Topics in Pre-Big Bang Cosmology

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Abstract

In the current paradigm of contemporary cosmology, the universe begins with an initial singularity known as the Big Bang. However, the corresponding standard model of cosmology suffers from a number of unresolved conceptual issues. The aim of this thesis is thus to explore aspects of cosmology that go beyond the current paradigm with regard to the evolution of the very early universe and the nature of the initial Big Bang singularity. In particular, this involves the study of many models proposing the existence of a 'pre-Big Bang' universe, i.e., models that postulate that the generation of today's structures in the universe occurs before the Big Bang. Moreover, additional theories explore the possibility of resolving the initial Big Bang singularity into a non-singular bouncing cosmology. A common methodology in this thesis consists in using cosmological perturbation theory to connect such alternative theories and models of the very early universe to observational constraints from, e.g., the cosmic microwave background radiation. In particular, the primordial curvature perturbation and primordial gravitational wave power spectra, as well as the scalar bispectrum, are computed in several contexts. For instance, it is found that a certain class of models, known as single field matter bounce cosmology, suffers from a no-go theorem, which invalidates those models, i.e., they cannot agree with observations. How general the no-go theorem can be applied is explored, and it is discovered that a theory of massive gravity can evade the theorem and can resolve several issues in single field matter bounce cosmology. In a slightly different context, it is shown that black holes can generally form in a contracting universe before the Big Bang, representing an interesting prediction. It is then investigated whether a string theory inspired model for the very early universe could take advantage of such primordial black holes in its evolution. Finally, theoretical aspects such as the stability of modified gravity theories resolving the initial Big Bang singularity are explored. It is found that certain models are better suited in terms of describing gravity up to higher energy and curvature scales.

Abrégé

Dans le paradigme actuel de la cosmologie contemporaine, l'univers commence avec une singularité initiale connue sous le nom de *Big Bang*. Par contre, le modèle standard de cosmologie correspondant souffre d'un grand nombre de problèmes conceptuels non résolus. Le but de cette thèse est donc d'explorer des aspects de la cosmologie qui vont au-delà du paradigme actuel par rapport à l'évolution de l'univers primordial et la nature de la singularité initiale du *Big Bang*. En particulier, cela implique l'étude de plusieurs modèles proposant l'existence d'un univers «pré *Big Bang*», c'est-à-dire des modèles postulant que la génération des structures de l'univers d'aujourd'hui se produit avant le *Big Bang*. De plus, d'autres théories explorent la possibilité de résoudre la singularité initiale du *Big Bang* en un univers non singulier rebondissant. Une méthodologie récurrente dans cette thèse consiste à utiliser la théorie des perturbations cosmologiques pour établir un lien entre de telles théories et de tels modèles alternatifs de l'univers primordial et les contraintes observationnelles venant, par exemple, du fond diffus cosmologique. En particulier, les spectres de puissance des perturbations de courbures primordiales et des ondes gravitationnelles primordiales, ainsi que le bispectre scalaire, sont calculés dans plusieurs contextes. Par exemple, nous trouvons qu'une certaine classe de modèles, connue sous le nom de la cosmologie rebondissante dominée par un seul champ de matière à pression nulle, souffre d'un théorème de type *no-go* qui invalide ces modèles, c'est-à-dire qu'ils ne peuvent être en accord avec les observations. Nous explorons à quel point le théorème de type *no-go* peut être appliqué et nous découvrons qu'une théorie de gravité massive peut éluder le théorème et résoudre plusieurs problèmes en lien avec les modèles de la cosmologie rebondissante dominée par un seul champ de matière à pression nulle. Dans un contexte légèrement différent, nous démontrons que des trous noirs peuvent généralement se former dans un univers en contraction avant le *Big Bang*, représentant une prédiction intéressante. Nous investiguons ensuite si un modèle d'univers primordial inspiré de la théorie des cordes pourrait tirer profit de tels trous noirs primordiaux dans son évolution. Finalement, nous explorons certains aspects théoriques tels que la stabilité de théories de gravité modifiée qui résolvent la singularité initiale du *Big Bang*. Nous trouvons que certains modèles sont plus appropriés pour décrire la gravité jusqu'à des échelles d'énergie et de courbure plus élevées.

Contents

Preface 1

1 Introduction 4
- 1.1 The observational status of primordial cosmology . . . 5
- 1.2 Why pre-Big Bang cosmology? . . . 12
- 1.3 Outline of this thesis . . . 12
- 1.4 Notation and conventions . . . 14

I Review of contemporary primordial cosmology 16

2 Cosmological perturbations: computing observables from theory 17
- 2.1 Background . . . 17
  - 2.1.1 Homogeneous and isotropic cosmology . . . 18
  - 2.1.2 Anisotropic cosmology . . . 21
- 2.2 Arnowitt-Deser-Misner formalism . . . 21
- 2.3 Cosmological perturbation theory . . . 24
  - 2.3.1 Perturbed metric . . . 24
  - 2.3.2 Gauge degrees of freedom . . . 25
  - 2.3.3 Scalar degrees of freedom . . . 25
  - 2.3.4 Matter perturbations . . . 26
  - 2.3.5 Scalar fields . . . 29
  - 2.3.6 Popular scalar gauges . . . 30
  - 2.3.7 Vector degrees of freedom . . . 31
- 2.4 Perturbed Einstein equations in the Newtonian gauge . . . 31
- 2.5 Perturbed action in the comoving gauge and equations of motion . . . 34

2.6 Tensor perturbations 37
2.6.1 Generic evolution of the perturbations 39
2.7 Computing the two-point correlation function 40
2.8 Computing the three-point correlation function 46
2.8.1 Bispectrum 46
2.8.2 Shapes 47
2.8.3 In-in formalism and third-order perturbed action 49

3 The status of theories of the very early universe 51
3.1 Horizon and flatness problems 51
3.2 Inflationary cosmology 54
3.2.1 Predictions of inflation 57
3.2.2 Problems of inflation 58
3.3 Alternatives to inflation 60
3.3.1 Ekpyrotic cosmology 61
3.3.2 Matter bounce cosmology 65
3.3.3 Pre-Big Bang cosmology 71
3.3.4 String Gas Cosmology 71

4 The status of non-singular cosmology 74
4.1 The singularity theorems 74
4.2 Non-singular bouncing cosmology 76
4.3 Limiting curvature 82

II Publications 87

5 Evolution of cosmological perturbations and the production of non-Gaussianities through a nonsingular bounce: Indications for a no-go theorem in single field matter bounce cosmologies 88
5.1 Introduction 88
5.2 A brief review of cosmological perturbation theory 91
5.3 Outline of the no-go conjecture 94
5.3.1 Fluctuations in the matter bounce 94

5.3.2 Bound from the tensor-to-scalar ratio . . . 96
5.3.3 Bound from the bispectrum . . . 98
5.3.4 The no-go theorem . . . 100
5.4 A brief review of single field bouncing cosmology . . . 101
5.5 Evolution of curvature perturbations during the bounce . . . 104
5.5.1 Evolution in Regime I . . . 106
5.5.2 Evolution in Regime II . . . 106
5.5.3 Evolution in Regime III . . . 108
5.5.4 Discussion . . . 109
5.5.5 Comparison with tensor modes . . . 111
5.6 A comprehensive analysis of the production of primordial non-Gaussianities during the bounce phase . . . 112
5.7 Combination of the observational bounds on non-Gaussianities and on the tensor-to-scalar ratio . . . 119
5.8 Conclusions . . . 122
5.9 Curvature perturbations expanding about the singularity . . . 123
5.10 Perturbations outside the bounce phase . . . 124
5.11 Third order perturbed action . . . 126
5.11.1 Derivation of the general form of the third order action . . . 126
5.11.2 Third order perturbed action in the limit of the matter-dominated contracting phase . . . 131
5.12 Evaluating the shape function in the bounce phase . . . 133

6 Matter bounce cosmology with a generalized single field: non-Gaussianity and an extended no-go theorem 138
6.1 Introduction . . . 138
6.2 Setup and background dynamics . . . 141
6.3 Mode functions and two-point correlation functions . . . 144
6.4 Non-Gaussianity . . . 148
6.4.1 Cubic action . . . 148
6.4.2 Contributions to the shape function . . . 149
6.4.3 Summary of results . . . 153
6.5 Amplitude parameter of non-Gaussianities and implication for the no-go theorem 155

6.6 Conclusions and discussion 157
6.7 The ratio λ/Σ 159

7 Massive gravity and the suppression of anisotropies and gravitational waves in a matter-dominated contracting universe 162
7.1 Introduction 162
7.2 The modified gravity theory 165
7.2.1 Setup 165
7.2.2 Hamiltonian analysis 168
7.2.3 Recovering the space-time diffeomorphism invariance 170
7.3 Nonsingular bouncing cosmology 170
7.3.1 Background evolution 170
7.3.2 Reconstructing a potential for the time-like Stückelberg scalar field . 176
7.4 Cosmological perturbation analysis 177
7.4.1 Scalar perturbations 178
7.4.2 Vector perturbations 179
7.4.3 Tensor perturbations 180
7.5 Anisotropies 182
7.6 Solutions in a matter-dominated contracting phase 183
7.6.1 Evolution of anisotropies 184
7.6.2 Evolution of gravitational waves 185
7.7 Conclusions and discussion 195
7.8 Second-order perturbed action for scalar modes 196
7.9 Evolution of anisotropies in a general background 199

8 Black hole formation in a contracting universe 201
8.1 Introduction 201
8.2 Evolution of the gravitational potential in a contracting universe 203
8.2.1 General background setup 203
8.2.2 Cosmological perturbations 204
8.3 Density contrast, Jeans scale, and power spectrum 206
8.3.1 Density contrast 206
8.3.2 Jeans scale 207

8.3.3 Power spectrum . . . 210
8.4 Examples of initial conditions . . . 211
8.4.1 Quantum vacuum . . . 211
8.4.2 Thermal initial conditions . . . 215
8.5 Black hole formation . . . 218
8.5.1 General requirement for black hole collapse . . . 218
8.5.2 Smoothing . . . 219
8.5.3 Critical density contrast for black hole collapse . . . 220
8.5.4 Press-Schechter formalism and a condition for black hole collapse . . . 222
8.6 Conclusions and discussion . . . 228

9 Stringy black-hole gas in α'-corrected dilaton gravity 233
9.1 Introduction . . . 233
9.2 String holes . . . 236
9.2.1 Black hole/string correspondence . . . 236
9.2.2 String-hole gas . . . 237
9.3 Dynamics from dilaton gravity . . . 241
9.3.1 Tree-level dilaton gravity . . . 241
9.3.2 Action with α' corrections . . . 244
9.3.3 $O(d,d)$-invariant α'-corrected action . . . 248
9.4 Phase space analysis . . . 252
9.4.1 Stability of the fixed point with GMV's action . . . 252
9.4.2 Stability of the fixed point with Meissner's action . . . 257
9.5 Conclusions and discussion . . . 259
9.6 String-hole gas evolution in the Einstein frame . . . 262

10 Cosmological perturbations and stability of nonsingular cosmologies with limiting curvature 264
10.1 Introduction . . . 264
10.2 Nonsingular cosmology with limiting curvature . . . 267
10.2.1 Setup of the theory and background evolution . . . 267
10.2.2 Cosmological perturbations and stability analysis . . . 274
10.2.3 Equivalence with $f(R,\mathcal{G})$ gravity . . . 282

10.3 New nonsingular model with a new curvature scalar 283
10.3.1 Setup 283
10.3.2 Cosmological perturbations and stability analysis 285
10.3.3 Stability around an anisotropic background 289
10.3.4 Recovering Einstein gravity and the addition of matter sources 292
10.4 Conclusions and discussion 294

11 Conclusions 296
11.1 Future directions 297

A Maximal extensions and singularities in inflationary spacetimes 300
A.1 Introduction 300
A.2 Setup and past incompleteness of inflationary spacetimes 304
A.3 The parallely propagated curvature singularity in inflationary spacetimes . . 306
A.4 Coordinates beyond the past boundary 309
A.5 Examples 311
A.5.1 Exact de Sitter spacetime 311
A.5.2 Inextendible toy model 312
A.5.3 Extendible toy model 314
A.6 Implications for inflationary models 316
A.6.1 Energy condition for subleading component 316
A.6.2 Single field slow-roll inflation 319
A.6.3 Limiting curvature models 322
A.7 Summary and discussion 326
A.8 General extendibility 327

Bibliography 330

List of Figures

1.1 *Planck* 2015 CMB sky map. 6
1.2 *Planck* 2018 temperature-temperature angular power spectrum. 7
1.3 *Planck* 2018 curvature perturbation power spectrum reconstruction. 9

3.1 Space and time sketch of inflation and cosmological perturbations. 56
3.2 Same spacetime sketch as Fig. 3.1 with the regimes of ignorance highlighted. 59
3.3 Space and time sketch of a prototypical bouncing cosmology. 62

5.1 Sketch of the different regimes in the bounce phase. 105
5.2 Sketch of the evolution of curvature perturbation on super-Hubble scales. . . 109

6.1 The shape of the bispectrum for different sound speeds. 154

7.1 Background evolution for the matter bounce model. 173
7.2 Plots of the lapse-independent function in the effective field theory, the cosmological constant-like function and its time derivative. 175

8.1 Plots of the probability that a black hole has formed. 226
8.2 Same plots as in Fig. 8.1, but for thermal initial conditions. 227

9.1 Phase-space trajectories for the Gasperini-Maggiore-Veneziano action in an FLRW background with matter satisfying the continuity equation and $p = \sigma/2 = 0$. 254
9.2 Phase-space trajectories for Meissner's action in an FLRW background with matter satisfying the continuity equation and $p = \sigma/2 = 0$. 257

10.1 Background trajectories for the inflationary model. 270

10.2 Phase-space diagram showing different background inflationary trajectories. . 272
10.3 Background trajectories for the genesis model. 272
10.4 Phase-space diagram showing different background genesis trajectories. . . . 274
10.5 Phase-space diagram of the inflationary model with stability conditions overlaid. 280
10.6 Phase-space diagram of the genesis model with stability conditions overlaid. 282
10.7 Phase-space diagram of the inflationary model with stability conditions in the new theory overlaid. 287
10.8 Phase-space diagram of the genesis model with stability conditions in the new theory overlaid. 288

A.1 Penrose diagram of de Sitter spacetime. 303
A.2 Penrose diagram of the inextendible toy model. 313
A.3 Penrose diagram of the extendible toy model. 317
A.4 Trajectories in phase space. 325

Preface

This thesis contains seven peer-reviewed, published articles that are original. These are presented in their original form, due to copyright, in Chapters 5 to 10 and Appendix A. We state below the contribution of the author to each of the included works.

Contributions of the Author

Jerome Quintin, Zeinab Sherkatghanad, Yi-Fu Cai, and Robert H. Brandenberger, *Evolution of cosmological perturbations and the production of non-Gaussianities through a nonsingular bounce: Indications for a no-go theorem in single field matter bounce cosmologies*, Physical Review D **92**, no. 6, 063532 (2015) [arXiv:1508.04141 [hep-th]]. Ref. [562] in the bibliography.

This article is presented in Chapter 5. As the first author of this paper, I led the analysis, I carried out a major portion of the calculations (around 75%) and verified all other calculations in tandem with the co-authors, I produced all the figures, and I wrote most of the text (around 90%) based on shared notes with Zeinab Sherkatghanad. Yi-Fu Cai and Robert Brandenberger participated in the discussion, analysis, verification of the calculations, and proofreading of the manuscript.

Yu-Bin Li, Jerome Quintin, Dong-Gang Wang, and Yi-Fu Cai, *Matter bounce cosmology with a generalized single field: non-Gaussianity and an extended no-go theorem*, Journal of Cosmology and Astroparticle Physics **1703**, no. 03, 031 (2017) [arXiv:1612.02036 [hep-th]]. Ref. [438] in the bibliography.

This article is presented in Chapter 6. I designed the project and led the analysis equally with the co-authors Yu-Bin Li and Dong-Gang Wang. Hence, our names appear in alphabetical order. I performed around 60% of the calculations myself and verified all other calculations in tandem with the co-authors. Finally, I produced all the figures and wrote the entire manuscript myself, based on shared notes between all authors. Yi-Fu Cai participated in the discussion, analysis, and verification of the calculations.

Chunshan Lin, Jerome Quintin, and Robert H. Brandenberger, *Massive gravity and the suppression of anisotropies and gravitational waves in a matter-dominated contracting universe*, Journal of Cosmology and Astroparticle Physics **1801**, 011 (2018) [arXiv:1711.10472 [hep-th]]. Ref. [459] in the bibliography.

This article is presented in Chapter 7. I designed the project and led the analysis equally with the co-author Chunshan Lin. Hence, our names appear in alphabetical order. I performed around three fifths of the calculations myself (mainly those of Secs. 7.3, 7.6, and 7.9) and verified all other calculations in tandem with the co-authors. Finally, I produced all the figures and wrote the entire manuscript myself, based on shared notes with Chunshan Lin. Robert Brandenberger participated in the discussion, analysis, verification of the calculations, and proofreading of the manuscript.

Jerome Quintin and Robert H. Brandenberger, *Black hole formation in a contracting universe*, Journal of Cosmology and Astroparticle Physics **1611**, no. 11, 029 (2016) [arXiv: 1609.02556 [astro-ph.CO]]. Ref. [559] in the bibliography.

This article is presented in Chapter 8. As the first author of this paper, I led the analysis, performed all the calculations, produced all the figures, and wrote the entire manuscript. Robert Brandenberger participated in the discussion, analysis, verification of the calculations, and proofreading of the manuscript.

Jerome Quintin, Robert H. Brandenberger, Maurizio Gasperini, and Gabriele Veneziano, *Stringy black-hole gas in α'-corrected dilaton gravity*, Physical Review D **98**, no. 10, 103519 (2018) [arXiv:1809.01658 [hep-th]]. Ref. [560] in the bibliography.

This article is presented in Chapter 9. As the first author of the paper, I led the analysis, performed all the calculations, produced all the figures, and wrote the entire manuscript. Robert Brandenberger, Maurizio Gasperini, and Gabriele Veneziano participated in the discussion, analysis, verification of the calculations, and proofreading of the manuscript.

Daisuke Yoshida, Jerome Quintin, Masahide Yamaguchi, and Robert H. Brandenberger, *Cosmological perturbations and stability of nonsingular cosmologies with limiting curvature*, Physical Review D **96**, no. 4, 043502 (2017) [arXiv:1704.04184 [hep-th]]. Ref. [635] in the bibliography.

This article is presented in Chapter 10. The co-author Daisuke Yoshida designed the project, but we led the analysis equally. I performed around two fifths of the calculations myself and verified all other calculations in tandem with Daisuke Yoshida. I produced all the figures together with Daisuke Yoshida. I wrote a large fraction of the text myself, based on shared notes with Daisuke Yoshida, and I collaborated in the writing of the whole manuscript. Masahide Yamaguchi and Robert Brandenberger participated in the discussion, analysis, verification of the calculations, and proofreading of the manuscript.

Daisuke Yoshida and Jerome Quintin, *Maximal extensions and singularities in inflationary spacetimes*, Classical and Quantum Gravity **35**, no. 15, 155019 (2018) [arXiv:1803.07085 [gr-qc]]. Ref. [634] in the bibliography.

This article is presented in Appendix A. The co-author Daisuke Yoshida designed the project, but we led the analysis equally. I performed around two fifths of the calculations myself (mainly those of Secs. A.6 and A.8) and verified all other calculations in tandem with Daisuke Yoshida. Daisuke Yoshida produced all the figures. Finally, I wrote a large fraction of the text myself, based on shared notes with Daisuke Yoshida, and I collaborated in the writing of the whole manuscript.

Chapter 1

Introduction

Cosmology is the science of the universe as a whole, where one tries to understand its evolution (past, current, and future), as well as its structures on large scales. This thesis focuses on the evolution of the *very early universe*, also known as *primordial cosmology*, i.e., the universe at its inception in the far past. In the standard model of cosmology, the universe 'begins' about 13.8 billion years ago with a Big Bang, a singular point in time where the universe was infinitely dense and hot. In this sense, very early universe cosmology tackles Big Bang-related issues. More precisely, we attempt to answer fundamental questions such as "What happened at the time of the Big Bang?", "Was there a Big Bang?", "Where do the structures observed in the universe today originate from?" and "Why do they originate like that?" For lack of fully satisfying answers, we at least try to provide a small contribution to our understanding about these issues and try to make incremental progress in fundamental physics.

Before introducing the different theories of primordial cosmology and explaining why we study them, let us briefly digress in the following section and use this introduction to put the reader up to speed with respect to what we know about the very early universe, i.e., what are the latest observations that give us information about primordial cosmology. Indeed, while this thesis is mostly theoretical in nature and tackles abstract theories of gravity and cosmology, science only progresses by making testable predictions. For a lot of the work presented in this thesis, observational measurements serve as the 'judge' as to whether certain theories are viable or whether they are falsified, in which case they should be rejected. The latter situation is nevertheless encouraging, in the sense that it allows theorists to move forward and focus their attention on alternative approaches. This thesis covers many 'alternative' ideas, and hopefully, they will allow the reader to have a better understanding of where physics stands at the extremely high energy scales pertaining to the very early universe.

1.1 The observational status of primordial cosmology

The observational pillar of primordial cosmology is the *cosmic microwave background* (CMB), which is a unique prediction of the hot Big Bang model. Indeed, Big Bang cosmology proposes that the universe starts in a highly dense and hot state, where elementary particles interact strongly with one another. In fact, the universe is so dense that photons cannot travel freely in space and time, hence the universe is opaque. The Big Bang model predicts an expanding universe, so as the universe expands, matter cools down and interactions become more sparse. At some point, it becomes dilute enough for photons to start moving freely, i.e., with a mean free path at least the size of the observable universe. This corresponds to the moment nuclei and electrons combine to form atoms for the first time. This moment is called *recombination*, happening roughly $300,000$ years after the Big Bang, and the photons that are emitted at that time constitute the CMB, the oldest electromagnetic signal that can be observed in the universe. The signal is in the microwave band today because light redshifted from the time it was emitted due to the expansion of the universe. Also, it is a 'background' radiation in the sense that it is present homogeneously and isotropically across the universe.

The latest, state-of-the-art measurement of the CMB was performed by the *Planck* satellite [4, 6, 21]. In Fig. 1.1, we show the sky temperature map from the *Planck* 2015 data release [4]. The average temperature of the CMB is measured to be $\bar{T}_{\mathrm{cmb}} = 2.7255\,\mathrm{K}$ [290], and Fig. 1.1 shows the fluctuations in temperature of the microwave radiation about the average. We note that the temperature fluctuations are of the order of

$$\frac{|\delta T|}{\bar{T}_{\mathrm{cmb}}} \sim 10^{-4}\,, \tag{1.1.1}$$

and this is a first observational measurement that should be explained by primordial cosmology. In fact, explaining the origin of the fluctuations in the CMB is paramount in primordial cosmology.

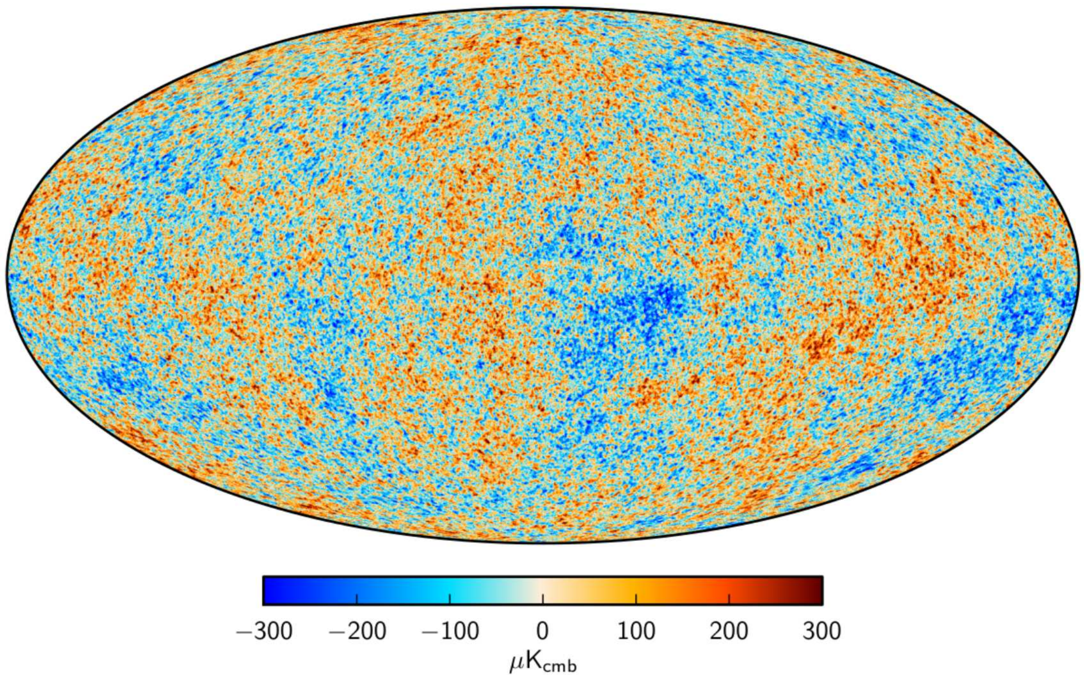

Figure 1.1 *Planck* 2015 CMB sky map taken from Ref. [4]. The color code indicates the fluctuations in temperature in μK.

To be more quantitative about the information that Fig. 1.1 encodes, we decompose the temperature fluctuations as follows,

$$\frac{\delta T}{\bar{T}_{\rm cmb}} = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\theta, \phi) \,, \tag{1.1.2}$$

where $Y_{\ell m}(\theta, \phi)$ are the spherical harmonics, which are functions of the spherical angles θ and ϕ on the sky. The parameters $a_{\ell m}$ encode the size of the fluctuations, and under the assumption of isotropy, i.e., assuming there is no preferred direction in the universe, we can sum over the index m and define

$$C_\ell^{TT} \equiv \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} \langle a_{\ell m}^* a_{\ell m} \rangle \,. \tag{1.1.3}$$

In the above, angular brackets can be thought of as the expectation value of a random variable. Hence, the $a_{\ell m}$'s have a vanishing mean value, $\langle a_{\ell m} \rangle = 0$, but non-zero variance, $\langle a_{\ell m}^* a_{\ell m} \rangle \neq 0$. The resulting parameters, the C_ℓ^{TT}'s, thus represent a measure of the temperature variance or *two-point correlation function*, $\langle \delta T\, \delta T \rangle$, as a function of the multipole moment ℓ. Hence, C_ℓ^{TT} is also called the *temperature-temperature angular power spectrum.*

In Fig. 1.2, we show the results from the *Planck* 2018 data release [18]. The plot shows the data points for $\mathcal{D}_\ell^{TT} \equiv \ell(\ell+1)C_\ell^{TT}/(2\pi)$ with their 1σ error bars. Also, the blue curve represents the best fit model, called ΛCDM, which will be further discussed below. Fig. 1.2 constitutes a very good example of what precision cosmology has become today. As such, any successful model of primordial cosmology should be in agreement with the data points in Fig. 1.2.

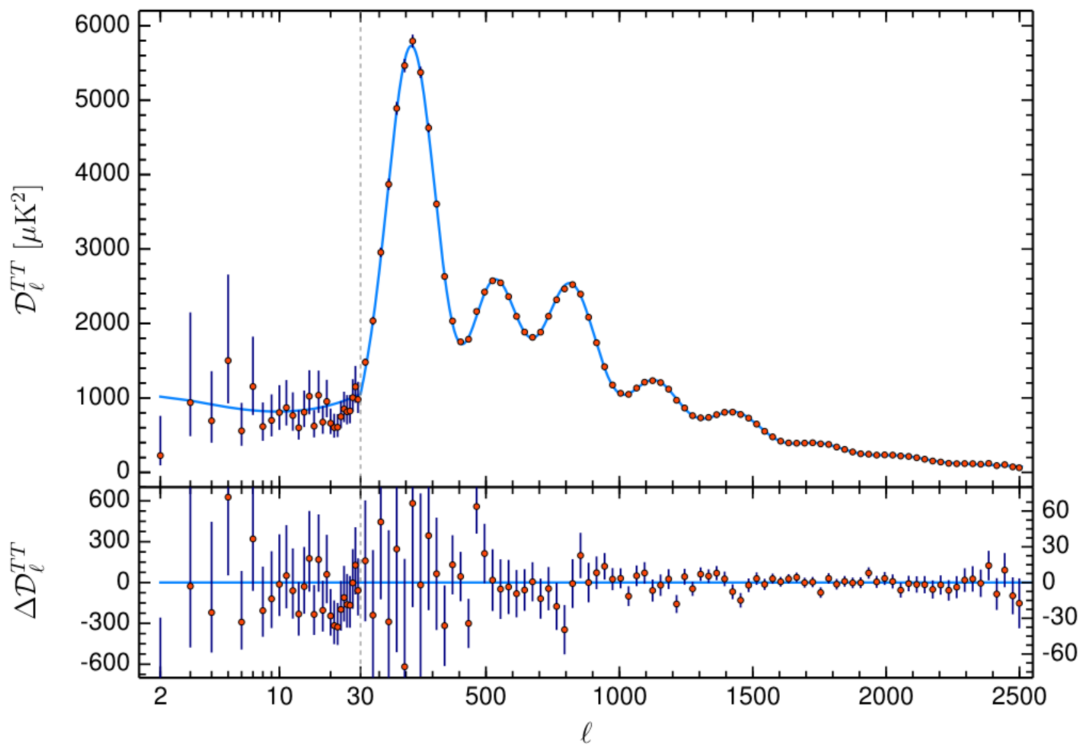

Figure 1.2 *Planck* 2018 temperature-temperature angular power spectrum taken from Ref. [18]. The red dots are the observational measurements and the purple lines are the 1σ error bars. The blue curve shows the best fit ΛCDM model from Ref. [18] and the bottom panel shows the residual, i.e., $\Delta\mathcal{D}_\ell^{TT} \equiv \mathcal{D}_\ell^{TT}$[measured] $- \mathcal{D}_\ell^{TT}$[best fit ΛCDM].

The CMB represents a snapshot of the universe at the time of recombination, but at $300,000$ years after the Big Bang, this is still very far from the very early times right after the Big Bang. Fortunately, physics in the approximate time range $[10^{-12}\,\mathrm{s},\, 3\times 10^5\,\mathrm{yr}]$ is relatively well understood (see, e.g., Refs. [258, 393] for reviews and details). The universe is first radiation dominated, going through the quark, hadron, and lepton epochs. Then, Big Bang nucleosynthesis occurs, where the basic atomic nuclei (mainly hydrogen and helium-4) form. After nucleosynthesis, the universe is dominated by a plasma composed of atomic nuclei, electrons, and photons until recombination occurs. As gravity and the microscopic physics are relatively well understood during these epochs, one can compute how initial fluctuations coming from the very early universe (anything happening before the onset of radiation-dominated expansion) evolve and compare them to the observed fluctuations in the CMB (see, e.g., Ref. [434] and references therein).

The reversed process can also be performed: starting from the observed data at the time of the CMB, such as the temperature-temperature angular power spectrum shown in Fig. 1.2, one can evolve physics backward until the time of the very early universe. The result is depicted in Fig. 1.3, which is a plot of the *primordial curvature perturbation power spectrum* (or two-point correlation function), denoted $\mathcal{P}_\zeta(k)$, where k is the wavenumber of the fluctuations. The concept of curvature perturbation will be defined in the next chapter, but for now, one can understand this quantity as being a combination of both spacetime and matter perturbations, and it is such primordial fluctuations that can evolve to give rise to CMB temperature fluctuations. Fig. 1.3 shows that the primordial power spectrum is nearly scale invariant across a wide range of scales, meaning that $\mathcal{P}_\zeta$ is almost independent of the wavenumber k. Yet, considering for example the 1σ confidence region in Fig. 1.3, rather than being horizontal the power spectrum has a slight tilt toward the red (i.e., toward larger wavenumbers/smaller wavelengths). Hence, we say that the power spectrum is *nearly* scale invariant and has a *red tilt*. As a consequence, it is useful to parameterize the primordial power spectrum as

$$\mathcal{P}_\zeta(k) = A_\mathrm{s}\left(\frac{k}{k_\mathrm{pivot}}\right)^{n_\mathrm{s}-1}, \tag{1.1.4}$$

where A_s is the amplitude of the scalar fluctuations, k_pivot is an arbitrary pivot scale for the parameterization, and n_s is the scalar tilt. With this definition, $n_\mathrm{s} = 1$ corresponds to exact scale invariance, and a red tilt means $n_\mathrm{s} < 1$ (though close to 1).

The current best fit cosmological model has only six parameters and is called ΛCDM. The letter Λ refers to the cosmological constant Λ that could explain the late-time acceleration of the universe (see, e.g., Refs. [543, 572]), and CDM stands for 'cold dark matter', a necessary ingredient to explain the formation of the large-scale structures in the universe and the rotation curves of galaxies (see, e.g., Refs. [74, 217, 243]). Among the six parameters of

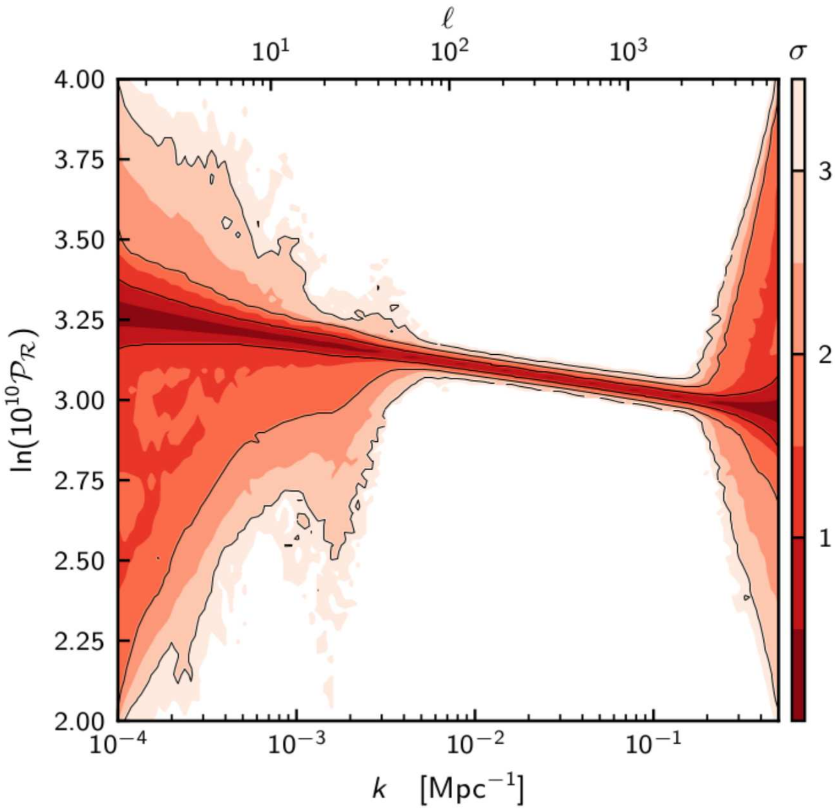

Figure 1.3 *Planck* 2018 curvature perturbation power spectrum reconstruction taken from Ref. [22]. The bottom horizontal axis shows the fluctuations' wavenumber k and the top horizontal axis shows the corresponding angular scale in the CMB as a function of the multipole number ℓ. The vertical axis is the rescaled curvature power spectrum $\mathcal{P}_\zeta$. Note that the *Planck* collaboration uses the variable $\mathcal{R}$ instead of ζ for the curvature perturbation. The color code gives the confidence level of the measurement. For example, the two darkest shades of red indicate a confidence level of up to 65% ($\leq 1\sigma$), the next two lighter shades of red (neutral red) indicate a confidence level of up to 95% ($\leq 2\sigma$), and the next two lighter shades of red (light red) indicate a confidence level of up to 99.7% ($\leq 3\sigma$). For the very lightest shade of red, the confidence level is greater than 99.7% ($> 3\sigma$).

ΛCDM, only two refer to the primordial universe: A_s and n_s. The current best measurements are [18]:

$$\ln(10^{10}A_\mathrm{s}) = 3.047 \pm 0.014 \qquad \text{and} \qquad n_\mathrm{s} = 0.9665 \pm 0.0038\,. \tag{1.1.5}$$

Hence, it is crucial for a theory of the very early universe to give rise to a curvature perturbation power spectrum in the form of Eq. (1.1.4) with an amplitude and a tilt in concordance with the numbers above.

The scalar amplitude and tilt are the only parameters about the very early universe that are known with very high precision. Yet, we can gain more information about the primordial universe from the CMB. For instance, beyond the two-point correlation function, we can compute the three-point correlation function, which can be thought of as skewness in a probability distribution. Similar to the discussion about the two-point function above, correlations of the form $\langle(\delta T)^3\rangle$ can be decoded from the CMB temperature fluctuation map, and one can transform this into knowledge about the primordial three-point function of the form $\langle\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle$, also known as the bispectrum. The amplitude of the primordial three-point function is then usually stated in terms of the f_NL parameters, which quantify how 'non-linear' the perturbations are or how much they deviate from a perfect Gaussian distribution. In this sense, the terms 'non-Gaussianity', 'three-point function', and 'bispectrum' are sometimes used interchangeably. The function f_NL typically depends on three wavenumbers, but we often focus on certain limits or 'shapes' of the three-point function, which have names such as 'local', 'equilateral', or 'orthogonal'. We will define the function f_NL as well as the various shapes appropriately in the next chapter, but for now, let us state the current best measurements coming from the *Planck* telescope [13]:

$$f_\mathrm{NL}^\mathrm{local} = 0.8 \pm 5.0\,, \qquad f_\mathrm{NL}^\mathrm{equil} = -4 \pm 43\,, \qquad f_\mathrm{NL}^\mathrm{ortho} = -26 \pm 21\,. \tag{1.1.6}$$

We see that the three quantities are currently consistent with zero given the 1σ uncertainties. This is why primordial non-Gaussianities do not enter in the 'standard model' of cosmology. Nevertheless, they constitute key information about the very early universe since models should be consistent with Gaussianity, or at least, predict only small deviations from Gaussianity. As will be clear from this thesis, this is very useful since models predicting too large non-Gaussianities can thus be ruled out.

There is one more piece of information that can be extracted from the CMB that is very useful for primordial cosmology. If primordial gravitational waves (also known as tensor perturbations) are produced in the very early universe, they can affect the polarization of photons in the CMB [582]. Beyond the measurement of the CMB temperature, the polarization of the CMB photons also carries a lot of information about the physics of the early and late universe. Polarization comes in two types: the 'gradient' or E-mode component and the 'curl' or B-mode component. Primordially, almost only[1] gravitational waves can induce B modes; density perturbations cannot. Accordingly, the BB correlation function is a direct probe of the primordial gravitational wave power spectrum.

The primordial tensor power spectrum is usually parameterized as follows,

$$\mathcal{P}_{\mathrm{t}}(k) = A_{\mathrm{t}} \left(\frac{k}{k_{\mathrm{pivot}}} \right)^{n_{\mathrm{t}}} , \tag{1.1.7}$$

where A_{t} is the amplitude and n_{t} is the tilt. We note the difference in convention compared to the scalar modes since now $n_{\mathrm{t}} = 0$ represents scale invariance. A measurement of both A_{t} and n_{t} would yield a lot of information about the very early universe. However, there is no confirmed detection of primordial gravitational waves. At this point, there exist only upper bounds on the amplitude. It is useful to define the tensor-to-scalar ratio,

$$r \equiv \frac{\mathcal{P}_{\mathrm{t}}}{\mathcal{P}_{\zeta}} , \tag{1.1.8}$$

since this is the quantity that is usually constrained by observations. Currently, the best measurements [11, 22] indicate

$$\begin{aligned} r &< 0.064 \qquad (95\,\%\ \mathrm{CL},\ k_{\mathrm{pivot}} = 0.002\,\mathrm{Mpc}^{-1}) \,, \\ r &< 0.070 \qquad (95\,\%\ \mathrm{CL},\ k_{\mathrm{pivot}} = 0.05\,\mathrm{Mpc}^{-1}) \,. \end{aligned} \tag{1.1.9}$$

Similarly to non-Gaussianities, while the measurements are still consistent with zero, such bounds nevertheless tell us a lot of information about very early universe models. Predicting a tensor-to-scalar ratio in serious excess of the observational bounds can serve as the basis to rule out certain models (as will become clear in this thesis).

[1] At leading order, vector perturbations can also contribute to form B-mode polarization in the CMB radiation. For instance, cosmic string wakes can lead to a strong BB correlation function [241]. Nevertheless, signals from vector modes (e.g., due to cosmic string wakes) and tensor modes (e.g., primordial gravitational waves) should leave relatively distinct features in the CMB BB angular power spectrum (see, e.g., Ref. [509]).

1.2 Why pre-Big Bang cosmology?

In the previous section, we lay down the current observational measurements that constrain the very early universe. As was stressed, successful theories of primordial cosmology should predict numbers in agreement with Eqs. (1.1.5), (1.1.6) and (1.1.9), as well as resolve theoretical issues. There exists a paradigm in contemporary cosmology: the universe begins with the Big Bang singularity, subsequently expands exponentially fast for a very short time interval, and then standard radiation-dominated expansion follows. The phase of exponential expansion is known as *inflation*. As we will explain in Chapter 3, inflationary cosmology is successful in solving many of the problems of standard Big Bang cosmology and makes several predictions in good agreement with the constraints outlined in the previous section. However, inflation has problems of its own and cannot resolve every issue of standard Big Bang cosmology. Just the fact that the universe begins with a singularity remains confounding. These facts motivate alternative views for the very early universe. As will become clear in Chapter 3, many of the alternative scenarios require the Big Bang singularity to be replaced by a non-singular cosmology, often a smooth transition from a contracting universe to an expanding universe. In those models, the processes of primordial cosmology occur before the 'Big Bang', hence we explore topics in 'pre-Big Bang cosmology'.

1.3 Outline of this thesis

The thesis is organized as follows: Part I serves as a review of contemporary primordial cosmology, while Part II presents original research articles published by the author of this thesis. The goal of Part I is to introduce the necessary tools and provide up-to-date theoretical developments to follow and appreciate Part II of the thesis. In other words, Part I presents the methodology often used in Part II, as well as sets the context in which the chapters of Part II lie.

Specifically, Chapter 2 of Part I begins[2] with a review of cosmological perturbation theory, which is the key theoretical approach to connect concepts of primordial cosmology to the observational constraints discussed earlier in this introductory chapter. Among other topics,

[2]Since it is not possible to review everything, one has to start somewhere. In this thesis, the author more or less assumes that the reader is comfortable with general relativity and certain concepts of cosmology at the level of, e.g., Refs. [189, 258]. Nevertheless, key concepts are reviewed, and an emphasis is given on notions that are used throughout the rest of the thesis.

we review basic background cosmology, the perturbed Einstein field equations, and how to compute two- and three-point correlation functions. The idea is that the reader should understand how one can compute quantities such as A_s, n_s, A_t, n_t, r, and f_NL. Chapter 3 then introduces the underlying principles of theories of the very early universe. We review the conditions that theories of primordial cosmology should satisfy and show how inflationary cosmology as well as models of bouncing cosmology can realize these conditions. For every class of models, we try to emphasize what are the theoretical predictions, their status with respect to the observations, and discuss the remaining theoretical issues. In the last chapter of Part I, Chapter 4, we present the current knowledge in the field of resolving the Big Bang singularity. We explain why this is a difficult task, what are the approaches that can be undertaken, and present a few theories that succeed in resolving the Big Bang singularity to yield a non-singular cosmology. An emphasis is also given on the remaining theoretical issues of those theories, mainly with respect to the stability of the cosmological perturbations.

The papers presented in Part II have been mentioned in the Preface, and the reader may notice that they are not ordered according to their publication date. Instead, we tried to regroup topics that fit together. In that sense, Part II starts with Chapter 5, the first article to propose the existence of a no-go theorem for single field matter bounce cosmology, an alternative to inflationary cosmology that proposes a matter-dominated contracting universe before the Big Bang. The no-go theorem involves the connection between the enhancement of curvature perturbations, the production of non-Gaussianities, and the suppression of the tensor-to-scalar ratio through a non-singular bounce phase. Chapter 6 is an immediate follow-up to Chapter 5 since it presents an extended no-go theorem, again connecting curvature perturbations, non-Gaussianities, and the tensor-to-scalar ratio. Chapter 7 presents an idea that could rescue the matter bounce scenario by evading the no-go theorem. This is done at the expense of modifying the gravitational theory, hence a focus is given on the evolution of the primordial gravitational waves (related to the tensor-to-scalar ratio) among other things.

We move on with Chapter 8, which is connected to the idea of the matter bounce scenario. However, the aspiration is greater: the goal is to describe the general evolution of cosmological perturbations in a contracting universe before the Big Bang dominated by a generic fluid (of which a matter-dominated contracting universe is a subcase). By characterizing the evolution of the cosmological fluctuations, we derive the conditions under which black holes can form from the collapse of large inhomogeneities. Chapter 9 is an immediate follow-up, which starts with the picture of a pre-Big Bang universe dominated by large black holes (possibly formed from the processes described in Chapter 8). The idea is to explore whether it is feasible to construct a possibly new scenario for the very early universe in which black holes would play a key role. The article presented in Chapter 9 is only a first step forward in that direction by exploring whether string theory (as a proposal for the theory of quantum gravity) can support a phase of cosmological evolution dominated by a gas of stringy black holes.

The remaining two articles presented in Chapter 10 and Appendix A focus on more theoretical issues related to the Big Bang singularity. In Chapter 10, we explore the concept of limiting curvature as an approach to construct non-singular cosmologies. As the name says, the theory proposes a way of bounding spacetime curvature to avoid reaching singularities. In the paper of Chapter 10, we explored the cosmological perturbations for different implementations of limiting curvature. In particular, we assessed the viability of several models by deriving the conditions to prevent catastrophic instabilities in the perturbations. Appendix A is separated from the other chapters since it lies in a slightly different context. We explore extendibility (very roughly speaking, whether a spacetime has a singularity or not) in the context of inflationary cosmology. It is related to the rest of the thesis since it tackles aspects of the Big Bang singularity and in particular since we apply the theorems derived to a model of limiting curvature presented in Chapter 10 to prove its extendibility[3].

Finally, we conclude with Chapter 11, where we summarize the key results of this thesis, essentially giving the take-home messages of the original research articles presented in Part II. We also end with a section on future research directions. We give a short summary of what are the big questions that remain unanswered and what the author believes deserves to be tackled in the future.

1.4 Notation and conventions

Throughout Part I of the thesis, we try to be consistent in our notation and conventions. Greek indices indicate spacetime coordinates, $\mu, \nu, ... \in \{0, 1, 2, 3\}$, while Latin indices indicate spatial coordinates, $i, j, ... \in \{1, 2, 3\}$, unless otherwise noted. Also, we use the mostly

[3]Anecdotally, the analysis presented in Appendix A was motivated by us following a question by the anonymous referee of the article presented in Chapter 10. In that sense, Appendix A was a follow-up to Chapter 10.

plus metric signature, $(-,+,+,+)$. We define the reduced Planck mass by $M_{\mathrm{Pl}} \equiv 1/\sqrt{8\pi G_{\mathrm{N}}}$, where G_{N} is Newton's gravitational constant. Unless otherwise stated, referring to the Planck mass implicitly means the reduced Planck mass. The speed of light and Planck's reduced constant are set to unity: $c = \hbar = 1$. Cosmic (or physical) time is denoted by t, and a derivative with respect to cosmic time is denoted by an overdot, $\dot{} \equiv \mathrm{d}/\mathrm{d}t$. Conformal time, which will soon be appropriately defined, is denoted by τ, and a derivative with respect to conformal time is denoted by a prime, $' \equiv \mathrm{d}/\mathrm{d}\tau$.

Disclaimer Note, however, that the notation may change from one chapter to another in Part II. We apologize for the inconvenience, but this is due to the fact that different conventions were used in different publications. As an example, Chapter 5 uses the mostly minus metric signature, $(+,-,-,-)$, the letter η for conformal time, and M_p for the Planck mass. Changes in convention should nevertheless be clearly identified and consistent within a given chapter.

Part I

Review of contemporary primordial cosmology

Chapter 2

Cosmological perturbations: computing observables from theory

In the introduction, we introduced a set of cosmological observables that carry a lot of information about the primordial universe. The main ones are the primordial curvature perturbation power spectrum (or two-point correlation function), the primordial scalar non-Gaussianities (or three-point correlation function), and the primordial tensor power spectrum. Finally, the ratio of the tensor and scalar two-point functions' amplitude is called the tensor-to-scalar ratio. One of the goals of this chapter is to explain how one computes these observables theoretically given a model for the very early universe. This chapter is based on many textbooks, lecture notes, and review articles; we particularly recommend Refs. [58–60, 121, 270, 403, 416, 448, 474, 482, 512, 518, 537, 547–549, 573], but many more exist.

2.1 Background

We now begin by reviewing background cosmology. The starting point is the Einstein field equations of general relativity[1],

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_{\mathrm{N}}T_{\mu\nu}\,, \tag{2.1.1}$$

[1] We assume that there is no cosmological constant, i.e., we set $\Lambda = 0$, unless otherwise stated.

where $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor, $g_{\mu\nu}$ is the metric tensor, $R \equiv g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar, and $T_{\mu\nu}$ is the energy-momentum tensor. The Einstein field equations can be derived by the variational principle starting with the Hilbert-Einstein action together with matter,

$$S = \int \mathrm{d}^4x \, \sqrt{-g}\frac{M_{\mathrm{Pl}}^2}{2}R + S_{\mathrm{matter}} \,, \tag{2.1.2}$$

where the energy-momentum tensor is then defined by

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{matter}}}{\delta g^{\mu\nu}} \,. \tag{2.1.3}$$

In the above, $g \equiv \det(g^{\mu}{}_{\nu})$ is the determinant of the metric tensor.

2.1.1 Homogeneous and isotropic cosmology

At the background level, the idea behind the cosmological principle is that the universe is homogeneous and isotropic, hence the metric is best described by the Friedmann-Lemaître-Robertson-Walker (FLRW) line element,

$$\bar{g}_{\mu\nu}\mathrm{d}x^{\nu}\mathrm{d}x^{\nu} = -\mathrm{d}t^2 + a(t)^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \,, \tag{2.1.4}$$

$$= a(\tau)^2(-\mathrm{d}\tau^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j) \,, \tag{2.1.5}$$

where $\bar{g}_{\mu\nu}$ indicates that this is the background expression for the metric tensor, a is the scale factor, which characterizes the size of the universe as a function of time, and δ_{ij} is the Kronecker delta function. Here and throughout this thesis, unless stated otherwise, we assume that spatial sections are perfectly flat at the background level. We wrote the FLRW line element in both physical and conformal time above. It is straightforward to see that the latter is defined by

$$\mathrm{d}\tau \equiv a^{-1}\mathrm{d}t \,. \tag{2.1.6}$$

It will be convenient to go form physical time to conformal time and vice versa throughout this thesis.

In the spirit of the cosmological principle, let us assume that the energy-momentum tensor is that of a homogeneous and isotropic perfect fluid with energy density $\bar{\rho}$ and pressure $\bar{p}$,

$$T^{\mu}{}_{\nu} = \mathrm{diag}(-\bar{\rho}, \bar{p}\,\delta^{i}{}_{j}) \,, \tag{2.1.7}$$

where at the background level, $\bar{\rho}$ and $\bar{p}$ are only functions of time. Substituting the above expressions for the energy-momentum tensor and FLRW metric in the Einstein field equations, we obtain the Friedmann equations,

$$3M_{\rm Pl}^2 H^2 = \bar{\rho}\,, \qquad 2M_{\rm Pl}^2 \dot{H} = -(\bar{\rho}+\bar{p})\,, \tag{2.1.8}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter. Equivalently, in conformal time,

$$3M_{\rm Pl}^2 \mathcal{H}^2 = a^2\bar{\rho}\,, \qquad 6M_{\rm Pl}^2 \mathcal{H}' = -a^2(\bar{\rho}+3\bar{p})\,, \tag{2.1.9}$$

where $\mathcal{H} \equiv a'/a$ is the conformal Hubble parameter. A useful identity that will be used throughout is

$$\mathcal{H} = aH\,. \tag{2.1.10}$$

From the Bianchi identity, $\nabla_\mu G^\mu{}_\nu = 0$, where ∇_μ denotes the covariant derivative, follows the conservation of the energy-momentum tensor, $\nabla_\mu T^\mu{}_\nu = 0$. In FLRW, the latter implies the following conservation equation,

$$\dot{\bar{\rho}} + 3H(\bar{\rho}+\bar{p}) = 0\,, \tag{2.1.11}$$

or equivalently,

$$\bar{\rho}' + 3\mathcal{H}(\bar{\rho}+\bar{p}) = 0\,, \tag{2.1.12}$$

which is also straightforward to derive by taking the time derivative of the first Friedmann equation and substituting in the second Friedmann equation.

We then assume that the pressure and energy density can be related through a barotropic equation of state of the form $\bar{p} = w\bar{\rho}$, where w is the equation of state parameter. With this assumption, the conservation equation can be integrated once to find

$$\bar{\rho} \propto a^{-3(1+w)}\,. \tag{2.1.13}$$

Substituting this result back into the first Friedmann equation and upon integration, one finds

$$a \propto |t|^{\frac{2}{3(1+w)}} \propto |\tau|^{\frac{2}{1+3w}}\,. \tag{2.1.14}$$

For example, typical matter contents include 'dust' with vanishing pressure, radiation with equation of state $\bar{p} = \bar{\rho}/3$, and a stiff fluid with equation of state $\bar{p} = \bar{\rho}$. In these instances, one finds

$$w = 0 \implies a \propto |t|^{2/3} \propto |\tau|^2 \,, \tag{2.1.15}$$

$$w = \frac{1}{3} \implies a \propto |t|^{1/2} \propto |\tau| \,, \tag{2.1.16}$$

$$w = 1 \implies a \propto |t|^{1/3} \propto |\tau|^{1/2} \,. \tag{2.1.17}$$

Such background solutions will be studied in greater detail throughout this thesis.

A useful variable that will often appear is

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = 1 - \frac{\mathcal{H}'}{\mathcal{H}^2} \,. \tag{2.1.18}$$

From the Friedmann equations, it is straightforward to check that

$$\epsilon = \frac{3}{2}\left(1 + \frac{\bar{p}}{\bar{\rho}}\right) = \frac{3}{2}(1 + w) \,, \tag{2.1.19}$$

where the last equality follows for an equation of state of the form $\bar{p} = w\bar{\rho}$. Accordingly, ϵ is another way of expressing the equation of state.

Lastly, let us notice that from the definition $w \equiv \bar{p}/\bar{\rho}$, the time derivative of the equation of state parameter can be expanded as

$$\dot{w} = 3H(1 + w)(w - c_{\mathrm{s}}^2) \,, \qquad w' = 3\mathcal{H}(1 + w)(w - c_{\mathrm{s}}^2) \,, \tag{2.1.20}$$

where the conservation equation (2.1.11) has been used and

$$c_{\mathrm{s}}^2 \equiv \frac{\dot{\bar{p}}}{\dot{\bar{\rho}}} = \frac{\bar{p}'}{\bar{\rho}'} \tag{2.1.21}$$

defines the adiabatic speed of sound squared. We thus notice that a fluid with constant equation of state, $\dot{w} = 0$, satisfies $w = c_{\mathrm{s}}^2$.

2.1.2 Anisotropic cosmology

Let us briefly comment on the inclusion of small anisotropies about an FLRW background. One can introduce deviations from isotropy by writing the metric as

$$g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\mathrm{d}t^2 + a(t)^2\sum_{i=1}^{3} e^{2\theta_i(t)}\mathrm{d}x^i\mathrm{d}x^i\,, \tag{2.1.22}$$

with the constraint $\sum_i \theta_i = 0$. This metric is in the Bianchi type-I form and bares the following physical interpretation: $a(t)$ becomes an 'average' scale factor, the θ_i's being corrections (i.e., the anisotropies) about the average expansion/contraction. As such, H is also an average Hubble parameter, while $H_i = H + \dot{\theta}_i$ is the value is the expansion rate in a given spatial direction x^i.

Substituting the above metric into the Einstein field equations yields

$$3M_{\mathrm{Pl}}^2H^2 = \bar{\rho} + \rho_\theta\,, \qquad 2M_{\mathrm{Pl}}^2\dot{H} = -(\bar{\rho}+\bar{p}) - (\rho_\theta + p_\theta)\,, \tag{2.1.23}$$

$$\ddot{\theta}_i + 3H\dot{\theta}_i = 0\,, \tag{2.1.24}$$

where[2]

$$\rho_\theta = \frac{M_{\mathrm{Pl}}^2}{2}\sum_{i=1}^{3}\dot{\theta}_i^2\,. \tag{2.1.26}$$

Solving $\ddot{\theta}_i + 3H\dot{\theta}_i = 0$ immediately yields $\dot{\theta}_i \propto a^{-3}$, hence $\rho_\theta \sim \dot{\theta}_i^2 \propto a^{-6}$. In comparison with Eq. (2.1.13), we notice that anisotropies thus behave as a perfect fluid with stiff equation of state of the form $p_\theta = \rho_\theta$ $(w = 1)$.

2.2 Arnowitt-Deser-Misner formalism

Before considering cosmological perturbations, let us introduce the Arnowitt-Deser-Misner (ADM) formalism [35] (also known as the $3+1$ formulation of general relativity; see, e.g.,

[2] We note that anisotropies are related to the concept of shear, with $\sigma^2 \sim \sum_i \dot{\theta}_i^2$, where σ is related to the shear tensor $\sigma_{\mu\nu}$ through the definition $\sigma^2 \equiv \sigma_{\mu\nu}\sigma^{\mu\nu}/2$. This is why the shear equation of motion is sometimes written as

$$\dot{\sigma}_{\mu\nu} + 3H\sigma_{\mu\nu} = 0\,. \tag{2.1.25}$$

Also, the energy density in anisotropies is then simply $\rho_\theta \sim \sigma^2$. For more details about shear, see, e.g., Ref. [270].

Refs. [63, 329, 537, 552, 616] for introductions and reviews) as an aside since it is a very useful tool in cosmology.

In the ADM formalism, the metric tensor is written as

$$\begin{aligned} g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu &= -N^2\mathrm{d}t^2 + \gamma_{ij}(\mathrm{d}x^i + N^i\mathrm{d}t)(\mathrm{d}x^j + N^j\mathrm{d}t) \\ &= -(N^2 + N_k N^k)\mathrm{d}t^2 + 2N_l \mathrm{d}x^l \mathrm{d}t + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j\,, \end{aligned} \tag{2.2.27}$$

where $N(t, \mathbf{x})$ and $N^i(t, \mathbf{x})$ are the lapse and shift functions, respectively. Also, $\gamma_{ij} = g_{ij}$ is the spatial metric (the metric on the spatial hypersurface), which is quite explicit from the above expression. Note, however, that $g^{ij} \neq \gamma^{ij}$. In fact, one can verify that $g^{00} = -N^{-2}$, $g^{i0} = g^{0i} = N^{-2}N^i$, and $g^{ij} = \gamma^{ij} - N^{-2}N^iN^j$. Defining $\gamma \equiv \det(\gamma^i{}_j)$ to be the determinant of the spatial metric, one can check that the following property holds

$$\sqrt{-g} = N\sqrt{\gamma}\,. \tag{2.2.28}$$

To be more precise, the idea of the $3+1$ formalism is to start with a four-dimensional spacetime $(\mathcal{M}, \boldsymbol{g})$ and to let $\Sigma \subset \mathcal{M}$ be a spacelike, three-dimensional hypersurface. The hypersurface Σ is the "3" and time evolution is the "1" in the $3+1$ formalism. With $\boldsymbol{n}$ being a future-pointing (timelike) unit normal vector, hence satisfying $g_{\mu\nu}n^\mu n^\nu = -1$, the *induced metric* on Σ is

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu\,. \tag{2.2.29}$$

Here, $\gamma_{\mu\nu}$ is purely spatial, i.e., it resides entirely in Σ with no piece along n^μ:

$$n^\mu\gamma_{\mu\nu} = n^\mu g_{\mu\nu} + n^\mu n_\mu n_\nu = n_\nu - n_\nu = 0\,. \tag{2.2.30}$$

This confirms that $g_{ij} = \gamma_{ij}$. For the ADM metric (2.2.27), we have $n_\mu = (-N, \mathbf{0})$ and $n^\mu = (N^{-1}, -N^{-1}N^i)$.

Let the hypersurface $(\Sigma, \boldsymbol{\gamma})$ have a covariant derivative $\boldsymbol{D}$ such that $D_\mu\gamma_{\nu\alpha} = 0$. The *extrinsic curvature* $\boldsymbol{K}$ of the hypersurface Σ is defined as

$$K_{\mu\nu} \equiv \frac{1}{2}\mathscr{L}_{\boldsymbol{n}}\gamma_{\mu\nu} = \gamma_\mu{}^\alpha\gamma_\nu{}^\beta\nabla_\alpha n_\beta = D_\nu n_\mu\,, \tag{2.2.31}$$

where $\mathscr{L}$ is the Lie derivative. The trace is often called the mean curvature of the hypersurface Σ:

$$K \equiv \gamma^{ij} K_{ij} \,. \tag{2.2.32}$$

The extrinsic curvature is useful in the sense that it allows us to relate curvature in the four-dimensional spacetime $\mathcal{M}$ (e.g., the Ricci tensor and scalars, ${}^{(4)}R_{\mu\nu}$ and ${}^{(4)}R$) to curvature in the three-dimensional hypersurface Σ (e.g., the spatial Ricci scalar, ${}^{(3)}R$) through the Gauss-Codazzi relations (see, e.g., Refs. [63, 329, 537, 552, 616] for expressions). We do not write all of them here, but for instance, the scalar Gauss relation reads

$${}^{(4)}R + 2{}^{(4)}R_{\mu\nu} n^\mu n^\nu = {}^{(3)}R + K^2 - K_{ij} K^{ij} \,, \tag{2.2.33}$$

and by combining different relations, one can show that

$${}^{(4)}R = {}^{(3)}R - K^2 + K_{ij} K^{ij} - 2\nabla_\mu (-K n^\mu + n^\nu \nabla_\nu n^\mu) \,. \tag{2.2.34}$$

That way, the full Einstein equations reduce to a set of constraint and evolution equations, known as the ADM equations. The constraint equations, 'living' entirely on Σ, are

$$\begin{aligned} {}^{(3)}R + K^2 - K_{ij} K^{ij} &= 16\pi G_{\mathrm{N}} \varepsilon \,, \\ D_j (K^{ij} - \gamma^{ij} K) &= 8\pi G_{\mathrm{N}} J^i \,, \end{aligned} \tag{2.2.35}$$

where $\varepsilon \equiv T_{\mu\nu} n^\mu n^\nu$ and $J^i \equiv T_\mu{}^i n^\mu$, and they are known as the Hamiltonian and momentum constraints, respectively. The evolution equations then tell us about $\partial_t K_{ij}$ (see, e.g., Refs. [63, 329, 537] for the expression) and the time evolution of the induced metric:

$$\partial_t \gamma_{ij} = 2N K_{ij} + D_i N_j + D_j N_i \,. \tag{2.2.36}$$

The same way we can 'split' the Einstein equations, the ADM formalism can be used to 'split' the spacetime into $3+1$ dimensions at the level of the action as follows:

$$S = \frac{M_{\mathrm{Pl}}^2}{2} \int \mathrm{d}^4 x \, \sqrt{-g}{}^{(4)}R = \frac{M_{\mathrm{Pl}}^2}{2} \int \mathrm{d}t \int \mathrm{d}^3\mathbf{x} \, \sqrt{\gamma} N \left({}^{(3)}R + K_{ij} K^{ij} - K^2 \right) \,. \tag{2.2.37}$$

Indeed, this immediately follows from Eqs. (2.2.28) and (2.2.34) and noting that the right-most term in Eq. (2.2.34) is a total divergence term, which is a vanishing boundary term in general relativity. The concepts of the $3+1$ decomposition such as the above action will be used, for instance, in Chapters 5 and 7.

2.3 Cosmological perturbation theory

2.3.1 Perturbed metric

We now go back to cosmology and implement perturbations about the cosmological background. The idea is to expand the full metric tensor as a Taylor series,

$$g_{\mu\nu}(t,\mathbf{x}) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(t,\mathbf{x}) + \mathcal{O}(\delta g_{\mu\nu}^2)\,, \tag{2.3.38}$$

introducing linear fluctuations of the form $\delta g_{\mu\nu}$ and ignoring higher-order fluctuations, schematically written as $\delta g_{\mu\nu}^2$. It is understood that $\bar{g}_{\mu\nu}$ is the FLRW background metric introduced in the first section of this chapter, and in order to have a valid perturbative expansion, it should always be true that the linear fluctuations remain small compared to the background, i.e., schematically $|\delta g_{\mu\nu}| \ll |\bar{g}_{\mu\nu}|$. It is then useful to parameterize the linearized metric as follows,

$$\begin{aligned} g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu =& -(1+2A)\mathrm{d}t^2 + 2a(t)(\partial_i B + S_i)\mathrm{d}x^i\mathrm{d}t \\ &+ a(t)^2[(1+2C)\delta_{ij} + 2\partial_i\partial_j E + 2\partial_{(i}F_{j)} + h_{ij}]\mathrm{d}x^i\mathrm{d}x^j\,. \end{aligned} \tag{2.3.39}$$

In the above, $A(t,\mathbf{x})$, $B(t,\mathbf{x})$, $C(t,\mathbf{x})$, and $E(t,\mathbf{x})$ are scalar perturbations, $S_i(t,\mathbf{x})$ and $F_i(t,\mathbf{x})$ (with the conditions that $\partial_i S^i = 0$ and $\partial_i F^i = 0$) are vector perturbations, and $h_{ij}(t,\mathbf{x})$ (with the conditions that $h^i{}_i = 0$ and $\partial_i h^i{}_j = 0$) is the transverse and traceless tensor perturbation. Note that parentheses in the indices indicate symmetrization, i.e., $\partial_{(i}F_{j)} \equiv (\partial_i F_j + \partial_j F_i)/2$. An important result of cosmological perturbation theory is that scalar, vector, and tensor perturbations evolve independently at linear level, so they are treated separately in what follows.

In comparison with the ADM metric (2.2.27), we note that we can think of the lapse, shift, and induced metric as receiving perturbations:

$$N(t,\mathbf{x}) = \bar{N}(t) + \delta N(t,\mathbf{x})\,, \quad N^i(t,\mathbf{x}) = \bar{N}^i(t) + \delta N^i(t,\mathbf{x})\,, \quad \gamma_{ij}(t,\mathbf{x}) = \bar{\gamma}_{ij}(t) + \delta\gamma_{ij}(t,\mathbf{x})\,. \tag{2.3.40}$$

With an FLRW background, we simply have $\bar{N} = 1$, $\bar{N}^i = 0$, and $\bar{\gamma}_{ij} = a(t)^2\delta_{ij}$, and the cosmological linear perturbations are $\delta N = A$, $\delta N_i = a(\partial_i B + S_i)$, and $\delta\gamma_{ij} = a^2(2C\delta_{ij} + 2\partial_i\partial_j E + 2\partial_{(i}F_{j)} + h_{ij})$. A calculation then shows that the Ricci scalar curvature on a spatial hypersurface is given by

$$^{(3)}R = -\frac{4}{a^2}\nabla^2 C \tag{2.3.41}$$

to linear order, where $\nabla^2 \equiv D^i D_i$ is the spatial Laplacian (or Laplace operator). Hence, we see that C is related to *curvature perturbations*.

2.3.2 Gauge degrees of freedom

In general relativity, coordinates such as $(t, \mathbf{x})$ carry no physical meaning. Therefore, when one performs a gauge transformation, i.e., a small transformation of the spacetime coordinates, fictitious fluctuating modes can appear, which need to be 'gauged away'. In general, the following coordinate transformation,

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu\,, \tag{2.3.42}$$

corresponds to four independent gauge degrees of freedom: the time component ξ^0, which is a scalar mode, as well as the space components, which can be decomposed as

$$\xi^i = \xi^i_{\text{transverse}} + \bar{\gamma}^{ij}\partial_j\xi\,, \tag{2.3.43}$$

where the transverse[3] three-vector $\xi^i_{\text{transverse}}$ carries two (vector) degrees of freedom and ξ is a second scalar mode. Note that $\bar{\gamma}_{ij}$ is the spatial background metric. In sum, there are two scalar gauge degrees of freedom as well as two vector gauge degrees of freedom, while tensor perturbations are gauge invariant.

2.3.3 Scalar degrees of freedom

Let us first consider scalar perturbations. The gauge transformations are

$$t \to t + \xi^0 \qquad \text{and} \qquad x^i \to x^i + a^{-2}\delta^{ij}\partial_j\xi\,. \tag{2.3.44}$$

[3] It is also sometimes called the divergenceless component since it must be that $\partial_i\xi^i_{\text{transverse}} = 0$.

Under these transformations, one can check that the scalar perturbations of the perturbed metric (2.3.39) transform according to

$$\begin{aligned} A &\to A - a\dot{\xi}^0 - aH\xi^0\,, \\ B &\to B + \xi^0 - a\dot{\xi}\,, \\ C &\to C - aH\xi^0\,, \\ E &\to E - \xi\,. \end{aligned} \tag{2.3.45}$$

We note that the combination[4] $\dot{E} - B/a$ transforms independently of the spatial gauge ξ; its transformation only depends on the temporal gauge ξ^0, similarly to the transformations for A and C. Therefore, one can construct the following quantities, known as the Bardeen potentials, in terms of the quantities A, C, and $\dot{E} - B/a$ such that they are gauge invariant:

$$\begin{aligned} \Phi &\equiv A - \partial_t[a^2(\dot{E} - B/a)]\,, \\ \Psi &\equiv -C + a^2 H(\dot{E} - B/a)\,. \end{aligned} \tag{2.3.48}$$

The most common approach, however, is to 'fix' the gauge by setting certain conditions on the perturbation variables. Several gauges exist, but we will review only a subset below, focusing on the ones that are used most often in this thesis.

2.3.4 Matter perturbations

At this point, we have only perturbed the metric. On top of that, the matter content should also fluctuate. For example, let us consider a perfect fluid with energy-momentum tensor

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}\,, \tag{2.3.49}$$

[4] In fact,

$$\dot{E} - B/a = \sigma/a^2\,, \tag{2.3.46}$$

where σ is the scalar component of the shear tensor,

$$\sigma_{\mu\nu} = \frac{1}{3}K\gamma_{\mu\nu} - K_{\mu\nu}\,, \tag{2.3.47}$$

when expanded to linear order.

where u^μ is the four-velocity of the fluid particles, satisfying $u_\mu u^\mu = -1$. The energy density and pressure are then perturbed as follows,

$$\rho(t,\mathbf{x}) = \bar{\rho}(t) + \delta\rho(t,\mathbf{x})\,, \qquad p(t,\mathbf{x}) = \bar{p}(t) + \delta p(t,\mathbf{x})\,. \tag{2.3.50}$$

The energy-momentum tensor is then perturbed as follows,

$$\begin{aligned} \delta T^0{}_0 &= -\delta\rho\,, \\ \delta T^0{}_i &= (\bar{\rho}+\bar{p})\delta u_i\,, \\ \delta T^i{}_j &= \delta p\,\delta^i{}_j\,, \end{aligned} \tag{2.3.51}$$

where $\delta u_i(t,\mathbf{x})$ is the velocity perturbation of the fluid. We assume that there is no anisotropic stress, i.e., $\delta T^i{}_j = 0$ for $i \neq j$, which is a valid approximation for the most common types of matter, including the ones studied in this thesis.

We note that $\delta\rho$ and δp are scalar perturbations, hence they transform under the temporal gauge as

$$\delta\rho \to \delta\rho - a\dot{\bar{\rho}}\xi^0\,, \qquad \delta p \to \delta p - a\dot{\bar{p}}\xi^0\,. \tag{2.3.52}$$

Accordingly,

$$\delta\rho^{(\mathrm{gi})} = \delta\rho - a^2\dot{\bar{\rho}}(\dot{E} - B/a) \qquad \text{and} \qquad \delta p^{(\mathrm{gi})} = \delta p - a^2\dot{\bar{p}}(\dot{E} - B/a) \tag{2.3.53}$$

are gauge-invariant constructions for the energy density and pressure perturbations.

Let us define the three-momentum density perturbations,

$$\delta q_i \equiv (\bar{\rho}+\bar{p})\delta u_i\,, \tag{2.3.54}$$

and its scalar part δq such that $\partial_i \delta q = \delta q_i$. Then, δq is also a scalar perturbation, and it transforms under the temporal gauge as

$$\delta q \to \delta q + a(\bar{\rho}+\bar{p})\xi^0\,. \tag{2.3.55}$$

Hence,

$$\delta q^{(\mathrm{gi})} = \delta q + a^2(\bar{\rho}+\bar{p})(\dot{E} - B/a) \tag{2.3.56}$$

$$\delta u_i^{(\mathrm{gi})} = \delta u_i + a^2\partial_i(\dot{E} - B/a) \tag{2.3.57}$$

$$\bar{\rho}\Delta \equiv \delta\rho - 3H\delta q = \delta\rho + \frac{\dot{\bar{\rho}}}{\bar{\rho}+\bar{p}}\delta q \tag{2.3.58}$$

are gauge invariant. We note that we simply used the conservation equation $\dot{\bar{\rho}}+3H(\bar{\rho}+\bar{p}) = 0$ in the last equality. The quantity Δ is called the *comoving gauge density contrast*, since as we will see below, in the comoving gauge $\delta q = 0$. In such a case, we find $\Delta = \delta\rho/\bar{\rho} \equiv \delta$, which is the definition of the *density contrast*.

Comparing with the transformation of the scalar metric perturbations, we can construct new gauge-invariant quantities, mixing metric and matter perturbations. For example,

$$\tilde{\mathcal{R}} \equiv -\mathcal{R} \equiv -C + \frac{H}{\dot{\bar{\rho}}}\delta\rho \tag{2.3.59}$$

defines the *curvature perturbation on uniform density hypersurfaces*, while

$$\zeta \equiv -C - \frac{H}{\bar{\rho}+\bar{p}}\delta q \tag{2.3.60}$$

defines the *comoving curvature perturbation*. Very often, $\tilde{\mathcal{R}}$ and ζ are equivalent to one another on large scales. Indeed, we notice that

$$\zeta - \tilde{\mathcal{R}} = -\frac{H}{\bar{\rho}+\bar{p}}\delta q - \frac{H}{\dot{\bar{\rho}}}\delta\rho = -\frac{H\bar{\rho}}{\dot{\bar{\rho}}}\Delta = \frac{1}{3(1+w)}\Delta\,, \tag{2.3.61}$$

and often the comoving gauge density contrast is vanishing on large scales. We will see that this is the case in Einstein gravity when we perturb the Einstein equations in the next section (in the Newtonian gauge as an example). The concept of curvature perturbations will be used repeatedly over this thesis.

2.3.5 Scalar fields

Scalar fields are a common form of matter to study, and they will occur frequently over the course of this thesis. A canonical scalar field ϕ has a Lagrangian density of the form

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\,, \tag{2.3.62}$$

where $V(\phi)$ is the potential. The Euler-Lagrange equation of motion for the scalar field (known as the Klein-Gordon equation) reads

$$\Box\phi - \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0\,, \tag{2.3.63}$$

where $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the d'Alembertian or d'Alembert operator. The energy-momentum tensor for the scalar field is

$$T_{\mu\nu} = \mathcal{L} g_{\mu\nu} + \partial_\mu\phi\partial_\nu\phi\,. \tag{2.3.64}$$

This is equivalent to a perfect fluid with

$$\rho = -\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\,, \qquad p = \mathcal{L}\,, \tag{2.3.65}$$

and with fluid velocity

$$u_\mu = \frac{\partial_\mu\phi}{\sqrt{-\partial_\nu\phi\partial^\nu\phi}}\,. \tag{2.3.66}$$

In particular, in an FLRW background one finds

$$\bar\rho = \frac{1}{2}\dot{\bar\phi}^2 + V(\bar\phi)\,, \qquad \bar p = \frac{1}{2}\dot{\bar\phi}^2 - V(\bar\phi)\,, \tag{2.3.67}$$

and the equation of state can be written as

$$w = \frac{\dot{\bar\phi}^2 - 2V(\bar\phi)}{\dot{\bar\phi}^2 + 2V(\bar\phi)}\,, \qquad \epsilon = \frac{\dot{\bar\phi}^2}{2M_{\mathrm{Pl}}^2 H^2}\,. \tag{2.3.68}$$

Also, the Friedmann equation can be expressed as

$$(3-\epsilon)M_{\mathrm{Pl}}^2 H^2 = V(\bar\phi)\,, \tag{2.3.69}$$

and the scalar field equation of motion becomes

$$\ddot{\bar{\phi}} + 3H\dot{\bar{\phi}} + \left.\frac{\mathrm{d}V}{\mathrm{d}\phi}\right|_{\phi=\bar{\phi}} = 0\,. \tag{2.3.70}$$

As for a generic perfect fluid, we introduce linear fluctuations such that

$$\phi(t, \mathbf{x}) = \phi(t) + \delta\phi(t, \mathbf{x})\,, \tag{2.3.71}$$

so $\delta\phi$ is a scalar perturbation. Accordingly, it transforms under the temporal gauge as

$$\delta\phi \to \delta\phi - a\dot{\bar{\phi}}\xi^0\,, \tag{2.3.72}$$

and $\delta\phi^{(\mathrm{gi})} = \delta\phi - a^2\dot{\bar{\phi}}(\dot{E} - B/a)$ is a gauge-invariant perturbation variable for the scalar field fluctuation. The comoving curvature perturbation then becomes

$$\zeta = -C - \frac{H}{\dot{\bar{\phi}}}\delta\phi\,. \tag{2.3.73}$$

2.3.6 Popular scalar gauges

Let us briefly review of a few of the gauges that will be used throughout this thesis. First, a popular gauge is the *conformal Newtonian gauge* (or *longitudinal gauge*). This gauge is defined by setting $B = E = 0$, so in particular, $\dot{E} - B/a = 0$. Therefore, from Eq. (2.3.48) we can write

$$\Phi = A\,, \qquad \Psi = -C\,, \tag{2.3.74}$$

hence the perturbed metric is simply

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -(1 + 2\Phi)\mathrm{d}t^2 + a(t)^2(1 - 2\Psi)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\,. \tag{2.3.75}$$

Also, from Eqs. (2.3.53) and (2.3.56) it follows that $\delta\rho$, δp, and δu_i (and equivalently δq) are all gauge-invariant matter perturbations. In this gauge, the comoving curvature perturbation is generally

$$\zeta = \Psi - \frac{H}{\bar{\rho} + \bar{p}}\delta q\,, \tag{2.3.76}$$

and it is

$$\zeta = \Psi - \frac{H}{\dot{\phi}}\delta\phi \tag{2.3.77}$$

for a scalar field.

A second gauge that is quite common is the *comoving gauge* (or *unitary gauge*), where one sets $\delta q = 0$ (or $\delta\phi = 0$ for a scalar field) and $E = 0$. In this case, we notice that $\zeta = -C$. Consequently, we can write the perturbed metric as

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -(1+2A)\mathrm{d}t^2 + 2a(t)\partial_i B \mathrm{d}x^i \mathrm{d}t + a(t)^2(1-2\zeta)\delta_{ij}\mathrm{d}x^i \mathrm{d}x^j \,. \tag{2.3.78}$$

A third gauge of interest is the *spatially-flat gauge*, where one sets $C = E = 0$. As a result, the perturbed metric is

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -(1+2A)\mathrm{d}t^2 + 2a(t)\partial_i B \mathrm{d}x^i \mathrm{d}t + a(t)^2\delta_{ij}\mathrm{d}x^i \mathrm{d}x^j \,, \tag{2.3.79}$$

which explicitly has a flat metric on spatial hypersurfaces. Matter perturbations remain and carry the information about curvature perturbations. For example, for a scalar field one is left with

$$\zeta = -\frac{H}{\dot{\phi}}\delta\phi \,. \tag{2.3.80}$$

2.3.7 Vector degrees of freedom

For vector modes, we recall that the gauge transformation is the following: $x^i \to x^i + \xi^i_{\text{transverse}}$. It follows that the vector perturbations transform as

$$S_i \to S_i + a\dot{\xi}_i^{\text{transverse}} \,, \qquad F_i \to F_i - \xi_i^{\text{transverse}} \,, \tag{2.3.81}$$

and so the combination $\dot{F}_i + S_i/a$ is gauge invariant. Alternatively, one can fix the gauge by setting, for example, $S_i = 0$. This gauge will be used in Chapter 7.

2.4 Perturbed Einstein equations in the Newtonian gauge

We have seen that there exist many gauges in which one can study cosmological perturbations. In this subsection, we pick the Newtonian gauge together with a generic perfect fluid as a example. We now want to find out the dynamical equations for the perturbations, and we do so by perturbing the Einstein equations:

$$\delta G^{\mu}{}_{\nu} = 8\pi G_{\rm N} \delta T^{\mu}{}_{\nu} \,. \tag{2.4.82}$$

Substituting the perturbed metric for the Newtonian gauge [Eq. (2.3.75)] into the Einstein tensor and expanding up to linear order, one finds the following perturbation equations of motion:

$$\begin{aligned} 3H(\dot{\Psi} + H\Phi) - a^{-2}\nabla^2\Psi &= -4\pi G_{\rm N}\delta\rho \,, \\ \partial_i(\dot{\Psi} + H\Phi) &= -4\pi G_{\rm N}(\bar{\rho} + \bar{p})\delta u_i \,, \\ \left[\ddot{\Psi} + 3H\dot{\Psi} + H\dot{\Phi} + (3H^2 + 2\dot{H})\Phi\right]\delta^i{}_j + \frac{1}{2a^2}\nabla^2(\Phi - \Psi)\delta^i{}_j & \\ - \frac{1}{2a^2}\delta^{ik}\partial_k\partial_j(\Phi - \Psi) &= 4\pi G_{\rm N}\delta p\, \delta^i{}_j \,, \end{aligned} \tag{2.4.83}$$

where we used Eq. (2.3.51) for $\delta T^{\mu}{}_{\nu}$. An immediate consequence of the third equation above is that $\Phi = \Psi$. Indeed, considering the case $i \neq j$, the right-hand side of the third equation vanishes since there is no anisotropic stress. The left-hand side must therefore vanish as well, which is generically only true if $\Phi = \Psi$. Consequently, the set of equations of motion becomes

$$3H(\dot{\Phi} + H\Phi) - a^{-2}\nabla^2\Phi = -4\pi G_{\rm N}\delta\rho \,, \tag{2.4.84}$$

$$\partial_i(\dot{\Phi} + H\Phi) = -4\pi G_{\rm N}(\bar{\rho} + \bar{p})\delta u_i \,, \tag{2.4.85}$$

$$\ddot{\Phi} + 4H\dot{\Phi} + (3H^2 + 2\dot{H})\Phi = 4\pi G_{\rm N}\delta p \,. \tag{2.4.86}$$

At this point, let us define the matter's speed of sound $c_{\rm s}$ according to

$$c_{\rm s}^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_S \,, \tag{2.4.87}$$

where the subscript S means at constant entropy. In particular, if the equation of state of matter is purely barotropic and independent of entropy, the pressure perturbations will be entirely adiabatic and we have

$$\delta p = c_{\rm s}^2 \delta\rho \,. \tag{2.4.88}$$

Substituting this expression into the right-hand side of Eq. (2.4.86) and using Eq. (2.4.84) to eliminate $\delta\rho$, after simplification we are left with

$$\ddot{\Phi} + H(4 + 3c_{\rm s}^2)\dot{\Phi} + \left[3H^2(1 + c_{\rm s}^2) + 2\dot{H}\right]\Phi - a^{-2}c_{\rm s}^2\nabla^2\Phi = 0\,. \tag{2.4.89}$$

Transforming physical time derivatives into conformal time derivatives, this equation can be rewritten as

$$\Phi'' + 3\mathcal{H}(1 + c_{\rm s}^2)\Phi' + \left[2\mathcal{H}' + \mathcal{H}^2(1 + 3c_{\rm s}^2)\right]\Phi - c_{\rm s}^2\nabla^2\Phi = 0\,. \tag{2.4.90}$$

This equation will be studied for example in Chapter 8.

Another useful equation can will be used in Chapter 8 can be derived: starting from Eq. (2.4.84), converting to conformal time, and solving for $\delta \equiv \delta\rho/\bar{\rho}$ yields

$$\delta = \frac{2}{3\mathcal{H}^2}\left[\nabla^2\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi)\right]\,, \tag{2.4.91}$$

where the Friedmann equation $3\mathcal{H}^2 = 8\pi G_{\rm N}a^2\bar{\rho}$ was used. Alternatively, solving Eq. (2.4.84) for the Laplacian of the Newtonian potential ($\nabla^2\Phi$) and using the scalar part of Eq. (2.4.85),

$$\dot{\Phi} + H\Phi = -4\pi G_{\rm N}\delta q\,, \tag{2.4.92}$$

we obtain

$$\nabla^2\Phi = 4\pi G_{\rm N}a^2(\delta\rho - 3H\delta q) = 4\pi G_{\rm N}a^2\bar{\rho}\Delta = \frac{3}{2}\mathcal{H}^2\Delta\,, \tag{2.4.93}$$

where the comoving gauge density contrast Δ was defined in Eq. (2.3.58). The above equation is known as the *Poisson equation*. It is then straightforward to see that on large scales[5] (i.e., as $\nabla^2\Phi/\mathcal{H}^2 \to 0$), the comoving gauge density contrast goes to zero. This implies that the

[5]This is easier to see upon transforming the perturbation Φ to Fourier space,

$$\Phi(\tau, \mathbf{x}) = \int \frac{{\rm d}^3\mathbf{k}}{(2\pi)^3}\Phi_{\mathbf{k}}(\tau)e^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{2.4.94}$$

in which case the Poisson equation becomes

$$\frac{k^2}{\mathcal{H}^2}\Phi_{\mathbf{k}} = -\frac{3}{2}\Delta\,, \tag{2.4.95}$$

where $k \equiv \sqrt{\mathbf{k}\cdot\mathbf{k}}$. Therefore, large scales correspond to $k \ll \mathcal{H}$, when the comoving wavelength of the fluctuations are large compared to the comoving Hubble radius:

$$\lambda_{\rm comoving} \propto k^{-1} \gg (aH)^{-1}\,. \tag{2.4.96}$$

right-hand side of Eq. (2.3.61) vanishes in that limit, and therefore,

$$\zeta \approx \tilde{\mathcal{R}} \tag{2.4.97}$$

on scales where $a^{-2}\nabla^2\Phi \ll H^2$. This is why it is said that the two definitions of curvature perturbation are often equivalent on large scales.

Finally, let us write down a few properties about how the comoving curvature perturbation relates to the Newtonian potential Φ in the Newtonian gauge. Recalling Eq. (2.3.76), using Eq. (2.4.92) to eliminate δq and using the background Friedmann equation, we find

$$\zeta = \Phi + \frac{2(\dot{\Phi} + H\Phi)}{3H(1+w)}\,. \tag{2.4.98}$$

Also, taking a time derivative of the above and using Eq. (2.4.89) to eliminate $\ddot{\Phi}$, we obtain

$$\dot{\zeta} = \frac{2}{3H(1+w)} a^{-2} c_\mathrm{s}^2 \nabla^2\Phi\,, \tag{2.4.99}$$

where the background Friedmann equations and Eq. (2.1.20) for $\dot{w}$ are also used for simplification. These equations will be useful in Chapter 8. An immediate implication of the above equation is that on large scales (as $\nabla^2\Phi/(aH)^2 \to 0$), we find $\dot{\zeta} \to 0$, meaning that the comoving curvature perturbation is *conserved* on large scales.

2.5 Perturbed action in the comoving gauge and equations of motion

As another example of application of cosmological perturbation theory, let us consider the comoving gauge for a canonical scalar field in general relativity. In this case, $\delta\phi = 0$ and the perturbed metric is given by Eq. (2.3.78). This time, rather than performing the expansion at the level of the equations of motion (i.e., perturbing the Einstein equations), we are going to do the expansion at the level of the action,

$$S = \int \mathrm{d}^4x\, \sqrt{-g} \left(\frac{M_\mathrm{Pl}^2}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu\phi \partial_\nu\phi - V(\phi) \right)\,. \tag{2.5.100}$$

Analyzing the perturbations in Fourier space will be done in more detail in the next sections.

Furthermore, let us set up the problem in the ADM formalism, so the action reduces to [recall Eq. (2.2.37)]

$$\begin{aligned} S = \int \mathrm{d}t \int \mathrm{d}^3\mathbf{x}\, \sqrt{\gamma} N \bigg[&\frac{M_{\mathrm{Pl}}^2}{2} \left({}^{(3)}R + K_{ij}K^{ij} - K^2 \right) + \frac{\dot{\phi}^2}{2N^2} - \frac{N^i \dot{\phi} \partial_i \phi}{N^2} \\ &- \frac{1}{2} \left(\gamma^{ij} - \frac{N^i N^j}{N^2} \right) \partial_i \phi \partial_j \phi - V(\phi) \bigg] . \end{aligned} \tag{2.5.101}$$

We then introduce linear perturbations in the unitary gauge, so already $\phi(t, \mathbf{x}) = \bar{\phi}(t)$ implies that terms with $\partial_i \phi$ go away:

$$S = \frac{1}{2} \int \mathrm{d}t \int \mathrm{d}^3\mathbf{x}\, \sqrt{\gamma} \Big[M_{\mathrm{Pl}}^2 N {}^{(3)}R + M_{\mathrm{Pl}}^2 N^{-1} \left(E_{ij}E^{ij} - E^2 \right) + N^{-1} \dot{\bar{\phi}}^2 - 2NV(\bar{\phi}) \Big] . \tag{2.5.102}$$

In the above, we defined $E_{ij} \equiv N K_{ij}$, so using Eq. (2.2.36) to solve for the extrinsic curvature, one has

$$E_{ij} = \frac{1}{2} \left(\dot{\gamma}_{ij} - 2D_{(i} N_{j)} \right) , \tag{2.5.103}$$

and $E \equiv \gamma^{ij} E_{ij} = NK$ (not to be confused with the metric perturbation E, which is zero in the comoving gauge anyway).

The lapse and shift act as Lagrange multipliers in the action above, so we can immediately deduce a set of constraint equations by varying Eq. (2.5.102) with respect to N and N^i:

$$\frac{\delta S}{\delta N} = 0 \implies M_{\mathrm{Pl}}^2 {}^{(3)}R - M_{\mathrm{Pl}}^2 N^{-2} (E_{ij}E^{ij} - E^2) - N^{-2} \dot{\bar{\phi}}^2 - 2V(\bar{\phi}) = 0 , \tag{2.5.104}$$

$$\frac{\delta S}{\delta N^i} = 0 \implies D_j \left[N^{-1} \left(E^j{}_i - \delta^j{}_i E \right) \right] = 0 . \tag{2.5.105}$$

We now perturb the lapse, shift, and induced metric in the comoving gauge as

$$N = 1 + A , \qquad N_i = a \partial_i B , \qquad \gamma_{ij} = a^2 (1 - 2\zeta) \delta_{ij} . \tag{2.5.106}$$

Also, the three-dimensional curvature expanded to linear order in the comoving gauge is given by [according to Eq. (2.3.41)]

$${}^{(3)}R = \frac{4}{a^2} \nabla^2 \zeta . \tag{2.5.107}$$

Substituting the perturbed functions into the definition of the rescaled extrinsic curvature E_{ij} and expanding to linear order yields

$$E_{ij} = a^2\delta_{ij}(H - 2H\zeta - \dot{\zeta}) - aD_{(i}\partial_{j)}B\,, \qquad E = 3H - 3\dot{\zeta} - a^{-1}\nabla^2 B\,, \tag{2.5.108}$$

hence we find

$$E_{ij}E^{ij} - E^2 = -6H^2 + 12H\dot{\zeta} + 4a^{-1}H\nabla^2 B\,, \tag{2.5.109}$$

$$E^j{}_i - \delta^j{}_i E = 2\delta^j{}_i(-H + \dot{\zeta}) + a^{-1}(\delta^j{}_i\nabla^2 B - D^{(j}\partial_{i)}B)\,. \tag{2.5.110}$$

Substituting everything into the constraint equations (2.5.104)–(2.5.105) and expanding to linear order again gives

$$\begin{aligned}&4M_{\rm Pl}^2 a^{-2}\nabla^2\zeta - M_{\rm Pl}^2(-6H^2 + 12H\dot{\zeta} + 4a^{-1}H\nabla^2 B + 12H^2 A) - \dot{\bar{\phi}}^2 + 2\dot{\bar{\phi}}^2 A - 2V(\bar{\phi}) = 0\,,\\ &2D_i\dot{\zeta} + 2HD_iA = 0\,.\end{aligned} \tag{2.5.111}$$

The constraint equations should be satisfied order by order. This is confirmed at the background level in the above equations since $3M_{\rm Pl}^2H^2 = \bar{\rho} = \dot{\bar{\phi}}^2/2 + V(\bar{\phi})$. There remains to solve the constraint equations at the linear perturbation level. This is done with

$$A = -\frac{\dot{\zeta}}{H} \qquad \text{and} \qquad aB = \frac{\zeta}{H} + a^2\epsilon\nabla^{-2}\dot{\zeta}\,, \tag{2.5.112}$$

where we recall ϵ is given in Eq. (2.3.68) for a scalar field and we introduced the inverse Laplacian ∇^{-2} such that $\nabla^{-2}(\nabla^2 f) = f$ for any scalar f.

Substituting these solutions for A and B (the linearized lapse and scalar component of the shift) into the action (2.5.102), integrating by parts and using the background Friedmann equations to simplify yields the following second-order perturbed scalar action:

$$S^{(2)}_{\rm scalar} = M_{\rm Pl}^2\int {\rm d}t{\rm d}^3\mathbf{x}\, a^3\epsilon\left(\dot{\zeta}^2 - a^{-2}(\vec{\nabla}\zeta)^2\right)\,, \tag{2.5.113}$$

where we use the notation $(\vec{\nabla}\zeta)^2 \equiv \partial_i\zeta\partial^i\zeta$. Equivalently in conformal time,

$$S^{(2)}_{\rm scalar} = M_{\rm Pl}^2\int {\rm d}\tau{\rm d}^3\mathbf{x}\, a^2\epsilon\left((\zeta')^2 - (\vec{\nabla}\zeta)^2\right)\,. \tag{2.5.114}$$

Let us introduce the Sasaki-Mukhanov [514, 577] variable defined by

$$v \equiv z\zeta\,, \qquad z^2 \equiv 2M_{\rm Pl}^2 a^2 \epsilon\,. \tag{2.5.115}$$

The advantage of the Sasaki-Mukhanov variable is that it allows us to rewrite the perturbed action in a canonical form,

$$S^{(2)}_{\rm scalar} = \frac{1}{2}\int \mathrm{d}\tau \mathrm{d}^3\mathbf{x}\left((v')^2 - (\vec{\nabla}v)^2 + \frac{z''}{z}v^2\right)\,. \tag{2.5.116}$$

Upon variation of the action, the perturbed equation of motion is

$$v'' - \frac{z''}{z}v - \nabla^2 v = 0\,, \tag{2.5.117}$$

or back in terms of ζ, we have

$$\zeta'' + 2\frac{z'}{z}\zeta' - \nabla^2\zeta = 0\,, \qquad \ddot{\zeta} + (3+\eta)H\dot{\zeta} - a^{-2}\nabla^2\zeta = 0\,, \tag{2.5.118}$$

where we defined

$$\eta \equiv \frac{\dot{\epsilon}}{H\epsilon}\,. \tag{2.5.119}$$

The above equation of motion for the comoving curvature perturbation ζ will be used repeatedly over the course of this thesis.

2.6 Tensor perturbations

We have so far focused on scalar perturbations, in particular in the Newtonian and comoving gauges. Let us now turn our attention to tensor perturbations, which we recall are gauge invariant. Thus, we can start with the Hilbert-Einstein action and substitute the perturbed metric

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + a^2(t)(\delta_{ij} + h_{ij})\mathrm{d}x^i \mathrm{d}x^j\,. \tag{2.6.120}$$

Expanding up to second order in h_{ij} yields

$$S^{(2)}_{\rm tensor} = \frac{M_{\rm Pl}^2}{8}\int \mathrm{d}t\mathrm{d}^3\mathbf{x}\, a^3\left(\dot{h}_{ij}^2 - a^{-2}(\vec{\nabla}h_{ij})^2\right) = \frac{M_{\rm Pl}^2}{8}\int \mathrm{d}\tau\mathrm{d}^3\mathbf{x}\, a^2\left(h_{ij}'^2 - (\vec{\nabla}h_{ij})^2\right)\,, \tag{2.6.121}$$

where it is understood that, e.g., $\dot{h}_{ij}^2 = \dot{h}_{ij}\dot{h}^{ij}$ and $(\vec{\nabla} h_{ij})^2 = \partial_l h_{ij}\partial^l h^{ij}$. We recall that the tensor perturbations h_{ij} are transverse and traceless, i.e., $\partial_i h^i{}_j = 0$ and $h^i{}_i = 0$, so it is useful to expand them into Fourier space as follows,

$$h_{ij}(\tau, \mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \sum_{\lambda=+,\times} \epsilon_{ij}^{(\lambda)} h_{\mathbf{k},\lambda}(\tau) e^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{2.6.122}$$

where the tensor quantities $\epsilon_{ij}^{(\lambda)}$ satisfy

$$k^i \epsilon_{ij}^{(\lambda)} = 0\,, \qquad g^{ij}\epsilon_{ij}^{(\lambda)} = 0\,, \qquad g^{im}g^{jn}\epsilon_{ij}^{(\lambda)}\epsilon_{mn}^{(\lambda')} = 2\delta^{\lambda\lambda'}\,, \tag{2.6.123}$$

to ensure the transverse and traceless properties of h_{ij}. Accordingly, $h_{\mathbf{k},+}(\tau)$ and $h_{\mathbf{k},\times}(\tau)$ become the two polarization states of the tensor modes. That way, the perturbed action for tensor modes becomes

$$S^{(2)}_{\mathrm{tensor}} = \frac{M_{\mathrm{Pl}}^2}{4} \sum_{\lambda=+,\times} \int \mathrm{d}\tau \mathrm{d}^3\mathbf{k}\, a^2 \left[(h'_{\mathbf{k},\lambda})^2 - k^2 (h_{\mathbf{k},\lambda})^2 \right]\,, \tag{2.6.124}$$

and we can define the Sasaki-Mukhanov variable

$$u_{\mathbf{k},\lambda} \equiv \frac{M_{\mathrm{Pl}}}{2} a h_{\mathbf{k},\lambda}\,, \tag{2.6.125}$$

which renders the perturbed action canonical:

$$S^{(2)}_{\mathrm{tensor}} = \frac{1}{2} \sum_{\lambda=+,\times} \int \mathrm{d}\tau \mathrm{d}^3\mathbf{k} \left[(u'_{\mathbf{k},\lambda})^2 - \left(k^2 - \frac{a''}{a} \right) (u_{\mathbf{k},\lambda})^2 \right]. \tag{2.6.126}$$

Upon variation, the resulting equation of motion is

$$u''_{\mathbf{k},\lambda} + \left(k^2 - \frac{a''}{a} \right) u_{\mathbf{k},\lambda} = 0 \tag{2.6.127}$$

for both polarization states $\lambda = +$ and $\lambda = \times$. Equivalently, in terms of $h_{\mathbf{k},\lambda}$ the equation of motion is

$$h''_{\mathbf{k},\lambda} + 2\mathcal{H}h'_{\mathbf{k},\lambda} + k^2 h_{\mathbf{k},\lambda} = 0\,, \qquad \ddot{h}_{\mathbf{k},\lambda} + 3H\dot{h}_{\mathbf{k},\lambda} + \frac{k^2}{a^2} h_{\mathbf{k},\lambda} = 0\,. \tag{2.6.128}$$

Let us drop the index λ from now on and keep in mind that the equations apply for both polarizations $\lambda = +, \times$. The equation of motion for the Sasaki-Mukhanov variable can be written as

$$u''_{\mathbf{k}} + \omega_k^2(\tau) u_{\mathbf{k}} = 0\,, \tag{2.6.129}$$

where

$$\omega_k^2(\tau) \equiv k^2 - \frac{a''}{a} \tag{2.6.130}$$

defines an effective frequency. We note that the equation depends only on $k \equiv |\mathbf{k}|$, hence a generic solution to the equation of motion is of the form

$$u_{\mathbf{k}}(\tau) = a^-_{\mathbf{k}} u_k(\tau) + a^+_{-\mathbf{k}} u_k^*(\tau)\,, \tag{2.6.131}$$

with $u_k(\tau)$, satisfying

$$u''_k + \omega_k^2(\tau) u_k = 0\,, \tag{2.6.132}$$

and its complex conjugate $u_k^*(\tau)$ being two linearly-independent solutions, which depend only on the magnitude of $\mathbf{k}$. The solutions are usually normalized such that

$$W[u_k, u_k^*] \equiv u'_k u_k^* - u_k u_k^{*\prime} = -i\,, \tag{2.6.133}$$

where $W[u_k, u_k^*]$ is the Wronskian.

2.6.1 Generic evolution of the perturbations

Let us consider the case where

$$a \propto |t|^p \propto |\tau|^{p/(1-p)} \tag{2.6.134}$$

for some real power p. Upon evaluating a''/a, the effective frequency is found to be

$$\omega_k^2(\tau) = k^2 - \frac{p(2p-1)}{(1-p)^2\tau^2}\,, \tag{2.6.135}$$

and the equation of motion to solve is

$$u''_k + \left(k^2 - \frac{\nu^2 - 1/4}{\tau^2}\right) u_k = 0\,, \tag{2.6.136}$$

where we defined

$$\nu^2 \equiv \frac{(1-3p)^2}{4(1-p)^2} \,. \tag{2.6.137}$$

The ordinary differential equation is recognized to be a special form of the Bessel equation, hence its solution is given by

$$u_k(\tau) = \sqrt{|\tau|}\left[C_{1,k} H^{(1)}_{|\nu|}(k|\tau|) + C_{2,k} H^{(2)}_{|\nu|}(k|\tau|)\right] \,, \tag{2.6.138}$$

where $H^{(1)}_{|\nu|}$ and $H^{(2)}_{|\nu|}$ are the Hankel functions[6] of the first and second kind, respectively, with index $|\nu|$. Such solutions will appear throughout the thesis. We will see in the next subsection how to set the initial (or boundary) conditions to determine the integration constants $C_{1,k}$ and $C_{2,k}$.

2.7 Computing the two-point correlation function

Let us describe the methodology to compute the two-point correlation function. We will do it for tensor perturbations, but the same method applies for scalar modes.

We recall that the solution to $u_{\mathbf{k}}(\tau)$ is of the form of Eq. (2.6.131). We can relate this to a real space perturbation $u(\tau, \mathbf{x})$ by the following Fourier transform:

$$u(\tau,\mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\left[a^-_{\mathbf{k}} u_k(\tau) + a^+_{-\mathbf{k}} u^*_k(\tau)\right] e^{i\mathbf{k}\cdot\mathbf{x}} = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\left[a^-_{\mathbf{k}} u_k(\tau) e^{i\mathbf{k}\cdot\mathbf{x}} + a^+_{\mathbf{k}} u^*_k(\tau) e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \,. \tag{2.7.139}$$

We then promote the canonical perturbation variable u to a quantum operator $\hat{u}$ such that the following commutation relations are satisfied,

$$[\hat{u}(\tau,\mathbf{x}), \hat{\Pi}(\tau,\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y}) \,, \qquad [\hat{u}(\tau,\mathbf{x}), \hat{u}(\tau,\mathbf{y})] = [\hat{\Pi}(\tau,\mathbf{x}), \hat{\Pi}(\tau,\mathbf{y})] = 0 \,, \tag{2.7.140}$$

where $\Pi(\tau,\mathbf{x}) \equiv u'(\tau,\mathbf{x})$ is the conjugate momentum to u. Hence, we have

$$\hat{u}(\tau,\mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\left[\hat{a}^-_{\mathbf{k}} u_k(\tau) e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}^+_{\mathbf{k}} u^*_k(\tau) e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \,, \tag{2.7.141}$$

[6] The same generic solution could be written in terms of the usual Bessel functions of the first and second kind, $J_{|\nu|}$ and $Y_{|\nu|}$. Indeed, this is completely equivalent since $H^{(1)}_{|\nu|} = J_{|\nu|} + iY_{|\nu|}$ and $H^{(2)}_{|\nu|} = J_{|\nu|} - iY_{|\nu|}$, so only the integration constants would be different. This does not matter as long as the integration constants remain arbitrary, i.e., as long as the boundary conditions have not been set.

and the commutation relations for $\hat{u}$ and $\hat{\Pi}$ imply that

$$[\hat{a}^-_{\mathbf{k}_1}, \hat{a}^+_{\mathbf{k}_2}] = (2\pi)^3 \delta^3(\mathbf{k}_1 - \mathbf{k}_2)\,, \qquad [\hat{a}^-_{\mathbf{k}_1}, \hat{a}^-_{\mathbf{k}_2}] = [\hat{a}^+_{\mathbf{k}_1}, \hat{a}^+_{\mathbf{k}_2}] = 0\,. \tag{2.7.142}$$

Accordingly, $\hat{a}^-_{\mathbf{k}}$ and $\hat{a}^+_{\mathbf{k}}$ can be interpreted as annihilation and creation operators, respectively. In particular, the action of the annihilation operator on the vacuum state $|0\rangle$ vanishes:

$$\hat{a}^-_{\mathbf{k}}|0\rangle = 0\,. \tag{2.7.143}$$

Then, the *two-point correlation function* can be evaluated as follows:

$$\begin{aligned}
\langle \hat{u}_{\mathbf{k}_1} \hat{u}_{\mathbf{k}_2} \rangle \equiv \langle 0|\hat{u}_{\mathbf{k}_1} \hat{u}_{\mathbf{k}_2}|0\rangle &= \langle 0|(\hat{a}^-_{\mathbf{k}_1} u_k + \hat{a}^+_{-\mathbf{k}_1} u^*_k)(\hat{a}^-_{\mathbf{k}_2} u_{\tilde{k}} + \hat{a}^-_{-\mathbf{k}_2} u^*_{\tilde{k}})|0\rangle \\
&= u_{k_1} u^*_{k_2} \langle 0|\hat{a}^-_{\mathbf{k}_1} \hat{a}^+_{-\mathbf{k}_2}|0\rangle \\
&= u_{k_1} u^*_{k_2} \langle 0|[\hat{a}^-_{\mathbf{k}_1}, \hat{a}^+_{-\mathbf{k}_2}]|0\rangle \\
&= u_{k_1} u^*_{k_1} (2\pi)^3 \delta^3(\mathbf{k}_1 + \mathbf{k}_2) \equiv P_u(k_1)(2\pi)^3 \delta^3(\mathbf{k}_1 + \mathbf{k}_2)\,.
\end{aligned} \tag{2.7.144}$$

Therefore,

$$P_u(k, \tau) \equiv |u_k(\tau)|^2 \tag{2.7.145}$$

defines the *power spectrum*, though we will make use of the dimensionless power spectrum more often, which is defined by

$$\mathcal{P}_u(k, \tau) \equiv \frac{k^3}{2\pi^2} P_u(k, \tau) = \frac{k^3}{2\pi^2} |u_k(\tau)|^2\,. \tag{2.7.146}$$

Unless otherwise stated, by 'power spectrum' we will implicitly mean the dimensionless power spectrum, and it is related to the two-point function as follows:

$$\langle \hat{u}_{\mathbf{k}_1} \hat{u}_{\mathbf{k}_2} \rangle = \frac{\mathcal{P}_u(k_1)}{2k_1^3} (2\pi)^5 \delta^3(\mathbf{k}_1 + \mathbf{k}_2)\,. \tag{2.7.147}$$

Since we recall $u_{\mathbf{k}} = \frac{1}{2} M_{\rm Pl} a h_{\mathbf{k}}$, the tensor perturbation two-point correlation function is

$$\langle \hat{h}_{\mathbf{k}_1} \hat{h}_{\mathbf{k}_2} \rangle = P_h(k_1)(2\pi)^3 \delta^3(\mathbf{k}_1 + \mathbf{k}_2)\,, \tag{2.7.148}$$

and the corresponding (dimensionless) power spectrum is

$$\mathcal{P}_h(k,\tau) = \frac{k^3}{2\pi^2}\frac{4}{M_{\rm Pl}^2}\frac{|u_k(\tau)|^2}{a(\tau)^2} = \frac{2k^3}{\pi^2 M_{\rm Pl}^2}\frac{|u_k(\tau)|^2}{a(\tau)^2}\,. \tag{2.7.149}$$

Finally, we recall that there are two polarization states for tensor modes, i.e., $h_{\mathbf{k},+}$ and $h_{\mathbf{k},\times}$, which evolve the same way. Therefore, the tensor power spectrum must be multiplied by two, the number of polarization states, to get the full contribution:

$$\mathcal{P}_{\rm t}(k,\tau) = 2\mathcal{P}_h(k,\tau) = \frac{4k^3}{\pi^2 M_{\rm Pl}^2}\frac{|u_k(\tau)|^2}{a(\tau)^2}\,. \tag{2.7.150}$$

As already mentioned, the same quantization procedure applies for scalar modes. In particular, the curvature perturbation two-point correlation function is

$$\langle\hat{\zeta}_{\mathbf{k}_1}\hat{\zeta}_{\mathbf{k}_2}\rangle = P_\zeta(k_1)(2\pi)^3\delta^3(\mathbf{k}_1+\mathbf{k}_2)\,, \qquad P_\zeta(k) \equiv |\zeta_k|^2\,, \tag{2.7.151}$$

and with the Sasaki-Mukhanov variable $v_k \equiv z\zeta_k$, we can write the power spectrum as

$$\mathcal{P}_\zeta(k,\tau) \equiv \frac{k^3}{2\pi^2}|\zeta_k(\tau)|^2 = \frac{k^3}{2\pi^2}\frac{|v_k(\tau)|^2}{z^2(\tau)}\,. \tag{2.7.152}$$

The mode function $v_k(\tau)$ is found by solving the Fourier transform of the equation of motion for $v(\tau,\mathbf{x})$ [recall Eq. (2.5.117)]:

$$v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0\,. \tag{2.7.153}$$

Similarly, ζ_k satisfies

$$\zeta_k'' + 2\frac{z'}{z}\zeta_k' + k^2\zeta_k = 0\,, \qquad \ddot{\zeta}_k + (3+\eta)H\dot{\zeta}_k + \frac{k^2}{a^2}\zeta_k = 0\,. \tag{2.7.154}$$

Let us show an example of how the tensor and scalar power spectra can be calculated. For tensor modes, we solved the equation of motion for a generic background evolution and the mode function $u_k(\tau)$ was given by Eq. (2.6.138). To find the appropriate boundary condition, we separate the equation of motion, $u_k'' + (k^2 - a''/a)u_k = 0$, on different scales of interest. First, we note that there are two obvious regimes:

$$\begin{aligned} k^2 \gg |\mathcal{H}^2 + \mathcal{H}'| &\implies u_k'' + k^2 u_k \simeq 0 \\ k^2 \ll |\mathcal{H}^2 + \mathcal{H}'| &\implies u_k'' - \frac{a''}{a} u_k \simeq 0 \,, \end{aligned} \tag{2.7.155}$$

where we used the fact that $a''/a = \mathcal{H}^2 + \mathcal{H}'$. Therefore, the first regime where the fluctuations' comoving wavelength satisfies[7] $\lambda_{\mathrm{comoving}} \propto k^{-1} \ll (a|H|)^{-1}$ is called the *sub-Hubble* regime, while the second regime in which $\lambda_{\mathrm{comoving}} \propto k^{-1} \gg (a|H|)^{-1}$ is called the *super-Hubble* regime since $|\mathcal{H}|^{-1} = (a|H|)^{-1}$ defines the comoving Hubble radius. Furthermore, the sub- and super-Hubble regimes are sometimes defined by the limits $k|\tau| \to \infty$ and $k|\tau| \to 0$, respectively, since the comoving Hubble scale satisfies $|\mathcal{H}| \sim |\tau|^{-1}$.

In the sub-Hubble regime, we immediately notice the equation of motion of a harmonic oscillator with solution

$$u_k(\tau) \overset{k|\tau|\to\infty}{=} c_{1,k} e^{-ik\tau} + c_{2,k} e^{ik\tau} \,. \tag{2.7.157}$$

In flat Minkowski spacetime, it can be shown (see, e.g., Refs. [77, 513]) that the appropriate vacuum mode is the positive frequency mode, i.e., $u_k \propto \exp(-ik\tau)$. Normalizing according to Eq. (2.6.133), which sets $|c_{1,k}|^2 = (2k)^{-1}$, defines the *Bunch-Davies vacuum* [158]:

$$u_k(\tau) \overset{k|\tau|\to\infty}{=} \frac{1}{\sqrt{2k}} e^{-ik\tau} \,. \tag{2.7.158}$$

This will serve as the initial/boundary condition for cosmological perturbations in most cases.

Back to the general solution (2.6.138) for $u_k(\tau)$, we can take the sub-Hubble limit[8],

$$u_k(\tau) \overset{k|\tau|\gg 1}{\simeq} \sqrt{\frac{2}{\pi k}} \left(C_{1,k} e^{-ik\tau} e^{i\vartheta_-} + i C_{2,k} e^{ik\tau} e^{i\vartheta_+} \right) \,, \tag{2.7.159}$$

[7] We note that

$$|\mathcal{H}^2 + \mathcal{H}'| = (aH)^2 |2 - \epsilon| \sim (aH)^2 \tag{2.7.156}$$

as long as ϵ is not excessively larger than $\mathcal{O}(1)$ and as long as $\epsilon \neq 2$. Usually, $0 \leq \epsilon \leq 3$ for typical equations of state with $-1 \leq w \leq 1$ [recall Eq. (2.1.19)]. In the case where $|w| \gg 1$, one should be cautious and instead compare the amplitude of k modes with the quantity $a|H|\sqrt{|\epsilon|}$. Radiation with $w = 1/3$ (so $\epsilon = 2$) is also a very special sub-case.

[8] An assumption that is made when doing the expansion is that $\tau < 0$. The sign matters to differentiate the positive frequency mode from the negative frequency mode. For all models of the very early universe that will be studied in this thesis, the vacuum initial conditions will be set in a regime where $\tau < 0$.

where $\vartheta_\pm \equiv \frac{\pi}{2}(\pm|\nu| - \frac{1}{2})$. Comparison with the Bunch-Davies vacuum immediately sets $C_{2,k} = 0$ and

$$C_{1,k} = \frac{\sqrt{\pi}}{2} e^{-i\vartheta_-} \,. \tag{2.7.160}$$

Therefore, the solution becomes

$$u_k(\tau) = \frac{\sqrt{\pi}}{2} e^{-i\vartheta_-} |\tau|^{1/2} H^{(1)}_{|\nu|}(k|\tau|) \,. \tag{2.7.161}$$

As we will see in the next chapter, cosmological perturbations have to evolve from sub-Hubble scales to super-Hubble scales, thus exiting the Hubble radius, in order to have a successful structure formation scenario. Accordingly, the primordial power spectrum constrained by CMB observations (i.e., the power spectrum that sets the initial conditions at the onset of radiation-dominated expansion in the standard model of cosmology) is the power spectrum of super-Hubble perturbations. Therefore, to evaluate $\mathcal{P}_\mathrm{t}$, we must expand the solution for $u_k(\tau)$ in the limit $k|\tau| \to 0$:

$$u_k(\tau) \overset{k|\tau|\ll 1}{\simeq} \frac{2^{-1-|\nu|} e^{-i\vartheta_-} |\tau|^{1/2}}{\sqrt{\pi}\Gamma(1+|\nu|)} \left(\pi[1 + i\cot(\pi|\nu|)](k|\tau|)^{|\nu|} - 2^{2|\nu|} i\Gamma(|\nu|)\Gamma(1+|\nu|)(k|\tau|)^{-|\nu|}\right) \,, \tag{2.7.162}$$

where Γ is the gamma function. We note that the two solutions behave very differently: as $|\tau| \to 0$ (this corresponds to forward time evolution when $\tau < 0$), $|\tau|^{|\nu|}$ decays (we say that it is a *decaying mode*), while $|\tau|^{-|\nu|}$ grows (we say that it is a *growing mode*). The growing mode dominates as time progresses, hence we keep only the growing mode to evaluate the power spectrum. The super-Hubble solution becomes

$$u_k(\tau) \overset{k|\tau|\ll 1}{\simeq} -\frac{i 2^{|\nu|-1} e^{-i\vartheta_-} \Gamma(|\nu|)}{\sqrt{\pi}} k^{-|\nu|} |\tau|^{1/2-|\nu|} \,, \tag{2.7.163}$$

and we can evaluate the tensor power spectrum as follows:

$$\mathcal{P}_\mathrm{t}(k,\tau) \simeq \frac{4k^3}{\pi^2 M_\mathrm{Pl}^2 a(\tau)^2} \frac{2^{2(|\nu|-1)}\Gamma(|\nu|)^2}{\pi} k^{-2|\nu|} |\tau|^{1-2|\nu|} = \frac{4^{|\nu|}\Gamma(|\nu|)^2}{\pi^3 M_\mathrm{Pl}^2} a(\tau)^{-2} |\tau|^{1-2|\nu|} k^{3-2|\nu|} \,. \tag{2.7.164}$$

The amplitude of the resulting tensor perturbation power spectrum is model dependent (because of the time dependence), but we can read off the generic spectral index:

$$n_{\mathrm{t}} \equiv \frac{\partial \ln \mathcal{P}_{\mathrm{t}}}{\partial \ln k} = 3 - 2|\nu| = 3 - \left| \frac{1-3p}{1-p} \right| . \tag{2.7.165}$$

Some models that will be reviewed in the next section have, e.g., $p = 0 \implies |\nu| = 1/2$ (Ekpyrotic), $p = 1/3 \implies |\nu| = 0$ (pre-Big Bang), $p = 2/3 \implies |\nu| = 3/2$ (matter bounce), and $p \gg 1 \implies |\nu| \simeq 3/2$ (inflation); therefore, they predict $n_{\mathrm{t}} = 2$, $n_{\mathrm{t}} = 3$, $n_{\mathrm{t}} = 0$, and $n_{\mathrm{t}} \simeq 0$, respectively.

Let us repeat the calculation for scalar perturbations in the particular case of a background evolution with constant equation of state (so $\epsilon =$ constant). In this situation, $\epsilon' = 0$ implies that

$$\frac{z''}{z} = \frac{a''}{a} \tag{2.7.166}$$

since we recall $z^2 = 2M_{\mathrm{Pl}}^2 a^2 \epsilon$. Consequently, the equation of motion for the scalar Sasaki-Mukhanov equation reduces to

$$v_k'' + \left(k^2 - \frac{a''}{a} \right) v_k = 0 \,, \tag{2.7.167}$$

which is exactly the same equation of motion as for tensor modes u_k. Thus, the same generic solution follows, and upon setting the same Bunch-Davies initial state, the same super-Hubble solution results. Only the amplitude of the final curvature perturbation power spectrum[9] differs:

$$\mathcal{P}_\zeta(k,\tau) = \frac{k^3}{2\pi^2} \frac{|v_k(\tau)|^2}{2M_{\mathrm{Pl}}^2 a(\tau)^2 \epsilon} \simeq \frac{4^{|\nu|-2}\Gamma(|\nu|)^2}{\pi^3 \epsilon M_{\mathrm{Pl}}^2} a(\tau)^{-2} |\tau|^{1-2|\nu|} k^{3-2|\nu|} \,. \tag{2.7.168}$$

Indeed, the scalar spectral index is the same as the tensor spectral index in this case:

$$n_{\mathrm{s}} - 1 \equiv \frac{\partial \ln \mathcal{P}_\zeta}{\partial \ln k} = 3 - 2|\nu| = 3 - \left| \frac{1-3p}{1-p} \right| . \tag{2.7.169}$$

[9] This assumes that all the curvature perturbations come from the adiabatic perturbations of a single fluid or scalar field. If fluctuations from several fluids or fields exist, then usually non-adiabatic or entropy perturbations will appear. Those have not been discussed in this review chapter (see, e.g., Refs. [326, 327, 619, 624] for an introduction), but can play key roles in certain very early universe models. In particular, entropy modes can in certain circumstances be converted into curvature perturbations, hence contributing to the primordial curvature perturbation power spectrum. In such a case, the expression (2.7.168) certainly does not apply.

Two cases of interest are $p \approx 2/3$ (matter-dominated universe) and $p \gg 1$ (accelerated universe), i.e., when $|\nu| \approx 3/2$, in which case $n_{\mathrm{s}}-1 \approx 0$, indicating near scale invariance of the curvature perturbation power spectrum, in agreement with the observational measurement[10] [recall Eq. (1.1.5)]. Finally, the tensor-to-scalar ratio reads

$$r = \frac{\mathcal{P}_{\mathrm{t}}}{\mathcal{P}_\zeta} = 16\epsilon\,. \tag{2.7.170}$$

In the case of acceleration, we have $\epsilon \ll 1$, hence the model predicts $r \ll 1$, while in the case of matter domination, we have $\epsilon \approx 3/2$, hence the model predicts $r \approx 24$. The latter is in strong tension with the current observational upper bound [recall Eq. (1.1.9)]. This will be a key issue in the matter bounce scenario, reviewed in the next chapter.

2.8 Computing the three-point correlation function

2.8.1 Bispectrum

Similar to the two-point function, the three-point correlation function is given by[11]

$$\langle \hat{\zeta}_{\mathbf{k}_1}\hat{\zeta}_{\mathbf{k}_2}\hat{\zeta}_{\mathbf{k}_3}\rangle = (2\pi)^3\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)B_\zeta(k_1,k_2,k_3)\,, \tag{2.8.171}$$

where, in analogy with the power spectrum, $B_\zeta(k_1,k_2,k_3)$ is the *bispectrum*. A convenient parameterization of the bispectrum is

$$B_\zeta(k_1,k_2,k_3) = (2\pi)^4\frac{\mathcal{P}_\zeta^2}{(k_1k_2k_3)^3}\mathcal{A}(k_1,k_2,k_3)\,, \tag{2.8.172}$$

where $\mathcal{A}(k_1,k_2,k_3)$ is called the shape function. The dimensionless shape function is

$$\mathcal{F}(k_1,k_2,k_3) = \frac{\mathcal{A}(k_1,k_2,k_3)}{k_1k_2k_3}\,, \tag{2.8.173}$$

[10] A good model should also predict $n_{\mathrm{s}}-1<0$ according to the observational measurement (1.1.5). How this is done in the cases of matter domination and acceleration will be discussed in the next chapter.

[11] We only discuss the scalar three-point function in this thesis. However, other correlation functions are possible, such as the scalar-scalar-tensor correlator, $\langle \hat{\zeta}_{\mathbf{k}_1}\hat{\zeta}_{\mathbf{k}_2}\hat{h}_{\mathbf{k}_3}\rangle$, scalar-tensor-tensor correlator, $\langle \hat{\zeta}_{\mathbf{k}_1}\hat{h}_{\mathbf{k}_2}\hat{h}_{\mathbf{k}_3}\rangle$, and tensor-tensor-tensor correlator, $\langle \hat{h}_{\mathbf{k}_1}\hat{h}_{\mathbf{k}_2}\hat{h}_{\mathbf{k}_3}\rangle$. Although those can be computed theoretically, hence providing new observational predictions, their signals are usually extremely faint. In fact, experiments have not succeeded in measuring the tensor two-point function, so we are far from measuring three-point correlators involving graviton legs.

and finally, the dimensionless amplitude parameter is

$$f_{\rm NL}(k_1,k_2,k_3) = \frac{10}{3}\frac{\mathcal{A}(k_1,k_2,k_3)}{k_1^3+k_2^3+k_3^3}\,. \tag{2.8.174}$$

2.8.2 Shapes

Particular shapes (i.e., relations among the k's) of the bispectrum bear particular names and physical interpretation. First, the real space curvature perturbation, $\zeta(\mathbf{x})$, can be expanded into a Gaussian contribution, $\zeta_{\rm G}(\mathbf{x})$, and a non-Gaussian contribution as follows,

$$\zeta(\mathbf{x}) = \zeta_{\rm G}(\mathbf{x}) + \frac{3}{5}f_{\rm NL}^{\rm local}\left[\langle\zeta_{\rm G}(\mathbf{x})\rangle^2 - \langle\zeta_{\rm G}(\mathbf{x})^2\rangle\right]\,, \tag{2.8.175}$$

where the parameter $f_{\rm NL}^{\rm local}$ quantifies the size of the non-Gaussianity. It is coined 'local non-Gaussianity' since the expansion is local in real space. Also, we understand that a positive $f_{\rm NL}^{\rm local}$ enhances the probability distribution of curvature perturbations at large ζ (for a fixed power spectrum), hence more non-linear structures are formed. This is why non-Gaussianities are also understood in terms of non-linearities. In momentum space, the above equation becomes

$$\zeta_{\mathbf{k}} = \zeta_{\mathbf{k}}^{\rm G} + \frac{3}{5}f_{\rm NL}^{\rm local}\int\frac{{\rm d}^3\tilde{\mathbf{k}}}{(2\pi)^3}\zeta_{\tilde{\mathbf{k}}}^{\rm G}\zeta_{\tilde{\mathbf{k}}-\mathbf{k}}^{\rm G}\,. \tag{2.8.176}$$

Substituting the above into $\langle\hat{\zeta}_{\mathbf{k}_1}\hat{\zeta}_{\mathbf{k}_2}\hat{\zeta}_{\mathbf{k}_3}\rangle$ and expanding yields the following bispectrum:

$$B_\zeta(k_1,k_2,k_3) = \frac{6}{5}f_{\rm NL}^{\rm local}\left[P_\zeta(k_1)P_\zeta(k_2) + P_\zeta(k_1)P_\zeta(k_3) + P_\zeta(k_2)P_\zeta(k_2)\right]\,. \tag{2.8.177}$$

In the case of a scale-invariant dimensionless power spectrum, $k^3P_\zeta(k)/(2\pi^2) = \mathcal{P}_\zeta =$ constant, one finds

$$B_\zeta(k_1,k_2,k_3) = (2\pi)^4\frac{3}{10}f_{\rm NL}^{\rm local}\mathcal{P}_\zeta^2\left[\frac{1}{k_1^3k_2^3} + \frac{1}{k_1^3k_3^3} + \frac{1}{k_2^3k_2^3}\right]\,, \tag{2.8.178}$$

from which we can read off

$$\begin{aligned}\mathcal{A}(k_1,k_2,k_3) &= \frac{3}{10}f_{\rm NL}^{\rm local}(k_1^3+k_2^3+k_3^3)\,,\\ \mathcal{F}(k_1,k_2,k_3) &= \frac{3}{10}f_{\rm NL}^{\rm local}\left(\frac{k_1^2}{k_2k_3} + \frac{k_2^2}{k_1k_3} + \frac{k_3^2}{k_1k_2}\right)\,.\end{aligned} \tag{2.8.179}$$

This matches the definition for $f_{\rm NL}(k_1, k_2, k_3)$ given in Eq. (2.8.174). We notice that the dimensionless shape function is largest in the case where one of the k's is much smaller than the other two. Without loss of generality, one could have

$$k_1 \ll k_2 \approx k_3 \qquad \text{(squeezed shape)}\,, \tag{2.8.180}$$

which defines the limit in which the *squeezed shape* of momentum-space non-Gaussianities is calculated. The fact that $k_2 \approx k_3$ in that limit comes from the momentum-conserving delta function. The shape is said to be 'squeezed' since forming a triangle with sides of lengths k_1, k_2 and k_3 produces a squeezed triangle (approximately isosceles), where one side is much smaller than the other two sides.

Another shape is called the *equilateral* configuration, where all three $\zeta_{\mathbf{k}_i}$'s have equal wavelength at the time of horizon exit:

$$k_1 = k_2 = k_3 \qquad \text{(equilateral shape)}\,. \tag{2.8.181}$$

The name comes from the fact that the corresponding triangle is equilateral. In the *folded* (or flattened) configuration, defined by

$$k_1 = 2k_2 = 2k_3 \qquad \text{(folded shape)}\,, \tag{2.8.182}$$

the triangle is exactly isosceles and very flat (hence the name). Finally, the *orthogonal* configuration is defined by

$$k_1 = \sqrt{k_2^2 + k_3^2} = \sqrt{2}k\,,\;\; k_2 = k_3 \equiv k \qquad \text{(orthogonal shape)}\,, \tag{2.8.183}$$

since it is orthogonal[12] to both the squeezed, equilateral, and folded shapes.

[12] Orthogonality is defined in the sense that the 'scalar product' of two bispectra, defined by

$$B_1 \cdot B_2 \equiv \sum_{\mathbf{k}_i} \frac{B_1(k_1,k_2,k_3)B_2(k_1,k_2,k_3)}{P_\zeta(k_1)P_\zeta(k_2)P_\zeta(k_3)}\,, \tag{2.8.184}$$

vanishes. In the above definition, the sum is over all momentum vectors $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ satisfying momentum conservation. See, e.g., Ref. [59] for details.

2.8.3 In-in formalism and third-order perturbed action

Practically, in general after computing the power spectrum, one knows the result for $\mathcal{P}_\zeta$, so upon evaluating the three-point function $\langle \hat{\zeta}_{\mathbf{k}_1}\hat{\zeta}_{\mathbf{k}_2}\hat{\zeta}_{\mathbf{k}_3}\rangle$, the shape function $\mathcal{A}$ is derived, from which immediately follows the dimensionless shape function $\mathcal{F}$ and amplitude parameter $f_{\rm NL}$. It is those quantities that are usually compared with observational constraints. This is the strategy used for instance in Chapters 5 and 6. The question is thus how to compute the three-point function $\langle \hat{\zeta}_{\mathbf{k}_1}\hat{\zeta}_{\mathbf{k}_2}\hat{\zeta}_{\mathbf{k}_3}\rangle$ explicitly. This is most often done using a key result from the in-in formalism of quantum field theory[13]: the expectation value of an operator $\hat{Q}(\tau)$ is given by the following expression to n-th order in perturbation theory,

$$\langle \hat{Q}(\tau)\rangle = i^n \int_{\tau_0}^{\tau} \mathrm{d}\tau_1 \int_{\tau_0}^{\tau_1} \mathrm{d}\tau_2 ... \int_{\tau_0}^{\tau_{n-1}} \mathrm{d}\tau_n \,\langle 0|[\mathsf{H}_{\rm int}(\tau_n), [\mathsf{H}_{\rm int}(\tau_{n-1}), ..., [\mathsf{H}_{\rm int}(\tau_1), Q^I(\tau)]...]]|0\rangle\,, \tag{2.8.185}$$

where τ_0 corresponds to the initial time at which the vacuum $|0\rangle$ is set (usually the Bunch-Davies vacuum, so often $\tau_0 \to -\infty$ at finite momentum k), $\mathsf{H}_{\rm int}$ is the interacting Hamiltonian, and Q^I is to be evaluated using interaction picture operators (see, e.g., Refs. [59, 201, 620]). In particular, to leading order the expression for the expectation value is

$$\langle \hat{Q}(\tau)\rangle = -i \int_{\tau_0}^{\tau} \mathrm{d}\tilde{\tau}\,\langle 0|[Q^I(\tau), \mathsf{H}_{\rm int}(\tilde{\tau})]|0\rangle\,. \tag{2.8.186}$$

In deriving this expression, the Hamiltonian is expanded into a background component plus perturbations, $\mathsf{H} = \bar{\mathsf{H}} + \delta\mathsf{H}$, and the perturbations themselves are expanded into a quadratic component and an interacting part:

$$\delta\mathsf{H} = \mathsf{H}^{(2)} + \mathsf{H}_{\rm int}\,. \tag{2.8.187}$$

If we want to compute the three-point function, then it is sufficient to expand the interacting Hamiltonian to third order in perturbations, so $\mathsf{H}_{\rm int} = \mathsf{H}^{(3)} + \mathcal{O}(\zeta^4)$. The third-order Hamiltonian is related to the third-order Lagrangian and Lagrangian density through

$$\mathsf{H}^{(3)} = -L^{(3)} = -\int \mathrm{d}^3\mathbf{x}\,\mathcal{L}^{(3)}\,, \tag{2.8.188}$$

[13] We do not review the formalism here, but details can be found, e.g., in Refs. [59, 201, 481, 620, 623]. See, e.g., Refs. [51, 302] for approaches that do not make use of the in-in formalism.

which are also related to the third-order action via

$$S^{(3)} = \int \mathrm{d}t\mathrm{d}^3\mathbf{x}\,\mathcal{L}^{(3)} = \int \mathrm{d}t\,L^{(3)}\,. \tag{2.8.189}$$

The methodology for expanding the action up to third order can be found, e.g., in Ref. [620]. In the case of Einstein gravity and a canonical scalar field, the result is found to be

$$\frac{\mathcal{L}^{(3)}}{M_{\rm Pl}^2} = a^3\left(\epsilon^2-\epsilon\eta-\frac{\epsilon^3}{2}\right)\zeta\dot{\zeta}^2+a\epsilon^2\zeta(\partial_i\zeta)^2-2a\epsilon\dot{\zeta}\partial_i\zeta\partial^i\chi-\frac{a\epsilon\eta}{2}\zeta^2\nabla^2\zeta+\frac{\epsilon}{2a}\zeta(\partial_i\partial_j\chi)^2\,, \tag{2.8.190}$$

where χ is defined such that it satisfies

$$\nabla^2\chi \equiv a^2\epsilon\dot{\zeta}\,, \tag{2.8.191}$$

so we can write $\chi = a^2\epsilon\nabla^{-2}\dot{\zeta}$. With the expression for $\mathcal{L}^{(3)}$ in hand, we then have all the information to compute the three-point function as follows:

$$\langle\hat{\zeta}_{\mathbf{k}_1}\hat{\zeta}_{\mathbf{k}_2}\hat{\zeta}_{\mathbf{k}_3}\rangle = i\int_{\tau_0}^{\tau_{\rm end}}\mathrm{d}\tau\,\langle 0|[\hat{\zeta}_{\mathbf{k}_1}(\tau_{\rm end})\hat{\zeta}_{\mathbf{k}_2}(\tau_{\rm end})\hat{\zeta}_{\mathbf{k}_3}(\tau_{\rm end}), L^{(3)}(\tau)]|0\rangle\,, \tag{2.8.192}$$

where $\tau_{\rm end}$ denotes the end of the primordial evolution. Expanding the right-hand side above and performing the integral is very model dependent, so we refer to Ref. [161, 181, 201, 481, 566, 597, 620] and references therein for further details. When non-Gaussianities are calculated in Chapters 5 and 6, this shall be done explicitly.

Chapter 3

The status of theories of the very early universe

3.1 Horizon and flatness problems

As we stressed in the introduction of this thesis, a successful theory of the very early universe should be able to explain the origin of the CMB temperature fluctuations. Since the CMB is observed to be nearly homogeneous in temperature, photons in the early universe must have been in causal contact within a volume of at least the size of the observed CMB patch in order to reach thermal equilibrium. As massless particles, photons follow null geodesics with $\mathrm{d}s^2 = 0$. Since $\mathrm{d}s^2 = a(\tau)^2(-\mathrm{d}\tau^2 + \mathrm{d}\mathbf{x}^2)$ in FLRW, null geodesics satisfy $\mathrm{d}\mathbf{x}^2 = \mathrm{d}\tau^2$, and the maximal comoving distance traveled by a photon (and therefore by any particle) is given by

$$d_{\mathrm{comoving}}(t) \equiv \Delta x = \Delta\tau \equiv \tau - \tau_{\mathrm{ini}} = \int_{t_{\mathrm{ini}}}^{t} \frac{\mathrm{d}\tilde{t}}{a(\tilde{t})} = \int_{a(t_{\mathrm{ini}})}^{a(t)} \frac{\mathrm{d}\ln a}{aH(a)}\,, \tag{3.1.1}$$

where we used the fact that $H = \mathrm{d}\ln a/\mathrm{d}t$ in the last equality.

In the case of an expanding universe with equation of state w, the scale factor evolution is given by Eq. (2.1.14), so

$$H = \frac{2}{3(1+w)t} \propto a^{-3(1+w)/2}\,, \tag{3.1.2}$$

hence we find $(aH)^{-1} \propto a^{(1+3w)/2}$ and

$$d_{\mathrm{comoving}}(t) \propto \int_{a(t_{\mathrm{ini}})}^{a(t)} \mathrm{d}a\, a^{(-1+3w)/2} \propto \frac{2}{1+3w} \left(a(t)^{(1+3w)/2} - a(t_{\mathrm{ini}})^{(1+3w)/2} \right) , \tag{3.1.3}$$

assuming $1+3w > 0$. In standard Big Bang cosmology, the universe begins its evolution at the initial Big Bang singularity with $a(t_{\mathrm{ini}}) = 0$. Consequently, $d_{\mathrm{comoving}}(t_{\mathrm{cmb}}) \propto a(t_{\mathrm{cmb}})^{(1+3w)/2}$, meaning that there is a maximal finite distance over which particles can interact to reach thermal equilibrium, and it can be checked that for simple radiation- and dust-dominated expansion from the Big Bang to the CMB, the corresponding value for d_{comoving} is much smaller that the actual observed comoving size of the CMB. This is known as the *horizon problem*. The question is how to make $d_{\mathrm{comoving}}(t_{\mathrm{cmb}})$ much bigger.

The resolution to the horizon problem comes from ensuring a sufficiently long phase of evolution over which

$$\frac{\mathrm{d}}{\mathrm{d}t}(a|H|)^{-1} < 0\,; \tag{3.1.4}$$

in other words, the comoving Hubble radius must shrink. There are two simple ways of achieving this. If one wants a universe that is always expanding, then consider having a universe with equation of state $1+3w < 0$ over the time interval $t \in [t_{\mathrm{ini}}, t_r]$. This corresponds to violating the strong energy condition (see, e.g., the textbooks [189, 348, 552, 616] and Refs. [233, 615] for a review of the energy conditions in general relativity). As a result, computing the comoving distance at the time of the CMB involves splitting the integral as

$$\int_{a(t_{\mathrm{ini}})}^{a(t_{\mathrm{cmb}})} = \int_{a(t_r)}^{a(t_{\mathrm{cmb}})} + \int_{a(t_{\mathrm{ini}})}^{a(t_r)} , \tag{3.1.5}$$

and the second integral contributes as

$$\int_{a(t_{\mathrm{ini}})}^{a(t_r)} \mathrm{d}a\, a^{(-1+3w)/2} \propto \frac{2}{|1+3w|} \left(a(t_{\mathrm{ini}})^{-|1+3w|/2} - a(t_r)^{-|1+3w|/2} \right) \tag{3.1.6}$$

when $1+3w < 0$. Therefore, as[1] $a(t_{\mathrm{ini}}) \to 0$, the comoving distance receives an infinite, positive contribution. Consequently, causal contact is ensured at the time of the CMB.

The second possibility to achieve condition (3.1.4) is to have a phase of contraction

[1] The evolutionary phase during which $1+3w < 0$ may not last all the way to $a(t_{\mathrm{ini}}) \to 0$, but as long as the phase is long enough, d_{comoving} can be made sufficiently large and the horizon problem can be solved.

(keeping $1+3w>0$) before the Big Bang (let it occur at the time t_{BB}). Indeed, when this is the case, computing the comoving distance at the time of the CMB involves splitting the integral as

$$\int_{a(t_{\mathrm{ini}})}^{a(t_{\mathrm{cmb}})} = \int_{a(t_{\mathrm{BB}})}^{a(t_{\mathrm{cmb}})} + \int_{a(t_{\mathrm{ini}})}^{a(t_{\mathrm{BB}})}\,, \tag{3.1.7}$$

and since contraction implies $\dot{a}<0 \implies H\equiv \dot{a}/a<0$, the second integral contributes as

$$\int_{a(t_{\mathrm{ini}})}^{a(t_{\mathrm{BB}})} \frac{\mathrm{d}\ln a}{aH} = -\int_{a(t_{\mathrm{ini}})}^{a(t_{\mathrm{BB}})} \frac{\mathrm{d}\ln a}{a|H|} \propto \frac{2}{1+3w}\left(a(t_{\mathrm{ini}})^{(1+3w)/2} - a(t_{\mathrm{BB}})^{(1+3w)/2}\right)\,, \tag{3.1.8}$$

The second term on the right-hand side vanishes as[2] $a(t_{\mathrm{BB}})=0$, but the first term can be arbitrarily large as long as one takes t_{ini} sufficiently far in the past — $a(t_{\mathrm{ini}})$ can be made sufficiently large because we are in a phase of contraction, where the larger the universe is the farther back in time one is. Finally, this shows that a sufficiently long phase of contraction before the Big Bang also solves the horizon problem.

We now turn to the flatness problem. Let us introduce curvature in the FLRW metric and express it as[3]

$$\bar{g}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\mathrm{d}t^2 + a(t)^2\left(\frac{\mathrm{d}r^2}{1-\mathsf{k}r^2} + r^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2)\right)\,. \tag{3.1.9}$$

Then, the Friedmann equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\mathsf{k}}{a^2} = \frac{1}{3M_{\mathrm{Pl}}^2}\bar{\rho}\,. \tag{3.1.10}$$

Defining the critical energy density,

$$\bar{\rho}_{\mathrm{c}} \equiv 3M_{\mathrm{Pl}}^2H^2\,, \tag{3.1.11}$$

[2]The conclusion still hold if the Big Bang is replaced by a non-singular bounce, in which case the corresponding value of a is non-zero at that point in time, but the contribution from that term remains small compared to the contribution coming from the $a(t_{\mathrm{ini}})$ term.

[3]The curvature is usually characterized by the parameter k, but to avoid confusion with the wavenumber of cosmological perturbations that we also denote by k, we use the variable k instead.

to be the flat space ($\mathsf{k} = 0$) energy density, we introduce the density parameter

$$\Omega \equiv \frac{\bar{\rho}}{\bar{\rho}_{\mathrm{c}}} = 1 + \frac{\mathsf{k}}{(aH)^2} \,. \tag{3.1.12}$$

We see that the Ω parameter is linked to the geometry of the universe: $\Omega = 1$ for a flat universe ($\mathsf{k} = 0$), $\Omega > 1$ for a closed universe ($\mathsf{k} > 0$), and $\Omega < 1$ for a open universe ($\mathsf{k} < 0$). Using the Friedmann equation (3.1.10) to eliminate H^2 in Ω and using the solution $\bar{\rho} = \rho_0 a^{-3(1+w)}$ for matter with an equation of state parameter w, we can find

$$\frac{\Omega - 1}{\Omega} = \frac{3M_{\mathrm{Pl}}^2 \mathsf{k}}{\rho_0} a^{1+3w} \,, \tag{3.1.13}$$

and upon taking the logarithm and derivative, we obtain

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\ln a} = (1+3w)\Omega(\Omega - 1) \implies \frac{\mathrm{d}|\Omega - 1|}{\mathrm{d}t} = (1+3w)\left(\frac{\dot{a}}{a}\right)\Omega|\Omega - 1| \,. \tag{3.1.14}$$

Therefore, if the universe is exactly flat ($\Omega = 1$), the above expression implies $\mathrm{d}|\Omega - 1|/\mathrm{d}\ln a = 0$, so the universe will remain flat as time evolves. However, this is an unstable fixed point solution. Indeed, for $1+3w > 0$ and $\dot{a} > 0$, we find $\mathrm{d}|\Omega - 1|/\mathrm{d}\ln a > 0$, meaning that any small perturbation about $\Omega = 1$ will lead to curvature moving away from flatness (either positive or negative). There are two possibilities to overcome this conclusion: either $1 + 3w < 0$ (the strong energy condition is violated) keeping $\dot{a} > 0$, or $\dot{a} < 0$ (the universe is contracting) keeping $1 + 3w > 0$. We notice that those are exactly the same two solutions to the horizon problem. Therefore, we will see in the next sections that models of the very early universe typically lie in one of these two classes.

3.2 Inflationary cosmology

As we saw in the previous section, a way of solving both the horizon and flatness problems is to have a period of expansion during which the strong energy condition is violated, i.e., $1 + 3w < 0$. Noting that the Friedmann equations (2.1.8) can be rewritten as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\bar{\rho}}{3M_{\mathrm{Pl}}^2} \,, \qquad \frac{\ddot{a}}{a} = -\frac{\bar{\rho}}{6M_{\mathrm{Pl}}^2}(1+3w) \,, \tag{3.2.15}$$

it is clear that strong energy condition violation is only possible if $\ddot{a} > 0$, meaning having accelerated expansion (we ignore the possibility $\bar{\rho} < 0$). Primordial cosmology with a phase of accelerated expansion is known as the inflationary paradigm (some original papers include Refs. [49, 150, 274, 333, 335, 347, 463–465, 468, 470, 517, 576–578, 590–593, 613]).

Let us assume for now that $1 + w \geq 0$ (in addition to $\bar{\rho} \geq 0$) so that at least the weak energy condition remains satisfied. Since $1 + 3w = 2(\epsilon - 1)$ and $1 + w = 2\epsilon/3$, the requirement $-1 \leq w < -1/3$ can be recast as $0 \leq \epsilon < 1$. Since $\epsilon \equiv -\dot{H}/H^2$, the condition $\epsilon < 1$ is equivalent to imposing the Hubble parameter to evolve slowly, i.e., to remain almost constant. For this to be satisfied over a long period of time, ϵ itself must not change too much as a function of time, hence we ask for $|\eta| \ll 1$, where we recall $\eta \equiv \dot{\epsilon}/(H\epsilon)$.

The lowest limit, $w = -1$ or $\epsilon = 0$, is the case of exponential accelerated expansion. Indeed, in this case $\dot{H} = 0$, $H = \text{constant} > 0$, and

$$a(t) \propto e^{Ht} \,. \tag{3.2.16}$$

In conformal time, the expression for the scale factor becomes

$$a(\tau) = -\frac{1}{H\tau} \,, \qquad \tau < 0 \,. \tag{3.2.17}$$

Exponential expansion in FLRW is also known as *de Sitter spacetime* [249].

A constant Hubble parameter immediately implies a constant Hubble radius H^{-1}, so it is nice to sketch a space and time diagram depicting the scales at play in the evolution of cosmological perturbations. In Fig. 3.1, we see that a typical perturbation with wavelength $\lambda \propto k^{-1}$ starts at the beginning of inflation deep on sub-Hubble scales ($\lambda \ll H^{-1}$). The perturbation is then stretched (redshifted) exponentially during inflation since $a(t) \propto e^{Ht}$ and the physical wavelength grows as $\lambda = a\lambda_{\text{comoving}}$. The perturbation exits the Hubble radius (the point where the red and blue curves intersect during inflation) and the perturbations become super-Hubble ($\lambda \gg H^{-1}$). When inflation ends, the time known as reheating[4], the universe enters the phases of standard Big Bang cosmological expansion with radiation

[4] The issue of reheating is not discussed in this thesis, but the idea is as follows. During inflation, any matter with typical equation of state is diluted exponentially (recall, e.g., $\rho^{(\text{dust})} \propto a^{-3}$ and $\rho^{(\text{radiation})} \propto a^{-4}$). Therefore, when inflation ends, the universe is empty of standard model particles that fill the universe today. Accordingly, the universe has to 'reheat' (i.e., particles must be produced) at the end of inflation. See, e.g., Refs. [25, 28, 405, 406, 585, 599] for reviews of how this is done. A similar process of particle production often has to occur in alternative models to inflation (for instance, see Refs. [192, 330, 342, 354, 561, 579]).

domination and then matter domination. The Hubble parameter starts growing, and in fact, it grows faster than the redshifting of the perturbations' wavelength. From Eq. (2.1.14),

$$H(t)^{-1} = \frac{3(1+w)}{2} t \,, \tag{3.2.18}$$

while $\lambda \sim a(t) \sim t^{2/(3(1+w))}$, which grows less fast as long as $w > -1/3$. Accordingly, the perturbations reenter the horizon (i.e., they become sub-Hubble again) at a later stage (the point where the red and blue curves intersect in the post-inflationary era). The perturbations can then explain the fluctuations in temperature in the CMB.

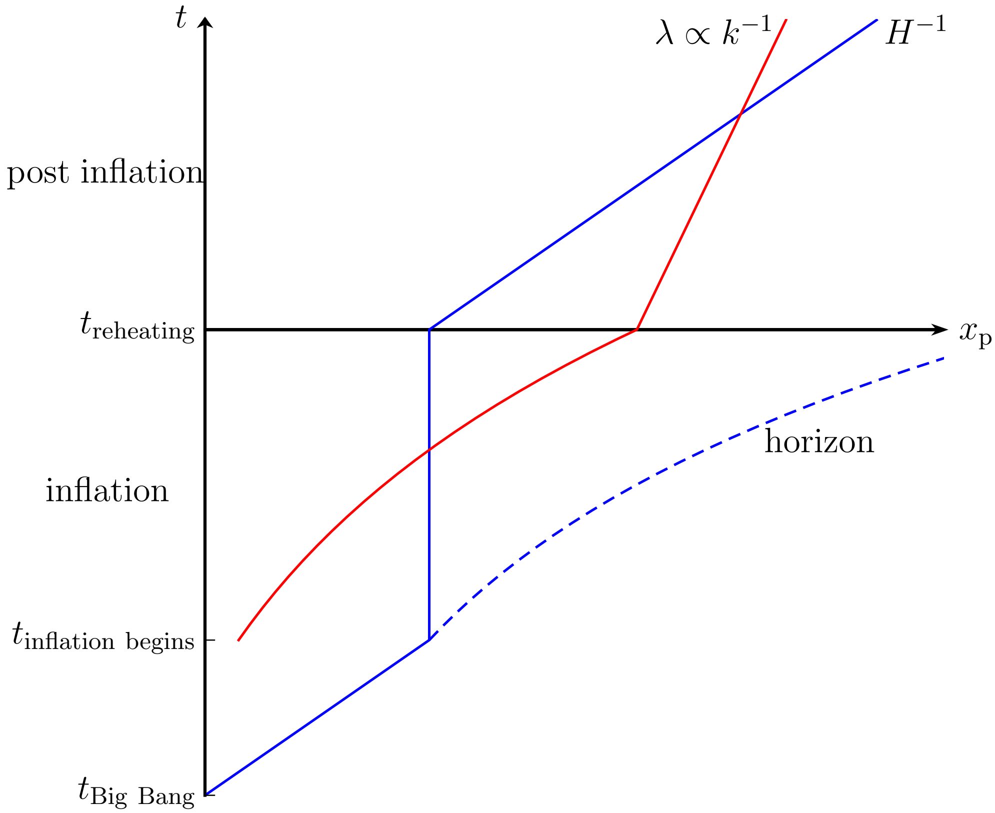

Figure 3.1 Space and time sketch of inflation and cosmological perturbations. The horizontal axis represents physical space coordinates, the vertical axis is physical time, the solid blue curve is the Hubble radius H^{-1}, the dashed blue curve is the particle horizon, and the solid red curve is the wavelength $\lambda \propto k^{-1}$ of a typical perturbation with wavenumber k. Various stages in the evolution of the universe are denoted on the vertical axis such as the time of the Big Bang, the beginning of inflation, and the end of inflation (reheating).

3.2.1 Predictions of inflation

Models of inflation and de Sitter space are mostly studied in the appendix of this thesis, so we shall not spend too much time reviewing them here. Nevertheless, let us state a few predictions that inflationary cosmology makes (for reviews of inflationary models and derivation of the predictions, see for instance Refs. [58, 59, 61, 110, 114, 115, 393, 466, 469, 471, 475, 532]). First, the curvature power spectrum is expected to be nearly scale invariant. Indeed, we saw that $a(t) \propto t^p$ with $p \gg 1$ (in which case $\ddot{a} > 0$) gives rise to a scalar spectral index very close to scale invariance. The exact number can be found to be

$$n_{\mathrm{s}} - 1 = -2\epsilon - \eta \tag{3.2.19}$$

under the slow-roll approximations, i.e., $\epsilon \ll 1$ and $|\eta| \ll 1$. In this case, we see that $n_{\mathrm{s}} - 1 \approx 0$ and $n_{\mathrm{s}} - 1 < 0$ in agreement with the observational measurement [Eq. (1.1.5)]. The scalar power spectrum amplitude is given by

$$A_{\mathrm{s}} = \frac{1}{8\pi^2} \frac{H^2}{M_{\mathrm{Pl}}^2 \epsilon}\,. \tag{3.2.20}$$

The tensor-to-scalar ratio, which was derived to be $r = 16\epsilon$, is predicted to be small, but non-zero. A wide range of models with different values for H (a few orders of magnitude below the Planck mass) and $\epsilon \ll 1$ can agree with the observations to various levels of confidence (see, e.g., Refs. [8, 14, 22, 208, 334, 461, 487, 488] and references therein). Let us note that a detection of primordial gravitational waves consistent with $r = 16\epsilon$ would reinforce inflation as the theory of the very early universe. The amplitude of the tensor power spectrum for inflation is predicted to be

$$A_{\mathrm{t}} = \frac{2}{\pi^2} \frac{H^2}{M_{\mathrm{Pl}}^2}\,. \tag{3.2.21}$$

For that reason, measuring the primordial gravitational wave spectrum would also give information about the energy scale of inflation, i.e., the value of H during inflation. The tensorial tilt is predicted to be

$$n_{\mathrm{t}} = -2\epsilon < 0\,, \tag{3.2.22}$$

which can be recast in the form of a consistency relation with the tensor-to-scalar ratio:

$$r = -8n_{\mathrm{t}}\,. \tag{3.2.23}$$

Verifying this consistency condition observationally in the future would be a really stringent test for inflation. Finally, different inflationary models can predict different shapes and amplitudes of non-Gaussianities. However, because of the model dependence we do not state any result here. We note that canonical single-field inflationary models typically predict very small non-Gaussianities, much below current $\mathcal{O}(1)$ constraints. Nevertheless, constraints on non-Gaussianities can still be used to constrain various models of inflation (see, e.g., Refs. [9, 13, 51, 201, 202, 620] and references therein).

3.2.2 Problems of inflation

While inflationary cosmology makes several predictions, most of them having been confirmed, it suffers from various conceptual issues. We review only a few issues here, but more extensive discussion can be found, e.g., in Refs. [99, 115–118, 120, 123, 126, 128, 133, 369, 370, 485].

A first problem that arises is known as the trans-Planckian problem. As it is clear from Fig. 3.1, since inflation lasts for a long period of time, the perturbations that account for the CMB fluctuations at late times start their evolution deep in the sub-Hubble regime. In fact, they are so short in wavelength that they can be in the trans-Planckian regime $\lambda \lesssim \ell_{\mathrm{Pl}}$, where $\ell_{\mathrm{Pl}} \equiv M_{\mathrm{Pl}}^{-1}$ is the Planck length. This is shown explicitly in Fig. 3.2. The problem with the trans-Planckian regime is that we do not have a good handle on the physics on these very small length scales. In fact, we certainly expect semi-classical quantum field theory in curved spacetime to break down and be replaced by a yet-to-be-found satisfying theory of quantum gravity. Accordingly, the derivation of the scalar and tensor power spectra outlined in the previous chapter would be invalid for those trans-Planckian fluctuations, hence the final results cannot be trusted. One can nevertheless estimate how much trans-Planckian physics can affect the predictions by parameterizing the unknown physics of the trans-Planckian regime. This has been explored, e.g., in Refs. [119, 142, 486] (we also refer to these papers for more details about the trans-Planckian problem in inflation).

A second problem that is relevant for inflationary cosmology is the initial Big Bang singularity. Indeed, it can be shown, as will be discussed in the next chapter, that inflationary cosmology is inevitably preceded by an initial cosmological singularity. In other words, while inflation is a candidate for the theory of the 'initial conditions' of our universe, it still does not tell us what happened before inflation. The period before inflation would correspond to very high energy scales and high densities. When $H \gtrsim M_{\mathrm{Pl}}$, we say that the universe is in

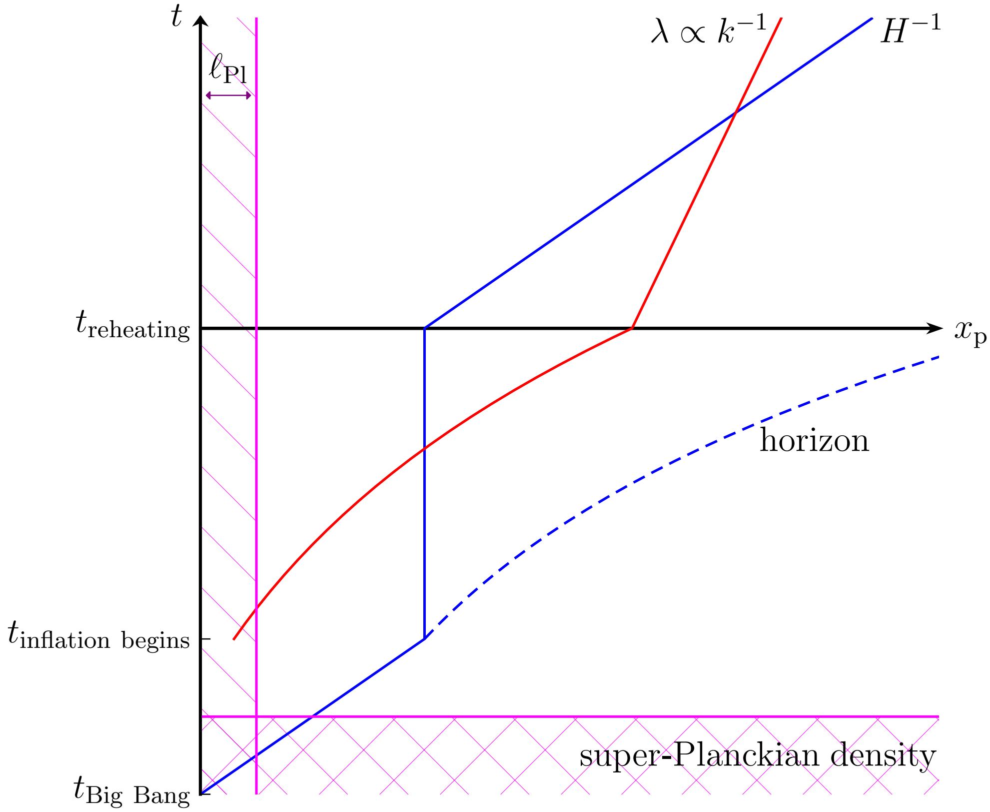

Figure 3.2 Same spacetime sketch as Fig. 3.1 with the regimes of ignorance (sub-Planckian distances and super-Planckian densities) highlighted by the hatched magenta regions.

the super-Planckian density regime, and again, it this regime we do not know what is the appropriate theory of quantum gravity that should prevail. In particular, it is still a question of ongoing research whether inflation can start generically from arbitrary initial conditions (see, e.g., Refs. [23, 103, 141, 188, 215, 216, 253, 287, 323, 324, 378, 462, 467] and references therein). This is not at all clear, especially that the universe in the trans-Planckian density regime could be very inhomogeneous and anisotropic.

We discussed two key issues of inflationary cosmology. We chose these two since they are generally avoided in alternatives to inflation (see the next section). However, we want to end by stressing again that there are more issues than just these two, which are still subject to intense debate (see, e.g., Refs. [100, 102, 113, 116, 120, 126, 334, 369–371, 461, 485]).

3.3 Alternatives to inflation

The problems of inflation can serve as motivation for exploring alternatives to inflation that could act as theories of the very early universe. Moreover, we saw that inflation is the not the only possibility to solve the horizon and flatness problems, and it is not the only theory predicting a nearly scale-invariant curvature power spectrum. Below, we review a few theories (namely, Ekpyrotic, matter bounce, pre-Big Bang and string gas cosmology), but we quickly note that there exist many other alternative scenarios, which are not discussed here, such as: the anamorphic scenario [330, 359, 364, 366]; conflation [279, 282]; adiabatic Ekpyrosis with rapidly-evolving equation of state [390, 391]; varying sound speed [62, 374, 389, 479]; varying speed of light [24, 221, 478]; the slowly-expanding universe [317, 318, 373, 385, 394]; the conformal scenario and other genesis models [223–226, 350–353, 439, 440, 442–446, 523, 524, 534, 574, 621].

Let us start by mentioning that many of the alternatives to inflation lie in a class of models known as *bouncing cosmology* (see, e.g., Refs. [56, 109, 452, 529] for generic reviews of the subject). The idea here is the the Big Bang was not the beginning of time, but rather the universe existed before the Big Bang. In other words, there existed a 'pre-Big Bang' universe, hence the title of this thesis (note, however, that the title is not meant to refer to the pre-Big Bang model only). In fact, bouncing cosmologies typically replace the Big Bang singularity by a bounce, i.e., a transition from a contracting universe to an expanding universe. As we saw in the first section of this chapter, if the contracting phase prior to the bounce is sufficiently long with matter satisfying the strong energy condition ($w \geq -1/3$), then both the horizon and flatness problems are resolved.

In Fig. 3.3, we plot the space and time sketch of a prototypical bouncing cosmology. The universe 'begins' its evolution at large negative times with a low-energy ($|H|$ is small), big universe ($|H|^{-1}$ is large). As the universe contracts, the Hubble radius shrinks (hence solving the horizon problem) and perturbations exit the Hubble radius (the first moment the solid red and blue curves intersect in Fig. 3.3). The universe keeps contracting and ultimately reaches a maximal energy scale, $|H(t_{B-})|$. At that point, $\dot{H} = 0$ and the universe enters the (non-singular) bounce phase. For this to happen, new physics beyond standard general relativity or exotic matter has to be invoked. This will be discussed briefly below and in greater detail in the next chapter. The bounce point, corresponding to $H(t_B) = 0$, marks the transition from contraction ($H < 0$) to expansion ($H > 0$). At that point, the Hubble radius goes to infinity. At t_{B+}, the bounce phase ends with another moment at which $\dot{H} = 0$, and the energy scale, $H(t_{B+})$, is high again. Standard Big Bang cosmological expansion follows with growing Hubble radius. At a later stage, the perturbations completely reenter the horizon.

In bouncing cosmology, let us note that perturbations of observational interest remain far from the trans-Planckian window at all times. Indeed, even though the perturbations originate deep on sub-Hubble scales, the Hubble radius is so large initially that all relevant perturbations remain on super-Planckian length scales. In other words, the physical wavelength of perturbations satisfies $\lambda \gg \ell_{\rm Pl}$, where we trust our low-energy effective theories of quantum field theory and general relativity. Therefore, the trans-Planckian problem that affected inflationary cosmology is completely avoided in bouncing cosmology.

Let us now discuss the details of some specific bouncing cosmology models.

3.3.1 Ekpyrotic cosmology

The original Ekpyrotic model [386, 387, 594, 595] was modeled by a 5-dimensional spacetime in string theory consisting of two $(3 + 1)$-dimensional branes that act as boundaries of the spacetime, separated by a finite dimension. In this setup, gravity lives in the 5-dimensional spacetime, whereas the other fundamental forces and matter live in the $(3 + 1)$-dimensional branes. The distance between the branes acts as a modulus of the theory, and the potential of the resulting scalar field is taken such that there is an attractive force between the two branes. Because of that, the branes approach one another (this is called the Ekpyrotic

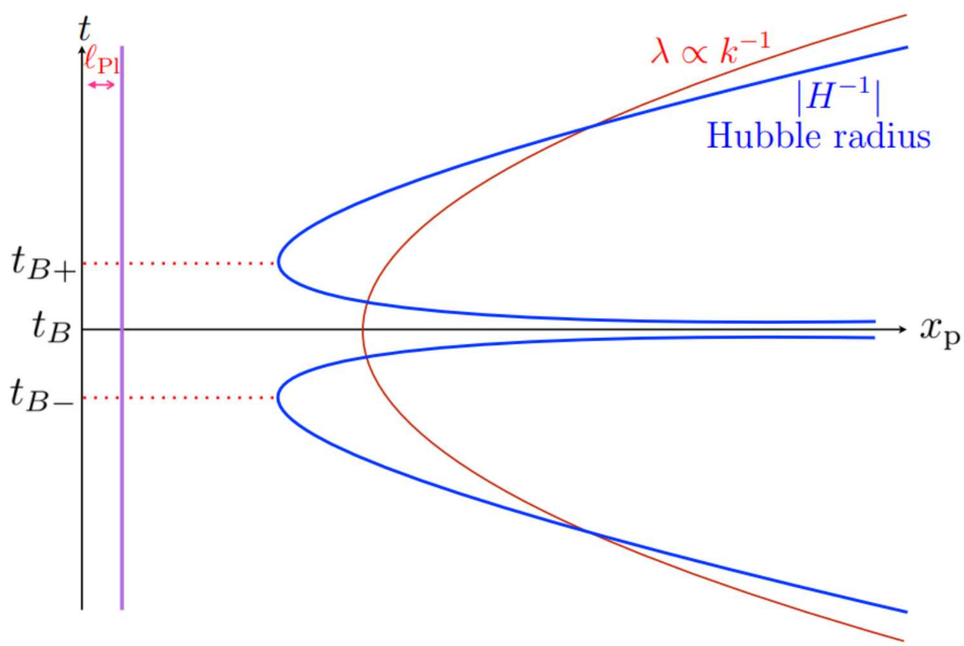

Figure 3.3 Space and time sketch of a prototypical bouncing cosmology adapted from Ref. [109]. The horizontal axis represents physical space coordinates and the vertical axis is physical time. The times t_{B-}, t_B, and t_{B+} indicate the beginning of the bounce phase, the bounce point, and the end of the bounce phase, respectively. The trans-Planckian regime is shown by the purple line. The physical wavelength λ of a perturbation with physical wavenumber k is shown in red. Finally, the blue curve depicts the physical Hubble radius, $|H|^{-1}$.

phase of contraction) and collide. By having a sufficiently long phase of contraction, the horizon and flatness problems are solved. When the branes pass one through the other, the branes experience a Big Crunch/Big Bang transition or a bounce. The collision of the branes produces matter and radiation (equivalent to reheating). Also, the branes are slightly rippled because of quantum fluctuations, so the collision does not happen everywhere at the same time. This explains the small temperature fluctuations in the CMB.

In our $(3+1)$-dimensional effective world, we can describe Ekpyrotic cosmology as a contracting universe with canonical scalar field and negative exponential potential of the form

$$V(\phi) = -V_0 e^{-c\phi}\,, \tag{3.3.24}$$

with $V_0 > 0$ and $c > \sqrt{6}$. Defining the constant

$$p \equiv \frac{2}{c^2}\,, \tag{3.3.25}$$

so $0 < p < 1/3$, the resulting solution to the Friedmann equations in FLRW is

$$a(t) \propto (-t)^{2/c^2} = (-t)^p\,, \qquad H(t) = -\frac{2}{c^2(-t)} = -\frac{p}{(-t)}\,, \tag{3.3.26}$$

which corresponds to a slowly contracting universe with equation of state

$$w = \frac{2}{3p} - 1 = \frac{c^2}{3} - 1 > 1\,, \qquad \epsilon = \frac{c^2}{2} = \frac{1}{p} > 3\,. \tag{3.3.27}$$

Also, the scalar field evolution is given by

$$\begin{aligned}
\phi(t) &= \frac{2}{c}\ln\left(\sqrt{\frac{c^4 V_0}{2(c^2-6)}}(-t)\right) = \sqrt{2p}\ln\left(\sqrt{\frac{V_0}{p(1-3p)}}(-t)\right)\,,\\
\dot\phi(t) &= -\frac{2}{c(-t)}\,,\\
\ddot\phi(t) &= -\frac{2}{c(-t)^2}\,,
\end{aligned} \tag{3.3.28}$$

and

$$V(\phi(t)) = -\frac{p(1-3p)}{(-t)^2}\,. \tag{3.3.29}$$

A key feature of the Ekpyrotic scenario is that it allows the universe to isotropize. To see this, let us write the Friedmann equation as

$$3M_{\mathrm{Pl}}^2H^2=\rho_\Lambda-\frac{3M_{\mathrm{Pl}}^2\mathsf{k}}{a^2}+\frac{\rho_0^{(\mathrm{dust})}}{a^3}+\frac{\rho_0^{(\mathrm{radiation})}}{a^4}+\frac{\rho_0^{(\mathrm{stiff})}}{a^6}+\frac{\sigma_0^2}{a^6}+\frac{\rho_0^{(\phi)}}{a^{3(1+w)}}\,, \tag{3.3.30}$$

where we include in the total background energy density the possible contributions from a cosmological constant Λ, curvature k, dust, radiation, a stiff fluid, anisotropies σ, and a scalar field ϕ with equation of state parameter w. In an expanding universe as $a\to\infty$, we see that the contribution from anisotropies decays very rapidly. In a contracting universe as $a\to 0$, it is the inverse: anisotropies grow and tend to dominate the energy content very rapidly. This is overcome, however, when there is a component with equation of state $w>1$. Accordingly, the energy density in the Ekpyrotic field with $w>1$ rapidly dominates over everything else, in particular anisotropies, thus anisotropies are diluted or 'washed out'. It is in that sense that Ekpyrotic contraction isotropizes the universe[5].

The problem of growing anisotropies is intrinsically linked to the Belinsky-Khalatnikov-Lifshitz (BKL) instability [67] (see also Refs. [64–66, 68, 235–238, 252, 450]). Roughly speaking, the BKL instability states that as the average volume of the universe contracts, the universe becomes highly anisotropic: it contracts in two directions while expanding in the other in a way that can be approximated by the Kasner solution to the Einstein field equations [384]. In fact, the contraction rates can abruptly change from one Kasner-like solution to another. This effect is known as the chaotic mixmaster oscillatory behavior [504, 505], rendering the universe also highly inhomogeneous. Ekpyrotic cosmology with $w>1$ has been shown to be free of these issues [273] as the chaotic mixmaster oscillations are suppressed. In this sense, Ekpyrotic cosmology does not only dilute anisotropies, but it resolves the whole issue of BKL instability, so the homogeneous and isotropic FLRW ansatz with $w>1$ is well justified in a contracting universe. This has been confirmed by a numerical study [306], where highly inhomogeneous and anisotropic initial conditions were implemented in an Ekpyrotic phase of contraction. It was found that homogeneous and isotropic regions with $w>1$ ultimately dominated the cosmological evolution. In this sense, Ekpyrotic cosmology is very robust with respect to arbitrary initial conditions (see also Ref. [433]).

[5]See, however, Ref. [50], which shows that contributions from anisotropic pressures with ultra-stiff equations of state tend to mitigate this conclusion. This will be briefly discussed in Chapter 7.

Looking at Eqs. (3.3.25) and (3.3.27), in the limit $c \gg \sqrt{6}$ one has $0 < p \ll 1/3$, $w \gg 1$, and $\epsilon \gg 3$. From the calculation of the previous chapter, it follows that the scalar and tensor spectral indices are given by [recall Eqs. (2.7.165)–(2.7.169)]

$$n_{\mathrm{s}} - 1 = n_{\mathrm{t}} = 3 - \left| \frac{1-3p}{1-p} \right| \approx 2\,. \tag{3.3.31}$$

The resulting power spectrum of curvature perturbations is thus predicted to be deeply blue [107], in clear contradiction with observations. This led to the proposal of the New Ekpyrotic model [155, 285, 427, 528], where the idea is to implement two (Ekpyrotic) scalar fields. Then, entropy modes can acquire a nearly scale-invariant power spectrum, which can subsequently be converted into curvature perturbations. However, the proposed model was found to be unstable [156, 433]. Therefore, the latest and most accepted Ekpyrotic model [281, 361, 436, 557] implements one Ekpyrotic scalar field together with a massless spectator scalar field kinetically coupled to the Ekpyrotic field. This model is stable [433] and has predictions in agreement with observations. In particular, the resulting curvature power spectrum can be nearly scale invariant, with a small red tilt (see, e.g., Ref. [431] for the expression) and with a sensible amplitude [283]. Also, the model predicts non-Gaussianities of order 1 to order 10 [283, 361]. A blue spectrum of primordial gravitational waves remains a prediction (i.e., $n_{\mathrm{t}} \approx 2$), which concretely means that there should be no detectable primordial gravitational waves on scales of observational interest (i.e., effectively $r \approx 0$).

More details about the Ekpyrotic scenario can be found in the specific reviews [366, 368, 424–426] as well as the general reviews [56, 109]. The Ekpyrotic scenario is not studied in detail in this thesis, but the concept appears at a number of instances. In particular, we will see in the next subsection about the matter bounce scenario that a phase of Ekpyrotic contraction could play an important role.

3.3.2 Matter bounce cosmology

Matter bounce cosmology is another alternative scenario which postulates a contracting universe before the Big Bang. The difference with the Ekpyrotic scenario is that the primordial curvature fluctuations are generated (i.e., they exit the Hubble radius) in a matter-dominated phase. The motivation comes from the fact that this automatically yields an exactly scale-invariant power spectrum of curvature perturbations [132, 286, 618]. Indeed, as we saw in the previous chapter, the equation of state of dust ($w = 0$) predicts $n_s - 1 = 0$ (recall Eq. (2.7.169) where $w = 0 \implies a(t)^p$ with $p = 2/3$).

In the original matter bounce model, the basic idea was to use a coherently oscillating, massive, canonical scalar field,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2\,, \tag{3.3.32}$$

in the regime $m \gg |H|$. For such a Lagrangian, the background energy density and pressure are given by

$$\bar{\rho} = \frac{1}{2}\dot{\bar{\phi}}^2 + \frac{1}{2}m^2\bar{\phi}^2\,, \qquad \bar{p} = \frac{1}{2}\dot{\bar{\phi}}^2 - \frac{1}{2}m^2\bar{\phi}^2\,, \tag{3.3.33}$$

and the scalar field equation of motion $\ddot{\bar{\phi}} + 3H\dot{\bar{\phi}} + m^2\bar{\phi} = 0$ is solved for

$$\bar{\phi}(t) = 2\sqrt{\frac{2}{3}}\frac{M_{\mathrm{Pl}}}{m|t|}\sin(mt)\,, \qquad H(t) = \frac{2}{3t}\,. \tag{3.3.34}$$

Also, the Friedmann equation $3M_{\mathrm{Pl}}^2H^2 = \bar{\rho}$ is satisfied to leading order in $m \gg |H|$, and the second Friedmann equation $2M_{\mathrm{Pl}}^2\dot{H} = -(\bar{\rho}+\bar{p})$ is satisfied *in average* to leading order in $m \gg |H|$. To be more specific, by satisfied in average we mean that

$$\left\langle \frac{-(\bar{\rho}+\bar{p})}{2M_{\mathrm{Pl}}^2\dot{H}} \right\rangle \overset{m\gg|H|}{\simeq} 2\langle\cos^2(mt)\rangle = 1\,, \tag{3.3.35}$$

where angled brackets here indicate the usual definition of time averaging,

$$\langle f(t)\rangle \equiv \frac{1}{P}\int_0^P \mathrm{d}t\, f(t)\,, \tag{3.3.36}$$

for some periodic function $f(t)$ with period P. In our case of interest, the period is $P = 2\pi/m$, which is much smaller than the Hubble time $|H|^{-1}$ in the large mass regime. The same applies for the equation of state parameter, which is found to be

$$\langle w\rangle \overset{m\gg|H|}{\simeq} 1 - 2\langle\sin^2(mt)\rangle = 0\,, \tag{3.3.37}$$

hence the average equation of state is that of dust.

The main issue with a model of matter bounce cosmology where $w = 0$ (at least in average) is that the curvature power spectrum is predicted to be *exactly* scale invariant. However, according to observations [recall Eq. (1.1.5)], $n_\mathrm{s}-1=0$ is ruled out by more than 5σ. Also, in the case of a constant equation of state (again, at least in average), we saw in the previous chapter that the tensor-to-scalar ratio is predicted to be $r=16\epsilon$, so for matter domination with $w=0 \implies \epsilon=3/2$, we find[6]

$$r=24\,. \tag{3.3.38}$$

Such a large tensor-to-scalar ratio is in strong tension with current upper bounds from observations [recall Eq. (1.1.9)]. Not all predicted observable quantities are in disagreement with observations though. Indeed, the scalar amplitude is given by (see, e.g., Ref. [438])

$$A_\mathrm{s}=\frac{1}{48\pi^2}\left(\frac{H(t_{B-})}{M_\mathrm{Pl}}\right)^2\,, \tag{3.3.39}$$

where t_{B-} is the beginning of the bounce phase and assuming curvature perturbations remain constant during the phase phase. Accordingly, the scalar amplitude can match the observational measurement (1.1.5) provided the energy scale of the bounce, $|H(t_{B-})|$, is a few orders of magnitude below the Planck scale. In turn, if there was strong evidence for matter bounce cosmology to be the theory of the very early universe, the scalar amplitude would immediately tell us the scale of 'new physics' where the bounce occurs. Another prediction in agreement with observations is regarding the three-point function. Indeed, non-Gaussianities are found to be relatively small, $\mathcal{O}(1)$. Specifically, Ref. [181] found that[7]

$$f_\mathrm{NL}^\mathrm{local}=-\frac{35}{16}\,,\qquad f_\mathrm{NL}^\mathrm{equil}=-\frac{255}{128}\,,\qquad f_\mathrm{NL}^\mathrm{folded}=-\frac{9}{8}\,, \tag{3.3.40}$$

in agreement with observational constraints [recall Eq. (1.1.6)]. Nevertheless, the discordance of n_s and r with observations renders such a basic model unviable as the theory of the very early universe.

Moreover, matter bounce cosmology suffers from an important theoretical issue. In the

[6] In Chapter 5, we note that the number $r=96\pi$ is stated as the result for matter bounce cosmology. However, this is due to an incorrect normalization of the Sasaki-Mukhanov variable for tensor perturbations. The qualitative results presented in the publication of Chapter 5 remain unaffected by an $\mathcal{O}(10)$ difference in the value of r.

[7] In Ref. [438], we corrected an algebra mistake from Ref. [181]. Therefore, the numbers presented here truly come from Ref. [438], which is presented in Chapter 6.

previous discussion about the Ekpyrotic scenario, it was mentioned that anisotropies grow proportionally to a^{-6} in a contracting universe. Therefore, it is clear that a phase of matter domination, where the energy density grows only proportionally to a^{-3}, is unstable to the growth of anisotropies. This is the BKL instability. The issue can be alleviated by introducing a phase of Ekpyrotic contraction subsequent to the matter-dominated contraction. This possibility was studied in Refs. [167, 175], where it is shown that, indeed, the Ekpyrotic phase can successfully act as an isotropization mechanism. However, it was shown in Ref. [432] that there remains a large level of fine-tuning for this to work.

Various modifications to the original matter bounce scenario have been proposed to resolve the main issues (scalar tilt and tensor-to-scalar ratio). The most obvious strategy to reduce the tensor-to-scalar ratio consists in amplifying curvature perturbations. In Ref. [562], which is presented in Chapter 5, we explored whether it is possible for curvature perturbations to be enhanced in the bounce phase. The physics in the bounce phase appears, at first, very model dependent. Indeed, we will discuss in the next chapter several approaches to have a non-singular cosmology. However, it turns out that it is generally difficult for perturbations of observational interest to grow significantly during the bounce phase. Under the strict conditions that allow curvature perturbations to grow in the bounce phase, it was then shown in Ref. [562] that the enhancement of scalar modes implies the production of large non-Gaussianities in the bounce phase, beyond observational constraints. Consequently, we conjectured that matter bounce cosmology suffers from a 'no-go' theorem, stating that it is impossible to satisfy the bound on r and the constraints on $f_{\rm NL}$ simultaneously, i.e, if one constraint is satisfied, the other is inevitably not met.

The conclusions were then extended in Ref. [438], which is presented in Chapter 6. The idea there is to use a k-essence scalar field, a scalar field with a non-canonical kinetic term, which allows for a speed of sound different from the speed of light. In particular, k-essence allows for $c_{\rm s} \ll 1$. In that situation, it is shown that, indeed, curvature perturbations are enhanced, and as a result, the tensor-to-scalar ratio is $r = 24c_{\rm s} \ll 1$, which can agree with observations. However, a small sound speed implies a smaller strong coupling scale [62], i.e., as the universe contracts and the energy scale of the universe rises, one quickly enters the strong coupling regime where $\mathcal{L}^{(3)} \sim \mathcal{L}^{(2)}$, indicating that we cannot trust the perturbative expansion and so the results derived from it. Equivalently, this means that non-Gaussianities become very large, e.g., $f_{\rm NL}^{\rm local} = -\frac{165}{16} + \frac{65}{8c_{\rm s}^2} \gg 1$, well beyond observational constraints. As such, the no-go theorem remained valid and was extended to include the possibility of a non-trivial speed of sound via a non-canonical scalar field.

Another proposal was coined the ΛCDM bounce model [174, 180]. The idea is to have a fluid or a combination of fluids, such as the mix of dark energy and dark matter, that has an effective equation of state $w_{\mathrm{eff}} < 0$, yet $|w_{\mathrm{eff}}| \ll 1$. For instance, one could have a ΛCDM-like contracting universe (hence the name of the model), and the transition from the Λ-dominated contracting phase to the CDM-dominated phase gives such an effective equation of state. This yields a red tilt for curvature perturbations in agreement with observations[8] ($n_{\mathrm{s}} - 1 = 12w^{\star}_{\mathrm{eff}}$). An issue with this model is that it predicts a large, positive running of the scalar spectral index, i.e., how much n_{s} changes across scales. The running is defined as

$$\alpha_{\mathrm{s}} \equiv \frac{\mathrm{d}n_{\mathrm{s}}}{\mathrm{d}\ln k}\,, \tag{3.3.41}$$

and it is observationally constrained as follows [12, 14, 18, 22]:

$$\alpha_{\mathrm{s}} = -0.005 \pm 0.013 \qquad (95\,\%\ \mathrm{CL})\,. \tag{3.3.42}$$

The running in the ΛCDM bounce model can be suppressed if interaction in the dark sector is introduced [171, 174]; then $\alpha_{\mathrm{s}} = (n_{\mathrm{s}} - 1)^2/2 > 0$ is quite small, within the above bound. Also, CDM typically has a very small sound speed, so the model predicts $r = 24c_{\mathrm{s}} \ll 1$, again in agreement with observations. However, just as for a scalar field with a small sound speed, this leads to further issues, namely the growth of inhomogeneities up to non-linear scales and the formation of black holes [200, 559]. Chapter 8, which corresponds to Ref. [559], tackles this issue. Finally, we note that the model predicts[9] $n_{\mathrm{t}} = 12w_{\mathrm{eff}} < 0$.

Some of the phenomenology of the ΛCDM bounce can be mimicked by having a scalar field with the appropriate potential. This idea is known as the *quasi matter bounce* model [341] since it leads to a quasi-matter-dominated contracting phase. This approach allows $w < 0$ and $|w| \ll 1$ by using a potential of the form

$$V(\phi) = V_0 e^{-\sqrt{3(1-\alpha)}|\phi|}\,, \tag{3.3.43}$$

[8] The star ($\star$) means that w_{eff} is evaluated at the time a mode with wavenumber $k_\star$ (the pivot scale for $n_{\mathrm{s}} - 1$ in this case) exits the horizon. This must be stressed because w_{eff} is not constant in this model. We drop the star from here on, but it is always implicitly assumed.

[9] As before, w_{eff} is evaluated at the time of horizon exit. We note that the horizon for tensor modes (the Hubble radius) and for scalar modes (the Jeans radius) is not the same if $c_{\mathrm{s}} \ll 1$. This implies $n_{\mathrm{s}} - 1 \leq n_{\mathrm{t}} < 0$. The concept of Jeans radius will be appropriately introduced in Chapter 8.

with $V_0 > 0$ and $0 < \alpha \ll 1$. Indeed, one can check that the above potential yields an FLRW solution with $w = -\alpha$, hence the model predicts $n_\mathrm{s} - 1 = -12\alpha$ and $\alpha_\mathrm{s} = 0$, in agreement with observations. Since the kinetic structure of the scalar field remains canonical, the sound speed remains unity and there are no issues with respect to strong coupling, large non-Gaussianities or inhomogeneities. However, the issue of large tensor-to-scalar ratio remains unsolved in this case.

An approach that allows to resolve many issues at once is to leave the scalar sector of the theory untouched and try to only tweak the tensor sector. This is achievable if gravity is modified such that gravitons acquire a mass. This possibility is explored in Ref. [459], which is presented in Chapter 7. While there exist a few different theories of *massive gravity*, the approach explored in Chapter 7 is to use a Lorentz-violating massive gravity theory, which allows for a non-zero graviton mass much larger than $|H|$ at all times during the contracting phase. As a result, the tensor power spectrum is shown to be blue, hence effectively unobservable on scales of cosmological interest, just as in Ekpyrotic cosmology. The issue of large tensor-to-scalar ratio is thus solved, and since scalar non-Gaussianities remain those predicted in the original theory (i.e., $\mathcal{O}(1)$ amplitude), the model evades the no-go theorem of Chapters 5 and 6. Moreover, adding a mass to the graviton changes the equation of motion for shear in an anisotropic universe: it adds a mass term to it, exactly equal to the mass of the graviton. Consequently, the solution for the energy density in anisotropies is altered and is found to grow as a^{-3}. This alleviates the fine-tuning issue related to the BKL instability. The remaining issues here are more conceptual, with regards to the implementation of the massive gravity theory.

Finally, another possibility to evade the no-go theorem of Chapters 5 and 6 is to consider the addition of a second scalar field, more specifically a canonical, spectator scalar field with non-zero mass that generates a scale-invariant power spectrum of entropy modes. This is known as the *matter bounce curvaton model* [168] since the mechanism is similar to the inflationary curvaton model [52, 476]. The entropy modes are then converted into curvature perturbations, thus enhancing the latter. Consequently, one can get $r \ll 1$, and it is claimed that non-Gaussianities remain order 1 to 10 [168, 283]. Therefore, the no-go theorem is evaded. However, the fine-tuning of anisotropies remains for this model.

3.3.3 Pre-Big Bang cosmology

The pre-Big Bang model (see, e.g., Refs. [308, 314–316, 449, 609]) is another alternative to inflation. We will not review it in great detail here, but aspects of the model will be mentioned in Chapter 9. For now, let us simply note that the pre-Big Bang model is motivated by string theory and its dualities. In the Einstein frame (i.e., the frame in which gravity looks like Einstein gravity) and in $(3+1)$ spacetime dimensions, the model proposes a contracting universe dominated by a massless scalar field, $\mathcal{L} = -\partial_\mu\phi\partial^\mu\phi/2$, which behaves as a stiff fluid with equation of state $p = \rho$ (recall Eq. (2.3.65) and set $V(\phi) = 0$). Consequently, in FLRW one has $a(t) \propto \sqrt{-\tau}$, $\tau < 0$, and as seen in the previous chapter, the corresponding scalar and tensor power spectra are blue ($n_\mathrm{s} - 1 = n_\mathrm{t} = 3$). Near scale invariance of the curvature perturbation power spectrum can nevertheless be achieved when taking into account the possible contribution from the string theory axion [220, 263, 308, 315, 612].

3.3.4 String Gas Cosmology

To end this chapter, let us briefly mention string gas cosmology [147] as an alternative to inflation. In the string gas scenario, the universe starts in a quasi-static phase, dominated by a gas of fundamental, closed superstrings. The temperature of the gas is assumed to hover at the string theory Hagedorn temperature,

$$T_\mathrm{Hag} = \frac{1}{\sqrt{8\pi}\ell_\mathrm{s}}\,, \tag{3.3.44}$$

where[10] ℓ_s is the fundamental string length. The Hagedorn temperature is the maximal temperature for the gas to be in thermal equilibrium. The string gas scenario further assumes that the spatial sections are compact. In the early phase, the universe is dominated by strings winding those compact spatial sections, but as these winding strings annihilate, string loops are produced and this leads to a transition from the quasi-static phase to a phase of radiation-dominated expansion of the universe (the onset of standard Big Bang cosmology). The intersection and annihilation of the string winding modes can only occur with three large spatial dimensions, hence explaining why the observable universe has three dimensions of space. The rest of the string theory spatial dimensions remain compact and unobservable.

[10] Note that we could define a 'reduced' string length by $L_\mathrm{s} \equiv \sqrt{8\pi}\ell_\mathrm{s}$, in a similar fashion to the reduced Planck mass definition. Then, the Hagedorn temperature is simply $T_\mathrm{Hag} = L_\mathrm{s}^{-1}$.

In the early quasi-static phase, the Hubble radius is very large, so the thermal fluctuations of the gas originate on sub-Hubble scales and their physical wavelength remains constant. As the Hagedorn phase ends, the Hubble radius shrinks to a very small value and the fluctuations exit the Hubble radius. This is the transition to standard cosmology in which the universe becomes radiation dominated, so the Hubble radius starts increasing, and the fluctuations reenter the Hubble radius later on.

The primordial spectra are all determined by the evolution of the gas temperature as a function of the fluctuations' wavenumber, $T(k)$. The main results are derived in Refs. [108, 139, 144, 146, 199, 422, 520], and we summarize them as follows. The curvature perturbation power spectrum is given by

$$\mathcal{P}_\zeta(k) = \left(\frac{\ell_{\mathrm{Pl}}}{\ell_{\mathrm{s}}}\right)^4 \frac{T(k)}{T_{\mathrm{Hag}}} \frac{1}{1 - T(k)/T_{\mathrm{Hag}}}\,, \tag{3.3.45}$$

so the scalar spectral index presents a small red tilt:

$$n_{\mathrm{s}} - 1 = k \frac{\mathrm{d}(T(k)/T_{\mathrm{Hag}})}{\mathrm{d}k} \left(1 - \frac{T(k)}{T_{\mathrm{Hag}}}\right)^{-1} < 0\,. \tag{3.3.46}$$

The running α_{s} is proportional to $n_{\mathrm{s}} - 1 < 0$. The tensor-to-scalar ratio is small:

$$r = \left(1 - \frac{T(k)}{T_{\mathrm{Hag}}}\right)^2 \ln^2 \left(\frac{1 - T(k)/T_{\mathrm{Hag}}}{\ell_{\mathrm{s}}^2 k^2}\right) \ll 1\,. \tag{3.3.47}$$

The tensor spectral index shows as small *blue tilt*:

$$n_{\mathrm{t}} = -(n_{\mathrm{s}} - 1)\left(2\frac{T(k)}{T_{\mathrm{Hag}}} - 1\right) > 0\,. \tag{3.3.48}$$

Finally, non-Gaussianities are practically vanishing on scales of observational interest:

$$f_{\mathrm{NL}}^{\mathrm{local}} \simeq \left(\frac{\ell_{\mathrm{s}}}{\ell_{\mathrm{Pl}}}\right)^2 \left(\frac{k}{k_{\mathrm{pivot}}}\right) \times 10^{-30} \ll 1\,. \tag{3.3.49}$$

In summary, there is no tension between the predictions and the observational constraints. Moreover, there are new predictions, e.g., the tensor blue tilt, that could serve as strong tests for the theory in the future.

The issues with string gas cosmology are more theoretical. For example, deriving the required background evolution from full (i.e., non-perturbative) string theory is a difficult task (see, however, Refs. [73, 104–106] for recent developments). Specific details about string gas cosmology will not be discussed in this thesis, but the concept will be mentioned at a number of instances. More detail can be found in the reviews [57, 109, 122, 124, 125, 127, 130, 131, 134, 135, 145].

Chapter 4

The status of non-singular cosmology

As we saw in the previous chapter, many alternative models to inflation are 'pre-Big Bang' or bouncing cosmologies, in the sense that the primordial evolution happens before a Big Crunch/Big Bang transition or a non-singular bounce. The goal of this chapter is to review how the latter can be done, i.e., how we can resolve the Big Bang singularity to obtain a non-singular bounce.

4.1 The singularity theorems

The difficulty in resolving cosmological singularities comes from the fact that singularities are hard to circumvent in general relativity. Indeed, there exist rigorously-proved mathematical theorems that imply geodesic incompleteness (either past or future incompleteness), meaning that a particle moving along a geodesic inevitably has a starting point (past incompleteness) or an end point (future incompleteness), often a singularity in the spacetime. An example of these theorems is the following (attributed to Penrose [544]; see also Refs. [348, 616] for good reviews):

Theorem 4.1. *Let us consider a spacetime* $(\mathcal{M}, \boldsymbol{g})$ *in general relativity. If:*

(a) the Null Convergence Condition holds;

(b) there exists a non-compact connected Cauchy surface in $\mathcal{M}$*;*

(c) and there exists a closed trapped null surface in $\mathcal{M}$*;*

then $\mathcal{M}$ cannot be null geodesically complete.

We shall not dissect every hypothesis of the theorem (see, e.g., Ref. [348] for details), but let us mention how the conditions could be evaded to avoid geodesic incompleteness. A first possibility is to allow for closed timelike curves. We shall exclude this possibility on causality grounds. Another possibility is that there exists no non-compact connected Cauchy surface in $\mathcal{M}$. This technical condition can actually be relaxed, and it turns out to be not so important (see the discussion in Ref. [348]). A third (more likely) possibility is that the correct underlining theory is simply not general relativity. An obvious solution would be quantum gravity, especially that singularities often occur where we expect quantum effects to become important. It is certainly valid to explore this avenue, but it is difficult. Possible solutions (string theory, loop quantum gravity, etc.) are still incomplete and have their own issues. An easier approach may be to construct an effective field theory. Such avenues shall be developed in the next sections. Finally, there exists the possibility of violating the Null Convergence Condition (NCC).

The Null Convergence Condition stipulates that

$$\forall \boldsymbol{k} = k^\mu \boldsymbol{e}_\mu \ni g_{\mu\nu}k^\mu k^\nu = 0\,, \quad R_{\mu\nu}k^\mu k^\nu \geq 0 \qquad (\mathrm{NCC})\,, \tag{4.1.1}$$

i.e., for every future-pointing null vector field $\boldsymbol{k}$ with components k^μ, $R_{\mu\nu}k^\mu k^\nu \geq 0$. We recall that the Einstein field equations can be written as

$$R_{\mu\nu} = 8\pi G_\mathrm{N}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) + \Lambda g_{\mu\nu}\,, \tag{4.1.2}$$

where $T \equiv T^\mu{}_\mu$ is the trace of the energy-momentum tensor and where we reinserted the possibility of a cosmological constant Λ. Accordingly, it is manifest that the NCC is equivalent to the Null Energy Condition (NEC):

$$\forall \boldsymbol{k} = k^\mu \boldsymbol{e}_\mu \ni g_{\mu\nu}k^\mu k^\nu = 0\,, \quad T_{\mu\nu}k^\mu k^\nu \geq 0 \qquad (\mathrm{NEC})\,. \tag{4.1.3}$$

In the case of a perfect fluid with $T_{\mu\nu} = (\rho+p)u_\mu u_\nu + pg_{\mu\nu}$, the NEC reduces to the condition

$$p + \rho \geq 0 \qquad (\mathrm{NEC,\ perfect\ fluid})\,, \tag{4.1.4}$$

and furthermore, for an equation of state $p = w\rho$, this becomes $1 + w \geq 0$ (with $\rho > 0$)

or $\rho < 0$ (with $w < -1$). Therefore, in the case of general relativity it becomes clear that violating the NCC can only be done by violating the NEC, e.g., by having matter with either an equation state parameter $w < -1$ or negative energy density.

A first implication of the above is that inflationary cosmology with $-1 \leq w < -1/3$, so with matter obeying the NEC, is inevitably past incomplete (see Refs. [87–90]). The conclusion was generalized to any inflationary cosmology as long as the appropriately averaged Hubble parameter in the past is positive (see Ref. [86]). This will be reviewed in Appendix A of this thesis, which presents Ref. [634]. In this paper, we make the link between the implication of past (in)completeness and the issue of spacetime singularity. Indeed, the two concepts do not necessarily imply one another. A good example is de Sitter spacetime, which is geodesically past incomplete the flat FLRW patch, but nevertheless the full spacetime is non-singular in the global patch. Ref. [634] addresses when it is possible to *extend* the spacetime beyond the point or boundary where it would normally appear geodesically incomplete. We shall see in Appendix A that the condition for *extendibility* is related to the presence or absence of a *parallely propagated curvature singularity*, which will be appropriately defined. This type of singularity is not necessarily a *scalar curvature singularity*, which is what is usually implicitly meant by a singularity. A scalar curvature singularity occurs when any scalar curvature-invariant quantity such as

$$R\,,\ R_{\mu\nu}R^{\mu\nu}\,,\ R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\,,\ C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}\,,\ \text{etc.}\,, \tag{4.1.5}$$

where $C_{\mu\nu\alpha\beta}$ is the Weyl tensor, diverges. For example, the Big Bang singularity is a scalar curvature singularity. As we will see in Appendix A, there are cases where the spacetime has no scalar curvature singularity, but still a parallely propagated curvature singularity; the singularity is then said to be an intermediate or non-scalar singularity. For a review of the different types of singularities, see Refs. [270, 348, 616].

4.2 Non-singular bouncing cosmology

As it was made clear in the previous section, one needs to either modify Einstein gravity or include matter violating the NEC to construct a non-singular bouncing cosmology. In this section, we review one approach in that direction that is studied in this thesis, but many more exist that will not be reviewed here (vast literature is cited, e.g., in Chapters 5, 6, and

10, though this remains far from exhaustive).

An approach is to consider Horndeski theory [355]. The idea is to modify Einstein gravity by adding a new degree of freedom to the theory, specifically a scalar field ϕ. The most generic covariant action for the resulting scalar-tensor theory of gravity yielding at most second-order equations of motion[1] is then Horndeski theory[2]:

$$S[g_{\mu\nu},\phi]=\int \mathrm{d}^4x\,\sqrt{-g}\sum_{n=1}^{5}\mathcal{L}_n\,, \tag{4.2.6}$$

with

$$\mathcal{L}_2 = G_2(\phi,X)\,, \tag{4.2.7}$$
$$\mathcal{L}_3 = -\,G_3(\phi,X)\Box\phi\,, \tag{4.2.8}$$
$$\mathcal{L}_4 = G_4(\phi,X)R + G_{4,X}\left[(\Box\phi)^2-(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)\right], \tag{4.2.9}$$
$$\begin{aligned}\mathcal{L}_5 = {} & G_5(\phi,X)G^{\mu\nu}\nabla_\mu\nabla_\nu\phi-\frac{1}{6}G_{5,X}\left[(\Box\phi)^3-3\Box\phi(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)\right.\\ & \left.+\,2(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\alpha\phi)(\nabla_\alpha\nabla^\nu\phi)\right],\end{aligned} \tag{4.2.10}$$

where

$$X\equiv-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\,, \tag{4.2.11}$$

$\Box\equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the d'Alembertian, $G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor, and a coma in the index denotes a partial derivative, e.g.,

$$G_{n,X}(\phi,X)\equiv\frac{\partial G_n(\phi,X)}{\partial X}\,. \tag{4.2.12}$$

In particular: $G_2=G_3=G_5=0$ and $G_4=M_{\mathrm{Pl}}^2/2$ yields vacuum general relativity; $G_2=X-V(\phi)$, $G_3=G_5=0$ and $G_4=M_{\mathrm{Pl}}^2/2$ is a canonical scalar field; arbitrary $G_2(\phi,X)$, $G_3=G_5=0$ and $G_4=M_{\mathrm{Pl}}^2/2$ is known as k-essence [32–34, 307]; $G_3=2(\omega/\phi)X$,

[1] A theory yielding higher-than-second-order equations of motion is known to suffer from *Ostrogradski instabilities* (see, e.g., Refs. [511, 628, 629] and references therein). This is why we generally restrict our attention to classes of theories such as Horndeski theory. Alternatively, the presence of an Ostrogradski instability can be used as a tool to discredit a certain theory. This approach will be used for example in Chapter 10.

[2] We more or less follow the notation of Ref. [401] here. See also Ref. [401] for more details, including the general equations of motion in FLRW.

$G_4 = \phi$ and $G_5 = 0$ is Brans-Dicke theory [148, 276, 284]; arbitrary $G_2(\phi, X)$ and $G_3(\phi, X)$, $G_5 = 0$ and $G_4 = M_{\rm Pl}^2/2$ is known as kinetic gravity braiding [251, 555].

To see how Horndeski theory can allow for non-singular bouncing solutions, consider the sub-case of k-essence with

$$G_2(\phi, X) = \Lambda_2^{-1}(X - C^2)^2 - V(\phi)\,, \tag{4.2.13}$$

where C and $\Lambda_2 > 0$ are constants with dimensions of mass square and mass to the power four, respectively. Upon expanding the bracket squared in the above, we notice that the kinetic part of the theory is of the form $-2\Lambda_2^{-1}C^2X + \Lambda_2^{-1}X^2$, where the leading term in X appears to have the wrong sign, indicating a runaway quantum instability (known as the ghost instability [214]). However, the above Lagrangian has a non-trivial minimum at $X = C^2$, which is called the ghost condensate [31], about which fluctuations have the correct kinetic sign. The energy density and pressure of a k-essence scalar field in FLRW are given by

$$\bar{\rho} = 2\bar{X}G_{2,X}(\bar{\phi}, \bar{X}) - G_2(\bar{\phi}, \bar{X})\,, \qquad \bar{p} = G_2(\bar{\phi}, \bar{X})\,, \tag{4.2.14}$$

so for the above example of ghost condensate Lagrangian, we have

$$\bar{\rho} = \Lambda_2^{-1}(3\bar{X}^2 - 2C^2\bar{X} - C^4) + V(\bar{\phi})\,, \qquad \bar{p} = \Lambda_2^{-1}(\bar{X} - C^2)^2 - V(\bar{\phi})\,, \tag{4.2.15}$$

so $\bar{\rho} + \bar{p} = 4\Lambda_2^{-1}\bar{X}(\bar{X} - C^2)$. When $0 < \bar{X} < C^2$, we notice that $\bar{\rho} + \bar{p} < 0$. Thus, the NEC is violated and a non-singular bounce is achievable.

In a more realistic model, the ghost condensate can form dynamically, i.e., there can be a phase where the ghost condensate forms and the NEC is violated, while outside that phase, the NEC remains satisfied and standard cosmology follows. For example, we could have

$$G_2(\phi, X) = [1 - g(\phi)]X + M_{\rm Pl}^{-4}\beta X^2 - V(\phi)\,, \tag{4.2.16}$$

where $\beta > 0$ is a dimensionless constant and $g(\phi)$ is a dimensionless, Gaussian-shaped function peaked at $g_0 \equiv \max_{\phi\in\mathbb{R}} g(\phi) > 1$. That way, as $g(\phi) < 1$ and $|X| \ll M_{\rm Pl}^4$, we simply have a canonical scalar field, while for $g(\phi) > 1$ and $|X| \sim M_{\rm Pl}^4$, the kinetic part of the Lagrangian is of the form $-|1 - g_0|X + M_{\rm Pl}^{-4}\beta X^2$, i.e., it is a ghost condensate. This is the type of model that is studied in Chapter 5 (Ref. [562]; see references therein).

A key issue with regard to such non-singular constructions is that of stability. Upon perturbing the Horndeski theory action in the comoving gauge, the tensor and scalar second-order actions are found to be [401], respectively,

$$S_T^{(2)} = \frac{1}{8}\int \mathrm{d}t\mathrm{d}^3\mathbf{x}\, a^3\mathcal{G}_T\left[\dot{h}_{ij}^2 - \frac{c_T^2}{a^2}(\vec{\nabla}h_{ij})^2\right], \tag{4.2.17}$$

$$S_S^{(2)} = \int \mathrm{d}t\mathrm{d}^3\mathbf{x}\, a^3\mathcal{G}_S\left[\dot{\zeta}^2 - \frac{c_S^2}{a^2}(\vec{\nabla}\zeta)^2\right], \tag{4.2.18}$$

where

$$c_T^2 \equiv \frac{\mathcal{F}_T}{\mathcal{G}_T} \qquad \text{and} \qquad c_S^2 \equiv \frac{\mathcal{F}_S}{\mathcal{G}_S} \tag{4.2.19}$$

are the definitions for the tensor and scalar sound speed squared, respectively. Furthermore, we defined

$$\mathcal{F}_T \equiv 2[G_4 - X(\ddot{\phi}G_{5,X} + G_{5,\phi})], \tag{4.2.20}$$

$$\mathcal{G}_T \equiv 2[G_4 - 2XG_{4,X} - X(H\dot{\phi}G_{5,X} - G_{5,\phi})], \tag{4.2.21}$$

$$\mathcal{F}_S \equiv \frac{1}{a}\frac{\mathrm{d}\xi}{\mathrm{d}t} - \mathcal{F}_T, \tag{4.2.22}$$

$$\mathcal{G}_S \equiv \frac{\Sigma}{\Theta^2}\mathcal{G}_T^2 + 3\mathcal{G}_T, \tag{4.2.23}$$

$$\xi \equiv \frac{a}{\Theta}\mathcal{G}_T^2, \tag{4.2.24}$$

$$\begin{aligned}\Theta \equiv &-\dot{\phi}XG_{3,X} + 2HG_4 - 8HXG_{4,X} - 8HX^2G_{4,XX} + \dot{\phi}G_{4,\phi} + 2X\dot{\phi}G_{4,\phi X}\\ &- H^2\dot{\phi}(5XG_{5,X} + 2X^2G_{5,XX}) + 2HX(3G_{5,\phi} + 2XG_{5,\phi X})\end{aligned} \tag{4.2.25}$$

$$\begin{aligned}\Sigma \equiv\ &XG_{2,X} + 2X^2G_{2,XX} + 12H\dot{\phi}XG_{3,X} + 6H\dot{\phi}X^2G_{3,XX} - 2XG_{3,\phi} - 2X^2G_{3,\phi X} - 6H^2G_4\\ &+ 6[H^2(7XG_{4,X} + 16X^2G_{4,XX} + 4X^3G_{4,XXX}) - H\dot{\phi}(G_{4,\phi} + 5XG_{4,\phi X} + 2X^2G_{4,\phi XX})]\\ &+ 30H^3\dot{\phi}XG_{5,X} + 26H^3\dot{\phi}X^2G_{5,XX} + 4H^3\dot{\phi}X^3G_{5,XXX}\\ &- 6H^2X(6G_{5,\phi} + 9XG_{5,\phi X} + 2X^2G_{5,\phi XX}).\end{aligned} \tag{4.2.26}$$

The conditions to avoid ghost instabilities in the tensor and scalar sectors are

$$\mathcal{G}_T > 0, \qquad \mathcal{G}_S > 0 \qquad \text{(no-ghost-instability conditions)}, \tag{4.2.27}$$

while we must further ensure no gradient instability by having

$$\mathcal{F}_T > 0\,, \qquad \mathcal{F}_S > 0 \qquad \text{(no-gradient-instability conditions)}\,, \tag{4.2.28}$$

in the tensor and scalar sectors, respectively. If the no-ghost-instability conditions are satisfied, then the no-gradient-instability conditions are equivalent to requiring real sound speeds, i.e., $c_T^2 > 0$ and $c_S^2 > 0$. Let us see why an imaginary sound speed leads to a gradient instability by considering scalar modes as an example. First note that the equation of motion from the variation of the above second-order scalar action is

$$\frac{\mathrm{d}^2\zeta_k}{\mathrm{d}y^2} + 2\frac{\mathrm{d}\ln z}{\mathrm{d}y}\frac{\mathrm{d}\zeta_k}{\mathrm{d}y} + k^2\zeta_k = 0\,, \tag{4.2.29}$$

where $z \equiv \sqrt{2}a(\mathcal{F}_S\mathcal{G}_S)^{1/4}$, $\mathrm{d}y \equiv a^{-1}c_S\mathrm{d}t$, and noting that we transformed to Fourier space. On small scales ($|c_S|k \gg aH$) and in the case $c_S^2 < 0$, the solution is found to be

$$\zeta_k(t) \sim \exp\left(k\int^t \mathrm{d}\tilde{t}\,\frac{|c_S(\tilde{t})|}{a(\tilde{t})}\right)\,, \tag{4.2.30}$$

indicating an exponential growth of the perturbations, an instability due to the sign of the gradient term in the action. Such an instability is present in the model of Chapter 5, but it is usually not necessarily catastrophic since the time scale of the instability remains small if c_S^2 is not negative for a too long period of time given a certain value of k. Furthermore, the length scales where the instability occurs are usually very small, possibly outside the regime of validity of the effective field theory[3] (see Ref. [404]). Nevertheless, a more successful theory of non-singular cosmology should be able to avoid instabilities altogether.

Let us show why this may be challenging in the case of Horndeski theory. Let us first assume that the no-ghost- and no-gradient-instability conditions are satisfied in the tensor sector. From the definition of $\mathcal{G}_T$ and $\mathcal{F}_T$, there certainly is parameter space in $G_4(\phi, X)$ and $G_5(\phi, X)$ over which this assumption is met. Similarly, let us assume the condition $\mathcal{G}_S > 0$ is met, so there is no ghost instability in the scalar sector. Again, from the definition of $\mathcal{G}_S$ and

[3] We are not trying to state that gradient instabilities are generally not an issue. The point is simply that, case-by-case, one should compare the time duration of the instability as a function of the scale k up to the strong coupling scale, usually the ultraviolet cutoff of the effective field theory, k_{UV}. This will vary from one model to another. The result for the model of Chapter 5, as studied in Ref. [404], is that for $k < k_{\mathrm{UV}}$ the gradient instability is only very small, if not completely absent.

Σ above, there certainly is parameter space in the $G_n(\phi, X)$'s for this to be valid. Ref. [398] (see also Refs. [20, 441]) then showed that under these assumptions, it is impossible to have both full geodesic completeness and avoid gradient instabilities in the scalar sector. The argument is as follows: to avoid gradient instabilities in the scalar sector, we want $\mathcal{F}_S > 0$, which can be written as (using the assumption $\mathcal{F}_T > 0$ and the fact that $a(t) \geq 0 \; \forall t$)

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} > a\mathcal{F}_T \geq 0 \iff \xi(t) - \xi(t_0) > M_{\mathrm{Pl}}^2 \int_{t_0}^{t} \mathrm{d}\tilde{t}\, a(\tilde{t}) \geq 0\,, \qquad \forall t \geq t_0\,, \tag{4.2.31}$$

where without loss of generality we set[4] $\mathcal{F}_T = M_{\mathrm{Pl}}^2$. An immediate implication of the above condition is that $\xi(t)$ must be a monotonically increasing function of time $\forall t \geq t_0$. Then, for the spacetime to be geodesically complete, we require past and future completeness of the FLRW cosmology. We will show the argument for past completeness, but it is easy to repeat for future completeness. Past completeness is equivalent to

$$\lim_{t_0 \to -\infty} \int_{t_0}^{t} \mathrm{d}\tilde{t}\, a(\tilde{t}) = \infty\,, \tag{4.2.32}$$

which is satisfied for instance in bouncing cosmology. Taking the limit $t_0 \to -\infty$ of Eq. (4.2.31), we find that we must have

$$\lim_{t_0 \to -\infty} \xi(t_0) = -\infty\,, \tag{4.2.33}$$

together with the fact that $\xi(t)$ needs to cross 0 at some time $t_\star \in (t_0, \infty)$, so that $\xi(t > t_\star) > 0$ since $\xi(t)$ is monotonically growing with t. Recalling $\xi(t) = M_{\mathrm{Pl}}^4 a(t)/\Theta(t)$, since we work in the frame where $\mathcal{G}_T = M_{\mathrm{Pl}}^2$, we find that $\xi(t)$ can cross 0 at some time $t_\star$ only if:

(a) $\lim_{t \to t_\star^\pm} \Theta(t) = \pm\infty \implies \lim_{t \to t_\star^\pm} \xi(t) = 0^\pm$;

(b) $\Theta(t)$ has a finite discontinuity at $t_\star$;

(c) or $\lim_{t \to t_\star^\pm} \Theta(t_\star) = 0^\pm \implies \lim_{t \to t_\star^\pm} \xi(t) = \pm\infty$.

[4] The reason this can be done is that one can always perform a conformal transformation of the metric to move from the Jordan frame, where the coefficient in front of the Hilbert-Einstein term in the action, $G_4(\phi, X)$, is not $M_{\mathrm{Pl}}^2/2$, to the Einstein frame, where $G_4(\phi, X) = M_{\mathrm{Pl}}^2/2$ and $G_5 = 0$ (hence $\mathcal{G}_T = \mathcal{F}_T = 2G_4 = M_{\mathrm{Pl}}^2 \implies c_T^2 = 1$). See Ref. [227] for details of how this is done. Note, however, that the argument carries without this simplification.

If (a) occurs, then since for instance $\Theta \supset 2M_{\rm Pl}^2 H$, it means that a physical quantity like H blows up as $\Theta \to \pm\infty$. If (b) occurs, then the discontinuity in Θ implies a discontinuity in some physical quantity like H, in which case $\dot{H} \to \pm\infty$. Finally, if (c) occurs, then it means that the perturbed action blows up as $\mathcal{G}_S = M_{\rm Pl}^2(3 + M_{\rm Pl}^2\Sigma/\Theta^2) \to \infty$, indicating infinitely strong coupling, i.e., the perturbative expansion breaks down. In sum, it appears impossible to realize a geodesically complete, non-singular cosmology with stable perturbations (and not infinitely strongly coupled). This was claimed to be a no-go theorem for non-singular bouncing cosmologies in Horndeski theory [20, 162, 165, 227, 398, 407, 441, 502].

One of the only ways of evading the resulting theorem lies in condition (c) above. Indeed, it is still possible to have Θ-crossing[5] (i.e., Θ goes through zero), while keeping $\mathcal{G}_S$ and $\mathcal{F}_S$ finite. The key is to move away from the Einstein frame to allow time dependence in $\mathcal{G}_T$ (and $\mathcal{F}_T$), so that $\mathcal{G}_T^2(t)$ scales the same way as $\Theta(t)^2$ close to the Θ-crossing time. That way, the divergence is canceled out and the action remains finite (and within perturbative control). This is the approach applied in Refs. [42, 360, 367], which find fully stable non-singular cosmologies within Horndeski theory.

Another possibility to avoid the no-go theorem is to go beyond Horndeski theory (see, e.g., Refs. [37, 69, 230–232, 299, 300, 321, 322, 418, 420, 421, 458, 510]). We do not develop on how one can go beyond Horndeski theory here, but essentially, higher-order terms can be added to Horndeski's action as long as they satisfy particular degeneracy or constraint equations. With these higher-order terms, it has been possible to construct several models of stable, non-singular cosmology (see, e.g., Refs. [162–165, 227, 402, 407, 408, 503, 631]).

4.3 Limiting curvature

In the previous section, we showed one example of non-singular cosmology construction within Horndeski theory. There exist several more within the realm of scalar-tensor theories of gravity, and a key issue with many of them is always concerning stability. The approach is often to start with an action known to yield second-order equations of motion and see where in parameter space the NEC can be violated for the scalar field ϕ. We review another approach in this section, known as limiting curvature, which is the topic of Chapter 10 (presenting Ref. [635]), where the idea is to build a tailor-made theory for non-singular

[5] Note that some authors, such as in Refs. [360, 365, 367, 407, 503, 565, 631], use the variable γ instead of Θ, so they call this γ-crossing.

cosmology at the level of the action.

The principle of limiting curvature (see, e.g., Refs. [111, 112, 143, 320, 507, 516, 600, 633, 635] and references therein) goes as follows: under the assumption that a successful theory of quantum gravity (or at least of classical gravity up to high enough energy scales) should resolve spacetime singularities, there should exist a fundamental length scale ℓ_f in that theory that bounds all (infinitely many) curvature-invariant functions of the spacetime manifold, e.g.,

$$|R| \leq \ell_f^{-2}\,,\ |R_{\mu\nu}R^{\mu\nu}| \leq \ell_f^{-4}\,,\ |\nabla_\alpha R_{\mu\nu}\nabla^\alpha R^{\mu\nu}| \leq \ell_f^{-6}\,,\ |C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}| \leq \ell_f^{-8}\,,\ \text{etc.} \tag{4.3.34}$$

For instance, the fundamental length could be of the order of the Planck length, $\ell_f \sim \ell_{\mathrm{Pl}}$ (more generally, $\ell_f \geq \ell_{\mathrm{Pl}}$). However, this is difficult to obtain in practice since one could bound one or more curvature invariants and still have other curvature-invariant functions blowing up. For example, at the center of a Schwarzschild black hole, we have $R = 0$ and $R_{\mu\nu}R^{\mu\nu} = 0$ (in fact $R = 0$ and $R_{\mu\nu} = 0$ everywhere in Schwarzschild spacetime), but $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \to \infty$. Therefore, the approach of limiting curvature is to ensure the finiteness of a finite number of curvature-invariant functions and arrange for the spacetime to approach known non-singular spacetimes asymptotically, where all curvature invariants are bounded.

Before setting up the theory of limiting curvature in gravity, let us do a quick aside about limiting curvature in special relativity as an analogy. Let us start from the action of a point particle in classical mechanics:

$$S = \int \mathrm{d}t\,\frac{1}{2}m|\dot{\mathbf{x}}|^2\,. \tag{4.3.35}$$

To go from classical mechanics to special relativity, we need to impose the bound that speeds can never exceed the speed of light. This can be thought of as a 'limiting speed hypothesis'. To do so, let us introduce a Lagrange multiplier φ multiplying the speed squared $|\dot{\mathbf{x}}|^2$ together with a 'potential' $V(\varphi)$, so the action becomes

$$S = m\int \mathrm{d}t\,\left(\frac{1}{2}|\dot{\mathbf{x}}|^2 + \varphi|\dot{\mathbf{x}}|^2 - V(\varphi)\right)\,. \tag{4.3.36}$$

Varying the action with respect to φ gives the constraint equation

$$|\dot{\mathbf{x}}|^2 = \frac{\mathrm{d}V}{\mathrm{d}\varphi}\,. \tag{4.3.37}$$

We see that if the potential $V(\varphi)$ is chosen appropriately, i.e., if $\mathrm{d}V/\mathrm{d}\varphi \leq 1\ \forall\varphi$, then we realize the limiting speed hypothesis, i.e., $v \equiv |\dot{\mathbf{x}}|$ is always less than or equal to the speed of light ($v \leq c = 1$). For example, we could choose

$$V(\varphi) = \frac{2\varphi^2}{1+2\varphi} \implies |\dot{\mathbf{x}}|^2 = \frac{\mathrm{d}V}{\mathrm{d}\varphi} = 1 - \frac{1}{(1+2\varphi)^2}\,, \tag{4.3.38}$$

and indeed, it follows that $|\dot{\mathbf{x}}| \leq 1\ \forall\varphi \in \mathbb{R}$. Moreover, if we solve the above result for φ in terms of $|\dot{\mathbf{x}}|^2$ and substitute this back into the action (4.3.36), we obtain

$$S = m \int \mathrm{d}t\, \sqrt{1 - |\dot{\mathbf{x}}|^2} \tag{4.3.39}$$

up to an irrelevant constant, which is exactly the action of a point particle in special relativity (see, e.g., Ref. [415]).

A similar approach can be applied in gravity. We start with the Hilbert-Einstein action and add to it a (finite) number of Lagrangian multipliers,

$$S = \frac{M_{\mathrm{Pl}}^2}{2} \int \mathrm{d}^4x\, \sqrt{-g} \left[R + \sum_{i=1}^{n} \chi_i I_i - V(\chi_1, \chi_2, ..., \chi_n) \right] . \tag{4.3.40}$$

Thus, the theory has n dimensionless scalar field Lagrange multipliers χ_i with potential $V(\chi_1, \chi_2, ..., \chi_n)$, and the I_i's are curvature-invariant functions (with dimensions of mass squared in this convention), i.e., any scalar polynomial constructed from the Riemann curvature tensor $R^{\mu}{}_{\nu\alpha\beta}$, contractions thereof (with the metric tensor $g_{\mu\nu}$), and covariant derivatives thereof. Some examples were given in Eqs. (4.1.5) and (4.3.34). From the variation of the action with respect to χ_i, we obtain the following set of constraint equations[6]:

$$I_i = \frac{\partial V}{\partial \chi_i}\,. \tag{4.3.41}$$

Manifestly, if we solve the above constraint equations for the χ_i's as

$$\chi_i = \chi_i(I_1, I_2, ..., I_n)\,, \tag{4.3.42}$$

[6]To avoid confusion, let us clarify that i, j, ..., are not space indices here, but just labels.

we can write the theory as a pure higher-order tensor theory of modified gravity,

$$S = \frac{M_{\rm Pl}^2}{2} \int \mathrm{d}^4x \sqrt{-g} \left[R + F(I_1, I_2, ..., I_n) \right] , \tag{4.3.43}$$

where F stands as the Legendre transformation:

$$F(I_1, I_2, ..., I_n) = \sum_{i=1}^{n} \chi_i(I_1, I_2, ..., I_n) I_i - V(\chi_1(I_1, ..., I_n), ..., \chi_n(I_1, ..., I_n)) \, . \tag{4.3.44}$$

As an example, with only one Lagrange multiplier ($n = 1$) and $I_1 = R$, we obtain $F(R)$ gravity, where the exact functional form depends on the potential $V(\chi_1)$.

As in the analogy with special relativity, it is clear from the constraint equation (4.3.41) that the idea is then to have $|V_{,\chi_i}| \leq \ell_f^{-2}$ so that the curvature-invariant functions $|I_i|$ are also bounded from above. We further require the solution to approach a well-known non-singular spacetime when the I_i's take their limiting value (e.g., ℓ_f^{-2}) as $|\chi_i| \to \infty$. As an example, let us consider the case of two Lagrange multipliers ($n = 2$), so the constraints are

$$I_1 = \frac{\partial V}{\partial \chi_1} \qquad \text{and} \qquad I_2 = \frac{\partial V}{\partial \chi_2} \, . \tag{4.3.45}$$

The requirement on $V(\chi_1, \chi_2)$ is then $|V_{,\chi_1}| < \infty$ and $|V_{,\chi_2}| < \infty$. Moreover, let us consider the case where we force the spacetime to approach de Sitter asymptotically (i.e., in FLRW, $H \to$ constant and $\dot{H} \to 0$), which is known to be non-singular (in the global patch). Therefore, it would be useful to construct I_1 and I_2 such that in an FLRW background

$$\bar{I}_1 \propto H^2 \qquad \text{and} \qquad \bar{I}_2 \propto \dot{H} \, , \tag{4.3.46}$$

whence we impose

$$\frac{\partial V}{\partial \chi_1} \to \text{constant} \qquad \text{and} \qquad \frac{\partial V}{\partial \chi_2} \to 0 \qquad \text{as } \chi_1, \chi_2 \to \pm\infty \, . \tag{4.3.47}$$

This was the approach explored in Refs. [143, 516] (see also Chapter 10), where the theory was confirmed to yield non-singular background cosmologies with either asymptotically de Sitter spacetime or Minkowski spacetime (the sub-case where $H \to 0$).

To end this section, let us mention that the question that naturally arises is whether such non-singular constructions have well behaved perturbations. This is the topic of Chapter 10, where we explore the (tensor, vector and scalar) cosmological perturbations. In particular, we quantify when the resulting equations of motion are at most second order in derivatives and when the theory is (in)stable with respect to ghost and gradient instabilities.

Part II

Publications

Chapter 5

Evolution of cosmological perturbations and the production of non-Gaussianities through a nonsingular bounce: Indications for a no-go theorem in single field matter bounce cosmologies

5.1 Introduction

As was realized in [286, 618], there is a duality between the evolution of curvature fluctuations in an exponentially expanding universe and in a contracting universe with the equation of state of matter. In both cases, curvature fluctuations which originate as quantum vacuum perturbations on sub-Hubble scales acquire a scale-invariant spectrum at later times on super-Hubble scales. The observed small red tilt of the spectrum of curvature perturbations which has now been confirmed by observations (see e.g. [7, 12]) can be obtained in an expanding universe by a slow decrease of the Hubble constant during the period of inflation [517], whereas in a matter-dominated phase of contraction a small cosmological constant (with magnitude comparable to what is needed to explain today's dark energy) yields the same tilt [180] (see alternatively [341]). To avoid reaching a singularity at the end of the contracting phase, it is necessary to either modify gravity or consider matter violating the null energy condition (NEC). Then it is possible to obtain nonsingular bouncing cosmologies which have the potential to yield an explanation for the structures in the universe which we now observe. This scenario of structure formation alternative to inflation is called the "matter bounce" scenario (see e.g. [128, 132] for reviews).

Examples of modified gravity models which yield bouncing cosmologies include the "non-singular Universe" construction of [143, 516], nonlocal gravity actions like [81], or Hořava-Lifshitz gravity [101]. It is in general very hard to study the evolution of fluctuations in these models. We will hence focus on models in which the bounce is obtained from the matter sector. One method of obtaining a nonsingular bounce with a single scalar field involves the formation of a ghost condensate during the bounce phase (see [155, 228, 267, 455, 546, 556] for initial developments). A general problem for bouncing cosmologies is the Belinsky-Khalatnikov-Lifshitz (BKL) instability [67], the fact that the energy density in the form of anisotropies will explode and destroy the homogeneous bounce [167]. This problem can be "solved" by endowing the scalar matter field with a negative potential which leads to an Ekpyrotic phase of contraction before the bounce [172, 175, 561] and hence can mitigate the anisotropy problem [273] [1].

In the matter bounce scenario, primordial quantum fluctuations exit the Hubble horizon while the universe is in a matter-dominated contracting phase and the resulting power spectrum of curvature perturbations is scale-invariant [286, 618]. On the other hand, the gravitational wave mode obeys the same equation of motion on super-Hubble scales as the curvature perturbations (considering the canonical variables in each case). Hence, before the bounce phase the tensor-to-scalar ratio r would be of order unity. Thus, if the perturbations passed through the nonsingular bounce unchanged, it would imply that curvature perturbations and primordial gravitational waves would have the same amplitude after the bounce. In terms of the tensor-to-scalar ratio, it would mean that $r \sim \mathcal{O}(1)$, well above current observational bounds [7, 10, 12].

Since the curvature fluctuations couple nontrivially to matter during the bounce phase, whereas the tensor perturbations are determined simply by the evolution of the scale fac-

[1] Such a negative potential may arise from the standard model Higgs field since, based on the recent Higgs and top quark mass measurements, the standard model Higgs develops an instability at large field values (in the absence of new physics) [136].

tor $a(t)$, one may expect that the curvature perturbations would be enhanced relative to the tensor modes during the bounce. In fact, early calculations indicated that curvature perturbations grew exponentially during the bounce phase, hence suppressing the tensor-to-scalar ratio [172, 178]. A more recent study [54] numerically explored the evolution of scalar fluctuations through a nonsingular bounce model similar to the one studied in [172] and found no enhancement of curvature perturbations through the bounce. In light of the relevance of a possible enhancement of the curvature fluctuations for the predicted value of the tensor-to-scalar ratio, the growth of curvature fluctuations during a nonsingular bounce needs to be reconsidered. This is what we aim to do in this paper.

The second goal of this paper is to carefully track the evolution of the three-point function (bispectrum) of curvature perturbations through the bounce. In earlier work [181] it was shown that the bispectrum of curvature fluctuations before the bounce phase has an amplitude of the order $f_{\rm NL} \sim \mathcal{O}(1)$ with a specific shape. As we argued above, if the perturbations were to pass through the nonsingular bounce unchanged, it would imply a large tensor-to-scalar ratio in excess of the observational bounds. On the other hand, if curvature perturbations were to experience a nontrivial growth through the bounce, one should expect additional nonzero contributions to the bispectrum coming from the bounce phase, and there would then be the danger that the final amplitude of the bispectrum exceeds the observational upper bounds from [9, 13]. Thus, a potential conflict looms: either the tensor-to-scalar ratio is too large, or else the non-Gaussianities exceed observational bounds. This problem has indeed already been found in a model of a nonsingular bouncing cosmology in which a nonvanishing positive spatial curvature is responsible for the bounce [302, 303]. We will study this issue in the context of the more realistic models in which the nonsingular bounce is generated by the matter sector. In particular, we will explore the question in the context of a ghost-condensate bounce.

We will indeed demonstrate that, at least in our model, the evolution of the curvature perturbations in the bounce phase connects the value of the tensor-to-scalar ratio with the amplitude of non-Gaussianities. The suppression of the tensor-to-scalar ratio to restore compatibility with the observational bounds requires an enhancement of the curvature fluctuations during the bounce phase. Such an enhancement will increase the magnitude of the non-Gaussianities to a level inconsistent with the observational bounds on the amplitude of the bispectrum. Based on our result we conjecture that there exists a “no-go” theorem in single field nonsingular matter bounce cosmologies which relates the tensor-to-scalar ratio and non-Gaussianities, preventing these models to satisfy the current observational bounds. A tensor-to-scalar ratio below current observational bounds would imply a too large amplitude of non-Gaussianities, whereas non-Gaussianities of order $f_{\rm NL} \sim \mathcal{O}(1)$ would imply a too large amplitude of the primordial gravitational wave spectrum. Therefore, a single field nonsingular matter bounce cannot be made consistent with current observations if the primordial perturbations arise from vacuum initial conditions.

Our analysis assumes that both curvature perturbations and gravitational waves originate as quantum vacuum fluctuations in the initial phase of contraction. A model with thermal fluctuations (as obtained for example in the context of string gas cosmology [79, 147]) will easily avoid our "no-go" theorem. As shown in [144–146, 520], we obtain a tensor-to-scalar ratio much smaller than order unity while obtaining non-Gaussianities which are negligible on cosmological scales [199].

The paper is organized as follows. We first start with a short review of cosmological perturbation theory in Sec. 5.2. We then motivate the idea of the no-go theorem proposed in this paper in Sec. 5.3. In Sec. 5.4, we briefly review the general picture of bouncing cosmology in terms of a single scalar field of Galileon type. After that, in Sec. 5.5 we analyze the perturbation equation for primordial curvature perturbations at linear order during the nonsingular bouncing phase. We point out under which conditions there can be an enhancement of their amplitude. Then in Sec. 5.6, we perform a detailed analysis of the bispectrum generated in the bouncing phase of our specific model. We combine the analyses of scalar and tensor perturbations together with non-Gaussianities in Sec. 5.7, and we show how current observational bounds severely constrain the parameter space of the single field bouncing model. The analysis is expected to hold quite generally for single field matter bounce cosmologies. We conclude with a discussion in Sec. 5.8. Throughout this paper, we adopt the mostly minus convention for the metric and define the reduced Planck mass as $M_p^2 \equiv 1/8\pi G_{\rm N}$ where $G_{\rm N}$ is Newton's gravitational constant.

5.2 A brief review of cosmological perturbation theory

Linear perturbations of the metric about a homogeneous and isotropic background space-time can be decomposed into scalar, vector, and tensor modes (see [518] for a review of the theory of cosmological perturbations and [121] for an introductory overview). The scalar modes are those which couple to matter energy density and pressure perturbations. We call these the *cosmological perturbations*. Tensor modes exist in the absence of matter - they correspond to gravitational waves. In the case of matter without anisotropic stress at linear order in the amplitude of the fluctuations, there is only one physical degree of freedom for the scalar fluctuations. For the purpose of computations it is often convenient to work in the conformal Newtonian gauge (coordinate system) in which the perturbed metric for scalar modes reads

$$ds^2 = a^2(\eta)\left([1+2\Phi(\eta,\vec{x})]\,d\eta^2 - [1-2\Phi(\eta,\vec{x})]\,d\vec{x}^2\right) , \qquad (5.2.1)$$

where η denotes conformal time, $a(\eta)$ is the cosmological scale factor, $\vec{x}$ represents comoving spatial coordinates, and Φ denotes the gravitational potential. For tensor modes, the perturbed metric reads

$$ds^2 = a^2(\eta)\left(d\eta^2 - [\delta_{ij} + h_{ij}(\eta,\vec{x})]\,dx^i dx^j\right) , \qquad (5.2.2)$$

where h_{ij} is trace-free and divergenceless.

Let us consider the matter content to be described by a single scalar field of canonical form with Lagrangian density

$$\mathcal{L}_m = \frac{1}{2}M_p^2 g^{\mu\nu}\nabla_\mu\phi\nabla_\mu\phi - V(\phi)\,. \qquad (5.2.3)$$

Note that we take the scalar field to be dimensionless throughout this paper as a convention. Linear perturbations of the scalar field then have the form

$$\phi(\eta,\vec{x}) = \phi_0(\eta) + \delta\phi(\eta,\vec{x})\,, \qquad (5.2.4)$$

where ϕ_0 is the unperturbed homogeneous part of ϕ. In the scalar sector, metric and matter perturbations couple to one another, so it is useful to define a linear combination of these perturbations,

$$\mathcal{R} \equiv \frac{\mathcal{H}}{\phi_0'}\delta\phi + \Phi \ . \qquad (5.2.5)$$

There are two reasons for focusing on this variable. First of all, it gives the curvature fluctuation in comoving coordinates (coordinates in which the matter field is uniform), and is hence the variable we are interested in computing. Second, it is simply related to the Sasaki-Mukhanov [515, 577] variable v in terms of which the action for cosmological perturbations has canonical form. Note that in the above, $\mathcal{H} \equiv a'/a$ is the conformal Hubble parameter and a prime denotes a derivative with respect to conformal time. In fact, the Sasaki-Mukhanov variable is

$$v \equiv z\mathcal{R}\,, \tag{5.2.6}$$

with

$$z = a\frac{\phi_0'}{\mathcal{H}}M_p\,. \tag{5.2.7}$$

The equation of motion that results from expanding the perturbed action for gravity and matter to second order is given by

$$v_k'' + \left(c_s^2k^2 - \frac{z''}{z}\right)v_k = 0\,. \tag{5.2.8}$$

The equation is written in Fourier space, where k represents the comoving wave number of the curvature perturbations, and c_s is the speed of sound which is equal to one for a scalar field with canonical action (5.2.3). Similarly, for tensor modes the Mukhanov variable is

$$\mu \,\equiv\, ah\,, \tag{5.2.9}$$

where h is the amplitude of the polarization tensor h_{ij} (the two polarization states evolve independently at linear order and obey the same equation of motion) and the resulting equation of motion is

$$\mu_k'' + \left(c_s^2k^2 - \frac{a''}{a}\right)\mu_k = 0\,. \tag{5.2.10}$$

Alternatively, without the use of the Mukhanov variables, the equation of motion for curvature and tensor perturbations can be written as

$$\mathcal{R}_k'' + 2\frac{z'}{z}\mathcal{R}_k' + c_s^2k^2\mathcal{R}_k = 0\,, \tag{5.2.11}$$

$$h_k'' + 2\frac{a'}{a}h_k' + c_s^2k^2h_k = 0\,, \tag{5.2.12}$$

respectively.

Finally, let us introduce the scalar perturbation variable

$$\zeta \equiv \Phi + \frac{2}{3}\frac{\Phi' + \mathcal{H}\Phi}{\mathcal{H}(1+w)}\,, \tag{5.2.13}$$

where $w \equiv P/\rho$ is the equation of state parameter (P is the pressure and ρ is the energy density). On super-Hubble scales, i.e. for $k \ll \mathcal{H}$, this variable is equivalent to the curvature perturbation variable $\mathcal{R}_k$ [48]. In other words, $\mathcal{R}_k = \zeta_k$, and thus, throughout the rest of this paper, we will use $\mathcal{R}_k$ and ζ_k interchangeably to denote curvature perturbations on super-Hubble scales.

5.3 Outline of the no-go conjecture

As explained in the introduction, a careful study of the evolution of curvature perturbations and the production of non-Gaussianities during a nonsingular bounce may lead to a "no-go" theorem, the impossibility of obtaining a sufficiently small tensor-to-scalar ratio while maintaining a bispectrum with an amplitude smaller than the current observational bounds. In this section we will provide a qualitative analysis of this problem by giving simple estimates of the tensor-to-scalar ratio and of the amplitude of the bispectrum assuming that the curvature fluctuations undergo some growth through the bounce phase. We first start by setting up the matter bounce formalism.

5.3.1 Fluctuations in the matter bounce

In the matter bounce, primordial quantum fluctuations originate on sub-Hubble scales during a matter-dominated contracting phase and exit the Hubble radius during this phase. The perturbations then remain on super-Hubble scales as the universe contracts and passes through the bounce phase, except for a very small time interval right at the bounce point (at which time the Hubble radius goes to infinity). The fluctuations with wavelength of cosmological interest today will then reenter the Hubble radius in the standard radiation or matter-dominated expanding phases. If the bounce is completely symmetric, then fluctuations which exit the Hubble radius in the matter phase of contraction reenter the Hubble radius in the matter phase of expansion. However, we expect the bounce to be asymmetric and entropy to be generated during the bounce. In this case, the radiation phase of expansion is longer than the radiation phase of contraction.

To understand the evolution of quantum fluctuations in a contracting universe, one needs to determine the form of the variable z and then solve Eq. (5.2.8). Using the Friedmann equations, the time derivative of the Hubble parameter is given by

$$\dot{H} = -\frac{\dot{\phi}_0^2}{2}\,, \tag{5.3.14}$$

where a dot denotes a derivative with respect to cosmic time, t, and the subscript 0 indicates that we are referring to the background field. Defining the parameter ϵ,

$$\epsilon \equiv -\frac{\dot{H}}{H^2}\,, \tag{5.3.15}$$

and using Eq. (5.3.14), one finds

$$z = a\frac{\dot{\phi}_0}{H}M_p \;=\; a\sqrt{2\epsilon}M_p\,. \tag{5.3.16}$$

It is straightforward to show from the Friedmann equations that

$$\epsilon = \frac{3}{2}(1+w)\,, \tag{5.3.17}$$

so for a matter-dominated contracting universe with $w = 0$, we have $\epsilon = 3/2$. As a consequence, $z = a\sqrt{3}M_p$ and

$$\frac{z''}{z} \;=\; \frac{a''}{a}\,, \tag{5.3.18}$$

and we conclude that the scalar and tensor fluctuations evolve in exactly the same way. This is not true in general since w can vary in time. For example, in the case of inflationary cosmology, we recognize ϵ as the slow-roll parameter and it is time-dependent.

In a matter-dominated contracting universe, the scale factor scales as $a \sim (-t)^{2/3} \sim \eta^2$, and since $c_s^2 = 1$ for a canonical scalar field, the equation for the Sasaki-Mukhanov variable is

$$v_k'' + \left(k^2 - \frac{2}{\eta^2}\right)v_k = 0\,. \tag{5.3.19}$$

On super-Hubble scales, the k^2 term is negligible, and so the solution reads

$$v_k(\eta) \;=\; c_1\eta^2 + c_2\eta^{-1}\,. \tag{5.3.20}$$

Using the fact that $v_k = z\zeta_k$, the first term yields $\zeta_k \sim$ constant, but in a contracting universe, the second term is the dominant solution,

$$\zeta_k \sim \eta^{-3}\,, \tag{5.3.21}$$

which implies that curvature perturbations grow in a contracting universe. In fact, the growth rate is precisely the correct one to convert an initial vacuum spectrum into a scale-invariant one (see e.g. [128] for a review).

5.3.2 Bound from the tensor-to-scalar ratio

The tensor-to-scalar ratio is defined as

$$r \equiv \frac{\mathcal{P}_t(k_*)}{\mathcal{P}_\zeta(k_*)}\,, \tag{5.3.22}$$

where k_* is the pivot scale which is used to parametrize the power spectra for tensor and curvature perturbations. The individual power spectra are defined by [447]

$$\mathcal{P}_t(k) = 2\mathcal{P}_h(k) \equiv 2 \times 16\pi \frac{k^3}{2\pi^2}|h_k|^2 = 16\pi \frac{k^3}{\pi^2}\frac{|\mu_k|^2}{a^2}\,, \tag{5.3.23}$$

$$\mathcal{P}_\zeta(k) \equiv \frac{k^3}{2\pi^2}|\zeta_k|^2 = \frac{k^3}{2\pi^2}\frac{|v_k|^2}{z^2}\,, \tag{5.3.24}$$

respectively. The factor of 2 in the first step of the first line comes from the two polarization states of gravitons and the factor of 16π is a convention reflecting the fact that it is $16\pi M_p h$ which yields the canonical action of a free scalar field in an expanding background [518].

As we found in the previous subsection, $z = a\sqrt{3}M_p$ for the matter bounce, so the scalar power spectrum becomes

$$\mathcal{P}_\zeta(k) = \frac{k^3}{6\pi^2}\frac{|v_k|^2}{a^2 M_p^2}\,, \tag{5.3.25}$$

and furthermore, the tensor-to-scalar ratio becomes

$$r \;=\; 96\pi \left|\frac{\mu_{k_*}}{v_{k_*}}\right|^2 M_p^2\,. \tag{5.3.26}$$

where the factor M_p^2 reflects the fact that we have defined v_k to have dimensions of mass, whereas μ_k is dimensionless.

Since $z''/z = a''/a$ for the matter bounce, the evolution of scalar and tensor modes given by Eqs. (5.2.8) and (5.2.10), respectively, will be identical. In addition, if they originate from the same quantum vacuum, then $v_k(\eta) = M_p\mu_k(\eta)$. Consequently, we find that $r = 96\pi$. If perturbations passed through the bounce unchanged, it would result in $r = 96\pi$ at the beginning of the standard big bang cosmology phase which is three orders of magnitude larger than the current observational upper bound.

To gain some intuition on the effect of passing through the bounce phase, let us assume that curvature perturbations are enhanced by an amount $\Delta\zeta_k$ through the bounce, i.e.

$$\zeta_k(\eta_{B+}) = \zeta_k(\eta_{B-}) + \Delta\zeta_k \,, \tag{5.3.27}$$

where $\eta_{B\pm}$ denote the conformal time before $(-)$ and after $(+)$ the bounce. Then, the tensor-to-scalar ratio measured after the bounce becomes

$$r(\eta_{B+}) = 96\pi \left| \frac{h_{k_*}(\eta_{B+})}{\zeta_{k_*}(\eta_{B-}) + \Delta\zeta_{k_*}} \right|^2 . \tag{5.3.28}$$

Assuming that tensor modes remain constant through the bounce, i.e. $h_k(\eta_{B-}) = h_k(\eta_{B+})$, one finds that

$$\left| 1 + \frac{\Delta\zeta_{k_*}}{\zeta_{k_*}(\eta_{B-})} \right|^2 = \frac{r(\eta_{B-})}{r(\eta_{B+})} . \tag{5.3.29}$$

Taking the value of the tensor-to-scalar ratio before the bounce to be what we found earlier, i.e. $r(\eta_{B-}) = 96\pi$, and demanding that the tensor-to-scalar ratio is sufficiently suppressed after the bounce so that it satisfies the observational bound $r(\eta_{B+}) < 0.12$ (95% CL from [10, 14]), we find that curvature perturbations must be sufficiently enhanced during the bounce phase so that

$$\left| 1 + \frac{\Delta\zeta_{k_*}}{\zeta_{k_*}(\eta_{B-})} \right| \gtrsim 50.1 \,, \tag{5.3.30}$$

or using the triangle inequality,

$$\left| \frac{\Delta\zeta_{k_*}}{\zeta_{k_*}(\eta_{B-})} \right| \gtrsim 49.1 \,. \tag{5.3.31}$$

5.3.3 Bound from the bispectrum

The primordial bispectrum, B_ζ, is defined in terms of the three-point function as

$$\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\rangle \equiv (2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)B_\zeta(k_1,k_2,k_3)\,, \tag{5.3.32}$$

which we can rewrite as

$$\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\rangle = (2\pi)^7\delta^{(3)}\left(\sum_i \vec{k}_i\right)\frac{\mathcal{P}_\zeta^2}{\prod_i k_i^3}\mathcal{A}(k_1,k_2,k_3)\,, \tag{5.3.33}$$

where $k_i = |\vec{k}_i|$ and where the index i runs from 1 to 3. The function $\mathcal{A}(k_1,k_2,k_3)$ is known as the shape function and its amplitude defines the nonlinear parameter $f_{\rm NL}$ via

$$f_{\rm NL}(k_1,k_2,k_3) = \frac{10}{3}\frac{\mathcal{A}(k_1,k_2,k_3)}{\sum_i k_i^3}\,. \tag{5.3.34}$$

Of particular interest is the local form of non-Gaussianities for which one of the three modes exits the Hubble radius much earlier than the other two, i.e. $k_1 \ll k_2 = k_3$. For this case, one can write

$$\zeta(\vec{x}) = \zeta_g(\vec{x}) + \frac{3}{5}f_{\rm NL}^{\rm local}\zeta_g(\vec{x})^2\,, \tag{5.3.35}$$

where ζ_g is the Gaussian part of ζ.

In order to compute $f_{\rm NL}$, one must evaluate the three-point function. To leading order in the interaction coupling constant, the three-point function is related to the interaction Lagrangian, $L_{\rm int}$, via [481]

$$\langle\zeta(t,\vec{k}_1)\zeta(t,\vec{k}_2)\zeta(t,\vec{k}_3)\rangle = i\int_{t_i}^{t} d\tilde{t}\ \langle[\zeta(t,\vec{k}_1)\zeta(t,\vec{k}_2)\zeta(t,\vec{k}_3), L_{\rm int}(\tilde{t})]\rangle\,, \tag{5.3.36}$$

where the square brackets denote the commutator and where t_i denotes the initial time before which there is no non-Gaussianity. The interaction Lagrangian is obtained by evaluating the action up to third order in perturbation theory

$$L_{\rm int}(t) = \int d^3\vec{x}\ \mathcal{L}_3(t,\vec{x})\,, \tag{5.3.37}$$

and for a canonical scalar field, the Lagrangian density for ζ to cubic order is given by [481]

$$\frac{\mathcal{L}_3}{M_p^2} = \left(\epsilon^2 - \frac{\epsilon^3}{2}\right) a^3\zeta\dot{\zeta}^2 + \epsilon^2 a\zeta(\partial\zeta)^2 - 2\epsilon^2 a^3\dot{\zeta}(\partial\zeta)(\partial\chi) + \frac{\epsilon^3}{2}a^3\zeta(\partial_i\partial_j\chi)^2 + f(\zeta)\frac{\delta\mathcal{L}_2}{\delta\zeta}\,, \tag{5.3.38}$$

$$f(\zeta) = \frac{1}{4(aH)^2}(\partial\zeta)^2 - \frac{1}{4(aH)^2}\partial^{-2}\partial_i\partial_j(\partial_i\zeta\partial_j\zeta) - \frac{1}{H}\zeta\dot{\zeta} - \frac{\epsilon}{2H}\partial_i\zeta\partial_i\chi + \frac{\epsilon}{2H}\partial^{-2}\partial_i\partial_j(\partial_i\chi\partial_j\zeta)\,, \tag{5.3.39}$$

where ∂^{-2} is the inverse Laplacian and where we define $\chi \equiv \partial^{-2}\dot{\zeta}$. Also, the equation of motion for ζ coming from the second order perturbed Lagrangian density $\mathcal{L}_2$ is given by

$$\frac{\delta\mathcal{L}_2}{\delta\zeta} = \frac{\partial}{\partial t}(az^2\dot{\zeta}) - \frac{c_s^2z^2}{a}\partial^2\zeta\,. \tag{5.3.40}$$

As we saw in Sec. 5.3.1, curvature perturbations grow on super-Hubble scales during the matter-dominated contracting phase until the bounce phase. While on super-Hubble scales the spatial gradient terms are negligible, i.e. $\partial_i\zeta$, $\partial_i\chi \simeq 0$, the growth in ζ implies that the interaction Lagrangian is dominated by

$$\frac{\mathcal{L}_3}{M_p^2} \simeq \left(\epsilon^2 - \frac{\epsilon^3}{2}\right) a^3\zeta\dot{\zeta}^2 - \frac{1}{H}\zeta\dot{\zeta}\frac{\partial}{\partial t}(az^2\dot{\zeta})\,. \tag{5.3.41}$$

As was first shown in [181], the production of non-Gaussianities on a comoving scale k is dominated by the period between when the scale crosses the Hubble radius in the phase of matter contraction until the onset of the bounce phase, and the resulting non-Gaussianities are of order $f_{\rm NL} \sim \mathcal{O}(1)$. For example, for the local shape, the authors of [181] found $f_{\rm NL}^{\rm local} = -35/16$.

Following what was done in the previous subsection, let us now assume that curvature perturbations grow during the bounce phase. For simplicity, let us assume that they grow linearly in time with constant rate

$$\dot{\zeta} = \frac{\Delta\zeta}{\Delta t_B}\,, \tag{5.3.42}$$

where the duration of the bounce is given by $\Delta t_B \equiv t_{B+} - t_{B-}$. Then, in the limit $k \to 0$ on super-Hubble scales, the contribution to the three-point function coming from the bounce phase is schematically given by

$$\langle\zeta(t_{B+})^3\rangle_{\text{bounce}} \sim \frac{\zeta(t_{B+})^3}{M_p}\left(\frac{\Delta\zeta}{\Delta t_B}\right)^2\int_{t_{B-}}^{t_{B+}} dt\ a(t)^3\left[\epsilon(t)^2-\frac{\epsilon(t)^3}{2}\right]\left[\zeta(t_{B-})+\frac{\Delta\zeta}{\Delta t_B}(t-t_{B-})\right]\,, \tag{5.3.43}$$

and one expects that the dominant contribution to f_{NL} that results from evaluating the three-point function would scale as

$$f_{\text{NL}} \sim \frac{(\Delta\zeta)^2}{\Delta t_B}M_p^2\,, \tag{5.3.44}$$

plus terms of order $\Delta\zeta^1$ which would be subdominant for a large amplification $\Delta\zeta$.

We already see that a growth in the curvature perturbations during the bounce, $\Delta\zeta$, would enhance f_{NL}. From the previous subsection, we expect $\Delta\zeta$ to have a lower bound to match current observational bounds on r, and thus, we expect to find a lower bound on the amount of non-Gaussianities that are produced during the bounce phase. However, we cannot determine whether this contribution will be significant to $f_{\text{NL}} \sim \mathcal{O}(1)$ and whether the resulting lower bound will exceed current observational bounds without going into the details of the calculation.

5.3.4 The no-go theorem

Now, let us state our conjecture.

Conjecture 5.1. *For quantum fluctuations originating from a matter-dominated contracting universe, an upper bound on the tensor-to-scalar ratio (r) is equivalent to a lower bound on the amplification of curvature perturbations ($\Delta\zeta/\zeta$) which in turn is equivalent to a lower bound on the amount of primordial non-Gaussianities (f_{NL}). Furthermore, if the initial quantum vacuum is a canonical Bunch-Davies vacuum with $c_s = 1$, if the nonsingular bounce phase is due to a single NEC violating scalar field, and if general relativity holds at all energy scales, then satisfying the current observational upper bound on the tensor-to-scalar ratio cannot be done without contradicting the current observational upper bounds on f_{NL} (and vice-versa).*

In the rest of this paper, we will give an example of realization of this conjecture.

5.4 A brief review of single field bouncing cosmology

In the context of Einstein gravity, matter which violates the null energy condition must be introduced in order to obtain a cosmological bounce. A simple toy model is quintom cosmology, i.e. a model in which a scalar field with opposite sign in the action compared to a usual scalar field is introduced, and it is arranged that this field comes to dominate late in the contracting phase, thus yielding a nonsingular bounce [176]. A specific realization of this can be obtained in the Lee-Wick theory [177]. These models, however, suffer from a ghost instability [214]. To avoid this instability (at least at the perturbative level) one can make use of the ghost condensation mechanism [455] or the Galileon construction [267, 556].[2] These mechanisms involve a modified kinetic term in the action.

As mentioned in the introduction, bouncing models typically also suffer from the anisotropy problem, and to mitigate this problem, one can build into the scenario an Ekpyrotic phase of contraction which occurs at some point after the matter phase of contraction. Specifically, one can use a single scalar field with a kinetic term designed to yield a nonsingular bounce, and a potential energy function with a negative potential over some range of field values which is designed to yield Ekpyrotic contraction [172]. In this approach, a second scalar field with canonical kinetic term and with quadratic potential can be used to represent the regular matter of the Universe [175]. In this paper we will not consider the role which this second scalar field may play (for some ideas see [168]) but only consider the field ϕ which generates the Ekpyrotic contraction and the nonsingular bounce.

Throughout this paper, we assume only Einstein gravity plus matter. Thus, the action is given by

$$S = \int d^4x \, \sqrt{-g} \left(-\frac{M_p^2}{2} R + \mathcal{L}_m \right) , \tag{5.4.45}$$

where g is the determinant of the metric, R is the Ricci scalar, and $\mathcal{L}_m$ is the matter Lagrangian. We assume that the matter content is dominated by only one scalar field (ϕ) before reheating (the energy density of matter created via reheating becomes only important after the bounce phase – see [561]). Thus, for the dynamics of the matter-dominated contracting era and the bounce phase to be described by second order equations of motion, we consider

[2] Alternative possibilities of alleviating this instability may be achieved by considering various modified gravity implementations such as models of extended $F(R)$ gravity [40, 526], modified Gauss-Bonnet gravity [41], and torsion gravity scenarios [29, 170].

a Lagrangian of the most general form [522]

$$\mathcal{L}_m \;=\; K(\phi, X) + G(\phi, X)\Box\phi + \mathcal{L}_4 + \mathcal{L}_5\,, \tag{5.4.46}$$

where the kinetic variable X is defined as

$$X \equiv \frac{1}{2} g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\,, \tag{5.4.47}$$

and where the d'Alembertian operator is defined as

$$\Box\phi \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu\phi\,. \tag{5.4.48}$$

We do not write down the explicit form that $\mathcal{L}_4$ and $\mathcal{L}_5$ can take here, but the key point is that they involve higher order derivatives. If we assume that the energy scale at which the bounce occurs is low enough so that higher order derivative terms in the Lagrangian are negligible, then we can assume that $\mathcal{L}_4$, $\mathcal{L}_5 \approx 0$.

For the bounce to be nonsingular, the above Lagrangian must violate the null energy condition (NEC) at high energies. To do so, we assume the first term of the Lagrangian to have the form

$$K(\phi, X) = M_p^2[1 - g(\phi)]X + \beta X^2 - V(\phi)\,, \tag{5.4.49}$$

where β is some positive constant. We see from Eq. (5.4.49) that when $g(\phi) > 1$, the sign of the kinetic term is reversed and a ghost condensate which violates the NEC is formed [155, 228, 267, 455, 556]. For this reason, one typically chooses the function $g(\phi)$ to have the form

$$g(\phi) = \frac{2g_0}{e^{-\sqrt{2/p}\phi} + e^{b_g\sqrt{2/p}\phi}}\,, \tag{5.4.50}$$

where p and b_g are positive constants. As $\phi \to 0$ at the bounce point, $g(\phi) \to g_0$, and the constant g_0 is naturally chosen to be $g_0 > 1$ to allow the NEC violation. We can also see from the form of $g(\phi)$ above that as ϕ goes away from 0 and as the kinetic variable X becomes small outside the bounce phase, $g(\phi)$ rapidly goes to 0 and the Lagrangian recovers its canonical form.

The potential $V(\phi)$ can be chosen in order to obtain an Ekpyrotic phase of contraction. This can be done by means of a potential which is negative for small values of $|\phi|$, but which approaches $V = 0$ exponentially at large positive and negative field values. Specifically, we have chosen the potential

$$V(\phi) = -\frac{2V_0}{e^{-\sqrt{2/q}\phi} + e^{b_V\sqrt{2/q}\phi}}, \tag{5.4.51}$$

where V_0, q, and b_V are positive constants. Without the second term in the denominator, one obtains the potential postulated in the Ekpyrotic scenario [387].

One can then parametrize the background evolution during the bounce phase as follows. The Hubble parameter grows linearly in time, passing through zero at the time $t = t_B$ (the bounce point),

$$H(t) = \Upsilon(t - t_B), \tag{5.4.52}$$

where Υ is a positive constant. The scale factor immediately follows,

$$a(t) = a_B e^{\Upsilon(t-t_B)^2/2}. \tag{5.4.53}$$

Also, the scalar field evolves as

$$\dot{\phi}(t) = \dot{\phi}_B e^{-(t-t_B)^2/T^2}. \tag{5.4.54}$$

Since a_B and t_B can be arbitrarily redefined, we see that the parameters which describe the bounce phase are Υ, $\dot{\phi}_B$, t_{B-} (or t_{B+} assuming a symmetric bounce), and T. First, Υ gives the growth rate of the Hubble parameter. Second, $\dot{\phi}_B$ gives the maximal growth rate of the scalar field. Third, $\Delta t_B/T$ gives the dimensionless duration of the bounce. They can be related to the Lagrangian parameters via (see [172, 175])

$$\dot{\phi}_B \simeq \sqrt{\frac{2(g_0-1)}{3\beta}} M_p, \tag{5.4.55}$$

$$T \simeq \frac{H_{B+}}{\Upsilon}\sqrt{\frac{2}{\ln(\dot{\phi}_B^2/6H_{B+}^2)}}, \tag{5.4.56}$$

where $H_{B+} = \Upsilon(t_{B+} - t_B)$.

Given the model we have discussed and the bounce solution which we have given in parametric form, we will now follow the evolution of the curvature fluctuation variable ζ through the nonsingular bounce phase.

5.5 Evolution of curvature perturbations during the bounce

As we saw in Sec. 5.2, the equation of motion for curvature perturbations [Eq. (5.2.11)] can be written as

$$\mathcal{R}_k'' + \frac{(z^2)'}{z^2}\mathcal{R}_k' + c_s^2 k^2 \mathcal{R}_k = 0\,. \tag{5.5.57}$$

For a noncanonical Lagrangian of the form of Eq. (5.4.46), the variable z and the sound speed are given by [172]

$$z^2 = \frac{2M_p^4 a^2 \dot{\phi}^2 \mathcal{P}}{(2M_p^2 H - G_{,X}\dot{\phi}^3)^2}\,, \tag{5.5.58}$$

$$c_s^2 = \frac{1}{\mathcal{P}}\left[K_{,X} + 4H\dot{\phi}G_{,X} - \frac{G_{,X}^2\dot{\phi}^4}{2M_p^2} - 2G_{,\phi} + G_{,X\phi}\dot{\phi}^2 + (2G_{,X} + G_{,XX}\dot{\phi}^2)\ddot{\phi}\right]\,, \tag{5.5.59}$$

where a comma denotes a partial derivative and where we defined

$$\mathcal{P} \equiv K_{,X} + \dot{\phi}^2 K_{,XX} + \frac{3}{2M_p^2}\dot{\phi}^4 G_{,X}^2 + 6H\dot{\phi}G_{,X} + 3H\dot{\phi}^3 G_{,XX} - 2G_{,\phi} - \dot{\phi}^2 G_{,\phi X}\,. \tag{5.5.60}$$

As explained in Sec. 5.3.1, the perturbation modes that are of cosmological interest today were on super-Hubble scales during the bounce phase (except in the immediate vicinity of the bounce point), and thus we are most interested in the infrared (IR) regime of Eq. (5.5.57). In the limit $k \ll \mathcal{H}$, and recalling that $\mathcal{R}_k$ and ζ_k are equivalent quantities in this limit, the equation that we want to solve is

$$\frac{d\zeta'}{d\eta} + \frac{(z^2)'}{z^2}\zeta' = 0\,, \tag{5.5.61}$$

where we drop the k index when it is clear that we are on super-Hubble scales. It is obvious from the above equation that one solution is the constant mode solution, $\zeta' = 0$, that one expects on super-Hubble scales, e.g. in inflation [49, 138] (see, however, [423]). More generally, the solution to Eq. (5.5.61) can be written as

$$\zeta'(\eta) = \zeta'(\eta_i)\frac{z^2(\eta_i)}{z^2(\eta)}\,, \tag{5.5.62}$$

where η_i denotes the initial time where the initial conditions are set. The evolution of ζ is thus governed by the evolution of z^2, and we notice from the denominator of Eq. (5.5.58) that the evolution of z^2 has different regimes of interest:

$$\text{Regime I}: \ 2M_p^2|H(t)| \gg |G_{,X}(t)|\dot{\phi}^3(t)\,, \tag{5.5.63}$$

$$\text{Regime II}: \ 2M_p^2|H(t)| \ll |G_{,X}(t)|\dot{\phi}^3(t)\,, \tag{5.5.64}$$

$$\text{Regime III}: \ 2M_p^2 H(t) \approx G_{,X}(t)\dot{\phi}^3(t)\,. \tag{5.5.65}$$

We represent these different regimes in Fig. 5.1, and we explore the consequences of each regime in the following subsections.

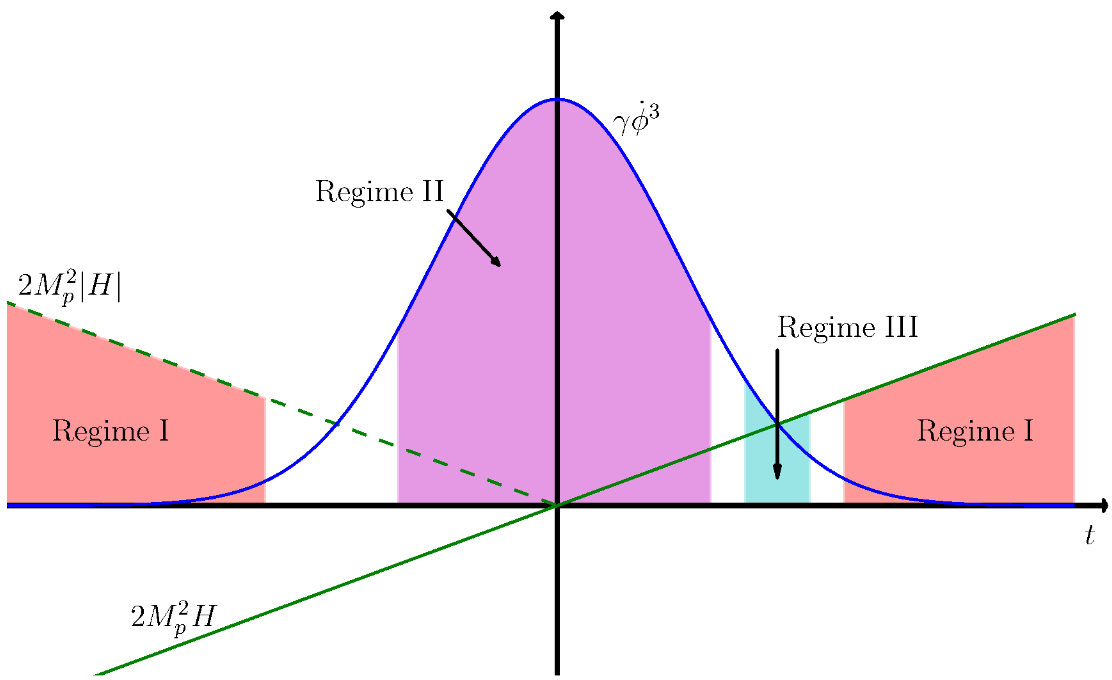

Figure 5.1 Sketch of the different regimes in the bounce phase (not to scale). The horizontal axis represents physical time. The green solid curve shows $2M_pH(t)$ and the dashed version depicts its absolute value. The bell-shaped blue curve represent $|G_{,X}(t)|\dot{\phi}^3(t)$, where we take $G_{,X} = \gamma$ to be a positive constant for simplicity. Regimes I, II, and III, defined by Eqs. (5.5.63), (5.5.64), and (5.5.65), are depicted by the pink, purple, and cyan regions, respectively.

5.5.1 Evolution in Regime I

When Eq. (5.5.63) is valid, the expression for z^2 reduces to

$$z^2 \simeq \frac{M_p^2 a^2 \dot{\phi}^2}{H^2} \left(\frac{1 - g(\phi)}{2} \right) . \tag{5.5.66}$$

Since the bounce phase is defined by $g(\phi) > 1$ and since z^2 must be positive to avoid ghost instabilities, the model parameters must be chosen such that this regime does not occur during the bounce phase. Outside the bouncing phase, the equation of motion in Regime 1 reduces to the standard one.

5.5.2 Evolution in Regime II

As the bounce point approaches, $H(t)$ goes to zero and we can expect Eq. (5.5.64) to be valid. To explore this regime, let us simplify the treatment by setting

$$G(\phi, X) = \gamma X \tag{5.5.67}$$

for some positive constant γ, so the regime becomes

$$2M_p^2 |H(t)| \ll \gamma \dot{\phi}^3(t) . \tag{5.5.68}$$

Using the parametrizations introduced in the previous section, this condition can be rewritten as

$$|\Delta t| e^{3(\Delta t)^2/T^2} \ll \frac{\gamma \dot{\phi}_B^3}{2M_p^2 \Upsilon} , \tag{5.5.69}$$

where we defined $\Delta t \equiv t - t_B$. Since $\Delta t_B/T$ determines the dimensionless duration of the bounce, remaining close to the bounce is equivalent to demanding that $|\Delta t|/T \ll 1$. In particular, if we demand that

$$|\Delta t| \ll \min \left\{ \frac{T}{\sqrt{3}}, \frac{\gamma \dot{\phi}_B^3}{2M_p^2 \Upsilon} \right\} , \tag{5.5.70}$$

then it is ensured that we are in the regime set by Eq. (5.5.68). Thus, the expression for z^2 given in Eq. (5.5.58) reduces to

$$z^2(t) \simeq \frac{3\beta M_p^4}{\gamma^2}\frac{a^2(t)}{\dot{\phi}^2(t)} \tag{5.5.71}$$

in this regime. In fact, there exists a time interval, which we define as $[t_{\mathrm{amp}-}, t_{\mathrm{amp}+}]$ with $t_{\mathrm{amp}\pm} \equiv t_B \pm \Delta t_{\mathrm{amp}}$, where the above approximation for $z^2(t)$ is certainly valid. We note that this expression is everywhere finite in that interval, so the solution to Eq. (5.5.61) can be directly written as

$$\dot{\zeta}(t) = \dot{\zeta}(t_i)\frac{a(t_i)z^2(t_i)}{a(t)z^2(t)}\,, \tag{5.5.72}$$

where the initial condition must be taken in the interval, i.e. $t_i \in [t_{\mathrm{amp}-}, t_{\mathrm{amp}+}]$, so logically we take $t_i = t_{\mathrm{amp}-}$. Also, the solution will only be valid up to $t_{\mathrm{amp}+}$. Inserting Eq. (5.5.71) and using the parametrizations introduced in the previous section, one finds

$$\begin{aligned}
\zeta(t) &\simeq \zeta(t_{\mathrm{amp}-}) + \dot{\zeta}(t_{\mathrm{amp}-})\int_{t_{\mathrm{amp}-}}^{t} d\tilde{t}\,\left(\frac{a(t_{\mathrm{amp}-})}{a(\tilde{t})}\right)^3\left(\frac{\dot{\phi}(\tilde{t})}{\dot{\phi}(t_{\mathrm{amp}-})}\right)^2 \\
&= \zeta(t_{\mathrm{amp}-}) + \dot{\zeta}(t_{\mathrm{amp}-})\left(\frac{a(t_{\mathrm{amp}-})}{a_B}\right)^3\left(\frac{\dot{\phi}_B}{\dot{\phi}(t_{\mathrm{amp}-})}\right)^2 \\
&\quad\times\int_{t_{\mathrm{amp}-}}^{t} d\tilde{t}\,\exp\left[-\left(\frac{2}{T^2}+\frac{3}{2}\Upsilon\right)(\tilde{t}-t_B)^2\right] \\
&= \zeta(t_{\mathrm{amp}-}) + \dot{\zeta}(t_{\mathrm{amp}-})\left(\frac{a(t_{\mathrm{amp}-})}{a_B}\right)^3\left(\frac{\dot{\phi}_B}{\dot{\phi}(t_{\mathrm{amp}-})}\right)^2 T\sqrt{\frac{\pi}{8+6T^2\Upsilon}} \\
&\quad\times\left[\mathrm{erf}\left(\frac{t-t_B}{T}\sqrt{2+\frac{3T^2\Upsilon}{2}}\right) - \mathrm{erf}\left(\frac{t_{\mathrm{amp}-}-t_B}{T}\sqrt{2+\frac{3T^2\Upsilon}{2}}\right)\right]\,.
\end{aligned} \tag{5.5.73}$$

Close to the bounce point, the scale factor remains nearly constant, so $a(t) \simeq a_B$. This implies that $\Upsilon(\Delta t)^2 \ll 2$, or in other words, that $H(t)\Delta t \ll \mathcal{O}(1)$. We will assume this to be valid throughout the rest of this paper whenever we are in the time interval $|\Delta t| \leq \Delta t_{\mathrm{amp}}$.

Therefore, the solution for $\zeta(t)$ reduces to

$$\zeta(t) \simeq \zeta(t_{\mathrm{amp}-}) + \dot{\zeta}(t_{\mathrm{amp}-}) \left(\frac{\dot{\phi}_B}{\dot{\phi}(t_{\mathrm{amp}-})} \right)^2 \frac{T\sqrt{2\pi}}{4} \left[\mathrm{erf}\left(\frac{t - t_B}{T}\sqrt{2} \right) - \mathrm{erf}\left(\frac{t_{\mathrm{amp}-} - t_B}{T}\sqrt{2} \right) \right] . \tag{5.5.74}$$

From the above solution, we see the constant mode and the growing mode. Whether the constant or the growing mode is dominant depends on many factors. For instance, the duration of this regime and the growth rate will play a crucial role. From the properties of the error function, we note that the growing mode grows at most linearly in time. Furthermore, the growth rate is maximal at the bounce point t_B and it is given by $\dot{\zeta}_{\max} \simeq \dot{\zeta}(t_{\mathrm{amp}-})[\dot{\phi}_B/\dot{\phi}(t_{\mathrm{amp}-})]^2$.

5.5.3 Evolution in Regime III

One can notice from Eq. (5.5.62) that if $z^2 \to \infty$, then $\zeta' \to 0$, and curvature perturbations remain constant on super-Hubble scales. One can see from Eq. (5.5.58) that this happens at some physical time t_s (or η_s in conformal time) when

$$2M_p^2 H(t_s) = G_{,X}(t_s)\dot{\phi}^3(t_s) . \tag{5.5.75}$$

At this point, the equation of motion for the curvature perturbations becomes singular, and furthermore, the Mukhanov variable $v_k = z\mathcal{R}_k$ diverges. For this reason, the evolution of the curvature perturbations has been explored in another gauge, the harmonic gauge (first introduced in the context of cosmological perturbation theory in [630]), where this singularity may disappear. Using the harmonic gauge, it has been shown in [54] that at η_s,

$$\left. \frac{d\mathcal{R}_k}{d\eta} \right|_{\eta=\eta_s} = 0 \tag{5.5.76}$$

for all k modes. Carefully dealing with the singular equation of motion in the conformal Newtonian gauge, one can find that in the IR limit, the solution in conformal time close to the singular time η_s is (see Appendix 5.9)

$$\zeta(\eta) = \zeta(\eta_i) + \zeta'(\eta_i) \left(\frac{(\eta - \eta_s)^3 + (\eta_s - \eta_i)^3}{3(\eta_s - \eta_i)^2} \right) . \tag{5.5.77}$$

This indicates that perturbations can grow before the singular point (coming from Regime II), and that they could grow after the singular point (toward Regime I), but we saw that this regime is not present in the bounce phase (Sec. 5.5.1), so the bounce phase will end shortly after the singular point η_s.

5.5.4 Discussion

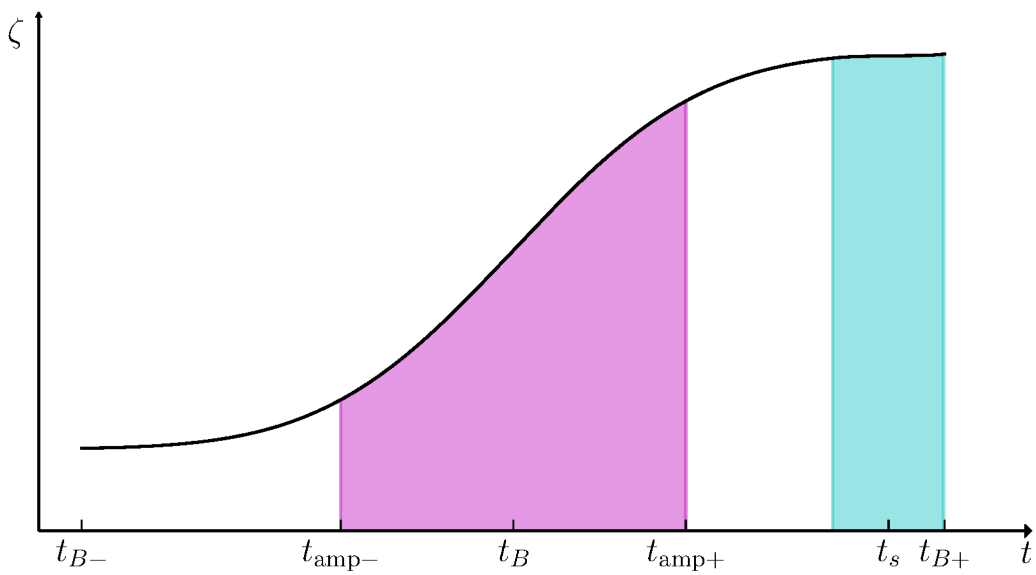

Figure 5.2 Sketch of the evolution of curvature perturbation ζ on super-Hubble scales as a function of physical time t (not to scale). The beginning of the bounce phase, the bounce point, and the end of the bounce phase are denoted by t_{B-}, t_B, and t_{B+}, respectively. We defined $t_{\text{amp}\pm} \equiv t_B \pm \Delta t_{\text{amp}}$, and t_s is the time at which $z^2 \to \infty$. The purple region corresponds to Regime II of Fig. 5.1, where ζ grows at most linearly. The cyan region corresponds to Regime III of Fig. 5.1, where ζ is almost constant.

Let us summarize the evolution of curvature perturbations on super-Hubble scales through the bounce phase. Figure 5.2 is a sketch of the evolution of ζ according to the results found above. If ζ enters the bounce phase with a nonvanishing time derivative[3], then we find that curvature perturbations can grow at most linearly in some time interval $[t_{\text{amp}-}, t_{\text{amp}+}]$ and that the growth is maximal at the bounce point. This happens in what we call Regime II. We highlight this regime in purple in Fig. 5.2. Regime III follows Regime II at which point

[3] If ζ enters the bounce phase with a vanishingly small time derivative, then curvature perturbations will remain constant throughout and exit the bounce phase unaffected.

z^2 blows up and becomes infinite at some time t_s. At this point, curvature perturbations become constant, and the bounce phase ends shortly after. We highlight this regime in cyan in Fig. 5.2.

In the end, the amplification that ζ receives is dominated by the growth during the interval $[t_{\mathrm{amp}-}, t_{\mathrm{amp}+}]$. Between the beginning of the bounce phase and the beginning of the amplification phase, we expect little growth of the curvature perturbations, and so, the initial time derivative of ζ at the beginning of the amplification phase should be of the same order as the time derivative of ζ at the beginning of the bounce phase, which we expect to be small. In fact, in the Ekpyrotic phase of contraction (where $w \gg 1$) which precedes the bounce phase, the dominant mode of ζ is constant in time while the second mode is decaying (as shown in Appendix 5.10). Hence, the amplitude of ζ at the end of the period of Ekpyrotic contraction is the same as the amplitude at the end of the matter phase of contraction (assuming for a moment that there is no intermediate radiation phase). Consequently, this could lead to a suppression of $\dot{\zeta}(t_{B-})$, and hence to a suppression of the growth of ζ in the bounce phase since, as we argued, $\dot{\zeta}(t_{B-}) \simeq \dot{\zeta}(t_{\mathrm{amp}-})$.

The reason why we can match ζ and $\dot{\zeta}$ at the end of the Ekpyrotic phase of contraction with the beginning of the bounce phase comes from the matching conditions of cosmological perturbations [177, 254, 358]. These conditions impose that the gravitational potential $\Phi_k(\eta)$ and the modified curvature perturbation variable $\hat{\zeta}_k(\eta)$ are continuous across any transition (e.g. from the Ekpyrotic phase of contraction to the bounce phase). The variable $\hat{\zeta}_k$ is defined as [177]

$$\hat{\zeta}_k \equiv \zeta_k + \frac{1}{3} c_s^2 \left(\frac{k}{\mathcal{H}}\right)^2 \Phi_k \left(1 - \frac{\mathcal{H}'}{\mathcal{H}^2}\right)^{-1} . \tag{5.5.78}$$

On super-Hubble scales ($k \ll \mathcal{H}$), we note that the second term of the above expression is suppressed, so $\hat{\zeta}_k \simeq \zeta_k$. Thus, ζ_k must also be continuous across a transition. That is why the values of ζ_k and $\dot{\zeta}_k$ at the end of the Ekpyrotic phase of contraction are taken as the initial conditions of the bounce phase.

At this point, we note that the maximal growth rate for ζ is given by

$$\dot{\zeta}_{\max} \simeq \dot{\zeta}(t_{B-}) \left(\frac{\dot{\phi}_B}{\dot{\phi}(t_{\mathrm{amp}-})}\right)^2 , \tag{5.5.79}$$

and that ζ grows at most linearly in time. Therefore, one can say that

$$\zeta(t_{\rm amp+}) - \zeta(t_{\rm amp-}) \lesssim \dot{\zeta}(t_{B-}) \left(\frac{\dot{\phi}_B}{\dot{\phi}(t_{\rm amp-})} \right)^2 (t_{\rm amp+} - t_{\rm amp-}) \, . \tag{5.5.80}$$

Furthermore, since ζ receives essentially no amplification outside the interval $[t_{\rm amp-}, t_{\rm amp+}]$, we can place an upper bound on the total growth that curvature perturbations on super-Hubble scales receive from the bounce phase,

$$\frac{\Delta\zeta}{\zeta(t_{B-})} \equiv \frac{\zeta(t_{B+}) - \zeta(t_{B-})}{\zeta(t_{B-})} \lesssim \frac{\dot{\zeta}(t_{B-})}{\zeta(t_{B-})} \left(\frac{\dot{\phi}_B}{\dot{\phi}(t_{\rm amp-})} \right)^2 2\Delta t_{\rm amp} \, , \tag{5.5.81}$$

where we divide the growth $\Delta\zeta$ by the initial size of ζ before the bounce to get a dimensionless quantity.

5.5.5 Comparison with tensor modes

We recall the equation of motion for tensor modes given by Eq. (5.2.12), which in the IR limit on super-Hubble scales reduces to

$$h'' + 2\frac{a'}{a}h' \ = \ 0 \, . \tag{5.5.82}$$

Once again, we drop the k index when it is clear that the modes are in the IR limit. Close to the bounce point, we recall that the scale factor is almost constant, i.e. $a(\eta) \simeq a_B$. Thus, we are left with the equation $h'' \simeq 0$, and consequently,

$$h(\eta) \ \simeq \ h(\eta_i) + h'(\eta_i)(\eta - \eta_i) \, , \tag{5.5.83}$$

or, equivalently,

$$h(t) \ \simeq \ h(t_i) + \dot{h}(t_i)(t - t_i) \, . \tag{5.5.84}$$

Thus, as in the case of curvature fluctuations in Region II in the vicinity of the bounce point, there is a linearly growing mode. Dimensional analysis, however, tells us that this growing mode will not overwhelm the constant mode. The argument is as follows: we can estimate $\dot{h}(t_i)$ to be of the order $Mh(t_i)$, where M is the mass scale at the bounce. On the other hand, we expect the time interval of the bounce phase to be of the order M^{-1}, and hence we expect the linearly growing term to be comparable at the end of the bounce phase to the constant mode.

Comparing the coefficients of the linearly growing modes of the curvature fluctuations and the tensor modes, i.e. Eq. (5.5.79) and the coefficient of the growing mode in Eq. (5.5.84), respectively, we see that it is the extra factor of $[\dot{\phi}_B/\dot{\phi}(t_{\mathrm{amp}-})]^2$ in the coefficient of the scalar modes which leads to the enhancement of the scalar power spectrum relative to the tensor power spectrum.

5.6 A comprehensive analysis of the production of primordial non-Gaussianities during the bounce phase

Now that we have identified the conditions under which the tensor-to-scalar ratio can be suppressed, we turn to the study of how the bispectrum evolves during the bounce phase. We make use of the formalism developed in [481] (see also [201, 620]).

Our starting point is the expression (5.3.36) for the three-point function. From this expression it is clear that the bispectrum builds up over time, which is to say that the three-point function after the bounce equals the three-point function before the bounce plus the result of integrating the right-hand side of (5.3.36) over the time interval of the bounce. From the form (5.3.38) of the interaction Lagrangian it follows that the terms which dominate the three-point function in the infrared are given by three powers of ζ and two powers of its time derivative. As shown explicitly in [181] in the computation of the three-point function in the matter-dominated contracting phase, the absolute amplitude of ζ cancels out in the definition of the shape function. Furthermore, Cai *et al.* [181] show that the bispectrum at the end of the period of matter contraction has an amplitude of the order 1 with a shape which is different from what is obtained in simple inflationary models. Since the dominant mode of ζ is constant during the Ekpyrotic phase of contraction, no additional contribution to the bispectrum is generated during that phase. We have not computed the contribution generated during a possible radiation phase of contraction between the end of the matter period and beginning of the Ekpyrotic period. This calculation could be done using the methods of [181] and we would find again a contribution with amplitude of the order of one and with a shape similar to that generated in the matter phase of contraction and different from that in simple inflationary models, the reason being that the same terms which dominate the bispectrum in the matter phase will also dominate in the radiation phase, and they are terms which are slow-roll suppressed during inflation.

Hence, we now turn to the evaluation of the contribution of the bouncing phase to the three-point function. However, we must keep in mind that the equations of [481], in particular the third order perturbed Lagrangian given by Eq. (5.3.38), are only valid for a canonical scalar field. We must generalize the analysis to the case of the matter Lagrangian studied here (this generalization will not affect the evolution of the three-point function outside of the bounce phase because the extra terms which we derive below are negligible except in the bounce phase). This has already been done in the case of inflation for very general Lagrangians (see, e.g., [206, 301]).

For the Lagrangian given by Eq. (5.4.46), perturbations up to third order in ζ yield the action

$$S_3 = \int d^4x \ \Big(B_1\left[\partial\zeta\partial\chi\partial^2\zeta - \zeta\partial_i\partial_j(\partial_i\zeta\partial_j\chi)\right] + B_2\dot{\zeta}^2\partial^2\zeta + B_3\dot{\zeta}\partial\zeta\partial\chi + B_4\zeta(\partial_i\partial_j\chi)^2 + B_5\zeta(\partial\zeta)^2 + B_6\dot{\zeta}^3 + B_7\zeta\dot{\zeta}^2 - 2f(\zeta)\frac{\delta\mathcal{L}_2}{\delta\zeta}\Big) \,, \qquad (5.6.85)$$

where

$$f(\zeta) = \frac{A_{20}a^2}{4M_p^2}\left[(\partial\zeta)^2 - \partial^{-2}\partial_i\partial_j(\partial_i\zeta\partial_j\zeta)\right] + \frac{A_{18}a^2}{M_p^2}\left[\partial\zeta\partial\chi - \partial^{-2}\partial_i\partial_j(\partial_i\zeta\partial_j\chi)\right] - \frac{2A_4a^3 - C_1}{2z^2c_s^2}a\zeta\dot{\zeta}\,. \qquad (5.6.86)$$

The derivation of this action and the form of the functions A_n, B_n, and C_n $(n = 1, ...)$ can be found in Appendix 5.11.1. As expected, this action is equivalent to the action given by Eq. (5.3.38) in the limit where the Lagrangian (5.4.46) is canonical in a matter-dominated contracting universe. This is shown in Appendix 5.11.2.

In order to cancel the last term in Eq. (5.6.85), we make a field redefinition in Fourier space $\zeta(\eta,\vec{k}) \to \zeta(\eta,\vec{k}) - f(\eta,\vec{k})$ in the third order Lagrangian. This way, there will be two contributions to the three-point function. The first part of the three-point function is the third order Lagrangian without the last term and the second part is related to the field redefinition terms where $\zeta(\eta,\vec{k})$ is replaced by $f(\eta,\vec{k})$. Using the Lagrangian formalism, we note that in Fourier space, we can canonically express the modes $\zeta(\eta,\vec{k})$ as follows,

$$\zeta(\eta,\vec{k}) = \zeta_k(\eta)a^\dagger_{\vec{k}} + \zeta^*_k(\eta)a_{-\vec{k}}\,, \qquad (5.6.87)$$

where $a_{\vec{k}}|0\rangle = 0$, so $a_{\vec{k}}$ is the annihilation operator, and $a^{\dagger}_{\vec{k}}$ is the respective creation operator. Then, if we consider the interaction picture, the three-point function to leading order in the interaction coupling constant is given by

$$\langle\zeta(\eta,\vec{k}_1)\zeta(\eta,\vec{k}_2)\zeta(\eta,\vec{k}_3)\rangle_{\text{int}} = i\int_{\eta_i}^{\eta} d\tilde{\eta}\ \langle[\zeta(\eta,\vec{k}_1)\zeta(\eta,\vec{k}_2)\zeta(\eta,\vec{k}_3), L_{\text{int}}(\tilde{\eta})]\rangle\,, \tag{5.6.88}$$

where η_i corresponds to the initial time before which there is no non-Gaussianity. Also, L_{int} is associated with the third order action (5.6.85) without its last term.

Here, we are interested in the production of non-Gaussianities during the bounce phase, so we consider the initial time to be the beginning of the bounce phase and we consider the end time at which the three-point function is evaluated to be the end of the bounce phase. However, as we saw in the previous section, curvature perturbations are nearly constant, and hence do not contribute to the three-point function, except during the small time interval $[\eta_{\text{amp}-}, \eta_{\text{amp}+}]$ where ζ grows. Thus, the integration bounds are taken to be from $\eta_{\text{amp}-}$ to $\eta_{\text{amp}+}$, and the evolution of the curvature perturbations is taken to be

$$\zeta_k(\eta) = \zeta_k^{\text{m}}(\eta_{B-}) + \zeta_k^{\text{m}\prime}(\eta_{B-})\left(\frac{\phi'_B}{\phi'(\eta_{\text{amp}-})}\right)^2(\eta-\eta_{\text{amp}-})\,. \tag{5.6.89}$$

The above expression follows from taking the maximal linear growth rate given by Eq. (5.5.79) throughout the amplification interval $[\eta_{\text{amp}-}, \eta_{\text{amp}+}]$. This expression slightly underestimates ζ_k for $\eta_{\text{amp}-} \leq \eta < \eta_B$ and slightly overestimates ζ_k for $\eta_B < \eta \leq \eta_{\text{amp}+}$ but it is a good approximation on average over the small interval $[\eta_{\text{amp}-}, \eta_{\text{amp}+}]$.

We recall that curvature perturbations are more or less constant during the Ekpyrotic phase of contraction that precedes the bounce phase. Therefore, it is natural to take the end conditions of the matter-dominated phase of contraction as the initial conditions of the bounce phase. As shown in Sec. 5.5.4, ζ_k and ζ'_k must be continuous across any transition on super-Hubble scales. Hence for the initial conditions of the bounce phase, we put the superscript "m" which denotes the matter bounce solution [181]

$$\zeta_k^{\text{m}}(\eta) = \frac{iAe^{ik(\eta-\tilde{\eta}_{B-})}[1-ik(\eta-\tilde{\eta}_{B-})]}{\sqrt{2k^3}(\eta-\tilde{\eta}_{B-})^3}\,, \tag{5.6.90}$$

where $\tilde{\eta}_{B-}$ is the conformal time at the singularity if the matter-dominated contracting phase were to continue to arbitrary densities (i.e. without NEC violating matter). Also, A is a normalization constant which is determined from the quantum vacuum condition at Hubble radius crossing in the contracting phase, and it is found to be

$$A = \frac{(\Delta\eta_{B-})^2}{\sqrt{3}a_B M_p}\,, \tag{5.6.91}$$

where $\Delta\eta_{B-} \equiv \tilde{\eta}_{B-} - \eta_{B-}$.

Let us comment on the wave number dependence of Eq. (5.6.89). We first solved the equation of motion in the bouncing phase in the limit where $k \ll \mathcal{H}$ to 0^{th} order. Then, matching the solution at the beginning of the bouncing phase with the one at the end of the matter contraction phase, we introduced some wave number dependence since the solution in the matter contraction phase has higher order terms in $k/\mathcal{H}$. Thus, one may worry that obtaining the correct k-dependent solution in the bounce phase up to leading order requires one to solve the full k-dependent equation of motion. However, we note that we will be interested in the IR limit again when evaluating the three-point function. Thus, any k dependence not included in the above solution is suppressed during the bounce phase as long as k remains much smaller than the largest energy scale attained during the bounce, i.e. as long as $k \ll \mathcal{H}_{B-}, \mathcal{H}_{B+}$, and as long as the corresponding wavelength of the fluctuations remains much larger than the bounce length scale, which can be reformulated as $k \ll (\Delta\eta_B)^{-1}$.

Substituting the interaction Lagrangian L_{int} associated with the action (5.6.85) without its last term into Eq. (5.6.88) and using Eq. (5.6.87), we find

$$
\begin{aligned}
&\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\rangle_{\rm int} = \\
&(2\pi)^3\delta^{(3)}\left(\sum_{i=1}^{3}\vec{k}_i\right)\zeta^*_{k_1}(\eta_+)\zeta^*_{k_2}(\eta_+)\zeta^*_{k_3}(\eta_+) \\
&\times i\int_{\eta_-}^{\eta_+} d\eta\left[\frac{B_1(\eta)z^2(\eta)}{M_p^2}\left(\frac{\vec{k}_1\cdot\vec{k}_3}{k_3^2}k_2^2+\frac{[\vec{k}_2\cdot(\vec{k}_2-\vec{k}_3)][\vec{k}_3\cdot(\vec{k}_2+\vec{k}_3)]}{k_3^2}\right)\zeta_{k_1}(\eta)\zeta_{k_2}(\eta)\zeta'_{k_3}(\eta)\right. \\
&+\left(\frac{B_2(\eta)}{a_B}k_1^2+\frac{B_3(\eta)z^2(\eta)}{M_p^2a_B}\frac{\vec{k}_1\cdot\vec{k}_3}{k_3^2}+\frac{B_4(\eta)z^4(\eta)}{M_p^4a_B}\frac{(\vec{k}_2\cdot\vec{k}_3)^2}{k_3^2k_2^2}+\frac{B_7(\eta)}{a_B}\right)\zeta_{k_1}(\eta)\zeta'_{k_2}(\eta)\zeta'_{k_3}(\eta) \\
&\left.+B_5(\eta)a_B(\vec{k}_1\cdot\vec{k}_2)\zeta_{k_1}(\eta)\zeta_{k_2}(\eta)\zeta_{k_3}(\eta)+\left(\frac{B_6(\eta)}{a_B^2}\right)\zeta'_{k_1}(\eta)\zeta'_{k_2}(\eta)\zeta'_{k_3}(\eta)\right]+(5\text{ permutations})\,.
\end{aligned}
\tag{5.6.92}
$$

Moreover, the contribution from the field redefinition is

$$
\begin{aligned}
&-\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)f(\vec{k}_3)\rangle_{\rm redef} = \\
&(2\pi)^3\delta^{(3)}\left(\sum_{i=1}^{3}\vec{k}_i\right) \\
&\times\left[\frac{A_{20}(\eta_+)a_B^2}{4M_p^2}\left(-\vec{k}_1\cdot(\vec{k}_3-\vec{k}_1)+\frac{(\vec{k}_1\cdot\vec{k}_3)[(\vec{k}_3-\vec{k}_1)\cdot\vec{k}_3]}{k_3^2}\right)|\zeta_{k_1}(\eta_+)|^2|\zeta_{k_2}(\eta_+)|^2\right. \\
&-\frac{A_{18}(\eta_+)a_Bz^2(\eta_+)}{M_p^4}\left(\frac{\vec{k}_1\cdot(\vec{k}_3-\vec{k}_1)}{k_1^2}-\frac{(\vec{k}_1\cdot\vec{k}_3)[(\vec{k}_3-\vec{k}_1)\cdot\vec{k}_3]}{k_1^2k_3^2}\right)\zeta'_{k_1}(\eta_+)\zeta^*_{k_1}(\eta_+)|\zeta_{k_2}(\eta_+)|^2 \\
&\left.+\left(\frac{2A_4(\eta_+)a_B^3-C_1(\eta_+)}{2z^2(\eta_+)c_s^2}\right)\zeta'_{k_1}(\eta_+)\zeta^*_{k_1}(\eta_+)|\zeta_{k_2}(\eta_+)|^2\right]+(5\text{ permutations})\,.
\end{aligned}
\tag{5.6.93}
$$

The permutations that we refer to are over the $\vec{k}_i$ vectors for $i=1,2,3$. We note that, to simplify the notation, we set $\eta_\pm\equiv\eta_{\rm amp\pm}$. The general form of the full three-point function can be expressed as

$$
\begin{aligned}
\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\rangle &= \langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\rangle_{\rm int}+\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)f(\vec{k}_3)\rangle_{\rm redef} \\
&=(2\pi)^7\delta^{(3)}\left(\sum_i\vec{k}_i\right)\frac{\mathcal{P}_\zeta^2}{\prod_i k_i^3}\mathcal{A}(k_1,k_2,k_3)\,,
\end{aligned}
\tag{5.6.94}
$$

and so, if we substitute Eqs. (5.6.92) and (5.6.93) into the above, we find the shape function to be given by

$$
\begin{aligned}
&\mathcal{A}(k_1,k_2,k_3) = \\
&\frac{k_3^3\zeta^*_{k_1}(\eta_+)\zeta^*_{k_2}(\eta_+)\zeta^*_{k_3}(\eta_+)}{4\zeta^*_{k_1}(\eta_+)\zeta^*_{k_2}(\eta_+)\zeta_{k_1}(\eta_+)\zeta_{k_2}(\eta_+)} \\
&\times i\int_{\eta_-}^{\eta_+} d\eta \left[B_1(\eta)z^2(\eta)\left(\frac{\vec{k}_1\cdot\vec{k}_3}{k_3^2}k_2^2 - \frac{[\vec{k}_2\cdot(\vec{k}_2+\vec{k}_3)][\vec{k}_3\cdot(\vec{k}_2+\vec{k}_3)]}{k_3^2}\right)\zeta_{k_1}(\eta)\zeta_{k_2}(\eta)\zeta'_{k_3}(\eta)\right. \\
&+\left(\frac{B_2(\eta)}{a_B}k_1^2 + \frac{B_3(\eta)z^2(\eta)}{M_p^2a_B}\frac{\vec{k}_1\cdot\vec{k}_3}{k_3^2} + \frac{B_4(\eta)z^4(\eta)}{M_p^4a_B}\frac{(\vec{k}_2\cdot\vec{k}_3)^2}{k_3^2k_2^2} + \frac{B_7(\eta)}{a_B}\right)\zeta_{k_1}(\eta)\zeta'_{k_2}(\eta)\zeta'_{k_3}(\eta) \\
&\left.+ B_5(\eta)a_B(\vec{k}_1\cdot\vec{k}_2)\zeta_{k_1}(\eta)\zeta_{k_2}(\eta)\zeta_{k_3}(\eta) + \frac{B_6(\eta)}{a_B^2}\zeta'_{k_1}(\eta)\zeta'_{k_2}(\eta)\zeta'_{k_3}(\eta)\right] \\
&+\frac{A_{20}(\eta_+)a_B^2k_3^3}{4M_p^2}\left(-\vec{k}_1\cdot(\vec{k}_3-\vec{k}_1) + \frac{(\vec{k}_1\cdot\vec{k}_3)[(\vec{k}_3-\vec{k}_1)\cdot\vec{k}_3]}{k_3^2}\right) \\
&-\frac{A_{18}(\eta_+)a_Bz^2(\eta_+)k_3^3}{M_p^4}\left(\frac{\vec{k}_1\cdot(\vec{k}_3-\vec{k}_1)}{k_1^2} - \frac{(\vec{k}_1\cdot\vec{k}_3)[(\vec{k}_3-\vec{k}_1)\cdot\vec{k}_3]}{k_1^2k_3^2}\right)\frac{\zeta'_{k_1}(\eta_+)\zeta^*_{k_1}(\eta_+)}{|\zeta_{k_1}(\eta_+)|^2} \\
&+k_3^3\left(\frac{2A_4(\eta_+)a_B^3 - C_1(\eta_+)}{2z^2(\eta_+)c_s^2}\right)\frac{\zeta'_{k_1}(\eta_+)\zeta^*_{k_1}(\eta_+)}{|\zeta_{k_1}(\eta_+)|^2} + (5\text{ permutations})\,. \qquad (5.6.95)
\end{aligned}
$$

At this point, we should note that the contributions coming from the terms with coefficients B_1, B_2, B_5, and A_{20} are of order $\mathcal{O}(k^5)$, and consequently, these terms are vanishingly small compared to other terms, which are of order $\mathcal{O}(k^3)$, on super-Hubble scales. Therefore, the three main contributions to the shape function are the $\zeta\zeta'^2$ term, the ζ'^3 term, and the field redefinition term. We evaluate each of these terms separately in Appendix 5.12 and we find the general expression for the shape function after the bounce phase [see Eq. (5.12.165)].

Three important forms of non-Gaussianity in cosmological observations are the local form, the equilateral form, and the orthogonal form. The local form of non-Gaussianity requires that one of the three momentum modes exits the Hubble radius much earlier than the other two, i.e. $k_1 \ll k_2 = k_3$. Evaluating the shape function (5.12.165) in this limit yields

$$\begin{aligned} f_{\rm NL}^{\rm local} &= \frac{10}{3}\frac{\mathcal{A}(k_1 \ll k_2 = k_3)}{\sum_i k_i^3} \\ &\simeq \frac{10}{3}\Bigg[\frac{-A^2}{8a_B\Delta\eta_{B-}^4}\left(\frac{\phi_B'}{\phi'(\eta_-)}\right)^2\left(\frac{B_4(\eta_B)z_B^4}{M_p^4} + B_7(\eta_B)\right) \\ &\quad - \frac{3A^2}{8\Delta\eta_{B-}^4\Delta\eta_{\rm amp}}\left(\frac{\phi_B'}{\phi'(\eta_-)}\right)^2\frac{B_6(\eta_B)}{a_B^2} + \frac{A_{18}(\eta_+)a_Bz^2(\eta_+)}{8M_p^4\Delta\eta_{\rm amp}} + \frac{2A_4(\eta_+)a_B^3 - C_1(\eta_+)}{8z^2(\eta_+)c_s^2\Delta\eta_{\rm amp}}\Bigg]. \end{aligned} \tag{5.6.96}$$

The equilateral form of non-Gaussianity requires that $k_1 = k_2 = k_3$, so one finds

$$\begin{aligned} f_{\rm NL}^{\rm equil} &= \frac{10}{3}\frac{\mathcal{A}(k_1 = k_2 = k_3)}{\sum_i k_i^3} \\ &\simeq \frac{10}{3}\Bigg[\frac{A^2}{16a_B\Delta\eta_{B-}^4}\left(\frac{\phi_B'}{\phi'(\eta_-)}\right)^2\left(\frac{B_3(\eta_B)z_B^2}{M_p^2} - \frac{B_4(\eta_B)z_B^4}{2M_p^4} - 2B_7(\eta_B)\right) \\ &\quad - \frac{3A^2}{8\Delta\eta_{B-}^4\Delta\eta_{\rm amp}}\left(\frac{\phi_B'}{\phi'(\eta_-)}\right)^2\frac{B_6(\eta_B)}{a_B^2} + \frac{3A_{18}(\eta_+)a_Bz^2(\eta_+)}{16M_p^4\Delta\eta_{\rm amp}} + \frac{2A_4(\eta_+)a_B^3 - C_1(\eta_+)}{8z^2(\eta_+)c_s^2\Delta\eta_{\rm amp}}\Bigg]. \end{aligned} \tag{5.6.97}$$

Finally, the orthogonal form of non-Gaussianity requires that $k_1 = \sqrt{k_2^2 + k_3^2} = \sqrt{2}k$, so one finds

$$\begin{aligned} f_{\rm NL}^{\rm ortho} &= \frac{10}{3}\frac{\mathcal{A}(k_1 = \sqrt{k_2^2 + k_3^2} = \sqrt{2}k)}{\sum_i k_i^3} \\ &\simeq \frac{10}{3}\Bigg[\frac{A^2}{16a_B\Delta\eta_{B-}^4}\left(\frac{\phi_B'}{\phi'(\eta_-)}\right)^2\Bigg((4 - 3\sqrt{2})\frac{B_3(\eta_B)z_B^2}{M_p^2} + (4 - 2\sqrt{2})\frac{B_4(\eta_B)z_B^4}{M_p^4} \\ &\quad - 2B_7(\eta_B)\Bigg) - \frac{3A^2}{8\Delta\eta_{B-}^4\Delta\eta_{\rm amp}}\left(\frac{\phi_B'}{\phi'(\eta_-)}\right)^2\frac{B_6(\eta_B)}{a_B^2} + (1 + \sqrt{2})\frac{A_{18}(\eta_+)a_Bz^2(\eta_+)}{16M_p^4\Delta\eta_{\rm amp}} \\ &\quad + \frac{2A_4(\eta_+)a_B^3 - C_1(\eta_+)}{8z^2(\eta_+)c_s^2\Delta\eta_{\rm amp}}\Bigg]. \end{aligned} \tag{5.6.98}$$

Substituting in some values for the model parameters (Υ, T, ϕ_B', γ, etc., introduced in Sec. 5.4) would yield specific numbers for the amount of non-Gaussianities that has been produced during the bounce. However, instead of giving exact values now, we will try to constrain the parameter space from observations. This is what we do in the next section.

5.7 Combination of the observational bounds on non-Gaussianities and on the tensor-to-scalar ratio

Let first rewrite the expression for $f_{\rm NL}^{\rm local}$ using Eqs. (5.6.91), (5.12.156), (5.12.157), (5.12.159), (5.12.163), and (5.12.164),

$$\begin{aligned} f_{\rm NL}^{\rm local} \simeq & -\frac{5}{12\gamma^4\phi_B'^8}\left(3a_B^2\beta\gamma^2M_p^2\phi_B'^6+3\gamma^4\phi_B'^8+12a_B^8\beta M_p^6\Upsilon+16a_B^6\gamma^2M_p^4\phi_B'^2\Upsilon\right)\left(\frac{\phi_B'}{\phi'(\eta_-)}\right)^2 \\ & -\frac{25a_B^2M_p^2}{6\gamma^3\phi_B'^5}\left(2a_B^2\beta M_p^2+3\gamma^2\phi_B'^2\right)\frac{1}{\Delta\eta_{\rm amp}}\left(\frac{\phi_B'}{\phi'(\eta_-)}\right)^2+\frac{5a_B^4\beta M_p^4}{4\gamma^3\phi_B'^5}\frac{1}{\Delta\eta_{\rm amp}}\left(\frac{\phi_B'}{\phi'(\eta_-)}\right)^5 \\ & +\frac{5a_B^2M_p^2}{\gamma\phi_B'^3}\frac{1}{\Delta\eta_{\rm amp}}\left(\frac{\phi_B'}{\phi'(\eta_-)}\right)^3+\frac{10\gamma}{3\beta\phi_B'}\frac{1}{\Delta\eta_{\rm amp}}\left(\frac{\phi_B'}{\phi'(\eta_-)}\right)\,. \end{aligned} \tag{5.7.99}$$

The equilateral and orthogonal $f_{\rm NL}$ have similar expressions, only with different coefficients.

At this point, we do not want to insert specific values for the model parameters. Yet, in order to have a healthy bounce, i.e. one that yields a bounce free of ghost instabilities, we expect the model parameters to lie in specific regimes. From [172, 175, 178], we expect that $\phi_B' < a_BM_p$, $\Upsilon \ll M_p^2$, $\beta \sim \mathcal{O}(1)$, and $\gamma \ll 1$. Also, from Eq. (5.4.54), it is obvious that $\phi_B'/\phi'(\eta_-) > 1$. Therefore, keeping only dominant terms, the expression for $f_{\rm NL}^{\rm local}$ reduces to

$$f_{\rm NL}^{\rm local} \simeq \frac{5a_B^4\beta M_p^4}{\gamma^3\phi_B'^5}\left(\frac{\phi_B'}{\phi'(\eta_-)}\right)^2\left[-\frac{a_B^4M_p^2\Upsilon}{\gamma\phi_B'^3}-\frac{5}{3\Delta\eta_{\rm amp}}+\frac{1}{4\Delta\eta_{\rm amp}}\left(\frac{\phi_B'}{\phi'(\eta_-)}\right)^3\right]\,. \tag{5.7.100}$$

In the square bracket, the three terms come from $\mathcal{A}_{\zeta\zeta'^2}$, $\mathcal{A}_{\zeta'^3}$, and $\mathcal{A}_{\rm redef}$, respectively. However, let us recall from Appendix 5.12 that the results for $\mathcal{A}_{\zeta\zeta'^2}$ and $\mathcal{A}_{\zeta'^3}$ were actually upper bounds in absolute value. Since we expect that $\Delta\eta_{\rm amp} \sim \mathcal{O}(1/a_BM_p)$ from Eq. (5.5.70), it results that the field redefinition term is dominant over the other ones, just like in the regular matter bounce [181], so we can write

$$f_{\rm NL}^{\rm local} \simeq \frac{5a_B^4\beta M_p^4}{4\gamma^3\phi_B'^5}\frac{1}{\Delta\eta_{\rm amp}}\left(\frac{\phi_B'}{\phi'(\eta_-)}\right)^5\,. \tag{5.7.101}$$

In order to combine the bound on curvature perturbations and the above result, it is useful to rewrite the expressions for $f_{\rm NL}$ in terms of the ratio $\Delta\zeta/\zeta(\eta_{B-})$. In Sec. 5.5, Eq.

(5.5.81) told us that

$$\frac{\Delta\zeta}{\zeta(\eta_{B-})} \lesssim 2\frac{\zeta'(\eta_{B-})}{\zeta(\eta_{B-})}\left(\frac{\phi'_B}{\phi'(\eta_-)}\right)^2 \Delta\eta_{\mathrm{amp}}\,. \tag{5.7.102}$$

In the previous section, we argued that the initial conditions at η_{B-} were given by the end conditions of the matter-dominated phase of contraction, so we can say that

$$\zeta(\eta_{B-}) \overset{k/\mathcal{H}\to 0}{\simeq} \zeta_k^{\mathrm{m}}(\eta_{B-})\,, \qquad \zeta'(\eta_{B-}) \overset{k/\mathcal{H}\to 0}{\simeq} \zeta_k^{\mathrm{m}\prime}(\eta_{B-})\,. \tag{5.7.103}$$

Recalling that ζ_k^{m} is given by Eq. (5.6.90), we find that

$$\frac{\zeta'(\eta_{B-})}{\zeta(\eta_{B-})} \simeq \lim_{k/\mathcal{H}\to 0}\frac{\zeta_k^{\mathrm{m}\prime}(\eta_{B-})}{\zeta_k^{\mathrm{m}}(\eta_{B-})} = \frac{3}{\Delta\eta_{B-}}\,, \tag{5.7.104}$$

and thus Eq. (5.7.102) becomes

$$\frac{1}{6}\left(\frac{\Delta\eta_{B-}}{\Delta\eta_{\mathrm{amp}}}\right)\left(\frac{\Delta\zeta}{\zeta(\eta_{B-})}\right) \lesssim \left(\frac{\phi'_B}{\phi'(\eta_-)}\right)^2 . \tag{5.7.105}$$

This allows us to place a lower bound on Eq. (5.7.101) as follows,

$$f_{\mathrm{NL}}^{\mathrm{local}} \gtrsim \frac{5a_B^4\beta M_p^4}{144\sqrt{6}\gamma^3\phi_B'^5}\frac{1}{\Delta\eta_{\mathrm{amp}}}\left(\frac{\Delta\eta_{B-}}{\Delta\eta_{\mathrm{amp}}}\right)^{5/2}\left(\frac{\Delta\zeta}{\zeta(\eta_{B-})}\right)^{5/2} . \tag{5.7.106}$$

Our initial estimation in Sec. 5.3.3 showed that we expected f_{NL} to have terms of order $(\Delta\zeta/\zeta)^1$ and $(\Delta\zeta/\zeta)^2$. The terms of order $(\Delta\zeta/\zeta)^1$ in the full calculation corresponded to terms of order $[\phi'_B/\phi'(\eta_-)]^2$ in our approximation scheme and they originated from $\mathcal{A}_{\zeta\zeta'^2}$ and $\mathcal{A}_{\zeta'^3}$. A term of order $(\Delta\zeta/\zeta)^2$, i.e. of order $[\phi'_B/\phi'(\eta_-)]^4$, could have originated from $\mathcal{A}_{\zeta\zeta'^2}$ but the full calculation showed that it did not have any real component [see Eq. (5.12.152)]. Instead, the full calculation showed the presence of terms of order $(\Delta\zeta/\zeta)^{5/2}$, $(\Delta\zeta/\zeta)^{3/2}$, and $(\Delta\zeta/\zeta)^{1/2}$ coming from the field redefinition contribution to the shape function. In the large amplification limit, we are left with one term of order $(\Delta\zeta/\zeta)^{5/2}$ as shown in Eq. (5.7.106).

Let us recall that $\Delta\zeta/\zeta$ is bounded from below in order to satisfy the current observational bound on the tensor-to-scalar ratio r. Using the bound (5.3.31), we can further constrain the bound (5.7.106),

$$f_{\rm NL}^{\rm local} \gtrsim 240 \left(\frac{\beta}{\gamma^3}\right) \left(\frac{a_B M_p}{\phi'_B}\right)^5 \left(\frac{(a_B M_p)^{-1}}{\Delta\eta_{\rm amp}}\right) \left(\frac{\Delta\eta_{B-}}{\Delta\eta_{\rm amp}}\right)^{5/2} . \tag{5.7.107}$$

Let us note that $\Delta\eta_{B-} \sim \mathcal{H}_{B-}^{-1}$, and since the bounce energy scale is taken to be much less than the Planck scale, it results that $\Delta\eta_{B-} \gg (a_B M_p)^{-1}$. Thus, since every dimensionless ratio in Eq. (5.7.107) at least of order 1 or much greater than 1, it results that $f_{\rm NL}^{\rm local} \gtrsim 240$. Including the negative contribution to $f_{\rm NL}^{\rm local}$ from the matter-dominated contracting phase which is of order 1 [181] and the negative contributions from $\mathcal{A}_{\zeta\zeta'^2}$ and $\mathcal{A}_{\zeta'^3}$ would reduce this bound, but really not significantly.

The best observational bounds on primordial non-Gaussianities currently come from the Planck experiment, which reports [13]

$$f_{\rm NL}^{\rm local} = 0.8 \pm 5.0 \,, \qquad f_{\rm NL}^{\rm equil} = -4 \pm 43 \,, \qquad f_{\rm NL}^{\rm ortho} = -26 \pm 21 \,, \tag{5.7.108}$$

at 68% CL. We see that the lower bound on $f_{\rm NL}^{\rm local}$ coming from the bounce phase is already excluded by the observations at very high confidence level. Following the same analysis as above for the equilateral and orthogonal shapes yields the bounds $f_{\rm NL}^{\rm equil} \gtrsim 359$ and $f_{\rm NL}^{\rm ortho} \gtrsim 289$, which are also excluded at very high confidence level, although to a smaller extent than $f_{\rm NL}^{\rm local}$.

To summarize, in this section we took the non-Gaussianity results derived in the previous section and imposed that there had been a sufficient amplification of curvature perturbations in order to satisfy the current observational bound on the tensor-to-scalar ratio. As a result, the theoretical lower bounds on $f_{\rm NL}^{\rm local}$, $f_{\rm NL}^{\rm equil}$, and $f_{\rm NL}^{\rm ortho}$ are excluded at high confidence level by the current observational constraints on non-Gaussianities. This shows that the model suffers from the "no-go" theorem that we conjectured in Sec. 5.3.4.

Looking at Eq. (5.7.107), we see that this could be alleviated if, for instance, the amplification period was very long compared to the Planck time, or if the model parameters were such that $\beta/\gamma^3 \ll 1$ or $a_B M_p/\phi'_B \ll 1$. However, these conditions seem unlikely to occur in a physically admissible matter bounce scenario.

5.8 Conclusions

In the present paper, we have studied in detail the nonlinear dynamics of primordial curvature perturbations during the nonsingular bouncing phase in a matter bounce model described by a single generic scalar field minimally coupled to Einstein gravity. This type of model can be made consistent with the observational bound on the tensor-to-scalar ratio by realizing an enhancement of the curvature perturbations in the bouncing phase. We derived the conditions on the model parameters for which such an enhancement can be achieved. We then expanded the action for perturbations up to the third order, computed the full set of three point correlation functions and then derived the nonlinearity parameters $f_{\rm NL}$ in the cases of specific shapes of observational interest. Our results show that if the primordial non-Gaussianities mainly arise from a manifest growth of curvature perturbations during the bounce, then the nonlinearity parameter would become dangerously large and lead to a disagreement with the observational constraints from cosmic microwave background (CMB) data[4]. Specifically, we examined the relation between the nonlinearity parameter in the local, equilateral, and orthogonal limits and the growth of the curvature perturbations and explicitly showed that the observational bounds on the tensor-to-scalar ratio and the CMB bispectrum cannot be simultaneously satisfied. This leads us to conjecture that there is a "no-go" theorem for single field matter bounce cosmologies, assuming that the nonsingular bounce is realized by a generic scalar field minimally coupled to Einstein gravity, *which would rule out a large class of matter bounce models.*

We note that this "no-go" theorem might be circumvented by dropping certain assumptions imposed above. For instance, if one introduces more than one degree of freedom such as in the matter bounce curvaton mechanism [166, 168], the constraints from the tensor-to-scalar ratio as well as from the primordial non-Gaussianities can be satisfied at the same time, the reason being that in the curvaton scenario the scalar fluctuations are generated by a different mechanism than the tensors. As another example, if the initial Bunch-Davies vacuum is noncanonical (e.g., in the ΛCDM bounce [180], the initial quantum vacuum has $c_s \ll 1$), the initial ratio of the tensor modes to the scalar modes can be suppressed, in which case there is no need for the curvature perturbations to be enhanced during the bounce.

Our analysis also does not immediately apply to nonsingular matter bounce models in

[4] We recall that it has also been found in [302, 303] that non-Gaussianities could become dangerously large in a certain nonsingular bouncing cosmology and it has been conjectured that this could be generic to a large family of nonsingular bouncing cosmologies.

which the violation of the null energy condition is obtained by changes in the gravitational action (e.g., in Loop Quantum Cosmology [179, 626], Hořava-Lifshitz gravity [101], extended $F(R)$ gravity [40, 526], modified Gauss-Bonnet gravity [41], or torsion gravity scenarios [29, 170]). We might expect that the no-go theorem will extend to modified gravity matter bounce models which have the same number of degrees of freedom as General Relativity, in which case the tensor-to-scalar is generically large [178]. However, if the gravity model contains new degrees of freedom, then we might be in a situation similar to what happens in the curvaton scenario: the new degrees of freedom source the scalar modes but not the tensor modes, thus suppressing the tensor-to-scalar ratio during the bounce phase. Yet, it would be interesting to explicitly analyze the conditions under which the bispectrum constraints can be made consistent with the observed bound on the tensor-to-scalar ratio in such models.

5.9 Curvature perturbations expanding about the singularity

The equation of motion for curvature perturbations in the IR limit is [see Eq. (5.2.11)]

$$\frac{d\zeta'}{d\eta} + \frac{(z^2)'}{z^2}\zeta' = 0\,. \tag{5.9.109}$$

Let us parametrize z^2 close to the singular point η_s as

$$z^2(\eta) \sim \frac{1}{(\eta - \eta_s)^2}\,, \tag{5.9.110}$$

so the equation of motion becomes

$$\frac{d\zeta'}{d\eta} = \frac{2}{\eta - \eta_s}\zeta'\,. \tag{5.9.111}$$

Since after the singular time we have $\eta > \eta_s > \eta_i$, we integrate as follows,

$$\ln\left(\frac{\zeta'(\eta)}{\zeta'(\eta_i)}\right) = 2\int_{\eta_i}^{\eta} \frac{d\tilde{\eta}}{\tilde{\eta} - \eta_s} = 2\left(\int_{\eta_i}^{\eta_s-\epsilon} + \int_{\eta_s-\epsilon}^{\eta_s+\epsilon} + \int_{\eta_s+\epsilon}^{\eta}\right)\frac{d\tilde{\eta}}{\tilde{\eta} - \eta_s}\,, \tag{5.9.112}$$

for some constant ϵ. As we take the limit $\epsilon \to 0$, the second integral vanishes by definition and we are left with the first and third integral. Evaluating them, we find

$$\begin{aligned}\ln\left(\frac{\zeta'(\eta)}{\zeta'(\eta_i)}\right) &= 2\lim_{\epsilon\to 0}\left[\ln\left(\frac{(\eta_s-\epsilon)-\eta_s}{\eta_i-\eta_s}\right)+\ln\left(\frac{\eta-\eta_s}{(\eta_s+\epsilon)-\eta_s}\right)\right] \\ &= 2\lim_{\epsilon\to 0}\ln\left(\frac{-\epsilon(\eta-\eta_s)}{(\eta_i-\eta_s)\epsilon}\right) \\ &= 2\ln\left(\frac{\eta-\eta_s}{\eta_s-\eta_i}\right) . \end{aligned} \tag{5.9.113}$$

Therefore,

$$\zeta'(\eta) = \zeta'(\eta_i)\left(\frac{\eta-\eta_s}{\eta_s-\eta_i}\right)^2 \tag{5.9.114}$$

as expected if there were no singularity. A final integration yields

$$\zeta(\eta) = \zeta(\eta_i) + \zeta'(\eta_i)\left(\frac{(\eta-\eta_s)^3+(\eta_s-\eta_i)^3}{3(\eta_s-\eta_i)^2}\right) . \tag{5.9.115}$$

As expected, we recover the constant mode solution $\zeta' = 0$ as $\eta \to \eta_s$.

5.10 Perturbations outside the bounce phase

Let us consider matter with an equation of state $P = w\rho$ with w independent of time. In this case $z(t) \sim a(t)M_p$ (see Sec. 5.3.1). Then, the solution to the long wavelength curvature perturbations is given by [see Eq. (5.5.72)]

$$\zeta(t) = \zeta(t_i) + \dot{\zeta}(t_i)a(t_i)z(t_i)^2\int_{t_i}^{t}\frac{d\tilde{t}}{a(\tilde{t})z^2(\tilde{t})} \tag{5.10.116}$$

$$= \zeta(t_i) + \dot{\zeta}(t_i)a(t_i)^3\int_{t_i}^{t}\frac{d\tilde{t}}{a^3(\tilde{t})} . \tag{5.10.117}$$

For a constant $w \neq -1$, the solution to the scale factor is given by

$$a(t) = a_0 t^{2/3(1+w)} , \tag{5.10.118}$$

for some positive constant a_0, so we find

$$\zeta(t) = \zeta(t_i) + \dot{\zeta}(t_i)t_i^{\frac{2}{1+w}} \int_{t_i}^{t} d\tilde{t}\ \tilde{t}^{-\frac{2}{1+w}} \tag{5.10.119}$$

$$= \zeta(t_i) + \dot{\zeta}(t_i)t_i^{\frac{2}{1+w}} \left(\frac{w+1}{w-1}\right)\left(t^{\frac{w-1}{w+1}} - t_i^{\frac{w-1}{w+1}}\right), \tag{5.10.120}$$

as long as $|w| \neq 1$. Thus, for matter with $|w| > 1$, the solution for ζ exhibits a constant mode and a mode which is growing in an expanding universe (decaying in a contracting background), whereas for matter with $|w| < 1$, it exhibits a constant mode and a mode which is decaying in an expanding universe (and growing in a contracting background). For example, this implies that an Ekpyrotic phase of contraction in which $w \gg 1$ has a constant mode and a decaying mode.

For $w = -1$, one would recover the standard result of inflation where the constant mode is dominant on super-Hubble scales in an expanding background, and the second mode dominates in a contracting space.

The $w = 1$ case of fast roll expansion is relevant for the dynamics of our nonsingular bouncing cosmology right after the bounce phase. A phase of fast roll expansion occurs if the Lagrangian for the scalar field is dominated by its kinetic term, i.e. $V(\phi) \ll \dot{\phi}^2/2$. It then follows that the solution for the curvature perturbations in this case is (here in conformal time)

$$\zeta(\eta) \ = \ \zeta(\eta_i) + \zeta'(\eta_i)a(\eta_i)^2 \int_{\eta_i}^{\eta} \frac{d\tilde{\eta}}{a^2(\tilde{\eta})}\,. \tag{5.10.121}$$

Solving the background dynamics tells us that the solution to the scale factor in a phase of fast roll expansion is

$$a(\eta) \ = \ c_E(\eta - \eta_E)^{1/2}\,, \tag{5.10.122}$$

where c_E and η_E are some constants. Thus,

$$\zeta(\eta) = \zeta(\eta_i) + \zeta'(\eta_i)(\eta_i - \eta_E) \int_{\eta_i}^{\eta} \frac{d\tilde{\eta}}{\eta - \eta_E} \tag{5.10.123}$$

$$= \zeta(\eta_i) + \zeta'(\eta_i)(\eta_i - \eta_E) \ln\left(\frac{\eta - \eta_E}{\eta_i - \eta_E}\right). \tag{5.10.124}$$

So, for $w = 1$, curvature perturbations exhibit a constant mode solution and a logarithmically growing mode, i.e. $\zeta(\eta) \sim \ln \eta$. We note that this is also true in physical time since $a \sim$

$t^{1/3} \sim \eta^{1/2}$ implies that $\zeta(t) \sim \ln t^{2/3} \sim \ln t$.

5.11 Third order perturbed action

5.11.1 Derivation of the general form of the third order action

To study the three point correlation function in this matter bounce model, we have to evaluate the action up to third order in perturbation theory. We use the metric in the Arnowitt-Deser-Misner (ADM) form (see, e.g., [620])

$$ds^2 = N^2 dt^2 - \gamma_{ij}(N^i dt + dx^i)(N^j dt + dx^j)\,, \tag{5.11.125}$$

where $N_i = \gamma_{ij} N^j$ is the shift vector and N is the lapse function. The tensor γ_{ij} is the metric of the 3-dimensional spacelike hypersurfaces in this $3+1$ decomposition. It is related to the full 4-dimensional space-time metric tensor $g_{\mu\nu}$ via $\sqrt{-g} = N\sqrt{\gamma}$, where g and γ are the determinants of the tensors $g_{\mu\nu}$ and γ_{ij}, respectively. The action (5.4.45) in this ADM decomposition is given by

$$S = \int d^4x\, \sqrt{-g}\left[\frac{M_p^2}{2}\left(R^{(3)} + \kappa_{ij}\kappa^{ij} - \kappa^2\right) + K(\phi, X) + G(\phi, X)\Box\phi\right], \tag{5.11.126}$$

where $R^{(3)}$ is the three-dimensional Ricci scalar and the extrinsic curvature is defined by

$$\kappa_{ij} \equiv \frac{1}{2N}\left(\dot{\gamma}_{ij} - D_i N_i - D_j N_i\right). \tag{5.11.127}$$

We define the covariant derivative D_i on the spacelike hypersurfaces such that it is torsion-free and satisfies

$$D_i \gamma^{ij} = 0\,. \tag{5.11.128}$$

Then, $R^{i(3)}_{\ \ jkl}$ is the Riemann tensor associated with this connection, and

$$R^{(3)}_{ij} = R^{k(3)}_{\ \ ikj}\,, \tag{5.11.129}$$

$$R^{(3)} = \gamma^{ij} R^{(3)}_{ij}\,, \tag{5.11.130}$$

are the Ricci tensor and Ricci scalar, respectively. In the uniform field gauge where

$$\delta\phi = 0\,, \tag{5.11.131}$$

$$\gamma_{ij} = a^2 e^{2\zeta}\delta_{ij}\,, \tag{5.11.132}$$

one can use the Hamiltonian and momentum constraints to determine the scalar contributions to the lapse function and shift vector,

$$N = 1+\alpha\,,\ \ N_i = \partial_i\sigma\,, \tag{5.11.133}$$

up to leading order. Substituting Eq. (5.11.133) into the metric [Eq. (5.11.125)] and expanding the action [Eq. (5.11.126)] up to third order, we obtain the following,

$$\begin{aligned} S_3 = \int d^4x\ a^3 \Bigg[& a_1\alpha^3 + a_2\zeta\alpha^2 + a_3\dot{\zeta}\alpha^2 + a_4\frac{\partial^2\sigma}{a^2}\alpha^2 + a_5\frac{\partial\zeta\partial\sigma}{a^2}\alpha + a_6\alpha\dot{\zeta}\zeta + a_7\alpha\zeta\frac{\partial^2\sigma}{a^2} \\ & + 3M_p^2\alpha\dot{\zeta}^2 - \frac{M_p^2}{2}\ \frac{(\partial_i\partial_j\sigma)^2-(\partial^2\sigma)^2}{a^4}\alpha + 2M_p^2 H\zeta\frac{\partial\zeta\partial\sigma}{a^2} - 2M_p^2\dot{\zeta}\alpha\frac{\partial^2\sigma}{a^2} - 2M_p^2\zeta\alpha\frac{\partial^2\zeta}{a^2} \\ & - M_p^2\zeta^2\frac{\partial^2\zeta}{a^2} - M_p^2\alpha\frac{(\partial\zeta)^2}{a^2} - M_p^2\zeta\frac{(\partial\zeta)^2}{a^2} - 9M_p^2\dot{\zeta}^2\zeta + 2M_p^2\dot{\zeta}\frac{\partial\zeta\partial\sigma}{a^2} + M_p^2 H\zeta^2\frac{\partial^2\sigma}{a^2} \\ & + 2M_p^2\dot{\zeta}\zeta\frac{\partial^2\sigma}{a^2} - \frac{M_p^2}{2}\ \frac{(\partial_i\partial_j\sigma)^2-(\partial^2\sigma)^2}{a^4}\zeta - 2M_p^2\frac{\partial_i\zeta\partial_j\sigma\partial_i\partial_j\sigma}{a^4}\Bigg]\,, \end{aligned} \tag{5.11.134}$$

where we defined the following coefficients,

$$
\begin{aligned}
a_1 \equiv\, & 3M_p^2H^2 - \dot\phi^2\left(\frac{1}{2}K_{,X} + \dot\phi^2K_{,XX} + \frac{1}{6}\dot\phi^4K_{,XXX}\right) \\
& - 2H\dot\phi^3\left(5G_{,X} + \frac{11}{4}\dot\phi^2G_{,XX} + \frac{1}{4}\dot\phi^4G_{,XXX}\right) + \dot\phi^2\left(G_{,\phi} + \frac{7}{6}\dot\phi^2G_{,X\phi} + \frac{1}{6}\dot\phi^4G_{,\phi XX}\right)\,, \\
a_2 \equiv & - 9M_p^2H^2 + 3\dot\phi^2\left(\frac{1}{2}K_{,X} + \frac{1}{2}\dot\phi^2K_{,XX}\right) + 18H\dot\phi^3\left(G_{,X} + \frac{1}{4}\dot\phi^2G_{,XX}\right) \\
& - 3\dot\phi^2\left(G_{,\phi} + \frac{1}{2}\dot\phi^2G_{,\phi X}\right)\,, \\
a_3 \equiv & - 6M_p^2H + 6\dot\phi^3\left(G_{,X} + \frac{1}{4}\dot\phi^2G_{,XX}\right)\,, \\
a_4 \equiv\, & 2M_p^2H - 2\dot\phi^3\left(G_{,X} + \frac{1}{4}\dot\phi^2G_{,XX}\right)\,, \\
a_5 \equiv & - 2M_p^2H + 3\dot\phi^3G_{,X}\,, \\
a_6 \equiv & - 9\left(-2M_p^2H + \dot\phi^3G_{,X}\right)\,, \\
a_7 \equiv & - 2M_p^2H + 3\dot\phi^3G_{,X}\,.
\end{aligned}
$$

We note that the Hamiltonian and momentum constraints yield (these can also be obtained by varying the action above with respect to α and σ)

$$
\alpha = \frac{2M_p^2\dot\zeta}{u}\,, \tag{5.11.135}
$$

$$
\partial^2\sigma = a_8\partial^2\zeta + \partial^2\chi\,, \tag{5.11.136}
$$

respectively, where we defined

$$
u \equiv 2M_p^2H - \dot\phi^3G_{,X}\,, \tag{5.11.137}
$$

$$
a_8 \equiv -\frac{2M_p^2}{u}\,, \tag{5.11.138}
$$

and where

$$\partial^2\chi \equiv \frac{z^2\dot{\zeta}}{M_p^2} = 3a^2\dot{\zeta} + \frac{2M_p^2a^2\dot{\zeta}}{u^2}\Big(-6M_p^2H^2 + \dot{\phi}^2K_{,X} + \dot{\phi}^4K_{,XX} + 12H\dot{\phi}^3G_{,X} + 3H\dot{\phi}^5G_{,XX} - 2\dot{\phi}^2G_{,\phi} - \dot{\phi}^4G_{,X\phi}\Big)\,. \tag{5.11.139}$$

If we substitute Eqs. (5.11.135) and (5.11.136) into the third order perturbed action [Eq. (5.11.134)], we obtain

$$\begin{aligned} S_3 = \int d^4x\ a^3\Big[& A_1\dot{\zeta}^3 + A_2\zeta\dot{\zeta}^2 + A_3(\partial^2\zeta)\dot{\zeta}^2 + A_4\dot{\zeta}(\partial\zeta)^2 + A_5\partial\zeta\partial\chi\dot{\zeta} + A_6\zeta\dot{\zeta}\partial^2\zeta + A_7\zeta(\partial\zeta)^2 \\ & + A_8\partial\zeta\partial\chi\zeta + A_9\dot{\zeta}^2\partial^2\chi + A_{10}\zeta^2\partial^2\zeta + A_{11}\zeta^2\partial^2\chi + A_{12}\partial_i\zeta\partial_j\zeta\partial_i\partial_j\zeta + A_{13}\partial_i\zeta\partial_j\zeta\partial_i\partial_j\chi \\ & + A_{14}\partial_i\zeta\partial_j\chi\partial_i\partial_j\zeta + A_{15}\partial_i\zeta\partial_j\chi\partial_i\partial_j\chi + (A_{16}\dot{\zeta} + A_{17}\zeta)(\partial_i\partial_j\zeta)^2 + (A_{18}\dot{\zeta} + A_{19}\zeta)(\partial_i\partial_j\chi)^2 \\ & + (A_{20}\dot{\zeta} + A_{21}\zeta)\partial_i\partial_j\zeta\partial_i\partial_j\chi + (A_{22}\dot{\zeta} + A_{23}\zeta)(\partial^2\zeta)^2 + (A_{24}\dot{\zeta} + A_{25}\zeta)(\partial^2\chi)^2 \\ & + (A_{26}\dot{\zeta} + A_{27}\zeta)\partial^2\zeta\partial^2\chi + A_{28}\zeta\dot{\zeta}\partial^2\chi\Big] + S_b\,, \end{aligned} \tag{5.11.140}$$

where we defined the following,

$$\begin{aligned} & A_1 \equiv \frac{(2M_p^2)^3}{u^3}a_1 + \frac{(2M_p^2)^2}{u^2}a_3 + 6\frac{M_p^4}{u}\,, \quad A_2 \equiv \frac{(2M_p^2)^2}{u^2}a_2 + \frac{2M_p^2}{u}a_6 - 9M_p^2\,, \\ & A_3 \equiv \frac{(2M_p^2)^2}{u^2a^2}a_4a_8 - \frac{(2M_p^2)^2}{ua^2}a_8\,, \quad A_4 \equiv \frac{2M_p^2}{ua^2}a_5a_8 - \frac{2M_p^4}{ua^2} + \frac{2M_p^2}{a^2}a_8\,, \\ & A_5 \equiv \frac{2M_p^2}{ua^2}a_5 + \frac{2M_p^2}{a^2}\,, \quad A_6 \equiv \frac{2M_p^2}{ua^2}a_8a_7 - \frac{(2M_p^2)^2}{ua^2} + \frac{2M_p^2}{a^2}a_8\,, \\ & A_7 \equiv \frac{2M_p^2}{a^2}Ha_8 - \frac{M_p^2}{a^2}\,, \quad A_8 \equiv \frac{2M_p^2H}{a^2}\,, \quad A_9 \equiv \frac{(2M_p^2)^2}{u^2a^2}a_4 - \frac{(2M_p^2)^2}{ua^2}\,, \\ & A_{10} \equiv -\frac{M_p^2}{a^2} + \frac{M_p^2}{a^2}Ha_8\,, \quad A_{11} \equiv \frac{M_p^2H}{a^2}\,, \quad \frac{A_{12}}{a_8} \equiv A_{13} \equiv A_{14} \equiv -\frac{2M_p^2}{a^4}a_8\,, \\ & A_{15} \equiv -\frac{2M_p^2}{a^4}\,, \quad A_{16} \equiv -\frac{M_p^4}{ua^4}a_8^2\,, \quad A_{17} \equiv -\frac{M_p^2}{2a^4}a_8^2\,, \quad A_{18} \equiv -\frac{M_p^4}{ua^4}\,, \\ & A_{19} \equiv -\frac{M_p^2}{2a^4}\,, \quad A_{20} \equiv -\frac{2M_p^4}{ua^4}a_8\,, \quad A_{21} \equiv -\frac{M_p^2}{a^4}a_8\,, \quad A_{22} \equiv \frac{M_p^4}{ua^4}a_8^2\,, \\ & A_{23} \equiv \frac{M_p^2}{2a^4}a_8^2\,, \quad A_{24} \equiv \frac{M_p^4}{ua^4}\,, \quad A_{25} \equiv \frac{M_p^2}{2a^4}\,, \quad A_{26} \equiv \frac{2M_p^4}{ua^4}a_8\,, \\ & A_{27} \equiv \frac{M_p^2}{a^4}a_8\,, \quad A_{28} \equiv \frac{2M_p^2}{a^2} + \frac{2M_p^2}{ua}a_7\,. \end{aligned}$$

We note that S_b is a short-hand notation for all the boundary terms, which do not make a contribution to the calculation at 3rd order. After many integrations by part, we obtain

$$S_3 = \int d^4x \, \Big(B_1 \left[\partial\zeta\partial\chi\partial^2\zeta - \zeta\partial_i\partial_j(\partial_i\zeta\partial_j\chi) \right] + B_2\dot{\zeta}^2\partial^2\zeta + B_3\dot{\zeta}\partial\zeta\partial\chi + B_4\zeta(\partial_i\partial_j\chi)^2 + B_5\zeta(\partial\zeta)^2 + B_6\dot{\zeta}^3 + B_7\dot{\zeta}^2\zeta - 2f(\zeta)\frac{\delta\mathcal{L}_2}{\delta\zeta} \Big) \, , \tag{5.11.141}$$

where

$$\frac{\delta\mathcal{L}_2}{\delta\zeta} = \partial_t(az^2\dot{\zeta}) - \frac{c_s^2 z^2}{a}\partial^2\zeta \, , \tag{5.11.142}$$

and where

$$f(\zeta) = \frac{A_{20}a^2}{4M_p^2}\left[(\partial\zeta)^2 - \partial^{-2}\partial_i\partial_j(\partial_i\zeta\partial_j\zeta) \right] + \frac{A_{18}a^2}{M_p^2}\left[\partial\zeta\partial\chi - \partial^{-2}\partial_i\partial_j(\partial_i\zeta\partial_j\chi) \right] - \frac{2A_4a^3 - C_1}{2z^2c_s^2}a\zeta\dot{\zeta} \, . \tag{5.11.143}$$

We also introduced the following,

$$\begin{aligned}
B_1 \equiv & - A_{21}a^3 - \frac{1}{2}\partial_t(A_{20}a^3) + \frac{A_{20}}{2}a^3H - 2A_{18}z^2c_s^2a \, , \\
B_2 \equiv & \; A_3a^3 + (A_{26} + A_{20})\frac{z^2a^3}{M_p^2} \, , \\
B_3 \equiv & \; A_5a^3 + A_{15}\frac{z^2a^3}{M_p^2} \, , \\
B_4 \equiv & - \partial_t(A_{18}a^3) - 3A_{19}a^3 + 2A_{18}Ha^3 \, , \\
B_5 \equiv & \; \partial_t\left(A_4 + A_{13}\frac{z^2}{2M_p^2} \right)a^3 - 2A_{10}a^3 + A_7a^3 \, , \\
B_6 \equiv & \; A_1a^3 + (A_{18} + A_{24})\left(\frac{z^2}{M_p^2} \right)^2 a^3 + A_9\frac{z^2a^3}{M_p^2} + \frac{2A_4a^3 - C_1}{2c_s^2}a^2 \, , \\
B_7 \equiv & \; A_2a^3 + \left[A_{15}a^3 + \partial_t(A_{18}a)a^2 - \partial_t(A_{18}a^3) - 3A_{19}a^3 + 2A_{18}Ha^3 + A_{25}a^3 \right]\left(\frac{z^2}{M_p^2} \right)^2 \\
& + \frac{az^2}{2}\partial_t\left[\frac{(2A_4a^3 - C_1)a}{c_s^2z^2} \right] - \partial_t\left(\frac{az^2}{2} \right)\frac{(2A_4a^3 - C_1)a}{c_s^2z^2} - B_4\left(\frac{z^2}{M_p^2} \right)^2 \, .
\end{aligned} \tag{5.11.144}$$

Furthermore,

$$C_1 \equiv A_6 a^3 + (A_{27} - A_{21}) \frac{z^2 a^3}{M_p^2} \,, \tag{5.11.145}$$

and c_s is the speed of sound introduced earlier in Eq. (5.5.59).

5.11.2 Third order perturbed action in the limit of the matter-dominated contracting phase

Let us evaluate the third order action given by Eq. (5.11.141) in a matter-dominated contracting phase when $G(\phi, X) = 0$ and $K(\phi, X) = M_p^2 X - V(\phi)$. In this case, we have

$$\begin{aligned}
&a_1 = 3M_p^2 H^2 - \frac{1}{2} M_p^2 \dot{\phi}^2 \,, \quad a_2 = -9M_p^2 H^2 + \frac{3}{2} M_p^2 \dot{\phi}^2 \,, \quad a_3 = -6M_p^2 H \,, \\
&a_4 = 2M_p^2 H \,, \quad a_5 = -2M_p^2 H \,, \quad a_6 = 18 M_p^2 H \,, \quad a_7 = -2M_p^2 H \,,
\end{aligned}$$

together with $u = 2M_p^2 H$, $a_8 = -2M_p^2/u$, and $\partial^2 \chi = z^2 \dot{\zeta}/M_p^2 = a^2 \dot{\phi}^2 \dot{\zeta}/2H^2$. Then,

$$\begin{aligned}
&A_1 = -\frac{M_p^2 \dot{\phi}^2}{2H^3} \,, \quad A_2 = \frac{3M_p^2 \dot{\phi}^2}{2H^2} \,, \quad A_3 = 0 \,, \quad A_4 = -\frac{M_p^2}{a^2 H} \,, \quad A_5 = 0 \,, \quad A_6 = \frac{2M_p^2}{a^2 H} \,, \\
&A_7 = -\frac{3M_p^2}{a^2} \,, \quad A_8 = \frac{2M_p^2 H}{a^2} \,, \quad A_9 = 0 \,, \quad A_{10} = -\frac{2M_p^2}{a^2} \,, \quad A_{11} = \frac{M_p^2 H}{a^2} \,, \\
&\frac{A_{12}}{a_8} = A_{13} = A_{14} = \frac{2M_p^2}{a^4 H} \,, \quad A_{15} = -\frac{2M_p^2}{a^4} \,, \quad A_{16} = -\frac{M_p^4}{h a^4} a_8^2 \,, \quad A_{17} = -\frac{M_p^2}{2a^4} a_8^2 \,, \\
&A_{18} = -\frac{M_p^2}{2a^4 H} \,, \quad A_{19} = -\frac{M_p^2}{2a^4} \,, \quad A_{20} = -\frac{2M_p^4}{h a^4} a_8 \,, \quad A_{21} = -\frac{M_p^2}{a^4} a_8 \,, \quad A_{22} = \frac{M_p^4}{h a^4} a_8^2 \,, \\
&A_{23} = \frac{M_p^2}{2a^4} a_8^2 \,, \quad A_{24} = \frac{M_p^4}{h a^4} \,, \quad A_{25} = \frac{M_p^2}{2a^4} \,, \quad A_{26} = \frac{2M_p^4}{h a^4} a_8 \,, \quad A_{27} = \frac{M_p^2}{a^4} a_8 \,, \\
&A_{28} = \frac{2M_p^2}{a^2} + \frac{2M_p^2}{ha} a_7 \,.
\end{aligned}$$

Thus,

$$
\begin{aligned}
B_1(t) &= M_p^2 \frac{\dot{\phi}(t)^2 + 2\dot{H}(t)}{2a(t)H(t)^3} = 0\,, \\
B_2(t) &= 0\,, \\
B_3(t) &= -\frac{M_p^2 a(t)\dot{\phi}(t)^2}{H(t)^2} = -2\epsilon(t)M_p^2 a(t)\,, \\
B_4(t) &= -\frac{M_p^2 \dot{H}(t)}{2a(t)H(t)^2} = \frac{M_p^2\epsilon(t)}{2a(t)}\,, \\
B_5(t) &= M_p^2 a(t) \frac{2H(t)^2\dot{H}(t) + 2\dot{\phi}(t)H(t)\ddot{\phi}(t) - \dot{\phi}(t)^2\left[3\dot{H}(t) - H(t)^2\right]}{2H(t)^4} = M_p^2\epsilon(t)^2 a(t)\,, \\
B_6(t) &= 0\,, \\
B_7(t) &= \frac{M_p^2 a(t)^3\dot{\phi}(t)\left\{\dot{\phi}(t)\left[\dot{\phi}(t)^2 + 4H(t)^2\right]\dot{H}(t) - 8H(t)^3\ddot{\phi}(t)\right\}}{8H(t)^6} \\
&= -\frac{1}{2}M_p^2[\epsilon^3(t) - 2\epsilon^2(t)]a(t)^3\,.
\end{aligned}
$$

Here, we consider $\dot{\phi}(t)^2 = 2\epsilon(t)H(t)^2$, $\epsilon(t) = -\dot{H}(t)/H(t)^2$, and $c_s = 1$. Therefore, we find that

$$
\begin{aligned}
S_3 = \int d^4x \Bigg[&-2\epsilon M_p^2 a(t)\partial\zeta\dot{\zeta}\partial\chi + \frac{M_p^2\epsilon}{2a(t)}\zeta(\partial_i\partial_j\chi)^2 + M_p^2\epsilon^2 a(t)\zeta(\partial\zeta)^2 \\
&- \frac{1}{2}M_p^2(\epsilon^3 - 2\epsilon^2)a(t)^3\dot{\zeta}^2\zeta - 2f(\zeta)\frac{\delta\mathcal{L}}{\delta\zeta}\Bigg]\,,
\end{aligned}
\tag{5.11.146}
$$

and

$$
\begin{aligned}
f(\zeta) = &\frac{1}{4a(t)^2H(t)^2}\left[(\partial\zeta)^2 - \partial^{-2}\partial_i\partial_j(\partial_i\zeta\partial_j\zeta)\right] + \frac{1}{2a(t)^2H(t)}\left[-\partial\zeta\partial\chi + \partial^{-2}\partial_i\partial_j(\partial_i\zeta\partial_j\chi)\right] \\
&- \frac{1}{H(t)}\zeta\dot{\zeta}\,.
\end{aligned}
\tag{5.11.147}
$$

This is equivalent to the third order action given in [181] noting that we defined $\partial^2\chi = a^2\epsilon\dot{\zeta}$ whereas they considered $\partial^2\chi = \dot{\zeta}$.

5.12 Evaluating the shape function in the bounce phase

We want to evaluate Eq. (5.6.95) which, as explained in the text, has three dominant terms: the $\zeta\zeta'^2$ term, the ζ'^3 term, and the field redefinition term. Let us start with the contribution from the $\zeta\zeta'^2$ term to the shape function, which is

$$\mathcal{A}_{\zeta\zeta'^2} = \frac{\zeta^*_{k_3}(\eta_+)}{4\zeta_{k_1}(\eta_+)\zeta_{k_2}(\eta_+)} i \int_{\eta_-}^{\eta_+} d\eta \left[\left(\frac{B_3(\eta) z^2(\eta)}{M_p^2 a_B} k_3^3 \frac{\vec{k}_1 \cdot \vec{k}_3}{k_3^2} + \frac{B_4(\eta) z^4(\eta)}{M_p^4 a_B} k_3^3 \frac{(\vec{k}_2 \cdot \vec{k}_3)^2}{k_3^2 k_2^2} + \frac{B_7(\eta)}{a_B} k_3^3 \right) \zeta_{k_1}(\eta) \zeta'_{k_2}(\eta) \zeta'_{k_3}(\eta) \right], \tag{5.12.148}$$

where we omit the 5 additional permutations for now. Using Eq. (5.6.89) for $\zeta_{k_i}(\eta)$, we get

$$\mathcal{A}_{\zeta\zeta'^2} = \frac{i k_3^3}{4 a_B} \frac{\zeta^*_{k_3}(\eta_+) \zeta^{\mathrm{m}\prime}_{k_2}(\eta_{B-}) \zeta^{\mathrm{m}\prime}_{k_3}(\eta_{B-})}{\zeta_{k_1}(\eta_+)\zeta_{k_2}(\eta_+)} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^4 \int_{\eta_-}^{\eta_+} d\eta \left[\left(\frac{B_3(\eta) z^2(\eta)}{M_p^2} \frac{\vec{k}_1 \cdot \vec{k}_3}{k_3^2} + \frac{B_4(\eta) z^4(\eta)}{M_p^4} \frac{(\vec{k}_2 \cdot \vec{k}_3)^2}{k_3^2 k_2^2} + B_7(\eta) \right) \zeta_{k_1}(\eta) \right]. \tag{5.12.149}$$

We note that taking ζ'_{k_i} from Eq. (5.6.89) actually gives an upper bound on $\mathcal{A}_{\zeta\zeta'^2}$ since Eq. (5.6.89) used the maximal growth rate (5.5.79) for the full range $[\eta_-, \eta_+]$. This introduces some small error in the final result but this will turn out to be unimportant since, as we will see, the field redefinition contribution to the shape function will dominate over this upper bound on $\mathcal{A}_{\zeta\zeta'^2}$.

The time-dependent terms that remain inside the integral are B_3, B_4, B_7, z^2, and ζ_{k_1}. The latter, ζ_{k_1}, may experience a nontrivial enhancement during the interval $[\eta_-, \eta_+]$, and consequently, it may contribute significantly to the integral. The other terms, i.e. $B_3 z^2$, $B_4 z^4$, and B_7 defined in Eqs. (5.11.144) and (5.5.71), contribute as almost constant terms in the integral over the range $[\eta_-, \eta_+]$. Therefore, we rewrite Eq. (5.12.149) in the following form,

$$\mathcal{A}_{\zeta\zeta'^2} \simeq \frac{i k_3^3}{4 a_B} \frac{\zeta^*_{k_3}(\eta_+) \zeta^{\mathrm{m}\prime}_{k_2}(\eta_{B-}) \zeta^{\mathrm{m}\prime}_{k_3}(\eta_{B-})}{\zeta_{k_1}(\eta_+)\zeta_{k_2}(\eta_+)} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^4 \left(\frac{B_3(\eta_B) z_B^2}{M_p^2} \frac{\vec{k}_1 \cdot \vec{k}_3}{k_3^2} + \frac{B_4(\eta_B) z_B^4}{M_p^4} \frac{(\vec{k}_2 \cdot \vec{k}_3)^2}{k_3^2 k_2^2} + B_7(\eta_B) \right) \int_{\eta_-}^{\eta_+} d\eta \left[\zeta^{\mathrm{m}}_{k_1}(\eta_{B-}) + \zeta^{\mathrm{m}\prime}_{k_1}(\eta_{B-}) \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^2 (\eta - \eta_-) \right], \tag{5.12.150}$$

where we denote $z_B \equiv z(\eta_B)$. Performing the integral and using Eq. (5.6.89) for $\zeta_{k_i}(\eta_+)$

(again, this contributes to obtaining an upper bound for $\mathcal{A}_{\zeta\zeta'^2}$), we obtain

$$\begin{aligned}\mathcal{A}_{\zeta\zeta'^2} \simeq \frac{ik_3^3}{2a_B}\left(\frac{\phi'_B}{\phi'(\eta_-)}\right)^4\left(\frac{B_3(\eta_B)z_B^2}{M_p^2}\frac{\vec{k}_1\cdot\vec{k}_3}{k_3^2}+\frac{B_4(\eta_B)z_B^4}{M_p^4}\frac{(\vec{k}_2\cdot\vec{k}_3)^2}{k_3^2k_2^2}+B_7(\eta_B)\right)\frac{\zeta_{k_2}^{\mathrm{m}\prime}\zeta_{k_3}^{\mathrm{m}\prime}\zeta_{k_3}^{\mathrm{m}*}}{\zeta_{k_2}^{\mathrm{m}}}\\ \times\Delta\eta_{\mathrm{amp}}\left[1+\frac{\zeta_{k_1}^{\mathrm{m}\prime}}{\zeta_{k_1}^{\mathrm{m}}}\left(\frac{\phi'_B}{\phi'(\eta_-)}\right)^2\Delta\eta_{\mathrm{amp}}\right]\left[1+2\frac{\zeta_{k_3}^{\mathrm{m}\prime*}}{\zeta_{k_3}^{\mathrm{m}*}}\left(\frac{\phi'_B}{\phi'(\eta_-)}\right)^2\Delta\eta_{\mathrm{amp}}\right]\\ \times\left[1+2\frac{\zeta_{k_1}^{\mathrm{m}\prime}}{\zeta_{k_1}^{\mathrm{m}}}\left(\frac{\phi'_B}{\phi'(\eta_-)}\right)^2\Delta\eta_{\mathrm{amp}}\right]^{-1}\left[1+2\frac{\zeta_{k_2}^{\mathrm{m}\prime}}{\zeta_{k_2}^{\mathrm{m}}}\left(\frac{\phi'_B}{\phi'(\eta_-)}\right)^2\Delta\eta_{\mathrm{amp}}\right]^{-1},\end{aligned} \qquad (5.12.151)$$

where the modes $\zeta_{k_i}^{\mathrm{m}}$ are implicitly evaluated at η_{B-} and where we recall that $2\Delta\eta_{\mathrm{amp}} = \eta_+ - \eta_-$. At this point, one could substitute $\zeta_{k_i}^{\mathrm{m}}(\eta_{B-})$ with Eq. (5.6.90) and write the full expression for $\mathcal{A}_{\zeta\zeta'^2}$. However, to satisfy the observational bound on the tensor-to-scalar ratio, we expect there to be a large amplification of curvature perturbations during the interval $[\eta_-,\eta_+]$. In fact, from Eqs. (5.3.31) and (5.5.81), it must be that $|\zeta_{k_i}^{\mathrm{m}\prime}/\zeta_{k_i}^{\mathrm{m}}|[\phi'_B/\phi'(\eta_-)]^2\Delta\eta_{\mathrm{amp}} \gg \mathcal{O}(1)$. In that limit, the shape function (5.12.151) reduces to (to leading order)

$$\mathcal{A}_{\zeta\zeta'^2} \simeq \frac{ik_3^3}{4a_B}\left(\frac{\phi'_B}{\phi'(\eta_-)}\right)^4\left(\frac{B_3(\eta_B)z_B^2}{M_p^2}\frac{\vec{k}_1\cdot\vec{k}_3}{k_3^2}+\frac{B_4(\eta_B)z_B^4}{M_p^4}\frac{(\vec{k}_2\cdot\vec{k}_3)^2}{k_3^2k_2^2}+B_7(\eta_B)\right)\Delta\eta_{\mathrm{amp}}\left|\zeta_{k_3}^{\mathrm{m}\prime}\right|^2, \qquad (5.12.152)$$

which is purely imaginary, and hence, does not physically contribute to the physical shape function. The next-to-leading order terms are

$$\begin{aligned}\mathcal{A}_{\zeta\zeta'^2} \simeq \frac{ik_3^3}{8a_B}\left(\frac{\phi'_B}{\phi'(\eta_-)}\right)^2\left(\frac{B_3(\eta_B)z_B^2}{M_p^2}\frac{\vec{k}_1\cdot\vec{k}_3}{k_3^2}+\frac{B_4(\eta_B)z_B^4}{M_p^4}\frac{(\vec{k}_2\cdot\vec{k}_3)^2}{k_3^2k_2^2}+B_7(\eta_B)\right)\\ \times\frac{\zeta_{k_2}^{\mathrm{m}\prime}\zeta_{k_3}^{\mathrm{m}\prime}\zeta_{k_3}^{\mathrm{m}*}}{\zeta_{k_2}^{\mathrm{m}}}\left[\frac{\frac{\zeta_{k_1}^{\mathrm{m}\prime}}{\zeta_{k_1}^{\mathrm{m}}}+2\frac{\zeta_{k_3}^{\mathrm{m}\prime*}}{\zeta_{k_3}^{\mathrm{m}*}}}{\frac{\zeta_{k_1}^{\mathrm{m}\prime}\zeta_{k_2}^{\mathrm{m}\prime}}{\zeta_{k_1}^{\mathrm{m}}\zeta_{k_2}^{\mathrm{m}}}}-\frac{\zeta_{k_1}^{\mathrm{m}\prime}\zeta_{k_3}^{\mathrm{m}\prime*}}{\zeta_{k_1}^{\mathrm{m}}\zeta_{k_3}^{\mathrm{m}*}}\left(\frac{\zeta_{k_1}^{\mathrm{m}2}\zeta_{k_2}^{\mathrm{m}}}{\zeta_{k_1}^{\mathrm{m}\prime 2}\zeta_{k_2}^{\mathrm{m}\prime}}+\frac{\zeta_{k_1}^{\mathrm{m}}\zeta_{k_2}^{\mathrm{m}2}}{\zeta_{k_1}^{\mathrm{m}\prime}\zeta_{k_2}^{\mathrm{m}\prime 2}}\right)\right].\end{aligned} \qquad (5.12.153)$$

Using Eq. (5.6.90) for $\zeta_{k_i}^{\mathrm{m}}(\eta_{B-})$ and taking the limit $k \ll \mathcal{H}$, we find the leading order real-valued contribution to be

$$\mathcal{A}_{\zeta\zeta'^2} \simeq -\frac{A^2}{16a_B\Delta\eta_{B-}^4}\left(\frac{\phi'_B}{\phi'(\eta_-)}\right)^2(-k_1^3+k_2^3+k_3^3)\Bigg(\frac{B_3(\eta_B)z_B^2}{M_p^2}\frac{\vec{k}_1\cdot\vec{k}_3}{k_3^2}+\frac{B_4(\eta_B)z_B^4}{M_p^4}\frac{(\vec{k}_2\cdot\vec{k}_3)^2}{k_3^2k_2^2} + B_7(\eta_B)\Bigg)\,. \tag{5.12.154}$$

The $B_n(\eta_B)$ terms can be evaluated using Eqs. (5.11.144), (5.5.71), and (5.4.55):

$$\frac{B_3(\eta_B)z_B^2}{M_p^2} \simeq -\frac{6\beta M_p^4a_B^5\left(3\beta M_p^2a_B^2+2\gamma^2\phi_B'^2\right)}{\gamma^4\phi_B'^4}\,, \tag{5.12.155}$$

$$\frac{B_4(\eta_B)z_B^4}{M_p^4} \simeq -\frac{9\beta^2M_p^6a_B^7\left(4M_p^4\Upsilon a_B^6-3\gamma^2\phi_B'^6\right)}{2\gamma^6\phi_B'^{10}}\,, \tag{5.12.156}$$

$$B_7(\eta_B) \simeq \frac{3M_p^2a_B^3}{2\gamma^6\phi_B'^{10}}\Bigg(-9a_B^4\beta^2\gamma^2M_p^4\phi_B'^6+6a_B^2\beta\gamma^4M_p^2\phi_B'^8+6\gamma^6\phi_B'^{10}+12a_B^{10}\beta^2M_p^8\Upsilon + 24a_B^8\beta\gamma^2M_p^6\phi_B'^2\Upsilon+32a_B^6\gamma^4M_p^4\phi_B'^4\Upsilon\Bigg)\,. \tag{5.12.157}$$

Similarly, we can use the previous procedure to find the contribution from the ζ'^3 term to the shape function (again, omitting the additional permutations for now),

$$\begin{aligned}
\mathcal{A}_{\zeta'^3} &= \frac{ik_3^3\zeta_{k_3}^*(\eta_+)}{4\zeta_{k_1}(\eta_+)\zeta_{k_2}(\eta_+)}\int_{\eta_-}^{\eta_+}d\eta\,\left(\frac{B_6(\eta)}{a_B^2}\zeta'_{k_1}(\eta)\zeta'_{k_2}(\eta)\zeta'_{k_3}(\eta)\right)\\
&= \frac{ik_3^3\zeta_{k_3}^*(\eta_+)}{2\zeta_{k_1}(\eta_+)\zeta_{k_2}(\eta_+)}\frac{B_6(\eta_B)}{a_B^2}\zeta_{k_1}^{\mathrm{m}\prime}\zeta_{k_2}^{\mathrm{m}\prime}\zeta_{k_3}^{\mathrm{m}\prime}\left(\frac{\phi'_B}{\phi'(\eta_-)}\right)^6\Delta\eta_{\mathrm{amp}}\\
&= \frac{ik_3^3B_6(\eta_B)}{2a_B^2}\left(\frac{\phi'_B}{\phi'(\eta_-)}\right)^4\left\{|\zeta_{k_3}^{\mathrm{m}\prime}|^2+\frac{\zeta_{k_3}^{\mathrm{m}\prime}\zeta_{k_3}^{\mathrm{m}*}}{2\Delta\eta_{\mathrm{amp}}}\left(\frac{\phi'(\eta_-)}{\phi'_B}\right)^2\left[1-\frac{\zeta_{k_3}^{\mathrm{m}\prime*}}{\zeta_{k_3}^{\mathrm{m}*}}\left(\frac{\zeta_{k_1}^{\mathrm{m}}}{\zeta_{k_1}^{\mathrm{m}\prime}}+\frac{\zeta_{k_2}^{\mathrm{m}}}{\zeta_{k_2}^{\mathrm{m}\prime}}\right)\right]+\ldots\right\}\\
&\simeq -\frac{A^2B_6(\eta_B)}{16a_B^2\Delta\eta_{\mathrm{amp}}\Delta\eta_{B-}^4}\left(\frac{\phi'_B}{\phi'(\eta_-)}\right)^2(k_1^3+k_2^3+k_3^3)\,.
\end{aligned} \tag{5.12.158}$$

The ellipsis in the third line denotes higher-order terms in $|\zeta_{k_i}^{\mathrm{m}}/\zeta_{k_i}^{\mathrm{m}\prime}|[\phi'(\eta_-)/\phi'_B]^2(\Delta\eta_{\mathrm{amp}})^{-1}$, and in the fourth line, we took the leading order real-valued term in the expansion. From Eq. (5.11.144), the B_6 term is given by

$$B_6(\eta_B) \simeq \frac{10M_p^4a_B^6\left[2\beta M_p^2a_B^2+3\gamma^2\phi_B'^2\right]}{\gamma^3\phi_B'^5}\,. \tag{5.12.159}$$

From the same argument as for $\mathcal{A}_{\zeta\zeta'^2}$, the result (5.12.158) is actually an upper bound (in absolute value) for $\mathcal{A}_{\zeta'^3}$, which we will comment on later.

The contribution from the field redefinition term to the shape function is (again, omitting the additional permutations for now)

$$\begin{aligned}
\mathcal{A}_{\rm redef} &= k_3^3\left[-\frac{A_{18}(\eta_+)a_Bz^2(\eta_+)}{4M_p^4}\left(\frac{\vec{k}_1\cdot(\vec{k}_3-\vec{k}_1)}{k_1^2}-\frac{(\vec{k}_1\cdot\vec{k}_3)[(\vec{k}_3-\vec{k}_1)\cdot\vec{k}_3]}{k_1^2k_3^2}\right)\right.\\
&\quad\left.+\frac{2A_4(\eta_+)a_B^3-C_1(\eta_+)}{8z^2(\eta_+)c_s^2}\right]\frac{\zeta'_{k_1}(\eta_+)}{\zeta_{k_1}(\eta_+)}\\
&= k_3^3\left[-\frac{A_{18}(\eta_+)a_Bz^2(\eta_+)}{4M_p^4}\left(\frac{\vec{k}_1\cdot(\vec{k}_3-\vec{k}_1)}{k_1^2}-\frac{(\vec{k}_1\cdot\vec{k}_3)[(\vec{k}_3-\vec{k}_1)\cdot\vec{k}_3]}{k_1^2k_3^2}\right)\right.\\
&\quad\left.+\frac{2A_4(\eta_+)a_B^3-C_1(\eta_+)}{8z^2(\eta_+)c_s^2}\right]\frac{\zeta^{\rm m\prime}_{k_1}}{\zeta^{\rm m}_{k_1}}\left(\frac{\phi'_B}{\phi'(\eta_-)}\right)^2\left[1+\frac{\zeta^{\rm m\prime}_{k_1}}{\zeta^{\rm m}_{k_1}}\left(\frac{\phi'_B}{\phi'(\eta_-)}\right)^2 2\Delta\eta_{\rm amp}\right]^{-1}\\
&\simeq \frac{k_3^3}{2\Delta\eta_{\rm amp}}\left[-\frac{A_{18}(\eta_+)a_Bz^2(\eta_+)}{4M_p^4}\left(\frac{\vec{k}_1\cdot(\vec{k}_3-\vec{k}_1)}{k_1^2}-\frac{(\vec{k}_1\cdot\vec{k}_3)[(\vec{k}_3-\vec{k}_1)\cdot\vec{k}_3]}{k_1^2k_3^2}\right)\right.\\
&\quad\left.+\frac{2A_4(\eta_+)a_B^3-C_1(\eta_+)}{8z^2(\eta_+)c_s^2}\right]\,, \qquad (5.12.160)
\end{aligned}$$

where we took the leading order term in $|\zeta^{\rm m}_{k_1}/\zeta^{\rm m\prime}_{k_1}|[\phi'(\eta_-)/\phi'_B]^2(\Delta\eta_{\rm amp})^{-1}$. Here,

$$\frac{A_{18}(\eta_+)a_Bz^2(\eta_+)}{M_p^4}\simeq\frac{3a_B^4\beta M_p^4}{\gamma^3\phi'^5(\eta_+)}\,, \qquad (5.12.161)$$

$$\frac{2A_4(\eta_+)a_B^3-C_1(\eta_+)}{z^2(\eta_+)c_s^2}\simeq\frac{4}{\beta\gamma\phi'^3(\eta_+)}\left[3a_B^2M_p^2\beta+2\gamma^2\phi'^2(\eta_+)\right]\,. \qquad (5.12.162)$$

Recalling Eq. (5.4.54) and the fact that $|\eta_+-\eta_B|=|\eta_--\eta_B|=\Delta\eta_{\rm amp}$, we have $\phi'(\eta_+)=\phi'(\eta_-)$, and so we can rewrite the terms above as

$$\frac{A_{18}(\eta_+)a_Bz^2(\eta_+)}{M_p^4}\simeq\frac{3a_B^4\beta M_p^4}{\gamma^3\phi_B'^5}\left(\frac{\phi'_B}{\phi'(\eta_-)}\right)^5\,, \qquad (5.12.163)$$

$$\frac{2A_4(\eta_+)a_B^3-C_1(\eta_+)}{z^2(\eta_+)c_s^2}\simeq\frac{12a_B^2M_p^2}{\gamma\phi_B'^3}\left(\frac{\phi'_B}{\phi'(\eta_-)}\right)^3+\frac{8\gamma}{\beta\phi'_B}\left(\frac{\phi'_B}{\phi'(\eta_-)}\right)\,. \qquad (5.12.164)$$

Combining the different contributions (including all permutations), the general form of the total shape function is found to be

$$
\begin{aligned}
&\mathcal{A}(k_1,k_2,k_3) \simeq \\
&-\frac{A^2}{16\Delta\eta_{B-}^4}\left(\frac{\phi_B'}{\phi'(\eta_-)}\right)^2\frac{1}{\prod_i k_i^2}\left[\frac{B_3(\eta_B)z_B^2}{2M_p^2 a_B}\left(2\sum_{i\neq j}k_i^7k_j^2-2\sum_{i\neq j}k_i^5k_j^4-\sum_{i\neq j\neq \ell}k_i^5k_j^2k_\ell^2\right)\right.\\
&+\frac{B_4(\eta_B)z_B^4}{4M_p^4 a_B}\left(-\sum_i k_i^9+2\sum_{i\neq j}k_i^6k_j^3+6\sum_{i\neq j}k_i^7k_j^2-6\sum_{i\neq j}k_i^5k_j^4-2\sum_{i\neq j\neq \ell}k_i^4k_j^3k_\ell^2\right.\\
&\left.\left.+2\sum_{i\neq j\neq \ell}k_i^5k_j^2k_\ell^2\right)\right]-\left[\frac{A^2}{8\Delta\eta_{B-}^4}\left(\frac{\phi_B'}{\phi'(\eta_-)}\right)^2\frac{B_7(\eta_B)}{a_B}+\frac{3A^2}{8\Delta\eta_{B-}^4\Delta\eta_{\mathrm{amp}}}\left(\frac{\phi_B'}{\phi'(\eta_-)}\right)^2\frac{B_6(\eta_B)}{a_B^2}\right.\\
&\left.-\frac{2A_4(\eta_+)a_B^3-C_1(\eta_+)}{8z^2(\eta_+)c_s^2\Delta\eta_{\mathrm{amp}}}\right]\sum_i k_i^3-\frac{A_{18}(\eta_+)a_B z^2(\eta_+)}{32M_p^4\Delta\eta_{\mathrm{amp}}}\frac{1}{\prod_i k_i^2}\left(\sum_{i\neq j}k_i^7k_j^2\right.\\
&\left.-2\sum_{i\neq j}k_i^5k_j^4-2\sum_{i\neq j\neq \ell}k_i^5k_j^2k_\ell^2+\sum_{i\neq j}k_i^6k_j^3-\sum_{i\neq j\neq \ell}k_i^4k_j^3k_\ell^2\right). \qquad (5.12.165)
\end{aligned}
$$

Chapter 6

Matter bounce cosmology with a generalized single field: non-Gaussianity and an extended no-go theorem

6.1 Introduction

Matter bounce cosmology [132] is a very early universe structure formation scenario alternative to the paradigm of inflationary cosmology (see, e.g., [129] for a review of inflation, its problems and its alternatives). The idea is that quantum fluctuations exit the Hubble radius in a matter-dominated contracting phase before the Big Bang, which generates a scale-invariant power spectrum of curvature perturbations [286, 618]. The contracting phase is then followed by a bounce and the standard phases of hot Big Bang cosmology. This construction solves the usual problems of standard Big Bang cosmology such as the horizon and flatness problems, but in addition, it is free of the trans-Planckian corrections that plague inflationary cosmology [486], and one can naturally avoid reaching a singularity at the time of the Big Bang (contrary to standard[1] inflation [86, 87]) under the assumption that new physics appears at high energy scales [129, 132]. Nonsingular bounces can be constructed in various ways using matter violating the Null Energy Condition (NEC), with a modified

[1]The singularity before inflation could be avoided with, for example, bounce inflation (e.g., [617]).

gravity action, or within a quantum theory of gravity (see the reviews [56, 109, 132, 166, 529] and references therein).

A typical way of constructing a nonsingular matter bounce cosmology is to assume the existence of a new scalar field. With a canonical Lagrangian, the oscillation of the scalar field can drive a matter-dominated contracting phase when the ratio of the pressure to the energy density averages zero. As the energy scale of the universe increases, new terms can appear in the Lagrangian that violate the NEC and drive a nonsingular bounce. For example, using a Galileon scalar field [522] (or equivalently, in Horndeski theory [355]), one can construct a stable NEC violating nonsingular bounce [54, 167, 172, 267, 535, 556] that may be free of ghost and gradient instabilities [365, 367] (see, however, the difficulties in doing so as pointed out by [165, 227, 398, 441]).

To distinguish the matter bounce scenario from inflation observationally, studying primordial non-Gaussianities is a useful tool[2]. In the case of inflation, after the calculation of the bispectra generated in single field slow-roll models [481], there have been many studies in the past decade trying to extend the simplest result, which largely enriched the phenomenology of nonlinear perturbations (see [201, 620] for reviews). In particular, one important progress has been to generalize the canonical inflaton to a k-essence scalar field [33, 34], such as k-inflation [32, 307] and DBI models [27, 586], which are collectively known as general single field inflation [202]. In these models, due to the effects of a small sound speed, the amplitude of the bispectrum is enhanced and interesting shapes emerge [201, 202, 204, 527, 581, 620]. In a matter-dominated contracting phase, the calculation of the bispectrum has only been done by [181] for the original matter bounce model with a canonical scalar field. A natural extension is thus to consider a k-essence scalar field[3] similarly to what has been done in inflationary cosmology, especially since the appearance of a noncanonical field is quite common in the literature of nonsingular bouncing cosmology in order to violate the NEC as explained above. Because the perturbations behave differently in matter bounce cosmology compared to inflation, in particular due to the growth of curvature perturbations on super-Hubble scales during the matter-dominated contracting phase, the canonical matter bounce yields non-Gaussianities with negative sign and order one amplitude, which differs from the results in canonical single field inflation. It would be interesting to explore how these non-

[2] Another observable quantity, besides non-Gaussianities, that would allow one to differentiate between inflation and the matter bounce scenario is the running of the scalar spectral index (see [174, 431]).

[3] This could be easily further generalized to a Galileon field [250], which has also been done for inflation (see, e.g., [160, 222, 304, 400, 401]).

Gaussianity results change when one generalizes the original matter bounce scenario to be based on a k-essence scalar field.

Besides non-Gaussianity, another interesting observable for very early universe models is the tensor-to-scalar ratio r. In the original matter bounce scenario, this ratio is predicted to be very large [177, 178]. Indeed, the scalar and tensor power spectra share the same amplitude, and accordingly, the tensor-to-scalar ratio is naturally of order unity [562]. This is well beyond the current observational bound from the Cosmic Microwave Background (CMB), which states that $r < 0.07$ at 95% confidence [11].

A resolution to this problem is to allow for the growth of curvature perturbations during the bounce phase, which suppresses the tensor-to-scalar ratio. However, curvature perturbations tend to remain constant through the bounce phase on super-Hubble scales [54, 630]. In fact, amplification can only be achieved under some tuning of the parameters, and the overall growth is still limited[4] [562]. Yet, if the scalar modes are amplified, another problem follows in that it leads to the production of large non-Gaussianities [562], a problem that might be generic to a large class of nonsingular bounces [302, 303]. Again, these large non-Gaussianities are excluded by current measurements from the CMB [13]. This leads to conjecture that single field matter bounce cosmology suffers from a no-go theorem [562], which states that one cannot satisfy the bound on r without violating the bounds on non-Gaussianities and vice versa.

There is another way to suppress the tensor-to-scalar ratio if the sound speed of the perturbations can be smaller than the speed of light during the matter-dominated contracting phase. For example, in the ΛCDM bounce scenario [180] (and its extension [171]; see the review [174]), if there exists a form of dark matter with a small sound speed that dominates the contracting phase when the scale-invariant power spectra are generated, then the tensor-to-scalar ratio is already suppressed proportionally to the sound speed. Therefore, this provides another motivation to study non-Gaussianities when the sound speed is small during the matter-dominated contracting phase. An immediate question is whether the no-go theorem still holds true in this case or whether it can be circumvented. In this work, we

[4] The studies of Refs. [54, 562, 630] have been carried out for models where the nonsingular bounce is attributed to a noncanonical scalar field. Loop quantum cosmology (LQC) provides an alternative class of nonsingular bouncing models that could suppress r during the bounce. In LQC, the amplitude of the suppression depends on the equation of state during the bounce; if it is close to zero, then the suppression is very strong (see [178, 180, 627] and references therein for a discussion of LQC effects in nonsingular bouncing cosmology).

want to explore this possibility of having a k-essence scalar field that would mimic dust-like matter with a small sound speed at low energies and that could play the role of the NEC violating scalar field during the bounce.

In this paper, we will evaluate the bispectrum produced by a k-essence scalar field in a matter-dominated contracting universe. This more general setup will yield richer features, which have the potential to be detected by future non-Gaussianity observations. In particular, the shapes, amplitudes, and scaling behaviors will be studied systematically. We will show that a small sound speed implies a large amplitude associated with the three-point function. Accordingly, we will claim that the no-go theorem is not circumvented but rather extended: in single field matter bounce cosmology, one cannot suppress the tensor-to-scalar ratio, either from the onset of the initial conditions in the matter contracting phase or from the amplification of the curvature perturbations during the bouncing phase, without producing large non-Gaussianities.

The outline of the paper is as follows. In section 6.2, we first introduce the background dynamics of the matter bounce scenario and introduce the class of k-essence scalar field models that we study in this paper. In section 6.3, we calculate the power spectra of curvature perturbations and tensor modes and show how a small sound speed coming from the k-essence scalar field allows for the suppression of the tensor-to-scalar ratio. We then consider the primordial non-Gaussianity in section 6.4. Using the in-in formalism, we evaluate every contribution to the three-point function and give a detailed analysis of the size and shapes of the resulting bispectrum. In section 6.5, we compute the amplitude parameter of non-Gaussianities in different limits and finally combine these results with the bound on the sound speed from section 6.3 to show that the no-go theorem in matter bounce cosmology is extended. We summarize our results in section 6.6. Throughout this paper, we use the mostly plus metric convention, and we define the reduced Planck mass to be $M_{\mathrm{Pl}} = (8\pi G_{\mathrm{N}})^{-1/2}$, where G_{N} is Newton's gravitational constant.

6.2 Setup and background dynamics

The idea of the matter bounce scenario is to begin with a matter-dominated contracting phase. At the background level, this corresponds to having a scale factor as a function of physical time given by

$$a(t) = a_B \left(\frac{t - \tilde{t}_B}{t_B - \tilde{t}_B} \right)^{2/3} , \tag{6.2.1}$$

and the Hubble parameter follows,

$$H(t) = \frac{2}{3(t - \tilde{t}_B)} , \tag{6.2.2}$$

where t_B corresponds to the time of the beginning of the bounce phase and $\tilde{t}_B$ corresponds to the time at which the singularity would occur if no new physics appeared at high energy scales. Accordingly, a_B is the value of the scale factor at t_B. In terms of the conformal time τ defined by $\mathrm{d}\tau = a^{-1}\mathrm{d}t$, the scale factor is given by

$$a(\tau) = a_B \left(\frac{\tau - \tilde{\tau}_B}{\tau_B - \tilde{\tau}_B} \right)^{2} , \tag{6.2.3}$$

where τ_B and $\tilde{\tau}_B$ are the conformal times corresponding to t_B and $\tilde{t}_B$. Throughout the rest of this paper, the scale factor is normalized such that $a_B = 1$.

One can define the usual "slow-roll" parameters of inflation by

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{3}{2}(1 + w) , \qquad \eta \equiv \frac{\dot{\epsilon}}{H\epsilon} , \tag{6.2.4}$$

where a dot denotes a derivative with respect to physical time, and $w \equiv p/\rho$ is the equation of state parameter with p and ρ denoting pressure and energy density, respectively. In the case of the matter bounce, the matter contracting phase implies that pressure vanishes, which is to say that

$$w = 0 , \qquad \epsilon = \frac{3}{2} , \qquad \eta = 0 . \tag{6.2.5}$$

If the pressure does not vanish exactly but is still very small, i.e. $|p/\rho| \ll 1$, then the values for w, ϵ, and η in equation (6.2.5) are only valid as leading order approximations, and they will be time dependent rather than constant. In this paper, we will work in the limit where equation (6.2.5) is valid.

In the usual matter bounce scenario, one would introduce a canonical scalar field to drive the matter-dominated contracting phase and describe the cosmological fluctuations. In this paper, we aim for more generality and assume that the perturbations are introduced by a

k-essence scale field ϕ with Lagrangian density of the form[5]

$$\mathcal{L}_\phi = P(X, \phi) \ , \tag{6.2.6}$$

where $X \equiv -\partial_\mu\phi\partial^\mu\phi/2$, and we assume that the scalar field is minimally coupled to gravity. The energy density and pressure of this scalar field are then given by

$$\rho = 2XP_{,X} - P \ , \qquad p = P \ , \tag{6.2.7}$$

where a comma denotes a partial derivative, e.g. $P_{,X} \equiv \partial P/\partial X$. Thus, the Friedmann equations read

$$3M_{\rm Pl}^2 H^2 = 2XP_{,X} - P \ , \qquad M_{\rm Pl}^2 \dot{H} = -XP_{,X} \ . \tag{6.2.8}$$

Since we want a matter-dominated contracting phase, the pressure of the scalar field should vanish (at least in average), and $\rho = 2XP_{,X} \propto a^{-3}$.

It is helpful to have one specific example where a k-essence field drives the matter contraction. Let us consider the following Lagrangian density:

$$\mathcal{L}_\phi = K(X) = \frac{1}{8}(X - c^2)^2 \ . \tag{6.2.9}$$

This type of Lagrangian belongs to a subclass of k-essence models $P(X, \phi)$ where the kinetic terms $K(X)$ are separate from the potential terms $V(\phi)$, i.e. $P(X, \phi) = K(X) - V(\phi)$. Moreover, the above Lagrangian has vanishing potential. Then, the ghost condensate solution is given by $X = c^2$ and $\phi(t) = ct + \pi(t)$, with $\dot{\pi}(t) \ll c$. In this case, the background equations yield $p \simeq 0$ and $\rho \sim \dot{\pi} \propto a^{-3}$, which exactly corresponds to a matter-dominated universe. More details about this model can be found in [455]. We note that there should be also other forms of $P(X, \phi)$ that can drive a matter contraction, and remarkably, the analysis that follows in this paper is done in a *model-independent* way and does *not* rely on the specific model of equation (6.2.9).

[5] For an introduction to such a Lagrangian in early universe cosmology with the derivation of the background equations of motion and the definition of the different parameters, see, e.g., [32, 202, 307, 581].

The sound speed and another "slow-roll" parameter are defined by[6]

$$c_{\rm s}^2 \equiv \frac{\partial p}{\partial \rho} = \frac{P_{,X}}{P_{,X} + 2XP_{,XX}} ~, \qquad s \equiv \frac{\dot{c}_{\rm s}}{c_{\rm s}H} ~. \tag{6.2.10}$$

Calculations will be done for a general sound speed, but as we will argue, we will be interested in the small sound speed limit, which can be realized with the appropriate form for $P(X,\phi)$. For instance, the explicit example given by equation (6.2.9) yields $c_{\rm s} \simeq \dot{\pi}/c \ll 1$. Furthermore, we will generally assume later that the sound speed remains nearly constant, which is to say that $|s| \ll 1$. We also define two other variables for later convenience,

$$\Sigma \equiv XP_{,X} + 2X^2P_{,XX} = \frac{M_{\rm Pl}^2H^2\epsilon}{c_{\rm s}^2} ~, \tag{6.2.11}$$

and

$$\lambda \equiv X^2P_{,XX} + \frac{2}{3}X^3P_{,XXX} = \frac{X}{3}\Sigma_{,X} - \frac{1}{3}\Sigma ~. \tag{6.2.12}$$

The ratio λ/Σ will be of particular interest in the following sections. For inflation, it depends on the specific realization of the general single field, such as DBI and k-inflation models. For the matter bounce scenario, it can be obtained in an approximately model-independent way. The detailed calculation is in Appendix 6.7, where we find that the ratio λ/Σ can be expressed in terms of the sound speed, as shown by equation (6.7.81).

6.3 Mode functions and two-point correlation functions

We begin with an action of the form

$$S = \int \mathrm{d}^4x \, \sqrt{-g} \left(\frac{1}{2}M_{\rm Pl}^2R + \mathcal{L}_\phi \right) ~, \tag{6.3.13}$$

where g is the determinant of the metric tensor and R is the Ricci scalar. Importantly, we assume that the matter Lagrangian $\mathcal{L}_\phi$ has the general form of equation (6.2.6), but we do not restrict our attention to any specific model. By perturbing up to second order the above

[6]We assume that the cosmological perturbations will remain adiabatic throughout the matter-dominated contracting phase.

action, one finds[7]

$$S^{(2)} = \int \mathrm{d}\tau \mathrm{d}^3\vec{x}\ z^2 \left[\zeta'^2 - c_\mathrm{s}^2(\vec{\nabla}\zeta)^2\right] \ , \tag{6.3.14}$$

where $\zeta(\tau, \vec{x})$ denotes the curvature perturbation in the comoving gauge, i.e. on slices where fluctuations of the scalar field vanish ($\delta\phi = 0$). Also, a prime represents a derivative with respect to conformal time, $\vec{\nabla} = \partial_i$ is the spatial gradient, and we define $z^2 \equiv 2\epsilon a^2 M_\mathrm{Pl}^2/c_\mathrm{s}^2$. Transforming to Fourier space, the second-order perturbed action becomes

$$S^{(2)} = \int \mathrm{d}\tau \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\ z^2 \left[\zeta'(\vec{k})\zeta'(-\vec{k}) - c_\mathrm{s}^2 k^2 \zeta(\vec{k})\zeta(-\vec{k})\right] \ , \tag{6.3.15}$$

where $k^2 \equiv \vec{k}\cdot\vec{k} = |\vec{k}|^2$. Upon quantization of the curvature perturbation, one has

$$\hat{\zeta}(\tau, \vec{k}) = \hat{a}^\dagger_{\vec{k}} u_k(\tau) + \hat{a}_{-\vec{k}} u_k^*(\tau) \ , \tag{6.3.16}$$

where the annihilation and creation operators satisfy the usual commutation relation $[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}] = (2\pi)^3\delta^{(3)}(\vec{k} - \vec{k}')$. The equation of motion of the mode function is then given by

$$v_k'' + \left(c_\mathrm{s}^2 k^2 - \frac{z''}{z}\right) v_k = 0 \ , \tag{6.3.17}$$

where the mode function is rescaled as $v_k = z u_k$ (v_k is called the Mukhanov-Sasaki variable). Together with the commutation relation $[\hat{\zeta}(\vec{k}_1), \hat{\zeta}'(\vec{k}_2)] = (2\pi)^3\delta^{(3)}(\vec{k}_1 + \vec{k}_2)$, one finds (see, e.g., [181])

$$u_k(\tau) = \frac{iA[1 - ic_\mathrm{s}k(\tau - \tilde{\tau}_B)]}{2\sqrt{\epsilon c_\mathrm{s} k^3}(\tau - \tilde{\tau}_B)^3} e^{ic_\mathrm{s}k(\tau - \tilde{\tau}_B)} \tag{6.3.18}$$

$$u_k'(\tau) = \frac{iA}{2\sqrt{\epsilon c_\mathrm{s} k^3}} \left(\frac{-3[1 - ic_\mathrm{s}k(\tau - \tilde{\tau}_B)]}{(\tau - \tilde{\tau}_B)^4} + \frac{c_\mathrm{s}^2 k^2}{(\tau - \tilde{\tau}_B)^2}\right) e^{ic_\mathrm{s}k(\tau - \tilde{\tau}_B)} \tag{6.3.19}$$

to be the solution to the equation of motion (6.3.17) in the context of a matter-dominated contracting universe as described in the previous section. Here, A is a normalization constant that is determined by the quantum vacuum condition at Hubble radius crossing in the contracting phase, which is given by $A = (\tau_B - \tilde{\tau}_B)^2/M_\mathrm{Pl}$.

[7] Again, see, e.g., [201, 202, 307, 581] for a derivation of the perturbation equations in k-essence early universe cosmology.

The general two-point correlation functions are given by

$$\langle\hat{\zeta}(\tau_1,\vec{k}_1)\hat{\zeta}(\tau_2,\vec{k}_2)\rangle = (2\pi)^3\delta(\vec{k}_1+\vec{k}_2)u^*_{k_1}(\tau_1)u_{k_1}(\tau_2)\ , \tag{6.3.20}$$

$$\langle\hat{\zeta}(\tau_1,\vec{k}_1)\hat{\zeta}'(\tau_2,\vec{k}_2)\rangle = (2\pi)^3\delta(\vec{k}_1+\vec{k}_2)u^*_{k_1}(\tau_1)u'_{k_1}(\tau_2)\ , \tag{6.3.21}$$

and in particular, the power spectrum, evaluated at the bounce point τ_B (well after Hubble radius exit), is given by

$$\langle\hat{\zeta}(\tau_B,\vec{k})\hat{\zeta}(\tau_B,\vec{k}')\rangle = (2\pi)^3\delta^{(3)}(\vec{k}+\vec{k}')\frac{2\pi^2}{k^3}\mathcal{P}_\zeta(\tau_B,k)\ , \tag{6.3.22}$$

where

$$\mathcal{P}_\zeta(\tau_B,k) = \frac{A^2}{8\pi^2\epsilon c_\mathrm{s}(\tau_B-\tilde{\tau}_B)^6} = \frac{1}{12\pi^2 c_\mathrm{s}M_\mathrm{Pl}^2(\tau_B-\tilde{\tau}_B)^2}\ . \tag{6.3.23}$$

The scale invariance of the power spectrum in matter bounce cosmology is thus explicit from the above.

The above focused only on the scalar perturbations, but as mentioned in the introduction, the matter bounce scenario also generates a scale-invariant power spectrum of tensor perturbations. Considering the transverse and traceless perturbations to the spatial metric, $\delta g_{ij} = a^2 h_{ij}$, which can be decomposed as

$$h_{ij}(\tau,\vec{x}) = h_+(\tau,\vec{x})e^+_{ij} + h_\times(\tau,\vec{x})e^\times_{ij} \tag{6.3.24}$$

with two fixed polarization tensors e^+_{ij} and $e^\times_{ij}$, the second-order perturbed action has contributions of the form

$$S^{(2)} \supset \frac{M_\mathrm{Pl}^2}{4}\int \mathrm{d}\tau\mathrm{d}^3\vec{x}\ a^2\left[h'^2 - (\vec{\nabla}h)^2\right] \tag{6.3.25}$$

for each polarization state h_+ and $h_\times$. By normalizing each state as $\mu = aM_\mathrm{Pl}h/2$, the second-order perturbed action is of canonical form (μ is the Mukhanov-Sasaki variable), and the resulting equation of motion for each state is

$$\mu''_k + \left(k^2 - \frac{a''}{a}\right)\mu_k = 0\ , \tag{6.3.26}$$

where the equation is written in Fourier space. Since $a \sim \tau^2$ in a matter-dominated contracting phase, one has $a''/a = 2/\tau^2$, and so, one expects a scale-invariant power spectrum just as in de Sitter space. The tensor power spectrum is given by

$$\mathcal{P}_{\rm t} = 2\mathcal{P}_h = 2\left(\frac{2}{aM_{\rm Pl}}\right)^2 \frac{k^3}{2\pi^2}|\mu_k|^2 \ , \tag{6.3.27}$$

where the first factor of 2 accounts for the two polarizations $+$ and $\times$, and the factor $[2/(aM_{\rm Pl})]^2$ comes from the normalization of μ. Upon matching with quantum vacuum initial conditions at Hubble radius crossing similar to the above treatment for scalar modes, one finds the power spectrum of tensor modes at the bounce point to be given by

$$\mathcal{P}_{\rm t}(\tau_B, k) = \frac{2}{\pi^2 M_{\rm Pl}^2(\tau_B - \tilde{\tau}_B)^2} \ , \tag{6.3.28}$$

which is indeed independent of scale.

The tensor-to-scalar ratio is then defined to be

$$r \equiv \frac{\mathcal{P}_{\rm t}}{\mathcal{P}_\zeta} \ . \tag{6.3.29}$$

It follows from equations (6.3.23) and (6.3.28) that

$$r = 24c_{\rm s} \tag{6.3.30}$$

in the context of matter bounce cosmology with a general k-essence scalar field[8]. On one hand, this highlights the problem of standard matter bounce cosmology, which is driven by a canonical scalar field with $c_{\rm s} = 1$, in which case $r = 24$. On the other hand, the above result provides a natural mechanism to suppress the tensor-to-scalar ratio provided the k-essence scalar field has an appropriately small sound speed. For example, satisfying the observational bound [11] $r < 0.07$ at 95% confidence imposes a bound on the sound speed of the order of

$$c_{\rm s} \lesssim 0.0029 \ . \tag{6.3.31}$$

[8]Of course, this assumes that the perturbations remain constant on super-Hubble scales after the matter contraction phase, in particular through the bounce and until the beginning of the radiation-dominated expanding phase of standard Big Bang cosmology.

6.4 Non-Gaussianity

The previous section showed that a k-essence scalar field could yield a small tensor-to-scalar ratio in the context of the matter bounce scenario. This is done at the expense of having a small sound speed. In what follows, the goal is to compute the bispectrum and see how a small sound speed affects the results.

6.4.1 Cubic action

To evaluate the three-point correlation function, we must expand the action (6.3.13) up to third order. Let us recall the result of [202], the third-order interaction action of a general single scalar field[9],

$$
\begin{aligned}
S^{(3)} = \int \mathrm{d}t\mathrm{d}^3\vec{x}\, \Bigg\{ &- a^3\Big[\Sigma\Big(1-\frac{1}{c_\mathrm{s}^2}\Big)+2\lambda\Big]\frac{\dot{\zeta}^3}{H^3}+\frac{a^3\epsilon}{c_\mathrm{s}^4}(\epsilon-3+3c_\mathrm{s}^2)\zeta\dot{\zeta}^2 \\
&+\frac{a\epsilon}{c_\mathrm{s}^2}(\epsilon-2s+1-c_\mathrm{s}^2)\zeta(\partial\zeta)^2-2a\frac{\epsilon}{c_\mathrm{s}^2}\dot{\zeta}(\partial\zeta)(\partial\chi)+\frac{a^3\epsilon}{2c_\mathrm{s}^2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\eta}{c_\mathrm{s}^2}\right)\zeta^2\dot{\zeta} \\
&+\frac{\epsilon}{2a}(\partial\zeta)(\partial\chi)\partial^2\chi+\frac{\epsilon}{4a}(\partial^2\zeta)(\partial\chi)^2+2f(\zeta)\left.\frac{\delta L}{\delta\zeta}\right|_1\Bigg\}\ ,
\end{aligned}
\tag{6.4.32}
$$

where it is understood that $(\partial\zeta)^2=\partial_i\zeta\partial^i\zeta$, $(\partial\zeta)(\partial\chi)=\partial_i\zeta\partial^i\chi$, $\partial^2\zeta=\partial_i\partial^i\zeta$, and where we define χ such that $\partial^2\chi=a^2\epsilon\dot{\zeta}$. Also, we have

$$
\left.\frac{\delta L}{\delta\zeta}\right|_1 = a\left(\frac{\mathrm{d}\partial^2\chi}{\mathrm{d}t}+H\partial^2\chi-\epsilon\partial^2\zeta\right)\ , \tag{6.4.33}
$$

$$
\begin{aligned}
f(\zeta) = &\frac{\eta}{4c_\mathrm{s}^2}\zeta^2+\frac{1}{c_\mathrm{s}^2H}\zeta\dot{\zeta}+\frac{1}{4a^2H^2}\{-(\partial\zeta)(\partial\zeta)+\partial^{-2}[\partial_i\partial_j(\partial^i\zeta\partial^j\zeta)]\} \\
&+\frac{1}{2a^2H}\{(\partial\zeta)(\partial\chi)-\partial^{-2}[\partial_i\partial_j(\partial^i\zeta\partial^j\chi)]\}\ ,
\end{aligned}
\tag{6.4.34}
$$

where ∂^{-2} is the inverse Laplacian.

The first and second terms in the last line of equation (6.4.32) can be reexpressed as

$$
\frac{\epsilon}{2a}(\partial\zeta)(\partial\chi)\partial^2\chi+\frac{\epsilon}{4a}(\partial^2\zeta)(\partial\chi)^2=-\frac{a^3\epsilon^3}{2}\zeta\dot{\zeta}^2+\frac{\epsilon}{2a}\zeta(\partial_i\partial_j\chi)(\partial^i\partial^j\chi)+\mathcal{K}\ , \tag{6.4.35}
$$

[9] From here on, we take $M_\mathrm{Pl}=1$ for convenience.

where the boundary term is given by

$$\mathcal{K} = \partial_i \left[\zeta(\partial^i\chi)(\partial^2\chi) + \frac{1}{2}(\partial^i\zeta)(\partial\chi)^2 - \zeta(\partial^i\partial^j\chi)(\partial_j\chi) \right] . \tag{6.4.36}$$

Since the $\partial_i[...]$ term above does not contribute to the three-point function, the third-order action, equation (6.4.32), is equivalent to

$$\begin{aligned} S^{(3)} = \int \mathrm{d}t\mathrm{d}^3\vec{x} \, \Big\{ & - a^3 \Big[\Sigma \Big(1 - \frac{1}{c_\mathrm{s}^2} \Big) + 2\lambda \Big] \frac{\dot{\zeta}^3}{H^3} + \frac{a^3\epsilon}{c_\mathrm{s}^4}(\epsilon - 3 + 3c_\mathrm{s}^2)\zeta\dot{\zeta}^2 \\ & + \frac{a\epsilon}{c_\mathrm{s}^2}(\epsilon - 2s + 1 - c_\mathrm{s}^2)\zeta(\partial\zeta)^2 - 2a\frac{\epsilon}{c_\mathrm{s}^2}\dot{\zeta}(\partial\zeta)(\partial\chi) + \frac{a^3\epsilon}{2c_\mathrm{s}^2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\eta}{c_\mathrm{s}^2}\right)\zeta^2\dot{\zeta} \\ & - \frac{a^3\epsilon^3}{2}\zeta\dot{\zeta}^2 + \frac{\epsilon}{2a}\zeta(\partial_i\partial_j\chi)(\partial^i\partial^j\chi) + 2f(\zeta) \left. \frac{\delta L}{\delta\zeta} \right|_1 \Big\} . \end{aligned} \tag{6.4.37}$$

In the case of a canonical field with $c_\mathrm{s} = 1$, this action returns to equation (15) of [181]. Meanwhile, as usual the last term in this action is removed by performing the field redefinition

$$\zeta \to \tilde{\zeta} + f(\tilde{\zeta}) , \tag{6.4.38}$$

where $\tilde{\zeta}$ denotes the field after redefinition.

6.4.2 Contributions to the shape function

In this section, we calculate the three-point correlation function using the in-in formalism (to leading order in perturbation theory; see, e.g., [201, 481, 620] for the methodology),

$$\langle O(t) \rangle = -2 \ \mathrm{Im} \int_{-\infty}^{t} \mathrm{d}\bar{t} \ \langle 0|O(t)L_\mathrm{int}(\bar{t})|0\rangle , \tag{6.4.39}$$

where O represents a set of operators of the form $\hat{\zeta}^3$ in our case of interest. Then, the shape function, $\mathcal{A}$, is defined such that[10]

$$\langle \zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\zeta_{\vec{k}_3} \rangle = (2\pi)^7\delta^{(3)}\Big(\sum_i \vec{k}_i \Big) \frac{\mathcal{P}_\zeta^2}{\prod_i k_i^3} \mathcal{A}(\vec{k}_1, \vec{k}_2, \vec{k}_3) . \tag{6.4.40}$$

[10] We use $\zeta_{\vec{k}_i}$ to refer to $\hat{\zeta}(\tau, \vec{k}_i)$ to simplify the notation from here on.

In what follows, we list all the contributions to the shape function coming from the field redefinition and the interaction action (6.4.37). It is easy to check that, when taking the limit $c_{\rm s} = 1$, one recovers the results of [181] for the matter bounce with a canonical scalar field as expected.

Contribution from the field redefinition

In momentum space, the field redefinition can be written as

$$\zeta_{\vec{k}} \to \tilde{\zeta}_{\vec{k}} + \int \frac{d^3\vec{k}_1}{(2\pi)^3}\left[-\frac{3}{2c_{\rm s}^2} - \frac{3\epsilon}{4}\left(\frac{\vec{k}_1\cdot(\vec{k}-\vec{k}_1)}{k_1^2} - \frac{(\vec{k}\cdot\vec{k}_1)[\vec{k}\cdot(\vec{k}-\vec{k}_1)]}{k^2k_1^2}\right)\right]\tilde{\zeta}_{\vec{k}_1}\tilde{\zeta}_{\vec{k}-\vec{k}_1}\ . \quad (6.4.41)$$

This redefinition has the following contribution to the three-point correlation function,

$$\langle\zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\zeta_{\vec{k}_3}\rangle_{\rm redef} = \int \frac{d^3\vec{k}'}{(2\pi)^3}\left[-\frac{3}{2c_{\rm s}^2} - \frac{3\epsilon}{4}\left(\frac{\vec{k}'\cdot(\vec{k}_3-\vec{k}')}{k'^2} - \frac{(\vec{k}_3\cdot\vec{k}')(\vec{k}_3\cdot[\vec{k}_3-\vec{k}'])}{k_3^2k'^2}\right)\right]$$
$$\times\left(\zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\zeta_{\vec{k}'}\zeta_{\vec{k}_3-\vec{k}'}\right) + (2\ \text{permutations})\ , \quad (6.4.42)$$

and accordingly, the contribution to the shape function is

$$\mathcal{A}_{\rm redef} = \left(\frac{3\epsilon}{16} - \frac{3}{4c_{\rm s}^2}\right)\sum_i k_i^3 + \frac{3\epsilon}{64}\sum_{i\neq j}k_ik_j^2 - \frac{3\epsilon}{64\prod_i k_i^2}\left(\sum_{i\neq j}k_i^7k_j^2 + \sum_{i\neq j}k_i^6k_j^3 - 2\sum_{i\neq j}k_i^5k_j^4\right)\ . \quad (6.4.43)$$

When $c_{\rm s}^2 \ll 1$, this contribution is enhanced compared to the canonical case.

Contribution from the $\zeta\dot{\zeta}^2$ term

The term $\zeta\dot{\zeta}^2$ in equation (6.4.37) yields the following contribution to the bispectrum

$$\langle\zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\zeta_{\vec{k}_3}\rangle_{\zeta\dot{\zeta}^2} = -2\times 2\ {\rm Im}\int_{-\infty}^{\tau_B} d\bar{\tau}\ (2\pi)^3\delta\Big(\sum_i \vec{k}_i\Big)a^2\left[\frac{\epsilon}{c_{\rm s}^4}(\epsilon - 3 + 3c_{\rm s}^2) - \frac{\epsilon^3}{2}\right]$$
$$\times\, u_{k_1}^*(\tau_B)u_{k_1}(\bar{\tau})u_{k_2}^*(\tau_B)u_{k_2}'(\bar{\tau})u_{k_3}^*(\tau_B)u_{k_3}'(\bar{\tau}) + (2\ \text{permutations}). \quad (6.4.44)$$

To leading order in $c_{\rm s}k_i(\tau_B - \tilde{\tau}_B) \ll 1$, i.e. on scales larger than the sound Hubble radius[11], and recalling the solutions for u_k and u'_k [equations (6.3.18) and (6.3.19)], we get the following contribution to the shape function,

$$\mathcal{A}_{\zeta\dot{\zeta}^2} = -\frac{c_{\rm s}^2}{8}\left[\frac{1}{c_{\rm s}^4}(\epsilon - 3 + 3c_{\rm s}^2) - \frac{\epsilon^2}{2}\right]\sum_i k_i^3 \ . \tag{6.4.45}$$

Again, when $c_{\rm s}^2 \ll 1$, this contribution is enhanced compared to the canonical case.

Contribution from the $\dot{\zeta}\partial\zeta\partial\chi$ term

A similar computation for this term yields the following contribution to the shape function

$$\mathcal{A}_{\dot{\zeta}\partial\zeta\partial\chi} = -\frac{\epsilon}{8}\sum_i k_i^3 + \frac{\epsilon}{8\prod_i k_i^2}\Big(\sum_{i\neq j} k_i^7 k_j^2 - \sum_{i\neq j} k_i^4 k_j^5\Big). \tag{6.4.46}$$

We note that this contribution is independent of $c_{\rm s}$.

Contribution from the $\zeta(\partial_i\partial_j\chi)^2$ term

For this term, the contribution to the shape function is given by

$$\begin{aligned}\mathcal{A}_{\zeta(\partial_i\partial_j\chi)^2} =& -\frac{c_{\rm s}^2\epsilon^2}{32}\sum_i k_i^3 + \frac{c_{\rm s}^2\epsilon^2}{64}\sum_{i\neq j} k_i^2 k_j \\ &+ \frac{c_{\rm s}^2\epsilon^2}{64\prod_i k_i^2}\Big(\sum_i k_i^9 - \sum_{i\neq j} k_i^6 k_j^3 + 3\sum_{i\neq j} k_i^5 k_j^4 - 3\sum_{i\neq j} k_i^7 k_j^2\Big) \ .\end{aligned} \tag{6.4.47}$$

When $c_{\rm s}^2 \ll 1$, this contribution is suppressed compared to the canonical case.

Contribution from the $\dot{\zeta}^3$ term

The $\dot{\zeta}^3$ term is a new element in the Lagrangian caused by the nontrivial sound speed, which does not show up in the cubic action of canonical fields. Its contribution to the bispectrum

[11] This is also called the Jeans radius; see [180, 559] for an explicit definition of this scale and its role in matter bounce cosmology when $c_{\rm s} \neq 1$.

is

$$\langle\zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\zeta_{\vec{k}_3}\rangle_{\dot{\zeta}^3} = -6\times 2\ \mathrm{Im}\int_{-\infty}^{\tau_B} \mathrm{d}\bar{\tau}\ (2\pi)^3\delta^{(3)}\Big(\sum_i \vec{k}_i\Big)\Big(-\frac{aM_{\mathrm{Pl}}^2\epsilon}{Hc_{\mathrm{s}}^2}\Big)\Big(1-\frac{1}{c_{\mathrm{s}}^2}+2\frac{\lambda}{\Sigma}\Big) \times u_{k_1}^*(\tau_B)u'_{k_1}(\bar{\tau})u_{k_2}^*(\tau_B)u'_{k_2}(\bar{\tau})u_{k_3}^*(\tau_B)u'_{k_3}(\bar{\tau})\ , \tag{6.4.48}$$

where we have used the expression for Σ, equation (6.2.11). Then the contribution to the shape function is expressed as

$$\mathcal{A}_{\dot{\zeta}^3} = -\frac{9}{2}\left(1-\frac{1}{c_{\mathrm{s}}^2}+2\frac{\lambda}{\Sigma}\right)\sum_i k_i^3\ . \tag{6.4.49}$$

Since this is a new contribution compared to the canonical case, it vanishes for $c_{\mathrm{s}}^2 = 1$. Indeed, when $c_{\mathrm{s}}^2 = 1$, $\lambda/\Sigma \simeq (1-c_{\mathrm{s}}^2)/(6c_{\mathrm{s}}^2) = 0$ (see equation (6.7.81) in Appendix 6.7) and $1-1/c_{\mathrm{s}}^2 = 0$. We note though that when $c_{\mathrm{s}}^2 \ll 1$, this contribution is large.

Secondary contributions

The contribution from the term

$$\frac{a^3\epsilon}{2c_{\mathrm{s}}^2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\eta}{c_{\mathrm{s}}^2}\Big)\zeta^2\dot{\zeta}$$

in equation (6.4.37) is exactly zero since $\eta = 0$ during the matter contraction. We can also neglect the contribution from the term

$$\frac{a\epsilon}{c_{\mathrm{s}}^2}(\epsilon-2s+1-c_{\mathrm{s}}^2)\zeta(\partial\zeta)^2$$

since the leading order term of the resulting bispectrum is proportional to $c_{\mathrm{s}}^2 k_i^2(\tau_B-\tilde{\tau}_B)^2$, which means that it is suppressed outside the sound Hubble radius.

The above results differ from the ones of general single field inflation. As pointed out in [181], two main reasons account for the different non-Gaussianities between matter bounce cosmology and inflation. First, here the "slow-roll" parameter ϵ is of order one rather than being close to zero, so the amplitudes are larger and the higher-order terms in ϵ are not suppressed. Second, curvature perturbations grow on super-Hubble scales in a matter-dominated contracting universe, and this behaviour manifests itself in the integral of equation

(6.4.39), while for inflation, ζ usually remains constant after horizon-exit, so there is no such contribution.

In what follows, we summarize the above results and give a detailed analysis of the bispectrum. In particular, the differences with the canonical single field matter bounce scenario are discussed.

6.4.3 Summary of results

One can gather all the contributions above and get the total shape function,

$$\mathcal{A}_{\rm tot} = \left(-\frac{105}{32} + \frac{39}{16c_{\rm s}^2} + \frac{9c_{\rm s}^2}{128}\right)\sum_i k_i^3 + \frac{3}{256}(3c_{\rm s}^2+6)\sum_{i\neq j} k_i^2 k_j + \frac{3}{256\prod_i k_i^2} \times \left[3c_{\rm s}^2\sum_i k_i^9 + (10-9c_{\rm s}^2)\sum_{i\neq j}k_i^7k_j^2 - (3c_{\rm s}^2+6)\sum_{i\neq j}k_i^6k_j^3 + (9c_{\rm s}^2-4)\sum_{i\neq j}k_i^5k_j^4\right] , \tag{6.4.50}$$

where we have used $\epsilon = 3/2$ and $\lambda/\Sigma = (1-c_{\rm s}^2)/6c_{\rm s}^2$ for the matter contraction stage. Now the only free parameter in the total shape function is the sound speed $c_{\rm s}$. In what follows, we shall discuss several interesting aspects of this result.

Amplitude

The size of non-Gaussianity is depicted by the dimensionless amplitude parameter

$$f_{\rm NL}(\vec{k}_1,\vec{k}_2,\vec{k}_3) = \frac{10}{3}\frac{\mathcal{A}_{\rm tot}(\vec{k}_1,\vec{k}_2,\vec{k}_3)}{\sum_i k_i^3} . \tag{6.4.51}$$

As one can see in equation (6.4.50), for most values of $c_{\rm s}\in(0,1]$, the first term dominates the total shape function, and roughly, $f_{\rm NL}$ becomes

$$f_{\rm NL} \simeq -\frac{175}{16} + \frac{65}{8c_{\rm s}^2} + \frac{15c_{\rm s}^2}{64} , \tag{6.4.52}$$

which yields $f_{\rm NL}<0$ for $0.87 \lesssim c_{\rm s} \leq 1$ and $f_{\rm NL}>0$ for $c_{\rm s}\lesssim 0.87$. Thus, besides the negative amplitude in the canonical case [181], a small sound speed in matter bounce cosmology can produce a positive $f_{\rm NL}$. In the next section, we shall further discuss its behaviour in different limits to confront observations.

Shape

The shape of non-Gaussianity is described by the dimensionless shape function

$$\mathcal{F}(k_1/k_3, k_2/k_3) = \frac{\mathcal{A}_{\rm tot}}{k_1 k_2 k_3} \ . \tag{6.4.53}$$

Then, the first term in equation (6.4.50) gives exactly the form of the local shape. Thus, when the prefactor of the first term is nonvanishing ($c_{\rm s} \not\approx 0.87$), the shape function is dominated by the local form, while the remaining terms just give some corrections. The total shape of non-Gaussianity is shown in the left panel of Figure 6.1, which looks very similar to the plots in [181] for the canonical matter bounce except that the amplitude is much larger here with $c_{\rm s}$ small.

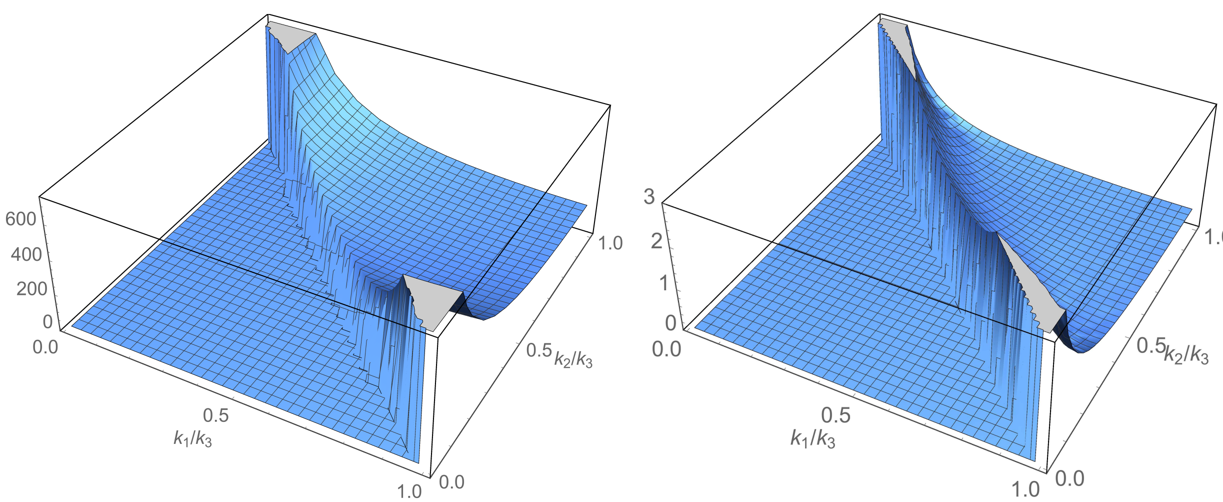

Figure 6.1 The shape of $\mathcal{F}(k_1/k_3, k_2/k_3)$ for $c_{\rm s} = 0.2$ (left panel) and $c_{\rm s} = 0.87$ (right panel).

At the same time, this result differs from the one of general single field inflation, where the equilateral form dominates the shape of non-Gaussianity for $c_{\rm s} \ll 1$ [202]. This is mainly caused by the different generation mechanisms of non-Gaussianity in these two scenarios. For the matter bounce scenario, the growth of curvature perturbations after Hubble radius exit makes a significant contribution to the final bispectrum. Meanwhile, the local form is usually thought to be generated on super-Hubble scales since "local" means that the non-Gaussianity at one place is *disconnected* with the one at other places. For general single field inflation, the dominant contribution is due to the enhanced interaction at horizon-crossing. Thus, these two scenarios behave quite differently with a small sound speed.

It is also interesting to note that for $c_s \approx 0.87$, the first term in equation (6.4.50) vanishes, so the shape function is dominated by the remaining terms. The shape of non-Gaussianity is plotted in the right panel of Figure 6.1 for this case, which is a new form different from the local one. To the best of our knowledge, no other scenario can give rise to such a kind of shape, thus it can be seen as a distinguishable signature of matter bounce cosmology for probes of non-Gaussianity.

The squeezed limit

Usually people are interested in the squeezed limit of the bispectrum ($k_1 \ll k_2 = k_3 = k$), since its scaling behaviour is helpful for clarifying the shapes of non-Gaussianity analytically. Here in the squeezed limit ($k_1/k \to 0$), the dimensionless shape function can be expanded as

$$\mathcal{F}(k_1/k_3, k_2/k_3) \simeq \frac{3}{8}\left(-\frac{33}{2}+\frac{13}{c_s^2}\right)\frac{k}{k_1}+\frac{3}{64}\left(1+6c_s^2\right)\frac{k_1}{k}+\mathcal{O}\left(\left(\frac{k_1}{k}\right)^2\right) . \tag{6.4.54}$$

The leading order term gives the scaling $\mathcal{F} \sim k/k_1$ and

$$\langle \zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\zeta_{\vec{k}_3}\rangle_{\text{squeezed}} \sim \frac{1}{k_1^3} , \tag{6.4.55}$$

which is consistent with the dominant local form. The only exception is when the coefficient of the first term vanishes ($c_s = \sqrt{26/33}$) and another scaling, $\mathcal{F} \sim k_1/k$, follows from the next-to-leading order term.

6.5 Amplitude parameter of non-Gaussianities and implication for the no-go theorem

There are three forms of the amplitude parameter $f_{\rm NL}$ that are of particular interest for cosmological observations. They are called the "local form", the "equilateral form", and the "folded form". The local form requires that one of the three momentum modes exits the Hubble radius much earlier than the other two, e.g., $k_1 \ll k_2 = k_3$. In this limit, one can simplify the total shape function, equation (6.4.50), to find

$$f_{\rm NL}^{\rm local} \simeq -\frac{165}{16} + \frac{65}{8c_{\rm s}^2} \ . \tag{6.5.56}$$

The equilateral form requires that the three momenta form an equilateral triangle, i.e. $k_1 = k_2 = k_3$. In this case, we obtain

$$f_{\rm NL}^{\rm equil} \simeq -\frac{335}{32} + \frac{65}{8c_{\rm s}^2} + \frac{45c_{\rm s}^2}{128} \ . \tag{6.5.57}$$

The folded form has $k_1 = 2k_2 = 2k_3$, hence

$$f_{\rm NL}^{\rm folded} \simeq -\frac{37}{4} + \frac{65}{8c_{\rm s}^2} \ . \tag{6.5.58}$$

As a result, in the limit where $c_{\rm s}^2 \ll 1$, we find that

$$f_{\rm NL}^{\rm local} \approx f_{\rm NL}^{\rm equil} \approx f_{\rm NL}^{\rm folded} \approx \frac{65}{8c_{\rm s}^2} \gg 1 \ . \tag{6.5.59}$$

Let us recall from section 6.3 that in order to satisfy the observational bound on the tensor-to-scalar ratio, we must impose $c_{\rm s} \lesssim 0.0029$. This immediately implies

$$f_{\rm NL}^{\rm local} \approx f_{\rm NL}^{\rm equil} \approx f_{\rm NL}^{\rm folded} \gtrsim 9.55 \times 10^5 \gg 1 \ . \tag{6.5.60}$$

This amplitude of primordial non-Gaussianity is clearly ruled out according to the observations [13],

$$f_{\rm NL}^{\rm local} = 0.8 \pm 5.0 \ , \quad f_{\rm NL}^{\rm equil} = -4 \pm 43 \ , \quad f_{\rm NL}^{\rm ortho} = -26 \pm 21 \ , \tag{6.5.61}$$

thus ruling out the viability of the class of models studied here.

Alternatively, if one requires that, e.g., $-9.2 \lesssim f_{\rm NL}^{\rm local} \lesssim 10.8$ (i.e., imposing $f_{\rm NL}^{\rm local}$ to be within the measured 2σ error bars), then one would need[12] $c_{\rm s} \gtrsim 0.62$. However, this lower bound on the sound speed yields a tensor-to-scalar ratio $r \gtrsim 14.88$, which is again clearly ruled out by observations [11].

In summary, there is no region of parameter space where $c_{\rm s}$ can give a good, small tensor-

[12] Note that this constraint does not exclude $c_{\rm s} \approx 0.87$, for which the new shape of non-Gaussianity in the right panel of Figure 6.1 emerges.

to-scalar ratio (i.e., of order 0.1 at most) and good, small non-Gaussianities (i.e., of order 10 at most). Therefore, independent of what happens during the bounce, we extend the no-go theorem conjectured in [562] to the following one:

No-Go Theorem 6.1. *For quantum fluctuations generated during a matter-dominated contracting phase, an upper bound on the tensor-to-scalar ratio (r) is equivalent to a lower bound on the amount of primordial non-Gaussianities (f_{NL}). Furthermore, if*

- *the matter contraction phase is due to a single (not necessarily canonical) scalar field,*
- *the same single scalar field allows for the violation of the NEC to produce a nonsingular bounce,*
- *and General Relativity holds at all energy scales,*

then satisfying the current observational upper bound on the tensor-to-scalar ratio cannot be done without contradicting the current observational upper bounds on f_{NL} (and vice versa).

6.6 Conclusions and discussion

In this paper, we computed the two- and three-point correlation functions produced by a generic k-essence scalar field in a matter-dominated contracting universe. Comparing the power spectra of scalar and tensor modes, we found that the tensor-to-scalar ratio can be appropriately suppressed if the sound speed associated with the k-essence scalar field is sufficiently small. In turn, we showed that the amplitude of the bispectrum is amplified by the smallness of the sound speed[13]. As a result, it seems incompatible to suppress the tensor-to-scalar ratio below current observational bounds without producing excessive non-Gaussianities. This leads us to extend the conjecture of the no-go theorem, which effectively rules out a large class of nonsingular matter bounce models.

Although this seriously constrains nonsingular matter bounce cosmology as a viable alternative scenario to inflation, there remain several classes of models that are not affected

[13] With a small sound speed, one may also reach the strong coupling regime where the perturbative analysis breaks down. This is known as the strong coupling problem [62, 374], which affects many non-inflationary scenarios (see in particular Appendix C of [62], which focuses on non-attractor models). It represents a general independent theoretical constraint, but in the context of the matter bounce scenario, our no-go theorem is more constraining due to current observational bounds.

by this no-go theorem. Indeed, one could still evade the no-go theorem assuming certain modified gravity models as stated in [562] (see references therein) or with the introduction of one or several new fields. For example, in the matter bounce curvaton scenario [168] (see also [5, 26, 556] for other nonsingular bouncing models using the curvaton mechanism) and in the two-field matter bounce scenario [175], entropy modes are generated by the presence of an additional scalar field, which are then converted to curvature perturbations. In both models near the bounce, the kinetic term of the entropy field varies rapidly, which acts as a tachyonic-like mass that amplifies (in a controlled way) the entropy fluctuations while not affecting the tensor modes. As a result, the tensor-to-scalar ratio is suppressed (see [166, 178] for reviews of this process). Furthermore, the production of non-Gaussianities in the matter bounce curvaton scenario has been estimated in [168], and it indicated that sizable, negative non-Gaussianities appeared, yet still in agreement with current observations. Accordingly, such a curvaton scenario does not appear to suffer from a no-go theorem. However, there still remains to do an appropriate extensive analysis of the production of non-Gaussianities when general multifields are included in the matter bounce scenario.

A similar curvaton mechanism is used in the new Ekpyrotic model [155, 427] (extended in [436, 437, 557, 625]), which generates a nearly scale-invariant power spectrum of curvature perturbations. In this case, however, the smallness of the observed tensor-to-scalar ratio must be attributed to the fact that the tensor modes have a blue power spectrum when they exit the Hubble radius in a contracting phase with $w \gg 1$. The new Ekpyrotic model originally predicted large non-Gaussianities [156, 157, 428–430] (see also the reviews [424, 425]), but some more recent extensions can resolve this issue [280, 281, 283, 361, 433]. Thus, here as well, it appears that these types of models do not suffer from a similar no-go theorem[14].

We note that one might be able to prove the no-go conjecture of this paper borrowing similar techniques to the effective field theory of inflation [204], i.e. by constructing an effective field theory of nonsingular bouncing cosmology (e.g., see the recent work of [165, 227]). In complete generality, this could allow us to find the exact and explicit relation between the tensor-to-scalar ratio (which involves the power spectra of curvature and tensor modes) and the bispectrum. In fact, the goal would be to find a consistency relation for the three-point function in single field nonsingular bouncing cosmology similar to what has been

[14]Furthermore, Ekpyrotic models are robust against the growth of anisotropies in a contracting universe. This is another challenge with the matter bounce scenario (see [167, 432]) that will have to be overcome to have a viable theory.

done in inflation [205, 229, 481]. This will be explored in a follow-up study.

Finally, we would like to emphasize that, for matter bounce cosmology, although the simplest k-essence model is ruled out by the no-go theorem, the bispectrum with $c_{\rm s} \neq 1$ (as an independent result of this paper) remains to be a probable target for future probes of non-Gaussianity. This possibility relies on the aforementioned bouncing models that can evade the no-go theorem with other mechanisms. In those cases, a nontrivial sound speed may still lead to the same behaviour of non-Gaussianities found in this paper, which potentially can be detected by future observations. Particularly, we predict a new shape with an amplitude still consistent with current observational limits, which can serve as the distinctive signature of matter bounce cosmology and help us distinguish it from other very early universe theories.

6.7 The ratio λ/Σ

Let us recall the definition of Σ and λ in equations (6.2.11) and (6.2.12). Their ratio is thus given by

$$\frac{\lambda}{\Sigma} = \frac{1}{3}\left(X\frac{\Sigma_{,X}}{\Sigma} - 1\right) . \tag{6.7.62}$$

Recalling the definition of $c_{\rm s}^2$ in equation (6.2.10), we note that

$$\Sigma = X(P_{,X} + 2XP_{,XX}) = X\frac{P_{,X}}{c_{\rm s}^2} . \tag{6.7.63}$$

Also, recalling the expression for ρ and p in equation (6.2.7), we find that $2XP_{,X} = \rho + p$, and so, the above expression for Σ becomes

$$\Sigma = \frac{\rho + p}{2c_{\rm s}^2} . \tag{6.7.64}$$

Consequently,

$$X\frac{\Sigma_{,X}}{\Sigma} = X\frac{\rho_{,X} + p_{,X}}{\rho + p} - 2X\frac{c_{{\rm s},X}}{c_{\rm s}} . \tag{6.7.65}$$

Working in the limit where $p = 0$, we note that $\rho = 2XP_{,X}$, and so, $p_{,X} = P_{,X} = \rho/(2X)$, which implies that $p_{,X}/\rho = 1/(2X)$. Also, $\rho_{,X} = p_{,X}/c_{\rm s}^2$ from the definition of the sound speed, and thus,

$$\frac{\rho_{,X}}{\rho} = \frac{p_{,X}}{\rho c_{\rm s}^2} = \frac{1}{2c_{\rm s}^2 X} . \tag{6.7.66}$$

Therefore, equation (6.7.65) in the limit where $p=0$ becomes

$$X\frac{\Sigma_{,X}}{\Sigma}=\frac{1}{2c_{\rm s}^2}+\frac{1}{2}-2X\frac{c_{{\rm s},X}}{c_{\rm s}}\ . \tag{6.7.67}$$

Alternatively, one can evaluate the ratio λ/Σ as

$$\frac{\lambda}{\Sigma}=\frac{1}{3}\left(\frac{\Sigma_{,X}}{\Sigma}X-1\right)=\frac{1}{3}\left(\frac{\dot\Sigma}{\Sigma}\frac{X}{\dot X}-1\right)\ . \tag{6.7.68}$$

Since we can write $\Sigma=H^2M_{\rm Pl}^2\epsilon/c_{\rm s}^2$ and recalling the definition of the slow-roll parameters in section 6.2, we get

$$\frac{\dot\Sigma}{H\Sigma}=-2\epsilon+\eta-2s\ . \tag{6.7.69}$$

Now, we note that we can write

$$\eta=\frac{\dot\epsilon}{H\epsilon}=\frac{\ddot H}{H\dot H}-2\frac{\dot H}{H^2}=\frac{\ddot H}{H\dot H}+2\epsilon\ . \tag{6.7.70}$$

Also, the Friedmann equation $M_{\rm Pl}^2\dot H=-XP_{,X}$ implies that

$$\frac{\ddot H}{H\dot H}=\frac{1}{H}\left(\frac{\dot X}{X}+\frac{\dot P_{,X}}{P_{,X}}\right)\ , \tag{6.7.71}$$

and so,

$$\frac{\dot X}{HX}=\eta-2\epsilon-\frac{\dot P_{,X}}{P_{,X}}\ . \tag{6.7.72}$$

Therefore, combining equation (6.7.69) and the above yields

$$\frac{\dot\Sigma}{\Sigma}\frac{X}{\dot X}=\frac{-2\epsilon+\eta-2s}{-2\epsilon+\eta-\frac{\dot P_{,X}}{P_{,X}}}\ . \tag{6.7.73}$$

In the limit where $p=0$, we recall that $\epsilon=3/2$ and $\eta=0$, and as a result,

$$\frac{\dot\Sigma}{\Sigma}\frac{X}{\dot X}=\frac{3+2s}{3+\frac{\dot P_{,X}}{P_{,X}}}\ . \tag{6.7.74}$$

Comparing the above with equation (6.7.67), since $(\dot{\Sigma}/\Sigma)(X/\dot{X}) = X\Sigma_{,X}/\Sigma$, we find

$$\frac{3+2s}{3+\frac{\dot{P}_{,X}}{P_{,X}}} = \frac{1}{2c_{\rm s}^2} + \frac{1}{2} - 2X\frac{c_{{\rm s},X}}{c_{\rm s}} \ , \tag{6.7.75}$$

but

$$-2X\frac{c_{{\rm s},X}}{c_{\rm s}} = -2\frac{X}{\dot{X}}\frac{\dot{c}_{\rm s}}{c_{\rm s}} = -2s\frac{HX}{\dot{X}} = \frac{-2s}{\eta - 2\epsilon - \frac{\dot{P}_{,X}}{P_{,X}}} \ , \tag{6.7.76}$$

where the last equality follows from equation (6.7.72). Thus, equation (6.7.75), with $\epsilon = 3/2$ and $\eta = 0$, leaves us with

$$\frac{3}{3+\frac{\dot{P}_{,X}}{P_{,X}}} = \frac{1}{2c_{\rm s}^2} + \frac{1}{2} \ , \tag{6.7.77}$$

and consequently,

$$\frac{\dot{P}_{,X}}{P_{,X}} = -3\left(\frac{1-c_{\rm s}^2}{1+c_{\rm s}^2}\right) \ . \tag{6.7.78}$$

As a result, equation (6.7.74) becomes

$$\frac{\dot{\Sigma}}{\Sigma}\frac{X}{\dot{X}} = X\frac{\Sigma_{,X}}{\Sigma} = \frac{1}{2c_{\rm s}^2}\left(1+\frac{2}{3}s\right)(1+c_{\rm s}^2) \ , \tag{6.7.79}$$

and in the end, (6.7.68) is equivalent to

$$\frac{\lambda}{\Sigma} = \frac{1}{3}\left[\frac{1}{2c_{\rm s}^2}\left(1+\frac{2}{3}s\right)(1+c_{\rm s}^2) - 1\right] \ . \tag{6.7.80}$$

In the limit where $|s| \ll 1$, this reduces to

$$\frac{\lambda}{\Sigma} \simeq \frac{1}{3}\left[\frac{1+c_{\rm s}^2}{2c_{\rm s}^2} - 1\right] = \frac{1-c_{\rm s}^2}{6c_{\rm s}^2} \ . \tag{6.7.81}$$

In comparison, DBI inflation has $\lambda/\Sigma = (1-c_{\rm s}^2)/(2c_{\rm s}^2)$ (see [202]).

Chapter 7

Massive gravity and the suppression of anisotropies and gravitational waves in a matter-dominated contracting universe

7.1 Introduction

The inflationary scenario [150, 274, 333, 578, 590] is the current paradigm of very early universe cosmology. It solves a number of conceptual problems within standard Big Bang cosmology and makes predictions for the structure in the Universe, which are confirmed to great precision by observations, in particular the slight red tilt [517] in the spectrum of scalar cosmological perturbations [14]. However, current realizations of inflation have some conceptual problems (see, e.g., [115, 129]), in particular the Trans-Planckian problem for cosmological perturbations [142, 486]. Hence, it is of interest to consider possible alternative very early universe scenarios.

In fact, alternatives to cosmological inflation exist (see, e.g., [109, 132, 133] for reviews). In particular, bouncing cosmologies [56, 166, 529] may provide alternatives to cosmological inflation. The Ekpyrotic scenario [387] is one candidate scenario. This scenario is based on postulating the existence of a new form of matter with an equation of state (EoS) $p \gg \rho$, where p and ρ are the pressure and energy density, respectively. In this case, an isotropic phase of contraction is a local attractor in initial condition space [273, 306] (see also [433] regarding the fine-tuning of the initial conditions), in the same way as for inflationary cosmology an inflating expanding background is a local attractor in initial condition space [103, 141, 278]. However, the spectrum of adiabatic cosmological perturbations has a deep blue spectrum [107, 388, 473], and one needs to make use of entropy fluctuations to produce a spectrum of nearly scale-invariant curvature perturbations at late times [155, 228, 255, 285, 427, 528] (see [424] for a review).

The *matter bounce* scenario is another alternative to inflation. It has opposite strengths and weaknesses compared to the Ekpyrotic scenario. On one hand, one does not need to specify any new forms of matter (except new physics required to obtain a non-singular cosmological bounce). The idea is that the universe starts in a homogeneous and isotropic contracting phase with the same matter content of the current expanding universe, i.e. with cold matter, radiation, and possibly a very small cosmological constant, required to explain the currently observed dark energy component. Then, it can be shown [286, 618] that adiabatic fluctuations with comoving wavelengths which originate in their quantum vacuum state and exit the Hubble radius during the matter-dominated phase of contraction acquire an almost scale-invariant spectrum at late times in the contracting phase. In fact, the presence of the dark energy component leads to a slight red tilt of the scalar spectrum [174, 180], in agreement with the observed spectrum. On the other hand, the homogeneous and isotropic contracting trajectory is not an attractor in initial condition space. In fact, the energy density in anisotropies grows faster than the energy density in the matter components, leading to an instability of the model [167], known as the Belinsky-Khalatnikov-Lifshitz (BKL) instability [67]. This problem is usually evaded with the inclusion of either higher-order curvature terms in the gravity action [500], a phase of Ekpyrotic contraction (see the models studied in [167, 172, 175, 561]), or another source with ultra-stiff EoS [98]. However, these resolutions are often fine-tuned [98, 432] or simply not robust to all types of anisotropies [50].

A second problem for matter bounce scenarios is that the scalar cosmological perturbations and gravitational waves grow at the same rate on super-Hubble scales since the squeezing factors in their mode equations of motion (EOMs) are the same (see, e.g., [121, 518] for reviews of the theory of cosmological perturbations). Because of that, the scalar and tensor power spectra have the same amplitude at the end of the contracting phase, i.e., the tensor-to-scalar ratio r is of order unity (see, e.g., [177, 178, 438, 562]). In addition, the amplitude of non-Gaussianities (characterized by the quantity $f_{\rm NL}$) is of order unity [181] (see, however, [302, 303]). Although it is possible to construct single field bounce models which boost the scalar fluctuations relative to the gravitational waves, these mechanisms typically also boost the non-Gaussianities to a level which is in contradiction with the current limits, which led to the conjecture of a no-go theorem for the class of single field matter bounce models [438, 562]. We call this the *large r problem*.

In this paper, we suggest a possible solution to both of these problems of the matter bounce scenario in the context of a modified gravity model with a massive graviton. The idea of modifying general relativity so that the graviton acquires a non-trivial mass has been extensively studied (see, e.g., the review [567] and also [234, 244] in the context of cosmology), especially as an attempt to explaining the accelerated expansion of the universe (see, e.g., [337, 338]). Massive gravity has also been studied in the context of very early universe cosmology, e.g., during inflation, in which case the propagation of gravitational waves would be affected by the non-trivial mass of the graviton [336, 412, 460] (see also [259]). Using this setup in the context of matter bounce cosmology, we find that the fluctuation equation for the gravitational waves has a mass term which prevents the squeezing of the modes on super-Hubble scales and hence solves the large r problem. As it turns out, the massive graviton also leads to a mass term in the EOM for the anisotropy parameter, and hence provides a natural isotropization mechanism in the contracting phase.

In the following, we first introduce the modified gravity theory we will be using. A Hamiltonian analysis shows that the theory is free of ghosts at the fully non-perturbative level and contains only two propagating gravitational modes. In section 7.3 we study the background evolution and show that the functions appearing in the gravitational action can be chosen such that a non-singular cosmological bounce results. In section 7.4 we study the evolution of cosmological fluctuations in the theory, discussing scalar, vector, and tensor modes. In particular, we show how the mass term due to the non-trivial graviton mass arises in the gravitational wave EOMs. In section 7.5 we show how a mass term also arises in the EOMs governing the anisotropies. Section 7.6 presents explicit solutions for the anisotropies and for the graviton spectrum in a matter phase of contraction. We discuss both the limits of large and small graviton masses and connect the results with observations. We summarize and discuss our results in section 7.7.

A word concerning notation: we work in natural units ($c = \hbar = 1$) with the reduced Planck mass and time defined by $M_{\rm Pl} \equiv t_{\rm Pl}^{-1} \equiv 1/\sqrt{8\pi G_{\rm N}}$, and we assume for computational simplicity that the universe is spatially flat.

7.2 The modified gravity theory

7.2.1 Setup

Our goal is to construct a modified theory of gravity that allows for a non-trivial graviton mass, while being as close as possible to Einstein gravity. In order to achieve this, we work with the Arnowitt-Deser-Misner (ADM) [35] decomposition of the four-dimensional metric $g_{\mu\nu}$,

$$g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -N^2\mathrm{d}t^2 + h_{ij}\left(\mathrm{d}x^i + N^i\mathrm{d}t\right)\left(\mathrm{d}x^j + N^j\mathrm{d}t\right) , \qquad (7.2.1)$$

where $N(t,\mathbf{x})$ and $N^i(t,\mathbf{x})$ are the lapse and shift functions, respectively, and h_{ij} is the three-dimensional induced metric tensor on the spatial hypersurface. In our theory, the graviton mass arises[1] from the non-trivial vacuum expectation value of four Stückelberg scalar fields,

$$\varphi^0(t,\mathbf{x}) = M^2 f(t)\,, \qquad \varphi^a(t,\mathbf{x}) = M^2 x^i \delta_i{}^a\,, \qquad (7.2.2)$$

where $a \in \{1,2,3\}$ and the Stückelberg scalar fields have mass dimension ($f(t)$ has dimension $[-1]$ and M is a mass scale). In the scalar field configuration, the following internal symmetries are imposed [260, 261],

$$\varphi^a \to \Lambda^a{}_b\varphi^b\,, \qquad \varphi^a \to \varphi^a + \Xi^a(\varphi^0)\,, \qquad (7.2.3)$$

where $\Lambda^a{}_b$ is the $SO(3)$ rotation operator, and $\Xi^a(\varphi^0)$ are three generic functions of their argument. The internal symmetry between the space-like Stückelberg scalar fields φ^a and the time-like Stückelberg scalar field φ^0, i.e. $\varphi^a \to \varphi^a + \Xi^a(\varphi^0)$, is crucial to eliminate the vector modes in the gravity sector, as we will see later in section 7.4.2. This symmetry projects out all temporal derivative terms of φ^a, and therefore, one finds at the end that these scalar fields are actually non-dynamical. A more rigorous proof will also be given with the Hamiltonian analysis in the next sub-section (see section 7.2.2).

At the first derivative level, we have the following quantity that respects the symmetries of equation (7.2.3) (see [260, 261]),

$$Z^{ab} = g^{\mu\nu}\partial_\mu\varphi^a\partial_\nu\varphi^b - \frac{(g^{\mu\nu}\partial_\mu\varphi^0\partial_\nu\varphi^a)(g^{\kappa\xi}\partial_\kappa\varphi^0\partial_\xi\varphi^b)}{X}\,, \qquad (7.2.4)$$

[1] The following setup to generate a non-trivial graviton mass is a generalization of the Lorentz-violating massive gravity theory of [260, 261].

where $X \equiv g^{\mu\nu}\partial_\mu\varphi^0\partial_\nu\varphi^0$ represents the kinetic term of the time-like Stückelberg scalar field φ^0. We note that in unitary gauge, one has $Z^{ab} = M^4 h^{ij}\delta_i{}^a\delta_j{}^b$. Then, the graviton mass term can be written as a generic function of Z^{ab}. However, it is useful to first define the following traceless tensor [456, 460],

$$\bar{\delta}Z^{ab} \equiv \frac{Z^{ab}}{Z} - 3\frac{Z^a{}_c Z^{cb}}{Z^2}\,, \tag{7.2.5}$$

where the internal indices are raised or lowered with δ_{ab}, i.e., $Z \equiv Z^{ab}\delta_{ab}$, $Z^a{}_c \equiv Z^{ad}\delta_{dc}$, and $Z_{ab} \equiv Z^{cd}\delta_{ca}\delta_{db}$. The graviton mass is then written in terms of the contraction of this traceless tensor,

$$\mathcal{L}_{\mathrm{mass}} \sim M_{\mathrm{Pl}}^2 m_g^2 \bar{\delta}Z^{ab}\bar{\delta}Z_{ab}\,, \tag{7.2.6}$$

which is a term that breaks space-time diffeomorphism invariance in unitary gauge, and $m_g = m_g(t, Z^{ab}, \delta_{ab})$ is a scalar function of its arguments in the internal space. The resulting action of our minimally-modified effective field theory (EFT) of gravity reads

$$S = \int \mathrm{d}^3\mathbf{x}\,\mathrm{d}t\,\sqrt{h}\left[N\left(\frac{M_{\mathrm{Pl}}^2}{2}{}^{(4)}R - \Lambda_1(t)\right) - \frac{9}{8}M_{\mathrm{Pl}}^2 m_g^2\bar{\delta}Z^{ab}\bar{\delta}Z_{ab} - \Lambda_2(t)\right]\,, \tag{7.2.7}$$

where $\Lambda_1(t)$ and $\Lambda_2(t)$ are some functions of time, which will be fixed later by the EOMs. The four-dimensional Ricci scalar is denoted by ${}^{(4)}R$, and according to the ADM formalism, it can be decomposed as

$${}^{(4)}R = K_{ij}K^{ij} - K^2 + {}^{(3)}R + \text{total derivatives}\,, \tag{7.2.8}$$

where K_{ij} is the extrinsic curvature tensor, $K \equiv h^{ij}K_{ij}$ is its trace, and ${}^{(3)}R$ is the three-dimensional Ricci scalar on the spatial hypersurface. Accordingly, due to the broken temporal diffeomorphism invariance, the action could actually be written in even more generality as

$$\begin{aligned} S = \int \mathrm{d}^3\mathbf{x}\,\mathrm{d}t\,\sqrt{h}\Bigg[N\Bigg(\frac{M_{\mathrm{Pl}}^2}{2}\Big[c_1(t)\left(K_{ij}K^{ij} - K^2\right) + c_2(t){}^{(3)}R\Big] - \Lambda_1(t)\Bigg) \\ - \frac{9}{8}M_{\mathrm{Pl}}^2 m_g^2\bar{\delta}Z^{ab}\bar{\delta}Z_{ab} - \Lambda_2(t)\Bigg]\,, \end{aligned} \tag{7.2.9}$$

of which phase space has even dimension, hence it is free from pathological inconsistencies [457]. With the above action, we would find that the propagation speed of gravitational waves is given by $c_g(t) = \sqrt{c_2(t)/c_1(t)}$. However, in what follows, we will often set $c_1 = c_2 = 1$ so that[2] $c_g = 1$. The m_g^2 term is the graviton mass term, and it does not contribute to the background evolution. Its non-trivial contribution only appears in the equations for perturbations. In particular, this theory is a generalization of one of the theories discussed in [457] (see also [453]).

In the action of equation (7.2.7), we note that Λ_1 is akin to a cosmological constant as in ΛCDM cosmology. However, we allow it to be time-dependent, which can be done in a consistent manner as long as the appropriate constraint equation is satisfied as we will see in section 7.2.3. This is similar to the constraint equation that follows from allowing the cosmological constant, Newton's constant, and the speed of light to actually be non-constant (see, e.g., [221, 294]). Here, the key is that we allow for another cosmological constant-like function, Λ_2, which is independent of the lapse function in the EFT and which can also be time dependent. This type of function can appear in different modified theories of gravity, for instance in Cuscuton cosmology [16, 17, 325].

Note that we are working in unitary gauge. As always, rather than starting from an EFT, it is possible to recover the four-dimensional general covariance by introducing a scalar field, e.g. $\Lambda_1(t) \to \Lambda_1(\varphi^0(t, \mathbf{x}))$ and so on. From the point of view of such a covariant description of the same theory, the action of equation (7.2.7) written in terms of N, N^i, h_{ij}, and without the scalar field, is nothing but the action in the so-called unitary gauge, in which the time coordinate is chosen to agree with a fixed monotonic function of the scalar field.

As we will see later, this theory is ghost free and able to realize a non-singular bounce, and at the same time, it yields a mass correction to gravitational waves and anisotropies. It also has only two degrees of freedom (DOFs) in the gravity sector as we will now demonstrate.

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[2] Thus, the model is consistent with the constraints coming from the joint observations of gravitational waves and electromagnetic signals from GW170817 [3].

7.2.2 Hamiltonian analysis

The conjugate momenta of the theory are given by

$$\pi^{ij} \equiv \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \frac{M_{\rm Pl}^2}{2}\sqrt{h}\left(K^{ij} - Kh^{ij}\right)\,, \tag{7.2.10}$$

$$\pi_N \equiv \frac{\partial \mathcal{L}}{\partial \dot{N}} = 0\,, \tag{7.2.11}$$

$$\pi_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{N}^i} = 0\,, \tag{7.2.12}$$

where the Lagrangian density $\mathcal{L}$ can be read off from the action of equation (7.2.7), and a dot denotes a derivative with respect to physical time t. The Hamiltonian then reads

$$\mathsf{H} = \int \mathrm{d}^3\mathbf{x}\ \left(\pi^{ij}\dot{h}_{ij} - \mathcal{L} + \lambda_N\pi_N + \lambda^i\pi_i\right) \tag{7.2.13}$$

$$= \int \mathrm{d}^3\mathbf{x}\ \left[N\mathcal{C} + N^i\mathcal{H}_i + \lambda_N\pi_N + \lambda^i\pi_i - \sqrt{h}\,G(h^{ij},t)\right]. \tag{7.2.14}$$

In the above, λ_N and the λ^i's are Lagrange multipliers, and so, we have

$$\pi_N \approx 0 \qquad \text{and} \qquad \pi_i \approx 0 \tag{7.2.15}$$

as primary constraints. Also, we defined

$$G(h^{ij},t) \equiv -\frac{9}{8}M_{\rm Pl}^2 m_g^2 \bar{\delta}Z^{ab}\bar{\delta}Z_{ab} - \Lambda_2(t)\,, \tag{7.2.16}$$

which represents the lapse-independent terms in the action, and

$$\mathcal{C} \equiv \frac{2}{M_{\rm Pl}^2\sqrt{h}}\left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2\right) - \frac{M_{\rm Pl}^2}{2}\sqrt{h}\,{}^{(3)}\!R + \sqrt{h}\,\Lambda_1(t)\,, \tag{7.2.17}$$

$$\mathcal{H}_i \equiv -2\sqrt{h}D_j\left(\frac{\pi_i{}^j}{\sqrt{h}}\right). \tag{7.2.18}$$

In the above, D_i is the covariant derivative on the three-dimensional spatial hypersurface, and the indices of π^{ij} are lowered or raised with the three-dimensional metric tensor, i.e., $\pi \equiv h_{ij}\pi^{ij}$, $\pi_i{}^j \equiv h_{il}\pi^{lj}$, etc. The consistency conditions of the four primary constraints in equation (7.2.15) give us

$$\frac{\mathrm{d}\pi_N}{\mathrm{d}t} = \{\pi_N, \mathsf{H}\} = -\mathcal{C} \approx 0\,, \tag{7.2.19}$$

$$\frac{\mathrm{d}\pi_i}{\mathrm{d}t} = \{\pi_i, \mathsf{H}\} = -\mathcal{H}_i \approx 0\,, \tag{7.2.20}$$

which represent the Hamiltonian and momentum constraints, respectively, and they are the secondary constraints. The consistency condition of the Hamiltonian constraint gives us the following tertiary constraint,

$$\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t} = \frac{\partial\mathcal{C}}{\partial t} + \{\mathcal{C}, \mathsf{H}\} = \sqrt{h}\,\frac{\partial\Lambda_1}{\partial t} + \frac{4}{M_{\mathrm{Pl}}^2\sqrt{h}}\left(\pi_{ij} - \frac{1}{2}\pi h_{ij}\right)\frac{\partial}{\partial h_{ij}}\left(\sqrt{h}\,G(h^{ij}, t)\right) \equiv \mathcal{C}_3 \approx 0\,. \tag{7.2.21}$$

On the other hand, the consistency conditions of the momentum constraints give us the following three tertiary constraints,

$$\frac{\mathrm{d}\mathcal{H}_i}{\mathrm{d}t} = \frac{\partial\mathcal{H}_i}{\partial t} + \{\mathcal{H}_i, \mathsf{H}\} = \{\mathcal{H}_i, \mathsf{H}\} \equiv \mathcal{H}_{3,i} \approx 0\,. \tag{7.2.22}$$

We can explicitly check that the one Hamiltonian constraint, three momentum constraints, as well as the corresponding four tertiary constraints are all of second class. On the other hand, the four primary constraints in equation (7.2.15) are of first class. Therefore, since each first class constraint eliminates two phase space degrees of freedom and each second class constraint eliminates one such degree of freedom, the total number of configuration space DOFs is

$$\#\mathrm{DOFs} = \frac{1}{2}\left[2 \times 10 - 2 \times (1+3) - (1+3+1+3)\right] = 2\,. \tag{7.2.23}$$

Consequently, the graviton has only two tensor polarizations, and there is no scalar or vector gravitons in the theory. This is in agreement with the general analysis of [218]. This will also be explicit from the analysis of metric perturbations in section 7.4. Another similar example in which the graviton has only two polarizations can be found in [247].

7.2.3 Recovering the space-time diffeomorphism invariance

The space-time diffeomorphism symmetry is broken in the action of equation (7.2.7). To recover the space-time diffeomorphism invariance, Z^{ab} should be rewritten in terms of scalar fields, i.e. as in equation (7.2.4), and let us also write

$$\Lambda_1(t) \to V(\varphi^0)\,, \tag{7.2.24}$$

$$\Lambda_2(t) \to M^2 N\sqrt{-X}\,, \tag{7.2.25}$$

$$m_g^2(t, Z^{ab}, \delta_{ab}) \to N\sqrt{-X}F(\varphi^0, Z^{ab}, \delta_{ab})\,, \tag{7.2.26}$$

where $\varphi^0(t,\mathbf{x}) = M^2 f(t) + \delta\varphi^0(t,\mathbf{x})$, F is a generic dimensionless function of its arguments, and we assume that $f(t)$ is a monotonic function of time (otherwise there may be some ambiguity in the unitary gauge). Then, in a Friedmann-Lemaître-Robertson-Walker (FLRW) background,

$$g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\mathrm{d}t^2 + a(t)^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\,, \tag{7.2.27}$$

and assuming that $\dot\varphi^0 > 0$ (thus $X = -N^{-2}(\dot\varphi^0)^2$ and $\Lambda_2 = M^2\dot\varphi^0$), the background action of the time-like Stückelberg scalar field reads

$$S_{\varphi^0} = -\int \mathrm{d}^3\mathbf{x}\,\mathrm{d}t\ a^3\left[V(\varphi^0) + M^2\dot\varphi^0\right]\,. \tag{7.2.28}$$

Taking the variation of the above action with respect to φ^0, we get

$$3H = \frac{V'(\varphi^0)}{M^2} = \frac{\dot V}{M^2\dot\varphi^0} = \frac{\dot\Lambda_1}{\Lambda_2}\,, \tag{7.2.29}$$

where we use the chain rule in the second equality, and $H \equiv \dot a/a$ represents the Hubble parameter. We note that equation (7.2.29) is a constraint equation that determines the allowed functional form of Λ_1 and Λ_2 in the EFT for a given FLRW background.

7.3 Nonsingular bouncing cosmology

7.3.1 Background evolution

We are interested in studying the matter bounce scenario with the action of equation (7.2.7). We first study the background evolution in this context, which should consist of a least a matter-dominated contracting phase and a non-singular bouncing phase. The bouncing phase serves as a transition from contraction to expansion, and we should recover standard Big Bang cosmology in the expanding phase.

In an FLRW background, the consistency condition of the Hamiltonian constraint, i.e. equation (7.2.21), reduces to[3]

$$\dot{\Lambda}_1 = 3H\Lambda_2. \tag{7.3.31}$$

This equation is actually the EOM of the non-dynamical Stückelberg scalar field φ^0 if we want to recover the general covariance of the theory. Indeed, we notice that it is the same as equation (7.2.29) that one finds when recovering the space-time diffeomorphism invariance of the action.

To represent the matter sector of the theory, one can introduce a canonical scalar field ϕ minimally coupled to gravity,

$$S_{\mathrm{matter}} = \int \mathrm{d}^4x\, \sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - U(\phi)\right). \tag{7.3.32}$$

The Einstein equations then read[4]

$$3M_{\mathrm{Pl}}^2H^2 = \frac{1}{2}\dot{\phi}^2 + U(\phi) + \Lambda_1\,, \tag{7.3.35}$$

$$M_{\mathrm{Pl}}^2\dot{H} = -\frac{1}{2}\dot{\phi}^2 + \frac{\Lambda_2}{2}\,, \tag{7.3.36}$$

and the variation of equation (7.3.32) with respect to ϕ yields

$$\ddot{\phi} + 3H\dot{\phi} + U'(\phi) = 0\,. \tag{7.3.37}$$

[3] The generalization with the action of equation (7.2.9) is

$$\dot{\Lambda}_1 + 6M_{\mathrm{Pl}}^2H^2\dot{c}_1 - 3H\Lambda_2 = 0\,. \tag{7.3.30}$$

[4] Again, for the sake of completeness, these can be generalized to

$$3M_{\mathrm{Pl}}^2c_1H^2 = \frac{1}{2}\dot{\phi}^2 + U(\phi) + \Lambda_1\,, \tag{7.3.33}$$

$$M_{\mathrm{Pl}}^2c_1\dot{H} = -\frac{1}{2}\dot{\phi}^2 - M_{\mathrm{Pl}}^2H\dot{c}_1 + \frac{\Lambda_2}{2}\,. \tag{7.3.34}$$

One can check that the two Friedmann equations (7.3.35) and (7.3.36) are consistent with each other, provided the constraint equation (7.3.31) and the EOM of matter [equation (7.3.37)] are satisfied. We can input the time dependence of $\Lambda_2(t)$ in the first place, and then, $\Lambda_1(t)$ evolves according to equation (7.3.31). In order to transition from a contracting universe ($H < 0$) to an expanding universe ($H > 0$) through a non-singular bounce, one needs to violate the Null Energy Condition (NEC), which in this case is equivalent to requiring the condition $\dot{H} > 0$. Thus, from equation (7.3.36), the condition is simply $\Lambda_2 > \dot{\phi}^2$ during the non-singular bouncing phase.

Alternatively, one could describe the background evolution by simply introducing a general fluid with energy-momentum tensor $T_{\mu\nu} \equiv (2/\sqrt{-g})\delta S_{\mathrm{matter}}/\delta g^{\mu\nu}$ such that at the background level $T_\mu{}^\nu = \mathrm{diag}(-\rho(t), p(t)\delta_i{}^j)$. Then, the EOMs are (with $c_1 \equiv 1$)

$$3M_{\mathrm{Pl}}^2 H^2 = \rho + \Lambda_1\,, \tag{7.3.38}$$

$$2M_{\mathrm{Pl}}^2 \dot{H} = -(\rho + p) + \Lambda_2\,, \tag{7.3.39}$$

together with the conservation equation

$$\dot{\rho} + 3H(\rho + p) = 0\,. \tag{7.3.40}$$

Then, in order to consider the matter bounce scenario, let us assume that the matter content has an approximately vanishing pressure, i.e. $p \approx 0$, in which case the conservation equation immediately implies that $\rho \propto a^{-3}$. Also, this means that the NEC can be violated if $\Lambda_2 > \rho$.

Let us now construct a specific example that reproduces the desired background evolution. Let us take the following ansatz,

$$\Lambda_2(t) = \Lambda_{2,0} \exp\left(-\frac{(t - t_B)^2}{\sigma^2}\right)\,, \tag{7.3.41}$$

where $\Lambda_{2,0}$, σ, and t_B are free constant parameters at this point. It is clear that in the limit where $|t - t_B| \gg \sigma$, one finds that $\Lambda_2 \simeq 0$, and thus, $\dot{\Lambda}_1 = 3H\Lambda_2 \simeq 0$, hence $\Lambda_1 \simeq$ constant (let us call this constant $\Lambda_{1,0}$). In the limit where $|t - t_B| \ll \sigma$, one finds that $\Lambda_2 \simeq \Lambda_{2,0}$. This suggests the existence of three regimes.

First, as $t \to -\infty$, we have $a \to \infty$ and $\rho \propto a^{-3} \to 0$, so $\Lambda_1 \to \Lambda_{1,0} \gg \rho$, and thus in this limit equations (7.3.38) and (7.3.39) become

$$H \to -\frac{1}{M_{\rm Pl}}\sqrt{\frac{\Lambda_{1,0}}{3}}\,, \qquad \dot{H} \to 0\,. \tag{7.3.42}$$

This is the cosmological constant-dominated contracting phase.

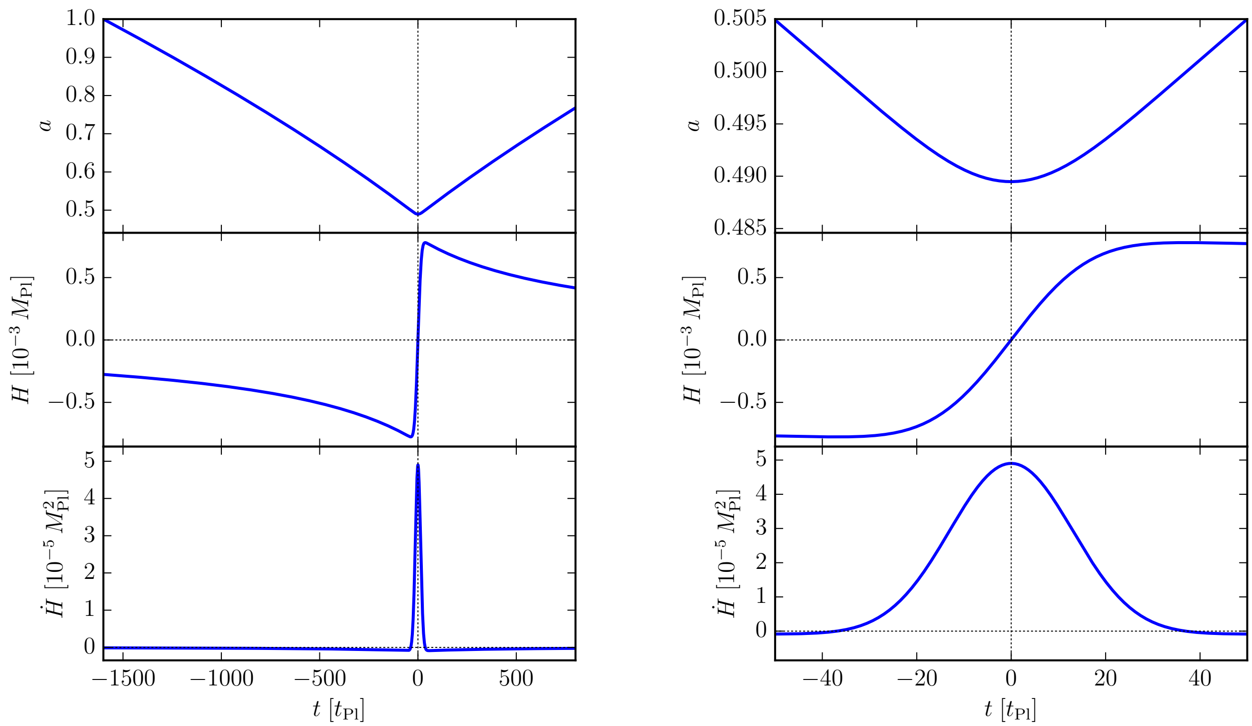

Figure 7.1 Background evolution for the matter bounce model presented in section 7.3.1. In particular, $\Lambda_2(t)$ is taken according to equation (7.3.41) with $\Lambda_{2,0} = 1.0 \times 10^{-4}\,M_{\rm Pl}^4$ and $\sigma = 18.441\,t_{\rm Pl}$. These numbers are chosen such that $|H(t_{B-})| = 8.0 \times 10^{-4}\,M_{\rm Pl}$, where t_{B-} represents the time when the NEC starts being violated (so $\dot{H}(t_{B-}) = 0$). In other words, $H(t_{B-})$ is the value of the Hubble parameter at the end of the contracting phase, just before the bouncing phase, so it is the maximum of the absolute value of the Hubble parameter. Note that we set the bounce point at $t_B = 0$. Then, one can solve equations (7.3.38) and (7.3.39) together with equation (7.3.31) with $p = 0$ and $\rho = \rho_0(a_0/a)^3$. The initial conditions are taken at the time $t_{\rm ini} = -1.6 \times 10^3\,t_{\rm Pl}$ as follows: $a(t_{\rm ini}) = 1$, $H(t_{\rm ini}) = -2.758 \times 10^{-4}\,M_{\rm Pl}$, and $\rho(t_{\rm ini}) = 2.282 \times 10^{-7}\,M_{\rm Pl}^4$. The plots above show the evolution of the scale factor a as well as the Hubble parameter $H = \dot{a}/a$ and its time derivative $\dot{H}$. The plots on the left show the evolution from the time at which the initial conditions are set until slightly after the bouncing phase. The plots on the right show a blowup of the same evolution restricted to the bouncing phase.

Second, there is a regime during which $|t - t_B| \gg \sigma$, so $\Lambda_1 \simeq \Lambda_{1,0}$ and $\Lambda_2 \simeq 0$, but where we still have $\rho \gg \Lambda_1$. If we write $\rho(t) = \rho_0[a_0/a(t)]^3$, then there is a time $t_{\rm eq}$ when $\rho(t_{\rm eq}) = \Lambda_{1,0}$, and thus, $\rho(t) \gg \Lambda_1$ for $\sigma \ll |t - t_B| \ll t_{\rm eq}$. In that case, $3M_{\rm Pl}^2 H^2 \simeq \rho \propto a^{-3}$, and so, $a(t) \propto (-t)^{2/3}$ and $H \sim 2/(3t)$. This is the matter-dominated contracting phase.

Third, in the regime when $|t - t_B| \ll \sigma$, we have $\Lambda_2 \simeq \Lambda_{2,0}$, and so, if $\Lambda_{2,0} \gg \rho(t)$ in that time interval, we have $2M_{\rm Pl}^2 \dot{H} \simeq \Lambda_{2,0}$. Consequently,

$$H(t) \simeq \frac{\Lambda_{2,0}}{2M_{\rm Pl}^2}(t - t_B)\,, \tag{7.3.43}$$

where we set the integration constant such that $H(t_B) = 0$, i.e., $t = t_B$ is the bounce point where the transition from $H < 0$ to $H > 0$ occurs. Therefore, this is the non-singular bouncing phase. In that case, $\dot{\Lambda}_1 = 3H\Lambda_2 \simeq 3\Lambda_{2,0}^2(t - t_B)/(2M_{\rm Pl}^2)$, which implies

$$\Lambda_1(t) \simeq \frac{3\Lambda_{2,0}^2}{4M_{\rm Pl}^2}(t - t_B)^2 + C\,, \tag{7.3.44}$$

where C is another integration constant. Requiring $H(t_B) = 0$ implies $\rho(t_B) + \Lambda_1(t_B) = 0$, hence $C = -\rho(t_B) = -\rho_0(a_0/a_B)^3$ with $a_B \equiv a(t_B)$.

The above solutions can be verified numerically. Setting the initial conditions in the second regime with a non-vanishing but initially sub-dominant cosmological constant $\Lambda_{1,0}$ and taking the ansatz for Λ_2 according to equation (7.3.41), we can numerically integrate the Friedmann equations (7.3.38) and (7.3.39) together with the constraint equation (7.3.31). The resulting dynamics is shown in figures 7.1 and 7.2. The background evolution clearly exhibits the phases of the matter bounce scenario with a non-singular bounce. There is a first phase when $H < 0$ and $\dot{H} < 0$ where the universe is matter dominated and contracting (see the plots on the left in figure 7.1 for $t \lesssim -40\,t_{\rm Pl}$). Then around $t \approx -40\,t_{\rm Pl}$, Λ_2 becomes non-negligible (see figure 7.2), and it triggers the NEC violating phase, i.e. the bouncing phase, during which $\dot{H} > 0$ and H transitions from being negative to positive (see the plots on the right in figure 7.1). After the bounce, the universe is expanding as in standard ΛCDM cosmology since $\Lambda_2 \to 0$ and $\dot{\Lambda}_1 \to 0$ (see figure 7.2), i.e., the action reduces to standard Einstein gravity with a small positive constant.

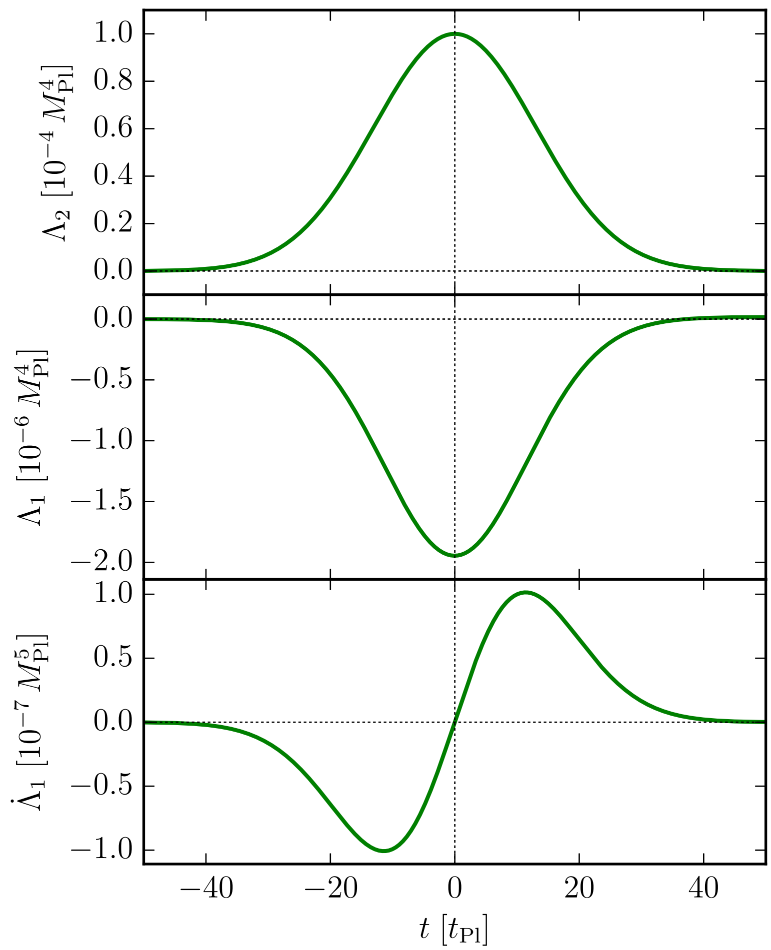

Figure 7.2 Plots of the lapse-independent function in the EFT $\Lambda_2(t)$, the cosmological constant-like function $\Lambda_1(t)$ and its time derivative $\dot{\Lambda}_1(t)$ for a time interval more or less corresponding to the bouncing phase. The time axis is the same as in the plots on the right in figure 7.1. As explained in the caption of figure 7.1, $\Lambda_2(t)$ is taken according to equation (7.3.41) with $\Lambda_{2,0} = 1.0 \times 10^{-4}\,M_{\mathrm{Pl}}^4$, $\sigma = 18.441\,t_{\mathrm{Pl}}$, and $t_B = 0$. Then, the evolution of $\Lambda_1(t)$ and its time derivative follows from solving the Friedmann equations and the constraint equation (7.3.31).

7.3.2 Reconstructing a potential for the time-like Stückelberg scalar field

We saw in section 7.2.3 that we can associate $\Lambda_2 = M^2\dot{\varphi}^0$ and $\Lambda_1 = V(\varphi^0)$ together with the constraint $3H = V'(\varphi^0)/M^2$. With the ansatz of equation (7.3.41) for $\Lambda_2(t)$, this suggests $M^4\dot{f}(t) = \Lambda_{2,0}e^{-(t-t_B)^2/\sigma^2}$ [recall that at the background level $\varphi^0 = M^2 f(t)$]. After integration, this implies

$$f(t) = f_i + \frac{\sqrt{\pi}}{2}\frac{\sigma\Lambda_{2,0}}{M^4}\left[\mathrm{erf}\left(\frac{t-t_B}{\sigma}\right) + \mathrm{erf}\left(\frac{t_B-t_i}{\sigma}\right)\right], \tag{7.3.45}$$

where we let $f_i \equiv f(t_i)$. This is a monotonic function of t as wanted. Then, in the limit where $|t-t_B| \gg \sigma$, one finds that $f(t) \simeq$ constant, and so, almost any potential $V(\varphi^0)$ will lead to $\Lambda_1 \simeq$ constant in that limit. Thus, there is no important constraint on the potential from that regime. Near the bounce, i.e. for $|t-t_B| \ll \sigma$, equation (7.3.45) gives

$$f(t) \simeq \tilde{f}_i + \frac{\Lambda_{2,0}}{M^4}(t-t_B)\,, \tag{7.3.46}$$

where we let $\tilde{f}_i \equiv f_i + (\sqrt{\pi}/2)(\sigma\Lambda_{2,0}/M^4)\mathrm{erf}[(t_B-t_i)/\sigma]$. Then, a good ansatz for the potential is

$$V(\varphi^0) = A + \frac{1}{2}m_{\varphi^0}^2\left(\varphi^0\right)^2\,. \tag{7.3.47}$$

Indeed, this implies

$$\Lambda_1(t) = V(M^2 f(t)) \simeq A + \frac{1}{2}m_{\varphi^0}^2\left(M^2\tilde{f}_i + \frac{\Lambda_{2,0}}{M^2}(t-t_B)\right)^2\,. \tag{7.3.48}$$

In comparison with equation (7.3.44), which is an expression for $\Lambda_1(t)$ in the same limit, we see that we must set $A = C = -\rho_0(a_0/a_B)^3$, $m_{\varphi^0}^2\Lambda_{2,0}^2/(2M^4) = 3\Lambda_{2,0}^2/(4M_{\mathrm{Pl}}^2)$, and $\tilde{f}_i = 0$. This last condition is equivalent to demanding $f_i = -(\sqrt{\pi}/2)(\sigma\Lambda_{2,0}/M^4)\mathrm{erf}[(t_B-t_i)/\sigma]$, which is to say that we set the integration constant in equation (7.3.45) such that $f(t_B) = 0$. In sum, a good potential is

$$V(\varphi^0) = \frac{1}{2}m_{\varphi^0}^2\left(\varphi^0\right)^2 - \rho_0\left(\frac{a_0}{a_B}\right)^3\,, \tag{7.3.49}$$

where the 'mass' of the time-like Stückelberg scalar field is $m_{\varphi^0} = \sqrt{3/2}M^2/M_{\rm Pl}$; and the time evolution of the field $\varphi^0 = M^2 f(t)$ is given by

$$f(t) = \frac{\sqrt{\pi}}{2}\frac{\sigma\Lambda_{2,0}}{M^4}\mathrm{erf}\left(\frac{t-t_B}{\sigma}\right) . \tag{7.3.50}$$

We can check that equation (7.3.49) implies $V'(\varphi^0) = 3M^4\varphi^0/(2M_{\rm Pl}^2)$, and so, the constraint equation $3H = V'(\varphi^0)/M^2$ gives

$$3H(t) = \frac{3M^4 f(t)}{2M_{\rm Pl}^2} \simeq \frac{3\Lambda_{2,0}}{2M_{\rm Pl}^2}(t-t_B) \tag{7.3.51}$$

in the limit where $|t-t_B| \ll \sigma$. This is in agreement with equation (7.3.43). Let us also note that by combining equations (7.3.49) and (7.3.50) we find

$$\Lambda_1(t) = V(M^2 f(t)) = \frac{3\pi\sigma^2\Lambda_{2,0}^2}{16M_{\rm Pl}^2}\mathrm{erf}^2\left(\frac{t-t_B}{\sigma}\right) - \rho_0\left(\frac{a_0}{a_B}\right)^3 . \tag{7.3.52}$$

Thus, in the limit where $|t-t_B| \gg \sigma$, one finds $\Lambda_1 \simeq 3\pi\sigma^2\Lambda_{2,0}^2/(16M_{\rm Pl}^2) - \rho_0(a_0/a_B)^3$, which can be a small positive constant if the parameters are tuned appropriately. This is what we see in figure 7.2 where the shape of Λ_1 resembles an error function squared, and Λ_1 is asymptotically a positive but small constant.

7.4 Cosmological perturbation analysis

We now consider the linear cosmological perturbations of the theory about an FLRW background. Due to the $SO(3)$ rotational symmetry of the background spacetime, one can decompose the metric perturbations into scalar, vector, and tensor modes, and the helicities completely decouple at the linear perturbation level. We define the metric perturbations as follows,

$$\begin{aligned} g_{00} &= -\left(1+2\alpha\right) , \\ g_{0i} &= a\left(S_i + \partial_i\beta\right) , \\ g_{ij} &= a^2\Big(\delta_{ij} + 2\psi\delta_{ij} + \partial_i\partial_j E + \frac{1}{2}(\partial_i F_j + \partial_j F_i) + \gamma_{ij}\Big) , \end{aligned} \tag{7.4.53}$$

where α, β, ψ and E are scalar perturbations, S_i and F_i are vector perturbations, and the γ_{ij}'s are tensor perturbations. Vector modes satisfy the transverse conditions,

$$\partial_i S^i = \partial_i F^i = 0\,, \tag{7.4.54}$$

and tensor modes satisfy the transverse and traceless conditions,

$$\partial_i \gamma^{ij} = 0\,, \qquad \gamma_i{}^i = 0\,. \tag{7.4.55}$$

Since we work in unitary gauge, the perturbations of all four Stückelberg scalar fields are turned off.

As we will see, the perturbation analysis reveals that there are only 2 tensor modes in the gravity sector, and there is no scalar and vector graviton. This is consistent with the Hamiltonian analysis of section 7.2.2.

7.4.1 Scalar perturbations

The derivation of the second-order perturbed action for scalar modes can be found in appendix 7.8. Consistent with the Hamiltonian analysis in section 7.2.2, no helicity-0 mode of the graviton is spotted in our perturbative expansion. The only scalar perturbation in our theory is the one from the matter sector, which is represented by a canonical scalar field. The resulting perturbed action in Fourier space is

$$S^{(2)}_{\text{scalar}} = \int \mathrm{d}^3\mathbf{k}\,\mathrm{d}t\ \left(\mathcal{K}\dot{\mathcal{R}}_k^2 - \tilde{\Omega}\mathcal{R}_k^2\right)\,, \tag{7.4.56}$$

where $\mathcal{R}_k$ is the Fourier transform of the curvature perturbation [defined in equation (7.8.123)] with wavenumber k. The expressions for $\mathcal{K}$ and $\tilde{\Omega}$ are given in equations (7.8.127) and (7.8.128).

During a matter contracting phase, if we assume that $\Lambda_2 \ll M_{\rm Pl}^2\dot{H}$ and thus $M_{\rm Pl}^2\dot{H} \simeq -\frac{1}{2}\dot{\phi}^2$, the quadratic action of scalar perturbations simplifies to

$$S^{(2)}_{\text{scalar}} \simeq M_{\rm Pl}^2 \int \mathrm{d}^3\mathbf{k}\,\mathrm{d}t\ a^3\epsilon\left(\dot{\mathcal{R}}_k^2 - \frac{k^2}{a^2}\mathcal{R}_k^2\right)\,, \tag{7.4.57}$$

where $\epsilon = \dot{\phi}^2/(2M_{\rm Pl}^2H^2)$. This is the standard perturbed action for curvature perturbations in general relativity with a canonical scalar field (see, e.g., [58, 121]). In other words, the scalar perturbation is the same as the one in general relativity as long as $\Lambda_1, \Lambda_2 \ll M_{\rm Pl}^2 H^2$ during the matter contracting phase, no matter how large the graviton mass m_g^2 is.

During a bouncing phase, one would need to know the exact time dependence of $\mathcal{K}$ and $\tilde{\Omega}$ in order to solve the EOM that results from varying the action of equation (7.4.56). However, since the curvature perturbations of observational interest remain mostly on super-Hubble scales during a non-singular bouncing phase, they tend to remain constant [54, 562, 630], i.e., their amplitude and spectral shape are unaffected. One can understand this fact by realizing that the duration of the bouncing phase is usually much shorter than the wavelength of the perturbations that are considered, hence they cannot receive significant amplification. In fact, it has been shown that the curvature perturbations can grow at most linearly in time[5], and the amplification that can be received is therefore bounded from above by the duration of the bouncing phase [562]. Consequently, we do not perform the full analysis of the evolution of curvature perturbations during the non-singular bouncing phase is what follows and only assume that they remain unchanged through the bounce.

7.4.2 Vector perturbations

Using the same methodology as the one described in appendix 7.8 for scalar modes, one finds that the quadratic action of vector perturbations reads (again in momentum space, but we omit the subscript k to simplify the notation)

$$S^{(2)}_{\rm vector} = \frac{1}{4} M_{\rm Pl}^2 \int \mathrm{d}^3\mathbf{k}\, \mathrm{d}t \, \left(\frac{1}{4} k^2 a^3 \dot{F}_i \dot{F}^i - k^2 a^2 S_i \dot{F}^i + k^2 a S_i S^i - \frac{1}{4} m_g^2 a^3 F_i F^i \right) . \quad (7.4.58)$$

[5] One caveat, though, is the possible presence of gradient instabilities (see, e.g., [404]), in which case modes with shorter wavelength would grow exponentially with time with an exponent which increases as k increases. In the model presented here, gradient instabilities are expected to be absent from the EFT point of view. From that perspective, the graviton is a gauge boson which couples to the density fluctuations from the matter sector (in fact, the matter fluctuation could be considered as a Nambu-Goldston boson), and the coupling would occur at the mixing scale corresponding to $\Lambda_{\rm mix}^2 \sim \dot{H}$. These two sectors are well decoupled on scales well above the mixing scale $\Lambda_{\rm mix}$. Indeed, an explicit computation tells us that in the limit where $(k/a)^2 \gg \dot{H}$, we have a free field theory at leading order with positive sound speed equal to unity during the bouncing phase, hence the theory is free from the possibly pathological ultraviolet (UV) gradient instabilities.

The vector mode S_i does not have a kinetic term, and it gives us the constraint equation

$$S_i = \frac{1}{2} a \dot{F}_i \,. \tag{7.4.59}$$

Substituting the above solution back into the action of equation (7.4.58), one finds

$$S^{(2)}_{\text{vector}} = -\frac{1}{16} M_{\text{Pl}}^2 \int \mathrm{d}^3\mathbf{k}\, \mathrm{d}t\ m_g^2 k^2 a^3 F_i F^i \,. \tag{7.4.60}$$

This clearly shows that the kinetic term for vector perturbations has canceled out. It is by no means an accident, because a kinetic term of vector modes is prohibited by the internal symmetry $\varphi^a \to \varphi^a + \Xi^a(\varphi^0)$ [recall equation (7.2.3)]. After integrating out F_i, the whole action for vector perturbations vanishes, and there is no vector mode left in the theory. This result is also consistent with the Hamiltonian analysis of section 7.2.2.

7.4.3 Tensor perturbations

Following the methodology described in appendix 7.8 but for tensor modes, one finds that the perturbed action for tensor perturbations reads[6]

$$S^{(2)}_{\text{tensor}} = \frac{M_{\text{Pl}}^2}{8} \int \mathrm{d}^3\mathbf{x}\, \mathrm{d}t\ a^3 \left[c_1 \dot{\gamma}_{ij}^2 - c_2 \frac{(\partial_l \gamma_{ij})^2}{a^2} - m_g^2 \gamma_{ij}^2 \right] . \tag{7.4.61}$$

Converting to conformal time τ defined by $\mathrm{d}\tau \equiv a^{-1}\mathrm{d}t$, we can write

$$S^{(2)}_{\text{tensor}} = \frac{M_{\text{Pl}}^2}{8} \int \mathrm{d}^3\mathbf{x}\, \mathrm{d}\tau\ a^2 \left[c_1 \gamma_{ij}'^2 - c_2 (\partial_l \gamma_{ij})^2 - m_g^2 a^2 \gamma_{ij}^2 \right] . \tag{7.4.62}$$

Letting

$$\gamma_{ij}(\tau, \mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}} \sum_{\lambda=+,\times} \epsilon_{ij}^{(\lambda)} \gamma_{\mathbf{k},\lambda}(\tau) e^{i\mathbf{k}\cdot\mathbf{x}} \,, \tag{7.4.63}$$

where $\epsilon_i^{(\lambda)i} = 0$, $k^i \epsilon_{ij}^{(\lambda)} = 0$, and $\epsilon_{ij}^{(\lambda)} \epsilon^{(\lambda')ij} = 2\delta_{\lambda\lambda'}$, the action becomes

$$S^{(2)}_{\text{tensor}} = \frac{M_{\text{Pl}}^2}{4} \sum_\lambda \int \mathrm{d}^3\mathbf{k}\, \mathrm{d}\tau\ a^2 \left[c_1 (\gamma_{\mathbf{k},\lambda}')^2 - \left(c_2 k^2 + m_g^2 a^2 \right) (\gamma_{\mathbf{k},\lambda})^2 \right] , \tag{7.4.64}$$

[6] In this sub-section, we perform the analysis starting with the gravity action of equation (7.2.9), which allows for a non-trivial propagation speed of gravitational waves.

where $\gamma_{\mathbf{k},\lambda}$ represents the two polarization states of the tensor modes, the $+$ and $\times$ polarizations. Varying the above action and defining $c_g^2 \equiv c_2/c_1$, one obtains the EOM

$$\gamma_k'' + 2\frac{a'}{a}\gamma_k' + \frac{c_1'}{c_1}\gamma_k' + c_g^2 k^2 \gamma_k + \frac{1}{c_1} m_g^2 a^2 \gamma_k = 0 \tag{7.4.65}$$

for each polarization state λ. From here on, let us assume that c_1 is a constant, and let us normalize it to unity, so $c_1 \equiv 1$. Therefore,

$$\gamma_k'' + 2\frac{a'}{a}\gamma_k' + c_g^2 k^2 \gamma_k + m_g^2 a^2 \gamma_k = 0\,, \tag{7.4.66}$$

where $c_g(\tau) = \sqrt{c_2(\tau)}$. Then, defining the Mukhanov-Sasaki variable

$$u_{\mathbf{k},\lambda} \equiv \frac{M_{\mathrm{Pl}}}{2} a \gamma_{\mathbf{k},\lambda}\,, \tag{7.4.67}$$

we can rewrite the action as

$$S^{(2)}_{\mathrm{tensor}} = \frac{1}{2}\sum_\lambda \int \mathrm{d}^3\mathbf{k}\,\mathrm{d}\tau\, \left[(u_{\mathbf{k},\lambda}')^2 - \left(c_g^2 k^2 + m_g^2 a^2 - \frac{a''}{a}\right)(u_{\mathbf{k},\lambda})^2\right]. \tag{7.4.68}$$

Upon variation, the EOM becomes

$$u_k'' + \left(c_g^2 k^2 + m_g^2 a^2 - \frac{a''}{a}\right) u_k = 0\,, \tag{7.4.69}$$

again for each polarization state λ. We note that the differential equation is of the form $u_k'' + \omega_k^2 u_k = 0$ with effective time-dependent frequency given by

$$\omega_k^2(\tau) = c_g^2 k^2 + m_g^2 a^2 - \frac{a''}{a}\,. \tag{7.4.70}$$

In comparison with the tensor mode EOM in general relativity [121], we notice that our theory allows for a non-trivial speed of sound for the gravitational waves, but most importantly, the effective frequency picks up a mass term. Therefore, while the mass term in the action of the form of equation (7.2.6) did not affect the background dynamics and did not introduce additional DOFs for scalar, vector, and tensor perturbations, we notice that it can affect the evolution of gravitational waves. This will be crucial in the context of the matter bounce scenario as we will soon see in section 7.6.2.

7.5 Anisotropies

While the previous section explored the characteristics of inhomogeneities about an FLRW background, one may also consider the theory in an anisotropic background. Let us slightly deform the FLRW metric by introducing anisotropies as follows,

$$g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\mathrm{d}t^2 + a(t)^2\sum_{i=1}^{3} e^{2\theta_i(t)}\mathrm{d}x^i\mathrm{d}x^i\,, \tag{7.5.71}$$

with the constraint equation $\sum_i \theta_i = 0$. This represents a metric of Bianchi type-I form. One can think of $a(t)$ as the average scale factor of the universe, and the $\theta_i(t)$'s are corrections (or anisotropies) to the average expansion or contraction. Accordingly, $H = \dot{a}/a$ is the mean Hubble parameter, and the Hubble parameter along a given spatial direction x^i is [167] $H_i = H + \dot{\theta}_i$.

Linearly expanding our action of equation (7.2.7) with the metric of equation (7.5.71), where we think of the θ_i's as small anisotropies about an FLRW background, we find

$$S_\theta \supset M_{\mathrm{Pl}}^2 \int \mathrm{d}^3\mathbf{x}\,\mathrm{d}t\; a^3\left(\dot{\theta}_i^2 - m_g^2\theta_i^2\right)\,, \tag{7.5.72}$$

where we see that the anisotropies receive a mass term (just like tensor modes). The corresponding EOM reads

$$\ddot{\theta}_i + 3H\dot{\theta}_i + m_g^2\theta_i = 0\,, \tag{7.5.73}$$

and if one associates an energy-momentum tensor with the variation of S_θ, then the anisotropies carry an energy density given by

$$\rho_\theta = \frac{M_{\mathrm{Pl}}^2}{2}\sum_{i=1}^{3}\left(\dot{\theta}_i^2 + m_g^2\theta_i^2\right)\,. \tag{7.5.74}$$

It makes sense that the anisotropies pick up a mass term just like tensor modes because the two are very similar. Indeed, one can think of the anisotropies as traceless ($\sum_i \theta_i = 0$) and transverse ($\partial_i\theta^i = 0$) just like tensor modes [c.f. equation (7.4.55)]. Accordingly, taking the limit $k \to 0$ of equation (7.4.66) for tensor modes and converting from conformal time to physical time, one finds the same EOM as equation (7.5.73) for anisotropies.

In comparison with the case where the graviton is massless, the metric of equation (7.5.71) in general relativity yields the Lagrangian density $\mathcal{L}_\theta \supset M_{\rm Pl}^2 \dot{\theta}_i^2$ and the EOM $\ddot{\theta}_i + 3H\dot{\theta}_i = 0$, which usually suggests that $\dot{\theta}_i \propto a^{-3}$, hence $\rho_\theta \sim M_{\rm Pl}^2 \dot{\theta}_i^2 \propto a^{-6}$ (see, e.g., [167]). This is reminiscent of a massless canonical scalar field, e.g., call it Θ with Lagrangian $\mathcal{L} = (1/2)(\partial_\mu\Theta)^2$, which at the background level has an effective EoS $p_\Theta = \rho_\Theta = (1/2)\dot{\Theta}^2$ (in agreement with the energy density scaling as a^{-6}). With the massive gravity action of equation (7.2.7), a comparison with equation (7.5.72) tells us that the anisotropies behave like a massive scalar field with Lagrangian $\mathcal{L} = (1/2)(\partial_\mu\Theta)^2 - (1/2)m_\Theta^2\Theta^2$, where the 'mass' of the anisotropies is the mass of the graviton, i.e. $m_\Theta = m_g$. In that case, the energy density of the massive scalar field is $\rho_\Theta = (1/2)\dot{\Theta}^2 + (1/2)m_\Theta^2\Theta^2$ [in agreement with equation (7.5.74)], and so the energy density no longer necessarily grows as a^{-6} in a contracting universe. In fact, if $m_\Theta \gg |H|$, we expect to recover the result of a coherently oscillating massive scalar field in cosmology, in which case the energy density would scale as a^{-3}. In other words, the anisotropies would behave as pressureless matter. This will be shown explicitly in the next section.

7.6 Solutions in a matter-dominated contracting phase

We derived the general EOMs for tensor perturbations and anisotropies in the previous sections for our massive gravity theory. We now want to solve these equations during a matter-dominated contracting phase, so let us set up the background evolution during such a phase. The scale factor is given by $a(t) \propto (-t)^{2/3}$ and so the Hubble parameter is

$$H(t) = \frac{2}{3(t - \tilde{t}_{B-})} \tag{7.6.75}$$

for $t < \tilde{t}_{B-}$. Here, $\tilde{t}_{B-}$ denotes the time at which the singularity would be reached if no new physics appeared at high energy scales to violate the NEC and avoid a Big Crunch. Without loss of generality, we set $\tilde{t}_{B-} = 0$ in what follows.

7.6.1 Evolution of anisotropies

Let us first solve the EOM for anisotropies [equation (7.5.73)] in a matter-dominated contracting phase. Since $H = 2/(3t)$, the differential equation is

$$\ddot{\theta}_i + \frac{2}{t}\dot{\theta}_i + m_g^2\theta_i = 0\,, \tag{7.6.76}$$

whose general solution is

$$\theta_i(t) = \frac{1}{(-t)}\left\{C_1\cos[m_g(-t)] + C_2\sin[m_g(-t)]\right\}\,, \tag{7.6.77}$$

where we have assume that the graviton mass m_g^2 is a constant. For simplicity, let us consider one of two modes only, e.g., $\theta_i(t) = C_1\cos(m_g t)/t$ (we would find the same result below if we considered the other mode or both). Then, according to equation (7.5.74), the energy density for the anisotropy θ_i is given by

$$\begin{aligned}\rho_{\theta_i} &= \frac{M_{\rm Pl}^2}{2}\left(\dot{\theta}_i^2 + m_g^2\theta_i^2\right)\\ &= \frac{M_{\rm Pl}^2 m_g^2}{2}\left(\frac{C_1}{-t}\right)^2\left[1 + \frac{3}{2}\left(\frac{H}{m_g}\right)\sin[2m_g(-t)] + \frac{9}{4}\left(\frac{H}{m_g}\right)^2\cos^2[m_g(-t)]\right]\,,\end{aligned} \tag{7.6.78}$$

where we used the fact that $H = 2/(3t)$. If we consider the limit where $m_g \gg |H|$, then it follows that

$$\rho_{\theta_i} \simeq \frac{M_{\rm Pl}^2 m_g^2}{2}\left(\frac{C_1}{-t}\right)^2 \propto a^{-3} \tag{7.6.79}$$

since $a(t) \propto (-t)^{2/3}$. Therefore, if the mass of the graviton m_g is larger than the absolute value of the Hubble parameter in some time interval in a matter-dominated contracting universe, then we find that the total energy density in anisotropies $\rho_\theta = \sum_i \rho_{\theta_i}$ scales as a^{-3}, i.e., it grows at the same rate as the background energy density of the pressureless matter.

The above result implies that as long as the anisotropies are sub-dominant at some initial time in the far past, they will always remain sub-dominant (in the regime when $m_g \gg |H|$). Furthermore, one can even show that this result is independent of the background EoS. Indeed, as long as $m_g \gg |H|$, one finds that $\rho_\theta \propto a^{-3}$ for any background EoS parameter $w \equiv p/\rho \geq 0$. The proof can be found in appendix 7.9. For example, if the model included radiation in addition to pressureless matter, radiation would get to dominate at higher energy scales, and the already sub-dominant anisotropies would be washed out since the energy density of radiation ($\rho_{\rm rad} \propto a^{-4}$) would grow faster than that of anisotropies ($\rho_\theta \propto a^{-3}$). As a result, the model is free of the BKL instability.

We note that the above resolution to the BKL instability problem in a contracting universe is also free of the problems that one encounters in trying to resolve the BKL instability with an Ekpyrotic phase of contraction as in [167, 172, 175, 561]. In that context, the Ekpyrotic scalar field has an EoS parameter $w \gg 1$, and thus, the background energy density in an Ekpyrotic phase of contraction scales as $a^{-3(1+w)}$, i.e., it grows much faster than the energy density in anisotropies that scales as a^{-6} when the graviton is massless. While this appropriately washes out the anisotropies for an Ekpyrotic field with isotropic pressure [167, 273], it has been shown in [50] that the presence of dominant ultra-stiff pressure anisotropies, which one should certainly expect in an anisotropic background, does not necessarily lead to isotropization on approach to a bounce. Indeed, the presence of anisotropic pressures in the background fluid would add a source term on the right-hand side of equation (7.5.73), which could enhance the growth of anisotropies. It would be interesting to reproduce the analysis of [50] with our massive gravity theory, but at least in the context of a matter-dominated contracting universe, one would not expect important anisotropic pressures to source the right-hand side of equation (7.5.73) since a pressureless fluid does not have any pressure by definition. Therefore, our resolution of the BKL instability with massive gravity should be robust with regards to that issue in the matter bounce scenario.

Resolving the BKL instability with an Ekpyrotic phase of contraction after a matter-dominated contracting phase also introduces a fine-tuning problem. As explained in [432], in order to generate N e-folds of matter contraction with sub-dominant anisotropies, one needs the initial ratio of the energy density in anisotropies to that in matter to be smaller than e^{-6N}, which can be an extremely small number. With our massive gravity theory, we showed that the energy densities in anisotropies and matter grow at the same rate, so their ratio remains constant. Thus, one only has to require the initial ratio to be smaller than unity, which is a much smaller fine-tuning requirement compared to a factor of order e^{-6N}.

7.6.2 Evolution of gravitational waves

To solve for the evolution of tensor modes in a matter-dominated contracting phase, we need to solve equation (7.4.69), which we recall is of the form $u_k'' + \omega_k^2 u_k = 0$ with effective frequency given by equation (7.4.70). Converting the scale factor evolution $a(t) \propto (-t)^{2/3}$ to conformal time with $\mathrm{d}\tau = a^{-1}\mathrm{d}t$, one finds

$$a(\tau) = \left(\frac{-\tau}{\tau_m}\right)^2 , \qquad \tau < 0 , \tag{7.6.80}$$

where τ_m is some constant at which $a(\tau_m) = 1$ by convention. Then,

$$\mathcal{H} \equiv \frac{a'}{a} = \frac{2}{\tau} \tag{7.6.81}$$

is the conformal Hubble parameter. It is also useful to calculate

$$\frac{a''}{a} = \frac{2}{\tau^2} . \tag{7.6.82}$$

Noting that $\mathcal{H} = aH$ and using equations (7.6.81) and (7.6.82), we note that

$$\frac{1}{a^2}\frac{a''}{a} = \frac{H^2}{\mathcal{H}^2}\frac{2}{\tau^2} = \frac{H^2}{2} , \tag{7.6.83}$$

and thus, the physical effective frequency of equation (7.4.70) can be written as

$$\frac{\omega_k^2}{a^2} = c_g^2\frac{k^2}{a^2} + m_g^2 - \frac{1}{a^2}\frac{a''}{a} = c_g^2\frac{k^2}{a^2} + m_g^2 - \frac{H^2}{2} . \tag{7.6.84}$$

This suggests that the size of the graviton mass compared to the Hubble parameter (in absolute value) will determine the evolutionary behavior of the tensor modes. Thus, we separate the analysis depending on whether the graviton mass is small or large compared to $|H|$.

Small graviton mass regime

In this subsection, we will assume that $m_g \ll |H|$ within some time interval during which the nearly scale-invariant power spectrum of curvature perturbations is generated, i.e. during which the scales of observational interest exit the Hubble radius.

Equation (7.6.84) suggests that we can write the effective frequency as

$$\omega_k^2 = c_g^2 k^2 - \left(\frac{1}{2} - \frac{m_g^2}{H^2}\right)\mathcal{H}^2 , \tag{7.6.85}$$

using the property $\mathcal{H} = aH$ once again. Recalling equation (7.6.81), we can finally write

$$\omega_k^2 = c_g^2 k^2 - \frac{\nu_t^2 - \frac{1}{4}}{\tau^2}\,, \tag{7.6.86}$$

where we defined the index

$$\nu_t \equiv \sqrt{\frac{9}{4} - 4\frac{m_g^2}{H^2}}\,. \tag{7.6.87}$$

Provided $m_g \ll |H|$, we can assume that ν_t is approximately constant (and not complex) in the time interval of interest. Let us further assume that c_g is a constant, i.e. $c_g'(\tau) = 0$. In fact, we will most often set $c_g = 1$, but we do the calculation in more generality. Therefore, the solutions to the EOM,

$$u_k'' + \left(c_g^2 k^2 - \frac{\nu_t^2 - \frac{1}{4}}{\tau^2}\right) u_k = 0\,, \tag{7.6.88}$$

are the Hankel (or, equivalently, Bessel) functions of the first and second kind:

$$u_k(\tau) = \sqrt{-\tau}\left\{A_k H_{\nu_t}^{(1)}[c_g k(-\tau)] + B_k H_{\nu_t}^{(2)}[c_g k(-\tau)]\right\}\,. \tag{7.6.89}$$

In the limit where $c_g k|\tau| \gg 1$, we impose Bunch-Davies initial conditions, $u_k(\tau) \simeq \frac{e^{-ic_g k\tau}}{\sqrt{2c_g k}}$, which sets the integration constants as follows: $A_k = (\sqrt{\pi}/2)e^{i\vartheta}$ and $B_k = 0$, with $\vartheta \equiv (\pi/2)(\nu_t - 3/2)$. Therefore, the above solution becomes

$$u_k(\tau) = \frac{\sqrt{\pi}}{2}e^{i\vartheta}\sqrt{-\tau}H_{\nu_t}^{(1)}[c_g k(-\tau)]\,. \tag{7.6.90}$$

Expanding on large scales, where $c_g k|\tau| \ll 1$, and keeping only the growing mode of the Hankel function that scales as $(-\tau)^{-\nu_t}$ (there is also a decaying mode that scales as $(-\tau)^{\nu_t}$; recall here that $-\tau \to 0^+$ in a contracting universe, and from equation (7.6.87), $\nu_t > 0$), one finds

$$u_k(\tau) \simeq -\frac{i2^{\nu_t - 1}e^{i\vartheta}\Gamma(\nu_t)}{\sqrt{\pi}}c_g^{-\nu_t}k^{-\nu_t}(-\tau)^{1/2-\nu_t}\,, \tag{7.6.91}$$

where Γ denotes the gamma function. Then, the corresponding power spectrum of tensor perturbations defined on large scales is

$$\mathcal{P}_t(k,\tau) = 2 \times \mathcal{P}_\gamma(k,\tau) = 2 \times \left(\frac{2}{aM_{\mathrm{Pl}}}\right)^2 \times \mathcal{P}_u(k,\tau)\,, \tag{7.6.92}$$

where

$$\mathcal{P}_u(k,\tau) \equiv \frac{k^3}{2\pi^2}\left|u_k(\tau)\right|^2\,, \tag{7.6.93}$$

and $u_k(\tau)$ is given by equation (7.6.91). The first factor of 2 in equation (7.6.92) accounts for the two polarization states of the gravitational waves, and the factor of $2/(aM_{\mathrm{Pl}})$ is due to the conversion from the Mukhanov-Sasaki variable to the variable γ [recall equation (7.4.67)]. The resulting power spectrum of tensor perturbations on large scales is

$$\mathcal{P}_t(k,\tau) = \frac{4^{\nu_t}\Gamma(\nu_t)^2}{\pi^3 c_g^{2\nu_t}}\frac{k^{3-2\nu_t}(-\tau)^{1-2\nu_t}}{a^2M_{\mathrm{Pl}}^2}\,. \tag{7.6.94}$$

Already, we can calculate the tensor spectral tilt:

$$n_t \equiv \frac{\mathrm{d}\ln\mathcal{P}_t}{\mathrm{d}\ln k} = 3 - 2\nu_t = 3 - 2\sqrt{\frac{9}{4} - 4\frac{m_g^2}{H^2}}\,. \tag{7.6.95}$$

Since we are working in the limit where $m_g \ll |H|$, the above can be simplified to

$$n_t \simeq \frac{8}{3}\frac{m_g^2}{H^2}\,. \tag{7.6.96}$$

It is thus clear that the tensor tilt is small and positive, i.e., it is a blue tilt (opposite to the tilt one obtains in inflationary cosmology [58], but the same sign as the tilt obtained in string gas cosmology [146]). Also, in the limit where $m_g \to 0$, we recover $n_t \to 0$, i.e. a scale-invariant power spectrum, as expected.

Using the property $\mathcal{H} = aH$ and equation (7.6.81), one can re-express the amplitude of the power spectrum as

$$\mathcal{P}_t(k,t) = \frac{2\,\Gamma(\nu_t)^2}{\pi^3 c_g^{2\nu_t}}\left(\frac{k}{aM_{\mathrm{Pl}}}\right)^{3-2\nu_t}\left(\frac{|H(t)|}{M_{\mathrm{Pl}}}\right)^{2\nu_t-1}\,. \tag{7.6.97}$$

We note that k is the comoving wavenumber here, so k/a represents a physical wavenumber. In the limit where $m_g \to 0$, we have $\nu_t \to 3/2$, and at the time t_{B-} at the end of the contracting phase (or right before the bouncing phase), one finds

$$\mathcal{P}_t(k,t_{B-})\to\frac{1}{2\pi^2c_g^3}\left(\frac{H_B}{M_{\mathrm{Pl}}}\right)^2\,, \tag{7.6.98}$$

where we defined $H_B\equiv H(t_{B-})$, i.e., it is the energy scale of the bounce. This matches the usual result (see, e.g., [438]) with $c_g=1$. We evaluate the power spectrum at the end of the contracting phase because this is the point at which the amplitude stops increasing. As we argued in section 7.4.1, the perturbations will remain more or less constant during the non-singular bouncing phase, so we can immediately match the above power spectrum with the primordial power spectrum after the bounce, i.e., at the beginning of standard Big Bang cosmology. Thus, we will drop the argument t_{B-} when it is clear that we are talking about the primordial power spectrum that can be connected with observations.

In comparison to the tensor power spectrum, the primordial power spectrum of curvature perturbations generated during a matter-dominated contracting phase on super-Hubble scales is given by (see, e.g., [438])

$$\mathcal{P}_{\mathcal{R}}(k)=\frac{1}{48\pi^2c_s}\left(\frac{H_B}{M_{\mathrm{Pl}}}\right)^2\,. \tag{7.6.99}$$

In order to avoid the over-production of scalar non-Gaussianities in the matter-dominated contracting phase, it has been shown that c_s cannot be too small. In fact, in order to satisfy the current constraints on f_{NL}, one must have [438] $c_s\gtrsim 0.62$. Imposing $\mathcal{P}_{\mathcal{R}}(k)=A_s\approx 2.2\times10^{-9}$ (to match the amplitude observed by Planck [12]), one would therefore need $|H_B|\gtrsim 8.0\times10^{-4}\,M_{\mathrm{Pl}}$. Comparing equations (7.6.97) and (7.6.99), the tensor-to-scalar ratio is given by

$$r_\star\equiv\frac{\mathcal{P}_t(k_\star)}{\mathcal{P}_{\mathcal{R}}(k_\star)}=\frac{96\,\Gamma(\nu_t)^2}{\pi}\frac{c_s}{c_g^{2\nu_t}}\left(\frac{k_\star}{M_{\mathrm{Pl}}}\right)^{3-2\nu_t}\left(\frac{|H_B|}{M_{\mathrm{Pl}}}\right)^{2\nu_t-3}\,, \tag{7.6.100}$$

where $k_\star$ represents the physical wavenumber at which the observation is made. Let us set $c_g=1$, $c_s=0.62$, and $|H_B|=8.0\times10^{-4}\,M_{\mathrm{Pl}}$ to have an idea of the size of the above tensor-to-scalar ratio. Furthermore, let us say that we consider a mode with physical wavenumber $k_\star$ that exits the Hubble radius at a time $t_\star$ at which point $m_g/|H|=0.13$. Then, $\nu_t\approx1.477$, and the above expression for $r_\star$ becomes

$$r_\star \approx 20.546 \left(\frac{k_\star}{M_{\text{Pl}}}\right)^{0.0454} \approx 0.048 \left(\frac{k_\star}{0.05\,\text{Mpc}^{-1}}\right)^{0.0454} . \tag{7.6.101}$$

Thus, we see that for modes of observational interest, the tensor-to-scalar ratio is suppressed within the current observational bound (for $k_\star = 0.05\,\text{Mpc}^{-1}$, $r_\star < 0.07$ at 95% confidence [11]). However, this is true only if m_g is sufficiently non-zero, because as $m_g \to 0$, one finds $\nu_t \to 3/2$ and $r \to 24c_s/c_g^3$ on all scales and independently of the value of H_B. With the constraint $c_s \gtrsim 0.62$, this cannot satisfy current observational bounds without tuning $c_g \gtrsim 5.968$, and such a large super-luminal propagation speed of gravitational waves does not sound very realistic.

Large graviton mass regime

Let us now explore the possibility that $m_g \gg |H|$, and as before, we assume that this is valid within some time interval during which the nearly scale-invariant power spectrum of curvature perturbations is generated. Then, there are three regimes to be considered (we assume that $c_g = 1$ from here on):

1. $(k/a)^2 \gg m_g^2 \gg H^2 \implies \omega_k^2 \simeq k^2$;
2. $m_g^2 \gg (k/a)^2 \gg H^2 \implies \omega_k^2 \simeq m_g^2 a^2$;
3. $m_g^2 \gg H^2 \gg (k/a)^2 \implies \omega_k^2 \simeq m_g^2 a^2$.

Regimes (1) and (2) represent sub-Hubble modes and regime (3) represents super-Hubble modes in the conventional sense. However, the scale of interest here is m_g. It separates the evolution of the perturbations into actually only two regimes. There are the 'sub-graviton' scales, where $(k/a)^2 \gg m_g^2$ (this is regime (1) above), in which case the modes are deeply sub-Hubble; and there are the 'super-graviton' scales, where $(k/a)^2 \ll m_g^2$.

On sub-graviton scales, the EOM for tensor modes reads $u_k'' + k^2 u_k = 0$, so the general solution is $u_k(\tau) = C_{1,k}e^{-ik\tau} + C_{2,k}e^{ik\tau}$. We require the usual Bunch-Davies normalization, which sets the integration constants $C_{1,k} = (2k)^{-1/2}$ and $C_{2,k} = 0$, so that

$$u_k(\tau) = \frac{1}{\sqrt{2k}} e^{-ik\tau} . \tag{7.6.102}$$

On super-graviton scales, the EOM reads $u_k'' + m_g^2 a^2 u_k = 0$, hence the differential equation becomes

$$u_k'' + m_g^2 \left(\frac{\tau}{\tau_m}\right)^4 u_k = 0\,, \tag{7.6.103}$$

and the solutions are Bessel functions,

$$u_k(\tau) = \left(\frac{m_g(-\tau)^3}{6\tau_m^2}\right)^{1/6} \left[A_k J_{-1/6}\left(\frac{m_g(-\tau)^3}{3\tau_m^2}\right) + B_k J_{1/6}\left(\frac{m_g(-\tau)^3}{3\tau_m^2}\right)\right]\,, \tag{7.6.104}$$

where A_k and B_k are two integration constants. One can show that $m_g(-\tau)^3/(3\tau_m^2) = (2/3)(m_g/|H|)$, so in the limit where $m_g \gg |H|$, the argument of the Bessel functions is large compared to unity. Thus, the Bessel functions can be expanded to find

$$u_k(\tau) \simeq \frac{1}{\sqrt{\pi}} \left(\frac{6\tau_m^2}{m_g(-\tau)^3}\right)^{1/3} \left[A_k \cos\left(\frac{\pi}{6} - \frac{m_g(-\tau)^3}{3\tau_m^2}\right) + B_k \sin\left(\frac{\pi}{6} + \frac{m_g(-\tau)^3}{3\tau_m^2}\right)\right] \tag{7.6.105}$$

in the limit where $m_g \gg |H|$.

We now match the solutions to the EOM on sub- and super-graviton regimes at "graviton horizon crossing", i.e. when the perturbation wavelength is equal to the graviton Compton wavelength. This is the case when $k/a = m_g$, which happens at the time $(-\tau) = \tau_m\sqrt{k/m_g}$ for a given mode with comoving wavenumber k. Specifically, we equate equation (7.6.105) with the Bunch-Davies vacuum equation (7.6.102) at the time of graviton horizon crossing, and we do the same thing with their conformal time derivatives. One obtains a set of two equations that can be solved for the unknowns A_k and B_k to find

$$A_k = \frac{\sqrt{2\pi}\tau_m^{1/3}}{6^{1/3}m_g^{1/6}} \exp\left[\frac{i}{3}\left(4\vartheta_k + \frac{\pi}{2}\right)\right]\,, \tag{7.6.106}$$

$$B_k = -\frac{\sqrt{2\pi}\tau_m^{1/3}}{6^{1/3}m_g^{1/6}} \exp\left[\frac{i}{3}\left(4\vartheta_k + \pi\right)\right]\,, \tag{7.6.107}$$

where we defined $\vartheta_k \equiv k^{3/2}\tau_m/m_g^{1/2}$. Therefore, the solution to the EOM on super-graviton scales, which reduces to the properly normalized Bunch-Davies vacuum on sub-graviton scales, is given by

$$u_k(\tau) \simeq \sqrt{\frac{2}{m_g}} \frac{\tau_m}{(-\tau)} e^{4i\vartheta_k/3} \left[e^{i\pi/6} \cos\left(\frac{\pi}{6} - \frac{m_g(-\tau)^3}{3\tau_m^2}\right) - e^{i\pi/3} \sin\left(\frac{\pi}{6} + \frac{m_g(-\tau)^3}{3\tau_m^2}\right)\right] , \tag{7.6.108}$$

and what is physically relevant for the power spectrum is the modulus squared:

$$|u_k(\tau)|^2 \simeq \frac{2}{m_g} \left(\frac{\tau_m}{-\tau}\right)^2 \left(\frac{1}{4}\right) = \frac{1}{2m_g a(\tau)} . \tag{7.6.109}$$

Finally, the power spectrum is [recall the definition from equations (7.6.92) and (7.6.93)]

$$\mathcal{P}_t(k,t) = \frac{8}{a^2 M_{\rm Pl}^2} \frac{k^3}{2\pi^2} \frac{1}{2m_g a(t)} = \frac{2}{\pi^2} \frac{(k/a)^3}{M_{\rm Pl}^2 m_g} , \tag{7.6.110}$$

which is a highly blue spectrum (the tilt is $n_t = 3$). Dividing by equation (7.6.99), we get the tensor-to-scalar ratio:

$$r_\star = 96 c_s \frac{k_\star^3}{m_g H_B^2} . \tag{7.6.111}$$

For instance, if we set $c_s = 0.62$ and $|H_B| = 8.0 \times 10^{-4}\, M_{\rm Pl}$, we have

$$r_\star \approx 2 \times 10^{-163} \left(\frac{k_\star}{0.05\,\mathrm{Mpc}^{-1}}\right)^3 \left(\frac{10^{-3}\, M_{\rm Pl}}{m_g}\right) . \tag{7.6.112}$$

Therefore, with a typical pivot scale $k_\star = 0.05\,\mathrm{Mpc}^{-1}$ [and more generally for physically observable scales in the cosmic microwave background (CMB)] and with a large graviton mass of the order of $|H_B|$, the tensor-to-scalar ratio is highly suppressed, well below current observational bounds. The model would effectively predict no observable primordial B-mode polarization in the CMB, similar to the prediction in Ekpyrotic cosmology [424] and pre-Big Bang cosmology [314, 315].

Connection with observations

Let us now connect the above results with cosmological observations. Currently, we note that the most constraining bound on the graviton mass is[7] [568]

$$m_g < 7.2 \times 10^{-23}\,\mathrm{eV}\ (95\%\ \mathrm{C.\,L.})\,. \tag{7.6.113}$$

Let us first consider the possibility that $m_g =$ constant throughout comic history, including the matter-dominated contracting phase in the context of the matter bounce scenario. Let us say that $m_g = 7.0 \times 10^{-23}\,\mathrm{eV}$, i.e. right below current constraints. Then, this means that for $|H(t)| < 7.0 \times 10^{-32}\,\mathrm{GeV}$, the mass of the graviton is effectively large compared to the background Hubble parameter in absolute value, and so, the primordial power spectrum of gravitational waves is given by equation (7.6.110), which we can rewrite as follows,

$$\mathcal{P}_t(k_\mathrm{p}) \approx 1.269 \times 10^{-127} \left(\frac{k_\mathrm{p}}{0.05\,\mathrm{Mpc}^{-1}}\right)^3\,, \tag{7.6.114}$$

where $k_\mathrm{p} \equiv k/a$ represents the physical wavenumber at the time of Hubble radius crossing. The above thus applies only for modes with $k_\mathrm{p} < m_g = 7.0 \times 10^{-32}\,\mathrm{GeV} \approx 5.489 \times 10^7\,\mathrm{Mpc}^{-1}$. Therefore, this applies for really all of the observables modes in the CMB (the CMB includes modes in the approximate range $[10^{-4}, 10^0]\,\mathrm{Mpc}^{-1}$). Therefore, we see from the above that the primordial gravitational wave spectrum is highly suppressed: even for the observable modes on the smallest CMB length scales, e.g., $k_\mathrm{p} \sim 1\,\mathrm{Mpc}^{-1}$, we have $\mathcal{P}_t \sim 10^{-123}$. It is only in the far UV, for $k_\mathrm{p} > m_p$, that the blue spectrum becomes closer to scale invariant (but still with a blue tilt). This corresponds to the regime where modes exit the Hubble radius when the graviton mass is effectively small compared to the background Hubble parameter in absolute value, so the primordial gravitational wave power spectrum has a blue tilt given by equation (7.6.96). In fact, the power spectrum is asymptotically scale invariant for $k_\mathrm{p} \to \infty$ [recall equation (7.6.98)].

In the previous setup though, i.e. with a constant graviton mass that satisfies equation (7.6.113), the anisotropies grow in a controlled way only for $|H(t)| \ll m_g$, but for $|H(t)| \gtrsim m_g$ they can rapidly dominate over the background and lead to the known BKL instability before the bounce. Indeed, we recall that the results of section 7.6.1, in particular equation (7.6.79),

[7]Latest bounds from gravitational waves observations [1, 2, 39] are of the same order.

are only valid if $m_g \gg |H|$. Therefore, it would be preferable if one had $m_g \gg |H(t)|$ for the whole contracting phase, i.e. for $0 < |H(t)| \leq |H_B|$. In that case, it is natural to take $m_g > |H_B|$. As a result, the anisotropies are under control for the whole contracting phase (all the way to the bounce), and the gravitational wave power spectrum is given by equation (7.6.110), i.e., it is suppressed and scales as k_{p}^3 across all scales (for $0 < k_{\mathrm{p}} < |H_B|$). However, since we expect $|H_B|$ to be relatively large, m_g cannot be constant throughout cosmic history in that case; m_g would have to be a time-dependent function in the EFT. For example, if we take $|H_B| \sim 8 \times 10^{-4}\, M_{\mathrm{Pl}}$, then we could have $m_g = m_{g,0} \sim 10^{-3}\, M_{\mathrm{Pl}}$ for the whole contracting phase and m_g would have to go to zero [or at least below the constraint of equation (7.6.113)] rapidly before or after the bounce[8].

The action of equation (7.2.7) allows for a time-dependent function $m_g(t, Z^{ab}, \delta_{ab})$. However, the analyses in sections 7.6.1, 7.6.2, and 7.6.2 assumed that m_g was constant in time, so we must set the functional form of $m_g(t, Z^{ab}, \delta_{ab})$ so that m_g is approximately constant (and large) before the bounce, quickly transitions at the beginning of the bounce and becomes very small (or zero) during the bouncing phase and for the rest of cosmic evolution. For instance, we could have

$$m_g(t, Z^{ab}, \delta_{ab}) = \frac{m_{g,0}}{2} \left[1 - \mathrm{erf}\left(\frac{t - t_B}{\sigma}\right)\right] . \tag{7.6.115}$$

in which case for $|t - t_B| \gg \sigma$ and $t < t_B$ (i.e. before the bounce), we get $m_g(t) \simeq m_{g,0}$, and similarly, for $|t - t_B| \gg \sigma$ and $t > t_B$ (i.e. after the bounce), we get $m_g(t) \simeq 0$. The transition time between the two constant mass phases would be of the order of σ, i.e. of the order of the duration of the bouncing phase.

One can then recover the space-time diffeomorphism invariance of the action by constructing the appropriate function $F(\varphi^0, Z^{ab}, \delta_{ab})$ so that equation (7.2.26) matches the above functional form of $m_g(t, Z^{ab}, \delta_{ab})$. It is straightforward to reconstruct such a function with a similar analysis to the one performed in section 7.3.2, although the resulting form of F might be complicated due to the appearance of the error function. However, we recall that the Gaussian ansatz in equation (7.3.41) as well as the subsequent error functions were taken for simplicity to make the different limits explicit. One could very well reconstruct the

[8]Most current constraints on the graviton mass apply only today in cosmic history, but a non-trivial graviton mass across time in standard Big Bang cosmology would still leave observable imprints in different CMB observations (see, e.g., [149]). Therefore, the graviton mass must already be sufficiently small after the bounce.

diffeomorphism-invariant action with different choices of functions such as rational functions that still yield the desired background dynamics.

7.7 Conclusions and discussion

We introduced a particular modified gravity model with a massive graviton, and we showed that it only propagates two gravitational degrees of freedom and is free of ghosts. We also showed that this model admits homogeneous and isotropic cosmological solutions with a non-singular bounce. We further studied the evolution of cosmological perturbations in this model. Whereas the scalar cosmological fluctuations grow on super-Hubble scales in the contracting phase like in other models, the finite graviton mass suppresses the growth of the gravitational waves. The finite graviton mass also enters in the EOM for anisotropies and leads to the conclusion that the energy density in anisotropies scales like pressureless matter. Hence, our model admits a realization of the *matter bounce* scenario which is free of two of the key problems of such scenarios: the *large r problem* and the anisotropy problem (also known as the BKL instability). Indeed, our model predicts a naturally highly suppressed tensor-to-scalar ratio on observational scales, and there is no anisotropy problem (no BKL instability).

While the analysis performed in this paper assumed a particular model of massive gravity, there is no *a priori* reason why the main conclusions should not hold with a different theory of gravity, as long as it admits a non-trivial mass for the graviton. For instance, the theories of [343, 344, 419, 569, 570] could all represent massive gravity theories in which the matter bounce scenario could be embedded to solve the large r and anisotropy problems. However, due to the existence of scalar and vector polarizations of the graviton in these theories, the scale invariance and near Gaussianity of the primordial curvature perturbations might be spoiled by the additional DOFs in the gravity sector. It would be interesting to investigate these theoretical possibilities.

In this paper, we did not study the evolution of the cosmological perturbations through the non-singular bouncing phase, although we argued that they should remain more or less unchanged. Yet, a proper analysis should be done in a follow-up paper. Accordingly, one may also wish to properly compute the strong energy scale of the theory, which determines the range of validity of the EFT. This is of particular interest when studying perturbations in non-singular bouncing cosmology in the context of EFT (see, e.g., [404, 571]). Finally, it may also be interesting to extend the analysis of this paper to explore non-Gaussianities from primordial perturbations generated during the matter contracting phase. We expect that at the three-point function level, similar to the case studied in [259], the graviton mass term will only contribute to the scalar-scalar-tensor and scalar-tensor-tensor couplings. However, due to the highly suppressed tensor perturbations, we do not expect sizable three-point correlation functions arising from those graviton mass terms. Thus, the amplitude of non-Gaussianity at the three-point function level should be the same as the one in the literature [181], i.e. $f_{\rm NL} \sim \mathcal{O}(1)$. Nevertheless, at the four-point function level, there could be interesting and sizable observational effects due to the non-trivial graviton mass. It would be interesting to investigate these non-linear effects and find new distinguishable features for very early universe models.

7.8 Second-order perturbed action for scalar modes

We start with the full action given by equation (7.2.7) plus the matter action of equation (7.3.32), i.e., we parametrize the matter content by a canonical scalar field ϕ with potential $U(\phi)$ for simplicity. Then, we linearly perturb the metric as in equation (7.4.53) and consider only the scalar perturbations. Similarly, we linearly perturb the scalar field as

$$\phi(t,\mathbf{x}) = \bar{\phi}(t) + \delta\phi(t,\mathbf{x})\,. \tag{7.8.116}$$

Then, the second-order perturbed action of scalar perturbations can be decomposed as In what follows, we drop the overbar on $\bar{\phi}(t)$ when it is clear that we are only referring the background evolution of ϕ.

$$S^{(2)}_{\rm scalar} = S^{(2)}_{\rm gravity} + S^{(2)}_{\rm matter}\,, \tag{7.8.117}$$

with

$$
\begin{aligned}
S^{(2)}_{\text{gravity}} = M_{\text{Pl}}^2 \int \mathrm{d}^3\mathbf{k}\,\mathrm{d}t \Bigg[&- \frac{1}{12} m_g^2 k^4 E^2 - 3a^3\dot{\psi}^2 + ak^2\psi^2 + a^3k^2\dot{E}\dot{\psi} - \frac{1}{2}a^3Hk^2\psi\dot{E} \\
&- 2a^2k^2\beta\dot{\psi} - 3a^3H^2\alpha^2 + \alpha\left(2a^2Hk^2\beta + 2ak^2\psi - a^3Hk^2\dot{E} + 6a^3H\dot{\psi}\right) \\
&- \frac{1}{2}E\left(3a^3H^2k^2\psi + a^3Hk^2\dot{\psi}\right) - \frac{1}{2}a^3\dot{H}k^2E\psi \Bigg] ,
\end{aligned} \tag{7.8.118}
$$

$$
\begin{aligned}
S^{(2)}_{\text{matter}} = \int \mathrm{d}^3\mathbf{k}\,\mathrm{d}t \Bigg\{ &\frac{1}{2}a^3\delta\dot{\phi}^2 - \frac{1}{2}a^3\dot{\phi}\left(k^2E + 2\alpha - 6\psi\right)\delta\dot{\phi} + \frac{1}{2}a^3\dot{\phi}^2\alpha^2 \\
&- \frac{1}{2}a\delta\phi^2\left[k^2 + a^2U''(\phi)\right] - \frac{1}{2}\delta\phi\left[2a^2\dot{\phi}k^2\beta + \left(k^2E - 2\alpha - 6\psi\right)\partial_t\left(a^3\dot{\phi}\right)\right] \Bigg\} ,
\end{aligned} \tag{7.8.119}
$$

where we recall that $2M_{\text{Pl}}^2\dot{H} = -\dot{\phi}^2 + \Lambda_2$. We note that the above action has been transformed to Fourier space where $\mathbf{k}$ represents the wavevector with magnitude $k \equiv |\mathbf{k}|$, also known as the wavenumber, and each perturbation variable represents its own Fourier transform, i.e., we omit the subscript k on each perturbation variable to simplify the notation in this appendix. Varying $S^{(2)}_{\text{scalar}}$ with respect to α and β, one obtains

$$
\alpha = \frac{\dot{\psi}}{H} + \frac{\dot{\phi}\delta\phi}{2M_{\text{Pl}}^2 H} , \tag{7.8.120}
$$

$$
\beta = -\frac{\psi}{aH} + \frac{1}{2}a\dot{E} + \frac{a\dot{\phi}\delta\dot{\phi}}{2k^2M_{\text{Pl}}^2H} - \frac{a\dot{\phi}^3\delta\phi}{4k^2M_{\text{Pl}}^4H^2} - \frac{a\dot{\phi}^2\dot{\psi}}{2k^2M_{\text{Pl}}^2H^2} - \frac{a\ddot{\phi}\delta\phi}{2k^2M_{\text{Pl}}^2H} , \tag{7.8.121}
$$

which represent the Hamiltonian and momentum constraints, respectively. Substituting equations (7.8.120) and (7.8.121) back into the quadratic action of scalar perturbations, one finds

$$
\begin{aligned}
S^{(2)}_{\rm scalar} = \int \mathrm{d}^3\mathbf{k}\,\mathrm{d}t \, \Bigg\{ & -\frac{1}{12}M_{\rm Pl}^2 m_g^2 k^4 E^2 + M_{\rm Pl}^2 \frac{k^2 a\dot{H}}{H^2}\psi^2 + \frac{1}{2}a^3\delta\dot{\phi}^2 + \frac{a^3\dot{\phi}^2}{2H^2}\dot{\psi}^2 \\
& + a^3\delta\dot{\phi}\left(3\psi\dot{\phi} - \frac{\dot{\phi}\dot{\psi}}{H}\right) + \frac{1}{2}a^3\delta\phi^2\Bigg[-\frac{k^2}{a^3} + \frac{\dot{\phi}^4}{4M_{\rm Pl}^4 H^2} + \frac{(-\dot{H}+6H^2)\dot{\phi}^2 + 4H\ddot{\phi}\dot{\phi}}{2M_{\rm Pl}^2 H^2} \\
& + \frac{3\dot{H}\dot{\phi} + 3H\ddot{\phi} + \dddot{\phi}}{\dot{\phi}}\Bigg] + a^3\delta\phi\Bigg[\psi\left(\frac{k^2\dot{\phi}}{Ha^2} + 9H\dot{\phi} + 3\ddot{\phi}\right) \\
& + \dot{\psi}\left(\frac{\dot{\phi}^3}{2M_{\rm Pl}^2 H^2} + \frac{3H\dot{\phi} + \ddot{\phi}}{H}\right)\Bigg]\Bigg\}\,.
\end{aligned}
\tag{7.8.122}
$$

We find that E does not have a kinetic term, and thus, it yields the constraint $E = 0$. Then, we introduce the curvature perturbation variable,

$$
\mathcal{R} \equiv \psi - \frac{H\delta\phi}{\dot{\phi}}\,, \tag{7.8.123}
$$

so the above action can be rewritten as

$$
\begin{aligned}
S^{(2)}_{\rm scalar} = \int \mathrm{d}^3\mathbf{k}\,\mathrm{d}t \; \Bigg\{ & \frac{a^3\dot{\phi}^2}{2H^2}\dot{\mathcal{R}}^2 + \Omega\mathcal{R}^2 + \psi\left[\frac{a^3\Lambda_2\dot{\phi}^2}{2M_{\rm Pl}^2 H^3}\dot{\mathcal{R}} - \frac{a^3\Lambda_2\dot{\phi}^2(6M_{\rm Pl}^2H^2 + \Lambda_2)}{4M_{\rm Pl}^4 H^4}\mathcal{R}\right] \\
& + \psi^2\left(\frac{a^3\Lambda_2^2\dot{\phi}^2}{8M_{\rm Pl}^4H^4} + \frac{ak^2\Lambda_2}{2H^2} + \frac{3a\Lambda_2 a^2\dot{\phi}^2}{4M_{\rm Pl}^2H^2}\right)\Bigg\}\,,
\end{aligned}
\tag{7.8.124}
$$

where Ω is a function of the background quantities $\dot{\phi}$, Λ_2, H, etc. Although the above action depends on both $\mathcal{R}$ and ψ, we note that there is only one scalar DOF in the theory, i.e. the one from matter fluctuations. The graviton itself does not have a helicity-0 component. This is consistent with the Hamiltonian analysis in section 7.2.2. The presence of ψ, which does not appear in general relativity, is a consequence of the broken temporal diffeomorphism of the theory. The variable ψ is actually a Lagrangian multiplier, and it gives us the following constraint:

$$
\psi = \frac{\dot{\phi}^2\left(6M_{\rm Pl}^2H^2\mathcal{R} + \Lambda_2\mathcal{R} - 2M_{\rm Pl}^2H\dot{\mathcal{R}}\right)}{4M_{\rm Pl}^4H^2\frac{k^2}{a^2} + 6M_{\rm Pl}^2H^2\dot{\phi}^2 + \Lambda_2\dot{\phi}^2}\,. \tag{7.8.125}
$$

Substituting the above constraint back into the perturbed action of equation (7.8.124), one finds

$$S^{(2)}_{\rm scalar} = \int \mathrm{d}^3\mathbf{k}\,\mathrm{d}t\ \left(\mathcal{K}\dot{\mathcal{R}}^2 - \tilde{\Omega}\mathcal{R}^2\right)\,, \tag{7.8.126}$$

where $\mathcal{K}$ and $\tilde{\Omega}$ are given by

$$\mathcal{K} = \frac{2k^2M_{\rm Pl}^4a^3\dot{\phi}^2 + 3M_{\rm Pl}^2a^5\dot{\phi}^4}{4k^2M_{\rm Pl}^4H^2 + a^2\left(6M_{\rm Pl}^2H^2 + \Lambda_2\right)\dot{\phi}^2}\,, \tag{7.8.127}$$

$$\begin{aligned}\tilde{\Omega} = &\frac{k^2M_{\rm Pl}^2a^5\dot{\phi}^2}{\left(4k^2M_{\rm Pl}^4H^2 + a^2\left(6M_{\rm Pl}^2H^2+\Lambda_2\right)\dot{\phi}^2\right)^2}\Bigg[\Lambda_2^2\left(\frac{4k^2M_{\rm Pl}^2}{a^2} + 2\dot{\phi}^2\right) \\ &- M_{\rm Pl}^2H\dot{\Lambda}_2\left(\frac{4k^2M_{\rm Pl}^2}{a^2} + 6\dot{\phi}^2\right) - M_{\rm Pl}^2H^2\Lambda_2\Bigg(\frac{2k^2\dot{\phi}^2}{a^2H^2} + \left(\frac{8\ddot{\phi}}{H\dot{\phi}} + 24\right)\frac{k^2M_{\rm Pl}^2}{a^2} + \frac{3\dot{\phi}^4}{M_{\rm Pl}^2H^2} \\ &+ 24\dot{\phi}^2\Bigg) + 2M_{\rm Pl}^2H^2\left(\frac{2k^2M_{\rm Pl}^2}{a^2} + 3\dot{\phi}^2\right)^2\Bigg]\,,\end{aligned} \tag{7.8.128}$$

i.e., they are two functions of the background quantities $\dot{\phi}$, Λ_2, H, etc. We immediately notice from the above that the scalar mode is always ghost free, because $\mathcal{K} > 0$. Also, in the UV limit where $k^2/a^2 \gg \dot{H}$, we have

$$S^{(2)}_{\rm scalar} \simeq M_{\rm Pl}^2\int \mathrm{d}^3\mathbf{k}\,\mathrm{d}t\ a^3\epsilon\left(\dot{\mathcal{R}}^2 - \frac{k^2}{a^2}\mathcal{R}^2\right)\,, \tag{7.8.129}$$

where $\epsilon = \dot{\phi}^2/(2M_{\rm Pl}^2H^2)$, and thus, the theory is free from UV gradient instabilities.

7.9 Evolution of anisotropies in a general background

We consider a general contracting universe dominated by matter with EoS $p = w\rho$, assuming $w \geq 0$. In this case, the scale factor and Hubble parameter are given by

$$a(t) \sim (-t)^{\frac{2}{3(1+w)}}\,, \qquad H(t) = \frac{2}{3(1+w)t}\,. \tag{7.9.130}$$

Then, the general solution to equation (7.5.73) is

$$\theta_i(t) = (-t)^{\nu_a}\left\{\left[C_1J_{\nu_a}[m_g(-t)] + C_2Y_{\nu_a}[m_g(-t)]\right\}\,, \tag{7.9.131}$$

and its time derivative is

$$\dot{\theta}_i(t) = -m_g(-t)^{\nu_a}\left\{[C_1 J_{\nu_a-1}[m_g(-t)] + C_2 Y_{\nu_a-1}[m_g(-t)]\right\}\,, \tag{7.9.132}$$

where J_{ν_a} and Y_{ν_a} are the Bessel functions of the first and second kind and where we defined the index

$$\nu_a \equiv \frac{1}{2} - \frac{1}{1+w}\,. \tag{7.9.133}$$

In the limit where $m_g \gg |H|$, we note that $m_g(-t) \gg 1$, and so, the asymptotic form of all the above Bessel functions is $[m_g(-t)]^{-1/2}$, ignoring constant factors and oscillatory functions of time, i.e., we only stress the time dependence of the overall amplitude. Thus, equations (7.9.131) and (7.9.132) give

$$\theta_i(t) \sim m_g^{-1/2}(-t)^{\nu_a-1/2}\,, \qquad \dot{\theta}_i(t) \sim m_g^{1/2}(-t)^{\nu_a-1/2}\,, \tag{7.9.134}$$

and so, the energy density given by equation (7.5.74) becomes

$$\rho_\theta \sim \dot{\theta}_i^2 + m_g^2\theta_i^2 \sim m_g(-t)^{2\nu_a-1}\,, \tag{7.9.135}$$

where the power of the time dependence can be expanded in terms of the EoS parameter w as

$$2\nu_a - 1 = -\frac{2}{1+w}\,. \tag{7.9.136}$$

Using equation (7.9.130) to convert the time dependence back into a scale factor dependence, one finds

$$\rho_\theta \sim (-t)^{-\frac{2}{1+w}} \sim \left(a^{\frac{3(1+w)}{2}}\right)^{-\frac{2}{1+w}} = a^{-3}\,. \tag{7.9.137}$$

Therefore, this shows that the energy density in anisotropies grows as a^{-3} in a contracting universe no matter what is the EoS of the dominant background matter content provided $m_g \gg |H|$.

Chapter 8

Black hole formation in a contracting universe

8.1 Introduction

The latest observations of the Cosmic Microwave Background indicate that a good theoretical model for the very early universe should predict a nearly scale-invariant power spectrum of curvature perturbations with a small red tilt [12], a small tensor-to-scalar ratio [11], and small non-Gaussianities [13]. Inflationary cosmology [49, 333, 463, 517] currently stands up as the best candidate for explaining these observations [14]. Yet, it is still an incomplete theory conceptually [102, 128, 129], because, for example, it suffers from a singularity at the time of the Big Bang [86, 87]. Thus, in addition to trying to resolve the issues of inflation, it is helpful to study competitive or complementary ideas that could enlighten our understanding of the very early universe.

One such idea is bouncing cosmology: one assumes that the universe existed forever before the Big Bang in a contracting phase, after which it transitioned into the expending universe that we observe today. In addition to solving the usual flatness and horizon problems of standard Big Bang cosmology, assuming that quantum cosmological perturbations exit the Hubble horizon in a matter-dominated contracting phase leads to a scale-invariant power spectrum of curvature perturbations [286, 618]. Furthermore, there exist many models that can avoid reaching a singularity at the time of the Big Bang, hence leading to nonsingular bouncing cosmologies (see [132, 166, 529] and references therein). Yet, it is still hard to construct models that can agree with all observational constraints (see, e.g., [562] and also [56, 109] for reviews).

An additional difficulty with bouncing cosmology comes from the fact that it appears less robust against certain instabilities as many unwanted features tend to grow in a contracting universe. One example is anisotropies: as $a \to 0$, anisotropies grow as $\rho \propto a^{-6}$, whereas the background matter and radiation evolve according to $\rho \propto a^{-3}$ and $\rho \propto a^{-4}$, respectively. This is known as the Belinsky-Khalatnikov-Lifshitz (BKL) instability [67]. This can be resolved if the background before the bounce can satisfy $\rho \propto a^{-q}$ with $q \gg 6$ [167, 273], which naturally occurs within the Ekpyrotic model [387, 388] (see also [424] and references therein).

There is another type of instability, always in a contracting universe, that has not been explored in as much detail, namely the growth of inhomogeneities. This type of instability was already known from the 1960s [451], but it is only in the 2000s that the work was extended [44], and it suggested that the growth of inhomogeneities in a contracting universe could lead to the formation of black holes.

The goal of this paper is thus to revisit the analysis of the growth of inhomogeneities in a contracting universe, and more specifically, characterize the formation of black holes. On one hand, we want to determine in which cases a contracting universe is robust or not against the formation of large inhomogeneities and black holes. This will determine in which cases it is justified to ignore the growth of inhomogeneities and allow us to claim which corresponding models remain healthy or not. On the other hand, we want to determine in which cases a contracting universe inevitably leads to the formation of black holes. These cases could be relevant in light of other alternative theories of the very early universe in which black holes could be the seeds of the current universe.

The outline of this paper is as follows. First, in section 8.2, we begin by setting the general framework in which we work, and we solve for the evolution of the gravitational potential in a contracting universe, aiming for generality. In section 8.3, we move on to find the density contrast in a generic contracting universe, and we comment on its evolution over the different length scales of interest. We also determine the power spectrum of the perturbations over the different scales of interest. In section 8.4, we explore two types of possible initial conditions for the fluctuations, quantum vacuum initial conditions and thermal initial conditions, and we find the power spectra in each cases. We also determine when the perturbations become non-linear. Then, in section 8.5, we derive the condition for black hole collapse, and we use the Press-Schechter formalism to determine which cases lead to the formation of black holes.

We also describe the black holes that form. Finally, in section 8.6, we summarize our results regarding the models that are robust (and those that are not) against the formation of black holes. We end by suggesting possible alternative theories that could take advantage of the formation of black holes. Throughout this paper, we adopt the mostly minus convention for the metric, and we define the reduced Planck mass by $M_{\rm Pl} \equiv (8\pi G_{\rm N})^{-1/2}$ where $G_{\rm N}$ is Newton's gravitational constant.

8.2 Evolution of the gravitational potential in a contracting universe

8.2.1 General background setup

We begin by finding the general evolution of the cosmological perturbations in a contracting universe. We try to be as generic as possible, and we do not specify any initial conditions for now. We start with an action of the form

$$S = -\frac{1}{16\pi G_{\rm N}} \int \mathrm{d}^4x \, \sqrt{-g} R + S_{\rm m} \, , \tag{8.2.1}$$

where $g_{\mu\nu}$ is the metric tensor, $g \equiv \det(g_{\mu\nu})$, R is the Ricci scalar, and $S_{\rm m}$ is the action for matter. We work in a flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe, so the background metric is

$$\mathrm{d}s^2 = g^{(0)}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = a(\eta)^2(\mathrm{d}\eta^2 - \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j) \, , \tag{8.2.2}$$

where a is the scale factor, η is the conformal time (defined by $\mathrm{d}\eta \equiv a^{-1}\mathrm{d}t$, where t is the physical time), and the x^i's represent the Cartesian comoving coordinates. The energy-momentum tensor is defined by

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta_g S_{\rm m}}{\delta g^{\mu\nu}} \, , \tag{8.2.3}$$

and we assume that it takes the form $T_\mu{}^\nu = \mathrm{diag}(\rho, -p\delta_i{}^j)$, where p represents the pressure and ρ the energy density. Accordingly, the background equations of motion (EOMs) are

$$\mathcal{H}^2 = \frac{8\pi G_\mathrm{N}}{3}a^2\rho \ , \tag{8.2.4}$$

$$\mathcal{H}' = -\frac{4\pi G_\mathrm{N}}{3}a^2\rho(1+3w) \ , \tag{8.2.5}$$

where $' \equiv \mathrm{d}/\mathrm{d}\eta$ and $\mathcal{H} \equiv a'/a$ is the conformal Hubble parameter. Furthermore, $w \equiv p/\rho$ is the equation of state (EoS) parameter.

From here on, we assume that the action for matter takes the form

$$S_\mathrm{m} = -\int \mathrm{d}^4x \ \sqrt{-g}\rho \ , \tag{8.2.6}$$

which is to say that *we will work in a hydrodynamical fluid setup*. The fluid has an EoS parameter w, and its sound speed is defined by

$$c_\mathrm{s}^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_s \ , \tag{8.2.7}$$

i.e. it is the variation of the pressure with respect to the energy density at constant entropy density, s. We note that we will ignore entropy perturbations throughout, i.e. *we assume that the fluid has only adiabatic fluctuations.*

8.2.2 Cosmological perturbations

Let us introduce linear scalar perturbations about the background introduced above. The perturbed metric written in the longitudinal (or conformal Newtonian) gauge with no anisotropic stress (i.e. $\delta T_{ij} = 0$ for $i \neq j$) is

$$\mathrm{d}s^2 = a(\eta)^2 \left\{[1+2\Phi(\eta,\mathbf{x})]\,\mathrm{d}\eta^2 - [1-2\Phi(\eta,\mathbf{x})]\,\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right\} \ . \tag{8.2.8}$$

The perturbation Φ is the Newtonian gravitational potential. The resulting EOM from the perturbed Einstein equations gives rise to the following partial differential equation [518],

$$\Phi'' + 3\mathcal{H}(1+c_\mathrm{s}^2)\Phi' + [2\mathcal{H}' + (1+3c_\mathrm{s}^2)\mathcal{H}^2]\Phi - c_\mathrm{s}^2\nabla^2\Phi = 0 \ , \tag{8.2.9}$$

where $\nabla^2 \equiv \partial_i \partial^i$ is the spacial Laplacian associated with the comoving space coordinates x^i. Alternatively, using the Friedmann equations (8.2.4) and (8.2.5) and transforming to Fourier space, the EOM can be written as

$$\Phi_k'' + 3\mathcal{H}(1+c_\mathrm{s}^2)\Phi_k' + 3(c_\mathrm{s}^2 - w)\mathcal{H}^2\Phi_k + c_\mathrm{s}^2 k^2 \Phi_k = 0 \ , \tag{8.2.10}$$

where k represents the magnitude of the comoving wavenumber associated with the perturbations.

From here on, we will assume that we can split the cosmological evolution into one or more separate phases of constant equation of state (EoS) parameter and constant sound speed. Therefore, for a *fixed (time-independent) EoS parameter* $w =$ constant, the solution to the background FLRW EOMs is[1]

$$a \propto (-\eta)^{\frac{2}{1+3w}} \ , \tag{8.2.11}$$

so

$$\mathcal{H} = -\frac{2}{1+3w}(-\eta)^{-1} \ , \tag{8.2.12}$$

and

$$\mathcal{H}' = -\frac{2}{1+3w}(-\eta)^{-2} \ . \tag{8.2.13}$$

The resulting EOM for the gravitational potential is

$$\Phi_k'' - \frac{6(1+c_\mathrm{s}^2)}{1+3w}\frac{1}{(-\eta)}\Phi_k' + \left(c_\mathrm{s}^2 k^2 + \frac{12(c_\mathrm{s}^2 - w)}{(1+3w)^2}\frac{1}{(-\eta)^2}\right)\Phi_k = 0 \ . \tag{8.2.14}$$

For $w =$ constant and for a *fixed (time-independent) sound speed* $c_\mathrm{s} =$ constant, the general solution to the above ordinary differential equation (ODE) is[2]

$$\Phi_k(\eta) = \left[2(1+3w)(-\eta)\right]^{\nu_1}\left[C_{1,k} J_{\nu_2}(-c_s k\eta) + C_{2,k} Y_{\nu_2}(-c_s k\eta)\right] \ , \tag{8.2.15}$$

[1]Since we are interested in a contracting universe, we consider the physical time to be negative, i.e. $t < 0$. The time $t = 0$ would correspond to a possible Big Crunch, Big Bang, or bounce. A negative physical time is equivalent to having a negative conformal time, $\eta < 0$, when $w < -1$ or $w > -1/3$, hence we have $(-\eta)$ in the scale factor since this quantity is positive. We can safely restrict ourself to matter with $w > -1/3$ for the rest of this paper and ignore exotic matter which could have $w < -1$. The case where $-1 \leq w \leq -1/3$ should be analyzed separately, but it will not be of interest in this paper.

[2]We note that the above ODE is invariant under $\eta \to -\eta$. Thus, the general solution is valid for both η and $-\eta$. We take the $-\eta$ branch of the solution for a contracting universe.

where $C_{1,k}$ and $C_{2,k}$ are two constants of integration that will of set by the initial conditions. Also, $J_\nu(x)$ and $Y_\nu(x)$ are the Bessel functions of the first and second kind, respectively. Finally, for shorthand notation, we define the indices

$$\nu_1 \equiv -\frac{5+6c_\mathrm{s}^2-3w}{2(1+3w)} \qquad \text{and} \qquad \nu_2 \equiv \frac{\sqrt{25+12c_\mathrm{s}^2+36c_\mathrm{s}^4+18w-36c_\mathrm{s}^2w+9w^2}}{2(1+3w)} \ . \tag{8.2.16}$$

From the definition of w and c_s and from the usual conservation equation [which follows from equations (8.2.4) and (8.2.5)],

$$\rho' + 3\mathcal{H}\rho(1+w) = 0 \ , \tag{8.2.17}$$

it is straightforward to show that

$$w' = 3\mathcal{H}(1+w)(w-c_\mathrm{s}^2) \ . \tag{8.2.18}$$

Since a constant EoS parameter means $w'=0$, the above equation implies that $w=c_\mathrm{s}^2$ under our assumptions. In this situation, the indices defined above simplify to become

$$\nu_1 = -\frac{5+3w}{2(1+3w)} \qquad \text{and} \qquad \nu_2 = \frac{5+3w}{2(1+3w)} \ , \tag{8.2.19}$$

and hence, we define the index $\nu \equiv \nu_2 = -\nu_1$ to simplify the notation from here on.

8.3 Density contrast, Jeans scale, and power spectrum

8.3.1 Density contrast

The gauge-invariant density contrast in a flat universe, $\delta(\eta,\mathbf{x})$, is related to the gravitation potential via [518]

$$\delta \equiv \frac{\delta\rho^{(\mathrm{gi})}}{\rho^{(0)}} = \frac{2}{3\mathcal{H}^2}\left(\nabla^2\Phi - 3\mathcal{H}\Phi' - 3\mathcal{H}^2\Phi\right) \ , \tag{8.3.20}$$

where $\delta\rho(\eta,\mathbf{x})$ denotes the energy density fluctuations and $\rho^{(0)}(\eta)$ denotes the background energy density. In Fourier space, this becomes

$$\delta_k \equiv \frac{\delta\rho_k^{(\mathrm{gi})}}{\rho^{(0)}} = -\frac{2}{3}\left(\frac{k^2}{\mathcal{H}^2}\Phi_k + \frac{3}{\mathcal{H}}\Phi_k' + 3\Phi_k\right) \ . \tag{8.3.21}$$

Using equation (8.2.12), we have

$$\delta_k(\eta) = -\frac{(1+3w)^2}{6}k^2(-\eta)^2\Phi_k(\eta) + (1+3w)(-\eta)\Phi_k'(\eta) - 2\Phi_k(\eta)\ , \tag{8.3.22}$$

and given the general solution for $\Phi_k(\eta)$, equation (8.2.15), we get

$$\begin{aligned}\delta_k(\eta) =\ & -\frac{1}{3\cdot 2^{\nu+1}(1+3w)^\nu(-\eta)^\nu}\Big\{6(1+3w)x[C_{1,k}J_{\nu-1}(x) + C_{2,k}Y_{\nu-1}(x)] \\ & + \left[12 - 6(5+3w) + \frac{(1+3w)^2x^2}{c_\mathrm{s}^2}\right]\Big[C_{1,k}J_\nu(x) + C_{2,k}Y_\nu(x)\Big]\Big\}\ ,\end{aligned} \tag{8.3.23}$$

where we further define $x \equiv c_\mathrm{s}k(-\eta)$ for shorthand notation.

8.3.2 Jeans scale

We will be interested in characterizing the formation of physical black holes, so we will primarily be interested in the sub-Hubble limit of the above density contrast. Since we are working with a fluid with a sound speed c_s possibly different from the speed of light, there is another scale of interest, the Jeans scale. It is defined to have a comoving wavenumber k_J such that the physical wavenumber is

$$\frac{k_\mathrm{J}}{a} \equiv \frac{\sqrt{4\pi G_\mathrm{N}\rho}}{c_\mathrm{s}} = \sqrt{\frac{3}{2}}\frac{|H|}{c_\mathrm{s}}\ , \tag{8.3.24}$$

or alternatively, we can write

$$k_\mathrm{J} = \sqrt{\frac{3}{2}}\frac{|\mathcal{H}|}{c_\mathrm{s}} = \frac{\sqrt{6}}{1+3w}\frac{1}{c_\mathrm{s}|\eta|}\ . \tag{8.3.25}$$

The associated comoving wavelength is $\lambda_\mathrm{J} \equiv 2\pi/k_\mathrm{J}$. Thus, the sub-Jeans scales correspond to the limit $\lambda \ll \lambda_\mathrm{J}$ or $k \gg k_\mathrm{J}$, which is equivalent to the limit where x is large; the super-Jeans scales correspond to the limit $\lambda \gg \lambda_\mathrm{J}$ or $k \ll k_\mathrm{J}$, which is equivalent to the limit where x is small. For dust, we have $w = c_\mathrm{s}^2 \to 0$, and so $\lambda_\mathrm{J} \to 0$. In other words, there is no sub-Jeans scales asymptotically, only super-Jeans/sub-Hubble and super-Hubble scales. For radiation, we have $w = c_\mathrm{s}^2 = 1/3$, and so $k_\mathrm{J} = 3/(\sqrt{2}|\eta|)$. In comparison, the Hubble scale is given by

$$k_H \equiv |\mathcal{H}| = \frac{2}{(1+3w)|\eta|} \,, \tag{8.3.26}$$

and so, for radiation, it is $k_H = 1/|\eta|$. Thus, although we still have $\lambda_{\mathrm{J}} < \lambda_H$ in this case, the two scales are really of the same order and nearly equal. Thus, there are very few scales in the super-Jeans/sub-Hubble regime. Most scales are either sub-Jeans or super-Hubble in this case.

Evolution below the Jeans length

We can expand the density contrast [equation (8.3.23)] to leading order in the limit where x is large ($\lambda \ll \lambda_{\mathrm{J}}$) to find

$$\begin{aligned}\delta_k(\eta) \overset{\lambda\ll\lambda_{\mathrm{J}}}{\simeq} &-\frac{(1+3w)^{2-\nu}k^{3/2}(-\eta)^{3/2-\nu}}{6\cdot 2^{\nu}\sqrt{\pi c_{\mathrm{s}}}} \\ &\times\left[(C_{1,k}-C_{2,k})\cos\left(c_{\mathrm{s}}k(-\eta)-\frac{\pi\nu}{2}\right)+(C_{1,k}+C_{2,k})\sin\left(c_{\mathrm{s}}k(-\eta)-\frac{\pi\nu}{2}\right)\right] \,.\end{aligned} \tag{8.3.27}$$

Thus, we see that the density contrast oscillates with frequency $\omega_k \equiv c_{\mathrm{s}}k$. However, we are more interested in the amplitude which goes as $(-\eta)^{3/2-\nu}$. Recalling the definition of ν in equation (8.2.19), we note that

$$\frac{3}{2}-\nu = \frac{3w-1}{3w+1} \,. \tag{8.3.28}$$

As physical time evolves in a contracting universe, $\eta \to 0^-$ or $(-\eta) \to 0^+$ for $w > -1/3$. Thus, we see that the amplitude of the density contrast *grows* in a contracting universe if

$$\frac{3}{2}-\nu < 0 \iff -\frac{1}{3} < w < \frac{1}{3} \,; \tag{8.3.29}$$

the amplitude of the density contrast is *constant* if

$$\frac{3}{2}-\nu = 0 \iff w = \frac{1}{3} \,; \tag{8.3.30}$$

and the amplitude of the density contrast *decreases* in a contracting universe if

$$\frac{3}{2}-\nu > 0 \iff w > \frac{1}{3} \,. \tag{8.3.31}$$

Consequently, in a dust-dominated contracting universe with $w = c_s^2 = 0$, we see that the amplitude of the density contrast *grows* as $(-\eta)^{-1}$. However, one needs to be careful since taking the limit $c_s \to 0$ would also imply that the density contrast blows up while the sub-Jeans regime of validity vanishes. In fact, for normal baryonic matter (or even for dark matter), we expect the sound speed and the EoS parameter to be small, but non-vanishing, i.e. $0 < w \ll 1$ and $0 < c_s \ll 1$. For a radiation-dominated contracting universe ($w = c_s^2 = 1/3$), we find that the amplitude of the density contrast is constant.

Evolution on super-Jeans/sub-Hubble scales

On one hand, on super-Jeans scales, x is small, and so, to leading order, equation (8.2.15) for the gravitational potential becomes

$$\Phi_k(\eta) \simeq \frac{-C_{2,k}\Gamma(\nu)}{\pi(1+3w)^\nu c_s^\nu k^\nu (-\eta)^{2\nu}} , \tag{8.3.32}$$

where $\Gamma(\nu)$ is the gamma function. On the other hand, on super-Hubble scales, $k/\mathcal{H}$ is large in equation (8.3.21), and so, the density contrast reduces to

$$\delta_k(\eta) \simeq -\frac{(1+3w)^2}{6} k^2 (-\eta)^2 \Phi_k(\eta) . \tag{8.3.33}$$

Therefore, substituting the super-Jeans solution for Φ_k into the sub-Hubble regime for δ_k yields

$$\delta_k(\eta) \simeq \frac{(1+3w)^{2-\nu}\Gamma(\nu)}{6\pi c_s^\nu} C_{2,k} k^{2-\nu} (-\eta)^{2(1-\nu)} . \tag{8.3.34}$$

The amplitude of the density contrast goes as $(-\eta)^{2(1-\nu)}$. Recalling the definition for ν is equation (8.2.19), we note that

$$2(1-\nu) = \frac{3(w-1)}{3w+1} . \tag{8.3.35}$$

Thus, we see that the amplitude of the density contrast *grows* in a contracting universe if

$$2(1-\nu) < 0 \iff -\frac{1}{3} < w < 1 ; \tag{8.3.36}$$

the amplitude of the density contrast is *constant* if

$$2(1-\nu)=0 \iff w=1\ ; \tag{8.3.37}$$

and the amplitude of the density contrast *decreases* in a contracting universe if

$$2(1-\nu)>0 \iff w>1\ . \tag{8.3.38}$$

Consequently, in a dust-dominated and in a radiation-dominated contracting universe, we find that the super-Jeans/sub-Hubble modes of the density contrast grow in amplitude as they approach a possible bounce.

Evolution on super-Hubble scales

On super-Hubble scales, the general form for the density contrast is $\delta_k(\eta)\simeq-2(\Phi_k'/\mathcal{H}+\Phi_k)$. Substituting in equation (8.3.32), the super-Hubble (and necessarily super-Jeans) solution for the density contrast is

$$\delta_k(\eta)\simeq\frac{2[1-\nu(1+3w)]\Gamma(\nu)}{\pi(1+3w)^{\nu}c_{\mathrm{s}}^{\nu}}C_{2,k}k^{-\nu}(-\eta)^{-2\nu}\ . \tag{8.3.39}$$

This time, the amplitude goes as $(-\eta)^{-2\nu}$, but $-2\nu<0$ given our assumption that $w>-1/3$, indicating growth in the amplitude of the perturbations in all cases.

8.3.3 Power spectrum

We saw above that the resulting density contrast oscillates with a time-varying amplitude on sub-Jeans scales. Since we will be primarily interested in the amplitude, let us average out the oscillations. Moreover, the general solution in Fourier space is generally complex, so let us take the magnitude squared to get the more physically meaningful real amplitude. Thus, equation (8.3.27) becomes

$$\left\langle|\delta_k(\eta)|^2\right\rangle\overset{\lambda\ll\lambda_{\mathrm{J}}}{\simeq}\frac{(1+3w)^{2(2-\nu)}}{36\cdot 2^{2\nu}\pi c_{\mathrm{s}}}(-\eta)^{3-2\nu}k^3\left(|C_{1,k}|^2+|C_{2,k}|^2\right)\ . \tag{8.3.40}$$

Here, $\langle\cdot\rangle$ really means averaging over the oscillations, i.e. $\langle\cos[\omega_k(-\eta)-\pi\nu/2]\sin[\omega_k(-\eta)-\pi\nu/2]\rangle=0$ and $\langle\cos^2[\omega_k(-\eta)-\pi\nu/2]\rangle=\langle\sin^2[\omega_k(-\eta)-\pi\nu/2]\rangle=1/2$.

On super-Jeans/sub-Hubble scales, we simply have

$$|\delta_k(\eta)|^2 \simeq \frac{(1+3w)^{2(2-\nu)}\Gamma(\nu)^2}{36\pi^2 c_{\mathrm{s}}^{2\nu}}|C_{2,k}|^2 k^{2(2-\nu)}(-\eta)^{4(1-\nu)} , \tag{8.3.41}$$

and on super-Hubble scales, it is

$$|\delta_k(\eta)|^2 \simeq \frac{4[1-\nu(1+3w)]^2\Gamma(\nu)^2}{\pi^2(1+3w)^{2\nu} c_{\mathrm{s}}^{2\nu}}|C_{2,k}|^2 k^{-2\nu}(-\eta)^{-4\nu} . \tag{8.3.42}$$

One can interpret the above quantities as the power spectra of the density contrast, i.e. $P_\delta(k,\eta) \equiv |\delta_k(\eta)|^2$. In dimensionless form,

$$\mathcal{P}_\delta(k,\eta) \equiv \frac{k^3}{2\pi^2}P_\delta(k,\eta) = \frac{k^3}{2\pi^2}|\delta_k(\eta)|^2 . \tag{8.3.43}$$

The averaged power spectrum on sub-Jeans scales is then identified with equation (8.3.40), so denoting the average by a bar, we have

$$\bar{\mathcal{P}}_\delta(k,\eta) = \frac{k^3}{2\pi^2}\left\langle|\delta_k(\eta)|^2\right\rangle \overset{\lambda\ll\lambda_{\mathrm{J}}}{\simeq} \frac{(1+3w)^{2(2-\nu)}}{72\cdot 2^{2\nu}\pi^3 c_{\mathrm{s}}}(-\eta)^{3-2\nu}k^6\left(|C_{1,k}|^2+|C_{2,k}|^2\right) . \tag{8.3.44}$$

Equivalently, on super-Jeans/sub-Hubble scales, the power spectrum is

$$\mathcal{P}_\delta(k,\eta) = \frac{k^3}{2\pi^2}|\delta_k(\eta)|^2 \simeq \frac{(1+3w)^{2(2-\nu)}\Gamma(\nu)^2}{72\pi^4 c_{\mathrm{s}}^{2\nu}}|C_{2,k}|^2 k^{7-2\nu}(-\eta)^{4(1-\nu)} , \tag{8.3.45}$$

and on super-Hubble scales, we have

$$\mathcal{P}_\delta(k,\eta) \simeq \frac{2[1-\nu(1+3w)]^2\Gamma(\nu)^2}{\pi^4(1+3w)^{2\nu} c_{\mathrm{s}}^{2\nu}}|C_{2,k}|^2 k^{3-2\nu}(-\eta)^{-4\nu} . \tag{8.3.46}$$

8.4 Examples of initial conditions

8.4.1 Quantum vacuum

At this point, we did not specify the initial conditions, which is why the above resulting power spectra still depend on the integration constants $C_{1,k}$ and $C_{2,k}$. A typical initial condition would be a quantum (Bunch-Davies) vacuum,

$$v_k^{(\mathrm{ini})}(\eta) = \frac{\mathrm{e}^{-\mathrm{i}c_\mathrm{s}k\eta}}{\sqrt{2c_\mathrm{s}k}}\ , \qquad c_\mathrm{s}k(-\eta)\to\infty\ , \tag{8.4.47}$$

where v is the Mukhanov-Sasaki variable. For example, in matter bounce cosmology, matching the super-Hubble evolution of the cosmological perturbations in a matter-dominated contracting universe at Hubble crossing with quantum vacuum initial conditions yields a scale-invariant power spectrum of curvature perturbations [286, 618]. Initially, one has an oscillating quantum vacuum state, but at Hubble radius crossing, the quantum fluctuations 'squeeze' and emerge as classical fluctuations on super-Hubble scales (see, e.g., [55, 550, 553, 589]). Similarly, in the case where the Jeans length is different than the Hubble radius, we associate quantum vacuum fluctuations with the sub-Jeans regime, and the squeezing of the fluctuations at Jeans length crossing leads to classical perturbations on super-Jeans scales, the growth of which might lead to the formation of black holes. Since we showed above that the Jeans and Hubble lengths are nearly equal for radiation, the quantum vacuum will be of more interest for dust when $w = c_\mathrm{s}^2 \ll 1$. Yet, we must ensure that we do not set $c_\mathrm{s} = 0$, since otherwise, the Jeans length would vanish, and we would no longer be able to define a quantum vacuum state on sub-Jeans scales.

We need to relate the variable v with the density contrast for which we computed the power spectrum. The Mukhanov-Sasaki variable is related to the gravitational potential via [518]

$$\nabla^2\Phi = -\frac{\beta}{\sqrt{2}M_\mathrm{Pl}c_\mathrm{s}^2\mathcal{H}}\left(\frac{v}{z}\right)'\ , \tag{8.4.48}$$

where

$$z \equiv \frac{a\sqrt{\beta}}{c_\mathrm{s}\mathcal{H}}\ , \tag{8.4.49}$$

and

$$\beta \equiv \mathcal{H}^2 - \mathcal{H}'\ . \tag{8.4.50}$$

Upon transforming to Fourier space, the initial condition in terms of the gravitational potential reads

$$-k^2\Phi_k^{(\mathrm{ini})} = -\frac{\beta}{\sqrt{2}M_\mathrm{Pl}c_\mathrm{s}^2\mathcal{H}}\left(\frac{v_k^{(\mathrm{ini})}}{z}\right)'\ . \tag{8.4.51}$$

Using equation (8.4.47), we have $v_k^{(\mathrm{ini})\prime} = -\mathrm{i}c_\mathrm{s}kv_k^{(\mathrm{ini})}$, and so, the above becomes

$$\Phi_k^{(\mathrm{ini})} = -\frac{\beta v_k^{(\mathrm{ini})}}{\sqrt{2}M_\mathrm{Pl}c_\mathrm{s}^2\mathcal{H}k^2 z}\left(\mathrm{i}c_\mathrm{s}k + \frac{z'}{z}\right) = -\frac{\mathrm{i}\beta(-\eta)^{3/2}}{2M_\mathrm{Pl}\mathcal{H}z}\frac{\mathrm{e}^{\mathrm{i}x}}{x^{3/2}}\left(1 - \frac{z'}{z}\frac{\mathrm{i}(-\eta)}{x}\right) . \tag{8.4.52}$$

where again, $x \equiv c_\mathrm{s}k(-\eta)$. To leading order when x is large, i.e. on sub-Jeans scales, the initial condition for Φ_k becomes

$$\Phi_k^{(\mathrm{ini})} \overset{x\to\infty}{\simeq} -\frac{\mathrm{i}\beta(-\eta)^{3/2}}{2M_\mathrm{Pl}\mathcal{H}z}\frac{\mathrm{e}^{\mathrm{i}x}}{x^{3/2}} . \tag{8.4.53}$$

We recall that we found the general solution for $\Phi_k(\eta)$ in equation (8.2.15). We now demand that the large x limit of equation (8.2.15) matches the above expression for $\Phi_k^{(\mathrm{ini})}$. To leading order, equation (8.2.15) becomes

$$\Phi_k(\eta) \overset{x\to\infty}{\simeq} \frac{\mathrm{e}^{\mathrm{i}(x-\frac{\nu\pi}{2}-\frac{\pi}{4})}(C_{1,k} - \mathrm{i}C_{2,k}) + \mathrm{e}^{-\mathrm{i}(x-\frac{\nu\pi}{2}-\frac{\pi}{4})}(C_{1,k} + \mathrm{i}C_{2,k})}{2^{\nu+\frac{1}{2}}\sqrt{\pi}(1+3w)^\nu(-\eta)^\nu x^{1/2}} . \tag{8.4.54}$$

For the above to match with the initial condition [equation (8.4.53)], which only goes as $\mathrm{e}^{\mathrm{i}x}$, it is clear that we must have $C_{1,k} + \mathrm{i}C_{2,k} = 0$, so that the term $\mathrm{e}^{-\mathrm{i}x}$ goes to 0 in the above. Thus, $C_{1,k} = -\mathrm{i}C_{2,k}$, and the above becomes

$$\Phi_k(\eta) \overset{x\to\infty}{\simeq} -\mathrm{i}\frac{2^{\frac{1}{2}-\nu}\mathrm{e}^{\mathrm{i}(x-\frac{\nu\pi}{2}-\frac{\pi}{4})}}{\sqrt{\pi}(1+3w)^\nu(-\eta)^\nu x^{1/2}}C_{2,k} . \tag{8.4.55}$$

Equating this to equation (8.4.53) imposes

$$\begin{aligned} C_{2,k} &= \frac{2^{\nu-\frac{3}{2}}\sqrt{\pi}(1+3w)^\nu(-\eta)^{\nu+\frac{1}{2}}\beta}{c_\mathrm{s}M_\mathrm{Pl}\mathcal{H}zk}\mathrm{e}^{\mathrm{i}(\nu+\frac{1}{2})\frac{\pi}{2}} \\ &= \frac{2^{\nu-\frac{3}{2}}\sqrt{\pi}(1+3w)^\nu(-\eta)^{\nu+\frac{1}{2}}\sqrt{\mathcal{H}^2-\mathcal{H}'}}{M_\mathrm{Pl}ak}\mathrm{e}^{\mathrm{i}(\nu+\frac{1}{2})\frac{\pi}{2}} , \end{aligned} \tag{8.4.56}$$

where we use the definition of β and z to simplify the second equality. Using equations (8.2.12) and (8.2.13) for $\mathcal{H}$ and $\mathcal{H}'$, the integrating constant further simplifies to become

$$C_{2,k} = \frac{2^{\nu-\frac{3}{2}}\sqrt{6\pi(1+w)}(1+3w)^{\nu-1}(-\eta)^{\nu-\frac{1}{2}}}{M_\mathrm{Pl}ak}\mathrm{e}^{\mathrm{i}(\nu+\frac{1}{2})\frac{\pi}{2}} . \tag{8.4.57}$$

The left-over time-dependent factor is actually just a constant after simplification since

$$\frac{(-\eta)^{\nu-\frac{1}{2}}}{a(\eta)} = \frac{\eta_0^{\frac{2}{1+3w}}}{a_0} \tag{8.4.58}$$

if we normalize the scale factor as $a(\eta) = a_0(-\eta/\eta_0)^{\frac{2}{1+3w}}$. However, it will be more convenient to keep the scale factor in the expression. In the end, we are left with

$$|C_{2,k}|^2 = \frac{3\pi(1+w)(1+3w)^{2(\nu-1)}(-\eta)^{2\nu-1}}{2^{2(1-\nu)}M_{\mathrm{Pl}}^2 a^2 k^2} \ . \tag{8.4.59}$$

Substituting the above integration constant found for quantum vacuum initial conditions on sub-Jeans scales into the general power spectrum on super-Jeans/sub-Hubble scales [equation (8.3.45)] leads to

$$\mathcal{P}_\delta(k,\eta) \simeq \frac{2^{2\nu}(1+w)(1+3w)^2\Gamma(\nu)^2}{96\pi^3 c_{\mathrm{s}}^{2\nu} M_{\mathrm{Pl}}^2 a^2} k^{5-2\nu}(-\eta)^{3-2\nu} \ . \tag{8.4.60}$$

As we saw earlier, the super-Jeans/sub-Hubble regime is valid for dust, but not so much for radiation. Thus, to leading order when $w = c_{\mathrm{s}}^2 \ll 1$, we obtain

$$\mathcal{P}_\delta(k,\eta) \simeq \frac{3}{16\pi^2 c_{\mathrm{s}}^5 M_{\mathrm{Pl}}^2 a^2}(-\eta)^{-2} = \frac{3\mathcal{H}^2}{64\pi^2 c_{\mathrm{s}}^5 M_{\mathrm{Pl}}^2 a^2} \ , \tag{8.4.61}$$

or

$$\mathcal{P}_\delta(k,t) \simeq \frac{3H^2(t)}{64\pi^2 c_{\mathrm{s}}^5 M_{\mathrm{Pl}}^2} \ , \tag{8.4.62}$$

where we use the fact that $\mathcal{H} = aH$ with $H \equiv \mathrm{d}\ln a/\mathrm{d}t$ being the physical Hubble parameter. As a result, the density contrast power spectrum on super-Jeans/sub-Hubble scales is scale-invariant (independent of k) and grows in amplitude as time evolves ($|H|$ grows in time in a contracting universe).

We will soon be interested in describing the formation of black holes. A necessary condition (but not sufficient) for black hole formation is $\mathcal{P}_\delta(k,t) > 1$, which can be viewed as defining the scale of non-linearity. However, since we obtain a scale-invariant power spectrum, there is no specific non-linear scale, but rather, a non-linear time after which all super-Jeans/sub-Hubble modes become non-linear. In terms of the Hubble parameter, non-linearity is reached when

$$|H| \gtrsim \frac{8}{\sqrt{3}}\pi c_{\rm s}^{5/2} M_{\rm Pl}\ . \tag{8.4.63}$$

Hence, all super-Jeans/sub-Hubble scales become non-linear when the energy scale of the universe becomes larger than a fraction of the Planck scale. The fraction may be very small depending on the smallness of the sound speed, and so, this may occur well before a Planck time before a possible bounce.

8.4.2 Thermal initial conditions

As another example of initial conditions, let us consider the case of thermal fluctuations. In this situation, the averaged energy density fluctuations on sub-Jeans scales for a thermal statistical system of characteristic size L and temperature T is given by (see [78, 182, 480] and also [542])

$$\langle\delta^2\rangle_L = \frac{T^2}{L^3\rho^2}\frac{\partial\rho}{\partial T}\ . \tag{8.4.64}$$

In Fourier space, the averaged density fluctuations become

$$|\delta_k|^2 = \frac{\gamma_{\rm f}^2 T^2}{a^3\rho^2}\frac{\partial\rho}{\partial T}\ , \tag{8.4.65}$$

where the constant $\gamma_{\rm f}$ depends on the choice of window function when doing the Fourier transformation (see [78] for details).

In our context of an ideal fluid with EoS parameter w, one can express the energy density as a function of temperature by (see [78])

$$\rho(T) = \frac{m_T^4}{w}\left(\frac{T}{m_T}\right)^{\frac{1+w}{w}}\ , \tag{8.4.66}$$

where m_T is a preferred mass scale associated with the fluid. We note that the above expression is only valid for $0 < w < 1$, so when we consider dust, we will take the limit $0 < w \ll 1$ as before. For radiation ($w = 1/3$), we recover the Stefan-Boltzmann law $\rho(T) \propto T^4$, where there is no preferred mass scale. For a general EoS, taking $\rho(a) = \rho_0(a/a_0)^{-3(1+w)}$ and using the above expression for $\rho(T)$, equation (8.4.65) becomes

$$|\delta_k|^2 = \frac{\gamma_{\rm f}^2(1+w)}{a_0^3 m_T^3}\left(\frac{m_T^4}{w\rho_0}\right)^{\frac{1}{1+w}} . \tag{8.4.67}$$

Setting $w = 1/3$ for radiation, the resulting dimensionless power spectrum on sub-Jeans scales is

$$\mathcal{P}_\delta(k) = \frac{2\gamma_{\rm f}^2}{3^{1/4}\pi^2}\frac{1}{\rho_0^{3/4}}\left(\frac{a}{a_0}\right)^3\left(\frac{k}{a}\right)^3 = \frac{2\gamma_{\rm f}^2}{3\pi^2 M_{\rm Pl}^{3/2}|H|^{3/2}}\left(\frac{k}{a}\right)^3 , \tag{8.4.68}$$

where in the second equality, we use the Friedmann equation $3M_{\rm Pl}^2H^2 = \rho$ and equation (8.4.66) (the Stefan-Boltzmann law for radiation). We note that the power spectrum is blue, and also, it is time independent for a fixed comoving wavenumber k. It follows that the scales that are non-linear ($\mathcal{P}_\delta > 1$) must satisfy

$$\frac{k}{a} > \left(\frac{3}{2}\right)^{1/3}\left(\frac{\pi}{\gamma_{\rm f}}\right)^{2/3}\sqrt{M_{\rm Pl}|H|} . \tag{8.4.69}$$

Thus, at later times, when the energy scale $|H|$ of the universe is higher, there are fewer physical scales k/a that become non-linear. However, as $|H| \to 0$ in the infinite past, it would appear that all physical scales become non-linear, which seems to render unphysical this choice of initial conditions. Yet, there is a subtlety that allows us to still consider thermal initial conditions.

We note that thermal fluctuations can be interpreted as a Poisson process, which presupposes a set of regions with coherence length ℓ_C (see [480]). In fact, we can express the averaged density contrast in position space as

$$\langle\delta^2\rangle_L = \left(\frac{\ell_C}{L}\right)^3 , \tag{8.4.70}$$

where the temperature-dependent coherence length is given by $\ell_C^3 = (T/\rho)^2(\partial\rho/\partial T)$. For example, for radiation, we find that $\ell_C \propto T^{-1} \propto a$. With this interpretation, a requirement for the fluctuations to be non-linear is that the coherence length must be larger than the scale of the thermal system, i.e. $\ell_C > L$. However, a requirement for thermalization is that $\ell_C \ll L$, and so, when the thermal fluctuations become non-linear, it must follow that they are no more thermal. In particular, this implies that in the far past when ℓ_C is large, one cannot consider thermal fluctuations on arbitrary small length scales. In this sense, the thermal initial state is well defined in the far past as long as we consider length scales that are large enough compared to the coherence length at that time. These set the initial conditions, and as the fluctuations evolve gravitationally without interactions, they loose their thermality, and in particular, they may become non-linear.

For dust in the limit $w \ll 1$, the density contrast squared, equation (8.4.67), should go to

$$|\delta_k|^2 \simeq \frac{\gamma_\mathrm{f}^2 m_T}{w\rho_0 a_0^3} \tag{8.4.71}$$

on sub-Jeans scales. In comparison, the general solution for dust on sub-Jeans scales, equation (8.3.40) with $w = c_\mathrm{s}^2 \ll 1$, is

$$\langle|\delta_k|^2\rangle \simeq \frac{k^3}{1152\pi c_\mathrm{s}(-\eta)^2}\left(|C_{1,k}|^2 + |C_{2,k}|^2\right) \ . \tag{8.4.72}$$

Thus, if the $C_{1,k}$ and $C_{2,k}$ terms contribute equally, it follows that we must take

$$|C_{2,k}|^2 = \frac{576\pi\gamma_\mathrm{f}^2 c_\mathrm{s} m_T(-\eta_\mathrm{ini})^2}{w\rho_0 a_0^3 k^3} \ , \tag{8.4.73}$$

where η_ini is the initial conformal time at which the initial conditions are set. In other words, at η_ini, we set the hydrodynamical cosmological perturbations to have the amplitude and spectrum of thermal fluctuations. From that moment onward, and especially as we consider the super-Jeans regime, the fluctuations are no more thermal.

On super-Jeans/sub-Hubble scales, the power spectrum for dust (equation (8.3.41) with $w = 0$) is

$$|\delta_k(\eta)|^2 \simeq \frac{|C_{2,k}|^2}{64\pi c_\mathrm{s}^5 k(-\eta)^6} \ . \tag{8.4.74}$$

Substituting in equation (8.4.73) yields

$$\mathcal{P}_\delta(k,t) \simeq \frac{3\gamma_\mathrm{f}^2}{32\pi^2 c_\mathrm{s}^4 w}\left(\frac{a_\mathrm{ini}}{a}\right)\left(\frac{H}{M_\mathrm{Pl}}\right)^2 m_T\left(\frac{a}{k}\right) \ , \tag{8.4.75}$$

where we use the relations $\rho = \rho_0(a/a_0)^{-3} = 3M_\mathrm{Pl}^2H^2$, $(-\eta) = -2/\mathcal{H} = -2/(aH)$, and $a/a_\mathrm{ini} = (\eta/\eta_\mathrm{ini})^2$ to simplify the expression, and we note that we define $a_\mathrm{ini} \equiv a(\eta_\mathrm{ini})$. It follows that non-linearity occurs when

$$\frac{k}{a} < \frac{3\gamma_{\mathrm{f}}^2}{32\pi^2 c_{\mathrm{s}}^4 w}\left(\frac{a_{\mathrm{ini}}}{a}\right)\left(\frac{H}{M_{\mathrm{Pl}}}\right)^2 m_T \ . \tag{8.4.76}$$

Thus, as $|H|/M_{\mathrm{Pl}}$ and a_{ini}/a grow in a contracting universe, more physical scales become non-linear on super-Jeans/sub-Hubble scales. Also, it appears that the largest length scales become non-linear first. This suggests that larger black holes form before smaller ones. We will confirm this result in the following section.

8.5 Black hole formation

The evolution and the spectrum of the cosmological perturbations found in the previous sections allow us to address the question of black hole formation. We found that the amplitude of the perturbations increases in many instances, and so, we expect some of the overdensities to collapse to form black holes as one approaches a possible bounce. However, we need to determine under what conditions one can claim that a black has formed.

8.5.1 General requirement for black hole collapse

Let us consider an element of physical volume $\mathrm{d}V = \mathrm{d}^3\mathbf{q}$ at some physical time t and physical position $\mathbf{q}$. Then, the amount of mass excess enclosed in this physical volume element at position $\mathbf{q}$ as a function of time is given by

$$\mathrm{d}\delta M(t,\mathbf{q}) = \mathrm{d}^3\mathbf{q}\ \delta\rho(t,\mathbf{q}) \ . \tag{8.5.77}$$

We argue that a black hole forms when an amount of mass excess $\delta M \geq M_s$ is found inside a ball of radius $R \leq R_s$, where R_s is the (physical) Schwarzschild radius given by $R_s = 2M_s G_{\mathrm{N}}$ for a black hole of mass M_s. This appears to be a fair requirement assuming that the hoop conjecture holds (the original idea of the hoop conjecture comes from [505, 598] ; see also subsequent papers on the subject, e.g. [291, 319]). Therefore, a black hole forms, i.e. an event horizon appears, if

$$\int_{R\leq R_s} \mathrm{d}\delta M \geq M_s \ . \tag{8.5.78}$$

More precisely, using equation (8.5.77) for the mass excess element, we say that the condition for a black hole to form at the point $\mathbf{q}_\star$ and time $t_\star$ is

$$\int_{\mathcal{B}(R_s,\mathbf{q}_\star)} \mathrm{d}^3\mathbf{q}\ \delta\rho(t_\star,\mathbf{q}) \geq \frac{R_s}{2G_\mathrm{N}}\ , \tag{8.5.79}$$

where the integral is over the volume of a ball of radius R_s centered at $\mathbf{q}_\star$, or more formally, over the region

$$\mathcal{B}(R_s,\mathbf{q}_\star) \equiv \left\{\mathbf{q}\in\mathbb{R}^3 \middle| |\mathbf{q}-\mathbf{q}_\star| \leq R_s\right\}\ . \tag{8.5.80}$$

8.5.2 Smoothing

The goal is thus to evaluate the integral on the left-hand side of equation (8.5.79). In order to do so, let us review the idea of smoothing. In general, the definition of a smoothed perturbation δ over a characteristic scale $\mathscr{R}$ is

$$\delta(t,\mathbf{x};\mathscr{R}) \equiv \frac{1}{\mathscr{V}(\mathscr{R})}\int \mathrm{d}^3\tilde{\mathbf{x}}\ W(|\mathbf{x}-\tilde{\mathbf{x}}|/\mathscr{R})\delta(t,\tilde{\mathbf{x}})\ , \tag{8.5.81}$$

where W is the window function, assumed to be spherically symmetric with comoving radius $\mathscr{R}$. The comoving volume associated with the smoothing region of characteristic scale $\mathscr{R}$ is defined by

$$\mathscr{V}(\mathscr{R}) \equiv \int \mathrm{d}^3\mathbf{x}\ W(|\mathbf{x}|/\mathscr{R}) = 4\pi\mathscr{R}^3\int \mathrm{d}y\ y^2W(y)\ , \tag{8.5.82}$$

where $\mathbf{y}\equiv\mathbf{x}/\mathscr{R}$, $y\equiv|\mathbf{y}|$. Then, in Fourier space, $\delta_\mathbf{k}(t;\mathscr{R}) = \mathcal{W}(k\mathscr{R})\delta_\mathbf{k}(t)$, where we denote the Fourier transform of W by $\mathcal{W}$. Also, the variance is related to the power spectrum by

$$\sigma^2(\mathscr{R},t) \equiv \langle[\delta(t,\mathbf{x};\mathscr{R})]^2\rangle = \int_0^\infty \frac{\mathrm{d}k}{k}\ \mathcal{W}^2(k\mathscr{R})\mathcal{P}_\delta(k,t) = \frac{1}{2\pi^2}\int_0^\infty \mathrm{d}k\ k^2\mathcal{W}^2(k\mathscr{R})|\delta_k(t)|^2\ . \tag{8.5.83}$$

A commonly used window function is the top-hat window function,

$$W(y) = \begin{cases} 1 & \text{for } 0\leq y\leq 1\ (|\mathbf{x}|\leq\mathscr{R}) \\ 0 & \text{for } y>1\ (|\mathbf{x}|>\mathscr{R})\ . \end{cases} \tag{8.5.84}$$

Its Fourier transform is

$$\mathcal{W}(k\mathscr{R}) = \frac{3[\sin(k\mathscr{R}) - k\mathscr{R}\cos(k\mathscr{R})]}{(k\mathscr{R})^3} \ . \tag{8.5.85}$$

Using the top-hat window function, we recognize that smoothing the perturbation $\delta\rho$ with characteristic scale[3] R_s/a yields

$$\delta\rho(t, \mathbf{x}; R_s/a) = \frac{1}{\mathscr{V}(R_s/a)} \int \mathrm{d}^3\tilde{\mathbf{x}}\ W(a|\mathbf{x} - \tilde{\mathbf{x}}|/R_s)\delta\rho(t, \tilde{\mathbf{x}}) \tag{8.5.86}$$

$$= \frac{3a^3}{4\pi R_s^3} \int_{|\mathbf{x}-\tilde{\mathbf{x}}| \leq \frac{R_s}{a}} \mathrm{d}^3\tilde{\mathbf{x}}\ \delta\rho(t, \tilde{\mathbf{x}}) \ , \tag{8.5.87}$$

but since comoving coordinates $\mathbf{x}$ are related to physical coordinates $\mathbf{q}$ by $\mathbf{q} = a\mathbf{x}$, we get

$$\delta\rho(t, \mathbf{x}; R_s/a) = \frac{3}{4\pi R_s^3} \int_{|\mathbf{q}-\tilde{\mathbf{q}}| \leq R_s} \mathrm{d}^3\tilde{\mathbf{q}}\ \delta\rho(t, \tilde{\mathbf{q}}) \equiv \delta\rho(t, \mathbf{q}; R_s) \ . \tag{8.5.88}$$

Therefore, we notice that the left-hand side of equation (8.5.79) is simply related to the smoothed perturbation $\delta\rho(t, \mathbf{q}; R_s)$. Specifically, at a fixed time and position, we find

$$\int_{\tilde{\mathbf{q}} \in \mathcal{B}(R_s, \mathbf{q}_\star)} \mathrm{d}^3\tilde{\mathbf{q}}\ \delta\rho(t_\star, \tilde{\mathbf{q}}) = \frac{4\pi}{3} R_s^3 \delta\rho(t_\star, \mathbf{q}_\star; R_s) \ . \tag{8.5.89}$$

8.5.3 Critical density contrast for black hole collapse

Combining equations (8.5.79) and (8.5.89), we say that a black hole forms when

$$\delta\rho(t_\star, \mathbf{q}_\star; R_s) \geq \frac{3}{8\pi G_{\mathrm{N}} R_s^2} \ . \tag{8.5.90}$$

Dividing by the background energy density on both sides and recalling the Friedmann equation $H(t)^2 = 8\pi G_{\mathrm{N}}\rho^{(0)}(t)/3$, the condition becomes

$$\frac{\delta\rho}{\rho^{(0)}}(t_\star, \mathbf{q}_\star; R_s) \geq \left(\frac{H^{-1}(t_\star)}{R_s}\right)^2 \ . \tag{8.5.91}$$

[3] R_s is the physical Schwarzschild radius, so we divide by the scale factor to have a comoving quantity.

We define the critical density contrast as a function of physical size R and time t to be

$$\delta_{\rm c}(R,t) \equiv \left(\frac{H^{-1}(t)}{R}\right)^2 \ . \tag{8.5.92}$$

Alternatively, as a function of conformal time and for a comoving scale $\mathscr{R} = R/a$,

$$\delta_{\rm c}(\mathscr{R},\eta) = \left(\frac{\mathcal{H}^{-1}(\eta)}{\mathscr{R}}\right)^2 \ . \tag{8.5.93}$$

Finally, the condition to form a black hole of Schwarzschild radius R_s at any time t and position $\mathbf{q}$ is

$$\delta(t,\mathbf{q};R_s) \geq \delta_{\rm c}(R_s,t) \ ; \tag{8.5.94}$$

or at conformal time η and comoving position $\mathbf{x}$, a black hole forms if $\delta(\eta,\mathbf{x};\mathscr{R}_s) \geq \delta_{\rm c}(\mathscr{R}_s,\eta)$, where $\mathscr{R}_s \equiv R_s/a$ is the comoving Schwarzschild radius.

From the form of our critical density contrast equation (8.5.92), we notice that a necessary (but not sufficient) condition for black hole formation is $\delta > 1$ on scales where $R < |H|^{-1}$, i.e. on sub-Hubble scales. This is to be expected since black holes are highly non-linear objects. In general, the smaller R is compared to the Hubble radius, then the larger $\delta_{\rm c}$ is, and so, the larger the density contrast δ needs to be to form a black hole of size R. In other words, the smaller the black hole we want to form, the more difficult it becomes. However, since δ is a smoothed quantity in the condition $\delta > \delta_{\rm c}$, i.e. integrated over space, its particular spectrum will affect the condition for black hole formation. For example, more power on smaller scales could lead to the production of smaller black holes before larger black holes. This is why we focused on computing the spectrum of δ in Fourier space in the previous sections.

We point out that equation (8.5.92) is only valid for $R \leq |H|^{-1}$. Naively, it is obvious that this equation cannot hold for super-Hubble perturbations since for long wavelength fluctuations, the critical density contrast for black hole formation would become very small, which would imply that small fluctuations would collapse into large black holes. In fact, if one takes $R \to \infty$, then $\delta_{\rm c} \to 0$, and it would seem to imply that *any* fluctuation would collapse into a black hole, which is physically inadmissible. The underlying reason comes from the fact that no black hole horizon can actually form above the cosmological apparent horizon. Indeed, any observer inside the cosmological horizon cannot know about the existence of the formation of a black hole if this black hole's horizon is greater than the cosmological horizon. Yet, there can still be large density fluctuations on super-Hubble scales, and these can form black holes if they re-enter the cosmological horizon at later times. In the context of a nonsingular bounce[4], large density fluctuations that exit the Hubble radius in the contracting phase could collapse into black holes once they re-enter the Hubble radius in the expanding phase. This is similar to the formation of primordial black holes [184, 187] in inflation where large density fluctuations can exit the Hubble radius during the inflationary phase and re-enter the Hubble radius in the subsequent radiation-dominated expanding phase, at which point the large density fluctuations can collapse into black holes (see, e.g., [305, 632]). The fact that no black hole horizon can form on length scales larger than the cosmological horizon is also explicit in general relativistic constructions such as in Schwarzschild-de Sitter spacetime (see, e.g., [96, 275, 584] and references therein) or McVittie spacetime (see, e.g., [275, 277, 379] and references therein).

8.5.4 Press-Schechter formalism and a condition for black hole collapse

Following the idea of the Press-Schechter formalism [554], we say that $\delta(\eta, \mathbf{x})$ is a Gaussian random field, and thus, the fraction of mass in spheres of radius $\mathscr{R}$ with overdensity $\delta > \delta_{\mathrm{c}}$ has a Gaussian probability,

$$\mathscr{P}(\mathscr{R}, \eta) = \frac{1}{\sqrt{2\pi}\sigma(\mathscr{R}, \eta)} \int_{\delta_{\mathrm{c}}(\mathscr{R}, \eta)}^{\infty} \mathrm{d}\delta \ \exp\left[-\frac{\delta^2}{2\sigma^2(\mathscr{R}, \eta)}\right] = \frac{1}{2} \ \mathrm{erfc}\left[\frac{\delta_{\mathrm{c}}(\mathscr{R}, \eta)}{\sqrt{2}\sigma(\mathscr{R}, \eta)}\right] \ , \tag{8.5.95}$$

where we recall that the variance $\sigma^2(\mathscr{R}, \eta)$ is given by equation (8.5.83) (simply replacing physical time with conformal time in this case). To account for the fact that there is an equal amount of matter in underdense as in overdense regions, relative to the background, we say that the actual probability is

$$F(\mathscr{R}, \eta) = 2\mathscr{P}(\mathscr{R}, \eta) = \mathrm{erfc}\left[\frac{\delta_{\mathrm{c}}(\mathscr{R}, \eta)}{\sqrt{2}\sigma(\mathscr{R}, \eta)}\right] \ . \tag{8.5.96}$$

Accordingly, the probability to form a black hole of comoving size $\mathscr{R}$ at conformal time η is large when the ratio $\delta_{\mathrm{c}}(\mathscr{R}, \eta)/\sigma(\mathscr{R}, \eta)$ is small. In fact, $F \to 1$ as $\delta_{\mathrm{c}}/\sigma \to 0$. Therefore, it is

[4] However, in the context of a nonsingular bounce, one would need to consider the possible effect of the formation of sub-Hubble black holes, so it is not yet clear how density fluctuations would evolve through a nonsingular bounce.

fair to say that black holes of characteristic radius $\mathscr{R}$ can only form in significant numbers when $\sigma(\mathscr{R},\eta) \gtrsim \delta_{\rm c}(\mathscr{R},\eta)$ (see, e.g., [506]). This makes sense intuitively since we found earlier that a black hole was formed at position $\mathbf{x}$ when $\delta(\eta,\mathbf{x};\mathscr{R}) \geq \delta_{\rm c}(\mathscr{R},\eta)$. Now, we say that a necessary condition is $\sigma(\mathscr{R},\eta) = \sqrt{\langle[\delta(\eta,\mathbf{x};\mathscr{R})]^2\rangle} \gtrsim \delta_{\rm c}(\mathscr{R},\eta)$, which is more or less equivalent.

In general, we evaluate σ^2 as follows:

$$\sigma^2(\mathscr{R},\eta) = \int_0^\infty \frac{\mathrm{d}k}{k}\mathcal{W}^2(k\mathscr{R})\mathcal{P}_\delta(k,\eta) \simeq \int_{k_H}^{k_{\rm J}} \frac{\mathrm{d}k}{k}\mathcal{W}^2(k\mathscr{R})\mathcal{P}_\delta(k,\eta) + \int_{k_{\rm J}}^\infty \frac{\mathrm{d}k}{k}\mathcal{W}^2(k\mathscr{R})\mathcal{P}_\delta(k,\eta)\ . \tag{8.5.97}$$

We note that instead of integrating from $k=0$, we set an infrared cutoff at the Hubble scale $k_H = |\mathcal{H}| = a|H|$ since we argue that no black holes could form on super-Hubble scales. In general, on super-Jeans/sub-Hubble scales, $\mathcal{P}_\delta(k,\eta)$ is given by equation (8.3.45), and on sub-Jeans scales, $\mathcal{P}_\delta(k,\eta)$ is given by equation (8.3.44). Accordingly, one could determine the general expression for the variance on arbitrary scales and for arbitrary matter, but the two most interesting cases, dust and radiation, are only applicable on distinct scales. Thus, we put generality aside, and we only consider them separately below.

Dust on super-Jeans/sub-Hubble scales

Let us begin with dust with quantum vacuum initial conditions. In this case, the density contrast power spectrum on super-Jeans/sub-Hubble scales is given by equation (8.4.62), and so, the variance is found to be

$$\sigma^2(\mathscr{R},t) = \int_{k_H}^{k_{\rm J}} \frac{\mathrm{d}k}{k}\ \mathcal{W}^2(k\mathscr{R})\mathcal{P}_\delta(k,t) \simeq \frac{3H^2}{64\pi^2 c_{\rm s}^5 M_{\rm Pl}^2}\int_{k_H}^{k_{\rm J}} \mathrm{d}k\ \frac{\mathcal{W}^2(k\mathscr{R})}{k}\ . \tag{8.5.98}$$

Taking equation (8.5.85) for the top-hat window function, the integral reduces to

$$\int_{k_H}^{k_{\rm J}} \mathrm{d}k\ \frac{\mathcal{W}^2(k\mathscr{R})}{k} = \frac{7}{4} - \gamma - \ln(2k_H\mathscr{R}) + \frac{(k_H\mathscr{R})^2}{10} + \mathcal{O}[(k_H\mathscr{R})^4] + \mathcal{O}[(k_{\rm J}\mathscr{R})^{-4}]\ , \tag{8.5.99}$$

where we use the fact that $k_{\rm J}\mathscr{R} \gg 1$ on super-Jeans scales and $k_H\mathscr{R} \ll 1$ on sub-Hubble scales. In the above, $\gamma \approx 0.577$ is the Euler-Mascheroni constant, which appears in the series expansion of the cosine integral. Keeping only the constant and logarithmic terms to leading order, the variance is found to be

$$\sigma^2(R,t) \simeq \frac{3H^2}{64\pi^2 c_\mathrm{s}^5 M_\mathrm{Pl}^2}\left[\frac{7}{4}-\gamma-\ln(2|H|R)\right] , \tag{8.5.100}$$

where we use $k_H\mathscr{R} = |\mathcal{H}|\mathscr{R} = a|H|\mathscr{R} = |H|R$. Then, recalling equation (8.5.92), the condition for black hole formation, $\sigma \gtrsim \delta_\mathrm{c}$, reads

$$\frac{3H^6R^4}{64\pi^2 c_\mathrm{s}^5 M_\mathrm{Pl}^2}\left[\frac{7}{4}-\gamma-\ln 2-\frac{1}{2}\ln(H^2R^2)\right] \gtrsim 1 . \tag{8.5.101}$$

This expression cannot be reduced analytically, so let us consider the formation of black holes which have a radius equal to a fraction of the Hubble radius, i.e. let $R = \alpha|H|^{-1}$ for some constant $\alpha \leq 1$ not too small so that we remain on super-Jeans scales. In this case, the above condition for black hole formation reduces to

$$|H| \gtrsim \frac{8\pi c_\mathrm{s}^{5/2} M_\mathrm{Pl}}{\sqrt{3}\alpha^2}\left[\frac{7}{4}-\gamma-\ln 2-\ln(\alpha)\right]^{-1/2} . \tag{8.5.102}$$

We see that the larger α is, the smaller the expression on the right-hand side of the above condition, which implies that a smaller energy scale (smaller $|H|$) needs to be reached to form black holes of size $R = \alpha|H|^{-1}$. In particular, this implies that Hubble-size black holes, i.e. black holes with Schwarzschild radius $R = |H|^{-1}$, form first when

$$|H| \simeq \frac{8\pi c_\mathrm{s}^{5/2} M_\mathrm{Pl}}{\sqrt{3(7/4-\gamma-\ln 2)}} , \tag{8.5.103}$$

a small fraction of the Planck scale when c_s is small. In comparison, we found in equation (8.4.63) that we entered the non-linear regime when $|H| \simeq 8\pi c_\mathrm{s}^{5/2} M_\mathrm{Pl}/\sqrt{3}$.

For dust with thermal initial conditions, the density contrast power spectrum on super-Jeans/sub-Hubble scales is given by equation (8.4.75), and so, the variance is found to be

$$\sigma^2(\mathscr{R},t) \simeq \frac{3\gamma_\mathrm{f}^2 a_\mathrm{ini} m_T}{32\pi^2 c_\mathrm{s}^4 w}\left(\frac{H}{M_\mathrm{Pl}}\right)^2 \int_{k_H}^{k_\mathrm{J}} \mathrm{d}k\, \frac{\mathcal{W}^2(k\mathscr{R})}{k^2} . \tag{8.5.104}$$

Using the top-hat window function again, the integral reduces to

$$\int_{k_H}^{k_{\rm J}} {\rm d}k \frac{\mathcal{W}^2(k\mathscr{R})}{k^2} \simeq \mathscr{R}\left\{-\frac{9\pi}{35}+\mathcal{O}[(k_{\rm J}\mathscr{R})^{-5}]+\frac{1}{k_H\mathscr{R}}+\frac{k_H\mathscr{R}}{5}-\frac{(k_H\mathscr{R})^3}{175}+\mathcal{O}[(k_H\mathscr{R})^5]\right\} , \tag{8.5.105}$$

in the limits $k_H\mathscr{R} \ll 1$ and $k_{\rm J}\mathscr{R} \gg 1$. Keeping the leading order terms and converting to physical quantities, the variance reduces to

$$\sigma^2(R,t) \simeq \frac{3\gamma_{\rm f}^2}{32\pi^2 c_{\rm s}^4 w}\left(\frac{a_{\rm ini}}{a}\right) m_T \left(\frac{H}{M_{\rm Pl}}\right)^2 \left(\frac{1}{|H|}-\frac{9\pi R}{35}\right) . \tag{8.5.106}$$

The condition for black hole formation $\sigma \gtrsim \delta_{\rm c}$ becomes

$$\frac{3\gamma_{\rm f}^2}{32\pi^2 c_{\rm s}^4 w}\left(\frac{a_{\rm ini}}{a}\right) \frac{m_T H^6 R^4}{M_{\rm Pl}^2}\left(\frac{1}{|H|}-\frac{9\pi R}{35}\right) \gtrsim 1 . \tag{8.5.107}$$

As before, let us consider black holes with radius $R = \alpha|H|^{-1}$, i.e. a fraction $\alpha \leq 1$ of the Hubble radius. Also, we note that, since $\rho \propto a^{-3}$ and $\rho \propto H^2$, we have $a/a_{\rm ini} = (H_{\rm ini}/H)^{2/3}$. Thus, the condition reduces to

$$|H| \gtrsim \frac{8}{3^{3/5}}\left(\frac{\pi}{\gamma_{\rm f}}\right)^{6/5} c_{\rm s}^{12/5} w^{3/5} \alpha^{-12/5}\left(1-\frac{9\pi\alpha}{35}\right)^{-3/5} \frac{H_{\rm ini}^{2/5} M_{\rm Pl}^{6/5}}{m_T^{3/5}} . \tag{8.5.108}$$

We see that the larger the fraction α is, the earlier black holes form. Consequently, Hubble-size black holes form first once again in this case. Associating the preferred mass scale of the fluid m_T with the energy scale at the time at which the initial conditions are taken, i.e. letting $m_T = H_{\rm ini}$, we find that

$$|H| \simeq 8\left(\frac{\pi}{\gamma_{\rm f}}\right)^{6/5}\left[3\left(1-\frac{9\pi}{35}\right)\right]^{-3/5} c_{\rm s}^{12/5} w^{3/5}\left(\frac{M_{\rm Pl}}{H_{\rm ini}}\right)^{1/5} M_{\rm Pl} \tag{8.5.109}$$

corresponds to the Hubble parameter at the time that the first (Hubble-size) black holes are formed. On one hand, since $c_{\rm s}^{12/5} w^{3/5} = c_{\rm s}^{18/5} \ll 1$ for dust, the critical time for black hole formation may be well before a Planck time before a possible bounce. On the other hand, we normally consider initial conditions such that $H_{\rm ini} \ll M_{\rm Pl}$, so this pushes the critical time closer to the Planck time.

To visualize the above results, we plot the probability of black hole formation, equation

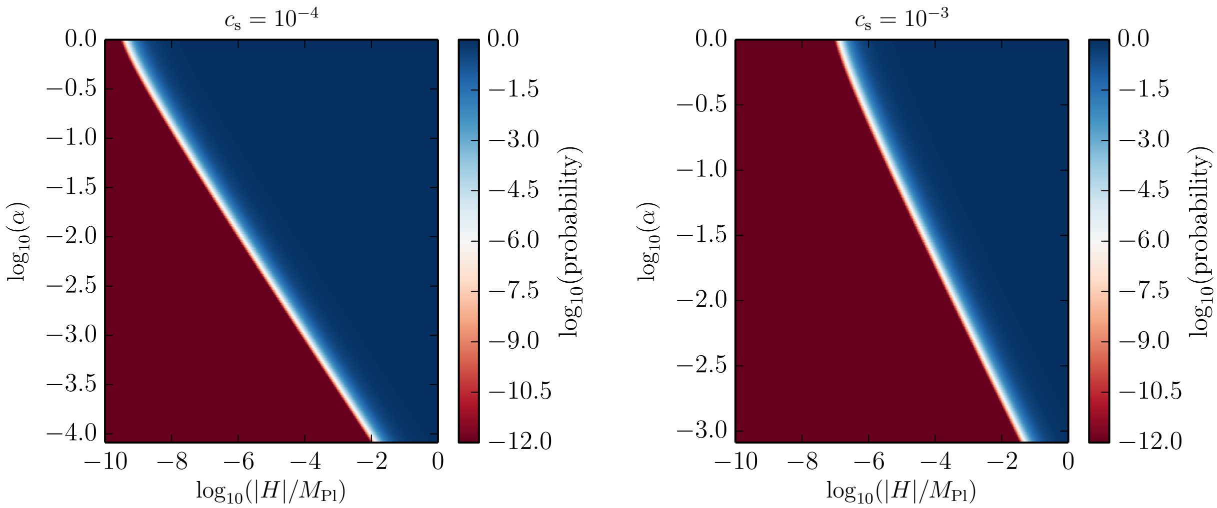

Figure 8.1 Plots of the probability $F = \mathrm{erfc}[\delta_c/(\sqrt{2}\sigma)]$ that a black hole has formed with dimensionless radius $\alpha = R/|H|^{-1}$ (vertical axis) and at Hubble parameter $|H|$ (horizontal axis). The probability is color coded on a $\log_{10}$ scale. The left and right plots show the probability for a sound speed c_s of 10^{-4} and 10^{-3}, respectively. The form of σ is taken from equation (8.5.100) for quantum vacuum initial conditions.

(8.5.96), in figures 8.1 and 8.2 for different cases. The plots are color coded on a $\log_{10}$ scale in terms of the probability F ranging from 1 (100% chance that a black hole has formed ; in dark blue) to[5] 0 (no chance that a black hole has formed ; in dark red). In each plot, the vertical axis represents the radius of the black hole as a fraction of the Hubble radius, i.e. $\alpha = R/|H|^{-1}$, on a $\log_{10}$ scale, ranging from the Jeans scale $\alpha = a|H|/k_J = \sqrt{2/3}c_s$ to the Hubble scale $\alpha = 1$. The horizontal axis represents the Hubble parameter as a fraction of the Planck scale on a $\log_{10}$ scale.

In figure 8.1, we show the probability in the case of quantum vacuum initial conditions, so we take equation (8.5.100) for σ^2. The left plot shows the result with a sound speed of $c_s = 10^{-4}$ and the right plot shows $c_s = 10^{-3}$. In all cases, we see that the probability to form a black hole changes abruptly from nearly 0 (dark red) to nearly 1 (dark blue). In the left plot, the lowest energy scale at which this occurs is around $|H| \sim 10^{-9}M_{\rm Pl}$, at which point $\alpha \sim 1$, meaning that the first black holes that form are of Hubble size. With a larger sound speed, in the right plot, the same is true, but it occurs when the Hubble parameter is $|H| \sim 10^{-7}M_{\rm Pl}$. In other words, the first black holes would form later with a larger sound

[5] We actually put a hard cutoff at $\log_{10} F = -12$, because as the probability goes to 0, the log scale would go to $-\infty$. As $F \leq 10^{-12}$, the probability is negligible, and we associate it with 0 probability.

speed.

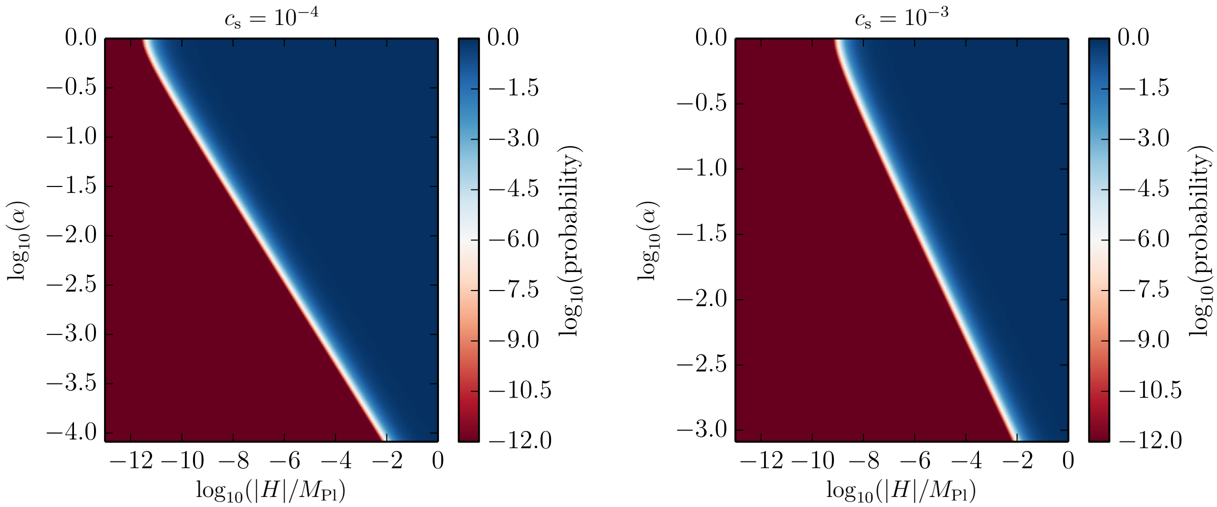

Figure 8.2 Same plots as in figure 8.1, but for thermal initial conditions, so the form of σ is taken from equation (8.5.106). Also, we take $H_{\rm ini} = 10^{-16} M_{\rm Pl}$ and $\gamma_{\rm f} = 2\sqrt{2}\pi^{3/4}$.

In figure 8.2, we show the probability in the case of thermal initial conditions, so we take equation (8.5.106) for σ^2. In addition, we pick the Hubble parameter at the initial time to be $H_{\rm ini} = 10^{-16} M_{\rm Pl}$, and following [78], we take $\gamma_{\rm f} = 2\sqrt{2}\pi^{3/4}$. Again, the left and right plots show $c_{\rm s} = 10^{-4}$ and $c_{\rm s} = 10^{-3}$, respectively. Just like in figure 8.1, the transition from a low probability of finding black holes to a high probability is abrupt, and the plots show that Hubble-size black holes ($\alpha = 1$) form first. The critical value of the Hubble parameter is quantitatively different but remains well below the Planck scale for a small sound speed.

Radiation on sub-Jeans scales

For radiation with thermal initial conditions, the density contrast power spectrum on sub-Jeans scales is given by equation (8.4.68), and so, the variance is found to be

$$\sigma^2(\mathscr{R}, t) \simeq \frac{2\gamma_{\rm f}^2}{3\pi^2 a^3 M_{\rm Pl}^{3/2} |H|^{3/2}} \int_{k_{\rm J}}^{\infty} {\rm d}k\ \mathcal{W}^2(k\mathscr{R}) k^2\ . \tag{8.5.110}$$

Taking the top-hat window function, equation (8.5.85), the integral reduces to

$$\int_{k_{\rm J}}^{\infty} {\rm d}k\ \mathcal{W}^2(k\mathscr{R}) k^2 \simeq \mathscr{R}^{-3} \left\{ \frac{3\pi}{2} - \frac{1}{3}(k_{\rm J}\mathscr{R})^3 + \mathcal{O}[(k_{\rm J}\mathscr{R})^5] \right\} \tag{8.5.111}$$

in the sub-Jeans limit $k_{\mathrm{J}}\mathscr{R} \ll 1$. Keeping only the leading order term, the variance becomes

$$\sigma^2(R,t) \simeq \frac{\gamma_{\mathrm{f}}^2}{\pi M_{\mathrm{Pl}}^{3/2}|H|^{3/2}R^3} \ . \tag{8.5.112}$$

Therefore, the condition for black hole formation $\sigma \gtrsim \delta_{\mathrm{c}}$ reduces to

$$|H| \gtrsim \left(\frac{\pi^2 M_{\mathrm{Pl}}^3}{\gamma_{\mathrm{f}}^4 R^2}\right)^{1/5} \ . \tag{8.5.113}$$

Thus, the larger R is, the smaller the quantity on the right-hand of the above expression, and the earlier black hole formation occurs. Since this is only valid on sub-Jeans scales, it implies that Jeans-size black holes form first. Taking $R = (a/k_{\mathrm{J}}) = \sqrt{2}/(3|H|)$ for radiation, these black holes form when

$$|H| \simeq \left(\frac{9\pi^2}{2\gamma_{\mathrm{f}}^4}\right)^{1/3} M_{\mathrm{Pl}} \ . \tag{8.5.114}$$

The numerical constant $[9\pi^2/(2\gamma_{\mathrm{f}}^4)]^{1/3}$ is only of[6] $\mathcal{O}(1)$, and consequently, the first black holes in a radiation-dominated contracting universe with thermal initial conditions form only when the energy scale reaches the Planck scale.

In nonsingular bouncing cosmologies, one usual assumes that new physics appears in the effective theory well below the Planck scale, and thus, one could expect to enter the bounce without forming any black hole. In this sense, a nonsingular bouncing cosmology in which the radiation-dominated contracting phase starts early in its cosmological evolution is robust against the formation of black holes. However, it remains to be shown that such an early transition from matter domination to radiation domination does not spoil the scale invariance of the power spectrum of curvature perturbations at the scales of observational interest in the usual matter bounce.

8.6 Conclusions and discussion

In this paper, we studied the adiabatic cosmological perturbations of a hydrodynamical fluid with constant EoS parameter and constant sound speed in a flat[7] contracting universe. We

[6] For example, it is shown in [78] that a Gaussian window function yields $\gamma_{\mathrm{f}} = 2\sqrt{2}\pi^{3/4}$, and so, $[9\pi^2/(2\gamma_{\mathrm{f}}^4)]^{1/3} \approx 0.28$, which is marginally smaller than $\mathcal{O}(1)$.

[7] Although we did not include the possible effects of spatial curvature in our analysis, we believe that our results would not be greatly affected by those effects since the contribution of spatial curvature decreases in

found the general evolution of the density contrast over the different regimes of interest: sub-Jeans scales, super-Jeans/sub-Hubble scales, and super-Hubble scales. The key results that are independent of the initial conditions can be summarized as follows:

- for a radiation-dominated contracting universe, the amplitude of the density contrast on sub-Jeans scales is constant in time;
- for a matter-dominated contracting universe, the amplitude of the density contrast on super-Jeans/sub-Hubble scales grows with time as one approaches a possible bounce.

We then considered two sets of initial conditions: quantum vacuum initial conditions and thermal initial conditions. This allowed us to find the general form of the power spectrum, and the main results are given below.

- By setting quantum vacuum initial conditions at Jeans crossing, the density contrast power spectrum in a matter-dominated contracting universe on super-Jeans/sub-Hubble scales is scale invariant, grows as $H^2(t)/M_{\rm Pl}^2$, and is enhanced by the smallness of the sound speed ($\mathcal{P}_\delta \sim c_{\rm s}^{-5}$). In addition, we find that non-linearity is reached well before the Planck scale when the sound speed is small.
- By setting thermal initial conditions at a fixed time on sub-Jeans scales, the density contrast power spectrum in a radiation-dominated contracting universe (on sub-Jeans scales) is blue ($\mathcal{P}_\delta \sim k^3$), and, in a matter-dominated contracting universe (on super-Jeans/sub-Hubble scales), it is red ($\mathcal{P}_\delta \sim k^{-1}$). Accordingly, for radiation, non-linearity occurs first on smaller length scales (Planck scale), whereas for matter, non-linearity occurs first on larger length scales (Hubble scale).

Then, under the assumption that the hoop conjecture is valid, we derived a general requirement for black hole collapse. By smoothing out the density contrast power spectrum and using the Press-Schechter formalism to describe the probability of black hole formation, we arrived at the following final results.

- For a matter-dominated contracting universe with quantum vacuum initial conditions, Hubble-size black holes, i.e. black holes with Schwarzschild radius $R = |H|^{-1}$, form first when the Hubble parameter reaches $|H| \sim c_{\rm s}^{5/2} M_{\rm Pl}$, a small fraction of the Planck scale for $c_{\rm s} \ll 1$.

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a contracting universe.

- We find the same results when we take thermal initial conditions instead of a quantum state, except that the critical energy scale for black hole formation goes as $|H| \sim c_{\rm s}^{18/5}(M_{\rm Pl}/H_{\rm ini})^{1/5}M_{\rm Pl}$, which depends on the value of the Hubble parameter at the time that the initial conditions are taken. Yet, in most cases, this is still a small fraction of the Planck scale.

- For a radiation-dominated contracting universe with thermal initial conditions, no black hole can form before the Hubble parameter reaches $|H| \simeq [9\pi^2/(2\gamma_{\rm f}^4)]^{1/3}M_{\rm Pl} \sim M_{\rm Pl}$, i.e. order the Planck scale.

In light of these results, we showed in this paper that nonsingular bouncing cosmology is robust against the formation of black holes if the sound speed is large enough. In particular, for a radiation-dominated contracting universe with $c_{\rm s}^2 = 1/3$, we found that no black hole could form before reaching a Planck time before the bounce. Equivalently, we expect this result to hold for even stiffer equations of state. In particular, this goes in line with the results of [521] according to which no black hole can form in an Ekpyrotic contracting phase where $w \gg 1$. However, one needs to be slightly careful in applying our results to a model where the background is driven by a scalar field[8] since it may have $w \neq c_{\rm s}^2$, or equivalently, w may be time dependent.

As we mentioned in the text, there remains to show that models of nonsingular bouncing cosmology which could have a mixture of matter and radiation (e.g., the ΛCDM bounce and its extensions [171, 180]) can still agree with observations. To avoid the formation of black holes, radiation needs to dominate early enough, and in turn, this will affect the perturbation modes that are of observational interest today and that acquire a nearly scale-invariant power spectrum of curvature perturbations in the matter-dominated contracting phase. In fact, it is known that the transition from matter domination to radiation domination would produce a break in the power spectrum from scale invariance to a very blue spectrum. Such a break is highly constrained from observations, and it implies that the radiation-dominated contracting phase must be shorter than in our expanding universe [435]. Yet, it appears to be still possible for these models to satisfy the observational constraints on the power

[8] We conjecture that an oscillating scalar field with $c_{\rm s}^2 = 1$ would not lead to the formation of black holes. Accordingly, the original matter bounce scenario would be stable against this type of instability. The situation is less obvious for a scalar field with a non-canonical kinetic term in its action (e.g., a k-essence scalar field), which could result in $c_{\rm s}^2 \ll 1$. In this case, the result might be closer to that of hydrodynamical pressureless matter where black holes are produced.

spectrum and avoid the formation of black holes in the contracting phase.

In this paper, we also showed that bouncing cosmologies that are solely driven by matter with $w = c_\mathrm{s}^2 \ll 1$ (or for which the matter-dominated contracting phase lasts long enough before radiation dominates) are not robust against the formation of black holes. Since we find that these black holes form well before reaching the Planck scale, the corresponding nonsingular bouncing cosmologies cannot ignore the formation of these black holes. This agrees with the results of [44] which find an unstable growth of inhomogeneities and the formation of black holes, hence the name "black crunch" that they gave to describe this scenario.

Finally, we showed that when the conditions for black hole formation are satisfied, the first black holes that form are of Hubble size (the Schwarzschild radius is equal to the Hubble radius). Once these Hubble-size black holes form, our perturbative analysis breaks down, hence we did not present the subsequent evolution of the universe. Still, we can comment on a number of possible outcomes.

It is argued in [44] that such Hubble-size black holes behave as a $w = 1$ fluid. This leads to an alternative scenario to inflationary cosmology, named holographic cosmology [43, 45–47], in which the so-called dense $p = \rho$ "black hole gas" serves as the seed to the observed large scale structure of our universe. Also, in line with our motivation coming from bouncing cosmology, it is suggested in [610] that such a dense black hole gas could lead to a model for the "big bounce". The idea is that, in string theory, the black holes would evolve to become a dense gas of "string holes", string states that lie along the correspondence curve between black holes and strings, as the string coupling evolves. Furthermore, it is believed that the Hubble-size string holes saturate the conjectured cosmological entropy bound (see, e.g., [93, 289, 608], the review [97] and references therein), and thus, the entropy associated with the Hubble radius would be proportional to the area. Since this is the same holographic scaling of the entropy and of the specific heat that is found in string gas cosmology, one may hope to have a successful structure formation scenario just as in string gas cosmology (see [139, 144–147, 520] and also [127, 131, 134] for reviews). Alternatively, [533] proposes the idea that a black hole could serve as a nucleation cite of a false vacuum bubble that could tunnel, under some conditions and assumptions, to an inflationary universe, and thus, the black holes that naturally form in a matter-dominated contracting universe could undergo such a tunneling and lead to inflationary universes. At last, it could be that the black holes that are produced in the contracting universe simply "pass through" any given model of nonsingular bounce and form primordial black holes when they re-enter the Hubble radius, as suggested by [185, 186]. These would leave specific imprints in today's universe, in the form of, e.g., dark matter or gravitational waves (see, e.g., [76, 211, 212]), which could allow us to constrain the given model.

In summary, although the formation of black holes in a contracting universe is an undesired feature in typical bouncing cosmologies, it seems to be of particular interest in many alternative scenarios of the very early universe and may allow us to probe new physics and lead to the emergence of new ideas. Consequently, we plan to expand upon the possible outcomes outlined above in more detail in a follow-up paper [558].

Note added: While this paper was under preparation, we were informed that a similar study had been undertaken by an independent group. This study reaches similar conclusions to ours with a slightly different approach [200].

Chapter 9

Stringy black-hole gas in α'-corrected dilaton gravity

9.1 Introduction

Since the rise of string theory as an effort to unify quantum field theory and general relativity, there has been a number of attempts to construct very early Universe cosmological scenarios embedded in string theory. Notable string cosmologies include string gas cosmology [131, 147], pre-Big Bang cosmology [310, 314–316, 609] (see also the review [449]), and Ekpyrotic cosmology [387, 388, 424]. There has also been a lot of effort put into trying to build a stringy realization of inflationary cosmology (see, e.g., Refs. [61, 169, 375–377, 587]), though with limited success, given the difficulty of finding (quasi-)de Sitter solutions in the string landscape (see, e.g., Refs. [242, 413] and also [82, 240] and [19, 530]). Overall, current string cosmologies have led to interesting predictions, but the theories often remain incomplete, or conceptual issues persist. Nevertheless, studying string cosmology might be one of the best approaches to test the validity of string theory.

A common feature of many string cosmologies is that they do not start with an initial Big Bang singularity. In string gas cosmology and pre-Big Bang cosmology, it is the T-duality of string theory that protects the models from reaching a singularity. T-duality roughly states that a small value of the 'radius of the Universe' (R) is equivalent to a large value of the radius. More precisely, the symmetry goes as $R \to \alpha'/R$, where $\alpha' \sim \ell_{\mathrm{s}}^2$ is the string theory dimensionful parameter related to the fundamental string length ℓ_{s}. Thus, one expects $R \sim \ell_s$ to define a minimal length scale at which point the Universe experiences a curvature bounce, i.e., a transition from growing to decreasing spacetime curvature. Details of how this is realized dynamically remains a challenge, but there has been recent progress in the context of string gas cosmology [104, 105]. In pre-Big Bang cosmology, the duality is called the scale factor duality [583, 602, 603, 606], and the symmetry goes as $a \to 1/a$, where a is the scale factor. Again, resolving the singularity dynamically in this context is nontrivial but can be realized, for instance, with a nonlocal potential [312, 314], with quantum loop corrections [152, 191, 604], or with limiting curvature [137, 266] (see also the reviews [308, 309, 315]), though the latter might be unstable to cosmological perturbations [635]. A key difference between the T-duality of string gas cosmology and the scale factor duality of pre-Big Bang cosmology is that the former requires space to be initially compact, while the latter does not need compactification as the Universe can be infinitely large.

The approach of this paper is to consider a generic universe before the Big Bang, so generally a contracting universe in the Einstein frame. The goal is to describe the state of matter and the corresponding cosmological evolution at very high densities, when the energy scale is of the order of the string mass, $M_s \equiv \ell_s^{-1}$, from the point of view of string theory. As the universe contracts, one expects matter that satisfies the usual energy conditions of general relativity to clump and become inhomogeneous. In fact, the overdensities can be such that matter undergoes collapse and forms black holes. More precisely, it was shown in Ref. [559] (see also Ref. [200]) with the theory of cosmological perturbations that in a contracting universe hydrodynamical matter with small sound speed suffers from the Jeans instability and collapses into Hubble-size black holes well before a bounce is reached. This instability in a generic contracting universe was first studied in Ref. [451], an analysis that was extended by Ref. [44] to argue that the final state of a contracting universe is a dense gas of black holes with a stiff equation of state (in which the pressure equates the energy density). In the context of string theory, it was shown in Ref. [159] that the past-trivial string vacuum of the tree-level low-energy effective gravidilaton action is also generically prone to gravitational instability, leading to the formation of black holes. All these studies thus indicate that the state of a contracting universe at high densities is composed of many black holes.

When the universe reaches the string scale, the black holes are then expected to become more stringy in nature. In fact, the state of a 'black-hole gas' is argued in Ref. [610] to become a 'string-hole gas'. String holes represent marginal black holes with mass equal to $M_s g_s^{-2}$ (see

Refs. [596, 605] as well as [239, 610, 611]), where g_s is the string coupling. This represents a correspondence curve along which the physical properties of black holes and strings match spectacularly well (see, e.g., Refs. [239, 339, 340, 356, 357]). In particular, the Schwarzschild radius and Hawking temperature of a string hole are given by the string length and mass, respectively. Therefore, string holes naturally describe the state of collapsed matter at the string scale. Correspondingly, a string-hole gas is the logical outcome of a contracting universe in the Einstein frame at high curvature. The challenge that is tackled in this paper is to find a string-motivated action that can describe the dynamics of a string-hole gas in agreement with its properties. In the string frame, Ref. [610] argued that a string-hole gas should have vanishing pressure and be described by a constant Hubble parameter and constant dilaton velocity, though it was not shown explicitly how these properties can arise from a string theory action.

The outline of this paper is as follows. We first review in Sec. 9.2.1 the concept of string holes and carefully derive in Sec. 9.2.2 the properties of a string-hole gas, both in the Einstein frame and string frame. We then show in Sec. 9.3.1 that with tree-level dilaton gravity as a low-energy effective action of string theory dynamics that matches the properties of a string-hole gas is only obtained in finely tuned situations. It is only when α' corrections are included that we find more appropriate solutions. We study two different first-order α'-corrected actions. First, we extend the work of Ref. [310] in Sec. 9.3.2 to include the contribution from matter in the dynamical equations. Second, in Sec. 9.3.3, we study the $O(d,d)$-invariant action of Ref. [497]. In Sec. 9.4, we perform a phase space analysis to judge the stability of the string-hole gas solutions for both α'-corrected actions, and we comment on the overall evolutionary scheme. In particular, we address the issue of connectivity to the string perturbative vacuum. We summarize the main conclusions in Sec. 9.5. The section is also devoted to a discussion about the possible subsequent fate of a string-hole gas and its role in leading to a nonsingular bouncing cosmology, and we mention future research directions.

Throughout this paper, we work with $\hbar = c = k_{\mathrm{B}} = 1$, and the reduced Planck mass and length are defined, respectively, by $M_{\mathrm{Pl}}^{2-D} \equiv 8\pi G$ and $\ell_{\mathrm{Pl}} \equiv M_{\mathrm{Pl}}^{-1}$, where G (also denoted G_D) is Newton's gravitational constant in $D = d+1$ spacetime dimensions. The number of spatial dimensions is denoted by d, and we assume that it is an integer greater than or equal to 3 throughout.

9.2 String holes

9.2.1 Black hole/string correspondence

One defines a string hole (SH) as an object that has the mass of a Schwarzschild black hole (BH) confined within a radius given by the string length, i.e. $M_{\rm SH} = M_{\rm BH} \sim R_{\rm BH}^{D-3}/G$ and $R_{\rm SH} = R_{\rm BH} = \ell_{\rm s}$, so $M_{\rm SH} \sim \ell_{\rm s}^{D-3}/G$. (For a review of D-dimensional black holes, see, e.g., Ref. [272]). Introducing the string mass given by the inverse of the string length, $M_{\rm s} = \ell_{\rm s}^{-1}$, the string coupling $g_{\rm s}$, and the dilaton ϕ, we recall the following relation that holds in the weak-coupling regime of the closed string sector (see, e.g., Ref. [308]):

$$\left(\frac{\ell_{\rm Pl}}{\ell_{\rm s}}\right)^{D-2} = \left(\frac{M_{\rm s}}{M_{\rm Pl}}\right)^{D-2} = g_{\rm s}^2 = e^{\phi} \ll 1\,. \tag{9.2.1}$$

From this relation, one can say that a string hole lies along the correspondence curve [239, 596, 605, 610, 611]

$$M_{\rm SH} \sim M_{\rm s} g_{\rm s}^{-2}\,. \tag{9.2.2}$$

It follows that the properties of strings and black holes match impressively well along this correspondence curve [239, 339, 340, 356, 357]. For instance, the black hole's Bekenstein-Hawking temperature,

$$T_{\rm BH} = \frac{D-3}{4\pi R_{\rm BH}}\,, \tag{9.2.3}$$

and the string's Hagedorn temperature (see, e.g., Ref. [36] or [636] for an introduction),

$$T_{\rm Hag} = \frac{1}{4\pi\sqrt{\alpha'}}\,, \tag{9.2.4}$$

both scale as $\ell_{\rm s}^{-1}$ for string holes, where $2\pi\alpha' = \ell_{\rm s}^2$. Similarly, the black hole's Bekenstein-Hawking entropy for a string hole,

$$S_{\rm BH} = \frac{\Omega_{D-2} R_{\rm BH}^{D-2}}{4G} \sim \frac{\ell_{\rm s}^{D-2}}{\ell_{\rm Pl}^{D-2}} \sim g_{\rm s}^{-2}\,, \tag{9.2.5}$$

where Ω_{D-2} is the area of a unit $(D-2)$-sphere, is of the same order as the entropy of a string,

$$S_{\rm str} = 4\pi\sqrt{\alpha'}E \sim \ell_{\rm s} M_{\rm SH} \sim g_{\rm s}^{-2}\,, \tag{9.2.6}$$

where we make use of Eq. (9.2.2) in the last proportionality for a string hole.

From the above correspondence, it is natural to expect a black hole that reaches the size of a fundamental string to become a string hole. Furthermore, if a contracting universe is populated with a dense gas of black holes, then the appropriate description of the gas at the string scale must be a string-hole gas. Hence, the main subject of this paper is the study of a string-hole gas as the state of matter at the string scale at the end of an Einstein-frame contracting cosmology. The main thermodynamic properties of a string-hole gas are derived in the next subsection.

9.2.2 String-hole gas

Let us consider a gas composed of N string holes. Considering a dense gas, the string holes have negligible momentum, and the energy of one string hole can be expressed as $E_{\rm SH} = M_{\rm SH} \sim \ell_{\rm s}^{-1} g_{\rm s}^{-2} = \ell_{\rm s}^{-1} e^{-\phi}$ by use of Eqs. (9.2.1) and (9.2.2). The gas with N string holes thus has total energy

$$E_{\rm gas} \equiv E = N E_{\rm SH} \sim N \ell_{\rm s}^{-1} e^{-\phi} \,. \tag{9.2.7}$$

In the same way, the entropy of one string hole is $S_{\rm SH} \sim g_{\rm s}^{-2} = e^{-\phi}$, so for a gas of N string holes, one finds

$$S_{\rm gas} \equiv S = N S_{\rm SH} \sim N e^{-\phi} \,. \tag{9.2.8}$$

Let the physical volume of the gas be given by $V_{\rm gas} \equiv V = f N V_{\rm SH}$, where one string hole has volume $V_{\rm SH} \sim \ell_{\rm s}^{D-1}$ and where f is a function that quantifies the separation of the string holes (e.g., $f = 1$ for a densely packed string-hole gas, while $f \gg 1$ for a dilute gas). Here, we consider a dense gas, so we take f to be of order unity and nearly constant. Thus, $N \sim V \ell_{\rm s}^{1-D}$, and the energy and entropy of the string-hole gas are, respectively, given by

$$E \sim V \ell_{\rm s}^{-D} e^{-\phi} \sim V \ell_{\rm s}^{-2} G^{-1} \tag{9.2.9}$$

and

$$S \sim V \ell_{\rm s}^{1-D} e^{-\phi} \sim V \ell_{\rm s}^{-1} G^{-1} \,, \tag{9.2.10}$$

where one uses Eq. (9.2.1) to express $e^{\phi} \sim G \ell_{\rm s}^{2-D}$. Accordingly, the energy and entropy densities are given by

$$\rho \equiv \frac{E}{V} \sim \ell_{\rm s}^{-D} e^{-\phi} \sim \ell_{\rm s}^{-2} G^{-1}\,, \tag{9.2.11}$$

$$s \equiv \frac{S}{V} \sim \ell_{\rm s}^{1-D} e^{-\phi} \sim \ell_{\rm s}^{-1} G^{-1}\,, \tag{9.2.12}$$

respectively.

Einstein-frame properties

At this point, there are several ways in which one can relate the energy and entropy together. Let us consider the Einstein frame in which the fundamental constant is Newton's constant, i.e., $G =$ constant, while the string length can vary as a function of time. From this point of view, one can eliminate $\ell_{\rm s}$ from Eqs. (9.2.9) and (9.2.10) and relate the energy and entropy through the expression

$$S \sim \sqrt{\frac{EV}{G}}\,, \tag{9.2.13}$$

or equivalently, from Eqs. (9.2.11) and (9.2.12), the densities are related by $s \sim \sqrt{\rho/G}$. We note that these equations correspond to the entropy and entropy density equations of a black-hole gas (see Refs. [489–491] as well as [43–47]). This makes sense; when viewed in the Einstein frame, the string-hole gas is dominated by its gravitational nature, i.e., the strings behave more like black holes, at least thermodynamically.

We note that the entropy equation (9.2.13) has been shown [490] to be the only formula that is manifestly invariant under the S- and T-dualities of string theory, at the same time as approaching the standard Bekenstein-Hawking black-hole entropy at small densities. This entropy expression also appears in different high-energy physics contexts (see Refs. [489–491] and references therein).

Using the thermodynamic identity $T^{-1} = (\partial S/\partial E)_V$, keeping G constant since we are in the Einstein frame, and using Eq. (9.2.11), one finds

$$T \sim \sqrt{\frac{EG}{V}} = \sqrt{\rho G} \sim \ell_{\rm s}^{-1}\,, \tag{9.2.14}$$

and one notes that the temperature is proportional to the Hagedorn temperature (9.2.4). Furthermore, using the identity $p = T(\partial S/\partial V)_E$ for the pressure, and using Eq. (9.2.14) for the temperature, one finds the equation of state (EOS)

$$p = \rho \,. \tag{9.2.15}$$

This matches the EOS of a black-hole gas (see Refs. [489–491] as well as [43–47]).

Similarly, if one considers a Friedmann-Lemaître-Robertson-Walker (FLRW) universe with scale factor a and if one requires the entropy in a comoving volume $V a^{-d}$ to be constant, then it follows from Eq. (9.2.13) that $E \sim V^{-1} \sim a^{-d}$ and furthermore

$$\rho \sim a^{-2d} \,. \tag{9.2.16}$$

Consequently, from Eq. (9.2.11), this implies

$$a \sim \ell_{\mathrm{s}}^{1/d} \sim e^{-\frac{\phi}{d(d-1)}} \,, \tag{9.2.17}$$

where one uses again the fact that G is a constant in the Einstein frame. Therefore, if one considers a string-hole gas in a contracting universe, then the scale factor, the string length, and the size of the string holes become smaller as time progresses, while the string coupling, the dilaton, the energy density, and the (Hagedorn) temperature grow.

String-frame properties

Let us consider an alternative point of view: the string frame in which the fundamental constant is the string length, i.e., $\ell_{\mathrm{s}} =$ constant, while the gravitational constant can vary as a function of time. From this point of view, one can eliminate G from Eqs. (9.2.9) and (9.2.10) and relate the energy and entropy through the expression

$$S \sim \ell_{\mathrm{s}} E \,, \tag{9.2.18}$$

and equivalently, it follows that $s \sim \ell_{\mathrm{s}}\rho$. From $T^{-1} = (\partial S/\partial E)_V$ and keeping the string length constant, it is straightforward to see that

$$T \sim \ell_{\mathrm{s}}^{-1} \sim T_{\mathrm{Hag}} \,, \tag{9.2.19}$$

which is a constant temperature. Furthermore, from $p = T(\partial S/\partial V)_E$, it follows that

$$p = 0\,. \tag{9.2.20}$$

This confirms the result of Ref. [610] and again matches what one could have guessed: in the string frame, the string-hole gas is dominated by its stringy nature, and this is why the EOS is that of a string gas with equal contribution from momentum and winding modes (see, e.g., Ref. [308]). Also, the expression (9.2.18) matches the leading-order behavior of the entropy of a string gas (see, e.g., Refs. [127, 147]).

Similarly, if one requires adiabaticity ($S =$ constant) in a constant comoving volume in FLRW, then it follows that the energy must be constant; hence,

$$\rho \sim a^{-d}\,. \tag{9.2.21}$$

From the standard conservation equation (more on this in the next section), this is in agreement with an EOS $p = 0$. With Eq. (9.2.11), this implies

$$a \sim G^{1/d} \sim e^{\phi/d}\,, \tag{9.2.22}$$

where one uses again the fact that ℓ_{s} is a constant in the string frame. Taking the time derivative of the above, this further implies

$$H = \frac{\dot{\phi}}{d}\,, \tag{9.2.23}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter and a dot denotes a derivative with respect to the (string-frame) cosmic time t.

To be consistent with the fact that the size of the string holes is constant in the string frame ($R_{\mathrm{SH}} = \ell_{\mathrm{s}} =$ constant), there are two possible cosmological evolutionary paths consistent with the constraint (9.2.23). First, it could be that the universe is static in the string frame ($H = 0$), similar to the (quasi)static Hagedorn phase of string gas cosmology [147] (see also Refs. [57, 122, 131] for reviews that highlight the challenges in that context). Second, it could be that the radius of the string holes is of the order of the Hubble radius ($R_{\mathrm{SH}} \sim H^{-1}$) with the string-frame Hubble parameter being constant ($H \sim \ell_{\mathrm{s}}^{-1}$). In that case, a dense string-hole gas coincides with having one string hole per Hubble volume. This last avenue was conjectured in Ref. [610] to correspond to the string phase in pre-Big Bang cosmology, and this is what we explore in the rest of this paper. We note that a dilute gas could also be possible with less than one string hole per Hubble volume in average, but naively, in this situation, curvature would continue to grow until the gas becomes dense. Conversely, an 'overdense' gas with more than one string hole per Hubble volume is most likely forbidden by entropy considerations. Indeed, a string-hole gas as defined above exactly saturates the appropriate entropy bound [38, 93–95, 97, 151, 154, 268, 607, 608] (see Refs. [315, 610] and additional references therein). This is also confirmed in the Einstein frame in which saturation occurs when the EOS is $p = \rho$ [43, 45–47, 490], and this is the only safe outcome with respect to entropy bounds in a contracting FLRW cosmology (see, e.g., Refs. [154, 610] but also [491]). These entropic considerations also reinforce a string-hole gas to be the state of matter at high densities.

In summary, assuming expansion in the string frame, the evolution of a string-hole gas corresponds to a constant Hubble parameter equal to the string mass, while the dilaton grows linearly with string-frame time according to the constraint (9.2.23). We note that expansion in the string frame is consistent with contraction in the Einstein frame; this is shown explicitly in Appendix 9.6. The goal is then to find a string-theoretic effective action that can support the evolution of a string-hole gas, i.e., an action of which the equations of motion (EOM) have a phase of string-hole gas evolution as a solution.

9.3 Dynamics from dilaton gravity

9.3.1 Tree-level dilaton gravity

We first study the string-frame, tree-level, low-energy effective string theory action (see, e.g., Refs. [308, 315])

$$S_0 = -\frac{1}{2\ell_{\mathrm{s}}^{d-1}} \int \mathrm{d}^{d+1}x \sqrt{|g|} e^{-\phi} \Big(R + g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + 2\ell_{\mathrm{s}}^{d-1} U(\phi) \Big) , \tag{9.3.24}$$

where $g \equiv \det(g^\mu{}_\nu)$ is the determinant of the metric tensor, $U(\phi)$ is the potential energy of the dilaton field, and R denotes the Ricci scalar in this section. Since we focus on the gravidilaton sector of the effective string theory action, we set to zero the potential contribution from the antisymmetric field strength coming from the Neveu-Schwarz/Neveu-Schwarz 2-form.

The above action represents the effective action for vacuum string theory, but we want to consider the addition of matter; hence, we take the total action to be $S = S_0 + S_{\rm m}$, where $S_{\rm m}$ represents the matter action. The energy-momentum tensor associated with $S_{\rm m}$ is defined as usual by $T_{\mu\nu} \equiv 2|g|^{-1/2}\delta S_{\rm m}/\delta g^{\mu\nu}$. The matter action may also depend on the dilaton, so

$$\sigma \equiv -\frac{2}{\sqrt{|g|}}\frac{\delta S_{\rm m}}{\delta\phi} \tag{9.3.25}$$

defines the dilaton (scalar) charge density.

Varying the action (9.3.24) in a homogeneous, isotropic, and flat FLRW spacetime,

$$g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \mathrm{d}t^2 - a(t)^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\,, \tag{9.3.26}$$

a set of dilaton-gravity background EOM in the string frame can be written as (see, e.g., Refs. [308, 315])

$$d(d-1)H^2 + \dot{\phi}^2 - 2dH\dot{\phi} = 2\ell_{\rm s}^{d-1}\left(e^\phi\rho + U(\phi)\right)\,, \tag{9.3.27}$$

$$\dot{H} - H\dot{\phi} + dH^2 = \ell_{\rm s}^{d-1}\left(e^\phi\left(p - \frac{\sigma}{2}\right) - U_{,\phi}\right)\,, \tag{9.3.28}$$

$$2\ddot{\phi} - \dot{\phi}^2 + 2dH\dot{\phi} - 2d\dot{H} - d(d+1)H^2 = 2\ell_{\rm s}^{d-1}\left(e^\phi\frac{\sigma}{2} - U(\phi) + U_{,\phi}\right)\,, \tag{9.3.29}$$

where one assumes that the energy-momentum tensor can be decomposed as a perfect fluid[1] with $T^\mu{}_\nu = \mathrm{diag}(\rho, -p\delta^i{}_j)$. Combining Eqs. (9.3.27)–(9.3.29), one can derive the fluid's conservation equation, which goes as

$$\dot{\rho} + dH(\rho + p) = \frac{1}{2}\sigma\dot{\phi}\,. \tag{9.3.30}$$

General power-law solutions to these equations are well known (see, e.g., Refs. [308, 309, 314, 315]) but mostly for vanishing potential, vanishing dilaton charge, and an EOS of the form $p = w\rho$. We want to consider a string-hole gas, in which these assumptions may not all be met. From Eqs. (9.2.11) and (9.2.21), a string-hole gas in the string frame has energy density

$$\rho = C\ell_{\rm s}^{-d-1}e^{-\phi} = \rho_0 a^{-d}\,, \tag{9.3.31}$$

[1] We comment on the possible presence of viscosity as a deviation from a perfect fluid description later in this section.

where C is a dimensionless positive constant and ρ_0 is a positive constant with dimensions of energy density. As seen in the previous section, this implies the constraint equation $H = \dot{\phi}/d$. Substituting this constraint and Eq. (9.3.31) into the conservation equation (9.3.30), one finds

$$\sigma = 2p\,, \tag{9.3.32}$$

independent of the EOS (only assuming $H \neq 0$). Therefore, one notices that if the dilaton charge density vanishes, the pressure is zero, which is the naive EOS for a string-hole gas in the string frame as shown in the previous section. Conversely, if we expect the pressure to vanish from thermodynamic arguments, then this tells us that the string-hole gas matter action should have no explicit ϕ dependence, so the dilaton charge density vanishes.

Inserting the constraint $H = \dot{\phi}/d = \text{constant}$ (which implies $\ddot{\phi} = \dot{H} = 0$) and Eq. (9.3.32) into Eq. (9.3.28) immediately yields $U_{,\phi} = 0$. Therefore, a fixed-point solution satisfying the constraint $H = \dot{\phi}/d = \text{constant}$ is only possible with a constant potential independent of the dilaton. Then, Eqs. (9.3.27) and (9.3.29) further reduce to

$$-\frac{d}{2}H^2 = C\ell_{\mathrm{s}}^{-2} + \ell_{\mathrm{s}}^{d-1}U\,, \tag{9.3.33}$$

$$-\frac{d}{2}H^2 = wC\ell_{\mathrm{s}}^{-2} - \ell_{\mathrm{s}}^{d-1}U\,, \tag{9.3.34}$$

where we set the EOS to be of the form $p = w\rho$. For the above equations to yield a real solution for H, the only possibility is to have a constant negative potential,

$$U = -\frac{1}{\ell_{\mathrm{s}}^{d-1}}\left(\frac{d}{2}H_\star^2 + \frac{C}{\ell_{\mathrm{s}}^2}\right)\,, \tag{9.3.35}$$

where the positive constant $H_\star$ should be of the order of ℓ_{s}^{-1} to yield the solution $H = H_\star \sim \ell_{\mathrm{s}}^{-1}$. This is equivalent to introducing a fine-tuned negative cosmological constant, $\Lambda \sim -\mathcal{O}(\ell_{\mathrm{s}}^{-D})$, in the string frame[2]. Any other forms of the potential $U(\phi)$ generically cannot support a string-hole gas evolution with $H = \dot{\phi}/d = \text{constant}$. Furthermore, the potential (9.3.35), which yields the solution $H_\star$, is only consistent with Eqs. (9.3.33)–(9.3.34) provided the EOS is also tuned to be

$$w = -1 - \frac{d\ell_{\mathrm{s}}^2 H_\star^2}{C}\,, \tag{9.3.36}$$

[2]We note, however, that such a negative constant value of U may naturally appear in the tree-level string effective action, but this would require a noncritical number of dimensions (see, e.g., Ref. [308]).

which violates the null energy condition. In summary, this avenue does not seem particularly appealing, considering it would require tuning an *ad hoc* negative cosmological constant and the EOS to a physically unexpected value.

This conclusion generalizes to nonlocal potentials of the form $U(\bar{\phi})$, where

$$\bar{\phi} = \phi - \ln a^d \tag{9.3.37}$$

is the shifted dilaton. Indeed, we note that $\dot{\bar{\phi}} = \dot{\phi} - dH = 0$ for a string-hole gas satisfying the constraint $H = \dot{\phi}/d$. Thus, regardless of the modifications to the EOM for a nonlocal potential (see, e.g., Refs. [308, 309, 315] for the exact modified EOM), $\bar{\phi}$ has to remain constant during a string-hole gas evolution, so any potential $U(\bar{\phi})$ would simply be a constant, i.e., a cosmological constant.

In summary, it appears that one cannot support the evolution of a string-hole gas with tree-level dilaton gravity, no matter the form of the potential (unless it is a fine-tuned negative cosmological constant). Therefore, one should explore the possibility of higher-order corrections.

9.3.2 Action with α' corrections

The low-energy effective action S_0 introduced in the previous subsection is only compatible with the conformal invariance of quantized strings on a curved background to zeroth order in $\alpha' \sim \ell_{\rm s}^2$. When going to first order, conformal invariance allows new higher-derivative terms such that the effective action contains terms that scale as the square of the spacetime curvature and so on. As long as curvature is small, e.g., $\ell_{\rm s}^2 R \ll 1$, then the perturbative expansion is dominated by the zeroth-order action. However, when the curvature reaches the string scale, which is the case when $H \sim \ell_{\rm s}^{-1}$, then higher-order terms are necessary. In fact, when the perturbative expansion breaks down on substring scales, working with an effective action is no longer viable, and one would have to work with a proper conformal field theory that could account for α' corrections nonperturbatively (see, e.g., Ref. [395]). This approach, however, is beyond the scope of this study, and in what follows, we assume that a first-order α'-corrected effective action is a sufficient approximation when $H \sim \ell_{\rm s}^{-1}$.

Demanding general covariance and gauge invariance of the string effective action, one can write down many perfectly valid actions that are compatible with the condition of conformal invariance to first order in α'. Those actions are related by simple field redefinitions of the metric and dilaton; hence, it is ambiguous which action to choose (see, e.g., Refs. [308, 315] and references therein). For instance, the simplest consistent action to first order in α' is $S = S_0 + S_{\alpha'}$ with

$$S_{\alpha'} = \frac{k\alpha'}{8\ell_{\mathrm{s}}^{d-1}} \int \mathrm{d}^{d+1}x \sqrt{|g|} e^{-\phi} R_{\mu\nu\kappa\lambda} R^{\mu\nu\kappa\lambda} \,, \tag{9.3.38}$$

where $R_{\mu\nu\kappa\lambda}$ is the Riemann tensor and either $k = 1$ for bosonic strings or $k = 1/2$ for heterotic superstrings. However, working with the above action (i.e. with the square of the Riemann tensor) in a cosmological context is rather cumbersome, because the field equations contain, in general, higher than second derivatives of the metric tensor. Such a formal complication can be avoided, however, by performing an appropriate field redefinition [310] and considering the action with

$$S_{\alpha'} = \frac{k\alpha'}{8\ell_{\mathrm{s}}^{d-1}} \int \mathrm{d}^{d+1}x \sqrt{|g|} e^{-\phi} \left(\mathcal{G} - (\nabla_\mu\phi\nabla^\mu\phi)^2 \right) \,, \tag{9.3.39}$$

where $\mathcal{G} \equiv R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is the Gauss-Bonnet invariant, $R_{\mu\nu} \equiv g^{\kappa\lambda}R_{\kappa\mu\lambda\nu}$ is the Ricci tensor, and $R \equiv g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar. This was first considered by Gasperini, Maggiore & Veneziano [310] (GMV hereafter; also studied in Refs. [152, 190, 191, 477] and discussed in [308, 315]). Therefore, for a first attempt, we examine the action $S = S_0 + S_{\alpha'} + S_{\mathrm{m}}$ with $S_{\alpha'}$ given by Eq. (9.3.39), and for the rest of this paper, we assume that the dilaton has no potential; i.e., we set $U(\phi) = 0$ in S_0.

GMV already showed that this action admits no homogeneous and isotropic fixed-point solution with $\dot{\bar{\phi}} = 0$, i.e. with $H = \dot{\phi}/d =$ constant for a string-hole gas. However, GMV only considered the vacuum action with no matter, i.e. $S = S_0 + S_{\alpha'}$. To find dynamics for the string-hole gas, one must include the matter action S_{m} as before. The EOM that follow from varying the corresponding action in a FLRW background are

$$\begin{aligned}
\rho ={}& \frac{1}{2}\ell_{\mathrm{s}}^{1-d}e^{-\phi}\left(\dot{\phi}^2 + d(d-1)H^2 - 2dH\dot{\phi} - \frac{3k\alpha'}{4}\mathcal{F}_\rho(H,\dot{\phi})\right) \,, \\
\sigma ={}& -\ell_{\mathrm{s}}^{1-d}e^{-\phi}\left(-2\ddot{\phi} + 2d\dot{H} + \dot{\phi}^2 + d(d+1)H^2 - 2dH\dot{\phi} + \frac{k\alpha'}{4}\mathcal{F}_\sigma(H,\dot{\phi},\dot{H},\ddot{\phi})\right) \,, \\
p ={}& \frac{1}{2d}\ell_{\mathrm{s}}^{1-d}e^{-\phi}\Big(-2d(d-1)\dot{H} + 2d\ddot{\phi} - d^2(d-1)H^2 + 2d(d-1)H\dot{\phi} - d\dot{\phi}^2 \\
&+ \frac{k\alpha'}{4}\mathcal{F}_p(H,\dot{\phi},\dot{H},\ddot{\phi})\Big) \,,
\end{aligned} \tag{9.3.40}$$

where we define

$$
\begin{aligned}
\mathcal{F}_\rho(H,\dot\phi) &\equiv c_1H^4 + c_3H^3\dot\phi - \dot\phi^4\,,\\
\mathcal{F}_\sigma(H,\dot\phi,\dot H,\ddot\phi) &\equiv 3c_3\dot H H^2 - 12\ddot\phi\dot\phi^2 + (c_1+dc_3)H^4 - 4dH\dot\phi^3 + 3\dot\phi^4\,,\\
\mathcal{F}_p(H,\dot\phi,\dot H,\ddot\phi) &\equiv 12c_1\dot H H^2 + 3c_3\ddot\phi H^2 + 6c_3\dot H H\dot\phi + 3dc_1H^4 - 2(2c_1-dc_3)H^3\dot\phi - 3c_3H^2\dot\phi^2\\
&\quad + d\dot\phi^4\,,
\end{aligned}
\tag{9.3.41}
$$

and

$$
\begin{aligned}
c_1 &\equiv -\frac{d}{3}(d-1)(d-2)(d-3)\,,\\
c_3 &\equiv \frac{4d}{3}(d-1)(d-2)\,.
\end{aligned}
\tag{9.3.42}
$$

These equations generalize the EOM that were already derived, e.g., in Refs. [308, 310, 315], to include matter; the vacuum limit ($\rho = p = \sigma = 0$) reduces to the EOM in Refs. [308, 310, 315]. We note that the above three EOM are not all independent. Indeed, one can verify that the continuity equation

$$
\dot\rho + dH(\rho+p) = \frac{1}{2}\sigma\dot\phi
\tag{9.3.43}
$$

relates the three EOM.

We now seek to find solutions to the above EOM that could describe a string-hole gas. To do so, one sets $\rho = C\ell_\mathrm{s}^{-d-1}e^{-\phi}$, $\sigma = 2p$, and $H = \dot\phi/d$. Furthermore, we relate the pressure and energy density through an EOS of the form $p = w\rho$. We expect the EOS to be $p = 0$ for a string-hole gas in the string frame from the thermodynamic arguments of Sec. 9.2.2, so the EOS parameter w is set to zero later on. Nevertheless, the more crucial property for a string-hole gas is that $p_\mathrm{eff} \equiv p - \sigma/2 = 0$; thus, we perform a slightly more general analysis in what follows with a generic EOS parameter w. One then looks for fixed-point solutions with $y_1 \equiv H = \text{constant}$, $y_2 \equiv \dot\phi = \text{constant}$, and $\ddot\phi = \dot H = 0$. The constraint $H = \dot\phi/d$

implies $y_2 = dy_1$, and the three differential EOM reduce to three algebraic equations for y_1,

$$
\begin{aligned}
-dy_1^2\left(1-\frac{3k\alpha' y_1^2\Delta}{4}\right) &= 2C\ell_{\mathrm{s}}^{-2}\,,\\
dy_1^2\left(1-\frac{k\alpha' y_1^2\Delta}{4}\right) &= -2wC\ell_{\mathrm{s}}^{-2}\,,\\
-d^2y_1^2\left(1-\frac{k\alpha' y_1^2\Delta}{4}\right) &= 2dwC\ell_{\mathrm{s}}^{-2}\,,
\end{aligned}
\tag{9.3.44}
$$

where we define $\Delta \equiv 2d^2+d-2$. We note that Δ is strictly positive (in fact, $\Delta \geq 19$ for $d \geq 3$). The second and third equations above are completely equivalent, which is due to the fact that the three EOM are not independent. Therefore, one only has to solve the first and second equations for y_1. Requiring real solutions for y_1, one can show that these two equations yield the same nontrivial solutions,

$$
y_1 = H = \pm\frac{2}{\ell_{\mathrm{s}}}\sqrt{\frac{2\pi(1-w)}{k(1-3w)\Delta}}\,,
\tag{9.3.45}
$$

if and only if $w < 1/3$ and

$$
C = \frac{8\pi d(1-w)}{k(1-3w)^2\Delta}\,,
\tag{9.3.46}
$$

where we use $2\pi\alpha' = \ell_{\mathrm{s}}^2$ to simplify the expressions. The solution for $\dot{\phi}$ immediately follows by multiplying Eq. (9.3.45) by d.

A couple of comments are in order. One first notes that $|H| \sim \ell_{\mathrm{s}}^{-1}$ as expected. Second, one notices that the restrictions $w < 1/3$ and Eq. (9.3.46) impose $C > 0$, which means that no real and consistent solution (except the trivial solution $H = \dot{\phi} = 0$) would have followed from setting $C = 0$. This reproduces what was stated by GMV, i.e. that there exists no consistent nontrivial solution satisfying the constraint $\dot{\bar{\phi}} = 0$ (which is equivalent to $H = \dot{\phi}/d = \text{constant}$) in vacuum. In summary, the GMV α'-corrected action that includes a string-hole gas matter action does allow for consistent solutions with the properties of a string-hole gas for any $w < 1/3$ and provided ρ has the appropriate amplitude, with C given in Eq. (9.3.46).

The unique physical solution for the EOS $p = 0$ ($w = 0$) is then

$$H = \frac{\dot{\phi}}{d} = \frac{2}{\ell_{\rm s}}\sqrt{\frac{2\pi}{k\Delta}}\,, \tag{9.3.47}$$

taking the positive solution for expansion in the string frame. For instance, in $d = 3$ dimensions and for $k = 1$, the solution is $H = 2\ell_{\rm s}^{-1}\sqrt{2\pi/19}$. In the case $w = 0$, the physical solution is valid only if $C = 8\pi d/(k\Delta)$, which might appear as a fine-tuning problem. However, we recall that C is only an arbitrary constant amplitude for the energy density [c.f. Eq. (9.3.31)], and it is certainly tunable depending on the total energy density of the universe and the other matter contents prior to the string-hole gas phase. In sum, the α'-corrected action considered in this subsection has background EOM that have a unique and natural solution [Eq. (9.3.47)] corresponding to a string-hole gas evolution.

9.3.3 $O(d,d)$-invariant α'-corrected action

As we mentioned in the previous subsection, there are several consistent α'-corrected actions related through field redefinitions. In this subsection, we consider a different choice for $S_{\alpha'}$, specifically

$$S_{\alpha'} = \frac{k\alpha'}{8\ell_{\rm s}^{d-1}}\int {\rm d}^{d+1}x\,\sqrt{|g|}e^{-\phi}\Big(\mathcal{G} - (\nabla_\mu\phi\nabla^\mu\phi)^2 - 4G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + 2(\nabla_\mu\phi\nabla^\mu\phi)\Box\phi\Big)\,, \tag{9.3.48}$$

where $G_{\mu\nu} \equiv R_{\mu\nu} - Rg_{\mu\nu}/2$ is the Einstein tensor and $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the d'Alembertian. This action shares the Gauss-Bonnet and $(\nabla\phi)^4$ terms with the action (9.3.39), but the second in line in Eq. (9.3.48) is new; nevertheless, this action is still free from higher derivatives in the cosmological field equations. The actions (9.3.39) and (9.3.48) are related by a field redefinition (see Ref. [308]). This action was first introduced by Meissner [497] (see also Ref. [380, 381]) who showed that it is invariant under the $O(d,d)$ symmetry to first order in the α' expansion.

The $O(d,d)$ symmetry plays a key role in string theory and even more in the context of pre-Big Bang cosmology (see Refs. [308, 315] and references therein). Indeed, the cosmological scale factor duality $a \to 1/a$ [606] is actually extendable to a continuous symmetry, the transformation group of which is $O(d,d)$. It was found that the action of the group transforms known solutions to the effective cosmological string theory into new solutions

[311, 498, 499, 539, 583]. The symmetry was shown to be present for the low-energy action to zeroth order in α' with the presence of matter [313], but it was also argued to apply to all orders in α' [583, 606]. The action that has the symmetry to first order in α' is the one found by Meissner [497], and it is the one introduced above that we consider below.

Since S_0 is already invariant under $O(d,d)$ transformations [308, 313, 315], it is natural to consider the α'-corrected action (9.3.48) that bares the same symmetry. Let us comment on the nature of the symmetry for a string-hole gas. Considering an isotropic and homogeneous cosmology for simplicity, the EOM of the full action $S = S_0 + S_{\alpha'} + S_{\mathrm{m}}$ are $O(d,d)$ invariant under the transformations $a \to 1/a$, $\bar{\phi} \to \bar{\phi}$, $\bar{\rho} \to \bar{\rho}$, $\bar{p} \to -\bar{p}$, and $\bar{\sigma} \to \bar{\sigma}$, where the shifted dilaton is given by Eq. (9.3.37) and the other shifted variables are $\bar{\rho} = \rho a^d$, $\bar{p} = p a^d$, and $\bar{\sigma} = \sigma a^d$. Thus, for a string-hole gas with $p = \sigma/2 = 0$, we expect $\bar{\rho} = C \ell_{\mathrm{s}}^{-d-1} e^{-\bar{\phi}} = \rho_0$, $\bar{p} = 0$, and $\bar{\sigma} = 0$, and readily, we notice the $O(d,d)$ invariance. Let us mention that in general, though, deviations from a perfect fluid description could change this conclusion. Indeed, it was shown in Ref. [313] that a particular nontrivial action of the $O(d,d)$ group can transform a perfect fluid with a diagonal stress tensor into a fluid with nondiagonal elements in its stress tensor. More precisely, a perfect fluid with EOS $p = w\rho$ transforms into a pressureless fluid (so $p \to 0$) with shear viscosity given by $\eta = -w\rho/(2H)$. However, for a string-hole gas, the perfect fluid EOS is precisely expected to be that of a pressureless fluid to start with ($w = 0$), so the transformation turns out to be trivial, and no shear viscosity appears. Therefore, a string-hole gas with vanishing pressure in the string frame has a valid and consistent perfect fluid description from the point of view of $O(d,d)$ invariance of its action. If one allows $w \neq 0$ to describe a string-hole gas (but still with $p_{\mathrm{eff}} = p - \sigma/2 = 0$), then a more refined analysis should drop the perfect fluid description and include the possible effects of viscosity, as was first considered in Ref. [491]. We keep the exploration of this possibility for future work.

Let us now derive the EOM. We consider the FLRW metric

$$g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = N(t)^2 \mathrm{d}t^2 - e^{2\beta(t)} \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j \,, \tag{9.3.49}$$

where, in this subsection, we introduce the lapse function $N(t)$ [which we later set to $N(t) \equiv 1$]. Also, the scale factor is written as $a(t) = e^{\beta(t)}$, so the Hubble parameter becomes $H(t) = \dot{\beta}(t)$. This is only a matter of convenience to compute the EOM below. The action $S = S_0 + S_{\alpha'}$ thus reduces to the form $S = -(\ell_{\mathrm{s}}/2) \int \mathrm{d}t\ \mathcal{V}_t \mathcal{L}(t)$, where $\mathcal{V}_t \equiv \ell_{\mathrm{s}}^{-d} \int_{\Sigma_t} \mathrm{d}^d x$ is the volume of the spatial hypersurface of constant time Σ_t (at time t) in string units, and the Lagrangian density is

$$\begin{aligned}\mathcal{L}(t) = e^{d\beta-\phi}\Big\{&\frac{1}{N}\Big[-2d\ddot{\beta} - d(d+1)\dot{\beta}^2 + 2dF\dot{\beta} + \dot{\phi}^2\Big]\\ &-\frac{k\alpha'}{4N^3}\Big[-3c_3F\dot{\beta}^3 + (d+1)d(d-1)(d-2)\dot{\beta}^4 + 3c_3\ddot{\beta}\dot{\beta}^2 - 2d(d-1)\dot{\phi}^2\dot{\beta}^2 + 2\ddot{\phi}\dot{\phi}^2\\ &+ 2d\dot{\phi}^3\dot{\beta} - 2F\dot{\phi}^3 - \dot{\phi}^4\Big]\Big\}\,,\end{aligned} \tag{9.3.50}$$

where $F \equiv \dot{N}/N$. After integration by parts, the action reduces to

$$\begin{aligned}S = \frac{\ell_\mathrm{s}}{2}\int \mathrm{d}t\ \mathcal{V}_t e^{d\beta-\phi}\Big\{&\frac{1}{N}\Big[-\dot{\phi}^2 - d(d-1)\dot{\beta}^2 + 2d\dot{\beta}\dot{\phi}\Big] + \frac{k\alpha'}{4N^3}\Big[c_1\dot{\beta}^4 + c_3\dot{\phi}\dot{\beta}^3 - 2d(d-1)\dot{\phi}^2\dot{\beta}^2\\ &+ \frac{4}{3}d\dot{\phi}^3\dot{\beta} - \frac{1}{3}\dot{\phi}^4\Big]\Big\}\,.\end{aligned} \tag{9.3.51}$$

Let us add to the above action a matter action S_m described by an energy density ρ, pressure p, and dilaton charge density σ as before. Then, varying the total action with respect to N, ϕ, and β [and afterward setting $N(t) \equiv 1$], one finds three EOM, which are the same as the set of equations (9.3.40), except the functions $\mathcal{F}_\rho$, $\mathcal{F}_\sigma$, and $\mathcal{F}_p$ that are replaced by

$$\mathcal{F}_\rho(H,\dot{\phi}) = c_1H^4 + c_3H^3\dot{\phi} - 2d(d-1)H^2\dot{\phi}^2 + \frac{4}{3}dH\dot{\phi}^3 - \frac{1}{3}\dot{\phi}^4\,, \tag{9.3.52}$$

$$\begin{aligned}\mathcal{F}_\sigma(H,\dot{\phi},\dot{H},\ddot{\phi}) = {}&3c_3\dot{H}H^2 - 8d(d-1)\dot{H}H\dot{\phi} + 4d\dot{H}\dot{\phi}^2 - 4d(d-1)\ddot{\phi}H^2 + 8d\ddot{\phi}H\dot{\phi} - 4\ddot{\phi}\dot{\phi}^2\\ &+ (c_1+dc_3)H^4 - 4d^2(d-1)H^3\dot{\phi} + 2d(3d-1)H^2\dot{\phi}^2 - 4dH\dot{\phi}^3 + \dot{\phi}^4\,,\end{aligned} \tag{9.3.53}$$

$$\begin{aligned}\mathcal{F}_p(H,\dot{\phi},\dot{H},\ddot{\phi}) = {}&12c_1\dot{H}H^2 + 6c_3\dot{H}H\dot{\phi} - 4d(d-1)\dot{H}\dot{\phi}^2 + 3c_3\ddot{\phi}H^2 - 8d(d-1)\ddot{\phi}H\dot{\phi}\\ &+ 4d\ddot{\phi}\dot{\phi}^2 + 3dc_1H^4 - 2(2c_1-dc_3)H^3\dot{\phi} - (3c_3+2d^2(d-1))H^2\dot{\phi}^2\\ &+ 4d(d-1)H\dot{\phi}^3 - d\dot{\phi}^4\,.\end{aligned} \tag{9.3.54}$$

Note that we reexpressed the Hubble parameter $\dot{\beta}$ with H.

As in the previous subsection, we consider a string-hole gas with $\rho = C\ell_\mathrm{s}^{-d-1}e^{-\phi}$, $\sigma = 2p$, $p = w\rho$, and $H = \dot{\phi}/d$. One looks for fixed-point solutions with $y_1 \equiv H =$ constant, $y_2 \equiv \dot{\phi} =$ constant (so $\dot{H} = \ddot{\phi} = 0$), and $y_2 = dy_1$. The three EOM then reduce to two independent algebraic equations:

$$-dy_1^2\left(1-\frac{3(d-2)k\alpha'}{4}y_1^2\right)=\frac{2C}{\ell_s^2}\;; \tag{9.3.55}$$

$$dy_1^2\left(1-\frac{(d-2)k\alpha'}{4}y_1^2\right)=-\frac{2wC}{\ell_s^2}\,. \tag{9.3.56}$$

Those two equations share the same nontrivial solutions,

$$y_1=H=\pm\frac{2}{\ell_s}\sqrt{\frac{2\pi(1-w)}{k(d-2)(1-3w)}}\,, \tag{9.3.57}$$

if and only if the amplitude parameter satisfies

$$C=\frac{8\pi d}{k(d-2)}\frac{1-w}{(1-3w)^2} \tag{9.3.58}$$

and as long as $w<1/3$. These expressions are not the same as Eqs. (9.3.45) and (9.3.46), but they only differ by numerical factors that depend on the number of spatial dimensions. Essentially, $\Delta=2d^2+d-2$ in Eqs. (9.3.45) and (9.3.46) is replaced by $d-2$ in Eqs. (9.3.57) and (9.3.58). The solutions are certainly of the same order, and as before (as expected), $|H|\sim\ell_s^{-1}$. The physical solution with $w=0$ reduces to

$$H=\frac{\dot{\phi}}{d}=\frac{2}{\ell_s}\sqrt{\frac{2\pi}{k(d-2)}}\,, \tag{9.3.59}$$

and it requires $C=8\pi d/(k(d-2))$. As before, we argue that C is an arbitrary constant, so this does not represent fine-tuning. Therefore, the $O(d,d)$-invariant α'-corrected action of this subsection yields a unique and natural solution, which corresponds to a string-hole gas evolution but which is different from the solution of the previous subsection. The differences are due to the fact that the physical effects of the higher-curvature corrections are not invariant, in general, under field redefinitions truncated to first order in α'. Such an ambiguity affects all models truncated to any given finite order of the α' expansion and can be resolved, in principle, only by considering exact conformal models, which automatically include the corrections to all orders. In the following section, restricting our discussion to the first order in α', we perform a phase space analysis of the two previous solutions in order to find the most appropriate one to describe — in this approximation — the main properties of the string-hole gas and of its dynamical evolution.

9.4 Phase space analysis

At this point, two distinct solutions that correspond to a string-hole gas evolution given two different α'-corrected actions have been found: the solutions (9.3.47) and (9.3.59) follow from GMV's action and Meissner's action, respectively. In the perspective of a greater evolutionary scheme, we now seek to determine the stability of those fixed points in the whole phase space of cosmological solutions. For instance, the nontrivial fixed points found by GMV [310] in vacuum were shown to be attractors in phase space and smoothly connected to the string perturbative vacuum (i.e., to the asymptotic state with vanishing string coupling and flat spacetime, $g_{\mathrm{s}} \to 0$ and $H \to 0$). Conversely, the attractor fixed points from Meissner's action in vacuum are disconnected from the low-energy trivial fixed point (see Refs. [308, 315]). We analyze the phase space with the addition of matter and in particular for a string-hole gas in the subsequent subsections.

9.4.1 Stability of the fixed point with GMV's action

Recall the GMV EOM given by Eq. (9.3.40). In general, for an EOS of the form $p = w\rho$ and assuming that σ is also proportional to ρ, one can see that there are only three independent variables in configuration space: H, $\dot{\phi}$, and $e^{\phi}\rho$. One can choose to use the Hamiltonian constraint [the first equation of the set (9.3.40)] to eliminate $e^{\phi}\rho$ from the other two evolution equations. This amounts to projecting the configuration space onto a two-dimensional vector space, where the vectors are of the form $y^A = (H, \dot{\phi})$, $A \in \{1, 2\}$. One can thus reexpress the set (9.3.40) as two independent differential equations, written in vector form as $\dot{y}^A = (\dot{H}, \ddot{\phi}) = \mathcal{C}^A$, where $\mathcal{C}^1$ and $\mathcal{C}^2$ are functions of H and $\dot{\phi}$ only [i.e. $\mathcal{C}^A = \mathcal{C}^A(y^B)$]. For example, when $p = \sigma/2 = 0$ ($w = 0$), their expressions are

$$\mathcal{C}^1(H,\dot\phi) = -\frac{1}{\mathcal{D}}\Big\{16H(dH-\dot\phi) + \frac{2k\ell_s^2}{\pi}\Big[\frac{3c_3}{4}(d+3)H^4 - \frac{3c_3}{2d}H^3\dot\phi - 2(d-1)(2d-1)H^2\dot\phi^2$$
$$+ 2(2d-3)H\dot\phi^3 - \dot\phi^4\Big] + \frac{3k^2\ell_s^4}{4\pi^2}\Big[\frac{3c_3^2}{16d}(d+1)H^6 + 3c_1H^4\dot\phi^2 + \frac{c_3}{d}(2d-3)H^3\dot\phi^3$$
$$- \frac{9c_3}{4d}H^2\dot\phi^4 + \dot\phi^6\Big]\Big\}, \tag{9.4.60}$$

$$\mathcal{C}^2(H,\dot\phi) = -\frac{1}{\mathcal{D}}\Big\{8\Big[d(d-1)H^2 - \dot\phi^2\Big] + \frac{k\ell_s^2}{\pi}\Big[\frac{3c_3}{4}(d-7)H^4 + \frac{3c_3}{2d}(2d-3)H^2\dot\phi^2$$
$$- 8(d-1)^2H\dot\phi^3 + (4d-3)\dot\phi^4\Big] - \frac{3k^2\ell_s^4c_3}{16\pi^2 d}H\Big[3c_1H^5 + 6c_1H^4\dot\phi + 3c_3H^3\dot\phi^2$$
$$+ 4d(d-3)H^2\dot\phi^3 - 3(4d-3)H\dot\phi^4 + 6\dot\phi^5\Big]\Big\}, \tag{9.4.61}$$

where

$$\mathcal{D} \equiv 16 + \frac{3k\ell_s^2c_3}{\pi d}\left((d+3)H^2 + 2H\dot\phi - \frac{3}{d-2}\dot\phi^2\right)$$
$$+ \frac{3k^2\ell_s^4c_3}{4\pi^2 d}H\left(\frac{3c_3}{4}H^3 - 3(d-3)H\dot\phi^2 + 6\dot\phi^3\right). \tag{9.4.62}$$

For $w = 0$, we recall that the string-hole gas fixed point is given by Eq. (9.3.47), and here we denote it as

$$y_\star^A = (H_\star, \dot\phi_\star) = \frac{2}{\ell_s}\sqrt{\frac{2\pi}{k\Delta}}(1,d)\,. \tag{9.4.63}$$

One can check that $\mathcal{C}^1(H_\star,\dot\phi_\star) = \mathcal{C}^2(H_\star,\dot\phi_\star) = 0$ ($\mathcal{C}^A(y_\star^B) = 0$), so $\dot H_\star = \ddot\phi_\star = 0$ ($\dot y_\star^A = 0$) as expected.

The Jacobian matrix for the system of differential equations is then

$$J_A{}^B = \partial_A\mathcal{C}^B\,, \tag{9.4.64}$$

and its eigenvalues are

$$r_\pm = \frac{1}{2}\Big\{\partial_H\mathcal{C}^1 + \partial_{\dot\phi}\mathcal{C}^2 \pm \Big[\big(\partial_H\mathcal{C}^1 + \partial_{\dot\phi}\mathcal{C}^2\big)^2 - 4\big(\partial_H\mathcal{C}^1\partial_{\dot\phi}\mathcal{C}^2 - \partial_{\dot\phi}\mathcal{C}^1\partial_H\mathcal{C}^2\big)\Big]^{1/2}\Big\}. \tag{9.4.65}$$

After calculating the partial derivatives and evaluating at the fixed point $(H_\star, \dot\phi_\star)$, one finds

$$r_\pm = \pm\frac{2}{\ell_{\rm s}}\sqrt{\frac{2\pi d\Delta}{k\mathcal{Q}}}\,, \tag{9.4.66}$$

where

$$\mathcal{Q} \equiv 16d^5 - 32d^4 - 46d^3 + 47d^2 + 36d - 20\,. \tag{9.4.67}$$

Since $r_+ > 0 > r_-$, it follows that the fixed point $(H_\star, \dot\phi_\star)$ is a saddle point, and therefore, it is generally not stable and certainly not an attractor in phase space.

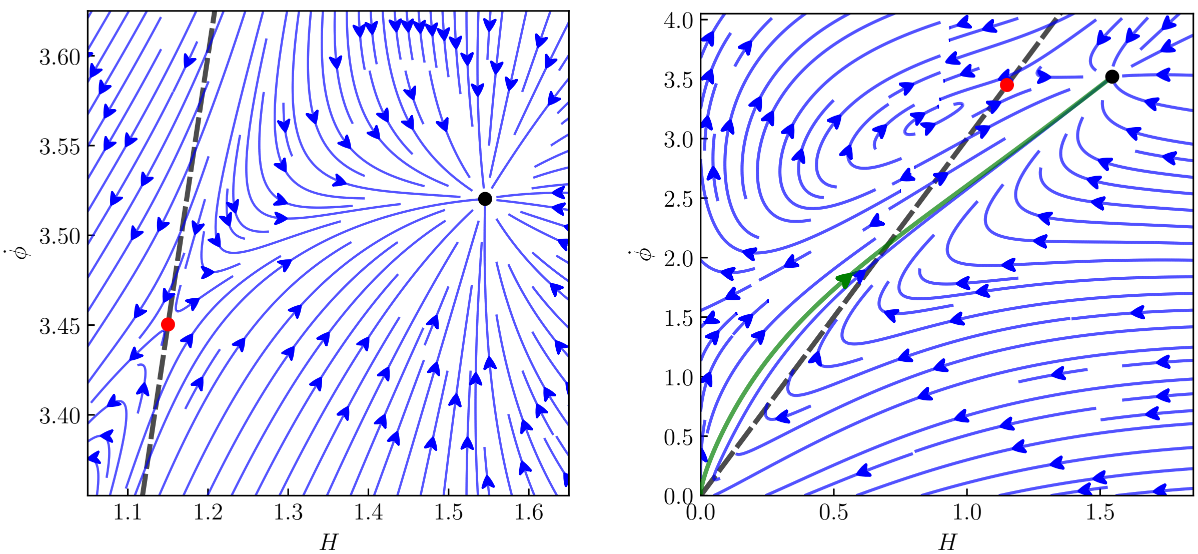

Figure 9.1 Phase space trajectories for GMV's action in a FLRW background with matter satisfying the continuity equation and $p = \sigma/2 = 0$. Setting $k = 1$, $\ell_{\rm s} = 1$, and $d = 3$, $\dot H$ and $\ddot\phi$ are computed from Eqs. (9.4.60) and (9.4.61), respectively. The red dot denotes the string-hole gas saddle point (9.4.63), and the black dot denotes the attractor fixed point of vacuum pre-Big Bang cosmology. The dashed gray curve depicts the line $\dot\phi = dH$, along which the saddle point is stable. The left and right plots show different ranges in H and $\dot\phi$. The left plot is a blowup of the right plot near the two nontrivial fixed points. In the right plot, the green line shows an example of trajectory that starts near the trivial fixed point at $H = \dot\phi = 0$ and goes to the attractor fixed point.

If one worries only about perturbations around the fixed point that preserve the condition $H = \dot\phi/d$, one may check the directional derivative of $\mathcal{C}^A$ with respect to the unit vector parallel to the line corresponding to $H = \dot\phi/d$. The unit vector is expressed as $u^A =$

$(1+d^2)^{-1/2}(1,d)$. The expression for the directional derivative is then

$$\boldsymbol{D_u}\mathcal{C}^A \equiv u^B\partial_B\mathcal{C}^A = \frac{\partial_H\mathcal{C}^A + d\partial_{\dot\phi}\mathcal{C}^A}{\sqrt{1+d^2}}\,, \tag{9.4.68}$$

and upon calculating the partial derivatives and evaluating at the fixed point $(H_\star, \dot\phi_\star)$, one finds

$$\begin{aligned}\boldsymbol{D_u}\mathcal{C}^1\big|_{(H_\star,\dot\phi_\star)} &= -\frac{\mathcal{P}^1}{\ell_{\mathrm{s}}\mathcal{Q}}\sqrt{\frac{2\pi}{k(2d^4+d^3+d-2)}}\,,\\ \boldsymbol{D_u}\mathcal{C}^2\big|_{(H_\star,\dot\phi_\star)} &= -\frac{\mathcal{P}^2}{\ell_{\mathrm{s}}\mathcal{Q}}\sqrt{\frac{2\pi\Delta}{k(1+d^2)}}\,,\end{aligned} \tag{9.4.69}$$

where

$$\begin{aligned}\mathcal{P}^1 &\equiv 4d(d+2)(2d-1)\Delta\,,\\ \mathcal{P}^2 &\equiv 8d^4 - 8d^3 - 18d^2 + 20d\,,\end{aligned} \tag{9.4.70}$$

Noting that $\mathcal{P}^A > 0$ and $\mathcal{Q} > 0$ for any $d \geq 3$, it follows that

$$\boldsymbol{D_u}\mathcal{C}^A\big|_{(H_\star,\dot\phi_\star)} < 0\,, \qquad A = 1,2\,, \tag{9.4.71}$$

and thus, the fixed point $(H_\star, \dot\phi_\star)$ is stable in the direction corresponding to the line $H = \dot\phi/d$. This implies that if one considers perturbations about the string-hole gas saddle point that respect the condition $H = \dot\phi/d$ the string-hole gas evolution is stable. However, for general perturbations about the saddle point, the trajectories might flow away from the string-hole gas evolution.

Further insight can be gained numerically. For example, setting $k=1$, $\ell_{\mathrm{s}}=1$, and $d=3$, one finds two real positive nontrivial fixed points that satisfy $\mathcal{C}^A(H,\dot\phi) = 0$: the string-hole gas fixed point with $y^A_\star = (2\sqrt{2\pi/19}, 6\sqrt{2\pi/19})$ and another fixed point approximately located at $(1.546, 3.520)$. The phase space trajectories are plotted in Fig. 9.1. The string-hole gas fixed point is depicted by the red dot, and visual inspection confirms that it is a saddle point (see the left plot of Fig. 9.1 for a close-up). The other fixed point, depicted by the black dot, is the attractor of standard (vacuum) pre-Big Bang cosmology[3] (see, e.g., Refs. [308, 310, 315]). We note that this is *exactly* the fixed point found by GMV, and it appears in the phase space no matter what the EOS parameter w is since $e^{\phi}\rho \to 0$ at that point.

The dashed gray curves in Fig. 9.1 depict the line $\dot{\phi} = dH$. When projecting the trajectories onto that line, it is clear from the left plot that the flow is attracted toward the string-hole gas saddle point in its vicinity. This is in agreement with the earlier (analytical) result that the string-hole gas saddle point is stable in the direction of the constraint $H = \dot{\phi}/d$.

In the right plot of Fig. 9.1, we show the phase space including the trivial fixed point $(H, \dot{\phi}) = (0, 0)$ corresponding to the string perturbative vacuum, and the green curve shows one trajectory passing infinitesimally close to that fixed point. We notice that it smoothly reaches the attractor fixed point (black dot), confirming the result of GMV[4] [310]. This also implies, however, that it is not possible for a trajectory to start near the string perturbative vacuum and evolve toward the string-hole gas fixed point smoothly. In the context of pre-Big Bang cosmology, the goal would be to start at the string perturbative vacuum and evolve toward a string-hole gas as the high-energy state of the universe before a bounce. Although GMV's α'-corrected action allows for a unique string-hole gas solution, it does not seem to be sufficient to describe the evolution of the universe thoroughly from the perturbative vacuum to the stringy state at high energies. This is not surprising because black-/string-hole formation is not a continuous process; rather, the holes collapse instantaneously from the vacuum fluctuations that have grown in amplitude. Therefore, asking for continuous trajectories connecting the vacuum to the string-hole gas fixed point is ill posed.

Nevertheless, there are arguments to support that a string-hole gas should be connected to the vacuum in some way. In a broader cosmological context, one could imagine starting asymptotically far in the past in a contracting universe (in the Einstein frame) which has 'normal' matter (e.g., a mix of dust [$w = 0$] and radiation [$w = 1/d$]). As shown in Ref. [559]

[3] Our numerical values differ from those of Refs. [308, 310, 315] simply due to the choice of units. We work with $k = \ell_{\rm s} = 1$, while Refs. [308, 310, 315] set $k\alpha' = 1$, so basically the numbers differ by a factor of $\sqrt{2\pi}$.

[4] This time, we note that this curve may not be *exactly* the solution found by GMV. However, it is close enough since $e^{\phi}\rho$ is subdominant at all times along the green trajectory. In particular, it shares its qualitative behavior: the perturbative evolution starts in the region $\dot{\bar{\phi}} = \dot{\phi} - dH > 0$ (above the dashed gray line), crosses the gray line (where $\dot{\bar{\phi}} = 0$), and ends at the attractor fixed point in the region $\dot{\bar{\phi}} < 0$ (below the gray line).

(see also Refs. [44, 200, 451]), starting with vacuum initial conditions, the pressureless matter would collapse into a black-hole gas, and as stated in the present work, it would evolve into a string-hole gas with EOS $p = \rho$. From that point of view, a string-hole gas with EOS $w = 1$ is naturally an attractor[5], and the same conclusion would necessarily follow in the string frame, although the physical intuition might be less obvious in the string frame. In that context, one cannot describe the entire cosmological evolution with the stringy actions studied in this paper; they would be applicable only at the time of formation of the string holes. In that case, when the condition $H = \dot{\phi}/d$ is met, as we showed above, the string-hole gas evolution is an attractor in the string frame.

9.4.2 Stability of the fixed point with Meissner's action

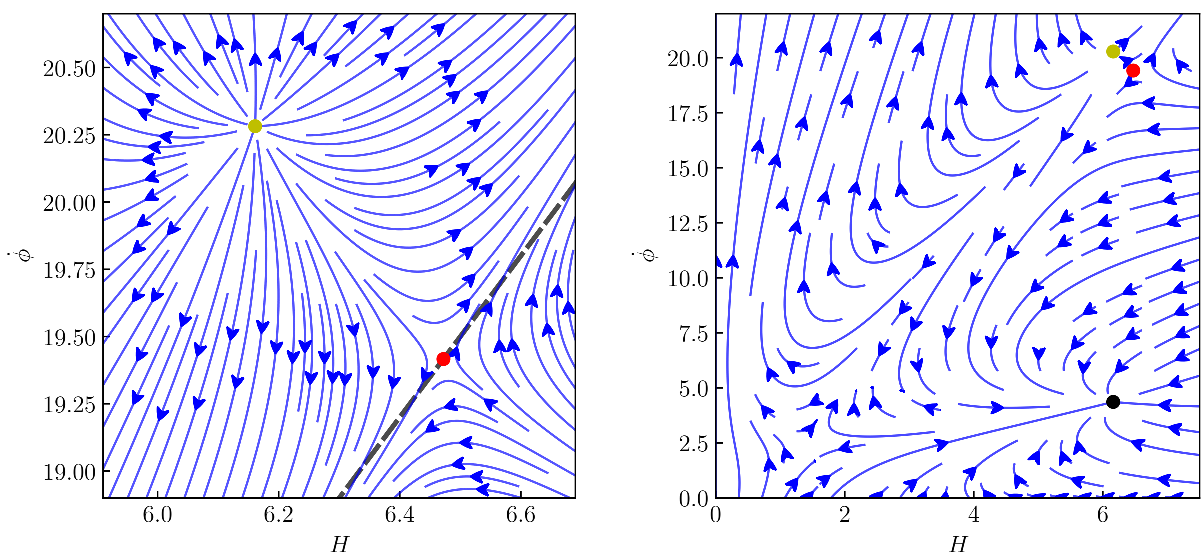

Figure 9.2 Phase space trajectories for Meissner's action in a FLRW background with matter satisfying the continuity equation and $p = \sigma/2 = 0$ (and setting $k = 1$, $\ell_s = 1$, and $d = 3$). The red dot denotes the string-hole gas saddle point (9.4.72), the yellow dot denotes the repeller fixed point, and the black dot denotes the attractor (see the text). The two plots show different ranges in H and $\dot{\phi}$. The left plot is a blowup of the right plot near the string-hole gas fixed point. Also in the left plot, the dashed gray curve depicts the curve $\dot{\phi} = dH$, along which the saddle point is unstable this time.

[5]Matter with the EOS $w = 1$ is generally (marginally) an attractor in a contracting universe, whether it is a black-/string-hole gas, anisotropies, or a massless scalar field. We use the word 'marginal' since any other component with EOS $w > 1$, e.g., an Ekpyrotic field with negative exponential potential, would overturn this conclusion and become the new attractor (see, e.g., Ref. [349]).

We now perform the same stability analysis as in the previous subsection, except starting with the EOM derived in Sec. 9.3.3 for Meissner's action and setting $p = \sigma/2 = 0$. Here the fixed point is [recall Eq. (9.3.59)]

$$y_\star^A = (H_\star, \dot{\phi}_\star) = \frac{2}{\ell_{\rm s}}\sqrt{\frac{2\pi}{k(d-2)}}(1, d)\,. \tag{9.4.72}$$

As before, we put the set of differential equations in the form $\dot{y}^A = (\dot{H}, \ddot{\phi}) = \mathcal{C}^A(H, \dot{\phi})$ and compute the eigenvalues of the corresponding Jacobian matrix $J_A{}^B = \partial_A \mathcal{C}^B$ evaluated at the fixed point $y_\star^A$. As a result, we find that there is one positive and one negative eigenvalue indicating that the fixed point is again a saddle point. This is confirmed by visual inspection of Fig. 9.2 (see the left plot for a close-up; Fig. 9.2 is generated the same way as Fig. 9.1, in particular, setting $k = 1$, $\ell_{\rm s} = 1$, and $d = 3$). Contrary to the saddle point of the previous subsection, though, the saddle point here turns out to be unstable in the direction of the string-hole gas constraint $H = \dot{\phi}/d$. Indeed, evaluating $\boldsymbol{D_u}\mathcal{C}^A$ at $y_\star^A$ yields two positive values. This is confirmed by looking at the direction of the flow along the dashed gray line in the left plot of Fig. 9.2, which depicts the line $\dot{\phi} = dH$; the trajectories are moving away from the fixed point (in red).

Additional fixed points are found by numerically solving $\mathcal{C}^A(H, \dot{\phi}) = 0$, and they are shown by the yellow and black dots in Fig. 9.2. The black dot is an attractor and was also found in the context of vacuum pre-Big Bang cosmology (see Ref. [308]). However, in this case, one notices that the attractor fixed point is disconnected from the trivial fixed point at the origin (see the right plot of Fig. 9.2), which confirms the results of Refs. [153, 308]. Furthermore, we find that the string-hole gas saddle point is also not connected to the string perturbative vacuum as it was the case with GMV's action. In fact, trajectories that start near the origin tend to grow rapidly in $\dot{\phi}$, while H remains small, and go nowhere near the fixed points.

In summary, the string-hole gas fixed point, which is a solution of Meissner's action, shares several characteristics with the solution of GMV's action: both are saddle points, disconnected from the string perturbative vacuum. However, the trajectories in the vicinity of the saddle points behave very differently for both actions. Indeed, the latter (GMV) is stable in the direction of the string-hole gas constraint $H = \dot{\phi}/d$, but the former (Meissner) is unstable. Therefore, Meissner's action appears very unlikely to be the physical action that can describe the evolution of a string-hole gas and of the universe at high energies.

Let us end by noting that, although the analysis outlined in this section focuses on the case $w = 0$, we found that the qualitative results about the characterization and (in)stability of the fixed points are the same for any value of $w \in (-1, 1/3)$. We do not include the quantitative details for a generic value of w for the sake simplicity and readability.

9.5 Conclusions and discussion

In this paper, we revisited the proposal that the stringy high-energy state of the Universe is a string-hole gas, i.e., a gas of black holes lying on the string-/black-hole correspondence curve. By analyzing its thermodynamic properties, we confirmed that a string-hole gas has the same EOS and entropy equation in the Einstein frame as a black-hole gas with $p = \rho$ and $S \sim \sqrt{EV/G}$. In the string frame, we found that a string-hole gas has vanishing pressure, and we derived the corresponding evolution to be given by $H = \dot{\phi}/d \sim \ell_{\mathrm{s}}^{-1}$. Our goal was then to find such a fixed point solution from the dynamical cosmological EOM of a string theory motivated action. We studied the gravidilaton sector of the low-energy effective action of string theory and found that, to zeroth order in the α' expansion, there is no string-hole gas solution without adding a tuned negative cosmological constant. However, going to first order in α', we studied two different actions, and both yielded a natural string-hole gas solution. Stability of those fixed point solutions was assessed by performing a phase space analysis. We found that both solutions are saddle points in $(H, \dot{\phi})$ phase space, but the solution coming from the action of GMV [310] tends to be better behaved since it is stable in the direction of the string-hole gas constraint $H = \dot{\phi}/d$. The solution coming from the action of Meissner [497] is unstable in the same direction and thus less appealing, even though it possesses the desired $O(d, d)$ symmetry of string cosmology to first order in α'. In summary, our results show that string theory consistently supports a string-hole gas phase of cosmological evolution, at least at the level of a gravidilaton effective action and minimally to first order in the α' expansion. Our stability analysis also indicates that a particular choice of action (GMV's action) is more appropriate at the level of our approximation.

We would like to point out some of the limitations of the current analysis. As mentioned before, the scale at which a string-hole gas forms and evolves is right at the limit of perturbative string theory in terms of the α' expansion. Our analysis showed that one needs an action that is valid at least to first order in α', but one could seek for a yet higher-order action (e.g., to second order in α') or an exact conformal model (valid to all orders in α') for a more robust implementation. Beforehand, it might be more straightforward to try to find a description of a string-hole gas such that its corresponding matter action has first-order α' corrections. Indeed, if first-order α' corrections are included in the gravity sector, they may as well be first-order α' corrections at the level of the matter action. For example, higher energy-momentum tensor corrections in the matter sector have been considered in Ref. [83] for Einstein gravity, but this has never been studied in the context of a string theory effective action or for any other theory beyond Einstein gravity. We note that such a possibility might also open the window to obtaining a nonsingular curvature bounce following the string-hole gas phase.

Another limitation comes from the fact that the current analysis was only performed within effective field theories of string theory and did not use perhaps the full 'strength' of string theory. As future work, one could try to construct the proper matter action for string holes from first principles rather than using a thermodynamic approach. At the level of general relativity, there has been recent progress in describing a black-hole lattice in cosmology (see, in particular, Ref. [213] in a nonsingular bouncing cosmological background as well as, e.g., Ref. [71, 72, 262] and references therein), which may be viewed as an approximation of a black-hole gas. Similar ideas with the addition of appropriate stringy ingredients could be used to develop a nonperturbative action for a string-hole gas.

Let us also mention the fact that black holes in string theory may not be best described by the semiclassical picture used in this paper. The singularity at the center of black holes may be resolved in full string theory, and even the concept of a black-hole horizon may need to be revised. For instance, a stringy black hole might be better described by a 'fuzzball' (see, e.g., Ref. [492–494] and references therein). In that context, a black-hole gas may be realized as a set of intersecting brane states [490], which is related to the concept of fractional brane gas (see, e.g., Ref. [207, 382, 383, 495, 496] and references therein).

Within the context of a string-hole gas as studied in this paper, we plan to extend the present work to determine what is the cosmological evolution subsequent to the string-hole gas phase and what the cosmological observable predictions intrinsic to the resulting very early Universe scenario are. First, the goal is to determine how a string-hole gas phase can be connected to standard Big Bang cosmology starting with radiation-dominated expansion. A string-hole gas phase is not expected to be stable for an infinitely long period of time. The gas will ultimately (Hawking) evaporate into radiation [610], a nonadiabatic process of entropy production that can be viewed as quantum particle creation in curved spacetime. Given that the string-hole gas is already saturating the appropriate entropy bound, the entropy release from the evaporation of the string holes cannot occur if the spacetime curvature remains constant or grows to a higher energy scale. Instead, the decay of the string holes must coincide with a (nonsingular) curvature bounce; in particular, the string- and Einstein-frames Hubble radii have to start growing. This would naturally coincide with the beginning of the expanding radiation-dominated phase of standard Big Bang cosmology.

Finding dynamics for the process of nonsingular curvature bounce shall be one of the key issues in follow-up work. Even though the actions studied in this paper contained higher-curvature corrections, they did not allow for nonsingular transitions from the string-hole gas phase to radiation expansion. Since the process of string-hole gas decay into radiation is quantum mechanical in nature, one may expect to find the desired dynamics from an action including quantum loop corrections. This is physically equivalent to taking into account the 'backreaction' from particle production due to quantum fluctuations in curved spacetime [315]. It is precisely this backreaction that might effectively violate the null energy condition, hence avoiding a Big Crunch singularity after the string-hole gas phase. Nonsingular bouncing backgrounds have already been found with string-theoretic loop corrections (see, e.g., Refs. [152, 191, 308, 315, 604] and references therein), but never in the context of a string-hole gas phase. Loop corrections might not be the only way, though, to obtain a nonsingular bounce in string theory. Another possibility, for instance, would be to consider an S-brane, a stringy object that can prevent the Universe from reaching a Big Crunch (see Ref. [140, 292, 409–411], also studied in Ref. [136]).

Finally, once a full very early Universe scenario has been developed at the background level, we shall be able to study the generation and evolution of the cosmological perturbations and determine what the observable predictions are. If fluctuations are seeded in the string-hole gas phase, one may find interesting results. On one hand, the quantum perturbations for a gas of black holes at the string scale may deviate considerably from the usual Bunch-Davies initial state. On the other hand, one shall not underestimate the effect of thermal fluctuations from the gas of string holes. Indeed, since the radius of the string holes equates the Hubble radius in the string frame, one may obtain holographic scaling of the specific heat capacity ($C_V \sim R^2$) on Hubble scales, similar to what is obtained from a string gas [127, 131]. It shall be interesting to see what spectra of primordial perturbations result and how they differ from the results of string gas cosmology (see, e.g., Refs. [127, 131] and references therein), pre-Big Bang cosmology (see, e.g., Refs. [308, 315, 316, 449] and references therein), and other very early Universe scenarios.

9.6 String-hole gas evolution in the Einstein frame

Given a consistent string-hole gas solution with $H = \dot{\phi}/d =$ constant in the string frame, one can derive the corresponding solution in the Einstein frame by using the relation (see, e.g., Refs. [308, 315])

$$\tilde{H} = \left(H - \frac{\dot{\phi}}{d-1}\right) e^{\phi/(d-1)} \,. \tag{9.6.73}$$

In this Appendix, a tilde denotes an Einstein-frame quantity, while no tilde means the string frame. The constraint $H = \dot{\phi}/d$ thus implies

$$\tilde{H} = -\frac{H}{d-1} e^{\phi/(d-1)} \,, \tag{9.6.74}$$

so one notices that for a constant-Hubble expanding phase in the string frame ($H > 0$), the Einstein-frame Hubble parameter must be negative ($\tilde{H} < 0$) and therefore contracting.

Let us recall that the Einstein-frame time is related to the string-frame time via (see, e.g., Ref. [308])

$$\mathrm{d}\tilde{t} = e^{-\phi/(d-1)} \mathrm{d}t \,. \tag{9.6.75}$$

Since $\dot{\phi} = dH =$ constant, where one now views $H = H_\star \sim \ell_{\mathrm{s}}^{-1}$ as one of the constant fixed-point solutions found, e.g., in Secs. 9.3.2 or 9.3.3, one can write

$$\phi(t) = dH(t - t_0) \tag{9.6.76}$$

for $t \leq t_0$. The integration constant t_0 is set such that $\phi(t_0) = 0$, at which point $g_{\mathrm{s}} = e^{\phi/2} = 1$, corresponding to strong coupling. Thus, the evolution in the perturbative regime (where $g_{\mathrm{s}} \ll 1$) translates to $t \ll t_0$. Upon integration of Eq. (9.6.75), one can then show that

$$\tilde{t} - \tilde{t}_0 = -\frac{(d-1)}{dH} \left(e^{-\phi(t)/(d-1)} - 1\right) \,, \tag{9.6.77}$$

for $\tilde{t} \leq \tilde{t}_0$, where $\tilde{t}_0$ in the Einstein-frame time equivalent to the string-frame time t_0. Let us choose $\tilde{t}_0 = -(d-1)/(dH) < 0$, so then

$$e^{\phi/(d-1)} = \frac{d-1}{dH}\frac{1}{(-\tilde{t})}\,. \tag{9.6.78}$$

Therefore, Eq. (9.6.74) becomes

$$\tilde{H}(\tilde{t}) = -\frac{1}{d(-\tilde{t})}\,, \tag{9.6.79}$$

which confirms $\tilde{H} < 0$ since $\tilde{t} \leq \tilde{t}_0 < 0$. The above expression further implies

$$\tilde{a}(\tilde{t}) \sim (-\tilde{t})^{1/d} \tag{9.6.80}$$

when integrating $\tilde{H} = \mathrm{d}\ln\tilde{a}/\mathrm{d}\tilde{t}$. Combining with Eq. (9.6.78), this implies $\tilde{a} \sim e^{-\phi/(d(d-1))}$, which is in agreement with how one expects the Einstein-frame scale factor to behave for a string-hole gas [recall Eq. (9.2.17)].

Chapter 10

Cosmological perturbations and stability of nonsingular cosmologies with limiting curvature

10.1 Introduction

One of the biggest problems with the classical theory of general relativity is the occurrence of singularities, which are inevitable under realistic assumptions [345, 348, 544] and which signify the breakdown and the incompleteness of the theory. The big bang singularity in cosmology and the singularity at the center of a black hole are two well-known instances of singularities in general relativity that one would like to resolve. These singularities often find themselves in regions of high density, high energy, and high curvature, where one may expect the breakdown of the classical theory and the emergence of quantum behavior. For this reason, there is hope that a quantum theory of gravity would provide the resolution to the otherwise pathological classical singularities.

Without a proper theory of quantum gravity, one may approach the problem with an effective theory that could mimic the low-energy behavior of the full quantum theory. The effective theory could be constructed with one or more new degrees of freedom, e.g. a new scalar field. This allows one to study nonsingular theories of the very early Universe within effective field theory (EFT) [162, 165, 227] as is done in, for instance, bouncing cosmologies [172, 267, 404, 556] or genesis scenarios [223, 226, 402, 523, 525, 551] with a generalized scalar field such as in Horndeski [355] and generalized Galileon [250] theories, whose equivalence was first proved in [401]. Alternatively, one may attempt to modify the Einstein-Hilbert gravity action to include higher-order curvature terms (e.g., [80, 81, 219]). Interestingly, this can serve as the basis for implementing the limiting curvature hypothesis, which seeks to incorporate the idea of a fundamental limiting length (that would be realized in the full theory of quantum gravity) into the effective theory for gravity.

In line with special relativity where the speed of light is bounded from above and quantum mechanics where the Heisenberg uncertainty relation holds, the idea of the limiting curvature hypothesis comes from the fact that one may expect quantum gravity to possess a fundamental length scale ℓ_f below which no measurement can be made and on which all physical observables must be smeared out. Presumably, this fundamental scale is at least of the order of the Planck length $\ell_{\rm Pl} = \sqrt{\hbar G/c^3}$, although it could be larger. Taking $\ell_f \sim \ell_{\rm Pl}$, it is straightforward to see that if all curvature invariants are bounded throughout the spacetime manifold ($|R| < \ell_{\rm Pl}^{-2}$, $|R_{\mu\nu}R^{\mu\nu}| < \ell_{\rm Pl}^{-4}$, $|\nabla_\rho R_{\mu\nu}\nabla^\rho R^{\mu\nu}| < \ell_{\rm Pl}^{-6}$, $|C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}| < \ell_{\rm Pl}^{-8}$, etc., where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $C_{\mu\nu\rho\sigma}$ is the Weyl tensor, and ∇ is the covariant derivative), then the spacetime is nonsingular. Indeed, in the well-known cases of the big bang and black hole singularities, some of the physically measurable curvature invariants such as R, $R_{\mu\nu}R^{\mu\nu}$, and $C^2 := C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ blow up; hence finding theories in which all invariants are bounded is certainly a necessary condition for constructing nonsingular cosmologies.

Unfortunately, bounding an infinite number of curvature invariants is rather nontrivial. Indeed, there are well-known instances where low-order curvature invariants are bounded while higher-order invariants are still unbounded (e.g., $|R_{\mu\nu}R^{\mu\nu}| < \ell_{\rm Pl}^{-4}$ while $|\nabla_\rho R_{\mu\nu}\nabla^\rho R^{\mu\nu}| \to \infty$). This is where the limiting curvature hypothesis comes in handy. The hypothesis states that [296, 297, 320, 483, 484] if one finds a theory that allows a finite number of curvature invariants to be bounded by an explicit construction (e.g., $|R| \leq \ell_{\rm Pl}^{-2}$ and $|R_{\mu\nu}R^{\mu\nu}| \leq \ell_{\rm Pl}^{-4}$), and when these invariants take on their limiting values, then any solution of the field equations reduces to a definite nonsingular solution (e.g., de Sitter space), for which all curvature invariants are automatically bounded. We note, though, that the assumptions of the limiting curvature hypothesis generally do not ensure that solutions avoid singularities when curvature scalars are not on their limiting values.

The limiting curvature hypothesis has been used and tested in the context of black hole physics [85, 111, 112, 196, 264, 295–297, 508, 600]. In this context, the geometry outside the black hole horizon is described by the usual Schwarzschild metric, but inside the event horizon, the black hole singularity is replaced with a nonsingular de Sitter spacetime, which, in turn, could be the source of a new "baby" Friedmann universe. Similarly, in a cosmological context [111, 112, 143, 265, 507, 516], a nonsingular universe can be constructed, in vacuum, such that it is asymptotically de Sitter in the past and Minkowski in the future (or vice versa). This is in line with Penrose's vanishing Weyl tensor conjecture [545] (see also the discussion in Ref. [346]), which states that the Weyl tensor should vanish at the beginning of the universe, since de Sitter space has $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = 0$. With the addition of matter sources, one can obtain asymptotically de Sitter and Friedmann cosmologies, remaining nonsingular throughout cosmic time. Recent works also show that the ideas of limiting curvature could allow one to construct nonsingular bouncing cosmologies [84, 197, 417].

In this paper, we want to revisit nonsingular cosmological models that make use of an effective theory for gravity in which the limit curvature hypothesis is realized. It was shown in Refs. [143, 516] that interesting background cosmologies can be found within this framework by constructing a theory in which the curvature invariants R and $4R_{\mu\nu}R^{\mu\nu} - R^2$ are bounded. However, these studies did not explore the cosmological perturbations [518] for the action containing the above curvature invariants. Recent developments in nonsingular cosmology within EFT [162, 165, 227] have shown that it is often rather difficult to avoid instabilities in the cosmological perturbations (e.g., see Refs. [20, 398, 441] for no-go theorems within Galileon and Horndeski theories; see, also, Refs. [363, 365, 367]). For this reason, one could tend to believe that nonsingular models constructed as in Refs. [143, 516] are going to be very unstable at the perturbation level, thus rendering the models unviable.

In this work, we will show that the naive models of Refs. [143, 516] are indeed generically unstable. We will see that minimal extensions in which one also includes the Weyl tensor squared, $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$, in the curvature invariants are more robust, i.e. there are fairly large regions of parameter space that are stable. Yet, there do not seem to exist nonsingular cosmological solutions that remain stable throughout cosmic history, and moreover, the theory will be shown to be equivalent to a $f(R, \mathcal{G})$ theory of gravity (where $\mathcal{G}$ is the Gauss-Bonnet term), which has unavoidable ghosts [248]. We will then construct a completely new curvature-invariant function and show that it allows for stable nonsingular cosmological solutions throughout time. There will remain some difficulties though in constructing a physically relevant model in certain cases.

The outline of this paper is as follows. In Sec. 10.2, we will review the construction of a nonsingular cosmology in which the limiting curvature hypothesis is realized, set up the action for the theory, and find the background equations of motion. In particular, we will discuss two specific scenarios: an inflationary scenario and a genesis scenario. We will then study the cosmological perturbations, determine the general stability conditions and check them for specific models. The equivalence with $f(R,\mathcal{G})$ gravity will also be demonstrated. In Sec. 10.3, we will construct a new model with a new curvature scalar, derive the resulting cosmological perturbations and the stability conditions, and comment on the case of an anisotropic background. We will then discuss the recovery of vacuum Einstein gravity and Friedmann cosmology with the addition of matter sources. We will end with a summary of the results and a discussion in Sec. 10.4.

10.2 Nonsingular cosmology with limiting curvature

10.2.1 Setup of the theory and background evolution

The approach taken in Refs. [143, 516] to implement the limiting curvature hypothesis consists of introducing a finite number of nondynamical scalar fields χ_i, or Lagrange multipliers, such that the action takes the form

$$S=\frac{M_{\mathrm{Pl}}^2}{2}\int \mathrm{d}^4x\,\sqrt{-g}\left[R+\sum_i \chi_i I_i - V(\chi_i)\right]\,, \tag{10.2.1}$$

where the I_i's are functions of curvature invariants that can depend on R, $R_{\mu\nu}$, $R^{\mu}{}_{\nu\rho\sigma}$, $C_{\mu\nu\rho\sigma}$ and combinations and derivatives thereof. Accordingly, given an appropriately chosen potential $V(\chi_i)$, one can rewrite this action into a general $F(R,R_{\mu\nu}R^{\mu\nu},...)$ effective theory of gravity. By virtue of the Lagrange multipliers, a given potential imposes constraints on the I_i's, hence the idea that the right choice of $V(\chi_i)$ can naturally bound the curvature invariants and satisfy the limiting curvature hypothesis asymptotically. We will give examples where this is realized below.

As is done in Refs. [143, 516], we are going to consider two nondynamical scalar fields and start with a general action of the form

$$S=\frac{M_{\mathrm{Pl}}^2}{2}\int \mathrm{d}^4x\,\sqrt{-g}\Big[R+\chi_1 I_1(\nabla,R^{\mu}{}_{\nu\rho\sigma})-V_1(\chi_1)+\chi_2 I_2(\nabla,R^{\mu}{}_{\nu\rho\sigma})-V_2(\chi_2)\Big]+S_{\mathrm{m}}\,, \tag{10.2.2}$$

where S_{m} is the action for possible matter sources. At this point, we do not make any assumption on the functional form of I_1 and I_2, but we want them to scale as R, so let us require that we recover a certain limit at the background level:

$$I_1^{(0)} = 12H^2\,, \qquad I_2^{(0)} = -6\dot{H}\,. \tag{10.2.3}$$

The superscript (0) refers to the metric of a flat[1] Friedmann-Lemaître-Robertson-Walker (FLRW) universe,

$$g_{\mu\nu}^{(0)}\mathrm{d}x^\mu\mathrm{d}x^\nu = -N(t)^2\mathrm{d}t^2 + a(t)^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\,, \tag{10.2.4}$$

where N is the lapse function and a is the scale factor. Accordingly, $H := \dot{a}/(Na)$ is the Hubble parameter, and a dot is a derivative with respect to physical time, t. At the background level, we can further ask that $\chi_1 = \chi_1^{(0)}(t)$ and $\chi_2 = \chi_2^{(0)}(t)$. The original action then becomes[2]

$$S^{(0)} = \frac{M_{\mathrm{Pl}}^2}{2}\int \mathrm{d}t\mathrm{d}^3\vec{x}\; Na^3\left[12\,(1+\chi_1)\left(\frac{\dot{a}}{Na}\right)^2 + 6\,(1-\chi_2)\frac{1}{N}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{a}}{Na}\right) - V_1 - V_2\right] + S_{\mathrm{m}}^{(0)}\,. \tag{10.2.5}$$

Varying $S^{(0)}$ with respect to χ_1 and χ_2 yields the equations of motion (EOMs)

$$12\left(\frac{\dot{a}}{a}\right)^2 = \frac{\mathrm{d}V_1}{\mathrm{d}\chi_1} \tag{10.2.6}$$

and

$$-6\left[\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2\right] = \frac{\mathrm{d}V_2}{\mathrm{d}\chi_2}\,, \tag{10.2.7}$$

respectively, where we set the lapse function to $N = 1$. Letting $T^\mu{}_\nu = \mathrm{diag}(-\varepsilon(t), p(t)\delta^i{}_j)$, where $T_{\mu\nu}$ is the stress-energy tensor associated with the matter action S_{m} and where ε is the energy density and p is the pressure, one can then vary the background action with respect to N to find

$$\frac{\varepsilon}{3M_{\mathrm{Pl}}^2} = (1 - 2\chi_1 - 3\chi_2)\left(\frac{\dot{a}}{a}\right)^2 - \dot{\chi}_2\left(\frac{\dot{a}}{a}\right) - \frac{1}{6}(V_1 + V_2)\,, \tag{10.2.8}$$

[1] It is straightforward to generalize this to include curvature (see Ref. [143]).

[2] We omit the superscript (0) for χ_1 and χ_2 when it is clear that they represent background quantities.

again setting $N = 1$. Finally, varying with respect to a gives (setting $N = 1$ once more)

$$-\frac{p}{M_{\rm Pl}^2} = (1-2\chi_1-3\chi_2)\left[2\left(\frac{\ddot{a}}{a}\right)+\left(\frac{\dot{a}}{a}\right)^2\right]-2(2\dot{\chi}_1+3\dot{\chi}_2)\left(\frac{\dot{a}}{a}\right)-\ddot{\chi}_2-\frac{1}{2}(V_1+V_2)\,. \quad (10.2.9)$$

Let us denote $V_1' := \mathrm{d}V_1/\mathrm{d}\chi_1$ and $V_2' := \mathrm{d}V_2/\mathrm{d}\chi_2$ for shorthand notation from here on. We can then summarize the set of EOMs as

$$12H^2 = V_1'(\chi_1)\,, \quad (10.2.10)$$

$$-6\dot{H} = V_2'(\chi_2)\,, \quad (10.2.11)$$

$$\frac{\varepsilon}{3M_{\rm Pl}^2} = (1-2\chi_1-3\chi_2)H^2 - \dot{\chi}_2 H - \frac{1}{6}[V_1(\chi_1)+V_2(\chi_2)]\,, \quad (10.2.12)$$

$$-\frac{p}{M_{\rm Pl}^2} = (1-2\chi_1-3\chi_2)[2\dot{H}+3H^2] - 2(2\dot{\chi}_1+3\dot{\chi}_2)H - \ddot{\chi}_2 - \frac{1}{2}[V_1(\chi_1)+V_2(\chi_2)]\,. \quad (10.2.13)$$

In the limit where $\chi_1 = \chi_2 = V_1 = V_2 = 0$, we note that we recover the usual Friedmann equations, as expected. Also, in the limit where $\varepsilon = p = 0$, one obtains the EOMs in vacuum. Demanding that $V_1'(\chi_1) > 0$ for all χ_1 values so that H is real and looking at an expanding universe (so $H > 0$; this could be generalized to a contracting universe with $H < 0$, in which case a minus sign would appear in certain equations), we can write the EOMs as

$$\dot{a} = a\sqrt{\frac{V_1'}{12}}\,, \quad (10.2.14)$$

$$\dot{\chi}_1 = -4\sqrt{\frac{V_1'}{12}}\frac{V_2'}{V_1''}\,, \quad (10.2.15)$$

$$\dot{\chi}_2 = -\sqrt{\frac{V_1'}{12}}\left[3\chi_2+2\chi_1-1+\frac{2(V_1+V_2)}{V_1'}+\frac{4\varepsilon}{V_1'M_{\rm Pl}^2}\right]\,. \quad (10.2.16)$$

Furthermore, the equations can be written in the following form:

$$\frac{\mathrm{d}\chi_2}{\mathrm{d}\chi_1} = \frac{V_1''}{4V_2'}\left[3\chi_2 + 2\chi_1 - 1 + \frac{2(V_1+V_2)}{V_1'} + \frac{4\varepsilon}{V_1' M_{\mathrm{Pl}}^2}\right], \tag{10.2.17}$$

$$\frac{\mathrm{d}\chi_1}{\mathrm{d}a} = -\frac{4}{a}\frac{V_2'}{V_1''}, \tag{10.2.18}$$

$$\frac{\mathrm{d}\chi_2}{\mathrm{d}a} = -\frac{1}{a}\left[3\chi_2 + 2\chi_1 - 1 + \frac{2(V_1+V_2)}{V_1'} + \frac{4\varepsilon}{V_1' M_{\mathrm{Pl}}^2}\right]. \tag{10.2.19}$$

Example of inflationary scenario

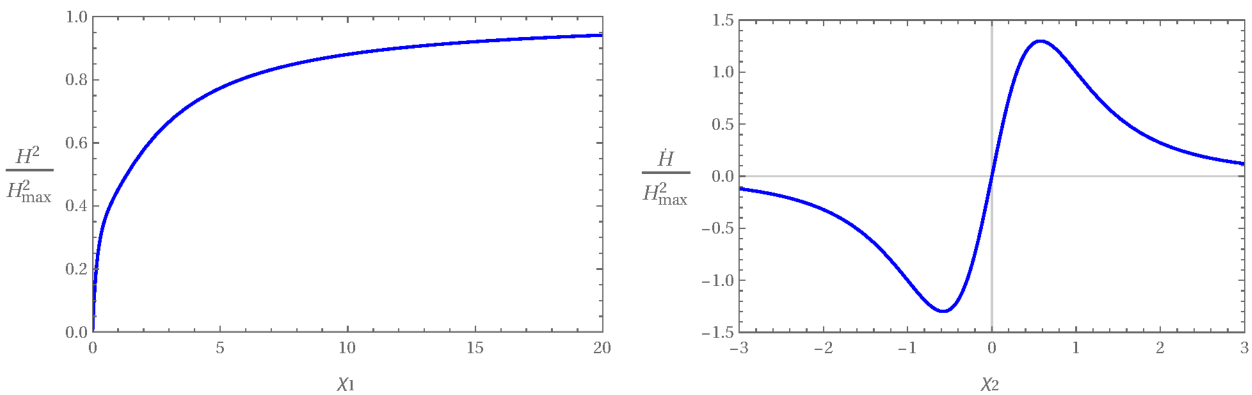

Figure 10.1 Background trajectories for the inflationary model given by the potentials (10.2.20) and (10.2.21). The left-hand plot shows H^2/H^2_{max} as a function of χ_1 as computed from Eq. (10.2.22), and the right-hand plot shows $\dot{H}/H^2_{\mathrm{max}}$ as a function of χ_2 as computed from Eq. (10.2.23). Note that χ_1 and χ_2 are dimensionless.

To solve the background EOMs, one needs to specify a form for the potentials $V_1(\chi_1)$ and $V_2(\chi_2)$. As a first example, one can consider

$$V_1(\chi_1) = 12H^2_{\mathrm{max}}\frac{\chi_1^2}{1+\chi_1}\left(1 - \frac{\ln(1+\chi_1)}{1+\chi_1}\right), \tag{10.2.20}$$

$$V_2(\chi_2) = -\,12H^2_{\mathrm{max}}\frac{\chi_2^2}{1+\chi_2^2}, \tag{10.2.21}$$

which is inspired from Ref. [143], and as we will see, this gives rise to a nonsingular inflationary scenario. In vacuum ($\varepsilon = p = 0$), the EOMs are given by

$$\frac{H^2}{H_{\max}^2} = \frac{V_1'(\chi_1)}{12H_{\max}^2} = \frac{2\chi_1 + 2\chi_1^2 + \chi_1^3 - 2\chi_1 \ln(1+\chi_1)}{(1+\chi_1)^3}\,, \tag{10.2.22}$$

$$\frac{\dot{H}}{H_{\max}^2} = -\frac{V_2'(\chi_2)}{6H_{\max}^2} = \frac{4\chi_2}{(1+\chi_2^2)^2}\,. \tag{10.2.23}$$

We plot these functions in Fig. 10.1. As we can see in the left-hand plot, the Hubble parameter is finite everywhere: as $\chi_1 \to 0$, the spacetime is asymptotically Minkowski ($H \to 0$; recall that we are in vacuum), whereas when $\chi_1 \to \infty$, the spacetime is asymptotically de Sitter since $H \to H_{\max}$. Similarly, looking at the right-hand plot, $\dot{H}$ is finite everywhere, and it is asymptotically vanishing as $\chi_2 \to \pm\infty$. Accordingly, this verifies the limiting curvature hypothesis as in Ref. [143].

We note at this point that since we regard our theory as a low-energy effective theory of a possible quantum theory of gravity, there should be a cutoff scale beyond which the EFT is no longer valid. The model here includes only two dimensionful parameters: M_{Pl} and $H_{\max}$. Therefore, the cutoff scale should naively be given by these parameters as $\Lambda_{\mathrm{cut}} = (M_{\mathrm{Pl}} H_{\max}^n)^{1/(1+n)}$ for a given integer $n \neq -1$, and in particular, it should be at least of the order of $H_{\max}$. Determining the exact value for Λ_{cut} involves a rather nontrivial computation for the given theory, but since the energy scale of our cosmological solutions is always less than $H_{\max}$ by construction, the validity of EFT is naturally ensured.

The phase-space diagram for the model is plotted in Fig. 10.2, where the arrows show the vectors with components $(\dot{\chi}_1, \dot{\chi}_2)$ computed from Eqs. (10.2.15) and (10.2.16) (in vacuum with $\varepsilon = 0$) with the potentials (10.2.20) and (10.2.21). We highlight a specific trajectory in green for illustration. In this case, the universe starts asymptotically at $\chi_1 \to \infty$ and $\chi_2 \to 0$ and ends asymptotically at $\chi_1 \to 0$ and $\chi_2 \to 0$, so as we saw from Fig. 10.1, the universe starts in a de Sitter spacetime and ends in a Minkowski spacetime.[3]

At this point, one may wonder how the given scenario evades the singularity theorems of [86, 87, 89] regarding the past incompleteness of inflationary cosmology. First, it is important to recall that the singularity theorem for inflation [87, 89] is proved under the assumption that gravity is given by the Einstein-Hilbert action and that inflation is driven by matter obeying the null energy condition. Our higher derivative gravity terms, when taken to the matter side of the equations of motion, act as matter violating the null energy condition.

[3] With the addition of matter sources, it would end in a FLRW spacetime as shown in Ref. [143].

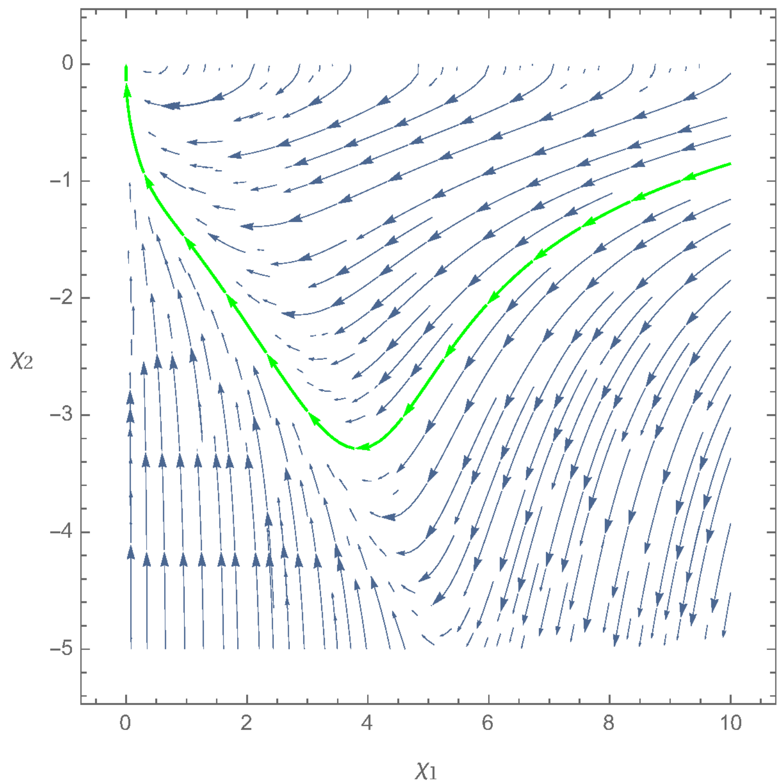

Figure 10.2 Phase-space diagram of (χ_1, χ_2) computed using Eqs. (10.2.15) and (10.2.16) showing different background inflationary trajectories. The arrows indicate the flow of time. The green curve illustrates a specific trajectory.

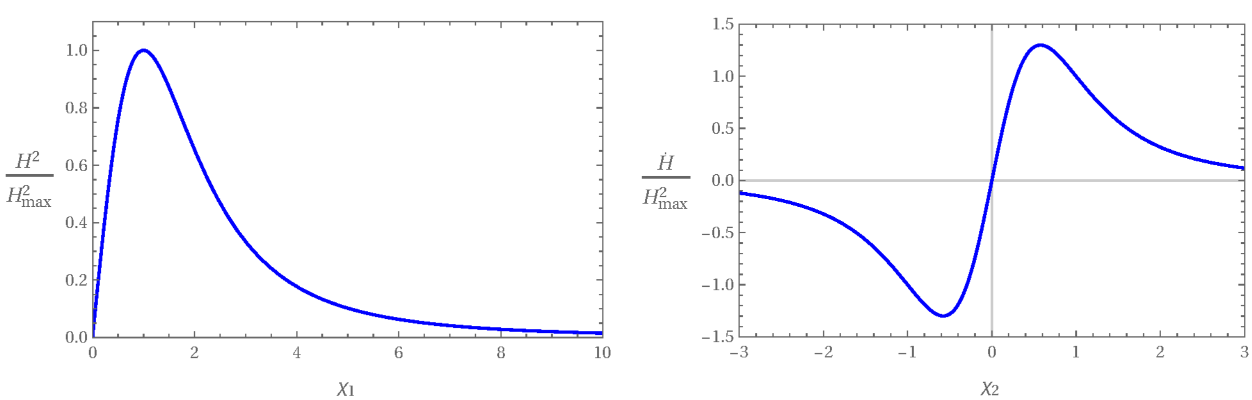

Figure 10.3 Background trajectories for the genesis model given by the potentials (10.2.24) and (10.2.25). The left-hand plot shows $H^2/H^2_{\rm max}$ as a function of χ_1 as computed from Eq. (10.2.26), and the right-hand plot shows $\dot{H}/H^2_{\rm max}$ as a function of χ_2 as computed from Eq. (10.2.27).

Hence the theorem does not apply in our setup. We also note that some of the past directed geodesics would have finite affine length (in agreement with the situation in Ref. [86]), but this is simply due to the fact that the flat FLRW chart does not cover the entire de Sitter space. One can extend the spacetime so that all geodesics are complete as in the case of de Sitter space. Thus, our inflationary universe is free from initial singularities.

Example of genesis scenario

As another example, let us consider

$$V_1(\chi_1) = -12H_{\rm max}^2\frac{8}{3+\chi_1^2}\,, \tag{10.2.24}$$

$$V_2(\chi_2) = -12H_{\rm max}^2\frac{\chi_2^2}{1+\chi_2^2}\,. \tag{10.2.25}$$

In vacuum, the EOMs become

$$\frac{H^2}{H_{\rm max}^2} = \frac{V_1'(\chi_1)}{12H_{\rm max}^2} = \frac{16\chi_1}{(3+\chi_1^2)^2}\,, \tag{10.2.26}$$

$$\frac{\dot H}{H_{\rm max}^2} = -\frac{V_2'(\chi_2)}{6H_{\rm max}^2} = \frac{4\chi_2}{(1+\chi_2^2)^2}\,. \tag{10.2.27}$$

We plot these functions in Fig. 10.3. As we can see in the left- and right-hand plots, the Hubble parameter and its time derivative are again everywhere finite: as $\chi_1 \to 0$ or $\chi_1 \to \infty$, the spacetime is asymptotically Minkowski with $H \to 0$, and $\dot H \to 0$ as $\chi_2 \to \pm\infty$. We note that $H_{\rm max}$ is now reached when $\chi_1 \to 1$. Thus, this is another type of scenario that verifies the limiting curvature hypothesis, namely a genesis scenario, which starts in Minkowski space rather than de Sitter space.

The phase-space diagram for this model is plotted in Fig. 10.4, where again, the arrows show the vectors with components $(\dot\chi_1, \dot\chi_2)$ computed from Eqs. (10.2.15) and (10.2.16) (in vacuum with $\varepsilon = 0$) with the genesis potentials (10.2.24) and (10.2.25). We highlight different trajectories in green, red, black, and purple for illustration. All of these curves either start or end at $\chi_1 \to \infty$, which corresponds to Minkowski spacetime. However, the green and red curves are pathological trajectories since they either end or start at $\chi_1 \to 1$, at which point it can be shown that $V_1'' \to 0$. Accordingly, from Eq. (10.2.15), one finds $\dot\chi_1 \to \pm\infty$ at that point. More interestingly, the black and purple curves start and end at $\chi_1 \to \infty$,

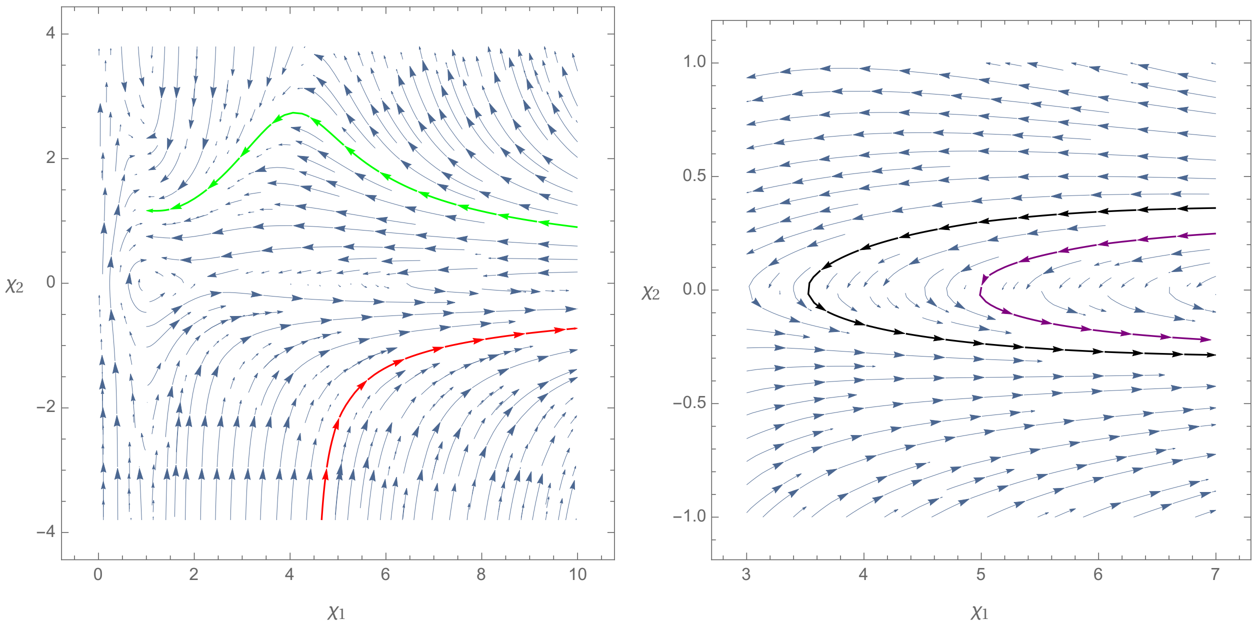

Figure 10.4 Phase-space diagram of (χ_1, χ_2) computed using Eqs. (10.2.15), (10.2.16), (10.2.24), and (10.2.25) showing different background genesis trajectories. Note that the right-hand plot is simply a zoomed-in version of the left-hand plot. The green, red, black, and purple curves show four different trajectories, which are discussed in the text.

and they turn around at some minimal $\chi_1 > 1$ value, so they never reach the "singularity" at $\chi_1 = 1$. Also, these trajectories always have $\chi_2 \ll 1$. When looking at the left-hand plot of Fig. 10.3, these trajectories suggest that the universe starts in the far right at $\chi_1 \to \infty$ (Minkowski spacetime), rolls up to the left but does not reach $H_{\max}$, and rolls back down to Minkowski spacetime again. In light of a structure formation scenario for the very early universe, one would like to have some form of reheating[4] near the maximal value that the Hubble parameter reaches. Thus, the universe would start as Minkowski spacetime, but it would end as a radiation- and then matter-dominated FLRW spacetime.

10.2.2 Cosmological perturbations and stability analysis

We now turn to the study of the cosmological perturbations for the action given by Eq. (10.2.2). At this point, one needs to specify the form of the curvature-invariant functions I_1 and I_2.

[4]For instance, reheating could occur via gravitational particle production [540, 541] (see Refs. [402, 561] for examples of gravitational particle production in nonsingular cosmologies).

Motivated by Refs. [143, 516], let us take

$$I_1 := R + \sqrt{12R_{\mu\nu}R^{\mu\nu} - 3R^2 + 3\kappa C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}}\,,$$
$$I_2 := \sqrt{12R_{\mu\nu}R^{\mu\nu} - 3R^2 + 3\kappa C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}}\,, \tag{10.2.28}$$

where at this point κ is just some real constant. In a flat FLRW background, these curvature invariants reduce to

$$I_1^{(0)} = 12H^2\,, \qquad I_2^{(0)} = -6\dot{H}^2\,, \tag{10.2.29}$$

under the assumption[5] that $\dot{H} < 0$, which was the hypothesis of Eq. (10.2.3) that allowed us to find the general background EOMs in Sec. 10.2.1. We note that the background expressions for the curvature invariants do not depend on the constant κ since flat FLRW spacetime is conformally flat, so the term proportional to the Weyl tensor squared does not affect the dynamics of background spacetime.

Tensor modes

Let us begin by studying the tensor fluctuations. We start by perturbing the metric linearly as

$$g^{(1)}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\mathrm{d}t^2 + a^2\left(\delta_{ij} + h_{ij} + \frac{1}{2}h_{ik}h^k{}_j\right)\mathrm{d}x^i\mathrm{d}x^j\,, \tag{10.2.30}$$

where the perturbation tensor h_{ij} is transverse and traceless, i.e. $h^i{}_i = \partial_i h^i{}_j = 0$ (adding the last term on the right-hand side does not change the linear equations but simplifies the derivation). We define the Fourier components of h_{ij} by

$$h_{ij} = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\left[h^+_{\vec{k}}e^+_{ij}(\vec{k}) + h^\times_{\vec{k}}e^\times_{ij}(\vec{k})\right]\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}\,, \tag{10.2.31}$$

where $\{e^+_{ij}, e^\times_{ij}\}$ represents the polarization basis. Given the curvature-invariant functions of Eq. (10.2.28), we can now perturb Eq. (10.2.2) to second order with the above metric to find

$$S^{(2)}_T = \frac{M^2_{\mathrm{Pl}}}{2}\int \mathrm{d}t\frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\sum_{I=+,\times} a^3\left(\mathcal{G}_T\ddot{h}^I_{\vec{k}}\ddot{h}^I_{-\vec{k}} + \mathcal{K}_T\dot{h}^I_{\vec{k}}\dot{h}^I_{-\vec{k}} - \mathcal{M}_T\frac{k^2}{a^2}h^I_{\vec{k}}h^I_{-\vec{k}}\right)\,, \tag{10.2.32}$$

[5] We note that the requirement $\dot{H} < 0$ restricts our attention to the regions of phase space in which $\chi_2 < 0$ for the examples given in Secs. 10.2.1 and 10.2.1.

where

$$\mathcal{G}_T = -(2+\kappa)\frac{\chi_1+\chi_2}{4\dot{H}}\,, \tag{10.2.33}$$

and the coefficients $\mathcal{K}_T$ and $\mathcal{M}_T$ will be specified shortly. We pause here to note that, in general, since the second-order action in the tensor sector has nondegenerate higher-derivative terms ($\propto \ddot{h}^2$), there appear to be ghost degrees of freedom according to Ostrogradski's theorem. The only way to avoid those Ostrogradski ghosts would be if $\mathcal{G}_T$ were to vanish identically ($\mathcal{G}_T \equiv 0$). We see that this is not possible for a generic real constant κ, but if one sets $\kappa = -2$, then the model is safe with regards to Ostrogradski instabilities. The original models of Refs. [143, 516] did not include $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ in their curvature invariants at all, so they had $\kappa = 0$. Accordingly, the above implies that these models are inherently unstable. Yet, the addition of the Weyl tensor squared in the invariants with a specific prefactor ($\kappa = -2$) avoids this conclusion while having no effect on the background evolution. Still, it does not mean that the theory is necessarily free of all types of instabilities as we will see shortly.

The other coefficients of Eq. (10.2.32) are given by

$$\mathcal{K}_T = \frac{1}{2}(1+\chi_1) - \left(\frac{1}{2} + \frac{H^2}{\dot{H}} + \frac{H\ddot{H}}{\dot{H}^2} - \frac{H}{\dot{H}}\frac{\mathrm{d}}{\mathrm{d}t}\right)(\chi_1+\chi_2)\,, \tag{10.2.34}$$

$$\mathcal{M}_T = \frac{1}{2}(1+\chi_1) - \left(\frac{1}{2} + \frac{H^2}{\dot{H}} - 2\frac{\ddot{H}^2}{\dot{H}^3} + \frac{\dddot{H}}{\dot{H}^2} + 2\frac{\ddot{H}}{\dot{H}^2}\frac{\mathrm{d}}{\mathrm{d}t} - \frac{1}{\dot{H}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right)(\chi_1+\chi_2)\,. \tag{10.2.35}$$

The criteria to avoid ghost and gradient instabilities are $\mathcal{K}_T > 0$ and $\mathcal{M}_T > 0$, respectively. By using the background EOMs (see Eq. (10.2.14), which can be rewritten as $H = \sqrt{V_1'/12}$, Eq. (10.2.15), and Eq. (10.2.16) in the case for vacuum with $\varepsilon = 0$), the conditions can be written solely in terms of the fields χ_1, χ_2 and their potentials $V_1(\chi_1)$, $V_2(\chi_2)$ as

$$\begin{aligned}
F_1 :=& \frac{V_2'^2}{V_1''} + \left[(\chi_1+\chi_2) - \frac{1}{4}(\chi_1+1) + \frac{1}{2}\frac{V_1+V_2}{V_1'}\right](V_2' - (\chi_1+\chi_2)V_2'') \\
&+ \frac{1}{4}(\chi_1+\chi_2)^2 V_2'' + \frac{1}{4}\frac{V_2'^2}{V_1'}(1-\chi_2) > 0\,, \qquad (10.2.36)\\
F_2 :=& -\frac{4\,(V_1+V_2)^{\,2}\,(\chi_1+\chi_2)\,(V_2'')^{\,2}}{V_2^3 V_1'} - \frac{(\chi_1+\chi_2)\,(2\chi_1+3\chi_2-1)^{\,2}V_1'\,(V_2'')^{\,2}}{V_2^3} \\
&+ \frac{(2\chi_1+3\chi_2-1)\,(7\chi_1+9\chi_2-2)\,V_1'V_2''}{2V_2^2} + \frac{2\,(6\chi_1+7\chi_2-1)\,V_1'V_2''}{V_2V_1''} \\
&+ \frac{2V_2{}^{(3)}\,(V_1+V_2)^{\,2}\,(\chi_1+\chi_2)}{V_2^2V_1'} - \frac{(5\chi_1+8\chi_2-3)\,V_1'}{2V_2} \\
&+ \frac{V_2{}^{(3)}\,(\chi_1+\chi_2)\,(2\chi_1+3\chi_2-1)^{\,2}V_1'}{2V_2^2} \\
&- \frac{4\,(V_1+V_2)\,[2\chi_1^2+(5\chi_2-1)\,\chi_1+\chi_2\,(3\chi_2-1)]\,(V_2'')^{\,2}}{V_2^3} \\
&+ \frac{(V_1+V_2)\,(11\chi_1+15\chi_2-4)\,V_2''}{V_2^2} + \frac{2\,[2\chi_1^2+(5\chi_2-1)\,\chi_1+\chi_2\,(3\chi_2-1)]\,V_2''}{V_2} \\
&+ \frac{2V_2{}^{(3)}\,(V_1+V_2)\,[2\chi_1^2+(5\chi_2-1)\,\chi_1+\chi_2\,(3\chi_2-1)]}{V_2^2} \\
&+ \frac{4\,(V_1+V_2)^{\,2}V_2''}{V_2^2V_1'} + \frac{8V_2V_1{}^{(3)}V_1'}{(V_1'')^{\,3}} - \frac{8V_1'}{V_1''} + \frac{4\,(V_1+V_2)\,V_2''}{V_2V_1''} - \frac{4V_2}{V_1''} - \frac{3\,(V_1+V_2)}{V_2} \\
&+ \frac{1}{2}\,(-8\chi_1-13\chi_2+5) > 0\,. \qquad (10.2.37)
\end{aligned}$$

These conditions will be tested for the different models of Secs. 10.2.1 and 10.2.1 shortly.

Vector modes

We shall consider vector fluctuations in the following gauge, where

$$g^{(1)}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\mathrm{d}t^2 + 2a\beta_i\mathrm{d}t\mathrm{d}x^i + a^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\,. \qquad (10.2.38)$$

Here, the vector perturbations β^i satisfy $\partial_i\beta^i = 0$. The Fourier components of β_i are then defined by

$$\beta_i = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\sum_{I=1,2}\left[\beta_I(\vec{k})e_i^I(\vec{k})\right]\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}\,, \qquad (10.2.39)$$

where $\{e_i^1, e_i^2\}$ are orthogonal spatial vectors perpendicular to $\vec{k}$. The second-order action for vector perturbations becomes

$$S_V^{(2)} = \frac{M_{\rm Pl}^2}{2} \int \mathrm{d}t \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \sum_{I=1,2} a^3 \frac{k^2}{a^2} \left[\mathcal{G}_V \left(\dot{\beta}_I^2 - \frac{k^2}{a^2}\beta_I^2\right) + \mathcal{K}_V \beta_I^2\right], \tag{10.2.40}$$

where $\mathcal{G}_V = \mathcal{G}_T$ and $\mathcal{K}_V = \mathcal{K}_T$. Accordingly, when $\kappa = -2$, which sets $\mathcal{G}_T = 0$ to avoid Ostrogradski ghosts, it turns out that $\mathcal{G}_V = 0$ as well, and as a result, there are no dynamical vector modes.

Scalar modes

We shall then focus on the scalar fluctuations in the spatially flat gauge, where $\chi_1 = \chi_1^{(0)} + \delta\chi_1$, $\chi_2 = \chi_2^{(0)} + \delta\chi_2$, and

$$g_{\mu\nu}^{(1)} \mathrm{d}x^\mu \mathrm{d}x^\nu = -(1+2\Phi)\mathrm{d}t^2 + 2a\partial_i B \mathrm{d}t\mathrm{d}x^i + a^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\,. \tag{10.2.41}$$

The second-order action for scalar modes is then given by

$$S_S^{(2)} = \frac{M_{\rm Pl}^2}{2} \int \mathrm{d}t \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\; a^3 \left[\frac{4}{3}\frac{k^4}{a^4}\mathcal{G}_S(\Phi + a\dot{B})^2 + 2\left(\frac{k^2}{a^2}aB - 3H\Phi\right)\delta\dot{\chi}_2 + M_{IJ}\Psi^I\Psi^J\right], \tag{10.2.42}$$

where $\mathcal{G}_S = \mathcal{G}_T$ and[6] $\Psi^I := (\Phi, B, \delta\chi_1, \delta\chi_2)$. The matrix M_{IJ} is given by[7]

$$M_{IJ} = \begin{pmatrix} \frac{8H^2(\chi_1+\chi_2)}{\dot{H}}\frac{k^2}{a^2} + 6H\left[H\left(2\chi_1 + 3\chi_2 - 1\right) + \dot{\chi}_2\right] & * & * & * \\ -\frac{4H(\chi_1+\chi_2)}{3\dot{H}}\frac{k^4}{a^3} - \left(4\chi_1 H + 6\chi_2 H - 2H + \dot{\chi}_2\right)\frac{k^2}{a} & \frac{2k^4(2A_1+2\chi_1+3\chi_2-1)}{3a^2} & * & * \\ -6H^2 - \frac{V_1'}{2} & \frac{4Hk^2}{a} & -\frac{V_1''}{2} & 0 \\ -9H^2 - \frac{V_2'}{2} - \frac{k^2}{a^2} & \frac{3Hk^2}{a} & 0 & -\frac{V_2''}{2} \end{pmatrix}, \tag{10.2.43}$$

where $*$ stands for symmetric components. Since no time derivatives of Φ, B, and $\delta\chi_1$ appear in the second-order action with $\kappa = -2$, these modes are nondynamical. Then, these variables can be eliminated by their equations of motion. After removing the nondynamical

[6] We omit the subscript $\vec{k}$ from the perturbation variables Ψ^I when it is clear that they represent the Fourier components.

[7] As before, we omit the superscript (0) for χ_1 and χ_2 when it is understood that they represent background quantities.

modes, the resulting action can be written solely in term of $\delta\chi_2$ as follows:

$$S_S^{(2)} = \frac{M_{\rm Pl}^2}{2}\int {\rm d}t\frac{{\rm d}^3\vec{k}}{(2\pi)^3}\; a^3\left[\mathcal{K}_S(\delta\dot{\chi}_2)^2 - \mathcal{M}_S(\delta\chi_2)^2\right]\,, \tag{10.2.44}$$

where

$$\begin{aligned}\mathcal{K}_S &= 12\mathcal{K}_T\Big[\Big(\frac{4}{3}\frac{k^2}{a^2}\frac{(\chi_1+\chi_2)}{\dot{H}} + \frac{\dot{\chi}_2}{H}\Big)^2 \\ &\qquad - 8\mathcal{K}_T\Big(\frac{4}{3}\frac{k^2}{a^2}\frac{(\chi_1+\chi_2)}{\dot{H}} + \frac{2H\dot{\chi}_1}{\dot{H}} + \frac{\dot{\chi}_2}{H} + 2\chi_1 + 3\chi_2 - 1\Big)\Big]^{-1}, \qquad (10.2.45)\\ \mathcal{M}_S &= \frac{\mathcal{K}_S^2}{\mathcal{K}_T^2}\left(C_8\frac{k^8}{a^8} + C_6\frac{k^6}{a^6} + C_4\frac{k^4}{a^4} + C_2\frac{k^2}{a^2} + C_0\right)\,, \qquad (10.2.46)\end{aligned}$$

with

$$C_8 = 192H^2\dot{H}^2(\chi_1+\chi_2)^2\frac{H}{\dot{\chi}_2}\Big[4H\dot{H}\dot{\chi}_2(\chi_1+\chi_2) - 4H\ddot{H}(\chi_1+\chi_2)^2 - \dot{H}\dot{\chi}_1\dot{\chi}_2\Big]\,. \tag{10.2.47}$$

We do not write down the form of the other C_n coefficients because they are not so relevant in the following stability analysis.

In the small scale limit ($k/a \to \infty$), one finds

$$\begin{aligned}\mathcal{K}_S &\simeq A_S\mathcal{K}_T, \qquad (10.2.48)\\ \mathcal{M}_S &\simeq B_S\mathcal{K}_S\frac{H}{\dot{\chi}_2}\Big[4H\dot{H}\dot{\chi}_2(\chi_1+\chi_2) - 4H\ddot{H}(\chi_1+\chi_2)^2 - \dot{H}\dot{\chi}_1\dot{\chi}_2\Big]\,, \qquad (10.2.49)\end{aligned}$$

where A_S and B_S are simply two real positive constants. Thus, the ghost instability in the scalar sector is avoidable when it is absent in the tensor sector, i.e. when $\mathcal{K}_T > 0$ is satisfied. In addition, the gradient instability is absent when $\mathcal{M}_S > 0$, so when

$$F_3 := \frac{H}{\dot{\chi}_2}\Big[4H\dot{H}\dot{\chi}_2(\chi_1+\chi_2) - 4H\ddot{H}(\chi_1+\chi_2)^2 - \dot{H}\dot{\chi}_1\dot{\chi}_2\Big] > 0\,. \tag{10.2.50}$$

By using the vacuum background EOMs, this condition can be rewritten as

$$F_3 = (\chi_1+\chi_2)[(\chi_1+\chi_2)V_2'' - V_2'] - \frac{(V_2')^2}{V_1''} > 0\,. \tag{10.2.51}$$

Regarding gradient instabilities which can occur on sub-Hubble scales ($k/a \gg H$), the

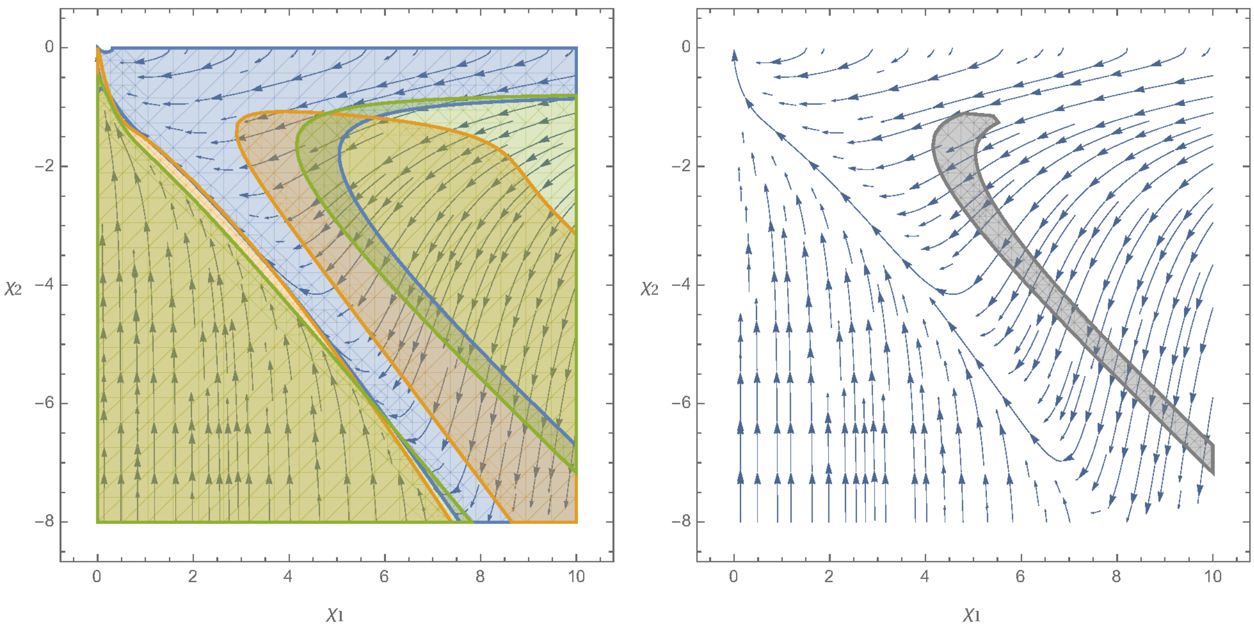

Figure 10.5 Phase-space diagram of the inflationary model of Sec. 10.2.1. On top of the phase-space trajectories, we show the regions that are stable according to the conditions that were derived in the text. In particular, in the left-hand plot, the blue, orange, and green regions show where the conditions $F_1 > 0$ [Eq. (10.2.36)], $F_2 > 0$ [Eq. (10.2.37)], and $F_3 > 0$ [Eq. (10.2.51)], respectively, are satisfied. The gray shaded area in the right-hand plot shows where all three conditions are met at the same time.

general procedure is to check for the stability of modes within the validity range of the EFT, i.e. for $H \ll k/a \leq \Lambda_{\mathrm{cut}}$ (see for instance Ref. [404]), where for our models $H \leq H_{\mathrm{max}} \leq \Lambda_{\mathrm{cut}}$. The condition given by Eq. (10.2.51) can be viewed as the one which ensures that the shortest wavelength modes ($k/a \sim \Lambda_{\mathrm{cut}}$) do not suffer from gradient instabilities. However, since the perturbed action exhibits a modified dispersion relation, i.e., since Eq. (10.2.46) has terms of order k^2, k^4, k^6, and k^8, longer wavelength modes (longer than $\Lambda_{\mathrm{cut}}^{-1}$, but still smaller than H^{-1}) could still suffer from gradient instabilities. As long as the duration for such gradient instabilities is not too long though, their amplification remains controllable in comparison to the smaller wavelength modes which easily blow up (within a time scale of the order of $\Lambda_{\mathrm{cut}}^{-1}$). This is why we only focus on the stability of the shortest wavelength modes.

Stability analysis

In summary, with $\kappa = -2$, we saw that there is no Ostrogradski instability. Then, we derived three conditions given by Eq. (10.2.36), which determines when the model is free of ghost instabilities in the tensor and scalar sectors, and Eqs. (10.2.37) and (10.2.51), which determine when the model is free of gradient instabilities in the tensor and scalar sector, respectively. The conditions depend on the potentials $V_1(\chi_1)$ and $V_2(\chi_2)$ and on the field values χ_1 and χ_2, so we need to study specific models to comment on the stability of the given theory.

Starting with the inflationary model of Sec. 10.2.1, where the potentials are given by Eqs. (10.2.20) and (10.2.21), we plot the regions of phase space that satisfy the three conditions in Fig. 10.5. The individual conditions are shown in the left-hand plot, and we see that there are large regions of phase space that can avoid ghost or gradient instabilities in the tensor or scalar sectors. However, when we look at the right-hand plot, which shows the combined region where all stability conditions are met, we see that there is, actually, only a very small region of phase space that is not unstable. In particular, the trajectories that correspond to asymptotically de Sitter and Minkowski (e.g., the green curve in Fig. 10.2) are generally unstable throughout their evolution, except in a very small region of phase space.

We show the same types of plots in Fig. 10.6 for the genesis scenario of Sec. 10.2.1. There as well, there are large regions of phase space that can avoid ghost and gradient instabilities in the tensor or scalar sectors, but it remains that only small portions of those can be stable with regards to all types of instabilities at the same time. In particular, the interesting

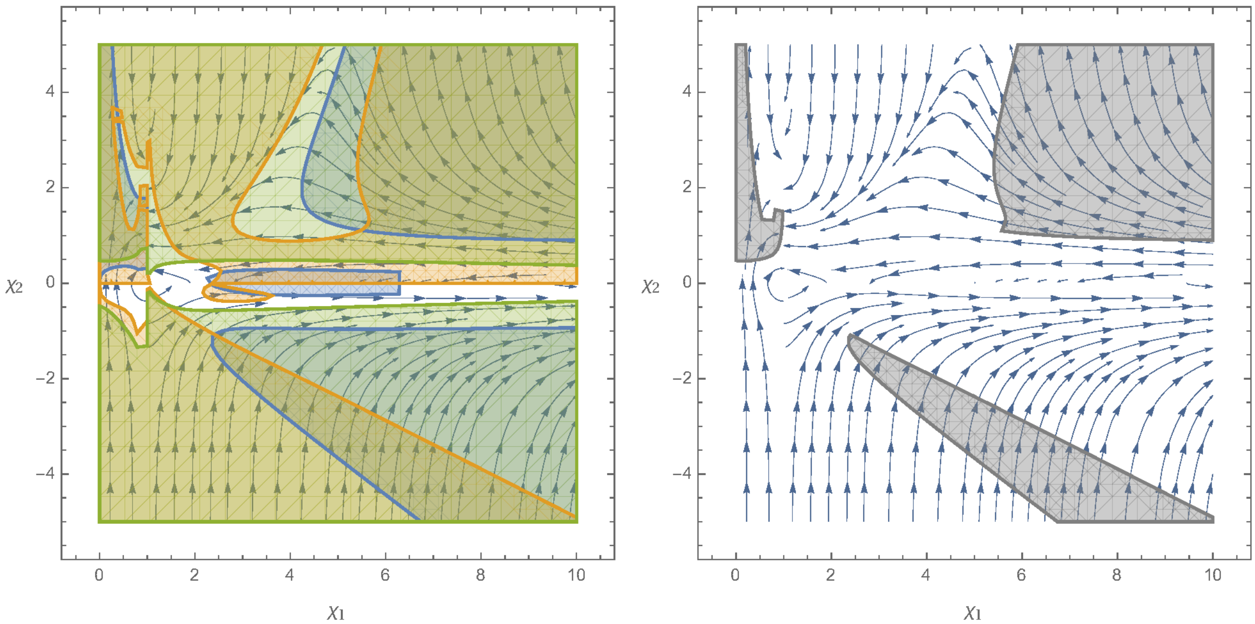

Figure 10.6 Phase-space diagram of the genesis model of Sec. 10.2.1. The convention used to illustrate the stable regions is described in Fig. 10.5.

trajectories (e.g., the black and purple curves of Fig. 10.4) are unstable throughout their evolution.

10.2.3 Equivalence with $f(R, \mathcal{G})$ gravity

We have shown that the Weyl square term with $\kappa = -2$ kills the dangerous degrees of freedom and can relax the instability around a flat FLRW background spacetime. This result can be understood because this theory is included by $f(R, \mathcal{G})$ gravity, whose cosmological solutions are perturbatively stable. However, it is shown by Ref. [248] that the perturbations of flat FLRW spacetime in $f(R, \mathcal{G})$ gravity lose a degree of freedom, which reappears as a ghost around anisotropic spacetimes. Therefore, our theory necessarily suffers from the same instability.

Let us demonstrate the equivalence between the above model and $f(R, \mathcal{G})$ gravity. The Gauss-Bonnet term $\mathcal{G}$ is defined by

$$\mathcal{G} := R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \,. \tag{10.2.52}$$

Plugging the relation

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + 2R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2 \tag{10.2.53}$$

into the definition of the Gauss-Bonnet term, we obtain

$$\mathcal{G} = \frac{2}{3}R^2 - 2R_{\mu\nu}R^{\mu\nu} + C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\,. \tag{10.2.54}$$

Then, we note that the second curvature-invariant function of Eq. (10.2.28) can be written as (with $\kappa = -2$)

$$I_2 = \sqrt{12R_{\mu\nu}R^{\mu\nu} - 3R^2 - 6C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}} = \sqrt{R^2 - 6\mathcal{G}}\,. \tag{10.2.55}$$

Accordingly, we see that I_2 and I_1 (which are given by $I_1 = I_2 + R = \sqrt{R^2 - 6\mathcal{G}} + R$) are functions of R and $\mathcal{G}$. Thus, the solutions of the EOMs for χ_1 and χ_2 can be written as

$$\chi_1 = \chi_1(R, \mathcal{G})\,, \qquad \chi_2 = \chi_2(R, \mathcal{G})\,, \tag{10.2.56}$$

and by plugging these solutions into Eq. (10.2.2), the original action becomes only a function of R and the Gauss-Bonnet term $\mathcal{G}$, i.e.

$$S = \int \mathrm{d}^4x\,\sqrt{-g}f(R, \mathcal{G})\,. \tag{10.2.57}$$

In conclusion, this theory is a specific model of $f(R, \mathcal{G})$ gravity.

10.3 New nonsingular model with a new curvature scalar

10.3.1 Setup

Let us investigate other curvature scalars, which reduce to H and $\dot{H}$ at the background level. In particular, we focus our attention on curvature scalars that are functions of the Ricci scalar and its derivatives.

Let us consider the following tensor constructed from the first derivative of R,

$$X^\mu{}_\nu := g^{\mu\rho}\nabla_\rho R\nabla_\nu R\,. \tag{10.3.58}$$

For a flat FLRW background, this quantity reduces to

$$X^{\mu}{}_{\nu}^{(0)} = -\left[6(4H\dot{H} + \ddot{H})\right]^2 \mathrm{diag}(1,0,0,0)\,. \tag{10.3.59}$$

The trace of the tensor $X^{\mu}{}_{\nu}$ is

$$X = X^{\mu}{}_{\mu} = \nabla_{\mu}R\nabla^{\mu}R\,, \tag{10.3.60}$$

and at the background level, it reduces to

$$X^{(0)} = -\left[6(4H\dot{H} + \ddot{H})\right]^2\,. \tag{10.3.61}$$

Since the second time derivative of the Hubble parameter appears here, we need to consider another curvature scalar to remove $\ddot{H}$. Thus, let us consider the second derivative of R,

$$\mathcal{R}^{\mu}{}_{\nu} := \nabla^{\mu}\nabla_{\nu}R\,, \tag{10.3.62}$$

whose expression in a FLRW spacetime reduces to

$$\mathcal{R}^{\mu}{}_{\nu}^{(0)} = -24H\mathrm{diag}\left[\frac{\dot{H}^2}{H} + \ddot{H} + \frac{\dddot{H}}{4H}, \Big(H\dot{H} + \frac{\ddot{H}}{4}\Big)\delta^{i}{}_{j}\right]\,. \tag{10.3.63}$$

Since $\dddot{H}$ appears only in the $(0,0)$ component, it can be removed by the tensor defined by

$$P^{\mu}{}_{\nu} := \delta^{\mu}{}_{\nu} - \frac{X^{\mu}{}_{\nu}}{X}\,, \tag{10.3.64}$$

whose FLRW limit now is $P^{\mu}{}_{\nu}^{(0)} = \mathrm{diag}(0,1,1,1)$. In fact, we can construct the scalar quantity $\mathrm{tr}[P\mathcal{R}P\mathcal{R}]$, which gives

$$\mathrm{tr}[P\mathcal{R}P\mathcal{R}]^{(0)} = -3XH^2\,, \tag{10.3.65}$$

at the background level. Therefore, the following quantity is a good candidate as a limiting curvature-invariant function,

$$
\begin{aligned}
I_1 :=& -12\frac{\mathrm{tr}[P\mathcal{R}P\mathcal{R}]}{3X} && (10.3.66)\\
=& -\frac{4}{X^3}\Big[X^2(\nabla_\mu\nabla_\nu R)(\nabla^\mu\nabla^\nu R) - 2X(\nabla^\mu R\nabla^\nu R)(\nabla_\mu\nabla_\rho R)(\nabla_\nu\nabla^\rho R)\\
&+(\nabla^\mu R\nabla^\nu R\nabla_\mu\nabla_\nu R)^2\Big] && (10.3.67)\\
=& -\frac{1}{X^3}\left[4X^2(\nabla\nabla R)^2 - 2X(\nabla X)^2 + (\nabla R\nabla X)^2\right] , && (10.3.68)
\end{aligned}
$$

which satisfies

$$I_1^{(0)} = 12H^2 \,, \tag{10.3.69}$$

as required. In order to bound $\dot{H}$, we introduce as before

$$I_2 := I_1 - R \,; \tag{10.3.70}$$

hence

$$I_2^{(0)} = -6\dot{H} \,, \tag{10.3.71}$$

again as required.

10.3.2 Cosmological perturbations and stability analysis

Tensor modes

By substituting the definition of tensor perturbations [Eq. (10.2.30)], the second-order action for tensor modes can be obtained as

$$S_T^{(2)} = \frac{M_{\mathrm{Pl}}^2}{2}\int \mathrm{d}t \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\sum_{I=+,\times} a^3\Big(\mathcal{K}_T\dot{h}^I_{\vec{k}}\dot{h}^I_{-\vec{k}} - \mathcal{M}_T\frac{k^2}{a^2}h^I_{\vec{k}}h^I_{-\vec{k}}\Big) \,, \tag{10.3.72}$$

where

$$\mathcal{K}_T = \frac{1+4\chi_1+3\chi_2}{2} \,, \tag{10.3.73}$$

$$\mathcal{M}_T = \frac{1-\chi_2}{2} \,. \tag{10.3.74}$$

Thus, the tensor sector does not include higher-derivative terms, so the number of degrees of freedom is 2, and no Ostrogradski instabilities appear. The stability conditions for ghost and gradient instabilities in the tensor sector are then given by

$$F_1 := 1 + 4\chi_1 + 3\chi_2 > 0\,, \tag{10.3.75}$$

$$F_2 := 1 - \chi_2 > 0\,. \tag{10.3.76}$$

Vector modes

Using Eq. (10.2.38), the second-order action for vector modes can be derived as

$$S_V^{(2)} = \frac{M_{\rm Pl}^2}{2} \int \mathrm{d}t \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} a^3 \sum_{I=1,2} \frac{k^2}{a^2} \mathcal{K}_V \beta_I^2, \tag{10.3.77}$$

with $\mathcal{K}_V = \mathcal{K}_T$. Therefore, no vector modes exist in this theory.

Scalar modes

Let us investigate the scalar perturbations defined in Eq. (10.2.41). Let us introduce a new perturbation variable φ in terms of Φ and B by the equation

$$\varphi_{\vec{k}} = \Phi_{\vec{k}} - \frac{k^2}{3aH} B_{\vec{k}}\,, \tag{10.3.78}$$

and we regard the components of $\Psi^I := (B, \delta\chi_1, \delta\chi_2, \varphi)$ as independent variables.

The second-order action for scalar modes can then be calculated as

$$S_S^{(2)} = \frac{M_{\rm Pl}^2}{2} \int \mathrm{d}t \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} a^3 \left[K\dot{\varphi}^2 + L_I \Psi^I \dot{\varphi} + M_{IJ} \Psi^I \Psi^J \right]\,, \tag{10.3.79}$$

where we omit writing down the specific form of the terms K, L_I, and M_{IJ}. Since the action does not include the time derivative of B, $\delta\chi_1$, and $\delta\chi_2$, we can remove these variables by the EOMs. The resulting action is given by

$$S_S^{(2)} = \frac{M_{\rm Pl}^2}{2} \int \mathrm{d}t \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\, a^3 \Big(\mathcal{K}_S \dot{\varphi}^2 - \mathcal{M}_S \varphi^2 \Big)\,, \tag{10.3.80}$$

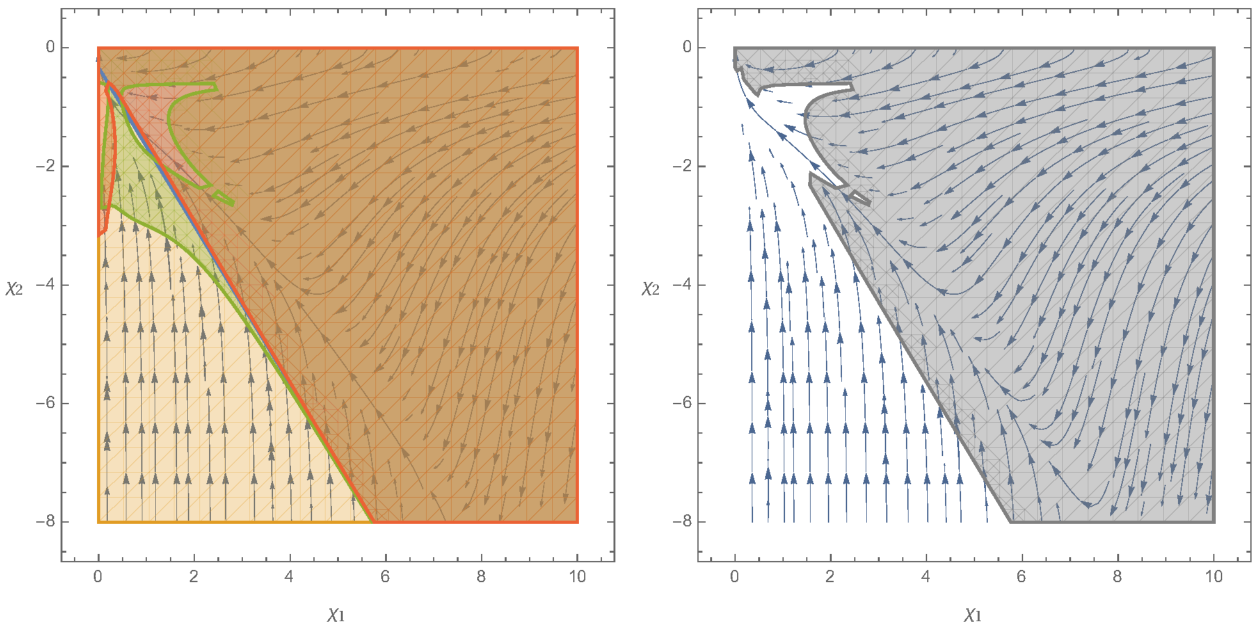

Figure 10.7 Phase-space diagram of the inflationary model of Sec. 10.2.1. On top of the phase-space trajectories, we show the regions that are stable according to the conditions that were derived in the text. In particular, in the left-hand plot, the blue, orange, green, and red regions show where the conditions $F_1 > 0$ [Eq. (10.3.75)], $F_2 > 0$ [Eq. (10.3.76)], $F_3 > 0$ [Eq. (10.3.83)], and $F_4 > 0$ [Eq. (10.3.84)], respectively, are satisfied. The gray shaded area in the right-hand plot shows where all four conditions are met at the same time.

where the coefficients, in the limit where $k/a \to \infty$, are given by

$$\mathcal{K}_S = \frac{a^4}{k^4} F_3 + \mathcal{O}\Big[\Big(\frac{k}{a}\Big)^{-6}\Big] , \tag{10.3.81}$$

$$\mathcal{M}_S = \mathcal{K}_S \frac{k^2}{a^2} F_4 + \mathcal{O}\Big[\Big(\frac{k}{a}\Big)^{0}\Big] , \tag{10.3.82}$$

with

$$\begin{aligned} F_3 = &- \frac{3V_1'}{8[3(\chi_1+\chi_2)V_1''V_2'' - 2V_1'(V_1''-V_2'')]} \Big\{ 2(4\chi_1+3\chi_2+1)(V_1')^2(V_1''-V_2'') \\ &+ 12(\chi_1+\chi_2)(V_2')^2 V_1'' + V_1' \Big[-16(\chi_1+\chi_2)V_2'V_1'' \\ &- \left[4\chi_1^2 + (5\chi_2+3)\chi_1 + \chi_2(\chi_2+3)\right] V_1''V_2'' + 8(V_2')^2 \Big] \Big\} , \end{aligned} \tag{10.3.83}$$

$$F_4 = - \frac{(\chi_2 - 1)\left[3(\chi_1+\chi_2)V_1''V_2'' + 2V_1'(V_1''+V_2'')\right]}{2(4\chi_1+3\chi_2+1)V_1'(V_1''+V_2'') + \left[4\chi_1^2 + (5\chi_2+3)\chi_1 + \chi_2(\chi_2+3)\right] V_1''V_2''} . \tag{10.3.84}$$

Stability analysis

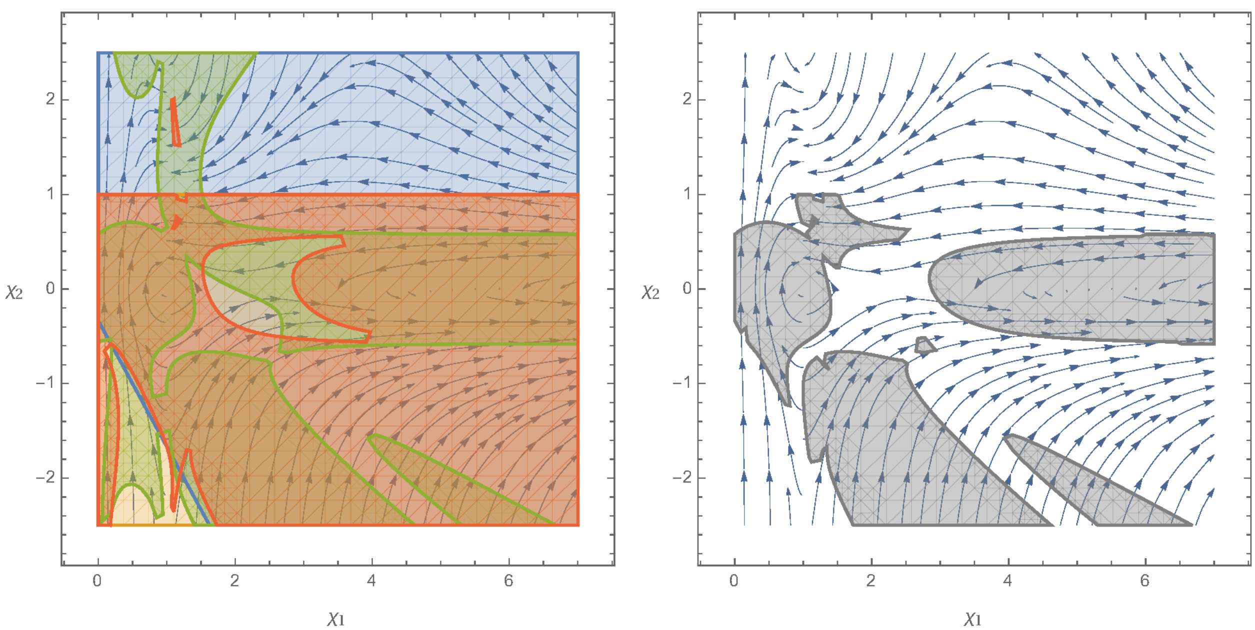

Figure 10.8 Phase-space diagram of the genesis model of Sec. 10.2.1. The convention used to illustrate the stable regions is described in Fig. 10.7.

In summary, we derived four conditions given by Eqs. (10.3.75)-(10.3.76) and Eqs. (10.3.83)-

(10.3.84), which determine when the model is free of ghost and gradient instabilities in the tensor and scalar sectors, respectively. Once again, the conditions depend on the potentials $V_1(\chi_1)$ and $V_2(\chi_2)$ and on the field values χ_1 and χ_2, so we need to study specific models to comment on the stability of the given theory.

Starting with the inflationary model of Sec. 10.2.1, we plot the regions of phase space that satisfy the four conditions in Fig. 10.7. Once again, there are large regions of phase space that can avoid ghost or gradient instabilities in the tensor or scalar sectors. The major difference with Fig. 10.5 is apparent in the right-hand plot of Fig. 10.7, which shows the combined region where all stability conditions are met. Indeed, whereas the previous theory had only a small strip of phase space that was stable, there is now a large portion of the phase space that is free of all types of instabilities. Furthermore, it is precisely in this region that we obtained the interesting background trajectories starting from de Sitter spacetime and ending in Minkowski spacetime (e.g., the green curve in Fig. 10.2). For this class of trajectories, there only seems to be a small region of phase space around $0 \lesssim \chi_1 \lesssim 2$ and $-0.5 \lesssim \chi_2 \lesssim -2.5$ where there still appears to be some instability. Yet, it seems clear that the curvature invariants given by Eqs. (10.3.66) and (10.3.70) lead to a much more stable theory compared to the curvature invariants of Eq. (10.2.28) that were analyzed in Sec. 10.2.2.

For the genesis model of Sec. 10.2.1, the stable phase-space regions are shown in Fig. 10.8. There as well, there now are larger regions of phase space that can avoid all types of instabilities. In particular, the interesting trajectories (e.g., the black and purple curves of Fig. 10.4) are *stable* throughout their evolution. This reinforces our conclusion that the limiting curvature theory with curvature invariants (10.3.66) and (10.3.70) is more stable and leads to interesting nonsingular cosmological scenarios that can remain stable throughout their evolution.

10.3.3 Stability around an anisotropic background

At this point, the new curvature-invariant function leads to generally stable nonsingular solutions around isotropic backgrounds. Still, one may worry that this new theory might still possess undesired ghosts. Indeed, it appears that the new curvature scalar of this section is not included in the ghost free theories found in Ref. [519], which can be further mapped into multifield extensions of Horndeski theories [399, 531, 536].

One situation where new ghosts may appear is around anisotropic backgrounds. In the model of Sec. 10.2, which we showed to be equivalent to a $f(R,\mathcal{G})$ theory of gravity, a degree of freedom is lost around a FLRW background and such a mode reappears as a ghost once the background is deformed to an anisotropic Bianchi type I universe. The purpose of this subsection is to investigate the stability of our new theory against such a nonperturbative deformation of the background spacetime. We shall investigate the perturbations around a Bianchi type I universe with the rotational symmetry in the $y-z$ plane, so the background metric is given by

$$g^{(0)}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\mathrm{d}t^2 + a(t)^2\left[\mathrm{e}^{-4\delta(t)}\mathrm{d}x^2 + \mathrm{e}^{2\delta(t)}(\mathrm{d}y^2+\mathrm{d}z^2)\right]. \tag{10.3.85}$$

At the background level, the curvature scalar invariants now include shear terms,

$$I_1^{(0)} = 12H^2 + 24\sigma^2\,, \tag{10.3.86}$$

$$I_2^{(0)} = -6\dot{H} + 18\sigma^2\,, \tag{10.3.87}$$

where the shear σ is defined by

$$\sigma(t) := \dot{\delta}(t)\,, \tag{10.3.88}$$

and so, the background dynamics is different from that analyzed in Sec. 10.2.1.

We focus on whether or not additional degrees of freedom of perturbations appear independently of the background dynamics. One would usually start with the action given by Eq. (10.2.2) with curvature invariants given by Eqs. (10.3.66) and (10.3.70) and perturb the action according to the scalar-vector-tensor decomposition of the perturbed metric given by Eqs. (10.2.30), (10.2.38), and (10.2.41). Following the methodology of [622] for perturbations around an anisotropic background, thanks to the axisymmetry of the background spacetime given by Eq. (10.3.85), the linear perturbations can actually be decomposed into scalar and vector modes only, with respect to rotations of the $y-z$ plane,[8]

$$\delta g_{\mu\nu} = \delta g^{\rm scalar}_{\mu\nu} + \delta g^{\rm vector}_{\mu\nu} \,. \tag{10.3.89}$$

The Fourier components of $\delta g^{\rm scalar}_{\mu\nu}$ and $\delta g^{\rm vector}_{\mu\nu}$ are given by

$$\delta g^{\rm scalar}_{\mu\nu,\vec{k}} = \begin{pmatrix} -2\Phi_{\vec{k}} & * & * & 0 \\ a(\mathrm{i}k_x B_{\vec{k}} - \mathrm{e}^{-4\delta}\frac{k_y}{k}\beta_{1,\vec{k}}) & a^2\mathrm{e}^{-6\delta}\frac{k_y^2}{k^2}h_{+,\vec{k}} & * & 0 \\ a(\mathrm{i}k_y B_{\vec{k}} + \mathrm{e}^{2\delta}\frac{k_x}{k}\beta_{1,\vec{k}}) & -a^2\frac{k_x k_y}{k^2}h_{+,\vec{k}} & a^2\mathrm{e}^{6\delta}\frac{k_x^2}{k^2}h_{+,\vec{k}} & 0 \\ 0 & 0 & 0 & -a^2\mathrm{e}^{2\delta}h_{+,\vec{k}} \end{pmatrix} , \tag{10.3.90}$$

$$\delta g^{\rm vector}_{\mu\nu,\vec{k}} = \begin{pmatrix} 0 & 0 & 0 & a\mathrm{e}^{\delta}\beta_{2,\vec{k}} \\ 0 & 0 & 0 & -a^2\mathrm{e}^{-4\delta}\frac{k_y}{k}h_{\times,\vec{k}} \\ 0 & 0 & 0 & a^2\mathrm{e}^{2\delta}\frac{k_x}{k}h_{\times,\vec{k}} \\ * & * & 0 & 0 \end{pmatrix} , \tag{10.3.91}$$

where k is defined by

$$k^2 := \mathrm{e}^{4\delta}k_x^2 + \mathrm{e}^{-2\delta}k_y^2. \tag{10.3.92}$$

Here we have fixed the arbitrariness of rotations in the $y-z$ plane so that $\vec{k} = (k_x, k_y, 0)$. We have also fixed the gauge degrees of freedom of perturbations. Our gauge choice corresponds to the spatially flat gauge [defined by Eqs. (10.2.30), (10.2.38), and (10.2.41)] in the limit where $\delta \to 0$ because the perturbed metric reduces to

$$\lim_{\delta\to 0}\delta g_{\mu\nu,\vec{k}} = U^T(\vec{k})\Delta(\vec{k})U(\vec{k}) \,, \tag{10.3.93}$$

[8] Since there is no spherical symmetry in an anisotropic background, the usual (three-dimensional) scalar-vector-tensor modes cannot develop independently. However, thanks to one rotational symmetry, the one in the $y-z$ plane, a (two-dimensional) scalar-vector decomposition works well. For example, the x component of a three-dimensional vector is just a scalar, and the y and z components can be decomposed into one scalar (gradient) mode and one vector (transverse) mode. The reason for the absence of (two-dimensional) tensor modes is just that 2×2 symmetric tensors can be written as two scalar modes and one vector mode only. In this sense, the ten components of the metric tensor can be decomposed into seven scalar modes and three vector modes generally. Since three scalar modes and one vector mode can be killed by gauge symmetry, we have four scalar modes and two vector modes in the metric tensor.

with

$$\Delta(\vec{k}) = \begin{pmatrix} -2\Phi_{\vec{k}} & a\mathrm{i}kB_{\vec{k}} & a\beta_1 & a\beta_2 \\ * & 0 & 0 & 0 \\ * & * & a^2h_+ & a^2h_\times \\ * & * & * & -a^2h_+ \end{pmatrix} . \tag{10.3.94}$$

The rotation matrix $U(\vec{k})$ is defined by

$$U(\vec{k}) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{k_x}{k} & \frac{k_y}{k} & 0 \\ 0 & -\frac{k_y}{k} & \frac{k_x}{k} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{10.3.95}$$

and it transforms the vector $(0, k_x, k_y, 0)$ into $(0, k, 0, 0)$.

Since the time derivative of $B_{\vec{k}}$ only appears through the following combination in the second-order action,

$$\varphi_{\vec{k}} := \Phi_{\vec{k}} - \frac{\mathrm{e}^{4\delta}k_x^2 + \mathrm{e}^{-2\delta}k_y^2}{3aH} B_{\vec{k}} \, , \tag{10.3.96}$$

it is useful to regard φ as a dynamical variable instead of Φ, analogous to Eq. (10.3.78). Thus, we now have six scalar mode perturbations (φ, B, β_1, h_+, $\delta\chi_1$, and $\delta\chi_2$), and two vector mode perturbations (β_2 and $h_\times$). By a straightforward calculation, one can show that the second-order action does not include any time derivatives of B, β_1, $\delta\chi_1$, $\delta\chi_2$, and β_2 after integration by parts. Thus, these variables are nondynamical and the remaining dynamical degrees of freedom are φ and h_+, which are scalar modes, and $h_\times$, which is a vector mode. This result shows that no additional degrees of freedom appear at least in this anisotropic background. Therefore, our new model is not disturbed by the deformation of the background spacetime given by Eq. (10.3.85). This is a crucial difference compared to the model of Sec. 10.2 or $f(R, \mathcal{G})$ gravity.

10.3.4 Recovering Einstein gravity and the addition of matter sources

At this point, it looks like the action (10.2.2) with curvature invariants (10.3.66) and (10.3.70) can lead to interesting nonsingular cosmological background models such as an inflationary model and a genesis model that would remain stable against all types of instabilities throughout most of their evolution. These models start in de Sitter or Minkowski spacetime, respectively, and both end up in Minkowski spacetime. To be viable structure formation scenarios for the very early universe, as we already pointed out, one needs a reheating mechanism that would produce radiation and matter in a sufficient amount after the universe has acquired its adiabatic, scale-invariant curvature perturbation power spectrum. At this point, let us suppose that such a reheating mechanism exists and that it produces matter and radiation in large amounts. Then, at the background level, one is left with the nonsingular theory described in Sec. 10.2.1 with nonzero energy density, and possibly, nonzero pressure as well. For the theory to successfully describe our universe, it is then necessary to recover the usual Einstein equations, i.e. the Friedmann equations in our context.

For the inflationary scenario, this is not a problem. Once matter is included, and as χ_1 and χ_2 (and their time derivatives) go to 0, one notes from Eqs. (10.2.20) and (10.2.21) that $V_1 \to 0$ and $V_2 \to 0$. Thus, we see that Eqs. (10.2.12) and (10.2.13) reduce to the Friedmann equations and can lead to the expected radiation- and matter-dominated era of our universe.

In the context of the genesis scenario, one runs into difficulty though. Once reheating has occurred and matter is included, one remains in the regime $\chi_2 \ll 1$, but $\chi_1 \to \infty$. From Eqs. (10.2.24) and (10.2.25), this still implies $V_1 \to 0$ and $V_2 \to 0$. Then, simply at the level of the action (10.2.2), one notices that recovering the Hilbert-Einstein term alone is only possible if $\chi_1 I_1 \to 0$, which is to say that I_1 vanishes faster than $\chi_1 \to \infty$. However, since $I_1 \sim \mathcal{O}(R)$, it is not possible to have a nonzero Hilbert-Einstein term while $I_1 \to 0$. One can also see that, for Eq. (10.2.12) to be valid as $\chi_1 \to \infty$, the only possibility is that H and ε vanish faster than $\chi_1 \to \infty$. Once again, this implies an empty Minkowski spacetime rather than a FLRW spacetime. Accordingly, it seems impossible, in the context of this genesis scenario, to have the higher-derivative terms from the curvature invariants vanish, i.e. to recover the Einstein equations, and be left with a nonempty Friedmann universe. Therefore, the genesis scenario within this theory remains at the level of a toy model.

It should be noted that nontrivial couplings between matter fields and χ_1 or χ_2 may relax this problem [266]. In this case, the bounds on the curvature are, however, weakened because I_1 and I_2 include matter fields. Such matter couplings must be subdominant at high energy scales in order to ensure the avoidance of curvature singularity, but they must be dominant at low energy scales in order to recover Einstein gravity.

10.4 Conclusions and discussion

In this paper, we revisited the nonsingular cosmologies of Refs. [143, 516], which implement the limiting curvature hypothesis. We extended the analysis beyond the background cosmology to include the linear cosmological perturbations and determined the criteria for stability. This showed that the original models of Refs. [143, 516] appear to have, generically, undesired additional degrees of freedom leading to Ostrogradski instabilities. These instabilities could be killed with the addition of the Weyl tensor squared in the curvature-invariant functions given the appropriate coefficient. Still, by exploring two nonsingular cosmological scenarios in which the limiting curvature hypothesis is realized (one inflationary and one genesis scenario), it appeared that the cosmologies inevitably possess either ghost or gradient instabilities through large portions of their evolution. Furthermore, we showed that the theory could be rewritten as a $f(R,\mathcal{G})$ theory of gravity, which is known to suffer from instabilities in anisotropic backgrounds.

We then constructed a new curvature-invariant function by taking a specific combination of covariant derivatives of the Ricci scalar. Given the same inflationary and genesis scenarios at the background level as before, we showed that the new curvature scalar could lead to stable cosmologies with respect to Ostrogradski ghosts, as well as ghost and gradient instabilities throughout most of their evolution. Furthermore, the theory does not possess additional new degrees of freedom around anisotropic backgrounds, contrary to $f(R,\mathcal{G})$ gravity.

In light of constructing a nonsingular theory for the very early universe, there remain some challenges though. If one starts in a vacuum universe (either de Sitter or Minkowski), one would need to provide some form of reheating mechanism, possibly via gravitational particle production, so that the universe can contain matter and radiation after the early epoch. Furthermore, one would need to assure that the theory reduces to the Einstein limit for gravity fast enough. This appears to be satisfied in our inflationary scenario, but it remains an issue in the genesis scenario. Finally, given a successful scenario at the background level and which is stable perturbatively, it would be straightforward to solve the perturbation equations to find the power spectra of these perturbations and compare with observations to validate the theory.

The analysis performed in this paper opens the window to construct and study other nonsingular cosmologies, e.g., bouncing cosmologies. As mentioned before, this has been explored in Ref. [197] (see also Refs. [84, 417]). However, other nonsingular models with limiting curvature [196, 197] could also suffer from instabilities as suggested by Ref. [397]. This might be due to the fact that Refs. [196, 197] implement the limiting curvature hypothesis within a mimetic theory [194, 195, 198, 580], whose stability (or instability) does not appear to have been settled yet (see, e.g., Refs. [53, 70, 193, 288, 564, 580]).

Finally, it would be interesting to study how the approach to implement the limiting curvature hypothesis used in this paper fits in the grand picture of general scalar-tensor theories of gravity (e.g., [37, 69, 70, 230–232, 245, 299, 300, 321, 322, 418, 420, 421, 453, 458, 510, 575]). In particular, it would be interesting to find general classes of curvature-invariant functions in which the limiting curvature hypothesis can be realized and where solutions are stable.

Chapter 11

Conclusions

Let us end this thesis with a short summary of the main results, i.e., what are the most important contributions presented in this thesis.

In Chapters 5 and 6, we showed that matter bounce cosmology suffers from a no-go theorem preventing it from matching with observations: the tensor-to-scalar ratio is naturally large, and when trying to suppress it by enhancing curvature perturbations, large scalar non-Gaussianities are produced. In Chapter 7, we saw that allowing for a non-zero graviton mass may resolve the no-go theorem. Indeed, by only changing the tensor sector, no additional scalar non-Gaussianities are produced and the primordial gravitational wave power spectrum becomes highly suppressed on observationally-relevant scales.

In Chapter 8, we studied the evolution of cosmological perturbations in a contracting universe dominated by a generic hydrodynamical fluid. We showed that the smaller the fluid's sound speed, the more likely black holes have formed at a given time, or put differently, black holes are most likely to form earlier on. Also, the results indicate that the largest black holes, those with a Hubble-size radius, are the first to form. Therefore, for the matter bounce scenario with a dust-like fluid, black hole formation is inevitable and may represent an issue. Alternatively, black holes may constitute a signature or key feature of other scenarios. In particular, in Chapter 9 we revisited the concept of string holes, i.e., black holes lying at the string scales. We studied how such a state of matter in the universe would behave and explored how it could be embedded in string theory. We found that a low-energy effective action of string theory can support the cosmological evolution of a gas of stringy black holes, as long as appropriate α' corrections are included.

In Chapter 10, we revisited the concept of limiting curvature as an approach to build an effective action for gravity that yields non-singular spacetime solutions, in particular non-singular cosmological background solutions. We studied the behavior of scalar, vector and tensor perturbations, characterizing the presence of Ostrogradski, ghost and gradient instabilities. We found that it is generally difficult to avoid all types of instabilities throughout cosmic evolution in a non-singular cosmology. We managed to construct curvature invariants that allow for more stable solutions, though at the expense of complicated functions. Finally, in Appendix A we studied the past extendibility of asymptotically de Sitter spacetimes. We derived necessary and sufficient conditions to ensure extendibility beyond a given boundary in a flat, exponentially-fast expanding FLRW background in the infinite past. In addition to toys models, the conditions were applied to physical inflationary proposals for the very early universe. For instance, small field inflation could be continuously extended to the infinite past, but such models remain unstable against initial condition fluctuations. As another example, Starobinsky inflation turned out to be strongly inextendible to the infinite past.

11.1 Future directions

While future directions were mentioned at the end of every publication included in this thesis, we gather here what we believe are truly the key big questions that remain unanswered and what should deserve to be tackled next.

Matter bounce cosmology This thesis severely constrained the matter bounce model. As already mentioned, the few surviving models include the matter bounce curvaton scenario (a two-field model, which loses the simplicity of the adiabatic, single-field model) and the original matter bounce scenario with massive gravity (which requires a large graviton mass only entering at the perturbative level). We note that there might be another possibility to solve both the BKL instability problem and the large tensor-to-scalar ratio issue. It has recently been shown that a negative anisotropic stress tensor can isotropize a bouncing universe [298]. In the context of a fluid, a non-zero shear viscosity could provide such an anisotropic stress, and thus isotropize the universe. Furthermore, shear viscosity would affect the propagation of gravitational waves (see, e.g., Ref. [328]) by adding damping. Consequently, shear viscosity could very well suppress the tensor-to-scalar ratio. It remains to be shown whether the magnitude of the effects (the isotropization and the damping of the gravitational waves) is quantitatively sufficient to solve the problems of the matter bounce scenario. Also, a microphysical explanation for the origin of the viscosity would be needed, although there are arguments and conjectures stating that there exists a generic lower bound on viscosity (see, e.g., Refs. [203, 293, 588]).

Black holes in bouncing cosmology Chapter 9 presented novel ideas where black holes played an interesting role in a possibly new (stringy) scenario of pre-Big Bang cosmology. However, the investigation remains at an early stage. In fact, in Sec. 9.5 we already presented many future directions to improve the analysis and bring it further. Let us add that more insight may be gained by revisiting certain issues in a different context. In fact, dealing with black holes in cosmology in a rigorous way is a difficult task. When several black holes are in the picture, it is practically impossible to mathematically solve the Einstein field equations analytically. Therefore, simpler but nevertheless enlightening problems could be tackled to progress in that direction. For instance, there are studies of dynamical black holes embedded in slowly-rolling scalar field cosmology (see, e.g., Refs. [331, 332]). In the context of a contracting universe, this may be a tractable problem, allowing us to understand what happens to black holes of cosmological size.

Non-singular cosmology The theories of limiting curvature explored in this thesis did not perform so well in terms of avoiding instabilities. As explained also in the introductory chapters, it is often quite difficult to obtain fully stable non-singular cosmologies. While it is achievable within quartic Horndeski theory or beyond-Horndeski theories (see references in Chapter 4), there remain many open questions with regard to general (possibly non-perturbative) stability (see, e.g., Refs. [362, 630] for first steps forward in that direction), strong coupling (see, e.g., Refs. [257, 571]), and more. It is therefore interesting to ask whether it could possible to construct new models of non-singular cosmology using the wealth of modified gravity models explored in the literature, trying to seek for simplicity and symmetry. A couple of ideas in that direction include the cuscuton [15–17, 75, 91, 325, 372] and a new class of minimally-modified theories of gravity [30, 183, 454, 457]. The cuscuton is simply a sub-case of k-essence scalar field, but it is actually a very peculiar model as its scalar degree of freedom does not propagate [15, 17, 246, 372] and the theory has infinitely many symmetries [538]. Moreover, the cuscuton is a limiting curvature theory (see Ref. [563]), and it is with no surprise that it can give rise to non-singular bouncing solutions [92]. As we are currently investigating [563], it turns out that the cuscuton shares many interesting properties with other models of non-singular cosmology, and it appears to be a robust conclusion that it is free of all types of instabilities. In the new class of minimally-modified gravity, the theories have the property of propagating only two degrees of freedom (the two tensor polarization states of gravitons) and nevertheless differ from general relativity. In particular, recent advances have shown the possibility of naturally violating the null energy condition [454]. It would therefore be very interesting to keep digging in that direction, obtaining non-singular bouncing solutions, studying the stability of the cosmological perturbations, and so on.

To end, we believe that theoretical primordial cosmology is at an interesting stage. On one hand, many alternatives to inflationary cosmology have been proposed, but for several reasons, none have really risen to gain acceptance and to be treated on an equal footing with inflation across the community. On the other hand, we continue to discover that the theory of inflationary cosmology, as it currently stands, cannot be the final answer to fundamental questions such as "Where does the universe come from?" and "How did it all begin?" Therefore, the field should be more stimulated than ever to go back to the blackboard and try to be creative and do new things.

Appendix A

Maximal extensions and singularities in inflationary spacetimes

A.1 Introduction

Spacetime singularities remain deep mysteries in gravitational theories. According to the singularity theorems of Penrose and Hawking [345, 348, 544], the occurrence of singularities is inevitable in General Relativity when the stress-energy tensor satisfies some energy conditions. For example, the singularity theorems ensure the presence of the initial Big Bang singularity in cosmology, also known as the initial singularity problem, which is characterized by incomplete timelike geodesics, provided the strong energy condition is satisfied.

Inflation [333] is a well-studied structure formation scenario for the very early universe, and it is supported by recent observations of the cosmic microwave background anisotropies [12, 14]. Since inflationary cosmology violates the strong energy condition, it was expected to solve the initial singularity problem. For example, Starobinsky's inflationary model [590] was originally proposed as a possible resolution to the initial singularity problem. However, even though the strong energy condition is violated, it does not guaranty the absence of a singularity. In fact, Borde and Vilenkin [87, 89] showed that eternal inflation[1] models are null-geodesically incomplete to the past provided the null energy condition is satisfied.

[1] Strictly speaking, the theorem was shown under the assumption that the volume of the past of an inflationary region is finite (assumption D in Ref. [89]) rather than the assumption of eternal inflation itself. However, this assumption is naturally satisfied in eternal inflation models based on the old inflation scenario as explained in Ref. [89].

After that, another interesting theorem was shown by Borde, Guth and Vilenkin [86]. They generalized the concept of Hubble parameter to a general (inhomogeneous and anisotropic) spacetime where comoving geodesic congruence is defined. Then, they showed that a geodesic is incomplete if the averaged Hubble parameter along this geodesic is positive. Interestingly, neither the null energy condition nor eternal inflation is assumed, and the theorem can be applied to a very wide class of inflationary models. This result motivates the exploration of alternative very early universe scenarios that could evade the assumptions of the theorem to be past complete (see, e.g., Refs. [143, 227, 516, 635] and references therein).

Let us quickly review the analysis of Ref. [86] with an emphasis on the case of a flat Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime. In this case, the averaged Hubble parameter along a null geodesic is defined by

$$H_{\rm av}[t_f, t_i] \equiv \frac{1}{\lambda(t_f) - \lambda(t_i)} \int_{\lambda(t_i)}^{\lambda(t_f)} \mathrm{d}\lambda\, H(\lambda)\,, \tag{A.1.1}$$

where $\lambda(t)$ is an affine parameter of a null geodesic at time t, and H is the Hubble parameter. One can show that the integral on the right-hand side is smaller or equal than unity by suitably choosing the affine parameter (see Ref. [86]). Thus, if $H_{\rm av}[t_f, -\infty] > 0$, one obtains

$$\begin{aligned} 0 &< \lim_{t_i \to -\infty} \frac{1}{\lambda(t_f) - \lambda(t_i)} \int_{\lambda(t_i)}^{\lambda(t_f)} \mathrm{d}\lambda\, H(\lambda) \\ &\leq \lim_{t_i \to -\infty} \frac{1}{\lambda(t_f) - \lambda(t_i)}\,. \end{aligned} \tag{A.1.2}$$

This means that the affine parameter $\lambda(t)$ has to be finite in the limit where $t \to -\infty$, and consequently, the corresponding flat FLRW spacetime is past incomplete. If the initial stage of the Universe is described by inflation, then the averaged Hubble parameter must be positive. Therefore, there appears to be no hope of avoiding the past incompleteness of any inflationary flat FLRW spacetime.

Nonetheless, it might be possible to *extend* the flat FLRW spacetime beyond the end points of the incomplete geodesics, which we call the past boundary $\mathscr{B}^-$. The important observation here is that the above discussion can be applied even when the spacetime is exactly flat de Sitter space. This implies that flat de Sitter space must be past incomplete. This does *not* contradict the fact that de Sitter space is free of singularities, because the flat patch of de Sitter space covers only half of the entire de Sitter space (see Fig. A.1).

Thus, even though flat de Sitter space is actually past incomplete, $\mathscr{B}^-$ is not singular, and the spacetime can be extended to the entire nonsingular de Sitter space beyond $\mathscr{B}^-$. In the context of the general setting of Ref. [86], this would happen when the comoving geodesic congruence, which defines the Hubble parameter, does not cover the entire spacetime.

Since inflationary spacetimes are effectively described by flat de Sitter space, it is natural to expect that the past incompleteness of inflation in a flat FLRW coordinate patch is just an apparent one and that there might exist a nonsingular maximal extension of it. The purpose of this paper is to explore such extendibility of inflationary spacetimes with flat spatial geometry.

The question that must be answered is how one can judge the extendibility of the past boundary $\mathscr{B}^-$. The classification of a boundary of spacetime was studied in [271] (see also [348]). In Ref. [209, 210], it was shown that a spacetime is locally inextendible (or inextensible) if and only if any component of the Riemann tensor and its covariant derivatives measured in a parallelly propagated (p. p.) tetrad basis diverges in the limit to the boundary. This kind of singularity is called a *p. p. curvature singularity* [348] or simply a *curvature singularity* [271]. We note that the more common *scalar curvature singularity*, where a scalar curvature invariant blows up, is necessarily also a p. p. curvature singularity. However, the converse is not always true, i.e., it is possible that a p. p. curvature singularity may not be a scalar curvature singularity. This kind of singularity is called an *intermediate singularity* [269] or a *nonscalar singularity* [271]. One notes that a locally extendible spacetime also includes a globally inextendible one. A boundary that can be extended locally but not globally is called a *locally extendible singularity* [269] or a *quasi-regular singularity* [271]. A conical singularity and the singularity in Taub-NUT spacetime correspond to locally extendible singularities (see, e.g., Ref. [271]). A globally extendible boundary is called a *regular boundary*, and there is no obstacle to extend spacetime beyond this boundary. An example of this is the boundary of flat de Sitter space discussed above. To summarize, the finiteness of the components of the Riemann tensor with respect to a p. p. tetrad basis is crucial to discuss the extendibility of spacetimes, at least locally.

Our paper is organized as follows. In the next section, we construct an affine parametrization of a null geodesic and explicitly demonstrate its past incompleteness as $t \to -\infty$. Then, in Sec. A.3, we explicitly construct a p. p. tetrad for flat FLRW spacetimes and discuss the extendibility of inflationary cosmologies. We find that a flat FLRW spacetime is continuously inextendible if the quantity $\dot{H}/a^2$ diverges in the limit $t \to -\infty$. On the other hand,

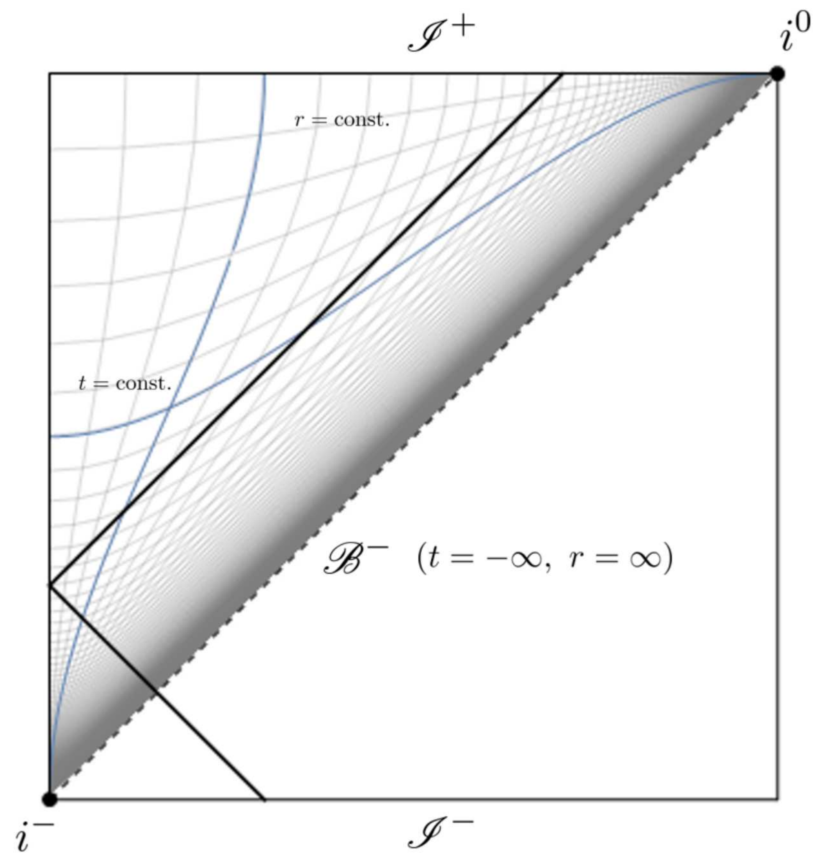

Figure A.1 Penrose diagram of de Sitter space: The upper half triangle region corresponds to the spacetime region which is covered by the flat FLRW coordinate patch. The bold line represents a null geodesic. The null geodesic is incomplete in the flat FLRW patch and reaches the past null boundary $\mathscr{B}^-$ (represented by the dashed line) at finite affine length. However, this geodesic is complete in the entire de Sitter spacetime. In the flat FLRW patch, the vertical gray lines represent surfaces of constant radius r, and the horizontal lines represent surfaces of constant time t. Examples are highlighted in blue. Also, i^0 and i^- denote spatial infinity and past timelike infinity, respectively, while $\mathscr{I}^+$ and $\mathscr{I}^-$ denote future and past null infinity, respectively.

it is continuously extendible if $\dot{H}/a^2$ is finite in the limit $t \to -\infty$. In Sec. A.4, we give a concrete method to extend the spacetime (when it is shown to be extendible). After that, we give examples of analytic spacetimes in Sec. A.5. There, we demonstrate the maximal extension of each inflationary spacetime or show the presence of the p. p. singularity explicitly. In Sec. A.6, we discuss the implications for inflationary models. Specifically, we derive the condition for single field slow-roll models to be continuously extendible, and we find that the Starobinsky model has a continuous p. p. singularity, but a small field inflation model does not. Moreover, we investigate the absence of p. p. curvature singularities in a model of modified gravity studied in Refs. [143, 516, 635]. The final section is devoted to the summary and discussion.

A.2 Setup and past incompleteness of inflationary spacetimes

Throughout this paper, we focus on flat FLRW spacetime,

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + a(t)^2\left(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2_{(2)}\right)\,, \tag{A.2.3}$$

where $\mathrm{d}\Omega^2_{(2)} = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$ is the metric of the unit 2-sphere. We assume that the early stage of cosmological evolution is described by inflationary exponential expansion, which leads to the past boundary $\mathscr{B}^-$ as we will see below. Precisely, we assume that the comoving geodesics (for which the spatial coordinates r, θ, and ϕ are constants) are past complete[2], i.e., the comoving time t is defined all the way to $t \to -\infty$, and the scale factor $a(t)$ approaches that of de Sitter space in the limit where $t \to -\infty$. Specifically, the assumption is that, asymptotically,

$$a(t) \simeq \bar{a}\, e^{H_\Lambda t} \qquad \text{as } t \to -\infty\,, \tag{A.2.4}$$

where $\bar{a}$ and H_Λ are positive constants. In that limit, H_Λ represents the Hubble parameter since $H \equiv \dot{a}/a \simeq H_\Lambda$. Throughout this paper, a dot denotes a derivative with respect to physical time t.

Under the above assumption, we can directly confirm the incompleteness of a null geodesic. In order to construct an affine parametrization of a null geodesic, it is useful to introduce

[2] We note that extensions of spacetimes where the comoving geodesics are past incomplete, e.g. cosmologies with an initial Big Bang singularity, was discussed in Ref. [396].

the conformal time η defined by

$$\eta(t) \equiv \int^t \frac{\mathrm{d}t'}{a(t')} \,. \tag{A.2.5}$$

Then, it is straightforward to see that a curve parametrized by $\tilde{\lambda}$ as follows,

$$x^\mu(\tilde{\lambda}) = \left(\eta(\tilde{\lambda}), r(\tilde{\lambda}), \theta(\tilde{\lambda}), \phi(\tilde{\lambda})\right) = \left(\tilde{\lambda}, -\tilde{\lambda}, 0, 0\right) \,, \tag{A.2.6}$$

is a null geodesic. However, the parameter $\tilde{\lambda}$ is not an affine parameter because the right-hand side of the geodesic equation,

$$\tilde{k}^\nu \nabla_\nu \tilde{k}^\mu = 2\frac{\partial_\eta a}{a}\tilde{k}^\mu \,, \tag{A.2.7}$$

does not vanish. Here, $\tilde{\boldsymbol{k}}$ is the tangent vector of the null geodesic (A.2.6), and it is given by

$$\tilde{\boldsymbol{k}} = \tilde{k}^\mu \boldsymbol{\partial}_\mu = \frac{\mathrm{d}x^\mu(\tilde{\lambda})}{\mathrm{d}\tilde{\lambda}}\boldsymbol{\partial}_\mu = \boldsymbol{\partial}_\eta - \boldsymbol{\partial}_r \,. \tag{A.2.8}$$

An affine parameter of the null geodesic (A.2.6) can be derived by re-parameterizing $\tilde{\lambda}$. Let us consider a new parameter $\lambda = \lambda(\tilde{\lambda})$ and its corresponding tangent vector $k^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$. Then, the tangent vector with respect to $\tilde{\lambda}$ can be expressed in terms of λ as

$$\tilde{k}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tilde{\lambda}} = \frac{\mathrm{d}\lambda}{\mathrm{d}\tilde{\lambda}}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \frac{\mathrm{d}\lambda}{\mathrm{d}\tilde{\lambda}}k^\mu \,, \tag{A.2.9}$$

and one can derive the geodesic equation for k^μ:

$$k^\nu \nabla_\nu k^\mu = -\frac{1}{(\partial_\eta \lambda)^2}\left(\partial_\eta^2 \lambda - 2\frac{\partial_\eta a}{a}\partial_\eta \lambda\right) k^\mu \,. \tag{A.2.10}$$

In order for λ to be an affine parameter, the right-hand side of Eq. (A.2.10) has to vanish. This is only the case if

$$\mathrm{d}\lambda \propto a^2\,\mathrm{d}\eta = a\,\mathrm{d}t \,. \tag{A.2.11}$$

Thus, the null geodesic (A.2.6) is past complete if and only if the affine parameter diverges in the limit where $t \to -\infty$, i.e., if the integral

$$\lambda[t_f, -\infty] = \int_{-\infty}^{t_f} \mathrm{d}t \; a(t) \tag{A.2.12}$$

is infinite for arbitrary t_f. In the case of inflation, with the assumption (A.2.4), the integral (A.2.12) converges:

$$\lambda[t_f, -\infty] \simeq \bar{a} \int_{-\infty}^{t_f} \mathrm{d}t \; e^{H_\Lambda t} = \frac{\bar{a}}{H_\Lambda} e^{H_\Lambda t_f} \,. \tag{A.2.13}$$

Therefore, flat FLRW spacetime with the assumption (A.2.4) has incomplete null geodesics, and there is a past boundary $\mathscr{B}^-$.

In the next section, we construct a p. p. tetrad basis along these incomplete null geodesics and discuss the extendibility.

A.3 The parallely propagated curvature singularity in inflationary spacetimes

To discuss the local extendibility of an inflationary spacetime, one needs to construct a p. p. tetrad basis along the null geodesic from the previous section since that is the basis which is well defined on the boundary $\mathscr{B}^-$. First, one can construct a simple tetrad $\hat{\boldsymbol{e}}^M$, which we call the FLRW tetrad, as follows:

$$\hat{\boldsymbol{e}}^0 = \mathbf{d}t = a\, \mathbf{d}\eta \,; \tag{A.3.14a}$$
$$\hat{\boldsymbol{e}}^1 = a\, \mathbf{d}r \,; \tag{A.3.14b}$$
$$\hat{\boldsymbol{e}}^2 = a\, r\, \mathbf{d}\theta \,; \tag{A.3.14c}$$
$$\hat{\boldsymbol{e}}^3 = a\, r \sin\theta\, \mathbf{d}\phi \,. \tag{A.3.14d}$$

Though the FLRW tetrad components are p. p. along comoving timelike geodesics, they are not parallel along the null geodesic of the previous section. This can be confirmed by calculating $\boldsymbol{\nabla}_{\boldsymbol{k}} \hat{\boldsymbol{e}}^M$. Indeed, the covariant derivatives of $\hat{\boldsymbol{e}}^0$ and $\hat{\boldsymbol{e}}^1$ along $\boldsymbol{k}$ are nonvanishing:

$$k^\mu \nabla_\mu \hat{\boldsymbol{e}}^0 = H\, \mathbf{d}r\,; \tag{A.3.15a}$$
$$k^\mu \nabla_\mu \hat{\boldsymbol{e}}^1 = H\, \mathbf{d}\eta\,. \tag{A.3.15b}$$

We note that only $\hat{\boldsymbol{e}}^2$ and $\hat{\boldsymbol{e}}^3$ are p. p. along $\boldsymbol{k}$.

Let us now construct a p. p. tetrad $\boldsymbol{e}^M$ simply by parallel transport of the FLRW tetrad at a point $(\eta, r) = (\eta_0, r_0)$ along the null geodesic. Since any two tetrad bases are related through a Lorentz transformation, we can write a p. p. tetrad basis $\boldsymbol{e}^M$ as a Lorentz transformation of the FLRW tetrad,

$$\boldsymbol{e}^0 = \cosh\zeta(\eta, r)\, \hat{\boldsymbol{e}}^0 + \sinh\zeta(\eta, r)\, \hat{\boldsymbol{e}}^1, \tag{A.3.16a}$$
$$\boldsymbol{e}^1 = \sinh\zeta(\eta, r)\, \hat{\boldsymbol{e}}^0 + \cosh\zeta(\eta, r)\, \hat{\boldsymbol{e}}^1, \tag{A.3.16b}$$

with $\boldsymbol{e}^2 = \hat{\boldsymbol{e}}^2$ and $\boldsymbol{e}^3 = \hat{\boldsymbol{e}}^3$. The Lorentz factor (or rapidity) ζ satisfies $\zeta(\eta_0, r_0) = 0$ so that $\boldsymbol{e}^M$ coincides with $\hat{\boldsymbol{e}}^M$ at the given point (η_0, r_0). Then, the covariant derivative of $\boldsymbol{e}^M$ along $\boldsymbol{k}$ can be expressed in terms of the Lorentz factor ζ as

$$k^\mu \nabla_\mu \boldsymbol{e}^0 = \frac{\partial_\eta a + a\,(\partial_\eta \zeta - \partial_r \zeta)}{a^3}\, \boldsymbol{e}^1\,, \tag{A.3.17a}$$
$$k^\mu \nabla_\mu \boldsymbol{e}^1 = \frac{\partial_\eta a + a\,(\partial_\eta \zeta - \partial_r \zeta)}{a^3}\, \boldsymbol{e}^0\,. \tag{A.3.17b}$$

For the $\boldsymbol{e}^M$'s to be parallely propagated, the right-hand sides in Eq. (A.3.17) have to vanish. Thus, one solution for the Lorentz factor $\zeta(\eta, r)$ is

$$\zeta = -\ln\left(\frac{a}{a_0}\right)\,, \tag{A.3.18}$$

where $a_0 \equiv a(\eta_0)$ is the value of the scale factor at $\eta = \eta_0$. In the limit where $a \to 0$, i.e. toward the boundary $\mathscr{B}^-$, the expression (A.3.18) tells us that the p. p. basis is obtained by parallel transport through an infinite boost from the FLRW tetrad basis. Since the p. p. tetrad is a well-defined basis to discuss the extendibility of the curve beyond $\mathscr{B}^-$, this fact also tells us that the FLRW tetrad basis is ill defined in the limit to the boundary $\mathscr{B}^-$. By plugging (A.3.18) into the definition of $\boldsymbol{e}^M$, we obtain concrete expressions for our p. p.

tetrad,

$$\boldsymbol{e}^0 = \frac{1}{2}\left(1+\frac{a^2}{a_0^2}\right)\frac{a_0}{a}\,\hat{\boldsymbol{e}}^0 + \frac{1}{2}\left(1-\frac{a^2}{a_0^2}\right)\frac{a_0}{a}\,\hat{\boldsymbol{e}}^1\,, \tag{A.3.19a}$$
$$\boldsymbol{e}^1 = \frac{1}{2}\left(1-\frac{a^2}{a_0^2}\right)\frac{a_0}{a}\,\hat{\boldsymbol{e}}^0 + \frac{1}{2}\left(1+\frac{a^2}{a_0^2}\right)\frac{a_0}{a}\,\hat{\boldsymbol{e}}^1\,. \tag{A.3.19b}$$

Inversely, the original FLRW tetrad can be expressed in terms of the p. p. tetrad as

$$\hat{\boldsymbol{e}}^0 = \frac{1}{2}\left(1+\frac{a^2}{a_0^2}\right)\frac{a_0}{a}\,\boldsymbol{e}^0 - \frac{1}{2}\left(1-\frac{a^2}{a_0^2}\right)\frac{a_0}{a}\,\boldsymbol{e}^1\,, \tag{A.3.20a}$$
$$\hat{\boldsymbol{e}}^1 = -\frac{1}{2}\left(1-\frac{a^2}{a_0^2}\right)\frac{a_0}{a}\,\boldsymbol{e}^0 + \frac{1}{2}\left(1+\frac{a^2}{a_0^2}\right)\frac{a_0}{a}\,\boldsymbol{e}^1\,. \tag{A.3.20b}$$

As studied in Refs. [209, 210, 271, 348], components of the Riemann tensor (and derivatives thereof) are crucial to discuss the local extendibility of a boundary. Since flat FLRW spacetime is conformally flat, any independent components of the Riemann tensor is described by the Ricci tensor. In the FLRW tetrad basis $\hat{\boldsymbol{e}}^M$, the Ricci tensor can be expanded as follows,

$$R_{\mu\nu}\mathbf{d}x^\mu \otimes \mathbf{d}x^\nu = -2\dot{H}\hat{\boldsymbol{e}}^0 \otimes \hat{\boldsymbol{e}}^0 + \left(3H^2+\dot{H}\right)\eta_{MN}\hat{\boldsymbol{e}}^M \otimes \hat{\boldsymbol{e}}^N\,, \tag{A.3.21}$$

where η_{MN} denotes the Minkowski metric tensor with tetrad indices. Since $H \to H_\Lambda$ and $\dot{H} \to 0$ by the condition (A.2.4), the components of the Ricci tensor with respect to the FLRW tetrad are finite. Thus, there appears to be no ill behavior for comoving timelike observers. Also, any scalar curvature invariant constructed from the Ricci tensor is finite in the limit to the boundary $\mathscr{B}^-$. Nevertheless, any component of the Ricci tensor with respect to the p. p. tetrad $\boldsymbol{e}^M$ could possibly diverge, because the p. p. tetrad is related to the FLRW tetrad through an infinite boost. Concretely, by using the expressions (A.3.20), we can write

$$\begin{aligned} R_{\mu\nu}\mathbf{d}x^\mu \otimes \mathbf{d}x^\nu = \frac{\dot{H}a_0^2}{2a^2}&\left[-\left(1+\frac{a^2}{a_0^2}\right)^2\boldsymbol{e}^0\otimes\boldsymbol{e}^0 + 2\left(1-\frac{a^4}{a_0^4}\right)\boldsymbol{e}^{(0}\otimes\boldsymbol{e}^{1)} - \left(1-\frac{a^2}{a_0^2}\right)^2\boldsymbol{e}^1\otimes\boldsymbol{e}^1\right] \\ &+\left(3H^2+\dot{H}\right)\eta_{MN}\boldsymbol{e}^M\otimes\boldsymbol{e}^N\,, \end{aligned} \tag{A.3.22}$$

where in general $\boldsymbol{e}^{(M}\otimes\boldsymbol{e}^{N)}$ is shorthand notation for $(\boldsymbol{e}^M\otimes\boldsymbol{e}^N + \boldsymbol{e}^N\otimes\boldsymbol{e}^M)/2$. From the above, we can see that the $(0,0)$, $(0,1)$, and $(1,1)$ components include the possibly divergent quantity $\dot{H}/a^2$ as $t \to -\infty$ and $a \to 0$. Therefore, we arrive at the following statement, the key result of the present work:

Theorem A.1. *The boundary $\mathscr{B}^-$ is a p. p. curvature singularity, or more precisely an intermediate singularity, if*

$$\left| \lim_{t\to-\infty} \frac{\dot{H}}{a^2} \right| = \infty \,, \tag{A.3.23}$$

and consequently, the corresponding inflationary spacetime cannot be extended beyond $\mathscr{B}^-$. On the other hand, if $\dot{H}/a^2$ converges, then the past boundary $\mathscr{B}^-$ is not singular, and it can be extendible at least locally.

The above statement relies solely on the behavior of the Ricci tensor (not on its derivatives), so the statement is pertaining to continuous (C^0) extendibility and C^0 p. p. curvature singularities. The precise definition of C^r p. p. curvature singularities and C^r extendibility for any integer $r \geq 0$ can be found in appendix A.8, and as expected, the criterion depends on the behavior of the r-th covariant derivative of the Ricci tensor. For example, continuously differentiable (C^1) extendibility is possible if the first covariant derivative of the Ricci tensor is continuous and does not diverge, i.e., $R_{\mu\nu}$ must be of class C^1. Similarly, smooth (infinitely differentiable or C^∞) extendibility requires the Ricci tensor to be smooth as well.

In what follows, we focus on continuous (C^0) extendibility, and that is what is implicitly meant unless specified. Parallely propagated curvature singularities of class C^1, C^2 and all the way to C^∞ are 'weaker' or 'milder' in the sense that they may only involve the divergence of quantities with higher derivatives of the Hubble parameter (see appendix A.8), e.g. quantities like $\ddot{H}/a^3$, $\dddot{H}/a^4$, etc.

A.4 Coordinates beyond the past boundary

In previous section, we found that an inflationary spacetime is possibly past extendible if the quantity $\dot{H}/a^2$ is finite all the way to the infinite past. The question that is raised in this case is how one can extend the spacetime beyond the boundary. In this section, we construct a new set of coordinates for the flat FLRW spacetime analogously to Eddington-Finkelstein coordinates in Schwarzschild spacetime. Let us consider a new set of coordinates $\{\lambda, v, \theta, \phi\}$

defined by

$$\lambda \equiv \lambda[t,-\infty] = \int_{-\infty}^{t} \mathrm{d}t'\, a(t')\,, \tag{A.4.24}$$

$$v \equiv \eta + r\,. \tag{A.4.25}$$

Note that, with these coordinates, the null geodesic (A.2.6) corresponds to the curve characterized by $v = \theta = \phi =$ constant. Now, the affine parameter is chosen so that $\lambda = 0$ corresponds to the past boundary $\mathscr{B}^-$. Using the relations

$$\mathrm{d}t = \frac{1}{a}\,\mathrm{d}\lambda\,, \tag{A.4.26}$$

$$\mathrm{d}r = \mathrm{d}v - \mathrm{d}\eta = \mathrm{d}v - \frac{1}{a^2}\,\mathrm{d}\lambda\,, \tag{A.4.27}$$

we can write the FLRW metric in terms of the new coordinates as

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -2\,\mathrm{d}\lambda\,\mathrm{d}v + a^2\,\mathrm{d}v^2 + a^2 r^2\,\mathrm{d}\Omega^2_{(2)}\,. \tag{A.4.28}$$

In the limit toward $\mathscr{B}^-$, the quantity $a\,r$ converges under the assumption (A.2.4), because it can be evaluated as

$$a\,r \sim -a\,\eta \sim -\left(\bar{a}\,e^{H_\Lambda t}\right)\left(-\frac{1}{\bar{a}\,H_\Lambda}e^{-H_\Lambda t}\right) = \frac{1}{H_\Lambda}\,. \tag{A.4.29}$$

Thus, the metric is regular at $\mathscr{B}^-$, and it is given by

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu|_{\mathscr{B}^-} = -2\,\mathrm{d}\lambda\,\mathrm{d}v + \frac{1}{H_\Lambda^2}\,\mathrm{d}\Omega^2_{(2)}\,. \tag{A.4.30}$$

We note that the metric components are regular on $\mathscr{B}^-$ even if $\mathscr{B}^-$ is a p. p. curvature singularity, because $\dot{H}/a^2$ does not appear in the expression (A.4.30). In the inextendible case, an ill behavior only appears in the components of the Ricci tensor, which can be expressed as

$$R_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -2\frac{\dot{H}}{a^2}\,\mathrm{d}\lambda^2 + \left(3H^2 + \dot{H}\right) g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta\,. \tag{A.4.31}$$

As it is clear from Eqs. (A.4.30) and (A.4.31), when $\dot{H}/a^2$ converges, there is no ill behavior at $\lambda = 0$. Thus, the spacetime can be extended to $\lambda \leq 0$ by making use of this coordinate system.

A.5 Examples

A.5.1 Exact de Sitter spacetime

As a first example of our general discussion, we first demonstrate the maximal extension of flat de Sitter space, where the scale factor is exactly given by

$$a(t) = \bar{a}\, e^{H_\Lambda t}\,. \tag{A.5.32}$$

Here, the FLRW time coordinate t is defined in the region $t \in (-\infty, \infty)$. Since the affine parameter λ can be evaluated as

$$\lambda = \frac{\bar{a}\, e^{H_\Lambda t}}{H_\Lambda}\,, \tag{A.5.33}$$

the coordinate region $t \in (-\infty, \infty)$ corresponds to $\lambda \in (0, \infty)$, hence the null geodesics are past incomplete. From Eq. (A.5.33), the scale factor can be written as a function of λ as

$$a(\lambda) = H_\Lambda \lambda\,. \tag{A.5.34}$$

Also, the conformal time can be evaluated as

$$\eta = -\frac{e^{-H_\Lambda t}}{\bar{a} H_\Lambda} = -\frac{1}{H_\Lambda^2 \lambda}\,. \tag{A.5.35}$$

Since the coordinate region $t \in (-\infty, \infty)$ corresponds to $\eta \in (-\infty, 0)$, flat de Sitter space is conformally isometric to the lower half of Minkowski spacetime as shown by the upper triangle of Fig. A.1. Using Eqs. (A.5.33) and (A.5.35), we can write the metric in the coordinates $\{\lambda, v, \theta, \phi\}$ as

$$g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = -2\, \mathrm{d}\lambda\, \mathrm{d}v + H_\Lambda^2 \lambda^2\, \mathrm{d}v^2 + \frac{1}{H_\Lambda^2} \left(1 + H_\Lambda^2 \lambda v\right)^2 \mathrm{d}\Omega_{(2)}^2\,. \tag{A.5.36}$$

Now, the metric tensor is well defined even for nonpositive values of λ. Thus, we can extend[3] flat de Sitter space with $\lambda \in (0, \infty)$ to the entire de Sitter spacetime with $\lambda \in (-\infty, \infty)$.

[3] In the language of appendix A.8, exact de Sitter space is one of the examples where the spacetime is actually smoothly (C^∞) extendible.

Since λ is nothing but the affine parameter of a null geodesic, this geodesic is now complete in the entire de Sitter spacetime. We note that with the closed FLRW coordinates of the entire de Sitter spacetime, also known as the global coordinates, the line element is given by

$$g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = -\mathrm{d}t_g^2 + \frac{\cosh^2(H_\Lambda t_g)}{H_\Lambda^2}\left(\mathrm{d}\psi^2 + \sin^2\psi\,\mathrm{d}\Omega_{(2)}^2\right) , \tag{A.5.37}$$

where t_g is the global time coordinate, and $\psi \in (0,\pi)$ is the third angle describing the 3-sphere with line element $\mathrm{d}\Omega_{(3)}^2 = \mathrm{d}\psi^2 + \sin^2\psi\,\mathrm{d}\Omega_{(2)}^2$. The above metric can be obtained from our coordinates by the following coordinate transformation:

$$\lambda = \frac{1}{H_\Lambda^2}\big(\cosh[H_\Lambda t_g]\cos\psi + \sinh[H_\Lambda t_g]\big) ; \tag{A.5.38}$$

$$v = -\frac{1 - e^{H_\Lambda t_g}\tan(\psi/2)}{e^{H_\Lambda t_g} + \tan(\psi/2)} . \tag{A.5.39}$$

A.5.2 Inextendible toy model

As a toy model, let us consider a flat FLRW spacetime with scale factor given by

$$a(t) = \frac{\bar{a}\,e^{H_\Lambda t}}{1 + \bar{a}\,e^{H_\Lambda t}} \simeq \begin{cases} \bar{a}\,e^{H_\Lambda t} & \text{as } t \to -\infty\,, \\ 1 & \text{as } t \to \infty\,. \end{cases} \tag{A.5.40}$$

This scale factor represents a universe which starts from de Sitter and approaches Minkowski at $t \to \infty$. One can evaluate the conformal time as

$$\eta = t - \frac{e^{-H_\Lambda t}}{\bar{a}H_\Lambda} , \tag{A.5.41}$$

and since the coordinate region $t \in (-\infty,\infty)$ corresponds to $\eta \in (-\infty,\infty)$, this spacetime is conformally isometric to the whole Minkowski spacetime. However, since

$$\frac{\dot{H}}{a^2} = -\frac{H_\Lambda^2}{\bar{a}\,e^{H_\Lambda t}} \to -\infty \qquad \text{as } t \to -\infty\,, \tag{A.5.42}$$

the past boundary $\mathscr{B}^-$ is singular (i.e. it is a p. p. curvature singularity) according to the key result of the previous section. The Penrose diagram of this spacetime is shown in Fig. A.2.

Let us explicitly see the inextendible nature of the spacetime by deriving the metric com-

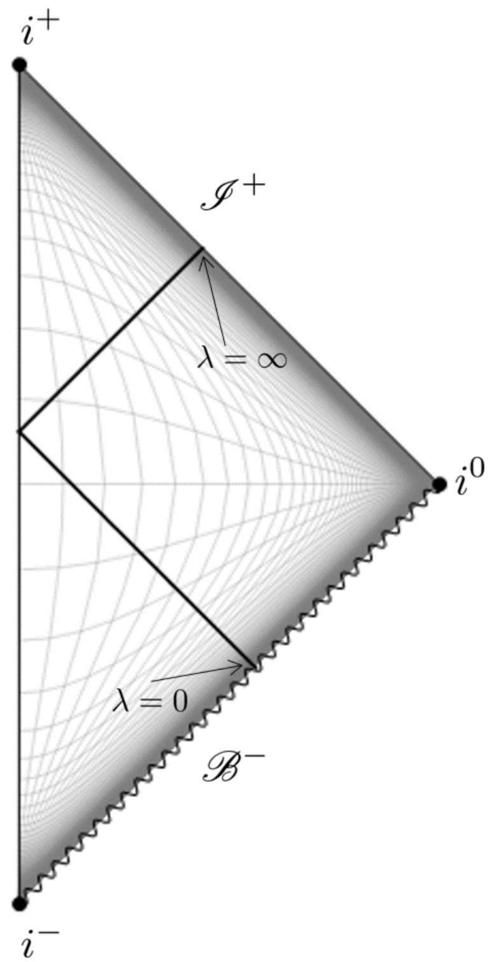

Figure A.2 Penrose diagram of the inextendible toy model with scale factor given by Eq. (A.5.40). The past boundary $\mathscr{B}^-$ is a p. p. curvature singularity, and consequently, the spacetime cannot be extended beyond $\mathscr{B}^-$. The bold line represents a null geodesic, which 'starts' on $\mathscr{B}^-$ at $\lambda \to 0$ and 'ends' on $\mathscr{I}^+$ at $\lambda \to \infty$. Note also that i^+ denotes future timelike infinity.

ponents in our new coordinate system $\{\lambda, v, \theta, \phi\}$. Following the definition in Eq. (A.4.24), λ can be written as

$$\lambda = H_\Lambda^{-1} \ln\left(1 + \bar{a}\, e^{H_\Lambda t}\right) . \tag{A.5.43}$$

Therefore, the scale factor a and the conformal time η can be written in terms of λ as

$$a(\lambda) = 1 - e^{-H_\Lambda \lambda} , \tag{A.5.44}$$

$$\eta(\lambda) = \frac{1}{H_\Lambda}\left(\ln\left[\frac{e^{H_\Lambda\lambda} - 1}{\bar{a}}\right] - \frac{1}{e^{H_\Lambda\lambda} - 1}\right) . \tag{A.5.45}$$

Thus, we can evaluate $a\,\eta$ as

$$a\,\eta = \frac{e^{-H_\Lambda\lambda}}{H_\Lambda}\left(\epsilon \ln\left[\frac{\epsilon}{\bar{a}}\right] - 1\right) , \tag{A.5.46}$$

where ϵ is a function of λ defined by

$$\epsilon(\lambda) \equiv e^{H_\Lambda\lambda} - 1 . \tag{A.5.47}$$

Then, the metric can be written as

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -2\,\mathrm{d}\lambda\,\mathrm{d}v + \left(1 - e^{-H_\Lambda\lambda}\right)^2 \mathrm{d}v^2 + \frac{e^{-2H_\Lambda\lambda}}{H_\Lambda^2}\left(\epsilon \ln\left[\frac{\epsilon}{\bar{a}}\right] - 1 - H_\Lambda \epsilon v\right)^2 \mathrm{d}\Omega_{(2)}^2 . \tag{A.5.48}$$

As one takes the limit $\lambda \to 0$ towards $\mathscr{B}^-$, the above expression becomes exactly equal to Eq. (A.4.30), and so the metric components are well defined in that limit. However, the metric is not differentiable in that limit because of the term of the form $\epsilon \ln \epsilon$. Thus, the Ricci tensor is not C^0, and the spacetime cannot be extended beyond $\lambda = 0$.

A.5.3 Extendible toy model

Let us consider another toy model with the scale factor given by

$$a(t) = \frac{\bar{a}}{e^{H_\Lambda t} + e^{-H_\Lambda t}} . \tag{A.5.49}$$

In the limit $t \to \pm\infty$, the scale factor behaves as that of a contracting or expanding flat de Sitter universe:

$$a(t) \simeq \begin{cases} \bar{a}\, e^{H_\Lambda t} & \text{as } t \to -\infty\,; \\ \bar{a}\, e^{-H_\Lambda t} & \text{as } t \to \infty\,. \end{cases} \tag{A.5.50}$$

Since the conformal time,

$$\eta = \frac{e^{H_\Lambda t} - e^{-H_\Lambda t}}{\bar{a} H_\Lambda}\,, \tag{A.5.51}$$

is defined in the parameter region $\eta \in (-\infty, \infty)$, the flat FLRW region $t \in (-\infty, \infty)$ is again conformally isometric to the whole Minkowski spacetime. Since $\dot{H}/a^2$ is now finite,

$$\frac{\dot{H}}{a^2} = -\frac{4H_\Lambda^2}{\bar{a}^2}\,, \tag{A.5.52}$$

any component of the Ricci tensor is well behaved in the limit to the past null boundary $\mathscr{B}^-$. Therefore, this spacetime is extendible[4] beyond the past null boundary the same way flat de Sitter space can be. The affine parameter λ is obtained as follows,

$$\lambda = \frac{\bar{a}}{H_\Lambda} \arctan e^{H_\Lambda t}\,, \tag{A.5.53}$$

and so the region $t \in (-\infty, \infty)$ corresponds to $\lambda \in (0, \bar{a}\pi/2H_\Lambda)$. Hence the original region is both future and past incomplete. Let us denote the future null boundary at $\lambda = \bar{a}\pi/2H_\Lambda$ by $\mathscr{B}^+$. The scale factor can then be written in terms of λ as

$$a(\lambda) = \frac{\bar{a}}{2} \sin\left(\frac{2H_\Lambda \lambda}{\bar{a}}\right)\,, \tag{A.5.54}$$

hence

$$a(\lambda)\eta(\lambda) = -\frac{1}{H_\Lambda} \cos\left(\frac{2H_\Lambda \lambda}{\bar{a}}\right), \tag{A.5.55}$$

[4] The toy model here is actually smoothly (C^∞) extendible just like exact de Sitter space. This is due to the fact that $\dot{H}/a^2 =$ constant, i.e. $\dot{H} \sim a^2$, and it is shown in appendix A.8 that C^∞ extendibility follows if $\dot{H} \sim a^q$ as $a \to 0$ with $q \in \mathbb{Z}_{\geq 2}$. The present case corresponds to $q = 2$.

and so the metric in the original region $\lambda \in (0, \frac{\bar{a}}{H_\Lambda}\frac{\pi}{2})$ can be written as

$$
\begin{aligned}
g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = &- 2\,\mathrm{d}\lambda\,\mathrm{d}v + \frac{\bar{a}^2}{4}\sin^2(2H_\Lambda\lambda/\bar{a})\,\mathrm{d}v^2 \\
&+ \frac{1}{4H_\Lambda^2}\left(2\cos\left[\frac{2H_\Lambda\lambda}{\bar{a}}\right] + \bar{a}H_\Lambda v\sin\left[\frac{2H_\Lambda\lambda}{\bar{a}}\right]\right)^2 \mathrm{d}\Omega_{(2)}^2\,.
\end{aligned}
\tag{A.5.56}
$$

The important point here is that the scale factor and the components of the metric are well defined in the whole parameter region $\lambda \in (-\infty, \infty)$, and therefore, we can extend the spacetime to that entire region. Since the scale factor is periodic, each spacetime region is characterized by the quantity

$$
\frac{2}{\pi}\frac{H_\Lambda}{\bar{a}}\lambda \in (n, n+1)\ , \tag{A.5.57}
$$

where $n \in \mathbb{Z}$ denotes the n-th copy of the original spacetime. The Penrose diagram of the entire spacetime is depicted in Fig. A.3.

The resulting maximally extended geodesically complete spacetime is a cyclic universe with periodically repeating phases of expansion and contraction, and the transitions from contraction to expansion at $\lambda = n\pi\bar{a}/2H_\Lambda$ are bounces. With $a(\lambda)$ given by Eq. (A.5.54), we notice that $a(\lambda = n\pi\bar{a}/2H_\Lambda) = 0$, so the scale factor vanishes at each bounce point. Yet, those bounces are nonsingular since the boundary region at $\lambda = n\pi\bar{a}/2H_\Lambda$ cannot be described by the usual FLRW coordinate system. Rather, we see that the metric of Eq. (A.5.56) reduces to Eq. (A.4.30) when $\lambda = n\pi\bar{a}/2H_\Lambda$, which is the correct description of the nonsingular bouncing surfaces. In other words, the 'physical' scale factor is the one with respect to the comoving observer in the flat FLRW spacetime, and it is meaningless on $\mathscr{B}^\pm$ (or on any other boundary region at $\lambda = n\pi\bar{a}/2H_\Lambda$). This is very similar to the case of de Sitter space, where at $\lambda = 0$, the scalar factor given by Eq. (A.5.34) would appear to be singular. However, there exists a coordinate system in which the bounce is explicitly nonsingular, as can be seen from Eq. (A.5.37).

A.6 Implications for inflationary models

A.6.1 Energy condition for subleading component

We have seen that convergence of the quantity $\dot{H}/a^2$ is crucial to extend an inflationary cosmology beyond the past null boundary $\mathscr{B}^-$. Since there is no contribution from vacuum

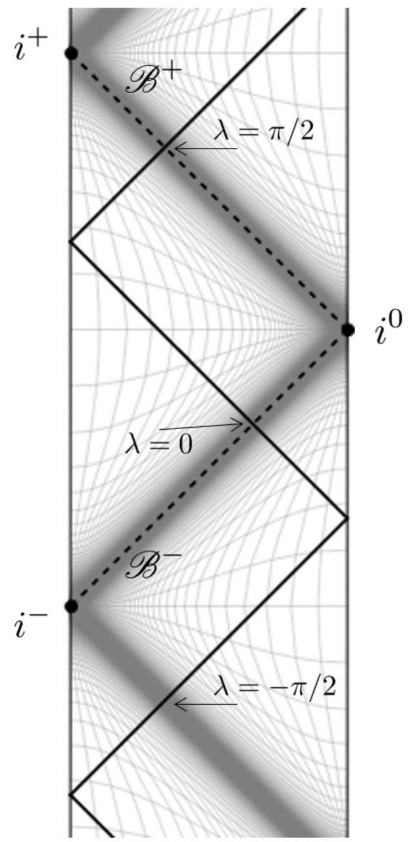

Figure A.3 Penrose diagram of the extendible toy model, where the scale factor is given by Eq. (A.5.49), and we take $\bar{a} = 1$ and $H_\Lambda = 1$ (in Planck units) for illustrative purposes. The original flat FLRW spacetime corresponds to the triangle region surrounded by dashed lines, with i^+ and i^- representing future and past timelike infinity for comoving observers in the original universe. The spacetime can be extended beyond the future and past boundaries $\mathscr{B}^\pm$, and the whole spacetime is geodesically complete.

energy to $\dot{H}$ in Einstein gravity, the behavior of $\dot{H}/a^2$ is determined by the next-to-leading order contribution to the energy density during inflation. In order to describe this situation effectively, let us consider Einstein gravity with a cosmological constant and a fluid which follows the equation of state,

$$P = w\rho\,, \tag{A.6.58}$$

where w is a constant, P and ρ are the pressure and the energy density of the fluid, respectively. We note, though, that a fluid description with the above equation of state might not be a fully realistic situation for inflation, but it serves as an effective description. By solving the conservation equation for the fluid as usual, we obtain

$$\rho \propto a^{-3(1+w)}\,. \tag{A.6.59}$$

We now assume that $w < -1$ so that the vacuum energy is dominant in the total energy density, i.e.

$$\rho_{\rm tot} = \Lambda + \rho \simeq \Lambda \qquad \text{as } a \to 0\,. \tag{A.6.60}$$

In that case, the evolution of the Hubble parameter is dominated by the vacuum energy Λ, and our assumption (A.2.4) is realized. Then using the Friedman equations, we can write the key quantity $\dot{H}/a^2$ as a function of the scale factor a,

$$\frac{\dot{H}}{a^2} = -\frac{1}{2M_{\rm Pl}^2}\frac{\rho + P}{a^2} \propto \frac{1+w}{a^{5+3w}}\,, \tag{A.6.61}$$

which converges in the limit $a \to 0$ only if $w = -1$ or $5 + 3w \leq 0$. The former case exactly corresponds to the vacuum energy Λ. Thus, if there is a correction to the vacuum energy, the equation of state of the next-to-leading order component has to satisfy the latter condition,

$$w \leq -\frac{5}{3}\,. \tag{A.6.62}$$

This clarifies the discontinuous nature between exact de Sitter space and inflationary cosmology satisfying Eq. (A.2.4): although the spacetime is nonsingular if w is exactly equal to -1, any arbitrary small deviation from $w = -1$ leads to a p. p. curvature singularity. The singularity is avoided only if the next-to-leading order component to the energy density goes to zero fast enough as $a \to 0$. The above result shows that it is not enough to violate the Null Energy Condition with $w < -1$ since an equation of state parameter in the range $w \in (-5/3, -1)$ would still lead to a p. p. curvature singularity.

We note that the condition $w \leq -5/3$ generally ensures C^0 extendibility. However, if $-5-3w$ is an integer in addition to the condition $w \leq -5/3$ (i.e. if $w \in \mathbb{Z}_{\leq -5}/3$), then the spacetime is C^∞ extendible as shown in appendix A.8. If w is not an integer multiple of $1/3$ but still $w \leq -5/3$, then the spacetime is at most $C^{\lfloor -3w \rfloor - 5}$ extendible.

A.6.2 Single field slow-roll inflation

Let us derive the expression for the key quantity $\dot{H}/a^2$ in the case of slow-roll inflation models driven by a canonical scalar field φ with a potential $V(\varphi)$. The complete set of equations of motion is given by[5]

$$H \simeq \sqrt{\frac{V(\varphi)}{3M_{\mathrm{Pl}}^2}}\,, \qquad \dot{\varphi} \simeq -\frac{V'(\varphi)}{3H}\,, \tag{A.6.63}$$

provided the following slow-roll approximations are satisfied:

$$\frac{|\ddot{\varphi}|}{3H|\dot{\varphi}|} \ll 1\,, \qquad \frac{\dot{\varphi}^2}{2V(\varphi)} \ll 1\,. \tag{A.6.64}$$

From these equations, we can write the scale factor as a function of φ,

$$a(\varphi) = a_e e^{-\mathcal{N}(\varphi)}\,, \tag{A.6.65}$$

with $\mathcal{N}(\varphi)$, the e-folding number, given by

$$\mathcal{N}(\varphi) \simeq \frac{1}{M_{\mathrm{Pl}}^2}\int_{\varphi_e}^{\varphi} \mathrm{d}\varphi\, \frac{V}{V'}\,. \tag{A.6.66}$$

Here, a_e and φ_e are the values of a and φ at the end of inflation respectively.

Using Eqs. (A.6.63) and (A.6.65), we can then write the key quantity $\dot{H}/a^2$ in terms of φ,

$$\frac{\dot{H}}{a^2} \simeq -\frac{1}{6a^2}\frac{(V')^2}{V} = -\frac{1}{6a_e^2}\frac{(V')^2}{V}e^{2\mathcal{N}(\varphi)} = -\frac{1}{6a_e^2}f(\varphi)\,, \tag{A.6.67}$$

[5] From here on, a prime denotes a derivative with respect to the argument of the function.

where the function f is defined by

$$f(\varphi) \equiv \frac{(V')^2}{V} e^{2\mathcal{N}(\varphi)} \simeq \frac{(V')^2}{V} \exp\left(\frac{2}{M_{\rm Pl}^2} \int_{\varphi_e}^{\varphi} \mathrm{d}\varphi \, \frac{V}{V'} \right) . \tag{A.6.68}$$

Thus, a necessary condition for a slow-rolling inflationary cosmology to be free of p. p. singularities can be written as

$$\lim_{\varphi \to \varphi(-\infty)} f(\varphi) = \text{finite} \,, \tag{A.6.69}$$

where $\varphi(-\infty)$ is the value of $\varphi(t)$ in the limit $t \to -\infty$. Thus, for any given potential $V(\varphi)$, we can judge the presence of a singularity by evaluating (A.6.68) and its limit. In the rest of this section, we evaluate (A.6.69) for the Starobinsky model and for a small field inflationary model.

Starobinsky model

Let us consider the Starobinsky model [590] with Einstein frame potential given by

$$V(\varphi) = \frac{3}{4} m^2 M_{\rm Pl}^2 \left(1 - e^{-\sqrt{\frac{2}{3}} \frac{\varphi}{M_{\rm Pl}}} \right)^2 , \tag{A.6.70}$$

where m is the 'mass' of the inflaton. Inflation occurs at large positive field values, and φ slowly rolls toward 0 as time t increases. Inversely, φ approaches $+\infty$ in the limit $t \to -\infty$. In that limit, $V(\varphi) \simeq 3m^2 M_{\rm Pl}^2/4 =$ constant, so the potential acts like a cosmological constant, i.e. the spacetime is asymptotically de Sitter. With the above potential, the e-folding number $\mathcal{N}$ can be evaluated following Eq. (A.6.66), and one finds

$$\mathcal{N}(\varphi) \simeq \frac{3}{4} e^{\sqrt{\frac{2}{3}} \frac{\varphi}{M_{\rm Pl}}} \qquad \text{as } \varphi \to \infty \,. \tag{A.6.71}$$

Since the quantity $(V')^2/V$ is given by

$$\frac{(V')^2}{V} = 2m^2 e^{-2\sqrt{\frac{2}{3}} \frac{\varphi}{M_{\rm Pl}}} \,, \tag{A.6.72}$$

it cannot suppress the factor of $e^{2\mathcal{N}}$ in the expression for $f(\varphi)$. Indeed,

$$f(\varphi) = 2m^2 e^{2\mathcal{N}(\varphi)-2\sqrt{\frac{2}{3}}\frac{\varphi}{M_{\rm Pl}}} \simeq 2m^2 \exp\left(\frac{3}{2}\exp\left[\sqrt{\frac{2}{3}}\frac{\varphi}{M_{\rm Pl}}\right]\right) \to \infty \qquad \text{(A.6.73)}$$

as $\varphi \to \infty$. We see from Eq. (A.6.72), which is proportional to $\dot{H}$, that the Hubble parameter approaches a constant exponentially fast as $\varphi \to \infty$ in field space. This matches the intuition that Starobinsky inflation rapidly approaches de Sitter in field space (at large field values). However, the scale factor reaches zero even faster than $\dot{H}$ as $a \sim \exp(-\exp(\varphi))$. The subtlety comes from the fact that the potential is very flat at large field values, which implies that large time intervals are needed for small field displacements. Consequently, de Sitter is actually approached only very slowly in physical time compared to the rate at which the scale factor goes to zero. Thus, the ratio $\dot{H}/a^2$ and equivalently $f(\varphi)$ blow up, and the spacetime is inextendible. We can conclude that if Starobinsky inflation starts from the infinite past at $t \to -\infty$ for comoving observers, then the past boundary $\mathscr{B}^-$ must be a p. p. curvature singularity.

We note, however, that Starobinsky inflation is unlikely to start from the infinite past in the first place. Indeed, this would require the initial field velocity to exactly vanish, which represents extreme fine-tuning. In general, the field equation of motion is $\ddot{\varphi} + 3H\dot{\varphi} \simeq 0$ for a nearly flat potential, and with $\dot{\varphi} \neq 0$ initially, this implies $\dot{\varphi} \propto a^{-3}$ and $\rho \sim a^{-6}$ (kinetic domination as $a \to 0$). Accordingly, the first slow-roll approximation in Eq. (A.6.64) would not be satisfied. This is known as ultra-slow-roll or non-attractor inflation (see, e.g., [173, 256, 392, 601]). In that situation, the effective equation of state parameter w would tend to unity as $a \to 0$, and the Universe would be past incomplete (standard Big Bang curvature singularity).

Another comment is in order: Starobinsky inflation with the Einstein frame potential $V(\varphi)$ given above is equivalent to an $f(R)$ modified theory of gravity of the form $f(R) = R + R^2/(6m^2)$ in the Jordan frame after a conformal transformation. It would be natural to extend the above analysis to inflationary scenarios with different $f(R)$ theories of gravity, e.g., slight deviations from Starobinsky inflation or generalizations thereof (see, e.g., [501]). This shall be the subject of a follow-up study.

Small Field inflation

Let us consider another slow-roll inflation model with a Higgs-like potential [472, 614] of the form

$$V(\varphi) = V_0 \left(1 - \left(\frac{\varphi}{2m}\right)^2\right)^2 , \tag{A.6.74}$$

where V_0 is a positive constant, and m is another mass scale. We would like to focus on small field inflation, which occurs when[6] $\varphi \ll m$. We note, however, that such small field inflation models are unstable against initial condition fluctuations [103, 324]. From the above potential, we find that the e-folding number is given by

$$\mathcal{N}(\varphi) = -\frac{m^2}{M_{\rm Pl}^2} \ln\left(\frac{\varphi}{\varphi_e}\right) + \frac{\varphi^2 - \varphi_e^2}{8M_{\rm Pl}^2} , \tag{A.6.75}$$

and it diverges in the limit $\varphi \to 0$. Thus, the limit $t \to -\infty$ corresponds to the limit $\varphi \to 0$. Then, $f(\varphi)$ can be evaluated as follows,

$$f(\varphi) = \frac{V_0}{m^4}\varphi^2 \left(\frac{\varphi}{\varphi_e}\right)^{-2\frac{m^2}{M_{\rm Pl}^2}} e^{\frac{\varphi^2-\varphi_e^2}{4M_{\rm Pl}^2}} , \tag{A.6.76}$$

and one finds that $f(\varphi)$ converges in the limit $\varphi \to 0$ when $m \leq M_{\rm Pl}$. Thus, if small field inflation starts from the infinite past at $t \to -\infty$ for comoving observers, then the Universe can be continuously (C^0) extended beyond the past boundary $\mathscr{B}^-$. In other words, noncomoving geodesics exit the original inflationary region sufficiently far in the past. However, the above does not tell us whether the past boundary $\mathscr{B}^-$ is C^r extendible for $r \geq 1$, and the spacetime could very well be C^1 inextendible. This remains to be verified, but the condition (A.6.69) would be much more complicated.

A.6.3 Limiting curvature models

In this subsection, we demonstrate how to evaluate the key quantity $\dot{H}/a^2$ in the limit corresponding to de Sitter in a class of modified gravity models. Specifically, we focus on a

[6] Chaotic inflation is also possible when φ is much larger than m, but we note that in the case of such chaotic inflation models (and more generally for $V(\varphi) \propto \varphi^p$, $p > 0$), the inflaton potential energy diverges as $\varphi \to \infty$ (i.e. $t \to -\infty$). Consequently, the effective description of the inflationary cosmology based on classical gravity would no longer be valid near $\mathscr{B}^-$. Thus, the present analysis cannot address the past extendibility of such inflationary models.

gravitational theory with limiting curvature, which is claimed to have nonsingular cosmological solutions, as proposed and investigated in Refs. [143, 516, 635] (see also references therein, and Refs. [197, 265, 266] and Refs. [196, 600, 633] for other applications to cosmological and black hole spacetimes respectively). Let us briefly review the theory and the inflationary solutions investigated in Ref. [635]. The action of this theory is given by

$$S = \frac{M_{\rm Pl}^2}{2} \int \mathrm{d}^4 x \sqrt{-g} \left(R + \sum_{j=1}^{2} \left[\chi_j I_j - V_j(\chi_j) \right] \right) , \tag{A.6.77}$$

where χ_1 and χ_2 are scalar fields, and I_1 and I_2 are curvature invariant functions, which reduce to

$$I_1^{\rm FLRW} = 12H^2 \,, \qquad I_2^{\rm FLRW} = -6\dot{H} \,, \tag{A.6.78}$$

in a flat FLRW spacetime. We note that there are many choices of curvature invariant functions which satisfy the condition (A.6.78), and the stability of the cosmological perturbations[7] strongly depends on the choice of I_1 and I_2. However, the background dynamics can be uniquely determined only from the condition (A.6.78).

Varying the action (A.6.77) with respect to χ_1 and χ_2 gives rise to the following constraint equations at the background level,

$$12H^2 = V_1'(\chi_1) \,, \qquad -6\dot{H} = V_2'(\chi_2) \,, \tag{A.6.79}$$

which ensure the finiteness of H and $\dot{H}$ if V_1' and V_2' are finite for any value of χ_1 and χ_2. This is the mechanism to limit the divergence of the curvature invariants in this theory. The other independent equation of motion is given by (see Ref. [635] for a derivation)

$$(1 - 2\chi_1 - 3\chi_2)H^2 - \dot{\chi}_2 H - \frac{V_1 + V_2}{6} = 0 \,. \tag{A.6.80}$$

The above equations of motion can be rewritten as a set of first-order differential equations only involving a, χ_1, and χ_2. In particular, the $\chi_1 - \chi_2$ phase space trajectories are governed

[7] See Ref. [635] for examples of covariant curvature invariant functions that reduce to Eq. (A.6.78) and for the analysis of the perturbations.

by the following equation:

$$\frac{\mathrm{d}\chi_2}{\mathrm{d}\chi_1} = \frac{V_1''}{4V_2'}\left(3\chi_2 + 2\chi_1 - 1 + \frac{2(V_1+V_2)}{V_1'}\right). \tag{A.6.81}$$

Similarly, the solutions in the $\chi_2 - a$ space satisfy the following equation:

$$\frac{\mathrm{d}\chi_2}{\mathrm{d}\ln a} = -\left(3\chi_2 + 2\chi_1 - 1 + \frac{2(V_1+V_2)}{V_1'}\right). \tag{A.6.82}$$

In order to determine if an inflationary solution [one which satisfies Eq. (A.2.4)] is past (in)complete, we need to check the limit of the ratio $\dot{H}/a^2$ as $a \to 0$ (which is equivalent to the limit $t \to -\infty$ when Eq. (A.2.4) is satisfied). In order for the spacetime to be asymptotically de Sitter, the potentials are going to be chosen as follows [143, 635]:

$$V_1(\chi_1) = 12H_{\mathrm{max}}^2 \frac{\chi_1^2}{1+\chi_1}\left(1 - \frac{\ln(1+\chi_1)}{1+\chi_1}\right), \tag{A.6.83}$$

$$V_2(\chi_2) = -12H_{\mathrm{max}}^2 \frac{\chi_2^2}{1+\chi_2^2}, \tag{A.6.84}$$

where H_{max} is a positive constant. Then, one can use Eq. (A.6.81) to draw the phase space trajectories as shown in Fig. A.4. As it is clear from the diagram[8] in the limit $t \to -\infty$, trajectories go to $\chi_1 \to$ constant and $\chi_2 \to \pm\infty$, and in that limit, the trajectories are asymptotically de Sitter [143]. Taking the limit $|\chi_2| \to \infty$ with χ_1 kept constant, Eq. (A.6.82) with the potentials (A.6.83) and (A.6.84) reduces to

$$\frac{\mathrm{d}\chi_2}{\mathrm{d}\ln a} \simeq -3\chi_2, \tag{A.6.85}$$

and upon integration, the solution is

$$\chi_2(a) \simeq \frac{K}{a^3}, \tag{A.6.86}$$

where K is an integration constant. Substituting this solution into Eq. (A.6.79) with the

[8] We note that only the region where $\chi_2 < 0$ was plotted in Ref. [635]. This region corresponds to the region $\dot{H} < 0$, because I_1 and I_2 satisfy the condition (A.6.78) only when $\dot{H} < 0$ in the model investigated there. However, in general, some trajectories enter the region where $\chi_2 \geq 0$ if I_1 and I_2 are appropriately defined in this region.

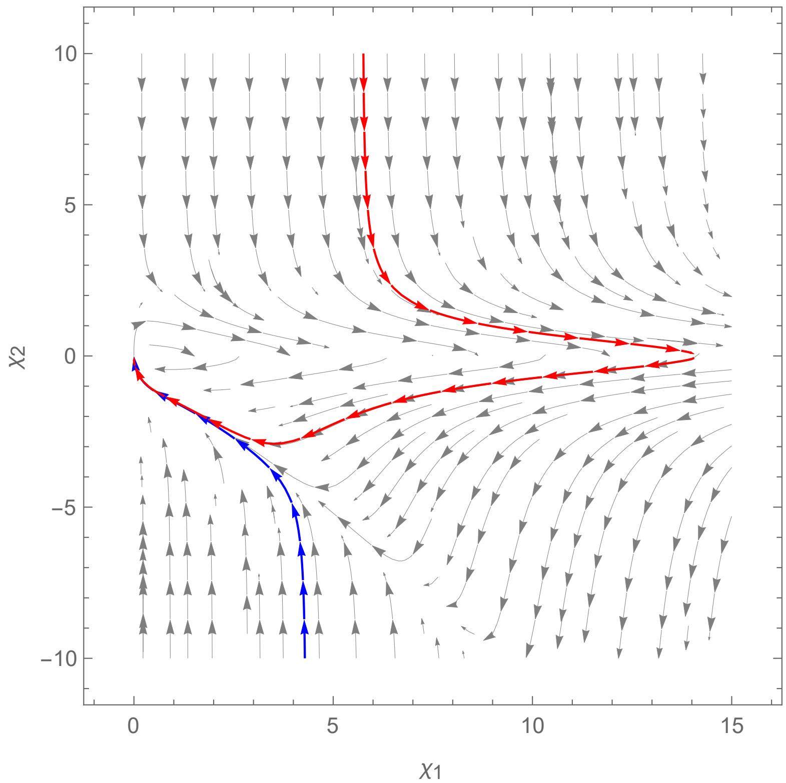

Figure A.4 Trajectories in the $\chi_1 - \chi_2$ phase space following Eq. (A.6.81) with the potentials (A.6.83) and (A.6.84). The arrows point forward in time. In the limit $t \to -\infty$, there are two kinds of trajectories: those that go to $\chi_1 \to$ constant and $\chi_2 \to +\infty$ (an example is highlighted in red) and those that go to $\chi_1 \to$ constant and $\chi_2 \to -\infty$ (an example is highlighted in blue).

potential of Eq. (A.6.84), one obtains

$$\frac{\dot{H}(a)}{a^2} \simeq \frac{1}{a^2}\frac{4KH_{\max}^2 a^9}{(K^2+a^6)^2} \simeq \frac{4H_{\max}^2}{K^3}a^7 \to 0 \qquad \text{as } a \to 0\,. \tag{A.6.87}$$

Therefore, the above trajectories that are asymptotically de Sitter are *not* past incomplete; one could construct an extension beyond the past boundary $\mathscr{B}^-$. More precisely, the spacetime is smoothly (C^∞) extendible since $\dot{H} \sim a^q$ with $q = 9 \in \mathbb{Z}_{\geq 2}$ (see appendix A.8).

Finally, we would like to comment on the stability of these solutions. The stability against cosmological perturbations strongly depends on the choice of I_1 and I_2, which satisfy the condition (A.6.78). In Ref. [635], the stability for two kinds of curvature invariant functions was investigated. From those results, it appears the trajectories that are asymptotically de

Sitter are unstable in the infinite past (when $\chi_1 \to$ constant and $\chi_2 \to \infty$). Therefore, although the solutions shown above are examples of past extendible inflationary cosmologies in modified gravity, they remain at the level of toy models that cannot describe the real Universe.

A.7 Summary and discussion

In the present paper, we showed that an inflationary spacetime with flat spatial curvature [i.e. a flat FLRW spacetime with a scale factor which satisfies the condition (A.2.4)] has a p. p. curvature singularity if the quantity $\dot{H}/a^2$ diverges in the limit $t \to -\infty$. On the other hand, if $\dot{H}/a^2$ converges, then the past boundary is regular and continuously (C^0) extendible. We presented concrete examples of both inextendible and extendible models in Sec. A.5. In the context of Einstein gravity with a cosmological constant and a perfect fluid, which follows the equation of state $p/\rho = w =$ constant, we found that the p. p. curvature singularity is only avoidable if $w \leq -5/3$. In the case of slow-roll inflation with a canonical scalar field, the key quantity $\dot{H}/a^2$ can be written in terms of the inflaton potential, and we derived the condition to judge whether the past boundary is singular or not for the given potential. By using this formula, we found that Starobinsky inflation has a C^0 p. p. curvature singularity, but a small field inflation model does not. Moreover, in the context of a theory of modified gravity with limiting curvature as investigated in Refs. [143, 635], we computed the asymptotic expression for $\dot{H}/a^2$ and determined that the inflationary solutions are smoothly (C^∞) extendible.

Throughout this paper, we have discussed extendibility of flat, asymptotically de Sitter, FLRW spacetimes. Of course, it could be possible that there is a noninflationary epoch before inflation or that inflation never occurs in the very early universe. This would necessarily happen if the vacuum energy became subdominant in the limit $a \to 0$. In Einstein gravity, candidates are, for example, spatial curvature with $\rho \propto a^{-2}$, dust with $\rho \propto a^{-3}$, radiation with $\rho \propto a^{-4}$, or anisotropies with $\rho \propto a^{-6}$. If there is a positive spatial curvature component, the early stages of inflation would be described by closed de Sitter space, where the universe enters a contracting phase sufficiently far in the past. In the case of a negative spatial curvature, we expect the situation to be similar to the flat case, because open de Sitter space also has a past boundary $\mathscr{B}^-$ that is extendible. If some thermal matter components or anisotropies are dominant over other components, it would lead to a Big Bang initial singularity or a Belinsky-Khalatnikov-Lifshitz singularity [67]. However, this might not be the case in a quantum theory of gravity (see, e.g., [414]). We stress here that the entire analysis performed in this paper is in the realm of classical General Relativity. The situation is certainly expected to be different when we better understand the quantum gravity effects at high energies.

We also only considered spacetimes that are perfectly homogeneous and isotropic. Thus, we did not include the effects of anisotropies or cosmological perturbations. The presence of cosmological perturbations would most likely change the criterion for past extendibility, as known in the case of eternal inflation models [468]. In such a case, our analysis is not applicable. Therefore, an interesting direction is to investigate how to develop the present analysis in the context of eternal inflation models, as it was done for the singularity theorems by Borde and Vilenkin [87, 89].

A.8 General extendibility

In this appendix, let us precisely define the concepts of singularity and extendibility. For fully rigorous mathematical definitions and theorems, we refer to Refs. [209, 210, 271, 348].

Let us call a spacetime $(\mathcal{M}, \boldsymbol{g})$, where $\mathcal{M}$ is the manifold and $\boldsymbol{g}$ the metric tensor, to be of class C^r when C^r ($r \in \mathbb{Z}_{\geq 0}$) is the differentiability class of the Riemann tensor. This is equivalent to the metric tensor being of class C^{r+2}. Then, a boundary $\mathscr{B} \subset \mathcal{M}$ is a C^r p. p. curvature singularity if any component of the r-th covariant derivative of the Riemann tensor is not of class C^0 in the limit toward $\mathscr{B}$ when measured in a p. p. tetrad basis $\boldsymbol{e}^M$. In particular, if the spacetime is conformally flat, i.e. if the Weyl tensor vanishes, then the Riemann tensor is fully determined by the Ricci tensor. In that case, $\mathscr{B}$ is a C^r p. p. curvature singularity if the quantity

$$\nabla_{\mu_1} \nabla_{\mu_2} \cdots \nabla_{\mu_r} R_{\mu_{r+1}\mu_{r+2}} \bigotimes_{j=1}^{r+2} \mathbf{d}x^{\mu_j} \tag{A.8.88}$$

expanded in terms of the $\boldsymbol{e}^M$'s diverges in the limit toward $\mathscr{B}$. Then, we say that the spacetime is C^r extendible if and only if there is no C^r p. p. curvature singularity. Similarly, the spacetime is C^r inextendible if and only if there is a C^r p. p. curvature singularity.

Following the above statements, the requirement derived at the end of Sec. A.3 for an inflationary spacetime is pertaining to a C^0 p. p. singularity and C^0 (continuous) (in)extendibility. However, the statement can be generalized. Starting with one covariant derivative of the Ricci tensor in FLRW, one can write

$$\begin{aligned}\nabla_\sigma R_{\mu\nu}\mathbf{d}x^\sigma \otimes \mathbf{d}x^\mu \otimes \mathbf{d}x^\nu =& -2(\ddot{H}-2\dot{H}H)(\hat{\boldsymbol{e}}^0)^{\otimes 3} + 4\dot{H}H\eta_{MN}\hat{\boldsymbol{e}}^M \otimes \hat{\boldsymbol{e}}^{(N} \otimes \hat{\boldsymbol{e}}^{0)} \\ &+ (\ddot{H}+6\dot{H}H)\eta_{MN}\hat{\boldsymbol{e}}^0 \otimes \hat{\boldsymbol{e}}^M \otimes \hat{\boldsymbol{e}}^N\end{aligned} \tag{A.8.89}$$

in the FLRW tetrad basis. Thus, when assumption (A.2.4) is satisfied, $\ddot{H} \to 0$ and $\dot{H}H \to 0$, and there appears to be no scalar polynomial curvature singularity. However, when transforming to the p. p. tetrad basis, one finds

$$\nabla_\sigma R_{\mu\nu}\mathbf{d}x^\sigma \otimes \mathbf{d}x^\mu \otimes \mathbf{d}x^\nu \simeq -\frac{(\ddot{H}-2\dot{H}H)a_0^3}{4a^3}(\boldsymbol{e}^0 - \boldsymbol{e}^1)^{\otimes 3} \tag{A.8.90}$$

to leading order in the limit $a \to 0$. Therefore, as the spacetime approaches de Sitter, the Hubble parameter is asymptotically a constant, and one needs to check the convergence of two quantities, $\ddot{H}/a^3$ and $\dot{H}/a^3$, in order to assess C^1 (continuously differentiable) extendibility. Strictly speaking, there are also subleading terms to Eq. (A.8.89) of the form $\ddot{H}/a$ and $\dot{H}/a$ that could generally diverge as $a \to 0$. However, if $\ddot{H}/a^3$ and $\dot{H}/a^3$ are shown to be convergent, then necessarily the quantities $\ddot{H}/a$ and $\dot{H}/a$ approach 0 as $a \to 0$.

Equivalently, one may evaluate the covariant derivative of the Ricci tensor in the coordinate system defined in Sec. A.4, where $\mathbf{d}\lambda \equiv a\,\mathbf{d}t$. That way, the Ricci tensor is given by Eq. (A.4.31), and its covariant derivative is found to be

$$\begin{aligned}\nabla_\sigma R_{\mu\nu}\mathbf{d}x^\sigma \otimes \mathbf{d}x^\mu \otimes \mathbf{d}x^\nu =& -\frac{2(\ddot{H}-2\dot{H}H)}{a^3}(\mathbf{d}\lambda)^{\otimes 3} \\ &+ \frac{2\dot{H}H}{a}g_{\alpha\beta}\left(\mathbf{d}x^\alpha \otimes \mathbf{d}x^\beta \otimes \mathbf{d}\lambda + \mathbf{d}x^\alpha \otimes \mathbf{d}\lambda \otimes \mathbf{d}x^\beta\right) + \frac{\ddot{H}+6\dot{H}H}{a}g_{\alpha\beta}\mathbf{d}\lambda \otimes \mathbf{d}x^\alpha \otimes \mathbf{d}x^\beta\,.\end{aligned} \tag{A.8.91}$$

It is straightforward to see that the coefficient of the $\mathrm{d}\lambda^3$ term follows from evaluating $\partial_\lambda F$, where $F \equiv -2\dot{H}/a^2$ is the coefficient of the $\mathrm{d}\lambda^2$ term in Eq. (A.4.31). For the second derivative of the form $\nabla_\omega\nabla_\sigma R_{\mu\nu}$, the most divergent component is the coefficient of the $\mathrm{d}\lambda^4$ term, and it is given by $\partial_\lambda^2 F$. In general, for the r-th covariant derivative, it is $\partial_\lambda^r F$.

In sum, we arrive at the following statement:

Theorem A.2. *A flat, asymptotically de Sitter, FLRW spacetime with boundary $\mathscr{B}^-$ at $t \to -\infty$ (equivalently $a \to 0$ or $\lambda \to 0$) is C^r inextensible and $\mathscr{B}^-$ is a C^r p. p. curvature singularity if*

$$\left| \lim_{\lambda\to 0} \frac{\partial^r}{\partial\lambda^r}\left(\frac{\dot{H}}{a^2}\right) \right| = \infty \,. \tag{A.8.92}$$

Alternatively, $\mathscr{B}^-$ is not a C^r p. p. curvature singularity and the spacetime is C^r extendible if the quantity $\partial_\lambda^r(\dot{H}/a^2)$ is finite as $\lambda \to 0$.

For $r = 0$, this is the statement given at the end of Sec. A.3. For $r = 1$, one needs to evaluate the limit of

$$\partial_\lambda\left(\frac{\dot{H}}{a^2}\right) = \frac{1}{a}\partial_t\left(\frac{\dot{H}}{a^2}\right) = \frac{\ddot{H} - 2\dot{H}H}{a^3}\,, \tag{A.8.93}$$

and so on. Alternatively, one can check that

$$\lim_{\lambda\to 0} \partial_\lambda^r\left(\frac{\dot{H}}{a^2}\right) = H_\Lambda^r \lim_{a\to 0} \partial_a^r\left(\frac{\dot{H}}{a^2}\right), \tag{A.8.94}$$

provided the spacetime has already been shown to be C^{r-1} extendible. Interestingly, this implies that C^∞ (infinitely differentiable) extendibility is possible if, as $a \to 0$, $\dot{H} \sim a^q$ with $q \in \mathbb{Z}_{\geq 2}$. Indeed, in that case, $\partial_a^r(a^{-2}\dot{H}) \sim a^{q-2-r} \to 0$ for $r < q-2$; $\partial_a^r(a^{-2}\dot{H}) \sim$ constant for $r = q-2$; and $\partial_a^r(a^{-2}\dot{H}) = 0$ for $r > q-2$. If $\dot{H} \sim a^q$ with q not an integer but still $q > 2$, then the spacetime is at most $C^{\lfloor q \rfloor - 2}$ extendible. As expected, exact de Sitter space with $\dot{H} \equiv 0$ is another example of C^∞ extendibility.

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