Aspects of power domains and power locales

by

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Abstract

Several equivalent approaches to power domains are presented; the naturality of this concept for denotational semantics is stressed and its modal interpretation is explained. We show how to solve equations involving powerdomains and apply the theory to an equation that leads to a characterization of bisimulations. Power locales are introduced, as the analogue of both power domains and power spaces. We study the monads defined by the power locales, their algebras and the points of the powerlocales.

Résumé

On presente plusieurs approches équivalents à la notion de domain des parties. On essaye de montrer qu'il s'agit d'un concept naturel pour la sémantique dénotationnelle et on en donne une interprétation modale. On montre comment résoudre des équations contenant le domain des parties et on applique cette théorie à une équation qui conduit à une charactérization des bisimulations. Les locaux des parties sont introduits, en tant que structure équivalente aux domains des parties et aux éspaces des parties. On étudie les monades définies par les locaux des parties, leur algèbres et les points des locaux des parties.

Contents

Introduction

Power domains and power locales are used in Computer Science to give a semantics to non-determinism ([SLG, Vic89]) and to model data-bases ([Vic92]). There are three well-established versions of power domains, named after Smyth, Hoare and Plotkin: they correspond, respectively, to the Smyth, Hoare and Vietoris power locales.

Power locales also have a mathematical interest of their own, since the Vietoris power locale stems from the theory of hyperspaces. More recently, M. Bunge ([Bun95]) showed that the *symmetric topos* (the topos-theoretic analogue of the Hoare power locale, see [BF96, BC94]) classifies distributions on a topos (a generalization of the classical notion of distribution for a topological space, see [Law92]).

This thesis intends to show how naturally the concepts of power domains and power locales arise when trying to give a semantics to non-determinism. We are reassured in this conviction also by the fact that there is no theoretical overhead when solving equations involving the power domains. As example, we present an equation in which the Plotkin power domain appears and which leads to a characterization of bisimulations. We also study the monads that derive from the power locales. This serves a double purpose. On one hand the monad defined by the Vietoris locale is the localic version of the classical Vietoris monad and hence it is of interest for the study of uniform locales. On the other hand the Smyth and Hoare monads display a remarkable symmetry: a good understanding of this fact should lead to an axiomatic theory of power locales.

The first power domain to appear in Computer Science was Plotkin's ([Plo76, Plo81]) and was an elaboration of an older construction due to Egli and Milner ([Mil73a, Mil73b]). It was rather involved and worked only for flat posets. This was clearly an unsatisfactory result, since the simplest structure of interest for semantics is the one of complete partial order (cpo, i.e. a poset where any directed subset has a join). Plotkin's idea was streamlined by Smyth ([Smy78]) who also introduced another power domain. His construction works for ω-domains (i.e. algebraic cpo's: algebraic means that every element is the directed join of the elements that are below it; ω means that the set of compact elements of the cpo is countable). In §1.1 both power domains are presented, together with the Hoare's, following the approach of [SLG]. They are constructed out of rooted trees, whose nodes correspond to the non-deterministic features of the program. Then it is shown how they can be obtained simply by taking the completion by ideals of the collection of finite non-empty sets of compact elements (of a given domain), endowed with a suitable order. The last characterization allows to treat equations involving the power domain constructions with the same method for solving equations based on D. Scott's "Limit-Colimit Coincidence Theorem" ([Sco72]). The general theory ([SP82, SLG]) is presented in §1.2.1. However the category ω-**Dom** of ω-domains, in which we have been working up to now, is not the most suitable from the point of view of denotational semantics, since it is not cartesian closed. We can restrict our attention to the largest cartesian closed full subcategory of ω-**Dom**, known as **SFP** ([Smy83]). Its objects, the SFP-domains, are exactly the colimits of ω-chains of the kind $\{D_n \xrightarrow{e_n} D_{n+1}\}_{n\in\mathbf{N}}$ where the D_n's are finite posets and the maps e_n are embeddings. **SFP** appears to be the right ambient for denotational semantics, since this category is closed under all the usual operations and under the power domain constructions. In §1.2.2 I spell out the details for equations in ω-**Dom** and **SFP** involving the power domains ([SP82, Smy78]) and apply this in §1.2.3 to the study of a particular equation, that can be used to give a characterization of bisimulations ([Abr91]). Let us remark at this point that several axiomatic approaches to domain theory have been developed by now ([Hyl91, Ros86, Fio94]). They provide the right setting for discussing partial maps and the notion of passage to the limit. However, none of them captures modality. This should perhaps be a subject of future investigation. The connection of power domains with Computer Science was made more explicit by Winskel ([Win85]): using the operators of possibility and inevitability he gave a modal interpretation of power domains (§1.3).

One year later Robinson ([Rob86]) gave the localic analogue of Winskel's modal constructions. He showed how the power locales so obtained can be considered as a generalization of power domains (§2.1). They are also the localic counterpart of power spaces, introduced by Smyth ([Smy83]) in his approach to non-determinism via multivalued functions (also in §2.1). Smyth's work relies on the theory of hyperspaces, i.e. spaces made up from subsets (see [Nad78, McA78]), which goes back to Vietoris ([Vie22], itself based on the Hausdorff metric, see [Hau14]). This theory has been used by Isbell when studying uniform spaces ([Isb64]) and later on uniform locales ([Isb72]). Indeed, the quest for a *hyperlocale* was one of the reasons that led Johnstone to the definition of the Vietoris monad on the category of compact regular locales ([Joh82, pp 111-118]) and then in [Joh85] to its generalization to **Loc**. I present it in §2.2 together with the monads generated by the other power locales. In the same article he also pointed out constructive difficulties in the characterization of the points of the Hoare and Vietoris power locales. Later, in [Joh89], he introduced the notion of weakly closed sublocale (the natural generalization of closed sublocale, from a constructive view-point). Using this notion, Bunge and Funk were able to identified the global points of the Hoare power locale of a locale D with the weakly closed sublocales of D with *open domain* ([BF96]). Afterwards Vickers ([Vic95b]) gave a unified version of the existing constructive proofs and phrased them in the setting of generalized (rather then just global) points (§2.3). Vickers has been pushing forwards this argument. He showed that sublocales with open domains and sublocales with ***compact domain*** (necessary in the characterization of the points of the Smyth power locale) are two instances of the same concept ([Vic95a]). This is the first step towards an axiomatic approach to power locale theory, that still has to be developed.

Chapter 1

Power Domains

1.1 Towards the definition of power domain

In this section we show how the power domain construction arises naturally when trying to give a semantics to non-deterministic programs ([SLG, ch. 11]).

Before starting, let us quickly review some basic notions for doing program semantics ([SLG, ch. 1-2]).

Definition 1 *Let D be a poset with a bottom element. We say that D is a* complete partial order *(cpo) if any directed subset A of D has a supremum $\bigvee A$.*

Computationally, we should think of $x \leq y$ $(x, y \in D)$ as "y has more information of x", or "x is an approximation of y".

Definition 2 *An element x of a cpo D is said to be* compact (or finite) *if whenever A is a directed subset of D and $\bigvee A \leq x$ then there is a in A such that $x \leq a$. The set of compact elements of D will be denoted by $Kpt(D)$.*

The compact elements are considered as the concrete elements on which we compute. A result of a computation, however, can also be denoted by an arbitrary element, provided that this can be seen as a join of finite elements: hence, computationally, the situation is satisfactory, when every element can be seen in this way.

Definition 3 *A* domain *D is an algebraic cpo, i.e. a cpo D in which, for every element x, the set $approx(x) = \{a \in Kpt(D) : a \leq x\}$ is directed and its supremum is x.*

Remark. Some authors prefer to use the term domain referring to cpo with a richer structure. I chose to use it to denote the minimal structure useful for doing denotational semantics. When we do use a richer structure this is said explicitly. For example in this section, unless otherwise stated, we will always work with an ω-domain, i.e. a domain whose set $Kpt(D)$ of compact elements is countable.

Let us denote by $\mathcal{P}$ a non-deterministic program. Assume that every single partial outcome of $\mathcal{P}$ can be represented by an element of D. When we run $\mathcal{P}$, at any time that the program has a choice we have a set of possible outcomes: this gives rise to a rooted tree, labeled by the elements of D.

Definition 4 *A* tree *T is a poset $(T, \leq)$ such that :*

- *it has a least element $\perp$; and*
- *for any x in T the set of predecessors of x is finite.*

The height *of x, i.e. the number of predecessors of x, will be denoted by σx.*

Definition 5 *Let T be a tree, $\zeta : T \rightarrow D$ a monotone map. We say that (T, ζ) is a* generating tree over *D.*

The nodes of the tree thus correspond to non-deterministic features of $\mathcal{P}$. It seems reasonable to assume that at any stage there are only finitely many choices;

therefore in the following our trees are always supposed to be finitary branching. The total outcomes that we get running $\mathcal{P}$ can be represented by the limits of the elements of D labeling each branch of the trees. With no loss of generality we may assume that the branches all have length ω (for, if there are branches of finite length, we can "prune" them). Hence the power domain ought to be defined out of the set

$$\mathcal{F}(D) = \{T_\omega^\zeta : (T, \zeta) \text{ is a generating tree over } D\}$$

where $T_\omega^\zeta = \{\bigvee_{x\in\gamma}\zeta(x) : \gamma \text{ is a branch of } T\}$ is the element generated by the tree (T, ζ).

Remark that $\mathcal{F}(D)$ contains any non-empty finite subset of D and that it is closed under unions. Moreover, if $f : D \longrightarrow D'$ is a continuous map (i.e. a map that preserves directed joins), then $f[A] = \{f(a)|a \in A\}$ is in $\mathcal{F}(D')$ for any A in $\mathcal{F}(D)$. Indeed, $f[T_\omega^\zeta] = T_\omega^{f\circ\zeta}$ for any generating tree (T, ζ) over D.

Since the compact elements in a domain D are considered to be the concrete elements on which we can compute, we would like to label our trees using just the compact elements of D. The following proposition shows that we can indeed do so, without any loss of generality.

Proposition 1 *For any generating tree $\mathcal{T} = (T, \zeta)$ over D there is a generating tree $\mathcal{T}' = (T, \zeta')$ such that:*

- $\zeta'[T] \subseteq Kpt(D)$; *and*
- $T_\omega^\zeta = T_\omega^{\zeta'}$.

Proof. Let $a_1, a_2, \ldots$ be an enumeration of the compact elements of D. For any node t of T, let A^t be the set

$$A^t = \{a_i \in Kpt(D) : a_i \leq \zeta(T), i \leq \sigma(t)\}.$$

Define a monotone map $\zeta' : T \to Kpt(D)$ such that for any t in T, one has $\zeta'(t) \leq \zeta(t)$. Use induction on the height $\sigma(t)$ of t:

- for $\sigma(t) = 0$ put $\zeta'(t) = \perp_D$;

- for $\sigma(t) > 0$, the induction hypothesis says that for any s in T such that $\sigma(s) < \sigma(t)$, $\zeta'(s)$ is in $Kpt(D)$ and $\zeta'(s) \leq \zeta(s)$. Then, if t' is the immediate predecessor of t, $A^t \cup \{\zeta'(t)\}$ is a finite subset of $approx(\zeta(t)) = \{a \in Kpt(D) : a \leq \zeta(t)\}$ and therefore there is in $approx(\zeta(t))$ an upper bound a of $A^t \cup \{\zeta'(t)\}$: choose such an a and put $\zeta'(t) = a$.

Then $\zeta' : T \to Kpt(D)$ is a monotone map, hence (T, ζ') is a generating tree over D. Moreover $T^\zeta_\omega = T^{\zeta'}_\omega$, since for any branch γ of T we have $\bigvee\{\zeta(T) : t \in \gamma\} = \bigvee\{\zeta'(T) : t \in \gamma\}$.

One inequality, namely

$$\bigvee\{\zeta'(T) : t \in \gamma\} \leq \bigvee\{\zeta(T) : t \in \gamma\}$$

is immediate from the definition of ζ'.

To prove the converse inequality, consider an a_n in $Kpt(D)$ such that $a_n \leq \bigvee_{t\in\gamma} \zeta(t)$. By compactness of a_n, there is t in γ such that $a_n \leq \zeta(t)$ and since ζ is monotone we can pick t so that we also have $n \leq \sigma(t)$, i.e. such that a_n is in A_t. By definition of ζ', we get $a_n \leq \zeta'(t) \leq \bigvee_{t\in\gamma} \zeta'(t)$.

Since D is algebraic we can conclude that

$$\bigvee\{\zeta(T) : t \in \gamma\} \leq \bigvee\{\zeta'(T) : t \in \gamma\}.$$

□

Consider the elements of the kind

$$T^\zeta_n = \{\zeta(x) : x \in T, \quad \sigma(x) = n\}$$

(called in the following the n^{th}*-level of the tree* T) for (T, ζ) generating tree over D and n any natural number. They denote the n^{th}-step of a non-deterministic computation: hence we would like them to be the compact elements of the power domain and to form ascending chains $\{T_n^\zeta\}_{n\in\mathbb{N}}$ with join given by T_ω^ζ. This will turn out to be true (except for a minor adjustment).
Remark that the elements of the kind T_n^ζ form the set

$$\mathcal{M}(D) = \{A \subseteq Kpt(D) : A \text{ is finite and non-empty}\}$$

and that this set is countable.
Several orders have been proposed for $\mathcal{F}(D)$. They lead to the three different versions of power domains. We start by defining relations on $\mathcal{M}(D)\times\mathcal{F}(D)$ that will be extended to orderings on $\mathcal{F}(D)$.

Definition 6 *Define the relations* $\sqsubseteq_i$ $(i = 0, 1, 2)$ *on* $\mathcal{M}(D)\times\mathcal{F}(D)$ *as follows; for* A *in* $\mathcal{M}(D)$ *and* S *in* $\mathcal{F}(D)$*:*

- $A \sqsubseteq_0 S$ *if* $\forall x \in S\,.\,\exists a \in A\,.\,a \leq x$ *(Smyth ordering);*
- $A \sqsubseteq_1 S$ *if* $\forall a \in A\,.\,\exists x \in X\,.\,a \leq x$ *(Hoare ordering);*
- $A \sqsubseteq_2 S$ *if* $A \sqsubseteq_0 S$ *and* $A \sqsubseteq_1 S$ *(Plotkin ordering).*

Definition 7 *For* S, S' *in* $\mathcal{F}(D)$ *let* $S \sqsubseteq_i S'$ *iff for any* A *in* $\mathcal{M}(D)$*, one has* $A \sqsubseteq_i S'$ *whenever* $A \sqsubseteq_i S$.

The meaning of these ordering is clear. In the Smyth case $A \sqsubseteq_0 S$ holds if any possible total outcome represented by an element of S comes from a partial outcome represented by an element of A. In the Hoare case, $A \sqsubseteq_1 S$ holds if any partial outcome represented by an element of A evolves into a total outcome represented by an element of S. The Plotkin ordering is the intersection of the previous two: therefore it is generally more interesting since it identifies fewer

programs (see the discussion about "demonic" and "angelic" non-determinism in [Vic89, ch. 11]).
It is clear from the definition that $\sqsubseteq_i$ are reflexive and transitive relations; since in generally they are not antisymmetric, we will consider the obvious quotient.

Definition 8 *For S, S' in $\mathcal{F}(D)$ let $S \equiv_i S'$ iff $S \sqsubseteq_i S'$ and $S' \sqsubseteq_i S$.*
The quotient $F_i(D) = \mathcal{F}(D)/\equiv_i$ is called the i^{th} power domain of *D (for $i = 0, 1, 2$ respectively the Smyth, Hoare and Plotkin power domain).*

Remark that for (T, ζ) generating tree over D we have

- if $m \leq n$ then $T^\zeta_m \sqsubseteq_i T^\zeta_n$;
- for all n in $\mathbb{N}$ one has $T^\zeta_n \sqsubseteq_i T^\zeta_\omega$

for $i = 0, 1, 2$.
Actually we can show that T^ζ_ω is the supremum of the chain $\{T^\zeta_n\}_{n\in\mathbb{N}}$ in $(\mathcal{F}(D), \sqsubseteq_i$).

Lemma 2 *Let (T, ζ) be a generating tree over D. For any A in $\mathcal{M}(D)$, if $A \sqsubseteq_i T^\zeta_\omega$ then $A \sqsubseteq_i T^\zeta_m$ for some m in $\mathbb{N}$ ($i = 0, 1, 2$).*

Proof. Let us start with the Smyth case, i.e. for $i = 0$. Let us verify the contrapositive of the statement: if $\forall m \in \mathbb{N}$ one has $A \not\sqsubseteq_0 T^\zeta_m$ then $A \not\sqsubseteq T^\zeta_\omega$. We will show this directly by defining a branch $\gamma = \{t_n : n \in N\}$ of T such that for all a in A one has $a \not\leq \bigvee\{\zeta(t_n) | t_n \in \gamma\}$.
For all $s \in T$, define the property

$$P(s) \quad \text{iff} \quad \forall m \in N\,.\,\exists t \geq s\,.\,(\sigma(t) \geq m \text{ and } \forall a \in A\,.\,a \not\leq \zeta(t)).$$

Then, by induction, we can define γ to be a branch

$$t_0 < t_1 < \ldots < t_n < \ldots$$

in T such that for all n in $\mathbb{N}$ the property $P(t_n)$ holds:

- let $t_0 = \perp_D$ be the root of T. Then $P(t_0)$ holds and $\sigma(t_0) = 0$:

- assume as induction hypothesis that we have defined

$$t_0 < t_1 < \ldots < t_n$$

such that $P(t_i)$ is true and $\sigma(t_i) = i$ for all $i = 0, \ldots, n$. Let $s_1, \ldots, s_k$ be the immediate successors of t_n.

If $P(s_i)$ does not hold, then there is an m_i in $\mathbb{N}$ such that

$$\forall t \geq s_i . (\forall a \in A . a \not\leq \zeta(t)) \implies \sigma(t) < m_i.$$

If $P(s_i)$ did not hold for $i = 1, \ldots, k$, then for $m = \max\{m_1, \ldots, m_k\}$ we would have

$$\forall t > t_n . (\forall a \in A . a \not\leq \zeta(t)) \implies \sigma(t) < m.$$

Comparing this result with $P(t_n)$ we get that $n = \sigma(t_n) > m$ which is in contradiction with the falseness of $P(s_1)$,...,$P(s_k)$. Hence there is an s_i such that $P(s_i)$ holds: let t_{n+1} be such one.

So, we have defined γ. Now suppose there is an $a \in A$ such that $a \leq \bigvee\{\zeta(t_n) : t_n \in \gamma\}$.

Then by compactness of a, there is t_n in γ such that $a \leq \zeta(t_n)$: this is not the case, since $P(t_n)$ true implies that for all a in A one has $a \not\leq \zeta(t_n)$. Therefore the branch γ has the desired property.

The other two cases are simpler. Let us start with the Hoare ordering (i.e. $i = 1$). Suppose $A \sqsubseteq_1 T^{\zeta}_{\omega}$: for any a in A, there is a branch γ_a in T such that

$a \leq \bigvee\{\zeta(t) : t \in \gamma_a\}$. By compactness of a we know there is t in γ_a, say of height m_a, such that $a \leq \zeta(t_a)$. Then $A \sqsubseteq T^\zeta_m$ where $m = \max\{m_a : a \in A\}$.
Finally to prove the statement for the Plotkin case we only need to use the previous steps of the proof. If $A \sqsubseteq_2 T^\zeta_\omega$ then, by definition, we have $A \sqsubseteq_0 T^\zeta_\omega$ and $A \sqsubseteq_1 T^\zeta_\omega$: hence there are m, n in $\mathbb{N}$ such that $A \sqsubseteq_0 T^\zeta_m$ and $A \sqsubseteq_1 T^\zeta_n$. Then $A \sqsubseteq_2 T^\zeta_p$ for $p = \max\{m, n\}$. □

Proposition 3 *Let (T, ζ) be a generating tree over D. Then T^ζ_ω is the supremum in $(\mathcal{F}(D), \sqsubseteq_i)$ of $\{T^\zeta_n : n \in N\}$ for $i = 0, 1, 2$.*

Proof. It has been already remarked that $T^\zeta_n \sqsubseteq_i T^\zeta_\omega$ for all n. Suppose now that an element Y of $\mathcal{F}(D)$ is an upper bound of $\{T^\zeta_n : n \in N\}$ with respect to $\sqsubseteq_i$: we want to show that $T^\zeta_\omega \sqsubseteq_i Y$ (and hence T^ζ_ω is the supremum). If $A \sqsubseteq T^\zeta_\omega$ then, by the previous lemma, $A \sqsubseteq T^\zeta_n$ for some n in $\mathbb{N}$ and hence $A \sqsubseteq_i Y$: therefore, by definition of $\sqsubseteq_i$ as preorder on $\mathcal{F}(D)$, $T^\zeta_\omega \sqsubseteq_i Y$. □

Before proving that the power domains are indeed domains, we need a couple of technical results.

Lemma 4 *If C is a countable preordered set and B is a directed subset of C, then there is an ω-chain $A \subseteq B$ such that: $\forall b \in B . \exists a \in A . b \leq a$.*
Therefore $\bigvee A$ exists iff $\bigvee B$ exists; if that is the case then $\bigvee A = \bigvee B$.

Proof. Let $b_0, b_1, \ldots$ be an enumeration of the elements of B. Put $B_n = \{b_i : i \leq n\}$ and define $A = \{a_n : n \in N\}$ by induction: for $n = 0$ put $a_0 = b_0$; now assume that we have defined $a_0 \leq a_1 \leq \ldots \leq a_n$ with $a_i \in B$ for $i = 0, \ldots, n$. Since $B_n \cup \{a_n\}$ is a finite subset of the directed set B, there is an element in B, call it a_{n+1}, which is an upper bound of $B_n \cup \{a_n\}$.
Then it is clear that the chain A we have defined has the desired property. □

Definition 9 *Let $(P, \leq)$ be a preorder. Consider the equivalence $\equiv$ defined by $x \equiv y$ iff $x \leq y$ and $y \leq x$; denote by $P/\equiv$ the poset obtained by taking the quotient of P by $\equiv$. We say that P is a* predomain *if $P/\equiv$ is a domain.*

Definition 10 *Let D be a preordered set. A set $C \subseteq Kpt(D)$ is a* set of compact elements of D up to equivalence *if*

$$\forall a \in Kpt(D). \exists c \in C. a \equiv c$$

where $\equiv$ is the equivalence relation induced by the preorder.

Lemma 5 *Let D be a preordered set and let C be a countable subset of D. Then D is a predomain with C as set of compact elements up to equivalence iff the following hold:*

1. *D has a least element; and*
2. *if $A \subseteq C$ is a chain, then A has a supremum in D; and*
3. *if x is in D there is a non-empty chain $A \subseteq C$ such that $x = \bigvee A$; and*
4. *if a is in C and A is an ω-chain in C such that $a \leq \bigvee A$ then $a \leq b$ for some $b \in A$.*

Proof. If D is a predomain with C as its set of compact elements up to equivalence, then the four properties clearly hold (in particular 3. is obtained using Lemma 4 applied to $B = \{y \in C : y \leq x\}$).

Let us prove the converse implication. Assume all four properties hold. Let us start with a remark: if $B \subseteq C$ is a directed set then there is by Lemma 4 an ω-chain $A \subseteq B$ such that $\bigvee B$ exists if and only if $\bigvee A$ exists and if this is the case then $\bigvee A = \bigvee B$. But for 2. $\bigvee A$ exists and therefore any directed subset of C has a supremum in D. For x in D put $C_x = \{a \in C : a \leq x\}$. Let us verify that C_x is directed and $x = \bigvee C_x$. For 3. there is a non-empty chain $A_x \subseteq C_x$ such that $x = \bigvee A_x$: for the remark we just did we can suppose without loss of generality that A_x is an ω-chain. If a and a' are in C_x, then by 4. there are b and b' in A_x such that $a \leq b$ and $a' \leq b'$. Since A_x is a chain, then $b \leq b'$ (or $b' \leq b$) and hence $a \leq b'$ and $a' \leq b'$, i.e. C_x is directed and therefore by the remark it has a supremum, namely $x = A_x$. Let us verify now that D is directed complete. Let B be a directed subset of D; consider $B' = \bigcup_{x \in B} C_x$: it is directed and it is contained in C. Hence B' has a supremum and $\bigvee B = \bigvee B'$.

Finally let us verify that C is a set of compact elements of D, up to equivalence. First of all, the elements of C are compact: let a be an element of C, B a directed subset of D such that $a \leq \bigvee B$. Since, as we have seen, $\bigvee B = \bigvee B'$ where $B' = \bigcup_{x \in B} C_x$ is directed and contained in C, there is an ω-chain $A \subseteq B'$ such that $\bigvee A = \bigvee B'$. Then using 4. from $a \leq \bigvee A$ it follows that there is $b' \in A$ such that $a \leq b'$: since b' is in C_x for some x of B, then for such an x we have $a \leq x$ and therefore a is compact. We are only left to verify that

$$\forall a \in Kpt(D) \,.\, \exists c \in C \,.\, a \equiv c.$$

Since $a = \bigvee C_a$, a is compact and C_a is directed there is c in C_a such that $a \leq c$; hence $a \equiv c$ and $c \in C$. $\square$

Remark that if D is a preordered set and C is a countable subset of D, then D with C satisfies the properties listed in Lemma 5 if and only if $D/\!\equiv$ together with $C/\!\equiv$ does. Hence, from the following Lemma 7, we get immediately the main result.

Theorem 6 *$F_i(D) = \mathcal{F}(D)/\!\equiv_i$ is an ω-domain with $\mathcal{M}(D)/\!\equiv_i$ as its set of compact elements for $i = 0, 1, 2$.*

Lemma 7 *Let D be a domain. Then $\mathcal{F}(D)$ is a predomain with $\mathcal{M}(D)$ as its set of compact elements up to equivalence with respect to $\sqsubseteq_i$ for $i = 0, 1, 2$.*

Proof. We only need to prove that the four properties stated in Lemma 5 hold. Let us see: 1. $\{\perp\}$ is a least element in $\mathcal{F}(D)$;
2. let $\{A_n\}_{n\in N} \subseteq \mathcal{M}(D)$ be a chain: we will define by induction a tree (T, ζ) such that $T^\zeta_\omega = \sup_{n\in N} A_n$. We already remarked that without loss of generality we can assume that $\{A_n\}_{n\in N}$ is an ω-chain. Let $\perp_D$ be the root of the tree T. Let us suppose that we have defined the n^{th} level of the tree T and that it is labeled by the elements of A_{n-1}. Now let us consider the Plotkin case: for any a_{n-1} in A_{n-1} and a_n in A_n add to the tree an arc: then the $(n+1)^{th}$–level is labeled by A_n. So the tree T is defined and clearly $\bigvee_{n\in N} A_n = T^\zeta_\omega$.
For the Smyth power domain we can apply the same procedure, with the proviso that an arc will be added only in the case that it gives rise to a branch of infinite length. Then $T^\zeta_{n+1} \subseteq A_n$ and hence $A_n \sqsubseteq_0 T^\zeta_{n+1} \sqsubseteq_0 T^\zeta_\omega$ for any n. If $Y \in \mathcal{F}(D)$ is an upper bound of $\{A_n : n \in N\}$, then $T^\zeta_\omega \sqsubseteq_0 Y$ and hence $T^\zeta_\omega = \sup\{A_n : n \in N\}$. Indeed, for any $A \in \mathcal{M}(D)$ such that $A \sqsubseteq_0 T^\zeta_\omega$, one has $A \sqsubseteq_0 T^\zeta_n$ for some n in $\mathbb{N}$ because of Lemma 2. But because of the way we constructed T there is m in $\mathbb{N}$ such that $T^\zeta_n \sqsubseteq_0 A_m$. Hence $A \sqsubseteq_0 Y$ and $T^\zeta_\omega \sqsubseteq_0 Y$.
Also for the Hoare power domain we can apply the same strategy as for the Plotkin one, but this time we will also add at any step $\perp_D$: so, if there is an a in A_{n+1} such that for all b in A_n one has $b \not\leq a$, one can add the arc $\perp \rightarrow a$.
Hence $T^\zeta_\omega = \bigvee_{n\in N}(A_n \cup \{\perp_D\})$ since $A_n \equiv_1 A_n \cup \{\perp_D\}$.
3. follows from Proposition 1 and 4. from Lemma 2. □

Recall now that a domain D is isomorphic to the set $Idl(Kpt(D))$ of ideals of $Kpt(D)$ ordered by subset inclusion ([SLG]). Then from Theorem 6 we have immediately the following result.

Theorem 8 *Let D be an ω-domain. Then, for $i = 0, 1, 2$ we have the isomorphisms:*

$$\mathcal{F}_i(D) \cong Idl(\mathcal{M}(D), \sqsubseteq_i).$$

Since the definition of $(\mathcal{M}(D), \sqsubseteq_i)$ makes sense also for domains, whose set $Kpt(D)$ of finite elements is not countable, we can drop this hypothesis and get a more general definition of power domain (observe that the hypothesis on the countability of $Kpt(D)$ has been used in Proposition 1).

Definition 11 *Let D be a domain (not necessarily an ω-domain). Then the* power domain of D *is the completion by ideals of the poset $(\mathcal{M}(D), \sqsubseteq_i)$ for $i = 0, 1, 2$.*

We can also give the so-called strict version of the power domains by allowing the empty set to be an element of $\mathcal{M}(D)$ (it will be used in §1.2 and in §1.3).

Definition 12 *Let D be a domain. Put*

$$\mathcal{M}^+(D) = \{A \subseteq Kpt(D) : A \textit{ is finite}\}.$$

Then the strict power domain *is defined by $\mathcal{F}_i^+(D) = Idl(\mathcal{M}^+(D), \sqsubseteq_i)$.*

1.2 Domain equations

In this section we want to show that the standard techniques for solving domain equations apply naturally also when the power domain constructions are involved. We start by recalling (§1.2.1) the method based on D. Scott's Limit-Colimit Coincidence Theorem ([SP82]). Then (§1.2.2.) we apply it to the case of equations in ω-**Dom** and **SFP** (see again [SP82]) and show that the last mentioned category is closed under the power domain constructions ([Smy78]). Finally (§1.2.3.) we apply this theory to the equation

$$D \cong F_2^+(\Sigma_{a \in Act} D_a)$$

which will be explained later and can be used to give a characterization of bisimulations ([Abr91]).

1.2.1 Solving domain equations

Initial fixed points

Solving domain equations plays a crucial role in denotational semantics. Hence we want to show a method that can be applied to all equations we might be concerned with. The first step is to consider any equation (for example

$$D \;\cong\; F_2^+(\Sigma_{n\in\mathbf{N}} D_n) \tag{1.1}$$

where Σ is the separated sum, cf. 1.7, and F_2^+ the strict Plotkin power domain, cf. Definition 12) as a particular instance of the generic equation

$$D \;\cong\; F(D) \tag{1.2}$$

where $\mathbf{C}$ is a category of some sort of domains and F and endofunctor on $\mathbf{C}$. In our example $\mathbf{C}$ will be **SFP** (cf. Definition 26) and F the composition of the functors:

$$F : \mathbf{C} \xrightarrow{<id_{\mathbf{C}}>_n} \prod_{n\in\mathbf{N}} \mathbf{C} \xrightarrow{\Sigma} \mathbf{C} \xrightarrow{F_2^+} \mathbf{C}.$$

Next we try to follow the analogy between partial orders and categories, thinking of F as a kind of order-preserving map.

In the posetal case a solution of the equation 1.2 would be a fixed point for F and we can look for the minimal one; also, the minimal prefixed point (if it exists) is the minimal prefixed point. Let us give the analogue for categories.

Definition 13 *A* fixed point *for a functor $F : \mathbf{C} \longrightarrow \mathbf{C}$ in a category $\mathbf{C}$ is a pair (A,α) where A is an object of $\mathbf{C}$ and $\alpha : FA \xrightarrow{\sim} A$ an isomorphism.*
*An F-*algebra *(or prefixed point for F) is a pair (A,α) where A is again an object of $\mathbf{C}$, but now α is just a morphism $\alpha : FA \rightarrow A$.*
An F-algebra homomorphism is a homomorphism between F-algebras (A,α) and (B,β) such that $\beta \circ f = \alpha \circ Ff$.

F-algebras and F-algebra homomorphisms form a category.

As for posets, we have the following result ([LS86]).

Proposition 9 *An initial F-algebra, if it exists, is also an initial fixed point.*

Proof. Let $(A, \alpha : FA \longrightarrow A)$ be an initial F-algebra, i.e. an initial object in the category of F-algebras. We want to prove that α is an isomorphism. Since $(FA, F\alpha : F^2A \longrightarrow FA)$ is also an F-algebra, by initiality of (A, α) there is an F-morphism f such that the diagram

$$\begin{tikzcd} FA \arrow[r, "\alpha"] \arrow[d, "Ff"'] & A \arrow[d, "f"] \\ F^2A \arrow[r, "F\alpha"'] & FA \end{tikzcd}$$

is commutative. Combining this with the diagram

$$\begin{tikzcd} F^2A \arrow[r, "F\alpha"] \arrow[d, "F\alpha"'] & FA \arrow[d, "\alpha"] \\ FA \arrow[r, "\alpha"'] & A \end{tikzcd}$$

and using again the initiality of (A, α), we get $\alpha \circ f = id_A$. Then we also have

$$\begin{aligned} f \circ \alpha &= F\alpha \circ Ff \\ &= F(\alpha \circ f) \\ &= F(id_A) \\ &= id_{FA}. \end{aligned}$$

Hence α is an isomorphism. $\square$

If $\mathbf{C}$ is a poset with a least element $\perp$, then the least fixed point of an order preserving map F can be constructed as the join of the sequence

$$\perp \leq F(\perp) \leq \ldots \leq F^n(\perp) \leq \ldots$$

(provided that this join exists in $\mathbf{C}$ and that F preserves it). Let us introduce the necessary terminology to generalize this result.

Definition 14 *An* ω-chain *in* $\mathbf{C}$ *is a functor* $\Delta : \omega \longrightarrow \mathbf{C}$, *i.e. a diagram of the form:*

$$D_0 \xrightarrow{f_0} D_1 \xrightarrow{f_1} \ldots \longrightarrow D_n \xrightarrow{f_n} \ldots$$

Dually an ω^{op}-chain *in* $\mathbf{C}$ *is a functor* $\Delta : \omega^{op} \longrightarrow \mathbf{C}$.

Notation. If $\Delta = (D_n, f_n)_{n\geq 0}$ is an ω-chain and $\mu = (\mu_n : D_n \to A)_{n\geq 0}$ is a cone over A, let us denote by:

- Δ^- the ω-chain $D_1 \xrightarrow{f_1} D_2 \xrightarrow{f_2} \ldots \longrightarrow D_n \xrightarrow{f_n} \ldots$; and
- μ^- the cone $(\mu_n : D_n \to A)_{n\geq 1}$.

Theorem 10 *Let* $\mathbf{C}$ *be a category with an initial object* $\perp_{\mathbf{C}}$ *and let* $F : \mathbf{C} \to \mathbf{C}$ *be a functor. Consider the* ω*-chain* $\Delta = < F^n(\perp_{\mathbf{C}}), F^n(!_{F\perp}) >$ *(where* $!_{F\perp} : \perp \to F\perp$ *is the unique such homomorphism). Suppose that* $\mu : \Delta \to A$ *is a colimiting cone and that* F *preserves it. Then the initial* F*-algebra exists and is* (A, α)*, where* α *is uniquely determined by the universality of the colimit* FA *of* μ^-.

Proof. Since $\perp_{\mathbf{C}}$ is initial in $\mathbf{C}$ and $A = colim_{\mathbf{C}}\Delta$, we also have $A = colim_{\mathbf{C}}\Delta^-$. Hence there is a unique morphism $\alpha : FA \to A$ such that $\mu_{n+1} = \alpha \circ F\mu_n$ for all n. Thus (A, α) is an F-algebra. Let $(A', \alpha' : FA' \to A')$ be any other F-algebra. We want to show that there is a unique morphism f of F-algebra such that

$$\begin{tikzcd} FA \arrow[r, "\alpha"] \arrow[d, "Ff"'] & A \arrow[d, "f"] \\ FA' \arrow[r, "\alpha'"'] & A' \end{tikzcd}$$

is a commutative diagram. Assume it does exist: let us verify the uniqueness of such an f. Consider the cone $\nu : \Delta \to A'$ defined as follows:

- $\nu_0 = !_{A'} : \perp_{\mathbf{C}} \longrightarrow A'$;

- $\nu_{n+1} = \alpha' \circ F\nu_n : F^n(\perp_{\mathbf{C}}) \longrightarrow A'$.

It is indeed a cone: we can show it by induction on n:

- $n = 0$: $\nu_0 = \nu_1 \circ !_{F\perp_{\mathbf{C}}}$ is trivially true;

- suppose that $\nu_n = \nu_{n+1} \circ F^n(!_{F\perp_{\mathbf{C}}})$: using the definition of ν_{n+1}, the induction hypothesis and the definition of ν_{n+2} we get:

$$\begin{aligned}\nu_{n+1} &= \alpha' \circ F\nu_n \\ &= \alpha' \circ F\nu_{n+1} \circ F^{n+1}(!_{F\perp_{\mathbf{C}}}) \\ &= \nu_{n+2} \circ F^{n+1}(!_{F\perp_{\mathbf{C}}}).\end{aligned}$$

Now, let us verify that for any n, $\nu_n = f \circ \mu_n$: then, by the universal property of the colimit construction, f is the unique such morphism:

- for $n = 0$ there is nothing to prove;

- suppose that $\nu_n = f \circ \mu_n$; then using the definition of ν_n, the induction hypothesis and the fact that f is an F-algebra morphism we get:

$$\begin{aligned}\nu_{n+1} &= \alpha' \circ F\nu_n \\ &= \alpha' \circ Ff \circ F\mu_n \\ &= f \circ \alpha \circ F\mu_n \\ &= f \circ \mu_{n+1}.\end{aligned}$$

Now we are left to prove the existence of f as a morphism of F-algebras: take f to be the unique map such that $\nu_n = f \circ \mu_n$ for any n. For any $n \geq 1$ we have:

$$\nu_n = f \circ \mu_n = f \circ \alpha \circ F\mu_{n-1}$$

(because of the definition of f and α) and

$$\begin{aligned}\nu_n &= \alpha' \circ F\nu_{n-1} \\ &= \alpha' \circ F(f \circ \mu_{n-1}) \\ &= \alpha' \circ Ff \circ F\mu_{n-1}.\end{aligned}$$

Hence, by the universal property of $FA = colim F\Delta$, we get $f \circ \alpha = \alpha' \circ Ff$ i.e. we have proved that f is a homomorphism of F-algebra. □

The previous proposition works, in particular, if **C** is a cpo, thought of as a category, and F an ω-continuous map. The generalization of this situation to categories is given in the following.

Definition 15 *A category* **C** *is an* ω-category *if it has an initial object and it has all colimits of* ω*-chains.*
A functor $F : \mathbf{C} \longrightarrow \mathbf{C}'$ *is* ω-continuous *if it preserves* ω*-colimits.*

Corollary 11 *Let* **C** *be an* ω*-category and* $F : \mathbf{C} \longrightarrow \mathbf{C}$ *an* ω*-continuous functor. Then there is an initial fixed point for* F*, given by* (A, α) *in the notations of Theorem 10.*

Remark that a denumerable product of ω-categories is an ω-category; moreover constant and projection functors are ω-continuous and composition and tupling preserve ω-continuity.
Hence to solve an equation as

$$D \equiv F_2^+(\sum_{n\in\mathbf{N}} D)$$

in an ω-category we only need to check the ω-continuity of the separated sum functor Σ and of the strict Plotkin power domain functor F_2^+.

Locally determined colimits

It is sometimes difficult to apply Theorem 10 directly as stated. However, in most of the domain-like categories, the hom-sets have naturally a posetal structure: this leads us to the study of O-categories , where a local notion of limit and colimit can be given (see [Wan79]). It also enables us to restate Theorem 10 in a more ready-to-use form.

Definition 16 *A category* **C** *is an* O-category *if it is a 2-category, whose hom-sets are dcpo's, i.e. if*

- *in every hom-sets any ascending chain has a least upper bound; and*
- *composition of morphisms is ω-continuous with respect to the order of the hom-sets.*

Remark that a product of O-categories is still an O-category. Also, if **C** is an O-category, so is $\mathbf{C}^{op}$: the order on hom-sets is given by $f^{op} \sqsubseteq g^{op}$ if and only if $f \sqsubseteq g$.

Since categories are regarded as the analogue of posets, we want to introduce some sort of ordering.

Definition 17 *Let* **C** *be an O-category (indeed it is enough that the hom-sets of* **C** *are posets). A couple of arrows*

$$A \underset{g}{\overset{f}{\rightleftarrows}} B$$

such that

- $g \circ f = id_A$*; and*
- $f \circ g \leq id_B$

is called a projection pair *from A to B; f is said to be an* embedding *, g a* projection.

We write $A \trianglelefteq B$ if there is a projection pair from A to B.

If $A \underset{g}{\overset{f}{\rightleftarrows}} B$ and $A \underset{g'}{\overset{f'}{\rightleftarrows}} B$ are both projection pairs then one obviously has $f \leq f'$ if and only if $g' \leq g$. Hence one of the morphisms of the projection pair determines the other. If f is an embedding we will write f^R for the corresponding projection; similarly if g is a projection we will write g^L for the corresponding embedding.

Definition 18 *If* $\mathbf{C}$ *is an O-category we can define the subcategory* $\mathbf{C}^E$*: it has the same objects of* $\mathbf{C}$ *and embeddings as morphisms. Similarly we have the subcategory* $\mathbf{C}^P$ *of projections.*

Because of the remark after Definition 16 we have:

$$\mathbf{C}^E \cong (\mathbf{C}^P)^{op}$$

and of course

$$\mathbf{C}^P \cong (\mathbf{C}^E)^{op}.$$

Notice that $\trianglelefteq$ is a preorder on the objects of the category $\mathbf{C}$. Let F be an endofunctor on $\mathbf{C}$. Solutions in $\mathbf{C}$ for the equation

$$FD \cong D \tag{1.3}$$

are fixed points of F. Continuing the analogy between posets and categories, we would like to define a minimal solution of the equation 1.3 to be a fixed point A of F such that for any other fixed point B one has $A \trianglelefteq B$. However this is not the right way to go since, in general, we can have objects A and B such that $A \trianglelefteq B$ and $B \trianglelefteq A$, without A and B being isomorphic: hence we would not achieve the uniqueness (up to isomorphisms) of minimal solutions. On the other hand requiring the uniqueness of an embedding e_B from A to any other fixed point B would lead to equations without any minimal solution. We can rescue the situation by requiring the uniqueness with respect to embeddings e_B, which are also F-algebra homomorphisms. If we denote by F^E the restriction of F to the category $\mathbf{C}^E$ we can restate this in the following way.

Definition 19 *A minimal solution for the equation 1.3 is an initial* F^E*-algebra.*

Before starting the investigation of the relation between $\mathbf{C}$ and $\mathbf{C}^E$, let us introduce the local notion of limit and colimit.

Definition 20 *Let* **C** *be an O-category. Let* $\mu : \Delta \longrightarrow A$ *be a cone in* $\mathbf{C}^E$, *where* Δ *is the* ω*-chain* $< A_n, f_n >$. *We say that* μ *is an* O-colimit *if:*

1. $< \mu_n \circ \mu_n^R >_n$ *is an increasing sequence in* $hom_C(A, A)$*; and*

2. $\bigsqcup_n(\mu_n \circ \mu_n^R) = id_A$.

Dually, if $\nu : A \longrightarrow \Delta$ *is a cone in* $\mathbf{C}^P$, *where* Δ *is the* ω^{op}*-chain* (A_n, f_n), *we say that* ν *is an* O-limit *if:*

1. $< \nu_n^L \circ \nu_n >_n$ *is an increasing sequence in* $hom_C(A, A)$*; and*

2. $\bigsqcup_n(\nu_n^L \circ \nu_n) = id_A$.

Proposition 12 *Let* **C** *be an O-category in which every* $hom(A, B)$ *has a least element* $\perp_{A,B}$*; suppose moreover that for any* f *in* $hom(A, B)$ *we have* $\perp_{B,C} \circ f = \perp_{A,C}$ *(this property is referred to as left-strictness of the composition). Then a terminal object* $\perp$ *in* **C** *is initial in* $\mathbf{C}^E$.

Proof. For every object A in **C** there is an embedding $\perp_{\perp,A}: \perp \longrightarrow A$ defined by $\perp_{\perp,A}^R = \perp_{A,\perp}$; indeed we have:

- $\perp_{A,\perp} \circ \perp_{\perp,A} = 1_\perp$ since $\perp$ is terminal in **C**;

- $\perp_{\perp,A} \circ \perp_{A,\perp} \sqsubseteq \perp_{A,A} \sqsubseteq id_A$.

The uniqueness of an embedding with domain $\perp$ comes from the uniqueness of its projection part as a map with $\perp$ as codomain. □

The next result (taken from [SP82], where an earlier idea found in [Sco72] is generalized) explains the relation between (co)limits and O-(co)limits.

Theorem 13 (The limit-colimit coincidence theorem) *Let* **C** *be an O-category and* $\Delta = (A_n, f_n)$ *an* ω*-chain in* $\mathbf{C}^E$. *Denote by* Δ^R *the* ω^{op}*-chain in* $\mathbf{C}^P$ *defined as* (A_n, f_n^R). *Then the following facts are equivalent:*

1. Δ *has a colimit in* $\mathbf{C}$*;*

2. Δ^R *has a limit in* $\mathbf{C}$*;*

3. Δ *has an O-colimit;*

4. Δ^R *has an O-limit.*

Moreover these facts imply:

5. Δ *has a colimit in* $\mathbf{C}^E$*;*

6. Δ^R *has a limit in* $\mathbf{C}^P$.

Proof. The equivalence of 3. and 4. comes directly from the definition of O-limit and O-colimit. To prove the rest of the theorem we will show:

A. 3. $\Longrightarrow$ 1. and 5.

B. 4. $\Longrightarrow$ 2. and 6.

C. 2. $\Longrightarrow$ 4.

D. 3. $\Longrightarrow$ 1.

A. Suppose $\mu : \Delta \longrightarrow A$ is an O-colimit. Let us prove that the cone μ has the universal property of colimits. Let $\mu' : \Delta \longrightarrow A'$ be any other cone. If there is a $\theta : A \longrightarrow A'$ such that $\mu'_n = \theta \circ \mu_n$ for any n then θ is uniquely determined; indeed we have:

$$\begin{aligned}
\theta &= \theta \circ \textstyle\bigsqcup_n (\mu_n \circ \mu_n^R) && \text{since } id_A \textstyle\bigsqcup_n (\mu_n \circ \mu_n^R) \\
&= \textstyle\bigsqcup_n (\theta \circ (\mu_n \circ \mu_n^R)) && \text{(composition is continuous)} \\
&= \textstyle\bigsqcup_n ((\theta \circ \mu_n) \circ \mu_n^R) && \text{(associativity of the composition)} \\
&= \textstyle\bigsqcup_n \mu'_n \circ \mu_n^R.
\end{aligned}$$

Now to prove the existence of θ, let us define $\theta = \bigsqcup_n \mu'_n \circ \mu_n^R$: this join exists, since $(\mu'_n \circ \mu_n^R)_n$ is increasing; indeed:

$$\begin{aligned}
\mu' \circ \mu_n^R &= (\mu'_{n+1} \circ f_n) \circ (f_n^R \circ \mu_{n+1}^R) \\
&= \mu'_{n+1} \circ (f_n \circ f_n^R) \circ \mu_{n+1}^R \\
&\sqsubseteq \mu'_{n+1} \circ \mu_{n+1}^R.
\end{aligned}$$

Clearly we also have $\theta \circ \mu_n = \mu'_n$, since

$$\begin{aligned}
\theta \circ \mu_n &= (\bigsqcup_m \mu'_m \circ \mu_m^R) \circ \mu_n \\
&= \bigsqcup_{m \geq n} (\mu'_m \circ \mu_m^R \circ \mu_n) \\
&= \bigsqcup_{m > n} (\mu'_m \circ \mu_m^R \circ \mu_m \circ f_{nm}) \\
&= \bigsqcup_{m > n} \mu'_m \circ f_{nm} \\
&= \bigsqcup_{m > n} \mu'_n \\
&= \mu'_n
\end{aligned}$$

where $f_{nm} = f_{m-1} \circ \ldots \circ f_n$ for $m > n$. Now we can also show that the universal property of the colimit holds not only in $\mathbf{C}$, but also in $\mathbf{C}^E$. In other words, we have to show that if $\mu' : \Delta \longrightarrow A'$ is a cone in $\mathbf{C}^E$, then the morphism $\theta = \bigsqcup_n \mu'_n \circ \mu_n^R$ is an embedding. Let us put $\theta^R = \bigsqcup_n \mu_n \circ \mu'^R_n$ (as before, this join exists since $(\mu_n \circ \mu'^R_n)$ is an ω-chain); the next two calculations show that (θ, θ^R) is a projection pair, i.e. that θ is an embedding:

$$\begin{aligned}
\theta^R \circ \theta &= (\bigsqcup_n (\mu_n \circ \mu'^R_n)) \circ (\bigsqcup_n (\mu'_n \circ \mu_n^R)) \\
&= \bigsqcup_n \mu_n \circ \mu'^R_n \circ \mu'_n \circ \mu_n^R \\
&= \bigsqcup_n \mu_n \circ \mu_n^R \\
&= id_A
\end{aligned}$$

$$\theta \circ \theta^R = (\bigsqcup_n (\mu'_n \circ \mu_n^R)) \circ (\bigsqcup_n (\mu_n \circ \mu'^R_n))$$

$$
\begin{aligned}
&= \bigsqcup_n \mu'_n \circ \mu_n^R \circ \mu_n \circ \mu'^R_n \\
&= \bigsqcup_n \mu'_n \circ \mu'^R_n \\
&\sqsubseteq id_{A'}.
\end{aligned}
$$

B. The proof is dual to the one in A.

C. Suppose $\nu : A \longrightarrow \Delta^R$ is a limiting cone in **C**. We want to show that:

- each ν_n is a projection;

- $(\nu_n^L \circ \nu_n)_n$ is increasing and $\bigsqcup_n (\nu_n^L \circ \nu_n) = id_A$.

In order to define ν_n^L, let us consider for any A_m in Δ the cone $\nu^{(m)} : A_m \longrightarrow \Delta^R$ in **C** defined as

$$
\nu_n^{(m)} = \begin{cases} f_{mn} & \text{if } m \leq n \\ f_{nm}^R & \text{if } m > n. \end{cases}
$$

Remark that $\nu_n^{(m)} = f_{nr}^R \circ f_{mr}$ for $r \geq \max(m, n)$, since:

- if $m \leq n$ then

$$
\begin{aligned}
f_{nr}^R \circ f_m r &= f_{nr}^R \circ f_{nr} \circ f_{mn} \\
&= f_{mn} \\
&= \nu_n^{(m)};
\end{aligned}
$$

- if $m > n$ then

$$
\begin{aligned}
f_{nr}^R \circ f_m r &= (f_{mr} \circ f_{nm})^R \circ f_{mr} \\
&= f_{nm}^R \circ f_{mr}^R \circ f_{mr} \\
&= f_{nm}^R \\
&= \nu_n^{(m)}.
\end{aligned}
$$

Now we are ready to show that $\nu^{(m)}$ is indeed a cone:

$$
f_n^R \circ \nu_{n+1}^{(m)} = f_n^R \circ (f_{n+1,r}^R \circ f_{m,r}) \quad \text{for } r \geq \max(m, n+1)
$$

$$\begin{aligned} &= (f_{n+1,r} \circ f_n)^R \circ f_{m,r} \\ &= f_{n,r}^R \circ f_{m,r} \\ &= \nu_n^{(m)}. \end{aligned}$$

Hence for any m there is a morphism $\nu_m^L : A_m \to A$ such that $\nu_n^{(m)} = \nu_n \circ \nu_m^L$. In particular for $n = m$ we get $id_{A_m} = \nu_m^{(m)} = \nu_m \circ \nu_m^L$. To show that ν_m is a projection we still need $\theta_m \circ \nu_m \sqsubseteq id_{A_m}$. Let us start by proving that $\nu_m^L = \nu_{m+1}^L \circ f_m$, which is obtained by showing that $\nu_{m+1}^L \circ f_m$ is the mediating morphism between ν and $\nu^{(m)}$:

$$\begin{aligned} \nu_n \circ (\nu_{m+1}^L \circ f_m) &= \nu_n^{(m+1)} \circ f_m && \text{since } \forall n, m : \nu_n^{(m)} = \nu_n \circ \nu_m^L \\ &= f_{nr}^R \circ f_{m+1,r} \circ f_m && \text{for } r \geq \max(m+1, n) \\ &= f_{nr}^R \circ f_{mr} \\ &= \nu_n^{(m)}. \end{aligned}$$

Now we can see that $(\nu_m^L \circ \nu_m)_m$ is increasing:

$$\begin{aligned} \nu_m^L \circ \nu_m &= \nu_{m+1}^L \circ f_m \circ \nu_m \\ &= \nu_{m+1}^L \circ f_m \circ f_m^R \circ \nu_{m+1} \\ &\sqsubseteq \nu_{m+1}^L \circ \nu_{m+1}. \end{aligned}$$

Hence we can define $\theta = \bigsqcup_m \nu_m^L \circ \nu_m$. Let us show that $\theta = id_A$ (and hence that $\nu_m^L \circ \nu_m \sqsubseteq id_A$); we can prove this by showing that θ is the mediating morphism between $\nu : A \longrightarrow \Delta^R$ and itself:

$$\begin{aligned} \nu_n \circ \theta &= \nu_n \circ \textstyle\bigsqcup_m \nu_m^L \circ \nu_m && \text{by definition of } \theta \\ &= \nu_n \circ \textstyle\bigsqcup_{m \geq n} \nu_m^L \circ \nu_m \\ &= \textstyle\bigsqcup_{m>n} \nu_n \circ \nu_m^L \circ \nu_m && \text{by continuity of the composition} \\ &= \textstyle\bigsqcup_{m>n} \nu_n^{(m)} \circ \nu_m && \text{since } \nu_m \circ \nu_m^L = \nu_n^{(m)} \forall n, m \\ &= \textstyle\bigsqcup_{m>n} f_{nm}^R \circ \nu_m && \text{by definition of } \nu_n^{(m)} \\ &= \nu_n && \text{since } \nu_n \text{is a cone .} \end{aligned}$$

□

The notion of O-colimit becomes particularly useful when not only the implication 3. $\Longrightarrow$ 5. for the previous theorem holds, but also its converse.

Definition 21 *An O-category* $\mathbf{C}$ *is said to have* locally determined ω-colimits of embeddings *when a cone* $\mu : \Delta \longrightarrow A$ *over an* ω*-chain* Δ *in* $\mathbf{C}^E$ *is colimiting in* $\mathbf{C}^E$ *if and only if* μ *is an O-colimit (or, equivalently, if and only if it is a colimit in* $\mathbf{C}$*).*

Corollary 14 *Let* $\mathbf{C}$ *be an O-category. Suppose* $\mathbf{C}$ *has all limits of* ω^{op}*-chains in* $\mathbf{C}^P$ *(or, equivalently, all colimits of* ω*-chains in* $\mathbf{C}^E$*). Then* $\mathbf{C}$ *has locally determined* ω*-colimits of embeddings.*

Proof. Let $\Delta = (A_n, f_n)$ be an ω-chain in $\mathbf{C}^E$ and suppose that $\mu : \Delta \longrightarrow A$ is a colimiting cone in $\mathbf{C}^E$: we want to prove that it is also a colimiting cone in $\mathbf{C}$. If we take in $\mathbf{C}$ the colimit of Δ, we know by Theorem 13 that it is also a colimit in $\mathbf{C}^E$ and therefore it is $\mu : \Delta \longrightarrow A$ (up to isomorphism). □

We conclude this section giving a sufficient condition for a functor $F : \mathbf{C} \longrightarrow \mathbf{C}'$ to be made into an ω-continuous functor $F^E : \mathbf{C}^E \longrightarrow \mathbf{C}'^E$. Though it is not necessary for solving the equation we are concerned with, for sake of completeness, we will include also the case of contravariant functors.
Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be O-categories. Let $F : \mathbf{A}^{op} \times \mathbf{B} \longrightarrow \mathbf{C}$ be a contravariant functor. The case F covariant (contravariant) is included by putting $\mathbf{A}$ ($\mathbf{B}$) equal to the trivial one-object category.

Definition 22 *The functor* F *is said to be* locally monotonic *if it is monotonic on the hom-sets; it is called* locally continuous *if it preserves suprema of* ω*-chains in every hom-set.*

Lemma 15 *Let* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ *be O-categories and suppose* $F : \mathbf{A}^{op} \times \mathbf{B} \longrightarrow \mathbf{C}$ *is locally monotonic. Then we can define a covariant functor*

$$F^E : \mathbf{A}^E \times \mathbf{B}^E \longrightarrow \mathbf{C}^E$$

by putting:

- $F^E(A, B) = F(A, B)$ *for objects* (A, B) *in* $\mathbf{A}^E \times \mathbf{B}^E$;
- $F^E(f, g) = F((f^R)^{op}, g)$ *for morphisms* (f, g) *in* $\mathbf{A}^E \times \mathbf{B}^E$.

Proof. Let us verify that $F^E(f, g)$ is an embedding with $(F^E(f, g))^R = F(f^{op}, g^R)$: indeed, we have:

$$\begin{aligned}
(F^E(f,g))^R \circ F^E(f,g) &= F(f^{op}, g^R) \circ F((f^R)^{op}, g) \\
&= F((f^R \circ f)^{op}, g^R \circ g) \\
&= F(id, id) \\
&= id
\end{aligned}$$

and

$$\begin{aligned}
F^E(f,g) \circ (F^E(f,g))^R &= F((f^R)^{op}, g) \circ F(f^{op}, g^R) \\
&= F(f^{R^{op}}, g) \circ F(f^{op}, g^R) \\
&= F(id, id) \\
&= id;
\end{aligned}$$

- F^E is a functor, since:

$$\begin{aligned}
F^E(id, id) &= F((id^{R^{op}}), id) \\
&= F(id^{op}, id) \\
&= id
\end{aligned}$$

and

$$
\begin{aligned}
F^E(f',g') \circ F^E(f,g) &= F(({f'}^R)^{op},g') \circ F((f^R)^{op},g) \\
&= F(((f' \circ f)^R)^{op}, g' \circ g) \\
&= F^E(f' \circ f, g' \circ g)
\end{aligned}
$$

□

Theorem 16 *With the hypothesis of the previous lemma, if moreover* **A** *and* **B** *have locally determined ω-colimits of embeddings and F is locally continuous, then F^E is ω-continuous.*

Proof. Let us show that F^E is ω-continuous. Consider an ω-chain

$$\Delta = ((A_n, B_n), (f_n, g_n))$$

in $\mathbf{A}^E \times \mathbf{B}^E$ and let $\mu : \Delta \longrightarrow (A, B)$ be a colimiting cone in $\mathbf{A}^E \times \mathbf{B}^E$ where $\mu = (\sigma_n, \tau_n)_n$. Then $\sigma = (\sigma_n)_n : (A_n, f_n) \longrightarrow A$ and $\tau = (\tau_n)_n : (B_n, g_n) \longrightarrow B$ are colimiting cones in $\mathbf{A}^E$ and, respectively, in $\mathbf{B}^E$; since **A** and **B** have locally determined ω-colimits of embeddings, then they are also O-colimits. We will use this to show that

$$F^E(\mu) : F^E(\Delta) \longrightarrow F^E(A, B)$$

is an O-colimit and therefore by Theorem 13 it is a colimit in $\mathbf{C}^E$ (i.e. F is ω-continuous). Remark that

$$
\begin{aligned}
F^E(\mu_n) \circ (F^E(\mu_n))^R &= F((\sigma_n^R)^{op}, \tau_n) \circ F(\sigma_n^{op}, \tau_n^R) \\
&= F((\sigma_n \circ \sigma_n^R)^{op}, \tau_n \circ \tau_n^R)
\end{aligned}
$$

Hence, since $(\sigma_n \circ \sigma_n^R)_n$ and $(\tau_n \circ \tau_n^R)_n$ are increasing, if we suppose that F is locally monotonic, also $(F^E(\mu_n) \circ (F^E(\mu_n))^R)_n$ is increasing. Moreover, by local continuity of F and since σ and μ are O-colimits, we have:

$$\begin{aligned}
\bigvee_n (F^E(\mu_n) \circ (F^E(\mu_n))^R) &= \bigvee_n F((\sigma_n \circ \sigma_n^R), \tau_n \circ \tau_n^R) \\
&= F(\bigvee_n (\sigma_n \circ \sigma_n^R), \bigvee_n (\tau_n \circ \tau_n^R)) \\
&= F(id_A, id_B) \\
&= id_{F(A,B)}.
\end{aligned}$$

□

1.2.2 Solving equations in ω-Dom and in SFP

Let us consider the category **Cpo** of cpo's and ω-continuous functions and its full subcategory ω-**Dom** of ω-algebraic cpo's. In both cases the hom-sets have a natural pointwise order: if f, g are in $hom_C(A, B)$ we say that $f \leq g$ if $f(a) \leq g(a)$ for all a in A. Similarly, any ω-chain $\{f_i\}_{i \in \mathbb{N}}$ of functions in $hom_C(A, B)$ has a least upper bound $\bigvee_i f_i$ defined pointwise as $(\bigvee_i f_i)(x) = \bigvee_i (f_i(x))$. Since composition is continuous with respect to this order **Cpo** and ω-**Dom** are O-categories.

The next two results, together with Corollary 14, show that these categories have locally determined ω-colimits of embeddings.

Proposition 17 *Let $\Delta = (A_n, f_n)$ be an ω^{op}-chain in* **Cpo**: *a limiting cone $\mu : A \longrightarrow \Delta$ can be defined as follows:*

- $A = \{(a_n)_{n \in \omega} : f_n(a_{n+1}) = a_n, a_n \in A_n \forall n\}$;
- $\mu_m : A \longrightarrow A_m$ *defined by* $\mu_m((a_n)_n)m$ *for all* m.

Proof. If we define an ordering on A as

$$(a_n)_n \leq (b_n)_n \quad \text{iff} \quad a_n \leq b_n \text{ for all } n \in \mathbb{N}$$

then $(A, \leq)$ is clearly a cpo with least element $(\bigvee_{n \geq m}(f_m \circ \ldots \circ f_{n-1})(\perp_{A_n}))_m$ (if the f_n's are projections, then they are strict maps and hence the least element is simply $(\perp_{A_m})_m$). Next, suppose $\nu_m : B \longrightarrow A_m$ is a cone. Then if there is a mediating morphism $\theta : B \longrightarrow A$ from ν to μ we have

$$\forall m \in \mathbb{N} \,.\, \forall b \in B \,.\, \mu_m \circ \theta(b) = \nu_m(b).$$

Hence $\theta(b) = (\nu_n(b))_{n<\omega}$. On the other hand we can assume this as definition of θ, since it is a continuous map. $\square$

Proposition 18 *Let $\Delta = (A_n, f_n)$ be an ω^{op}-chain in ω-$\mathbf{Dom}^P$. The construction of the limiting cone $\mu : A \longrightarrow \Delta$ of the previous proposition yields an ω-domain A; its set of compact elements is*

$$Kpt(A) = \bigcup_n \mu_n^L(Kpt(A_n)).$$

Proof. Set $B = \bigcup_n \mu_n^L(Kpt(A_n))$. We divide the proof into two steps.

1. If $f : D \longrightarrow E$ is an embedding and d is a compact element of D then $f(d)$ is compact in E.

2. B is a basis for A.

From 1. it follows that $B \subseteq Kpt(A)$, since the μ_n^L's are embeddings; but from 2. we also have $Kpt(A) \subseteq B$. Hence $B = Kpt(A)$.
Proof of 1. Suppose $f(d) \leq \bigvee S$ where S is a directed subset of E: then, considering the projection f^R associated to f, we get $d = f^R \circ f(d) \leq \bigvee f^R[S]$ and hence $d \leq f^R(s)$ for some s in S since d is compact: then $f(d) \leq f \circ f^R(s) \leq s$ for some s of S, i.e. $f(d)$ is compact.
Proof of 2. Consider x in A: because of Theorem 13 we have that $x = \bigvee_{n\in\omega} \mu_n^L \circ \mu_n(x)$ is an increasing sequence. Since we are supposing the A_n's to be ω-algebraic,

any $\mu_n(x)$ can be written as a directed join of compact elements of A_n, say $\mu_n(x) = \bigvee C_n$ with $C_n \subseteq Kpt(A_n)$ directed. Then $\bigcup_n \mu_n^L[C_n]$ is a directed set and its join is x. □

Another way to interpret the previous proposition is to say that ω-**Dom** has all colimits of ω-chains. Since any $hom(A, B)$ has a bottom element (the constant map $\perp_B$) and composition is left-strict, by Proposition 13, $\{\perp\}$ is a terminal object in ω-**Dom**. Hence this category is an ω-category: Corollary 15 gives a general method for finding a minimal solution to any equation $F(D) \cong D$ defined by an ω-continuous functor F. Moreover if the previous equation is given by F endofunctor on ω-**Dom**, in order to apply the mentioned method, we only need to check that F is locally continuous.
However, ω-**Dom** is not the most suitable category from the point of view of denotational semantics, since it is not cartesian closed (see [SLG, example 3.3.10, page 68] for a counter-example). We could restrict our attention to the full subcategory of ω-**Dom**, whose objects are consistently complete ω-domains.

Definition 23 *An ω-domain D is* consistently complete *if whenever two finite elements a, b of D have a common upper-bound they have a least common upper-bound $a \vee b$.*

This category is closed under the constructions of separated and smash sum, lifting and it is cartesian closed (see [SLG, pages 63-70]). Unfortunately things do not go so well when considering power domains.

Proposition 19 *If D is a consistently complete ω-domain so are $F_0(D)$ and $F_1(D)$.*

Proof. We only need to show that any two compact elements A, B which have an upper bound have a least one. Indeed for $i = 0$ the least upper bound can be defined as the equivalence class represented by $\{a \vee b : a \in A, b \in B$ and $\{a, b\}$ has an upper bound in D$\}$ and for $i = 1$ simply the class individuated by $A \cup B$. □

For the Plotkin case, however, this fails to be true (see [SLG, page 295] for a counter example). Fortunately the situation can be rescued by considering **SFP**, the largest cartesian closed full subcategory of ω-**Dom** (see [Smy83] for the proof of cartesian closedness). The category $\mathbf{SFP}^E$ is an example of an algebroidal category, a generalization of the concept of algebraic cpo (see [Smy78]).

Definition 24 *Let* **C** *be a category. An object* A *of* **C** *is* finite *in* **C** *if:*

- *for any* ω*- chain* $\Delta = (V_n, f_n)$ *in* **C** *with colimit* $\mu : \Delta \longrightarrow V$,
- *for any morphism* $v : A \longrightarrow V$, *and*
- *for any sufficiently large* n

there is a unique morphism $u_n : A \longrightarrow V_n$ *such that* $v = \mu_n \circ u_n$.

We will denote by $\mathbf{C_0}$ the full subcategory of **C** with objects the finite objects of **C**. Remark that if **C** is a poset thought of as a category, then the finite objects are exactly the finite elements.

Definition 25 *A category* **C** *is* algebroidal *if the following axioms hold:*

1. **C** *has an initial object and at most countably many finite objects;*
2. *every* ω*-chain of finite objects has a colimit in* **C***;*
3. *every object of* **C** *is a colimit of an* ω*-chain of finite objects.*

If **C** is a poset, 1. and 2. say that **C** is a cpo with at most countably many finite elements and 3. asserts that **C** is algebraic.

Theorem 20 *Every algebroidal category has all ω-colimits.*

Proof. Let $\mathbf{C}$ be an algebroidal category. Consider an ω-chain $\Delta(A_m, p_m)$ in $\mathbf{C}$: any A_m arises as a colimit $(i_m^n)_n : \Delta \longrightarrow A_m$ in $\mathbf{C}$, where $\Delta_m = (A_m^n, p_m^n)$ is an ω-chain in $\mathbf{C_0}$. We will define an ω-chain $\Gamma = (A_r^{s(r)}, q_r^{s(r+1)})$ in $\mathbf{C_0}$ so that there is an isomorphism between the category of cones from Γ and the one of cones from Δ; since under this isomorphism corresponding cones have the same vertex, the isomorphism also preserves cones: hence Δ has a colimit and it can be completed by calculating a colimit of Γ. For $r = 0$, put $s(0) = 0$: so $A_0^{s(0)} = A_0^0$. Now suppose that $A_0^{s(0)}, \ldots, A_r^{s(r)}$ have been defined so that $s(0) < s(1) < \ldots < s(r)$. For each $m = 0, \ldots, r$ let $q_m : A_m^{s(r)} \longrightarrow A_{r+1}$ be

$$q_m = p_r \circ \ldots \circ p_m \circ i_m^{s(r)}.$$

Since each $A_m^{s(r)}$ is finite, for sufficiently large t there are unique arrows $q_m^t : A_m^{s(r)} \longrightarrow A_{r+1}^t$ such that

$$q_m = i_{r+1}^t \circ q_m^t.$$

Call t_0 the least t such that the q_m^t's exist for all $m = 0, \ldots, r$. Put $s(r+1) = \max\{t_0, s(r)+1\}$. So we have defined $\Gamma = (A_r^{s(r)}, q_r^{s(r+1)})$. Observe that from the two previous equations we have in particular

$$p_r \circ i_r^{s(r)} = i_{r+1}^{s(r+1)} \circ q_r^{s(r+1)}. \tag{1.4}$$

Now let us set up the correspondence between the cones. If $(\mu_r)_r$ is a cone from $\Delta = (A_r, p_r)$ to an object X of $\mathbf{C}$, then putting $\nu_r = \mu_r \circ i_r^{s(r)}$ we get a cone $(\nu_r) : \Gamma = (A_r^{s(r)}, q_r^{s(r+1)}) \longrightarrow X$; indeed:

$$\begin{aligned}
\nu_r &= \mu_r \circ i_r^{s(r)} \\
&= \mu_{r+1} \circ p_r \circ i_r^{s(r)} && \text{since } (\mu_r)_r \text{ is a cone} \\
&= \mu_{r+1} \circ i_{r+1}^{s(r+1)} \circ q_r^{s(r+1)} && \text{because of equation 1.4} \\
&= \nu_{r+1} \circ q_r^{s(r+1)}.
\end{aligned}$$

Conversely, let $(\nu_r)_r : \Gamma \longrightarrow X$ be a cone. For every m we can define a cone $(\nu_m^n)_n : \Delta_m = (A_m^n, p_m^n) \longrightarrow X$. Put

$$\nu_m^n = \nu_{r+1} \circ q_m^{s(r+1)} \circ p_m^{s(r)-1} \circ \ldots \circ p_m^n$$

where r is such that $s(r) > n$ and $r \geq m$: the choice of r doesn't matter, because of the finiteness of the A_m^n's. Moreover if $n = s(m)$ we can also define

$$\begin{aligned} \nu_m^{s(m)} &= \nu_{m+1} \circ q_m^{s(m+1)} \\ &= \nu_m. \end{aligned} \tag{1.5}$$

Clearly each $(\nu_m^n)_n$ is a cone and therefore by the universal property of the colimiting cone $(i_m^n)_n : \Delta_m \longrightarrow A_m$ for every m there is exactly a $\mu_m : A_m \longrightarrow X$ such that

$$\nu_m^n = \mu_m \circ i_m^n \tag{1.6}$$

for all n. The collection of all the μ_m's forms a cone from Δ to X, i.e. for all m we have $\mu_m = \mu_{m+1} \circ p_m$; indeed, for some r such that $s(r) > n$, applying equations 1.5 and 1.6 and recalling that $(i_m^n) : \Delta_m \longrightarrow A_0$ is a cone, we have:

$$\begin{aligned} \nu_m^n &= \nu_{r+1} \circ q_m^{s(r+1)} \circ p_m^{s(r)-1} \circ \ldots \circ p_m^n \\ &= \nu_{r+1}^{s(r+1)} \circ q_m^{s(r+1)} \circ p_m^{s(r)-1} \circ \ldots \circ p_m^n \\ &= \mu_{r+1} \circ i_{r+1}^{s(r+1)} \circ q_m^{s(r+1) \circ p_m^{s(r)-1}} \circ \ldots \circ p_m^n \\ &= \mu_{r+1} \circ p_r \circ \ldots \circ p_m \circ i_m^{s(r)} \circ p_m^{s(r)-1} \circ \ldots \circ p_m^n \\ &= \mu_{r+1} \circ p_r \circ \ldots \circ p_m \circ i_m^n. \end{aligned}$$

Then by equation 1.6 we get $\mu_m = \mu_{r+1} \circ p_r \circ \ldots \circ p_m$. The latter arguments works also for μ_{m+1}, yielding $\mu_{m+1} = \mu_{r+1} \circ p_r \circ \ldots \circ p_{m+1}$. Hence we get $\mu_m = \mu_{m+1} \circ p_m$. Now the only thing that is left to show is that this correspondence is a bijection: if we start with $(\nu_r)_r : \Gamma \longrightarrow X$, then we get $(\mu_m) : \Delta \longrightarrow X$ and then the cone

$(\mu_m \circ i_m^{s(m)})_m : \Gamma \longrightarrow X$ is the $(\nu_r)_r$ we started with since

$$\begin{aligned} \mu_m \circ i_m^{s(m)} &= \nu_m^{s(m)} && \text{by } 1.6 \\ &= \nu_m && \text{by } 1.5; \end{aligned}$$

on the other hand if we start with $(\mu_m)_m : \Delta \longrightarrow X$ and consider $(\nu_r)_r : \Gamma \longrightarrow X$ where $\nu_r = \mu_r \circ i_r^{s(r)}$, then we can define $\nu_m^n = \nu_{r+1} \circ q_m^{s(r+1)} \circ p_m^{s(r)-1} \circ \ldots \circ p_m^n$ (with $r \geq m, s(r) > n$): remarking that

$$\begin{aligned} \nu_m^n &= \mu_{r+1} \circ i_{r+1}^{s(r+1)} \circ q_m^{s(r+1)} \circ p_m^{s(r)-1} \circ \ldots \circ p_m^n \\ &= \mu_{r+1} \circ p_r \circ \ldots \circ p_m \circ i_m^{s(r)} \circ p_m^{s(r)-1} \circ \ldots \circ p_m^n \\ &= \mu_m \circ i_m^{s(r)} \circ p_m^{s(r)-1} \circ \ldots \circ p_m^n \end{aligned}$$

and confronting it with 1.6 we see that the cone from Δ to X associated to $(\nu_r)_r$ is $(\mu_m)_m$, i.e. the one we started with. □

Let **FPO** be the full subcategory of ω-**Dom**, whose objects are finite posets.

Definition 26 *An* SFP-domain *is a colimit in* ω-**Dom** *of an* ω*-chain in* $\mathbf{FPO}^E$.

Let us denote by **SFP** the full subcategory of ω-**Dom**, whose objects are SFP-domains. From Proposition 17 we know, in particular, that every ω^{op}-chain in $\mathbf{Cpo}^P$ has a limit: hence any ω-chain in $\mathbf{Cpo}^E$ (and therefore in $\mathbf{Fpo}^E$) has a colimit.
We want to show now that $\mathbf{SFP}^E$ is an algebroidal category and its subcategory of finite objects is $\mathbf{Fpo}^E$. Indeed the only piece of information we still need is given by the following proposition.

Proposition 21 *The finite objects of* $\mathbf{SFP}^E$ *are the finite posets.*

Proof. Let us show that:

1. any finite poset is a finite object;

2. any finite object is a finite poset.

1. Let B be a finite poset; consider in $\mathbf{SFP}^E$ a colimiting cone $(\mu_n)_n : (A_n, p_n) \longrightarrow A$ and an embedding $f : B \longrightarrow A$: if there is an embedding $u_n : B \longrightarrow A_n$ such that $\mu_n \circ u_n = f$, then we have that $u_n = \mu_n^R \circ f$. Hence uniqueness is proved. To show the existence, remark that there is a t in $\mathbb{N}$ such that the restriction of $\mu_t \circ \mu_t^R$ to $f[B]$ is the identity (since every element of B is compact and embeddings preserve compactness). Now for any $n \geq t$ we can define $u_n = \mu_n^R \circ f$; it is an embedding with projection $u_n^R = f^R \circ \mu_n$, since:

$$\begin{aligned} u_n^R \circ u_n &= f^R \circ \mu_n \circ \mu_n^R \circ f \\ &= f^R \circ f \\ &= id \end{aligned}$$

and

$$\begin{aligned} u_n \circ u_n^R &= \mu_n^R \circ f \circ f^R \circ \mu_n \\ &= \mu_n^R \circ \mu_n \\ &= id. \end{aligned}$$

Since $\mu_n \circ u_n = f$, existence is proved as well. Hence B is a finite object.
2. Suppose now that B is a finite object: in particular B is an object of $\mathbf{SFP}^E$ and therefore is a colimit of an ω-chain (B_n, p_n) in $\mathbf{FPO}^E$. Hence there is an embedding $u_n : B \longrightarrow B_n$ (for n sufficiently large) such that $\mu_n \circ u_n = id$: this means that $u_n = \mu_n^R$; since $\mu_n^R \circ \mu_n = id_B$ we also have $u_n \mu_n = id_{B_n}$ and therefore $B \cong B_n$, i.e. B is a finite poset. □

Therefore $\mathbf{SFP}^E$ is an algebroidal category: by Theorem 20 it has all ω-colimits and, by Corollary 14, they are locally determined. Moreover $\{\bot\}$ is initial in $\mathbf{SFP}^E$. Then, by Theorem 10, we are able to solve any equation involving ω-continuous functors.

Let us show now that the category **SFP** is closed under the construction of the power domains ([Smy78]). Let us start by defining power domain functors:

$$F_i : \mathbf{Dom} \longrightarrow \mathbf{Dom} \qquad (i = 0, 1, 2)$$

as follows:

1. if D is a domain, put $F_i D = (Idl(\mathcal{M}(D), \sqsubseteq_i), \subseteq)$ (cf. Definition 11);
2. if $f : D \longrightarrow E$ is a continuous map let $F_i f$ be the extension by continuity of the following map defined on $Kpt(F_i D) = \{\downarrow u : u \in \mathcal{M}(D)\}$ as

$$F_i f(\downarrow u) = \{v \in \mathcal{M}(E) : v \sqsubseteq_i f[u]\}.$$

Proposition 22 *The functors* F_i^E : $\mathbf{Dom}^E \longrightarrow \mathbf{Dom}^E$ *are* ω*-continuous for* $i = 0, 1, 2$.

Proof. By Theorem 16 it is enough to check that the functors $F_i : \mathbf{Dom} \longrightarrow \mathbf{Dom}$ are locally continuous. Consider an increasing sequence $(f_n)_{n \in \mathbf{N}} : D \longrightarrow E$ of continuous functions with least upper bound the map f. We want to show that the least upper bound of $(F_i f_n)_{n \in \mathbf{N}}$ is $F_i f$; indeed for any $u \in \mathcal{M}(D)$ one has:

$$\begin{aligned} (\bigvee_n F_i f_n)(\downarrow u) &= \bigcup_n (F_i f_n(\downarrow u)) \\ &= \bigcup_n \{v \in \mathcal{M}(E) : v \sqsubseteq_i f_n[u]\} \\ &= \{v \in \mathcal{M}(E) : v \sqsubseteq_i f[u]\} \\ &= (F_i f)(\downarrow u) \end{aligned}$$

and hence $\bigvee_n F_i f_n = F_i f$. □

The ω-continuity of F_i on $\mathbf{Dom}^E$ implies in particular that F_i preserves the colimits of ω-chains in $\mathbf{Fpo}^E$ and hence we get that **SFP** is closed under the power domain constructions.

In the same way we can verify that **SFP** is a category closed under the usual operations and that they induce ω-continuous functors on $\mathbf{SFP}^E$: therefore the methods for solving equations explained before can be used.
Let us see for example that the separated sum give rise to an ω-continuous functor, referring the reader to [SLG, cc. 4, 11] for the other cases.
Define the functor

$$\sum : \prod_{n \in \mathbf{N}} \mathbf{Dom} \longrightarrow \mathbf{Dom} \tag{1.7}$$

in the following way:
- if $(D_n)_{n\in\mathbf{N}}$ are domains let $\sum_{n\in\mathbf{N}} D_n$ be the domain obtained by taking the disjoint union of the D_n's and adjoining a bottom element, i.e.

$$\sum_{n\in\mathbf{N}} D_n = (\bigcup_n \{< n, d >: d \in D_n\}) \bigcup \{\perp\}$$

and ordering it as follows:

- $\perp$ is the bottom element;
- $< n, d > \leq < m, d' >$ iff $n = m$ and $d \leq d'$.

- if $(f_n : D_n \longrightarrow E_n)_{n\in\mathbf{N}}$ are continuous maps, let

$$(\sum_n f_n) : \sum_n D_n \longrightarrow \sum_n E_n$$

be the map defined by:

- $(\sum_n f_n)(\perp) = \perp$;
- $(\sum_n f_n)(< n, d >) = < n, f_n(d) >$.

$\sum_n (f_n)$ is clearly a continuous map and $\sum$ a well defined functor; it is also immediate to see that it is locally continuous and therefore its restriction

$$\sum : \prod_n \mathbf{Dom}^E \longrightarrow \mathbf{Dom}^E$$

is ω-continuous. Therefore **SFP** is closed under this construction and the functor $\sum$ is ω-continuous on $\mathbf{SFP}^E$.

1.2.3 An application to the study of bisimulations

In this section, after recalling the notion of bisimulation (see [Par81, Mil83, HM85]) we define a domain D of synchronization trees (cf. Definition 29 in the following) by means of a domain equation. Hence we define a transition system $\mathcal{D}$ (cf. Definition 27), whose set of processes is D. We show that the maximal partial bisimulation $\sqsubseteq_B$ coincides with the order $\sqsubseteq$ on D. This can be used to define a logic that characterizes bisimulations (see [Abr91]).
Let us start by recalling the main concepts.

Definition 27 *A* transition system *is a 4-tuple*

$$(Proc, Act, \rightarrow, \uparrow)$$

where

- *Proc is a set of processes;*
- *Act is a set of actions;*
- $\rightarrow \subseteq Proc \times Act \times Proc$ *(notation* $p \xrightarrow{a} q$*);*
- $\uparrow \subseteq Proc$ *(notation* $p\uparrow$*).*

Moreover we write $p\downarrow$ *meaning* $\neg(p\uparrow)$.

We think of $p \xrightarrow{a} q$ as "p has the capability to do a and become q", $p\uparrow$ as "p may diverge" and $p\downarrow$ as "p definitely converges".

Definition 28 *A relation* $R \subseteq Proc \times Proc$ *is a* partial bisimulation *if for all* p, $q \in Proc$*: if* pRq *then* $\forall a \in Act$ *one has:*

- $p \xrightarrow{a} p' \implies \exists q'.\, q \xrightarrow{a} q'$ *and* $p'Rq'$*;*
- $p\downarrow \implies q\downarrow$ *and* $(\exists q'.\, q \xrightarrow{a} q' \implies \exists p'.\, p \xrightarrow{a} p'$ *and* $p'Rq')$.

If we consider the union of all the partial bisimulations, we obtain a relation $\sqsubseteq_B$ which is a partial bisimulation as well; so we have

$$p \sqsubseteq_B q \quad \text{iff} \quad \exists R \,.\, R \text{ partial bisimulation} \,.\, pRq.$$

There is another possibility to define $\sqsubseteq_B$, using ordinal recursion:

- $\sqsubseteq_0 = Proc \times Proc$;
- $p \sqsubseteq_{\alpha+1} q$ iff for all action A one has
  - $p \xrightarrow{a} p' \Longrightarrow (\exists q' \,.\, q \xrightarrow{a} q'$ and $p' \sqsubseteq_\alpha q')$;
  - $p \downarrow \Longrightarrow q \downarrow$ and $(q \xrightarrow{a} q' \Longrightarrow \exists p' \,.\, p \xrightarrow{a} p'$ and $p' \sqsubseteq_\alpha q')$;
- for limit ordinal λ: $\sqsubseteq_\lambda = \bigcap_{\alpha<\lambda} \sqsubseteq_\alpha$.

The sequence just defined is decreasing and bounded below by any partial bisimulation. Then it is eventually stationary, i.e. for some λ and for all $\alpha \geq \lambda$ one has $\sqsubseteq_\alpha = \sqsubseteq_\lambda$: for the least such ordinal, $\sqsubseteq_\lambda$ is a partial bisimulation; since it is the biggest one, it is exactly $\sqsubseteq_B$.

Definition 29 *Let Act be a countable set of actions. We call* domain of synchronization tree over Act *the initial solution in* **SFP** *of the domain equation*

$$D \cong F_2^+\Big(\sum_{a \in Act} D_a\Big)$$

where $D_a = D$ for all $a \in Act$ (F_2^+ being the strict Plotkin power domain, cf. Definition 8, and Σ the separated sum functor, defined by 1.7).

Now for such a domain D, consider the transition system $\mathcal{D} = (D, Act, \rightarrow, \uparrow)$ where

- $d \xrightarrow{a} d'$ iff $d \sqsubseteq d'$;

- $d\uparrow$ iff $\perp \in d$.

Proposition 23 *Let D be the domain of synchronization trees over Act. Then for any d_1, d_2 in D we have:*

$$d_1 \sqsubseteq_B d_2 \quad \textit{iff} \quad d_1 \sqsubseteq d_2.$$

Before going into the proof of the proposition, let us remark that if we want to apply the general theory for solving equations as previously exposed, we should prove that F_2^+ is an ω-continuous functor. But notice that $F_2^+(D) \cong (1)_\perp \oplus F_2[D]$ where 1 is a one-element domain, $(-)_\perp$ the lifting and $\oplus$ the amalgamated sum. Since the amalgamated sum is a locally continuous functor on the category of domains and continuous strict maps, so is F_2^+: then F_2^+ is ω-continuous on $\mathbf{SFP}^E$. In the proof we will need a characterization of the Plotkin power domain of an SFP-domain (see [Smy78]). Instead of working with equivalence classes of

$$F_2D = (\mathcal{F}(D)/\equiv_2, \sqsubseteq_2 /\equiv_2)$$

we can choose canonically a representative element for each class. Let D be an SFP-domain and let $\Delta =< D_n, p_n >$ be an ω-chain in $\mathbf{Fpo}^E$ such that there is a colimiting cone $(\mu_n)_n : \Delta \longrightarrow D$. For any $X \subseteq D$ define

$$X^+ = \{x \in D : \mu_n^R(x) \in \mu_n^R[X]\}.$$

Clearly $(-)^+$ is a closure operator on the power set of D (actually, it is the closure operator relative to the Lawson topology of D, see [Smy78, Appendix] for details). Moreover $X^+ \in \mathcal{F}(D)$, even if X is not since $X^+ = \bigvee_n \mu_n \circ \mu_n^R[X]$ (for all n, $\mu_n \circ \mu_n^R[X]$ is a finite set of compact elements, because each D_n is a finite poset).

Lemma 24 *If X and Y are in $\mathcal{F}(D)$ then*

1. $X \equiv_2 X^+$;

2. $X \sqsubseteq_2 Y$ *iff* $X^+ \sqsubseteq Y^+$.

Proof. 2. follows immediately from 1. Let us verify 1. Remark that if $f : E \longrightarrow E'$ is an embedding, then for $X \subseteq E$ and $X' \subseteq E'$ one has $f[X] \sqsubseteq_2 X'$ iff $X \sqsubseteq_2 f^R[X']$. Let $A \in \mathcal{M}(D)$: we want to prove that $A \sqsubseteq_2 X$ iff $A \sqsubseteq_2 X^+$. Take n in $\mathbb{N}$ large enough so that $\mu_n \circ \mu_n^R[A] = A$. Then we have:

$$\begin{aligned} \mu_n \circ \mu_n^R[A] = A \sqsubseteq_2 X &\iff \mu_n^R[A] \sqsubseteq_2 \mu_n[X] \\ &\iff \mu_n^R[A] \sqsubseteq_2 \mu_n[X^+] \\ &\iff A \sqsubseteq_2 X^+. \end{aligned}$$

□

Consider now the convex closure of X^+, denoted by X^* (recall that the convex closure of a subset $Y \subseteq D$ is defined as $Con(Y) = \{d \in D : \exists y_1, y_2 \in Y . y_1 \leq d \leq y_2\}$). Again, X^* is in $\mathcal{F}(D)$ since $X^* = \bigvee_n \mu_n[Con(\mu_n^R(X))]$.

Theorem 25 *Let D be an SFP-domain. Then there is an isomorphism:*

$$F_2 D \cong (\{X \subseteq D : \emptyset \neq X = X^*\}, \sqsubseteq_2).$$

Proof. Since $X \cong_2 Con(X)$ and $X \cong_2 Y$ iff $Con(X) = Con(Y)$ we have that $X \cong_2 X^*$ and $X \cong_2 Y$ iff $X^* = Y^*$: then for any class $[X]_{\equiv_2}$ in $\mathcal{F}(D)/\equiv_2$ we can choose as representant X^*. But from $X \cong_2 X^*$ we also get $X \sqsubseteq_2 Y$ iff $X^* \sqsubseteq_2 Y^*$ and therefore the ordering is respected. □

Now we can give the proof of Proposition 23.

Proof. Let $F : \mathbf{SFP}^E \longrightarrow \mathbf{SFP}^E$ be the functor defined by

$$F(D) = F_2^+(\sum_{a \in Act} D_a)$$

with $D_a = D$ for all $a \in Act$: it is an ω-continuous functor. Hence by the theory previously exposed the initial solution in $\mathbf{SFP}^E$ of the equation $D \cong F(D)$ can be calculated as the colimit

$$\begin{tikzcd}
\{\perp\} \arrow[r, "p_0"] \arrow[drr, "\mu_0"'] & F(\{\perp\}) \arrow[r, "p_1"] \arrow[dr, "\mu_1"] & \cdots \arrow[r] & F^k(\{\perp\}) \arrow[r, "p_k"] \arrow[dl, dashed, "\mu_k"'] & \cdots \\
 & & D & &
\end{tikzcd}$$

where

- $p_0 = !_{F(\{\perp\})}$, $p_{k+1} = F(p_k)$;
- $\mu_0 = !_D$, $\mu_{k+1} = F\mu_k$

for all $k \in \mathbb{N}$. Recall that the colimit is in $\mathbf{SFP}^E$: the μ_k's are embedding and therefore there are corresponding projections $\mu_k^R : D \longrightarrow F^n(\{\perp\})$. Then we can define the continuous maps $\pi_k = \mu_k \circ \mu_k^R$. Since the colimit is an O-colimit then we also have that:

(a) $\{\pi_k\}_k$ is an increasing sequence with join id_D;

(b) $\forall d_1, d_2 \in D : d_1 \leq d_2 \quad \text{iff} \quad \forall k \,.\, \pi_k d_1 \leq \pi_k d_2$

Now we shall prove:

1. $\forall k \,.\, d \sqsubseteq_k e \quad \Longrightarrow \quad \pi_k d \leq \pi_k e$;
2. $\leq \subseteq \sqsubseteq_B$.

In the following we will use the fact that $D \cong F(D)$; actually, for notational convenience, we will treat the isomorphism as an equality.

Because of the characterization of $\sqsubseteq_B$ via ordinals and of 1. we will have:

$$\sqsubseteq_B \subseteq \sqsubseteq_\omega \subseteq \sqsubseteq_k \quad \text{for any } k.$$

Hence, using (b), we get $\sqsubseteq_B \subseteq \leq$. Therefore, once proved 1. and 2. we will have shown that $\sqsubseteq_B = \leq$.

1. Let us show it by induction on k. For $k = 0$ there is nothing to prove since π_0 is the constant map $\perp$. Suppose (induction hypothesis) that if $d \sqsubseteq_k e$ then $\pi_k d \leq \pi_k e$ for any d and e. Assume $d \sqsubseteq_{k+1} e$, i.e. assume that for any action a:

- $< a, d' > \in d \quad \Longrightarrow \quad \exists < a, e' > \in e$ and $d' \sqsubseteq_k e'$;
- $\perp \notin d \quad \Longrightarrow \quad \perp \notin e$ and $(< a, e' > \in e \Rightarrow \exists d'. < a, d' > \in d$ and $d' \sqsubseteq_k e')$.

If $d = \emptyset$ then also $e = \emptyset$ and hence $\pi_k d = \pi_k e$. If $d = \perp_{FD}$ then $d \leq e$ and hence $\pi_k d = \pi_k e$. Suppose now $d \neq \emptyset \neq e$ and $d \neq \perp_{FD}$. Then

- $\pi_{k+1} d = (F\pi_k)(d) = X^*$ where

$$X = \{< a, \pi_k d' >:< a, d' > \in d\}(\cup\{\perp\})$$

(take the union with $\{\perp\}$ only if $\perp \in d$);

- $\pi_{k+1} e = (F\pi_k)(e) = Y^*$ where

$$Y = \{< a, \pi_k e' >:< a, e' > \in e\}(\cup\{\perp\}).$$

We want to show that $\pi_{k+1} d \leq \pi_{k+1} e$, i.e. that $X^* \sqsubseteq_2 Y^*$: then it is enough to show that $X \sqsubseteq_2 Y$. One has $X \sqsubseteq_1 Y$ since

$$\begin{array}{rlll} & < a, \pi_k d' > \in X & \text{i.e.} & < a, d' > \in d \\ \Longrightarrow & \exists e'. < a, e' > \in e & \text{and} & d' \sqsubseteq_k e' \\ \Longrightarrow & \exists < a, \pi_k e' > \in Y & \text{and} & \pi_k d' \leq \pi_k e' \end{array}$$

using in the last step the induction hypothesis and the definition of Y. To show that $X \sqsubseteq_0 Y$ is easier; indeed if $\perp \in X$ there is nothing to prove; if $\perp \notin X$ then $\perp \notin d$ implies that $\perp \notin e$ and

$$\forall < a, \pi_k e' > \in Y . \exists < a, \pi_k d' > \in X . < a, \pi_k d' > \leq < a, \pi_k e' > .$$

2. It is enough to show that $\leq$ is a partial bisimulation. The defining axiom holds: if $d \leq e$ then

- $d \sqsubseteq_1 c$, in particular $\forall < a, d' > \in d \,.\, \exists < a, c' > \in c \,.\, d' \leq c'$;
- $d \sqsubseteq_0 c$, in particular $\perp \notin d \implies \perp \notin c$ and $(\forall < a, c' > \in c \,.\, \exists < a, d' > \in d \,.\, d' \leq c')$

and hence for all actions a we have

- $d \xrightarrow{a} d' \implies \exists c' \,.\, c \xrightarrow{a} c' \,.\, d' \leq c'$;
- $d \downarrow \implies c \downarrow$ and $(c \xrightarrow{a} c' \implies \exists d' \,.\, d \xrightarrow{a} d' \,.\, d' \leq c')$. □

The domain of synchronization trees $\mathcal{D}$ that we have presented gives rise to a logic that is equivalent to the Hennessy-Milner logic in the infinitary case; hence it characterizes bisimulations (see [Abr91, sections 4,5]).

1.3 A modal interpretation of power domains

In this section we introduce small modal languages L_i (for $i = 0, 1, 2$) whose basic propositions are the compact elements of an ω-domain D. From their interpretation via Kripke forcing we can define relations on the collection $\mathcal{T}$ of generating trees over D and derive an alternative definition of power domains ([Win85]).

1.3.1 The Smyth power domain and the modal operator of inevitability

Let us think of the generic element of the Smyth power domain $\mathcal{F}_0(D)$ as the equivalence class determined by T^ζ_ω, where (T, ζ) is a generating tree over D. If A is as finite set of compact elements of D, by definition we have that $A \sqsubseteq_0 T^\zeta_\omega$ holds if

$$\forall x \in T^\zeta_\omega \,.\, \exists a \in A \,.\, a \leq x. \tag{1.8}$$

Since the elements of T^{ζ}_{ω} correspond to limits of the elements of $Kpt(D)$ labeling the branches of the tree T, 1.8 can be rephrased as:

$$\forall \gamma \text{ branch of } T \,.\, \exists a \in A \,.\, a \leq \bigvee_{t \in \gamma} \zeta(t). \tag{1.9}$$

Given the compactness of the elements of A, 1.9 can be expressed as:

$$\forall \gamma \text{ branch of } T \,.\, \exists t \in \gamma \,.\, \exists a \in A \,.\, a \leq \zeta(t). \tag{1.10}$$

This leads us to consider a language L_0 built out of $Kpt(D)$ using finite disjunctions (to cope with the existential quantification of the elements of A) and a modal operator of inevitability (to cope with the universal quantifier). Remark that the expressions $s \vee s'$ and $\Box s$ in the following definition are just formal combinations; in particular if $s, s' \in Kpt(D)$ then $s \vee s'$ is not the supremum of s and s' as elements of D.

Definition 30 *Let L_0 be the least set such that:*

- *$Kpt(D) \subseteq L_0$; and*
- *if $s, s' \in L_0$, then $s \vee s' \in L_0$; and*
- *if $s \in L_0$, then $\Box s \in L_0$.*

Next we interpret this language via Kripke forcing.

Let (T, ζ) be a generating tree over D (in the following we will write just T for short). Because of Proposition 1 we can suppose with no loss of generality that $\zeta(t)$ is a compact element of D for any t in T.

We will write $t \rightarrow t'$ meaning that there is an arc in T connecting the node t to t'.

Definition 31 *Let $\models_T$ be the least relation on $T \times L_0$ such that:*

1. *if* $a \in Kpt(D)$ *and* $a \leq \zeta(t)$ *then* $t \models_T a$*; and*

2. *if* $t \models_T s$ *or* $t \models_T s'$ *then* $t \models_T s \vee s'$*; and*

3. *if for all branch* γ *out of* t *there is* $t' \in \gamma$ *such that* $t' \models_T s$ *then* $t \models_T \Box s$.

Given the minimality of $\models_T$, the previous conditions are indeed if-and-only-if conditions. A neater definition of $\models_T$ can be obtained equivalently by replacing 3 with 3′

3′. if $t \models_T s$ or $(\forall t'. t \to t' \implies t' \models_T s)$ then $t \models_T \Box s$(to be read as t entails inevitably s).

We can now consider the statements of a non-deterministic computation (T, ζ) that are inevitably true and use this to order the collection $\mathcal{T}$ of generating trees over D.

Definition 32 *For* T, T' *in* $\mathcal{T}$ *define*

- $V_0(T) = \{\Box s \in L_0 : root(T) \models_T \Box s\}$;
- $T \preceq_0 T'$ *iff* $V_0(T) \subseteq V_0(T')$;
- $T \cong_0 T'$ *iff* $T \preceq_0 T'$ *and* $T' \preceq_0 T$.

Hence we can think of $\mathcal{T}/\cong_0$ as a poset: we will show that it is isomorphic to the Smyth power domain.
We need to define on L_0 an equivalence; for s, s' in L_0 say that $s \equiv s'$ if

$$\forall T \in \mathcal{T}. (\mathrm{root}(T) \models_T s \text{ iff } \mathrm{root}(T) \models_T s').$$

Then it is easy to see that this equivalence turns the formal operator $\vee$ of L_0 into a join in $L_0/\equiv$, which is indeed a semilattice. Moreover in $L_0/\equiv$ we have:

- $\Box(\Box s) \equiv \Box s$ and
- $\Box(s \vee \Box s') \equiv \Box(s \vee s')$.

Because of the definition of L_0, for any element s in L_0 there is an A in $\mathcal{M}(D)$ such that $\Box s \equiv \Box \bigvee A$. Hence we have immediately the following result.

Lemma 26 *For any generating tree (T, ζ) in $\mathcal{T}$ and any A in $\mathcal{M}(D)$ the following facts are equivalent:*

- $A \sqsubseteq_0 T_\omega^\zeta$;
- $root(T) \models_T \Box \bigvee A$.

Now we are ready for the main result.

Theorem 27 *The poset $\mathcal{T}/\cong_0$ is isomorphic to the Smyth power domain $F_0(D)$.*

Proof. Let us define a map $\varphi : F_0(D) \longrightarrow \mathcal{T}/\cong_0$.
The generic element of $F_0(D)$ can be thought of as the ideal, whose elements are the approximations of $T_\omega^\zeta \in \mathcal{F}(D)$ for some (T, ζ) generating tree over D, i.e. as the set $\mathcal{I} = \{A \in \mathcal{M}(D) : A \sqsubseteq_0 T_\omega^\zeta\}$.
Because of the previous lemma, we have immediately:

$$\mathcal{I} = \{A \in \mathcal{M}(D) : \mathrm{root}(T) \models_T \Box \bigvee A\}.$$

Hence we can define φ as the map that sends the equivalence class in $\mathcal{F}_0(D)$ determined by T_ω^ζ to the equivalence class in $\mathcal{T}/\cong_0$ determined by T. Then φ is a bijection and both φ and φ^{-1} preserve the order: hence φ is the desired isomorphism. $\Box$

1.3.2 The Hoare power domain and the modal operator of possibility

Let us proceed as we did in the previous section. The generic element of the Hoare power domain $\mathcal{F}_1(D)$ can be thought of as the equivalence class determined by T^ζ_ω, where (T, ζ) is a generating tree over D. For A in $\mathcal{M}(D)$ we have that $A \sqsubseteq_1 T^\zeta_\omega$ holds if

$$\forall a \in A \,.\, \exists x \in T^\zeta_\omega \,.\, a \leq x$$

which we can rephrase as:

$$\forall a \in A \,.\, \exists \gamma \text{ branch of } T \,.\, \exists t \in \gamma \,.\, a \leq \zeta(t).$$

Hence we need a language L_1 build out of $Kpt(D)$ and with a modal operator $\Diamond$ of possibility (to cope with the existential quantification of the branch γ).

Definition 33 *Let L_1 be the least set such that:*

- *$Kpt(D) \subseteq L_1$; and*
- *if s is in L_1 then so is $\Diamond s$*

(where $\Diamond s$ is just a formal expression).

This language can be interpreted as follows.

Definition 34 *For T in $\mathcal{T}$ define $\models_T$ to be the least relation on $T \times L_1$ such that:*

1. *if a is in $Kpt(D)$ and $a \leq \zeta(t)$ then $t \models_T a$; and*
2. *if there is a branch γ out of t and a t' in γ such that $t' \models_T s$ then $t \models_T \Diamond s$.*

Here too we can substitute 2. equivalently with the axiom 2′:

2'. if $t \models_T s$ or $\exists t'. t \to t'. t' \models_T \Diamond s$ then $t \models_T \Diamond s$.

Let us define on $\mathcal{T}$ a preorder, considering the propositions of the language L_1 that are possibly true for a generating tree (T, ζ) over D.

Definition 35 *For (T, ζ) and (T', ζ') generating trees in $\mathcal{T}$ put:*

- $V_1(T) = \{\Diamond s : root(T) \models_T \Diamond s\}$;
- $T \preceq_1 T'$ *if* $V_1(T) \subseteq V_1(T')$;
- $T \cong_1 T'$ *if* $T \preceq T'$ *and* $T' \preceq T$.

Now we can define an equivalence on L_1; for s, s' in L say that $s \equiv s'$ if

$$\forall T \in \mathcal{T} . (\mathrm{root}(T) \models_T s \quad \text{iff} \quad \mathrm{root}(T) \models_T s').$$

Then for any s in L_1 we have:

- $\Diamond(\Diamond s) \equiv \Diamond s$;
- there is a in $Kpt(D)$ such that $\Diamond s \equiv \Diamond a$.

Hence we have the following technical result.

Lemma 28 *For any T in $\mathcal{T}$ and A in $\mathcal{M}(D)$ the following facts are equivalent:*

- $A \sqsubseteq_1 T^{\zeta}_{\omega}$;
- $\forall a \in A . root(T) \models_T \Diamond a$.

Before going into the main result let us give a characterization of the Hoare power domain.

Proposition 29 *Let $\mathcal{L}(Kpt(D)) = \{X \subseteq Kpt(D) : X \neq \emptyset$ and $\downarrow X = X\}$ where $\downarrow X = \{y \in Kpt(D) : y \leq x$ for some $x \in X\}$. Then $\mathcal{L}(Kpt(D))$ partially ordered by inclusion is isomorphic to the Hoare power domain $F_1(D)$.*

Proof. The proof is very simple if one thinks of $F_1(D)$ as $Idl(\mathcal{M}(D), \sqsubseteq_1)$. Indeed we can define two order preserving maps $\varphi : F_1(D) \longrightarrow \mathcal{L}(Kpt(D))$ and $\psi : \mathcal{L}(Kpt(D)) \longrightarrow F_1(D)$, which are clearly mutually inverse. Just put:

$$\varphi(I) = \bigcup\{X \mid X \in I\} \quad \text{for} \quad I \quad \text{in} \quad F_1(D),$$

and

$$\psi(X) = \{A \in \mathcal{M}(D) : A \subseteq X\}.$$

□

Theorem 30 *The poset $\mathcal{T}/\equiv_1$ is isomorphic to the Hoare power domain $F_1(D)$.*

Proof. We shall prove that the map $\varphi : \mathcal{T}/\equiv_1 \longrightarrow \mathcal{L}(Kpt(D))$ defined by

$$\varphi([T]_{\equiv_1}) = \{a \in Kpt(D) : \mathrm{root}(T) \models_T \Diamond a\}$$

is an isomorphism. The theorem then follows from the previous proposition. Remark that $T \preceq_1 T'$ if and only if

$$\{a : \mathrm{root}(T) \models_T \Diamond a\} \subseteq \{a : \mathrm{root}(T') \models_{T'} \Diamond a\}$$

and therefore φ is well-defined, injective and order-preserving. Moreover it is onto because the generic element of $\mathcal{L}(Kpt(D))$ is of the form $\bigcup\{A \in \mathcal{M}(D) : A \sqsubseteq_1 T\zeta_\omega\}$ for some tree (T, ζ) and it reflects the order. Hence it is an isomorphism. □

1.3.3 The Plotkin power domain and the modal operators

Putting together what we have done in the two previous sections one can obtain the modal interpretation of the Plotkin power domain.

Definition 36 *Let L_2 be the least set such that:*

- $Kpt(D) \subseteq L_2$*; and*
- *if* $s, s' \in L_2$ *then* $s \vee s' \in L_2$*; and*
- *if* $s \in L_2$ *then* $\Box s \in L_2$*; and*
- *if* $s \in L_2$ *then* $\Diamond s \in L_2$.

Definition 37 *For T generating tree in $\mathcal{T}$ define $\models_T$ to be the least relation on $\mathcal{T} \times L_2$ such that:*

1. *if* $a \in Kpt(D)$ *and* $a \leq \zeta(t)$ *then* $t \models_T a$*;*
2. *if* $t \models_T s$ *or* $t' \models_T s$ *then* $t \models_T s \vee s'$*;*
3. *if for every branch γ out of t there is t' in γ such that $t' \models_T s$ then $t \models_T \Box s$;*
4. *if there is a branch γ out of t and there is t' in γ such that $t' \models s$ then $t \models_T \Diamond s$.*

Equivalently we can replace 3. and 4. with:

3'. if $t \models_T a$ or $(\forall t' . t \rightarrow t' \Rightarrow t' \models_T \Box s)$ then $t \models_T \Box s$;

4'. if $t \models_T s$ or $\exists t' . t \rightarrow t' . t' \models_T \Diamond s)$ then $t \models_T \Diamond s$.

Now, just as we did for the Smyth power domain, assume we are only interested in the statements that are inevitably true.

Definition 38 *For T and T' trees in $\mathcal{T}$ define:*

- $V_2(T) = \{\Box s \in L_2 : root(T) \models_T \Box s\}$;
- $T \preceq_2 T'$ *if and only if* $V_2(T) \subseteq V_2(T')$;
- $T \cong_2 T'$ *if and only if* $T \preceq_2 T'$ *and* $T' \preceq_2 T$.

Let us introduce an equivalence on L_2 as follows: say that $s \equiv s'$ if

$$\forall T.(\mathrm{root}(T) \models_T s \quad \text{iff} \quad \mathrm{root}(T) \models_T s').$$

Clearly $\equiv$ is an equivalence and $L_2/\equiv$ is a join semilattice where moreover we have:

- $\Diamond s \equiv \Diamond(\Diamond s) \equiv \Diamond(\Box s) \equiv \Box(\Diamond s)$;
- $\Box s \equiv \Box(\Box s)$;
- $\Diamond(s \vee s') \equiv \Diamond s \vee \Diamond s'$;
- $\Box(s \vee \Box s') \equiv \Box(s \vee s')$;
- $\Box(s \vee \Diamond s') \equiv \Box s \vee \Diamond s'$.

Then it is clear that any s in L_2 has a normal form where the modality operators are nested only one deep, namely s is equivalent to

$$a_0 \vee \ldots \vee a_n \vee \Diamond b_0 \vee \ldots \vee \Diamond b_m \vee \Box(c_{00} \vee \ldots \vee c_{0i_0}) \vee \ldots \vee \Box(c_{t0} \vee \ldots \vee c_{ti_t})$$

and therefore $\Box s$ is equivalent to

$$\Box(d_0 \vee \ldots \vee d_r) \vee \Diamond e_0 \vee \ldots \vee \Diamond e_t$$

for suitable $a_i, b_j, c_{ij}, d_l, e_k$ in $Kpt(D)$.

From this we obtain immediately the following result.

Lemma 31 *Let T and T' be trees in $\mathcal{T}$. Then the following facts are equivalent:*

- $T \preceq_2 T'$;
- $T \preceq_0 T'$ *and* $T \preceq_1 T'$.

Hence we can finally prove the main result of this section.

Theorem 32 *The poset $(\mathcal{T}/\cong_2, \preceq_2)$ is isomorphic to the Plotkin power domain* $< F_2(D)$.

Proof. The isomorphism is given by the map $\varphi : F_2(D) \longrightarrow \mathcal{T}/\cong_2$ defined as

$$\varphi([T^{\zeta}_{\omega}]_2) = [T]_{\cong_2}.$$

Since $T \preceq_2 T'$ is equivalent by the previous lemma to $T \preceq_0 T'$ and $T \preceq_1 T'$ and this in turn is equivalent to $T^{\zeta}_{\omega} \sqsubseteq_0 T'^{\zeta}_{\omega}$ and $T^{\zeta}_{\omega} \sqsubseteq_1 T'^{\zeta}_{\omega}$, i.e. to $T^{\zeta}_{\omega} \sqsubseteq_2 T'^{\zeta}_{\omega}$, φ is an isomorphism. □

Chapter 2

Power locales

2.1 Power locales as generalized power domains

In this section we define power locales as the localic analogue of the construction of power spaces ([Smy83]) and show how they can be seen as a generalization of power domains ([Rob86]).

The concept of power domain has been introduced in section 1.1 in order to give a semantics to non-determinism. Another possibility would have been to use many-valued functions between topological spaces (thinking of their opens as computable properties and of continuous maps as computable functions). We would need then a notion of continuity for a many-valued function. There are indeed several possibilities ([Ber59]).

Definition 39 *Let X and Y be topological spaces. A multifunction $\Gamma : X \longrightarrow Y$ is said to be:*

- upper semicontinuous *if $\Gamma^+(O) = \{x : \Gamma x \subseteq O\}$ is open in X whenever O is open in Y;*

- lower semicontinuous *if* $\Gamma^{-}(O) = \{x : \Gamma x \cap O \neq \emptyset\}$ *is open whenever* O *is open in* Y*;*
- continuous *if it is both upper and lower semicontinuous.*

In order to give a characterization of (semi-)continuous multifunctions Γ using the corresponding simple-valued function $\hat{\Gamma} : X \longrightarrow \wp Y$ defined by $\hat{\Gamma}x = \Gamma x$, we introduce the following topologies.

Definition 40 *Let* X *be a topological space,* $\mathcal{S}$ *a subset of* $\wp X$*. For* O *open of* X *put:*

$$\Box O = \{T \in \mathcal{S} : T \subseteq O\},$$

$$\Diamond O = \{T \in \mathcal{S} : T \cap O \neq \emptyset\}.$$

The upper topology on $\mathcal{S}$ *is generated by the base* $\{\Box O : O \in \mathcal{O}(X)\}$*, the* lower topology on $\mathcal{S}$ *by the subbase* $\{\Diamond O : O \in \mathcal{O}(X)\}$ *and the* Vietoris topology on $\mathcal{S}$ *by the subbase* $\{\Box O, \Diamond O : O \in \mathcal{O}(X)\}$.

Now it is easy to show ([SS3.b]) that a multifunction $\Gamma : X \longrightarrow Y$ with multivalues in $\mathcal{S}$ is upper semicontinuous (lower semicontinuous, continuous) if and only if $\hat{\Gamma} : X \longrightarrow \mathcal{S}$ is continuous with respect to the lower (upper, Vietoris) topology on $\mathcal{S}$.

Characterizations of power domains, as in Theorem 25, lead us to consider particular choices for $\mathcal{S}$, when trying to model non-determinism.

Notation. Let X be a topological space. We will use the following notations:

1. $CL(X)$ for the set of closed subsets of X;
2. $UC(X)$ for the set of upper-closed subsets of X;
3. $CONV(X)$ for the set of convex subsets of X (a subset S is convex if it is equal to the intersection of its topological closure $cl(S)$ and of its upper closure $\uparrow S$);

4. $COMP(X)$ for the set of compact subsets of X.

Of course, in 2. and 3., the upper-closure is referred to the specialization preorder on X, defined for x, y in X as:

$$x \leq y \quad \text{iff} \quad \forall O \in \mathcal{O}(X).\, x \in O \Longrightarrow y \in O.$$

Definition 41 *Let X be a topological space.*

- *The* upper power space $PS_0(X)$ of X *is* $COMP(X) \cap UC(X)$ *endowed with the upper topology.*
- *The* lower power space $PS_1(X)$ of X *is* $CL(X)$ *taken with the lower topology.*
- *The* Vietoris power space $PS_2(X)$ of X *is* $COMP(X) \cap CONV(X)$ *with the Vietoris topology.*

Now let us try to express the power spaces using only open subsets, in order to find axioms for defining the power locales.

The easiest is the lower power space: it can be defined equivalently (up to isomorphism) as

$$PS_1(X) = (\mathcal{O}(X), \text{topology with subbase } \{\Diamond O : O \in \mathcal{O}\})$$

where now $\Diamond O = \{U \in \mathcal{O}(X) : O \not\subseteq U\}$.

Remark that the following properties are verified:

- for any collection $\{U_i\}_{i \in I}$ of opens of X:

$$\begin{aligned} \Diamond \bigcup_i O_i &= \{U : \bigcup_i O_i \not\subseteq U\} \\ &= \{U : O_i \not\subseteq U \text{ for some } i\} \\ &= \bigcup_i \Diamond O_i; \end{aligned}$$

- $\Diamond\emptyset = \emptyset$.

For the upper power space we need some conditions. Assume that X is compact Hausdorff. Then the compact subsets correspond to the closed and all the sets are upper closed (for the last condition, we only need to assume X to be T_1). Hence we can define

$$PS_0(X) = (\mathcal{O}(X), \text{ topology with base } \{\Box O : O \in \mathcal{O}(X)\}),$$

where $\Box O = \{U \in \mathcal{O}(X) : U \cup O = X\}$.
Remark that the following properties hold:

- for any directed family $\{O_i\}_{i \in I}$ of opens of X:

$$\begin{aligned}
\Box \bigcup_i O_i &= \{U \in \mathcal{O}(X) : U \cup \bigcup_i O_i = X\} \\
&= \{U \in \mathcal{O}(X) : U \cup O_i = X \text{ for some } i\} \\
&= \bigcup_i \Box O_i
\end{aligned}$$

- for any O, O' opens of X:

$$\begin{aligned}
\Box(O \cap O') &= \{U \in \mathcal{O}(X) : U \cup (O \cap O') = X\} \\
&= \Box O \cap \Box O'
\end{aligned}$$

- $\Box X = \mathcal{O}(X)$.

Similarly for the Vietoris power space if we assume that X is T_1 and compact (so that $CONV(X) = CL(X) \subseteq COMP(X)$) then we have:

$$PS_2(X) = (\mathcal{O}(X), \text{ topology with subbase } \{\Diamond O, \Box O : O \in \mathcal{O}(X)\})$$

where $\Box O$ and $\Diamond O$ are defined as in the previous two cases.
Besides the properties already listed, for any O and O' open in X we also have:

- $\Box O \cup \Diamond O' \geq \Box(O \cup O')$;
- $\Box O \cap \Diamond O' \leq \Diamond(O \cap O')$.

Assuming these properties as axioms we get the definition of power locale of a locale A ([Rob86]). We will use the notation $Fr < G \mid R >$ to mean the universal solution to the problem of finding a frame containing the set G of generators and satisfying the presenting relations in R ([Vic89]): remark that this makes sense in any topos ([JT84]).

Definition 42 *The* Smyth (or upper) power locale $V_0(A)$ of A *is given by the frame:*

$$Fr < \Box a (a \in A) \mid \Box \bigvee S = \bigvee \{\Box s : s \in S\} \text{ for } S (\subseteq A) \text{ directed}, \\ \Box \wedge S = \wedge \{\Box s : s \in S\} \text{ for } S (\subseteq A) \text{ finite } > .$$

If we add the presenting relation $\Box \bot = \bot$, then we obtain the strict Smyth power locale $V_0^+(A)$.

Equivalently we could say that $\mathcal{O}(V_0(A))$ is the frame freely generated by $\mathcal{O}(A)$ qua preframe (a preframe is a poset with directed joins and finite meets, such that binary meets distribute over directed joins).
Note that if X is a compact Hausdorff space, then $V_0(\mathcal{O}(X)) \cong \mathcal{O}PS_0(X)$.

Definition 43 *The* Hoare (or lower) power locale $V_1(A)$ of A *is given by the frame:*

$$Fr < \Diamond a (a \in A) | \Diamond \bigvee S = \bigvee \{\Diamond s : s \in S\} \text{ for } S \subseteq A > .$$

If we add the presenting relation $\Diamond \top = \top$, then we have the strict Hoare power locale $V_1^+(A)$.

Equivalently we could say that $\mathcal{O}(V_1(A))$ is the frame freely generated by $\mathcal{O}(A)$ qua suplattice (we call suplattice a lattice with all suprema ([JT84])).
Note that for any topological space X, one has $V_1(\mathcal{O}(X)) \cong \mathcal{O}PS_1(X)$.

Definition 44 *The Vietoris power locale $V_2(A)$ of A is given by the frame*

$$\begin{aligned} Frm < \Box a, \Diamond a (a \in A) \ \mid \ & \Box \bigvee S = \bigvee \{\Box s : s \in S\} \text{ for } S (\subseteq A) \text{ directed}, \\ & \Box \wedge S = \wedge \{\Box s : s \in S\} \text{ for } S (\subseteq A) \text{ finite} , \\ & \Diamond \bigvee S = \bigvee \{\Diamond s : s \in S\} \text{ for } S \subseteq A, \\ & \Box a \vee \Diamond b \geq \Box (a \vee b), \\ & \Box a \wedge \Diamond b \leq \Diamond (a \wedge b). > \end{aligned}$$

The strict version of the Vietoris power locale is obtained by adding the presenting relation $\Box \perp = \perp$.

Remark. The presenting relations of the Vietoris locale will be referred to in the following as the axioms V1,V2,...,V5.

If X is T_1 and compact (in particular, compact Hausdorff) then $\mathcal{O}(PS_2(X)) \cong V_2(\mathcal{O}(X))$.

As the name suggests, there is a correspondence between power domains and power locales (though for historical reasons $V_2(A)$ is not referred to as the Plotkin power locale).

Let us denote by $\Sigma(D)$ the locale of Scott-opens of a domain D (recall that $U \subseteq D$ is Scott-open if it is upper-closed and inaccessible by directed joins).

Theorem 33 *If D is a domain then we have the following isomorphisms of locales (for $i = 0, 1, 2$): $V_i(\Sigma D) \equiv \Sigma F_i^+(D)$.*

Proof. We will show (see following lemmas) that we can define frame homomorphisms

$$\psi_i^* : \Sigma F_i^+(D) \longrightarrow V_i(\Sigma D)$$

and

$$\varphi_i^* : V_i(\Sigma D) \longrightarrow \Sigma F_i^+(D)$$

such that $\varphi_i^- \circ \psi_i^- = id_{\Sigma F_i^+(D)}$ and $\psi_i^- \circ \varphi_i^- = id_{V_i(\Sigma D)}$: therefore the isomorphisms will have been established.

Since the proofs of the cases $i = 0, 1$ are simplified versions of the proof of the case $i = 2$, they will be omitted.

Recall that the Scott topology on a domain (in our case $F_i^+(D)$) can be identified with the upward closed topology on its set of compact elements, in our case $(\mathcal{M}^+(D), \sqsubseteq_i)$. Hence we can define ψ_i^- just on the upward closure $\uparrow_i A$ in $(\mathcal{M}(D), \sqsubseteq_i)$ of a finite set A of compact elements of D and verify that ψ_i^- is indeed well defined (see Lemma 34 in the following); for $i = 0, 1, 2$ put respectively:

$$\psi_0^-(\uparrow_0 \{a_1, \dots, a_n\}) = \Box((\uparrow a_1) \vee \dots \vee (\uparrow a_n));$$
$$\psi_1^-(\uparrow_1 \{a_1, \dots, a_n\}) = \Diamond(\uparrow a_1) \wedge \dots \wedge \Diamond(\uparrow a_n);$$
$$\psi_2^-(\uparrow_2 \{a_1, \dots, a_n\}) = \Box((\uparrow a_1) \vee \dots \vee (\uparrow a_n)) \wedge \Diamond(\uparrow a_1) \wedge \dots \wedge \Diamond(\uparrow a_n)$$

where $\uparrow a_i = \{y \in D : y \geq a_i\}$.

Next let us remark that a basis of $V_i(\Sigma D)$ is constituted by the elements:

$i = 0 :$ $\{\Box((\uparrow a_1) \vee \dots \vee (\uparrow a_n)) : a_i \in Kpt(D)\}$;

$i = 1 :$ $\{\Diamond(\uparrow b) : b \in Kpt(D)\}$;

$i = 2 :$ $\{\Box((\uparrow a_1) \vee \dots \vee (\uparrow a_n)) \wedge \Diamond(\uparrow b_1) \wedge \dots \wedge \Diamond(\uparrow b_m) : a_i, b_j \in Kpt(D)\}$.

Indeed if U is a Scott-open subset of D, then we have

$$\begin{aligned} \Box U &= \Box \bigvee \{\uparrow a : a \in U, a \in Kpt(D)\} \\ &= \Box \bigvee \{(\uparrow a_1) \vee \dots \vee (\uparrow a_n) : a_i \in U, a_i \in Kpt(D), n \in N\} \\ &= \bigvee \{\Box(\uparrow a_1) \vee \dots \vee (\uparrow a_n) : \{a_1, \dots, a_n\} \in \mathcal{M}(D)\} \end{aligned}$$

(since $\Box$ preserves directed joins) and also

$$\begin{aligned} \Diamond U &= \Diamond \bigvee \{\uparrow b : b \in U, b \in Kpt(D)\} \\ &= \bigvee \{\Diamond \uparrow b : b \in U, b \in Kpt(D)\} \end{aligned}$$

(because $\Diamond$ preserves arbitrary joins).

Now we are ready to define the maps φ_i^-:

i=0: $\varphi_0^*(\Box(\uparrow a_1) \vee \ldots \vee \uparrow a_n)) = \bigcup\{\uparrow_0 X : X \subseteq \{a_1, \ldots, a_n\}\}$

i=1: $\varphi_1^*(\Diamond(\uparrow a)) = \uparrow_1 \{\perp, a\}$;

i=2: $\varphi_2^*(\Box(\uparrow a_1) \vee \ldots \vee \uparrow a_n)) = \bigcup\{\uparrow_2 X : X \subseteq \{a_1, \ldots, a_n\}\}$ and
$\varphi_2^*(\Diamond(\uparrow a)) = \uparrow_2 \{\perp, a\}$.

Once again, we will prove that φ_i^* is a frame homomorphism only in the case $i = 2$ (see Lemma 35 in the following). So, after reading the next three lemmas, the proof is complete. □

Lemma 34 *The map* $\psi_2^* : \Sigma F_2^+(D) \longrightarrow V_2(\Sigma D)$ *is well-defined.*

Proof. We only need to prove that if

1. $\uparrow_2 \{a_1, \ldots, a_n\} \subseteq \uparrow_2 \{b_1, \ldots, b_k\}$

then we have

2. $\Box((\uparrow a_1) \vee \ldots \vee (\uparrow a_n)) \wedge \Diamond(\uparrow a_1) \wedge \ldots \wedge \Diamond(\uparrow a_n)$
$\leq \Box((\uparrow b_1) \vee \ldots \vee (\uparrow b_k)) \wedge \Diamond(\uparrow b_1) \wedge \ldots \wedge \Diamond(\uparrow b_k)$

Remark that 1. is equivalent to

1'. $\{b_1, \ldots, b_k\} \sqsubseteq_2 \{a_1, \ldots, a_n\}$

i.e. to the two conditions

3. $\{b_1, \ldots, b_k\} \sqsubseteq_0 \{a_1, \ldots, a_n\}$

4. $\{b_1, \ldots, b_k\} \sqsubseteq_1 \{a_1, \ldots, a_n\}$

But 3. in turn is equivalent to

3'. $(\uparrow a_1) \vee \ldots \vee (\uparrow a_n) \leq (\uparrow b_1) \vee \ldots \vee (\uparrow b_k)$

and hence we get

5. $\diamond((\uparrow a_1) \vee \ldots \vee (\uparrow a_n)) \leq \Box((\uparrow b_1) \vee \ldots \vee (\uparrow b_k))$.

Similarly 4. is equivalent to

4'. $\forall j = 1, \ldots, k \,.\, \exists i_j \in \{1, \ldots, n\} \,.\, \uparrow a_{i_j} \subseteq \uparrow b_j$

from which we get

6. $\diamond(\uparrow a_1) \wedge \ldots \wedge \diamond(\uparrow a_n) \leq \diamond(\uparrow b_1) \wedge \ldots \wedge \diamond(\uparrow b_k)$.

Putting together 5. and 6. we get 2. $\Box$

Lemma 35 *The map $\varphi_2^- : V_2(\Sigma D) \longrightarrow \Sigma F_2^+(D)$ is well-defined.*

Proof. Let us start by showing that if $\alpha \leq \beta$ in $\Sigma(D)$ then

1. $\varphi_2^-(\Box\alpha) \leq \varphi_2^-(\Box\beta)$; and
2. $\varphi_2^-(\diamond\alpha) \leq \varphi_2^-(\diamond\beta)$.

Because of the remark in Theorem 33 when proving 1. we can assume without loss of generality that $\alpha = (\uparrow a_1) \vee \ldots \vee (\uparrow a_n)$ and $\beta = (\uparrow b_1) \vee \ldots \vee (\uparrow b_k)$. Hence $\{b_1, \ldots, b_k\} \sqsubseteq_0 \{a_1, \ldots, a_n\}$ (we have already observed in the previous lemma the equivalence of these two statements). If $X \subseteq \{a_1, \ldots, a_n\}$ then $\{a_1, \ldots, a_n\} \sqsubseteq_0 X$ and, by transitivity, $\{b_1, \ldots, b_k\} \sqsubseteq_0 X$: hence by refining $\{b_1, \ldots, b_k\}$ we can find $Y \subseteq \{b_1, \ldots, b_k\} \sqsubseteq_0 X$ such that $Y \sqsubseteq_2 X$. Then we have:

$$\begin{aligned}
\varphi^-(\Box((\uparrow a_1) \vee \ldots \vee (\uparrow a_n))) &= \bigcup\{\uparrow_2 X : X \subseteq \{a_1, \ldots, a_n\}\} \\
&\leq \bigcup\{\uparrow_2 Y : Y \subseteq \{b_1, \ldots, b_k\}\} \\
&= \varphi^-(\Box((\uparrow b_1) \vee \ldots \vee (\uparrow b_k))).
\end{aligned}$$

To verify 2. we may assume that $\alpha = \uparrow a$ and $\beta = \uparrow b$ (always because of the remark in Theorem 33). Then $\alpha \leq \beta$ iff $b \leq a$ iff $\{\perp, b\} \sqsubseteq_2 \{\perp, a\}$ iff $\uparrow_2\{\perp, a\} \leq \uparrow_2\{\perp, b\}$ i.e. $\varphi_2^*(\Diamond\alpha) \leq \varphi_2^*(\Diamond\beta)$. Hence φ_2^* is well defined on the element of the form

$$\Box(\uparrow a_1 \vee \ldots \vee \uparrow a_n) \wedge \Diamond(\uparrow b_1) \wedge \ldots \wedge \Diamond(\uparrow b_k).$$

Since this is a basis of $V_2(\Sigma D)$ we can extend φ_2^* "by linearity" to a map defined on $V_2(\Sigma D)$. Let us verify that the presenting relations of the Vietoris power locale hold:

(V1) and (V3) hold because of the definition of φ_2^*;

(V2): we need to show that:

$$\varphi_2^*(\Box(\alpha \wedge \beta)) \geq \varphi_2^*(\Box\alpha) \wedge \varphi_2^*(\Box\beta)$$

where we can suppose $\alpha = (\uparrow a_1) \vee \ldots \vee (\uparrow a_n)$ and $\beta = (\uparrow b_1) \vee \ldots \vee (\uparrow b_k)$. Then we have

$$\begin{aligned}
& \varphi_2^*(\Box(\alpha \wedge \beta)) \\
= \;& \varphi_2^*(\Box \bigvee\{\uparrow x_1 \vee \ldots \vee x_t : \{a_1, \ldots, a_n\}, \{b_1, \ldots, b_k\} \sqsubseteq_0 \{x_1, \ldots, x_t\}\}) \\
= \;& \bigvee\{\varphi_2^*(\Box(\uparrow x_1 \vee \ldots \vee \uparrow x_t)) : \{a_1, \ldots, a_n\}, \{b_1, \ldots, b_k\} \sqsubseteq_0 \{x_1, \ldots, x_t\}\} \\
= \;& \bigvee\{\uparrow_2 Y : Y \subseteq X; \{a_1, \ldots, a_n\}, \{b_1, \ldots, b_k\} \sqsubseteq_0 X\} \\
\supseteq \;& \bigvee\{Y : \{a_1, \ldots, a_n\}, \{b_1, \ldots, b_k\} \sqsubseteq_0 Y\} \\
= \;& \varphi_2^*(\Box\alpha) \wedge \varphi_2^*(\Box\beta)
\end{aligned}$$

where the last step holds since

$$X \sqsubseteq_2 Y \text{ and } X' \sqsubseteq_2 Y \text{ for some } X \subseteq A \text{ and } X' \subseteq A$$

is equivalent to

$$A \sqsubseteq_0 Y \text{ and } B \sqsubseteq_0 Y.$$

(V4): we need to show that:

$$\varphi_2^*(\Box(\alpha \vee \beta)) \leq \varphi_2^*(\Box\alpha) \vee \varphi_2^*(\Diamond b)$$

and again we can assume without loss of generality that $\alpha = \uparrow a_1 \vee \ldots \vee \uparrow a_n$ and $\beta = \uparrow b_1 \vee \ldots \vee \uparrow b_k$. Then if $Z \in \varphi_2^-(\Box(\alpha \vee \beta))$ i.e. if $X \cup Y \sqsubseteq_2 Z$ for some $X \subseteq \{a_1, \ldots, a_n\}$ and $Y \subseteq \{b_1, \ldots, b_k\}$ we can distinguish two possibilities:

- if $Y = \emptyset$ then $Z \in \varphi_2^-(\Box\alpha)$ since $X \sqsubseteq_2 Z$ where $X \subseteq \{a_1, \ldots, a_n\}$;
- if $Y \neq \emptyset$ then $Z \in \varphi_2^-(\Diamond)$ since $\exists b \in Y$ such that $\{\perp, b\} \sqsubseteq_2 Z$.

(V5): we need to prove:

$$\varphi_2^-(\Box\alpha) \wedge \varphi_2^-(\Diamond\beta) \leq \varphi_2^-(\Diamond(\alpha \wedge \beta))$$

where as usually $\alpha = (\uparrow a_1) \vee \ldots \vee (\uparrow a_n)$ and $\beta = (\uparrow b_1) \vee \ldots \vee (\uparrow b_k)$. Then if Z is in $\varphi_2^-(\Box\alpha) \wedge \varphi_2^-(\Diamond\beta)$ there are both an $X \subseteq \{a_1, \ldots, a_n\}$ and a $b_i \in \{b-1, \ldots, b_k\}$ such that $X \sqsubseteq_2 Z$ and $\{\perp, b_i\} \sqsubseteq_2 Z$ i.e. there is $z \in Z$ such that $a_j \leq z$ and $b_i \leq z$ for some $a_j \in \{a_1, \ldots, a_n\}$ and $b_i \in \{b_1, \ldots, b_k\}$: but then Z is in $\varphi_2^-(\Diamond(\alpha \wedge \beta))$.
$\Box$

Lemma 36 *The maps φ_2^- and ψ_2^- are mutually inverse.*

Proof. To prove that $\varphi_2^- \circ \psi_2^- = id_{\Sigma F_2^+(D)}$ we only need to check that

$$\varphi_2^- \circ \psi_2^-(\uparrow_2 \{a_1, \ldots, a_n\}) = \uparrow_2 \{a_1, \ldots, a_n\}$$

where $a_1, \ldots, a_n$ are compact elements of D. Indeed, we have

$$\begin{aligned}
& \varphi_2^- \circ \psi_2^-(\uparrow_2 \{a_1, \ldots, a_n\}) \\
= \;& \varphi_2^-(\Box((\uparrow a_1,) \vee \ldots \vee (\uparrow a_n)) \wedge \Diamond(\uparrow a_1) \wedge \ldots \wedge \Diamond(\uparrow a_n)) \\
= \;& \varphi_2^-(\Box((\uparrow a_1,) \vee \ldots \vee (\uparrow a_n))) \wedge \varphi_2^-(\Diamond(\uparrow a_1)) \wedge \ldots \wedge \varphi_2^-(\Diamond(\uparrow a_n)) \\
= \;& \bigcup\{\uparrow_2 X : X \subseteq \{a_1, \ldots, n\}\} \wedge \bigwedge_{i=1,\ldots,n} \uparrow_2 \{\perp, a_i\} \\
= \;& \uparrow_2 \{a_1, \ldots, a_n\}
\end{aligned}$$

where in the first three steps we are just applying the definition of φ_2^- and ψ_2^-, whereas the last step is motivated by the following:

- if $\{a_1,\ldots,a_n\} \sqsubseteq_2 Y$ then for all $i=1,\ldots,n$ one has $\{\bot,a_i\} \sqsubseteq_2 Y$;and
- if $X \sqsubseteq_2 Y$ for some $X \subseteq \{a_1,\ldots,a_n\}$ (and therefore $\{a_1,\ldots,a_n\} \sqsubseteq_2 Y$) and $\{\bot,a_i\} \sqsubseteq_2 Y$.

Similarly to prove that $\psi_2^* \circ \varphi_2^* = id_{V_1(\Sigma D)}$ we only need to show:

1. $\psi_2^* \circ \varphi_2^*(\Diamond(\uparrow a)) = \Diamond(\uparrow a)$; and

2. $\psi_2^* \circ \varphi_2^*(\Box(\uparrow a_1 \vee \ldots \uparrow a_n)) = \Box(\uparrow a_1 \vee \ldots \uparrow a_n)$.

Let us see 1.:

$$\begin{aligned}
\psi_2^* \circ \varphi_2^*(\Diamond(\uparrow a)) &= \psi_2^*(\uparrow_2\{\bot,a\}) \\
&= \Box((\uparrow\bot) \vee (\uparrow a)) \wedge \Diamond(\uparrow\bot) \wedge \Diamond(\uparrow a) \\
&= \Box\top \wedge \Diamond\top \wedge \Diamond(\uparrow a) \\
&= \Diamond \uparrow a
\end{aligned}$$

where the last step comes from the fact that $\Box\top = \top$.

We will prove 2. by induction on n; for $n=1$ we have:

$$\begin{aligned}
\psi_2^* \circ \varphi_2^*(\Box \uparrow a) &= \psi_2^*(\bigcup\{\uparrow_2 X : X \subseteq \{a\}\}) \\
&= \psi_2^*((\uparrow_2 \emptyset) \cup \uparrow_2 a)) \\
&= \psi_2^*((\uparrow_2 \emptyset) \vee \psi_2^*(\uparrow_2 a)) \\
&= \Box \bot \vee (\Box(\uparrow a) \wedge \Diamond(\uparrow a))
\end{aligned}$$

Let us assume, as induction hypothesis, that 2. holds for $k \leq n-1$; then

$$\begin{aligned}
&\psi_2^* \circ \varphi_2^*(\Box((\uparrow a_1) \vee \ldots \vee (\uparrow a_n))) \\
= \ & \psi_2^*(\bigcup\{\uparrow_2 X : X \subseteq \{a_1,\ldots,a_n\}\}) \\
= \ & \bigvee_{\{i_1,\ldots,i_n\} \subseteq \{1,\ldots,n\}} (\Box((\uparrow a_{i_1}) \vee \ldots \vee (\uparrow a_{i_r})) \wedge \Diamond(\uparrow a_{1_i}) \wedge \ldots \wedge \Diamond(\uparrow a_{i_r})) \\
= \ & (\Box((\uparrow a_1) \vee \ldots (\uparrow a_n)) \wedge \Diamond(\uparrow a_1) \wedge \ldots \wedge \Diamond(\uparrow a_n)) \vee \\
& \vee \bigvee_{\{i_1,\ldots,i_r\} \subset \{1,\ldots,n\}} (\Box((\uparrow a_1) \vee \ldots \vee (\uparrow a_r)) \wedge \Diamond(\uparrow a_{1_i}) \wedge \ldots \wedge \Diamond(\uparrow a_{i_r}))
\end{aligned}$$

Now applying the induction hypothesis to the second part of the disjunction, we get:

$$
\begin{aligned}
& \psi_2^* \circ \varphi_2^*(\Box((\uparrow a_1) \vee \ldots \vee (\uparrow a_n))) \\
= \; & (\Box((\uparrow a_1) \vee \ldots (\uparrow a_n)) \wedge \Diamond(\uparrow a_1) \wedge \ldots \wedge \Diamond(\uparrow a_n)) \\
& \vee \bigvee_{\substack{\{i_1,\ldots,i_r\}=\{1,\ldots,n\}\setminus\{j\} \\ \text{for } j\in\{1,\ldots,n\}}} \Box((\uparrow a_1) \vee \ldots \vee (\uparrow a_r)) \\
\geq \; & \Box((\uparrow a_1) \vee \ldots \vee (\uparrow a_n)) \wedge \bigwedge_{j=1,\ldots,n} (\Diamond(\uparrow a_j) \vee \Box(\bigvee_{\substack{i=1,\ldots,n \\ i\neq j}})(\uparrow a_i))) \\
\geq \; & \Box((\uparrow a_1) \vee \ldots \vee (\uparrow a_n)) \wedge \bigwedge_{j=1,\ldots,n} \Box((\uparrow a_1) \vee \ldots \vee (\uparrow a_n)) \\
= \; & \Box((\uparrow a_1) \vee \ldots \vee (\uparrow a_n)).
\end{aligned}
$$

On the other hand one obviously has

$$\Box((\uparrow a_1) \vee \ldots \vee (\uparrow a_n)) \leq \psi_2^* \circ \varphi_2^*(\Box((\uparrow a_1) \vee \ldots \vee (\uparrow a_n)))$$

and therefore the equality holds. $\Box$

If we consider the strict version of the power domains and we want ψ_i^* to be still an isomorphism, we are forced to replace the power locales with their strict versions.

Theorem 37 *If D is a domain we have the following isomorphism of locales:*

$$V_i^+(\Sigma D) \cong \Sigma F_i(D).$$

Now recall that a domain is sober in its Scott topology, i.e. it is isomorphic to the space of points of its Scott topology. Then we can finally show in which sense power locales are a generalization of power domains.

Theorem 38 *Let D be a domain. Then for $i = 0, 1, 2$ we have the isomorphisms:*

$$F_i(D) \cong pt(V_i^+(\Sigma D)).$$

2.2 Monads from power locales

In this section we show that the power locales give rise to monads over **Loc**. Johnstone proved that the algebras for the Victoris monad can be seen as a particular class of localic semilattices ([Joh85]). We extend this result to the other two power locales: when the proofs go along the same lines, they will be omitted.

Notation. For sake of readability, in the proofs of this section the index $i = 2$ will be dropped.

Proposition 39 *The assignments $A \mapsto V_i(A)$ (for $i = 0, 1, 2$) give rise to functors $V_i : \mathbf{Loc} \longrightarrow \mathbf{Loc}$; moreover there are natural transformations $\eta_i : Id_{\mathbf{Loc}} \longrightarrow V_i$ and $\mu_i : V_i^2 \longrightarrow V_i$ making (V_i, η_i, μ_i) into a monad on* **Loc**.
Similarly, the strict power locales $V_i^+(A)$ give rise to monads $(V_i^+, \eta_i^+, \mu_i^+)$ on **Loc**.

Proof. We start by defining $V(f)$ for $f : A \longrightarrow B$ morphism of locales: it is enough to specify the effect of $V(f)^*$ on the generators $\Box b$ and $\Diamond b$ of $V(B)$: put

- $V(f)^*(\Box b) = \Box(f^*(b))$;
- $V(f)^*(\Diamond b) = \Diamond(f^*(b))$.

Since f^* is a frame homomorphism, $V(f)^*$ preserves the presenting relations (V1)-(V5) and therefore it is a well-defined morphism on $V(B)$; moreover V is clearly functorial.
The unit η of the monad is defined by:

$$\eta_A^*(\Box a) = a \text{ and } \eta_A^*(\Diamond a) = a$$

and the multiplication μ by:

$$\mu_A^*(\Box a) = \Box(\Box a) \text{ and } \mu_A^*(\Diamond a) = \Diamond(\Diamond a).$$

It is straightforward to see that both maps preserve the relations (V1)-(V5) and are therefore well-defined. Also the axioms of monad follow immediately from the definitions. □

For $i = 0, 1$ one has $V_{i\eta_i} = \eta_{iV_i}$: hence the Smyth and Hoare monads have a particular structure, known as KZ-monad. We recall the definition of this concept just for a 2−category **C** enriched over posets, referring the reader to [Koc95] for the general theory.

Definition 45 *Let $T : \mathbf{C} \longrightarrow \mathbf{C}$ be a 2-functor (i.e. a functor that preserves the order of the hom-sets). A* KZ-monad on **C** *is a monad (T, η, μ) such that $T\eta_C \leq \eta_{TC}$ for all object C of* **C**.

Proposition 40 *Let (T, η, μ) be a KZ-doctrine on* **C**. *Then a T-algebra structure a on an object A of* **C**, *if it exists, is uniquely determined (up to isomorphism).*

Proof. From the axiom $\eta_A \circ a = id_A$ one has, by naturality of η, that $Ta \circ \eta_{TA} = id$. Since T defines a KZ-doctrine then one gets

$$\begin{aligned} \eta_A \circ a &= Ta \circ \eta_{TA} \\ &\geq Ta \circ T\eta_A \\ &= T(a \circ \eta_A) \\ &= id. \end{aligned}$$

Hence the structure a is a reflection left adjoint to η_A and therefore is uniquely determined. □

In particular, the Smyth and Hoare algebra structure of a locale, if it exists, is uniquely determined.

The Vietoris power locale, however, does not give rise to a KZ-monad for otherwise we should have $V_2\eta_2 \leq \eta_{2V_2}$ (and hence $\Box a = \Diamond a$ for all a) and this is clearly not the case.

In order to show that the algebras of the monads V_i have naturally a semilattices structure (in **Loc**), we need the following two lemmas about products of locales (for a characterization of products of locales see, for example, [Bor94]).

Lemma 41 *There is a natural isomorphism*

$$q_i : V_i(A) \otimes V_i(B) \xrightarrow{\sim} V_i(A \times B)$$

where $\otimes$ *and* $\times$ *denote the product and the coproduct in* **Loc**.

Proof. Let us define $q^* : V(A \times B) \longrightarrow V(A) \otimes V(B)$ setting:

- $q^*(\Box(a,b)) = \Box a \otimes \Box b$;
- $q^*(\Diamond(a,b)) = (\Diamond a \otimes 1) \vee (1 \otimes \Diamond b)$.

We have to verify that q^* preserves the presenting relations:
(V1): if $\{(a_i, b_i)\}_{i \in I}$ is a directed subset of $A \times B$, then:

$$\begin{aligned} q^*(\Box(\bigvee_i (a_i, b_i))) &= (\Box \bigvee_i a_i) \otimes (\Box \bigvee_j b_j) \\ &= (\bigvee_i \Box a_i) \otimes (\bigvee_j \Box b_j) \\ &= \bigvee_i (\Box a_i \otimes \Box b_i) \\ &= \bigvee_i q^*(\Box(a_i, b_i)); \end{aligned}$$

(V2): if $(a_i, b_i) \in A \times B$ for $i = 1, 2$, then:

$$\begin{aligned} q^*(\Box((a_1, b_1) \wedge (a_2, b_2))) &= \Box(a_1 \wedge a_2) \otimes \Box(b_1 \wedge b_2) \\ &= (\Box a_1 \wedge \Box a_2) \otimes (\Box b_1 \wedge \Box b_2) \\ &= (\Box a_1 \otimes \Box b_1) \wedge (\Box a_2 \otimes \Box b_2) \\ &= q^*(\Box(a_1, b_1)) \wedge q^*(\Box(a_2, b_2)); \end{aligned}$$

(V3): if $\{(a_i, b_i)\}_{i\in I}$ is any subset of $A \times B$, then:

$$\begin{aligned} q^*(\Diamond(\bigvee_i (a_i, b_i))) &= (\Diamond(\bigvee_i a_i) \otimes 1) \vee (1 \otimes \Diamond(\bigvee_j b_j)) \\ &= \bigvee_{i,j}((\Diamond a_i \otimes 1) \vee (1 \otimes \Diamond b_j)) \\ &= \bigvee_i((\Diamond a_i \otimes 1) \vee (1 \otimes \Diamond b_i)) \\ &= \bigvee_i q^*(\Diamond(a_i, b_i)); \end{aligned}$$

(V4): if (a_i, b_i) is in $A \times B$ for $i = 1, 2$ then:

$$\begin{aligned} & q^*(\Box((a_1, b_1) \vee (a_2, b_2))) \\ = \; & \Box(a_1 \vee a_2) \otimes \Box(b_1 \vee b_2) \\ \leq \; & (\Box a_1 \otimes \Box(b_1 \vee b_2)) \vee (\Diamond a_2 \otimes \Box(b_1 \vee b_2)) \\ \leq \; & (\Box a_1 \otimes \Box b_1) \vee (\Box a_1 \otimes \Diamond b_2) \vee (\Diamond a_2 \otimes \Box(b_1 \vee b_2)) \\ \leq \; & (\Box a_1 \otimes \Box b_1) \vee (1 \otimes \Diamond b_2) \vee (\Diamond a_2 \otimes 1) \\ = \; & q^*(\Box(a_1, b_1)) \vee q^*(\Diamond(a_2, b_2)) \end{aligned}$$

(V5): if (a_i, b_i) is in $A \times B$ for $i = 1, 2$ then:

$$\begin{aligned} & q^*(\Diamond((a_1, b_1) \wedge (a_2, b_2))) \\ = \; & (\Diamond(a_1 \wedge a_2) \otimes 1) \vee (1 \otimes \Diamond(b_1 \wedge b_2)) \\ \geq \; & ((\Box a_1 \otimes 1) \wedge (\Diamond a_2 \otimes 1)) \vee ((1 \otimes \Box b_1) \wedge (1 \otimes \Diamond b_2)) \\ \geq \; & ((\Box a_1 \otimes \Box b_1) \wedge (\Diamond a_2 \otimes 1)) \vee ((\Box a_1 \otimes \Box b_1) \wedge (1 \otimes \Diamond b_2)) \\ = \; & (\Box a_1 \otimes \Box b_1) \wedge ((\Diamond a_2 \otimes 1) \vee (1 \otimes \Diamond b_2)) \\ = \; & q^*(\Box(a_1, b_1)) \wedge q^*(\Diamond(a_2, b_2)) \end{aligned}$$

In the other direction we can define two maps:

$$r_1^* : VA \longrightarrow V(A \times B)$$

and

$$r_2^* : VB \longrightarrow V(A \times B)$$

by putting $r_1^*(\Box a) = \Box(a,1)$, $r_1^*(\Diamond a) = \Diamond(a,0)$ and similarly for r_2^*: the relations (V1)-(V5) are preserved (easy checking) and therefore we have a frame homomorphism $(r_1, r_2)^* : VA \otimes VB \longrightarrow V(a \times B)$ defined by:

$$(r_1, r_2)^*(x \odot y) = r_1^*(x) \wedge r_2^*(y).$$

Moreover

$$\begin{aligned}(r_1, r_2)^* \circ q^*(\Box(a,b)) &= (r_1, r_2)^*(\Box a \otimes \Box b) \\ &= r_1^*(\Box a) \wedge r_2^*(\Box b) \\ &= \Box(a,1) \wedge \Box(1,b) \\ &= \Box(a,b)\end{aligned}$$

and

$$\begin{aligned}(r_1, r_2)^* \circ q^*(\Diamond(a,b)) &= (r_1, r_2)^*((\Diamond a \otimes 1) \vee (1 \otimes \Diamond b)) \\ &= r_1^*(\Diamond a) \vee r_2^*(\Diamond b) \\ &= \Diamond(a,0) \vee \Diamond(0,b) \\ &= \Diamond(a,b)\end{aligned}$$

and therefore $(r_1, r_2)^* \circ q^* = id$.

Similarly we have $q^* \circ (r_1, r_2)^* = id$, since the maps $q_i^* \circ r_i^*$ are the projections p_i's of the product $VA \otimes VB$; for example for $i = 1$ one has:

$$\begin{aligned}q^* \circ r_1^*(\Box a) &= q^*(\Box(a,1)) \\ &= \Diamond a \otimes 1 \\ &= p_1^*(\Box a)\end{aligned}$$

and

$$\begin{aligned}q^* \circ r_1^*(\Diamond a) &= q^*(\Diamond(a,0)) \\ &= (\Diamond a \otimes 1) \vee (1 \otimes 0) \\ &= p_1^*(\Diamond a).\end{aligned}$$

Thus q is an isomorphism; its naturality follows from the definition. $\square$

Lemma 42 *There is a natural map $d : V_2(A) \otimes V_2(B) \longrightarrow V_2(A \times B)$ defined by:*

1. $d^*(\square c) = \bigvee\{\square a \otimes \square b : a \otimes b \leq c\}$

and

2. $d^*(\Diamond c) = \bigvee\{\Diamond a \otimes \Diamond b : a \otimes b \leq c\}$

for any c in $A \otimes B$.

Proof. Remark that if c is an open rectangle, say $a \otimes b$, then $d^*(\Diamond(a \otimes b)) = \Diamond a \otimes \Diamond b$ but $d^*(\square(a \otimes b)) \neq \square a \otimes \square b$ since we always have $0 \otimes 1 = 0 \leq a \otimes b$, but $\square 0 \otimes 1 \not\leq \square a \otimes \square b$ unless $b = 1$ because $\square 0 \neq 0$; this also shows that, when we work with the strict Vietoris power locale, we have $\square 0 = 0$ and therefore $d^*(\square(a \otimes b)) = \square a \otimes \square b$. Now let us verify that d^* preserves the presenting relations (V1)-(V5):

(V1) and (V3) are similar: let us see for example (V3): note that every element of $A \otimes B$ is the join of the open rectangles it containes. So to verify that d^* preserves relation (V3) it is enough to prove it for joins of the kind $(\bigvee_i a_i) \otimes b$ (and similarly $a \otimes (\bigvee_j b_j)$):

$$\begin{aligned} d^*(\Diamond((\bigvee_i a_i) \otimes b)) &= (\Diamond \bigvee_i a_i) \otimes (\Diamond b) \\ &= (\bigvee_i \Diamond a_i) \otimes (\Diamond b) \\ &= \bigvee_i ((\Diamond a_i) \otimes (\Diamond b)) \\ &= \bigvee_i d^*(\Diamond(a_i \otimes b_i)). \end{aligned}$$

(V2) and (V5) are similar: let us verify for example (V2): let c_1, c_2 be in $A \otimes B$; clearly we have

$$d^*(\square(c_1 \wedge c_2)) \leq d^*(\square c_1) \wedge d^*(\square c_2).$$

To prove the reverse inequality remark that if $a_1 \otimes b_1 \leq c_1$ and $a_2 \otimes b_2 \leq c_2$, then:

$$\begin{aligned}(\Box a_1 \otimes \Box b_1) \wedge (\Box a_2 \otimes \Box b_2) &= \Box(a_1 \wedge a_2) \otimes \Box(b_1 \wedge b_2) \\ &\leq d^*(\Box(c_1 \wedge c_2))\end{aligned}$$

and hence by the distributivity of the meet over arbitrary joins we get

$$d^*(\Box c_1) \wedge d^*(\Box c_2) \leq d^*(\Box(c_1 \wedge c_2))$$

and therefore the equality holds.
(V4) follows from the calculation:

$$\begin{aligned}\Box a \otimes \Box(b_1 \vee b_2) &\leq (\Box a \otimes \Box b_1) \vee (\Box a \otimes \Diamond b_2) \\ &\leq (\Box a \otimes \Box b_1) \vee (\Box 0 \otimes \Diamond b_2) \vee (\Diamond a \otimes \Diamond b_2) \\ &\leq (\Box a \otimes \Box b_1) \vee (\Box 0 \otimes 1) \vee (\Diamond a \otimes \Diamond b_2) \\ &\leq d^*(\Box(a \otimes b_1)) \vee d^*(\Diamond(a \otimes b_2)).\end{aligned}$$

□

Remark that the previous lemma holds also for the Smyth powerlocale (where d^* is defined just by 1.), the Hoare power locale (where d^* is defined just by 2.) and for all strict power locales.
Let us denote by **SLat(Loc)** the category of localic semilattices.

Proposition 43 *For any locale A, $V_i(A)$ can be given a structure of semilattice in* **Loc**, *i.e. we can think of the V_i's as functors*

$$V_i : \mathbf{Loc} \longrightarrow \mathbf{SLat(Loc)}$$

for $i = 0, 1, 2$.

Proof. Let us denote by 1 the terminal locale in the topos **Set**, i.e. the two elements locale. We can consider the point $p_0 : 1 \longrightarrow VA$ defined by $p_0^*(\Box a) = 1$

and $p_0^{\bullet}(\Diamond a) = 0$. As binary semilattice operation take:

$$n : VA \otimes VA \xrightarrow{q} V(A \times A) \xrightarrow{V(\nabla)} VA$$

where q is defined as in 41 and $\nabla : A \times A \to A$ is the codiagonal map. Since

$$(V\nabla)^{\bullet}(\Box a) = \Box(a, a) \quad \text{and} \quad (V\nabla)^{\bullet}(\Diamond a) = \Diamond(a, a)$$

we have $n^{\bullet}(\Box a) = \Box a \otimes \Box a$ and $n^{\bullet}(\Diamond a) = (\Diamond a \otimes 1) \vee (1 \otimes \Diamond a)$.

Commutativity and associativity of n follow directcly from the definition. Idempotency, i.e. the fact that $n \circ \Delta = id_{VA}$, comes from the following two calculations:

$$\begin{aligned}
\Delta^{\bullet} \circ n^{\bullet}(\Box a) &= \Delta^{\bullet}(\Box a \otimes \Box a) \\
&= \Box a \wedge \Box a \\
&= \Box a
\end{aligned}$$

and

$$\begin{aligned}
\Delta^{\bullet} \circ n^{\bullet}(\Diamond a) &= \Delta^{\bullet}((\Diamond a \otimes 1) \vee (1 \otimes \Diamond a)) \\
&= \Diamond a \vee \Diamond a \\
&= \Diamond a.
\end{aligned}$$

We are only left to prove that p_0 is a unit for n, i.e. that

$$n \circ (id \otimes p_0) : VA \cong VA \otimes 1 \to VA \otimes VA \to VA$$

is the identity. This is true, since:

$$\begin{aligned}
(id \otimes p_0)^{\bullet} \circ n^{\bullet}(\Box a) &= (id \otimes p_o)^{\bullet}(\Box a \otimes \Box a) \\
&= \Box a \otimes 1
\end{aligned}$$

which corresponds to $\Box a$ under the isomorphism $VA \cong VA \otimes 1$, and similarly:

$$\begin{aligned}
(id \otimes p_0)^{\bullet} \circ n^{\bullet}(\Diamond a) &= (id \otimes p_0)^{\bullet}((\Diamond a \otimes 1) \vee (1 \otimes 0)) \\
&= (id \otimes p_0)^{\bullet}(\Diamond a \otimes 1) \\
&= \Diamond a \otimes 1
\end{aligned}$$

which corresponds in VA to $\Diamond a$. $\Box$

We can generalize this result to all V_i-algebras.

Proposition 44 *Any V_i-algebra has a natural semilattice structure, i.e. the forgetful functor from V_i-algebras to locales factors through the category of localic semilattices.*

Proof. Given a V-algebra $(A, \alpha : VA \to A)$ we can consider the following maps:

$$x_0 : 1 \xrightarrow{p_0} VA \xrightarrow{\alpha} A$$

and

$$s : A \otimes A \xrightarrow{\eta_A \otimes \eta_A} VA \otimes VA \xrightarrow{n} VA \xrightarrow{\alpha} A.$$

Let us start with two remarks:

1. any V-algebra homomorphism $f : (A, \alpha) \longrightarrow (B, \beta)$ is a homomorphism for the operations x_0 and s (since p_0, η_A, n are natural and because $f \circ \alpha = \beta \circ Vf$);
2. the operations induced on a free V-algebra (VA, μ_A) as above are exactly the operations defined in the previous proposition, since

- $p_o = \mu_A \circ p_0 : 1 \to V^2A \to VA$;

- $n = \alpha \circ n \circ (\eta_{VA} \otimes \eta_{VA}) : VA \otimes VA \to V^2A \otimes V^2A \to V^2A \to VA.$

The first equality holds trivially. The second follows from:

$$\begin{aligned}
(\eta_{VA} \otimes \eta VA)^* \circ n^* \circ \mu^*(\Box a) &= (\eta_{VA} \otimes \eta VA)^* \circ n^*(\Box(\Box a)) \\
&= (\eta_{VA} \otimes \eta VA)^*(\Box(\Box a) \otimes \Box(\Box a)) \\
&= \Box a \otimes \Box a \\
&= n^*(\Box a)
\end{aligned}$$

and similar calculation for $\Diamond a$.

Remark that $\alpha : (VA, \mu_A) \longrightarrow (A, \alpha)$ is a homomorphism of V-algebras and therefore by the two remarks it is a homomorphism from (VA, η, p_0) to (A, s, x_0). Now, by axiom of V-algebra, α is split epi in **Loc** since $\alpha \circ \eta_A = id_A$. Therefore s and x_0 satisfy all the equations of the theory of semilattices. Let us check for example that s is idempotent, i.e. that $s \circ \nabla_A = id_A$. We can consider the following diagram:

$$\begin{tikzcd}
A \arrow[rr, "\Delta_A"] \arrow[d, "\eta_A"'] & & A \otimes A \arrow[d, "\eta_A \otimes \eta_A"] \\
VA \arrow[rr, "\Delta_{VA}"] \arrow[dr, "id_{VA}"'] & & VA \otimes VA \arrow[dl, "n"] \\
& VA \arrow[d, "\alpha"] & \\
& A &
\end{tikzcd}$$

Remark that

- the square commutes because of the naturality of η_A;
- the triangle commutes since n is idempotent;
- the composite $\alpha \circ id_{VA} \circ \eta_A$ is the identity;
- the composite map $\alpha \circ n \circ \eta_A \otimes \eta_A$ commutes by definition of s.

Hence we get that $s \circ \nabla_A = id_A$. □

Given the uniqueness of the algebra structure in the Smyth and Hoare case, we can conclude that the the algebras for these two monads are particular classes of localic semilattices, rather then semilattices with a richer structure. The same result holds for the Vietoris monad. We can simplify the problem remarking that, for any locale A, we have the following isomorphism:

$$\varphi : 1 \times V_2^+ A \longrightarrow V_2 A$$

defined by

- $\varphi^*(\Box a) = (1, \Box a)$, and
- $\varphi^*(\Diamond a) = (0, \Diamond a)$

This map is well defined since Ω can be obtained from $\mathcal{O}(VA)$ imposing the further relations $\Box 0 \sim 1$ and $\Diamond 1 \sim 0$. Moreover it is an isomorphism, since we can define its inverse φ^{-1} by putting:

- $(\varphi^{-1})^*(1, \Box a) = \Box a$;
- $(\varphi^{-1})^*(0, \Box a) = \Diamond 1 \wedge \Box a$;
- $(\varphi^{-1})^*(0, \Diamond a) = \Diamond a$;
- $(\varphi^{-1})^*(1, \Diamond a) = \Diamond a \vee \Box 0$.

Then, if $(A, \alpha : VA \longrightarrow A)$ is a V-algebra structure inducing the semilattice structure (A, s, x), we have the diagram

$$\begin{tikzcd}
1 \arrow[d, "p_0"'] \arrow[dr, "x"] & \\
VA \arrow[r, "\alpha"] & A \\
V^+A \arrow[u] \arrow[ur, "\alpha^+"'] &
\end{tikzcd}$$

where the upper triangle commutes by the definition of x. Hence α is uniquely determined if and only if α^+ is.

Now let us call $(C^+(A), c)$ the equalizer of the maps

$$\begin{tikzcd}
A \otimes V^+(A) \arrow[r, "\eta_A^+ \otimes 1"] \arrow[drr, "\pi_1"'] & V^+(A) \otimes V^+(A) \arrow[rr, "d"] & & V^+(A \otimes A) \arrow[r, "V^+(s)"] & V^+(A) \\
& & A \arrow[urr, "\eta_A^+"'] & &
\end{tikzcd}$$

where π_1 is the projection of the product on the first component.

Lemma 45 *The map* $< \alpha^+, 1 >: V^+(A) \longrightarrow A \otimes V^+(A)$ *factors through* c.

Proof. Since $C^+(A)$ has been defined as an equalizer it is enough to show that the following diagram is commutative:

$$\begin{tikzcd} V^+(A) \arrow[r, "\beta"] \arrow[drr, "\alpha^+"'] & V^+(A)\otimes V^+(A) \arrow[r, "d"] & V^+(A\otimes A) \arrow[r, "V^+(s)"] & V^+(A) \\ & & A \arrow[ur, "\eta^+_A"'] & \end{tikzcd}$$

where $\beta = <\eta^+_A \circ \alpha^+, 1>$. But we only need to prove the commutativity of this diagram for free V^+-algebras since the above one then will be obtained using naturality of α^+ and the fact that $V^+(\alpha)$ is split. Hence we want to show the commutativity of

$$\begin{tikzcd} V^{+2}(A) \arrow[r, "\beta"] \arrow[drr, "\mu^+_A"'] & V^{+2}(A)\otimes V^{+2}(A) \arrow[r, "d"] & V^+(V^+A\otimes V^+A) \arrow[r, "V^+(s)"] & V^{+2}(A) \\ & & V^+A \arrow[ur, "\eta^+_{V^+A}"'] & \end{tikzcd}$$

where $\beta = <\eta^+_{V^+A} \circ \mu^+_A, 1>$.

The proof is just a matter of (long!) calculations: the reader is referred to [Joh85, Lemma 2.3] □

Lemma 46 *The composite of the maps:*

$$\begin{tikzcd} C^+(A) \arrow[r, "\varsigma"] & A\otimes V^+(A) \arrow[r, "1\otimes\alpha^+"] \arrow[rr, bend right, "\pi_1"'] & A\otimes A \arrow[r, "s"] & A \end{tikzcd}$$

are equal.

Proof. Though no conceptual difficulty arises, the proof is a rather involved matter of diagram chasing: again the reader is referred to [Vic22, Lemma 2.6]. □

We are finally ready for the main result.

Theorem 47 *Let (A, s, x) be a localic semilattice. Then there is at most one V_2-algebra structure on A inducing s and x.*

Proof. As observed previously we only need to show that if (A, α_1^+) and (A, α_2^+) are V_2^+-algebra structures, then α_1^+ and α_2^+ are equal. Consider the diagram:

$$\begin{tikzcd}
V^+(A) \arrow[r, "(\alpha_1^+,1)"] \arrow[dr] & A \otimes V^+(A) \arrow[r, "1\otimes\alpha_2^+"] \arrow[rr, bend left, "\pi_1"] & A \otimes A \arrow[r, "s"] & A \\
 & C^+(A) \arrow[u] \arrow[urr, "\pi_1 \circ c"'] & &
\end{tikzcd}$$

Since the two smaller triangles are commutative because of the two previous lemmas, we have

$$\pi_1 \circ (\alpha_1^+, 1) = s \circ (1 \otimes \alpha_2^+) \circ (\alpha_1^+, 1)$$

and therefore

$$\alpha_1^+ = s \circ (\alpha_1^+ \otimes \alpha_2^+).$$

Interchanging α_1^+ and α_2^+, and recalling that s is commutative, we get:

$$\begin{aligned}
\alpha_1^+ &= s \circ (\alpha_1^+ \otimes \alpha_2^+) \\
&= s \circ (\alpha_2^+ \otimes \alpha_1^+) \\
&= \alpha_2^+
\end{aligned}$$

$\square$

2.3 Points of the power locales

In this section we present a constructively sound characterization of the generalized points of power locales ([Vic95b]). The points of $V_i(D)$ at stage E (i.e. the locale homomorphisms from E to $V_i(D)$) are identified with particular sublocales of the product $E \otimes D$.

The proofs rely onto two "coverage theorems", that enable us to convert frame presentations into equivalent suplattice and preframe presentations.

Theorem 48 (The Suplattice Coverage Theorem) *Let S be a $\wedge$-semilattice. Let C be a relation from $\wp S$ to S such that if (X, u) is in C (read as X covers u) then:*

- *if $x \in X$ then $x \leq u$; and*
- *if $a \in S$ then $\{x \wedge a : x \in X\}$ covers $u \wedge a$.*

Then:

$$Fr < S \text{ (qua semi-lattice)} | u \leq \bigvee X \text{ for } (X, u) \in C >$$

$$\cong sl < S \text{ (qua poset)} | u \leq \bigvee X \text{ for } (X, u) \in C >$$

(where Fr stands for frame and sl for complete suplattice).

Proof. See [AV93]. □

Theorem 49 (The Preframe Coverage Theorem) *Let S be a $\vee$-semilattice and let C be a relation from $\wp Fin(S)$ to $Fin(S)$ such that if (X, G) is in C (read as X covers G) then:*

- *if F is in X then $F \leq_S G$;*
- *X is inhabited;*
- *if F_1 and F_2 are in X then there is some F in X with $F_1 \leq_S F$ and $F_2 \leq_S F$;*
- *if a is in S then $\{\{x \vee a : x \in F\} : F \in X\}$ covers $\{y \vee a : y \in G\}$.*

($Fin(S)$ stands for the finite powerset of S; $\leq_S$ stands for the the Smyth ordering on $Fin(S)$, i.e. $F \leq_S G$ if $\forall y \in G \,.\, \exists x \in F \,.\, x \leq y$).
Then:

$$Fr < S \text{ (qua } \vee\text{-semilattice)} | \bigwedge G \leq \bigvee_{F \in X} \bigwedge F \text{ for } (X, G) \in C >$$

$$\cong pFr < S \text{ (qua poset)} \mid \bigwedge G \leq \bigvee_{F \in X} \bigwedge F \text{ for } (X, G) \in C >$$

(where pFr stands for preframe).

Proof. See [JV91]. □

2.3.1 Points of the Smyth power locale

Classically there is an order reversing bijection between the global points of the Smyth power locale $V_0(D)$ and the compact fitted sublocales of D (a locale is *fitted* if it can be expressed as an intersection of open locales).

Indeed, a global point of $V_0(D)$ (i.e. a homomorphism of locales $1 \longrightarrow V_0(D)$) corresponds to a preframe map $X : \mathcal{O}(D) \longrightarrow \Omega$, since $\mathcal{O}V_0(D)$ is the frame freely generated by $\mathcal{O}(D)$ qua preframe.

To the map X then we can associate the (compact fitted) sublocale D' presented by the relations $1 \leq b$ for any b in $\mathcal{O}(D)$ such that $X(b) = 1$. Vice versa, given a compact fitted sublocale D' with presenting relations $1 \leq b_i$ for $i \in I$, we can define a preframe map $X : \mathcal{O}(D) \longrightarrow \Omega$ by putting $X(a) = 1$ iff $a = b_i$ for some i in I.

Clearly, the bigger is X, the more relations have to be considered and therefore the smaller is D': hence the bijection is order reversing.

This bijection can be set up also by constructively sound methods (see [Joh85, Lemma 3.4]). We present a generalized version of this result, as in [Vic95b].

Definition 46 *Let $f : D \longrightarrow E$ be a map of locales. We say that D is* compact over *E iff f is a proper map.*

This means ([Ver94]) that the right adjoint $\forall_f$ of f^* is a preframe homomorphism that satisfies the Frobenius identity

$$\forall_f(a \vee f^*(b)) = \forall_f(a) \vee b \tag{2.1}$$

for $a \in \mathcal{O}(D)$, $b \in \mathcal{O}(E)$.

Definition 47 *Let $D' \xrightarrow{i} D$ be a sublocale and let $f : D \longrightarrow E$ be a map of locales. We say that the locale D' of D has* compact domain over E *if $f \circ i$ is proper; in the case $E = 1$, we simply say that D' has* compact domain.

Classically a locale is compact over 1 exactly when it is a compact locale; in particular, a sublocale has compact domain exactly when it is compact.

Definition 48 *A sublocale D' of D is* weakly fitted over E *iff it can be presented over $\mathcal{O}(D)$ by relations of the kind $f^*(b) \leq a$ for a in $\mathcal{O}(D)$ and b in $\mathcal{O}(E)$; when $E = 1$ we simply say that D' is* weakly fitted.

When $E = 1$ the relations $f^*(b) \leq a$ are classically equivalent to either $0 \leq a$ (which can be omitted) or $1 \leq a$: hence, we recover the definition of fitted sublocale.

Definition 49 *We will call the* weakly fitted hull of D' in D over E *the least sublocale of D, fitted over E, that contains D'.*

The fitted hull of D' in D over E is presented over $\mathcal{O}(D)$ by all the relations of the kind $f^*(b) \leq a$ for a in $\mathcal{O}(D)$ and b in $\mathcal{O}(E)$.

Let E and D be locales. We will write $c\S b$ for $c \otimes 1 \vee 1 \otimes b$ ($c \in \mathcal{O}(E), b \in \mathcal{O}(D)$): these elements generate $E \otimes D$ by finite unions and directed joins. Let us call $p : E \otimes D \longrightarrow E$ and $q : E \otimes D \longrightarrow D$ the projections of the product.

Lemma 50 *Let $i : D' \longrightarrow E \otimes D$ be a sublocale that is compact over E. Denote by $\forall_{pi} : \mathcal{O}(D') \longrightarrow \mathcal{O}(E)$ the preframe homomorphism that exists by compactness of D'. Let $X : \mathcal{O}(D) \longrightarrow \mathcal{O}(E)$ be the preframe map $X = \forall_{pi} \circ i^* \circ q^*$. Then the weakly fitted hull of D' in $E \otimes D$ over E is presented by the relations*

$$X(b) \otimes 1 \leq 1 \otimes b \tag{2.2}$$

for b in $\mathcal{O}(D)$. If D' is weakly closed, then the relations 2.2 present D' itself.

Proof. The weakly fitted hull of D' in $E \otimes D$ over E is presented by the relations

$$a \otimes 1 \leq u \tag{2.3}$$

that hold modulo D', i.e. such that

$$i^* \circ p^*(a) \leq i^*(u). \tag{2.4}$$

The generic element u of $E \otimes D$ can be written as a directed join

$$u = \bigvee\{\wedge_j(c_j \S b_j) : \wedge_j(c_j \S b_j) \leq u\}$$

(where the intersection are finite). Then we can equivalently rewrite the equation 2.4 as

$$a \leq \bigvee\{\wedge_j(c_j \vee X(b_j)) : \wedge_j(c_j \S b_j) \leq u\}. \tag{2.5}$$

Indeed, since $\forall_{ip}$ is right adjoint to $i^* \circ p^*$ and it satisfies the Frobenius identity 2.1, one has:

$$\begin{aligned}
a &\leq \forall_{ip} \circ i^*(u) \\
&\leq \forall_{ip} \circ i^*(\bigvee\{\wedge_j(c_j \S b_j) : \wedge_j(c_j \S b_j) \leq u\}) \\
&= \bigvee\{\wedge_j \forall_{ip} \circ i^*(c_j \S b_j) : \wedge_j(c_j \S b_j) \leq u\} \\
&= \bigvee\{\wedge_j \forall_{ip}(i^*(c_j \otimes 1) \vee i^*(1 \otimes b_j)) : \wedge_j(c_j \S b_j) \leq u\} \\
&= \bigvee\{\wedge_j \forall_{ip}(i^* \circ p^*(c_j) \vee i^* \circ q^*(b_j)) : \wedge_j(c_j \S b_j) \leq u\} \\
&= \bigvee\{\wedge_j(c_j \vee \forall_{ip} \circ i^* \circ q^*(b_j)) : \wedge_j(c_j \S b_j) \leq u\} \\
&= \bigvee\{\wedge_j(c_j \vee X(b_j)) : \wedge_j(c_j \S b_j) \leq u\}.
\end{aligned}$$

Hence the relations $X(b) \otimes 1 \leq 1 \otimes b$ are a particular case of 2.3, with $a = X(b)$ and $u = 1 \otimes b$ and they hold modulo D' since in the inequality 2.5 we can consider the meet $\wedge_j(c_j \vee X(b_j))$ where j ranges over the singleton, $c_j = 0$ and $b_j = b$.

Vice versa, if $a \otimes 1 \leq u$ holds modulo D' (i.e. if the equation 2.5 holds) we can deduce it from the relations of the kind $X(b) \otimes 1 \leq 1 \otimes b$:

$$\begin{aligned}
a \otimes 1 &\leq (\bigvee\{\wedge_j(c_j \vee X(b_j)) : \wedge_j(c_j \S b_j) \leq u\}) \otimes 1 \\
&= \bigvee\{\wedge_j(c_j \vee X(b_j)) \otimes 1 : \wedge_j(c_j \S b_j) \leq u\} \\
&= \bigvee\{\wedge_j((c_j \otimes 1) \vee (X(b_j) \otimes 1)) : \wedge_j(c_j \S b_j) \leq u\} \\
&\leq \bigvee\{\wedge_j(c_j \S b_j : \wedge_j(c_j \S b_j) \leq u)\} \\
&\leq u.
\end{aligned}$$

Therefore, the equations 2.2 present the weakly fitted hull of D'. Since the hull is the smallest weakly fitted sublocale of D containing D', it coincides with D' when D' is weakly fitted. □

Lemma 51 *Let $Y : \mathcal{O}(D) \longrightarrow \mathcal{O}(E)$ be a preframe homomorphism. Consider the sublocale D' presented over $\mathcal{O}(D)$ by the relations $Y(b) \otimes 1 \leq 1 \otimes b$ for b in $\mathcal{O}(D)$. Then the preframe map X, defined as in Lemma 50 ,coincides with Y and D' is a locale that, over E, is fitted with compact domain.*

Proof. We will show that:

1. there is a preframe homomorphism $\forall : \mathcal{O}(D') \longrightarrow \mathcal{O}(E)$ such that for any a in $\mathcal{O}(E)$ and b in $\mathcal{O}(D)$ one has $\forall \circ i^*(a\S b) = a \vee Y(b)$;

2. D' has compact domain over E since $\forall$ is a right adjoint to $i^* \circ p^*$ and it satisfies the Frobenius identity 2.1 (with $\forall = \forall_f$ and $f = p \circ i$).

Hence, choosing $a = 0$ in $\forall \circ i^*(a\S b) = a \vee Y(b)$ we see that $\forall \circ i^* \circ q^* = Y$. Then the map Y coincides with X, in the notation of the previous lemma (since $\forall = \forall_{pi}$) and D' is weakly fitted.

1. We can think of $\mathcal{O}(D')$ in the following ways:

$$
\begin{aligned}
&\cong Fr \quad < \mathcal{O}(E)\times\mathcal{O}(D) \quad (\text{qua } \vee\text{-semilattice}) \mid \\
&\qquad\qquad\qquad\qquad \S \text{ bilinear w.r.t. directed joins and finite meets} \\
&\qquad\qquad\qquad\qquad (a \vee Y(b'))\S b \leq a\S b \text{ for } b' \leq b > \\
&\cong pFr \quad < \mathcal{O}(E)\times\mathcal{O}(D) \quad (\text{qua poset}) \mid \text{same relations } > .
\end{aligned}
$$

The isomorphism is justified by Theorem 49, where the covering relation contains the elements of the kind:

- $(\{(\bigvee_i a_i, b)\}, \{\{(a_i, b)\} | i \in I\})$ for $\{a_i\}_{i\in I}$ directed family;
- $(\{(a, b), (a', b)\}, \{\{(a \wedge a', b)\}\})$;
- $(\{(a \vee Y(b'), b)\}, \{\{(a, b)\}\})$ where $b' \leq b$.

Hence we can define $\forall$ to be the only homomorphism of preframes making the following diagram commutative

$$
\begin{tikzcd}
\mathcal{O}(E)\times\mathcal{O}(D) \arrow[r] \arrow[dr, "e"'] & \mathcal{O}(E)\otimes\mathcal{O}(D) \arrow[r] & \mathcal{O}(D') \arrow[dl, "\forall"] \\
& \mathcal{O}E &
\end{tikzcd}
$$

where e is defined by $e(a, b) = a \vee Y(b)$ (remark that e preserves the order and the presenting relations). Hence 1. is proved.

2. To prove that $\forall$ is right adjoint to $i^* \circ p^*$ it is enough to show that $id_{\mathcal{O}(E)} \leq \forall \circ i^* \circ p^*$ and $i^* \circ p^* \circ \forall \leq id_{\mathcal{O}(D')}$. The first inequality follows from:

$$
\begin{aligned}
c &\leq c \vee Y(0) \\
&= \forall \circ i^*(c\S 0) \\
&= \forall \circ i^*(c\S 1) \\
&= \forall \circ i^* \circ p^*(c).
\end{aligned}
$$

To prove the second inequality, observe that the generic element of $\mathcal{O}(D')$ is of the form $i^*(a\S b)$. Hence we have:

$$
\begin{aligned}
i^* \circ p^* \circ \forall \circ i^*(c\S b) &= i^* \circ p^*(a \vee Y(b)) \\
&= i^*((a \vee Y(b)) \otimes 1) \\
&= i^*((a \otimes 1) \vee (Y(b) \otimes 1) \\
&= i^*((a\S 0) \vee (Y(b)\S 0)) \\
&= i^*(a \vee Y(b)\S 0) \\
&\leq i^*(a\S b).
\end{aligned}
$$

The Frobenius identity 2.1 we have to prove is:

$$\forall(a \vee i^* \circ p^*(b)) = \forall a \vee b$$

where a is in $\mathcal{O}(D')$ and b in $\mathcal{O}(E)$. Since a is of the form $i^*(c\S d)$ for suitable c in $\mathcal{O}(E)$ and d in $\mathcal{O}(D)$, we have

$$
\begin{aligned}
\forall(a \vee i^* \circ p^*(b)) &= \forall(i^*(c\S d) \vee i^* \circ p^*(b)) \\
&= \forall \circ i^*((c\S d) \vee (b \otimes 1) \\
&= \forall \circ i^*((c \vee b)\S d) \\
&= c \vee b \vee Y(d) \\
&= \forall \circ i^*(c\S d) \vee b \\
&= \forall a \vee b.
\end{aligned}
$$

□

Theorem 52 *There is an order reversing bijection, natural in E, between the points of $V_0(D)$ at stage E and the sublocales of $E \otimes D$ that, over E, are weakly fitted with compact domain.*

Proof. To prove the theorem we can equivalently show that there is an order reversing isomorphism, natural in E, between the poset of preframe homomorphisms from $\mathcal{O}(D)$ to $\mathcal{O}(E)$ and the poset of sublocales of $E \otimes D$ that, over E, are weakly fitted with compact domain. The bijection has been set up in the previous two lemmas:

- to any sublocale $i : D' \longrightarrow E \otimes D$ which, over E, is weakly fitted with compact domain, we can associate the preframe map X defined in Lemma 50;

- to any preframe homomorphism X we can associate the sublocale $D' \longrightarrow E \otimes D$ defined as in Lemma 51.

The bijection reverses the order since the bigger is X, the less restraining the presenting relations of D' are. Hence we are only left to verify the naturality of the isomorphism. Consider a homomorphism of locales, say $f : E' \longrightarrow E$. To a sublocale $i : D' \longrightarrow E \otimes D$ the map f associates the sublocale $D'' \longrightarrow E' \otimes D$ obtained by pulling back i along $f \otimes id_D$: then D'' is presented by the relations $1 \otimes b \leq f^* \circ X(b) \otimes 1$. This proves the naturality of the bijection, since f acts on preframe homomorphisms by associating to X the map $f^* \circ X$. □

Corollary 53 *There is an order reversing bijection between the global points of $V_0(D)$ and the weakly fitted compact sublocales of D.*

Proof. This is an immediate consequence of the previous theorem: we only need to put $E = 1$. □

2.3.2 Points of the Hoare power locale

Classically there is an order-preserving bijection between the global points of the Hoare power locale $V_1(D)$ and the closed sublocales of D. Indeed, since $\mathcal{O}V_1(D)$

can be seen as the frame freely generated by $\mathcal{O}(D)$ qua suplattice, global points of $V_1(D)$ correspond to suplattice maps $X : \mathcal{O}(D) \longrightarrow \Omega$; these in turn correspond to opens of $\mathcal{O}(D)$, according to the following order reversing bijection:

to any X associate $a_X = \bigvee\{b \in \mathcal{O}(D) : X(b) = 0\}$;

to any a in $\mathcal{O}(D)$ associate the map X defined as $X(b) = 0$ iff $b \leq a$.

To obtain an order preserving isomorphism we can identify a global point of $V_1(D)$ (corresponding to a suplattice map X) with the closed sublocale $D - a_X$.
Now, this locale is presented by the relations $b \leq !_D^* \circ X(b)$ (for b in $\mathcal{O}(D)$), where $!_D$ is the unique map of locales from D to 1. Classically this inequality is either $b \leq 1$, which can be disregarded, or $b \leq 0$. In a general topos, however, the map $!_D^*$ can assume values other then 0 and 1: hence the need of generalizing the notion of closed sublocale ([Joh89]).

Definition 50 *Let $f : D \longrightarrow E$ be a locale homomorphism, D' a sublocale of D. D' is* weakly closed over E *if its frame can be presented over $\mathcal{O}D$ by relations of the form $a \leq f^*(b)$, for a in $\mathcal{O}(D)$ and b in $\mathcal{O}E$; in the case $E = 1$ we simply say that D' is* weakly closed.

Classically, the notion of weakly closed locale is equivalent to the notion of closed locale.

Definition 51 *We will call* weak closure of D' over D *the smallest sublocale of D which is weakly closed over E and contains D'.*

The weak closure of D' over D is presented by all the relations $a \leq f^*(b)(a \in \mathcal{O}(D), b \in \mathcal{O}(E))$ that hold modulo D' (i.e. such that $i^*(a) \leq i^* \circ f^*(b)$).

However, this is not yet enough to characterize the points of $V_1(D)$. Bunge and Funk ([BF96]) proved that the poset $W(D)$ of weakly closed sublocales (which is a subcoframe of $Sub(D)$, see [JJ91]) is isomorphic to the poset of suplattice maps from $\mathcal{O}(D)$ to $Sub(1)$. The isomorphism is given by the map χ defined on a sublocales $i : B \longrightarrow D$ as

$$\chi_B(U) = ||B \cap U||$$

for any open U of $\mathcal{O}(D)$, where $||B \cap U||$ is the image of the map $B \cap U \longrightarrow B \xrightarrow{i} D \xrightarrow{!_D} 1$. The map χ_B gives rise to a suplattice map $\chi_{0_B} : \mathcal{O}(D) \longrightarrow \Omega$ when $||B \cap U||$ is an open sublocale for all open U of $\mathcal{O}(D)$: this happens exactly when B has "open domain".

Definition 52 *Let $f : D \longrightarrow E$ be a locale homomorphism. D is* open over E *iff f is an open map.*

This means (see [JT84]) that the frame map f^* has a left adjoint $\exists_f$ which is a homomorphism of $\mathcal{O}(E)$-modules, i.e. that, besides preserving suprema, satisfies the Frobenius identity

$$\exists_f(a \wedge f^*(b)) \;=\; \exists_f(a) \wedge b. \tag{2.6}$$

for a in $\mathcal{O}(D)$ and b in $\mathcal{O}(E)$.

Definition 53 *Let $f : D \longrightarrow E$ be a locale homomorphism, D' a sublocale of D. D' has* open domain over E *if the map $D' \longrightarrow D \xrightarrow{f} E$ is open; in the case $E = 1$, we simply say that D' has* open domain.

Classically, any locale has open domain.

The restriction of the map χ_0 yields an isomorphism between the poset of weakly closed locales with open domain and the poset of suplattice maps from $\mathcal{O}(D)$ to Ω (see [BF96, Theorem 2.1]). Since suplattice maps from $\mathcal{O}(D)$ to Ω correspond to frame maps from $\mathcal{O}(V_1(D))$ to Ω (i.e. points of $V_1(D)$), the restriction of the map in question yields an isomorphism between the poset of weakly closed sublocales of D with open domain and the poset of the points of $V_1(D)$.

We will show that this result holds also for the generalized points of $V_1(D)$ ([Vic92]).

Lemma 54 *Let $i : D' \longrightarrow E \otimes D$ be a sublocale open over E. Let $\exists_{pi} : \mathcal{O}(D') \longrightarrow \mathcal{O}(E)$ be the $\mathcal{O}(E)$-homomorphism that exists by openness of D'. Denote by X the suplattice homomorphism $X = \exists_{pi} \circ i^* \circ q^* : \mathcal{O}(D) \longrightarrow \mathcal{O}(E)$. Then the weak closure of D' in $E \otimes D$ over E can be presented by the relations*

$$1 \otimes b \leq X(b) \otimes 1 \tag{2.7}$$

for $b \in \mathcal{O}(D)$. If D' is weakly closed, then the relations 2.7 present D' itself.

Proof. The weak closure of D' in $E \otimes D$ over E is presented by the relations $x \leq p^*(a) = a \otimes 1$ (x in $\mathcal{O}(E) \otimes \mathcal{O}(D)$, a in $\mathcal{O}(E)$) that hold modulo D'. Since the elements of the kind $c \otimes b$ (c in $\mathcal{O}(E)$, b in $\mathcal{O}(D)$) form a base of $\mathcal{O}(E) \otimes \mathcal{O}(D)$, we can simply consider the relations of the form

$$c \otimes b \leq a \otimes 1 \tag{2.8}$$

that hold modulo D', i.e. the ones such that

$$i^*(c \otimes b) \quad \leq \quad i^*(a \otimes 1),$$

which can be rewritten as

$$i^* \circ p^*(c) \wedge i^* \circ q^*(b) \leq i^* \circ p^*(a).$$

This is equivalent to

$$\exists_{pi}(i^* \circ p^*(c) \wedge i^* \circ q^*(b)) \leq a$$

since $\exists_{pi}$ is left adjoint to $i^* \circ p^*$, and to

$$c \wedge X(b) \leq a \tag{2.9}$$

since $\exists_{pi}$ satisfies the Frobenius identity 2.6.
Now, the inequality 2.9 always holds for $c = 1$ and $a = X(b)$. Hence all the inequality of the kind $1 \otimes b \leq X(b) \otimes 1$ hold modulo D' and are a special case of the relations of the kind 2.8. On the other hand, if $c \otimes b \leq a \otimes 1$ holds modulo D', i.e. if $c \wedge X(b) \leq a$, we can recover it from $1 \otimes b \leq X(b) \otimes 1$, since by taking the meet with $c \otimes 1$ we get

$$c \otimes b \leq (c \wedge X(b)) \otimes 1 \leq a \otimes 1.$$

□

Lemma 55 *Let $Y : \mathcal{O}(D) \longrightarrow \mathcal{O}(E)$ be a suplattice homomorphism. Denote by $i : D' \longrightarrow E \otimes D$ the sublocale defined over $\mathcal{O}(D) \otimes \mathcal{O}(E)$ by the relations $1 \otimes b \leq Y(b) \otimes 1$ for b in $\mathcal{O}(D)$. Then the suplattice map X, defined as in Lemma 54, coincides with Y and D' is a sublocale that, over E, is open and weakly closed.*

Proof. We will show that:

1. there is a suplattice homomorphism $\exists : \mathcal{O}(D') \longrightarrow \mathcal{O}(E)$ such that for any a in $\mathcal{O}(E)$ and b in $\mathcal{O}(D)$ one has $\exists \circ i^*(a \otimes b) = a \wedge Y(b)$;

2. D' is open over E since the map $\exists$ is left adjoint to $i^* \circ p^*$ and it satisfies the Frobenius identity 2.6 (with $f^* = i^* \circ p^*$ and $\exists_f = \exists$).

Then, from $\exists \circ i^*(a \otimes b) = a \wedge Y(b)$ we see that $Y = \exists \circ i^* \circ q^*$ (just set $a = 1$). Since $\exists = \exists_{pi}$ (in the notation of the previous lemma), the map Y coincides with the map X and the weak closure of D' with D' itself: hence D' is weakly closed.
1. We can think of $\mathcal{O}(D')$ in the following ways:

$$
\begin{aligned}
& Fr < \mathcal{O}(E) \times \mathcal{O}(D) \text{ (qua meet-semilattice)} \mid \\
& \qquad \times \text{ is bilinear w.r.t. } \textstyle\bigvee : 1 \times b \leq Y(b) \times 1 > \\
\simeq\ & Fr < \mathcal{O}(E) \times \mathcal{O}(D) \text{ (qua meet-semilattice)} \mid \\
& \qquad \times \text{ is bilinear w.r.t. } \textstyle\bigvee : a \times b \leq (a \wedge Y(b')) \times b \text{ for } b \leq b' > \\
\cong\ & sl < \mathcal{O}(E) \times \mathcal{O}(D) \text{ (qua poset)} \mid \\
& \qquad \times \text{ is bilinear w.r.t. } \textstyle\bigvee : a \times b \leq (a \wedge Y(b')) \times b \text{ for } b \leq b' >
\end{aligned}
$$

where the last isomorphism holds because of Theorem 48, the elements of the covering relation being of the kind:

$$(\{(a_i, 1) : i \in I\}, (\textstyle\bigvee_{i \in I} a_i, 1));$$

$$(\{(1, b_j) : j \in J\}, (1, \textstyle\bigvee_{j \in J} b_j));$$

$$(\{(a \wedge Y(b'), b)\}, (a, b)) \text{ where } b \leq b'.$$

Then we can define $\exists$ to be the unique suplattice homomorphism completing the diagram

$$
\begin{tikzcd}
\mathcal{O}(E) \times \mathcal{O}(D) \arrow[r] \arrow[dr, "\epsilon"'] & \mathcal{O}(E) \otimes \mathcal{O}(D) \arrow[r] & \mathcal{O}(D') \arrow[dl, "\exists"] \\
& \mathcal{O}E &
\end{tikzcd}
$$

where the map ϵ is defined by $\epsilon(a, b) = a \wedge Y(b)$ (remark that it preserves the order and the presenting relations). Hence 1. has been proved.

2. To prove that $\exists$ is left adjoint to $i^* \circ p^*$ it is enough to show that $id_{\mathcal{O}(D')} \leq i^* \circ p^* \circ \exists$ and $\exists \circ i^* \circ p^* \leq id_{\mathcal{O}(E)}$. Since the generic element of $\mathcal{O}(D')$ is a join of elements of the kind $i^*(a \otimes b)$ (for a in $\mathcal{O}(E)$, b in $\mathcal{O}(D)$), the first inequality follows from:

$$
\begin{aligned}
i^*(a \otimes b) &\leq i^*(a \wedge Y(b) \otimes b) \\
&\leq i^*(a \wedge Y(b) \otimes 1) \\
&= i^* \circ p^*(a \wedge Y(b)) \\
&= i^* \circ p^* \circ \exists \circ i^*(a \otimes b).
\end{aligned}
$$

The second inequality follows from:

$$
\begin{aligned}
\exists \circ i^* \circ p^*(a) &= \exists \circ i^*(a \otimes 1) \\
&= a \wedge Y(1) \\
&\leq a.
\end{aligned}
$$

Now to verify that the Frobenius identity holds, we only need to check $\exists(i^* \circ p^*(a) \wedge i^*(c \otimes b)) = a \wedge \exists \circ i^*(c \otimes b)$ $(a, c \in \mathcal{O}(E), b \in \mathcal{O}(D))$, since all the maps involved are suplattices homomorphisms and the $i^*(c \otimes b)$'s generate $\mathcal{O}(D')$ by unions:

$$
\begin{aligned}
\exists(i^* \circ p^*(a) \wedge i^*(c \otimes b)) &= \exists(i^*(p^*(a) \wedge (c \otimes b)) \\
&= \exists \circ i^*((a \wedge c) \otimes b) \\
&= (a \wedge c) \wedge Y(b) \\
&= a \wedge \exists \circ i^*(c \otimes b).
\end{aligned}
$$

□

The following is a a generalized version, due to Vickers ([Vic95b]), of the Bunge-Funk theorem ([BF96]) constructively characterizing the points of the Hoare power locale $V_1(D)$.

Theorem 56 *There is an order isomorphism, natural in E, between the points of $V_1(D)$ at stage E and the sublocales of $E \otimes D$ that, over E, are weakly closed with open domain.*

Proof. To prove the theorem we can equivalently show that there is an order isomorphism, natural in E, between the poset of suplattice homomorphisms from $\mathcal{O}(D)$ to $\mathcal{O}(E)$ and the poset of sublocales of $E \otimes D$ that, over E, are weakly closed with open domain.

The bijection has been set up in the previous two lemmas:

to any sublocale $i : D' \longrightarrow E \otimes D$ which, over E, is open and weakly closed we can associate the suplattice map defined in Lemma 54;

to any suplattice map X, we can associate the sublocale $D' \longrightarrow E \otimes D$ defined in Lemma 55.

It is clear that the bijection preserves the order. Hence we are only left to verify the naturality of the isomorphism. Consider a homomorphism of locales, say $f : E' \longrightarrow E$. To a sublocale $i : D' \longrightarrow E \otimes D$ the map f associates the sublocale $D'' \longrightarrow E' \otimes D$ obtained by pulling back i along $f \otimes id_D$: then D'' is presented by the relations $1 \otimes b \leq f^* \circ X(b) \otimes 1$. This proves the naturality of the bijection, since f acts on suplattice homomorphisms by associating to X the map $f^* \circ X$. □

Corollary 57 *There is an order isomorphism between the global points of* $V_1(D)$ *and the weakly closed sublocales of D with open domain.*

Proof. The result follows immediately from the previous theorem, considering $E = 1$. □

2.3.3 Points of the Vietoris power locale

Global points of the Vietoris power locale have been studied classically by Johnstone ([John85, Theorem 3.7]): he identified them with the set of compact semifitted sublocales of D (a sublocale is semifitted if it is the intersection of a fitted sublocale and of a closed sublocale).

The result, however, is valid also constructively, provided that we introduce the necessary generalizations, as in the previous two sections ([Vic95b]).

Definition 54 *Let $D' \xrightarrow{i} D$ be a sublocale, $f : D \longrightarrow E$ a map of locales. We say that D' is* weakly semifitted over *E iff it is a meet of a weakly fitted sublocale and a weakly closed sublocale over E; when $E = 1$, we simply say that D' is weakly semifitted.*

In terms of presenting relations, D' is weakly semifitted if it can be presented by relations of the form

$$b \leq f^*(a) \tag{2.10}$$

$$f^*(a') \leq b' \tag{2.11}$$

for a, a' in $\mathcal{O}(E)$ and b, b' in $\mathcal{O}(D)$.

Remark that, classically, weakly semifitted coincides with semifitted.

If $f : E \longrightarrow D$ is an open map, we can consider the sup-preserving map $\exists_f$ which is left adjoint to f^*. Recalling that $\mathcal{O}V_1(D)$ is the frame freely generated by $\mathcal{O}(D)$ qua suplattice, we can define uniquely a map of frames l^* such that $l^*(\Diamond a) = \exists_f(a)$, i.e. such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{O}(D) \arrow[rr] \arrow[dr, "\exists_f"'] & & \mathcal{O}(V_1(D)) \arrow[dl, "l^*"] \\
& \mathcal{O}(E) &
\end{tikzcd}$$

Similarly, if $f : E \longrightarrow D$ is a proper map, we can consider the preframe homomorphism $\forall_f$ which is right adjoint to f^*. Then there is exactly one map of frames r^* such that $r^*(\Box a) = \forall_f a$, since $\mathcal{O}(V_0(D))$ is the frame freely generated by $\mathcal{O}(D)$ qua preframe and r^* makes the following diagram commute:

$$\begin{tikzcd}
\mathcal{O}(D) \arrow[rr] \arrow[dr, "\forall_f"'] & & \mathcal{O}(V_0(D)) \arrow[dl, "r^*"] \\
& \mathcal{O}(E) &
\end{tikzcd}$$

Lemma 58 *Let $f : D \longrightarrow E$ be an open and proper map. Then $< l, r >: E \longrightarrow V_1(D) \otimes V_0(D)$ factors through a point ξ of $V_2(D)$ at stage E.*

Proof. Consider the maps $\Downarrow_D: V_2(D) \longrightarrow V_1(D)$ and $\Uparrow_D: V_2(D) \longrightarrow V_0(D)$ defined as $\Downarrow_D^*(\Diamond a) = \Diamond a$ and $\Uparrow_D^*(\Box b) = \Box b$: they are well defined, since any presenting relation of $V_0(D)$ and $V_1(D)$ is also a presenting relation of $V_2(D)$. Then the homomorphism $<\Downarrow_D, \Uparrow_D>: V_2(D) \longrightarrow V_1(D) \otimes V_0(D)$ defines a sublocale, since any element of $V_2(D)$ is a join of elements of the kind $\Diamond a_1 \wedge \ldots \wedge \Diamond a_n \wedge \Box b$ for suitable a_i and b in $\mathcal{O}(D)$. Then there is exactly one map of locales $\xi : E \longrightarrow V_2(D)$ such that

$$\begin{tikzcd}
E \arrow[rr, "<l,r>"] \arrow[dr, "\xi"'] & & V_1(D) \otimes V_0(D) \\
& V_2(D) \arrow[ur, "<\Downarrow_D, \Uparrow_D>"'] &
\end{tikzcd}$$

is a commutative diagram. Indeed, putting $\xi^*(\Box a) = \forall_f(a)$ and $\xi^*(\Diamond b) = \exists_f(b)$ we define the required map. Because of the properties of $\exists_f$ and $\forall_f$ we only need to check the mixed relations:

$$\begin{aligned}
\xi^*(\Box a) \wedge \xi^*(\Diamond b) &= \forall_f(a) \wedge \exists_f(b) \\
&= \exists_f(f^* \circ \forall_f(a) \wedge b) \\
&\leq \exists_f(a \wedge b) \\
&= \xi^*(\Diamond(a \wedge b))
\end{aligned}$$

since $\exists_f$ satisfies the Frobenius identity 2.6 and $f^* \circ \forall_f \leq id$;

$$\begin{aligned}
\xi^*(\Box(a \vee b)) &= \forall_f(a \vee b) \\
&\leq \forall_f(a \vee f^* \circ \exists_f(b)) \\
&= \forall_f(a) \vee \exists_f(b) \\
&= \xi^*(\Box a) \vee \xi^*(\Diamond b)
\end{aligned}$$

since $id \leq f^* \circ \exists_f$ and $\forall_f$ satisfies the Frobenius identity 2.1. □

Lemma 59 *Let $i : D' \longrightarrow E \otimes D$ be a sublocale with compact, open domain. Consider the map of locales $\xi : E \longrightarrow V_2(D')$ defined as in the previous lemma.*

Let $X : E \longrightarrow V_2(D)$ be the point of $V_2(D)$ defined by $X = V_2q \circ V_2i \circ \xi$. The smallest weakly semifitted sublocale of $E \otimes D$ containing D' can be presented by the relations:

$$1 \odot b \;\leq\; X^*(\Diamond b) \odot 1 \tag{2.12}$$

$$X^*(\Box b) \odot 1 \;\leq\; 1 \odot b \tag{2.13}$$

for b in $\mathcal{O}(D)$.

In particular, if D' is weakly semifitted, the equations 2.12 and 2.13present D' itself.

Proof. The smallest weakly semifitted sublocale of $E \otimes D$ containing D' is clearly the intersection of the weak closure $Cl(D')$ of D' and the weakly fitted hull $H(D')$ of D'. Since D' has compact and open domain, we have the adjunctions $\exists_{pi} \dashv p^* \circ i^* \dashv \forall_{pi}$ and, applying lemmas 54 and 50, we can present $Cl(D')$ by the relations

$$1 \odot b \;\leq\; \exists_{pi} \circ i^* \circ q^*(b) \odot 1 \tag{2.14}$$

and $H(D')$ by the relations

$$\forall_{pi} \circ i^* \circ q^*(b) \odot 1 \;\leq\; 1 \odot b \tag{2.15}$$

for b in $\mathcal{O}(D)$. Hence D' is presented by the equation 2.14 and 2.15 together and they are equivalent to the equations 2.12 and 2.13. Indeed, since $\xi^*(\Box b) = \forall_{pi}(b)$ and $\xi^*(\Diamond b) = \exists_{pi}(b)$ (b in $\mathcal{O}(D)$), one has:

$$\begin{aligned} X^*(\Diamond b) &= \xi^* \circ V_2 i^* \circ V_2 q^*(\Diamond b) \\ &= \xi^* \circ V_2 i^*(\Diamond(1 \odot b)) \\ &= \xi^*(\Diamond i^*(1 \odot b)) \\ &= \exists_{pi} \circ i^*(1 \odot b) \\ &= \exists_{pi} \circ i^* \circ q(b) \end{aligned}$$

and

$$
\begin{aligned}
X^*(\Box b) &= \xi^* \circ V_2 i^* \circ V_2 q(\Box b) \\
&= \xi^* \circ V_2 i^*(\Box(1 \otimes b)) \\
&= \xi^*(\Box i^*(1 \otimes b)) \\
&= \forall_{pi} \circ i^*(1 \otimes b) \\
&= \forall_{pi} \circ i^* \circ q^*(b)
\end{aligned}
$$

□

Lemma 60 *Let $Y : E \longrightarrow V_2(D)$ be a map of locales. The sublocale $D' \xrightarrow{i} E \otimes D$ presented by the equations*

$$1 \otimes b \leq Y^*(\Diamond b) \otimes 1 \tag{2.16}$$

$$Y^*(\Box b) \otimes 1 \leq 1 \otimes b \tag{2.17}$$

(for b in $\mathcal{O}(D)$) is weakly semifitted over E and has compact and open domain. Moreover, the map X, defined as in Lemma 58 with $f = pi$, coincides with the map Y.

Proof. Consider the weakly semifitted sublocale $i : D' \longrightarrow E \otimes D$ defined by the relations

$$1 \otimes b \leq Y^*(\Diamond b) \otimes 1$$

and

$$Y^*(\Box b) \otimes 1 \leq 1 \otimes b$$

for b in $\mathcal{O}(D)$. We will show that

1. there is a suplattice homomorphism $\exists_{pi} : \mathcal{O}(D') \longrightarrow \mathcal{O}(E)$ left adjoint to $p^* \circ i^*$ such that $\exists_{pi} \circ i^*(a \otimes b) = a \wedge Y^*(\Diamond b)$ and the Frobenius identity 2.6 holds;

2. there is a preframe homomorphism $\forall_{pi} : \mathcal{O}(D') \longrightarrow \mathcal{O}(E)$ right adjoint to $p^* \circ i^*$ and such that $\forall_{pi} \circ i^*(a\S b) = a \vee Y^*(\Box b)$ and the Frobenius identity 2.1 holds.

Hence the domain of D' is open (because of 1.) and compact (because of 2.). Also, we can consider the points $l : E \longrightarrow V_1(D)$ and $r : E \longrightarrow V_0(D)$ (notations as in the previous lemma) and the point $\xi : E \longrightarrow V_2(D)$ such that $< l, r > = <\Downarrow_{D'}, \Uparrow_{D'}> \circ \xi$. We want to show that the point $X = V_2 q \circ V_2 i \circ \xi$ coincides with Y. This happens iff $<\Downarrow_D, \Uparrow_D> \circ X = <\Downarrow_D, \Uparrow_D> \circ Y$ (recall that $<\Downarrow_D, \Uparrow_D>$ is a monomorphism), i.e. iff $\Downarrow_D \circ X = \Downarrow_D \circ Y$ and $\Uparrow_D \circ X = \Uparrow_D \circ Y$. This is the case, since we have:

$$\begin{aligned}
X^* \circ \Downarrow_D^* (\Diamond b) &= X^*(\Diamond b) \\
&= \xi^* \circ V_2 i^* \circ V_2 q^*(\Diamond b) \\
&= \xi^*(\Diamond i^*(1 \otimes b)) \\
&= \exists_{pi} \circ i^*(1 \otimes b) \\
&= Y^*(\Diamond b) \\
&= Y^* \circ \Downarrow_D^* (\Diamond b)
\end{aligned}$$

and similarly for the other equality (just replace $\Downarrow$ by $\Uparrow$ and $\Diamond$ by $\Box$).

So, we are only left to prove statements 1. and 2.

1. The proof goes along the lines of Lemma 55. Applying Theorem 48, we can

think of $\mathcal{O}(D')$ in the following ways:

$$\begin{array}{rl} Fr < \mathcal{O}(E)\times\mathcal{O}(D) \text{ (qua} & \text{meet-semilattice)} \mid \\ & \odot \text{ is bilinear w.r.t. } \bigvee; \\ & 1 \odot b \le Y^*(\Diamond b) \odot 1; \\ & Y^*(\Box b) \odot 1 \le 1 \odot b > \\ \cong Fr < \mathcal{O}(E)\otimes\mathcal{O}(D) \text{ (qua} & \text{meet-semilattice)} \mid \\ & \otimes \text{ is bilinear w.r.t. } \bigvee; \\ & a \otimes b \le (a \wedge Y^*(\Diamond b')) \otimes b \text{ for } b \le b'; \\ & (a \wedge Y^*(\Box b')) \otimes b \le (a \wedge Y^*(\Box b')) \otimes (b \wedge b') > \\ \cong sl < \mathcal{O}(E)\otimes\mathcal{O}(D) \text{ (qua} & \text{poset)} \mid \text{same relations} > . \end{array}$$

Then we can define the poset map $e : \mathcal{O}(E)\times\mathcal{O}(D) \longrightarrow \mathcal{O}(E)$ by putting $e(a,b) = a \wedge Y^*(\Diamond b)$. This map respects the relations; the first two are just straightforward calculation; the third involves the presenting relation $\Box a \wedge \Diamond b \le \Diamond(a \wedge b)$ of $\mathcal{O}V_2(D)$:

$$\begin{aligned} e(a \wedge Y^*(\Box b'), b) &= a \wedge Y^*(\Box b') \wedge Y^*(\Diamond b) \\ &= a \wedge Y^*(\Box b' \wedge \Diamond b) \\ &\le a \wedge Y^*(\Diamond(b \wedge b') \wedge \Box b') \\ &= a \wedge Y^*(\Box b') \wedge Y^*(\Diamond(b \wedge b')) \\ &= e(a \wedge Y^*(\Box b').b \wedge b') \end{aligned}$$

Hence by the universal property of $\mathcal{O}(D')$ qua suplattice, there is exactly one suplattice map $\exists_{pi}$ such that $\exists_{pi} \circ i^*(a \otimes b) = a \wedge Y^*(\Diamond b)$. Then, just as in Lemma 55, we have that $\exists_{pi}$ is left adjoint to $i^* \circ p^*$ and it satisfies the Frobenius identity 2.6.

2. The proof goes along the lines of Lemma 49. We can think of $\mathcal{O}(D')$ in the

following ways:

$$\begin{aligned} Fr < \mathcal{O}(E)\times\mathcal{O}(D) \text{ (qua } & \text{join-semilattice)} \mid \\ & \S \text{ is bilinear w.r.t. finite meets and directed joins} \\ & a \vee Y^{*}(\Box b')\S b \leq a\S b \text{ for } b' \leq b \\ & a \vee Y^{*}(\Diamond b')\S(b \vee b') \leq (a \vee Y^{*}(\Diamond b'))\S b > \\ \cong pFr < \mathcal{O}(E)\times\mathcal{O}(D) \text{ (qua } & \text{poset)} \mid \text{same relations } > . \end{aligned}$$

Then we can consider the order preserving map $e : \mathcal{O}(E)\times\mathcal{O}(D) \longrightarrow \mathcal{O}(E)$ defined by $e(a,b) = a \vee Y^{*}(\Box b)$. It is straightforward to verify that e preserves the first two presenting relations. The last one follows from the axiom $\Box(a \vee b) \leq \Box a \vee \Diamond b$ of $V_2(D)$, since we have:

$$\begin{aligned} e(a \vee Y^{*}(\Diamond b'), b \vee b') &= a \vee Y^{*}(\Diamond b') \vee Y^{*}(\Box(b \vee b')) \\ &= a \vee Y^{*}(\Diamond b' \vee \Box(b \vee b')) \\ &\leq a \vee Y^{*}(\Diamond b' \vee \Diamond b' \vee \Box b) \\ &= a \vee Y^{*}(\Diamond b') \vee Y^{*}(\Box b) \\ &= e(a \vee Y^{*}(\Diamond b'), b) \end{aligned}$$

□

Theorem 61 *Let D and E be locales. Then there is a bijective correspondence, natural in E, between the points of $V_2(D)$ at stage E and the sublocales of $E \otimes D$ that, over E, are weakly semifitted with compact, open domain.*

Proof. The bijection has been set up in the previous two lemmas. □

Corollary 62 *The global points of $V_2(D)$ can be identified with the weakly semifitted sublocales of D with compact, open domain.*

Proof. It follows directly from the previous theorem, putting $E = 1$. □

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