = f(w)$ as in Eq.(2.39), using $S(f(w), w) = -\frac{1}{2}$ and we get

$$T_{cyl}(w) = \left(\frac{\partial f(w)}{\partial w}\right)^2 T\big(f(w)\big) + \frac{c}{12} S\big(f(w), w\big) = z^2 T(z) - \frac{c}{24}\,. \tag{3.4}$$

Then we Laurent expand the energy-momentum tensor on the cylinder and obtain

$$T_{cyl}(w) = \sum_{n\in\mathbb{Z}} z^{-n} L_n - \frac{c}{24} = \sum_{n\in\mathbb{Z}} \Big(L_n - \frac{c}{24}\delta_{n,0} \Big) e^{-nw} \ , \tag{3.5}$$

which tells us that the 0-th Laurent mode on the cylinder is given by $(L_{cyl})_0 = L_0 - \frac{c}{24}$ and similarly, $(\overline{L}_{cyl})_0 = \overline{L}_0 - \frac{\overline{c}}{24}$, where $c, \overline{c}$ denote the central charges. Thus, the Hamiltonian is given by

$$\begin{aligned} H_{cyl} &= (L_{cyl})_0 + (\overline{L}_{cyl})_0 = L_0 + \overline{L}_0 - \frac{c+\overline{c}}{24} \ , \\ P_{cyl} &= i\Big((L_{cyl})_0 - (\overline{L}_{cyl})_0 \Big) = i\bigg(L_0 - \overline{L}_0 - \frac{c-\overline{c}}{24} \bigg) \ , \end{aligned} \tag{3.6}$$

and the ground state energy is given by $E_0 = \langle T_{cyl00} \rangle = -\frac{c+\overline{c}}{24}$. Then we let $q = e^{2\pi i \tau}$, $\overline{q} = e^{-2\pi i \overline{\tau}}$ and write the partition function as

$$\mathcal{Z}(\tau, \overline{\tau}) = Tr_{\mathcal{H}} \Big(q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{\overline{c}}{24}} \Big) \ . \tag{3.7}$$

In addition, the partition function $\mathcal{Z}(\tau, \overline{\tau})$ has to be invariant under the action of the modular group since the torus is unchanged under $SL(2,\mathbb{Z})/\mathbb{Z}_2$ transformations. This is what we will use to study the modularity of two-dimensional CFTs in this section.

Now that we found the partition function for conformal field theories on the torus, we will study the partition functions for bosonic and fermionic CFTs in more detail in the following sections. So far, we studied the structures of conformal field theories without using their Lagrangian, but since Lagrangian formalism is the approach that appears naturally in string theory, in the following sections, let us consider bosonic and fermionic CFTs with given Lagrangian actions.

3.3 Free Bosons

Let us consider a massless scalar field $X(z, \overline{z})$ compactified on a one-dimensional torus, i.e. a circle of radius R. Consider the following action

$$
\begin{aligned}
\mathcal{S} &= \frac{1}{2\pi\alpha'} \int dz d\overline{z} \, \sqrt{|g|} g^{ab} \partial_a X \partial_b X \\
&= \frac{1}{2\pi\alpha'} \int dz d\overline{z} \, \partial X \cdot \overline{\partial} X \, ,
\end{aligned}
\tag{3.8}
$$

where the metric $g_{ab} = \begin{pmatrix} 0 & \frac{1}{2z\overline{z}} \\ \frac{1}{2z\overline{z}} & 0 \end{pmatrix}$, and $\frac{1}{2\pi\alpha'}$ is the string tension and the constant α' is called the *Regge slope* for which we conventionally choose to be 2. This action in fact has a connection to the action of a massless scalar field on a cylinder in string theory, which is obtained through mapping the cylinder to the complex plane with a change of variables where $z = e^{x^0 + ix^1}$.

By setting $\delta_X \mathcal{S} = 0$ as we do in the Appendix A.8, we get the equation of motion for the above action to be $\partial\overline{\partial} X(z, \overline{z}) = 0$. Therefore, we conclude that the currents

$$
\begin{aligned}
j(z) &= i\partial X(z, \overline{z}) = \sum_{n \in \mathbb{Z}} z^{-n-1} j_n \, , \\
\overline{j}(\overline{z}) &= i\overline{\partial} X(z, \overline{z}) = \sum_{n \in \mathbb{Z}} \overline{z}^{-n-1} \overline{j}_n
\end{aligned}
\tag{3.9}
$$

are chiral and anti-chiral, respectively. Also, the action $\mathcal{S}$ being invariant under conformal transformations implies that the field $X(z, \overline{z})$ has conformal dimensions $(h, \overline{h}) = (0, 0)$ as one can check by taking $X'(z, \overline{z}) = X(y, \overline{y})$ under the transformation $z \mapsto f(z) = y$. Thus, we conclude that $j(z)$ and $\overline{j}(\overline{z})$ are primary fields with conformal dimensions $(1, 0)$ and $(0, 1)$ respectively.

The current algebra for a free boson is the so-called $U(1)$ algebra, which we have presented in the previous chapter. When we have only one current field $j(z)$, Eq.(2.48) thus gives the following current algebra,

$$
[j_m, j_n] = m \delta_{m+n,0} \, . \tag{3.10}
$$

In addition, we can integrate Eq.(3.9) to find the field $X(z, \overline{z})$, which is given by

$$
X(z, \overline{z}) = x_0 - i\Big(j_0 \ln z + \overline{j}_0 \ln \overline{z}\Big) + i \sum_{n \neq 0} \frac{1}{n} \Big(j_n z^{-n} + \overline{j}_n \overline{z}^{-n}\Big) \, , \tag{3.11}
$$

where X_0 is a constant. From there, let us now study the consequences of the field being compactified on a circle of radius R.

3.3.1 Free Bosons Compactified on a Circle

For a free boson on a circle of radius R, we identify the field $X(z, \overline{z})$ with $X(z, \overline{z}) + 2\pi R n$ with $n \in \mathbb{Z}$, which means we have the relation where $X(e^{2\pi i}z, e^{-2\pi i}\overline{z}) = X(z, \overline{z}) + 2\pi R n$. Solve this equation using Eq.(3.11), we find that on a circle of radius R, the 0-th current modes for chiral and aniti-chiral fields satisfy $j_0 - \overline{j}_0 = nR$ with $n \in \mathbb{Z}$, which implies that under $j_0, \overline{j}_0$ the ground state is non-trivially charged. For $n \in \mathbb{Z}$, denote the ground state as $|\Gamma, n\rangle$, where we use Γ to label the eigenvalue of j_0 acting on this state, then we have $j_0 |\Gamma, n\rangle = \Gamma |\Gamma, n\rangle$ and $\overline{j}_0 |\Gamma, n\rangle = (\Gamma - nR) |\Gamma, n\rangle$. We will soon determine the value for Γ.

To find the partition function, we need the 0-th Laurent mode of the energy-momentum tensor, so let us now study the energy-momentum tensor in the bosonic case. We define the energy-momentum tensor for the action $\mathcal{S}$ as

$$T_{ab} = 4\pi\gamma \frac{\delta \mathcal{S}}{\sqrt{|g|}\delta g^{ab}} \ , \tag{3.12}$$

where γ is another normalization constant which we will determine soon. Then we use $\delta\sqrt{|g|} = -\frac{1}{2}\sqrt{|g|}g_{ab}\delta g^{ab}$ to compute entries of T_{ab}, we find the following results,

$$T_{ab} = \begin{pmatrix} \gamma\partial X \partial X & 0 \\ 0 & \gamma\overline{\partial} X \overline{\partial} X \end{pmatrix} , \tag{3.13}$$

where $a, b = z$ or $\overline{z}$. Detailed computations can be found in [16]. Note that it is important to take the normal ordered expression for T_{ab} since for a quantum theory, the expectation value of T_{ab} vanishes. Therefore, the chiral part of the energy-momentum tensor is

$$T_{zz} = T(z) = \gamma N(\partial X \partial X)(z) = \gamma N(jj)(z) \ , \tag{3.14}$$

and similarly, the anti-chiral part is given by $\overline{T}(\overline{z}) = \gamma N(\overline{j}\overline{j})(\overline{z})$.

Now we can determine γ using the fact that $j(z)$ is a primary field of conformal dimension $h = 1$. We Laurent expand both sides of $T(z) = \gamma N(jj)(z)$ and find

$$L_n = \gamma N(jj)_n = \gamma \sum_{k>-1} j_{n-k} j_k + \gamma \sum_{k\leq -1} j_k j_{n-k} \ . \tag{3.15}$$

Then the commutator $[L_m, j_n] = -2\gamma n j_{m+n}$ as shown in Appendix A.9. Recall that the Laurent modes of a primary field satisfies $[L_m, \phi_n]$ Eq.(2.43), therefore, with the choice of $\alpha' = 2$ we conclude that $\gamma = \frac{1}{2}$. Also, we claim that the CFT of a free boson has central charge $c = 1$ and we will verify this statement in the Appendix A.10.

Now we have the key ingredient to compute the partition function, taking $n = 0$ in Eq.(3.15), we get the 0-th Laurent mode of the chiral part $T(z)$ of the energy-momentum tensor, which is

$$L_0 = \frac{1}{2} \sum_{k>-1} j_{-k} j_k + \frac{1}{2} \sum_{k\leq -1} j_k j_{-k} = \frac{1}{2} j_0 j_0 + \sum_{k\geq 1} j_{-k} j_k \tag{3.16}$$

where we let index $l = -k$ for the second term. Since the Hilbert space of a free bosonic CFT consists states generated by the current modes $j_{-k}, \overline{j}_{-l}$ for $k, l \geq 1$ and the chiral and anti-chiral parts of a field decouple, we can express the states for the chiral part as $|n_1, n_2, n_3, \ldots\rangle = j_{-1}^{n_1} j_{-2}^{n_2} j_{-3}^{n_3} \ldots |\Gamma, n\rangle$ with $n_i \geq 0$ where $|\Gamma, n\rangle$ is the ground state. Now we can act the 0-th Laurent mode of the energy-momentum tensor on states in the Hilbert space following calculations in the Appendix A.11, then we obtain the result

$$\begin{aligned} L_0 |n_1, n_2, n_3, \ldots\rangle &= \left(\frac{1}{2}\Gamma^2 + \sum_{k\geq 1} n_k k\right) |n_1, n_2, n_3, \ldots\rangle \ , \\ \overline{L}_0 |n_1, n_2, n_3, \ldots\rangle &= \left(\frac{1}{2}\big(\Gamma - nR\big)^2 + \sum_{k\geq 1} n_k k\right) |n_1, n_2, n_3, \ldots\rangle \ . \end{aligned} \tag{3.17}$$

Then we can bring our results for $L_0, \overline{L}_0$ into Eq.(3.7) and compute the partition function of a free boson compactified on a circle of radius R as shown in the Appendix A.11. Let

$q = e^{2\pi i\tau}$ and $\overline{q} = e^{-2\pi i\overline{\tau}}$, we get that the partition function is given by

$$\mathcal{Z}_{boc}(\tau,\overline{\tau}) = \frac{1}{|\eta(\tau)|^2}\sum_{\Gamma,n} q^{\frac{1}{2}\Gamma^2}\overline{q}^{\frac{1}{2}(\Gamma-nR)^2} . \tag{3.18}$$

where we sum over $n \in \mathbb{Z}$ and discrete values of Γ. In the above expression, $\eta(\tau)$ is the Dedekind η-function, which is defined as

$$\eta(\tau) \equiv q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n) , \tag{3.19}$$

which behaves under the T-transformations and S-transformations as

$$\eta(\tau+1) = e^{\frac{\pi i}{12}}\eta(\tau) , \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau) . \tag{3.20}$$

One can find a detailed verification of these two relations in [15].

Now let us determine the value of Γ using the modular invariance of the partition function and the modular properties of the Dedekind η-function. For the modular T-transformation $\tau \mapsto \tau+1$ we have

$$\mathcal{Z}(\tau,\overline{\tau}) \mapsto \mathcal{Z}(\tau+1,\overline{\tau}+1) = \frac{1}{|\eta(\tau)|^2}\sum_{\Gamma,n} q^{\frac{1}{2}\Gamma^2}\overline{q}^{\frac{1}{2}(\Gamma-nR)^2}e^{2\pi i n(R\Gamma-\frac{1}{2}nR^2)} , \tag{3.21}$$

and we see that modular invariance implies that $R\Gamma - \frac{nR^2}{2} = m$ with $m \in \mathbb{Z}$ which gives $\Gamma = \frac{m}{R} + \frac{nR}{2}$. Thus, we can label a ground state as $|m,n\rangle$ with $m,n \in \mathbb{Z}$, and it has chiral ground state charge $\frac{m}{R} + \frac{nR}{2}$ and anti-chiral ground state charge $\frac{m}{R} - \frac{nR}{2}$. Here integers m and n are called the momentum and winding numbers, respectively. Our final result for the partition function of a free boson on a circle of radius R is given by

$$\begin{aligned}\mathcal{Z}_{boc}(\tau,\overline{\tau}) &= \frac{1}{|\eta(\tau)|^2}\sum_{m,n\in\mathbb{Z}} q^{\frac{1}{2}\left(\frac{m}{R}+\frac{nR}{2}\right)^2}\overline{q}^{\frac{1}{2}\left(\frac{m}{R}-\frac{nR}{2}\right)^2} \\ &= \frac{1}{|\eta(\tau)|^2}\sum_{m,n\in\mathbb{Z}} exp\left\{-\pi\tau_2\left(\frac{2m^2}{R^2}+\frac{n^2R^2}{2}\right)+2\pi i\tau_1 mn\right\} ,\end{aligned} \tag{3.22}$$

where the modular parameter $\tau = \tau_1 + i\tau_2$ as we have seen before.

We can also show that the partition function is modular invariant under the modular S-transformations, that is to say

$$\mathcal{Z}_{boc}\bigg(-\frac{1}{\tau},-\frac{1}{\overline{\tau}}\bigg)=\mathcal{Z}_{boc}(\tau,\overline{\tau})\ , \tag{3.23}$$

and this is a straightforward calculation which is shown in the Appendix A.12.

An interesting property of the partition function Eq.(3.22) of a free boson on a circle of radius R is that it remains unchanged when we replace R by α'/R, where $\alpha'=2$ in our case. This amounts to switch the momentum and winding numbers, and this is known as the T-duality. Thus, in string theory, we cannot distinguish whether the radius of the circle is R or $2/R$ for a closed string propagating in a circular background. The radius $R=\sqrt{2}$ is called the self-dual radius.

3.3.2 Free Bosons on $\mathbb{Z}_2$ Orbifold

In the previous subsection, we discussed the case of a free boson compactified on a circle, and in the following let us consider a variantion of it, which is a boson compactified on a $\mathbb{Z}_2$-orbifold of a circle with radius R, and the bosonic field $X(z,\overline{z})$ is identified with both $X(z,\overline{z})+2\pi R$ and $-X(z,\overline{z})$. Denote this $\mathbb{Z}_2$ symmetry as the orbifold action $\mathcal{R}$: $X(z,\overline{z})\mapsto -X(z,\overline{z})$, focusing on the chiral part, then the boundary condition can be expressed as

$$X(z+m\omega_1+n\omega_2)=e^{2\pi i(m\mu+n\nu)}X(z)\ , \tag{3.24}$$

where μ,ν take the value of either 0 or $\frac{1}{2}$, which denote "untwisted" and "twisted" boundary conditions, repectively. Note that here the four boundary conditions (μ,ν) come from the topological structure of the space on which the bosonic field lives, and their root is different from the root of the boundary conditions allowed by the nature of fermionic fields, which we will discuss in section 3.4.

Denote the partition function for a free boson with boundary condition (μ,ν) as $\mathcal{Z}_{\mu,\nu}$, then we have the following partition functions for each boundary conditions

$$\mathcal{Z}_{0,0} = Tr_{\mathcal{H}}\left(q^{L_0-\frac{1}{24}}\overline{q}^{\overline{L}_0-\frac{1}{24}}\right) , \qquad \mathcal{Z}_{0,\frac{1}{2}} = Tr_{\mathcal{H}}\left(\mathcal{R}q^{L_0-\frac{1}{24}}\overline{\mathcal{R}}\overline{q}^{\overline{L}_0-\frac{1}{24}}\right) ,$$
$$\mathcal{Z}_{\frac{1}{2},0} = Tr_{\mathcal{H}}\left(q^{L_0-\frac{1}{48}}\overline{q}^{\overline{L}_0-\frac{1}{48}}\right) , \qquad \mathcal{Z}_{\frac{1}{2},\frac{1}{2}} = Tr_{\mathcal{H}}\left(\mathcal{R}q^{L_0-\frac{1}{48}}\overline{\mathcal{R}}\overline{q}^{\overline{L}_0-\frac{1}{48}}\right) , \tag{3.25}$$

where the Hilbert space $\mathcal{H}$ contains only states that are invariant under $\mathcal{R}$.

To find the partition functions of free bosons on $\mathbb{Z}_2$-orbifold, all that is left to do is to determine how this orbifold action $\mathcal{R}$ acts on a general state, then following the same calculations carried out in the previous subsection, we will reach our results. Choose the action $\mathcal{R}$ such that the ground state $|\Gamma, n\rangle$ is left invariant, consider the orbifold action $\mathcal{R} : X(z,\overline{z}) \to -X(z,\overline{z})$ acting on a state $j_k |\Gamma, n\rangle$, we see that $\mathcal{R}j_k = -j_k\mathcal{R}$. Then for a general state $|n_1, n_2, n_3, \ldots\rangle$, we have

$$\mathcal{R}\,|n_1, n_2, n_3, \ldots\rangle = \mathcal{R}j_{-1}^{n_1}j_{-2}^{n_2}j_{-3}^{n_3}\ldots|\Gamma, n\rangle = (-1)^{n_1+n_2+n_3+\cdots}\,|n_1, n_2, n_3, \ldots\rangle \ . \tag{3.26}$$

Follow the similar calculations as we did for a free boson on a circle with the above expression, it is easy to find the partition function $Z_{0,0}$.

$$Z_{0,0} = Z_{boc}(\tau, \overline{\tau}) , \qquad Z_{0,\frac{1}{2}} = 2\left|\frac{\eta(\tau)}{\vartheta_2(\tau)}\right| ,$$
$$Z_{\frac{1}{2},0} = 2\left|\frac{\eta(\tau)}{\vartheta_3(\tau)}\right| , \qquad Z_{\frac{1}{2},\frac{1}{2}} = 2\left|\frac{\eta(\tau)}{\vartheta_4(\tau)}\right| , \tag{3.27}$$

where partition functions $Z_{0,\frac{1}{2}}, Z_{\frac{1}{2},0}$ and $Z_{\frac{1}{2},\frac{1}{2}}$ follow similar calculations for fermionic partition functions $Z_{R,NS}, Z_{NS,NS}$ and $Z_{NS,R}$ respectively, which are done in the next section, except in the orbifold boson case the boson numbers n_k at each mode can be any natural number, while in the fermionic case the fermion number n_k at each mode is restricted to 0 and 1 due to the nature of fermions. As we have discussed, the partition function needs to be modular invariant. Taking the modularity properties of theta functions into consideration, the total partition function of a free boson on a $\mathbb{Z}_2$-orbifold is the sum of $Z_{0,0}, Z_{0,\frac{1}{2}}, Z_{\frac{1}{2},0}$ and $Z_{\frac{1}{2},\frac{1}{2}}$,

$$Z_{orb}(\tau, \overline{\tau}) = \frac{1}{2}Z_{boc}(\tau, \overline{\tau}) + \left|\frac{\eta(\tau)}{\vartheta_2(\tau)}\right| + \left|\frac{\eta(\tau)}{\vartheta_3(\tau)}\right| + \left|\frac{\eta(\tau)}{\vartheta_4(\tau)}\right| , \tag{3.28}$$

where the factor of $\frac{1}{2}$ comes from projecting states onto orbifold invariant states, i.e. the states that are invariant under the orbifold action $\mathcal{R}$, because the Hilbert space contains only orbifold invariant states.

Then we can use the relation between theta functions and the Dedekind η-function as shown in Appendix B.2 to write the partition function as

$$Z_{orb}(\tau,\overline{\tau}) = \frac{1}{2}\left(Z_{boc}(\tau,\overline{\tau}) + \frac{|\vartheta_2(\tau)\vartheta_3(\tau)|}{|\eta(\tau)|^2} + \frac{|\vartheta_2(\tau)\vartheta_4(\tau)|}{|\eta(\tau)|^2} + \frac{|\vartheta_3(\tau)\vartheta_4(\tau)|}{|\eta(\tau)|^2} \right) . \tag{3.29}$$

3.4 Free Fermions

In this section, let us consider the CFTs of a free fermion on the torus. For a free Majorana fermion $\Psi = \begin{pmatrix} \psi(z,\overline{z}) \\ \overline{\psi}(z,\overline{z}) \end{pmatrix}$, the spinors $\psi(z,\overline{z}), \overline{\psi}(z,\overline{z})$ are both real fields, i.e.$\psi^\dagger = \psi$ and $\overline{\psi}^\dagger = \overline{\psi}$. Consider the following action

$$\begin{aligned} \mathcal{S} &= \frac{1}{2\pi\alpha'} \int dz d\overline{z}\, \sqrt{|g|} 2\Psi^\dagger \begin{pmatrix} \partial & 0 \\ 0 & \overline{\partial} \end{pmatrix} \Psi \\ &= \frac{1}{2\pi\alpha'} \int dz d\overline{z} (\psi\overline{\partial}\psi + \overline{\psi}\partial\overline{\psi}) , \end{aligned} \tag{3.30}$$

where the metric $g_{ab} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$ and α' is chosen to be 2 as before.

By setting $\delta_\psi \mathcal{S} = \delta_{\overline{\psi}} \mathcal{S} = 0$ as we do in the Appendix A.13, we get the equation of motion of the above action to be $\overline{\partial}\psi(z,\overline{z}) = \partial\overline{\psi}(z,\overline{z}) = 0$, which implies that $\psi(z,\overline{z}) = \psi(z)$ is a chiral field and $\overline{\psi}(z,\overline{z}) = \overline{\psi}(\overline{z})$ is a anti-chiral field. By requiring the action $\mathcal{S}$ to be invariant under conformal transformations, we get that the field $\psi(z)$ has conformal dimension $(h,\overline{h}) = (\frac{1}{2}, 0)$ and the field $\overline{\psi}(\overline{z})$ has conformal dimension $(h,\overline{h}) = (0, \frac{1}{2})$. One can easily verify this statement by taking $\psi'(z,\overline{z}) = \left(\frac{\partial y}{\partial z}\right)^{1/2} \psi(y,\overline{y})$ and $\overline{\psi}'(z,\overline{z}) = \left(\frac{\partial \overline{y}}{\partial \overline{z}}\right)^{1/2} \overline{\psi}(y,\overline{y})$ under the transformation $z \mapsto f(z) = y$, and check the invariance of the action. In addition, we know that fermionic fields have two possible behaviours under 2π rotations, to distinguish these possibilities, let us define the following. On the complex plane, for the chiral part we have the Neveu-Schwarz sector (NS) $\psi(e^{2\pi i}z) = +\psi(z)$ and the Ramond sector (R) $\psi(e^{2\pi i}z) = -\psi(z)$. Laurent expand the field $\psi(z)$, we get

$$\psi(z) = \sum_r z^{-r-\frac{1}{2}}\psi_r \quad \text{where} \quad \psi_r = \oint \frac{dz}{2\pi i} z^{r-\frac{1}{2}}\psi(z) \,. \tag{3.31}$$

We see that when $r \in \mathbb{Z} + \frac{1}{2}$ the field behaves as the NS sector does, and when $r \in \mathbb{Z}$ the field behaves as the R sector does on the complex plane. Note that this is the opposite of the case on a cylinder, where the NS sector is $\psi(w + 2\pi i) = -\psi(w)$ and the R sector is $\psi(w + 2\pi i) = \psi(w)$. The fundamental domain of a torus has two periods as we have seen, i.e. space period ω_1 and time period ω_2, and we can write the boundary conditions as

$$\psi(z + m\omega_1 + n\omega_2) = e^{2\pi i(m\mu + n\nu)}\psi(z) \,, \tag{3.32}$$

where μ, ν take values of either 0, the periodic (R) boundary condition, or $\frac{1}{2}$, the anti-periodic (NS) boundary condition. Thus, a fermionic field on a torus has four types of periodicity conditions or *spin structures*: (NS, NS), (NS, R), (R, NS), and (R,R), and we need to specify them when computing the partition function. Also, invariance under modular transformation which preserves the spin structure requires that the chiral and anti-chiral parts of a fermion have the same spin structure.

Recall that we introduced the radial ordering in section 2.3, and for fermionic fields it is defined as,

$$R(\psi(z)\theta(w)) := \begin{cases} +\psi(z)\theta(w) & \text{for } |z| > |w| \,, \\ -\theta(w)\psi(z) & \text{for } |z| < |w| \,. \end{cases} \tag{3.33}$$

where the minus sign is due to the fermionic nature of the fields. Then we can determine the OPE of the product of two fermionic fields by calculating the propagator $\langle\Psi_i(x)\Psi_j(y)\rangle$ with $i, j = 1, 2$ [14], we get that

$$\begin{aligned} \psi(z)\psi(w) &= \frac{\alpha'/2}{z - w} + \text{non-singular terms} \,, \\ \overline{\psi}(\overline{z})\overline{\psi}(\overline{w}) &= \frac{\alpha'/2}{\overline{z} - \overline{w}} + \text{non-singular terms} \,, \end{aligned} \tag{3.34}$$

with $\alpha' = 2$ as chosen. With this expression we will then determine the Laurent mode algebra by bringing Eq.(3.33) into the anti-commutator of the Laurent modes $\{\psi_r, \psi_s\}$ as shown in Appendix A.14, and we get

$$\{\psi_r, \psi_s\} = \delta_{r+s,0} \ . \tag{3.35}$$

In the NS sector, $r \in \mathbb{Z} + \frac{1}{2}$, but in the R sector, $r \in \mathbb{Z}$ and we see that there exists a 0-th mode, with $\psi_0^2 = \frac{1}{2}$ and since $\psi_k \psi_0 |0\rangle = 0$ for $k > 0$, we have that $\psi_0 |0\rangle$ is degenerate to a vacuum $|0\rangle$.

Our goal is to find the partition function of a free fermionic CFT on the torus. Similar as what we did for bosonic fields, in order to find the 0-th Laurent mode of the energy-momentum tensor, let us first study the canonical energy-momentum tensor for a theory with fields ϕ_i (where in our case the two fields are $\phi_1 = \psi, \phi_2 = \overline{\psi}$) and Lagrangian $\mathcal{L}$, which is defined as

$$T_{\mu\nu} = 4\pi\alpha'\gamma\left(- \eta_{\mu\nu}\mathcal{L} + \sum_i \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi_i)} \partial_\nu \phi_i \right) , \tag{3.36}$$

where γ is again some normalisation constant to be determined. One can easily find the energy-momentum tensor to be

$$T_{ab} = \begin{pmatrix} \gamma\psi\partial\psi & -\gamma\overline{\psi}\partial\overline{\psi} \\ -\gamma\psi\overline{\partial}\psi & \gamma\overline{\psi\partial\psi} \end{pmatrix} = \begin{pmatrix} \gamma\psi\partial\psi & 0 \\ 0 & \gamma\overline{\psi\partial\psi} \end{pmatrix} , \tag{3.37}$$

where the off diagonal entries vanish by the equations of motions $\overline{\partial}\psi = \partial\overline{\psi} = 0$. Now let us focus on the chiral part and write it using the normal ordered expression as $T_{zz} = T(z) = \gamma N(\psi\partial\psi)(z)$. We have that for fermionic fields the normal ordered product of two fields is given by

$$\begin{aligned} N(\psi\theta)_m &= - \sum_{k > -h^\theta} \psi_{m-k}\theta_k + \sum_{k \leq -h^\theta} \theta_k \psi_{m-k} \ . \\ N(\psi\partial\theta)_m &= - \sum_{k > -h^\theta - 1} (-h^\theta - k)\psi_{m-k}\theta_k + \sum_{k \leq -h^\theta - 1} (-h^\theta - k)\theta_k\psi_{m-k} \ . \end{aligned} \tag{3.38}$$

The derivation of this equation is similar to what we have done for the bosonic case as shown in Appendix A.7, the difference is that we will now use the radial ordering Eq.(3.34) for fermions, which leads to a minus sign for the first term.

By Laurent expanding both sides of the equation $T(z) = \gamma N(\psi\partial\psi)(z)$ and applying the method used to derive Eq.(2.57), we find

$$L_m = \gamma N(\psi\partial\psi)_m = \gamma \sum_{k>-\frac{3}{2}} \left(k+\frac{1}{2}\right)\psi_{m-k}\psi_k - \gamma \sum_{k\leq -\frac{3}{2}} \left(k+\frac{1}{2}\right)\psi_k\psi_{m-k} \,. \tag{3.39}$$

To determine the constant γ, we will compute the commutator $[L_m, \psi_r]$. We do the calculation in Appendix A.15 and find the following

$$[L_m, \psi_r] = \gamma(-m-2r)\psi_{m+r} \,. \tag{3.40}$$

Recall that the invariance of the fermionic action leads to the statement that the chiral part $\psi(z)$ is a primary field with conformal dimension $h = \frac{1}{2}$, then by comparing the above result with Eq.(A.8) where we found that for a primary field with conformal dimension h satisfies the commutation relation $[L_m, \phi_n] = [(h-1)m - n]\phi_{m+n}$, we get that the constant $\gamma = \frac{1}{2}$, which is the same as what we found in the bosonic case.

For the CFT of a free fermion, its central charge is $c = \frac{1}{2}$ and we will verify this claim in Appendix A.16.

To compute the partition functions for a free fermion on a torus using the operator formalism, it is important to take time-ordering into consideration for the same reason in any QFT. Due to the nature of fermionic fields, a minus sign is generated after each reordering of two fermions. To preserve this feature, we introduce an operator $(-1)^F$ where $F := F_0 + \sum_{k>0} \psi_{-k}\psi_k$ is the *world-sheet fermion number* (distinguish from the *space-time fermion number*) in the chiral sector. F_0 is defined in the periodic case in space direction, and equals 0 or 1 when acting on $|0\rangle$ or $\psi_0 |0\rangle$, respectively. Similarly, we have operator $(-1)^{\overline{F}}$ for the anti-chiral sector with $\overline{F} := \overline{F}_0 + \sum_{k>0} \overline{\psi}_{-k}\overline{\psi}_k$. We insert this operator into the definition of the partition function to the time-periodic case only because the time-antiperiodic case keeps this time-ordering feature naturally. That is to say, these operators are inserted into the partition functions with the (NS, R) and (R, R) boundary conditions, and the partition function is defined as

$$\mathcal{Z}_{\cdot\,,R} = Tr_{\mathcal{F}_X}((-1)^F q^{L_0-\frac{1}{48}}(-1)^{\overline{F}}\overline{q}^{\overline{L}_0-\frac{1}{48}})\,, \tag{3.41}$$

where $\cdot$ is taken to be NS or R.

Now let us compute the partition function for a free fermion CFT. To do so, we want the 0-th mode of the energy-momentum tensor. First we will consider the NS sector, by taking $m = 0$ in Eq.(3.39), we obtain the 0-th Laurent mode, which is given by

$$\begin{aligned} L_0 &= \frac{1}{2}\sum_{k>-\frac{3}{2}}\left(k+\frac{1}{2}\right)\psi_{-k}\psi_k + \frac{1}{2}\sum_{k\geq\frac{3}{2}}\left(k-\frac{1}{2}\right)\psi_{-k}\psi_k \\ &= \frac{1}{2}\psi_{-\frac{1}{2}}\psi_{\frac{1}{2}} + \sum_{k\geq\frac{3}{2}} k\psi_{-k}\psi_k \\ &= \sum_{k=\frac{1}{2}}^{\infty} k\psi_{-k}\psi_k \end{aligned} \tag{3.42}$$

where $k \in \mathbb{Z} + \frac{1}{2}$ and in the first line we replaced k by $-k$ in the second term. We see that the NS sector has zero vacuum energy by computing the expectation value of $T(z) = \frac{1}{2}N(\psi\partial\psi)(z)$ and using the normal ordering prescription as in ordinary QFTs

$$\langle T(z)\rangle = \frac{1}{2}\lim_{\epsilon\to 0}\bigg(\psi(z+\epsilon)\partial\psi(z) - \langle\psi(z+\epsilon)\partial\psi(z)\rangle\bigg) = 0\,, \tag{3.43}$$

where $\psi(z+\epsilon)\partial\psi(z) = \frac{1}{\epsilon^2}$ by Eq.(3.34) and the expectation value $\langle\psi(z+\epsilon)\partial\psi(z)\rangle = \frac{1}{\epsilon^2}$ is computed in Appendix A.17.

For the R sector, it is more interesting since we will see that the vacuum energy is non-zero. Due to the ambiguity in the normal ordering at the 0-th mode $\psi_0\psi_0$, we need to be careful and take the expectation value of the energy density into consideration when computing L_0. To obtain the vacuum energy density, we compute the expectation value of $\langle T(z)\rangle$ in the following, using what we found for $\langle\psi(z+\epsilon)\partial\psi(z)\rangle$ in the R sector in Appendix A.17,

$$
\begin{aligned}
\langle T(z)\rangle &= \frac{1}{2}\lim_{\epsilon\to 0}\Big(\psi(z+\epsilon)\partial\psi(z) - \langle\psi(z+\epsilon)\partial\psi(z)\rangle\Big) \\
&= \frac{1}{2}\lim_{\epsilon\to 0}\left[\frac{1}{2\epsilon^2}\left(2-\sqrt{\frac{z}{z+\epsilon}}-\sqrt{\frac{z+\epsilon}{z}}\right)+\frac{1}{4z\sqrt{z(z+\epsilon)}}\right] \\
&= \frac{1}{2}\lim_{\epsilon\to 0}\left[-\frac{1}{8z^2}+\mathcal{O}(\epsilon)+\frac{1}{4z\sqrt{z(z+\epsilon)}}\right] \\
&= \frac{1}{16z^2}\,,
\end{aligned}
\tag{3.44}
$$

where we Taylor expand $\sqrt{1+x}$ and $\frac{1}{\sqrt{1+x}}$ around $x=0$ in the second line with $x=\frac{\epsilon}{z}$.

By Eq.(2.37), we see that L_0 is the coefficient of the z^{-2} in the mode expansion of the energy-momentum tensor. Thus, we find the 0-th Laurent modes to be

$$
\begin{aligned}
L_0 &= \sum_{k=\frac{1}{2}}^{\infty} k\psi_{-k}\psi_k && \text{with } k\in\mathbb{Z}+\frac{1}{2} \text{ for the NS sector,} \\
L_0 &= \sum_{k=1}^{\infty} k\psi_{-k}\psi_k + \frac{1}{16} && \text{with } k\in\mathbb{Z} \text{ for the R sector.}
\end{aligned}
\tag{3.45}
$$

Then by Eq.(3.6), we see that the chiral part of the Hamiltonian on the cylinder is given by $H_{NS}=-\frac{1}{48}$, and $H_R=\frac{1}{24}$.

The Hilbert space of the free fermion CFT is the Fock space generated by $\psi_{-r},\overline{\psi}_{-s}$ for $r,s\geq\frac{1}{2}$ and each mode appears at most once due to the Fermi-statistics. For the NS sector, a general chiral state in this Hilbert space can be written as $|n_{\frac{1}{2}},n_{\frac{3}{2}},\ldots\rangle = (\psi_{-\frac{1}{2}})^{n_{1/2}}(\psi_{-\frac{3}{2}})^{n_{3/2}}\ldots|0\rangle$, with $n_k=0$ or 1 due to Fermi-statistics, and the action of L_0 on the state gives

$$
L_0\,|n_{\frac{1}{2}},n_{\frac{3}{2}},\ldots\rangle = \sum_{k=\frac{1}{2}}^{\infty} k\Big(\psi_{-\frac{1}{2}}\Big)^{n_{\frac{1}{2}}}\ldots n_k\Big(\psi_{-k}\psi_k\psi_{-k}\Big)\ldots|0\rangle = \sum_{k=\frac{1}{2}}^{\infty} n_k k\,|n_{\frac{1}{2}},n_{\frac{3}{2}},\ldots\rangle\ . \tag{3.46}
$$

For the R sector, a general chiral state is given by $|n_0,n_1,\ldots\rangle = (\psi_0)^{n_0}(\psi_{-1})^{n_1}\ldots|0\rangle$ with $n_k=0$ or 1 for the same reason, and we get

$$
L_0\,|n_0,n_1,\ldots\rangle = \left(\sum_{k=1}^{\infty} n_k k + \frac{1}{16}\right)|n_0,n_1,\ldots\rangle = \left(\sum_{k=0}^{\infty} n_k k + \frac{1}{16}\right)|n_0,n_1,\ldots\rangle\ . \tag{3.47}
$$

Using the above expressions we will compute the partition function of a free fermion with the (NS, NS) boundary coundition. Recall that the central charge is given by $c = \frac{1}{2}$ for a fermionic CFT, and we have

$$
\begin{aligned}
Tr_{\mathcal{F}_{NS}}(q^{L_0-\frac{c}{24}}) &= q^{-\frac{1}{48}} \sum_{n_{\frac{1}{2}}=0}^{1} \sum_{n_{\frac{3}{2}}=0}^{1} \cdots \left\langle n_{\frac{1}{2}}, n_{\frac{3}{2}}, \ldots \middle| e^{2\pi i \tau L_0} \middle| n_{\frac{1}{2}}, n_{\frac{3}{2}}, \ldots \right\rangle \\
&= q^{-\frac{1}{48}} \sum_{n_{\frac{1}{2}}=0}^{1} \sum_{n_{\frac{3}{2}}=0}^{1} \cdots \sum_{l=0}^{\infty} \frac{(2\pi i \tau)^l}{l!} \left\langle n_{\frac{1}{2}}, n_{\frac{3}{2}}, \ldots \middle| (L_0)^l \middle| n_{\frac{1}{2}}, n_{\frac{3}{2}}, \ldots \right\rangle \\
&= q^{-\frac{1}{48}} \sum_{n_{\frac{1}{2}}=0}^{1} \sum_{n_{\frac{3}{2}}=0}^{1} \cdots \sum_{l=0}^{\infty} \frac{(2\pi i \tau)^l}{l!} \left(\sum_{k=\frac{1}{2}}^{\infty} n_k k \right)^l \left\langle n_{\frac{1}{2}}, n_{\frac{3}{2}}, \ldots \middle| n_{\frac{1}{2}}, n_{\frac{3}{2}}, \ldots \right\rangle \\
&= q^{-\frac{1}{48}} \sum_{n_{\frac{1}{2}}=0}^{1} \sum_{n_{\frac{3}{2}}=0}^{1} \cdots \prod_{k=\frac{1}{2}}^{\infty} q^{n_k k} \\
&= q^{-\frac{1}{48}} \prod_{k=\frac{1}{2}}^{\infty} \left(1 + q^k\right) \\
&= q^{-\frac{1}{48}} \prod_{k=0}^{\infty} \left(1 + q^{k+\frac{1}{2}}\right) \\
&\equiv \sqrt{\frac{\vartheta_3(\tau)}{\eta(\tau)}} \,,
\end{aligned} \tag{3.48}
$$

where $\vartheta_3(\tau)$ is the Jacobi theta function defined in Eq.(B.18). Note that the second equality is a special case of the Jacobi triple product identity, which is proved in Appendix B.1. Similarly, for the anti-chiral part we have

$$
Tr_{\mathcal{F}_{NS}}(\overline{q}^{\overline{L}_0-\frac{\overline{c}}{24}}) = \sqrt{\frac{\vartheta_3(\overline{\tau})}{\eta(\overline{\tau})}} \,. \tag{3.49}
$$

Thus, the partition function of a free fermion with the (NS, NS) spin structure is simply the product of the traces,

$$
\mathcal{Z}_{NS,NS} = Tr_{\mathcal{F}_{NS}}(q^{L_0-\frac{1}{48}}\overline{q}^{\overline{L}_0-\frac{1}{48}}) = \left| \frac{\vartheta_3(\tau)}{\eta(\tau)} \right| . \tag{3.50}
$$

For the space-periodic and time-antiperiodic case, i.e. the (R, NS) boundary condition, we bring in the expression for L_0 given in Eq. (3.47) and do similar calculation as above, and get

$$\begin{aligned}\frac{1}{\sqrt{2}}Tr_{\mathcal{F}_R}(q^{L_0-\frac{c}{24}}) &= \frac{1}{\sqrt{2}}q^{-\frac{1}{48}}\sum_{n_0=0}^{1}\sum_{n_1=0}^{1}\dots\langle n_0,n_1,\dots|e^{2\pi i\tau L_0}|n_0,n_1,\dots\rangle \\ &= \frac{1}{\sqrt{2}}q^{\frac{1}{24}}\prod_{k=0}^{\infty}\left(1+q^k\right) \\ &= \sqrt{\frac{\vartheta_2(\tau)}{\eta(\tau)}}\,,\end{aligned} \tag{3.51}$$

where the $1/\sqrt{2}$ is a convention due to the definition of the modular function $\vartheta_2(\tau)$ as stated in Eq.(B.17). which is obtained by taking $\omega = q^{-\frac{1}{2}}$ in the Jacobi triple product identity Eq. (B.1). Then we obtain the partition function with (R, NS) boundary condition,

$$\mathcal{Z}_{R,NS} = 2Tr_{\mathcal{F}_R}(q^{L_0-\frac{1}{48}}\overline{q}^{\overline{L}_0-\frac{1}{48}}) = \left|\frac{\vartheta_2(\tau)}{\eta(\tau)}\right|, \tag{3.52}$$

where the factor of 2 is conventional and we will see the reason to include this factor when studying the modular properties of these partition functions.

For the (NS, R) boundary condition, recall that the operator $(-1)^F$ is needed since periodic time condition fails to capture the time-ordering feature. Apply the partition function defined in Eq.(3.41) and write $(-1)^F = e^{-\pi i F}$, we have

$$\begin{aligned}Tr_{\mathcal{F}_{NS}}((-1)^F q^{L_0-\frac{c}{24}}) &= q^{-\frac{1}{48}}\sum_{n_{\frac{1}{2}}=0}^{1}\sum_{n_{\frac{3}{2}}=0}^{1}\dots\left\langle n_{\frac{1}{2}},n_{\frac{3}{2}},\dots\middle|e^{-\pi i F}e^{2\pi i\tau L_0}\middle|n_{\frac{1}{2}},n_{\frac{3}{2}},\dots\right\rangle \\ &= q^{-\frac{1}{48}}\sum_{n_{\frac{1}{2}}=0}^{1}\dots\sum_{l=0}^{\infty}\frac{(2\pi i\tau)^l}{l!}\left(\sum_{k=\frac{1}{2}}^{\infty}\left(n_k k-\frac{n_k}{2\tau}\right)\right)^l\left\langle n_{\frac{1}{2}},\dots\middle|n_{\frac{1}{2}},\dots\right\rangle \\ &= q^{-\frac{1}{48}}\sum_{n_{\frac{1}{2}}=0}^{1}\sum_{n_{\frac{3}{2}}=0}^{1}\dots\prod_{k=\frac{1}{2}}^{\infty}(-1)^{n_k}q^{n_k k} \\ &= q^{-\frac{1}{48}}\prod_{k=0}^{\infty}\left(1-q^{k+\frac{1}{2}}\right)\end{aligned}$$

$$\equiv \sqrt{\frac{\vartheta_4(\tau)}{\eta(\tau)}} , \tag{3.53}$$

where the theta function $\vartheta_4(\tau)$ is defined in Eq.(B.19). Then we obtain the partition function with (NS, R) boundary condition:

$$\mathcal{Z}_{NS,R} = Tr_{\mathcal{F}_{NS}}((-1)^F q^{L_0-\frac{1}{48}}(-1)^{\overline{F}}\overline{q}^{\overline{L}_0-\frac{1}{48}}) = \left|\frac{\vartheta_4(\tau)}{\eta(\tau)}\right| , \tag{3.54}$$

Finally, for the case with periodic boundary condition in both time and space direction, i.e. the (R, R) boundary condition, we bring in the expression for L_0 given in Eq. (3.47) and do similar calculation as above, and get

$$\begin{aligned} Tr_{\mathcal{F}_R}(q^{L_0-\frac{c}{24}}) &= q^{-\frac{1}{48}} \sum_{n_0=0}^{1}\sum_{n_1=0}^{1} \ldots \langle n_0, n_1, \ldots | e^{-\pi i F} e^{2\pi i \tau L_0} | n_0, n_1, \ldots \rangle \\ &= q^{\frac{1}{24}} \prod_{k\geq 0}\left(1-q^k\right) = 0 , \end{aligned} \tag{3.55}$$

from this we reach the following result,

$$\mathcal{Z}_{R,R} = Tr_{\mathcal{F}_R}((-1)^F q^{L_0-\frac{1}{48}}(-1)^{\overline{F}}\overline{q}^{\overline{L}_0-\frac{1}{48}}) = 0 . \tag{3.56}$$

Taking modular invariance and the properties of theta functions as shown in Appendix B.3 into consideration, we see that for the fermionic partition function to be modular invariant, either it satisfies the (R, R) boundary condition, where the partition functino vanishes, or it is a combination of the rest three cases, where the partition function is given by

$$\mathcal{Z} = \mathcal{Z}_{R,NS} + \mathcal{Z}_{NS,NS} + \mathcal{Z}_{NS,R} = \left|\frac{\vartheta_2(\tau)}{\eta(\tau)}\right| + \left|\frac{\vartheta_3(\tau)}{\eta(\tau)}\right| + \left|\frac{\vartheta_4(\tau)}{\eta(\tau)}\right| , \tag{3.57}$$

which means that all three fields must be present in a modular invariant fermionic theory.

Chapter 4

Exploit Conformal Field Theory to Understand Quantum Gravity

Physicists have been attempting to find the connection between quantum field theory and gravitational theory for decades, and the difficulty in renormalizing the gravity theory has been one of the biggest issues. To get rid of the infinities arising in general relativity, string theory quickly drew the attention of both physicists and mathematicians. With the help of extra dimensions and the introduction of supersymmetry, perturbative string theory brought revolutionary insights to our understanding of quantum gravity. Later on, non-perturbative aspects of string theory began to emerge in 1990s. Among these non-perturbative aspects, the AdS/CFT correspondence discovered an unexpected relation between quantum field theory and gravity theory in one higher dimension, and has been a useful toolkit in fields of theoretical physics such as particle and condensed matter physics. In recent developments in the AdS/CFT correspondence, the averaging procedure has drawn more attention. For example, the 2D Jackiw-Teitelboim gravity is the dual of an ensemble average of random quantum mechanical systems [2]. It is natural to ask whether this duality holds in higher dimensions. To answer this question, we will study how 3D AdS gravity emerges from the averaging procedure of an ensemble of bosonic CFTs [4]. In this chapter, we will first give a brief introduction to the AdS/CFT correspondence. Then we will present how gravity emerges from the ensemble averaging procedure.

4.1 The AdS/CFT Correspondence

This section will give a very brief introduction to the AdS/CFT correspondence without going into mathematical details as its main purpose is to provide background knowledge on the formulation of the partition function of a gravitational theory in three dimensions.

The AdS/CFT correspondence, also known as gauge/gravity duality or holography, was proposed by Maldacena in 1997 [1]. It is a duality relating the classical dynamics of gravity to quantum physics of strongly coupled systems in one lower dimension. The original formulations related the geometry of a five-dimensional Anti-de Sitter space (in the bulk) to a four-dimensional CFT (on the boundary) [1, 17, 18]. More precisely, this equivalence between the type IIB string theory on asymtotically $\mathrm{AdS}_5 \times \mathrm{S}^5$ and the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with $SU(N)$ gauge group is indicated by the following relation,

$$\mathcal{Z}_{CFT}[\phi_0] \equiv \left\langle e^{-\int d^4x \phi_0(x)\mathcal{O}(x)} \right\rangle = \mathcal{Z}_{string}[\phi|_\partial = \phi_0] \,, \tag{4.1}$$

where $\mathcal{O}(x)$'s are operators of the field theory and in the gravity theory, source fields $\phi(x, z) = \phi_0(x)$ when evaluated on the boundary of the bulk. We can consider the holographic duality as a geometrization of the quantum dynamics of a system, which is described by the renormalization group flow, and we can identify couplings of the field theory with values of bulk fields in the gravity theory at the boundary of the bulk. In particular, a fixed point of the renormalization group flow, i.e. at which the β-function vanishes, corresponds to a theory with conformal invariance. This feature makes geometrizing CFTs much easier compared to how difficult it is in general to find a geometry associated to a QFT.

Since dual theories share the same Hilbert space and dynamics, the AdS/CFT correspondence is highly useful as we can translate one question to the dual perspective where the problem is, hopefully, less complicated to solve. In order to understand this duality better, we will first see how to match the degrees of freedom of the QFT and the dual gravity theory. As we know, the entropy of a system counts its degrees of freedom. In QFT, the entropy is extensive, thus it is proportional to the spatial volume. For example, in a d-dimensional spacetime with one temporal dimension, the QFT entropy satisfies the following relation,

$$S_{QFT} \propto vol\left(R^{d-1}\right) . \tag{4.2}$$

On the other hand, in a gravity theory with a $d+1$-dimensional spacetime, we know that the entropy of a system within a given size has an upper bound which is the entropy of a black hole fitting in this volume. The Bekenstein-Hawking formula tells us that this black hole entropy is proportional to its surface area,

$$S_{GR} \propto area\left(\partial R^{d}\right) = vol\left(R^{d-1}\right) , \tag{4.3}$$

which is exactly the entropy in the context of quantum field theory. This relation is why entropy also plays a crucial role in the study of the holographic duality.

In fact, there are three classical solutions that satisfy the relation in Eq.(4.1): small black holes, large black holes, and thermal AdS. The transition between the black hole phase and the thermal AdS phase is known as the Hawking-Page phase transition. Therefore, our approach above which relates entropy in QFT and that in general relativity is straightforward but actually incomplete. Below the Hawking-Page phase transition, where thermal AdS is the solution, the entropy of the gauge theory entropy is not captured by the black hole entropy but the thermal entropy. We will not go into details as it is not the focus of this thesis.

Now let us take a brief look at a more specific example. In the AdS_3/CFT_2 correspondence, for the case where the conformal boundary is a genus one surface, it is shown that the bulk partition function should be expressed as a sum over handlebodies [3]. These handlebodies can be obtained from decomposing the genus one surface into $S^1 \times S^1$ and filling in the first S^1 to make it a two-dimensional disk D^2. Different decompositions generate different handlebodies, and they are related by some modular transformations. In fact, given any handlebody obtained from the above decomposing-filling procedure, all of the rest handlebodies with the same boundary can be obtained by performing a modular transformation on the boundary. Then we can label each handlebody with an element of the modular group $SL(2, \mathbb{Z})$. In addition, we know that elements in the group

$$P = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\} \subseteq SL(2, \mathbb{Z}) \tag{4.4}$$

leave the handlebody invariant, so after removing this subgroup from $SL(2, \mathbb{Z})$, we are left with modular transformations that generate only unique handlebodies. In other words, all inequivalent handlebodies can be obtained by performing a modular transformation $\gamma \in SL(2, \mathbb{Z})/P$ on one handlebody $D^2 \times S^1$. In the next section, we will see how this expression for the partition function of quantum gravity in three-dimensions is related to the partition functions of two-dimensional CFTs. Note that for higher genus cases, non-handlebodies also contribute to the sum for the full partition function expression, which we will not consider in this thesis.

The AdS/CFT correspondence has been a success since it was discovered. First of all, it reveals the relationship between quantum field theory and string theory. Also, it provides an almost background independent representation of quantum gravity. We need to make a note here that this independence of the background metric holds almost everywhere except on the asymptotic boundary. That is to say that the definition of such a theory only requires the asymptotic boundary conditions to be fixed but the interior needs not. Even though a full non-perturbative definition of quantum gravity has not yet been found, the holographic duality serves a great method. When we study quantum gravity on an arbitrary curved space with AdS asymptotics using the AdS/CFT correspondence, non-perturbative results are obtained because one does not need to decompose a curved target space metric into a background (flat) metric and fluctuations, while the ordinary definition of string theory requires such decomposition.

In addition, The AdS/CFT correspondence brings new insights into quantum field theory, such as in integrability aspects, scattering amplitudes, correlation functions, etc. It also serves as an important step towards understanding a quantum version of gravitational theory. For example, through this relation we can identify microstates of certain black holes to pure states in a dual CFT.

Although it was originally discovered in the context of string theory, the AdS/CFT correspondence has been extended to very broad range of fields in physics. Not only does it have applications in the physics of black holes and quantum gravity, but it also helps to develop research in quantum chromodynamics, and in condensed matter physics such as superconductivity and quantum phase transition, etc. For reviews on different applications of the holographic duality, please see [19–21].

4.2 Gravity as an Ensemble Average

In this section, we will first determine the Narain moduli space exploiting the modularity properties of the CFT partition functions, and derive the averaged partition function of free bosonic CFTs over the Narain moduli space focusing on genus one. Finally, we will discuss the interpretation of this averaged CFT in the context of a gravitational theory.

4.2.1 The Narain Moduli Space for Bosonic CFT

In 1986, Narain gave a formulation of toroidal compactifications of heterotic string theory [22, 23], which also applies to CFTs of compactified bosons. We will first give an introduction to moduli spaces for CFTs, then we will present the derivation of the Narain moduli space for bosonic CFTs.

Let us consider the simplest case: a single boson compactified on a circle with a radius R as we discussed in Chapter 2, where different radius $R \in (0, \infty)$ gives different theories. For each theory, i.e. for a theory with a certain radius R, states of the boson are labelled by momentum number m's and winding numbers n's. In the lattice language in the context of abstract algebra, all states of a theory form a lattice, where lattice elements represent different states of the theory, and each is labelled by m and n. Then we can write the partition function in terms of m's and n's as a sum over all elements in the lattice just like the expression of the partition function of a boson compactified on a circle with radius R shown in Eq.(3.22). Now consider the set of all such compactified free bosonic theories with different parameter $R \in (0, \infty)$, these theories form a moduli space.

Consider a sigma-model of D free bosons with a target space of a D-dimensional torus with fields X^p where $p \in \{1, \dots, D\}$ as the coordinates, and a symmetric target space metric G_{pq} and an anti-symmetric tensor B_{pq}. On a Euclidean worldsheet with flat metric, the action of the theory is given by

$$S = \frac{1}{2\pi\alpha'} \int dz d\overline{z} \left(G_{pq} \partial X^p \overline{\partial} X^q + B_{pq} dX^p \wedge dX^q \right) . \tag{4.5}$$

Now let us construct the Narain moduli space $\mathcal{M}_D$ for D free bosons. Consider vertex operators $e^{ik\cdot X + i\overline{k}\cdot\overline{X}}$ as we discussed in Section 5.1. The spectrum of momenta $(k, \overline{k})$ form a lattice in a 2D-dimensional momentum space $\mathbb{R}^{D,D}$, and denote this lattice as $\mathbb{L}$. Note that in this case, the chiral and anti-chiral central charges are equal, i.e. $c = \overline{c} = D$. The OPE of two vertex operators id given by

$$: e^{ik\cdot X(z) + i\overline{k}\cdot\overline{X}(\overline{z})} :: e^{ik'\cdot X(0) + i\overline{k}'\cdot\overline{X}(0)} : \sim z^{k\cdot k'} \overline{z}^{\overline{k}\cdot\overline{k}'} : e^{i(k+k')\cdot X(0) + i(\overline{k}+\overline{k}')\cdot\overline{X}(0)} : . \tag{4.6}$$

For a boson compactified on a circle, the vertex operator is required to satisfy that the product picks up a phase of $e^{2\pi i (k\cdot k' - \overline{k}\cdot\overline{k}')} = 1$ when one vertex operator circles the other. Thus, all elements $\mathbf{k}, \mathbf{k}' \in \mathbb{L}$ must meet the following requirement,

$$\langle \mathbf{k}, \mathbf{k}' \rangle \equiv k \cdot k' - \overline{k} \cdot \overline{k}' \in \mathbb{Z} . \tag{4.7}$$

Recall the definition of the dual of a lattice. Let $\mathbb{L}$ be a lattice, its dual lattice $\mathbb{L}^*$ defined as

$$\mathbb{L}^* \equiv \{ \, \mathbf{v} \in \text{span}(\mathbb{L}) \mid \forall \, \mathbf{w} \in \mathbb{L} \, , f(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle \in \mathbb{Z} \, \} . \tag{4.8}$$

We see that the condition Eq.(4.7) is equivalent to say that $\mathbb{L} \subset \mathbb{L}^*$.

We have also shown in Section 3.3 that the partition function of the theory is modular invariant under T and S transformations. Let us investigate what constraints modular invariance imposes on the lattice $\mathbb{L}$. Under T transformation $\tau \to \tau + 1$, the invariance requires the difference between chiral and anti-chiral 0-th Laurent mode to be an integer, i.e. $L_0 - \overline{L}_0 \in \mathbb{Z}$. That is equivalent to say that $\forall\, \mathbf{k} \in \mathbb{L}$,

$$\langle \mathbf{k}, \mathbf{k} \rangle = f(\mathbf{k}) \in 2\mathbb{Z}\,, \tag{4.9}$$

which is exactly the definition of even lattices $\mathbb{L}$.Under S transformation $\tau \to -1/\tau$, extend Eq.(3.18) the partition function for D compactified bosons is given by

$$\mathcal{Z}_{D,\mathbb{L}}(\tau) = \frac{1}{|\eta(\tau)|^{2D}} \sum_{\mathbf{k} \in \mathbb{L}} e^{\pi i \tau k^2 - \pi i \overline{\tau}\overline{k}^2} \tag{4.10}$$

We apply the Poisson resummation formula as we did proving modular invariance in Section 3.3,

$$\sum_{\mathbf{k}' \in \mathbb{L}} \delta(\mathbf{k}' - \mathbf{k}) = \frac{1}{V_{\mathbb{L}}} \sum_{\mathbf{k}'' \in \mathbb{L}^*} e^{2\pi i \mathbf{k}'' \cdot \mathbf{k}}\,, \tag{4.11}$$

where $V_{\mathbb{L}}$ is the volume of a unit cell of lattice $\mathbb{L}$. Then we obtain the following

$$\begin{aligned}
\mathcal{Z}_{D,\mathbb{L}^*}\left(-\frac{1}{\tau}\right) &= \frac{1}{|\sqrt{-i\tau}\eta(\tau)|^{2D}} \sum_{\mathbf{k}'' \in \mathbb{L}^*} e^{-\pi i k''^2/\tau + \pi i \overline{k}''^2/\overline{\tau}} \\
&= \frac{1}{|\sqrt{-i\tau}\eta(\tau)|^{2D}} \sum_{\mathbf{k}'' \in \mathbb{L}^*} \int d^{2D}\mathbf{k} d^{2D}\mathbf{k}''\; e^{2\pi i \mathbf{k}'' \cdot \mathbf{k}} e^{-\pi i \mathbf{k}''^2/\tau + \pi i \overline{\mathbf{k}}''^2/\overline{\tau}} \\
&= \frac{1}{|\sqrt{-i\tau}\eta(\tau)|^{2D}} (-i\tau)^{D/2} (i\overline{\tau})^{D/2} \sum_{\mathbf{k}'' \in \mathbb{L}^*} \int d^{2D}\mathbf{k}\; e^{2\pi i \mathbf{k}'' \cdot \mathbf{k}} e^{\pi i \tau k^2 - \pi i \overline{\tau}\overline{k}^2} \\
&= \frac{1}{|\eta(\tau)|^{2D}} \sum_{\mathbf{k}'' \in \mathbb{L}^*} \int d^{2D}\mathbf{k}\; e^{2\pi i \mathbf{k}'' \cdot \mathbf{k}} e^{\pi i \tau k^2 - \pi i \overline{\tau}\overline{k}^2} \\
&= \frac{V_{\mathbb{L}}}{|\eta(\tau)|^{2D}} \sum_{\mathbf{k}' \in \mathbb{L}} \int d^{2D}\mathbf{k}\; \delta(\mathbf{k}' - \mathbf{k}) e^{\pi i \tau k^2 - \pi i \overline{\tau}\overline{k}^2} \\
&= \frac{V_{\mathbb{L}}}{|\eta(\tau)|^{2D}} \sum_{\mathbf{k}' \in \mathbb{L}} e^{\pi i \tau k'^2 - \pi i \overline{\tau}\overline{k}'^2} \\
&= V_{\mathbb{L}} \mathcal{Z}_{D,\mathbb{L}}(\tau)\,.
\end{aligned} \tag{4.12}$$

By modular invariance, we get that the unit cell has volume $V_{\mathbb{L}} = 1$, which means the determinant of the Gram matrix of the lattice is 1. This is exactly the definition of a unimodular (self-dual) lattice. Therefore, we conclude that the lattice $\mathbb{L}$ has to be self-dual,

$$\mathbb{L} = \mathbb{L}^* \,. \tag{4.13}$$

Therefore, the modular invariance of the partition function of compactified bosons requires the Narain lattice $\mathbb{L}$ defining a compactified bosonic theory with D bosons to be an even self-dual (unimodular) lattice in $\mathbb{R}^{D,D}$, represented by Eq.(4.9) and Eq.(4.13).

We notice that if $\mathbb{L}$ is an even self-dual lattice, then when a $O(D, D, \mathbb{R})$ transformation $\mathbf{\Lambda}$ acts on the lattice, the new lattice

$$\mathbb{L}' = \mathbf{\Lambda}\mathbb{L} \tag{4.14}$$

remains even and self-dual. That is to say that the evenness and self-duality conditions are invariant under Lorentz boosts of the $2D$-dimensional space, $O(D, D, \mathbb{R})$ transformations. In fact, all even self-dual lattices of a given Lorentzian signature can be obtained by acting $O(D, D, \mathbb{R})$ transformations on any single lattice, and most of these transformations produce inequivalent theories.

Also, a transformation $O(D, \mathbb{R}) \times O(D, \mathbb{R})$ acting on $k, \overline{k}$ separately leaves the mass-shell condition and operator products involving $k \cdot k'$ and $\overline{k} \cdot \overline{k}'$ invariant. In other words, redefining a vector $\mathbf{k} \in \mathbb{L}$ as $\mathbf{k}' = \mathbf{U}\mathbf{k}$ with an element $\mathbf{U} \in O(D, \mathbb{R}) \times O(D, \mathbb{R})$ leaves the partition function of compactified bosons unchanged, thus these transformations produce equivalent theories. Therefore, the moduli space is reduced to $O(D, D, \mathbb{R})/O(D, \mathbb{R}) \times O(D, \mathbb{R})$.

In addition, there is some subgroup of $O(D, D, \mathbb{R})$ that permutes elements of a lattice in the above moduli space but leaves the lattice itself unchanged. This subgroup is $O(D, D, \mathbb{Z})$, which corresponds to the over-counting due to the T-duality symmetry $R \to \alpha'/R$.

Thus, we conclude that the moduli space of inequivalent compactified bosonic theories is given by removing subgroups $O(D, \mathbb{R}) \times O(D, \mathbb{R})$ and $O(D, D, \mathbb{Z})$ from the space obtained by acting $O(D, D, \mathbb{R})$ transformations on a single lattice, written as

$$\mathcal{M}_D = O(D, D, \mathbb{Z}) \backslash O(D, D, \mathbb{R}) / O(D, \mathbb{R}) \times O(D, \mathbb{R}) \,. \tag{4.15}$$

When one performs an averaging procedure, it is important to define a measure of the moduli space that is being considered. The *natural metric* on a moduli space is defined by the kinetic term of the bosonic theory. For the Narain moduli space, the natural measure is the Zamolodchikov metric

$$ds^2 = G^{mn} G^{pq} (dG_{mp} dG_{nq} + dB_{mp} dB_{nq}) \ , \tag{4.16}$$

where a detailed derivation can be found in [24].

4.2.2 An Averaged Partition Function over the Narain Moduli Space

For each CFT in the moduli space $\mathcal{M}_D$, its partition function can be expressed as a sum of Siegel-Narain theta functions over lattice points as we will shown in later this section. To compute the average of the Siegel-Narain theta function over $\mathcal{M}_D$, we will apply the Siegel-Weil formula [25–27]. In number theory, the Siegel-Weil formula is an identity between an integral of a theta function and an Eisenstein series, it allows us to express the above average in terms of a real analytical Eisenstein series as we will discuss in the following.

In this subsection, we will derive the averaged partition function of free bosonic CFTs over the Narain moduli space focusing on genus one. First, let us consider the case $D = 1$ in Eq.(4.5), i.e. a free bosonic CFT on a moduli space $\mathcal{M}_1$ parametrized by the radius of a 1-torus $R \in (0, \infty)$, this is just a circle of circumference $2\pi R$. Due to the symmetry $R \to \alpha'/R$, we only need the range of $R \in [\sqrt{\alpha'}, \infty)$. The reason why we discuss the $D = 1$ case separately is that this case is special compared to its $D > 1$ friends as we will see shortly. The action is given in the following

$$S = \frac{R^2}{2\pi\alpha'} \int dz d\bar{z} \, \partial X \bar{\partial} X \ , \tag{4.17}$$

and the Zamolodchikov metric and the measure of the moduli space $\mathcal{M}_1$ are

$$ds^2 = 4 \frac{dR^2}{R^2} \ , \qquad \mu(R) = 2 \frac{dR}{R} \ . \tag{4.18}$$

For arbitrary curvilinear coordinates in N-dim $(v^1, \ldots, v^N)$ with metric tensor $g_{\mu\nu}$, the Laplace operator can be expressed as

$$\Delta = \frac{1}{\sqrt{det\ g}} \frac{\partial}{\partial v^\mu} \left(\sqrt{det\ g} g^{\mu\nu} \frac{\partial}{\partial v^\nu} \right) . \tag{4.19}$$

Recall that the natural metric of the upper half plane $\mathcal{H}$ which is given by

$$ds^2 = \frac{d\tau_1^2 + d\tau_2^2}{\tau_2^2} , \tag{4.20}$$

Thus the Laplacian of the moduli space $\mathcal{M}_1$ and the Laplacian on the upper half plane $\mathcal{H}$ are given by

$$\Delta_{\mathcal{M}_1} = \frac{1}{4} \left(R \frac{\partial}{\partial R} \right)^2 , \qquad \Delta_{\mathcal{H}} = \tau_2^2 \left(\frac{\partial^2}{\partial \tau_1^2} + \frac{\partial^2}{\partial \tau_2^2} \right) . \tag{4.21}$$

For a free boson on a circle of radius R, with momentum number m and winding number n, its partition function is given by Eq.(3.22). Define

$$\begin{aligned} \Theta(R, \tau) &= \sum_{m,n \in \mathbb{Z}} Q(m, n; R, \tau) , \\ Q(m, n; R, \tau) &= exp \left\{ -\pi \tau_2 \left(\frac{\alpha' m^2}{R^2} + \frac{n^2 R^2}{\alpha'} \right) + 2\pi i \tau_1 m n \right\} , \end{aligned} \tag{4.22}$$

where $\Theta(R, \tau)$ is the Siegel-Narain theta function for $D = 1$. Then we can write the partition function Eq.(3.22) as

$$\mathcal{Z}_{boc}(R, \tau) = \frac{\Theta(R, \tau)}{|\eta(\tau)|^2} . \tag{4.23}$$

It is easy to verify that $Q(m, n; R, \tau)$ with modular parameter $\tau = \tau_1 + i\tau_2$ satisfies the following differential equation,

$$\left(\tau_2^2 \left(\frac{\partial^2}{\partial \tau_1^2} + \frac{\partial^2}{\partial \tau_2^2} \right) + \tau_2 \frac{\partial}{\partial \tau_2} - \frac{1}{4} \left(R \frac{\partial}{\partial R} \right)^2 \right) Q(m, n; R, \tau) = 0 . \tag{4.24}$$

Then trivially we get the following differential equation for $\Theta(R, \tau)$, which can be expressed as the following equation,

$$\left(\Delta_{\mathcal{H}} + \tau_2 \frac{\partial}{\partial \tau_2} - \Delta_{\mathcal{M}_1}\right)\Theta(R, \tau) = 0 \, . \tag{4.25}$$

Before we solve the differential equation to find a solution to $\Theta(R, \tau)$, let us make a few observations first. Define a function $F_1(\tau)$ over $\mathcal{M}_1$ as

$$F_1(\tau) = 2\int_{\sqrt{\alpha'}}^{\infty} \frac{dR}{R}\, \Theta(R, \tau) \, . \tag{4.26}$$

Then by Eq.(4.25) we have

$$\left(\Delta_{\mathcal{H}} + \tau_2 \frac{\partial}{\partial \tau_2}\right) F_1(\tau) = \Delta_{\mathcal{M}_1} F_1(\tau) = \frac{R}{2}\frac{\partial \Theta(R, \tau)}{\partial R}\bigg|_{R=\sqrt{\alpha'}}^{R=\infty} . \tag{4.27}$$

We notice that the symmetry $R \to \alpha'/R$ gives $\Theta(R) = \Theta(\alpha'/R)$, i.e. $\Theta(R)$ is symmetric about $R = \sqrt{\alpha'}$. Then, $\partial_R \Theta|_{R=\sqrt{\alpha'}} = 0$. Therefore, the value of the right hand side of the above equation goes to 0 if the term also vanishes at $R = \infty$.

However, in the $D = 1$ case, the above equation does not vanish for $R \to \infty$. Thus, let us move on to cases with $D > 1$.

For D free bosons, we will parametrize the moduli space $\mathcal{M}_D$ with a symmetric constant metric G_{pq} and an anti-symmetric 2-form B_{pq}, and also denote moduli as m. For $D > 1$, the measure $\mu(m)$ of the moduli space can be normalized since when $D > 1$ the volume of the moduli space is finite, i.e. $vol(\mathcal{M}_D) < \infty$ (a more detailed explanation can be found in Appendix A.19). Then the partition function for D free bosons can be expressed as

$$Z_D(m, \tau) = \frac{\Theta(m, \tau)}{|\eta(\tau)|^{2D}} \, , \tag{4.28}$$

where the Siegel-Narain theta function $\Theta(m, \tau)$ is a sum over integer-valued momenta m_p and winding numbers n^q,

$$\Theta(m, \tau) = \sum_{m_p, n^p \in \mathbb{Z}^D} exp\left\{ -\frac{\pi \tau_2}{\alpha'}\left(G^{pq} v_p v_q + G_{pq} n^p n^q\right) + 2\pi i \tau_1 m_p n^q \right\} , \tag{4.29}$$

where $v_p = \alpha' m_p + B_{pq} n^q$. From the above expression we can see that $\lim_{\tau_2 \to \infty} \Theta(m, \tau) = 1$. In addition, the Siegel-Narain theta function $\Theta(m, \tau)$ obeys

$$\left(\Delta_{\mathcal{H}} + D\tau_2 \frac{\partial}{\partial \tau_2} - \Delta_{\mathcal{M}_D} \right) \Theta(m, \tau) = 0 \,, \tag{4.30}$$

where the Laplacian on the Narain moduli space $\mathcal{M}_D$ deduced from the metric Eq.(4.16) is given by

$$\begin{aligned} \Delta_{\mathcal{M}_D} &= G_{mn} G_{pq} \left(\widehat{\partial}_{G_{mp}} \widehat{\partial}_{G_{nq}} + \frac{1}{4} \partial_{B_{mp}} \partial_{B_{nq}} \right) + G_{mn} \widehat{\partial}_{G_{mn}} \,, \\ \text{where} \quad \widehat{\partial}_{G_{mn}} &= \frac{1}{2}(1 + \delta_{mn}) \partial_{G_{mn}} \,. \end{aligned} \tag{4.31}$$

Now we will average $\Theta(m, \tau)$ over $\mathcal{M}_D$. Define

$$F_D(\tau) = \int_{\mathcal{M}_D} d\mu(m) \; \Theta(m, \tau) \,. \tag{4.32}$$

Then, the following differential equation holds

$$\left(\Delta_{\mathcal{H}} + D\tau_2 \frac{\partial}{\partial \tau_2} \right) F_D(\tau) = \Delta_{\mathcal{M}_D} F_D(\tau) = 0 \,. \tag{4.33}$$

Note that even though the volume of the moduli space $vol(\mathcal{M}_D) < \infty$ for $D > 1$, $F_D(\tau)$ converges only for $D > 2$ due to the behaviour of $\Theta(m, \tau)$ at infinity.

Also, as we have seen above, since $\lim_{\tau_2 \to \infty} \Theta(m, \tau) = 1$, we have $\lim_{\tau_2 \to \infty} F_D(\tau) = 1$.

In addition, since the partition function is modular invariant, the modular invariance property of $F_D(\tau) |\eta(\tau)|^{-2D}$ gives that $F_D(\tau)$ has weights $(D/2, D/2)$ under modular transformations, since we have $f(\tau) = |\eta(\tau)|^2$ has modular weights $(1/2, 1/2)$.

Given that the function $f(\tau) = \tau_2$ has modular weights $(-1, -1)$, let us define the following modular invariant function,

$$W_D(\tau) \equiv \tau_2^{D/2} F_D(\tau) \,, \tag{4.34}$$

then it is trivial that $\lim_{\tau_2 \to \infty} W_D(\tau) = \tau_2^{D/2}$.

Also, we see that $W_D(\tau)$ is an eigenfunction of the upper half plane Laplacian $\Delta_{\mathcal{H}}$ with the eigenvalue of $D/2(D/2-1)$ as shown in Appendix A.20. Therefore, the function $W_D(\tau)$ for $D > 2$ satisfies the following conditions:

1. It is modular-invariant;
2. It behaves as the function $\tau_2^{D/2}$ as $\tau_2 \to \infty$;
3. It is an eigenfunction of $\Delta_{\mathcal{H}}$ with eigenvalue $D/2(D/2-1)$.

As it turns out, the non-holomorphic Eisenstein series of weight $D/2$ satisfies the requirements above,

$$E_{D/2}(\tau) = \sum_{\substack{(c,d)=1 \\ c,d \in \mathbb{Z}}} \frac{\tau_2^{D/2}}{|c\tau + d|^D} \,, \tag{4.35}$$

where the sum is over $c, d \in \mathbb{Z}$ which are coprimes up to a sign.

Therefore, the average of genus 1 partition function over the Narain moduli space $\mathcal{M}_D$ is found to be

$$\langle Z_D(m,\tau) \rangle = \frac{E_{D/2}(\tau)}{\tau_2^{D/2} |\eta(\tau)|^{2D}} \,. \tag{4.36}$$

4.2.3 Averaged Bosonic CFT and the Chern-Simons Theory

The D free boson CFT is highly symmetric with $U(1)^{2D}$ global symmetry, which should correspond to a bulk gravitational theory with $U(1)^{2D}$ gauge symmetry according to the AdS/CFT correspondence. Three-dimensional general relativity is a topological theory that can be written as the Chern-Simons theory at the level of perturbation, so let us consider the $U(1)^{2D}$ Chern-Simons theory with fields A^k, B^k and the following action,

$$S_{CS} = \sum_{k=1}^{D} \frac{1}{2\pi} \int_M A^k \wedge dB^k \,, \tag{4.37}$$

then we can compute the perturbative partition function of one handlebody $M = D^2 \times S^1$ by computing the trace $Tr \exp(-\beta H)$ in the Hilbert space of states obtained from quantizing the disk D^2 [3], then this partition function is given by

$$\mathcal{Z}_{CS,M}(\tau) = \frac{1}{|\eta(\tau)|^{2D}} \,. \tag{4.38}$$

For the reasons we explained at the end of section 4.1, to find the full partition function of the theory, we need to sum the above partition functions Eq.(4.38) over all inequivalent handlebodies obtained from modular transform M, we get the following expression of the full partition function

$$\begin{aligned} &\mathcal{Z}_{CS}(\tau) = \sum_{\gamma \in SL(2,\mathbb{Z})/P} \frac{1}{|\eta(\gamma\tau)|^{2D}}, \\ &\text{with } \ P = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\} \subseteq SL(2,\mathbb{Z}) \ \text{ and } \ P \cong \mathbb{Z} \,. \end{aligned} \tag{4.39}$$

The set of matrices P is removed since elements in P leave $\Im(\tau)$ invariant.

As we have discussed in previous subsection, because $|\eta(\tau)|^2$ and $\Im(\tau)$ have modular weight $(1/2, 1/2)$ and $(-1,-1)$, respectively, the function $f(\tau) = \Im(\tau)^{1/2}|\eta(\tau)|^2$ is modular invariant, i.e. $f(\gamma\tau) = f(\tau)$. Thus we can decompose the full partition function Eq.(4.39) into two parts with one of which being the modular invariant function $f(\tau)^D$,

$$\mathcal{Z}_{CS}(\tau) = \sum_{\gamma \in SL(2,\mathbb{Z})/P} \frac{\Im(\gamma\tau)^{D/2}}{\Im(\gamma\tau)^{D/2}|\eta(\gamma\tau)|^{2D}} = \sum_{\gamma \in SL(2,\mathbb{Z})/P} \frac{\Im(\gamma\tau)^{D/2}}{\Im(\tau)^{D/2}|\eta(\tau)|^{2D}} \,, \tag{4.40}$$

In fact, the non-holomorphic Eisenstein series $E_{D/2}(\tau)$ we introduced in the previous subsection can also be written as the sum over all modular images of the function $\Im(\tau)^{D/2}$ as expressed in the following equation,

$$E_{D/2}(\tau) = \sum_{\gamma \in SL(2,\mathbb{Z})/P} \Im(\gamma\tau)^{D/2} \quad , \tag{4.41}$$

therefore, the full partition function of the $U(1)^{2D}$ Chern-Simons theory can be written as

$$\mathcal{Z}_{CS}(\tau) = \frac{E_{D/2}(\tau)}{\tau_2^{D/2}|\eta(\tau)|^{2D}} = \langle \mathcal{Z}_D(m,\tau) \rangle \ , \tag{4.42}$$

which is exactly the averaged partition function of bosonic CFTs with D free bosons over the Narain moduli space in Eq.(4.36). Thus we conclude that an averaged bosonic CFT indeed has a gravitational interpretation.

We will not go into details for higher genus cases, but similar result holds except that the corresponding Eisenstein series diverges when the number of genus g is larger than $D-1$. In other words, the Siegel-Weil formula fails when $g > D-1$ as explained in [4].

Before we move on to the next chapter, we would like to make a comment here about the traditional AdS/CFT correspondence. Consider the original case in the traditional holographic paradigm, individual $\mathcal{N} = 4$ super Yang-Mills (SYM) theories are dual to type IIB string theory on asymtotically $\mathrm{AdS}_5 \times \mathrm{S}^5$. In fact, the $\mathrm{AdS}_5 \times \mathrm{S}^5$ supergravity is also the classical limit of ensemble averaged type IIB string theory. These two dualities hold simoutaneously as discussed in [6].

Chapter 5

Relations between the Bosonic and the Fermionic Theories

In the previous chapter, we presented how gravity emerges from the averaging procedure over the Narain moduli space for an ensemble of free bosonic CFTs using knowledge of CFT reviewed in chapter 2. Since all particles are either bosons or fermions, it leads to the following question: does an ensemble of fermionic CFTs give rise to some gravitational theory when it is averaged over some moduli space? Furthermore, what about an ensemble of supersymmetric CFTs? To investigate the first question, a key step is to determine the moduli space of fermionic CFTs. Due to the Grassmannian nature of fermions, fermionic CFTs are more complicated to study. One way to circumvent this difficulty is to exploit an equivalence between bosonic and fermionic theories known as bosonization. Inspired by the two-dimensional quantum field theory interpretation of a mathematical identity, the equivalence between free bosonic and fermionic theories was extended to an interacting fermionic theory by Coleman [10] and Mandelstam [28] in 1975. Besides in particle physics where applications of bosonization to string theory have been explored, bosonization has also been studied in parallel in the context of condensed matter physics, which we will not focus in this thesis.

In this chapter, we will discuss bosonization for complex fermions in the context of conformal field theory, where the building blocks are the exponential of bosonic fields known as vertex operators. Then, we will give an introduction to the theory of a massless fermionic field with a self-interaction which is called the Thirring model [8], and we will show how it is related to bosonic theories. In the end of this chapter, we will discuss how the couplings in $c = 1$ bosonic CFT and $c = 1$ fermionic CFT are related to each other [29], then build a map between bosonic and fermionic moduli spaces.

5.1 Vertex Operators and Bosonization

Before we study bosonization for complex fermions, let us first go back to a free bosonic CFT and present the concept of vertex operators, which were first introduced by Fubini and Veneziano [30]. The vertex operator is a primary field of conformal dimension $(h, \overline{h}) = (\alpha^2/2, \alpha^2/2)$ and it is defined as $V_\alpha(z, \overline{z}) \equiv: e^{i\alpha X(z,\overline{z})} :$, where $: \cdots :$ denotes the normal ordering. Such a construction is possible since a bosonic field $X(z, \overline{z})$ has vanishing conformal dimensions as we found before. One can easily verify the conformal dimension of the vertex operator by bringing Eq.(3.11) into the $V_\alpha(z, \overline{z})$ and computing the commutator $[L_0, V_\alpha(z, \overline{z})]$ of the 0-th Laurent mode and the vertex operator, and we will skip the verification here. Also, by evaluating the commutator $[j_0, V_\alpha(z, \overline{z})]$ we obtain that the eigenvalue of j_0 for the vertex operator is given by α.

For a free boson compactified on a circle of radius R, we require the vertex operator to satisify $V_\alpha =: e^{i\alpha X} :=: e^{i\alpha(X+2\pi Rn)} := V_\alpha e^{2\pi i\alpha Rn}$, thus we have

$$\alpha = \frac{m}{R} \qquad \text{with } m \in \mathbb{Z}\,. \tag{5.1}$$

Now consider a special case where $\alpha = \pm 1$, then the vertex operator $V_{\pm 1}(z, \overline{z})$ becomes a primary field of conformal dimension $(h, \overline{h}) = (1/2, 1/2)$. Focusing on the chiral part of the vertex operator, we will write the current as $j^\pm(z) =: e^{\pm iX(z)} :$ and determine the current algebra of $j^\pm(z)$. Recall that for quasi-primary fields, the current algebra is determined as $[j^\pm_m, j^\pm_n] = 0$ and $[j^+_m, j^-_n] = \delta_{m+n,0}$ using Eq.(2.48) where C^k_{ij} can be found by the three-point function Eq.(2.45) and $p_{ijk}(m, n)$ can be found by Eq.(2.47). Then we combine these commutation relations with the current algebra for $j(z) = i\partial X(z)$, and get

$$
\begin{aligned}
&[j_m, j_n] = m\delta_{m+n,0} \ , && [L_m, j_n] = -nj_{m+n} \ , \\
&[j_m, j_n^+] = +j_{m+n}^+ \ , && [j_m, j_n^-] = -j_{m+n}^- \ , \\
&[j_m^+, j_n^-] = \delta_{m+n,0} \ , && [j_m^+, j_n^+] = [j_m^-, j_n^-] = 0 \ .
\end{aligned} \tag{5.2}
$$

On the other hand, let us consider a system with a complex chiral fermion $\Psi(z)$ and its conjugate $\Psi^\dagger(z)$ where

$$
\begin{aligned}
\Psi(z) &= \frac{1}{\sqrt{2}}\Big(\psi^{(1)}(z) + i\psi^{(2)}(z)\Big) \ , \\
\Psi^\dagger(z) &= \frac{1}{\sqrt{2}}\Big(\psi^{(1)}(z) - i\psi^{(2)}(z)\Big) \ ,
\end{aligned} \tag{5.3}
$$

with two real chiral fermions $\psi^{(1)}(z)$ and $\psi^{(2)}(z)$. Laurent expand $\Psi(z)$ and $\Psi^\dagger(z)$ as we did for real chiral fermions, we get $\Psi(z) = \sum_r \Psi_r z^{-r-\frac{1}{2}}$ and $\Psi^\dagger(z) = \sum_r \Psi_r^\dagger z^{-r-\frac{1}{2}}$. The Laurent modes satisify the following relations,

$$
\begin{aligned}
&\{\Psi_r, \Psi_s\} = \{\Psi_r^\dagger, \Psi_s^\dagger\} = 0 \ , \\
&\{\Psi_r, \Psi_s^\dagger\} = \delta_{r+s,0} \ ,
\end{aligned} \tag{5.4}
$$

and one can verify the above relations using $\{\psi_r^{(i)}, \psi_s^{(j)}\} = \delta^{ij}\delta_{r+s,0}$, which is a more general anti-commutation relation than Eq.(3.35). Since $\Psi(z)$ and $\Psi^\dagger(z)$ are chiral fields of conformal dimension $h = 1/2$, we can construct a chiral field of conformal dimension $h = 1$ in the following way,

$$
j(z) = -N(\Psi\Psi^\dagger)(z) = iN(\psi^{(1)}\psi^{(2)})(z) \ . \tag{5.5}
$$

Then when we Laurent expand both sides of the equation we have the following relation for the Laurent modes,

$$
j_n = iN(\psi^{(1)}\psi^{(2)})_n = -i \sum_{k \in \mathbb{Z}+\frac{1}{2}} \psi_{n-k}^{(1)}\psi_k^{(2)} \ , \tag{5.6}
$$

where a detailed calculation can be found in Appendix A.18. Using the above expression and Eq.(3.40), we find that

$$\begin{aligned}
[L_m, j_n] &= \sum_{k\in\mathbb{Z}+\frac{1}{2}} [L_m^{(1)} + L_m^{(2)}, -i\psi_{n-k}^{(1)}\psi_k^{(2)}] \\
&= -i \sum_{k\in\mathbb{Z}+\frac{1}{2}} \Big([L_m^{(1)}, \psi_{n-k}^{(1)}]\psi_k^{(2)} + \psi_{n-k}^{(1)}[L_m^{(2)}, \psi_k^{(2)}] \Big) \\
&= -i \sum_{k\in\mathbb{Z}+\frac{1}{2}} \Big(\Big(-\frac{m}{2} - n + k\Big)\psi_{m+n-k}^{(1)}\psi_k^{(2)} + \Big(-\frac{m}{2} - k\Big)\psi_{n-k}^{(1)}\psi_{m+k}^{(2)} \Big) \\
&= -i \sum_{k\in\mathbb{Z}+\frac{1}{2}} \Big(\Big(-\frac{m}{2} - n + k\Big)\psi_{m+n-k}^{(1)}\psi_k^{(2)} + \Big(-\frac{m}{2} - k + m\Big)\psi_{m+n-k}^{(1)}\psi_k^{(2)} \Big) \\
&= ni \sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_{m+n-k}^{(1)}\psi_k^{(2)} \\
&= -nj_{m+n} \,.
\end{aligned} \tag{5.7}$$

It implies that $j(z)$ is a primary field of conformal dimension $h = 1$ by comparing $[L_m, j_n] = -nj_{m+n}$ to Eq.(2.43), and therefore we conclude that $j(z)$ is a current. Now we will determine its current algebra by computing the commutator $[j_m, j_n]$ using Eq.(5.6).

$$\begin{aligned}
[j_m, j_n] &= -\sum_{r,s\in\mathbb{Z}+\frac{1}{2}} [\psi_{m-s}^{(1)}\psi_s^{(2)}, \psi_{n-r}^{(1)}\psi_r^{(2)}] \\
&= -\sum_{r,s\in\mathbb{Z}+\frac{1}{2}} \Big(\psi_{m-s}^{(1)}\psi_s^{(2)}\psi_{n-r}^{(1)}\psi_r^{(2)} - \psi_{n-r}^{(1)}\psi_r^{(2)}\psi_{m-s}^{(1)}\psi_s^{(2)} \Big) \\
&\overset{*}{=} -\sum_{r,s\in\mathbb{Z}+\frac{1}{2}} \Big(-\psi_{m-s}^{(1)}\psi_{n-r}^{(1)}\psi_s^{(2)}\psi_r^{(2)} + \psi_r^{(2)}\psi_s^{(2)}\psi_{n-r}^{(1)}\psi_{m-s}^{(1)} \Big) \\
&\overset{**}{=} \sum_{r,s\in\mathbb{Z}+\frac{1}{2}} \Big(\psi_{m-s}^{(1)}\psi_{n-r}^{(1)}\psi_s^{(2)}\psi_r^{(2)} + \psi_{m-s}^{(1)}\psi_{n-r}^{(1)}\psi_r^{(2)}\psi_s^{(2)} - \psi_r^{(2)}\psi_s^{(2)}\psi_{m-s}^{(1)}\psi_{n-r}^{(1)} - \psi_r^{(2)}\psi_s^{(2)}\psi_{n-r}^{(1)}\psi_{m-s}^{(1)} \Big) \\
&= \sum_{r,s\in\mathbb{Z}+\frac{1}{2}} \Big(\psi_{m-s}^{(1)}\psi_{n-r}^{(1)}\{\psi_s^{(2)}, \psi_r^{(2)}\} - \psi_r^{(2)}\psi_s^{(2)}\{\psi_{m-s}^{(1)}, \psi_{n-r}^{(1)}\} \Big) \\
&= \sum_{r\in\mathbb{Z}+\frac{1}{2}} \Big(\psi_{m+r}^{(1)}\psi_{n-r}^{(1)} - \psi_r^{(2)}\psi_{m+n-r}^{(2)} \Big) \,,
\end{aligned} \tag{5.8}$$

where in $*$ and $**$ we used $\{\psi_m^{(i)}, \psi_n^{(j)}\} = \delta_{ij}\delta_{m+n,0}$. For the first term in Eq.(5.8), we find

$$\begin{aligned}
\sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_{m+r}\psi_{n-r} &= \sum_{r\leq n-\frac{1}{2}} \psi_{m+r}\psi_{n-r} + \sum_{r\geq n+\frac{1}{2}} \psi_{m+r}\psi_{n-r} \\
&= \sum_{r\leq n-\frac{1}{2}} \psi_{m+r}\psi_{n-r} - \sum_{r\geq n+\frac{1}{2}} \psi_{n-r}\psi_{m+r} + \sum_{r\geq n+\frac{1}{2}} \{\psi_{m+r}, \psi_{n-r}\} \\
&\overset{*}{=} \sum_{r\leq n-\frac{1}{2}} \psi_{m+r}\psi_{n-r} - \sum_{r\leq -m-\frac{1}{2}} \psi_{m+r}\psi_{n-r} + \sum_{r\geq n+\frac{1}{2}} \delta_{m+n,0} \\
&= \sum_{r=-m+\frac{1}{2}}^{n-\frac{1}{2}} \psi_{m+r}\psi_{n-r} + \sum_{r\geq n+\frac{1}{2}} \delta_{m+n,0} \qquad (5.9) \\
&= \psi_{\frac{1}{2}}\psi_{n+m-\frac{1}{2}} + \psi_{\frac{3}{2}}\psi_{n+m-\frac{3}{2}} + \cdots + \psi_{n+m-\frac{3}{2}}\psi_{\frac{3}{2}} + \psi_{n+m-\frac{1}{2}}\psi_{\frac{1}{2}} + \sum_{r\geq n+\frac{1}{2}} \delta_{m+n,0} \\
&\overset{**}{=} \psi_{\frac{1}{2}}\psi_{n+m-\frac{1}{2}} + \psi_{\frac{3}{2}}\psi_{n+m-\frac{3}{2}} + \cdots - \psi_{\frac{3}{2}}\psi_{n+m-\frac{3}{2}} - \psi_{\frac{1}{2}}\psi_{n+m-\frac{1}{2}} + \sum_{r\geq n+\frac{1}{2}} \delta_{m+n,0} \\
&= \sum_{r\geq n+\frac{1}{2}} \delta_{m+n,0}
\end{aligned}$$

where in $*$ we replaced r by $n-m-r$ and in $**$ we used the anti-commutation relation for fermions. Similarly, the second term in (5.8) is given by

$$\sum_{r,\in\mathbb{Z}+\frac{1}{2}} \psi_r\psi_{m+n-r} = \sum_{r\geq m+n+\frac{1}{2}} \delta_{m+n,0} \ , \qquad (5.10)$$

Combine the above two results, we find that

$$[j_m, j_n] = \sum_{r\geq n+\frac{1}{2}} \delta_{m+n,0} - \sum_{r\geq m+n+\frac{1}{2}} \delta_{m+n,0} = \sum_{r=n+\frac{1}{2}}^{m+n-\frac{1}{2}} \delta_{m+n,0} = m\delta_{m+n,0} \ . \qquad (5.11)$$

Recall that a current satifying the above algebra is a $U(1)$ current as we mentioned in the discussion of a free boson. In addition, we will determine the $U(1)$ charge of the complex fermion $\Psi(z)$ in the following,

$$[j_m, \Psi_s] = \sum_{r\in\mathbb{Z}+\frac{1}{2}} [-i\psi_{m-r}^{(1)}\psi_r^{(2)}, \frac{1}{\sqrt{2}}(\psi_s^{(1)} + i\psi_s^{(2)})] \qquad (5.12)$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2}} \sum_{r\in\mathbb{Z}+\frac{1}{2}} \Big([-i\psi^{(1)}_{m-r}\psi^{(2)}_r, \psi^{(1)}_s] + [-i\psi^{(1)}_{m-r}\psi^{(2)}_r, i\psi^{(2)}_s] \Big) \\
&= \frac{1}{\sqrt{2}} \sum_{r\in\mathbb{Z}+\frac{1}{2}} \Big(-i\psi^{(1)}_{m-r}\psi^{(2)}_r\psi^{(1)}_s + i\psi^{(1)}_s\psi^{(1)}_{m-r}\psi^{(2)}_r + \psi^{(1)}_{m-r}\psi^{(2)}_r\psi^{(2)}_s - \psi^{(2)}_s\psi^{(1)}_{m-r}\psi^{(2)}_r \Big) \\
&= \frac{1}{\sqrt{2}} \sum_{r\in\mathbb{Z}+\frac{1}{2}} \Big(i\{\psi^{(1)}_{m-r}, \psi^{(1)}_s\}\psi^{(2)}_r + \psi^{(1)}_{m-r}\{\psi^{(2)}_r, \psi^{(2)}_s\} \Big) \\
&= \frac{1}{\sqrt{2}} \sum_{r\in\mathbb{Z}+\frac{1}{2}} \Big(i\delta_{m+s-r,0}\psi^{(2)}_r + \psi^{(1)}_{m-r}\delta_{r+s,0} \Big) \\
&= \frac{1}{\sqrt{2}} \Big(\psi^{(1)}_{m+s} + i\psi^{(2)}_{m+s} \Big) \\
&= +\Psi_{m+s} \ ,
\end{aligned}
$$

thus $\Psi(z)$ has $U(1)$ charge +1. Similarly, the commutation relation $[j_m, \Psi^\dagger_s] = -\Psi^\dagger_{m+s}$ tells us that the $U(1)$ charge of the conjugate fermion $\Psi^\dagger(z)$ is -1.

Combine the above results we found, we have the following commutation and anti-commutation relations,

$$
\begin{aligned}
[j_m, j_n] &= m\delta_{m+n,0} \ , & [L_m, j_n] &= -nj_{m+n} \ , \\
[j_m, \overline{\Psi}_k] &= -\overline{\Psi}_{m+k} \ , & [j_m, \overline{\Psi}_k] &= -\overline{\Psi}_{m+k} \ , \\
\{\Psi_m, \overline{\Psi}_n\} &= \delta_{m+n,0} \ , & \{\Psi_m, \Psi_n\} &= \{\overline{\Psi}_m, \overline{\Psi}_n\} = 0 \ .
\end{aligned}
\tag{5.13}
$$

by comparing the above relations with Eq.(5.2), we see that the current algebra of a complex fermion and the current algebra of a free boson are the same.

To further prove the equivalence between the theory of two free complex fermions and the theory of a free boson compactified on a circle of radius $R = 1$ (or its dual radius $R = 2$), let us compare the patition functions for these two theories. Let $w = e^{2\pi i z}$ and define a charged character $\chi(\tau, z)$ as

$$
\chi(\tau, z) = Tr_{\mathcal{H}}\Big(q^{L_0 - \frac{c}{24}} w^{j_0} \Big) \ . \tag{5.14}
$$

For a free boson compactified on a circle of radius R, the primary fields for this theory are given by the vertex operators $V_{\pm\frac{m}{R}}(z) =: e^{\pm imX/R}:$ with the j_0 charge $\alpha = \frac{m}{R}$ and the conformal dimension $h = \frac{m^2}{2R^2}$ where $m \in \mathbb{Z}$. The states in the Hilbert space can be written as

$$|\alpha, n_1, n_2, \ldots\rangle = \lim_{z,\bar{z}\to 0} j_{-1}^{n_1} j_{-2}^{n_2} \ldots V_\alpha(z, \bar{z}) |0\rangle \ , \tag{5.15}$$

with $n_i \geq 0$. Then following similar calculations we did for the free boson on a circle case, for any α, the 0-th modes L_0 and j_0 acting on this state gives

$$\begin{aligned} L_0 |\alpha, n_1, n_2, \ldots\rangle &= \left(\sum_{k\geq 1} k n_k + \frac{\alpha^2}{2} \right) |\alpha, n_1, n_2, \ldots\rangle \ , \\ j_0 |\alpha, n_1, n_2, \ldots\rangle &= \alpha |\alpha, n_1, n_2, \ldots\rangle \ , \end{aligned} \tag{5.16}$$

Then we bring these results to the character and get

$$\chi_X(\tau, z) = \frac{1}{\eta(\tau)} \sum_{m\in\mathbb{Z}} q^{\frac{m^2}{2R^2}} w^{\frac{m}{R}} \ . \tag{5.17}$$

Now let us consider the system of a complex chiral fermion $\Psi(z)$ and its conjugate $\Psi^\dagger(z)$. Since their Hilbert spaces are independent of each other, we perform the same calculations as we did in subsection 3.4 for a free fermion, then for the character we get

$$\chi_{\Psi,\Psi^\dagger}(\tau, z) = \chi_\Psi(\tau, z)\chi_{\Psi^\dagger}(\tau, z) = q^{-\frac{1}{24}} \prod_{k\geq 0} \left(1 + q^{k+\frac{1}{2}} w\right)\left(1 + q^{k+\frac{1}{2}} w^{-1}\right) . \tag{5.18}$$

By the Jacobi triple product identity Eq.(B.1), we see that the following relation holds when the radius $R_0 = 1$,

$$\chi_X(\tau, z) = \chi_{\Psi,\Psi^\dagger}(\tau, z) \ , \tag{5.19}$$

and the same relation holds for the anti-chiral part, i.e. $\chi_X(\bar{\tau}, \bar{z}) = \chi_{\Psi,\Psi^\dagger}(\bar{\tau}, \bar{z})$. Therefore, the partition function $\mathcal{Z}_{boc}$ for a free boson on a circle with radius $R = 1$ and the partition function $\mathcal{Z}_{fot}$ for a free complex fermion on the torus equal, which leads to the conclusion that these two theories are indeed equivalent.

Therefore, we can express CFTs of complex fermions in term of CFTs of bosons, and this is called the *bosonization* of a complex fermion. This special equivalence between a theory of a boson and a theory of two fermions in CFTs has applications in string theory, for example in the covariant lattice approach, which is used to construct fermionic string theories in maximal dimensions and below [31, 32].

In this section, we discovered the equivalence between a compact free bosonic CFT at a certain radius R_0 and a free fermionic CFT through studying their current algebra and partition functions. In the next section, we will further the exploration in bosonization and discuss the equivalence between the free bosonic theory to an interacting fermionic theory, which is known as the Thirring model.

5.2 The Thirring Model and the Free Bosonic Theory

Let us consider the massless Thirring model and discuss how this theory is related to the free bosonic theory. The Thirring model consists of a massless Dirac fermion with a current-current (or quartic-self, four-fermion) interaction as defined in the following Lagrangian,

$$\begin{aligned} \mathcal{L}_{Thirring} &= i\overline{\Psi}\gamma^\mu\partial_\mu\Psi - \frac{g}{2}j^\mu j_\mu \ , \\ \text{where } \ j_\mu &= \overline{\Psi}\gamma_\mu\Psi \ , \end{aligned} \tag{5.20}$$

with g being the Thirring coupling. As we have seen before, $\overline{\Psi} = \Psi^\dagger\gamma^0$ is the Dirac adjoint of a complex fermion $\Psi = \begin{pmatrix} \psi \\ \overline{\psi} \end{pmatrix}$. The equations of motion for $\overline{\Psi}$ and Ψ are determined to be

$$\begin{aligned} i\gamma^\mu\partial_\mu\Psi - gj^\mu\gamma_\mu\Psi = 0 \ , \\ i\partial_\mu\overline{\Psi}\gamma^\mu + gj_\mu\overline{\Psi}\gamma^\mu = 0 \ , \end{aligned} \tag{5.21}$$

respectively. Then we find that the conservation law is given by

$$\partial_\mu j^\mu = \epsilon^{\mu\nu}\partial_\mu j_\nu = 0 \ . \tag{5.22}$$

Let X be a scalar field on a circle with radius R, and define its dual field $\widetilde{X}$ as

$$\partial_\mu \widetilde{X} = \epsilon_{\mu\nu} \partial^\nu X \ , \tag{5.23}$$

then the current can also be written as $j_\mu \sim \partial_\mu \widetilde{X}$ since the curl of the gradiant of a scalar field vanishes.

Solutions of the equations of motion Eq.(5.21) can be written as the following form

$$\Psi(x) = \sqrt{\frac{\mu}{2\pi}} : \exp\left\{ i\frac{\pi}{\lambda}\widetilde{X}(x) + i\lambda\gamma^5 X(x) \right\} : \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \tag{5.24}$$

where μ is the IR regulator mass of the scalar field X. Such a construction comes from taking two symmetries into consideration, which rotate the individual phases of the left and right-moving fields. For Dirac fermions, they are given by $\psi \to e^{ia}\psi$ and $\psi \to e^{ia\gamma^5}\psi$. This is called the chiral symmetry which preserves the numbers of the left or right-moving fermions. Using the expression Eq.(5.24), we can determine the exact form of the current j_μ, which is given by

$$j_\mu = \frac{\lambda}{\pi} \epsilon_{\mu\nu} \partial^\nu X \ . \tag{5.25}$$

In addition, since we have the equivalence relation between the free fermion and the compact free boson at a certain radius, we can easily verify that the kinetic term in the Lagrangian of the Thirring model and the kinetic term in the Lagrangian of the compact free boson on a circle are related by a factor of $1/2$.

Rewrite the Thirring model Lagrangian in terms of the scalar field X, it becomes

$$\mathcal{L}_{Thirring} = \frac{\lambda^2}{2\pi}\, \partial_\mu X \partial^\mu X \ , \tag{5.26}$$

which is exactly the Lagrangian for a free boson compactified on a circle with radius $R = 2\lambda/\sqrt{\pi}$.

Now let us bring the expression for the fermionic field Ψ stated in Eq.(5.24) into the equation of motion, through a straightforward calculation we find the following relation,

$$\frac{\lambda^2}{\pi} = \frac{\pi}{\pi + g} \, . \tag{5.27}$$

Replace λ by the radius R of the circle of a compact free bosonic theory, the relation Eq.(5.27) becomes

$$\frac{R^2}{4} = \frac{\pi}{\pi + g} \, , \tag{5.28}$$

which tells us how the Thirring coupling g relates to the compact free boson coupling (the radius R of the circle) in the equivalence between an interacting fermion and a compact free boson. Also, we see that this relation can be reduced to the equivalence we found in section 5.1. When the Thirring coupling vanishes, i.e. $g = 0$, the Thirring model becomes simply the free Dirac fermion. On the other hand, the radius of the equivalent compact free boson corresponding to $g = 0$ is $R = 2$, which, through the T-duality, is dual to the $R = 1$ case (recall that we have chosen $\alpha' = 2$). Therefore, the above relation reduces the duality we discussed in the previous section.

For more details on the equivalence between the Thirring model and the compact free boson, please see Coleman's paper [10] or this book on non-perturbative methods in two-dimensional quantum field theory [33].

5.3 Relation between the moduli spaces of $c = 1$ bosonic and fermionic CFTs

We have seen how a compact free boson is equivalent to the Thrring model through a relation between their couplings given in Eq.(5.28), in fact some examples of such a duality between bosonic and fermionic theories have been studied in two dimensions. Such equivalence emerges through coupling one side of the duality by a dynamical $\mathbb{Z}_2$ gauge field [29]. Note that dynamical $\mathbb{Z}_2$ gauge fields are different from background $\mathbb{Z}_2$ gauge fields, but a background gauge field can be promoted to a dynamical one by being coupled to another background gauge field. Table 5.1 gives a list of the dualities in quantum field theory. As we can see in this Table, equivalent theories through the bosonization

fermionic side of the duality	bosonic side of the duality
Majorana	Ising model / $\mathbb{Z}_2$
Majorana / $\mathbb{Z}_2$	Ising model
Dirac	(Ising model / $\mathbb{Z}_2)^2$
Dirac / $\mathbb{Z}_2$	XY-model

Table 5.1: Four sets of dual theories discussed in [29].

technique require either the bosonic theory or the fermionic theory to be coupled to a dynamical $\mathbb{Z}_2$ gauge field. In fact, this statement also holds for conformal field theories. Let us now build a map of couplings for bosonic and fermionic CFTs with central charge $c = 1$.

The moduli space for $c = 1$ bosonic CFTs has been discussed in [34]. In Figure 5.1a), the horizontal axis R_c represents the radius of the circle on which a bosonic theory is compact, and the vertical axis R_o is the radius of the bosonic orbifold. Due to the T-duality $R \to \alpha'/R$ with the conventional choice of the Regge slope $\alpha' = 2$, we can omit the portion of both axes with $R_c, R_o \in [0, \sqrt{2})$ where $R = \sqrt{2}$ is known as the self-dual radius. At the point $R_c = 2$, we have shown in section 5.1 that by bosonization, the theory is equivalent to a free Dirac fermionic CFT. In addition, the R_c axis and the R_o axis meet at a certain point, where the free bosonic theory on a circle is at $R_c = 2\sqrt{2}$ and the orbifold free bosonic theory is at $R_o = \sqrt{2}$. This equality between the partition function of a free boson orbifold given in Eq.(3.28) and the partition function of a free boson on a circle given in Eq.(3.22) holds due to the equivalence between the reflection $X \to -X$ and the half-period translation $X \to X + \pi R$ at the self-dual radius [35, 36], that is to say that we have

$$\mathcal{Z}_{orb}\Big(R_o = \sqrt{2}\Big) = \mathcal{Z}_{boc}\Big(R_c = 2\sqrt{2}\Big)\,. \tag{5.29}$$

An extra note we would like to make is that the above intersection point is known as the *Kosterlitz-Thouless* point, due to the fact that the partition function at this point corresponds to the Kosterlitz-Thouless point of the XY-model on the torus.

To construct the moduli space for $c = 1$ fermionic CFTs, let us refine the description of Coleman's original bosonization, which relates the Thirring coupling g of the Thirring model to the radius R of a compact free boson through Eq.(5.28). To keep track of the

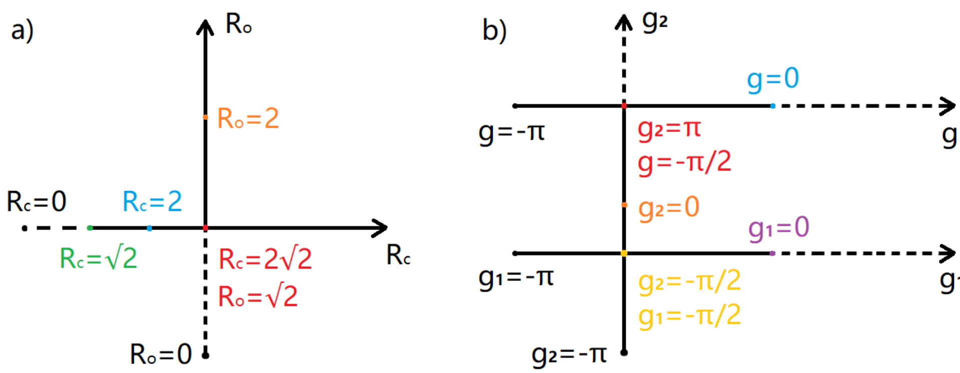

Figure 5.1: The moduli space of a) $c = 1$ bosonic CFTs, and b) $c = 1$ fermionic CFTs.

chiral symmetries of the Dirac fermion, couple it to two background fields corresponding to reflection $\mathbb{Z}_2^R$ and charge conjugation $\mathbb{Z}_2^C$ symmetries. Karch, Tong, and Turner [29] showed that at the self-dual radius $R_c = \sqrt{2}$, i.e. at the green point in Figure 5.1a), the compact boson is equivalent to a Dirac fermion coupled to a dynamical $\mathbb{Z}_2$ gauge field obtained through gauging the background reflection symmetry $\mathbb{Z}_2^R$. Then starting from this duality, they re-obtained the ungauged free Dirac fermion by coupling a dynamical $\mathbb{Z}_2$ gauge field to the bosonic side of the equivalence:

$$\text{Dirac}/\mathbb{Z}_2 \leftrightarrow \text{compact boson} \Longrightarrow \text{Dirac} \leftrightarrow \text{compact boson}/\mathbb{Z}_2 \,. \tag{5.30}$$

An essential change as a consequence of gauging the compact free boson is that now there is a shift in the T-duality by a factor of 2. That is to say,

$$\begin{aligned} \text{original T-duality:} \;\; R &\mapsto \frac{2}{R} \,, \\ \text{new T-duality:} \;\; R &\mapsto \frac{4}{R} \,. \end{aligned} \tag{5.31}$$

This change also shifts the self-dual point in the Thirring coupling g as one can check by replacing R with g using Eq.(5.28), we get that

$$\begin{aligned} \text{original T-duality:} \;\; g &\mapsto \frac{3\pi - g}{1 + g/\pi} \;\; \text{with self-dual point} \;\; g = \pi \,, \\ \text{new T-duality:} \;\; g &\mapsto -\frac{g}{1 + g/\pi} \;\; \text{with self-dual point} \;\; g = 0 \,. \end{aligned} \tag{5.32}$$

Therefore, even though the Thirring coupling g has the range of $[-\pi, \infty)$, the new T-duality allows us to omit the part where $g \in (0, \infty)$ as shown in Figure 5.1b). The g axis depicts the moduli space of the Thirring model of one Dirac fermion with Thirring coupling g.

We can also considered an intermediate case where either one of the two Majorana fermions consisting the Dirac fermion is coupled with a $\mathbb{Z}_2$ gauge field through gauging the background charge conjugation symmetry $\mathbb{Z}_2^C$. In other words, The intermediate case consists one Majorana fermion and one copy of the Ising model following the second duality in Table 5.1:

$$\text{Majorana} \times \text{Majorana} \Longrightarrow \text{Majorana} \times \text{Majorana}/\mathbb{Z}_2 \leftrightarrow \text{Majorana} \times \text{Ising}\,. \tag{5.33}$$

The moduli space of the Thirring coupling g_1 is shown in Figure 5.1b) as the g_1 axis. In this case, the T-duality is shifted by a factor of 2 as well, thus we only need to consider where $g_1 \in [-\pi, 0]$.

In addition, we can couple the two Majorana fermions with two $\mathbb{Z}_2$ gauge fields. The subtlety here is that we have two options. One is to gauge the $\mathbb{Z}_2^R$ symmetry for both Majorana fermions, the other way is to gauge the $\mathbb{Z}_2^R$ symmetry for one fermion and gauge the $\mathbb{Z}_2^C$ symmetry for the other fermion:

$$\text{Majorana} \times \text{Majorana} \Longrightarrow \begin{cases} \left(\text{Majorana}/\mathbb{Z}_2\right)^2 \leftrightarrow \left(\text{Ising}\right)^2\,, \\ \left(\text{Majorana} \times \text{Majorana}\right)/\left(\mathbb{Z}_2 \times \mathbb{Z}_2\right)\,. \end{cases} \tag{5.34}$$

For the first way of gauging, at the orbifold radius $R_o = 2$, i.e. at the orange point in Figure 5.1a), the orbifold free boson is dual to two copies of dynamical $\mathbb{Z}_2$ gauged Majorana fermions obtained through the first way. The second way of gauging produces a similar but different dual theory from the $\mathbb{Z}_2$ gauged compact boson in Eq.(5.30), but its dual bosonic theory is not exactly the orbifold boson shown in Figure 5.1a) either. In fact, its dual theory is the compact boson with switched coupling gauge fields $\mathbb{Z}_2^S : X \to X + \pi R$ and $\mathbb{Z}_2^C : X \to -X$. At the self-dual radius $R_o = \sqrt{2}$, there is a symmetry $S \leftrightarrow C$. For the second way of gauging, the moduli space of the Thirring coupling g_2 is depicted by the g_2 axis in Figure 5.1b). This theory still has the original T-duality $R \mapsto 2/R$, therefore the self-dual point remains at $g_2 = \pi$ and we omit $g \in (\pi, \infty)$.

In the previous three paragraphs, we have sketched out the moduli spaces for the three $c = 1$ fermionic CFTs, namely the Thirring model, the Majorana $\times$ Ising theory, and the two Majorana gauged by two $\mathbb{Z}_2$ theory. However, at this stage, even though we have been referring to Figure 5.1b) in our description of each moduli space, we have not yet determined whether they intersect. To do so, we simply need to find the couplings at which these theories share the same partition function. Without going into details, we will explain the main idea of the calculation. The key of this calculation is to exploit the above dualities and express the fermionic partition function in terms of the dual bosonic partition function with a term involving the Arf invairant which captures the effect of $\mathbb{Z}_2$ gauge fields on a theory [37, 38]. The results are given in the following,

$$\begin{aligned} \mathcal{Z}_{(Maj\times Maj)/(\mathbb{Z}_2\times\mathbb{Z}_2)}\big(g_2 = \pi\big) &= \mathcal{Z}_{Dirac}\big(g = -\pi/2\big) \ , \\ \mathcal{Z}_{Maj\times Maj/\mathbb{Z}_2}\big(g_1 = -\pi/2\big) &= \mathcal{Z}_{(Maj\times Maj)/(\mathbb{Z}_2\times\mathbb{Z}_2)}\big(g = -\pi/2\big) \ , \end{aligned} \tag{5.35}$$

which are exactly the results Karch, Tong, and Turner found plus that we applied the new T-duality to the Dirac theory and the Majorana $\times$ Ising theory, which tells us that $g = -\pi/2$ is dual to $g = \pi$. The reason why we choose to shift the intersections through the new T-duality is to draw a map between the moduli spaces of the $c = 1$ bosonic and fermionic CFTs, which is given in Figure 5.2.

To conclude this section, we will give a summary of how the $c = 1$ bosonic and fermionic moduli spaces are related, which also services as an instruction on how to read the map shown in the following diagram.

- Grey axes and black axes correspond to bosonic and fermionic moduli spaces, respectively.
- R_c and R_o labels theories with T-duality $R \mapsto 2/R$, g_2 labels a theory with T-duality $g \mapsto \frac{3\pi - g}{1+g/\pi}$, and g and g_1 label theories with T-duality $g \mapsto -\frac{g}{1+g/\pi}$.
- A portion of each axis is omitted due to T-duality.

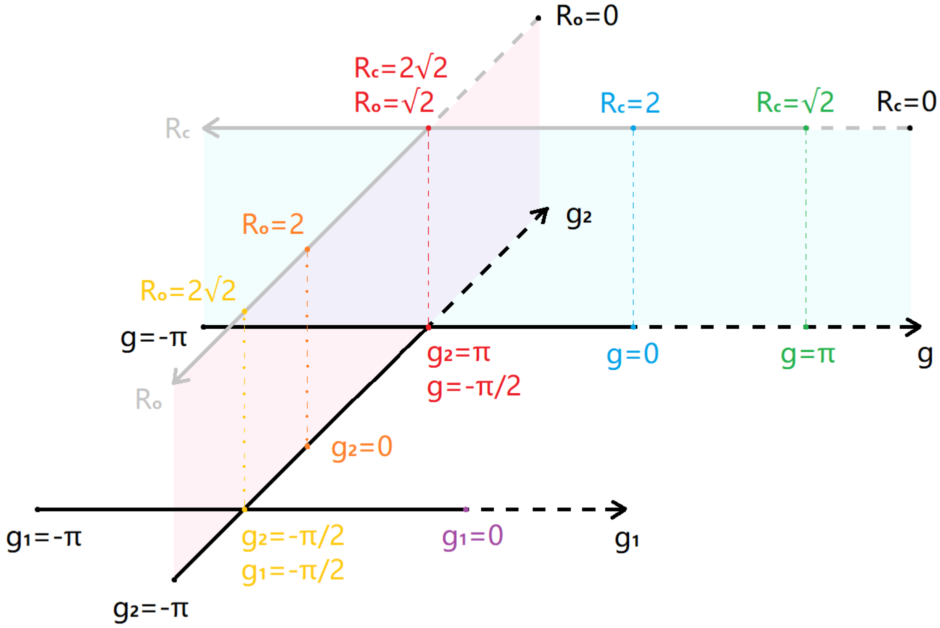

Figure 5.2: A map between the moduli spaces of $c = 1$ bosonic and fermionic CFTs.

• The Thirring coupling and the radius are related by Eq.(5.28).

• Duality between moduli spaces labelled by g and R_c with examples connected by a dashed line: Dirac$/\mathbb{Z}_2 \leftrightarrow$ compact boson or Dirac $\leftrightarrow$ compact boson$/\mathbb{Z}_2$.

• Duality between moduli spaces labelled by g_2 and R_o with examples connected by a dot-dashed line: (Maj $\times$ Maj)$/(\mathbb{Z}_2 \times \mathbb{Z}_2) \leftrightarrow$ compact boson/Z$_2$ with $\mathbb{Z}_2^S$ and $\mathbb{Z}_2^C$ switched or (Maj$/\mathbb{Z}_2$) $\times$ (Maj$/\mathbb{Z}_2$) $\leftrightarrow$ orbifold boson .

Therefore, using the equivalence between the moduli spaces of $c = 1$ bosonic and fermionic CFTs, we see that when $c = 1$, results we found in the bosonic version hold in the fermionic case as well. More specifically, when we average the CFTs of two interacting Majorana fermions over its moduli space, the result diverges like it is in the bosonic case. This makes sense since the bosonic CFTs of one compact free boson and the fermionic CFTs of two interacting Majorana fermions are related by bosonization/fermionization.

Chapter 6

Conclusion

In this thesis, to find a fermionic version of the duality between a gravitational theory and an ensemble average of $c \geq 1$ bosonic CFTs, we reviewed fundamentals in CFT and the AdS/CFT correpondence. Then we presented how the partition function of Chern-Simons gravity matches with the averaged partition function of an ensemble of compact free bosonic CFTs over the Narain moduli space in the case where the target space is a torus. Then we studied bosonization, and developed a relation between the couplings of the compact free bosonic CFT and the Thirring couplings of the interacting fermionic CFT. Using bosonization, we fermionized the moduli space of the $c = 1$ bosonic CFTs, and built a map between it and the $c = 1$ fermionic moduli space. Then we found that, for $c = 1$ fermionic CFTs of two Majorana fermions with a four-fermion interaction, averaging over the moduli space leads to a divergence in the averaged partition function just like it does in the bosonic case.

To continue this project, we want to carefully determine the moduli space for $c > 1$ fermionic CFTs. We can consider an interacting theory with N Majorana fermions with four-fermion interactions. In QFT, this model is known as the random Thirring model. To determine which theories in the random Thirring model are conformally invariant, we set the beta functions of the Thirring couplings to zero and solve for conditions on the couplings which make the theory a CFT. From there, we can determine the moduli space of the random Thirring CFTs. This method seems more complicated than applying the bosonization trick but it is powerful because it applies to arbitrary dimensions, even when bosonization cannot be applied (as bosonization has not yet been found in dimensions other than one or two).

The random Thirring model can be considered a $1+1$ dimensional generalization of the Sachdev–Ye–Kitaev (SYK) model, [39, 40] an exactly solvable strongly interacting system of N Majorana fermions in $0+1$ dimensions. In the IR, the SYK model exhibits approximate conformal symmetry, and this nearly conformal feature is similar to the situation where we consider near extremal black holes which develop a nearly AdS_2 region. Also, in the SYK model, the out-of-time-order correlators grow in a way which indicates chaotic dynamics [40], and it matches with the growth expected in a gravitational theory at relatively low energies. Thus it is worth to determine how the random Thirring CFTs, as a generalization of the SYK model, are related to quantum chaos. Therefore, studying random Thirring CFTs also leads to studies in black hole physics in AdS_3.

Due to similarities between the random Thirring model and the SYK model, we can apply techniques developed in the SYK context to the random Thirring case. Because black holes scramble information and back hole physics is naturally related to ideas of quantum chaos, random matrices, and random CFTs, future explorations of an ensemble average of the random Thirring CFTs using these techniques will contribute to black hole physics and quantum information theory.

Moreover, ensemble averaging also appears in condensed matter physics where it is related to quenched disorder in spin glasses, and some recent explorations indicate that the SYK model, spin glasses, and holography are closely related [41–44]. This is another reason why further investigations of the ensemble average of the random Thirring CFTs will deepen our understanding of the above connection and contribute to quantum gravity and studies of disordered systems in condensed matter physics.

We can also carry out the averaging procedure in two-dimensional rational conformal field theories (RCFTs). Two-dimensional RCFTs and Chern-Simons theory in three dimensions have a profound relationship, and this connection was in fact one of the first examples of the AdS/CFT correspondence. An important example of two-dimensional RCFTs is the Wess–Zumino–Witten (WZW) model. This model is related to bosonization as well. When applied to non-Abelian theories, the bosonization technique becomes more complicated because the symmetry transformations of the fermionic theory are non-local with respect to the bosonic fields. The work of Polyakov, Wiegmann [45, 46] and Witten [47] provided remarkable progress on non-Abelian bosonization. They started from different points of view and arrived at an equivalent bosonic action involving the action of the principal sigma model with a Wess-Zumino term. In addition, the Chern-Simons action plays a important role in the generalization of the boson-fermion equivalence to three-dimensional spacetime. Therefore, continued work on the averaging procedure in RCFTs will provide further insights into the AdS/CFT correpondence, string theory, condensed matter theory, and quantum computing as the WZW model has numerous applications in these fields.

Furthermore, we can return to the original starting point of this project and ask whether an ensemble of superconformal field theories (SCFTs) gives rise to some theories in gravity/supergravity when it is averaged over some moduli space. Moduli spaces of SCFTs are more or less well established, and it is a goal for further research.

Appendix A

Detailed Computations

A.1 Compute the following Jacobi identity using $p(n,0) = 0$ and $p(m,n) = -p(n,m)$

$$\begin{aligned}
0 &= [[L_m, L_n], L_0] + [[L_n, L_0], L_m] + [[L_0, L_m], L_n] \\
&= [(m-n)L_{m+n} + cp(m,n), L_0] + [nL_n + cp(n,0), L_m] \\
&\qquad + [-mL_m + cp(0,m), L_n] \\
&= (m-n)(m+n)L_{m+n} + (m-n)cp(m+n,0) \\
&\qquad + n(n-m)L_{m+n} + ncp(n,m) - m(m-n)L_{m+n} - mcp(m,n) \\
&= (m+n)p(n,m)\ .
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
0 &= [[L_{-n+1}, L_n], L_{-1}] + [[L_n, L_{-1}], L_{-n+1}] + [[L_{-1}, L_{-n+1}], L_n] \\
&= [(1-2n)L_1 + cp(-n+1,n), L_{-1}] + [(n+1)L_{n-1} \\
&\qquad + cp(n,-1), L_{-n+1}] + [(n-2)L_{-n} + cp(-1,-n+1), L_n] \\
&= 2(1-2n)L_0 + (n+1)(2n-2)L_0 + (n+1)cp(n-1,-(n-1)) \\
&\qquad - (n-2)2nL_0 + (n-2)cp(-n,n) \\
&= (n+1)cp(n-1,-(n-1)) - (n-2)cp(n,-n)
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
\implies p(n,-n) &= \left(\frac{n+1}{n-2}\right)p(n-1,-(n-1)) \\
&= \left(\frac{n+1}{n-2}\right)\left(\frac{n}{n-3}\right)p(n-2,-(n-2))
\end{aligned}$$

$$
\begin{aligned}
&= \quad \cdots \\
&= \frac{(n+1)n(n-1)\cdots 4}{(n-2)(n-3)(n-4)\cdots 1}\, p(2,-2) \\
&= \frac{1}{6}(n+1)n(n-1)p(2,-2) \\
&= \frac{1}{12}(n+1)n(n-1)\ .
\end{aligned}
$$

Conventionally, we take $p(2,-2) = \frac{1}{2}$ so the central charge for free bosons is $c = 1$.

A.2 To determine how a primary field transforms under infinitesimal conformal transformations $f(z) = z + \epsilon(z)$ with $\epsilon(z) \ll 1$,

$$
\begin{aligned}
\phi'(z,\overline{z}) &= \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \overline{f}}{\partial \overline{z}}\right)^{\overline{h}} \phi(f(z), \overline{f}(\overline{z})) \\
&= (1 + \partial_z \epsilon(z))^h (1 + \partial_{\overline{z}} \overline{\epsilon}(\overline{z}))^{\overline{h}} \phi(z + \epsilon(z), \overline{z} + \overline{\epsilon}(\overline{z})) \\
&= (1 + h\partial_z \epsilon(z) + \overline{h}\partial_{\overline{z}} \overline{\epsilon}(\overline{z}))(\phi(z,\overline{z}) + \epsilon \partial_z \phi(z,\overline{z}) + \overline{\epsilon}\partial_{\overline{z}} \phi(z,\overline{z})) \\
&= \phi(z,\overline{z}) + \Big(h\partial_z \epsilon + \epsilon \partial_z + \bar{h}\partial_{\bar{z}} \bar{\epsilon} + \bar{\epsilon}\partial_{\bar{z}}\Big)\phi(z,\bar{z})\ .
\end{aligned} \tag{A.3}
$$

A.3 To determine conditions on the energy-momentum tensor of a CFT, let us first consider when the conformal symmetry $x^\mu \mapsto x^\mu + \epsilon^\mu(x)$ has constant $\epsilon^\mu(x)$, we see that by using the conservation law and Eq.(2.18), in this case

$$
0 = \partial^\mu j_\mu = \partial^\mu (T_{\mu\nu}\epsilon^\nu) = (\partial^\mu T_{\mu\nu})\epsilon^\nu\ , \tag{A.4}
$$

which implies that $\partial^\mu T_{\mu\nu} = 0$. Now consider a general transformation $\epsilon^\mu(x)$, we have

$$
\begin{aligned}
\partial^\mu j_\mu &= (\partial^\mu T_{\mu\nu})\epsilon^\nu + T_{\mu\nu}(\partial^\mu \epsilon^\nu) \\
&= \frac{1}{2} T_{\mu\nu}(\partial^\mu \epsilon^\nu + \partial^\nu \epsilon^\mu) \\
&= \frac{1}{2}\frac{2}{d} T_{\mu\nu}\eta_{\mu\nu}(\partial \cdot \epsilon) \\
&= \frac{1}{d}(\partial \cdot \epsilon) T_\mu{}^\mu = 0\ .
\end{aligned} \tag{A.5}
$$

Since the transformation $\epsilon^\mu(x)$ is arbitrary, we conclude that $T_\mu{}^\mu = 0$.

A.4 Show that $\partial_{\bar{z}} T_{zz} = 0$

$$\begin{aligned}
\partial_{\bar{z}} T_{zz} &= \frac{1}{4}(\partial_0 + i\partial_1)(T_{00} - iT_{10}) \\
&= \frac{1}{4}(\partial_0 T_{00} + \partial_1 T_{10} + i\partial_1 T_{00} - i\partial_0 T_{10}) \\
&\overset{*}{=} \frac{1}{4}(\partial_0 T_{00} + \partial_1 T_{10} - i\partial_1 T_{11} - i\partial_0 T_{01}) \\
&= \frac{1}{4}(\partial_\mu T_{\mu 0} - i\partial_\mu T_{\mu 1}) \\
&= 0\,,
\end{aligned} \tag{A.6}$$

where in the equality $*$ we used Eq.(A.5) and the symmetry property of $T_{\mu\nu} = T_{\nu\mu}$. The last equality is obtained by the translational invariance Eq.(A.4).

A.5 In section 2.3 we stated the the OPE of the energy-momentum tensor, and we will verify the claim by computing the following commutation relations:

$$\begin{aligned}
[L_m, L_n] &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} z^{m+1} w^{n+1} [T(z), T(w)] \\
&= \oint_{\mathcal{C}(0)} \frac{dw}{2\pi i} w^{n+1} \oint_{\mathcal{C}(w)} \frac{dz}{2\pi i} z^{m+1} R\Big(T(z)T(w)\Big) \\
&= \oint_{\mathcal{C}(0)} \frac{dw}{2\pi i} w^{n+1} \oint_{\mathcal{C}(w)} \frac{dz}{2\pi i} z^{m+1} \left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} \right) \\
&= \oint_{\mathcal{C}(0)} \frac{dw}{2\pi i} w^{n+1} \left(\frac{c}{12} m(m^2-1) w^{m-2} + 2(m+1) w^m T(w) + w^{m+1} \partial_w T(w) \right) \\
&= \frac{c}{12}(m^3 - m) \oint \frac{dw}{2\pi i} w^{m+n-1} + 2(m+1) \oint \frac{dw}{2\pi i} w^{m+n+1} T(w) + \oint \frac{dw}{2\pi i} w^{m+n+2} \partial_w T(w) \\
&= \frac{c}{12}(m^3 - m)\delta_{m+n,0} + 2(m+1) \oint \frac{dw}{2\pi i} w^{m+n+1} T(w) \\
&\qquad + \oint_{\mathcal{C}(0)} \frac{dw}{2\pi i} \partial_w \Big(w^{m+n+2} T(w) \Big) - (m+n+2) \oint_{\mathcal{C}(0)} \frac{dw}{2\pi i} w^{m+n+1} T(w) \\
&= \frac{c}{12}(m^3 - m)\delta_{m+n,0} + [2(m+1) - (m+n+2)] \oint \frac{dw}{2\pi i} w^{m+n+1} T(w) \\
&= (m-n) L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}\,.
\end{aligned} \tag{A.7}$$

$$
\begin{aligned}
[L_m, \phi_n] &= \oint_{\mathcal{C}(0)} \frac{dw}{2\pi i} w^{n+h-1} \oint_{\mathcal{C}(w)} \frac{dz}{2\pi i} z^{m+1} R\Big(T(z)\phi(w)\Big) \\
&= \oint_{\mathcal{C}(0)} \frac{dw}{2\pi i} w^{n+h-1} \oint_{\mathcal{C}(w)} \frac{dz}{2\pi i} z^{m+1} \left(\frac{h\phi(w)}{(z-w)^2} + \frac{\partial_w \phi(w)}{z-w} \right) \\
&= \oint_{\mathcal{C}(0)} \frac{dw}{2\pi i} w^{n+h-1} \Big(h(m+1) w^m \phi(w) + w^{m+1} \partial_w T(w) \Big) \\
&= h(m+1) \oint \frac{dw}{2\pi i} w^{m+n+h-1} \phi(w) + \oint \frac{dw}{2\pi i} w^{m+n+h} \partial_w \phi(w) \\
&= h(m+1) \oint \frac{dw}{2\pi i} w^{m+n+h-1} \phi(w) \\
&\qquad + \oint_{\mathcal{C}(0)} \frac{dw}{2\pi i} \partial_w \Big(w^{m+n+h} \phi(w) \Big) - (m+n+h) \oint_{\mathcal{C}(0)} \frac{dw}{2\pi i} w^{m+n+1} \phi(w) \\
&= [h(m+1) - (m+n+h)] \oint \frac{dw}{2\pi i} w^{m+n+h-1} \phi(w) \\
&= [(h-1)m - n] \phi_{m+n} \, .
\end{aligned} \tag{A.8}
$$

A.6 To determine an asymptotic out-state, the complex coordinate $z = e^{x^0 + ix^1}$ has the hermitian conjugate $z \mapsto \frac{1}{\overline{z}}$, where we identify $\overline{z}$ with the complex conjugate z^*, then we will define the hermitian conjugate of the field $\phi(z, \overline{z})$ to be the following and Laurent expand it,

$$
\begin{aligned}
\phi^\dagger(z, \overline{z}) &= \overline{z}^{-2h} z^{-2\overline{h}} \phi(\frac{1}{z}, \frac{1}{\overline{z}}) \\
&= \overline{z}^{-2h} z^{-2\overline{h}} \sum_{n, \overline{n} \in \mathbb{Z}} \overline{z}^{n+h} z^{\overline{n}+\overline{h}} \phi_{n,\overline{n}} \\
&= \sum_{n, \overline{n} \in \mathbb{Z}} \overline{z}^{n-h} z^{\overline{n}-\overline{h}} \phi_{n,\overline{n}} \, .
\end{aligned} \tag{A.9}
$$

Compare this expression with Eq.(2.26) we get that $(\phi_{n,\overline{n}})^\dagger = \phi_{-n,-\overline{n}}$, now let $w = 1/\overline{z}$ and $\overline{w} = 1/z$, then we can write the asymptotic out-state as

$$
\langle \phi | = \lim_{z, \overline{z} \to 0} \langle 0 | \, \phi^\dagger(z, \overline{z}) = \lim_{w, \overline{w} \to \infty} w^{2h} \overline{w}^{2\overline{h}} \langle 0 | \, \phi(w, \overline{w}) \, , \tag{A.10}
$$

for this state to have non-singular at $w, \overline{w} \to \infty$, we require that $\langle 0 | \, \phi_{n,\overline{n}} = 0$ for all $n < h$ and $\overline{n} < \overline{h}$. Therefore, we reach the result Eq.(2.50).

A.7 To determine the Laurent modes of the normal ordered product of two fields χ and ϕ, we bring Eq. (2.53) into $N(\chi\phi)_n$,

$$\begin{aligned} N(\chi\phi)_n &= \oint_{\mathcal{C}(0)} \frac{dw}{2\pi i} w^{n+h^\chi+h^\phi-1} \oint_{\mathcal{C}(w)} \frac{dz}{2\pi i} \frac{\phi(z)\chi(w)}{z-w} \\ &= \oint_{\mathcal{C}(0)} \frac{dw}{2\pi i} w^{n+h^\chi+h^\phi-1} \left(\oint_{|z|>|w|} \frac{dz}{2\pi i} \frac{\phi(z)\chi(w)}{z-w} - \oint_{|z|<|w|} \frac{dz}{2\pi i} \frac{\chi(w)\phi(z)}{z-w} \right) \\ &= \oint_{\mathcal{C}(0)} \frac{dw}{2\pi i} w^{n+h^\chi+h^\phi-1} (\mathcal{I}_1 - \mathcal{I}_2) \,. \end{aligned} \tag{A.11}$$

We will first compute $\mathcal{I}_1$ and Laurent expand fields χ and ϕ in the expression

$$\begin{aligned} \mathcal{I}_1 &= \oint_{|z|>|w|} \frac{dz}{2\pi i} \frac{1}{z-w} \sum_{k,l\in\mathbb{Z}} z^{-l-h^\phi} w^{-k-h^\chi} \phi_l \chi_k \\ &= \oint_{|z|>|w|} \frac{dz}{2\pi i} \frac{1}{z} \sum_{p=0}^{\infty} \left(\frac{w}{z}\right)^p \sum_{k,l\in\mathbb{Z}} z^{-l-h^\phi} w^{-k-h^\chi} \phi_l \chi_k \\ &= \oint_{|z|>|w|} \frac{dz}{2\pi i} \sum_{p=0}^{\infty} \sum_{k,l\in\mathbb{Z}} z^{-l-h^\phi-p-1} w^{-k-h^\chi+p} \phi_l \chi_k \\ &= \sum_{p=0}^{\infty} \sum_{k,l\in\mathbb{Z}} \delta_{l+h^\phi+p,0} w^{-k-h^\chi+p} \phi_l \chi_k \\ &= \sum_{p=0}^{\infty} \sum_{k\in\mathbb{Z}} w^{-k-h^\chi+p} \phi_{-h^\phi-p} \chi_k \,, \end{aligned} \tag{A.12}$$

where we used $\frac{1}{z-w} = \frac{1}{z(1-w/z)} = \frac{1}{z}\sum_p (\frac{w}{z})^p$. Then the contour integral on $\mathcal{C}(0)$ is computed in the following,

$$\begin{aligned} \oint_{\mathcal{C}(0)} \frac{dw}{2\pi i} w^{n+h^\chi+h^\phi-1} \mathcal{I}_1 &= \oint_{\mathcal{C}(0)} \frac{dw}{2\pi i} \sum_{p=0}^{\infty} \sum_{k\in\mathbb{Z}} w^{n+h^\phi-k+p-1} \phi_{-h^\phi-p} \chi_k \\ &= \sum_{p=0}^{\infty} \delta_{k-h^\phi-n-p,0} \phi_{-h^\phi-p} \chi_k \\ &= \sum_{p=0}^{\infty} \phi_{-h^\phi-p} \chi_{h^\phi+n+p} \\ &= \sum_{s\le -h^\phi} \phi_s \chi_{n-s} \,, \end{aligned} \tag{A.13}$$

where in the last equality we let $s = -h^\phi - p$. Similarly, for $\mathcal{I}_2$, we write $\frac{1}{z-w} = \frac{1}{w(z/w-1)} = -\frac{1}{w}\sum_p(\frac{z}{w})^p$, then we obtain

$$\begin{aligned} I_2 &= \sum_{p=0}^{\infty}\sum_{k\in\mathbb{Z}} w^{-k-h^\chi-p-1}\chi_k\phi_{-h^\phi+p+1} \\ \oint_{\mathcal{C}(0)}\frac{dw}{2\pi i}w^{n+h^\chi+h^\phi-1}\mathcal{I}_2 &= \sum_{s>-h^\phi}\chi_{n-s}\phi_s \,. \end{aligned} \tag{A.14}$$

Therefore, we conclude that the Laurent modes $N(\chi\phi)_n$ is given by Eq.(2.55).

A.8 Obtain the equation of motion for the action Eq.(3.8) by varying the action $\mathcal{S}$ with respect to the field X, we get

$$\begin{aligned} \delta_X\mathcal{S} &= \frac{1}{2\pi\alpha'}\int dz d\overline{z}\,\Big(\partial(\delta X)\cdot\overline{\partial}X + \partial X\cdot\overline{\partial}(\delta X)\Big) \\ &= \frac{1}{2\pi\alpha'}\int dz d\overline{z}\,\Big(\partial(\delta X\cdot\overline{\partial}X) - \delta X\cdot\partial\overline{\partial}X \\ &\qquad\qquad + \overline{\partial}(\partial X\cdot\delta X) - \overline{\partial}\partial X\cdot\delta X\Big) \\ &= -\frac{1}{\pi\alpha'}\int dz d\overline{z}\,\delta X(\partial\overline{\partial}X) \\ &\overset{\text{set to}}{=} 0\,, \end{aligned} \tag{A.15}$$

since δX is an arbitrary variation, by setting $\delta_X\mathcal{S} = 0$, the equation of motion is $\partial\overline{\partial}X = 0$.

A.9 Compute the following commutator

$$\begin{aligned} [L_m, j_n] &= \gamma[N(jj)_m, j_n] \\ &= \gamma\sum_{k>-1}[j_{m-k}j_k, j_n] + \gamma\sum_{k\le -1}[j_k j_{m-k}, j_n] \\ &= \gamma\sum_{k>-1}\Big(j_{m-k}[j_k, j_n] + [j_{m-k}, j_n]j_k\Big) + \gamma\sum_{k\le-1}\Big(j_k[j_{m-k}, j_n] + [j_k, j_n]j_{m-k}\Big) \\ &= \gamma\sum_{k>-1}\Big(k j_{m-k}\delta_{k+n,0} + (m-k)\delta_{m-k+n,0}j_k\Big) \\ &\qquad + \gamma\sum_{k\le-1}\Big((m-k)j_k\delta_{m-k+n,0} + k\delta_{k+n,0}j_{m-k}\Big) \end{aligned} \tag{A.16}$$

$$
\begin{aligned}
&= \gamma \sum_{k\in\mathbb{Z}} \Big((m-k)\delta_{m-k+n,0} j_k + k\delta_{k+n,0} j_{m-k} \Big) \\
&= -2\gamma n j_{m+n} \ ,
\end{aligned}
$$

where we applied the current algebra as stated in Eq.(3.10) for $N = 1$ [11] and $\alpha' = 2$,

$$
[j_m, j_n] = \frac{\alpha'}{2} m \delta_{m+n,0} \ . \tag{A.17}
$$

A.10 Recall the commutation relation Eq.(2.17) for the Virasoro algebra with central charge c, then take $m = 2, n = -2$ and use $L_n |0\rangle = 0$ for $n > -2$, we get the following

$$
\langle 0|L_2 L_{-2}|0\rangle = \langle 0|[L_2, L_{-2}]|0\rangle = \frac{c}{2} \ . \tag{A.18}
$$

Then we apply Eq.(3.15) to express $\langle 0| L_2$ and $L_{-2}|0\rangle$ in terms of modes j_n, and we find

$$
\begin{aligned}
\langle 0| L_2 &= \frac{1}{2} \langle 0| (j_2 j_0 + j_1 j_1) = \frac{1}{2} \langle 0| j_1 j_1 \ , \\
L_{-2}|0\rangle &= \frac{1}{2} j_{-1} j_{-1} |0\rangle \ ,
\end{aligned} \tag{A.19}
$$

where we used $[j_m, j_n] = \frac{\alpha'}{2} m \delta_{m+n,0}$ [11] with $\alpha' = 2$. Now let us rewrite Eq.(A.17) as

$$
\begin{aligned}
\frac{c}{2} &= \langle 0|L_2 L_{-2}|0\rangle \\
&= \frac{1}{4} \langle 0|j_1 j_1 j_{-1} j_{-1}|0\rangle \\
&= \frac{1}{4} \langle 0|j_1 [j_1, j_{-1}] j_{-1}|0\rangle + \frac{1}{4} \langle 0|j_1 j_{-1} j_1 j_{-1}|0\rangle \\
&= \frac{1}{4} \langle 0|j_1 j_{-1}|0\rangle + \frac{1}{4} \langle 0|j_1 j_{-1} [j_1, j_{-1}]|0\rangle + \frac{1}{4} \langle 0|j_1 j_{-1} j_{-1} j_1|0\rangle \\
&= \frac{1}{2} \langle 0|[j_1, j_{-1}]|0\rangle + \frac{1}{2} \langle 0|j_{-1} j_1|0\rangle \\
&= \frac{1}{2} \ ,
\end{aligned} \tag{A.20}
$$

where we also used $j_n |0\rangle = 0$ for $n > -1$ and $\langle 0| j_n$ for $n < 1$. Therefore, we conclude that for a free boson CFT, the central charge $c = 1$.

A.11 Compute the eigenvalue of the 0-th Laurent mode of the energy-momentum tensor acting on a state in the Hilbert space of a free boson CFT. Applying Eq.(A.17) we have

$$[j_{-k}j_k, j_{-k}^{n_k}] = j_{-k}[j_k, j_{-k}^{n_k}] + [j_{-k}, j_{-k}^{n_k}]j_k = j_{-k}[j_k, j_{-k}^{n_k}] \; , \tag{A.21}$$

then we will prove $[j_k, j_{-k}^{n_k}] = n_k k j_{-k}^{n_k-1}$ by induction. For $n_k = 0$, it is trivial that $[j_k, 0] = 0$. Assume that the equality holds for integer $n_k = m > 0$, for which we have the equality $[j_k, j_{-k}^m] = mkj_{-k}^{m-1}$, then consider when $n_k = m+1$, we have

$$\begin{aligned}[j_k, j_{-k}^{m+1}] &= j_{-k}^m[j_k, j_{-k}] + [j_k, j_{-k}^m]j_{-k} \\ &= kj_{-k}^m + mj_{-k}^{m-1}j_{-k} \\ &= (m+1)kj_{-k}^{(m+1)-1} \; .\end{aligned} \tag{A.22}$$

Therefore, we have $[j_{-k}j_k, j_{-k}^{n_k}] = n_k k j_{-k}^{n_k}$. Now let us compute L_0 acting on a state $|n_1, n_2, n_3, \ldots\rangle = j_{-1}^{n_1}j_{-2}^{n_2}j_{-3}^{n_3}\ldots|\Gamma, n\rangle$ in the bosonic CFT Hilbert space,

$$\begin{aligned}L_0\,|n_1, n_2, n_3, \ldots\rangle &= \left(\frac{1}{2}j_0j_0 + \sum_{k\geq 1} j_{-k}j_k\right)j_{-1}^{n_1}j_{-2}^{n_2}j_{-3}^{n_3}\ldots|\Gamma, n\rangle \\ &= \frac{1}{2}j_0j_0j_{-1}^{n_1}j_{-2}^{n_2}j_{-3}^{n_3}\ldots|\Gamma, n\rangle + \sum_{k\geq 1} j_{-1}^{n_1}j_{-2}^{n_2}\ldots(j_{-k}j_k)j_{-k}^{n_k}\ldots|\Gamma, n\rangle \\ &= \frac{1}{2}\Gamma^2 j_{-1}^{n_1}j_{-2}^{n_2}j_{-3}^{n_3}\ldots|\Gamma, n\rangle + \sum_{k\geq 1} j_{-1}^{n_1}j_{-2}^{n_2}\ldots[j_{-k}j_k, j_{-k}^{n_k}]\ldots|\Gamma, n\rangle \\ &\qquad + \sum_{k\geq 1} j_{-1}^{n_1}j_{-2}^{n_2}\ldots j_{-k}^{n_k+1}j_k\ldots|\Gamma, n\rangle \\ &= \left(\frac{1}{2}\Gamma^2 + \sum_{k\geq 1} n_k k\right)j_{-1}^{n_1}j_{-2}^{n_2}\ldots j_{-k}^{n_k}\ldots|\Gamma, n\rangle \\ &= \left(\frac{1}{2}\Gamma^2 + \sum_{k\geq 1} n_k k\right)|n_1, n_2, n_3, \ldots\rangle \; .\end{aligned} \tag{A.23}$$

where we used $j_0\,|\Gamma, n\rangle = \Gamma\,|\Gamma, n\rangle$. Also, following similar computations and replacing j_k by $\bar{j}_k$, we find that $\bar{L}_0\,|n_1, n_2, n_3, \ldots\rangle = \left(\frac{1}{2}(\Gamma - nR)^2 + \sum_{k\geq 1} n_k k\right)|n_1, n_2, n_3, \ldots\rangle$. Now bring these results back to the expression for the partition function, and remind ourselves that the central charge $c = 1$ for a bosonic CFT, we get

$$
\begin{aligned}
Tr(q^{L_0-\frac{c}{24}}) &= q^{-\frac{1}{24}} \sum_{\Gamma,n} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \ldots \langle n_1, n_2, \ldots | e^{2\pi i \tau L_0} | n_1, n_2, \ldots \rangle \\
&= q^{-\frac{1}{24}} \sum_{\Gamma,n} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{l=0}^{\infty} \frac{(2\pi i \tau)^l}{l!} \Big\langle n_1, n_2, \ldots \Big| (L_0)^l \Big| n_1, n_2, \ldots \Big\rangle \\
&= q^{-\frac{1}{24}} \sum_{\Gamma,n} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{l=0}^{\infty} \frac{(2\pi i \tau)^l}{l!} \left(\frac{1}{2}\Gamma^2 + \sum_{k=1}^{\infty} n_k k \right)^l \langle n_1, n_2, \ldots | n_1, n_2, \ldots \rangle \\
&= q^{-\frac{1}{24}} \sum_{\Gamma,n} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \ldots q^{\frac{1}{2}\Gamma^2} \prod_{k=1}^{\infty} q^{n_k k} \\
&= q^{-\frac{1}{24}} \left(\prod_{k=1}^{\infty} \sum_{n_k=0}^{\infty} \left(q^k \right)^{n_k} \right) \sum_{\Gamma,n} q^{\frac{1}{2}\Gamma^2} \\
&= q^{-\frac{1}{24}} \prod_{k=1}^{\infty} \frac{1}{1-q^k} \sum_{\Gamma,n} q^{\frac{1}{2}\Gamma^2} \\
&= \frac{1}{\eta(\tau)} \sum_{\Gamma,n} q^{\frac{1}{2}\Gamma^2} ,
\end{aligned}
\tag{A.24}
$$

where $n \in \mathbb{Z}$, Γ is summed over discrete values, and we used $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. $\eta(\tau)$ is the Dedekind η-function as defined in Eq.(3.20). Similarly, the anti-chiral part is given by

$$
Tr(\overline{q}^{\overline{L}_0-\frac{\overline{c}}{24}}) = \frac{1}{\eta(\overline{\tau})} \sum_{\Gamma,n} q^{\frac{1}{2}(\Gamma-nR)^2} . \tag{A.25}
$$

Before the final step to reach the partition function, let us remind ourselves that for two operators A, B acting on different factors of a tensor product, the trace of the product equals to the product of traces, i.e. $\text{Tr}(AB) = \text{Tr}(A)\text{Tr}(B)$. Therefore, we find the partition function to be

$$
\mathcal{Z}_{boc}(\tau, \overline{\tau}) = Tr(q^{L_0-\frac{c}{24}}) Tr(\overline{q}^{\overline{L}_0-\frac{\overline{c}}{24}}) = \frac{1}{|\eta(\tau)|^2} \sum_{\Gamma,n} q^{\frac{1}{2}\Gamma^2} \overline{q}^{\frac{1}{2}(\Gamma-nR)^2} . \tag{A.26}
$$

A.12 We have shown that the partition function of a boson compactified on a circle of radius R is invariant under the modular T-transformation in our discussion, which in turn determines the value of Γ. Now, we would also like to verify that the partition function

Eq.(3.18) is invariant under the modular S-transformation, using the *Poisson resummation formula*

$$\sum_{n\in\mathbb{Z}} \exp\Big(-\pi a n^2 + bn\Big) = \frac{1}{\sqrt{a}}\sum_{k\in\mathbb{Z}} \exp\left(-\frac{\pi}{a}\left(k+\frac{b}{2\pi i}\right)^2\right). \tag{A.27}$$

For more details about this formula, please see Appendix B.3.

We will apply this formula twice to the partition function as shown below,

$$\begin{aligned}
\mathcal{Z}_{boc}\left(-\frac{1}{\tau},-\frac{1}{\tau}\right) &= \frac{1}{|\eta(-1/\tau)|^2}\sum_{\Gamma,n}\exp\left(-\frac{\pi i\Gamma^2}{\tau}+\frac{\pi i(\Gamma-nR)^2}{\overline{\tau}}\right) \\
&= \frac{1}{|\sqrt{-i\tau}\eta(\tau)|^2}\sum_{\Gamma}\exp\left(-\frac{\pi i\Gamma^2}{\tau}\right)\sum_{\Gamma-nR}\exp\left(\frac{\pi i(\Gamma-nR)^2}{\overline{\tau}}\right) \\
&= \frac{1}{|\sqrt{-i\tau}\eta(\tau)|^2}\sqrt{-i\tau}\sqrt{i\overline{\tau}}\sum_{\Gamma}\exp\Big(\pi i\tau\Gamma^2\Big)\sum_{\Gamma-nR}\exp\Big(-\pi i\overline{\tau}(\Gamma-nR)^2\Big) \\
&= \frac{1}{|\eta(\tau)|^2}\sum_{\Gamma,n} q^{\frac{1}{2}\Gamma^2}\overline{q}^{\frac{1}{2}(\Gamma-nR)^2} \\
&= \mathcal{Z}_{boc}(\tau,\overline{\tau}) ,
\end{aligned} \tag{A.28}$$

which shows the invariance under S-transformation.

A.13 We will obtain the equation of motion for the action Eq.(3.30) by varying the action $\mathcal{S}$ with respect to the field ψ, we get

$$\begin{aligned}
\delta_\psi\mathcal{S} &= \frac{1}{2\pi\alpha'}\int dz d\overline{z}\,\Big(\delta\psi\overline{\partial}\psi+\psi\overline{\partial}(\delta\psi)\Big) \\
&= \frac{1}{2\pi\alpha'}\int dz d\overline{z}\,\Big(\delta\psi\overline{\partial}\psi+\overline{\partial}(\psi\delta\psi)-(\overline{\partial}\psi)\delta\psi\Big) \\
&= \frac{1}{\pi\alpha'}\int dz d\overline{z}\;\delta\psi\overline{\partial}\psi \overset{\text{set to}}{=} 0 ,
\end{aligned} \tag{A.29}$$

where we used the anti-commutation property of fermions. Since the variation $\delta\psi$ is arbitrary, thus, we find the equation of motion for field ψ is $\overline{\partial}\psi = 0$. Similarly, the equation of motion for field $\overline{\psi}$ is found to be $\partial\overline{\psi} = 0$.

A.14 Compute the following anti-commutation relation for the Laurent modes of a fermionic field using the OPE of the product of two fermionic fields,

$$
\begin{aligned}
\{\psi_r, \psi_s\} &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \{\psi(x), \psi(w)\} z^{r-\frac{1}{2}} w^{s-\frac{1}{2}} \\
&= \oint \frac{dw}{2\pi i} w^{s-\frac{1}{2}} \left(\oint_{|z|>|w|} \frac{dz}{2\pi i} \psi(z)\psi(w) z^{r-\frac{1}{2}} - \oint_{|z|<|w|} \frac{dz}{2\pi i} - \psi(w)\psi(z) z^{r-\frac{1}{2}} \right) \\
&= \oint \frac{dw}{2\pi i} w^{s-\frac{1}{2}} \oint_{\mathcal{C}(w)} \frac{dz}{2\pi i} R(\psi(z)\psi(w)) z^{r-\frac{1}{2}} \\
&= \oint \frac{dw}{2\pi i} w^{s-\frac{1}{2}} \oint_{\mathcal{C}(w)} \frac{dz}{2\pi i} \frac{\alpha'/2}{z-w} z^{r-\frac{1}{2}} \\
&= \frac{\alpha'}{2} \oint \frac{dw}{2\pi i} w^{r+s-1} \\
&= \frac{\alpha'}{2} \, \delta_{r+s,0} \,,
\end{aligned}
\tag{A.30}
$$

where we choose the Regge slope to be $\alpha' = 2$.

A.15 Calculate the following commutator using Eq.(3.35), we have

$$
[L_m, \psi_r] = \gamma \sum_{k>-\frac{3}{2}} \left(k + \frac{1}{2}\right) [\psi_{m-k}\psi_k, \psi_r] - \gamma \sum_{k\leq -\frac{3}{2}} \left(k + \frac{1}{2}\right) [\psi_k \psi_{m-k}, \psi_r] \,, \tag{A.31}
$$

where we can compute the commutator $[\psi_{m-k}\psi_k, \psi_r]$ and $[\psi_k\psi_{m-k}, \psi_r]$ using Eq.(A.30)

$$
\begin{aligned}
[\psi_{m-k}\psi_k, \psi_r] &= \psi_{m-k}\psi_k\psi_r - \psi_r\psi_{m-k}\psi_k \\
&= \psi_{m-k}\psi_k\psi_r + \psi_{m-k}\psi_r\psi_k - \psi_{m-k}\psi_r\psi_k - \psi_r\psi_{m-k}\psi_k \\
&= \psi_{m-k}\{\psi_k, \psi_r\} - \{\psi_{m-k}, \psi_r\}\psi_k \\
&= \psi_{m-k}\delta_{k+r,0} - \psi_k\delta_{m-k+r,0} \,,
\end{aligned}
\tag{A.32}
$$

and similarly we have $[\psi_k\psi_{m-k}, \psi_r] = \psi_k\delta_{m-k+r,0} - \psi_{m-k}\delta_{k+r,0}$. Now bring these expressions back into $[L_m, \psi_r]$ and we get the following,

$$
[L_m, \psi_r] = \gamma \sum_{k>-\frac{3}{2}} \left(k + \frac{1}{2}\right) \left(\psi_{m-k}\delta_{k+r,0} - \psi_k\delta_{m-k+r,0}\right)
$$

$$
\begin{aligned}
&- \gamma \sum_{k \leq -\frac{3}{2}} \left(k + \frac{1}{2}\right)\left(\psi_k \delta_{m-k+r,0} - \psi_{m-k}\delta_{k+r,0}\right) \\
&= \gamma\left(- r + \frac{1}{2}\right)\psi_{m+r} - \gamma\left(m + r + \frac{1}{2}\right)\psi_{m+r} \\
&= \gamma\left(- m - 2r\right)\psi_{m+r} \,.
\end{aligned} \tag{A.33}
$$

A.16 Recall the commutation relation Eq.(2.17) for the Virasoro algebra with central charge c, then take $m = 2, n = -2$ and use $L_n |0\rangle = 0$ for $n > -2$, we get the following

$$
\langle 0|L_2 L_{-2}|0\rangle = \langle 0|[L_2, L_{-2}]|0\rangle = \frac{c}{2} \,. \tag{A.34}
$$

Then we apply Eq.(3.39) to express $\langle 0| L_2$ and $L_{-2} |0\rangle$ in terms of modes ψ_r, and we find

$$
\begin{aligned}
\langle 0| L_2 &= \frac{1}{2} \langle 0| \left(\psi_{\frac{3}{2}}\psi_{\frac{1}{2}} + 2\psi_{\frac{1}{2}}\psi_{\frac{3}{2}}\right) = \frac{1}{2} \langle 0| \left(\left\{\psi_{\frac{3}{2}}, \psi_{\frac{1}{2}}\right\} + \psi_{\frac{1}{2}}\psi_{\frac{3}{2}}\right) = \frac{1}{2} \langle 0| \psi_{\frac{1}{2}}\psi_{\frac{3}{2}} \,, \\
L_{-2} |0\rangle &= \frac{1}{2}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}} |0\rangle
\end{aligned} \tag{A.35}
$$

where we used $[\psi_r, \psi_s] = \frac{\alpha'}{2}\delta_{r+s,0}$ with $\alpha' = 2$. Now let us rewrite Eq.(A.17) as

$$
\begin{aligned}
\frac{c}{2} &= \langle 0|L_2 L_{-2}|0\rangle \\
&= \frac{1}{4} \langle 0|\psi_{\frac{1}{2}}\psi_{\frac{3}{2}}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}|0\rangle \\
&= \frac{1}{4} \langle 0|\psi_{\frac{1}{2}}\left\{\psi_{\frac{3}{2}}, \psi_{-\frac{3}{2}}\right\}\psi_{-\frac{1}{2}}|0\rangle - \frac{1}{4} \langle 0|\psi_{\frac{1}{2}}\psi_{-\frac{3}{2}}\psi_{\frac{3}{2}}\psi_{-\frac{1}{2}}|0\rangle \\
&= \frac{1}{4} \langle 0|\psi_{\frac{1}{2}}\psi_{-\frac{1}{2}}|0\rangle - 0 \\
&= \frac{1}{4} \langle 0|\left\{\psi_{\frac{1}{2}}, \psi_{-\frac{1}{2}}\right\}|0\rangle - \frac{1}{4} \langle 0|\psi_{-\frac{1}{2}}\psi_{\frac{1}{2}}|0\rangle \\
&= \frac{1}{4} \,,
\end{aligned} \tag{A.36}
$$

where we also used $\psi_r |0\rangle = 0$ for $r > -\frac{1}{2}$ and $\langle 0| \psi_r$ for $r < \frac{1}{2}$. Therefore, we conclude that for a free fermion CFT, the central charge $c = \frac{1}{2}$.

A.17 We will compute the expectation value of $\psi(z+\epsilon)\partial\psi(z)$ in the following,

$$\begin{aligned}
\langle \psi(w)\partial_z\psi(z)\rangle &= \partial_z \langle \psi(w)\psi(z)\rangle \\
&= \partial_z \left(\sum_{k,l\in\mathbb{Z}+\frac{1}{2}} w^{-k-\frac{1}{2}} z^{-l-\frac{1}{2}} \langle \psi_k\psi_l\rangle \right) \\
&= \partial_z \left(\sum_{k=\frac{1}{2}}^{\infty} w^{-k-\frac{1}{2}} z^{k-\frac{1}{2}} \right) \\
&= \partial_z \left[\frac{1}{w} \sum_{n\in\mathbb{N}} \left(\frac{z}{w}\right)^n \right] \\
&= \partial_z \left(\frac{1}{w-z} \right) \\
&= \frac{1}{(w-z)^2} \ ,
\end{aligned} \tag{A.37}$$

where we used $\langle \psi_k\psi_l\rangle = \delta_{k+l,0}$ with $k>0$ and 0 otherwise, and let $n = k-\frac{1}{2}$ in the third line, and $\frac{1}{1-x} = \sum_{n\in\mathbb{N}} x^n$. Then we let $w = z+\epsilon$ and get $\langle \psi(z+\epsilon)\partial\psi(z)\rangle = \frac{1}{\epsilon^2}$.

$$\begin{aligned}
L_0 &= \frac{1}{2}\sum_{k>-\frac{3}{2}} \left(k+\frac{1}{2}\right)\psi_{-k}\psi_k + \frac{1}{2}\sum_{k\geq\frac{3}{2}} \left(k-\frac{1}{2}\right)\psi_{-k}\psi_k \\
&= -\frac{1}{4}\psi_1\psi_{-1} + \frac{1}{4}\psi_0\psi_0 + \frac{3}{4}\psi_{-1}\psi_1 + \sum_{k\geq\frac{3}{2}} k\psi_{-k}\psi_k \\
&= -\frac{1}{4}\{\psi_1,\psi_{-1}\} + \frac{1}{8} + \psi_{-1}\psi_1 + \sum_{k\geq\frac{3}{2}} k\psi_{-k}\psi_k \\
&= \sum_{k=1}^{\infty} k\psi_{-k}\psi_k - \frac{1}{8} \ .
\end{aligned} \tag{A.38}$$

For the R sector, the Laurent modes take $k\in\mathbb{Z}$, we have

$$\begin{aligned}
\langle \psi(w)\partial_z\psi(z)\rangle &= \partial_z \langle \psi(w)\psi(z)\rangle \\
&= \partial_z \left(\sum_{k,l\in\mathbb{Z}} w^{-k-\frac{1}{2}} z^{-l-\frac{1}{2}} \langle \psi_k\psi_l\rangle \right)
\end{aligned}$$

$$= \partial_z \left[\frac{1}{2\sqrt{zw}} + \frac{1}{\sqrt{zw}} \sum_{k=1}^{\infty} \left(\frac{z}{w} \right)^k \right] \tag{A.39}$$
$$= \partial_z \left(\frac{1}{2\sqrt{zw}} \frac{w+z}{w-z} \right)$$
$$= \frac{-w^2 + 4zw + z^2}{4z\sqrt{zw}(w-z)^2} \ ,$$

where we used $\frac{1+x}{1-x} = 1 + 2\sum_{n=1}^{\infty} x^n$, now let $w = z + \epsilon$, then we have

$$\langle \psi(z+\epsilon) \partial_z \psi(z) \rangle = \frac{1}{2\epsilon^2} \left(\sqrt{\frac{z}{z+\epsilon}} + \sqrt{\frac{z+\epsilon}{z}} \right) - \frac{1}{4z\sqrt{z(z+\epsilon)}} \ , \tag{A.40}$$

then we will bring this result back to $\langle T(z) \rangle$ to find the vacuum expectation value.

A.18 Construct a chiral field of conformal dimension $h = 1$ as $j(z) = -N(\Psi\Psi^\dagger)(z) = iN(\psi^{(1)}\psi^{(2)})(z)$, To find an expression for the Laurent mode j_n we will first prove the following relation

$$N(\psi^{(a)}\psi^{(b)})_n = -\sum_{k>-\frac{1}{2}} \psi^{(a)}_{n-k}\psi^{(b)}_k + \sum_{k\leq-\frac{1}{2}} \psi^{(b)}_k \psi^{(a)}_{n-k} \tag{A.41}$$
$$\overset{k\to n-k}{=} \sum_{k\geq n+\frac{1}{2}} \psi^{(b)}_{n-k}\psi^{(a)}_k - \sum_{k<n+\frac{1}{2}} \psi^{(a)}_k \psi^{(b)}_{n-k}$$
$$= \sum_{k>-\frac{1}{2}} \psi^{(b)}_{n-k}\psi^{(a)}_k - \sum_{k\leq-\frac{1}{2}} \psi^{(a)}_k \psi^{(b)}_{n-k} - \sum_{k>-\frac{1}{2}}^{n-\frac{1}{2}} \psi^{(b)}_{n-k}\psi^{(a)}_k - \sum_{k>-\frac{1}{2}}^{n-\frac{1}{2}} \psi^{(a)}_k \psi^{(b)}_{n-k}$$
$$= -N(\psi^{(b)}\psi^{(a)})_n - \sum_{k>-\frac{1}{2}}^{n-\frac{1}{2}} \{\psi^{(a)}_k, \psi^{(b)}_{n-k}\}$$
$$= -N(\psi^{(b)}\psi^{(a)})_n - \sum_{k>-\frac{1}{2}}^{n-\frac{1}{2}} \delta^{ab}\delta_{n,0}$$
$$= -N(\psi^{(b)}\psi^{(a)})_n \ ,$$

where the last equality holds since we have $\sum_{k=a}^{b} f(k) = 0$ for $b < a$. Using the above relation, we can write the Laurent mode j_n as

$$
\begin{aligned}
j_n &= iN(\psi^{(1)}\psi^{(2)})_n \\
&= -i \sum_{k>-\frac{1}{2}} \psi^{(1)}_{n-k}\psi^{(2)}_k + i \sum_{k\leq -\frac{1}{2}} \psi^{(2)}_k \psi^{(1)}_{n-k} \\
&= -i \sum_{k>-\frac{1}{2}} \psi^{(1)}_{n-k}\psi^{(2)}_k - i \sum_{k\leq -\frac{1}{2}} \psi^{(1)}_{n-k}\psi^{(2)}_k \\
&= -i \sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi^{(1)}_{n-k}\psi^{(2)}_k \, .
\end{aligned} \tag{A.42}
$$

A.19 Consider the case $D = 2$, since locally $SO(2,2,\mathbb{Z}) \cong SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})/\mathbb{Z}_2$ and $SO(2,2,\mathbb{R}) \cong SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/\mathbb{Z}_2$, then the Narain moduli space is given by

$$
\begin{aligned}
\mathcal{M}_2 &= SO(2,2;\mathbb{Z})\backslash SO(2,2,\mathbb{R})/SO(2)\times SO(2) \\
&= \Big(SL(2,\mathbb{Z})\times SL(2,\mathbb{Z})/\mathbb{Z}_2\Big)\Big\backslash\Big(SL(2,\mathbb{R})\times SL(2,\mathbb{R})/\mathbb{Z}_2\Big)\Big/U(1)\times U(1) \\
&= \Big(SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/U(1)\Big)\times\Big(SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/U(1)\Big)\, ,
\end{aligned} \tag{A.43}
$$

where $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/U(1) = SL(2,\mathbb{Z})\backslash\mathcal{H}$ with $\mathcal{H}$ being the upper half plane is isomorphic to the moduli space of a genus 1 Riemann surface. Thus the Narain moduli space $\mathcal{M}_2$ is two copies of the moduli space of the compact torus, which has finite volume. When D increases, the volume converges faster.

A.20 Let us show that the function $W_D(\tau) = \tau_2^{D/2}F_D(\tau)$ is an eigenfunction of the upper half plane Laplacian $\Delta_{\mathcal{H}}$ with eigenvalue of $D/2(D/2-1)$.

$$
\begin{aligned}
\Delta_{\mathcal{H}}W_D(\tau) &= \tau_2^2\left(\frac{\partial^2}{\partial\tau_1^2}+\frac{\partial^2}{\partial\tau_2^2}\right)\tau_2^{\frac{D}{2}}F_D(\tau) \\
&= \tau_2^2\left(\tau_2^{\frac{D}{2}}\frac{\partial^2}{\partial\tau_1^2}+\frac{\partial}{\partial\tau_2}\left(\frac{D}{2}\tau_2^{\frac{D}{2}-1}+\tau_2^{\frac{D}{2}}\frac{\partial}{\partial\tau_2}\right)\right)F_D(\tau) \\
&= \tau_2^2\left(\tau_2^{\frac{D}{2}}\frac{\partial^2}{\partial\tau_1^2}+\frac{D}{2}\left(\frac{D}{2}-1\right)\tau_2^{\frac{D}{2}-2}+D\tau_2^{\frac{D}{2}-1}\frac{\partial}{\partial\tau_2}+\tau_2^{\frac{D}{2}}\frac{\partial^2}{\partial\tau_2^2}\right)F_D(\tau) \\
&= \tau_2^{\frac{D}{2}}\left(\Delta_{\mathcal{H}}+D\tau_2\frac{\partial}{\partial\tau_2}\right)F_D(\tau)+\frac{D}{2}\left(\frac{D}{2}-1\right)\tau_2^{\frac{D}{2}}F_D(\tau) \\
&\overset{4.33}{=} \frac{D}{2}\left(\frac{D}{2}-1\right)W_D(\tau)\, .
\end{aligned} \tag{A.44}
$$

Appendix B

Theta Functions and Their Properties

B.1 Jacobi Triple Product Identity

We will prove the beautiful Jacobi triple product identity in the following. The Jacobi triple product identity is given by

$$\prod_{r\geq 0}\Big(1-q^{r+1}\Big)\Big(1+q^{r+\frac{1}{2}}w\Big)\Big(1+q^{r+\frac{1}{2}}w^{-1}\Big)=\sum_{m\in\mathbb{Z}}q^{\frac{m^2}{2}}w^m\ , \tag{B.1}$$

where $|q|<1$ and $w\neq 0$. To prove this identity, we will first derive the Eulers' formulas stated below, my derivation is inspired by a derivation in [48]. The Euler's formulas are given by

$$\begin{aligned}\prod_{n\in\mathbb{N}}\Big(1+x^n w\Big)&=\sum_{n\in\mathbb{N}}\frac{x^{n(n-1)/2}w^n}{(1-x)\dots(1-x^n)} && \text{with } |x|<1\ ,\\ \prod_{n\in\mathbb{N}}\Big(1+x^n w\Big)^{-1}&=\sum_{n\in\mathbb{N}}\frac{(-1)^n w^n}{(1-x)\dots(1-x^n)} && \text{with } |x|<1,\ |w|<1\ ,\end{aligned} \tag{B.2}$$

and then simply let $q=x^{\frac{1}{2}}$. Let

$$f(w)=\prod_{k=0}^{N-1}(1+x^k w)\ , \tag{B.3}$$

then it is easy to see that

$$f(w) = (1+w)\prod_{k=1}^{N-1}(1+x^k w) = (1+w)\prod_{k=0}^{N-2}\Big(1+x^k(xw)\Big) = \frac{1+w}{1+x^N w}f(xw)\ , \tag{B.4}$$

or we can express it as $(1+x^N w)f(w) = (1+w)f(xw)$. On the other hand, we can write $f(w)$ as a power series of w with coefficients given by a function of x,

$$f(w) = \sum_{n=0}^{N} a_n(x) w^n\ , \tag{B.5}$$

let $a_n(x) = 0$ for $n < 0$, then Eq.(B.4) gives

$$\begin{aligned}(1+x^N w)\sum_{n=0}^{N} a_n(x) w^n &= (1+w)\sum_{n=0}^{N} a_n(x) x^n w^n\ ,\\ \sum_{n=0}^{N} a_n(x) w^n + \sum_{n=0}^{N} a_n(x) x^N w^{n+1} &= \sum_{n=0}^{N} a_n(x) x^n w^n + \sum_{n=0}^{N} a_n(x) x^n w^{n+1}\ ,\\ \sum_{n=0}^{N}\Big(a_n(x) + a_{n-1}(x) x^N\Big) w^n &= \sum_{n=0}^{N}\Big(a_n(x) x^n + a_{n-1}(x) x^{n-1}\Big) w^n\ ,\end{aligned} \tag{B.6}$$

comparing the coefficient of w^n, we get the recursion relation

$$\begin{aligned} a_n(x) &= \left(\frac{1-x^{N-n+1}}{1-x^n}\right) x^{n-1} a_{n-1}(x)\\ &= \left(\frac{1-x^{N-n+1}}{1-x^n}\frac{1-x^{N-n+2}}{1-x^{n-1}}\dots\frac{1-x^N}{1-x}\right)\Big(x^{n-1}\dots x^0\Big) a_0(x)\\ &= \frac{(1-x^{N-n+1})\dots(1-x^N)}{(1-x)\dots(1-x^n)} x^{n(n-1)/2}\\ &\equiv \frac{(1-x^N)!!!\ x^{n(n-1)/2}}{(1-x^n)!!!(1-x^{N-n})!!!}\ ,\end{aligned} \tag{B.7}$$

where $a_0(x) = 1$ by the definition of $f(w)$, and we denote $(1-x^n)\dots(1-x) = (1-x^n)!!!$ for convenience. Therefore, we get

$$\prod_{k=0}^{N-1}(1+x^k w) = \sum_{n=0}^{N}\frac{(1-x^N)!!!\ x^{n(n-1)/2} w^n}{(1-x^n)!!!(1-x^{N-n})!!!}\ . \tag{B.8}$$

By taking $N \to \infty$ then $x^N \to 0$ since $|x| < 1$, we obtain the first formula in Eq.(B.2).

For the second Euler's identity, we let $w = -1$ in Eq.(B.8) and multiply both sides by $(-1)^{N-2n}[(1-x^N)!!!]^{-1}$, then we get

$$0 = \sum_{n=0}^{N} \frac{x^{n(n-1)/2}(-1)^{N-n}}{(1-x^n)!!!(1-x^{N-n})!!!} \qquad \text{for } N \geq 1 \,. \tag{B.9}$$

Now let

$$\begin{aligned} g(w) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(1-x^n)!!!} w^n = \sum_{n=0}^{\infty} g_n(x) w^n \,, \\ h(w) &= \sum_{n=0}^{\infty} \frac{x^{n(n-1)/2}}{(1-x^n)!!!} w^n = \sum_{n=0}^{\infty} h_n(x) w^n \,, \end{aligned} \tag{B.10}$$

then the product $g(w)h(w)$ can be expressed as

$$\begin{aligned} g(w)h(w) &= \sum_{N=0}^{\infty} f_N(x) w^N \qquad \text{where } f_0(x) = 1 \,, \\ \text{and } f_N(x) &= \sum_{n=0}^{N} g_n(x) h_{N-n}(x) = \sum_{n=0}^{N} \frac{x^{n(n-1)/2}(-1)^{N-n}}{(1-x^n)!!!(1-x^{N-n})!!!} = 0 \qquad \text{for } N \geq 1 \end{aligned} \tag{B.11}$$

by Eq.(B.9), then we have $g(w)h(w) = 1$ and thus $h(w) = 1/g(w)$. We notice that

$$h(w) = \sum_{n \in \mathbb{N}} \frac{x^{n(n-1)/2} w^n}{(1-x^n)!!!} = \prod_{n \in \mathbb{N}} \Big(1 + x^n w\Big) \tag{B.12}$$

by the first Euler's identity, therefore, we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(1-x^n)!!!} w^n = g(w) = \prod_{n \in \mathbb{N}} \Big(1 + x^n w\Big)^{-1} \,, \tag{B.13}$$

which is exactly the second Euler's identity in Eq.(B.2).

Now that we have proved the Euler's identities, we will use them to verify the Jacobi triple product identity by a simple calculation. To do so, we replace x by x^2 and replace

w by xw, the first Euler's identity can be written as

$$
\begin{aligned}
\prod_{n\in\mathbb{N}}\left(1+x^{2n+1}w\right) &= \sum_{n\in\mathbb{N}} \frac{x^{n(n-1)}x^n w^n}{(1-x^2)\dots(1-x^{2n})} \\
&= \sum_{n\in\mathbb{N}} x^{n^2} w^n \frac{\prod_{k\in\mathbb{N}}(1-x^{2k+2+2n})}{\prod_{k\in\mathbb{N}}(1-x^{2k+2})} \\
&\overset{*}{=} \prod_{k\in\mathbb{N}}\left(1-x^{2k+2}\right)^{-1} \sum_{n\in\mathbb{Z}} x^{n^2} w^n \prod_{k\in\mathbb{N}}\left(1-x^{2k+2+2n}\right) \\
&\overset{**}{=} \prod_{k\in\mathbb{N}}\left(1-x^{2k+2}\right)^{-1} \sum_{n\in\mathbb{Z}} x^{n^2} w^n \sum_{m\in\mathbb{N}} \frac{(-1)^m x^{m(m-1)} x^{(2n+2)m}}{(1-x^2)\dots(1-x^{2m})} \\
&= \prod_{k\in\mathbb{N}}\left(1-x^{2k+2}\right)^{-1} \sum_{m\in\mathbb{N}} \frac{(-1)^m (xw^{-1})^m}{(1-x^2)\dots(1-x^{2m})} \sum_{n\in\mathbb{Z}} x^{(n+m)^2} w^{n+m} \\
&= \prod_{k\in\mathbb{N}}\left(1-x^{2k+2}\right)^{-1} \prod_{j\in\mathbb{N}}\left(1+x^{2j+1}w^{-1}\right)^{-1} \sum_{n\in\mathbb{Z}} x^{n^2} w^n
\end{aligned}
\tag{B.14}
$$

where in $*$ all terms of the sum with $n < 0$ are zero, in $**$ we applied the first Euler's identity with x replaced by x^2 and we let $w = x^{2n+2}$, and in the last equality we applied the second Euler's identity with $x \to x^2$ and $w \to -xw^{-1}$ and replace $n+m$ by n in the last sum over $n \in \mathbb{N}$. Then we obtain the equality

$$
\prod_{k\in\mathbb{N}}\left(1-x^{2k+2}\right)\left(1+x^{2k+1}w\right)\left(1+x^{2k+1}w^{-1}\right) = \sum_{n\in\mathbb{Z}} x^{n^2} w^n \,, \tag{B.15}
$$

which is exactly Eq.(B.1) with $q = x^2$, thus the proof is complete.

B.2 Relation between Theta Functions and the Dedekind η-function

The Jacobi Theta functions are modular functions defined as the following,

$$
\vartheta_1(\tau) \equiv -i \sum_{n\in\mathbb{Z}+\frac{1}{2}} (-1)^{n-\frac{1}{2}} q^{\frac{n^2}{2}} = -i\eta(\tau) q^{\frac{1}{12}} \prod_{k=0}^{\infty}\left(1-q^k\right)\left(1-q^{k+1}\right), \tag{B.16}
$$

$$\vartheta_2(\tau) \equiv \sum_{n\in\mathbb{Z}+\frac{1}{2}} q^{\frac{n^2}{2}} = \eta(\tau) q^{\frac{1}{12}} \prod_{k=0}^{\infty} \left(1+q^k\right)\left(1+q^{k+1}\right) = \frac{1}{2}\eta(\tau) q^{\frac{1}{12}} \prod_{k=0}^{\infty} \left(1+q^k\right)^2 , \tag{B.17}$$

$$\vartheta_3(\tau) \equiv \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}} = \eta(\tau) q^{-\frac{1}{24}} \prod_{k=0}^{\infty} \left(1+q^{k+\frac{1}{2}}\right)^2 , \tag{B.18}$$

$$\vartheta_4(\tau) \equiv \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{n^2}{2}} = \eta(\tau) q^{-\frac{1}{24}} \prod_{k=0}^{\infty} \left(1-q^{k+\frac{1}{2}}\right)^2 , \tag{B.19}$$

where the second equality in each line comes from the Jacobi triple product identity and the Dedekind η-function is defined in Eq.(3.19). Compute the product of the last three Theta functions and we get

$$\begin{aligned}
\vartheta_2\vartheta_3\vartheta_4(\tau;q) &= \frac{1}{2}\eta^3(\tau) \prod_{k=0}^{\infty} \left(1+q^k\right)^2 \left(1+q^{k+\frac{1}{2}}\right)^2 \left(1-q^{k+\frac{1}{2}}\right)^2 \\
&= 2\eta^3(\tau) \prod_{k=1}^{\infty} \left(1+q^k\right)^2 \left(1-q^{2k-1}\right)^2 \\
&= 2\eta^3(\tau) \left(\prod_{k=1}^{\infty} \left(1+q^k\right)^2 \right) \left(\prod_{k=1}^{\infty} \left(1-q^{2k-1}\right)^2 \right) \\
&= 2\eta^3(\tau) \left(\prod_{k=1}^{\infty} \left(1+q^{2k-1}\right)^2 \left(1+q^{2k}\right)^2 \right) \left(\prod_{k=1}^{\infty} \left(1-q^{2k-1}\right)^2 \right) \\
&= 2\eta^3(\tau) \prod_{k=1}^{\infty} \left(1+q^{2k}\right)^2 \left(1-q^{4k-2}\right)^2 ,
\end{aligned} \tag{B.20}$$

by comparing the second and the last lines we see that $\vartheta_2\vartheta_3\vartheta_4(\tau;q) = \vartheta_2\vartheta_3\vartheta_4(\tau;q^2)$. Then, from $\vartheta_2\vartheta_3\vartheta_4(\tau;0) = 2\eta^3(\tau)$ we obtain that $\vartheta_2\vartheta_3\vartheta_4(\tau;q) = 2\eta^3(\tau)$ if $|q| < 1$. Therefore, we have the following identity,

$$\vartheta_2(\tau)\vartheta_3(\tau)\vartheta_4(\tau) = 2\eta^3(\tau) \, . \tag{B.21}$$

B.3 Modular Transformations of Theta Functions

As we have seen, partition functions of CFTs on the torus can be expressed using theta functions, and to better study their modularity properties, let us now investigate the behaviour of theta functions under the modular transformations. First of all, let us consider the shift $\tau \to \tau + 1$. Recall that $q = e^{2\pi i\tau}$, use Eq.(B.17, B.18, and B.19) and modular properties of the Dedekind η-function as shown in Eq.(3.19), we have

$$\vartheta_2(\tau+1) = \frac{1}{2} e^{\frac{\pi i}{12}} \eta(\tau) e^{\frac{\pi i}{6}} q^{\frac{1}{12}} \prod_{k=0}^{\infty} \left(1 + e^{2\pi i k} q^k\right)^2 = e^{\frac{\pi i}{4}} \vartheta_2(\tau) \,, \tag{B.22}$$

$$\vartheta_3(\tau+1) = e^{\frac{\pi i}{12}} \eta(\tau) e^{-\frac{\pi i}{12}} q^{-\frac{1}{24}} \prod_{k=0}^{\infty} \left(1 + e^{2\pi i k + \pi i} q^{k+\frac{1}{2}}\right)^2 = \vartheta_4(\tau) \,, \tag{B.23}$$

$$\vartheta_4(\tau+1) = e^{\frac{\pi i}{12}} \eta(\tau) e^{-\frac{\pi i}{12}} q^{-\frac{1}{24}} \prod_{k=0}^{\infty} \left(1 - e^{2\pi i k + \pi i} q^{k+\frac{1}{2}}\right)^2 = \vartheta_3(\tau) \,. \tag{B.24}$$

Before we study the behaviours of theta functions under the modular transformation $\tau \to -1/\tau$, we will first introduce the *Poisson summation formula*, which is a useful tool in studying modular properties of functions. It relates the sum of a function over a lattice and the sum of its Fourier transformation over Fourier modes. The formula can be expressed as the following,

$$\sum_{n\in\mathbb{Z}} \exp(-\pi a n^2 + bn) = \frac{1}{\sqrt{a}} \sum_{k\in\mathbb{Z}} \exp\left(-\frac{\pi}{a}\left(k + \frac{b}{2\pi i}\right)^2 \right) . \tag{B.25}$$

Take $a = -i\tau$ and $b = i\pi$, we have

$$\sum_{n\in\mathbb{Z}} (-1)^n \exp\left(\pi i \tau n^2\right) = \frac{1}{\sqrt{-i\tau}} \sum_{k\in\mathbb{Z}} \exp\left(-\frac{\pi i}{\tau}\left(k + \frac{1}{2}\right)^2 \right) , \tag{B.26}$$

then bring the above expression into theta functions and the transformation is straightforward to see using Euler's formula,

$$\vartheta_2\left(-\frac{1}{\tau}\right) \equiv \sum_{n\in\mathbb{Z}+\frac{1}{2}} \exp\left(-\frac{\pi i n^2}{\tau}\right) = \sqrt{-i\tau} \sum_{n\in\mathbb{Z}} (-1)^n \exp\left(\pi i \tau n^2\right) = \sqrt{-i\tau}\,\vartheta_4(\tau) \,. \tag{B.27}$$

Take $a = -i\tau$ and $b = 0$, we have

$$\sum_{n\in\mathbb{Z}} \exp\left(\pi i \tau n^2\right) = \frac{1}{\sqrt{-i\tau}} \sum_{k\in\mathbb{Z}} \exp\left(-\frac{\pi i k^2}{\tau}\right) , \tag{B.28}$$

then we get

$$\vartheta_3\left(-\frac{1}{\tau}\right) \equiv \sum_{n\in\mathbb{Z}} \exp\left(-\frac{\pi i n^2}{\tau}\right) = \sqrt{-i\tau}\sum_{n\in\mathbb{Z}} \exp\left(\pi i \tau n^2\right) = \sqrt{-i\tau}\vartheta_3(\tau)\,. \tag{B.29}$$

For $\vartheta_4(\tau)$, we just need to apply modular transformation to Eq.(B.27) one more time, then we get

$$\vartheta_4\left(-\frac{1}{\tau}\right) = \frac{1}{\sqrt{i/\tau}}\vartheta_2\left(-\frac{1}{-\frac{1}{\tau}}\right) = \sqrt{-i\tau}\vartheta_2(\tau)\,. \tag{B.30}$$

These modular transformation properties of theta functions will be handy for computing CFTs partition functions on a torus.

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