Progress in low-dimensional anti-de Sitter gravity

Matthew Hodel, Department of Physics
McGill University, Montreal
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Abstract

We discuss recent developments in theories of quantum gravity in two and three spacetime dimensions.

We begin by considering the Jackiw-Teitelboim (JT) theory, a model of pure 2d gravity, providing a new perspective on a perturbative expansion in the quantum version of this theory, inspired by random matrix theory. Random matrix theory (RMT) captures universal properties of the very late-time dynamics of an extremely wide class of chaotic quantum systems. JT gravity goes farther by agreeing precisely with a random matrix ensemble, beyond universal quantities. We take a Lagrangian approach to quantization of JT theory. We find that we must perform path integrals over 2d geometries of arbitrary genus. Integrating first over the dilaton field restricts to constant negative curvature surfaces. A connection is made with the topological recursion relations of Eynard and Orantin, which allow the computation of such path integrals for arbitrary-genus surfaces by cutting them apart into pairs of pants. The resulting asymptotic series for JT observables matches onto the genus expansion familiar from random matrix theory.

Recent work has also appeared deriving a boundary graviton description of pure AdS_3 gravity. Pure gravity was thought for some time to be trivial in three spacetime dimensions since there are no local graviton degrees of freedom. Nevertheless global effects render the theory nontrivial. We will use standard path integral methods to approximate the gravity partition function around a semiclassical background geometry. The resulting boundary graviton scalar field action describes global excitations around background AdS_3 spacetime. It serves as a 2d generalization of the 1d Schwarzian theory, which itself describes boundary fluctuations of the JT theory.

Abrégé

Nous passons en revues les développements récents dans la théorie de la gravité quantique dans un espace-temps à deux (2) et trois (3) dimensions.

Nous considérons d'abord la théorie gravitationnelle de Jackiw-Teitelboim (JT), un modèle de gravité pure à deux dimensions. Inspirée par la théorie des matrices aléatoires (TMA), elle présente une nouvelle perspective sur une expansion perturbative dans la version quantique de la théorie. La TMA saisit les propriétés universelles de la dynamique à temps très tardive d'une énorme classe de systèmes quantiques chaotiques. Il semble que la théorie JT est en parfait accord avec la TMA, saisissant plus d'information que les propriétés universelles. Nous adoptons une approche lagrangienne à la quantification de la théorie JT. Nous constatons que nous devons effectuer des intégrales de chemins sur des géométries 2D de genre arbitraire. En intégrant d'abord le champ de dilaton, nous sommes restreints à des surfaces à courbures négatives constantes. Nous établissons alors un lien avec des relations de récursivité topologique d'Eynard et d'Orantin qui permettent de calculer des intégrales de chemins pour des surfaces à genres arbitraires en les coupant en paires de culottes. La série asymptotique des observables de la théorie JT correspond alors à une série d'expansion en genre de la TMA.

Des travaux récents ont également dérivé une description de graviton de frontière pour la théorie gravitationnelle pure de AdS_3. La théorie pure en trois dimensions a longtemps été considérée triviale puisqu'il n'y a pas de de gravitons à degrés de liberté locaux. Néanmoins, des effets globaux rendent la théorie non-triviale. Nous utiliserons des méthodes d'intégration de chemin standard afin de se rapprocher de la fonction de partition gravitationnelle autour d'une géométrie de fond semi-classique. L'action d'un graviton de frontière avec un dilaton obtenu par cette méthode décrit des excitations globales autour d'un espace-temps AdS_3 de fond. Elle sert d'une généralisation à deux dimensions de la théorie Schwarzian à une dimension, qui elle décrit les fluctuations à la bordure de la théorie JT.

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Chapter 1

Preface

Studying gravity in fewer spacetime dimensions than the four we inhabit has long been a fruitful playground for making headway on the otherwise largely intractable problem of quantizing a theory of gravity.

In what follows we will consider two such lower-dimensional theories of gravity—the Jackiw-Teitelboim model in two dimensions, and AdS_3 gravity in three dimensions. In both cases we will focus on the case of pure gravity, in which the only dynamical field is the metric, with no additional matter content added. At first this may seem puzzling—since the metric, a massless spin-2 particle, transforms in a traceless symmetric tensor representation of its $SO(d-2)$ little group, in d spacetime dimensions it has $(d-2)(d-1)/2 - 1 = d(d-3)/2$ independent degrees of freedom. We notice immediately that in $d = 3$ this amounts to zero total degrees of freedom. Even worse, for $d = 2$ the formula says that there are -1 degrees of freedom. Thus it appears that pure Einstein gravity is trivial in three dimensions, and perhaps undefined in two spacetime dimensions.

Despite appearances, pure 3d gravity is not a trivial theory. Though our analysis correctly demonstrates that there are no local degrees of freedom in this theory, there nevertheless exist global degrees of freedom that act nontrivially at the boundary of spacetime. We will encounter such boundary graviton degrees of freedom in Chapter 5.

Pure two-dimensional Einstein gravity, on the other hand, requires modifications to be rendered non-trivial. When evaluating the Einstein-Hilbert action on some spacetime manifold $\mathcal{M}$, at least in Euclidean signature the Gauss-Bonet theorem tells us that the action simply evaluates to a topological invariant of the manifold, the Euler characteristic $\chi_{\mathcal{M}}$. To add dynamics to the theory we must add one extra degree of freedom to place ourselves in the same situation as the 3d case. The simplest way to do so is to add a scalar field Φ, known as the dilaton. Such an addition leads to the Jackiw-Teitelboim model of gravity, which we review in section 2.1.

Though both these theories of gravity may be simpler and more tractable than the ordinary four-dimensional Einstein theory of gravity, they would not be worth studying if they were not interesting in their own right. What makes both these theories of gravity interesting is that they contain black holes. This fact is simplest to see in Euclidean signature, where what is usually meant by a black hole is a geometry in which the thermal circle is contractible. To see this, consider that in the Euclidean continuation of an ordinary empty spacetime, the Euclidean time direction is an S^1 directly fibered over all the spatial directions, and is thus non-contractible. Physically this corresponds to a thermal gas at temperature β^{-1}, where β is the radius of the S^1. Working in polar coordinates, the spatial S^{d-2} is of course contractible in this spacetime. For Euclidean black holes these two swap roles—there is a black hole horizon in the way, obstructing any attempt to contract the spatial S^{d-2}, and conversely the thermal circle shrinks to zero size at the black hole horizon, thus becoming contractible. [1]

Once the existence of black hole solutions is established one important goal is to understand their entropy. The laws of black hole mechanics, written down by analogy to the laws of statistical mechanics, suggest that black holes have an entropy proportional to their surface area. Indeed, in Hawking's seminal semiclassical treatment of black hole radiation, it was shown that black holes are thermal objects with a temperature that is conjugate to this entropy in the first law of thermodynamics.[2] There therefore must be some discrete spectrum of microstates leading to this entropy, if we believe black holes constitute ordinary thermodynamic systems. There have previously been enumerations of these microstates from string theory constructions for particular

[1] For $d = 2$ of course there is no such spatial sphere but by analogy with the higher dimensional cases we require the same criterion of contractibility of the Euclidean time circle.

[2] Here we use the language "conjugate" in the sense of a Legendre transform exchanging the independent variables T and S.

supersymmetric black holes, but significant progress toward such microstate counting in generic non-supersymmetric cases is still lacking. One reason to be interested in the lower-dimensional theories of pure gravity we consider is that one might plausibly be able to identify such microstates in these simplified cases, though we will not make contact with this subject herein.

Rather than being two isolated examples of tractable theories of quantum gravity, Jackiw-Teitelboim gravity and pure AdS_3 gravity are in fact related to one another. The 2d Jackiw-Teitelboim theory in asymptotically-AdS_2 space can actually be arrived at in an appropriate limit of the AdS_3 discussion. This is done by performing a Kaluza-Klein compactification on the extra spatial direction of the 3d geometry. In both cases, the original motivation was to simplify the analysis of quantum gravity and in particular black holes by posing the problem in fewer spacetime dimensions. This can be somewhat of a double-edged sword, since special features such as an infinite dimensional algebra of asymptotic symmetries arise in 3d that are not shared by higher-dimensional theories; such specialization is even more severe in the 2d formulation. Importantly, however, our increasingly sophisticated picture of these two examples allows us to compare them and extract common features which are expected to persist in higher dimensions. Thus these exciting developments pave the way to plausibly uncovering a deeper understanding of quantum features of black holes and gravity in generic dimensions in the presence of a negative cosmological constant.

The remainder of this thesis will be organized as follows. In Chapter 2 we review the basics of Jackiw-Teitelboim gravity. In Chatper 3 we situate the mathematical subject of random matrix theory as it appears in physics to package universal properties of chaotic quantum systems. Our main discussion is separated into two parts: in Chapter 4 we consider 2d gravitational physics, and in Chapter 5 we consider 3d gravitational physics. Both sections attempt to quantize classical theories of gravity—the Jackiw-Teitelboim theory in the 2d case, and AdS_3 gravity in the 3d case. Again, we consider both these theories as pure gravity theories, treating the dynamics of the metric alone with no added matter content. Conclusions and outlook are presented in Chapter 6.

This thesis is a review of recent literature and does not constitute a contribution to original knowledge on the part of the author. As such there are no coauthors to credit.

Chapter 2

Gravity Background

2.1 Jackiw-Teitelboim Gravity

This section follows [1]. Jackiw-Teitelboim gravity, first introduced in [2], [3], is by some measures the simplest possible theory of gravity. As discussed in the preface, the number of graviton degrees of freedom in two spacetime dimensions is formally -1. This means the theory is overconstrained and requires an additional degree of freedom to be rendered non-trivial. We can see this manifestly at the level of the action. The ordinary Einstein-Hilbert action (including boundary terms) on some Euclidean 2d manifold $\mathcal{M}$ is

$$\mathrm{I} \stackrel{?}{=} \int_{\mathcal{M}} \sqrt{g}R + 2\int_{\partial\mathcal{M}} \sqrt{h}K \tag{2.1}$$

with R the Ricci scalar, h the boundary (1d) metric, and K the extrinsic curvature of the boundary. The combination of both terms simply evaluates to $\chi_{\mathcal{M}}$[1], the Euler characteristic of $\mathcal{M}$. While we can still allow such a term in our action, to obtain a non-trivial theory we introduce a single (necessarily scalar field) degree of freedom which we will call the dilaton, and write the first term in a Taylor expansion about $\Phi = 0$:

$$\mathrm{I} = -\Phi_0 \left[\int_{\mathcal{M}} \sqrt{g}R + 2\int_{\partial\mathcal{M}} \sqrt{h}K\right] - \left[\int_{\mathcal{M}} \sqrt{g}\Phi(R+2) + 2\int_{\partial\mathcal{M}} \sqrt{h}\Phi(K-1)\right] + \mathcal{O}(\Phi^2). \tag{2.2}$$

[1] Recall the Euler characteristic of a Riemann surface with g handles and h boundaries is given by $\chi_{\mathcal{M}} = 2 - 2g - h$.

The JT partition function is then given by the Euclidean signature path integral

$$Z(\beta) = \int_{\mathcal{M}_\beta} DgD\Phi \; e^{-I[g,\Phi]}. \tag{2.3}$$

The dependence on β enters through the boundary conditions on the dynamical fields $g_{\mu\nu}, \Phi$, which we must specify in order to make this quantity well defined. Following the standard holographic renormalization procedure, we introduce a cutoff ϵ and require that $\mathcal{M}_\beta$ have a single boundary of length β/ϵ and that the dilaton takes on a fixed value $\Phi|_{\partial\mathcal{M}_\beta} = \Phi_r/\epsilon$. We take $\epsilon \to 0$ at the end of the computation.

We begin by performing the $D\Phi$ integral along a contour $\mathcal{C}$ parallel to the imaginary axis, so that Φ acts as a Lagrange multiplier enforcing that $\mathcal{M}_\beta$ have constant negative curvature $R = -2$.[2] The spacetime manifold $\mathcal{M}_\beta$ can have any genus. In the simplest case of genus zero, it is simply a cut out portion of the Poincaré disc (subject to the aforementioned boundary conditions). We use global coordinates so that the metric of the disc is $ds^2 = d\rho^2 + \sinh^2\rho d\theta^2$. This region is defined by its boundary. One expedient way to parameterize the boundary is to write the angle it sweeps out as a function of the proper length u along the boundary, $\theta(u)$. On the disc, the JT action so far evaluates to

$$\mathrm{I} = -\Phi_0 \underbrace{\left[\int_{\mathcal{M}} \sqrt{g}R + 2\int_{\partial\mathcal{M}} \sqrt{h}K\right]}_{4\pi} + 2\int_{\partial\mathcal{M}} \sqrt{h}\Phi_r(K-1) \tag{2.4}$$

A computation in local coordinates shows that the extrinsic curvature of the boundary takes the form

$$K = 1 + \epsilon^2 \mathrm{Sch}(\tan\theta/2, u) \tag{2.5}$$

with Sch the Schwarzian derivative.[3] The action is thus

$$-\mathrm{I} = 4\pi\Phi_0 + 2\Phi_r \int_0^\beta du \; \mathrm{Sch}(\tan\theta/2, u). \tag{2.6}$$

[2]Recall the standard representation of a delta function as $\delta(x-x_0) = \int_{\mathbb{R}} e^{i(x-x_0)y} dy$.

[3]The Schwarzian derivative is defined as follows: $\mathrm{Sch}(f(u), u) = (f''(u)/f'(u))' - (1/2)(f''(u)/f'(u))^2 = (f'''(u)/f'(u)) - (3/2)(f''(u)/f'(u))^2$.

At this point we can identify a saddle point in the path integral—the linear solution $\theta(u) = \frac{2\pi\epsilon}{\beta}u$ has vanishing Schwarzian derivative. It is not unique, however, as translations of this region in the Poincaré disc give the same contribution. We choose to not integrate over this mode, which amounts to quotienting the space $\mathrm{Diff}(S^1)$ of allowable boundaries by $PSL(2,\mathbb{R})$, the isometry group of the Poincaré disc.

The Schwarzian path integral featuring the action (2.6) arrived at for classical JT theory on the Poincaré disc turns out to be one-loop exact [4]. Rather than reproduce this result, we consider the following toy model of a one-loop exact integral on the 2-sphere to illustrate the power of these methods. We first perform the integration exactly by a change of variables:

$$I \equiv \int_{S^2} (d\theta d\phi \sin(\theta)) e^{-N\cos\theta} = 2\pi \int_{z=\cos(\theta)}^{z\in[-1,1]} dz e^{-Nz} = 4\pi \sinh(N)/N. \tag{2.7}$$

Alternatively we can try to understand this integral from a saddle point analysis. The saddle point condition is $(\cos(\theta))' = 0$, resulting in saddle points at the north and south poles. Considering for now the north pole, we replace $\sin(\theta)$ and $\cos(\theta)$ by their Taylor expansions around $\theta = 0$

$$I \approx 2\pi \int d\theta(\theta + \dots) e^{-N(1-\theta^2/2+\dots)} \tag{2.8}$$

$$= 2\pi e^{-N} \int d(\theta^2/2) e^{+N\theta^2/2} \tag{2.9}$$

$$= 2\pi e^{-N}(1/N) \left[e^{N\theta^2/2} \right]_{\theta=0} \tag{2.10}$$

$$= 2\pi e^{-N}/N. \tag{2.11}$$

In just the same way we find a contribution from the south pole $-2\pi e^{+N}/N$. Summing the two contributions gives us back precisely the exact answer (2.7). That is, it turns out we can forget about the higher order terms in θ entirely and still get the right answer! This is what it means for an integral to be one-loop exact.

There exists a mathematical theorem, the Duistermaat-Heckman formula (whose proof uses techniques of supersymmetry localization), stating the following. In general, integrals of the form

$$\int_{\text{symplectic } \mathcal{M}} e^{\omega} e^{NH} \tag{2.12}$$

are one-loop exact, where ω is the symplectic form of $\mathcal{M}$ and H generates a Hamiltonian vector flow on $\mathcal{M}$. In the above case, the sphere is a symplectic manifold with Hamiltonian vector flow given by rotation in the ϕ direction, ∂_ϕ.

We now return to the path integral with action (2.6),

$$Z(\beta) = \int_{SL(2,\mathbb{R})\backslash \mathrm{Diff}(S^1)} D\theta \exp\left[4\pi\Phi_0 + 2\Phi_r \int_0^\beta du \mathrm{Sch}(\tan(\theta/2, u))\right]. \tag{2.13}$$

The manifold in question $SL(2,\mathbb{R})\backslash \mathrm{Diff}(S^1)$ is indeed symplectic,[4] and the Hamiltonian vector flow is given by translations in u as $\theta(u) \mapsto \theta(u+\epsilon)$. The symplectic form of the Schwarzian theory is (writing $\theta = \theta_0 + \delta\theta$)

$$\omega = \int du \left(\delta\theta'' \wedge \delta\theta' + \mathrm{Sch}(\tan(\theta/2), u)\delta\theta' \wedge \delta\theta\right). \tag{2.14}$$

Having identified all the necessary prerequisites to invoke the theorem alluded to above, for now we simply cite the answer for the Schwarzian partition function (which coincides with the JT partition function $Z(\beta)$ on the Poincaré disc)

$$Z(\beta) = e^{4\pi\Phi_0} \frac{e^{\pi^2/\beta}}{4\sqrt{\pi}\beta^{3/2}}. \tag{2.15}$$

We will later arrive at this result by taking advantage of our knowledge that the path integral is one-loop exact in Section 4.2.4.

[4]The notation here emphasizes the fact that we have performed a left-quotient by $SL(2,\mathbb{R})$ rather than a right-quotient.

Chapter 3

Mathematics Background

3.1 Random Matrix Theory

3.1.1 Random Matrix Universality

For an authoritative review of this subject, refer to the excellent lecture notes [5]. We will follow the discussion given in [1].

Random matrix theory has a long history as a tool in studies of quantum chaos, dating back to the work of Wigner, Dyson, and Mehta [6]. Indeed, one can make exceedingly general claims about the spectral statistics of most quantum systems, which are generically chaotic.[1] One considers the density of states $\rho(E)$ of such a system, defined simply as a sum of delta functions of the (by assumption) discrete energy eigenvalues $\{E_i\}$:

$$\rho(E) = \sum_n \delta(E - E_n). \tag{3.1}$$

Here we imagine some averaging procedure which allows us to approximate the discrete spectrum by some continuous function $\rho(E)$. If one considers a microcanonical ensemble around a specific energy E and plots the distribution of these energy levels, the claim of quantum chaos is that, provided the system is chaotic, these energy levels are distributed

[1]That is to say, heuristically speaking, integrable quantum systems are extremely rare in the space of all quantum systems.

like the eigenvalues of a random matrix. For integrable systems one will typically instead encounter Poisson statistics.

One should appreciate the generality of this claim—it posits that such random matrix statistics occur in each superselection sector of nearly any quantum theory.[2] This is general enough that it ought to be true for black holes as well, since we expect their full quantum-gravitational description to amount to an ordinary quantum system, thanks to the clues discussed in the Preface.

It is helpful to have a particular observable in mind to compute that is sensitive to this fine-grained chaotic behavior. It is important to note that most all quantities we ordinarily consider are *not* such observables.

The observable we will focus on is the spectral form factor, given in terms of the analytically continued partition function as an expectation value over some disorder average $\langle\cdot\rangle$ as

$$\langle Z(\beta+it)Z(\beta-it)\rangle = \left\langle \sum_{n,m} e^{-(\beta+it)E_n} e^{-(\beta-it)E_m} \right\rangle . \tag{3.2}$$

In the remainder of this section we will focus on random matrix theory quantities and take $\langle\cdot\rangle = \langle\cdot\rangle_{\text{RME}}$ to be an expectation value in a given random matrix ensemble RME. The physical significance of this quantity is that it exposes the statistics of the energy eigenvalues, as desired. Additionally, as $t\to\infty$ it agrees with the two point function in a thermal ensemble $\langle\phi(t)\phi(0)\rangle_{\beta\mapsto\beta/2}$ (this is one way to state the Eigenstate Thermalization Hypothesis [7]).

Chaotic systems display a characteristic behavior for the spectral form factor which can be conveniently captured on a log-log plot (Fig. 3.1).[8] It includes three regions associated with different time-scales—first, a "slope" region during which the system relaxes to its semiclassical equilibrium state, then a "ramp" region in which the form factor grows *linearly* with t evincing so-called "spectral rigidity" of random matrices, and finally a flat "plateau" region which displays the finite dimension of the quantum Hilbert space.

[2] A small disclaimer is that there are in fact several such universality classes based on the discrete symmetries of the problem.

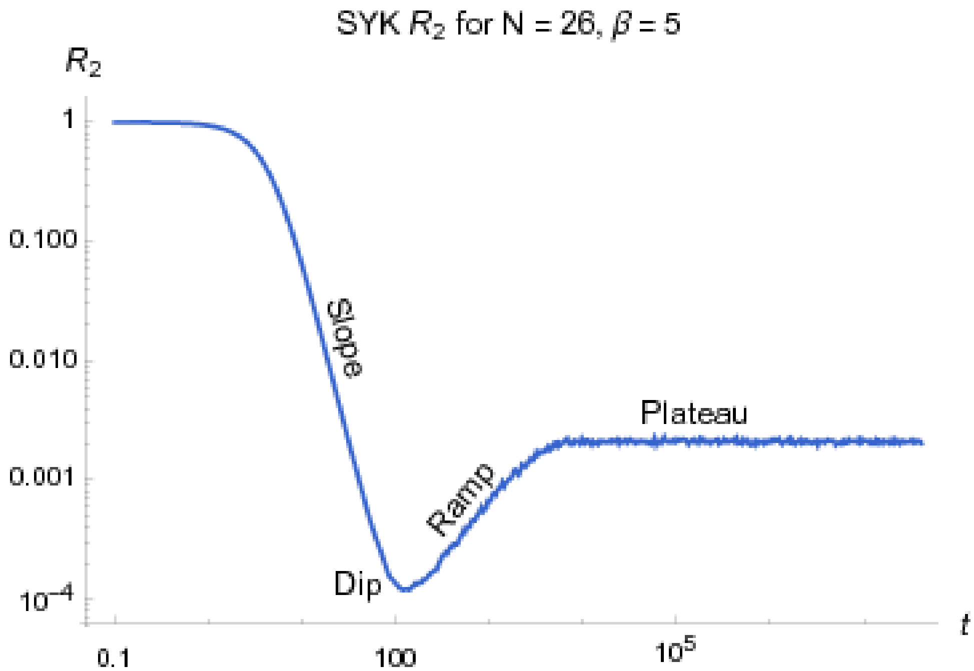

Figure 3.1: Example spectral form factor for the SYK model. Figure taken from [9].

In more detail, if the density of states $\rho(E)$ really were a continuous function, the form factor would decay to zero as it does in the slope region and remain there forever. Thus the ramp is a signature of the underlying discreteness of the spectrum. Spectral rigidity will be explained in more detail below. The transition from ramp to plateau occurs at the "Heisenberg time" t_H of the system—in the case of an $L \times L$ random matrix $t_H \sim L$.[3] To see why the plateau occurs we consider a late time limit of (3.2), in which the RME average kills off-diagonal oscillatory terms:

$$\langle Z(\beta+it)Z(\beta-it)\rangle_{\text{RME}} \overset{t\to\infty}{\to} \left\langle \sum_{n,m=1}^{L} e^{-itE_n}e^{+itE_m} \right\rangle_{\text{RME}} \to \left\langle \sum_{n=1}^{L} e^{-itE_n}e^{+itE_n} \right\rangle_{\text{RME}} \sim L. \tag{3.3}$$

So we see that the late time behavior of the form factor is to become a constant of order L, the size of the random matrix. This is to be contrasted with the initial ($t = 0$) value of the

[3]When modeling a superselection sector of some chaotic quantum system by random matrices, the rank of the matrix in question is the number of energy eigenvalues contained in the spectral window being considered. Thus $L \sim e^S$ where S is the entropy of the system.

form factor which is of order L^2:

$$\langle Z(\beta+it)Z(\beta-it)\rangle_{\text{RME}} \overset{t\to 0}{\to} \left\langle \left(\sum_{n=1}^{L} e^{-\beta E_n}\right)^2 \right\rangle \sim L^2. \tag{3.4}$$

The random matrix partition function[4] is given by

$$\mathcal{Z}_{\text{RMT}} = \int_{RME} [dH] e^{-L\text{Tr}V(H)}. \tag{3.5}$$

Here we use H to represent a random Hermitian matrix variable anticipating that it will ultimately be the Hamiltonian of a quantum system. The matrix potential $V(H)$ specifies the random matrix ensemble we are using, with $V(H) = -H^2/2$ corresponding to the most commonly used "GUE" (Gaussian Unitary Ensemble). The Haar measure $[dH]$ on $L \times L$ Hermitian matrices H is the unique one invariant under the transformation $H \to UHU^\dagger$ for any unitary U.

This redundancy can be thought of as a gauge symmetry under which the gauge invariant information is the eigenvalue spectrum of H. One can then proceed with the standard Fadeev-Popov prodedure for gauge-fixing path integrals as follows. Set $U = e^{ih}$ in the above transformation so that

$$H \to e^{ih} H e^{-ih} \sim H + i[h, H]. \tag{3.6}$$

We will imagine H to be diagonal, $H = \sum_a \lambda_a |a\rangle\langle a|$, and choose an off-diagonal $h = |c\rangle\langle d| + |d\rangle\langle c|$. In this case $[h, H] = (\lambda_d - \lambda_c)h$, and the Jacobian factor picked up in the Fadeev-Popov procedure is

$$\det\left(\frac{\partial H_{ab}}{\partial h_{cd}}\right) = \prod_{a<b} (\lambda_a - \lambda_b)^2 \equiv \Delta(\{\lambda_a\})^2 \tag{3.7}$$

This quantity, which plays a key role in the random matrix literature, is known as the Vandermonde determinant. The resulting probability distrubtion of eigenvalues of the

[4]Not to be confused with the above quantity $Z(\beta) = \text{Tr}(e^{-\beta H})$, which will instead act as an observable we can insert into the RMT partition function.

random matrix is

$$\mathbb{P}(\{\lambda_a\}) = \prod_{a<b}(\lambda_a - \lambda_b)^2 e^{-L\sum_a V(\lambda_a)}. \tag{3.8}$$

As will soon become apparent, the two factors in this determinant compete—the exponential damping factor wants the eigenvalues to all sit at the minimum of the random matrix potential $V(\lambda)$, while the Vandermonde determinant factor leads to a repulsion between eigenvalues. It is this repulsion that underlies spectral rigidity.

We proceed with a mean field analysis. Consider the effective potential felt by a single eigenvalue

$$V_{\text{eff}}(\lambda_a) = LV(\lambda_a) - \sum_{b\neq a} \log(\lambda_a - \lambda_b)^2 \tag{3.9}$$

and the asymptotic spectral density in the limit of large matrix rank L[5]

$$\rho_0(\lambda) = \lim_{L\to\infty} \mathbb{P}(\lambda). \tag{3.10}$$

Under a continuous spectrum approximation we can replace the sum in (3.11) by an integral

$$V_{\text{eff}}(\lambda) = LV(\lambda) - \int d\lambda' \left[\log(\lambda - \lambda')^2\right] \rho_0(\lambda'). \tag{3.11}$$

The resulting equations of motion $V'_{\text{eff}}(\lambda) = 0$ are

$$V'(\lambda) = 2\int d\lambda' \frac{\rho_o(\lambda')}{\lambda - \lambda'}. \tag{3.12}$$

Given the explicit form of $V(\lambda)$, one can solve this implicit equation to get an explicit equation for $\rho_0(\lambda)$.

For the GUE case $V(\lambda) = -\lambda^2/2$ the resulting spectral density is the so-called "Wigner semicircle law" $\rho_0(\lambda) = \sqrt{4 - \lambda^2}$. More sophisticated matrix potentials will produce more elaborate spectral densities.

[5]As usual in matrix perturbation theory, the perturbative parameter is $1/L$ so that taking $L \to \infty$ corresponds to some classical limit of the matrix integral, which can be thought of as a zero-dimensional "path integral".

The spectral density alone is not enough to suit our purposes, however. To probe the statistics of the spectrum we will require the two-point funtion of spectral densities $\langle \rho_0(\lambda)\rho_0(\lambda') \rangle$.

To proceed we will use the method of orthogonal polynomials—there are other routes to the spectral form factor however, such as the loop equations.[10]

We begin with an explicit representation of the Vandermonde determinant

$$\prod_{a<b}(\lambda_a - \lambda_b) = \det \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 \dots \\ 1 & \lambda_2 & \lambda_2^2 & \lambda_2^3 \dots \\ 1 & \lambda_3 & \lambda_3^2 & \lambda_3^3 \dots \\ 1 & \lambda_4 & \lambda_4^2 & \lambda_4^3 \dots \end{pmatrix}. \tag{3.13}$$

Recalling that determinants are invariant under elementary row operations in which multiples of another row are added to a given row, we can replace each term in this matrix by any *monic* (that is, with leading coefficient one) polynomial in λ_i of the same degree.

$$\prod_{a<b}(\lambda_a - \lambda_b) = \det \begin{pmatrix} 1 & p_1(\lambda_1) & p_2(\lambda_1) & p_3(\lambda_1) \dots \\ 1 & p_1(\lambda_2) & p_2(\lambda_2) & p_3(\lambda_2) \dots \\ 1 & p_1(\lambda_3) & p_2(\lambda_3) & p_3(\lambda_3) \dots \\ 1 & p_1(\lambda_4) & p_2(\lambda_4) & p_3(\lambda_4) \dots \end{pmatrix}. \tag{3.14}$$

It will be to our advantage to choose these polynomials $p_n(\lambda)$ according to the following orthogonality condition in the matrix ensemble given by $V(H)$

$$\int e^{-LV(\lambda)} p_n(\lambda) p_m(\lambda) d\lambda \overset{\text{require}}{\propto} \delta_{n,m}. \tag{3.15}$$

We will focus our attention on the GUE case ($V(H) = -H^2/2$) in which case these are simply the Hermite polynomials.[6] We can also rewrite the eigenvalue probability distribution

[6] It is worth noting how remarkable it is that the Hermite polynomials show up in the context of random matrix theory and quantum chaos. They are famously related to the wavefunctions of the energy eigenstates of the harmonic oscillator, the most notoriously *non*-chaotic system in physics! Nevertheless, what the current discussion is revealing is that, despite that the spectrum of the harmonic oscillator is completely integrable, the distribution of the energy eigenstates' roots displays random matrix statistics.

as

$$\mathbb{P}(\{\lambda\}) = \Delta(\{\lambda\})^2 e^{-L\sum_n V(\lambda_n)} \tag{3.16}$$

$$= \det\left(\sum_{m=1}^{L} \psi_m(\lambda_i)\psi_m(\lambda_j)\right) \tag{3.17}$$

where we have introduced the "wavefunctions"[7]

$$\psi_m(\lambda) = p_m(\lambda) e^{-LV(\lambda)/2}. \tag{3.18}$$

This procedure makes it easy to integrate out eigenvalues at our leisure. The expression (3.17) above is called the "Slater determinant" and represents the wavefunction for identical fermions (recall that eigenvalues repel each other due to spectral rigidity and thereby obey some analog of a Pauli-exclusion principle).

We can now write a more systematic expression for the probability distribution of any number of eigenvalues,

$$\mathbb{P}(\lambda_1, \ldots, \lambda_k) \propto \det(K(\lambda_i, \lambda_j))_{1\leq i,j\leq k} \tag{3.19}$$

$$K(\lambda, \lambda') = \sum_m \psi_m(\lambda)\psi_n(\lambda'). \tag{3.20}$$

For our current purposes (finding the spectral form factor) we only require $\mathbb{P}(\lambda_1, \lambda_2)$. In the GUE case the kernel is simply the projector onto the first L eigenstates of the harmonic oscillator,

$$K = \sum_{m=1}^{L} |m\rangle\langle m|. \tag{3.21}$$

There then follows an operator equation requiring that the eigenvalues of the Hamiltonian in question are no more than the L-th energy eigenvalue of the harmonic oscillator Hamiltonian,

$$\left(-\frac{1}{L}\partial_\lambda^2 + \frac{1}{4}\lambda^2\right) \leq L + \frac{1}{2}. \tag{3.22}$$

[7]Again, in the GUE case of current interest, these literally are the harmonic oscillator eigenfunctions.

We proceed by doing perturbation theory in $1/L$ around the situation in which both eigenvalues are close to each other. We thus set

$$\lambda = \lambda_0 + \frac{x}{L\rho_0(\lambda_0)} \tag{3.23}$$
$$\lambda' = \lambda_0 + \frac{x'}{L\rho_0(\lambda_0)}. \tag{3.24}$$

The constraint (3.22) then becomes[8].

$$(-L\rho_0(\lambda_0)^2\partial_x^2 + \frac{L}{4}\lambda_0^2) \leq L \tag{3.25}$$
$$\iff -\frac{1}{(2\pi)^2}(4-\lambda^2)\partial_x^2 \leq 1 - \frac{\lambda_0^2}{4} \tag{3.26}$$
$$\iff -\partial_x^2 \leq \pi^2. \tag{3.27}$$

From the last line we can see that the wave number k of the solutions must take on values in the range $[-\pi, \pi]$. Thus we have as a resolution of the identity

$$\mathbb{1} = \int_{-\pi}^{\pi} dk|k\rangle\langle k|. \tag{3.28}$$

The Dyson sine kernel is simply given by $\langle x|x'\rangle$:

$$K(x,x') = \langle x|x'\rangle \tag{3.29}$$
$$= \langle x| \underbrace{\left(\int_{-\pi}^{\pi} dk|k\rangle\langle k|\right)}_{\mathbb{1}} |x'\rangle \tag{3.30}$$
$$= \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ik(x-x')} \tag{3.31}$$
$$= \frac{\sin(\pi(x-x'))}{\pi(x-x')}. \tag{3.32}$$

One can then obtain the probability distribution for two eigenvalues (that is, the resulting marginal probability distribution upon integrating out all but two eigenvalues) as a

[8]As we are taking the $L \to \infty$ limit it is appropriate to drop the $1/2$ on the right-hand side of (3.22)

determinant of this kernel, as promised.[9]

$$\mathbb{P}(x, x') = \det \begin{pmatrix} K(x,x) & K(x,x') \\ K(x',x) & K(x',x') \end{pmatrix} \tag{3.33}$$

$$= 1 - \frac{\sin^2(\pi(x-x'))}{\pi^2(x-x')^2}. \tag{3.34}$$

Reverting back to the matrix eigenvalues variables from the variables x, x' we arrive at

$$\mathbb{P}(\lambda, \lambda') = \rho_0(\lambda, \lambda') - \frac{\sin^2(\pi L \rho_0(\lambda_0)(\lambda - \lambda'))}{\pi^2 L^2 (\lambda - \lambda')^2}. \tag{3.35}$$

We now use this eigenvalue distribution to compute the spectral form factor

$$\langle Z(\beta + it) Z(\beta - it) \rangle_{\text{RME}} = L^2 \int d\lambda d\lambda' \mathbb{P}(\lambda, \lambda') e^{-\lambda(\beta+it) - \lambda'(\beta - it)} \tag{3.36}$$

$$+ L \int d\lambda \mathbb{P}(\lambda) e^{-2\lambda\beta}. \tag{3.37}$$

The top line is the continuum representation of off-diagonal terms in the sum defining spectral form factor, whereas the second line captures diagonal terms in which the two eigenvalues coincide.

This expression, combined with the distribution (3.35), contains the full content of the spectral form factor. Indeed, consider rewriting $\sin^2(x)$ as $1/2 - \cos(2x)/2$. Keeping just the first term (ignoring the sinusoidal oscillations) produces the ramp. One can see this as follows: with $\sin^2()$ approximated by the constant value $1/2$, we have a term proportional to $(\lambda - \lambda')^{-2}$. Setting $\beta = 0$ for simplicity, the integrals in (3.36) become Fourier transforms, with a result proportional to t. This is the origin of the ramp in random matrix theory.

The plateau arises from the remaining $\cos(2x)/2$ term, which when properly accounted for interrupts the linearly growing ramp with a flat curve. The crossover from ramp to plateau is a sharp kink, as one would expect for the Fourier transform of the specific frequency occurring in $\cos(2(\lambda - \lambda') \ldots)$.

[9] Note that when x, x' coincide the expression $\sin^2(x-x')/(x-x')^2$ approaches 1 as can be seen by making a Taylor approximation to $\sin(x)$ or by using L'Hopital's rule.

Finally, we can identify the slope as well by examining the first term in (3.35),

$$\mathbb{P}(\lambda, \lambda') \sim \rho_0(\lambda, \lambda') \sim \rho_0(\lambda)\rho_0(\lambda'). \tag{3.38}$$

Though it is not of such great importance since it is non-universal in quantum chaos, the specific power of the slope's decay (t^{-3}) in random matrix theory can be seen in

$$Z(\beta + it) = \int d\lambda \rho_0(\lambda) e^{-\lambda(\beta+it)}. \tag{3.39}$$

This Fourier transform is dominated by the region of steepest descent of the function $\rho_0(\lambda)$, which occurs near the edge of the semicircle where $\rho_0(\lambda) \sim \sqrt{\lambda}$, yielding

$$Z(\beta + it) \sim \int dx \sqrt{x} e^{-x(\beta+it)} \propto \left(\frac{1}{\beta + it}\right)^{3/2} \tag{3.40}$$

As promised, this gives a spectral form factor $\langle |Z|^2 \rangle \sim 1/t^3$.

As a final general remark about the spectral form factor, we note here that, if one does not perform the ensemble average but instead samples a single realization of the random matrix H, the slope will look the same (we say it is "self-averaging") but the ramp and plateau will be replaced by wild oscillations about their average values (see Fig. 3.2). In technical terms, "wild" here amounts to having amplitude the same magnitude as their mean and occurring on a timescale $\Delta E_{\mathrm{RME}}^{-1}$ controlled by the spectral window defining our random matrix ensemble.

3.1.2 Resolvent, Loop Equations, Double-Scaling Limits

So far we have presented random matrix theory in the context of random matrix universality of chaotic quantum systems. The following discussion will go beyond this to probe non-universal features of random matrix ensembles. The reason for this is that our theory of interest, JT gravity, in fact agrees with a random matrix ensemble *on the nose* at the quantum level. To properly address this rather astonishing result, we will con-

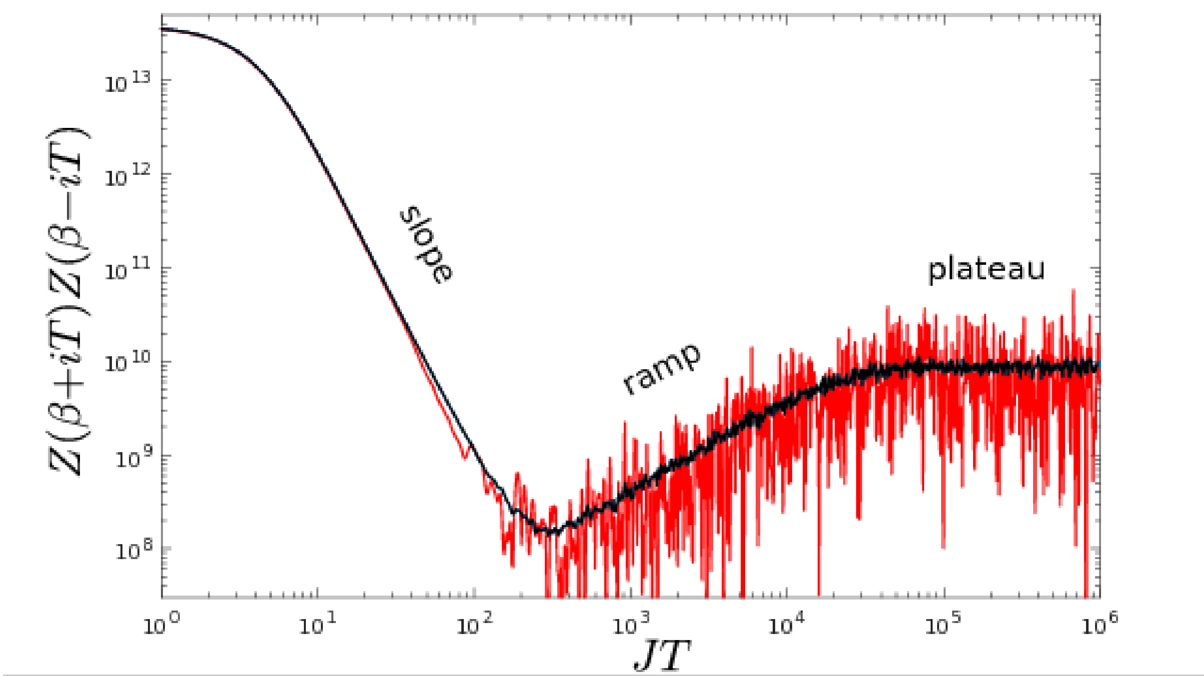

Figure 3.2: Spectral form factor for one sample of SYK. Figure taken from [11].

sider observables in RMT beyond the spectral form factor, beginning with the so-called "resolvent."

We recall the definition (3.5) of the RMT partition function

$$\mathcal{Z}_{\text{RMT}} = \int_{RME} [dH] e^{-L\text{Tr}V(H)}, \tag{3.41}$$

where as before H is an $L \times L$ random Hermitian matrix variable. When considered in the "classical" $L \to \infty$ limit, the eigenvalues form a distribution $\rho_0(\lambda)$ which can be obtained directly from the matrix potential $V(H)$. In fact, in what follows, we will require only a certain form of the leading density of eigenvalues $\rho_0(\lambda)$, and be happy to accept any matrix potential $V(H)$ whose large-L limit reproduces $\rho_0(\lambda)$.

The observables we will consider to probe beyond universal (that is, $V(H)$-independent) information are correlators of matrix resolvents:

$$f(E_1, E_2, \ldots, E_n) \equiv \left\langle \mathrm{Tr}\frac{1}{E_1 - H} \ldots \mathrm{Tr}\frac{1}{E_n - H} \right\rangle_{\text{connected}} \tag{3.42}$$

$$\simeq \sum_{g=0}^{\infty} \frac{R_{g,n}(E_1, \ldots, E_n)}{L^{2g+n-2}}. \tag{3.43}$$

The meaning of $\simeq$ in the above equation is that, though what is on its left-hand side is well-defined, what appears on its right-hand side is only an asymptotic series in $1/L$. Our ultimate goal will be to derive a recursion relation for the quantities $R_{g,n}$.

We now introduce the *loop equations* [10], an alternative to the method of orthogonal polynomials used above. These equations are simply Schwinger-Dyson equations for the random matrix integral of a total (matrix) derivative,

$$0 = \int [dH] \frac{\partial}{\partial H_{ij}} \left(\left(\frac{1}{E_1 - H} \right)_{ij} \mathrm{Tr}\frac{1}{E_2 - H} \ldots \mathrm{Tr}\frac{1}{E_n - H} e^{-L\mathrm{Tr}V(H)} \right). \tag{3.44}$$

The derivative pulls down terms from each of the factors in parentheses to give a relation between the observables under consideration, the resolvents.[10]

We now make our departure from a general discussion about a generic matrix ensemble and consider instead the specific matrix ensemble that appears in JT gravity. Our starting point will be the leading eigenvalue density $\rho_0(\lambda)$, which we can derive from the one-loop exact expression for the JT partition function $Z(\beta)$ on the Poincaré disc written

[10]This is a somewhat subtle matter since not all of the quantities under consideration are immediately recognizable as resolvents. The resolution to this has been worked out in the literature and we will soon cite an example of such recursion relations.

down in Section 2.1.[11]

$$Z(\beta) = e^{S_0} \frac{e^{\pi^2}/\beta}{4\sqrt{\pi}\beta^{3/2}} \tag{3.45}$$

$$\equiv \int dE e^{-\beta E} \rho_0(E) \tag{3.46}$$

$$\implies \rho_0(E) = \frac{e^{S_0}}{4\pi^2} \sinh(2\pi\sqrt{E}), \tag{3.47}$$

where the last line is obtained from the inverse Laplace transformation of the line above. Alarmingly, the eigenvalue density is not normalizable, apparently necessitating an infinite number of eigenvalues. This situation is encountered in RMT in what are called "double-scaling limits."

Heuristically, such double-scaling limits consist simply of "zooming in" on a particular region of the leading eigenvalue density. Though the density looks non-normalizable, we imagine there is a finite entropy S_0 and, despite that we are taking $L \to \infty$, we do perturbation theory in e^{-S_0} rather than $1/L$. [12] One caveat is that the matrix potential $V(H)$ actually diverges in such a limit—only the effective potential for a single eigenvalue including the repulsive interactions with all the other eigenvalues remains finite—but since what is of interest to us in rather the quantity $\rho_0(E)$ this poses no problem.

3.1.3 Topological Recursion

As noted above, we aim to derive recursion relations for the quantities $R_{g,n}(E_1, \ldots, E_n)$ in the random matrix integral genus expansion. It turns out to be more convenient to work instead with another, related, set of quantities, defined as

$$W_{g,n}(z_1, \ldots, z_n) \equiv (-1)^n 2^n z_1 \ldots z_n R_{g,n}(-z_1^2, \ldots, -z_n^2). \tag{3.48}$$

[11] Note that one can pass between $Z(\beta)$ and the resolvent via an integral transform as $\int_0^\infty d\beta Z(\beta) e^{\beta E} = \int_0^\infty d\beta \mathrm{Tr}(e^{-\beta H}) e^{\beta E} = \mathrm{Tr} \frac{1}{E-H}$.

[12] This procedure bears some resemblance to "zooming in" to some particular energy window in passing to the microcanonical ensemble. In such a situation the rank of the Hamiltonian will be none other than e^{S_0}, providing good motivation for using random matrices of this rank.

There are two exceptions to the above definition for the lowest values of the parameters g, n:

$$W_{0,1}(z) = 2zy(z) \tag{3.49}$$

$$W_{0,2}(z_1, z_2) = \frac{1}{(z_1 - z_2)^2}, \tag{3.50}$$

where $y(z) = e^{-S_0} i\pi\rho_0(-z^2) = \sin(2\pi z)/4\pi$. This quantity $y(z)$ is called the "spectral curve." The purpose for changing coordinates from the E_i to the z_i as in (3.48) is to make $y(z)$ a single-valued fucntion. Recall that the spectral density $\rho_0(E) \propto \sinh(2\pi\sqrt{E})$ written as a function of E is double-valued. The recursion relations, which ultimately are nothing other than a sophisticated rewriting of the loop equations above, then take the form

$$\begin{aligned} W_{g,n}(z_1, \overbrace{z_2, \ldots, z_n}^{\text{``}J\text{''}}) = \mathrm{Res}_{z=0} \Bigg\{ \frac{1}{z_1^2 - z^2} \frac{1}{4y(z)} \Big[& W_{g-1,n+1}(z, -z, J) \\ & + \sum_{\substack{I \cup I' = J \\ h + h' = g}} W_{h,1+|I|}(z, I) W_{h',1+|I'|}(-z, I') \Big] \Bigg\}. \end{aligned} \tag{3.51}$$

The residue appearing here helps patch up the concerns alluded to in footnote [10] regarding whether the loop equations can be written exclusively in terms of resolvents (or alternatively the variables $W_{g,n}$). The simplest nontrivial example (in which only the first term on the right-hand side of (3.51) appears) is

$$W_{1,1}(z_1) = \mathrm{Res}_{z=0} \left\{ \frac{1}{z_1^2 - z^2} \frac{\pi}{\sin(2\pi z)} \frac{1}{4z^2} \right\} = \frac{3 + 2\pi^2 z_1^2}{25 z_1^4}. \tag{3.52}$$

As a reminder about the purpose of considering such objects, consider the RME expectation value of a single resolvent,

$$\left\langle \mathrm{Tr} \frac{1}{E_1 - H} \right\rangle = e^{S_0} R_{0,1} + e^{-S_0} R_{1,1} + \mathcal{O}(e^{-3S_0}). \tag{3.53}$$

Thus $W_{1,1}$, defined in terms of $R_{1,1}$, tells us about the first subleading term in the the genus expansion for this single resolvent operator. More generally, $W_{g,n}$ tells us about the g-th subleading term in the genus expansion of an observable consisting of n resolvents. Note that all such $W_{g,n}$ with the sole exception of $W_{0,1}$ are rational functions of the z_i variables.

Incredibly, and crucially for the discussion below on the genus expansion in JT gravity, these quantities $W_{g,n}$ are related to the volumes $V_{g,n}(b_1, \dots, b_n)$ of moduli space of genus-g Riemann surfaces with n geodesic boundaries of respective lengths $b_1, \dots, b_n$. This relation was proved by Eynard and Orantin in [12], who showed that one can go back and forth between the two quantities by a Laplace transform. Upon doing so, the recursion relations (3.51) for $W_{g,n}$ become Mirzakhani's recursion relations for the volumes of moduli space of bordered Riemann surfaces. [13]. We collect the results for both $W_{g,n}$ and $V_{g,n}$ below[14]:

$$\begin{array}{ccc} W_{0,1} = 2z_1 \frac{\sin(2\pi z_1)}{4\pi}, & W_{0,2} = \frac{1}{(z_1 - z_2)^2} & W_{0,3} = \frac{1}{z_1^2 z_2^2 z_3^2} \\ W_{1,1} = \frac{3 + 2\pi^2 z_1^2}{24 z_1^4} & W_{2,1} = \left(\frac{105}{128 z_1^1 0} + \frac{203\pi^2}{192 z_1^8} + \frac{139\pi^4}{192 z_1^6} + \frac{169\pi^6}{480 z_1^4} + \frac{29\pi^8}{192 z_1^2} \right) & \end{array}$$

The corresponding expressions for the volumes of moduli space of bordered Riemann surfaces are

$$\begin{array}{ccc} V_{0,1} = \text{undefined}, & V_{0,2} = \text{undefined} & V_{0,3} = 1 \\ V_{1,1} = \frac{1}{48}(b_1^2 + 4\pi^2) & V_{2,1} = \frac{(4\pi^2 + b_1^2)(12\pi^2 + b_1^2)}{2211840}(6960\pi^4 + 384\pi^2 b_1^2 + 5b_1^4) & \end{array}$$

One can check explicitly in these examples that the two quantities really are related via Laplace transformations as

$$W_{g,n}(z_1, \dots, z_n) = \int_0^\infty b_1 db_1 e^{-b_1 z_1} \cdots \int_0^\infty b_n db_n e^{-b_n z_n} V_{g,n}(b_1, \dots, b_n). \tag{3.54}$$

Chapter 4

Two Dimensions - Jackiw-Teitelboim Gravity

4.1 JT Gravity as a BF Theory

Though there is no description of JT gravity as a Chern-Simons theory as in the case of 3d gravity, without matter fields JT gravity is still a topological theory and as such can be classically formulated as a type of topological quantum field theory (TQFT) known as a BF theory.[1] This section follows [14].

We begin by passing from the second order metric formulation of JT gravity to the first order frame field and spin connection variables, defined as

$$e^a = e^a_i dx^i \tag{4.1}$$

$$\omega^a_b = \epsilon^a_b \omega \tag{4.2}$$

[1]See for example [15].

with $\epsilon_2^1 \equiv 1 = -\epsilon_1^2$. The second order equations of motion again become in the first order language the no torsion constraint and the definition of the curvature tensor as

$$de^a = -\omega^a_b \wedge e^b \tag{4.3}$$

$$R^a_b = d\omega^a_b + \underbrace{\omega^a_c \wedge \omega^c_b}_{\sim\omega\wedge\omega=0} = d\omega^a_b. \tag{4.4}$$

Moreover, in two spacetime dimensions, the curvature tensor is entirely determined by the Ricci scalar R, so that

$$d\omega^a_b = \frac{1}{2} R e^a \wedge e_b. \tag{4.5}$$

We can furthermore translate what appears in the second order action into first order language as

$$\sqrt{g} d^2 x R = 2 d\omega^1_2 = 2 d\omega \tag{4.6}$$

$$\sqrt{g} d^2 x = e^1 \wedge e_2 = e^1 \wedge e^2 \tag{4.7}$$

and thus the bulk JT term becomes

$$\frac{1}{2}\int_{\mathcal{M}} \sqrt{g}\Phi(R+2) \mapsto \int_{\mathcal{M}} \Phi(d\omega + e^1 \wedge e^2) + \Phi_a(de^a + \epsilon^a_b \omega \wedge e^b) \tag{4.8}$$

$$\equiv i \int_{\mathcal{M}} \mathrm{Tr}(BF) \tag{4.9}$$

where Φ_a is a Lagrange multiplier enforcing the no torsion condition $de + \omega \wedge e = 0$. In the last line we have repackaged $e^a, \omega, \Phi, \Phi^a$ into matrix variables B, A (with $F = DA = dA + A \wedge A$ the curvature of the gauge field A) given by

$$B = -i \begin{pmatrix} -\Phi^1 & \Phi^2 + \Phi \\ \Phi^2 - \Phi & \Phi^1 \end{pmatrix}, \tag{4.10}$$

$$A = +\frac{1}{2} \begin{pmatrix} -e^1 & e^2 - \omega \\ e^2 + \omega & e^1 \end{pmatrix}. \tag{4.11}$$

In BF theories, the matrix B functions not as a dynamical degree of freedom but as a Lagrange multiplier enforcing the flatness of the connection, $F = 0$. The factors of i in the definition of B and outside the BF action arrange for the imaginary contours of Φ, Φ^1, Φ^2 to produce a *real* contour for the matrix B.

We have thus recast JT gravity into a path integral over the space of flat connections $DA = 0$, with measure induced by the symplectic form on the space of gauge fields:[16]

$$\Omega(\sigma, \eta) = 2\alpha \int \mathrm{Tr}(\sigma \wedge \eta). \tag{4.12}$$

In this expression σ, η live in the tangent space to the space of gauge fields—they are "evaluation" one-forms on phase space parametrizing infinitesimal variations of A. Of key importance is that Ω is Kahler compatible with the metric on the space of one forms:

$$g(\sigma, \eta) = \Omega(\sigma, J\eta) \tag{4.13}$$

where J is a complex structure satisfying $J^2 = -\mathbb{1}$ and $\Omega(\sigma, \eta) = \Omega(J\sigma, J\eta)$.[17]

In BF theory we can take $J = \star$ to be the Hodge star operator, thus giving a metric on the space of infinitesimal variations of A to be

$$g(\sigma, \eta) = 2\alpha \int \mathrm{Tr}(\sigma \wedge \star\eta). \tag{4.14}$$

The above expression works for BF theories with compact gauge-groups. In the JT gravity case, however, the gauge group $SL(2, \mathbb{R})$ is non-compact, and we must make the following modification,

$$g(\sigma, \eta) = 2\alpha \int \mathrm{Tr}(\sigma \wedge \star T\eta). \tag{4.15}$$

T reverses the sign of the negative directions in the Lie algebra metric. Here we must take $J = \star T$ for Kahler compatibility.

4.2 Genus Expansion of JT Path Integral

Recall the expression (2.2) for the JT action

$$\mathrm{I}_{\mathrm{JT}} = -\Phi_0 \left[\int_{\mathcal{M}} \sqrt{g} R + 2 \int_{\partial \mathcal{M}} \sqrt{h} K \right] - \left[\int_{\mathcal{M}} \sqrt{g} \Phi (R+2) + 2 \int_{\partial \mathcal{M}} \sqrt{h} \Phi (K-1) \right]. \quad (4.16)$$

The observables we will consider are connected correlators of partition functions

$$\langle Z(\beta_1) \dots Z(\beta_n) \rangle_{\text{conn.}} \quad (4.17)$$

where each partition function is given by $Z(\beta) = \mathrm{Tr}(e^{-\beta H})$. These observables are related to the resolvents introduced above via an integral transform. From a "bulk" perspective we can compute $\langle Z(\beta_1) \dots Z(\beta_n) \rangle_{\text{conn.}}$ by imposing boundary conditions on our bulk Euclidean spacetime manifold to have n geodesic boundaries of lengths $\beta_1, \dots, \beta_n$. We integrate over the moduli[2] of the bulk spacetime subject to this condition, and moreover allow spacetimes of arbitrary genus, summing over such choices with a weighting factor $(e^{-S_0})^{\chi_{\text{Euler}}}$ coming from the topological term in the JT action (2.2). All told,

$$\langle Z(\beta_1) \dots Z(\beta_n) \rangle_{\text{conn.}} \simeq \sum_{g=0}^{\infty} \frac{Z_{g,n}(\beta_1, \dots, \beta_n)}{(e^{S_0})^{2g+n-2}}. \quad (4.18)$$

An example of such a geometry is pictured in Fig. 4.1

The terms $Z_{g,n}(\beta_1, \dots, \beta_n)$ in the genus expansion are then

$$Z_{g,n}(\beta_1, \dots, \beta_n) = \int d(\text{bulk moduli}) \int D(\text{boundary wiggles}) e^{\int_{\partial \mathcal{M}} \sqrt{h} \Phi (K-1)}. \quad (4.19)$$

We have already encountered the "boundary wiggles" part of this path integral in the case of $g = 0, n = 1$ for JT gravity on the Poincaré disc in the Background section. There the boundary wiggles were described by the Schwarzian action. We can cut apart the geometry in 4.1 at minimal geodesics to separate the bulk Riemann surface from the regions

[2]For the uninitiated, "moduli" here simply refers to parameters that control the shape of the surface subject to the constant negative curvature constraint.

$$Z_{g=2,n=3}(\beta_1, \beta_2, \beta_3) \quad =$$

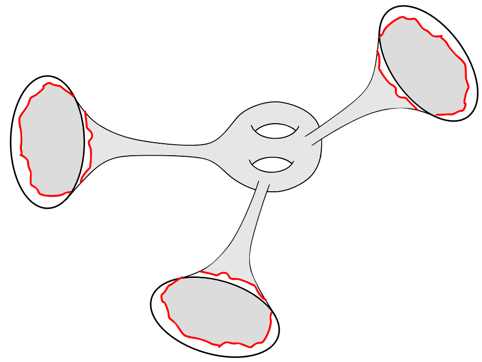

Figure 4.1: The $g = 2$ contribution to the observable $\langle Z(\beta_1)Z(\beta_2)Z(\beta_3)\rangle_{\text{conn.}}$. Figure taken from [14].

resembling trumpets that extend to the asymptotic boundary. For these trumpets there is a similar Schwarzian-like action whose path integral is also one-loop exact.

The difficult part of the computation (4.19) is in fact the finite-dimensional integral over bulk moduli. To perform this integral we first must determine the measure—that is, what, precisely, is meant by $d(\text{bulk moduli})$.

We begin by introducing the symplectic form on the space of bulk moduli. Of key importance is that any constant negative curvature Riemann surface can be constructed by gluing together pairs of pants (that is, topologically speaking, three-holed spheres). The requirement of constant negative curvature fixes all the freedom except the lengths $\tilde{b}_i$ and relative twists τ_i of the boundaries ("pant cuffs") as they are glued together. The coordinates $\{\tilde{b}_i, \tau_i\}$, with τ_i measured as a geodesic length rather than as an angle, are called Fenchel–Nielsen (FN) coordinates on moduli space. Since there are $k = 3g + n - 3$ gluings required to form a surface of genus g and with n boundaries, these coordinates are $2k = 6g + 2n - 6$ in number. The symplectic form on moduli space, known as the Weil-Petersson form, is then given by

$$\Omega = \alpha \sum_{i=1}^{3g+n-3} d\tilde{b}_i \wedge d\tau_i, \tag{4.20}$$

with α a coefficient yet to be determined. From here we can write the standard volume form $\text{Vol} = \Omega^k/k!$.

4.2.1 Weil-Petersson measure from BF theory

Our aim in this section is to use the symplectic form $\Omega(\sigma, \eta)$ on the space of flat $SL(2,\mathbb{R})$ connections to derive the Weil-Petersson symplectic measure on the moduli space of complex curves.[3]

Recall that σ, η live in the tangent space to A in the space of flat $SL(2,\mathbb{R})$ connections,

$$\sigma, \eta \in \{\delta A | \text{ for flat connections } A, A' \equiv A + \epsilon\delta A \text{ still flat}\}. \tag{4.21}$$

The condition $DA' = dA' + A' \wedge A' = 0$ leads to the a condition on δA

$$d(\delta A) + A \wedge \delta A + \delta A \wedge A = 0. \tag{4.22}$$

To better explain the language "evaluation two-form" used above, note that $\Omega(\cdot,\cdot)$ is an object that accepts two infinitesimal gauge field variations and returns a number, $\Omega(\delta_1 A, \delta_2 A) \in \mathbb{R}$.

We now show that Ω is gauge invariant. Under the gauge transformation

$$\delta_2 A \to \delta_2 A + d\Theta + [A, \Theta] \tag{4.23}$$

$$\Omega(\delta_1 A, \delta_2 A) \to \Omega(\delta_1 A, \delta_2 A) + 2\alpha \int \text{Tr}(\delta_1 A \wedge (d\Theta + [A, \Theta])). \tag{4.24}$$

The second term in (4.24) vanishes after integration by parts since A is flat.

Considering for now two pant legs being glued together at their cuffs, we choose coordinates[4] ρ, x such that $|\rho|$ measures the distance away from the cuff and the sign of ρ specifies which pant leg we are on, while x measures distance around the cuff, normalized

[3]Note that an object of complex dimension one has real dimension two, explaining the term "complex curve" commonly used interchangeably with Riemann surface in the math literature.

[4]Note that these are coordinates on the Riemann surface in question itself, rather than coordinates on its moduli space.

such that $x \sim x + 1$. The metric in these coordinates takes the form

$$ds^2 = d\rho^2 + \cosh^2 \rho[bdx + \tau\delta(\rho)d\rho]^2. \tag{4.25}$$

The delta function serves to account for a possibly non-zero twist in the gluing. Indeed, one can see that as a particle moves from one pant cuff to the other it experiences a "jump" by τ along the x direction as follows. Define a coordinate y to be what is written in brackets in (4.25),

$$dy = bdx + \tau\delta(\rho)d\rho. \tag{4.26}$$

Then by integrating we find

$$y = bx + \tau\theta(\rho). \tag{4.27}$$

As one crosses from $\rho < 0$ to $\rho > 0$ the theta function turns on and, since the twisted coordinate y is continuous, x must jump by τ to compensate. This discontinuity is what it means for there to be a nonzero twist.

We can make an analogous statement in first order (that is, BF theory) language. As before we take A to be given by (4.11), where the frame field and spin connection here take the form

$$e^1 = d\rho, \qquad e^2 = \cosh\rho[bdx + \tau\delta(\rho)d\rho], \qquad \omega = -b\sinh\rho dx. \tag{4.28}$$

An explicit calculation then shows

$$\mathrm{Tr}(\delta_1 A \wedge \delta_2 A) = \frac{1}{2}[\delta_1 b \wedge \delta_2\tau - \delta_2 b \wedge \delta_1\tau]\delta(\rho)dxd\rho, \tag{4.29}$$

from which we find

$$\Omega(\delta_1 A, \delta_2 A) \equiv 2\alpha \int_{x=0,\rho=-\infty}^{x=1,\rho=\infty} \mathrm{Tr}(\delta_1 A \wedge \delta_2 A) \tag{4.30}$$

$$= \alpha[\delta_1 b \wedge \delta_2\tau - \delta_2 b \wedge \delta_1\tau]. \tag{4.31}$$

This is none other than the Weil-Petersson symplectic form (4.20), which we have now derived from the perspective of $SL(2,\mathbb{R})$ BF theory.

4.2.2 Asymptotic Boundary Conditions

To proceed we require the measure over the Schwarzian modes as well (D(boundary wiggles) from (4.19)). To get there we must present the asymptotic boundary conditions of JT theory in the first order formulation. We will choose these boundary conditions on the $SL(2,\mathbb{R})$ gauge field A in order to reproduce the boundary conditions in the second order formulation of JT reviewed in section 2.1,

$$g_{uu}|_{\text{bdy}} = \frac{1}{\epsilon^2}, \qquad \Phi|_{\text{bdy}} = \frac{\Phi_r}{\epsilon}, \qquad \epsilon \to 0 \tag{4.32}$$

Recall the coordinate system r, u on the Poincaré disc used here with r the distance toward the boundary and u the geodesic length along the boundary. We set the holographic renormalization parameter $\epsilon = 2e^{-r}$ (so that as the boundary is taken to $r = \infty$ we are taking the $\epsilon \to 0$ limit), the boundary condition (2.5) on the extrinsic curvature of the boundary becomes[5]

$$K = 1 + 4e^{-2r}\text{Sch}(\theta(u)/2, u). \tag{4.33}$$

This extrinsic curvature arises from the following asymptotic form of the metric

$$ds^2 = dr^2 + \left(\frac{1}{4}e^{2r} - \text{Sch}\Big(\frac{\tan(\theta(u))}{2}, u\Big) + \dots\right) du^2. \tag{4.34}$$

We now aim to translate (4.34) into a condition on the gauge field A in BF theory. A standard boundary condition in BF theory is

$$B + icA_u|_{\text{bdy}} = 0 \tag{4.35}$$

[5]Recall that $\theta(u)$ unconventionally parameterizes the boundary by the angle swept out as a function of its geodesic length u.

for some undetermined constant c. This adds a boundary term to the BF action of the form

$$I_{BF} = -i \int_{\mathcal{M}} +\text{Tr}(BF) + \frac{i}{2} \int_{\partial\mathcal{M}} \text{Tr}(BA). \tag{4.36}$$

The condition (4.35) descends from 3d when viewing BF theory as a dimensional reduction of Chern-Simons theory. To see this, consider decomposing the 3d Chern-Simons gauge field as $A^{(3d)} = A + Bdx^3$. Then 4.35 corresponds to setting a linear combination of the 3d gauge field components to zero at the boundary, as in done e.g. in the Brown-Heanneaux boundary conditions.[6] [19]

Here we simply translate the constraint (4.34) directly into first order language, with the result [7]

$$e^1 = dr, \qquad e^2 = \frac{1}{2}e^r - \text{Sch}(u)du, \qquad \omega = -\frac{1}{2}e^r + \text{Sch}(u)du. \tag{4.37}$$

Then the BF gauge field behaves at large r as

$$A \xrightarrow{r\to\infty} \frac{dr}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{du}{2}\begin{pmatrix} 0 & e^r \\ -2e^{-r}\text{Sch}(u) & 0 \end{pmatrix}. \tag{4.38}$$

Note that here, and above, $\text{Sch}(u)$ is an arbitrary function parameterizing the shape of the wiggly boundary.

To fix the constant in (4.35) we impose $\Phi|_{\text{bdy}} = \Phi_r/\epsilon = (1/2)e^r\Phi_r$ which leads to $c = 2\Phi_r$. Then solving (4.35) as $B = -2i\Phi_r A_u|_{\text{bdy}}$, the boundary term in I_{BF} becomes

$$I_{\partial\mathcal{M}} = \Phi_r \int du \text{Tr}(A_u)^2 = -\Phi_r \int du \text{Sch}(u). \tag{4.39}$$

4.2.3 The Schwarzian boundary mode and its symplectic form

The mode controlling the dynamics of the boundary arises from "large gauge tranformations" $\Theta(r,u)$ that do not vanish at the boundary, where they act physically. We will

[6] As we will see in Chapter 5, such a choice reduces the 3d Chern-Simons action to a chiral Wess-Zumino-Witten model.[18]

[7] We switch now to a shorthand $\text{Sch}(u) \equiv \text{Sch}(\frac{\tan\theta(u)}{2}, u)$

see the same phenomenon in the 3d case in Chapter 5—this is how Chern-Simons theory reduces to a chiral WZW "boundary graviton" mode.

A generic gauge transformation $\Theta(r,u)$ contains three independent parameters, but enforcing compatibility of the transformed gauge field $A' = A + d\Theta + [A,\Theta]$ with the boundary conditions (4.38) reduces this freedom to a single degree of freedom $\varepsilon(u)$. The resulting asymptotic condition on the behavior of $\Theta(u,r)$ is

$$\Theta(u,r) \xrightarrow{r\to\infty} \begin{pmatrix} \frac{1}{2}\varepsilon'(u) & \frac{1}{2}e^{r}\varepsilon(u) \\ -e^{-r}[\mathrm{Sch}(u)\varepsilon(u)+\varepsilon''(u)] & -\frac{1}{2}\varepsilon'(u) \end{pmatrix}. \tag{4.40}$$

Such a gauge transformation acts on the $\mathrm{Sch}(u)$ appearing in A as

$$\mathrm{Sch}(u) \mapsto \mathrm{Sch}(u) + \varepsilon'''(u) + \varepsilon(u)\mathrm{Sch}'(u) + 2\varepsilon'(u)\mathrm{Sch}(u). \tag{4.41}$$

This matches the behavior of $\mathrm{Sch}(u) = \mathrm{Sch}(f(u),u)$ under infinitesimal reparametrizations $u \to u+\varepsilon(u)$. Thus $\Theta(u,r)$ induces the boundary wiggles, which are nothing other than such reparametrizations (i.e. elements of $\mathrm{Diff}(S^1)$).

We can now nail down the measure for the Schwarzian mode by evaluating the symplectic form on a pair $\delta_i A = d\Theta_i + [A,\Theta_i]$ of pure gauge infinitesimal gauge field variations. Integrating by parts to pick up a boundary term and imposing (4.40) we find

$$\Omega(\delta_1 A, \delta_2 A) = 2\alpha \int_{\mathcal{M}} \mathrm{Tr}(\delta_1 A \wedge \delta_2 A) \tag{4.42}$$

$$= 2\alpha \int_{\partial\mathcal{M}} \mathrm{Tr}(\Theta_1(d\Theta_2 + [A,\Theta_2])) \tag{4.43}$$

$$\overset{(4.40)}{=} \alpha \int_0^\beta du[\varepsilon_1'(u)\varepsilon_2''(u) - \mathrm{Sch}(u)(\varepsilon_1(u)\varepsilon_2'(u) - \varepsilon_1'(u)\varepsilon_2(u))]. \tag{4.44}$$

This result can be recast geometrically as

$$\Omega = \frac{\alpha}{2}\int_0^\beta du[d\varepsilon'(u)\wedge d\varepsilon''(u) - 2\mathrm{Sch}(u)d\varepsilon(u)\wedge d\varepsilon'(u)]. \tag{4.45}$$

Note that the same α appears here as in the Weil-Petersson measure for the bulk moduli (4.20) since both are derived from the general form (4.12).

4.2.4 Path integrals for Schwarzian boundary modes

There are two cases to consider for the boundary Schwarzian fluctuations ("boundary wiggles") at play in the JT path integral. We will first derive the result for the Poincaré disc geometry, quoted in Section 2.1, and then perform a similar calculation for the trumpet regions pictured in Fig. 4.1.

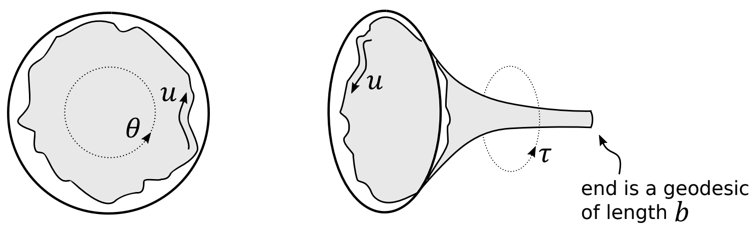

Figure 4.2: The boundary Schwarzian mode in Poicare disc (left) and trumpet (right) geometries. Figure taken from [14].

The one-loop exactness of both these path integrals allows us to compute them in perturbation theory around their respective classical saddles, and nevertheless get the correct answer.

Considering first the Poincaré disc case, we will use the coordinates $ds^2 = d\rho^2 + \sinh^2 \rho d\theta^2$. We parametrie the boundary as before by the angle $\theta(u)$ swept out as a function of its proper length. The other coordinate, $\rho(u)$ is determined by from the condition $g_{uu}|_{\text{bdy}} = 1/\epsilon^2$. The JT action in this geometry reduces to (see (2.6))

$$\mathrm{I}_{\text{JT}}|_{\text{disc}} = -\Phi_r \int du \,\text{Sch}\left(\tan\frac{\theta}{2}, u\right) = -\frac{\Phi_r}{2}\int du \left(\left(\frac{\theta''}{\theta'}\right)^2 - \theta'^2\right), \tag{4.46}$$

where we have explicitly evaluated the Schwarzian derivative. The Schwarzian path integral we want to consider is then

$$Z^{\text{disc}}_{\text{Sch}}(\beta) = \int \frac{d\mu[\theta]}{SL(2,\mathbb{R})} \exp\left[-\frac{\Phi_r}{2}\int_0^\beta du \left(\left(\frac{\theta''}{\theta'}\right)^2 - \theta'^2\right)\right]. \tag{4.47}$$

The point of the preceding subsection was to help find the measure $d\mu[\theta]$—it is the one induced by the symplectic form (4.45). First note, however, the $SL(2,\mathbb{R})$ quotient. Its purpose is to remove the zero modes of the boundary wiggles—translations of the region within the Poincaré disc represent degenerate contributions to the path integral and we do not want to double-count such contributions.[8]

Taking advantage of the fact that the integral is known to be one-loop exact, we consider small fluctuations around the saddle $\theta(u) = 2\pi u/\beta$,

$$\theta(u) = \frac{2\pi}{\beta}(u + \varepsilon(u)). \tag{4.48}$$

To quadratic order, ε has three zero modes, the $SL(2,\mathbb{R})$ generators $\varepsilon = 1, e^{\pm\frac{2\pi}{\beta}iu}$. We can perform the $SL(2,\mathbb{R})$ quotient by simply not integrating over these zero modes. Thus we take

$$\varepsilon(u) = \sum_{|n|\geq 2} e^{-\frac{2\pi}{\beta}inu}(\varepsilon_n^{(\mathrm{R})} + i\varepsilon_n^{(\mathrm{I})}). \tag{4.49}$$

Requiring $\epsilon(u) \in \mathbb{R}$ fixes $\varepsilon_n^{(\mathrm{R})} = \varepsilon_{-n}^{(\mathrm{R})}$ and $\varepsilon_n^{(\mathrm{I})} = -\varepsilon_{-n}^{(\mathrm{I})}$. Using this form of $\varepsilon(u)$ in (4.45) leads to

$$\Omega = 2\alpha\frac{(2\pi)^2}{\beta^2}\sum_{n\geq 2}(n^3 - n)d\varepsilon_n^{(\mathrm{R})} \wedge d\varepsilon_n^{(\mathrm{I})}. \tag{4.50}$$

Thus at last we have for the symplectic measure

$$\frac{d\mu[\theta]}{SL(2,\mathbb{R})} = \prod_{n\geq 2} 2\alpha\frac{(2\pi^3)}{\beta^2}(n^3 - n)d\varepsilon_n^{(\mathrm{R})}d\varepsilon_n^{(\mathrm{I})}. \tag{4.51}$$

[8] Recall that $SL(2,\mathbb{R})$ is the isometry group of the Poincaré disc.

Expanding the Schwarzian action to quadratic order in ε, we find for the disc partition function[9]

$$\boxed{Z^{\text{disc}}_{\text{Sch}}(\beta)} = e^{\pi^2/\beta} \prod_{n\geq 2} 2\alpha \frac{(2\pi)^3}{\beta^2}(n^3-n) \int d\varepsilon_n^{(\text{R})} d\varepsilon_n^{(\text{I})} \exp\left[-(2\pi)^4 \frac{(\varepsilon_n^{(\text{R})})^2+(\varepsilon_n^{(\text{I})})^2}{2\beta^3}(n^4-n^2)\right] \tag{4.52}$$

$$= e^{\pi^2/\beta} \prod_{n\geq 2} 2\alpha \frac{(2\pi)^3}{\beta^2}(n^3-n) \frac{2\pi\beta^3}{(n^4-n^2)(2\pi)^4} \tag{4.53}$$

$$= e^{\pi^2/\beta} \prod_{n\geq 2} \frac{2\alpha\beta}{n} = \boxed{e^{\pi^2/\beta} \frac{1}{4(\alpha\beta)^{3/2}\sqrt{\pi}}}. \tag{4.54}$$

The divergent product is treated using zeta-function regularization in the final step, and the divergent term is discarded. Alternatively one could turn the product into a sum by taking a logarithm, expanding the log of the product as a sum of logs, and re-exponentiating the result. Introducing a smooth cutoff in this fashion will lead to the same result upon discarding the divergent piece. This matches the result (2.15) for $\alpha^{3/2} = e^{-4\pi\Phi_0}$.

We now turn to the case of the trumpet geometries. We use the plural since there is a distinct trumpet for each value b of the "minimal length geodesic" at its short end. We can create such geometries by periodically identifying a section of the Poincaré disc:

$$ds^2 = d\sigma^2 + \cosh^2\sigma d\tau^2; \quad \tau \sim \tau + \beta. \tag{4.55}$$

The periodic identification breaks the $SL(2,\mathbb{R})$ symmetry of the disc down to a $U(1)$ subgroup given by rotation around the τ direction. The shape of the boundary is now specified by a function $\tau(u)$. The boundary action becomes

$$I^{\text{trumpet}}_{\text{bdy}} = -\Phi_r \int du\, \text{Sch}(e^{-\tau}, u), \tag{4.56}$$

[9] In what follows we set $\Phi_r = 1/2$.

leading to the following expression for the path integral over the boundary Schwarzian modes for the trumpet geometries

$$Z_{\text{Sch}}^{\text{trumpet}}(\beta, b) = \int \frac{d\mu[\tau]}{U(1)} \exp\left[-\frac{\Phi_r}{2}\int_0^\beta du\, \left((\frac{\tau''}{\tau'})^2 + \tau'^2\right)\right]. \tag{4.57}$$

Again leveraging the one-loop exactness we proceed in perturbation theory around the saddle point $\tau(u) = \frac{b}{\beta}u$,

$$\tau(u) = \frac{b}{\beta}(u + \varepsilon(u)). \tag{4.58}$$

The saddle point satisfies $\mathrm{Sch}(ub/\beta, u) = 0$. Evaluating the symplectic form as before, and writing the associated measure $d\mu[\tau]$, end up with the path integral

$$\boxed{Z_{\text{Sch}}^{\text{trumpet}}(\beta, b)} = e^{\frac{-b^2}{4\beta}} \prod_{n\geq 1} 2\alpha\frac{(2\pi)^3}{\beta^2}(n^3 + \frac{b^2}{(2\pi)^2}n) \int d\varepsilon_n^{(\text{R})} d\varepsilon_n^{(\text{I})} \exp\left[-(2\pi)^4\frac{(\varepsilon_n^{(\text{R})})^2 + (\varepsilon_n^{(\text{I})})^2}{2\beta^3}(n^4 + \frac{b^2}{(2\pi)^2}n^2)\right] \tag{4.59}$$

$$= e^{\frac{-b^2}{4\beta}} \prod_{n\geq 1} 2\alpha\frac{(2\pi)^3}{\beta^2}(n^3 + \frac{b^2}{(2\pi)^2}n)\frac{2\pi\beta^3}{(2\pi)^4(n^4 + \frac{b^2}{(2\pi)^2}n^2)} \tag{4.60}$$

$$= e^{\frac{-b^2}{4\beta}} \prod_{n\geq 1} \frac{2\alpha\beta}{n} = \boxed{e^{\frac{-b^2}{4\beta}}\frac{1}{2\sqrt{\pi\alpha\beta}}}. \tag{4.61}$$

4.2.5 Fixed genus contributions to JT path integral

Let us for the moment focus on computing single boundary quantities. That is, our observable will be the JT partition function $Z(\beta)$, rather than the multi-point correlators $\langle Z\beta_1 \ldots Z(\beta_n)\rangle_{\text{conn.}}$. In this case the genus expansion (4.18) takes the form[10]

$$Z(\beta) = \sum_{g=0}^{\infty} \frac{Z_{g,1}(\beta)}{(e^{S_0})^{2g-1}} = e^{S_0}Z_{0,1}(\beta) + e^{-S_0}Z_{1,1}(\beta) + e^{-3S_0}Z_{2,1}(\beta) + \ldots \tag{4.62}$$

This asymptotic series in e^{-S_0} is presented schematically in Fig. 4.3. Our object of study thusfar has primarily been the $\mathcal{O}(e^{S_0})$ term in this expansion, which in the present context we would call the disc contribution to the JT path integral $Z(\beta)$.

[10]Note that the Euler characteristic of a surface with g handles and $h = 1$ boundary is $\chi = 1 - 2g$.

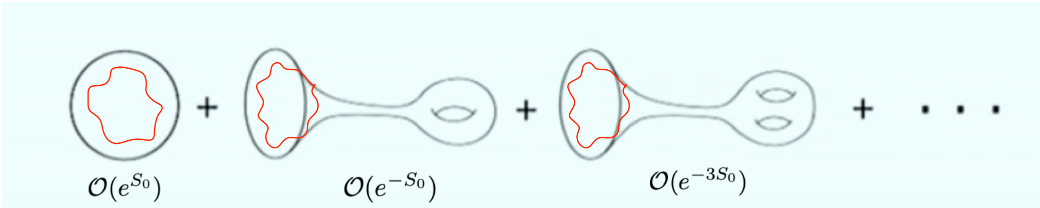

Figure 4.3: JT genus expansion for $Z(\beta)$. Figure adapted from [20].

Higher order terms in the expansion consist of higher genus compact Riemann surfaces, glued on to a trumpet geometry at a minimal geodesic boundary (the"throat" at the end of the trumpet). All the geometries pictured admit constant curvature metrics, as required by the dilaton equations of motion in the JT path integral. The diagram really is schematic since it suppresses an integral over the length b of this minimal geodesic, as well as integrals over the moduli of the higher genus surfaces.

Note that the terms in this expansion are *not* classical saddle points—they do not solve the JT equations of motion

$$\nabla_m \nabla_n \Phi + g_{mn}\Phi - g_{mn}\nabla^2\Phi = 0 \tag{4.63}$$

since they imply the existence of a Killing vector field $\xi^m = \epsilon^{mn}\partial_n\Phi$, which is not present for any of the pictured geometries.[11] The terms with $g \geq 1$ in the expansion (4.62) are given by

$$Z_{g,1}(\beta) = \int_0^\infty b\, db \underbrace{\left(\frac{e^{\frac{-b^2}{4\beta}}}{2\sqrt{\pi\beta}}\right)}_{Z_{\text{Sch}}^{\text{trumpet}}(\beta,b)} V_{g,1}(b), \tag{4.64}$$

where $V_{g,1}(b)$ is the volume of moduli space of genus-g Riemann surfaces with a single geodesic boundary of length b. The factor in parentheses is simply the one-loop exact trumpet partition function that appears in (4.61).[12]

Note the measure bdb in the integral over the length of the minimal geodesic. This results directly from the symplectic form (4.20). The twist coordinate τ does not make an

[11]Were we to consider the spectral form factor, the "cylinder" contribution does indeed have such a Killing isometry, but nevertheless it still does not satisfy the equations of motion.

[12]It is this factor, with its exponential damping by b^2, that causes the cylinder geometry to not satisfy the equations of motion. It represents a pressure shrinking the minimal geodesic length b to zero.

appearance in the integrand and thus we may trivially integrate over it. Recalling that τ is measured in geodesic length rather than as an angular coordinate, the appropriate bounds on the integral are from 0 to b, so that

$$\int_{\tau\in[0,b]} \Omega = \alpha db \int_0^b d\tau = \alpha b db. \tag{4.65}$$

We can immediately generalize the formula (4.62) to find a result for all the terms in the genus expansion (4.18) of generic JT observables. For $Z_{g,n}(\beta_1, \dots, \beta_n)$ we simply include the appropriate number of trumpet partition functions along with an additional factor of the volume of the moduli space of the bulk Riemann surface, as

$$Z_{g,n}(\beta_1, ..., \beta_n) = \int_0^\infty b_1 db_1 ... \int_0^\infty b_n db_n Z_{\text{Sch}}^{\text{trumpet}}(\beta_1, b_1) ... Z_{\text{Sch}}^{\text{trumpet}}(\beta_n, b_n) V_{g,n}(b_1, ..., b_n). \tag{4.66}$$

There are two special cases we must separate out—$(g,n) = (0,1)$ corresponds to the Poincaré disc, and $(g,n) = (0,2)$ corresponds to a cylinder geometry built by gluing together two trumpets:

$$Z_{0,1}(\beta) = Z_{\text{Sch}}^{\text{disc}}(\beta), \tag{4.67}$$

$$Z_{0,2}(\beta_1, \beta_2) = \int_0^\infty b db Z_{\text{Sch}}^{\text{trumpet}}(\beta_1, b) Z_{\text{Sch}}^{\text{trumpet}}(\beta_2, b). \tag{4.68}$$

4.2.6 Topological recursion and JT gravity

The data $Z_{g,n}$ constitute a solution to Eynard and Orantin's topological recursion, introuduced at the end of section 3.1.3. Recall that the spectral curve $y(z)$ determines universal information about random matrix theory observables—we would like to identify what form it takes in the case of JT gravity. Indeed, we can obtain it from the semiclassical approximation $\rho_0(E)$ to the density of states, which appears in the formula

$$Z_{0,1}(\beta) = \int_0^\infty \rho_0(E) e^{-\beta E} dE. \tag{4.69}$$

We refer to the result (4.52) for $Z_{0,1}(\beta) = Z^{\text{disc}}_{\text{Sch}}(\beta)$, which leads to an expression[13]

$$\rho_0(E) = \frac{1}{4\pi^2}\sinh(2\pi\sqrt{E}). \tag{4.70}$$

Translating $\sqrt{-E} = i\sqrt{E} = z$, we find for the spectral curve

$$y(z) = \frac{1}{4\pi}\sin(2\pi z). \tag{4.71}$$

Recall from section 3.1.3 that topological recursion is formulated in terms of variables $W_{g,n}$ defined in terms of the coefficients $R_{g,n}$ appearing in the RMT genus expansion by (3.48). Moreover, these $W_{g,n}$ (built out of resolvents) are related to the $Z_{g,n}$ (built out of partition functions) by the integral transform

$$W_{g,n}(z_1, ...z_n) = 2^n z_1 \dots z_n \int_0^\infty d\beta_1 e^{-\beta_1 z_1^2} ... \int_0^\infty d\beta_n e^{-\beta_n z_n^2} Z_{g,n}(\beta_1, ..., \beta_n). \tag{4.72}$$

After explicitly substituting the expression (4.61) for the trumpet partition function into the definition (4.66) of the quantities $Z_{g,n}$, and performing the β integrals we find

$$W_{g,n}(z_1, \dots, z_n) = \int_0^\infty b_1 db_1 e^{-b_1 z_1} \cdots \int_0^\infty b_n db_n e^{-b_n z_n} V_{g,n}(b_1, \dots, b_n). \tag{4.73}$$

This matches with our expectations from random matrix theory arrived at in (3.54).

[13] As above, we set $\Phi_r = 1/2$.

Chapter 5

Three Dimensions - Pure AdS$_3$ Gravity

5.1 Chern-Simons Formulation of AdS$_3$ Gravity

We consider 3d Lorentzian gravity with negative cosmological constant and no matter content. Our goal is to show this theory is equivalent to a Chern-Simons theory with gauge group $SL(2,\mathbb{R})$, as first done in [21] [22]. [1] To do so we work in the so-called first order formulation of general relativity (GR), where the metric is written in terms of a frame field e^a_i as $ds^2 = e^a_i e^b_j \delta_{ab} dx^i dx^j$. Here i, j are spacetime indices, raised and lowered with g_{ij}, and a, b are flat tangent space indices, raised and lowered with δ_{ab}. In the ordinary (second-order) formulation of GR, the degrees of freedom are the dynamical metric components and their derivatives (that is, the Christoffel symbols). Going from second- to first- order amounts to the transition from Lagrangian to Hamiltonian mechanics, and as such we consider the Christoffel symbols to be independent degrees of freedom (conjugate momenta), rather than derivatives of the metric, christening the resulting object the spin connection $\omega^a_{i\,b}$.[2] Geometrically, we view ω (with no indices) as a $SO(2,2)$-valued tangent bundle on our 3d spacetime, as this is the isometry group of Lorentzian AdS$_3$.

Out of the objects $e^a_i, \omega^a_{i\,b}$ we can define the curvature tensor as

$$R^a_{ijb} = \partial_i \omega^a_{j\,b} - \partial_j \omega^a_{i\,b} + [\omega_i, \omega_j]^a_b. \tag{5.1}$$

[1] We also found the review [23] to be very helpful.

[2] Concretely, the connection Γ appearing in the ordinary GR covariant derivative $D = \partial + \Gamma$ now takes the form $\omega^{ab} = \omega^{ab}_i dx^i$.

Without using local coordinates this expression simply reads $R = d\omega + \omega \wedge \omega$.

The Einstein-Hilbert action (ignoring for now the cosmological constant) written in terms of these degrees of freedom is

$$I_{\mathrm{E}H} = \frac{1}{2}\int_{\mathcal{M}} \epsilon^{ijk}\epsilon_{abc}e_i^a R_{jk}^{bc} = \frac{1}{2}\int_{\mathcal{M}} \epsilon^{ijk}\epsilon_{abc}e_i^a \left(\partial_j \omega_{kc}^b - \partial_k \omega_{jc}^b + [\omega_j, \omega_k]_c^b\right) \tag{5.2}$$

Imagining e, ω to be gauge fields we can superficially see the similarity to the Chern-Simons action $\int A \wedge (dA + A^2)$. Note here that in higher dimensions we would require additional factors of the frame field e out front, schematically $\int A \wedge \cdots \wedge A \wedge (dA + A^2)$, thereby ruining any similarity to Chern-Simons.

To check this more carefully we introduce a dual notation which takes advantage of the Hodge duality between 1-forms and 2-forms in a 3d manifold:

$$R_a \equiv \frac{1}{2}\epsilon_{abc}R^{bc} \leftrightarrow R^{ab} \equiv -\epsilon^{abc}R_c \tag{5.3}$$

$$\omega_a \equiv \frac{1}{2}\epsilon_{abc}\omega^{bc} \leftrightarrow \omega^{ab} \equiv -\epsilon^{abc}\omega_c. \tag{5.4}$$

The above action, now including the cosmological constant term, can be rewritten as

$$I_{\mathrm{EH}} = \frac{1}{2}\int_{\mathcal{M}} \left(2e^a \wedge R_a[\omega] - \frac{\Lambda}{3}\epsilon_{abc}e^a \wedge e^b \wedge e^c\right). \tag{5.5}$$

We now introduce an $SL(2,\mathbb{R})$-valued[3] gauge fields $A^a = e^a + \omega^a$, $\bar{A} = -e^a + \omega^a$. These components are the coefficients of $SL(2,\mathbb{R})$ generators J_a, so that the connection takes the form $A = A^a J_a$. Evaluating the terms in the standard Chern-Simons action

$$I_{CS}[A] = \int_{\mathcal{M}} \mathrm{Tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A) \tag{5.6}$$

[3] In Lorentzian signature, $SO(2,2) \cong SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ is the isomoetry group of AdS_3. We are currently considering each of these two chiral factors individually by treating A and $\bar{A}$ separately.

(and similarly for $\bar{A}$) we find

$$\begin{aligned}\mathrm{Tr}[A \wedge dA] &= 2e^a \wedge d\omega_a \\ \frac{2}{3}\mathrm{Tr}[A \wedge A \wedge A] &= \frac{1}{3}\epsilon_{abc} e^a \wedge \left(e^b \wedge e^c + 3\omega^b \wedge \omega^c\right).\end{aligned}$$

Putting these together with proper normalization we find

$$I_{CS}[A] = \frac{1}{4}\int_{\mathcal{M}} \left(2e^a \wedge R_a[\omega] + \frac{1}{3}\epsilon_{abc} e^a \wedge e^b \wedge e^c\right) \tag{5.7}$$

and thus $I_{\mathrm{CS}}[A] = (1/2)I_{\mathrm{E}H}$ as in (5.5) with $\Lambda = -1$. We find an identical result for $-I_{\mathrm{CS}}[\bar{A}]$ and thus have shown that in 3d asymptotically AdS$_3$ spacetime manifolds $I_{\mathrm{E}H} = I_{\mathrm{CS}}[A] - I_{\mathrm{CS}}[\bar{A}]$—the pure gravity action reduces to a difference of Chern Simons actions with gauge group $SL(2,\mathbb{R})$ (in Lorentzian signature).

5.2 From Chern-Simons to Boundary Gravitons

Our goal in this section is to obtain a boundary effective action for fluctuations around the vacuum AdS$_3$ solution to pure 3d gravity. We follow [24] and [25], who correct the previous treatment in [26]. [4]

In what follows we work temporarily in Lorentzian signature, where Greek indices take on values $\mu, \nu = t, r, \theta$. We will use the shorthand $\dot{f} \equiv \partial_t f$ and $f' \equiv \partial_\theta f$. "Light-cone" coordinates $x^\pm = \theta \pm t$ will also be used with derivatives $\partial_\pm = (1/2)(\partial_\theta \pm \partial_t)$.

Topologically, global AdS$_3$ is a solid cylinder. We use global coordinates $x^{0,1,2} = t, \theta, r$ that cover the entire geometry. The metric in these coordinates takes the form

$$g = -(r^2+1)dt^2 + r^2 d\theta^2 + \frac{dr^2}{r^2+1}. \tag{5.8}$$

[4] Effective actions for other geometries, including for examples conical deficits and both the one-sided and two-sided BTZ black holes, are straightforward generalizations of what follows, but we will not present them here.

In passing from the second order to first order formalism as discussed above, the frame field and spin connection take on the values

$$e^0 = \sqrt{r^2+1}dt, \qquad e^1 = rd\theta, \qquad e^2 = \frac{dr}{\sqrt{r^2+1}}, \tag{5.9}$$

$$\omega^0 = \sqrt{r^2+1}, d\theta \qquad \omega^1 = rdt, \qquad \omega^2 = 0. \tag{5.10}$$

Next we combine the frame field and spin connection as above into the chiral and antichiral gauge fields $A, \bar{A}$. As above, the gauge group is $SL(2,\mathbb{R})$ gauge group, coming from the decomposition of the Lorentzian AdS$_3$ isometry group $SO(2,2) = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$.[5] The result is

$$A = (e^a + \omega^a)J_a = \sqrt{r^2+1}dx^+ J_0 + rdx^+ J_1 + \frac{dr}{\sqrt{r^2+1}} J_2, \tag{5.11}$$

$$\bar{A} = (-e^a + \omega^a)J_a = \sqrt{r^2+1}dx^- J_0 - rdx^- J_1 - \frac{dr}{\sqrt{r^2+1}} J_2. \tag{5.12}$$

Written out in matrix notation these become

$$A = \begin{pmatrix} \frac{dr}{2\sqrt{r^2+1}} & -\frac{(\sqrt{r^2+1}-r)dx^+}{2} \\ \frac{(\sqrt{r^2+1}+r)dx^+}{2} & -\frac{dr}{2\sqrt{r^2+1}} \end{pmatrix}, \ \bar{A} = \begin{pmatrix} -\frac{dr}{2\sqrt{r^2+1}} & -\frac{(\sqrt{r^2+1}+r)dx^-}{2} \\ \frac{(\sqrt{r^2+1}-r)dx^-}{2} & \frac{dr}{2\sqrt{r^2+1}} \end{pmatrix}. \tag{5.13}$$

The equations of motion (in the first order formalism, the Einstein equations written in terms of the spin-connection, plus the torision-free constraint) imply that $A, \bar{A}$ are both locally flat. Thus we can write both as pure gauge expressions, $A = g^{-1}dg$ and $\bar{A} = \bar{g}^{-1}d\bar{g}$, where $g, \bar{g}$ are independent $SL(2,\mathbb{R})$ elements. There is a redundancy in such a description—it is invariant under $g \mapsto h \cdot g$ for any constant $SL(2,\mathbb{R})$ element h. Nevertheless, we can pick representatives for $g, \bar{g}$ that do reproduce the form of $A, \bar{A}$ given

[5]Explicitly, we take as generators of $SL(2,\mathbb{R})$ matrices J^a of the form

$$J^0 = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}, \ J^1 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \ J^2 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}.$$

above. A convenient set of such representative is the following:

$$g = \begin{pmatrix} \rho c_+ & -\rho^{-1} s_+ \\ \rho s_+ & \rho^{-1} c_+ \end{pmatrix}, \qquad \bar{g} = \begin{pmatrix} \rho^{-1} c_- & -\rho s_- \\ \rho^{-1} s_- & \rho c_- \end{pmatrix}, \tag{5.14}$$

where $c_\pm \equiv \cos(x^\pm/2),\ s_\pm \equiv \sin(x^\pm/2)$, and $\rho \equiv \sqrt{\sqrt{r^2+1}+r}$. Under $\theta \mapsto \theta + 2\pi$ we find $(g, \bar{g}) \mapsto (-g, -\bar{g})$. This implies that the holonomy of the gauge field around the θ circle is $\mathcal{P}e^{\int_0^{2\pi} A_\theta d\theta} = -I$. We would instead like for the result to be $+I$—the minus sign indicates singularities in $A, \bar{A}$ when treated as $SL(2,\mathbb{R})$ gauge fields.

To remedy this situation we take a closer look at the gauge group. Locally we know its form to be $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$ but a priori globally it could be any cover of $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$, where $PSL(2,\mathbb{R}) \cong SL(2,\mathbb{R})/\mathbb{Z}_2$. Our computation of the holonomy of A nails it down to be exactly $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$, thereby identifying $-g$ with g and in particular the holonomy $-I$ with I. Thus we see that $A, \bar{A}$ are in fact non-singular when considered as $PSL(2,\mathbb{R})$ connections.

Additionally, we note that the fundamental group of $PSL(2,\mathbb{R})$ is $\pi_1(PSL(2,\mathbb{R})) = \mathbb{Z}$. Having accounted for the $\mathbb{Z}_2$ quotient above, we see that in global AdS$_3$ g winds exactly once around the geometry's contractible θ-circle. The situation is analogous to a compact boson restricted to its winding number one superselection sector.

We now proceed to the quantization of the classical action (5.6). To do so we follow the "constrain first, then quantize" paradigm of [27]. Accordingly we separate out the gauge field A into spatial and temporal components as

$$A = A_0 dt + \tilde{A}_i dx^i, \tag{5.15}$$

$$\bar{A} = \bar{A}_0 dt + \tilde{\bar{A}}_i dx^i. \tag{5.16}$$

The full action includes a boundary term S_∂ whose presence is required to implement the variational principle consistently with AdS$_3$ boundary conditions, which we will introduce momentarily. After implementing the above decomposition of A we find

$$S_{\text{grav}} = S[A] - S[\bar{A}] + S_{\partial}, \tag{5.17}$$

$$S[A] = \frac{1}{8\pi G}\int_{\mathcal{M}} dt \wedge \text{Tr}\left(-\frac{1}{2}\tilde{A}\wedge\dot{\tilde{A}} + A_0\tilde{F}\right), \tag{5.18}$$

$$S_{\partial} = -\frac{1}{16\pi G}\int_{\partial\mathcal{M}} d^2x \left(\text{Tr}(A_\theta^2) + \text{Tr}(\bar{A}_\theta^2)\right). \tag{5.19}$$

The expression for $S[\bar{A}]$ simply substitutes $\bar{A}$ for A everywhere in (5.18). $\tilde{F} \equiv \tilde{d}\tilde{A} + \tilde{A}\wedge\tilde{A}$ is the spatial field strength, where we have decomposed the exterior derivative into spatial and temporal parts as $d = dt\partial_t + \tilde{d}$.

The aforementioned AdS_3 boundary conditions were first presented in [19] and take the form[6]

$$A \xrightarrow{r\to\infty} \begin{pmatrix} \frac{dr}{2r} + \mathcal{O}(r^{-2}) & \mathcal{O}(r^{-1}) \\ r dx^+ + \mathcal{O}(r^{-1}) & -\frac{dr}{2r} + \mathcal{O}(r^{-2}) \end{pmatrix}, \tag{5.20}$$

$$\bar{A} \xrightarrow{r\to\infty} \begin{pmatrix} -\frac{dr}{2r} + \mathcal{O}(r^{-2}) & -r dx^- + \mathcal{O}(r^{-1}) \\ \mathcal{O}(r^{-1}) & \frac{dr}{2r} + \mathcal{O}(r^{-2}) \end{pmatrix}. \tag{5.21}$$

The content of these equations is that the gauge field must match the leading terms in each $SL(2,\mathbb{R})$ component as $r\to\infty$, and is allowed to fluctuate around these values at the prescribed powers in r. The leading order terms can be found by simply taking the $r\to\infty$ limit of the AdS_3 solution (5.8), whereas identifying the powers in the subleading terms requires using the field equations.

On-shell variation of the action (importantly, including S_∂) leads to an expression

$$\delta S_{\text{grav}} = -\frac{1}{4\pi G}\int_{\partial\mathcal{M}} d^2x \left[\text{Tr}(A_-\delta A_\theta) + \text{Tr}(\bar{A}_+\delta\bar{A}_\theta)\right] \tag{5.22}$$

which vanishes under the boundary conditions (5.20). Indeed, one can check $A_- = \bar{A}_+ = 0$ at leading order.

[6]The authors of [19] wrote these as conditions on the metric components themselves rather than in the Chern-Simons language.

Note that, as visible already in (5.18), the temporal components A_0 and $\bar{A}_0$ appear as Lagrange multipliers inside S_{grav} enforcing the condition $\tilde{F} = 0$, thereby restricting the path integral to be over *flat* $PSL(2,\mathbb{R})$ spatial connections $\tilde{A}$. Typically Lagrange multipliers are allowed to take any value, but here we require their compatibility with the boundary conditions (5.20). This amounts to making the gauge choice

$$A_0 = \begin{pmatrix} 0 & -\frac{\sqrt{r^2+1}-r}{2} \\ \frac{\sqrt{r^2+1}+r}{2} & 0 \end{pmatrix}, \qquad \bar{A}_0 = \begin{pmatrix} 0 & \frac{\sqrt{r^2+1}+r}{2} \\ -\frac{\sqrt{r^2+1}-r}{2} & 0 \end{pmatrix}, \tag{5.23}$$

in which A_0 and $\bar{A}_0$ take the values they do in the global AdS$_3$ solution (5.8).

After integrating out A_0 and $\bar{A}_0$, the remaining functional integral is over the moduli space of flat connections on the disc (which here functions as a Cauchy slice of global AdS$_3$). Flatness ensures that, as above for $(A, \bar{A})$ in global AdS$_3$, we can write $(\tilde{A}, \tilde{\bar{A}})$ as pure gauge solutions

$$\tilde{A} = g^{-1}\tilde{d}g, \qquad \tilde{\bar{A}} = \bar{g}^{-1}\tilde{d}\bar{g}, \tag{5.24}$$

where $g(\vec{x},t), \bar{g}(\vec{x},t)$ are independent $PSL(2,\mathbb{R})$ elements. As before, this decomposition is redundant—$g(\vec{x},t)$ and $h(t)g(\vec{x},t)$ give the same $\tilde{A}$ for any $h(t) \in PSL(2,\mathbb{R})$. To remove the redundancy we identify such gauge group elements g, amounting to a "quasilocal" $PSL(2,\mathbb{R})$ quotient on g.[7] It is in performing this quotient that [24] departs from the analysis of [26].

Rewriting S_{grav} in terms of $g, \bar{g}$, we find a difference of chiral Wess-Zumino-Witten actions,

$$S = S_-[g] - S_+[\bar{g}] \tag{5.25}$$

$$S_\pm[g] = \frac{1}{8\pi G}\left[\int_{\partial\mathcal{M}} d^2x \mathrm{Tr}\Big((g^{-1})'\partial_\pm g\Big) \pm \frac{1}{6}\int_{\mathcal{M}} \mathrm{Tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg)\right]. \tag{5.26}$$

To proceed we must (1) enforce the aforementioned boundary condition that g winds exactly once around the contractible θ-circle of $PSL(2,\mathbb{R}) \cong SO(2,1)$ and (2) translate the

[7]The language "quasilocal" here refers to the fact that $h(t)$ is not necessarily a constant $PSL(2,\mathbb{R})$ element but instead is allowed to depend on time.

AdS$_3$ boundary conditions above into boundary conditions on $g, \bar{g}$. To do so we utilize the following Gauss parametrization of the $PSL(2,\mathbb{R})$ elements g,[8]

$$g = \begin{pmatrix} 1 & 0 \\ +F & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & \Psi \\ 0 & 1 \end{pmatrix}. \tag{5.27}$$

We now evaluate $\tilde{A} = g^{-1}\tilde{d}g$ to find

$$\tilde{A} = \begin{pmatrix} \tilde{d}\ln\lambda - \Psi(\lambda^2\tilde{d}F) & 2\Psi\tilde{d}\ln\lambda - \Psi^2(\lambda^2\tilde{d}F) + \tilde{d}\Psi \\ \lambda^2\tilde{d}F & -\tilde{d}\ln\lambda + \Psi(\lambda^2)\tilde{d}F \end{pmatrix}. \tag{5.28}$$

Comparing this expression with the boundary conditions (5.20) and matching components leads to the constraints

$$\lambda^2 = \frac{r}{F'}, \qquad\qquad \Psi = -\frac{F''}{2rF'}. \tag{5.29}$$

Thus we see the solution is entirely determined in terms of F. Though our Gauss decomposition of F extends into the bulk, we parametrize the boundary value of F as $F|_\partial = \tan(\phi/2)$. The single-valuedness of g then implies $\phi' \neq 0$, and we can consistently choose $\phi' > 0$ everywhere. Furthermore, the winding property of g now translates to the condition $\phi(\theta + 2\pi, t) = \phi(\theta, t) + 2\pi$. Together, these two requirements mean that $\phi(t) \in \mathrm{Diff}(S^1)$ for fixed t.

At this point one might attempt to tie all of the above together and write an effective action for the mode $\phi(\theta, t)$, but doing so would be premature. We first must account for the aforementioned quasilocal $SL(2,\mathbb{R})$ quotient $g(\vec{x}, t) \sim h(t)g(\vec{x}, t)$. Tracing the effect of this left $PSL(2,\mathbb{R})$ action through the Gauss parametrization, we find that it acts on F by a Möbius transformation $F \mapsto \frac{aF+b}{cF+d}$, where $h(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \in PSL(2,\mathbb{R})$. This translates

[8] For the rest of the section we focus on the chiral sector g, A. The antichiral sector $\bar{g}, \bar{A}$ sees completely analogous results.

simply into an action on the boundary mode ϕ leading us to identify

$$\tan\left(\frac{\phi(\theta,t)}{2}\right)\sim\frac{a(t)\tan\left(\frac{\phi(\theta,t)}{2}\right)+b(t)}{c(t)\tan\left(\frac{\phi(\theta,t)}{2}\right)+d(t)},\qquad h(t)=\begin{pmatrix} a(t) & b(t)\\ c(t) & d(t)\end{pmatrix}\in PSL(2,\mathbb{R}). \tag{5.30}$$

That is to say, at fixed time t, ϕ is an element of the quotient space $PSL(2,\mathbb{R})\backslash\text{Diff}(S^1)$.[9]

Having properly accounted for the quasilocal $PSL(2,\mathbb{R})$ quotient we can now at last plug the parameterization (5.27) into the Wess-Zumino-Witten action (5.26), properly implementing the constraints (5.29) to arrive at the promised "boundary graviton" action

$$\boxed{S_\pm[\phi]=\frac{C}{24\pi}\int d^2x\left[\frac{\phi''\partial_\pm\phi'}{\phi'^2}-\phi'\partial_\pm\phi\right],\qquad \phi\in\frac{\text{Diff}(S^1)}{PSL(2,\mathbb{R})}.} \tag{5.31}$$

Here $C=3/2G$ is the Brown-Heanneaux central charge.[10] Each chiral half of this action appears Lorentz non-covariant, but the total action $S[\phi]=S_+[\phi]-S_-[\phi]$ consisting of two chiral halves is Lorentz-covariant.

This exotic scalar field action produces fourth-order equations of motion (refer to the following section) and is thus manifestly non-local. As such it renders an *effective* field theory on the boundary of AdS_3 describing the chiral boundary graviton modes, corresponding to lumps of metric excitations forever circulating to the left or right at spatial infinity.

Indeed, we can view $S_\pm[\phi]$ as a higher-dimensional generalization of the 1d Schwarzian action (2.6) which captures the boundary metric excitations of the 2d JT gravity. In section 5.4.1 below we explicitly verify that the Kaluza-Klein reduction of one of the two chiral pieces of the boundary graviton action we have found in fact coincides with the Schwarzian theory on the Poincaré disc.

In this light it is not at all surprising that our result $S_\pm[\phi]$ is non-local. As made painfully clear in the discussion of the genus-expansion of the JT path integral in section 4.2, the Schwarzian action is *not* a UV complete formulation of the 2d gravitational

[9] The notation here reflects the fact that we taking a *left* quotient by $PSL(2,\mathbb{R})$.

[10] Note that the minus sign in (5.31) corresponds to an analogous result that can be derived in the antichiral sector. When taking the minus sign one should thus replace ϕ by $\bar{\phi}$.

physics but rather the first term in an asymptotic series expansion of gravitational observables. We anticipate a similar state of affairs in AdS_3 gravity, though as of yet we do not know the full UV-complete answer in this case.

One final (but important!) note is to specify what is the measure $D\phi$ in the path integral over ϕ in which the action $S_\pm[\phi]$ appears. The measure DA in the bulk Chern-Simons path integral simply translates into the standard bi-invariant Haar measure $[dg]$ in the chiral Wess-Zumino-Witten model. After implementing the Gauss parametrization and constraints this becomes

$$[dg] = \prod_{\theta,t} d\Psi d\lambda dF \lambda \mapsto \prod_{\theta,t} dF \int d\lambda d\Psi \delta(\lambda^2 F' - r)\delta(\Psi + \frac{F''}{2rF'}) = \prod_{\theta,t} \frac{dF}{F'} = \prod_{\theta,t} \frac{d\phi}{\phi'}. \quad (5.32)$$

5.3 Equations of Motion from Boundary Graviton Action

Our result $S_+[\phi] \equiv \int d^2x \mathcal{L}[\phi, \partial_\mu \phi, \partial_{\mu\nu}]$ for the (chiral) boundary graviton action (5.31) describing right-moving gravitons circulating around the boundary of vacuum AdS_3 spacetime is non-local as it is only an effective description of the bulk dynamics at long distances (in the "IR", in the language of the renormalization group). Here, to compute the resulting equations of motion, we must remember to include higher order terms in the variation of parameters.

$$0 = \delta S = \int d^2x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \phi)} \partial_\mu \partial_\nu \delta\phi \right) \quad (5.33)$$

$$= \int d^2x \; \delta\phi \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} + \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \phi)} \right). \quad (5.34)$$

We compute the various derivatives appearing in this expression to find

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{5.35}$$

$$\frac{\partial \mathcal{L}}{\partial \phi'} = -\frac{2\phi''\partial_+\phi'}{\phi'^3} - \partial_+\phi - \frac{\phi'}{2} \tag{5.36}$$

$$\frac{\partial \mathcal{L}}{\partial \phi''} = \frac{\partial_+\phi'}{\phi'^2} + \frac{\phi''}{2\phi'^2} \tag{5.37}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = -\frac{\phi'}{2} \tag{5.38}$$

$$\frac{\partial \mathcal{L}}{\partial \ddot{\phi}} = \frac{\phi''}{2\phi'^2}. \tag{5.39}$$

As usual when computing equations of motion from an action principle, since $\delta\phi$ in (5.34) is an arbitrary function, we require that the quantity in parentheses vanish identically. Thus,

$$0 = \left(\frac{2\phi''\partial\phi'}{\phi'^3}\right)' + \partial_+\phi' + \frac{\phi''}{2} + \frac{\dot{\phi}'}{2} + \left(\frac{\partial_+\phi''}{\phi'^2} - \frac{2\phi''\partial_+\phi'}{\phi'^3} + \frac{\phi'''}{2\phi'^2} - \frac{\phi''^2}{\phi'^3}\right)' \tag{5.40}$$

$$= \left(\frac{2\phi''\partial\phi'}{\phi'^3}\right)' + 2\partial_+\phi' + \left(\frac{2\partial_+\phi''}{\phi'^2} - \frac{4\phi''\partial_+\phi'}{\phi'^3}\right)' \tag{5.41}$$

$$= 2\left[\partial_+\phi' + \left(\frac{\partial_+\phi''}{\phi'^2} - \frac{\phi''\partial_+\phi'}{\phi'^3}\right)'\right] \tag{5.42}$$

$$= 2\partial_+\left[\phi' + \frac{\phi''}{\phi'^2}\right]' \tag{5.43}$$

$$= \partial_+\left(\frac{1}{\phi'}\left(\left(\frac{\phi''}{\phi'}\right)' - \frac{1}{2}\left(\frac{\phi''}{\phi'}\right)^2 + \frac{\phi'^2}{2}\right)\right) \tag{5.44}$$

$$= \partial_+\left(\frac{1}{\phi'}\left(\mathrm{Sch}(\phi,\theta) + \frac{\phi'^2}{2}\right)\right) \tag{5.45}$$

$$= \partial_+\left(\frac{1}{\phi'}\mathrm{Sch}\left(\tan\frac{\phi}{2},\theta\right)\right) \tag{5.46}$$

$$= \partial_+\left(\frac{1}{\phi'}T[\phi]\right). \tag{5.47}$$

We see that the equations of motion are none other than the conservation of the stress tensor $T[\phi] = \mathrm{Sch}(\tan\phi/2, \theta)$ of the theory. In this sense the boundary graviton action amounts to a non-dissipative hydrodynamics, in that it is simply the theory of a conserved current (the stress tensor) and observables built out of it.

5.4 Dimensional Reduction

5.4.1 From Boundary Gravitons to Schwarzian Action

Here we show how, starting from the right-moving boundary graviton action describing (a chiral part of) pure 3d gravity in global AdS_3 spacetime, to perform a Kaluza-Klein reduction to arrive at the Schwarzian action, itself describing boundary fluctuations of the 2d Jackiw-Teitelboim theory of gravity.

Indeed, we start by Wick-rotating the answer (5.31) $S_+[\phi]$ to Euclidean signature analog $S_{\mathrm{E}}[\phi]$ via $t = -iy$ so as to better compare with the presentation of the Schwarzian action in Euclidean signature in Chapter 4. We find

$$S_{\mathrm{E}}[\phi] = \frac{C}{24\pi} \int_0^{2\pi} d\theta \int_0^{\Delta y} dy \left(\frac{(\bar{\partial}\phi')\phi''}{\phi'^2} - (\bar{\partial}\phi)\phi' \right), \tag{5.48}$$

where under the Wick-rotation $\partial_+ = (1/2)(\partial_\theta + \partial_t)$ becomes $\bar{\partial} = (1/2)(\partial_\theta + i\partial_y)$.

The Kaluza-Klein reduction is carried out by sending the size of the thermal circle $\Delta y \mapsto 0$ while keeping fixed the product $C\Delta y \equiv C'$. We must also imagine $\phi(\theta, y) \approx \phi(\theta)$ is approximately constant along the y-direction as $\Delta y \mapsto 0$, so that $\partial_y \phi \mapsto 0$. Thus we can replace $\bar{\partial}$ by $(1/2)\partial_\theta$ and trivially perform the y-integral to find

$$S_{\mathrm{E}}[\phi] \mapsto \frac{C\Delta y}{24\pi} \int_0^{2\pi} d\theta \frac{1}{2} \left(\frac{\phi''^2}{\phi'^2} - \phi'^2 \right) \tag{5.49}$$

$$= \frac{C'}{48\pi} \int_0^{2\pi} d\theta \left(\mathrm{Sch}\left(\tan\frac{\phi}{2}, \theta \right) \right). \tag{5.50}$$

We recognize this final result as the Schwarzian action previously introduced in (2.6).

Chapter 6

Conclusions

We have considered two theories of gravity with negative cosmological constant in fewer than four spacetime dimensions and taken steps toward their quantization. In two dimensions we treated pure Jackiw-Teitelboim gravity, which can be recast as a type of topological gauge theory called a BF theory. We started by showing how this theory, when considered on a topologically trivial spacetime—in Euclidean signature, the Poincaré disc—can be captured by an effective 1d boundary action, the Schwarzian action. In three dimensions we studied pure AdS_3 gravity, which itself can be recast as a topological gauge theory, in this case a Chern-Simons theory. In both cases the gauge group $PSL(2, \mathbb{R})$ is non-compact. When defined on a topologically trivial spacetime—vacuum AdS_3—pure AdS_3 gravity can similarly be reduced to an effective boundary description, the 2d bounary graviton model. These effective boundary presentations of both 2d and 3d gravity correspond to semiclassical descriptions of the relevant physics, capturing global fluctuations of the spacetime away from its vacuum configuration. We note that in both cases similar effective boundary descriptions exist describing the fluctuations around other, topologically non-trivial, background spacetimes. We present such a modification in the 2d case, for the trumpet geometries of Figure 4.1, but omit any such examples in the 3d case.

These boundary effective actions are related to one another. The 2d boundary graviton model for AdS_3 gravity in vacuum AdS spacetime consists of two chiral pieces. Considering one of them in isolation, and performing a Kaluza-Klein compactification on either the thermal circle or the spatial θ-circle, one lands directly on the 1d Schwarzian model describing JT gravity on the Poincaré disc. Similar dimensional reductions should be possible for boundary effective actions describing other geometries, e.g. the 2-sided AdS_3-Schwarzschild solution and its reduction to the 2-sided Jackiw-Teitelboim wormhole solution. In such cases it is likely more important to insist on performing the compactification along the spatial θ-circle.

For Jackiw-Teitelboim gravity we can go further than this semiclassical treatment, and indeed we derive an asymptotic series representation of the quantized theory's path integral, with successive terms arising from spacetime geometries of increasing genus. The language and formalism of random matrix theory is implemented, where similar genus expansions generically appear in computations of quantities of interest. This perturbative structure paves the way toward an exact solution of the quantum JT theory. What is missing is a UV completion to the asymptotic series. Though not treated herein, the literal interpretation of JT gravity as a random matrix model with a spectral curve given by the Schwarzian density of states makes such a UV completion possible. Non-perturbative effects that are well-understood in the random matrix theory literature can be directly translated into non-perturbative effects in JT gravity, regulating the divergences in the large-genus asymptotics of our perturbative expression.

Most striking about this answer, that the Euclidean signature Jackiw-Teitelboim path integral is precisely described by a particular random matrix ensemble, is that it seems to suggest that at the quantum level JT gravity amounts to an ensemble average of ordinary quantum theories rather than a single one. This is a somewhat surprising result in light of the holographic principle and its concrete realization in AdS/CFT, in which a theory of quantum gravity in $d+1$ spacetime dimensions is identified with a single ordinary quantum field theory in d spacetime dimensions. In all known examples of such holographic correspondences, the gravitational theory under consideration originates from top-down string theoretic constructions and contains significant matter content in addition to geometrical degrees of freedom. The two theories of *pure* gravity considered herein lack any such matter content, and at least the JT case leads to qualitatively different results in the form of an ensemble averaged boundary description.

A natural question is whether a similar result holds in three dimensions, for pure AdS_3 gravity. Performing a direct generalization of the 2d analysis will be difficult. Doing so would require computing the AdS_3 path integral exactly, presumably by summing over 3-manifolds and their moduli, a subject in the mathematical literature that is less well-understood than the study of Riemann surfaces and their moduli, especially by physicists. If such an answer does exist, one would expect the $(0+0)d$ random matrix ensemble of JT theory to be replaced by a $(0+1)d$ matrix quantum mechanics ensemble for AdS_3. Thinking more abstractly, there has been a long history of attempting to identify a 2d CFT dual to pure AdS_3 gravity, or alternatively to show that no such dual CFT exists. A potential lesson one could draw from the JT analysis is that pure AdS_3 gravity may be dual not to any particular conformal field theory, but rather to an ensemble average of CFTs.

Finally, we can gain another perspective on the meaning of the ensemble average as follows. Considering any of the more familiar examples of holographic correspondences with matter content in higher dimensions, we can artificially introduce a disorder average in its boundary description, for example by averaging over small windows of time or over nearby values of the couplings. Doing so washes out the wild oscillations in the late time behavior of the theory's spectral form factor, pictured in Figure 3.2, and replaces them by the smooth ramp and plateau structure pictured in Figure 3.1. In this way we expect to be able to leverage the machinery of random matrix ensembles to study chaotic properties of all holographic theories. We expect this procedure to produce strictly less insight than it did in the JT case treated herein, since the disorder average conceals the underlying unitarity of the boundary description. Moreover, unlike JT gravity which precisely matches onto a particular random matrix ensemble, these higher dimensional gravitational theories can only be expected to match random matrix behavior in universal quantities, in the sense of quantum chaos.

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