An interior curvature estimate for a class of Weingarten curvature equations

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DEDICATION

To Christopher

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I would never really know how to properly appreciate or show my appreciation for Prof. Pengfei Guan, my supervisor; I could only try in words and fail. Besides his sheer mathematical brilliance that comes through in every sentence, I sincerely thank his amazing kindness and steadfast patience.

ABSTRACT

An interior principal curvature estimate for a class of Weingarten curvature equations is proved. An emphasis is placed on the completeness of the presentation, and accordingly most of the common assertions made in literature when treating the subject are carefully proved as preliminaries.

Given a hypersurface in the Euclidean space whose principal curvatures satisfy a Weingarten-type curvature equation, an *a priori* interior bound for the principal curvatures can be given; that is, the bound is not dependent on the particular form of the solution or its behaviour near the boundary, and only depends on the preset equation in the interior.

Afterwards, some theoretical context for the estimate and a discussion of its significance in application are provided.

ABRÉGÉ

Une estimation de courbure principale à l'intérieur pour une classe d'équations de Weingarten de courbure est prouvée. L'accent est mis sur l'exhaustivité de la présentation, et en conséquence la plupart des affirmations ordinairement faites dans la littérature lors du traitement du sujet sont soigneusement prouvée comme préliminaires.

Étant donné une hypersurface dans l'espace euclidien dont les courbures principales satisfont une équation de type Weingarten de courbure, une borne *à priori* à l'intérieur des courbures principales peut être donnée ; autrement dit, la borne ne dépend pas de la forme particulière de la solution ou de son régularité près du bord, et ne dépend que de l'équation prédéfinie à l'intérieur.

Ensuite, un certain contexte théorique pour l'estimation et une discussion sur son importance dans l'application sont fournis.

TABLE OF CONTENTS

CHAPTER 1
Introduction

1.1 Introduction

Our problem of concern will be the *prescribed Weingarten curvature problem* in the Euclidean space. *Weingarten curvatures* are elementary symmetric functions of the principal curvatures $\kappa_1, \kappa_2, ..., \kappa_n$. Roughly speaking, the problem asks to find an embedded hypersurface $M \in \mathbb{R}^{n+1}$ whose Weingarten curvatures satisfy a given condition, typically in the form of an equation involving a function given in the ambient space or express parametrization.

We would like to establish an *a priori* estimate of the principal curvatures given such an equation: bound on the principal curvatures on the solution if one exists, with no more information than the given equation. Specifically, the rigorous setting for our problem is as follows. One considers a local, strictly star-shaped solution M to a problem

$$Q(\kappa_1(X), \kappa_2(X), ..., \kappa_n(X)) = f(X) > 0, X \in M \tag{1.1}$$

where Q is a given positive C^2 function invariant under switching the order of its arguments, conditions on which to be laid out in the next section. By uniformly star-shaped we mean that the *support function* $u := \langle X, \nu(X) \rangle$ has a positive lower bound:

$$u = \langle X, \nu \rangle > 2c_0 > 0 \tag{1.2}$$

.

By local solution, we mean that M is a slice of a hypersurface in $\mathbb{R}^{n+1}$ and X is a parametrization from $B_1 \subset \mathbb{R}^n$. We specialize so that M is the graph $\{(x, X(x)) : x \in B_1 \subset \mathbb{R}^n\}$, but the estimate process will not involve the setting except that f is given on the same domain B_1. Our estimate is of the form

$$|\kappa_i(X(0))| \leq C(|f|_{C^2(B_1)}, |X|_{C^1(B_1)}, |f^{-1}|_{C^0(B_1)}, c_0, Q), \forall i \tag{1.3}$$

It should be clear that this result in fact imply the general interior estimate on a compact subset $\Omega' \Subset \Omega$ for a solution on a general domain Ω:

$$|\kappa_i|_{C^0(\Omega')} \leq C(|f|_{C^2(\Omega)}, |X|_{C^1(\Omega)}, |f^{-1}|_{C^0(\Omega)}, c_0, \Omega', \Omega, Q), \forall i \tag{1.4}$$

Note that although our setting is simple, there are many ways to formulate a similar problem, for most of which our methodology will work. Our proof essentially reproduces that in [11], with some minor admissibility conditions modified in the result. The graph can be defined on (a subset of) $\mathbb{S}^n$ as in [3] or [7]; instead of being defined on the parametrization domain, f can be defined in the ambient set within $\mathbb{R}^{n+1}$ where the hypersurface is contained in as in [4]. f can also depend on the normal to the surface, some cases of which is treated in [8]. In addition, once the hypersurface is assumed to be represented in some way as a graph of a function u, one can also formulate a version where Q takes as its arguments simply the eigenvalues of the Hessian D^2u as in [2].

We prove the estimate by employing a maximum principle: set up a suitable test function, and the estimate will follow from the fact that the test function achieves a maximum within a bounded domain. The estimate is on the curvature, a geometric quantity, proved using primarily geometric identities, and thus has the advantage of being applicable in general settings as noted above. One of the primary applications of a curvature estimate is in establishing existence by method of continuity (e.g. in [3]); there one converts the curvature estimate into a Hessian estimate of a function, where the specificity of a setting comes into play.

The resulting interior estimate (1.4) does not depend on the shape of or the behaviour of the hypersurface on the boundary of Ω, and thus is distinct from the global estimates presented in the above-cited sources. It will trivially imply the global estimate in case of a compact domain with no boundary, e.g. in [3].

1.2 Statement of the Result

We first clarify the conditions to be imposed on the C^2 symmetric function Q. The following are based on the conditions assumed in [4], but include further restrictions meant to be used in the estimate procedure. Its domain will be denoted $\Gamma \subset \mathbb{R}^n$; Γ shall be an open convex cone, invariant under switching of its components, with the following size restriction:

$$\{\forall i, \lambda_i > 0\} \subset \Gamma \subset \{\sum_i \lambda_i > 0\} \tag{1.5}$$

If a solution of (1.1) has principal curvatures at every point in Γ, it shall be called *admissible.*

For Q itself, we start by noting complications arising from computing Q. What we actually have is the suitably differentiable components of the second fundamental form matrix (h_{ij}); extracting individual eigenvalues of the matrix in order to plug into Q is often a convoluted process. Without delving into the relevant algebraic geometry, we simply (and reasonably, since Q is symmetric in its arguments) assume that Q can be calculated in a C^2 manner as a function of symmetric matrices; that is, if $\lambda := (\lambda_1, \lambda_2, ..., \lambda_n) \in \Gamma$ is the eigenvalue of a symmetric matrix W, we have a C^2 function Q on W such that

$$Q(W) = Q(\lambda) \tag{1.6}$$

where we abuse the notation Q to mean both the function on the matrix and the eigenvalues. Note $Q(W)$ is invariant under orthonormal conjugations on W, since the eigenvalues do not change.

On Γ, Q is supposed to satisfy:

$$\frac{\partial Q}{\partial \lambda_i} > 0, \forall i \tag{1.7}$$

$$Q \text{ is concave and positive.} \tag{1.8}$$

This is common to most settings of the problem. (1.8) in fact implies a useful upper bound: from concavity, we know the tangent line in the direction of λ at λ, stays above the graph: $\forall s > 0$,

$$0 < Q(s\lambda) \leq Q(\lambda) + (s-1)\sum_i \frac{\partial Q}{\partial \lambda_i}\lambda_i$$

$$\sum_i \frac{\partial Q}{\partial \lambda_i}\lambda_i \leq Q(\lambda) \text{ from setting } s \to 0 \text{ above} \tag{1.9}$$

Then for the lower bound we assume that there exists a positive, strictly increasing function ϕ on the positive reals (particular to a specific Q) such that

$$\sum_i \frac{\partial Q}{\partial \lambda_i}\lambda_i > \phi \circ Q > 0 \tag{1.10}$$

this particular uniformity condition follows [3]. If, in addition to being symmetric, Q is a homogeneous function of degree m, we have in fact

$$\sum_i \frac{\partial Q}{\partial \lambda_i}\lambda_i = mQ$$

which is Taylor's formula. This, coupled with (1.9), implies that any non-degree 1 homogeneous Q should be powered to 1 in order to hope for concavity. Now we can assume one of the following conditions in order to ensure the estimate:

$$\exists K > 0, \alpha < 1 : \sum_i \frac{\partial Q}{\partial \lambda_i}(\lambda) < K(1 + |\sum_i \frac{\partial Q}{\partial \lambda_i}\lambda_i^2|^\alpha) \tag{1.11}$$

$$\exists K, N > 0, \forall i : \frac{\partial Q}{\partial \lambda_i}(\lambda) < K\lambda_i^{2N} \tag{1.12}$$

In particular, (1.11), in addition to the aforementioned (1.6), (1.7), (1.8), and (1.10), is satisfied by $Q := \frac{\sigma_k}{\sigma_{k-1}}, 1 < k \leq n$, where σ_i is the i-th elementary symmetric function. Q is defined on $\Gamma_k := \{\lambda \in \mathbb{R}^n : \sigma_i(\lambda) > 0, i \leq k\}$. The precise definition thereof and the proof of its eligibility will be presented in Chapter 2.

Now we are ready to give the rigorous form of the estimate.

Theorem 1.2.1. *Let C^2 symmetric function Q defined in a suitable cone Γ satisfy (1.6), (1.7), (1.8), (1.10), and at least one of (1.11) and (1.12). Then given a C^2 positive function f and a C^4 function u defined on the ball $B_1 \subset \mathbb{R}^n$ such that its graph $M = \{x, u(x) : x \in B_1 \subset \mathbb{R}^n\}$ in $\mathbb{R}^{n+1}$ is an admissible uniformly star-shaped solution to (1.1), the principal curvatures of M satisfy an interior estimate of the form (1.3).*

And as noted above, the immediate corollary:

Corollary 1.2.2. *Let C^2 symmetric function Q defined in a suitable cone Γ satisfy (1.6), (1.7), (1.8), (1.10), and at least one of (1.11) and (1.12). Then given a C^2 positive function f and a C^4 function u defined on a domain $\Omega \subset \mathbb{R}^n$ such that its graph $M = \{x, u(x) : x \in B_1 \subset \mathbb{R}^n\}$ in $\mathbb{R}^{n+1}$ is an admissible uniformly star-shaped* *solution to (1.1), given a compact subset $\Omega' \Subset \Omega$ the principal curvatures of M satisfy an interior estimate of the form (1.4).*

Proof. Since Ω' is compact, there exists $r > 0$ such that $B_r(x) \subset \Omega$ for all $x \in \Omega'$. Then given x in Ω', by precomposing a map $B_1 \to B_r(x)$ we get an estimate at x of the form (1.3); since r only depends on Ω', Ω, and C^n bounds on $B_r(x)$ are trivially bounded by bounds on Ω, (1.4) holds. □

1.3 Organization of the Thesis

The organization of this thesis is as follows. We recap preliminaries to the subject in Chapter 2; this is not only to recall theory, but also to fix various conventions such as signs and normalizations and thus ground the discussion in rigorous terms. The main computations are in Chapter 3, where we prove the main estimate. The final chapter will be devoted to putting some context to the result and highlighting its significance, including some possibilities of application.

CHAPTER 2
Preliminaries

In this chapter, we present important backgrounds to the subject, split into two sections. First we make precise the notions of principal curvature and related quantities in recalling extrinsic notions in differential geometry. Then we study the elementary symmetric functions, in particular proving that their quotients indeed satisfy the conditions (1.7)-(1.11) as asserted in the introduction. Much literature (e.g. [2] [7]) concerning the prescribed curvature problem presents, or refers to, the general theory of hyperbolic polynomials by Lars Gårding [5] which implies the same results; there it provides some context for the general conditions laid out in the introduction, such as the convexity and positivity of the admissible cone. We instead opt to give straightforward elementary proofs to results we need.

2.1 Extrinsic Riemannian Geometry

This section recalls classical theory of surfaces, which is often only presented in 3-dimensions but immediately generalized to hypersurfaces in $(n+1)$-dimensions (for example, see [1]). We assume basic definitions in Riemannian geometry.

A *hypersurface* M in $\mathbb{R}^{n+1}$ is an embedded n-dimensional submanifold $i : M \hookrightarrow \mathbb{R}^{n+1}$. In the context of Riemannian geometry, it is natural to equip M with the pullback of the Euclidean metric on $\mathbb{R}^{n+1}$. That is, with a parametrization X, the tangent space $T_X M$ embeds as the hyperplane

$$T_X M := \text{span}_i \left\langle \frac{\partial}{\partial x_i} X = X_i \right\rangle < \mathbb{R}^{n+1}$$

as normed vector spaces. This means in particular that we can calculate the unit normal to the surface in the familiar Euclidean manner. As long as M is connected and orientable, the *Gauss map* $\nu : M \to \mathbb{R}^{n+1}$ is uniquely determined up to a sign by

$$|\nu(X)| = 1, \nu(X) \perp T_X M \tag{2.1}$$

The choice of the sign represents a choice of orientation.

Then we define the *second fundamental form* W, which is a bilinear form on the tangent space $T_X M$, by

$$W(v, w) = \langle d\nu_X(v), w \rangle$$

This is dependent on the choice of orientation–one can make it independent of it by having it be the coefficient of the normal ν, as is done usually ([1]). However, one still obtains a scalar value by contracting with the normal again when calculating the principal curvatures, therefore the principal curvatures are dependent on the orientation in any case.

W is symmetric, which is the easiest to see in local coordinates as above. Note differentiating $\langle \nu(X), X_i \rangle \equiv 0$ with respect to x_j gives $\langle (\nu \circ X)_j, X_i \rangle = - \langle \nu, X_{ij} \rangle$:

$$W(X_i, X_j) = \langle d\nu_X(X_i), X_j \rangle = \langle (\nu \circ X)_i, X_j \rangle = - \langle \nu, X_{ij} \rangle = W(X_j, X_i) \qquad (2.2)$$

So we can diagonlize W in an orthonormal basis; the resulting eigenvalues of W are called *principal curvatures.* Concretely, if $\{X_i\}$ is positively oriented and orthonormal, the second fundamental form is given by the symmetric matrix (h_{ij}) with the principal curvatures as its eigenvalues, where the components h_{ij} are given by

$$h_{ij} = \langle d\nu_X(X_i), X_j \rangle = \langle \nu_i, X_j \rangle$$

There is accordingly a divergence within the literature on the sign in the definitions of W and the principal curvatures, and we, keeping with conventions in [7], fix the definitions so that we do away with optional negative signs. For example, [3] has opposite sign conventions to us, so that they work with negative principal curvatures.

We now prove a few identities relating to the aforementioned quantities which will be useful for us. First, we recall the definition and existence of local normal coordinates ([1], p115):

Lemma 2.1.1. *Let M be an n-dimensional differentiable manifold. Given $m \in M$, there exists a local coordinate X which maps into a neighbourhood containing m such that the metric is trivial and the covariant derivatives coincide with partial derivatives at m, or $g_{ij}(m) = \delta_{ij}$ and $\Gamma^k_{ij}(m) = 0 \Leftrightarrow \partial_k g_{ij}(m) = 0, \forall i, j, k$.*

Now we comment on some notational features which will be employed from now on. We extend the subscript notation, which has already been employed to signify partial derivatives of the coordinate and Gauss maps, to in fact mean general covariant derivatives on tensors. Some care is needed, as in the proof of (2.5). We will also use the Einstein summation convention from now on: a dummy index repeated in a term is summed over unless otherwise stated.

We now prove a few identities that will be useful for us in the next chapter. This particular collection follows [7].

Proposition 2.1.2. *For the second fundamental form* (h_{ij}) *and the Gauss map* ν *written in local normal coordinates* X *around* m*, the following holds at* m*:*

$$X_{ij} = -h_{ij}\nu \tag{2.3}$$

$$v_i = h_i^j X_j = h_{ij} X_j \tag{2.4}$$

$$h_{ijk} = h_{ikj} \tag{2.5}$$

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk} \tag{2.6}$$

where R *is the Riemann curvature tensor. Note* X_{ij}, v_i *are partial differentiated; all the additional subscripts will mean covariant differentiation. Since the last two identities are tensor identities, they in fact hold in arbitrary coordinates.*

Proof.

(2.3, Gauss formula): Differentiating $\langle \nu, \nu \rangle = 1$ gives

$$\nu_j \perp \nu \Rightarrow \nu_j \in \mathrm{span}_i \langle X_i \rangle$$

Then the coefficient of X_i in the expansion of ν_j is given $\langle \nu_j, X_i \rangle = -h_{ij}$ since $\{X_i\}$ is orthonormal.

(2.4, Weingarten equation): Differentiating $\langle \nu, X_i \rangle = 0$ with respect to x_j gives $\langle \nu, X_{ij} \rangle = -\langle \nu_j, X_i \rangle = -h_{ij}$. It turns out there is no other component other than in the ν direction by (2.1.1).

(2.5, Codazzi equation): Diffrentiating $h_{ij} = -\langle \nu, X_{ij} \rangle$

$$\begin{aligned} h_{ijk} &= -\langle \nu, X_{ij} \rangle_k = -\langle \nu_k, X_{ij} \rangle - \langle \nu, X_{ijk} \rangle = -\langle \nu, X_{ijk} \rangle \\ &= -\langle \nu, \nabla_k \partial_j X_i \rangle = -\langle \nu, \partial_k \partial_j X_i \rangle = -\langle \nu, \partial_j \partial_k X_i \rangle = -\langle \nu, \partial_j X_{ik} \rangle \\ &= -\langle \nu, \nabla_j X_{ik} \rangle = -\langle \nu_j, X_{ik} \rangle - \langle \nu, X_{ikj} \rangle = -\langle \nu, X_{ik} \rangle_j = h_{ikj} \end{aligned}$$

(2.6, Gauss equation): $R_{abcd} = \partial_c \Gamma_{abd} - \partial_d \Gamma_{acb}$. Using $g_{ij} = \langle X_i, X_j \rangle$

$$\begin{aligned} \Gamma_{abd} &= \frac{1}{2} [\langle X_{ab}, X_d \rangle + \langle X_a, X_{db} \rangle + \langle X_{ad}, X_b \rangle \\ &\quad + \langle X_a, X_{bd} \rangle - \langle X_{ba}, X_d \rangle - \langle X_b, X_{da} \rangle] \\ &= \frac{1}{2} [\langle X_a, X_{db} \rangle + \langle X_a, X_{bd} \rangle] \end{aligned}$$

in differentiating the Christoffel symbol, we use $\langle X_{ij}, X_{kl} \rangle = h_{ij} h_{kl}$ and the fact that the subscripts of X_{ijk} can be exchanged as in the proof of (2.5).

$$\partial_c \Gamma_{abd} = \frac{1}{2}\left[\langle X_{ac}, X_{db}\rangle + \langle X_{ac}, X_{bd}\rangle + \langle X_a, X_{dbc}\rangle + \langle X_a, X_{bdc}\rangle\right]$$
$$\partial_d \Gamma_{acb} = \frac{1}{2}\left[\langle X_{ad}, X_{bc}\rangle + \langle X_{ad}, X_{cb}\rangle + \langle X_a, X_{bcd}\rangle + \langle X_a, X_{cbd}\rangle\right]$$
$$\therefore R_{abcd} = h_{ac}h_{db} - h_{ad}h_{bc}$$

□

Now we prove the existence of an even more specific form of local normal coordinates: the one that diagonalizes the second fundamental form. This simple proposition is used widely in maximum principle arguments involving the matrix elements of the second fundamental form.

Proposition 2.1.3. *Let M be an n-dimensional differentiable manifold. Given $m \in M$, there exists a local normal coordinate X around m such that the second fundamental form (h_{ij}) with respect to it is diagonal.*

Proof. Suppose $X' : U \to M, 0 \in U \subset \mathbb{R}^n$ is any local normal coordinate at m such that $X'(0) = m$. If $(h'_{ij}(0))$ is the matrix of the second fundamental form in the coordinate, it is symmetric by (2.2), and thus diagonalizable by a real orthogonal matrix A: $A^T(h'_{ij})A$ is diagonal at m. Then $X := X' \circ A : A^T U \to M$, $\nu := \nu' \circ A : A^T U \to M$ satisfies

$$X_i = X'_k A_{ki}$$
$$\nu_j = \nu'_l A_{lj}$$

and thus

$$h_{ij} = \langle X_i, \nu_j \rangle = A_{ki}A_{lj} \langle X'_k, \nu'_l \rangle = h'_{kl}A_{ki}A_{lj}$$

which means precisely that $(h_{ij}(0)) = A^T(h'_{ij}(0))A$ is diagonal.

The fact that X is another local normal coordinate is immediate since the transformation is linear and A is orthogonal.

$\square$

2.2 Elementary Symmetric Functions

We introduce a few properties of and notations involving elementary symmetric functions. Our normalization for the symmetric functions will be, for $1 \leq k \leq n$, $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$,

$$\sigma_k(\lambda) := \sum_{1 \leq i_1 < i_2 < ... < i_k \leq n} \lambda_{i_1}\lambda_{i_2} \cdots \lambda_{i_k}$$

If $\lambda \in \mathbb{R}^n$, and $1 \leq i_1 < i_2 < ... < i_m \leq n$, we define $(\lambda|i_1, i_2, ..., i_m) \in \mathbb{R}^{n-m}$ by

$$(\lambda|i_1, i_2, ..., i_m) = (\lambda_1, ..., \hat{\lambda_{i_1}}, ..., \hat{\lambda_{i_m}}, ..., \lambda_n)$$

where hat denotes omission of the element. Note the simple but useful facts:

$$\sigma_k(\lambda) = \sigma_k(\lambda|i) + \lambda_i\sigma_{k-1}(\lambda|i)$$
$$\frac{\partial \sigma_k}{\partial \lambda_i} = \sigma_{k-1}(\lambda|i)$$

Now recall $\Gamma_k = \Gamma_k(\mathbb{R}^n)$ is the open cone defined by $\Gamma_k := \{\lambda \in \mathbb{R}^n : \sigma_i(\lambda) > 0, \forall i \leq k\}$. It is convex [7, Lem 2.4]:

Proposition 2.2.1. *For $1 \leq k \leq n$, the cone $\Gamma_k(\mathbb{R}^n)$ is convex.*

We now prove an important lemma:

Lemma 2.2.2. *Suppose for $k > 1$ $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \Gamma_k(\mathbb{R}^n)$. Then $(\lambda|i) \in \Gamma_{k-1}(\mathbb{R}^{n-1})$ for each $i \in \{1, 2, ..., n\}$.*

Proof. Without loss of generality, show $(\lambda|1) \in \Gamma_{k-1}$. Since Γ_k is open, for some $\delta > 0$ $(\lambda_1 - \delta, \lambda_2 - \delta, ..., \lambda_n - \delta) \in \Gamma_k$. Note, $\forall R_i > 0, (R_1, R_2, R_3, ..., R_n) \in \Gamma_k$. By convexity the sum $\lambda(R) := (\lambda_1 - \delta + R_1, \lambda_2 - \delta + R_2, ..., \lambda_n - \delta + R_n) \in \Gamma_k$. Since

$$\begin{aligned}\sigma_k(\lambda(R)) &= \sigma_k(\lambda(R)|1) + (\lambda_1 + R_1 - \delta)\sigma_{k-1}(\lambda(R)|1) \\ &= \sigma_k(\lambda(R)|1) + (\lambda_1 + R_1 - \delta)\sigma_{k-1}(\lambda(R)|1) > 0, \forall R_i > 0\end{aligned}$$

thus we have $\sigma_{k-1}(\lambda(R)|1) \geq 0$. But $\sigma_{k-1}(\lambda(R)|1)$ is nonconstant (take R^1, R^2 large enough to have $\lambda(R^i) = (K_i, K_i, ..., K_i)$ and $K_1 \neq K_2$) and linear in each of R_i, so it cannot attain zero and stay nonnegative.

□

Now we present classical inequalities, attributed to Newton and MacLaurin [7]. The proofs are standard, which we reproduce without a specific source.

Proposition 2.2.3. *Suppose* $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \Gamma_k(\mathbb{R}^n)$, $2 < k < n$. *Then*

$$(n-k+1)(k+1)\sigma_{k-1}(\lambda)\sigma_{k+1}(\lambda) \leq k(n-k)\sigma_k^2(\lambda) \tag{2.7}$$

$$\sigma_{k+1}(\lambda) \leq \frac{\binom{n}{k+1}}{\binom{n}{k}^{\frac{k+1}{k}}}\sigma_k^{\frac{k+1}{k}}(\lambda) \tag{2.8}$$

Proof.

(2.7): Re-normalize the symmetric functions as follows:

$$W_l(\lambda) = \frac{\sigma_l(\lambda)}{\binom{n}{l}}$$

then it suffices to prove

$$W_{k-1}(\lambda)W_{k+1}(\lambda) \leq W_k^2(\lambda) \tag{2.9}$$

We first prove (2.7) for $k = n-1$. If $W_{k+1}(\lambda) \leq 0$, the result is trivial. If not, since the inequality is homogeneous, we can normalize so that $\sigma_{k+1}(\lambda) = \lambda_1 \cdots \lambda_n = 1$. Then

$$\begin{aligned}
\sigma_k^2(\lambda) = \left(\frac{1}{\lambda_1} + ... + \frac{1}{\lambda_n}\right)^2 &= \frac{1}{\lambda_1^2} + ... + \frac{1}{\lambda_n^2} + 2\sum_{i<j}\frac{1}{\lambda_i\lambda_j} \\
&\geq \frac{2}{n-1}\sum_{i<j}\frac{1}{\lambda_i\lambda_j} + 2\sum_{i<j}\frac{1}{\lambda_i\lambda_j} \qquad (2.10) \\
&= \frac{2n}{n-1}\sum_{i<j}\frac{1}{\lambda_i\lambda_j} = \frac{2n}{n-1}\sigma_{k+1}(\lambda)\sigma_{k-1}(\lambda)
\end{aligned}$$

where (2.10) follows from the expansion

$$0 \leq \sum_{i<j}\left(\frac{1}{\lambda_i} - \frac{1}{\lambda_j}\right)^2 = (n-1)\sum_i \frac{1}{\lambda_i^2} - 2\sum_{i<j}\frac{1}{\lambda_i\lambda_j}$$

For $k < n-1$, we rely on the following fact: there exists $\lambda' = (\lambda'_1, ..., \lambda'_{n-1}) \in \mathbb{R}^{n-1}$ such that for $i \le n-1$

$$W_i(\lambda') = W_i(\lambda)$$

Then coupled with $n = k-1$ case we will be able to inductively increase $n-k-1$ for which the result holds. Define and differentiate

$$P(t) = (t+\lambda_1)\cdots(t+\lambda_n) = t^n + \sigma_1(\lambda)t^{n-1} + ... + \sigma_n(\lambda)$$
$$P'(t) = nt^{n-1} + (n-1)\sigma_1(\lambda)t^{n-1} + ... + \sigma_{n-1}(\lambda)$$

by Rolle's theorem and explicit differentiation in case of nontrivial multiplicities, it is clear that P' has $n-1$ real roots. So we have for some $\lambda'_1, ..., \lambda'_{n-1} \in \mathbb{R}$

$$P'(t) = n(t+\lambda'_1)\cdots(t+\lambda'_{n-1}) = nt^{n-1} + n\sigma_1(\lambda')t^{n-1} + ... + n\sigma_{n-1}(\lambda')$$

or

$$\forall i \le n-1 : n\sigma_i(\lambda') = (n-i)\sigma_i(\lambda) \Longleftrightarrow W_i(\lambda') = W_i(\lambda)$$

(2.8): It suffices to prove for $1 \le i \le k$

$$W_{i+1}(\lambda) \le W_i^{\frac{i+1}{i}}(\lambda) \tag{2.11}$$

If $W_{k+1}(\lambda) < 0$, the inequality is trivial. If not, as in (2.10)

$$\sigma_2(\lambda) \le \frac{n-1}{2n}\sigma_1^2(\lambda) \Leftrightarrow W_2(\lambda) \le W_1^2(\lambda)$$

Now, if (2.11) holds for $i < k$, as in $W_{i+1}^{\frac{i}{i+1}}(\lambda) \leq W_i(\lambda)$

$$W_{i+2}(\lambda) \leq \frac{W_{i+1}^2(\lambda)}{W_i(\lambda)} \leq W_{i+1}^{\frac{i+2}{i+1}}(\lambda)$$

so by induction the result holds.

□

We now prove the conditions (1.6), (1.7), (1.8), (1.11) for $Q := \frac{\sigma_k}{\sigma_{k-1}}$ ((1.10) follows since Q is homogeneous of degree 1). For (1.6), we have the following formula: if $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ represent an ordering of the eigenvalues of the symmetric matrix $W = (W_{ij})$,

$$\sigma_k(\lambda) = \frac{1}{k!} \sum_{1 \leq i_1,...,i_k,j_1,...,j_k \leq n} \delta(i_1, ..., i_k; j_1, ..., j_k) W_{i_1 j_1} \cdots W_{i_k j_k}$$

where $\delta(i_1, ..., i_k; j_1, ..., j_k)$ is defined so that

$$\delta(i_1, ..., i_k; j_1, ..., j_k) = \begin{cases} (-1)^{\mathrm{sgn}(\sigma)} & \sigma \in S_k : \sigma(i_1, ..., i_k) = (j_1, ..., j_k) \\ & \text{otherwise; in particular} \\ 0 & \text{if } (i_m) \text{ not all distinct or} \\ & (j_m) \text{ not all distinct} \end{cases}$$

The identity follows from explicitly calculating $\det(W + tI)$ and extracting the coefficient of t^k, which is precisely $\sigma_k(\lambda)$.

The following proposition and proof closely follows [7].

Proposition 2.2.4. *For $1 < k \leq n$, write $Q(\lambda) = \frac{\sigma_k(\lambda)}{\sigma_{k-1}(\lambda)}$ for $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n$. Then $Q^i := \frac{\partial Q}{\partial \lambda_i}$ satisfies*

$$0 < Q^i \leq n - k + 1 \text{ in } \Gamma_k, \tag{2.12}$$

and Q is concave in Γ_{k-1}, or, given $\lambda \in \Gamma_{k-1}$, $\xi \in \mathbb{R}^n$,

$$\frac{\partial^2}{\partial t^2} Q(\lambda + t\xi)|_{t=0} \leq 0 \tag{2.13}$$

for small enough t such that $\lambda + t\xi \in \Gamma_{k-1}$, since Γ_{k-1} is open.

Proof.

$$\begin{aligned} Q^i &= \frac{\sigma_{k-1}(\lambda)\partial_{\lambda_i}\sigma_k(\lambda) - \sigma_k(\lambda)\partial_{\lambda_i}\sigma_{k-1}(\lambda)}{\sigma_{k-1}^2(\lambda)} \\ &= \frac{\sigma_{k-1}(\lambda)\sigma_{k-1}(\lambda|i) - \sigma_k(\lambda)\sigma_{k-2}(\lambda|i)}{\sigma_{k-1}^2(\lambda)} \\ &\leq \frac{\sigma_{k-1}(\lambda|i)}{\sigma_{k-1}(\lambda)} \leq \sum_i \frac{\sigma_{k-1}(\lambda|i)}{\sigma_{k-1}(\lambda)} = n - k + 1 \end{aligned}$$

where we use the fact that $\lambda \in \Gamma_k, (\lambda|i) \in \Gamma_{k-1}$. On the other hand, by (2.7)

$$\begin{aligned} Q^i &= \frac{(\sigma_{k-1}(\lambda|i) + \lambda_i\sigma_{k-2}(\lambda|i))\,\sigma_{k-1}(\lambda|i) - (\sigma_k(\lambda|i) + \lambda_i\sigma_{k-1}(\lambda|i))\,\sigma_{k-2}(\lambda|i)}{\sigma_{k-1}^2(\lambda)} \\ &= \frac{\sigma_{k-1}(\lambda|i)^2 - \sigma_k(\lambda|i)\sigma_{k-2}(\lambda|i)}{\sigma_{k-1}^2(\lambda)} \geq \frac{n}{k(n-k+1)} \frac{\sigma_{k-1}^2(\lambda|i)}{\sigma_{k-1}^2(\lambda)} > 0 \end{aligned}$$

Now we turn to concavity. We can calculate explicitly for $Q = \frac{\sigma_2}{\sigma_1}$; given $\lambda \in \Gamma_2$, $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$,

$$\begin{aligned}
\partial_t Q(\lambda + tx) &= \frac{\sum_i x_i \sigma_1(\lambda + tx|i)}{\sigma_1(\lambda + tx)} - \frac{\sigma_2}{\sigma_1^2}(\lambda + tx)\sigma_1(x) \\
\partial_t^2 Q(\lambda + tx)|_{t=0} &= \frac{\sum_i x_i \sum_{j \neq i} x_j}{\sigma_1(\lambda)} - \frac{\sigma_1(x)\sum_i x_i \sigma_1(\lambda|i)}{\sigma_1^2(\lambda)} - \frac{\sigma_1(x)\sum_i x_i \sigma_1(\lambda|i)}{\sigma_1^2(\lambda)} + 2\sigma_1^2(x)\frac{\sigma_2}{\sigma_1^3}(\lambda) \\
&= \frac{2\sigma_2(x)\sigma_1^2(\lambda) - 2\sigma_1(x)\sigma_1(\lambda)\sum_i x_i\left(\sigma_1(\lambda) - \lambda_i\right) + 2\sigma_1^2(x)\sigma_2(\lambda)}{\sigma_1^3(\lambda)} \\
&= \frac{2\sigma_1(x)\sigma_1(\lambda)\sum_i x_i\lambda_i + \left(2\sigma_2(x) - \sigma_1^2(x)\right)\sigma_1^2(\lambda) + \sigma_1^2(x)\left(2\sigma_2(\lambda) - \sigma_1^2(\lambda)\right)}{\sigma_1^3(\lambda)} \\
&= -\frac{\sum_i\left[x_i^2\sigma_1^2(\lambda) - 2\sigma_1(x)\sigma_1(\lambda)x_i\lambda_i + \lambda_i^2\sigma_1^2(x)\right]}{\sigma_1^3(\lambda)} = -\frac{\sum_i\left[x_i\sigma_1(\lambda) - \lambda_i\sigma_1(x)\right]^2}{\sigma_1^3(\lambda)}
\end{aligned}$$

and thus nonnegative. Now we induct on $m \geq 2 : Q_{m+1} := \frac{\sigma_{m+1}}{\sigma_m}$. We note, combinatorially, it is immediate that

$$\sum_i \sigma_{m-1}(\lambda|i)\lambda_i^2 = \sigma_1(\lambda)\sigma_m(\lambda) - (m+1)\sigma_{m+1}(\lambda)$$

then

$$\begin{aligned}
(m+1)Q_{m+1}(\lambda) &= \sigma_1(\lambda) - \sum_i \frac{\sigma_{m-1}(\lambda|i)\lambda_i^2}{\sigma_m(\lambda)} \\
&= \sigma_1(\lambda) - \sum_i \frac{\sigma_{m-1}(\lambda|i)\lambda_i^2}{\sigma_m(\lambda|i) + \lambda_i\sigma_{m-1}(\lambda|i)} \\
&= \sigma_1(\lambda) - \sum_i \frac{\lambda_i^2}{Q_m(\lambda|i) + \lambda_i} =: \sigma_1(\lambda) - \sum_i g_i(t)
\end{aligned}$$

We now differentiate after plugging in $\lambda = \lambda + tx$. $\sigma_1(\lambda + tx)$ is linear in t, so it vanishes in the second derivative. We write $\partial_t Q_m(\lambda) := \partial_t Q_m(\lambda + tx)|_{t=0}$ and so on.

$$
\begin{aligned}
g_i'(t) &= \frac{2\left(\lambda_i + tx_i\right)x_i}{Q_m(\lambda+tx|i)+\lambda_i+tx_i} - \frac{\left(\lambda_i+tx_i\right)^2\left(\partial_t Q_m(\lambda+tx|i)+x_i\right)}{\left(Q_m(\lambda+tx|i)+\lambda_i+tx_i\right)^2} \\
g_i''(0) &= \frac{2x_i^2}{Q_m(\lambda|i)+\lambda_i} - \frac{2\lambda_i x_i\left(\partial_t Q_m(\lambda|i)+x_i\right)}{\left(Q_m(\lambda|i)+\lambda_i\right)^2} \\
&\quad - \frac{2\lambda_i x_i\left(\partial_t Q_m(\lambda|i)+x_i\right)+\lambda_i^2\partial_t^2 Q_m(\lambda|i)}{\left(Q_m(\lambda|i)+\lambda_i\right)^2} + 2\times\frac{\lambda_i^2\left(\partial_t Q_m(\lambda|i)+x_i\right)^2}{\left(Q_m(\lambda|i)+\lambda_i\right)^3} \\
&= \frac{-\lambda_i^2\partial_t^2 Q_m(\lambda|i)}{\left(Q_m(\lambda|i)+\lambda_i\right)^2} + \frac{2x_i^2\left(Q_m(\lambda|i)+\lambda_i\right)^2}{\left(Q_m(\lambda|i)+\lambda_i\right)^3} \\
&\quad + \frac{-4\lambda_i x_i\left(\partial_t Q_m(\lambda|i)+x_i\right)\left(Q_m(\lambda|i)+\lambda_i\right)+2\lambda_i^2\left(\partial_t Q_m(\lambda|i)+x_i\right)^2}{\left(Q_m(\lambda|i)+\lambda_i\right)^3} \\
&= \frac{-\lambda_i^2\partial_t^2 Q_m(\lambda|i)}{\left(Q_m(\lambda|i)+\lambda_i\right)^2} + \frac{2\left(x_i\left(Q_m(\lambda|i)+\lambda_i\right)-\lambda_i\left(\partial_t Q_m(\lambda|i)+x_i\right)\right)^2}{\left(Q_m(\lambda|i)+\lambda_i\right)^3} \geq 0
\end{aligned}
$$

where we note $(\lambda|i) \in \Gamma_m \Rightarrow \partial_t^2 Q_m(\lambda|i) \leq 0$ from the induction hypothesis, $Q_m(\lambda|i) + \lambda_i = \frac{\sigma_m(\lambda)}{\sigma_{m-1}(\lambda|i)} \geq 0$. So we have shown $\frac{\partial^2}{\partial t^2} Q(\lambda + tx)|_{t=0} \leq 0$ for all m.

□

All in all, we have proved:

Theorem 2.2.5. *For $1 < k \leq n$, the curvature equation (1.1) with $Q := \frac{\sigma_k}{\sigma_{k-1}}$ defined on Γ_k satisfies Theorem 1.2.1 and Theorem 1.2.2.*

CHAPTER 3
The Estimate

This chapter is devoted to the proof of (1.2.1). In the first section we present relevant facts before moving on to proving the result in the second section.

3.1 Groundwork

We first explore the consequences arising from calculating Q from the second fundamental form (h_{ij}). We assume (h_{ij}) is diagonal, using (). Then the diagonal elements $(h_{11}, h_{22}, ..., h_{nn})$ are precisely the eigenvalues $\kappa := (\kappa_1, \kappa_2, ..., \kappa_n)$, and thus we have

$$Q^{ii}(h_{ij}) := \frac{\partial Q}{\partial W_{ii}}(h_{ij}) = \frac{\partial}{\partial t}Q((h_{ij}) + tI^{ij})|_{t=0} = \frac{\partial Q}{\partial \lambda_i}(\kappa) := Q_i \tag{3.1}$$

where I^{ij} denotes the diagonal matrix whose only nonzero entry is one at the (i, j) component. Similarly, if $i \neq j$, $(h_{ij}) + tI^{ij}$ is either upper- or lower-triangular, so does not affect the eigenvalues; $Q^{ij}(h_{ij}) := \frac{\partial Q}{\partial W_{ij}}(h_{ij}) = 0$.

A deeper consequence is that Q's concavity as a function on admissible eigenvalues implies its concavity as a function on symmetric matrices with admissible eigenvalues. This was shown in [2]; we reproduce it below.

The core ideas are that the map from a matrix to its *smallest* eigenvalue is rather well behaved, and is in fact concave, and that a concave function can be written as an infimum on a set of linear functions. The latter is true for any concave function, but we note in our case it follows easily because we can take the set of tangent planes. We check the former: by the spectral theorem if W is a symmetric matrix, with eigenvalues $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ and respective eigenvectors $v_1, v_2, ..., v_n$, we have the characterization

$$\lambda_1 = \min_{v \neq 0} \frac{\langle Wv, v \rangle}{\langle v, v \rangle}$$

which is an infimum of linear (thus concave) functions, and thus concave. In fact, by taking the tensor power $W^{\otimes k}$ and making it act on the exterior power $\Lambda^k \mathbb{R}^n$, which is spanned by the eigenvectors $v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k}$ with eigenvalues $\lambda_{i_1} + \lambda_{i_2} + ... + \lambda_{i_k}$, we see we can construct a symmetric operator whose smallest eigenvalue is $\lambda_1 + \lambda_2 + ... + \lambda_k$ for each k; thus the sum of first k smallest eigenvalues is concave in W for each k

Now, if for some sets $S \in \mathbb{R}^n \times \mathbb{R}$

$$Q(\lambda) = \inf_{(v,t) \in S} \langle v, \lambda \rangle + t$$

we can just take the subset S' of S where the first component, the vector $v = (v_1, v_2, ..., v_n)$ is ordered in the decreasing order since Q is symmetric. Then for each $(v,t) \in S$

$$\langle v, \lambda \rangle + t = \sum_{i=j}^{n-1} (v_j - v_{j+1})(\lambda_1 + ... + \lambda_j) + v_n (\lambda_1 + ... + \lambda_n) + t$$

which is concave in W. Then Q is an infimum of concave functions in W, so thus is concave in W.

We now derive a few identities which will be used in the proof of the main estimate in the next section. Recall the support function $u = \langle \nu, X \rangle$: differentiating it gives

$$u_i = \langle \nu, X \rangle_i = \langle \nu_i, X \rangle + \langle \nu, X_i \rangle = h_{ik} \langle X_k, X \rangle \tag{3.2}$$

$$\begin{aligned} u_{ij} &= [h_{ik} \langle X_k, X \rangle]_j = h_{ikj} \langle X_k, X \rangle + h_{ik} \langle X_{kj}, X \rangle + h_{ik} \langle X_k, X_j \rangle \\ &= h_{ijk} \langle X_k, X \rangle - h_{ik} h_{kj} u + h_{ik} \delta_{jk} \end{aligned} \tag{3.3}$$

Especially, if (h_{ij}) is diagonal, (3.3) becomes

$$u_{ij} = h_{ijk} \langle X_k, X \rangle + \left(-h_{ii}^2 u + h_{ii} \right) \delta_{ij} \tag{3.4}$$

Now, we explore the implications of 1.1. Assuming the equation, we differentiate

$$f_a = Q(h_{ij})_a = Q^{ij} h_{ija} \tag{3.5}$$

$$f_{aa} = Q^{ij,kl} h_{ija}, h_{kla} + Q^{ij} h_{ijaa} \leq Q^{ij} h_{ijaa} \tag{3.6}$$

where the term involving the second derivative of Q is non-negative due to its concavity.

Then it will be necessary to switch the order of the indices of h_{ijaa} in the last term; from Gauss' equation for the Riemann curvature tensor (2.6)

$$\begin{aligned} h_{ijkl} = h_{kijl} &= h_{kilj} + h_{mi}R_{mkjl} + h_{km}R_{mijl} \\ &= h_{klij} + h_{mi}(h_{mk}h_{lj} - h_{ml}h_{kj}) + h_{km}(h_{mi}h_{lj} - h_{ml}h_{ij}) \end{aligned} \tag{3.7}$$

We finally note the existence of a suitable cutoff function ρ on B_1. In particular, we would like to bound ρ' and ρ'' by a power $\rho^{1-\delta}$ for given $\delta \in (0,1)$. We define, for large $N \in \mathbb{N}$,

$$\eta(x) := (1 - |x|^2)^N \tag{3.8}$$

then

$$\begin{aligned} |\partial_{x_1}\eta| &= \left|N(1-|x|^2)^{N-1}(-2x_1)\right| \leq C_N \eta^{1-\frac{1}{N}} \\ \left|\partial_{x_1}^2\eta\right| &= \left|N(N-1)(1-|x|^2)^{N-2}(4x_1^2) - 2N(1-|x|^2)^{N-1}\right| \leq C_N \eta^{1-\frac{2}{N}} \end{aligned}$$

for some suitable $C_N = O(N)$. So we can take $N > \frac{1}{\delta}$.

3.2 Curvature Estimate

First, note that because of (1.5) an upper bound will imply a lower bound. Thus our goal will be to estimate $\max_i \kappa_i(0)$ from above.

Recall the definitions (1.2) and (3.8). We define a C^2 test function

$$\theta = \frac{\eta \max_i \kappa_i}{u - c_0}$$

on B_1, which vanishes on the boundary. θ achieves a maximum in its domain, and since $\max_i \kappa_i, \eta > 0$ and $u - c_0 > c_0 > 0$, the maximum is achieved in the interior at some $m \in B_1$. By (3.1), we can obtain a local normal frame around m such that $\max_i \kappa_i = h_{11}$ and $(h_{ij}(m))$ is diagonal. In what follows, all the function values are assumed to be at m unless said otherwise.

Since log is increasing, $\log \theta$ achieves its maximum at m. Differentiating twice

$$0 = (\log \theta)_i = \frac{\eta_i}{\eta} + \frac{h_{11i}}{h_{11}} - \frac{u_i}{u - c} \tag{3.9}$$

$$0 \geq_{\mathbb{R}^{n\times n}} (\log \theta)_{ij} = \frac{\eta_{ij}}{\eta} - \frac{\eta_i \eta_j}{\eta^2} + \frac{h_{11ij}}{h_{11}} - \frac{h_{11i}h_{11j}}{h_{11}^2} - \frac{u_{ij}}{u - c} + \frac{u_i u_j}{(u - c)^2} \tag{3.10}$$

contracting each side of (3.10) with (Q^{ij}), which, as noted in (3.1), is positive definite and diagonal,

$$\begin{aligned} 0 &\geq Q^{ij}\left[\frac{\eta_{ij}}{\eta} - \frac{\eta_i\eta_j}{\eta^2}\right] + \frac{Q^{ij}h_{11ij}}{h_{11}} - \frac{Q^{ij}h_{11i}h_{11j}}{h_{11}^2} - \frac{Q^{ij}u_{ij}}{u - c_0} + \frac{Q^{ij}u_i u_j}{(u - c_0)^2} \\ &\geq Q_i\left[\frac{\eta_{ii}}{\eta} - \frac{\eta_i^2}{\eta^2}\right] + \frac{f_{11} + h_{11}^2 Q_i h_{ii} - h_{11} Q_i h_{ii}^2}{h_{11}} - \frac{Q_i h_{11i}^2}{h_{11}^2} - \frac{Q_i u_{ii}}{u - c_0} + \frac{Q_i u_i^2}{(u - c_0)^2} \end{aligned}$$

where we apply (3.7) then (3.6); now using (1.9), (1.10), (3.9), (3.2), (3.4),

$$\geq Q_i\left[\frac{\eta_{ii}}{\eta} - \frac{2\eta_i^2}{\eta^2}\right] + \frac{f_{11} + h_{11}^2\phi(f)}{h_{11}} + \frac{2Q_i u_i \eta_i}{(u - c_0)\eta} - \sum_i \frac{f_i \langle X, X_i\rangle}{u - c_0} - \frac{f}{u - c_0} + \frac{uQ_i h_{ii}^2}{u - c_0} - Q_i h_{ii}^2$$

the last 2 terms combine. Also we have the following AM-GM inequality $\forall \epsilon > 0$:

$$\frac{2u_i\eta_i}{(u-c_0)\,\eta} \geq -\frac{\eta_i^2}{(u-c_0)^2\,\eta^2\epsilon} - \epsilon u_i^2$$

so taking $\epsilon > 0$ small such that $\epsilon u_i^2 \leq \epsilon h_{ii}^2 \langle X, X_i\rangle^2 \leq \frac{uh_{ii}^2}{2(u-c_0)}$

$$\geq Q_i\left[\frac{\eta_{ii}}{\eta} - C(c_0, |X|_{C^1})\frac{\eta_i^2}{\eta^2}\right] + h_{11}\phi(f) + \frac{c_0 Q_i h_{ii}^2}{2(u-c_0)} - C(c_0, |X|_{C^1}, |f|_{C^2})$$

$$\geq C(c_0, |X|_{C^1}, \delta)\sum_i \frac{Q_i}{\eta^\delta} + h_{11}\phi(f) + \frac{c_0 Q_i h_{ii}^2}{2(u-c_0)} - C(c_0, |X|_{C^1}, |f|_{C^2})$$

Now we branch into options (1.11) or (1.12). If (1.11) holds, we take $\beta > 1$ close to 1 so that (specifically, $\beta\alpha < 1$) $\left(\sum_i Q_i\right)^\beta \leq K' + \frac{c_0}{2(u-c_0)}Q_i h_{ii}^2$ and then δ small enough so that $\frac{\delta\beta}{1-\beta} = 1$. Then using Young's inequality with $p = \beta, q = \frac{\beta}{\beta-1}$:

$$C(c_0, |X|_{C^1}, \delta)\sum_i \frac{Q_i}{\eta^\delta} \geq -C(c_0, |X|_{C^1}, \delta, K)\frac{1}{\eta} - (\sum_i Q_i)^\beta$$

we have

$$0 \geq -C(c_0, |X|_{C^1}, Q)\frac{1}{\eta} + h_{11}\phi(f) - C(c_0, |X|_{C^1}, |f|_{C^2})$$

and thus we have a bound $\eta h_{11} \leq C(c_0, |X|_{C^1}, |f|_{C^2}, Q)$.

If (1.12) holds, we use Young's inequality on each Q_i with $p = 1 + \beta, q = \frac{1+\beta}{\beta}$ such that $\beta N = 1$. Again, take $\delta > 0$ small that $\frac{\delta(1+\beta)}{\beta} = 1$ Then

$$\begin{aligned} C(c_0, |X|_{C^1}, \delta)\frac{Q_i}{\eta^\delta} &\geq -C(c_0, |X|_{C^1}, Q)\frac{1}{\eta} - \frac{c_0}{2K^\beta(u - c_0)}Q_iQ_i^\beta \\ &\geq -C(c_0, |X|_{C^1}, Q)\frac{1}{\eta} - \frac{c_0}{2(u - c_0)}Q_ik_i^2 \end{aligned}$$

and we have

$$0 \geq -C(c_0, |X|_{C^1}, Q)\frac{1}{\eta} + h_{11}\phi(f) - C(c_0, |X|_{C^1}, |f|_{C^2})$$

yet again.

Now, a bound on ηh_{11} means bound on θ; that in particular means a bound on

$$\frac{h_{11}(0)}{u(0) - c_0} = \theta(0)$$

and we finally have our upper bound.

CHAPTER 4
Applications

This chapter caps off earlier discussion and result by placing them in context. The estimate presented clearly offers distinct information compared to estimates, say, obtained in [3] or [4], where one assumes a solution regular up to the boundary. We present some types of arguments in which a local estimate like this might be applied in this section.

4.1 Local Estimates

One case where a local estimate might be used is the local pre-compactness argument to establish existence or higher regularity. Since existence can be considered locally and smoothness is a local property, it often suffices to show the existence of a solution or a derivative locally, coupled with a kind of uniqueness argument.

First, recall the Arzel-Ascoli threorem [10, Thm 7.25]: a uniformly bounded and equicontinuous family of functions on a compact set is sequentially compact, that is any sequence has a uniformly convergent subsequence. Then we set the argument up so that such a limit will be the derivative required to show regularity. There are other distinct but similar arguments which translate uniform bounds to existence of topological limits: for example, method of continuity [6, Thm 17.8]. widely used in nonlinear elliptic PDE, and Banach-Alaoglu theorem [9, Thm 3.15], used in variational techniques.

We first consider, for a model of the argument, probably the most basic and fundamental problem in elliptic partial differential equation theory: Poisson's equation. As a linear uniformly elliptic PDE, it satisfies the following interior estimate [6, Thm 6.2]

Theorem 4.1.1. *Suppose $u \in C^{2,\alpha}(\Omega)$ is a solution of the equation $\Delta u = f$ on $\Omega \subset \mathbb{R}^n$ where $f \in C^\alpha(\Omega)$. Then for any $\Omega' \Subset \Omega$*

$$|u|_{C^{2,\alpha}(\Omega')} \leq C(\Omega', |f|_{C^\alpha(\Omega)}, |u|_{C^0(\Omega)}) \tag{4.1}$$

This estimate can be used to show that in this setting increasing regularity in f increases regularity in u by defining for small h, a unit vector v, the *difference operator*

$$\Delta_h u(x) := \frac{u(x + hv) - u(x)}{h}$$

which approximates the yet-$C^{1,\alpha}$ $\partial_v u$ as h becomes smaller. If $f \in C^{1,\alpha}$ then (4.1) can be used to establish local uniform bounds on $\partial_i \Delta_h u$, and $\partial_{ij} \Delta_h u$ and thus extract a C^α third derivative from the set of $\partial_{ij} \Delta_h u$ for small h. We can inductively continue this process, the resulting interior regularity theorem [6, Thm 6.17] being

Theorem 4.1.2. *In the setting of Theorem 4.1.1, $f \in C^{k,\alpha}$ implies $u \in C^{k+2,\alpha}$. In particular, $f \in C^\infty$ implies $u \in C^\infty$*

With a similar approximating argument, [11, Thm 1.2] proves from their interior curvature estimate an existence theorem; an interior gradient estimate is assumed, especially since the curvature estimate depends on it. Noting that their interior curvature estimate only slightly in the assumptions from ours, we can state the following application of our result, deferring to them the proof.

Theorem 4.1.3. *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain, and consider the equation (1.1) for a graph of the function u over Ω. If $f \in C^{1,1}(\bar{\Omega})$ there is an admissible subsolution $\bar{u} \in C^2(\Omega) \cap C^{0,1}(\bar{\Omega})$ such that its graph and its principal curvatures $\bar{\kappa}_i$ satisfy*

$$Q(\bar{\kappa}_1, ..., \bar{\kappa}_n) \geq f$$

then there exists an admissible solution $u \in C^{3,\alpha}(\Omega) \cap C^{0,1}(\bar{\Omega})$ of (1.1) with $u = 0$ on $\partial\Omega$.

The domain Ω is not assumed to have more than a C^1 boundary for u to have C^3 regularity, which is where the local estimate comes into play. As above, the proof extracts a subsequence from the sequence of already existing u_ϵ for small $\epsilon > 0$ to get the wanted solution u.

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