GNSS Modulation: A Unified Statistical Description with Application to Tracking Bounds

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Abstract

A unifying framework for all signals belonging to the Global Positioning System (GPS) and Galileo system is presented and applied to assess the potential code tracking performance of modernized satellite radionavigation signals. The framework reconciles, under a single analytical formulation, subcarrier signaling schemes, including the Binary Offset Carrier (BOC), Multiplexed Binary Offset Carrier (MBOC), and Alternative Binary Offset Carrier (ALTBOC). The new formulation allows for the derivation of closed form equations for the Auto-Correlation Function (ACF) and Power Spectral Density (PSD) containing, as special cases, the corresponding functions for GPS and Galileo signals. The analytical expressions are used to obtain new bounds on code tracking accuracy based on the Ziv-Zakai Bound (ZZB). Although the code tracking performance of GPS and Galileo signals is typically investigated using the Cramér-Rao Bound (CRB), the approach is heuristic. The CRB does not adequately describe the potential code tracking performance of weak or wideband signals and does not account for tracking biases. On the other hand, there are no such restrictions for Bayesian bounds such as the ZZB. However, because the CRB is easier to evaluate, it is advantageous to quantitatively identify when the CRB is a meaningful benchmark before having to resort to the ZZB. Therefore, thresholds on signal energy are provided to indicate necessary conditions for the use of the CRB. In agreement with informationtheoretic developments, the thresholds reveal that a large signal bandwidth cannot reliably compensate for low signal energy in order to sustain code tracking performance.

Sommaire

Un cadre unifié pour tous les signaux appartenant au système de positionnement global (GPS) et au système Galileo est présenté et appliqué afin de révéler la précision potentielle de la poursuite du code des signaux modernes de la radionavigation par satellite. Le cadre réconcilie, sous une formulation analytique, les modulations codées du type BOC, MBOC et ALTBOC (dites respectivement Binary Offset Carrier, Multiplexed Binary Offset Carrier et Alternative Binary Offset Carrier). La nouvelle formulation permet d'obtenir des équations fermées propres aux fonctions d'autocorrélation (ACF) et de densité spectrale de puissance (PSD) contenant, à titre de cas spéciaux, les expressions correspondantes pour les signaux GPS et Galileo. Ces expressions conduisent à de nouvelles limites pour la précision de la poursuite du code basées sur la borne de Ziv-Zakai (ZZB). Quoique la performance de la poursuite du code des signaux GPS et Galileo est ordinairement évaluée en utilisant la borne de Cramér-Rao (CRB), l'approche n'est pas rigoureuse. La CRB n'est pas adéquate pour des signaux faibles ou à large bande passante, et ne peut refléter le comportement d'une poursuite biaisée. D'autre part, ces restrictions ne s'appliquent pas aux bornes bayésiennes, telle la ZZB. Or, la CRB peut être évaluée plus aisément que la ZZB, ainsi il est avantageux d'identifier concrètement les conditions supportant son utilisation avant d'avoir recours à la ZZB. Pour délimiter la validité de la CRB, un seuil sur l'énergie des signaux est établi. Soutenu par des développements basés sur la théorie de l'information, ce seuil révèle qu'une large bande passante ne peut compenser avec fiabilité la piètre précision d'une poursuite de signaux faibles.

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List of Acronyms

ABOC	Augmented Binary Offset Carrier
ACF	Auto-Correlation Function
ALTBOC	Alternative Binary Offset Carrier
AMS	Asymptotically Mean Stationary
BLUE	Best Linear Unbiased Estimator
BOC	Binary Offset Carrier
BPSK	Binary Phase Shift Keying
CASM	Coherent Adaptive Subcarrier Modulation
CBCS	Composite Binary Coded Symbol
CBOC	Composite Binary Offset Carrier
CDBOC	Complex Double Binary Offset Carrier
cdf	Cumulative Distribution Function
CDMA	Code Division Multiple Access
CNDR	Carrier-to-Noise Density Ratio
CRB	Cramér-Rao Bound
\mathbf{CS}	Commercial Service
DBOC	Double Binary Offset Carrier
DCT	Dominated Convergence Theorem
\mathbf{DFT}	Discrete Fourier Transform
DSSS	Direct Sequence Spread Spectrum
GB	Gabor Bandwidth
GBOC	Generalized Binary Offset Carrier

GNSS	Global Navigation Satellite System		
GPS	Global Positioning System		
Ι	Inphase		
iid	Independent and Identically Distributed		
LOC	Linear Offset Carrier		
LOS	Line-Of-Sight		
MBOC	Multiplexed Binary Offset Carrier		
MI	Mutual Information		
MSE	Mean Square Error		
OS	Open Service		
\mathbf{pdf}	Probability Density Function		
PPS	Precise Positioning Service		
PRN	Pseudo-Random Noise		
PRS	Public Regulated Service		
PSD	Power Spectral Density		
\mathbf{Q}	Quadriphase		
QPSK	Quadrature Phase Shift Keying		
RMS	Root Mean Square		
SAR	Search-And-Rescue		
SOL	Safety-Of-Life		
SPS	Standard Positioning Service		
\mathbf{SSB}	Single Side Band		
TMBOC	Time Multiplexed Binary Offset Carrier		
ZZB	Ziv-Zakai Bound		

Chapter 1

Introduction

A Global Navigation Satellite System (GNSS) provides a continuous worldwide access to radionavigation signals with the aim of delivering a passive positioning and timing solution. Currently driven by the Global Positioning System (GPS) and the Galileo system, GNSS modernization was initiated over a decade ago to fulfill civilian mass-market demands and meet military security needs. Bandwidth consumption is a major concern for GNSS modernization and entails a considerable increase in signal complexity. To spectrally separate modernized signals from legacy signals while reusing frequencies well suited for space-based radionavigation, the next generation GNSS employs Direct Sequence Spread Spectrum (DSSS) modulations characterized by multilevel code symbols. The underlying multilevel waveforms, known as subcarriers, afford various degrees of spectral disparity by variably distributing signal power to the edges of the frequency bands, away from that of heritage signals.

In providing efficient bandwidth usage, the subcarriers also yield variably sharp correlation peaks, offering multipath resilience and high resolution code tracking [1]. On the other hand, the enhanced modulations encumber receiver designs and performance studies. Harnessing the precision offered by the next generation GNSS requires receivers capable of accurate subcarrier synchronization, and depending on the modulation, markedly different receiver performance may ensue. The subcarriers produce multiple correlation peaks, albeit with a sharp center peak, that complicate unambiguous code (and subcarrier) tracking [2]. Typical receivers with code tracking loops driven by discriminator functions based on correlation peak detection may falsely lock onto a side peak [2].

As such, the second-order statistical properties of the signals, such as the Auto-Correlation Function (ACF) and Power Spectral Density (PSD), indicate spectral occupancy and, importantly, describe code tracking performance. Currently, the most popular measure for the potential code tracking accuracy of GNSS signals is the Cramér-Rao Bound (CRB) [3], [4]. The CRB sets a lower bound on the variance of an unbiased time delay estimator [5]. However, the CRB is limited because it does not incorporate inherent tracking biases due to the signal structures and prior time delay information [6], [7]. Of greater concern for the practitioner, the literature has not indicated the necessary quantitative conditions for the suitability of the CRB for GNSS. Therefore, performance comparisons and conclusions obtained using the CRB remain theoretically ungrounded.

To address a system-wide evaluation of code tracking performance, a consolidated mathematical theory of all GPS and Galileo signals and their statistical properties becomes instrumental. However, due to the complexity of the modulations, precise analytical expressions for the signal structures and the second-order statistical properties of the signals are currently only known for simple cases [1], [8], [9], [10]. Other cases require approximations [11], [12]. Consequently, system designers are faced with multiple case specific formulations and approximations [13], [14].

The objective of this thesis is to provide a unified analytical statistical description of GNSS signals and to leverage the resulting theory to reassess code tracking performance. The approach employed to reach this objective has yielded the following chief contributions:

• A general analytical description of GNSS signal structures

Results presented in an antecedent publication by the author [15] have been extended to obtain a mathematical framework that generalizes all existing signaling schemes used by GPS and Galileo.

• Single equations for the ACF and PSD of GNSS signals

Single closed form equations have been derived to describe the ACF and PSD containing, as special cases, the corresponding functions for all GPS and Galileo signals.

• A bound on the code tracking accuracy of GNSS signals

The Ziv-Zakai Bound (ZZB) has been used to obtain a lower bound on the time delay estimation error for all GPS and Galileo signals. The ZZB is specially suited for time delay estimation [16], [17] and more accurately describes the potential code tracking accuracy when compared to the widely used CRB.

Code tracking thresholds of GNSS signals

Closed form expressions have been derived to indicate the necessary conditions that support the use of the CRB to describe the potential code tracking accuracy of GNSS signals. The corresponding thresholds indicate that an increase in signal bandwidth cannot compensate for low signal energy in order to provide accurate code tracking.

This thesis is organized as follows. In Chapter 2, we provide a mathematical overview of all existing modulations used by GPS and Galileo. In Chapter 3, we generalize the GPS and Galileo signal structures by combining all characteristics of the modulations. The integrated analysis leads to a closed form description for the modulations, which include the Binary Offset Carrier (BOC), Multiplexed Binary Offset Carrier (MBOC), and Alternative Binary Offset Carrier (ALTBOC). We refer to the generalization as *GNSS modulation*. In Chapter 4, we provide a unified statistical description of GNSS modulation through the ACF and PSD of the generalized signal. The agreement between the proposed theory and practice is also illustrated. In Chapter 5, we provide measures that help generalize the code tracking performance of signals in the presence of noise and interference. In Chapter 6, we derive an information-theoretic model that assesses the potential code tracking accuracy of GNSS modulation. We subsequently propose the ZZB as an improved benchmark for code tracking performance. The ZZB accurately bounds time delay estimation errors and is employed to obtain thresholds that delineate proper use of the CRB. Simulations are conducted to compare the code tracking thresholds of all GPS and Galileo signals.

Chapter 2

Background and Motivation

The goal of this chapter is to investigate the structures of the GNSS signals and to provide descriptions for their corresponding modulations. In our treatment of these signals, we use an unconventional, and to the best of our knowledge, new description for the modulations. The new description will facilitate the transition to a general framework for the GNSS signals. For mathematical convenience, to describe the GNSS signals, we resort to a signal representation using analytic time functions [18], with all signals compactly defined via complex envelopes¹ and all filters defined in baseband.

2.1 Signal Components and Navigation Signals

We start by introducing the signal terminology which is used throughout this thesis. We refer to GNSS signals as either signal components or navigation signals. Modernized GPS (GPS II and III) and Galileo satellites will transmit a combination of 21 signal components on 7 carrier frequencies. The purpose of the different signal components is to provide an access to a diversified array of navigation services. The services can be arranged into five groups: 1) Safety-of-Life (SOL), 2) Open Service (OS) or Standard Positioning Service (SPS), 3) Commercial Service (CS), 4) Public Regulated Service (PRS) or Precise Positioning Service (PPS), and 5) Search-And-Rescue (SAR) [20]. Signal components contain data associated

¹The specific definition of the complex envelopes used in this work is due to Zhang and Miller [19]. We note that for GNSS signals, the complex envelope representation is valid because most signal power is confined to a limited bandwidth.

to one particular service and are transmitted on either an Inphase (I) or Quadriphase (Q) carrier; when multiplexed together, they form navigation signals [21]. The navigation signals are simultaneously transmitted by different satellites and share the same frequency bands.

To differentiate the signal components, GPS and Galileo employ Code Division Multiple Access (CDMA) where all signal components are phase-modulated by a unique spreading code. The spreading codes allow a receiver to access different signal components transmitted by the same satellite or similar components transmitted simultaneously by different satellites. The spreading codes not only differentiate the signal components but also provide a means of ranging. To determine the satellite-to-receiver distances, a receiver tracks the codes in the signal components and thus estimates the signal propagation times. The data in the navigation signals describes the position of the satellites and, with the ranging, provides the necessary information to determine the position of the receiver [20]. The calculation of the receiver position based on the information distilled from the navigation signals is outside the scope of this thesis. The reader is referred to, e.g., Borre et al. [22], for a detailed description of the positioning methods. Our goal here is to completely characterize the GNSS signal structures and to investigate of the potential code tracking accuracy offered by the corresponding modulations.

2.2 GNSS Signal Definition

Each GNSS signal component r(t) consists of three parts: 1) a data-carrying signal d(t), 2) a spreading sequence c(t), and 3) a subcarrier s(t). Each of these signals is detailed below.

The real and imaginary parts of the waveform c(t) consist of Pseudo-Random Noise (PRN) codes. The PRN codes are composed of time aligned chips of duration T_c and polarity ± 1 , and each form a unique periodic signal of period MT_c ($M \in \mathbb{Z}^+$). Similarly, the real and imaginary parts of the data d(t) are composed of bits of duration DT_c ($D/M \in \mathbb{Z}^+$) and polarity ± 1 which are time aligned with the PRN codes. The real and imaginary parts of the data are, in that order, each multiplied by the real and imaginary parts of the complex spreading sequence to produce a spread data signal given by

$$x(t) = \Re\{c(t)\} \Re\{d(t)\} + \jmath \Im\{c(t)\} \Im\{d(t)\}.$$
(2.1)

In (2.1), $j \triangleq \sqrt{-1}$ represents the imaginary unit and the operators $\Re(\cdot)$ and $\Im(\cdot)$ extract the real and imaginary parts, respectively. The GNSS signal components are characterized by a subcarrier s(t) that is time-aligned with the PRN codes and is given by a combination of quantized sinusoids:

$$s(t) = a_0 \operatorname{sgn} \left\{ \cos(2n\pi f_r t + \varphi_0) \right\} + a_1 \operatorname{sgn} \left\{ \cos(2m\pi f_r t + \varphi_1) \right\} + a_2 \left[\operatorname{sgn} \left\{ \cos(2m\pi f_r t + \varphi_1 + \varphi_2) \right\} + \operatorname{sgn} \left\{ \cos(2m\pi f_r t + \varphi_1 - \varphi_2) \right\} \right].$$
(2.2)

In (2.2), $\operatorname{sgn}(\cdot)$ is the signum function defined as $\operatorname{sgn}(b) \triangleq -1$ if b < 0, or 1 if $b \ge 0$ ($b \in \mathbb{R}$). A subcarrier of the form given by (2.2) is a 2-level or 4-level wave depending on the choices of the weights $a_k \in \mathbb{R}$ (k = 0, 1, 2) and phases $\varphi_k \in [0, 2\pi)$ (k = 0, 1, 2). The parameter m indicates the rate of the subcarrier, mf_r , with respect to a reference frequency f_r (the frequency f_r is set by convention to the legacy GPS SPS PRN code rate of 1.023 MHz [20]). Similarly, the parameter n indicates the rate of the PRN codes, nf_r , again with respect to f_r . Both parameters are real numbers satisfying the condition that 2m/n is an integer. By convention, $1/(mf_r)$ represents one subcarrier period (a full oscillation), while $1/(nf_r)$ represents one code chip T_c [1]. We note that the subcarrier rate may change over time; for conciseness, the time dependency of the parameter m remains implicit. The subcarrier described in (2.2) is the fundamental building block of the modulations used in GPS and Galileo. In fact, the modulations are simply a formalized way of identifying the employed subcarriers. Fig. 2.1 shows two examples of GNSS subcarriers produced by (2.2). The figure shows how the different parameters of (2.2) yield subcarriers of different rate, phase, and level structure.



Fig. 2.1 Examples of subcarriers used by GPS and Galileo modulations: (a) MBOC; (b) ALTBOC.

By modulating the subcarrier s(t) by the real or imaginary parts of the spread data x(t), we obtain the signal component r(t), i.e.,

$$r(t) = \Re \{x(t)\} s(t)$$
 or $r(t) = \Im \{x(t)\} s(t).$ (2.3)

Several signal components can be combined using a multiplexing scheme. The multiplexed components form navigation signals which are transmitted over particular frequency bands.

2.3 GPS and Galileo Modulations

The modulations used by the GNSS navigation signals can be organized according to signal structure and signal statistics. Three basic families exist: BOC, MBOC, and ALTBOC. In this section, we separately describe the modulation families.

2.3.1 Binary Offset Carrier Modulation

The BOC modulation is denoted by BOC(m, n). As described earlier, the parameter m indicates the subcarrier rate, and the parameter n indicates the PRN code rate. The signal component subcarrier is further described as sine-phased or cosine-phased (the cosine-phased subcarrier is sometimes known as staggered-BOC [23], although, strictly speaking, the original BOC formulation due to Betz [3] only considered a sine-phased subcarrier). In this thesis, to resolve any ambiguity caused by the subcarrier phase, the notation $BOC_{sin}(m, n)$ and $BOC_{cos}(m, n)$ is employed. The original sine-phased BOC modulation is a generalization of the well known Manchester encoding scheme in which m = n. For the case where 2m = n, the component modulation is equivalent, in terms of second order statistics, to Binary Phase Shift Keying (BPSK) denoted by BPSK(n) [1], where n indicates the PRN code rate. It is important to realize, however, that by the definition in (2.2), BOC modulation may be absorbed into the code chip polarities leading to the aforementioned relation between BOC and BPSK modulation.

Mathematically, a BOC modulated signal can be expressed as

$$y_{\text{BOC}}(t) = x(t)s_{\text{BOC}(m,n)}(t).$$
(2.4)

The spread data signal x(t) may take complex values to accommodate quadrature multiplexing (the real and imaginary parts of (2.4) are the BOC signal components). In comparison to a BPSK signal, the inclusion of a BOC subcarrier $s_{BOC(m,n)}(t)$ with 2m > n evenly redistributes power from the main frequency band (centered around the carrier frequency) to higher and lower sidebands, permitting spectral diversity. The extent of the power redistribution and the separation among the sidebands is larger when a cosine-phased subcarrier is employed instead of a sine-phased subcarrier [8], [9]. As an added benefit, the BOC subcarrier produces signals with a sharper center ACF peak, which implies a wider Gabor bandwidth and thus enhanced code tracking performance [24] and, particularly, robustness to multipath [1]. However, a finer center peak comes at the price of more pronounced side peaks, a situation that complicates receiver designs [2].

2.3.2 Multiplexed Binary Offset Carrier Modulation

To enjoy the benefits of spectral diversity and multipath robustness while maintaining compatibility with BOC receivers, MBOC modulation is used [25]. The MBOC modulation employs a subcarrier formed by combining two sine-phased BOC subcarriers of different rates so that the PSD of the resulting signal is a mixture of the BOC subcarrier power spectra:

$$G_{\text{MBOC}}(f) = p \, G_{\text{BOC}_{\sin}(m,n)}(f) + (1-p) \, G_{\text{BOC}_{\sin}(n,n)}(f), \quad p \in (0,1).$$
(2.5)

The notation MBOC(m, n, p) specifies the multiplexing of a $\text{BOC}_{\sin}(m, n)$ and $\text{BOC}_{\sin}(n, n)$ subcarrier with the fraction of power allocated to the former being $p \in (0, 1)$. Although various approaches may be used to obtain a time domain signal with a PSD described by (2.5), two notably distinct methods have attracted most of the system designers' interest. These are the Composite Binary Offset Carrier² (CBOC) and the Time Multiplexed Binary Offset Carrier (TMBOC). The former is based on a weighted summation of two different BOC subcarriers, while the latter is based on a temporal multiplexing of two different BOC subcarriers.

The CBOC signal component modulation is denoted by either CBOC(m, n, p, +) or CBOC(m, n, p, -). In this notation, '+' and '-' indicate, respectively, that the $\text{BOC}_{\sin}(m, n)$ subcarrier is added to, or subtracted from, the $\text{BOC}_{\sin}(n, n)$ subcarrier. In this respect, CBOC(m, n, p, +) subcarriers are known as in-phase, while CBOC(m, n, p, -) subcarriers are known as anti-phase. In either case, the weighted sum of the two subcarriers yields a 4-level net subcarrier. By specifying the weights as \sqrt{p} and $\sqrt{1-p}$, the expression for the subcarrier used by the CBOC signal component modulation is given by

$$s_{\text{CBOC}(m,n,p,\pm)}(t) = \sqrt{1-p} \, s_{\text{BOC}_{\sin}(n,n)}(t) \pm \sqrt{p} \, s_{\text{BOC}_{\sin}(m,n)}(t). \tag{2.6}$$

To achieve the PSD described by (2.5), it is necessary to form the MBOC signal from a sum of two CBOC signal components, one with an in-phase subcarrier and another with

²CBOC belongs to a class of signaling schemes known as Composite Binary Coded Symbol (CBCS) [26].

an anti-phase subcarrier. For this reason, MBOC can be used as a means for multiplexing two different spread data signals, say $x_1(t)$ and $x_2(t)$ defined via (2.1), while producing a constant envelope aggregate signal (required to minimize distortions caused by non-linear satellite payload amplifiers). In this case, the MBOC modulated signal can be expressed from CBOC signal components as

$$y_{\text{MBOC}}(t) = x_1(t) s_{\text{CBOC}(m,n,p,+)}(t) - x_2(t) s_{\text{CBOC}(m,n,p,-)}(t).$$
(2.7)

In (2.7), the spread data signals are real. It should be noted that, in practice, $x_1(t)$ is a data signal, while $x_2(t)$ a pilot signal [27]. Employing this configuration, one obtains a multiplexing scheme denoted by CBOC(m, n, p).

The TMBOC signal component modulation is similarly denoted by TMBOC(m, n, p). The TMBOC subcarrier configuration is repetitive with a period BT_c $(M/B \in \mathbb{Z}^+)$. Within each period, A (out of B) PRN code chips employ a BOC_{sin}(m, n) subcarrier and the remaining B - A chips employ a BOC_{sin}(n, n) subcarrier. In this case, the parameter psignifies the ratio A/B. Denoting by \mathcal{T}_1 the set of chips where the subcarrier is of the BOC_{sin}(m, n) kind and by \mathcal{T}_2 the set of chips where the subcarrier is of the BOC_{sin}(n, n)kind, the TMBOC subcarrier is given by

$$s_{\text{TMBOC}(m,n,A/B)}(t) = \begin{cases} s_{\text{BOC}_{\sin}(m,n)}(t), & \lfloor (t \mod BT_{c})/T_{c} \rfloor \in \mathcal{T}_{1}, \\ s_{\text{BOC}_{\sin}(n,n)}(t), & \lfloor (t \mod BT_{c})/T_{c} \rfloor \in \mathcal{T}_{2}. \end{cases}$$
(2.8)

In (2.8), mod denotes the modulo operator defined as $a \mod b \triangleq a - b \lfloor \frac{a}{b} \rfloor$ $(a, b \in \mathbb{R})$. The subcarrier is then used to form the TMBOC version of the MBOC modulated signal. The TMBOC modulated signal can be expressed as

$$y_{\text{MBOC}}(t) = x(t)s_{\text{TMBOC}(m,n,p)}(t).$$
(2.9)

In (2.9), the spread data signal is real. By convention, the multiplexed signal described by (2.7) and the signal component described by (2.9) are both defined as MBOC modulated signals because their PSD adheres to (2.5).

2.3.3 Alternative Binary Offset Carrier Modulation

As previously alluded to, BOC modulated signals can be used for quadrature multiplexing. Particularly, in legacy GPS, each satellite can transmit up to two BPSK(n) signal components over the same frequency band (one inphase and the other in quadriphase). This approach to signal component multiplexing is equivalent to Quadrature Phase Shift Keying (QPSK) denoted here by QPSK(n) [20], where n indicates the PRN code rate. Moreover, the CBOC multiplexing scheme, described earlier, combines two signal components to carefully increase bandwidth and facilitate tracking. However, to combine more than two signal components while maintianing a constant envelope aggregate signal, the modernized GPS and Galileo employ two other schemes: Interplex and ALTBOC modulation. Interplex modulation is also referred to as Coherent Adaptive Subcarrier Modulation (CASM) and is used to simultaneously transmit individually receivable signal components within the same frequency band while providing a constant envelope and control of signal component power [28], [29]. However, in the rest of this thesis, we will not consider Interplex modulation because it is transparent to the receiver — this is illustrated by its omission in the Galileo Open Service Signal-In-Space Interface Control Document (OS SIS ICD) [30]. On the other hand, ALTBOC modulation produces a modular signal structure that, in addition to providing Interplex-like transmission advantages, promotes flexible receiver designs. With ALTBOC, one can choose to use simple designs for the reception of individual BPSK-like signal components, or more complicated, but more accurate, wideband designs for the reception of all signal components contained in the ALTBOC modulated signal [31], [32].

ALTBOC modulation employs finite-level subcarriers which are cosine-phased and sinephased. Such orthogonal subcarriers are combined to produce a Single Side Band (SSB) waveform. Modulating SSB waveforms with spread data produces an effect that is very similar to the frequency shift caused by multiplication with a complex exponential in the time domain but suffers from harmonic distortion due to the finite levels of the subcarriers [32], [33]. If (smooth) sine and cosine functions are used as the subcarriers³, the harmonics vanish but the signal envelope is not constant. ALTBOC modulation reaches a middle ground by

³Sinusoidal subcarriers are used in a class of signaling schemes known as Linear Offset Carrier (LOC) [3].

employing 4-level subcarriers to create 8-ary Phase Shift Keying (8-PSK) SSB waveforms that allow for a constant envelope and minimize spectral regrowth.

By using SSB waveforms, the spectra of the modulated signal components are translated to either higher or lower frequencies. In contrast to BOC and MBOC, ALTBOC modulation places different signal components on either side of the split-spectrum in the same I-Q phase plane. The multiplexing technique provided by the ALTBOC modulation is preferable over separately injecting signals on closely adjacent center frequencies because it eliminates the use of narrowband filtering and thus reduces distortion within the desired band [32]. ALTBOC is given the designation ALTBOC(m, n), where, once again, the parameters mand n indicate the subcarrier and code rates, respectively.

The ALTBOC subcarriers are formed by the sum of two differently quantized sinusoids of the same rate [see (2.2) and Fig. 2.1(b)]. Like CBOC, the combination of waveforms results in an in-phase or anti-phase subcarrier (in ALTBOC, the in-phase and anti-phase subcarriers are sometimes known as single and product subcarriers, respectively [32]). The ALTBOC subcarriers can be considered as special cases of the so-called Augmented BOC (ABOC) subcarrier [15]. The ABOC subcarrier is a 4-level BOC subcarrier whose extra (augmented) levels are adjustable in width. The generalization permitted by the ABOC subcarrier is presented herein for the first time and, as we will see, is the first step towards a generalized representation of all modulation families used in GNSS. The ABOC subcarrier is denoted by ABOC $(m, n, p/q, a_1, a_2, \pm)$. The parameters a_1 and a_2 $(a_1, a_2 \in \mathbb{R}^+)$ parameterize the levels of the subcarrier. These levels are $\pm a_1$ with the possible addition of $\pm(a_1+a_2)$ or $\pm(a_1-a_2)$ levels, depending on whether the subcarrier is in-phase or anti-phase, respectively. The parameter $p/q \in (0,1]$ $(p/q \in \mathbb{Q}^+)$ signifies the fraction of the subcarrier period for which the $\pm(a_1 \pm a_2)$ levels are attained. As in BOC, the ABOC subcarrier may be sine-phased or cosine-phased, and the disambiguation $ABOC_{sin}(m, n, p/q, a_1, a_2, \pm)$ and $ABOC_{cos}(m, n, p/q, a_1, a_2, \pm)$ is employed when needed. For the ALTBOC modulation, most ABOC parameters remain implicit; the parameter p/q is set to 1/2, while the parameters a_1 and a_2 are fixed to $\sqrt{2}/8$ and 1/4, respectively. The ALTBOC signal is formed from ABOC

subcarriers as follows:

$$y_{\text{ALTBOC}}(t) = x_{1}(t) \Big[s_{\text{ABOC}_{\text{COS}}(m,n,\frac{1}{2},\frac{\sqrt{2}}{8},\frac{1}{4},+)}(t) - \jmath s_{\text{ABOC}_{\text{Sin}}(m,n,\frac{1}{2},\frac{\sqrt{2}}{8},\frac{1}{4},+)}(t) \Big] \\ + x_{2}(t) \Big[s_{\text{ABOC}_{\text{COS}}(m,n,\frac{1}{2},\frac{\sqrt{2}}{8},\frac{1}{4},+)}(t) + \jmath s_{\text{ABOC}_{\text{Sin}}(m,n,\frac{1}{2},\frac{\sqrt{2}}{8},\frac{1}{4},+)}(t) \Big] \\ + x_{3}(t) \Big[s_{\text{ABOC}_{\text{COS}}(m,n,\frac{1}{2},\frac{\sqrt{2}}{8},\frac{1}{4},-)}(t) - \jmath s_{\text{ABOC}_{\text{Sin}}(m,n,\frac{1}{2},\frac{\sqrt{2}}{8},\frac{1}{4},-)}(t) \Big] \\ + x_{4}(t) \Big[s_{\text{ABOC}_{\text{COS}}(m,n,\frac{1}{2},\frac{\sqrt{2}}{8},\frac{1}{4},-)}(t) + \jmath s_{\text{ABOC}_{\text{Sin}}(m,n,\frac{1}{2},\frac{\sqrt{2}}{8},\frac{1}{4},-)}(t) \Big].$$
(2.10)

In (2.10), the signals $x_1(t)$ and $x_2(t)$ are spread data signals which, in practice, both convey ranging information and pilot signals in carrier phase quadrature. The last two signals, $x_3(t)$ and $x_4(t)$, consist of intermodulation terms formed by the products of spread data streams in $x_1(t)$ and $x_2(t)$. These redundant components ensure an overall constant envelope signal⁴ [32], [34].

Below, Table 2.1 and Table 2.2 summarize the modulations used by GPS and Galileo.

	1	1	
Modulation	Multiplexing	Signal component	Example
family	scheme	modulation	subcarrier
	QPSK(n)	BPSK(n)	
$\mathrm{BOC}(m,n)$		$BOC_{sin}(m,n)$	
		$\operatorname{BOC}_{\operatorname{cos}}(m,n)$	
	CBOC(<i>m</i> , <i>n</i> , <i>p</i>)	$\operatorname{CBOC}(m, n, p, +)$	ارس سا
$\operatorname{MBOC}(m,n,p)$		CBOC(m, n, p, -)	- M
		TMBOC (m, n, p)	
		$ABOC_{sin}(m,n,\frac{1}{2},\frac{\sqrt{2}}{8},\frac{1}{4},+)$	
	$\operatorname{ALTBOC}(m,n)$	ABOC _{sin} $(m, n, \frac{1}{2}, \frac{\sqrt{2}}{8}, \frac{1}{4}, -)$	
$\operatorname{ALTBOC}(m, n)$		ABOC _{cos} $(m, n, \frac{1}{2}, \frac{\sqrt{2}}{8}, \frac{1}{4}, +)$	
		ABOC _{cos} $(m, n, \frac{1}{2}, \frac{\sqrt{2}}{8}, \frac{1}{4}, -)$	JUL

 Table 2.1
 GPS and Galileo Modulations and Subcarriers

⁴In general, an ALTBOC-type signal where $p/q \neq 1/2$ produces a constant envelope if the sum of the values of p/q assigned to any sine-phased and cosine-phased ABOC subcarrier equals 1, e.g., $ABOC_{sin}(m, n, \frac{1}{3}, a_1, a_2, \pm)$ and $ABOC_{cos}(m, n, \frac{2}{3}, a_1, a_2, \pm)$.

Navigation signal	Multiplexing scheme	Signal component	Signal component modulation	Service			
GPS II &	GPS II & III						
		L1Cd L1Cp ^{a} (pilot)	$\frac{BOC_{sin}(1,1)}{TMBOC(6,1,\frac{4}{33})}$	SPS			
L1	Interplex	L1P(Y)	BPSK(10)	PPS			
		L1M	$\mathrm{BOC}_{\mathrm{sin}}(10,5)$	PPS			
		L1C/A	BPSK(1)	SPS			
		L2C	BPSK(1)	SPS			
L2	Interplex	L2P(Y)	BPSK(10)	PPS			
		L2M	$\mathrm{BOC}_{\mathrm{sin}}(10,5)$	PPS			
L5	QPSK(10)	L5-I L5-Q (pilot)	BPSK(10)	SPS, SOL			
Galileo							
E1	Interplex with $CBOC(6, 1, \frac{1}{11})$	E1-B E1-C (pilot) E1-A	CBOC(6, 1, $\frac{1}{11}$, +) CBOC(6, 1, $\frac{1}{11}$, -) BOC _{cos} (15, 2.5)	OS, CS, SAR, SOL PRS			
L6		$L6^{b}$		SAR			
E6	Interplex	E6-B E6-C (pilot)	BPSK(5)	CS			
		E6-A	$\mathrm{BOC}_{\mathrm{cos}}(10,5)$	PRS			
E5b	QPSK(10)	E5b-I E5b-Q (pilot)	BPSK(10)	OS, CS, SOL			
E5	ALTBOC(15, 10)	E5a-I E5a-Q (pilot) E5b-I E5b-Q (pilot)	ABOC _{sin} (15, 10, $\frac{1}{2}, \frac{\sqrt{2}}{8}, \frac{1}{4}, +$) and ABOC _{cos} (15, 10, $\frac{1}{2}, \frac{\sqrt{2}}{8}, \frac{1}{4}, +$)	OS, CS, SOL			
E5a	QPSK(10)	E5a-I E5a-Q (pilot)	BPSK(10)	OS, CS			

 Table 2.2
 GPS and Galileo Modulations and Services

 a L1Cp is transmitted with 3/4 of the total power of the L1C signals to produce a PSD that matches the Galileo E1-B/E1-C multiplexed signal [35].

 ${}^{b}L6$ is a dedicated downlink to control stations; the signal details are omitted for receiver usage [27].

Specifically, Table 2.1 shows the hierarchy of the modulations used by GPS and Galileo, while Table 2.2 points out where each signal component modulation fits within the navigation signals; a list of the particular navigation signals and signal components along with their corresponding designations and services is also provided [20], [21], [27], [29], [31], [36].

2.4 Motivation for a General Framework

Because code tracking performance depends on the signal component modulation and multiplexing scheme, it is advantageous to coalesce all previously described modulations within a single framework. The goal of a generalized signal structure is to facilitate the analytical characterization of the statistical properties of GNSS signals. In addition to streamlining the study of code tracking performance, this has important practical consequences. In particular, the analytical expressions for the statistical properties, such as the ACF and PSD, are useful for incorporation within robust delay discriminator designs [37], [38], [39] as well as for rapid receiver simulations [15], [13].

However, until now, little work has been devoted to a general signal theory for GPS and Galileo. There have been a few attempts at a generalization but only for select GNSS signal classes. Pratt and Owen [8], Hein et al. [9] and, especially, Lohan et al. [10] describe a generalization for particular subcarrier modulations. As a result, receiver designs addressing the ambiguity challenges posed by the new waveforms, i.e., false tracking of side peaks [2], become applicable to a wider range of signals [38]. The existing theoretical formulations, however, require approximations and specific extensions for particular modernized signaling schemes [11], [12] and fall short in precisely incorporating the ALTBOC and MBOC subcarriers. Consequently, recent works have resorted to approximate analytical formulations to investigate code tracking performance [13], [14]. In the remainder of this section, we survey the GNSS signal theories published by Lohan et al. [10], [11], [12]. Some of these theories are applied in this thesis for comparison with the general framework that is developed in later chapters.

Lohan et al. proposed a modulation family known as the Double Binary Offset Carrier (DBOC) [10] and demonstrated its relationship to BOC modulation. The DBOC modulation employs a 2-level subcarrier that can be expressed as

$$s_{\text{DBOC}(N_1,N_2)}(t) = \Pi\left(\frac{t}{T_c/(N_1N_2)} - \frac{1}{2}\right) \star \sum_{i=-\infty}^{\infty} \sum_{j=0}^{N_1-1} \sum_{k=0}^{N_2-1} (-1)^{j+k} \,\delta\left(t - \frac{jT_c}{N_1} - \frac{kT_c}{N_1N_2} - iT_c\right).$$

$$(2.11)$$

In (2.11), $\Pi(t)$ is the unit-height rectangular pulse function of support [-1/2, 1/2), \star denotes convolution⁵, and $\delta(t)$ is the Dirac delta function. The subcarrier is represented as a convolution between a rectangular function and a train of scaled Dirac delta functions. The parameters N_1 and N_2 are nonzero positive integers referred to as modulation numbers of the first and second stage, respectively. The modulation number of the first stage divides the PRN code chips into N_1 antipodal subchips, while the modulation number of the second stage divides the subchips into N_2 antipodal parts. For $N_1 = 2m/n$ and $N_2 = 1$, the result corresponds to a BOC_{sin}(m, n) subcarrier. If instead $N_2 = 2$, a BOC_{cos}(m, n) subcarrier is generated. The DBOC modulated signal can be expressed as

$$y_{\text{DBOC}(N_1,N_2)}(t) = x(t)s_{\text{DBOC}(N_1,N_2)}(t).$$
(2.12)

In (2.12), the spread data x(t) may take complex values. The analytical expressions for the ACF and PSD derived using (2.12) provide a good match to simulation results.

To accommodate complex subcarriers, Lohan et al. introduced the Complex Double Binary Offset Carrier (CDBOC) [11]. The CDBOC modulated signal uses two DBOC subcarriers and two complex spread data signals, so that

$$y_{\text{CDBOC}(N_1,N_2,N_3,N_4)}(t) = x_1(t)s_{\text{DBOC}(N_1,N_2)}(t) + jx_2(t)s_{\text{DBOC}(N_3,N_4)}(t).$$
(2.13)

In (2.13), the subcarriers are formed from different rectangular functions [cf. (2.11)]. This complicates the derivation of the statistical properties of the signals. To express both subcarriers using a fixed reference rectangular function, Lohan et al. adjusted the rate of the train of Dirac delta functions in (2.11). If N_3N_4 is a divisor of N_1N_2 , both subcarriers

⁵We adopt the following notation: $x(t) \star y(t) \triangleq \int_{-\infty}^{\infty} x(\lambda)y(t-\lambda) d\lambda = \int_{-\infty}^{\infty} x(t-\lambda)y(\lambda) d\lambda$ and thus $x(t) \star y(-t) = \int_{-\infty}^{\infty} x(\lambda)y(-t+\lambda) d\lambda = \int_{-\infty}^{\infty} x(\lambda+t)y(\lambda) d\lambda$ $(x(t), y(t) \in \mathbb{C})$.

can be defined using a rectangular function of width $T_c/(N_1N_2)$. In general, the subcarriers used in CDBOC modulation can be expressed as

$$s_{\text{CDBOC}(N_1,N_2,N_3,N_4)}(t) = \Pi\left(\frac{t}{T_c/(N_1N_2)} - \frac{1}{2}\right) \star \sum_{i=-\infty}^{\infty} \sum_{j=0}^{N_3-1} \sum_{k=0}^{N_4-1} \sum_{l=0}^{\frac{N_1N_2}{N_3N_4} - 1} (-1)^{j+k} \times \delta\left(t - \frac{jT_c}{N_3} - \frac{kT_c}{N_3N_4} - \frac{lT_c}{N_1N_2} - iT_c\right).$$
(2.14)

As a result, the CDBOC modulated signal is equivalently expressed as

$$y_{\text{CDBOC}(N_1,N_2,N_3,N_4)}(t) = x_1(t)s_{\text{CDBOC}(N_1,N_2,N_1,N_2)}(t) + \jmath x_2(t)s_{\text{CDBOC}(N_1,N_2,N_3,N_4)}(t). \quad (2.15)$$

For distinct complex spread data signals $x_1(t)$ and $x_2(t)$, the statistical properties of a CDBOC modulated signal can be used to approximate those of an ALTBOC modulated signal. However, (2.15) does not match the ALTBOC signal definition provided by (2.10). Contrary to ALTBOC, CDBOC uses 2-level subcarriers (as opposed to 4-level subcarriers) and has no intermodulation terms. Therefore, the expressions for the ACF and PSD of the ALTBOC modulated signal derived using (2.15) are approximations.

Lohan and Renfors later extended the CDBOC modulation theory to include MBOC [12]. They reasoned that the sum of two subcarriers, one scaled by \sqrt{p} and the other by $\sqrt{1-p}$ $(p = A/B \in \mathbb{Q}^+)$, is statistically equivalent to time multiplexing, where the first subcarrier occurs for A out of B PRN code chips. As a result, to derive the statistical properties, it was claimed that an MBOC modulated signal can be generally modeled as [cf. (2.5)]

$$y_{\text{MBOC}(N_1,N_2,p)}(t) = x(t) \left[\sqrt{p} \, s_{\text{CDBOC}(N_1,1,N_1,1)}(t) + \sqrt{1-p} \, s_{\text{CDBOC}(N_1,1,N_2,1)}(t) \right].$$
(2.16)

However, this expression does not provide an accurate statistical description for CBOC or TMBOC modulated signals. For CBOC modulation, (2.16) does not include the anti-phase component and, for TMBOC modulation, does not capture the temporal separation between the multiplexed subcarrier variants, thus creating a false statistical dependence between the subcarrier variants. Again, the expressions for the ACF and PSD of the MBOC modulated signal derived using (2.16) are approximations.

The statistical properties of the signals defined by Lohan et al. admit closed form expressions. The authors obtained the ACF via the expected value of the finite-time autocorrelation⁶ of the functions defined by (2.12), (2.15), and (2.16); the assumption that the spread data is composed of uncorrelated zero mean code chips simplifies the calculations [10], [11], [12]. The reference rectangular function [cf. (2.11) and (2.14)] allows the ACF to be expressed as a summation of shifted triangular functions. Finally, the PSD is calculated from the squared magnitude spectrum of the signals by applying the temporal shift property to the Fourier transform of the reference rectangular function.

In summary, the works of Lohan et al. are important because they introduce a signal representation using convolution which greatly facilitates the derivation of the statistical properties of GNSS signals. However, for most existing GNSS signals, the published results are case specific and provide only approximations. In this work, we seek to eliminate the inaccuracies resulting from these approximations by deriving a highly modular GNSS signal model based on the convolution approach that incorporates all subcarriers, data signals, and combinations thereof.

2.5 GNSS Channel Model

Having introduced the modulations used by GPS and Galileo, we now characterize the GNSS signal channel. The channel model presented here is used throughout this thesis and establishes the fundamental assumptions required to investigate the code tracking performance of next generation GPS and Galileo signals.

Let us consider the channel depicted in Fig. 2.2. In the figure, y(t) is an infinite bandwidth reference signal intended for transmission by a GPS or Galileo satellite. The signal y(t) consists of either a signal component or a multiplexed signal (a mathematical description is provided later in the context of a general framework for the GNSS signals [see Section 3.2]). Before transmission, the reference signal is filtered by the satellite transmitter chain and then propagates through the channel which eventually produces the received

⁶In the original papers [10], [11], [12], the finite-time autocorrelation is sometimes referred to as a convolution.



 $^a\mathrm{The}$ effective filter contains the receive and transmit filters.

 ${}^b\mathrm{The}$ noise and interference is shaped by the receive filter.

Fig. 2.2 Block diagram of the GNSS channel model.

Line-Of-Sight (LOS) signal. The LOS signal is of primary interest because it is what a receiver ideally synchronizes to. In this work, we consider only the LOS component (with negligible excess scattering delay) and thus assume that the channel exhibits flat-fading [40]. Furthermore, we assume that the Doppler-induced spectral broadening is minimal, and only a Doppler shift caused by the relative motion of the transmitting satellites is considered. Under such conditions, the channel exhibits slow-fading [40]. Consequently, the channel parameters are assumed fixed (or change very slowly) over an observation interval [0, T). In this respect, the transmitted signal undergoes a time delay t_0 , complex attenuation α_0 , and phase shift ϕ_0 . In this work, t_0 is the channel parameter of interest. The time delay is sometimes referred to as the code delay because it is estimated by the receiver as it tracks the PRN code (and subcarrier) of the received signal. We assume that T is sufficiently large to produce an interval in which the delay can be unambiguously distinguished [see Section 4.1].

Upon reception, the transmitted signal is filtered by the receiver front-end. The combination of the receive and transmit filters is modeled by the impulse response z(t). Therefore, in the absence of noise and interference, the received signal is given by

$$\hat{y}(t,t_0) = \int_{-\infty}^{\infty} \alpha_0 e^{j\phi_0} y(\lambda - t_0) z(t - \lambda) \,\mathrm{d}\lambda.$$
(2.17)

The explicit inclusion of t_0 in the function $\hat{y}(t, t_0)$ underlines the fact that we are especially interested in the time delay. We remark that the observation interval is sufficiently large so that the contribution of the reference signal y(t) to (2.17) is negligible outside of the interval. Furthermore, the Fourier transform of $\hat{y}(t, t_0)$ computed over the time interval [0, T) is equivalent to that computed over any interval of the same length [41]. Consequently, $\hat{y}(t, t_0)$ has a power spectrum given by

$$G_{\hat{y}}(f) = \frac{\kappa}{T} |Y(f)Z(f)|^2.$$
(2.18)

In (2.18), the attenuation coefficient scales the total signal power with $\kappa \triangleq |\alpha_0|^2$. In addition, Y(f) is the Fourier transform of y(t) computed over [0, T), and Z(f) is the Fourier transform of z(t) which implements a lowpass filter that makes $G_{\hat{y}}(f)$ negligible outside of a bandwidth [-W/2, W/2]. In addition to bandlimiting the signal, the effective filter degrades the power of the received signal. For a bandlimited GNSS signal, the output power of the receiver front-end is largely due to the subcarriers. The subcarriers are implicitly included in $\hat{y}(t, t_0)$ and contribute an aggregate power of P_S . The power of the signal due to the subcarriers and the attenuation coefficient is defined as the carrier power:

$$C \triangleq \frac{1}{2} \kappa P_S. \tag{2.19}$$

By convention, the carrier power is associated to the bandpass (as opposed to baseband) signal⁷. The power degradation due to the filtering is referred to as the correlation loss⁸ [10], [42] defined via

$$\psi \triangleq \frac{1}{2C} \int_{-W/2}^{W/2} G_{\hat{y}}(f) \,\mathrm{d}f.$$
(2.20)

Fig. 2.3 illustrates the correlation loss by comparing the ACF of a bandlimited signal to

⁷By definition, complex envelope signals have twice the power of their bandpass counterparts [19]. ⁸Note that the term "loss" refers to a decibel scale.



Fig. 2.3 Example of the ACF of a bandlimited signal component: $CBOC(6, 1, \frac{1}{11}, -)$.

the ACF of an infinite bandwidth signal (both normalized to 2C). In (2.20), ψ represents the power containment, i.e., the fraction of carrier power C available beyond the receiver front-end. The unit-power spectrum can then be defined as

$$\breve{G}_{\hat{y}}(f) \triangleq \frac{1}{2\psi C} G_{\hat{y}}(f).$$
(2.21)

The use of the unit-power spectrum permits a separate treatment for the correlation loss, carrier power, and signal spectrum.

Referring again to Fig. 2.2, the transmitted signal is corrupted by noise and interference w(t). The signal w(t) is the outcome of a stationary circular symmetric (zero mean) complex Gaussian process [43] with a PSD denoted by $G_w(f)$ that implicitly incorporates the filtering due to the receiver front-end. We assume that w(t) has real and imaginary parts that are independent at all times. In addition, over the receiver front-end bandwidth, we assume that there are no frequencies for which $G_w(f)$ is zero. The Gaussian noise and interference is added to $\hat{y}(t, t_0)$ to form the received signal. The result, as shown in Fig. 2.2, is given by

$$\tilde{y}(t) = \hat{y}(t, t_0) + w(t).$$
(2.22)

In order to investigate the statistical properties of GNSS signals, it will be necessary to

consider the ensemble of all possible outcomes of (2.22), where select channel and signal parameters are random. As a convention, in this work we denote random variables and processes using the underlining (Van Dantzig) notation, e.g., $\tilde{y}(t)$ is a realization of the random process $\tilde{y}(t)$.

2.6 Cramér-Rao Bound

The common approach to study the potential code tracking accuracy of GNSS signals (irrespective of receiver designs) is to evaluate the CRB. The CRB sets a lower bound on the error, in terms of variance, of any unbiased parameter estimator. For simplicity, let us consider a time delay estimate, say \hat{t}_0 , obtained by observing $\hat{y}(t, t_0)$ in the presence of ideal bandlimited white Gaussian noise with $G_w(f) = 2N_0$ over the band of interest [19]. We note that $N_0/2$ is the double-sided PSD of the additive white Gaussian noise corrupting the bandpass signal from which arises the complex envelope⁹. In this case, the CRB can be expressed as [18]

$$[\sigma_{\hat{t}_0}^2]_{\rm \tiny CRB} = \frac{N_0}{4\pi^2 T \int_{-W/2}^{W/2} f^2 G_{\hat{y}}(f) \,\mathrm{d}f}.$$
(2.23)

The CRB is inversely related to the power and spread of the frequencies in $\hat{y}(t, t_0)$. This indicates that signals with spectral power concentrated at the band edges potentially provide better code tracking accuracy. However, as discussed in following section, the CRB has several important limitations.

2.7 Motivation for Improved Code Tracking Bounds

The main feature of the modulations used by the modernized GPS and Galileo is the flexible distribution of the signal power across the frequency bands. Specifically, modernized signals manifest split-spectra which help reduce the amount of frequency overlap between signals in common bands and provide accurate code tracking [3]. In comparison to legacy GPS

⁹It has been shown that the complex envelope of white noise with a double-sided PSD of $N_0/2$ is, itself, not white [44] but is circularly symmetric with a PSD of $2N_0u(f + f_0)$, where $u(\cdot)$ is the unit step function and f_0 is the central frequency of the bandpass signal [45]. However, as long as $W \leq 2f_0$, it is correct to say that over the band of interest $G_w(f) = 2N_0$.

signals, the improvement in code tracking accuracy is due to the concentration of power on the sidebands, which suggests that frequency spread is inversely related to the Mean Square Error (MSE) of the time delay estimator. This is consistent with signals of sufficiently high energy observed in the presence of ideal bandlimited white Gaussian noise because the MSE adopts the form of the CRB [17]. In fact, existing receiver designs have been shown to achieve a performance close to the CRB [46].

However, the CRB is restricted to an unbiased parameter estimator and assumes that the receiver maintains code tracking lock [47]. In addition, the CRB does not account for the ambiguity introduced by the multi-peak ACF of the modernized GNSS signals [6], nor does it consider prior time delay information. As a result, exclusive use of the CRB has lead to theoretically ungrounded conclusions on potential code tracking accuracy, especially for weak wideband signals [48], [49]. Consequently, practical receiver performance can significantly depart from the CRB [50].

To overcome the limitations of the CRB, it becomes essential to establish a baseline that more closely describes the potential code tracking accuracy and to quantitatively determine the conditions required for the suitability of the CRB. This can be accomplished via Bayesian bounds such as the ZZB. The ZZB has been used extensively to evaluate the performance of time delay estimation [7], [16], [17]. However, to the author's knowledge, it has never been employed to characterize the code tracking performance of GNSS signals. The ZZB is mentioned by Krasner [50] in the context of legacy GPS signals but is not applied to obtain results.

This work proposes the use of the ZZB as a new benchmark for the code tracking performance of GNSS signals. Because combined noise and interference is a major contributor to the GNSS error budget, and the ambiguity caused by the ACF side peaks is a leading obstacle for practical code tracking loop designs, our goal is to extend the ZZB to the case of colored Gaussian noise and to incorporate the signal structures in the theoretical development.

Chapter 3

Generalized GNSS Signal Structure

This chapter presents an analytical framework that describes the GNSS signals using a systematic convolution approach. The derivation primarily focuses on incorporating all characteristics of the existing modulation families (BOC, MBOC, and ALTBOC) and leads to a generalized signal that corresponds to the infinite bandwidth reference signal y(t) considered in the last chapter. In addition to avoid dealing with separate case specific formulas, the analytical framework provides a stronger understanding of GNSS signal structures, as well as provision for accurately investigating other signals not yet defined, e.g., an ALTBOC variant with parameter $p/q \neq 1/2$. From here on, we consider the modulation families under a unification designated as GNSS modulation [15].

3.1 Generalized Subcarrier

Let us first analytically describe the subcarriers used in GNSS modulation. Although we have already introduced a generalized subcarrier in (2.2), the expression does not lend itself well to the mathematical investigation of the statistical properties of the GNSS modulation signal. Instead, the mathematical description of the ABOC subcarrier $s_{\text{ABOC}(m,n,\frac{p}{q},a_1,a_2,\pm)}(t)$ can be modified to describe all subcarriers.

Without loss of generality, we consider a single segment of the ABOC subcarrier over an interval $[0, T_c)$ denoted by the majuscular notation $S_{ABOC(m,n,\frac{p}{q},a_1,a_2,\pm)}(t)$. This segment has 4 levels: $\{\pm a_1, \pm (a_1 + a_2)\}$ or $\{\pm a_1, \pm (a_1 - a_2)\}$, depending on whether the subcarrier is in-

phase or anti-phase, respectively. The ratio of the subcarrier and code rate, $N = 2m/n \ge 1$, is known as a modulation number [15], [11] and represents the number of subcarrier halfperiods present in the interval $[0, T_c)$. Consistent with the definition in (2.2), N also indicates whether two consecutive subcarrier segments are the same (even N) or antipodal (odd N) [3]. We recall that the rational number p/q relates to the width of the $\pm (a_1 \pm a_2)$ (augmented) levels added to a subcarrier with $\pm a_1$ levels. In provision for an augmented subcarrier, we introduce two other modulation numbers, η and η' , set to q and p (in the ABOC case), respectively. The modulation number η is an integer greater than or equal to 1 which evenly divides the subcarrier half-periods into time intervals of length $T_c/(2\eta N)$. These intervals are the widest possible such that over their span, the subcarrier levels remain constant. In conjunction with the ABOC parameter p/q, for every subcarrier half period, $2\eta'$ intervals (out of 2η) form the augmented levels. Fig. 3.1 exemplifies the modulation numbers.

Having partitioned the general subcarrier waveform into segments of width $T_{\rm c}/(2\eta N)$, we



Fig. 3.1 Synthesis of the ABOC subcarrier segment: (a) $\operatorname{ABOC}_{\operatorname{sin}}(m, n, \frac{p}{q}, a_1, a_2, +)$; (b) $\operatorname{ABOC}_{\operatorname{cos}}(m, n, \frac{p}{q}, a_1, a_2, +)$.
can now express the subcarrier segment as a train of shifted and scaled rectangular pulses of width $T_c/(2\eta N)$. It is well known that this procedure may be carried out as a convolution between a rectangular pulse and a sequence of appropriately positioned and scaled Dirac delta functions. The required set of impulses can be described separately for the upper and lower halves of the ABOC subcarrier segment denoted by U(t) and L(t), respectively. In this regard, examples are shown in Fig. 3.1, where the upper subcarrier halves appear as shaded and the lower halves as non-shaded. As for the GNSS modulation subcarriers introduced in the previous chapter, we employ the phase disambiguation $L_{sin}(t)$ and $L_{cos}(t)$, or $U_{sin}(t)$ and $U_{cos}(t)$. The impulses defining the lower non-shaded region of the sine-phased subcarrier segment of Fig. 3.1(a) can be described by

$$L_{\rm sin}(t) = a_1 \sum_{n=0}^{N-1} \sum_{m=0}^{\eta-1} (-1)^n \left[\delta \left(t - \frac{nT_{\rm c}}{N} - \frac{mT_{\rm c}}{2\eta N} \right) + \delta \left(t - \frac{nT_{\rm c}}{N} - \frac{T_{\rm c}}{2N} - \frac{mT_{\rm c}}{2\eta N} \right) \right], \qquad (3.1)$$

while those defining the upper shaded regions of Fig. 3.1(a) can be described by

$$U_{\rm sin}(t) = a_2 \sum_{n=0}^{N-1} \sum_{m=0}^{\eta'-1} (-1)^n \left[\delta \left(t - \frac{nT_{\rm c}}{N} - \frac{mT_{\rm c}}{2\eta N} - \frac{(\eta - \eta')T_{\rm c}}{2\eta N} \right) + \delta \left(t - \frac{nT_{\rm c}}{N} - \frac{T_{\rm c}}{2N} - \frac{mT_{\rm c}}{2\eta N} \right) \right].$$
(3.2)

On the other hand, the impulses defining the lower non-shaded region of the cosine-phased subcarrier segment of Fig. 3.1(b) can be described by

$$L_{\rm cos}(t) = a_1 \sum_{n=0}^{N-1} \sum_{m=0}^{\eta-1} (-1)^n \left[\delta \left(t - \frac{nT_{\rm c}}{N} - \frac{mT_{\rm c}}{2\eta N} \right) - \delta \left(t - \frac{nT_{\rm c}}{N} - \frac{T_{\rm c}}{2N} - \frac{mT_{\rm c}}{2\eta N} \right) \right], \tag{3.3}$$

while those defining the upper shaded regions of Fig. 3.1(b) can be described by

$$U_{\rm cos}(t) = a_2 \sum_{n=0}^{N-1} \sum_{m=0}^{\eta'-1} (-1)^n \left[\delta \left(t - \frac{nT_{\rm c}}{N} - \frac{mT_{\rm c}}{2\eta N} \right) - \delta \left(t - \frac{nT_{\rm c}}{N} - \frac{T_{\rm c}}{2N} - \frac{mT_{\rm c}}{2\eta N} - \frac{(\eta - \eta')T_{\rm c}}{2\eta N} \right) \right].$$
(3.4)

A GNSS modulation signal is formed from several subcarrier segment halves of possibly

different modulation numbers and amplitudes. Therefore, it is beneficial to encapsulate all subcarrier segment halves into a single parameterized expression. Conveniently, the ABOC subcarrier segment halves described by equations (3.1) - (3.4) are analytically similar, and thus can be consolidated. The general expression for the set of impulses defining any subcarrier segment half is given by

$$W_{h,j}^{(l)}(t) = a_{h,j}^{(l)} \sum_{n=0}^{N_{h,j}^{(l)}-1} \sum_{z=0}^{\left[\frac{\zeta Z \eta_{h,j}^{(l)}}{N_{h,j}^{(l)} \eta_{h,1}^{(l)}}\right]^{-1}} \sum_{m=0}^{1} (-1)^{n+m(1+\nu_h)} \times \delta\left(t - \frac{nT_c}{N_{h,j}^{(l)}} - \frac{mT_c}{2N_{h,j}^{(l)}} - \frac{[m(1-2\nu_h) + \nu_h](\eta_{h,1}^{(l)} - \eta_{h,j}^{(l)})T_c}{2\eta_{h,1}^{(l)} N_{h,j}^{(l)}} - \frac{zT_c}{2\zeta Z}\right).$$
(3.5)

In the above equation, the subcarrier phase is related to the index h through the indicator function given by

$$\nu_{h} = \begin{cases} 1, & h \text{ even (sine-phased subcarrier)}, \\ 0, & h \text{ odd (cosine-phased subcarrier)}. \end{cases}$$
(3.6)

Moreover, we have considered that a GNSS modulation signal may simultaneously employ up to four different subcarriers (h = 1, 2, 3, 4), each of which consist of two time multiplex variants (l = 1, 2) produced by the sum of two subcarrier segment halves (j = 1, 2). Therefore, the modulation numbers are parameterized as $N_{h,j}^{(l)}$ and $\eta_{h,j}^{(l)}$, and the subcarrier levels are parametrized as $a_{h,j}^{(l)}$. In (3.5), the parameters Z and ζ regularize the rate of the resulting impulse train and are set to any common multiples of the required¹ values of $N_{h,j}^{(l)}$ (h = 1, 2, 3, 4, j = 1, 2, and l = 1, 2) and $\eta_{h,1}^{(l)}$ (h = 1, 2, 3, 4 and l = 1, 2), respectively. Although we refer to Z and ζ as a parameters, we must emphasize that they are, in fact, functions of other parameters. As we will see later, the time multiplexing structure is required to form the complete subcarriers. As mentioned previously, this information is conveyed by the ratio A/B where, over a period BT_c , A (out of B) PRN code chips employ an l = 1 subcarrier and the remaining B - A chips employ an l = 2 subcarrier.

For conciseness, we can store the subcarrier parameters as entries in vectors $\boldsymbol{\vartheta}$ and $\boldsymbol{\varrho}_h^{(l)}$.

¹In many cases, the subcarriers are sufficiently described without having to specify all modulation numbers (see footnote a in Table 3.1).

The parameter vector $\boldsymbol{\vartheta} \in \mathbb{R}^3$ contains variables specific to the overall GNSS modulation signal and is defined as

$$\boldsymbol{\vartheta} \triangleq \left[\frac{A}{B}, \zeta, Z\right],\tag{3.7}$$

while the parameter vector $\boldsymbol{\varrho}_h^{(l)} \in \mathbb{R}^6$ is unique to each subcarrier making up the overall signal and is defined as

$$\boldsymbol{\varrho}_{h}^{(l)} \triangleq \left[N_{h,1}^{(l)}, N_{h,2}^{(l)}, \eta_{h,1}^{(l)}, \eta_{h,2}^{(l)}, a_{h,1}^{(l)}, a_{h,2}^{(l)} \right].$$
(3.8)

To reinforce the fact that the set of impulses defining any subcarrier segment is generally produced by the sum of two $W_{h,j}^{(l)}(t)$ waveforms (j = 1, 2), the parameters indexed by jare explicitly listed in the same $\boldsymbol{\varrho}_{h}^{(l)}$ vector. Effectively, the present formulation ensures that $W_{h,1}^{(l)}(t)$ takes the form of L(t) and that $W_{h,2}^{(l)}(t)$ takes the form of either U(t) or L(t). In the first case, the sum of $W_{h,1}^{(l)}(t)$ and $W_{h,2}^{(l)}(t)$ results in an ABOC subcarrier segment, while in the second case, the sum results in a CBOC subcarrier segment. Moreover, when considered alone, $W_{h,1}^{(l)}(t)$ represents a BOC or TMBOC subcarrier segment. Consequently, the novel formulation depicted by (3.5) allows us to formulate a generalized subcarrier segment, referred to as the Generalized Binary Offset Carrier (GBOC) subcarrier segment:

$$S_{\text{GBOC}(\vartheta, \varrho_h^{(l)})}(t) = \Pi\left(\frac{t}{T_c/(2\zeta Z)} - \frac{1}{2}\right) \star \sum_{j=1}^2 W_{h,j}^{(l)}(t).$$
(3.9)

Each GNSS subcarrier segment is a special case of the GBOC subcarrier segment obtained by specifying the parameter vectors $\boldsymbol{\vartheta}$ and $\boldsymbol{\varrho}_h^{(l)}$. Table 3.1 lists the specific parameter vectors required to describe all subcarriers used by GPS and Galileo signals.

3.2 Generalized Signal

In the previous section, we obtained a general expression for any GNSS modulation subcarrier over the interval $[0, T_c)$. This was defined as the GBOC subcarrier segment. We now consider the whole GNSS modulation signal.

As shown in Fig. 3.2, the GNSS modulation signal is a linear combination of shifted and

Modulation	Parameter vectors	
	$\boldsymbol{\vartheta} = [\frac{A}{B}, \zeta, Z]$	$\boldsymbol{\varrho}_{h}^{(l)} = [N_{h,1}^{(l)}, N_{h,2}^{(l)}, \eta_{h,1}^{(l)}, \eta_{h,2}^{(l)}, a_{h,1}^{(l)}, a_{h,2}^{(l)}]^{a}$
Component modulation		
BPSK(n)	$\boldsymbol{\vartheta} = [1, 1, 1]$	$\boldsymbol{\varrho}_2^{(1)} = [1, \backsim, 1, \backsim, 1, 0]$
$\operatorname{BOC}_{\operatorname{sin}}(m,n)$	$\boldsymbol{\vartheta} = [1, 1, \frac{2m}{n}]$	$\boldsymbol{\varrho}_2^{(1)} = [\tfrac{2m}{n}, \backsim, 1, \backsim, 1, 0]$
$\operatorname{BOC}_{\operatorname{cos}}(m,n)$	$\boldsymbol{\vartheta} = [1, 1, \frac{2m}{n}]$	$\boldsymbol{\varrho}_1^{(1)} = [\tfrac{2m}{n}, \backsim, 1, \backsim, 1, 0]$
$\operatorname{CBOC}(m, n, p, +)$	$\boldsymbol{\vartheta} = [1, 1, \frac{4m}{n}]$	$\boldsymbol{\varrho}_2^{(1)} = [2, \frac{2m}{n}, 1, 1, \sqrt{1-p}, \sqrt{p}]$
$\operatorname{CBOC}(m,n,p,-)$	$\boldsymbol{\vartheta} = [1, 1, \frac{4m}{n}]$	$\boldsymbol{\varrho}_{2}^{(1)} = [2, \frac{2m}{n}, 1, 1, \sqrt{1-p}, -\sqrt{p}]$
$ ext{TMBOC}(m,n,p)^{b}$	0 [1 $4m$]	$\boldsymbol{\varrho}_{2}^{(1)} = \left[\frac{2m}{n}, \backsim, 1, \backsim, 1, 0\right]$
	$\boldsymbol{\vartheta} = [p, 1, \frac{4m}{n}]$	$\boldsymbol{\varrho}_{2}^{(2)} = [2, \sim, 1, \sim, 1, 0]$
ABOC _{sin} $(m, n, \frac{p}{q}, a_1, a_2, +)$	$\boldsymbol{\vartheta} = [1, q, \frac{2m}{n}]$	$\boldsymbol{\varrho}_{2}^{(1)} = [\frac{2m}{n}, \frac{2m}{n}, q, p, a_{1}, a_{2}]$
$\operatorname{ABOC}_{\operatorname{sin}}(m,n,\frac{p}{q},a_1,a_2,-)$	$\boldsymbol{\vartheta} = [1, q, \frac{2m}{n}]$	$\boldsymbol{\varrho}_{2}^{(1)} = [\frac{2m}{n}, \frac{2m}{n}, q, p, a_{1}, -a_{2}]$
ABOC _{cos} $(m, n, \frac{p}{q}, a_1, a_2, +)$	$\boldsymbol{\vartheta} = [1, q, \frac{2m}{n}]$	$\boldsymbol{\varrho}_1^{(1)} = \left[\frac{2m}{n}, \frac{2m}{n}, q, p, a_1, a_2\right]$
$\operatorname{ABOC}_{\operatorname{cos}}(m, n, \frac{p}{q}, a_1, a_2, -)$	$\boldsymbol{\vartheta} = [1, q, \frac{2m}{n}]$	$\boldsymbol{\varrho}_1^{(1)} = [\frac{2m}{n}, \frac{2m}{n}, q, p, a_1, -a_2]$
Multiplexing modulation		
$\operatorname{QPSK}(n)^c$		$\boldsymbol{\varrho}_2^{(1)} = [1, \backsim, 1, \backsim, 1, 0]$
	$\boldsymbol{\vartheta} = [1, 1, 1]$	$\boldsymbol{\varrho}_4^{(1)} = [1, \backsim, 1, \backsim, 1, 0]$
$\operatorname{CBOC}(m,n,p)$	$\boldsymbol{\vartheta} = [1, 1, \frac{4m}{2}]$	$\boldsymbol{\varrho}_{2}^{(1)} = [2, \frac{2m}{n}, 1, 1, \sqrt{1-p}, \sqrt{p}]$
		$\underline{\boldsymbol{\varrho}}_4 = [\underline{\boldsymbol{z}}, \underline{\underline{\boldsymbol{n}}}, 1, 1, -\sqrt{1-p}, \sqrt{p}]$
		$\boldsymbol{\varrho}_1^{(1)} = [\frac{2m}{n}, \frac{2m}{n}, 2, 1, \frac{\sqrt{2}}{8}, \frac{1}{4}]$
$\operatorname{ALTBOC}(m, n)$	$\eta = \begin{bmatrix} 1 & 2 & \frac{2m}{2} \end{bmatrix}$	$\boldsymbol{\varrho}_{2}^{(1)} = [\frac{2m}{n}, \frac{2m}{n}, 2, 1, \frac{\sqrt{2}}{8}, \frac{1}{4}]$
		$\boldsymbol{\varrho}_{3}^{(1)} = [\frac{2m}{n}, \frac{2m}{n}, 2, 1, \frac{\sqrt{2}}{n}, -\frac{1}{4}]$
		$\boldsymbol{\varrho}_4^{(1)} = [\frac{2m}{n}, \frac{2m}{n}, 2, 1, \frac{\sqrt{2}}{8}, -\frac{1}{4}]$

 Table 3.1
 GNSS Modulation Parameters

^{*a*}The parameter vectors not contributing to the signal assume the form $\boldsymbol{\varrho}_{h}^{(l)} = [\sim, \sim, \sim, \sim, 0, 0]$ (nulling the subcarrier) and are omitted from the table; the symbol \sim represents nonzero "don't-care" values.

^bTo exactly construct the TMBOC signal from the GBOC subcarrier segment, the time multiplexing sets \mathcal{T}_1 and \mathcal{T}_2 are needed. Note that the statistical properties of the TMBOC signal depend not on \mathcal{T}_1 and \mathcal{T}_2 , but rather on the cardinalities $|\mathcal{T}_1|$ and $|\mathcal{T}_2|$ contained in $p = A/B = |\mathcal{T}_1|/(|\mathcal{T}_1| + |\mathcal{T}_2|)$. ^cQPSK(n) is sufficiently described by the parameters of BPSK(n).

scaled complex subcarrier segments. The complex subcarrier segments generalize the SSB waveforms present in the ALTBOC signal and are given by

$$\varsigma^{(l)}(t) = S_{\text{GBOC}(\vartheta, \varrho_h^{(l)})}(t) \pm j S_{\text{GBOC}(\vartheta, \varrho_{h'}^{(l)})}(t).$$
(3.10)

A complex subcarrier segment is made from a selection of GBOC subcarrier segments, as indicated by h and h' (h = 1, 2, 3, 4 and h' = 1, 2, 3, 4), and incorporates time multiplex variants, as indicated by l (l = 1, 2). Non-overlapping shifted copies of $\varsigma^{(l)}(t)$ are appropriately scaled by the spread data signal x(t) to form the (baseband) GNSS modulation signal. To explicitly incorporate the specific time multiplexing configuration [36], we introduce the indicator function given by

$$\lambda^{(l)}(t) = \begin{cases} 1, & \lfloor (t \mod BT_c)/T_c \rfloor \in \mathcal{T}_l, \\ 0, & \text{otherwise.} \end{cases}$$
(3.11)

In (3.11), the binary transitions are time aligned with those of the spread data signal, and \mathcal{T}_l indicates the set of PRN code chips for which the *l*th time multiplex variant is in effect.





Fig. 3.2 Synthesis of the GNSS modulation signal component.

A spread data signal with periodic nulling can then be expressed as

$$\hat{x}^{(l)}(t) = \lambda^{(l)}(t) \, x(t), \qquad (3.12)$$

where x(t) is given by (2.1). As illustrated in Fig. 3.2, the time multiplex subcarrier variants $\varsigma^{(l)}(t)$ and the spread data with periodic nulling $\hat{x}^{(l)}(t)$ are combined to produce a GNSS signal component.

In general, a GNSS modulation signal is composed of K complex subcarrier segments and spread data signals denoted by $\varsigma_i^{(l)}(t)$ (i = 1, ..., K) and $\hat{x}_i^{(l)}(t)$ (i = 1, ..., K), respectively. To mathematically describe the GNSS modulation signal, the complex subcarrier segments and spread data are first stacked as vector-valued functions defined as

$$\boldsymbol{\varsigma}^{(l)}(t) \triangleq \left[\varsigma_1^{(l)}(t), \dots, \varsigma_K^{(l)}(t)\right]^{\mathsf{T}} \quad \text{and} \quad \hat{\boldsymbol{x}}(t) \triangleq \left[\hat{x}_1(t), \dots, \hat{x}_K(t)\right]^{\mathsf{T}}.$$
 (3.13)

In the above definitions, $(\cdot)^{\mathsf{T}}$ is the transpose operator. The length of the vectors in (3.13) depends on the specific GNSS modulation, e.g., K = 1 for a BPSK modulated signal, while K = 4 for an ALTBOC modulated signal. Finally, we can express the generalized GNSS modulation signal as

$$y(t) = \sum_{k=-\infty}^{\infty} \sum_{l=1}^{2} \hat{\boldsymbol{x}}^{(l)} (kT_{\rm c})^{\mathsf{T}} \boldsymbol{\varsigma}^{(l)} (t - kT_{\rm c}).$$
(3.14)

Note that (3.14) implicitly incorporates all possible subcarrier segments and time multiplex variants via $\boldsymbol{\varsigma}^{(l)}(t)$ but only reveals the effective subcarriers (that result from the time multiplexing) by exploiting the periodic nulling provided by $\hat{\boldsymbol{x}}^{(l)}(t)$. In this work, y(t) is the infinite bandwidth baseband reference GNSS signal [see Section 2.5].

Chapter 4

Statistical Properties of GNSS Signals

Having derived a general expression that describes the GNSS modulation signal, we now study the statistical properties of interest, namely the ACF and PSD. We will see that the ACF of the GNSS modulation signal contains a well defined dominant term. An analytical expression for this dominant term allows one to investigate the code tracking performance provided by GNSS modulation and can be used to improve receiver code tracking accuracy. As we will demonstrate, the Fourier transform of the dominant ACF term corresponds to the envelope of the PSD. An analytical expression for this envelope allows one to investigate the spectral occupancy of the GNSS modulation signal and is useful for filter design. In this chapter, we derive the analytical expressions that describe the dominant ACF term and the PSD envelope of the GNSS modulation signal.

4.1 Stochastic Signal Definition

To characterize the statistical properties of the GNSS modulation signal, we must first define the received signal. The received signal follows the description given in the previous chapters and is summarized by the block diagram in Fig. 4.1. In the figure, y(t) is the infinite bandwidth reference signal that consists of either a signal component or a multiplexed signal. The reference signal is formed by spreading the real and imaginary parts of the



Fig. 4.1 Block diagram of the GBOC modulator and channel model.

data signal $d_i(t)$ (i = 1, ..., K) with the real and imaginary parts of the spreading sequence $c_i(t)$ (i = 1, ..., K). In this chapter, the real and imaginary parts of the data are modeled as realizations of independent and identically distributed (iid) bits, while the real and imaginary parts of the codes consist of deterministic (pseudorandom) chips. The resulting spread data $x_i(t)$ (i = 1, ..., K) modulates the deterministic GBOC subcarriers to produce the reference signal y(t) that is filtered by the satellite transmitter chain and receiver front-end (these filtering operations are comprehensively represented by a single impulse response z(t)) and is transmitted across the channel. The channel parameter of interest for code tracking is the time delay t_0 . Since usually there is no prior knowledge regarding the time delay, it is appropriate to assume that t_0 is chosen uniformly over a range in which it can be resolved unambiguously with respect to some reference, e.g., the data bit transitions. The range of the possible delays is given by [0, T), where $T = LT_c$ and LT_c is set to an integer multiple of the data bit period DT_c . Since we are concerned with signals of fixed power, we assume that the complex attenuation α_0 is finite and deterministic. In addition, the initial phase of the specular component contained within ϕ_0 is chosen uniformly over $[0, 2\pi)$ and is independent from the phases of the accompanying diffuse waves. We note that all random quantities are drawn independently from the noise and interference w(t). Referring to (2.22), the resulting random process can be expressed as

$$\tilde{y}(t) = \hat{y}(t, \underline{t}_0) + \underline{w}(t). \tag{4.1}$$

Since (4.1) incorporates the filtering of $y(t-\underline{t}_0)$ [see (2.17)], the implicit stochastic integral is interpreted in the mean square sense; its existence is due to the properties of Asymptotically Mean Stationary (AMS) processes [see Section 4.3] [51], [52].

4.2 Auto-Correlation Function

The ACF of the received GNSS modulation signal $\tilde{y}(t)$ is given by

$$R_{\tilde{y}}(t_1, t_2) = \mathsf{E}\{\tilde{y}(t_1)\,\tilde{y}(t_2)^*\}.$$
(4.2)

In (4.2), $\mathsf{E}(\cdot)$ is the expectation operator and $(\cdot)^*$ denotes the conjugate. Equivalently, the ACF of the received signal can be expressed in terms of the ACF of the delayed signal and the ACF of the noise and interference. By employing the signal model described by (4.1) and referring to (2.17) and results from linear systems with stochastic inputs [51], we obtain

$$R_{\tilde{y}}(t_1, t_2) = \kappa R_{\tilde{y}}(t_1, t_2) \star z(t_1) \star z(t_2)^* + R_w(t_1 - t_2).$$
(4.3)

In (4.3), $\kappa = |\alpha_0|^2$, $R_w(t_1 - t_2)$ is the ACF of $\underline{w}(t)$, and $R_{\underline{y}}(t_1, t_2)$ is the ACF of $y(t - \underline{t}_0)$. The latter is given by equation (A.3) in Appendix A.1 and can be expressed as

$$R_{\check{y}}(t_{1}, t_{2}) = \mathsf{E}\Biggl\{\sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{l_{1}=1}^{2} \sum_{l_{2}=1}^{2} \mathsf{tr}\left(\underline{\hat{x}}^{(l_{1})}(kT_{c} + mT_{c})\underline{\hat{x}}^{(l_{2})}(kT_{c})^{\mathsf{H}} \times \mathbf{\varsigma}^{(l_{2})}(t_{2} - kT_{c} - \underline{t}_{0})^{*} \mathbf{\varsigma}^{(l_{1})}(t_{1} - mT_{c} - kT_{c} - \underline{t}_{0})^{\mathsf{T}}\right)\Biggr\}.$$
(4.4)

In (4.4), $tr(\cdot)$ denotes the trace and $(\cdot)^{H}$ is the conjugate transpose operator. By propagating the expectation inside the summations and evaluating with respect to \underline{t}_{0} , we obtain

$$R_{\check{y}}(t_{1}, t_{2}) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{l_{1}=1}^{2} \sum_{l_{2}=1}^{2} \operatorname{tr} \left\{ \mathsf{E} \left\{ \underline{\hat{x}}^{(l_{1})} (kT_{\mathrm{c}} + mT_{\mathrm{c}}) \underline{\hat{x}}^{(l_{2})} (kT_{\mathrm{c}})^{\mathsf{H}} \right\} \\ \times \frac{1}{LT_{\mathrm{c}}} \int_{0}^{LT_{\mathrm{c}}} \boldsymbol{\varsigma}^{(l_{2})} (t_{2} - kT_{\mathrm{c}} - t_{0})^{*} \boldsymbol{\varsigma}^{(l_{1})} (t_{1} - mT_{\mathrm{c}} - kT_{\mathrm{c}} - t_{0})^{\mathsf{T}} \, \mathrm{d}t_{0} \right\}.$$
(4.5)

The integral in (4.5) is evaluated element-wise on the matrix integrand. By employing the substitution $\xi = t_2 - kT_c - t_0$, we obtain

$$R_{\check{y}}(t_{1}, t_{2}) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{l_{1}=1}^{2} \sum_{l_{2}=1}^{2} \operatorname{tr} \left\{ \mathsf{E} \left\{ \underline{\hat{x}}^{(l_{1})}(kT_{\mathrm{c}} + mT_{\mathrm{c}})\underline{\hat{x}}^{(l_{2})}(kT_{\mathrm{c}})^{\mathsf{H}} \right\} \times \frac{1}{LT_{\mathrm{c}}} \int_{t_{2}-kT_{\mathrm{c}}-LT_{\mathrm{c}}}^{t_{2}-kT_{\mathrm{c}}} \boldsymbol{\varsigma}^{(l_{2})}(\xi)^{*} \boldsymbol{\varsigma}^{(l_{1})}(t_{1} - t_{2} - mT_{\mathrm{c}} + \xi)^{\mathsf{T}} \, \mathrm{d}\xi \right\}.$$
(4.6)

By using the identity provided by (A.4) in Appendix A.2 while exploiting the linearity of (4.6) and replacing $t_1 - t_2$ by τ and t_2 by t, the ACF of the delayed signal can be expressed as

$$R_{\check{y}}(\tau,t) = \sum_{m=-\infty}^{\infty} \sum_{l_{1}=1}^{2} \sum_{l_{2}=1}^{2} \operatorname{tr} \left\{ \int_{-\infty}^{\infty} \mathsf{E} \left\{ \sum_{k=1}^{L-1} \hat{\underline{x}}^{(l_{1})} (mT_{c} + kT_{c} - LT_{c} + t) \hat{\underline{x}}^{(l_{2})} (kT_{c} - LT_{c} + t)^{\mathsf{H}} + \hat{\underline{x}}^{(l_{1})} (mT_{c} - LT_{c} + t) \hat{\underline{x}}^{(l_{2})} (-LT_{c} + t)^{\mathsf{H}} [u(\xi - t \mod T_{c})] + \hat{\underline{x}}^{(l_{1})} (mT_{c} + t) \hat{\underline{x}}^{(l_{2})} (t)^{\mathsf{H}} [1 - u(\xi - t \mod T_{c})] \right\} \\ \times \frac{1}{LT_{c}} \boldsymbol{\varsigma}^{(l_{2})}(\xi)^{*} \boldsymbol{\varsigma}^{(l_{1})} (\tau - mT_{c} + \xi)^{\mathsf{T}} \, \mathrm{d}\xi \right\}.$$

$$(4.7)$$

In (4.7), $u(\cdot)$ represents the unit step function.

A difficulty arises in further evaluating (4.7) because we cannot express the behavior of the deterministic spreading codes within $\underline{\hat{x}}^{(l)}(t)$ (l = 1, 2) analytically. Instead, we resort to the basic properties of the spread data used in GNSS. In doing so, it is possible to extract the *tractable* terms from (4.7). Any term is tractable if it does not depend on correlations of the spread data and therefore is analytically expressible. Thus, the tractable terms in (4.7) occur only when m = 0 and therefore $l = l_1 = l_2$; in such a case, the product of like PRN code chips and data bits contained in the real and imaginary parts of $\underline{\hat{x}}^{(l)}(t)$ is always 1. By grouping the remaining terms into a function called $\epsilon(\tau, t)$, the ACF of the delayed signal is given by

$$R_{\check{y}}(\tau,t) = \frac{1}{LT_{\rm c}} \sum_{l=1}^{2} \operatorname{tr} \left(\mathsf{E} \left\{ \sum_{k=0}^{L-1} \underline{\hat{x}}^{(l)}(t-kT_{\rm c})\underline{\hat{x}}^{(l)}(t-kT_{\rm c})^{\mathsf{H}} \right\} \int_{-\infty}^{\infty} \boldsymbol{\varsigma}^{(l)}(\xi)^{*} \boldsymbol{\varsigma}^{(l)}(\tau+\xi)^{\mathsf{T}} \, \mathrm{d}\xi \right) + \epsilon(\tau,t).$$

$$(4.8)$$

For a nonzero spread data vector¹, the diagonal entries of $\underline{\hat{x}}^{(l)}(t - kT_c)\underline{\hat{x}}^{(l)}(t - kT_c)^{H}$ correspond to the modulus of the entries in the spread data vectors, ρ ($\rho = 1$ or 2, depending on whether the spread data is real or complex, respectively). Conversely, the off-diagonal entries only consist of spread data correlation-dependent cross-terms and have no fixed real component. This ensures a predominant ACF center peak (a desirable property for code tracking) and is due to the pseudorandom nature of the spreading codes in $\underline{\hat{x}}^{(l)}(t)$. In evaluating $\sum_{k=0}^{L-1} \underline{\hat{x}}^{(l)}(t - kT_c)\underline{\hat{x}}^{(l)}(t - kT_c)^{H}$, the spread data spans L/B time multiplexing repetition periods each containing $|\mathcal{T}_l|$ PRN code chips ($|\cdot|$ denotes set cardinality) and the spread data correlation-dependent cross-terms are expected to combine mostly destructively, again, due to the properties of the PRN codes. Therefore, the dominant contribution is given by the diagonal entries of the resulting matrix which are equal to $\rho |\mathcal{T}_l|L/B$. By redefining $\epsilon(\tau, t)$ from (4.8) so that it includes the remaining intractable terms, we obtain

$$R_{\check{y}}(\tau,t) = \sum_{l=1}^{2} \frac{\rho|\mathcal{T}_{l}|}{BT_{c}} \operatorname{tr}\left(\int_{-\infty}^{\infty} \boldsymbol{\varsigma}^{(l)}(\xi)^{*} \boldsymbol{\varsigma}^{(l)}(\tau+\xi)^{\mathsf{T}} \,\mathrm{d}\xi\right) + \epsilon(\tau,t).$$
(4.9)

The tractable term in (4.9) consists of a linear combination of finite-time cross-correlations of the subcarrier segments found in $\boldsymbol{\varsigma}^{(l)}(t)$. A simplification is possible by writing the inner-product of the complex subcarrier segments as

$$\boldsymbol{\varsigma}^{(l)}(\tau+\xi)^{\mathsf{H}}\boldsymbol{\varsigma}^{(l)}(\xi) = \gamma \sum_{h=1}^{4} S_{\mathrm{GBOC}(\boldsymbol{\vartheta},\boldsymbol{\varrho}_{h}^{(l)})}(\xi) S_{\mathrm{GBOC}(\boldsymbol{\vartheta},\boldsymbol{\varrho}_{h}^{(l)})}(\tau+\xi), \quad \gamma \in \mathbb{Z}^{+}.$$
(4.10)

¹Recall that $\hat{\underline{x}}^{(l)}(t)$ is nonzero if and only if the *l*th time variant is in effect, i.e., $\lambda^{(l)}(t) = 1$.

Combined with (4.9), the simplification provided by (4.10) is consistent with existing modulations and holds due to the presence of subcarrier complex conjugate pairs in $\boldsymbol{\varsigma}^{(l)}(t)$ [cf. (2.10)]. Therefore, the tractable term of (4.9) can be expressed as an autocorrelation coefficient defined as

$$R_{S}(\tau) \triangleq \frac{\rho\gamma}{P_{S}T_{c}} \sum_{h=1}^{4} \sum_{l=1}^{2} \left(l - 1 - (-1)^{l} \frac{A}{B} \right) S_{\text{GBOC}(\vartheta, \varrho_{h}^{(l)})}(\tau) \star S_{\text{GBOC}(\vartheta, \varrho_{h}^{(l)})}(-\tau).$$
(4.11)

In (4.11), we have explicitly shown the set cardinality $|\mathcal{T}_l|$ in terms of the time multiplexing parameter A/B. The correlation in (4.11) ensures the maximum value $R_S(0) = 1$ (the scaled power $P_S/(\rho\gamma)$ is defined below [see (4.21)]). Conversely, the intractable contribution to (4.9) is referred to as the *parasitic correlations*. The parasitic correlations are deeply related to the phenomenon of self-noise that corrupts the output of finite-time correlators (or delay discriminators) used in practical receivers [24], [53]. The ACF of the received GNSS modulation signal can be obtained, from (4.3), (4.9), and (4.11), in terms of the subcarrier autocorrelation coefficient:

$$R_{\tilde{y}}(\tau,t) = \kappa \left[P_S R_S(\tau) \star z(\tau) \star z(-\tau)^* + \epsilon(\tau,t) \right] + R_w(\tau). \tag{4.12}$$

In (4.12), $\epsilon(\tau, t)$ is again redefined from (4.9) to incorporate any filtering due to z(t). The tractable behavior of $R_{\tilde{y}}(\tau, t)$ is described by the autocorrelation coefficient $R_S(\tau)$. Actually, $R_S(\tau)$ is the dominant analytical term portraying the ACF within the interval² $[-T_c, T_c]$. The expression for $R_S(\tau)$ is examined in greater detail in the sections that follow. The intractable term of $R_{\tilde{y}}(\tau, t)$ is described by $\epsilon(\tau, t)$. Although $\epsilon(\tau, t)$ contains the parasitic correlations, it is given no analytical description. Qualitatively, the parasitic correlations are primarily composed of a linear combination of distorted finite-time autocorrelations of the subcarrier segments centered at $\tau = mMT_c$ (*m* being a nonzero integer), where the distortion is due to the distribution of the data bits. Long PRN codes (large values of *M*) minimize the occurrence of these finite-time autocorrelations. Moreover, $\epsilon(\tau, t)$ also consists of a superposition of small distorted finite-time cross-correlations of the subcarrier segments

²The ACF delay interval $[-T_c, T_c]$ is the interval of interest when tracking a GNSS modulation signal.

centered at $\tau = nT_c$ $(n \in \mathbb{Z})$. Good PRN codes suppress these finite-time cross-correlations.

The notion of the parasitic correlations enunciates two general statistical features of the GNSS PRN codes. First, codes chosen for a non-multiplexed scheme (where A = B and $\lambda^{(2)}(t) = 0$) do not behave the same as for a multiplexed scheme; this was also mentioned by Rushanan [35] and Hein et al. [25]. Indeed, the correlation behavior of the PRN codes largely depends on the time multiplexed code portions rather than on the whole code sequences. Secondly, signals employing subcarriers with odd $N_{h,1}^{(l)}$ require that the PRN codes incorporate periodic phase flips with period T_c . The phase flips are needed to fulfill the original subcarrier definition [3], [54]. Therefore, to limit the contribution of the parasitic correlations, the PRN codes must account for time multiplexing and phase flips, although the latter have little effect on the ACF because they cancel in the dominant contribution. In fact, for ideal codes (where $\epsilon(\tau, t) = 0$) periodic phase flips have no impact on the second order statistics of the GNSS modulation signal [1].

4.3 Power Spectral Density

The nature of the parasitic correlations suggests that the GNSS modulation signal is not a stationary random process. However, since the process is steady enough over time (bounded and not converging to zero), the asymptotic time average (mean) of its statistical properties exist and is not everywhere zero [52]. This concept is summarized in the following theorem.

Theorem 4.3.1 Let $\underline{\tilde{y}}(t)$ be a GNSS modulation signal with a time varying ACF $R_{\tilde{y}}(\tau, t)$. Then $\underline{\tilde{y}}(t)$ is Asymptotically Mean Stationary (AMS) in the wide sense, that is, there exists a nonzero time average given by

$$\bar{R}_{\tilde{y}}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} R_{\tilde{y}}(\tau, t) \,\mathrm{d}t.$$
(4.13)

Proof See Appendix B.1.

We have seen that the GNSS modulation signal employs periodic time multiplexing with period BT_c , periodic PRN codes with period MT_c , and iid data bits with period DT_c . Therefore, it is not surprising that this periodic nature manifests itself in the probabilistic descriptions. It is important to note that in the absence of a random delay \underline{t}_0 , the GNSS modulation signal is wide sense cyclostationary with period DT_c . This means that if L = D, the presence of \underline{t}_0 makes the GNSS modulation signal $\underline{\tilde{y}}(t)$ wide sense stationary [52]. For the general case where $L \neq D$, the signal $\underline{\tilde{y}}(t)$ is wide sense cyclostationary with period LT_c .

The AMS property is needed to define the PSD of the GNSS modulation signal.

Corollary 4.3.1 The PSD $G_{\tilde{y}}(f)$ of a GNSS modulation signal $\underline{\tilde{y}}(t)$ is given by the Fourier transform of the asymptotic time averaged ACF $\bar{R}_{\tilde{y}}(\tau)$ of $\tilde{y}(t)$:

$$G_{\tilde{y}}(f) = \int_{-\infty}^{\infty} \bar{R}_{\tilde{y}}(\tau) \mathrm{e}^{-j2\pi f\tau} \,\mathrm{d}\tau.$$
(4.14)

Proof The above relation follows by noting that the PSD of a signal is defined as the expectation of its average energy spectrum per unit time as time grows infinitely large [52], [55]. Note that if $\underline{\tilde{y}}(t)$ is wide sense stationary, then $R_{\tilde{y}}(\tau, t) = \overline{R}_{\tilde{y}}(\tau) \forall t$ and (4.14) reduces to the Wiener-Khinchin Theorem.

The asymptotic time averaged ACF of the GNSS modulation signal can be expressed in terms of the dominant ACF term $R_S(\tau)$ and the time average of the parasitic correlations over an interval $[0, LT_c)$ defined as $\bar{\epsilon}(t)$, i.e.,

$$\bar{R}_{\tilde{y}}(\tau) = \kappa \left[P_S R_S(\tau) \star z(\tau) \star z(-\tau)^* + \bar{\epsilon}(\tau) \right] + R_w(\tau).$$
(4.15)

Finally, the PSD of the GNSS modulation signal can be obtained, analogously to (4.12), in terms of a subcarrier power spectrum:

$$G_{\tilde{y}}(f) = \kappa \left[P_S G_S(f) \left| Z(f) \right|^2 + \mathscr{F} \left\{ \bar{\epsilon}(\tau) \right\} \right] + G_w(f).$$

$$(4.16)$$

In (4.16), $\mathscr{F}(\cdot)$ denotes Fourier transformation, $G_S(f)$ represents the power spectrum of the GBOC subcarrier, i.e., $G_S(f) = \mathscr{F}\{R_S(\tau)\}$, and $G_w(f)$ represents the PSD of the noise and interference, i.e., $G_w(f) = \mathscr{F}\{R_w(\tau)\}$. The tractable behavior of $G_{\tilde{y}}(f)$ is described by the spectrum $G_S(f)$. Actually, $G_S(f)$ is the unit-power envelope (where $\int_{-\infty}^{\infty} G_S(f) df = R_S(0) = 1$) of the PSD of the GNSS modulation signal. The expression for $G_S(f)$ is examined in greater detail in the sections that follow. The intractable behavior of $G_{\tilde{y}}(f)$ is described by $\mathscr{F}\{\bar{\epsilon}(\tau)\}$. Qualitatively, for infinite bandwidth signals (where $z(t) = \delta(t)$), the contribution of $\mathscr{F}\{\bar{\epsilon}(\tau)\}$ to the PSD corresponds to closely spaced spectral lines. These spectral lines are separated by a frequency of $1/MT_c$ and have amplitudes approximately following the envelope described by $G_S(f)$. In fact, due to the presence of small finite-time correlations centered at $\tau = nT_c$ ($n \in \mathbb{Z}$) in $\bar{\epsilon}(\tau)$, the PSD is formed from a sequence of lines whose magnitudes are sometimes superior or inferior to $G_S(f)$ [24].

4.4 Extension to Bandpass

Since our analysis has been conducted using complex envelopes, it is desirable to mention how the results apply to a bandpass signal. Representing the carrier frequency of the GNSS modulation signal by f_0 and the maximum Doppler shift by f_D , we note that for GNSS $f_D \ll 1/T_c \ll f_0$ [20]. Therefore, the dominant ACF term of the bandpass signal incorporates the fine structure of the carrier under an envelope approximately given by $R_S(\tau)$. In this case, the ACF normalization is with respect to the physical subcarrier power, as opposed to that of its complex envelope. Therefore, the corresponding unit-power bandpass PSD envelope can be approximated by adding symmetrically shifted $(\pm f_0)$ copies of $G_S(f)$ and scaling the result by 1/2. We note that the extension to bandpass is not needed for the study of coherent code tracking loops [24] since the complex envelope represents a carrier-stripped signal.

4.5 Analytical Description of the ACF and PSD

Until now, we have based the statistical properties of GNSS modulation on the dominant ACF term and PSD envelope. However, we have not yet provided explicit analytical expressions for these functions. The chief results of the foregoing theory are the analytical expressions for the dominant ACF term and PSD envelope of the GNSS modulation signal.

The dominant ACF term is obtained by defining a scaled subcarrier power $\dot{P}_S \triangleq P_S/(\rho\gamma)$ and combining expressions (3.5), (3.9), and (4.11). The parameterized expression for the dominant ACF term of the GNSS modulation signal is given by

$$R_{S}(\tau) = \frac{1}{2\check{P}_{S}\zeta Z} \sum_{h=1}^{4} \sum_{l=1}^{2} \sum_{j=1}^{2} \sum_{j'=1}^{2} \left(l - 1 - (-1)^{l} \frac{A}{B} \right) a_{h,j}^{(l)} a_{h,j'}^{(l)} \Omega_{(\vartheta, \boldsymbol{\varrho}_{h}^{(l)})}^{(j,j')}(\tau), \quad (4.17)$$

where

$$\Omega_{(\vartheta, \varrho_{h}^{(l)})}^{(j,j')}(\tau) = \sum_{n=0}^{N_{h,j}^{(l)}-1} \sum_{n'=0}^{N_{h,j'}^{(l)}-1} \sum_{z=0}^{\left\lfloor \frac{\zeta z \eta_{h,j'}^{(l)}}{N_{h,j'}^{(l)} \eta_{h,1}^{(l)}} \right\rfloor^{-1}} \sum_{z'=0}^{1} \sum_{m=0}^{1} \sum_{m'=0}^{1} (-1)^{(n+n')+(m+m')(\nu_{h}+1)} \\
\times \Lambda \left(\frac{\tau}{T_{c}/(2\zeta Z)} - \frac{Z\zeta}{N_{h,j}^{(l)} \eta_{h,1}^{(l)}} \left[\zeta m + 2\zeta n + z N_{h,j}^{(l)}/Z + [m(1-2\nu_{h}) + \nu_{h}](\eta_{h,1}^{(l)} - \eta_{h,j}^{(l)}) \right] \\
+ \frac{Z\zeta}{N_{h,j'}^{(l)} \eta_{h,1}^{(l)}} \left[\zeta m' + 2\zeta n' + z' N_{h,j'}^{(l)}/Z + [m'(1-2\nu_{h}) + \nu_{h}](\eta_{h,1}^{(l)} - \eta_{h,j'}^{(l)}) \right] \right).$$
(4.18)

In (4.18), $\Lambda(\tau)$ is the unit-height triangular function of support [-1, 1). The above equation indicates that $R_S(\tau)$ can be formed from a sum of shifted and scaled triangular functions as exemplified in Fig. 4.2.



Fig. 4.2 Synthesis of the GNSS modulation signal ACF.

The PSD envelope results from the Fourier transform of the dominant ACF term provided by (4.17) which amounts to evaluating $\Upsilon^{(j,j')}_{(\vartheta,\boldsymbol{\varrho}_h^{(l)})}(f) = \mathscr{F}\left\{\Omega^{(j,j')}_{(\vartheta,\boldsymbol{\varrho}_h^{(l)})}(\tau)\right\}$. The parameterized expression for the PSD envelope of the GNSS modulation signal is given by

$$G_{S}(f) = \frac{1}{4\check{P}_{S}T_{c}\pi^{2}f^{2}} \sum_{h=1}^{4} \sum_{l=1}^{2} \sum_{j=1}^{2} \sum_{j'=1}^{2} \left(l - 1 - (-1)^{l} \frac{A}{B} \right) a_{h,j}^{(l)} a_{h,j'}^{(l)} \Upsilon_{(\vartheta, \varrho_{h}^{(l)})}^{(j,j')}(f), \quad (4.19)$$

where

$$\begin{split} \Upsilon_{(\vartheta,\boldsymbol{\varrho}_{h}^{(j)})}^{(j,j')}(f) &= e^{-\jmath \pi f T_{c} \nu_{h}} \Big[(1-\eta_{h,j}^{(l)}/\eta_{h,1}^{(l)})/N_{h,j}^{(l)} - (1-\eta_{h,j'}^{(l)}/\eta_{h,1}^{(l)})/N_{h,j'}^{(l)} \Big] \\ &\times \Big(1 - (-1)^{\nu_{h}} e^{-\jmath \pi f T_{c} [1+(1-2\nu_{h})(1-\eta_{h,j'}^{(l)}/\eta_{h,1}^{(l)})]/N_{h,j}^{(l)}} \Big) \\ &\times \Big(1 - (-1)^{\nu_{h}} e^{\jmath \pi f T_{c} [1+(1-2\nu_{h})(1-\eta_{h,j'}^{(l)}/\eta_{h,1}^{(l)})]/N_{h,j'}^{(l)}} \Big) \\ &\times \Big(1 - (-1)^{N_{h,j}^{(l)}} e^{-\jmath 2\pi f T_{c}} \Big) \Big(1 - e^{-\jmath \pi f T_{c} \eta_{h,j'}^{(l)}/(\eta_{h,1}^{(l)}N_{h,j}^{(l)})} \Big) \Big(1 + e^{-\jmath 2\pi f T_{c}/N_{h,j}^{(l)}} \Big)^{-1} \\ &\times \Big(1 - (-1)^{N_{h,j'}^{(l)}} e^{\jmath 2\pi f T_{c}} \Big) \Big(1 - e^{\jmath \pi f T_{c} \eta_{h,j'}^{(l)}/(\eta_{h,1}^{(l)}N_{h,j'}^{(l)})} \Big) \Big(1 + e^{\jmath 2\pi f T_{c}/N_{h,j'}^{(l)}} \Big)^{-1}. \end{split}$$
(4.20)

The motivation for using complex exponentials to express the PSD is clear if we consider that (4.20) is obtained by evaluating the summation in (4.18) after subjecting the constituent spectrum $\mathscr{F}\{\Lambda(\tau)\} = (1 - e^{j2\pi f})(1 - e^{-j2\pi f})/(2\pi f)^2$ to the linearity, scaling, and shifting properties of the Fourier transform. The PSD is strictly real and symmetric since it arises from the Fourier transform of a real and symmetric function. We note that singularities occur at frequencies $f = nN_{h,j}^{(l)}/T_c$ where $n \in \{0, m-1/2\}$ ($m \in \mathbb{Z}$) for all $N_{h,j}^{(l)}$ (h = 1, 2, 3, 4, j = 1, 2, and l = 1, 2) not assigned to "don't-care" values in Table 3.1. If required, the PSD at these points can be evaluated separately.

The normalized subcarrier power \check{P}_S must also be expressed analytically. Considering the general subcarrier segment of (3.9) and the dominant ACF term provided by (4.11), it can be determined that

$$\check{P}_{S} = \sum_{h=1}^{4} \sum_{l=1}^{2} \left(l - 1 - (-1)^{l} \frac{A}{B} \right) \left\{ \left(a_{h,1}^{(l)} \right)^{2} + \frac{\eta_{h,2}^{(l)}}{\eta_{h,1}^{(l)}} \left(a_{h,2}^{(l)} \right)^{2} + \frac{\eta_{h,2}^{(l)}}{\eta_{h,1}^{(l)}} \left(a_{h,1}^{(l)} a_{h,2}^{(l)} \right) \right. \\
\left. \times 2 \left[\left(1 - N_{h,2}^{(l)} \mod N_{h,1}^{(l)} \right) \mod \frac{2N_{h,1}^{(l)}}{N_{h,2}^{(l)}} \right] \right\}.$$
(4.21)

By using equations (4.17) - (4.21) with the parameters in Table 3.1, the dominant term of the ACF and the PSD envelope of all GPS and Galileo signals can be generated precisely.

4.6 Simulation of the GNSS Modulation Theory

This section describes the validation of the proposed analytical expressions for the dominant ACF term and the PSD envelope of the GNSS modulation signal.

4.6.1 Simulation Methodology

The simulation model used to verify the statistical theory developed in this work employs existing Gold codes and subcarriers to construct segments of real GPS and Galileo signals³. Following common practice, the signal segments are noise-less, data-less, carrier-stripped, and sampled at a suitable rate⁴ $v_{\rm s}$. An ideal lowpass filter then limits the bandwidth of the simulated signal segments and the statistical properties are obtained using time averages.

The empirical ACF [52] is obtained from the circular autocorrelation of the simulated signal segment of length MT_c . When normalized with respect to the maximum magnitude, the empirical ACF can be compared with the analytical result provided by (4.17) for delays within $[-T_c, T_c]$. This congruity is due to the the dominant ACF term.

The empirical PSD [52] is obtained from the MT_cv_s -point Discrete Fourier Transform (DFT) of the normalized empirical ACF. When multiplied by $1/v_s$ (relating the magnitude of the DFT to that of the Fourier transform), the empirical PSD is implicitly scaled with respect to twice the physical carrier power $2C = \kappa P_S$ and can be compared with the analytical result provided by (4.19) for frequencies within $[-v_s/2, v_s/2]$. However, in practice, the PSD is composed of a sequence of spectral lines whose magnitude fluctuates about the analytical envelope. This fluctuation is due to a frequency power reshuffling caused by the PRN codes; since we ignored the effect of the parasitic correlations, the fluctuation does not appear in the analytical result. Therefore, it suffices to offset the simulated PSD to obtain a fair comparison with the theory. Experimental results show that Gold codes (with M = 1023) produce spectra that can be consistently adjusted by -6 dB (although strong spectral lines

³The codes are GPS PRN1-4 [20] where M = 1023, and the subcarriers are simulated using (2.2) where, for BOC subcarriers, $a_0 = a_2 = 0$, $a_1 = 1$, and $\varphi_1 = 0, 3\pi/2$; for CBOC subcarriers, $a_0 = \sqrt{1-p}$, $a_1 = \pm\sqrt{p}$, $a_2 = 0$, and $\varphi_0 = \varphi_1 = 3\pi/2$; for TMBOC subcarriers, $a_0 = a_2 = 0$, $a_1 = 1$, and $\varphi_1 = 3\pi/2$; and for ALTBOC subcarriers, $a_0 = 0$, $a_1 = \sqrt{2}/8$, $a_2 = \pm 1/8$, $\varphi_1 = 0, 3\pi/2$, and $\varphi_2 = \pi/4$.

⁴A suitable sampling rate is one where the aliasing effects are considered negligible; the simulations use $v_s = 511.5$ MHz.

can overshoot the envelope by about 8 dB). Although the PSD envelope offset is known [15], [24], past theoretical frameworks [11], [54] omit its mention.

4.6.2 Results

The developed statistical theory of GNSS modulation is verified here for a select number of infinite bandwidth GNSS signal components and navigation signals. The theoretical and simulated ACF and PSD of select CBOC and ALTBOC multiplexed (navigation) signals are shown in Fig. 4.3 and Fig. 4.4, while those of select TMBOC and CBOC signal components are shown in Fig. 4.5 and Fig. 4.6. For select ABOC signal components (specifically those



Fig. 4.3 Statistical properties of the $CBOC(6, 1, \frac{1}{11})$ multiplexed signal.



Fig. 4.4 Statistical properties of the ALTBOC(15, 10) multiplexed signal.



Fig. 4.5 Statistical properties of the TMBOC $(6, 1, \frac{4}{33})$ signal component.



Fig. 4.6 Statistical properties of select CBOC signal components: (a) $CBOC(6, 1, \frac{1}{11}, +)$; (b) $CBOC(6, 1, \frac{1}{11}, -)$.

necessary for the ALTBOC multiplexing scheme), the comparison between theory and simulation is made in Fig. 4.7.



Fig. 4.7 Statistical properties of select ABOC signal components: (a) ABOC_{cos}(15, 10, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{\sqrt{2}}{2}$, +); (b) ABOC_{sin}(15, 10, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{\sqrt{2}}{2}$, +); (c) ABOC_{cos}(15, 10, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{\sqrt{2}}{2}$, -); (d) ABOC_{sin}(15, 10, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{\sqrt{2}}{2}$, -).

In the figures, the normalized magnitude of the ACF is unit-less and the magnitude of the PSD has a unit⁵ of dBr/Hz. It can be seen that the analytical model agrees with

 $^{^5\}mathrm{The}$ unit dBr/Hz measures decibels relative to 2C. If 2C corresponds to 1 W, then dBr/Hz is equivalent to dBW/Hz.

the simulated results. In general, this congruity can be verified for all existing GPS and Galileo signals. To the author's knowledge no other theoretical formulation is capable of simultaneously (and precisely) describing the ACF and PSD of ALTBOC, CBOC and TMBOC signals. Moreover, the constituent statistics of the ALTBOC multiplexed signals, i.e., the ACF and PSD of the ABOC signal components, are herein illustrated for the first time — widening our perspective on the wideband ALTBOC signal.

The statistical theory of GNSS modulation is also verified for bandlimited signals by accounting for the filtering effect on the dominant ACF term in (4.12). The theoretical and simulated ACF of the Galileo ALTBOC signal is shown in Fig. 4.8 for ideal bandlimiting. The deviation of the result from the infinite bandwidth case is due to the correlation loss.



Fig. 4.8 ACF of the ALTBOC(15, 10) multiplexed signal of select bandwidths: (a) 30 MHz; (b) 50 MHz; (c) 90 MHz (double-sided).

Finally, it is important to compare the proposed theory with prior analytical formulations [11], [12]. In Fig. 4.9, we demonstrate the accuracy of the GNSS modulation theory in describing the dominant ACF term of the Galileo ALTBOC and MBOC modulated signals. While the proposed theory closely follows the simulations, prior analytical formulations are less accurate because they contravene the signal definitions [see Section 2.4]. In investigating code tracking performance, the use of prior analytical expressions may lead to discrepant results and, in practice, may hinder receiver designs that employ ACF replicas [56], [38]. Until now, ALTBOC and MBOC modulation were the two (out of three) modulation families with approximate theoretical statistical descriptions.

In the remainder of this thesis, we use the GNSS modulation theory to study the potential code tracking accuracy inherent in the GNSS modulation signal.



Fig. 4.9 Comparison of analytical formulations describing the ACF: (a) ALTBOC(15, 10); (b) TMBOC(6, 1, $\frac{4}{33}$); (c) CBOC(6, 1, $\frac{1}{11}$).

Chapter 5

Code Tracking Performance Measures

In this chapter, we derive the signal characteristics that are useful to study the code tracking performance of the GNSS modulation signal. Two ostensibly different characteristics are responsible for accurate time delay estimation: 1) the Carrier-to-Noise Density Ratio (CNDR), that measures the quality of the received signal with respect to noise and interference, and 2) the Gabor Bandwidth (GB), that describes the frequency spread of the received signal. These characteristics help define the CRB. The CRB sets a lower bound on the error, in terms of variance, of an unbiased time delay estimator. The CNDR, GB, and CRB are the code tracking performance measures that are discussed in this chapter. These measures are all considered for signals in the presence of noise and interference. The inclusion of interference in our study is possible due to an important theorem that is described at the onset of this chapter.

5.1 Generalization of Signals in the Presence of Noise and Interference

In the remainder of the thesis, we consider, without loss of information, a discrete version of the received signal $\tilde{y}(t)$. The signal $\tilde{y}(t)$ is sampled with a sampling period¹ $T_{\rm s} = 1/W$

¹Recall that $\tilde{y}(t)$ is bandlimited to [-W/2, W/2].

over the time interval [0, T). The resulting N samples $(N = T/T_s)$ are stacked in the vector given by

$$\tilde{\boldsymbol{y}} = \hat{\boldsymbol{y}}(t_0) + \boldsymbol{w}. \tag{5.1}$$

In accordance with the continuous time description, the noise and interference \boldsymbol{w} is drawn from a stationary circular symmetric Gaussian process with covariance² $\boldsymbol{\Sigma} \in \mathbb{R}^{N \times N}$. The vector $\tilde{\boldsymbol{y}}$ is observed by the receiver over the time interval [0, T), where T is also known as the integration time of the receiver. The explicit inclusion of t_0 in the vector-valued function $\hat{\boldsymbol{y}}(t_0)$ emphasizes the importance of the time delay.

Consider that T is much larger than the coherence time of \boldsymbol{w} so that the covariance matrix $\boldsymbol{\Sigma}$ is well approximated by a circulant matrix³ [57]. As a result, hermitian inner products of the derivatives of $\hat{\boldsymbol{y}}(t_0)$ with respect to t_0 computed under the metric⁴ $\boldsymbol{\Sigma}^{-1}$ establish a relationship with the spectral moments of the frequency distribution of $\hat{\boldsymbol{y}}(t_0)$. The following theorem ensues.

Theorem 5.1.1 Let $\hat{\boldsymbol{y}}(t_0) \in \mathbb{C}^N$ be *m*-times differentiable at $t_0 \in \mathbb{R}$ and contain the Nyquist samples of a bandlimited signal with time delay t_0 and power spectrum $G_{\hat{y}}(f)$ ($|f| \leq W/2$). Let the signal be observed in the presence of wide sense stationary bandlimited noise with a PSD $G_w(f)$ ($|f| \leq W/2$), and let the noise samples have an invertible covariance $\boldsymbol{\Sigma} \in \mathbb{R}^{N \times N}$. Then, for a sufficiently large sample size N,

$$\left[\frac{\partial^m}{\partial t_0^m}\,\hat{\boldsymbol{y}}(t_0+\tau)\right]^{\mathsf{H}}\boldsymbol{\Sigma}^{-1}\left[\frac{\partial^m}{\partial t_0^m}\,\hat{\boldsymbol{y}}(t_0)\right] \simeq \frac{N}{W}\int_{-W/2}^{W/2} (2\pi f)^{2m}\,\frac{G_{\hat{\boldsymbol{y}}}(f)}{G_w(f)}\,\mathrm{e}^{j2\pi f\tau}\,\mathrm{d}f.$$
(5.2)

Proof See Appendix B.2.

The covariance matrix Σ is invertible [see footnote 4], and thus can be decomposed as

$$\Sigma = V \Lambda V^{\mathsf{H}}.$$
(5.3)

²The covariance is real because the real and imaginary parts of w are assumed independent [see Section 2.5], [45].

³In a circulant matrix, each column is a downward cyclic shift of the column to its left.

⁴Here, the covariance matrix is invertible because it is circulant, and the PSD of the noise process is always nonzero [see Section 2.5], [57].

In (5.3), $\mathbf{V} \in \mathbb{C}^{N \times N}$ is a unitary matrix and $\mathbf{\Lambda} \in \mathbb{R}^{N \times N}$ is a diagonal matrix with positive diagonal entries. Therefore, the Mahalanobis (whitening) transformation exists and is given by [57]

$$\tilde{\boldsymbol{y}}' = \boldsymbol{V} \boldsymbol{\Lambda}^{-\frac{1}{2}} \boldsymbol{V}^{\mathsf{H}} \tilde{\boldsymbol{y}} = \hat{\boldsymbol{y}}'(t_0) + \boldsymbol{w}'.$$
(5.4)

The above transformation converts $\hat{\boldsymbol{y}}(t_0)$ into $\hat{\boldsymbol{y}}'(t_0)$ by changing the signal basis so that the samples of the combined noise and interference are uncorrelated and have unit variance. With this transformation, the left hand side of (5.2), called the *m*th order hermitian inner product, is defined as

$$R^{(m)}(\tau) \triangleq \left[\frac{\partial^m}{\partial t_0^m} \,\hat{\boldsymbol{y}}'(t_0 + \tau)\right]^{\mathsf{H}} \left[\frac{\partial^m}{\partial t_0^m} \,\hat{\boldsymbol{y}}'(t_0)\right].$$
(5.5)

The definition provided by (5.5) reveals that Theorem 5.1.1 is due to Parseval's theorem where $T_s G_{\hat{y}}(f)/G_w(f)$ is the power spectrum of the whitened signal $\hat{y}'(t, t_0)$ whose samples correspond to the entries in $\hat{y}'(t_0)$ [see the alternative proof of Theorem 5.1.1 in Appendix B.3]. In this respect, the whitening transformation is done by power spectrum inversion, i.e., $G_w(f)^{-\frac{1}{2}}$ is the whitening filter.

By employing the definition provided by (5.5), Theorem 5.1.1 can be used to describe the code tracking performance of the GNSS modulation signal. Indeed, as we will see in the following sections, Theorem 5.1.1 provides a general description for the CNDR, GB, and CRB in the presence of noise and interference.

5.2 Effective Carrier-to-Noise Density Ratio

In order to obtain a general measure for the quality of the GNSS modulation signal observed in the presence of bandlimited noise, we are typically interested in the ratio of the received signal energy to noise PSD. In the presence of ideal bandlimited white noise with a PSD⁵ of $2N_0$ over a finite double-sided bandwidth, the ratio of signal energy to noise density is given by $\psi TC/N_0$, where C/N_0 is referred to as the CNDR [20], and we recall that C is the

⁵Recall that the complex envelope of bandlimited white noise with a double-sided PSD of $N_0/2$ has a PSD of $2N_0$ over the band of interest [45].

carrier power and ψ corresponds to the correlation loss. According to Theorem 5.1.1, the CNDR is proportional to the norm of the transformed vector⁶ $\hat{y}'(t_0)$ [58]. Therefore, an effective measure for signal quality in the presence of noise and interference can be obtained from the norm of the signal vector, considered in a basis where the samples of the combined noise and interference are uncorrelated and have unit variance, and is given by the 0th hermitian inner product described by (5.5), i.e.,

$$R^{(0)}(0) \simeq \psi T C / N_w. \tag{5.6}$$

In (5.6), N_w is the effective noise density which can be defined in terms of the unit-power spectrum $\check{G}_{\hat{y}}(f)$ of the GNSS modulation signal:

$$N_w \triangleq \left[2 \int_{-W/2}^{W/2} \frac{\breve{G}_{\hat{y}}(f)}{G_w(f)} \,\mathrm{d}f \right]^{-1}.$$
(5.7)

The effective noise density indicates the density of the ideal bandlimited white noise required to obtain the same CNDR as produced by arbitrary noise and interference. Therefore, we refer to C/N_w as the effective CNDR.

5.3 Effective Gabor Bandwidth

Better code tracking performance is expected when the GNSS modulation signal allocates more power to frequencies located on the edges of the occupied bands. A measure for the spread of the frequencies over the signal bandwidth is given by the GB. The GB is also referred to as the Root Mean Square (RMS) bandwidth and provides a measure for the standard deviation of the PSD; it is typically expressed as $\beta_0 = \sqrt{\int_{-W/2}^{W/2} f^2 \check{G}_{\hat{y}}(f) df}$ [1], [10]. In general, according to Theorem 5.1.1, the GB can be related to the 1st order hermitian inner product described by (5.5), i.e.,

$$R^{(1)}(0) \simeq 4\pi^2 \beta_w^2 R^{(0)}(0).$$
(5.8)

⁶In general, for a fixed carrier power and time delay, the norm of $\hat{y}'(t_0)$ corresponds to $\max_{t \in \mathbb{R}} \frac{|\hat{y}(t,t_0)|^2}{\mathsf{E}\{|w(t)|^2\}}$.

In (5.8), β_w is the effective GB. The effective GB is defined for signal vectors placed in a basis where the combined noise and interference samples are uncorrelated and have unit variance, so that

$$\beta_w \triangleq \left[2N_w \int_{-W/2}^{W/2} f^2 \frac{\breve{G}_{\hat{y}}(f)}{G_w(f)} \,\mathrm{d}f \right]^{\frac{1}{2}}.$$
(5.9)

The effective GB incorporates the combined noise and interference spectrum to obtain a measure for the spread of frequencies.

5.4 Generalized Cramér-Rao Bound

In describing the code tracking performance of the GNSS modulation signal, the main use of Theorem 5.1.1 is for the derivation of the CRB. Under the assumption of small fluctuations due to noise and interference, the signal vector $\tilde{\boldsymbol{y}}$ can be linearized about an arbitrary time delay. If $\tilde{\boldsymbol{y}}$ is linearized in the vicinity of the true time delay, an estimate for the time delay, say \hat{t}_0 , can be obtained by using the Best Linear Unbiased Estimator (BLUE). In fact, typical delay lock loop discriminators are based on a linearized model [24], [47]. Then, provided that the obtained estimate lies within the linear region, the variance of the BLUE is given by⁷ [5]

$$[\sigma_{\hat{t}_0}^2]_{\text{\tiny BLUE}} = \frac{1}{2R^{(1)}(0)}.$$
(5.10)

Theorem 5.1.1 reveals that the variance of the BLUE does not depend on the actual value of the time delay t_0 . Therefore, in the case of Gaussian noise, (5.10) corresponds to the CRB [5], [59]. Finally, by evaluating (5.10) via (5.6) and (5.8), we obtain a convenient expression for the CRB [cf. (2.23)]:

$$[\sigma_{\hat{t}_0}^2]_{\rm \tiny CRB} \simeq \frac{1}{8\pi^2 \psi T \beta_w^2 C/N_w}.$$
(5.11)

The CRB indicates that the ranging potential of the GNSS modulation signal depends on signal quality and frequency content but not on the detailed structure of the signal. The CRB is well known for Gaussian white noise [3], [58], [60], yet remains largely undisclosed

⁷The factor of 1/2 in the variance of the BLUE is due to the estimation of a real parameter from complex observations.

in the case of colored noise [41]. The strategy presented herein allows for a generalization in terms of the effective noise density and the effective GB.

The CRB establishes a lower bound for the code tracking error of receivers operating within a linear region, and thus only gives a local measure of the estimator variance. However, due to its relevance in practice, a linearized model is often assumed [24], [46]. As a result, the CRB has become the de facto performance standard when describing code tracking accuracy in GNSS [3], [4]. However, if a signal is falsely detected and is linearized about a point far removed from the true time delay, the CRB may no longer describe optimal performance. Furthermore, if the time delay search interval is finite, the estimator is biased [7], and again the CRB may fail to indicate true performance. Therefore, unwarranted use of the CRB may easily lead to ungrounded conclusions, especially when applied to conditions that challenge the linearized model, such as weak and/or wideband signal tracking. These conditions are foreseeable in the next generation GNSS [32], [48], and thus it is important to employ a benchmark that more accurately describes the potential code tracking accuracy of the GNSS modulation signal. In the next chapter, we propose the use of an alternative benchmark for the code tracking performance of the GNSS modulation signal. This new

Chapter 6

Code Tracking Information

In this chapter, we develop an information-theoretic description for the quantity of time delay information contained in the GNSS modulation signal. Based on the quantity of time delay information, we will see that the code tracking performance is markedly categorized into two regimes, one where the time delay estimation error is small and another where it is large. To delimit the code tracking regimes, we define thresholds on the CNDR. Our study remains independent of code tracking implementation.

6.1 Code Tracking Regimes

In this section, we are primarily concerned with the code tracking performance of constant power signals and consider the ensemble produced by (5.1) where only the time delay and noise and interference are random. As in Chapter 4, we assume that the time delay \underline{t}_0 is uniformly distributed over [0, T) and is independent from the noise and interference \underline{w} . Therefore, the probability density function (pdf) of \underline{t}_0 can be expressed as

$$p(t_0) = \frac{1}{T} [u(t_0) - u(t_0 - T)], \qquad (6.1)$$

where $u(\cdot)$ is the unit step function. The initial uncertainty in the time delay can also be measured using the entropy defined as $h(\underline{t}_0) \triangleq -\mathsf{E}\{\ln(p(\underline{t}_0))\} = \ln(T)$. Naturally, the larger the observation interval, the greater the uncertainty. Suppose that the true time delay is given by \check{t}_0 . In reality, \check{t}_0 remains hidden from direct observation, and only the signal $\tilde{\boldsymbol{y}} = \hat{\boldsymbol{y}}(\check{t}_0) + \boldsymbol{w}$ is observed. Because the time delay is modeled as a random variable \underline{t}_0 , we wish to obtain the realization t_0 that corresponds to \check{t}_0 . By invoking the Gaussianity of $\underline{\boldsymbol{w}}$, the joint pdf of the entries in $\underline{\tilde{\boldsymbol{y}}}$, conditioned on t_0 , is given by

$$p(\tilde{\boldsymbol{y}}|t_0) = \frac{1}{\pi^N |\boldsymbol{\Sigma}|} e^{\left(-[\tilde{\boldsymbol{y}} - \hat{\boldsymbol{y}}(t_0)]^{\mathsf{H}} \boldsymbol{\Sigma}^{-1}[\tilde{\boldsymbol{y}} - \hat{\boldsymbol{y}}(t_0)]\right)}.$$
(6.2)

In (6.2), $|\cdot|$ denotes the determinant of the matrix argument, and we recall that N is the number of samples in \tilde{y} and that Σ is the covariance matrix of \underline{w} . By spectrally decomposing Σ as in (5.3) and invoking (5.4), the above pdf can be written as

$$p(\tilde{\boldsymbol{y}}|t_0) = \frac{1}{\pi^N |\boldsymbol{\Sigma}|} e^{\left(-\tilde{\boldsymbol{y}}'^{\mathsf{H}} \tilde{\boldsymbol{y}}' - \hat{\boldsymbol{y}}'(t_0)^{\mathsf{H}} \hat{\boldsymbol{y}}'(t_0) + 2\Re\{\tilde{\boldsymbol{y}}'^{\mathsf{H}} \hat{\boldsymbol{y}}'(t_0)\}\right)}.$$
(6.3)

Next, by employing Bayes' theorem and absorbing any terms not depending on t_0 into a normalization constant K, the posterior time delay pdf can be expressed as

$$p(t_0|\,\tilde{\boldsymbol{y}}) = K[u(t_0) - u(t_0 - T)] e^{\left(2\Re\{\tilde{\boldsymbol{y}}'^{\mathsf{H}}\hat{\boldsymbol{y}}'(t_0)\}\right)}.$$
(6.4)

A clairvoyant form of the posterior time delay pdf can be obtained by expressing \tilde{y}' in terms of the true time delay, i.e., $\tilde{y}' = \hat{y}'(\check{t}_0) + w'$, and evaluating (6.4) via (5.5):

$$p(t_0|\,\tilde{\boldsymbol{y}}) = K[u(t_0) - u(t_0 - T)] e^{\left(2\Re\{R^{(0)}(\check{t}_0 - t_0)\} + \chi(t_0)\right)}.$$
(6.5)

The first term in the exponent of (6.5) is the finite-time ACF of the GNSS modulation signal computed under the metric of the inverse noise covariance and is provided by Theorem 5.1.1. For ideal bandlimited white Gaussian noise, the finite-time ACF is well described by lowpass filtering and scaling the function described by (4.17). This term reaches a maximum given by $2\psi TC/N_w$ at $t_0 = \check{t}_0$ [see (5.6)] to indicate the location of the true time delay. The second term in the exponent of (6.5) is a noise function denoted by $\chi(t_0)$. Adversely, $\chi(t_0)$ tends to displace the global maximum away from $t_0 = \check{t}_0$. Hence, in (6.5) the finite-time ACF and noise function assume opposing roles. The posterior uncertainty in the time delay can be measured by the conditional entropy defined as $h(\underline{t}_0 | \underline{\tilde{y}}) \triangleq -\mathsf{E}\{\ln (p(\underline{t}_0 | \underline{\tilde{y}}))\}$. Consequently, the average reduction of uncertainty in the time delay due to the observation of \tilde{y} is given by the Mutual Information (MI) $I(\underline{t}_0; \underline{\tilde{y}}) = h(\underline{t}_0) - h(\underline{t}_0 | \underline{\tilde{y}})$ that signifies the maximum quantity of information about the time delay that can be reliably retrieved after observing $\underline{\tilde{y}}$. Rate distortion theory states that in order to increase the fundamental accuracy of any time delay estimator, the observations must deliver more information concerning \underline{t}_0 [61]. Unfortunately, direct evaluation of the MI is cumbersome, requiring a multi-dimensional integration over the unspecified normalization constant K.

To circumvent the computational complexity introduced by the posterior time delay pdf, we adopt a statistical interpretation that describes the behavior of the time delay information at high and low effective CNDR. By considering the posterior time delay pdf over an ensemble of \tilde{y} in which \check{t}_0 is invariant, $\chi(t_0)$ can be modeled as a realization of a random variable defined as

$$\underline{\chi}(t_0) \triangleq 2\Re\{\underline{\boldsymbol{w}}^{\prime \mathsf{H}} \hat{\boldsymbol{y}}^{\prime}(t_0)\}.$$
(6.6)

Because \underline{w}' is a circular symmetric Gaussian random vector, the argument of $\Re(\cdot)$ in (6.6) is a complex Gaussian random variable with iid real and imaginary parts [43]. Therefore, $\underline{\chi}(t_0)$ is normally distributed with zero mean and variance given by half that of the latter. The variance of $\underline{\chi}(t_0)$ can be expressed as¹ $\sigma_{\chi}^2 = 2R^{(0)}(0)$ and, by referring to Theorem 5.1.1, is given by

$$\sigma_{\chi}^2 \simeq 2\psi T C/N_w. \tag{6.7}$$

We note that σ_{χ}^2 is equal to the maximum value of the ACF term in (6.5). In practice, although the expected value of $\underline{\chi}(t_0)$ is zero, $\underline{\chi}(t_0)$ may produce values that displace the global maximum of (6.5) away from \check{t}_0 . However, if the maximum value of the finite-time ACF sufficiently exceeds the standard deviation of the noise function, then the noise peaks no longer pose an ambiguity for the time delay estimation [60]. As a reference, we can

¹In general, $\chi(t_0)$ has an ACF given by $2\Re\{R^{(0)}(\tau)\}$.

define a minimum (MIN) effective CNDR threshold for which $\sigma_{\chi} = \sigma_{\chi}^2$, resulting in

$$[C/N_w]_{\text{MIN}} = \frac{1}{2\psi T}.$$
 (6.8)

Therefore, the ACF term dominates the exponent of (6.5) whenever $C/N_w \gg [C/N_w]_{\text{MIN}}$ and, conversely, the ACF term is swamped by the noise function whenever $C/N_w \ll [C/N_w]_{\text{MIN}}$. Consequently, there exist two distinct regimes of time delay estimation. In one case, the effective CNDR is high and the maximum of the posterior time delay pdf accurately indicates the true time delay. In the other case, the effective CNDR is low and ambiguous peaks severely hinder accurate detection. The first case is known as the small-error regime, and the second case is known as the large-error regime. Fig. 6.1 provides an example for the posterior time delay pdf in the two code tracking regimes as well as in the transition between the regimes². In the example, the multi-peak ACF structure is not apparent due to the bandlimiting, the presence of the noise function, and the exponentiation in (6.5). Moreover, the transition is abrupt [58]. The remainder of this section is divided into two parts which are devoted to the small-error regime and the large-error regime.



Fig. 6.1 Code tracking regimes: (a) large-error regime; (b) transition; (c) small-error regime.

²In Fig. 6.1, the correlated noise function $\underline{\chi}(t_0)$ [see footnote 1] is simulated using the method presented by Beaulieu et al. [62] and the ACF $R^{(0)}(\tau)$ is that of the example in Fig. 2.3.

6.1.1 Small-Error Regime

In the small-error code tracking regime, the posterior time delay pdf $p_{\rm s}(t_0 | \tilde{y})$ can be expressed by neglecting the effect of the noise function in (6.5), so that

$$p_{\rm s}(t_0|\,\tilde{\boldsymbol{y}}) = K_{\rm s}[u(t_0) - u(t_0 - T)] e^{\left(2\Re\{R^{(0)}(\check{t}_0 - t_0)\}\right)}.$$
(6.9)

By using Theorem 5.1.1, $\Re\{R^{(0)}(\check{t}_0 - t_0)\}\$ can be expressed as

$$\Re\{R^{(0)}(\check{t}_0 - t_0)\} \simeq 2\psi TC \int_{-W/2}^{W/2} \frac{\check{G}_{\hat{y}}(f)}{G_w(f)} \cos(2\pi f[\check{t}_0 - t_0]) \,\mathrm{d}f.$$
(6.10)

Moreover, due to the concave nature of the ACF, the exponentiation in (6.9), and the small-error assumption, it is appropriate to employ the second degree series expansion $\cos(2\pi f[\check{t}_0 - t_0]) \simeq 1 - 2\pi^2 f^2[\check{t}_0 - t_0]^2$ in (6.10). Therefore, as an approximation, we have

$$p_{\rm s}(t_0 | \, \tilde{\boldsymbol{y}}) \simeq K_{\rm s}[u(t_0) - u(t_0 - T)] e^{\left(4\psi TC \int_{-W/2}^{W/2} \frac{\tilde{G}_{\tilde{\boldsymbol{y}}}(f)}{G_w(f)} (1 - 2\pi^2 f^2 [\check{t}_0 - t_0]^2) \, \mathrm{d}f\right)}.$$
 (6.11)

Finally, by evaluating the integral in the exponent using (5.9) and (5.11) while absorbing any terms not depending on t_0 into the normalization constant K_s , the small-error posterior time delay pdf can be expressed as

$$p_{\rm s}(t_0|\,\tilde{\boldsymbol{y}}) \simeq K_{\rm s}[u(t_0) - u(t_0 - T)] \mathrm{e}^{\left(-\frac{(t_0 - \tilde{t}_0)^2}{2[\sigma_{\tilde{t}_0}^2]_{\rm CRB}}\right)}.$$
(6.12)

The small-error posterior time delay distribution resembles a Gaussian with a mean given by the true time delay and a variance given by the CRB. For large effective CNDR, the time delay uncertainty is well described by the entropy of a Gaussian random variable with variance $[\sigma_{\hat{t}_0}^2]_{CRB}$ [58], i.e.,

$$h_{\rm s}(\underline{t}_0|\,\underline{\tilde{\boldsymbol{y}}}) \simeq \frac{1}{2} \ln \left(2\pi \mathrm{e}[\sigma_{\hat{t}_0}^2]_{\rm \tiny CRB}\right). \tag{6.13}$$

Naturally, the larger $[\sigma_{\hat{t}_0}^2]_{\text{\tiny CRB}}$, the greater the uncertainty. By using the prior time delay entropy $\ln(T)$ with (5.11) and (6.13), the MI between the time delay and the observed

GNSS modulation signal in the small-error regime is given by

$$I_{\rm s}(\underline{t}_0; \underline{\tilde{\boldsymbol{y}}}) \simeq \frac{1}{2} \ln \left(4\pi \mathrm{e}^{-1} \psi T^3 \beta_w^2 C / N_w \right). \tag{6.14}$$

The time delay information is pumped through the channel at a logarithmic rate with respect to the effective CNDR and the effective GB. In the small-error regime, the time delay is already known quite precisely and increases in the effective CNDR and effective GB only provide diminishing returns in time delay information [58]. This information limits the time delay estimation error in accordance with the CRB.

6.1.2 Large-Error Regime

In the large-error code tracking regime, the posterior time delay pdf $p_{\ell}(t_0|\tilde{y})$ can be approximated by neglecting the ACF term in (6.5), so that

$$p_{\ell}(t_0|\tilde{\boldsymbol{y}}) = K_{\ell}[u(t_0) - u(t_0 - T)] e^{\chi(t_0)}.$$
(6.15)

In this case, a linearization of the signal [see Section 5.4] may not be practical due to the signal detection ambiguity caused by the noise function. To further evaluate the posterior time delay pdf, a fixed normalization constant K_{ℓ} (not depending on \tilde{y}) can be adopted as an approximation [60]. A fixed K_{ℓ} is appropriate whenever the observation interval is large enough to produce an adequate sample of the variations of the posterior time delay pdf [see Fig. 6.1(a)] and is reinforced at a low effective CNDR, where the variations are moderate and the resulting pdf is nearly uniform. If a fixed K_{ℓ} is assumed, then the average of the posterior time delay pdf evaluated over the ensemble of observations must yield the prior time delay pdf. The average posterior time delay pdf can be expressed as

$$\mathsf{E}\{p_{\ell}(t_0|\,\underline{\tilde{\boldsymbol{y}}})\} = \int_{-\infty}^{\infty} K_{\ell}[u(t_0) - u(t_0 - T)] \mathrm{e}^{\chi(t_0)} \,\frac{\mathrm{e}^{\left(-\frac{\chi(t_0)^2}{2\sigma_{\chi}^2}\right)}}{\sqrt{2\pi}\sigma_{\chi}} \,\mathrm{d}\chi(t_0). \tag{6.16}$$

By completing the square of the exponential arguments and employing the substitution $\lambda = (\chi(t_0)/\sigma_{\chi} - \sigma_{\chi})/\sqrt{2}$, the Gaussian integral $\int_{-\infty}^{\infty} e^{-\lambda^2} d\lambda = \sqrt{\pi}$ [63] can be used to obtain

$$\mathsf{E}\{p_{\ell}(t_0|\,\tilde{\boldsymbol{y}})\} = K_{\ell}[u(t_0) - u(t_0 - T)] \mathrm{e}^{\frac{1}{2}\sigma_{\chi}^2}.$$
(6.17)

This represents the uniform time delay pdf with a normalization constant given by

$$K_{\ell} = \frac{1}{T} e^{-\frac{1}{2}\sigma_{\chi}^2}.$$
 (6.18)

Finally, by substituting (6.18) into (6.15), an approximation for the large-error posterior time delay pdf is given by

$$p_{\ell}(t_0|\tilde{\boldsymbol{y}}) \simeq \frac{1}{T} [u(t_0) - u(t_0 - T)] e^{\left(\chi(t_0) - \frac{1}{2}\sigma_{\chi}^2\right)}.$$
(6.19)

As originally postulated, (6.19) is increasingly uniform for vanishing effective CNDR [cf. (6.7)]. The large-error time delay uncertainty is described by the entropy given by

$$h_{\ell}(\underline{t}_{0}|\,\underline{\tilde{\boldsymbol{y}}}) = -\int_{0}^{T}\int_{-\infty}^{\infty} \ln\left(K_{\ell}\,\mathrm{e}^{\chi(t_{0})}\right) K_{\ell}\,\mathrm{e}^{\chi(t_{0})}\,\frac{\mathrm{e}^{\left(-\frac{\chi(t_{0})^{2}}{2\sigma_{\chi}^{2}}\right)}}{\sqrt{2\pi}\sigma_{\chi}}\,\mathrm{d}\chi(t_{0})\,\mathrm{d}t_{0}.$$
(6.20)

To evaluate (6.20), we use the fact that the realization $\chi(t_0)$ does not depend on the particular value of t_0 . Furthermore, for a fixed K_{ℓ} , expanding the logarithm allows us to use the Gaussian integral, as in (6.16), to obtain

$$h_{\ell}(\underline{t}_{0}|\,\underline{\tilde{\boldsymbol{y}}}) \simeq -TK_{\ell} \big[\ln\left(K_{\ell}\right) + \sigma_{\chi}^{2} \big] \mathrm{e}^{\frac{1}{2}\sigma_{\chi}^{2}} + TK_{\ell} \int_{-\infty}^{\infty} \big[\sigma_{\chi}^{2} - \chi(t_{0}) \big] \mathrm{e}^{\chi(t_{0})} \frac{\mathrm{e}^{\left(-\frac{\chi(t_{0})^{2}}{2\sigma_{\chi}^{2}}\right)}}{\sqrt{2\pi}\sigma_{\chi}} \,\mathrm{d}\chi(t_{0}). \tag{6.21}$$

In (6.21), the integral equals zero because the integrand is antisymmetric over the integration region. Finally, by referring to (6.18), we obtain

$$h_{\ell}(\underline{t}_0|\,\underline{\tilde{\boldsymbol{y}}}) \simeq \ln\left(T\right) - \frac{1}{2}\sigma_{\chi}^2.$$
 (6.22)

By using the prior time delay entropy $\ln(T)$ with (6.7) and (6.22), the MI between the random time delay and the observed GNSS modulation signal in the large-error regime is
given by

$$I_{\ell}(\underline{t}_0; \underline{\tilde{y}}) \simeq \psi T C / N_w. \tag{6.23}$$

When considered in a basis in which the noise samples are uncorrelated, the information rate described by (6.23) is the maximum possible rate sustainable by the channel³ [58], [60]. In this respect, at low effective CNDR the channel conveys about as much information as it can support. As a result, by operating deeply in the large-error regime, the system quenches the time delay uncertainty with increasing effective CNDR. It does so at a faster rate than in the small-error regime. Here, the concern is detection rather than estimation, a situation which does not benefit from the frequency distribution of the signal.

6.1.3 Mutual Information Threshold

There are several ways to define the start of the small-error regime. One way is to find the value of the effective CNDR required to yield the same amount of time delay information in the small-error regime than in the large-error regime [58]. The effective CNDR at which this occurs is called the MI threshold. It is obtained by intersecting (6.14) with (6.23), i.e.,

$$\psi TC/N_w = \frac{1}{2} \ln \left(4\pi e^{-1} \psi T^3 \beta_w^2 C/N_w \right).$$
 (6.24)

In solving for C/N_w , we note that (6.24) has several solutions, but the solution of interest can be obtained by recalling that the small-error regime requires $C/N_w \gg [C/N_w]_{\text{MIN}}$ [see (6.8)]. Consequently, the threshold can be expressed as

$$[C/N_w]_{\rm MI} = -\frac{1}{2\psi T} W_{-1} \left(\frac{-\mathrm{e}}{2\pi T^2 \beta_w^2}\right).$$
(6.25)

In (6.25), $W_{-1}(\cdot)$ is the secondary real valued branch of the Lambert W function W(z), defined as the inverse function of $we^w = z$ where $w = W(z) \leq -1$ ($-e^{-1} \leq z < 0$) [65]. Here, the MI threshold also requires $T\beta_w \geq e/\sqrt{2\pi}$ which is usually satisfied. To describe the code tracking potential using the CRB, the time delay estimator must operate in the

³For an arbitrarily large bandwidth, the capacity of an ideal bandlimited white Gaussian noise channel is $\lim_{W\to\infty} W \ln(1 + \frac{C\psi}{WN_w}) = \frac{C\psi}{N_w}$ with $N_w/2$ being the double-sided noise power spectral density and $C\psi$ the received physical carrier power [64].

small-error regime. Therefore, a necessary (although insufficient) condition that supports the use of the CRB is $C/N_w > [C/N_w]_{\text{MI}}$.

6.2 Code Tracking Mean Square Error

Up to this point we have described the amount of time delay information that is provided by the GNSS modulation signal and have described a boundary separating the small-error regime from the large-error regime. We now consider the error resulting from the time delay estimator.

The code tracking performance of the GNSS modulation signal can be quantified by the MSE, $\mathsf{E}\{|\underline{\hat{t}}_0 - \underline{t}_0|^2\}$, between the time delay estimator $\underline{\hat{t}}_0$ and the time delay \underline{t}_0 . It is well known that when the time delay estimator is the conditional mean, i.e., $\underline{\hat{t}}_0 = \mathsf{E}\{\underline{t}_0|\,\underline{\tilde{y}}\}$, then the MSE is minimized. Consequently, an asymptote for the MSE lower bound in the small-error regime can be obtained by considering (6.12), so that⁴

$$\lim_{C/N_w \to \infty} \mathsf{E}\{|\hat{\underline{t}}_0 - \underline{t}_0|^2\} = \lim_{C/N_w \to \infty} [\sigma_{\hat{t}_0}^2]_{\rm CRB}.$$
(6.26)

This asymptote was first obtained by Woodward [60] and thus (2.23) is sometimes referred to as the Woodward Equation [67]. In the large-error regime, the lower bound for the MSE is provided by the asymptote [see footnote 4]

$$\lim_{C/N_w \to 0} \mathsf{E}\{|\underline{\hat{t}}_0 - \underline{t}_0|^2\} = \sigma_{t_0}^2 = \frac{T^2}{12}.$$
(6.27)

In (6.27), $T^2/12$ is the variance corresponding to the prior uniform distribution of the time delay and the limit is evaluated using (6.19). For very low effective CNDR, the observed signal may not provide enough information to produce a better estimate than what is already possible without making an observation in the first place.

The asymptotic analysis reveals that the lower bound for the MSE undergoes a transition from $\sigma_{t_0}^2$ to $[\sigma_{\hat{t}_0}^2]_{_{\rm CRB}}$ with increasing effective CNDR. The transition is due to the change from

 $^{^4}$ The convergence towards the asymptote for a bounded MSE is due to the Dominated Convergence Theorem (DCT) [66].

the large-error regime to the small-error regime and is concurrent with the MI transition. Obtaining a description for the minimum MSE at a moderate effective CNDR requires an exact expression for the posterior time delay pdf and some involved integration. Instead, we draw on an existing lower bound specifically tailored for the problem of time delay estimation called the ZZB [16], [17].

6.2.1 Ziv-Zakai Bound

Let us consider a time delay detection problem that consists of choosing between two equiprobable hypotheses defined as

$$\mathcal{H}_{0} : \dot{t}_{0} = t_{0},
\mathcal{H}_{1} : \check{t}_{0} = t_{0} + \tau.$$
(6.28)

Above, τ is a nonnegative constant such that $t_0 + \tau \in [0, T)$. Let us consider two different detection strategies: an optimal one and a suboptimal one, whose corresponding probabilities of detection error are denoted by $P_{e,o}(\tau, t_0)$ and $P_{e,s}(\tau, t_0)$, respectively. By definition, the optimal strategy minimizes the probability of detection error, i.e., $P_{e,o}(\tau, t_0)$ serves as a lower bound to $P_{e,s}(\tau, t_0)$. Ziv and Zakai [16] used this fact to derive a lower bound, called the ZZB [6], [7], [17], on the MSE of the time delay estimator by relating the MSE to the probability of detection error of a (suboptimal) detector. In the rest of this section, we outline the derivation of the ZZB for the case of the GNSS modulation signal with a time delay characterized by a uniform prior distribution and where the signal is observed in the presence of Gaussian noise and interference.

First, we investigate the optimal detection strategy. The optimal detection strategy consists of choosing the most probable time delay hypothesis given the received signal. The test statistic is given by

$$V_{\rm o}(\tilde{\boldsymbol{y}}) = \frac{p(\tilde{\boldsymbol{y}} \mid t_0)}{p(\tilde{\boldsymbol{y}} \mid t_0 + \tau)},\tag{6.29}$$

and we decide that \mathcal{H}_1 is true if $V_{o}(\tilde{\boldsymbol{y}}) \leq 1$. By expressing the test statistic in logarithmic form, the probability that (6.29) fails to detect the correct time delay hypothesis over the

ensemble of received signals is given by

$$P_{e,o}(\tau, t_0) = \mathsf{P}\big\{\mathcal{H}_0\big\}\mathsf{P}\big\{\ln\big(V_o(\underline{\tilde{\boldsymbol{y}}})\big) \le 0 \,|\, \mathcal{H}_0\big\} + \mathsf{P}\big\{\mathcal{H}_1\big\}\mathsf{P}\big\{\ln\big(V_o(\underline{\tilde{\boldsymbol{y}}})\big) > 0 \,|\, \mathcal{H}_1\big\}.$$
(6.30)

In (6.30), $\mathsf{P}(\cdot)$ denotes the probability. By evaluating (6.29) using (6.3), the log-likelihood function taken over the ensemble of \tilde{y} is given by

$$\ln\left(V_{o}(\underline{\tilde{\boldsymbol{y}}})\right) = 2\Re\left\{\underline{\tilde{\boldsymbol{y}}}^{\prime\mathsf{H}}[\hat{\boldsymbol{y}}^{\prime}(t_{0}) - \hat{\boldsymbol{y}}^{\prime}(t_{0} + \tau)]\right\}.$$
(6.31)

The conditional probabilities in (6.30) can then be evaluated by inserting $\check{t}_0 = t_0$ and $\check{t}_0 = t_0 + \tau$ into $\tilde{y}' = \hat{y}'(\check{t}_0) + w'$ for \mathcal{H}_0 and \mathcal{H}_1 , respectively. By using Theorem 5.1.1 and assuming equiprobable hypothesis, the minimum probability of error can be expressed as

$$P_{e,o}(\tau, t_0) = \frac{1}{2} \left(\mathsf{P} \left\{ \Re\{R^{(0)}(0) - R^{(0)}(\tau)\} \le \underline{\zeta}(t_0) \right\} + \mathsf{P} \left\{ \Re\{R^{(0)}(\tau) - R^{(0)}(0)\} > \underline{\zeta}(t_0) \right\} \right).$$
(6.32)

In (6.32), $\underline{\zeta}(t_0)$ is a random variable defined as

$$\underline{\zeta}(t_0) \triangleq \Re\{\underline{\boldsymbol{w}}^{\prime \mathsf{H}}[\hat{\boldsymbol{y}}^{\prime}(t_0+\tau) - \hat{\boldsymbol{y}}^{\prime}(t_0)]\}.$$
(6.33)

Because \underline{w}' is a circular symmetric Gaussian random vector, $\underline{\zeta}(t_0)$ is Gaussian with zero mean and variance given by half that of the argument of $\Re(\cdot)$ in (6.33), i.e.,

$$\sigma_{\zeta}^2 = \Re\{R^{(0)}(0) - R^{(0)}(\tau)\}.$$
(6.34)

Because $\underline{\zeta}(t_0)$ is a normal random variable, we can express (6.32) using the Gaussian Q function defined as $Q(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{1}{2}\lambda^2} d\lambda$ [63], i.e.,

$$P_{\rm e,o}(\tau) = Q\left(\Re\{R^{(0)}(0) - R^{(0)}(\tau)\}^{\frac{1}{2}}\right).$$
(6.35)

In (6.35), we have used the symmetry of the pdf of $\zeta(t_0)$ and dropped t_0 from the notation

because the minimum probability of error does not depend on t_0 .

Next, we consider the suboptimal detection strategy. The suboptimal detection is made by first obtaining an estimate $\hat{t}_0 \in [0, T)$ for the time delay and to choose the hypothesis that corresponds to a time delay that is closer to \hat{t}_0 . The test statistic is given by

$$V_{\rm s}(\hat{t}_0) = \frac{|\hat{t}_0 - t_0 - \tau|}{|\hat{t}_0 - t_0|},\tag{6.36}$$

and we decide that \mathcal{H}_1 is true if $V_s(\hat{t}_0) \leq 1$. Over the ensemble of estimates, the probability that (6.36) fails to detect the correct hypothesis is given by

$$P_{e,s}(\tau, t_0) = \mathsf{P}\{\mathcal{H}_0\}\mathsf{P}\{V_s(\underline{\hat{t}}_0) \le 1 \,|\, \mathcal{H}_0\} + \mathsf{P}\{\mathcal{H}_1\}\mathsf{P}\{V_s(\underline{\hat{t}}_0) > 1 \,|\, \mathcal{H}_1\}.$$
(6.37)

With equiprobable hypotheses, it follows that

$$P_{\rm e,s}(\tau, t_0) = \frac{1}{2} \left(\mathsf{P}\left\{ \underline{\hat{t}}_0 - t_0 > \frac{\tau}{2} \right\} + \mathsf{P}\left\{ \underline{\hat{t}}_0 - t_0 - \tau < -\frac{\tau}{2} \right\} \right).$$
(6.38)

In (6.38), we have employed the continuity of $\underline{\hat{t}}_0$ to write the events as strict inequalities.

To determine a lower bound for the MSE of any time delay estimator, we need the following lemma.

Lemma 6.2.1 The MSE between the time delay \underline{t}_0 and the estimator $\underline{\hat{t}}_0$ is lower bounded such that

$$\mathsf{E}\{|\underline{\hat{t}}_{0} - \underline{t}_{0}|^{2}\} \ge \int_{0}^{T} \int_{0}^{T-\tau} \frac{\tau}{T} P_{\mathrm{e,s}}(\tau, t_{0}) \,\mathrm{d}t_{0} \,\mathrm{d}\tau, \tag{6.39}$$

where $P_{e,s}(\tau, t_0)$ is the probability of detection error of a (suboptimal) detector [see (6.38)].

Proof See Appendix B.4.

In this case, we note that the MSE bound depends on the particular estimator. To obtain a bound that applies to any estimator, we can replace $P_{\rm e,s}(\tau, t_0)$ by the minimum probability of error $P_{\rm e,o}(\tau)$. The lower bound for the MSE can then be expressed as

$$\mathsf{E}\{|\underline{\hat{t}}_{0} - \underline{t}_{0}|^{2}\} \ge \int_{0}^{T} \tau \Lambda\left(\frac{\tau}{T}\right) P_{\mathrm{e,o}}(\tau) \,\mathrm{d}\tau.$$
(6.40)

In (6.40), $\Lambda(\tau)$ is the unit height triangular function of support [-1, 1). Finally, explicit application of (6.35) produces the final bound:

$$\mathsf{E}\{|\underline{\hat{t}}_{0} - \underline{t}_{0}|^{2}\} \ge \int_{0}^{T} \tau \Lambda\left(\frac{\tau}{T}\right) Q\left(\Re\{R^{(0)}(0) - R^{(0)}(\tau)\}^{\frac{1}{2}}\right) \mathrm{d}\tau.$$
(6.41)

We note that the ZZB captures the multi-peak nature of the ACF via $R^{(0)}(\tau)$.

6.2.2 Mean Square Error Threshold

To more easily visualize the behavior of the ZZB, we can approximate (6.41) by loosening the inequality under the assumption that the effective CNDR is either very large or very small and that the GB is large. The approximation is verified for all GPS and Galileo signals via simulations in a subsequent section. The derivation of this approximation is lengthy and is relegated to Appendix A.3. The result is provided by (A.16) and is given by

$$\mathsf{E}\{|\underline{\hat{t}}_{0} - \underline{t}_{0}|^{2}\} \gtrsim 2\sigma_{t_{0}}^{2}Q(\sqrt{\psi TC/N_{w}}) + [\sigma_{\hat{t}_{0}}^{2}]_{\text{\tiny CRB}}P_{3/2}(\psi TC/N_{w}).$$
(6.42)

In (6.42), $P_a(x)$ denotes the incomplete Gamma function of a with upper limit x ($a, x \in \mathbb{R}$) defined as $P_a(x) \triangleq \frac{1}{\Gamma(a)} \int_0^x e^{-\lambda} \lambda^{a-1} d\lambda$, where $\Gamma(a) \triangleq \int_0^\infty e^{-\lambda} \lambda^{a-1} d\lambda$ [63]. Specifically, (6.42) indicates that for a low effective CNDR the MSE is largely due to $\sigma_{t_0}^2$, while for a high effective CNDR the MSE is dominated by $[\sigma_{t_0}^2]_{CRB}$. For moderate effective CNDR the MSE lies between $\sigma_{t_0}^2$ and $[\sigma_{t_0}^2]_{CRB}$; the transition from $\sigma_{t_0}^2$ to $[\sigma_{t_0}^2]_{CRB}$ is mostly due to the first term on the right of (6.42) which is controlled by the large-error MI [cf. (6.23)].

The effective CNDR for which the ZZB nearly coincides with the CRB is given by the MSE threshold. Mathematically, the MSE threshold can be defined as the effective CNDR that results when the MSE is 3 dB above⁵ $[\sigma_{\hat{t}_0}^2]_{\text{CRB}}$ [6]. By using (5.11), (6.27), and (6.42), and considering that near the threshold $P_{3/2} (\psi TC/N_w) \simeq 1$, the MSE threshold is obtained by setting

$$\frac{T^2}{6}Q(\sqrt{\psi TC/N_w}) = \frac{1}{8\pi^2 \psi T \beta_w^2 C/N_w}.$$
(6.43)

⁵Similarly, the MIN threshold defined by (6.8) also occurs when the MSE is about 3 dB below $\sigma_{t_0}^2$. It can be obtained by setting $\frac{T^2}{6}Q(\sqrt{\psi TC/N_w}) = \frac{T^2}{24}$.

The MSE threshold must occur in the small error regime, i.e., $C/N_w > [C/N_w]_{\text{MI}}$. Therefore, by using $Q(\sqrt{\psi TC/N_w}) \simeq \frac{e^{-\frac{1}{2}\psi TC/N_w}}{\sqrt{2\pi\psi TC/N_w}}$, the MSE threshold can be expressed as

$$[C/N_w]_{\rm \scriptscriptstyle MSE} = -\frac{1}{\psi T} W_{-1} \left(\frac{-9}{8\pi^3 T^4 \beta_w^4}\right). \tag{6.44}$$

It is not surprising that the MSE threshold adopts the same form as the MI threshold given by (6.25). The larger exponent (4 instead of 2) on $T\beta_w$ in (6.44) emphasizes that the MSE threshold must occur well inside the small-error regime. Both (6.25) and (6.44) illustrate that increasing the power containment and integration time significantly decreases the effective CNDR required to operate in the small-error regime and triggers the suitability of the CRB. On the other hand, the GB is (in practice) ancillary to the location of the code tracking thresholds. Consequently, although a large GB is often associated with accurate code tracking due to the CRB, the time delay estimation error may be underestimated by the CRB due to an insufficient effective CNDR. As a result, wideband signals intended for accurate code tracking may provide no benefit when it comes to high attenuation environments.

6.3 Simulation of the GNSS Code Tracking Bounds

This section illustrates the theoretical limits on the code tracking accuracy of the GNSS modulation signal. A brief reference is made to the ZZB, but the focus is the code tracking thresholds.

6.3.1 Simulation Methodology

The results used to illustrate the code tracking behavior of the GNSS modulation signal are obtained for an ideal bandlimited white Gaussian noise channel. To accommodate the wide range of GPS and Galileo signals, a double-sided front-end bandwidth of 8 MHz, 24 Mhz, and 96 MHz is used. Unless specified otherwise, the receiver employs a coherent integration time of 20 ms, corresponding to an observation interval that is equal to the data bit period of the legacy GPS SPS signal. Finally, it is assumed that parasitic correlations are negligible so that the subcarriers dominate the statistical properties of the GNSS modulation signal.

The ZZB is simulated using (6.41). In order to evaluate (6.41), the finite-time ACF of the bandlimited GNSS modulation signal is computed using the function provided by (4.17). The function is then bandlimited using an ideal lowpass filter and scaled by TC/N_0 .

The code tracking thresholds are obtained from (6.8), (6.25), and (6.44). The thresholds depend on the signal characteristics that include the correlation loss and GB. The correlation loss and GB are computed via (2.20) and (5.9), respectively, by employing the PSD envelope function provided by (4.19).

6.3.2 Results

The fundamental code tracking performance of the GNSS modulation signal is described in the figures that follow. For select multiplexed signals, the ZZB is shown in Fig. 6.2, while for select signal components, the ZZB is shown in Fig. 6.3.

The ZZB curves illustrate that the RMS time delay error decreases abruptly from the prior time delay standard deviation to the one-sigma error provided by the CRB. Fig. 6.4 explicitly shows the CRB and highlights the location of the MIN, MI, and MSE threshold.



Fig. 6.2 ZZB of select CBOC and ALTBOC multiplexed signals: (a) $CBOC(6, 1, \frac{1}{11})$; (b) ALTBOC(15, 10), for precorrelation bandwidths: (i) 8 MHz; (ii) 24 MHz; (iii) 96 MHz (double-sided).



Fig. 6.3 ZZB of select BOC and TMBOC signal components: (a) $BOC_{cos}(10,5)$; (b) TMBOC(6,1, $\frac{4}{33}$), for precorrelation bandwidths: (i) 8 MHz; (ii) 24 MHz; (iii) 96 MHz (double-sided).



Fig. 6.4 Code tracking thresholds of the ALTBOC(15, 10) multiplexed signal.

The sharp transition between the MI and MSE threshold indicates that efforts to increase the signal quality past the receiver front-end (e.g., by minimizing implementation losses) considerably reduce the time delay estimation errors when code tracking takes place between the thresholds. Conversely, small decreases in CNDR may force the receiver out of code tracking lock. However, once the CRB is reached, further increases in CNDR provide diminishing returns; code tracking accuracy is then relegated to the GB.

The main results of this chapter are summarized in Fig. 6.5. In the figure, the minimum CNDR required to achieve the CRB is illustrated for all modulations used by GPS and Galileo. Simulations using the ZZB (like those shown previously) are conducted to locate the MSE threshold which is compared to the threshold given by (6.44). The simulations and theoretical results agree well. This indicates that the multi-peak nature of the ACF of the GNSS modulation signal does not considerably disturb the MSE threshold. The large-error regime, as indicated by the MI threshold, is also highlighted.



Fig. 6.5 Required CNDR to achieve the CRB for GPS and Galileo modulations.

In most cases, the empirical 30 dB-Hz level often associated with the code tracking threshold of a conventional receiver (a typical worst case scenario) [68], lies between the MI and MSE threshold. In addition, typical strong signals of 40 – 45 dB-Hz [30], [68], [69], have no trouble achieving the CRB for a sufficiently large precorrelation bandwidth. However, attenuated and bandlimited signals are subject to the threshold effects.

For a small precorrelation bandwidth, the location of the MI and MSE threshold is predominantly due to the correlation loss. In Fig. 6.5, signals with a small precorrelation bandwidth (8 MHz) are numbered according to decreasing power containment. The trend predicted by the theory is that a large correlation loss increases the thresholds [see (6.25) and (6.44)]. Moreover, for a large precorrelation bandwidth, most of the carrier power passes the receiver front-end and the location of the MI and MSE threshold continues to increase with increasing GB. However, as we have previously pointed out, typical values for the GB produce a nearly constant MI and MSE threshold. As a result, further displacement of the threshold is due to the integration time. Fig. 6.6 summarizes this threshold behavior. We note that over a wide range of values for the GB and integration time, the separation between the thresholds remains nearly constant.



Fig. 6.6 Variation of code tracking thresholds: (i) MSE threshold; (ii) MI threshold; (iii) MIN threshold.

As a reference, Fig. 6.7 shows the ranging error at the MSE threshold. For a large precorrelation bandwidth (96 MHz), the results are numbered with increasing GB to show the effect of bandwidth on code tracking once the MSE threshold is reached.

The results presented here show that efforts to achieve a high code tracking accuracy at a low CNDR by choosing a modulation that produces a large GB are futile. If the CNDR is below the MI threshold, the GB has a negligible impact on the time delay estimation error. This contrasts with recent claims that modulations employing high subcarrier and code rates produce signals apt for indoor tracking [48]. To track attenuated signals, one must minimize the correlation loss and maximize the integration time. Finally, an increase in GB for signals operating beyond the MSE threshold improves accuracy, even if the resulting signals fall behind the MSE threshold.



Fig. 6.7 Ranging error at the MSE threshold for GPS and Galileo modulations.

8 MHz bandwidth

Chapter 7

Conclusion

The first part of this thesis developed an analytical framework for next generation satellite radionavigation signals. The proposed theory of GNSS modulation generalizes the signal structures used by the modernized GPS and Galileo. Because subcarriers take a central role in the signal structures, a significant consequence of the GNSS modulation theory is the GBOC subcarrier. The GBOC subcarrier is a time multiplexed multilevel waveform that generalizes the existing subcarriers. To facilitate the use of the framework, we provided a collection of parameters under which the GBOC subcarrier defines the BOC, CBOC, TMBOC, and the recently proposed ABOC component modulation, as well as the CBOC and ALTBOC multiplexing schemes. In addition, we used the analytical framework to obtain single equations for the ACF and PSD of all signal components and multiplexed signals. Simulations have confirmed the accuracy of the theory, and when compared to previous approaches, our model not only offers better accuracy, but applies to a wider range of signals.

Our statistical description suggests that the GNSS modulation signal, subject to slow flat fading, is generally nonstationary. Since the PSD is only defined for a wide sense stationary process, the result suggests that no fixed PSD exists. We solved this problem by showing that the GNSS modulation signal is AMS and thus possesses a convergent time averaged ACF and PSD. Moreover, we have shown that the ACF and PSD depend mostly on the GBOC subcarrier and are functions of its parameters.

Conclusion

The second part of this thesis applied the statistical description of GNSS modulation to provide an accurate assessment of the potential code tracking accuracy of all GPS and Galileo signals. To characterize the code tracking performance in the presence of bandlimited noise and interference, we obtained expressions for the effective CNDR and effective GB. By using the code tracking performance measures, we derived closed form expressions for the MI and MSE code tracking thresholds, which define, respectively, the necessary CNDR for the time delay estimator to operate in the small-error regime and for the CRB to serve as a meaningful benchmark.

Our information-theoretic analysis reveals that the code tracking thresholds depend primarily on the correlation loss and observation interval, while the specific signal structure does not significantly influence the code tracking thresholds. We have shown that the code tracking thresholds decrease with power containment and integration time, which are limited, respectively, by the receiver front-end bandwidth and the channel coherence time. Finally, we have shown that the CRB, which decreases with the effective GB, is precarious for small power containment and integration time. Therefore, the potential code tracking accuracy offered by wideband signals is only appreciable for signals of adequate strength.

In addition to noise and interference, multipath is a major contributor to the GNSS error budget. Although, the channel model employed in this thesis does not allow for multipath, as future work we intend to apply the GNSS modulation theory to extend the investigation of code tracking bounds to the case of multipath. Finally, it should be mentioned that receiver architectures have not been addressed in this work. We have also left it for future work to investigate the merits of the GNSS modulation theory to obtain code tracking loops for next generation satellite radionavigation receivers.

Appendix A

Derivation of Select Results

A.1 Delayed GNSS Modulation Signal ACF

In this appendix, we compute the ACF of the delayed GNSS modulation signal $y(t - \underline{t}_0)$. The delayed GNSS modulation signal can be expressed as [cf. (3.14)]

$$y(t - \underline{t}_0) = \sum_{k = -\infty}^{\infty} \sum_{l=1}^{2} \hat{\underline{x}}^{(l)} (kT_c)^{\mathsf{T}} \boldsymbol{\varsigma}^{(l)} (t - kT_c - \underline{t}_0).$$
(A.1)

The ACF of $y(t - \underline{t}_0)$ is given by

$$R_{\tilde{y}}(t_{1}, t_{2}) = \mathsf{E} \left\{ y(t_{1} - \underline{t}_{0}) y(t_{2} - \underline{t}_{0})^{*} \right\}$$

$$= \mathsf{E} \left\{ \left[\sum_{k_{1} = -\infty}^{\infty} \sum_{l_{1} = 1}^{2} \hat{\underline{x}}^{(l_{1})} (k_{1}T_{c})^{\mathsf{T}} \boldsymbol{\varsigma}^{(l_{1})} (t_{1} - k_{1}T_{c} - \underline{t}_{0}) \right] \right]$$

$$\times \left[\sum_{k_{2} = -\infty}^{\infty} \sum_{l_{2} = 1}^{2} \hat{\underline{x}}^{(l_{2})} (k_{2}T_{c})^{\mathsf{T}} \boldsymbol{\varsigma}^{(l_{2})} (t_{2} - k_{2}T_{c} - \underline{t}_{0}) \right]^{*} \right\}$$

$$= \mathsf{E} \left\{ \sum_{k_{1} = -\infty}^{\infty} \sum_{k_{2} = -\infty}^{\infty} \sum_{l_{1} = 1}^{2} \sum_{l_{2} = 1}^{2} \hat{\underline{x}}^{(l_{2})} (k_{2}T_{c})^{\mathsf{H}} \boldsymbol{\varsigma}^{(l_{2})} (t_{2} - k_{2}T_{c} - \underline{t}_{0})^{*} \times \boldsymbol{\varsigma}^{(l_{1})} (t_{1} - k_{1}T_{c} - \underline{t}_{0})^{\mathsf{T}} \hat{\underline{x}}^{(l_{1})} (k_{1}T_{c}) \right\}.$$
(A.2)

By employing the substitutions $m = k_1 - k_2$ and $k = k_2$ (the end effects of the re-indexing with m vanish due to the infinite extent of k) and using the identity $\mathbf{x}^{\mathsf{H}} \mathbf{A} \mathbf{y} = \mathsf{tr}(\mathbf{y} \mathbf{x}^{\mathsf{H}} \mathbf{A})$ $(\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^N, \text{ and } \boldsymbol{A} \in \mathbb{C}^{N \times N}), \text{ we obtain}$

$$R_{\check{y}}(t_{1}, t_{2}) = \mathsf{E}\Biggl\{\sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{l_{1}=1}^{2} \sum_{l_{2}=1}^{2} \mathsf{tr} \left(\underline{\hat{x}}^{(l_{1})} (kT_{c} + mT_{c}) \underline{\hat{x}}^{(l_{2})} (kT_{c})^{\mathsf{H}} \times \boldsymbol{\varsigma}^{(l_{2})} (t_{2} - kT_{c} - \underline{t}_{0})^{*} \boldsymbol{\varsigma}^{(l_{1})} (t_{1} - mT_{c} - kT_{c} - \underline{t}_{0})^{\mathsf{T}} \Biggr) \Biggr\}.$$
(A.3)

A.2 Summation of Overlapping Integrals

In this appendix, we prove the identity

$$\sum_{k=-\infty}^{\infty} f_k \int_{a-kT_c-AT_c}^{b-kT_c-BT_c} S(t) dt = \int_{-\infty}^{\infty} \left(\sum_{k=1}^{\lfloor b/T_c \rfloor - \lfloor a/T_c \rfloor - B} f_{k+\lfloor a/T_c \rfloor - A} + f_{\lfloor a/T_c \rfloor - A} \left[u(t-a \mod T_c) \right] + f_{\lfloor b/T_c \rfloor - B} \left[1 - u(t-b \mod T_c) \right] \right) S(t) dt.$$
(A.4)

In (A.4), f_k represents a discrete function of k, S(t) is an integrable function of support $[0, T_c)$, $a, b \in \mathbb{R}$, and $A, B \in \mathbb{Z}$. Let us consider the integral on the left hand side of (A.4). Since S(t) has a finite support, the integral is nonzero for only select values of k. These values of k belong to one of the following cases:

1) $\lfloor a/T_c \rfloor - A + 1 \le k \le \lfloor b/T_c \rfloor - B - 1$ 2) $k = \lfloor a/T_c \rfloor - A$ 3) $k = \lfloor b/T_c \rfloor - B$

By considering these cases individually, the left hand side of (A.4) can be expressed as

$$\sum_{k=-\infty}^{\infty} f_k \int_{a-kT_c-AT_c}^{b-kT_c-BT_c} S(t) \, \mathrm{d}t = \sum_{k=\lfloor a/T_c \rfloor - A+1}^{\lfloor b/T_c \rfloor - B-1} f_k \int_{-\infty}^{\infty} S(t) \, \mathrm{d}t + f_{\lfloor a/T_c \rfloor - A} \int_{a \mod T_c}^{\infty} S(t) \, \mathrm{d}t + f_{\lfloor b/T_c \rfloor - B} \int_{-\infty}^{b \mod T_c} S(t) \, \mathrm{d}t.$$
(A.5)

Alternatively, we can write the above in terms of a single integral by modifying the integrands using unit step functions. After interchanging the order of summation and integration, we obtain

$$\sum_{k=-\infty}^{\infty} f_k \int_{a-kT_c-AT_c}^{b-kT_c-BT_c} S(t) dt = \int_{-\infty}^{\infty} \left(\sum_{k=\lfloor a/T_c\rfloor-A+1}^{\lfloor b/T_c\rfloor-A-1} f_k S(t) + f_{\lfloor a/T_c\rfloor-A} S(t) [u(t-a \mod T_c)] + f_{\lfloor b/T_c\rfloor-B} S(t) [1-u(t-b \mod T_c)] \right) dt, \quad (A.6)$$

which, after redefining index k on the right hand side, is equivalent to (A.4).

A.3 Simplified ZZB

In this appendix, we obtain a simplifying approximation for the ZZB. The ZZB is given by (6.41) and is repeated here for convenience:

$$\mathsf{E}\{|\underline{\hat{t}}_{0} - \underline{t}_{0}|^{2}\} \ge \int_{0}^{T} \tau \Lambda\left(\frac{\tau}{T}\right) Q\left(\Re\left\{R^{(0)}(0) - R^{(0)}(\tau)\right\}^{\frac{1}{2}}\right) \,\mathrm{d}\tau.$$
(A.7)

Let $R^{(0)}(\tau)$ be negligible for $|\tau| > \alpha \ (1/(\pi\beta_w) < \alpha < T)$ so that the integral in (A.7) can be separated as

$$\mathsf{E}\{|\underline{\hat{t}}_{0} - \underline{t}_{0}|^{2}\} \geq \int_{0}^{\alpha} \tau \Lambda\left(\frac{\tau}{T}\right) Q\left(\Re\left\{R^{(0)}(0) - R^{(0)}(\tau)\right\}^{\frac{1}{2}}\right) d\tau + \int_{\alpha}^{T} \tau \Lambda\left(\frac{\tau}{T}\right) Q\left(\Re\left\{R^{(0)}(0)\right\}^{\frac{1}{2}}\right) d\tau.$$
(A.8)

By using Theorem 5.1.1, $\Re\{R^{(0)}(\tau)\}\$ can be expressed as

$$\Re \left\{ R^{(0)}(\tau) \right\} \simeq 2\psi TC \int_{-W/2}^{W/2} \frac{\breve{G}_{\hat{y}}(f)}{G_w(f)} \cos(2\pi f\tau) \,\mathrm{d}f.$$
(A.9)

By employing the inequality $\cos(2\pi f\tau) \ge \max_{\tau \in \mathbb{R}} \{1 - 2(\pi f\tau)^2, -1\}$, the right hand side of (A.9) can be lower-bounded; applying (5.7) and (5.9) to the bound, $\Re\{R^{(0)}(\tau)\}$ can be

expressed as

$$\Re\{R^{(0)}(\tau)\} \gtrsim \psi TC/N_w \begin{cases} 1 - 2(\pi\beta_w\tau)^2, & |\tau| < 1/(\pi\beta_w), \\ -1, & \text{elsewhere,} \end{cases}$$
(A.10)

By employing (A.10) to replace the expression for $\Re\{R^{(0)}(0)\}$ in (A.8) and by virtue of the (decreasing) monotonicity of the Gaussian Q function, we obtain

$$\mathsf{E}\{|\underline{\hat{t}}_{0} - \underline{t}_{0}|^{2}\} \gtrsim \int_{0}^{1/(\pi\beta_{w})} \tau \Lambda\left(\frac{\tau}{T}\right) Q\left(\pi\beta_{w}\tau\sqrt{2\psi TC/N_{w}}\right) \,\mathrm{d}\tau \\ + \int_{1/(\pi\beta_{w})}^{\alpha} \tau \Lambda\left(\frac{\tau}{T}\right) Q\left(\sqrt{2\psi TC/N_{w}}\right) \,\mathrm{d}\tau \\ + \int_{\alpha}^{T} \tau \Lambda\left(\frac{\tau}{T}\right) Q\left(\sqrt{\psi TC/N_{w}}\right) \,\mathrm{d}\tau,$$
(A.11)

By performing integration by parts on the first integral (using the fact that $\frac{dQ(x)}{dx} = \frac{-e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}$) and evaluating the other integrals directly, we obtain

$$\mathsf{E}\{|\underline{\hat{t}}_{0} - \underline{t}_{0}|^{2}\} \gtrsim \frac{T^{2}}{6} Q\left(\sqrt{\psi TC/N_{w}}\right) + \frac{\alpha^{2}}{2} \Lambda\left(\frac{2\alpha}{3T}\right) \left[Q\left(\sqrt{2\psi TC/N_{w}}\right) - Q\left(\sqrt{\psi TC/N_{w}}\right)\right] \\ + \int_{0}^{1/(\pi\beta_{w})} \frac{\tau^{2}}{2} \Lambda\left(\frac{2\tau}{3T}\right) \beta_{w} \sqrt{\pi\psi TC/N_{w}} e^{\left(-(\pi\beta_{w}\tau)^{2}\psi TC/N_{w}\right)} d\tau.$$
(A.12)

Next, by employing the substitution $\lambda = (\pi \beta_w \tau)^2 \psi T C / N_w$ we obtain, after some algebra,

$$\mathsf{E}\{|\underline{\hat{t}}_{0} - \underline{t}_{0}|^{2}\} \gtrsim \frac{T^{2}}{6} Q\left(\sqrt{\psi TC/N_{w}}\right) + \frac{\alpha^{2}}{2} \Lambda\left(\frac{2\alpha}{3T}\right) \left[Q\left(\sqrt{2\psi TC/N_{w}}\right) - Q\left(\sqrt{\psi TC/N_{w}}\right)\right] \\ + \int_{0}^{\psi TC/N_{w}} \frac{\mathrm{e}^{-\lambda}}{2\pi^{\frac{5}{2}} \beta_{w}^{2} \psi TC/N_{w}} \left(\frac{\sqrt{\lambda}}{2} - \frac{\lambda}{3\pi \beta_{w} T \sqrt{\psi TC/N_{w}}}\right) \mathrm{d}\lambda.$$
(A.13)

The integral in the above expression can be evaluated using the incomplete Gamma function $P_a(x) = \frac{1}{\Gamma(a)} \int_0^x e^{-\lambda} \lambda^{a-1} d\lambda, \text{ where } \Gamma(a) = \int_0^\infty e^{-\lambda} \lambda^{a-1} d\lambda \ (a, x \in \mathbb{R}) \ [63].$ The result is

$$\mathsf{E}\{|\underline{\hat{t}}_{0} - \underline{t}_{0}|^{2}\} \gtrsim \frac{T^{2}}{6}Q\left(\sqrt{\psi TC/N_{w}}\right) + \frac{\alpha^{2}}{2}\Lambda\left(\frac{2\alpha}{3T}\right)\left[Q\left(\sqrt{2\psi TC/N_{w}}\right) - Q\left(\sqrt{\psi TC/N_{w}}\right)\right] \\ + P_{3/2}\left(\psi TC/N_{w}\right)\left[\frac{1}{8\pi^{2}\beta_{w}^{2}\psi TC/N_{w}}\right] \\ - P_{2}\left(\psi TC/N_{w}\right)\frac{16}{3\sqrt{2\pi}T}\left[\frac{1}{8\pi^{2}\beta^{2}\psi TC/N_{w}}\right]^{\frac{3}{2}}.$$
(A.14)

By expressing the above in terms of the CRB and prior time delay variance, we obtain

$$\mathsf{E}\{|\underline{\hat{t}}_{0} - \underline{t}_{0}|^{2}\} \gtrsim 2\sigma_{t_{0}}^{2}Q\left(\sqrt{\psi TC/N_{w}}\right) + \frac{\alpha^{2}}{2}\Lambda\left(\frac{2\alpha}{3T}\right)\left[Q\left(\sqrt{2\psi TC/N_{w}}\right) - Q\left(\sqrt{\psi TC/N_{w}}\right)\right] + \left[\sigma_{\underline{\hat{t}}_{0}}^{2}\right]_{\text{CRB}}P_{3/2}\left(\psi TC/N_{w}\right) - \frac{16}{3\sqrt{2\pi}T}\left[\sigma_{\underline{\hat{t}}_{0}}^{3}\right]_{\text{CRB}}P_{2}\left(\psi TC/N_{w}\right).$$
(A.15)

Comparison can be made with the results obtained by Chazan et al. [17] and Bell et al. [7]. By assuming a sufficiently large or small effective CNDR (which is appropriate for determining thresholds), the second term on the right hand side of (A.15) is negligible. Moreover, for a sufficiently large effective GB (which is in agreement with practice), the last term is also negligible. Therefore, an approximation for the ZZB is given by

$$\mathsf{E}\{|\underline{\hat{t}}_{0} - \underline{t}_{0}|^{2}\} \gtrsim 2\sigma_{t_{0}}^{2}Q(\sqrt{\psi TC/N_{w}}) + [\sigma_{\underline{\hat{t}}_{0}}^{2}]_{\text{\tiny CRB}} P_{3/2}(\psi TC/N_{w}).$$
(A.16)

Appendix B

Miscellaneous Proofs

B.1 Proof of Theorem 4.3.1

In this appendix we show that the received GNSS modulation signal $\underline{\tilde{y}}(t)$ is AMS. We note that the delayed signal $y(t - \underline{t}_0)$ is AMS because there exists an asymptotic time averaged ACF given by

$$\bar{R}_{\check{y}}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} R_{\check{y}}(\tau, t) \,\mathrm{d}t.$$
(B.1)

The latter exists because $R_{\tilde{y}}(\tau, t)$ is a bounded function and is periodic over t with period $DT_{\rm c}$ [see (4.7)]. By using this fact we can show that the received signal $\tilde{y}(t)$ must also be AMS, i.e., there exists an asymptotic time averaged ACF given by [52]

$$\bar{R}_{\tilde{y}}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \mathsf{E}\left\{ \underline{\tilde{y}}\left(t + \frac{\tau}{2}\right) \underline{\tilde{y}}\left(t - \frac{\tau}{2}\right)^* \right\} \, \mathrm{d}t. \tag{B.2}$$

In terms of the effective channel filter z(t), the time averaged ACF is given by [cf. (2.17)]

$$\bar{R}_{\tilde{y}}(\tau) = \lim_{T \to \infty} \frac{\kappa}{T} \int_{-T/2}^{T/2} \mathsf{E}\left\{ \left[\int_{-\infty}^{\infty} z \left(t - u + \frac{\tau}{2} \right) y(u - \underline{t}_0) \, \mathrm{d}u \right] \left[\int_{-\infty}^{\infty} z \left(t - v - \frac{\tau}{2} \right) y(v - \underline{t}_0) \, \mathrm{d}v \right]^* \right\} \mathrm{d}t$$
$$= \lim_{T \to \infty} \frac{\kappa}{T} \int_{-T/2}^{T/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z \left(t - u + \frac{\tau}{2} \right) z \left(t - v - \frac{\tau}{2} \right)^* R_{\check{y}}(u, v) \, \mathrm{d}u \, \mathrm{d}v \, \mathrm{d}t. \tag{B.3}$$

By introducing the change variables t' = t - (u + v)/2 and $\tau' = v - u$ (with unit Jacobian), the time averaged ACF can be expressed as

$$\bar{R}_{\tilde{y}}(\tau) = \lim_{T \to \infty} \frac{\kappa}{T} \int_{-T/2}^{T/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z \left(t + \frac{\tau}{2} - \left[t - t' - \frac{\tau'}{2} \right] \right) z \left(t - \frac{\tau}{2} - \left[t - t' + \frac{\tau'}{2} \right] \right)^* \\ \times R_{\tilde{y}} \left(t - t' - \frac{\tau'}{2}, t - t' + \frac{\tau'}{2} \right) dt' d\tau' dt.$$
(B.4)

Above, the integrand is bounded and thus the DCT [66] allows us to interchange the limit and integration to obtain

$$\bar{R}_{\tilde{y}}(\tau) = \kappa \int_{-\infty}^{\infty} R_z(\tau + \tau') \bar{R}_{\tilde{y}}(\tau') \,\mathrm{d}\tau', \tag{B.5}$$

where $R_z(\lambda) = \int_{-\infty}^{\infty} z (t' + \lambda/2) z (t' - \lambda/2)^* dt'$. Finally, because $\bar{R}_{\tilde{y}}(\tau)$ exists, $\bar{R}_{\tilde{y}}(\tau)$ also exists. Therefore, $\tilde{y}(t)$ is AMS.

B.2 Proof of Theorem 5.1.1

In this appendix, we evaluate the hermitian inner product defined as

$$R^{(m)}(\tau) \triangleq \left[\frac{\partial^m}{\partial t_0^m} \,\hat{\boldsymbol{y}}(t_0 + \tau)\right]^{\mathsf{H}} \boldsymbol{\Sigma}^{-1} \left[\frac{\partial^m}{\partial t_0^m} \,\hat{\boldsymbol{y}}(t_0)\right]. \tag{B.6}$$

The vector $\hat{\boldsymbol{y}}(t_0) \in \mathbb{C}^N$ contains N entries corresponding to the samples of a bandlimited signal $\hat{y}(t, t_0)$ with a time delay t_0 obtained over an observation interval of length T in the presence of bandlimited stationary circular symmetric noise with covariance $\boldsymbol{\Sigma} \in \mathbb{R}^{N \times N}$. For a sufficiently large T (large N), the noise covariance admits a circulant form and thus has a spectral decomposition given by [57]

$$\Sigma \simeq V \Lambda V^{\mathsf{H}},$$
 (B.7)

where the *n*th (n = 0, ..., N - 1) column of the unitary matrix $\mathbf{V} \in \mathbb{C}^{N \times N}$ consists of the eigenvector given by

$$\boldsymbol{v}_n = \frac{1}{\sqrt{N}} \left[1, \mathrm{e}^{j2\pi n/N}, \dots, \mathrm{e}^{j2\pi n(N-1)/N} \right]^\mathsf{T}, \tag{B.8}$$

and the *n*th diagonal entry of the diagonal matrix $\Lambda \in \mathbb{R}^{N \times N}$ is the eigenvalue given by

$$\lambda_n = \frac{1}{T_{\rm s}} \sum_{h=-\infty}^{\infty} G_w \left(\frac{n}{T} + \frac{h}{T_{\rm s}} \right). \tag{B.9}$$

In (B.9), $G_w(f)$ ($|f| \le W/2$) is the PSD of the continuous time noise process from where originate the noise samples and $T_s = 1/W$ is the sampling period of the entries in the signal vectors. We note that there is no aliasing in (B.9) because $G_w(f)$ is bandlimited. The hermitian inner product can then be expressed as

$$R^{(m)}(\tau) \simeq \left[\frac{\partial^m}{\partial t_0^m} \,\hat{\boldsymbol{y}}(t_0 + \tau)\right]^{\mathsf{H}} \boldsymbol{V} \boldsymbol{\Lambda}^{-1} \boldsymbol{V}^{\mathsf{H}} \left[\frac{\partial^m}{\partial t_0^m} \,\hat{\boldsymbol{y}}(t_0)\right]$$
$$= \sum_{n=0}^{N-1} \lambda_n^{-1} \left[\frac{\partial^m}{\partial t_0^m} \,\hat{\boldsymbol{y}}(t_0 + \tau)\right]^{\mathsf{H}} \boldsymbol{v}_n \boldsymbol{v}_n^{\mathsf{H}} \left[\frac{\partial^m}{\partial t_0^m} \,\hat{\boldsymbol{y}}(t_0)\right]. \tag{B.10}$$

The adjacent hermitian inner products on the right hand side of (B.10) each implement the DFT. Indeed, by writing the hermitian inner products in terms of the vector entries we obtain

$$R^{(m)}(\tau) \simeq \sum_{n=0}^{N-1} \lambda_n^{-1} \left[\frac{1}{\sqrt{N}} \sum_{k_1=0}^{N-1} \frac{\partial^m}{\partial t_0^m} \hat{y}(k_1 T_{\rm s}, t_0 + \tau) \,\mathrm{e}^{-j2\pi k_1 n \frac{T_{\rm s}}{T}} \right]^* \\ \times \left[\frac{1}{\sqrt{N}} \sum_{k_2=0}^{N-1} \frac{\partial^m}{\partial t_0^m} \,\hat{y}(k_2 T_{\rm s}, t_0) \,\mathrm{e}^{-j2\pi k_2 n \frac{T_{\rm s}}{T}} \right].$$
(B.11)

Since the DFT is a sampled and scaled version of a periodically extended Fourier transform, (B.11) can be expressed in terms of the signal spectra, so that

$$R^{(m)}(\tau) \simeq \frac{1}{N} \sum_{n=0}^{N-1} \left[\frac{1}{T_{\rm s}} \sum_{h_1=-\infty}^{\infty} G_w(f) \Big|_{f=\frac{n}{T}+\frac{h_1}{T_{\rm s}}} \right]^{-1} \\ \times \left[\frac{\alpha_0 \mathrm{e}^{j\phi_0}}{T_{\rm s}} \sum_{h_2=-\infty}^{\infty} Y(f) Z(f) \frac{\partial^m}{\partial t_0^m} \mathrm{e}^{-j2\pi f t_0} \Big|_{f=\frac{n}{T}+\frac{h_2}{T_{\rm s}}} \right] \\ \times \left[\frac{\alpha_0 \mathrm{e}^{j\phi_0}}{T_{\rm s}} \sum_{h_3=-\infty}^{\infty} Y(f) Z(f) \frac{\partial^m}{\partial t_0^m} \mathrm{e}^{-j2\pi f(t_0+\tau)} \Big|_{f=\frac{n}{T}+\frac{h_3}{T_{\rm s}}} \right]^*.$$
(B.12)

In (B.12), $\mathscr{F}{\hat{y}(t,t_0)} = \alpha_0 e^{j\phi_0} Y(f) Z(f) e^{-j2\pi ft_0}$, and Z(f) incorporates the filtering used to produce the observed signal from an infinite bandwidth reference signal characterized by the spectrum Y(f) [see (2.17) and cf. (2.18)]. If Z(f) and $G_w(f)$ are bandlimited to $|f| \leq W/2$, for each value of the index n, only one value of the index h_i (i = 1, 2, 3) actually contributes to (B.12). Therefore, it is possible to set $h_1 = h_2 = h_3 = h$ and consolidate the summations to produce

$$R^{(m)}(\tau) \simeq \frac{\kappa}{T} \sum_{n=0}^{N-1} \sum_{h=-\infty}^{\infty} \left| \frac{\partial^m}{\partial t_0^m} e^{-j2\pi f t_0} \right|^2 \frac{|Y(f)Z(f)|^2}{G_w(f)} e^{j2\pi f \tau} \Big|_{f=\frac{n}{T}+\frac{h}{T_{\rm s}}} = \sum_{n=0}^{N-1} \sum_{h=-\infty}^{\infty} (2\pi f)^{2m} \frac{G_{\hat{y}}(f)}{G_w(f)} e^{j2\pi f \tau} \Big|_{f=\frac{n}{T}+\frac{h}{T_{\rm s}}}.$$
(B.13)

In (B.13), $\kappa = |\alpha_0|^2$ and we have used (2.18) to introduce the power spectrum $G_{\hat{y}}(f)$. The above equation can be simplified by exploiting the periodic extension of the spectra to eliminate the index h by redefining the index n. The result is

$$R^{(m)}(\tau) \simeq \sum_{n=-N/2}^{N/2-1} (2\pi f)^{2m} \frac{G_{\hat{y}}(f)}{G_w(f)} e^{j2\pi f\tau} \Big|_{f=\frac{n}{T}}.$$
 (B.14)

Finally, for a sufficiently large T, the summation in (B.14) can be replaced by an integral, i.e.,

$$R^{(m)}(\tau) \simeq T \int_{-W/2}^{W/2} (2\pi f)^{2m} \frac{G_{\hat{y}}(f)}{G_w(f)} e^{j2\pi f\tau} df, \qquad (B.15)$$

which, for T = N/W, proves Theorem 5.1.1.

B.3 Alternative Proof of Theorem 5.1.1

In this appendix we give an alternative proof of Theorem 5.1.1 which is based on Parseval's theorem. We employ the same notation that is used in Appendix B.2. Let us consider the signal vector $\hat{y}'(0) \in \mathbb{C}^N$ obtained by transforming the samples of the signal $\hat{y}(t,0)$, i.e.,

$$\hat{\boldsymbol{y}}'(0) = \boldsymbol{V} \boldsymbol{\Lambda}^{-\frac{1}{2}} \boldsymbol{V}^{\mathsf{H}} \hat{\boldsymbol{y}}(0).$$
(B.16)

The entries in $\hat{y}'(0)$ can also be obtained by sampling the signal $\hat{y}'(t,0)$. By expanding (B.16), the *n*th (n = 0, ..., N - 1) entry of $\hat{y}'(0)$ is given by

$$\hat{y}'(nT_{\rm s},0) = \frac{1}{T} \sum_{k=0}^{N-1} \sum_{h=-\infty}^{\infty} \hat{Y}'(f) \,\mathrm{e}^{j2\pi k n \frac{T_{\rm s}}{T}} \Big|_{f=\frac{k}{T}+\frac{h}{T_{\rm s}}},\tag{B.17}$$

where

$$\hat{Y}'(f) = \frac{\alpha_0 \mathrm{e}^{j\phi_0} Y(f) Z(f)}{\sqrt{WG_w(f)}}.$$
(B.18)

By definition, (B.17) is an inverse DFT which indicates that the Fourier transform of $\hat{y}'(t,0)$ is $\hat{Y}'(f)$. Now, for large N, the Whittaker-Shannon interpolation formula can be used to obtain the following hermitian inner product:

$$R^{(m)}(\tau) = \left[\frac{\partial^m}{\partial t_0^m} \hat{\boldsymbol{y}}'(t_0 + \tau)\right]^{\mathsf{H}} \left[\frac{\partial^m}{\partial t_0^m} \hat{\boldsymbol{y}}'(t_0)\right]$$
$$\simeq W \int_{-\infty}^{\infty} \left[\frac{\partial^m}{\partial t_0^m} \hat{\boldsymbol{y}}'(t, t_0 + \tau)\right]^* \left[\frac{\partial^m}{\partial t_0^m} \hat{\boldsymbol{y}}'(t, t_0)\right] \,\mathrm{d}t. \tag{B.19}$$

By applying Parseval's theorem and using (B.18), we obtain

$$R^{(m)}(\tau) \simeq W \int_{-\infty}^{\infty} \left[\hat{Y}'(f) \frac{\partial^m}{\partial t_0^m} e^{-j2\pi f(t_0+\tau)} \right]^* \left[\hat{Y}'(f) \frac{\partial^m}{\partial t_0^m} e^{-j2\pi ft_0} \right] df$$
$$= T \int_{-W/2}^{W/2} (2\pi f)^{2m} \frac{\kappa}{T} \frac{|Y(f)Z(f)|^2}{G_w(f)} e^{j2\pi f\tau} df.$$
(B.20)

Finally, by using (2.18), we obtain

$$R^{(m)}(\tau) \simeq T \int_{-W/2}^{W/2} (2\pi f)^{2m} \frac{G_{\hat{y}}(f)}{G_w(f)} e^{j2\pi f\tau} df, \qquad (B.21)$$

which, for T = N/W, proves Theorem 5.1.1.

B.4 Proof of Lemma 6.2.1

To prove Lemma 6.2.1, we first prove the identity

$$\mathsf{E}\{|\underline{\hat{t}}_{0} - \underline{t}_{0}|^{2}\} = \int_{0}^{2T} \frac{\tau}{2} \mathsf{P}\left\{|\underline{\hat{t}}_{0} - \underline{t}_{0}| > \frac{\tau}{2}\right\} \,\mathrm{d}\tau.$$
(B.22)

The right hand side of (B.22) can be written using the cumulative distribution function (cdf) of $|\underline{t}_0 - \underline{t}_0|$ defined as $F(\tau) \triangleq \mathsf{P}\left\{|\underline{t}_0 - \underline{t}_0| \leq \tau\right\}$, so that

$$\int_{0}^{2T} \frac{\tau}{2} \mathsf{P}\left\{\left|\underline{\hat{t}}_{0} - \underline{t}_{0}\right| > \frac{\tau}{2}\right\} \, \mathrm{d}\tau = \int_{0}^{2T} \frac{\tau}{2} \left[1 - F\left(\frac{\tau}{2}\right)\right] \, \mathrm{d}\tau. \tag{B.23}$$

By employing the substitution $\xi = \tau/2$ and integrating by parts (defining $f(\xi) \triangleq \frac{\mathrm{d}F(\xi)}{\mathrm{d}\xi}$ and recognizing that $F(T) = \mathsf{P}\{|\hat{\underline{t}}_0 - \underline{t}_0| \leq T\} = 1$), we obtain

$$\int_{0}^{2T} \frac{\tau}{2} \mathsf{P}\left\{ |\underline{\hat{t}}_{0} - \underline{t}_{0}| > \frac{\tau}{2} \right\} \, \mathrm{d}\tau = \int_{0}^{T} \xi^{2} f(\xi) \, \mathrm{d}\xi.$$
(B.24)

Because $f(\xi)$ is the pdf of $|\underline{\hat{t}}_0 - \underline{t}_0|$, the right hand side of (B.24) is the MSE and thus we have shown (B.22). Next, we relate the MSE to the probability of detection error of the suboptimal test statistic (6.36). We can re-express (B.22) by obtaining the mean of the probabilities conditioned on the realizations t_0 and $t_0 + \tau$ drawn from uniform distributions. The result is

$$\mathsf{E}\{|\hat{\underline{t}}_{0} - \underline{t}_{0}|^{2}\} = \int_{0}^{2T} \frac{\tau}{2} \left(\mathsf{P}\left\{\hat{\underline{t}}_{0} - \underline{t}_{0} > \frac{\tau}{2}\right\} + \mathsf{P}\left\{\hat{\underline{t}}_{0} - \underline{t}_{0} < -\frac{\tau}{2}\right\}\right) \,\mathrm{d}\tau$$

$$= \int_{0}^{2T} \frac{\tau}{2T} \left(\int_{0}^{T} \mathsf{P}\left\{\hat{\underline{t}}_{0} - t_{0} > \frac{\tau}{2}\right\} \,\mathrm{d}t_{0} + \int_{-\tau}^{T-\tau} \mathsf{P}\left\{\hat{\underline{t}}_{0} - t_{0} - \tau < -\frac{\tau}{2}\right\} \,\mathrm{d}t_{0}\right) \,\mathrm{d}\tau.$$

$$(B.25)$$

Because the integrands in (B.25) are nonnegative, we obtain the following result by employing (6.38):

$$\mathsf{E}\{|\underline{\hat{t}}_{0} - \underline{t}_{0}|^{2}\} \ge \int_{0}^{T} \int_{0}^{T-\tau} \frac{\tau}{T} P_{\mathrm{e,s}}(\tau, t_{0}) \,\mathrm{d}t_{0} \,\mathrm{d}\tau, \tag{B.26}$$

which proves Lemma 6.2.1.

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