On the Additive Graph Generated by a Subset of the Natural Numbers

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Abstract

In this thesis, we are concerned with finite simple graphs. Given a subset S of $\{3, 4, \ldots, 2n-1\}$, the additive graph generated by S has vertex set $V = [n]$ and edge set E, with $(i, j) \in E$ if and only if $i + j \in S$. The focus of this thesis is the relationship between generating sets S and monotone properties in the corresponding graphs. We make the first known investigation of the *Traversal by Prime Sum Problem*, in which the set S is the prime numbers and the property of interest is a Hamilton cycle. A number of new results are proved concerning both these graphs and the additive graphs for which the set S is the practical numbers.

For any subset S of $\{3, 4, \ldots, 2n-1\}$, we prove that the $|S|$-closure of the additive graph generated by S is the complete graph; this allows for the determination of tight thresholds for a number of monotone properties in terms of $|S|$ using results from closure theory. These graphs are shown to be the first known wide-ranging and representative subclass of graphs with complete k-closure, and they afford a new and simple construction of minimum graphs with complete k-closure. Finally, as an example of the number-theoretic interpretations of these graphs and their properties, we generalize a theorem by Cramer concerning prime numbers to a number of different sequences.

Abrégé

Étant donné un sous-ensemble S de $\{3, 4, \ldots, 2n-1\}$, le graphe additif engendré par S a un ensemble de sommets $V = [n]$ et un ensemble d'arrêtes E, telle que $(i, j) \in E$ si et seulement si $i+j \in S$. L'objectif de cette thèse est l'étude des relations entre l'ensemble S et les propriétés monotones du graphe additif correspondant. On effectue les premières recherches connue sur le *Traversal By Prime Sum Problem*, problème dans lequel l'ensemble S correspond à l'ensemble des nombres premiers et la propriété de graphe recherchée est l'existence d'un cycle hamiltonien. De nouveaux résultats sont établis pour ce problème ainsi que dans le cas où S est l'ensemble des entiers pratiques.

Pour un tel S quelconque, on démontre que la $|S|$-fermeture du graphe additif engendré par S est un graphe complet. Ainsi, en utilisant les résultats de la théorie de la fermeture on parvient à déterminer le seuil pour plusieurs propriétés monotones des graphes en terme de $|S|$. Ces graphes sont les premiers représentant connus d'une large sous-classe de graphe k-fermés complets. Ils permettent de donner une construction nouvelle et simple de graphe fermés et complets minimaux. Enfin, comme exemple d'interprétation arithmétique de ces graphes et de leurs propriétés, on généralise un théorème de Cramer sur les nombres premiers à

d’autres suites d’entiers.

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Chapter 1

Introduction

This thesis is concerned with a natural family of *finite simple graphs* recently suggested by Vašek Chvátal [8]. Fix any *natural number* n and any subset S of the natural numbers. Then the *additive graph generated by* S, denoted by $G(n, S)$, is the graph with *vertex set* $V = [n] = \{1, 2, \ldots, n\}$ and *edge set* E, with $(i, j) \in E$ if and only if $i + j \in S$. An element $s \in S$ *induces an edge* in $G(n, S)$ if there exist distinct vertices i and j such that $i + j = s$. For a fixed n, only elements in the set $\{1+2, 1+3, \ldots, (n-1)+n\}$ can induce edges, and hence we will require that S be a subset of $\{3, 4, \ldots, 2n-1\}$. Such a set S is referred to as a *generating set*. The additive graph of *order* 8 generated by $S = \{3, 4, 6, 8, 12, 14, 15\}$ is drawn as Figure 1.1.

The motivation for this definition was the *Traversal by Prime Sum Problem*, a long-standing open problem that asks whether for every $m \geq 2$ there is a *Hamilton cycle* in the graph $G(2m, P)$, where P is the set of *prime numbers* in $\{3, 4, \ldots, 4m-1\}$ [50]. In fact, these graphs were first considered briefly by Alladi *et al.* [1] thirty years ago in interpreting a result concerning additive *partitions* of the set of natu-

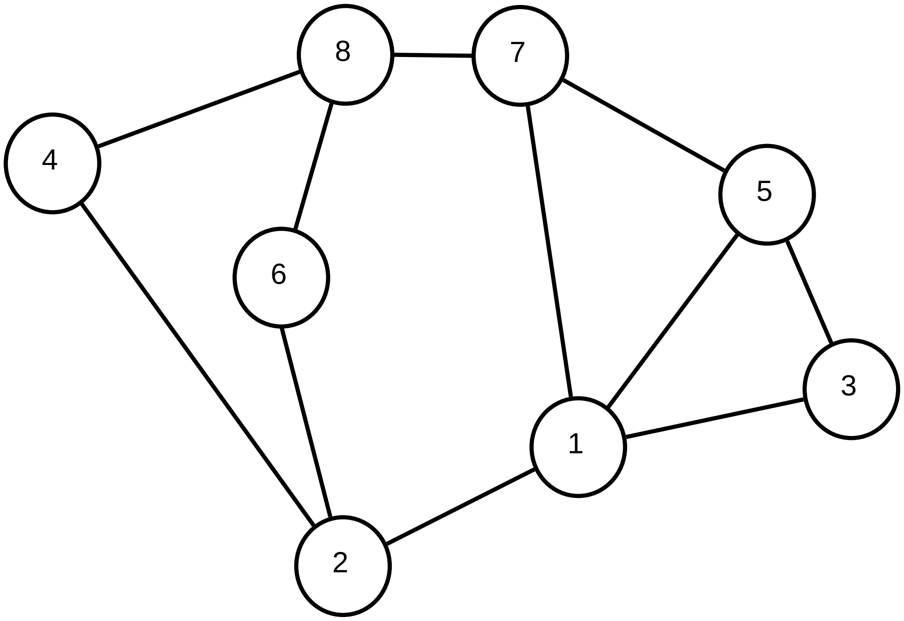

Figure 1.1: The additive graph $G(8, \{3, 4, 6, 8, 12, 14, 15\})$.

ral numbers. A subclass of these graphs was also considered around this time by Erdős and Silverman [14], with regard to a conjecture about the *chromatic number* of a graph whose vertices are the natural numbers and whose edges are precisely those pairs of vertices that sum to an r-th power. In both of these cases, the infinite analogues of the additive graphs defined by Chvátal were actually those of interest. Finite additive graphs were soon after studied by a small group of Russian mathematicians, but unfortunately only a few of the resulting papers are accessible and have been translated [21, 22, 23, 35]. Their notion of "arithmetic graphs" arose from the idea that graphs represented as additive graphs generated by subsets of the natural numbers require less computer memory for storage. They did not in general require that $V = [n]$, and they were primarily interested in the minimum cardinality of a set S needed to generate a given graph.

Despite their natural construction and the recognized utility of other *integer graphs* in number theory [39], the properties of these additive graphs do not appear to have been extensively studied. Only recently have graph-theoretic properties such as *connectivity* been considered in somewhat similar objects entitled addition Cayley graphs (see [24] and references therein). For a subset $\mathcal{S}$ of the abelian group $\mathcal{G}$, the *addition Cayley graph* induced by $\mathcal{S}$ on $\mathcal{G}$ is the graph with vertex set $\mathcal{G}$ and two group members *adjacent* if their sum is in $\mathcal{S}$. The graphs we are considering resemble addition Cayley graphs with $\mathcal{G} = (\mathbb{Z}_n, +)$, but devoid of any group structure our graphs possess very different properties.

This thesis is organized as follows. In Chapter 2, we are concerned with generating sets produced by infinite sequences. We begin by introducing the Traversal by Prime Sum Problem and tracing its number theory origins. Using some basic results from combinatorial number theory, we prove that the members of the subclass of additive graphs in question are *k-connected* for $1 \leq k \leq 50$ given a sufficient number of vertices B_k, and that they possess a highly structured 2-*factor*. Furthermore, via a simple heuristic we confirm that these graphs are *hamiltonian* when the number of vertices is less than 100. Comparable results are obtained for a problem of our own devising entitled the *Traversal by Practical Sum Problem*. We also briefly consider other sequences in order to unify the known results in the literature related to these additive graphs.

In Chapter 3, we extend our focus to arbitrary generating sets. Our main result of this chapter is that the $|S|$-closure of $G(n, S)$ is the *complete graph*. By finding bounds on the cardinality of the edge set of $G(n, S)$ in terms of $|S|$, we provide a new and simple construction of minimum graphs with n vertices and complete k-closure for all n and k. Finally, we use the known stability of many *monotone properties* to prove thresholds for these properties in this family of graphs. These graphs are thereby shown to be the first known wide-ranging and representative subclass of complete k-closure graphs.

In Chapter 4, we cite two alternate proofs of the threshold for connectivity and we determine the tight threshold for *triangles*. As an example of the number-theoretic interpretations of these graphs and their properties, a theorem by Cramer [11] concerning prime numbers is generalized to a number of different sequences. We conclude in Chapter 5 by suggesting several avenues for further work, including considerations of random generating sets and some algorithmic questions.

Standard terms from graph theory and number theory are italicized upon their first use, and definitions are provided in the glossary. All graph-theoretic definitions are consistent with those used in the text by Bondy and Murty [4], and all number-theoretic definitions are consistent with those used in the text by Rosen [42].

Chapter 2

Predetermined Sequences as Generating Sets

In this chapter we are interested in generating sets that are predetermined integer sequences, such as the sequence of prime numbers and the *Fibonacci sequence*. We begin by making the first known rigorous investigation of the Traversal by Prime Sum Problem, which concerns the existence of Hamilton cycles in the additive graphs generated by the prime numbers. There are many interesting similarities between the properties of prime numbers and the properties of practical numbers, and so we proceed to consider a related problem we dub the Traversal by Practical Sum Problem. In both cases we show that the additive graphs in question satisfy several necessary conditions for Hamilton cycles. We then briefly turn our attention to the Fibonacci sequence, general *linear recurrence* sequences, and the sequences of r-th powers, in order to unify the known results in the literature related to these additive graphs.

2.1 Traversal by Prime Sum Problem

The Traversal by Prime Sum Problem has its origins in number theory, but we focus on its graph-theoretic interpretation concerning a Hamilton cycle in the additive graph $G(2m, P)$ generated by the prime numbers. A simple heuristic is stated that allows us to find by hand a Hamilton cycle in $G(2m, P)$ for $2m \leq 100$, and we observe that if the twin prime conjecture[1] is true, then $G(2m, P)$ is hamiltonian for infinitely many values of m. Turning to the question of the existence of a Hamilton cycle in these graphs for all orders, we prove that some necessary conditions for Hamilton cycles are satisfied. These include the existence of a 2-factor, 2-connectedness, and the existence of long cycles.

2.1.1 Problem Formulation and a Cycle Extension Procedure

Antonio Filz [16] defined a *prime circle* of order $2m$ to be a circular permutation of the numbers from 1 to $2m$ with each adjacent pair summing to a prime number. A prime circle of order 8 is drawn as Figure 2.1. He posed the question of their existence for all $m \geq 2$. This question has been popularized in the number theory community by Richard Guy [26], who also asks for an asymptotic estimate of the number of such circles of a given order. A similar question was posed independently several years before this by Henry Larson [31]. A *prime chain* is defined to be a sequential arrangement of the integers 1 through n such that the sum of every pair of adjacent numbers is prime. Larson [31] confirmed that such chains exist for $n \leq 50$ and asked "What is the smallest value of n for which there is no prime chain?" Note that the existence of prime circles of all even orders would imply the

[1] There are infinitely many pairs of primes p and $p + 2$.

existence of prime chains for all n.

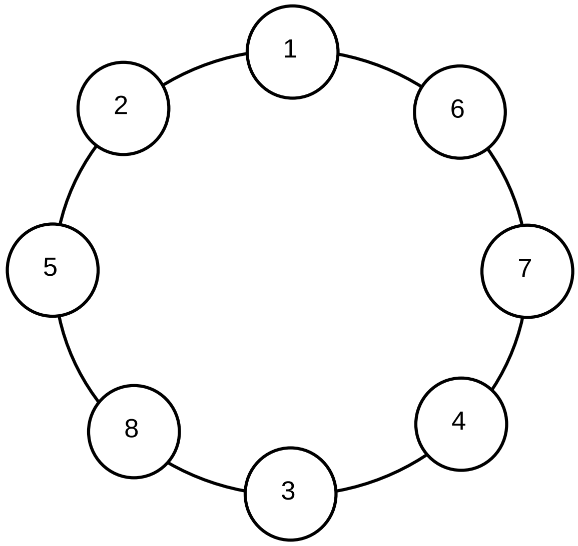

Figure 2.1: A prime circle of order 8.

An equivalent formulation of this problem in graph-theoretic terms is the following: Define $G(2m, P)$ to be the graph with vertex set $V = [2m]$ and $(i, j) \in E$ if and only if $i+j$ is prime. Then prime circles and prime chains correspond to Hamilton cycles and *Hamilton paths*, respectively, and we can ask whether $G(2m, P)$ is hamiltonian for $m \geq 2$. This question is known in the graph theory community, seemingly independent of its number theory origins, by Douglas West [50] and has been termed the *Traversal by Prime Sum Problem*. This is the formulation that we focus on herein.

There are almost no known results pertaining to this problem. West [50] observes that if $2m + 1$ and $2m + 3$ are both prime, then $G(2m, P)$ has the Hamilton cycle

$$(1, 2m, 3, 2m-2, 5, 2m-4, \ldots, 2m-1, 2, 1),$$

and thus if the twin prime conjecture is true, then $G(2m, P)$ is hamiltonian for infinitely many values of m. However, there is no proof of the existence of any infinite hamiltonian subfamily of $G(2m, P)$ graphs.

One procedure for constructing a Hamilton cycle in $G(2m, P)$ given a Hamilton cycle in $G(2m-2, P)$ that we have found works for intermediate values of m is the following. Note that vertex subscripts are modulo $2m-2$. This procedure applied to a Hamilton cycle in $G(16, P)$ is drawn as Figure 2.2.

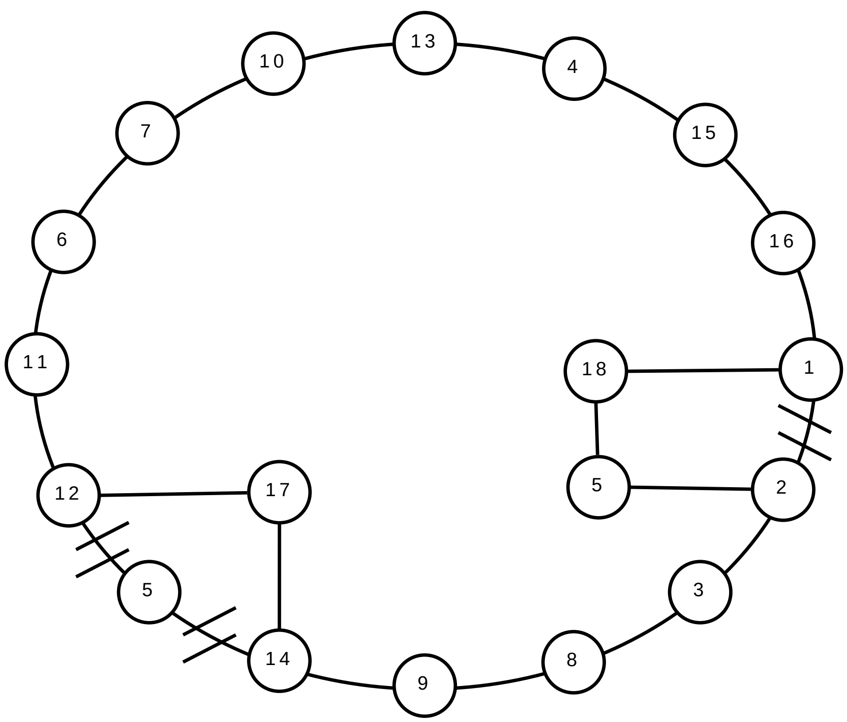

Figure 2.2: Extending a Hamilton cycle in $G(16, P)$ to a Hamilton cycle in $G(18, P)$.

- Input a Hamilton cycle $C = (v_1, v_2, \ldots, v_{2m-2}, v_1)$ in $G(2m-2, P)$.

- If there exist distinct subscripts i and j such that

$$\begin{aligned} v_{i-1} + 2m - 1 &\in P \\ v_{i+1} + 2m - 1 &\in P \\ v_i + 2m &\in P \\ v_j + 2m &\in P \\ v_i + v_{j+1} &\in P \end{aligned}$$

  then output the Hamilton cycle

$$(v_1, v_2, \ldots, v_{i-1}, 2m-1, v_{i+1}, \ldots, v_j, 2m, v_i, v_{j+1}, \ldots, v_{2m-2}, v_1)$$

  in $G(2m, P)$.

- Otherwise, if there exist distinct subscripts i and j such that

$$\begin{aligned} v_{i-1} + 2m &\in P \\ v_{i+1} + 2m &\in P \\ v_i + 2m - 1 &\in P \\ v_j + 2m - 1 &\in P \\ v_i + v_{j+1} &\in P \end{aligned}$$

  then output the Hamilton cycle

$$(v_1, v_2, \ldots, v_{i-1}, 2m, v_{i+1}, \ldots, v_j, 2m-1, v_i, v_{j+1}, \ldots, v_{2m-2}, v_1)$$

  in $G(2m, P)$.

Using this simple heuristic, we have found Hamilton cycles in $G(2m, P)$ for all $2m \leq 100$. The success of this method is not surprising when considered from a probabilistic perspective, under the assumptions that the elements in the Hamilton cycle C are distributed randomly and that the primes in $[n]$ are uniformly distributed with probability $\dfrac{n}{\ln n}$.

2.1.2 Properties of $G(n, P)$

We now consider some necessary conditions for the existence of Hamilton cycles in $G(2m, P)$ for all $m \geq 2$. Our main result is that $G(2m, P)$ has a highly structured 2-factor when m is even, which is constructed by taking the union of two disjoint perfect matchings. As well, we show that $G(n, P)$ is 2-connected for all $n \geq 6$ and that if the twin prime conjecture is true, then connectivity increases without bound as a function of n. We begin by making two important remarks about $G(n, P)$ graphs. Throughout, $\pi(x)$ is used to denote the *prime counting function*.

Remark 2.1.1. *$G(n, P)$ is bipartite for all n, as its vertex set can be partitioned into the set of odd numbers at most n and the set of even numbers at most n.*

Remark 2.1.2. *The degree of vertex i in $G(n, P)$ is exactly $\deg(i) = \pi(i + n) - \pi(i)$. Then, since*

$$\lim_{n\to\infty} \frac{\pi(n)}{n/\ln n} = 1$$

by the prime number theorem, it follows that

$$\deg(i) \sim \frac{n}{\ln n}$$

in $G(n, P)$.[2] *Dirac [12] proved that a graph G with minimum degree δ contains a cycle of length at least $\delta + 1$. Hence $G(n, P)$ has a cycle of length $\Omega\left(\frac{n}{\ln n}\right)$.*[3]

Although there exist a number of sufficient conditions for Hamilton cycles [17, 18], most are fairly strict degree conditions that are not applicable in light of Remark 2.1.2. Thus we focus instead on the simpler problem of showing that these

[2] $f(n) \sim g(n)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$.
[3] $f(n) \in \Omega(g(n))$ if $\exists N, C > 0$ such that $|f(n)| > |Cg(n)| \; \forall\, n \geq N$.

graphs are highly connected and that they contain structured 2-factors.

We adopt the notation used by Anderson and Walker [2]: Let $b > a$ have distinct parities, and denote the set of "nested pairs"

$$\{(a+k, b-k) \mid 0 \le k \le \tfrac{1}{2}(b-a-1)\}$$

by $[a;b]$. We refer to such a pairing $[a;b]$ as a *brick*. A *brick matching* in a graph $G(2m, S)$ is then a *perfect matching* of the form

$$[k_0+1; k_1] \cup [k_1+1; k_2] \cup \cdots \cup [k_{s-1}+1; k_s]$$

where $k_0 = 0$, $k_s = 2m$, and k_i is even for all $i = 1, \ldots, s-1$. In other words, a brick matching is a perfect matching composed of disjoint bricks. A brick matching in $G(18, P)$ is drawn as Figure 2.3.

Greenfield and Greenfield [20] showed that Bertrand's postulate[4] is essentially equivalent to the statement that the first $2m$ integers can always be arranged into m disjoint pairs so that the sum of the entries in each pair is prime. We provide a straightforward generalization of their result.

Theorem 2.1.3. *Consider any strictly increasing integer sequence $\{a_n\}$, where $a_1 \ge 3$. For a fixed n, let $S_a = \{a_i < 2n\}$. Then $G(2m, S_a)$ has a brick matching for all $m \ge 1$ if and only if for every $m \ge 1$ there is at least one odd a_i with $2m < a_i < 4m$.*

Proof. (⇒) Fix any $m \ge 1$. Consider the brick containing $2m$ in a brick matching M

[4] For every positive integer n with $n > 1$, there is a prime p such that $n < p < 2n$.

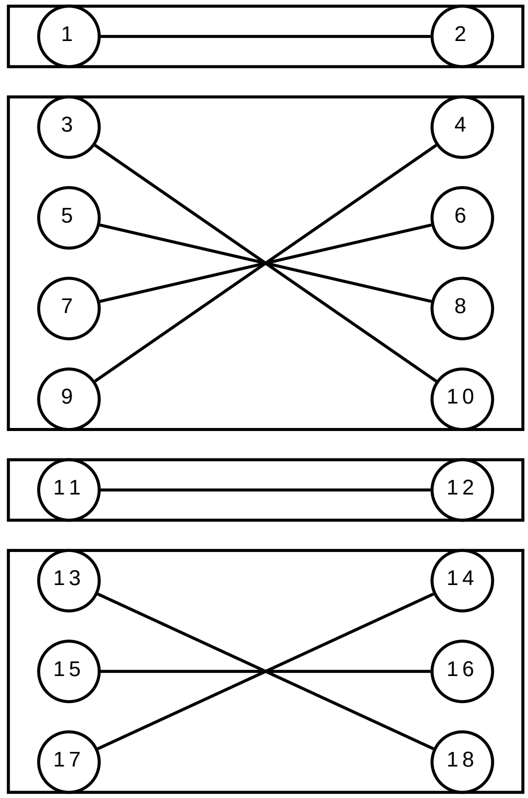

Figure 2.3: The brick matching $[1;2] \cup [3;10] \cup [11;12] \cup [13;18]$ in $G(18, P)$.

of $G(2m, S_a)$. The *neighbour* of $2m$ in this brick must be $a_i - 2m$ for some i, where $0 < a_i - 2m < 2m$. If a_i were even, then the vertex $\frac{1}{2}a_i$ would be *unmatched* in this brick and not *matched* in any other brick, contradicting the fact that M is a perfect matching.

($\Leftarrow$) We prove this by strong induction on m. The condition stated in the theorem implies that $a_1 = 3$, and hence the base case $m = 1$ is satisfied. Now consider the graph $G(2m, S_a)$ for some $m > 1$, and assume the induction hypothesis for all graphs $G(2j, S_a)$ with $1 \leq j < m$. By assumption there is an odd element in the generating set, say a_i, that lies between $2m$ and $4m$. Hence the brick $[a_i - 2m; 2m]$

is present in $G(2m, S_a)$. If $a_i = 2m + 1$, then we are done. Otherwise, the induction hypothesis implies that there is a brick matching M in $G(a_i - 2m - 1, S_a)$, and therefore $G(2m, S_a)$ has the brick matching $M \cup [a_i - 2m; 2m]$. □

Corollary 2.1.4. *(Theorem 1, [20]) The integers* $\{1, 2, \ldots, 2m\}$ *can be arranged into* m *disjoint pairs so that the sums of the elements in each pair is prime.* □

We will make repeated use of the following generalization of Bertrand's postulate, which is due to Erdős [13].

Theorem 2.1.5. *[13] There exists at least one prime of the form* $4k + 1$ *and at least one prime of the form* $4k' + 3$ *between* n *and* $2n$ *for all* $n > 6$. □

Theorem 2.1.5, and the correspondence between perfect matchings in $G(2m, P)$ and Bertrand's postulate noted by Greenfield and Greenfield [20], immediately suggests Lemma 2.1.6. The existence of two disjoint brick matchings in $G(2m, P)$ is of interest because their union gives a 2-factor in this graph.

Lemma 2.1.6. *Let* $G(2m, P_{4k+1})$ *and* $G(2m, P_{4k'+3})$ *be the subgraphs of* $G(2m, P)$ *induced by primes of the form* $4k + 1$ *and* $4k' + 3$*, respectively.*

(i) $G(2m, P_{4k'+3})$ *has a brick matching.*

(ii) If $G(2m, P_{4k+1})$ *does not have a brick matching, then the addition of either the brick* $[1; 2]$ *or the brick* $[1; 6]$ *will create one.*

Proof. Statement (i) is a corollary of Theorem 2.1.3, since by Theorem 2.1.5 there is at least one prime of the form $4k' + 3$ between $2m$ and $4m$ for all $m \geq 1$.

We prove statement (ii) by strong induction on m. The case $m = 1$ holds since the addition of $[1;2]$ gives a brick matching. The case $m = 2$ holds because of the brick $[1;4]$ in $G(4, P_{4k+1})$. The case $m = 3$ holds since the addition of $[1;6]$ gives a brick matching. Thus assume the induction hypothesis for all values less than some $m > 3$.

By Theorem 2.1.5, we know that there exists a prime of the form $4k+1$ between $2m$ and $4m$ for $m > 3$. This prime can be written as $2m + 2t + 1$ for some t, and it follows that $[2t + 1; 2m]$ is present in $G(2m, P_{4k+1})$. If $t = 0$, then we are done. Otherwise, by the induction hypothesis, either $G(2t, P_{4k+1})$ has a brick matching M or the addition of one of $[1;2]$ or $[1;6]$ creates such a brick matching. Clearly $M \cup [2t + 1; 2m]$ is a brick matching for $G(2m, P_{4k+1})$. □

Theorem 2.1.7. *Suppose that $G(2m, P)$ does not have a 2-factor that is the union of two disjoint brick matchings. Then m is odd and $G(2m, P)$ has two brick matchings that intersect only in the brick $[1;2]$.*

Proof. We say that a brick $[a;b]$ is *even* (resp. *odd*) if it consists of an even (resp. odd) number of edges. A simple parity argument shows that bricks in $G(2m, P_{4k+1})$ are even and that bricks in $G(2m, P_{4k'+3})$ are odd.

Note that a brick matching in $G(2m, P_{4k+1})$ and a brick matching in $G(2m, P_{4k'+3})$ are necessarily disjoint, and so their union gives a 2-factor in $G(2m, P)$. By Lemma 2.1.6, $G(2m, P_{4k'+3})$ has a brick matching for all $m \geq 1$. Thus it suffices to consider when $G(2m, P_{4k+1})$ has a brick matching.

Suppose that m is even and that $G(2m, P_{4k+1})$ does not have a brick matching. By Lemma 2.1.6, it suffices to add one of the bricks $[1;2]$ or $[1;6]$. Both of these bricks are odd, which implies that there must be another odd brick in the brick matching. This is a contradiction, and hence if m is even, then $G(2m, P_{4k+1})$ has a brick matching.

Suppose that m is odd, that $G(2m, P_{4k+1})$ does not have a brick matching, and that the addition of the brick $[1;6]$ creates a brick matching. If this brick matching M and a brick matching M' in $G(2m, P_{4k'+3})$ are not disjoint, then their intersection is either the brick $[3;4]$ or the brick $[1;6]$. In the former case, we can replace the section $[1;2] \cup [3;4]$ of M' with $[1;4]$. In the latter case, we can replace the brick $[1;6]$ of M' with $[1;4] \cup [5;6]$. Thus there exist two disjoint brick matchings in $G(2m, P)$, and $G(2m, P)$ has a 2-factor.

Hence the only case in which $G(2m, P)$ may not have a 2-factor is when m is odd and the addition of the brick $[1;2]$ is necessary for the creation of a brick matching M in $G(2m, P_{4k+1})$. In this instance, a brick matching M' in $G(2m, P_{4k'+3})$ that is not disjoint from M must contain the brick $[1;2]$. Since $[1;2]$ is the only odd brick in M, this is exactly the intersection. □

This 2-factor in $G(2m, P)$ (m even) need not be connected. For example, the union of the brick matching $[1;12]$ in $G(12, P_{4k+1})$ and the brick matching $[1;6] \cup [7;12]$ in $G(12, P_{4k'+3})$ is a 2-factor in $G(12, P)$ with three cycles.

Since hamiltonian graphs are necessarily 2-connected, we briefly consider the connectivity of $G(n, P)$. Proposition 2.1.8 is a well-known result.

Proposition 2.1.8. *(Lemma 9.3, [4]) If G is a k-connected graph and H is a graph obtained from G by adding a new vertex y with at least k neighbours in G, then H is also k-connected*

Proof. Consider any *separating set* U of H. If $y \in U$, then $U - y$ is a separating set of G and hence $|U| \geq k + 1$. If $y \notin U$ and all of the neighbours of y are in U, then $|U| \geq k$. Otherwise, U is a separating set of G and hence $|U| \geq k$. □

Theorem 2.1.9. *Consider any strictly increasing integer sequence $\{a_n\}$, where $a_1 \geq 3$. For a fixed n, recall that $S_a = \{a_i < 2n\}$. Then $G(n, S_a)$ is k-connected for all $n \geq N$ if and only if $G(N, S_a)$ is k-connected and for every $n \geq N$ there are at least k elements a_i with $n < a_i < 2n$.*

Proof. ($\Rightarrow$) $G(n, S_a)$ is k-connected for all $n \geq N$ implies that $\deg(n) \geq k$ in $G(n, S_a)$ for all $n \geq N$. Since for a fixed n we have that $\deg(n) = |S_a \cap \{n+1, n+2, \ldots, 2n-1\}|$, the claim follows.

($\Leftarrow$) We proceed by induction on n. The base case $n = N$ is assumed. Consider any $n > N$ and suppose that $G(n-1, S_a)$ is k-connected. By assumption, vertex n will have degree at least k in $G(n, S_a)$. The claim follows by Proposition 2.1.8. □

Corollary 2.1.10. *$G(n, P)$ is 2-connected for $n = 4$ and $n \geq 6$.*

Proof. $G(4, P)$ is a 4-*cycle* C_4, $G(6, P)$ is the $(3 \times 2)-$*grid graph*, and by Theorem 2.1.5 there are at least two primes between n and $2n$ for $n > 6$. □

Ramanujan [40] proved that for any positive integer k, there is a natural number $N(k)$ such that there are at least k primes between $\frac{1}{2}n$ and n for all $n \geq N(k)$.

The *k-th Ramanujan prime* is therefore defined to be the smallest integer R_k such that $\pi(n) - \pi(\frac{1}{2}n) \geq k$ for all $n \geq R_k$. Equivalently, we have that there are at least k primes between n and $2n$ provided $n \geq \lceil \frac{1}{2}R_k \rceil$. Denote by B_k the smallest integer at least $\lceil \frac{1}{2}R_k \rceil$ such that $G(B_k, P)$ is k-connected. It follows by Theorem 2.1.9 that $G(n, P)$ is k-connected for all $n \geq B_k$, and that this is best possible. The sequence $\{R_k\}$ can be found in the Online Encyclopedia of Integer Sequences [47] and the sequence $\{B_k\}$ ($k \leq 50$) was determined using the network analysis software Ucinet [5]. The results are summarized in Table 2.1 – for example, $G(n, P)$ is 50-connected for $n \geq 324$. It is not known whether there is an alternate way of determining the sequence $\{B_k\}$.

Theorem 2.1.11. *If the twin prime conjecture is true, then* $\lim_{n\to\infty} \kappa(G(n, P))$ *is unbounded.*

Proof. We have investigated the increasing connectivity of $G(n, P)$ for $n \leq 324$. Now suppose to the contrary that there exists an N and a k such that $G(n, P)$ is k-connected but not $(k + 1)$-connected for all $n \geq N$. Choose N to be the smallest integer with this property. $G(N, P)$ has a minimal separating set $U \subset [N]$ of cardinality $|U| = k$, which when removed separates the graph into two or more *components*. We can assume without loss of generality that U is a separating set in $G(n, P)$ for all $n \geq N$.

Upon removal of U there cannot always be exactly two components consisting of the odd labeled vertices not in U and the even labeled vertices not in U, as we can assume that every vertex has degree at least $k + 1$. Thus there must exist two vertices i and $i + 2$ that are in different components. If there exist infinitely many twin primes, then there exists a pair of prime numbers $(x, x+2)$ with $x \geq N+i+1$.

Table 2.1: Connectivity thresholds for $G(n, P)$.

k	R_k	$\lceil \frac{1}{2} R_k \rceil$	B_k	k	R_k	$\lceil \frac{1}{2} R_k \rceil$	B_k
1	2	1	1	26	281	141	150
2	11	6	6	27	307	154	156
3	17	9	10	28	311	156	160
4	29	15	16	29	347	174	174
5	41	21	22	30	349	175	180
6	47	24	24	31	367	184	186
7	59	30	30	32	373	187	190
8	67	34	36	33	401	201	202
9	71	36	40	34	409	205	210
10	97	49	50	35	419	210	220
11	101	51	54	36	431	216	222
12	107	54	58	37	433	217	232
13	127	64	66	38	439	220	234
14	149	75	76	39	461	231	240
15	151	76	78	40	487	244	246
16	167	84	84	41	491	246	250
17	179	90	90	42	503	252	258
18	181	91	96	43	569	285	286
19	227	114	114	44	571	286	288
20	229	115	118	45	587	294	294
21	233	117	120	46	593	297	304
22	239	120	126	47	599	300	310
23	241	121	130	48	601	301	316
24	263	132	138	49	607	304	318
25	269	135	144	50	641	321	324

It follows that i and $i+2$ have a common neighbour $x-i$ in $G(n, P)$ for sufficiently large n. This implies that $x-i > N$ must be in U, which is a contradiction. □

It is questioned whether there is a proof of Theorem 2.1.11 that does not require assuming the twin prime conjecture.

The results of this section are a first step towards resolving the Traversal by Prime Sum Problem. Many of our approaches to this problem, including finding Hamilton cycles for specific graph orders, investigating the existence of long cycles, and taking the union of special disjoint perfect matchings to create 2-factors, are reminiscent of attempts to solve the notorious middle levels problem[5] (see [28] and references therein). In this case, however, we are dealing with an infinite *nested sequence of graphs*. It is therefore possible that a Hamilton cycle can be found in $G(2m, P)$ by connecting cycles in the structured 2-factors of $G(2k, P)$, k even and $2 \leq k \leq m$.

2.2 Traversal by Practical Sum Problem

In this section we investigate the additive graph generated by the sequence of practical numbers. A *practical number* is a natural number n such that all smaller natural numbers can be represented as sums of distinct *divisors* of n. For example, 12 is a practical number because $5 = 2+3$, $7 = 3+4$, $8 = 2+6$, $9 = 3+6$, $10 = 4+6$, and $11 = 1+4+6$. It is easy to see that every practical number greater than 1 must

[5]Let $B(k)$ denote the bipartite graph whose vertices are all of the subsets of $\{1, \ldots, 2k+1\}$ of size k or $k+1$, and whose edges represent the inclusion between two such subsets. Is $B(k)$ Hamiltonian for all $k > 1$?

be even, and that every practical number greater than 2 is a multiple of 4 or 6. The first ten practical numbers are $\{1, 2, 4, 6, 8, 12, 16, 18, 20, 24\}$.

One reason for interest in practical numbers is that they have many properties in common with prime numbers. For example, if $p(x)$ is the *enumerating function* for the sequence of practical numbers, then

$$c_1 \frac{x}{\ln x} < p(x) < c_2 \frac{x}{\ln x},$$

where c_1 and c_2 are constants [43]. Moreover, theorems analogous to Goldbach's conjecture[6] [34], the twin prime conjecture [34], Legendre's conjecture[7] [27], and the conjecture of infinitely many *Fibonacci primes* [33] are all known to be true for practical numbers. This unexpected correspondence is particularly interesting because practical numbers are somewhat more predictable in their distribution. To demonstrate this, consider the following lemma.

Lemma 2.2.1. *(Lemma 1, [34]) Denote the sum of divisors function by σ. If m is a practical number and n is a natural number such that $1 \leq n \leq \sigma(m) + 1$, then mn is a practical number. In particular, for $1 \leq n \leq 2m$, mn is practical.* □

In light of the Traversal by Prime Sum Problem, it is reasonable to consider the following: Let $G(n, \mathcal{P})$ be the graph with vertex set $V = [n]$ and $(i, j) \in E$ if and only if $i + j$ is a practical number. What properties does this graph possess? This graph has at least two components for all $n > 1$, as practical numbers are even and hence no odd labeled vertex is adjacent to any even labeled vertex. However,

[6] Every even natural number greater than 2 can be written as the sum of two primes.
[7] For every natural number n, there exists a prime p with $n^2 < p < (n + 1)^2$.

looking at the components induced by the odd and even labeled vertices yields some noteworthy results.

Let $G_o(2m, \mathcal{P})$ and $G_e(2m, \mathcal{P})$ be the subgraphs of $G(2m, \mathcal{P})$ induced by the odd and even labeled vertices, respectively. We prove several results comparable to those proved for the graphs $G(n, P)$, beginning with the existence of infinite hamiltonian subfamilies of both $G_o(2m, \mathcal{P})$ graphs and $G_e(2m, \mathcal{P})$ graphs.

Theorem 2.2.2. *There are infinitely many m such that $G_o(2m, \mathcal{P})$ is hamiltonian and there are infinitely many m' such that $G_e(2m', \mathcal{P})$ is hamiltonian.*

Proof. Melfi [34] proved that there are infinitely many triples $(x-2, x, x+2)$ of practical numbers. Given such a triple $(2m-2, 2m, 2m+2)$, $G_o(2m, \mathcal{P})$ contains the Hamilton cycle

$$(1, 2m-1, 3, 2m-5, 7, \ldots, 2m-7, 5, 2m-3, 1)$$

and $G_e(2m-2, \mathcal{P})$ contains the Hamilton cycle

$$(2, 2m-2, 4, 2m-6, 8, \ldots, 2m-8, 6, 2m-4, 2). \qquad \square$$

Before continuing with other results similar to those we obtained regarding the additive graphs generated by the prime numbers, we use Lemma 2.2.1 to demonstrate that these additive graphs generated by the practical numbers may be more easily studied.

Proposition 2.2.3. *If $G_o(2m, \mathcal{P})$ has a 2-factor with k cycles and $G_e(2m, \mathcal{P})$ has a 2-factor with k' cycles, then $G_e(4m, \mathcal{P})$ has a 2-factor with $k + k'$ cycles.*

Proof. Consider the k cycles in the 2-factor of $G_o(2m, \mathcal{P})$. Lemma 2.2.1 implies that if $i + j$ is practical, then $2i + 2j$ is practical. By doubling the value of each vertex label, we arrive at k cycles in $G_e(4m, \mathcal{P})$ that *span* the vertices $i \equiv 2 \mod 4$. Similarly, by doubling the value of each vertex label in the k' cycles in the 2-factor of $G_e(2m, \mathcal{P})$, we arrive at k' cycles in $G_e(4m, \mathcal{P})$ that span the vertices $i \equiv 0 \mod 4$. Thus $G_e(4m, \mathcal{P})$ has a 2-factor with $k + k'$ cycles. □

Lemma 2.2.4 is motivated by Erdős' generalization of Bertrand's postulate (Theorem 2.1.5), and will be used in much the same way in order to prove the existence of 2-factors and 2-connectivity.

Lemma 2.2.4. *There exists at least one practical number of the form $8k$ and at least one practical number of the form $8k' + 4$ between $2m + 2$ and $4m$ for all $m \geq 3$.*

Proof. We prove that there is at least one practical number of the form $8k$ between $2m+2$ and $4m$ for all $m \geq 3$ by induction on m. In the interval $[8, 12]$, 8 is a practical number of the form $8k$. Thus assume there is a practical number p of this form in $[2(m-1)+2, 4(m-1)]$ and consider the interval $[2m+2, 4m]$, where $m > 3$. If $p > 2m$, then p is in the interval $[2m + 2, 4m - 4]$. If $p = 2m$, then by Lemma 2.2.1 $2p = 4m$ is practical. In either case, there is a practical number of the form $8k$ in $[2m+2, 4m]$.

We prove that there is at least one practical number of the form $8k' + 4$ between $2m + 2$ and $4m$ for all $m \geq 3$ by strong induction on m. In the intervals $[8, 12]$, $[10, 16]$, and $[12, 20]$, 12 is a practical number of the form $8k' + 4$. In the intervals $[14, 24]$, $[16, 28]$, and $[18, 32]$, 20 is a practical number of the form $8k' + 4$. Thus assume that there is at least one practical number of this form in $[2j + 2, 4j]$ for all

$3 \leq j < m$, where $m \geq 9$.

By the induction hypothesis, there is a practical number p' of the form $8k' + 4$ in $[2m' + 2, 4m']$, where $m' = \lfloor \frac{1}{3} m \rfloor$. Then $3p'$ is practical by Lemma 2.2.1, and is in the interval

$$\begin{aligned} &[2m + 6, 4m] && \text{if } m \equiv 0 \mod 3; \\ &[2m + 4, 4m - 4] && \text{if } m \equiv 1 \mod 3; \\ &[2m + 2, 4m - 8] && \text{if } m \equiv 2 \mod 3. \end{aligned}$$

Since $p' \equiv 4 \mod 8$, we have that $3p' \equiv 4 \mod 8$, and thus the induction claim is true for m. □

Lemma 2.2.5, Theorem 2.2.6, and Corollary 2.2.7 are direct analogues of Lemma 2.1.6, Theorem 2.1.7, and Corollary 2.1.10, respectively, and so we omit the proofs.

Lemma 2.2.5. *Let $G_o(2m, \mathcal{P}_{8k})$ and $G_o(2m, \mathcal{P}_{8k'+4})$ be the subgraphs of $G_o(2m, \mathcal{P})$ induced by practical numbers of the form $8k$ and $8k' + 4$, respectively.*

(i) $G_o(4m, \mathcal{P}_{8k'+4})$ has a brick matching[8]*.*

(ii) If $G_o(4m, \mathcal{P}_{8k})$ does not have a brick matching, then the addition of the edge $(1, 3)$ will create one. □

Theorem 2.2.6. *Suppose that $G_o(4m, \mathcal{P})$ does not have a 2-factor that is the union of two disjoint brick matchings. Then m is odd and $G_o(4m, \mathcal{P})$ has two brick matchings that intersect only in the edge $(1, 3)$.* □

Corollary 2.2.7. *$G_o(2m, \mathcal{P})$ is 2-connected for $m \geq 3$.* □

[8] In this case, a brick $[a; b]$ is the set of edges $\{(a + 2k, b - 2k) | \, 0 \leq k \leq \frac{1}{4}(b - a - 1)\}$.

Remark 2.2.8. *For $m \leq 8$, $G_e(4m, \mathcal{P})$ does not have two disjoint brick matchings when $m \in \{1, 2, 3, 4, 5, 8\}$.*

We finish this section by showing that the connectivity of $G_o(2m, \mathcal{P})$ increases without bound as a function of m. This is not surprising given Theorem 2.1.11 and the fact that there are infinitely many pairs of practical numbers p and $p + 2$ [34].

Theorem 2.2.9. $\lim\limits_{m\to\infty} \kappa(G_o(2m, \mathcal{P}))$ *is unbounded.*

Proof. Firstly, suppose that $G_o(2m, \mathcal{P})$ is k-connected. We claim that $G_o(2m + 2, \mathcal{P})$ is also k-connected. To see this, note that $\deg(2m - 1) \geq k$ in $G_o(2m, \mathcal{P})$. Suppose vertex 1 is adjacent to vertex $2m-1$. Then $2m$ is a practical number, and by Lemma 2.2.1 $4m$ is also a practical number. This implies that vertex $2m + 1$ is adjacent to vertex $2m - 1$ in $G_o(2m + 2, \mathcal{P})$. Suppose vertex i is adjacent to vertex $2m - 1$ and $i \geq 3$. Then vertex $2m + 1$ is adjacent to vertex $i - 2$ in $G_o(2m + 2, \mathcal{P})$. Hence $\deg(2m + 1) \geq \deg(2m - 1) \geq k$ in $G_o(2m + 2, \mathcal{P})$. It follows by Proposition 2.1.8 that $G_o(2m + 2, \mathcal{P})$ is also k-connected.

Note that $G_o(2, \mathcal{P})$ and $G_o(4, \mathcal{P})$ are connected. Now suppose to the contrary that there exists an M and a k such that $G_o(2m, \mathcal{P})$ is k-connected but not $(k + 1)$-connected for all $m \geq M$. Choose M to be the smallest integer with this property. $G_o(2M, \mathcal{P})$ has a minimal separating set $U \subset [2M]$ of cardinality $|U| = k$, which when removed separates the graph into two or more components. We can assume without loss of generality that U is a separating set for all $m \geq M$. There must exist two successive vertices i and $i + 2$ that are in different components. Melfi [34] proved that there exist infinitely many pairs of practical numbers $(x, x + 2)$, and hence there exists such a pair with $x \geq 2M + i + 1$. It follows that i and $i + 2$ have a common neighbour $x - i$ in $G_o(2m, \mathcal{P})$ for sufficiently large m. This implies that $x - i > 2M$ must be in U, which is a contradiction. □

The following is a natural question that we dub the Traversal by Practical Sum Problem. The answer to this may suggest the correct answer to the Traversal by Prime Sum Problem.

Question 2.2.10. *Is $G_o(2m, \mathcal{P})$ hamiltonian for $m \geq 3$? Is $G_e(2m, \mathcal{P})$ hamiltonian for $m \geq 5$?*

2.3 Other Sequences as Generating Sets

We conclude this chapter by briefly turning our attention to the structure of the additive graphs generated by the Fibonacci sequence, general linear recurrence sequences, and the sequences of r-th powers, in order to unify the known results in the literature related to these graphs. In particular, the question of when the additive graphs generated by linear recurrence sequences are bipartite is well studied [1,7,19], and perfect matchings in additive graphs generated by the sequences of squares and cubes have been previously investigated [2].

2.3.1 Linear Recurrence Sequences

Given the abundance of research related to the Fibonacci numbers, this integer sequence is a natural candidate for special consideration. Thus consider the graph $G(n, F)$, where F is the set of Fibonacci numbers in $\{3, 4, \ldots, 2n - 1\}$. The graph $G(35, F)$ is drawn as Figure 2.4. We prove some basic results about these additive graphs, which leads to a concise discussion of the theory of additive partitions.

Theorem 2.3.1. *$G(n, F)$ is connected for all n.*

Proof. Firstly, note that $G(1, F)$ and $G(2, F)$ are connected. For $n \geq 3$, the sum of the two largest Fibonacci numbers in the set $\{3, 4, \ldots, n\}$ must be strictly less than $2n$. The claim follows by Theorem 2.1.9. □

Remark 2.3.2. *The condition in Theorem 2.1.3 is necessary only for brick matchings, and not for perfect matchings in general. However, every third Fibonacci number being even suggests that not every graph $G(2m, F)$ has a perfect matching. A minimal counterexample is found by considering the first value of $2m$ for which there is no odd Fibonacci number in the interval $[2m+1, 4m-1]$, which is $2m = 22$. To see that $G(22, F)$ has no perfect matching, note that in such a matching vertex 17 must be matched with vertex 4 and vertex 21 must be matched with vertex 13. Since vertex 9 has only vertices 4 and 13 as neighbours, there is no such perfect matching.*

Theorem 2.3.3. *$G(n, F)$ has no odd cycles.*

Proof. We proceed by induction on n. The base case is trivial, so fix any $n > 1$ and assume that $G(n-1, F)$ has no odd cycles. It is clear that vertex n can have degree at most 2 in $G(n, F)$: If there exist distinct $i, j \in \{1, 2, \ldots, n-1\}$ such that $n+i$ and $n+j$ are in F, then the next Fibonacci number $(n+i)+(n+j)$ is strictly greater than $2n-1$ and hence does not contribute to the degree of vertex n. If $\deg(n) = 1$, then no odd cycles have been created and we are finished by the induction hypothesis.

Thus assume that $\deg(n) = 2$. Suppose $n+i$ and $n+j$ are both Fibonacci numbers, with $1 \leq i < j \leq n-1$. Then the sequence of Fibonacci numbers is

$$\{1, 1, 2, 3, \ldots, n+2i-j, j-i, n+i, n+j, \ldots\},$$

and the addition of vertex n simply creates the 4-cycle $(n, j, n + i - j, i, n)$. We cannot have that vertices i and j are adjacent, as this would imply a triangle in $G(n-1, F)$. It follows that if $G(n-1, F)$ has no odd cycles, then $G(n, F)$ also has no odd cycles. □

Remark 2.3.4. *The proof of Theorem 2.3.3 shows that $G(n, F)$ is never 2-connected: If $\deg(n) > 1$ in $G(n, F)$, and i and j are as in the proof, then $\deg(j - i) = 1$ because there is only one Fibonacci number in the set $\{j - i + 1, j - i + 2, \ldots, j - i + n\}$. As well, the proof technique can be used to show that $G(n, F)$ is planar for all n.*

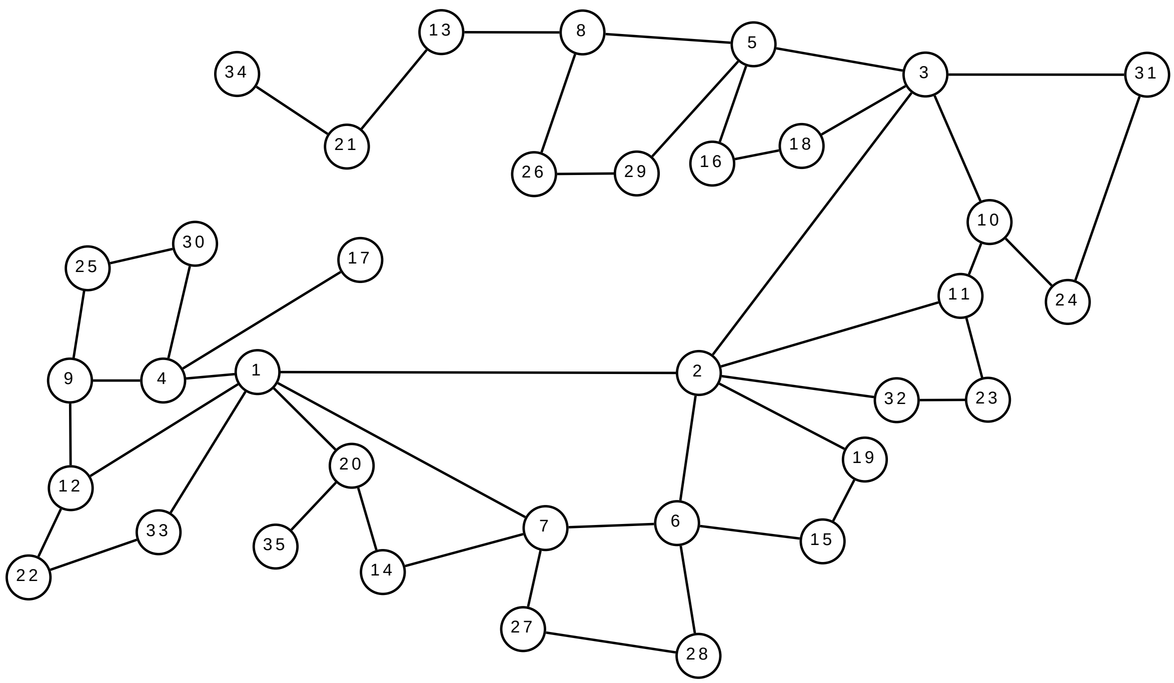

Figure 2.4: The additive graph $G(35, F)$ of order 35 generated by the Fibonacci numbers.

Theorem 2.3.3 was discovered independently by Silverman [46], who proved that the natural numbers have a division into two disjoint sets with the property that a natural number is a Fibonacci number if and only if it is not the sum of two distinct members of the same set. He also proved that this division is unique. This work inspired Alladi *et al.* [1] to study partitions induced by general linear recurrences. Given a set of natural numbers $A \subseteq \mathbb{N}$, they say that A generates an *additive partition* of $\mathbb{N}$ if there exists $A_1, A_2 \subseteq \mathbb{N}$ with $\mathbb{N} = A_1 \cup A_2$, $A_1 \cap A_2 = \varnothing$, such that for any distinct natural numbers a and b with $a, b \in A_1$ or $a, b \in A_2$ we have $a + b \notin A$. They proved the following interesting theorem by means of an inductive construction.

Theorem 2.3.5. *(Theorem 2.2, [1]) If $U = \{u_n\}$ is a linear recurrence sequence with $u_{n+2} = u_{n+1} + u_n$, $n \geq 1$, $u_1 = 1$, and $u_2 > 1$, then U generates a unique additive partition of $\mathbb{N}$.* □

Alladi *et al.* [1] provided a graph-theoretic interpretation of their result, which led to their consideration of additive graphs generated by a subset of the natural numbers. They note that a set $A \subseteq \mathbb{N}$ generates an additive partition of $\mathbb{N}$ if and only if the infinite additive graph generated by A is bipartite.

Corollary 2.3.6. *Suppose that the infinite sequence $\{a_n\}$ is defined by a linear recurrence $a_{n+2} = a_{n+1} + a_n$, $n \geq 1$, $a_1 = 1$, and $a_2 > 1$. Then $G(n, S_a)$ is bipartite for all n.* □

In the wake of several generalizations of this result to other specific recurrence relations, Chow and Long [7] recently established an unexpected connection between sets that generate additive partitions and the theory of continued fractions. Some of their results were subsequently enhanced by Grabiner [19]. The following theorem subsumes much of what was previously known. A brief introduction to continued fractions can be found in Chapter 12 of the text by Rosen [42].

Theorem 2.3.7. *(i) (Theorem 2, [7]) For any irrational $1 < \alpha < 2$, the set of numerators of all convergents of the continued fraction for α generates an additive partition.*

(ii) (Theorem 7, [19]) This partition is unique if and only if the continued fraction for α has infinitely many partial quotients equal to 1. □

2.3.2 Sequences of r-th Powers

Another fundamental integer sequence is the sequence of r-th powers $\{2^r, 3^r, 4^r, \ldots\}$ $(r \geq 2)$. For a given n, let S_r denote the set $\{i^r < 2n\}$. Anderson and Walker [2] considered the question of the existence of perfect matchings in $G(2m, S_2)$ and $G(2m, S_3)$. We conclude this chapter by rephrasing some of their results using our terminology and notation.

Theorem 2.3.8. *(Theorem 4, [2]) $G(2m, S_2)$ has a perfect matching if and only if $2m \notin \{2, 4, 6, 10, 12, 20, 22\}$.* □

Theorem 2.3.9. *[2] $G(2m, S_3)$ has a perfect matching if $2m \geq 238$.* □

In contrast to Remark 2.3.2, Anderson and Walker [2] completely characterized when the graph $G(2m, S_2)$ has a perfect matching but no brick matching.

Theorem 2.3.10. *(Theorem 5, [2]) $G(2m, S_2)$ has a brick matching if and only if $m \equiv 0 \mod 4$.* □

The conjecture by Erdős and Silverman [14] mentioned in Chapter 1 also concerns these graphs, and it can be phrased in the following way. This conjecture was made almost thirty years ago, and was repeated several times by Erdős, but no results are known.

Conjecture 2.3.11. *[14]* $\lim_{n\to\infty} \chi(G(n, S_r))$ *is unbounded for all* $r \geq 2$*, where* $\chi(G)$ *is the chromatic number of* G*.*

Chapter 3

Arbitrary Generating Sets

In this chapter we shift our focus to arbitrary generating sets. We begin by describing the motivation of our main result, which is that the $|S|$-closure of $G(n, S)$ is always the complete graph. By finding bounds on the cardinality of the edge set of $G(n, S)$ in terms of $|S|$, we discover a new and simple construction of minimum (with respect to edge set cardinality) graphs with n vertices and complete k-closure for all n and k. Finally, we use the known stability or complete stability of many monotone properties to prove thresholds for these properties in this family of additive graphs. These graphs are thereby shown to be the first known wide-ranging and representative subclass of complete k-closure graphs.

3.1 Motivation

We introduce in this section the important concepts of the k-closure of a graph and of the stability of a property. Our motivation was the Traversal by Prime Sum Problem, which was considered in the previous chapter. Specifically, Vašek Chvátal [8] posed the following two questions in an attempt to understand whether the presence or absence of a Hamilton cycle in $G(2m, P)$ is inevitable given the proportion of prime numbers in any interval $[3, 4m-1]$.

Question 3.1.1. *What is the maximum cardinality k such that $G(n, S)$ is not hamiltonian for all $|S| = k$?*

Question 3.1.2. *What is the minimum cardinality k such that $G(n, S)$ is hamiltonian for all $|S| = k$?*

In response to Question 3.1.1, the maximum cardinality k such that $G(n, S)$ is not hamiltonian for all $|S| = k$ is easily seen to be $k = 2$. Firstly, note that two generating set elements can induce at most $n - 1$ edges in a graph of order n. Secondly, Remark 3.1.3 contains two of the many examples of generating sets of size 3 that induce a Hamilton cycle.

Remark 3.1.3. *The generating set $S = \{3, n+1, n+3\}$ induces the Hamilton cycle*

$$(1, n, 3, n-2, 5, n-4, \dots, n-3, 4, n-1, 2, 1)$$

in $G(n, S)$, while the generating set $S' = \{n-1, n+1, 2n-1\}$ induces the Hamilton cycle

$$(n, 1, n-2, 3, n-4, 5, \dots, 4, n-3, 2, n-1, n)$$

in $G(n, S')$.

The solution to Question 3.1.2 is more involved. We will show that the minimum cardinality k such that $G(n, S)$ is hamiltonian for all $|S| = k$ is $k = n$. The proof is influenced by the following well-known result due to Bondy and Chvátal [3], which is a consequence of Ore's Theorem [37].

Theorem 3.1.4. *[3] G is hamiltonian if and only if $G + (u, v)$ is hamiltonian, where u and v are two nonadjacent vertices with $\deg(u) + \deg(v) \geq n$.* □

The graph obtained by continually adding in edges between vertices with this property in a graph G results in a well defined graph termed the *closure* of G. In particular, since the complete graph has a Hamilton cycle, Theorem 3.1.4 implies that a graph whose closure is complete is necessarily hamiltonian. It was subsequently shown that all of the classic sufficient degree conditions for a Hamilton cycle imply a complete closure [3].

More generally, let the *k-closure* of G, denoted by $cl_k(G)$, be the graph obtained by continually adding an edge (u, v) for every nonadjacent u, v with $\deg(u) + \deg(v) \geq k$. Bondy and Chvátal [3] proved that the k-closure is independent of the order of the addition of the edges.

As well, let P be a property defined on all graphs of order n, and let k be a non-negative integer. Then P is said to be *k-stable* if whenever $G + (u, v)$ has property P and $\deg(u) + \deg(v) \geq k$, then G itself has property P. The *stability* $s(P)$ of a property P is the smallest integer k such that P is k-stable. Thus Theorem 3.1.4 states that the property of having a Hamilton cycle is n-stable.

The stability of many other important monotone properties were also determined by Bondy and Chvátal [3], and this concept of the k-closure of a graph marked the beginning of a new era in the research on Hamilton cycles and other important graph-theoretic structures. Closure concepts now play an important role in results on the existence of cycles, paths, and other subgraphs in graphs [6]. The body of work inspired by Bondy and Chvátal's seminal paper has been dubbed *closure theory*.

Thus if $|S| = n$ implies that the n-closure of $G(n, S)$ is complete, then any graph $G(n, S)$ with $|S| \geq n$ is hamiltonian. In the next section we prove something much stronger, and in the final sections of this chapter we explore some of the other implications of this result.

3.2 On the $|S|$-closure of $G(n, S)$

The goal of this section is to prove Theorem 3.2.2, which states that the $|S|$-closure of $G(n, S)$ is complete (i.e., that $cl_{|S|}(G(n, S)) = K_n$). For example, a construction of the 7-closure of the additive graph of order 7 generated by $S = \{3, 4, 6, 7, 10, 12, 13\}$ is drawn as Figure 3.1. The proof technique is an induction argument nested within another induction argument. Lemma 3.2.1 serves as the base case of the outer induction, and its proof is highly suggestive of the proof of the general theorem.

For a fixed n, let the *region* of i be the set $\{i + 1, i + 2 \ldots, i + n\} - \{2i\}$. Denote this by $R(i)$, and let R denote the set $\{3, 4, \ldots, 2n - 1\}$. The motivation for this definition is that $s \in S \cap R(i)$ if and only if s contributes to the degree of vertex i in $G(n, S)$.

Lemma 3.2.1. *Let $|S| = N$. Then in the N-closure of $G(n, S)$, vertices 1 and n are adjacent to all other vertices.*

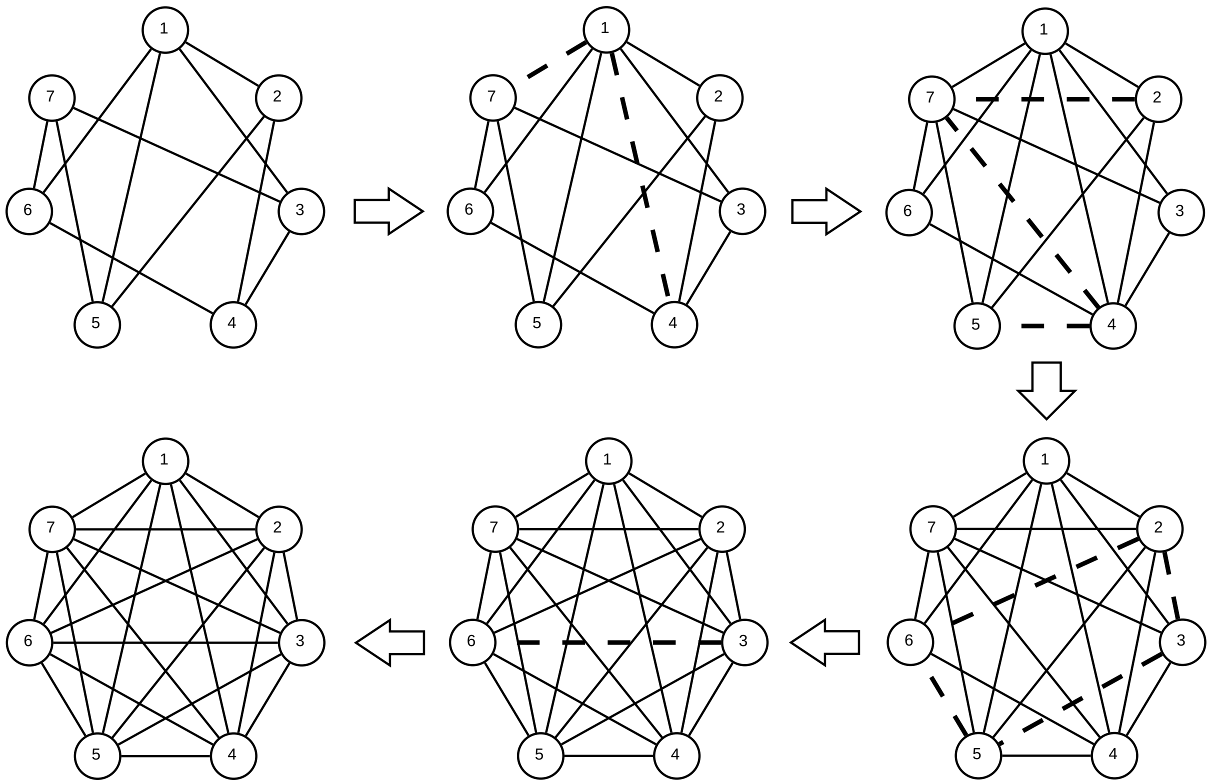

Figure 3.1: The 7-closure of $G(7, \{3, 4, 6, 7, 10, 12, 13\})$ is the complete graph.

Proof. Fix n and S, where $|S| = N$. We prove by induction on i that

$$\deg(1) + \deg(n - i + 1) \geq N$$

for all $i = 1, \ldots, n-1$ in $cl_N(G(n, S))$; the proof for vertex n is the mirror argument.

As the base case, consider vertices 1 and n. Every element of S is in $R(1) \cup R(n) = \{3, 4, \ldots, 2n - 1\} = R$, and hence contributes to the degree of at least one of these two vertices. Thus $|S| = N \Rightarrow \deg(1) + \deg(n) \geq N$. So either $n + 1 \in S$ or the edge $(1, n)$ is present in the N-closure but was not induced by a generating set element.

Now suppose that $\deg(1) + \deg(n-i+1) \geq N$ for $i < k$ in $cl_N(G(n,S))$, for some fixed $1 < k \leq n-1$. The overlap for the regions of vertices 1 and $n-k+1$ is

$$\begin{array}{rl} \{n-k+2, \ldots, n+1\} & \text{if } k < \frac{1}{2}(n+1); \\ \{n-k+2, \ldots, n+1\} - \{2(n-k+1)\} & \text{if } k \geq \frac{1}{2}(n+1). \end{array}$$

Suppose $k < \frac{1}{2}(n+1)$. Since $|R - (R(1) \cup R(n-k+1))| = k-1$, at least $|S| - k + 1$ elements in the generating set contribute to the degree of at least one of these two vertices. Hence $|S| = N \Rightarrow \deg(1) + \deg(n-k+1) \geq N - k + 1$. Now consider the elements in $R(1) \cap R(n-k+1) - \{n-k+2\}$. By the induction hypothesis, for $i < k$ either $(n-i+1) + 1 \in S$, in which case this can be counted twice towards the sum of the degrees of 1 and $n-k+1$ because it is in $R(1) \cap R(n-k+1)$, or $(1, n-i+1)$ is present in the N-closure and is not yet counted in the sum of the degrees. Thus we have that

$$\begin{aligned} \deg(1) + \deg(n-k+1) &\geq (N-k+1) + |R(1) \cap R(n-k+1) - \{n-k+2\}| \\ &= (N-k+1) + (k-1) \\ &= N. \end{aligned}$$

Suppose $k \geq \frac{1}{2}(n+1)$. Since $|R - (R(1) \cup R(n-k+1))| = k-2$, at least $|S| - k + 2$ elements in the generating set contribute to the degree of at least one of these two vertices. Hence $|S| = N \Rightarrow \deg(1) + \deg(n-k+1) \geq N - k + 2$. In this case, $|R(1) \cap R(n-k+1) - \{n-k+2\}| = k-2$. So by similar reasoning as above,

$$
\begin{aligned}
\deg(1) + \deg(n-k+1) &\geq (N-k+2) + |R(1) \cap R(n-k+1) - \{n-k+2\}| \\
&= (N-k+2) + (k-2) \\
&= N.
\end{aligned}
$$

Therefore, the induction hypothesis is true for $i = k$. It follows that in the N-closure of $G(n, S)$, vertex 1 is adjacent to all other vertices. □

Theorem 3.2.2. *The N-closure of $G(n, S)$ is complete, where $N = |S|$.*

Proof. Fix n and S, where $|S| = N$. We prove by induction on i, $i \leq \lceil \frac{1}{2}n \rceil$, that vertices

$$\{1, \ldots, i\} \cup \{n-i+1, \ldots, n\}$$

have degree $n-1$ in the N-closure. The base case $i = 1$ is proved as Lemma 3.2.1. Thus assume the induction hypothesis for $i = k-1$, and consider the set of vertices

$$\{1, \ldots, k\} \cup \{n-k+1, \ldots, n\}.$$

It suffices to consider vertices k and $n-k+1$. Note that a proof of the following two claims will show that vertices k and $n-k+1$ have degree $n-1$ in the N-closure, and hence verify the induction hypothesis. Therefore, Theorem 3.2.2 follows from:

Claim 3.2.3. *For a fixed k, $\deg(k) + \deg(n-k+2-j) \geq N$ for $j = 1, \ldots, n-2k+1$.*

Claim 3.2.4. *For a fixed k, $\deg(n-k+1) + \deg(k-1+j) \geq N$ for $j = 1, \ldots, n-2k+1$.*

We will prove Claim 3.2.3 by induction on j. Claim 3.2.4 is proved by a mirror argument. The base case $j = 1$ is shown by considering vertices k and $n - k + 1$, and is proved in much the same way as Lemma 3.2.1. The overlap for the regions of vertices k and $n - k + 1$ is

$$\begin{aligned} \{n-k+2,\ldots,n+k\} \quad & \text{if } k < \tfrac{1}{3}(n+2); \\ \{n-k+2,\ldots,n+k\} - \{2k, 2(n-k+1)\} \quad & \text{if } k \geq \tfrac{1}{3}(n+2). \end{aligned}$$

Suppose $k < \frac{1}{3}(n+2)$. Since $|R-(R(k) \cup R(n-k+1))| = 2k-2$, at least $|S|-2k+2$ elements in the generating set contribute to the degree of at least one of these two vertices. Hence $|S| = N \Rightarrow \deg(k) + \deg(n-k+1) \geq N - 2k + 2$. Now consider the elements in $R(k) \cap R(n-k+1) - \{n+1\}$. By the induction hypothesis of Theorem 3.2.2, we know that k and $n-i+1$ are adjacent in the N-closure for $i < k$. Thus for $i < k$ either $(n-i+1)+k \in S$, in which case this can be counted twice towards the sum of the degrees of k and $n-k+1$ because it is in $R(k) \cap R(n-k+1)$, or $(k, n-i+1)$ is present in the N-closure and is as of yet uncounted in the sum of the degrees. Similarly, we know that i and $n-k+1$ are adjacent in the N-closure for $i < k$. Thus we have that

$$\begin{aligned} \deg(k) + \deg(n-k+1) &\geq (N-2k+2) + |R(k) \cap R(n-k+1) - \{n+1\}| \\ &= (N-2k+2) + (2k-2) \\ &= N. \end{aligned}$$

Suppose $k \geq \frac{1}{3}(n+2)$. Since $|R - (R(k) \cup R(n-k+1))| = 2k-4$, at least $|S| - 2k + 4$ elements in the generating set contribute to the degree of at least one of these two vertices. Hence $|S| = N \Rightarrow \deg(k) + \deg(n-k+1) \geq N - 2k + 4$. In this case, $|R(k) \cap R(n-k+1) - \{n+1\}| = 2k-4$. So by similar reasoning as above,

$$\begin{aligned}
\deg(k) + \deg(n-k+1) &\geq (N-2k+4) + |R(k) \cap R(n-k+1) - \{n+1\}| \\
&= (N-2k+4) + (2k-4) \\
&= N.
\end{aligned}$$

Fix any $1 < \ell \leq n - 2k + 1$, assume the induction hypothesis for $j < \ell$, and consider vertices k and $n - k + 2 - \ell$. The overlap for the regions of vertices k and $n - k + 2 - \ell$ is

$$\begin{cases}
\{n-k+3-\ell, \ldots, n+k\} & \text{if } k < \frac{1}{3}(n+4-2\ell); \\
\{n-k+3-\ell, \ldots, n+k\} - \{2k, 2(n-k+2-\ell)\} & \text{if } k \geq \frac{1}{3}(n+3-\ell); \\
\{n-k+3-\ell, \ldots, n+k\} - \{2(n-k+2-\ell)\} & \text{otherwise.}
\end{cases}$$

Suppose $k < \frac{1}{3}(n+4-2\ell)$. Since $|R - (R(k) \cup R(n-k+2-\ell))| = 2k-3+\ell$, at least $|S| - 2k + 3 - \ell$ elements in the generating set contribute to the degree of at least one of these two vertices. Hence $|S| = N \Rightarrow \deg(k) + \deg(n-k+2-\ell) \geq N - 2k + 3 - \ell$. Now consider the elements in $R(k) \cap R(n-k+2-\ell) - \{n+2-\ell\}$. By the induction hypothesis of Claim 3.2.3, for all $j < \ell$ either $k + (n-k+2-j) \in S$, in which case this can be counted twice towards the sum of the degrees of k and $n-k+2-\ell$ because it is in $R(k) \cap R(n-k+2-\ell)$, or $(k, n-k+2-j)$ is present in the N-closure and is as of yet uncounted in the sum of the degrees. In addition, by the induction hypothesis of Theorem 3.2.2 we have that $(i, n-k+2-\ell)$ and $(k, n-i+1)$ are present in the N-closure for all $i < k$. Thus we have that

$$\begin{aligned}
\deg(k) + \deg(n-k+2-\ell) &\geq (N-2k+3-\ell) + \\
&\quad |R(k) \cap R(n-k+2-\ell) - \{n+2-\ell\}| \\
&= (N-2k+3-\ell) + (2k-3+\ell) \\
&= N.
\end{aligned}$$

Suppose $\frac{1}{3}(n+4-2\ell) \leq k < \frac{1}{3}(n+3-\ell)$. Since $|R-(R(k) \cup R(n-k+2-\ell))| = 2k-4+\ell$, at least $|S|-2k+4-\ell$ elements in the generating set contribute to the degree of at least one of these two vertices. Hence $|S| = N \Rightarrow \deg(k) + \deg(n-k+2-\ell) \geq N-2k+4-\ell$. In this case, $|R(k) \cap R(n-k+2-\ell) - \{n+2-\ell\}| = 2k-4+\ell$. So by similar reasoning as above,

$$\begin{aligned}
\deg(k) + \deg(n-k+2-\ell) &\geq (N-2k+4-\ell) + \\
&\quad |R(k) \cap R(n-k+2-\ell) - \{n+2-\ell\}| \\
&= (N-2k+4-\ell) + (2k-4+\ell) \\
&= N.
\end{aligned}$$

Lastly, suppose $k \geq \frac{1}{3}(n+3-\ell)$. Since $|R-(R(k) \cup R(n-k+2-\ell))| = 2k-5+\ell$, at least $|S|-2k+5-\ell$ elements in the generating set contribute to the degree of at least one of these two vertices. Hence $|S| = N \Rightarrow \deg(k)+\deg(n-k+2-\ell) \geq N-2k+5-\ell$. In this case, $|R(k) \cap R(n-k+2-\ell) - \{n+2-\ell\}| = 2k-5+\ell$. So by similar reasoning as above,

$$\begin{aligned}\deg(k) + \deg(n-k+2-\ell) &\geq (N - 2k + 5 - \ell) + \\ &\quad |R(k) \cap R(n-k+2-\ell) - \{n+2-\ell\}| \\ &= (N - 2k + 5 - \ell) + (2k - 5 + \ell) \\ &= N.\end{aligned}$$

Therefore, the induction hypothesis is true for $j = \ell$. Hence Claim 3.2.3 is true, and the theorem follows. □

Theorem 3.2.2 is best possible in the sense that there are many graphs $G(n, S)$ with $|S| = k - 1$ whose k-closure is not complete. In addition, this threshold for complete k-closure in terms of $|S|$ is surprisingly sharp, as shown in Theorem 3.2.5.

Theorem 3.2.5. *For any n, and any k with $1 \leq k \leq 2n-3$, there exists a generating set S of cardinality $k-1$ such that $cl_k(G(n,S)) = G(n,S)$.*

Proof. If $k = 1$, then the generating set of cardinality $k-1$ produces the *empty graph*. Otherwise, fix n and $2 \leq k \leq 2n-3$, and let

$$S = \{3, 4, \ldots, \lceil \tfrac{k-1}{2} \rceil + 2\} \cup \{2n - \lfloor \tfrac{k-1}{2} \rfloor, \ldots, 2n-2, 2n-1\}.$$

Suppose $k < 2n-3$. If k is odd, then $\Delta = \deg(1) = \frac{1}{2}(k-1)$, where Δ is the *maximum degree*. If k is even, then $\Delta = \deg(1) = \frac{1}{2}k$ and $\deg(i) < \deg(1)$ for all vertices $i \neq 1$. In either case, it follows that there are no two distinct vertices i and j with $\deg(i) + \deg(j) \geq k$.

Suppose $k = 2n - 3$. If n is even, then $\deg(i) = n - 2$ for all i. If n is odd, then $\Delta = \deg(\frac{1}{2}(n + 1)) = n - 1$ and $\deg(i) < \deg(\frac{1}{2}(n + 1))$ for all vertices $i \neq \frac{1}{2}(n + 1)$. It follows that there are no two distinct nonadjacent vertices i and j with $\deg(i) + \deg(j) \geq 2n - 3$. □

Using the main result of this section, we can now easily answer Question 3.1.2.

Corollary 3.2.6. *The minimum cardinality k such that $G(n, S)$ is hamiltonian for all $|S| = k$ is $k = n$.*

Proof. This follows from Theorem 3.1.4, Theorem 3.2.2, and the fact that $S = \{3, 4, \ldots, n + 1\}$ is a generating set of size $n - 1$ such that $G(n, S)$ is not hamiltonian ($\deg(n) = 1$). □

Remark 3.2.7. *The k-closure of a graph can be computed in time proportional to the size of the output [48]. In addition, a Hamilton cycle in the n-closure of a graph can be transformed in time $O(n^3)$ into a Hamilton cycle in the original graph [3]. Since finding a Hamilton cycle in the complete graph is trivial, Corollary 3.2.6 implies that if $|S| \geq n$, then a Hamilton cycle can be found in $G(n, S)$ in polynomial time.*

Corollary 3.2.6 does not imply anything about a Hamilton cycle in $G(2m, P)$, since there are far less than $2m$ primes in any interval $[3, 4m - 1]$. However, it is useful with regard to the construction of combinatorial Gray codes. Given a finite set $\mathcal{X}$ of combinatorial objects and some relation $\mathcal{R}$ on $\mathcal{X}$, a *cyclic combinatorial Gray code* for $\mathcal{X}$ is a circular permutation of its elements so that adjacent pairs are in $\mathcal{R}$ [44]. Therefore, we can say that given the set $\mathcal{X} = [n]$, with the relation $(i, j) \in \mathcal{R}$ if and only if $i + j \in S$, there is a cyclic combinatorial Gray code for $\mathcal{X}$ if $|S| \geq n$. Moreover, by Remark 3.2.7 this code can be found in polynomial time.

3.3 Edge Set Cardinality Bounds

In this section we determine tight bounds on the edge set cardinality of $G(n, S)$ in terms of $|S|$ using a straightforward counting argument. The corresponding bounds on *graph density* are plotted as Figure 3.2. Moreover, these bounds imply a new and simple construction of minimum graphs with n vertices and complete k-closure for all n and k.

Theorem 3.3.1. *If $|S| = k$ and $G(n, S) = (V, E)$, then $f(k) \leq |E| \leq g(n, k)$, where*

$$f(k) = \left\lfloor \frac{1}{8}(k+2)^2 \right\rfloor$$

and

$$g(n, k) = \binom{n}{2} - \left\lfloor \frac{1}{8}(2n-k-1)^2 \right\rfloor.$$

Proof. Given a fixed n, let $e(i)$ be the number of edges induced by generating set element i. The distribution of $e(i)$ is displayed as Table 3.1. As noted by Grigor'yan [21], it is easily shown that

$$e(i) = \begin{cases} \lfloor \frac{1}{2}(i-1) \rfloor & \text{if } i \leq n+1; \\ \lfloor \frac{1}{2}(2n-i+1) \rfloor & \text{if } i > n+1. \end{cases}$$

If $|S| = k$ induces the minimum number of edges over all possible generating sets of that cardinality, then

$$S = \{3, 4, \ldots, \lfloor \tfrac{k}{2} \rfloor + 2\} \cup \{2n - \lceil \tfrac{k}{2} \rceil, \ldots, 2n-2, 2n-1\}$$

Table 3.1: Distribution of the number of edges induced by generating set elements.

$i \in S$	3	4	5	6	7	8	9	$\dots$	$2n-5$	$2n-4$	$2n-3$	$2n-2$	$2n-1$
$e(i)$	1	1	2	2	3	3	4	$\dots$	3	2	2	1	1

and

$$|E| = \sum_{i=1}^{k} \left\lfloor \frac{1}{4}(i+3) \right\rfloor .$$

Evaluating this sum, we have that

- $k \equiv 0 \mod 4 \Rightarrow |E| = 2\frac{1}{4}k\frac{1}{4}(k+4) = \frac{1}{8}k(k+4)$;
- $k \equiv 1 \mod 4 \Rightarrow |E| = 2\frac{1}{4}(k-1)\frac{1}{4}(k+3) + \frac{1}{4}(k+3) = \frac{1}{8}(k+1)(k+3)$;
- $k \equiv 2 \mod 4 \Rightarrow |E| = 2\frac{1}{4}(k-2)\frac{1}{4}(k+2) + 2\frac{1}{4}(k+2) = \frac{1}{8}(k+2)^2$;
- $k \equiv 3 \mod 4 \Rightarrow |E| = 2\frac{1}{4}(k-3)\frac{1}{4}(k+1) + 3\frac{1}{4}(k+1) = \frac{1}{8}(k+3)(k+1)$.

Summarizing these results leads to the condensed formula

$$f(k) := \left\lfloor \frac{1}{8}(k+2)^2 \right\rfloor .$$

Note the following identity:

$$f(2n-3-k) + g(n,k) = \binom{n}{2}.$$

From this, it follows that

$$g(n,k) := \binom{n}{2} - \left\lfloor \frac{1}{8}(2n-k-1)^2 \right\rfloor . \qquad \square$$

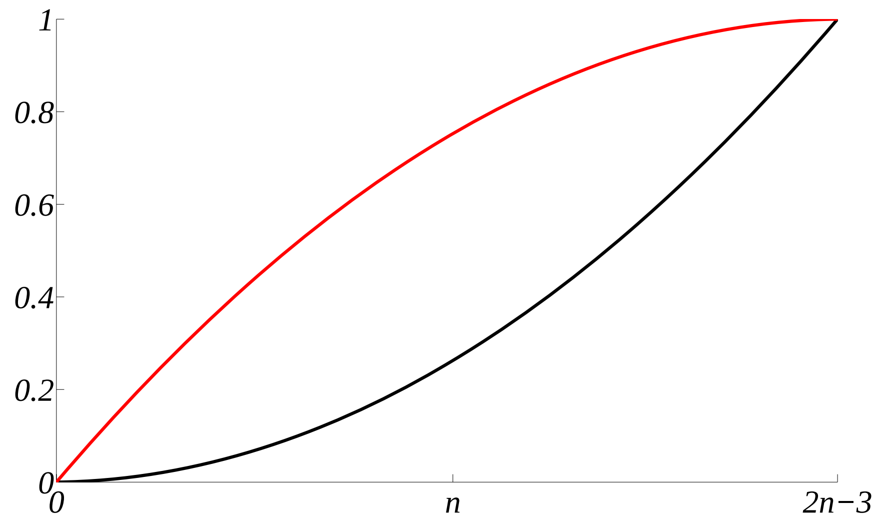

Figure 3.2: Range of graph density as a function of $|S|$.

It is interesting to note that the function $f(k)$ has no dependence on n. As well, Clark *et al.* [10] determined the minimum number of edges necessary for the k-closure of a graph of order n to be complete. They proved that this lower bound was best possible via a complicated construction of an infinite family of graphs. This lower bound function matches our lower bound function $f(k)$, and so we arrive at a useful corollary that greatly simplifies their construction. In particular, our construction is not recursive.

Corollary 3.3.2. *Fix any n, and any k with $0 \leq k \leq 2n-3$. Let*

$$S = \{3, 4, \ldots, \lfloor \tfrac{k}{2} \rfloor + 2\} \cup \{2n - \lceil \tfrac{k}{2} \rceil, \ldots, 2n-2, 2n-1\}.$$

Then $G(n, S)$ is a minimum graph with complete k-closure. □

3.4 Thresholds for Monotone Properties in $G(n, S)$

In order to explore some of the implications of Theorem 3.2.2, we can ask when an additive graph $G(n, S)$ always has a particular monotone property. However, closure theory is only useful in this respect when the presence or absence of a k-stable property in the k-closure of a graph is apparent. In the extreme case when $cl_k(G) = G$, no new information is obtained. Because of the ubiquity of properties present in the complete graph, the case when the k-closure of a graph is the complete graph is of particular interest. This led Faudree *et al.* [15] to introduce the concept of the complete closure number and the concept of complete stability, which are of special interest to us.

The *complete closure number* $cc(G)$ of a graph G of order n is defined to be the maximum integer $k \leq 2n - 3$ such that $cl_k(G) = K_n$. The *complete stability* $cs(P)$ of a property P defined on all graphs of order n and satisfied by K_n is the minimum integer k such that any graph G satisfies P when $cl_k(G)$ is complete. Not surprisingly, for many properties the value of the complete stability is lower than the value of the stability [15].

Remark 3.4.1. *Theorem 3.2.2 states that $cc(G(n, S)) \geq |S|$. Therefore, suppose a property P has $cs(P) \leq t$. Then $G(n, S)$ is assured of having property P whenever $|S| \geq t$.*

The goals of this section are twofold. Firstly, we wish to demonstrate that our family of additive graphs is a wide-ranging and representative subclass of graphs with complete k-closure. We accomplish this by proving that for many monotone properties, the thresholds in terms of $|S|$ for their appearances in $G(n, S)$ graphs that are a consequence of Remark 3.4.1 are tight. Secondly, determining these tight thresholds yields some interesting number-theoretic results.

The following mimics the exposition of Section 2.5 in the survey by Broersma *et al.* [6]. Every subsection follows a standard format: introduction of an important monotone property, citations of the known results concerning its complete stability, interpretation of these results from the point of view of thresholds in additive graphs using Remark 3.4.1, and, if possible, a limit example showing that this threshold is tight. The limit examples are mostly based on the failure of obvious necessary degree conditions, and so only in the more complicated cases are they explained in depth. The properties considered in this section are informally grouped into three categories: properties related to cycles, properties related to paths, and other well-known graph-theoretic properties.

We determine for the first time the complete stability of two properties[1], by using additive graph limit examples to show that the known stability of the property matches its complete stability. For the property P: "G has a P_4" we determine the tight threshold in additive graphs, which differs markedly from its complete stability. Some of the results of this section are summarized in Table 3.2.

[1] These properties are "G is k-factor-critical" and "G is k-matching-extendable."

Table 3.2: Tight thresholds for monotone properties in $G(n, S)$.

Property P	$\|S\|$	Limit Example	Reflects $cs(P)$?
k-connected	$n+k-2$	$\{3,4,\ldots,n+k-1\}$	Yes
k-edge-connected	$n+k-2$	$\{3,4,\ldots,n+k-1\}$	Yes
Contains a Hamilton path	$n-1$	$\{3,4,\ldots,n\}$	Yes
Contains a Hamilton cycle	n	$\{3,4,\ldots,n+1\}$	Yes
Contains a C_k (n and k odd, $n \gg k$)	n	$\{3,5,\ldots,2n-1\}$	Yes
k-hamiltonian	$n+k$	$\{3,4,\ldots,n+k+1\}$	Yes
k-edge-hamiltonian	$n+k$	$\{3,4,\ldots,n+k+1\}$	Yes
k-Hamilton-connected	$n+k+1$	$\{n-k,n-k+1,\ldots,2n-1\}$	Yes
k-leaf-connected ($k < n-1$)	$n+k-1$	$\{3,4,\ldots,n+k\}$	Yes
Edge-pancyclic	$\lceil\frac{1}{2}(3n-3)\rceil$	$\{3,4,\ldots,n,n+2,\ldots,2n-1\}$ (n odd) $\{3,4,\ldots,n,n+2,\ldots,2n-2,2n-1\}$ (n even)	Yes
Panconnected	$\lceil\frac{1}{2}(3n-3)\rceil$	$\{3,4,\ldots,n,n+2,\ldots,2n-1\}$ (n odd) $\{3,4,\ldots,n,n+2,\ldots,2n-2,2n-1\}$ (n even)	Yes
Contains a matching of size k ($2k \leq n$)	$2k-1$	$\{3,4,\ldots,2k\}$	Yes
k-factor-critical ($n-k$ even)	$n+k-1$	$\{3,4,\ldots,n+k\}$	Yes
k-matching-extendable (n even)	$n+2k-1$	$\{3,4,\ldots,n-2k,2n-4k,\ldots,2n-1\}$	Yes
Contains a k-factor ($1 \leq k \leq 3$, kn even)	$n+k-2$	$\{3,4,\ldots,n+k-1\}$	Yes
$\alpha \leq k$	$2n-2k-1$	$\{3,4,\ldots,2n-2k\}$	Yes
$\mu \leq k$	$n-k$	$\{3,4,\ldots,n-k+1\}$	Yes
Contains a K_3 ($n \geq 5$)	n	$\{3,5,\ldots,2n-1\}$	Yes (n odd); No (n even)
Contains a P_4 ($n \geq 6$)	5	$\{3,4,2n-2,2n-1\}$	No

3.4.1 G contains a cycle C_k

Having previously determined the threshold for a Hamilton cycle, we begin by determining thresholds for cycles of any length. A simple graph on n vertices is *pancyclic* if it contains at least one cycle of length ℓ for all $3 \leq \ell \leq n$.

Theorem 3.4.2. *(Theorem 4.1, [15]) The property P: "G is pancyclic" satisfies $cs(P) = n+1$ if n is even and $n \leq cs(P) \leq n+1$ if n is odd.* □

Corollary 3.4.3. *$|S| \geq n+1$ implies that $G(n, S)$ is pancyclic.* □

No example is known of an additive graph of order $n \geq 5$ with $|S| = n$ that is not pancyclic. This is reflected in the following conjecture, due to Schelten and Schiermeyer [45].

Conjecture 3.4.4. *[45] The property P: "G is pancyclic" satisfies $cs(P) = n$ if n is odd.*

This conjecture was inspired by the following result.

Theorem 3.4.5. *(Theorem 2, [45]) The property P: "G contains a cycle C_k" satisfies $cs(P) \leq n$ for each integer k between 3 and $\frac{1}{5}(n+13)$ if n is odd, $n \geq 32$.* □

We prove a tight threshold for triangles (and hence odd cycles) in these additive graphs in Section 4.2.

3.4.2 G is k-hamiltonian or k-edge-hamiltonian

There are many important graph-theoretic properties intimately related to a graph being hamiltonian. For example, a graph G is *k-hamiltonian* (resp. *k-edge-hamiltonian*) if the deletion of at most k vertices (resp. edges) from G results in a hamiltonian graph. This is an obvious generalization of k-connectivity (resp. k-edge-connectivity).

Theorem 3.4.6. *[15] The property P: "G is k-hamiltonian" and the property P': "G is k-edge-hamiltonian" satisfy $cs(P) = cs(P') = n + k$.* □

Corollary 3.4.7. *$|S| \geq n+k$ implies that $G(n, S)$ is k-hamiltonian and k-edge-hamiltonian, and this is best possible.* □

Limit Example 3.4.8. $G(n, S);\ S = \{3, 4, \ldots, n + k + 1\}$

3.4.3 G contains edge-disjoint Hamilton cycles

While a graph with complete n-closure typically possesses many Hamilton cycles, edge-disjoint Hamilton cycles are less common.

Theorem 3.4.9. *(Theorem 4.7, [15]) The property P: "G contains two edge-disjoint Hamilton cycles" satisfies $n + 2 \leq cs(P) \leq n + 4$.*
(Theorem 4.8, [15]) The property P': "G is hamiltonian and for every Hamilton cycle C, there exists another Hamilton cycle C' edge-disjoint from C" satisfies $n + 3 \leq cs(P') \leq n + 4$. □

Corollary 3.4.10. *$|S| \geq n+4$ implies that for any Hamilton cycle C in $G(n, S)$, the graph contains a Hamilton cycle edge-disjoint from C.* □

3.4.4 G is vertex-pancyclic or edge-pancyclic

In 2-connected *chordal graphs*, which are well-studied, every vertex and every edge is contained in a triangle. We can extend this concept beyond cycles of length 3, and say that G is *vertex-pancyclic* (resp. *edge-pancyclic*) if each vertex v (resp. edge e) of G belongs to a cycle of length ℓ for all $3 \leq \ell \leq n$.

Theorem 3.4.11. *(Theorem 28, [41]) The property P: "G is vertex-pancyclic" satisfies* $cs(P) = \lceil \frac{1}{3}(4n-3) \rceil$. □

Corollary 3.4.12. $|S| \geq \lceil \frac{1}{3}(4n-3) \rceil$ *implies that $G(n, S)$ is vertex-pancyclic.* □

Theorem 3.4.13. *(Theorem 30, [41]) The property P: "G is edge-pancyclic" satisfies* $cs(P) = \lceil \frac{1}{2}(3n-3) \rceil$. □

Corollary 3.4.14. $|S| \geq \lceil \frac{1}{2}(3n-3) \rceil$ *implies that $G(n, S)$ is edge-pancyclic, and this is best possible.* □

Limit Example 3.4.15. $G(n, S);\ S = \{3, 4, \ldots, n-1\} \cup \{n, n+2, \ldots, 2n-1\}$ *(n odd)*
$G(n, S);\ S = \{3, 4, \ldots, n-1\} \cup \{n, n+2, \ldots, 2n-2, 2n-1\}$ *(n even)*

In the above limit examples, the edge $(n-1, n)$ is not contained in a triangle. It suffices to note that vertices $n-1$ and n have no common neighbours, as such a neighbour would imply two consecutive generating set elements in $\{n, n+1, \ldots, 2n-2\}$.

3.4.5 G is cycle extendable or fully cycle extendable

If a cycle in a graph is not *chordless*, then there is a strictly smaller cycle in the graph on some subset of the vertices spanned by the original cycle. Inverting this idea, we say that a cycle C in G is *extendable* if there exists a cycle C' in G such that $V(C) \subset V(C')$ and $|V(C')| = |V(C)| + 1$. A graph is *cycle extendable* if G contains at least one cycle and every non-Hamilton cycle in G is extendable.

Theorem 3.4.16. *[41] The property P: "G is cycle extendable" satisfies* $cs(P) = \lceil \frac{3}{2}n \rceil - 2$. □

A graph G is *fully cycle extendable* if G is cycle extendable and every vertex of G lies on a triangle of G. The following corollary follows from Theorem 3.4.11 and Theorem 3.4.16.

Corollary 3.4.17. *[41] The property P: "G is fully cycle extendable" satisfies $cs(P) = \lceil \frac{3}{2}n \rceil - 2$.* $\square$

Corollary 3.4.18. $|S| \geq \lceil \frac{3}{2}n \rceil - 2$ *implies that $G(n, S)$ is fully cycle extendable.* $\square$

The best-known additive graph limit examples are generated by the sets $S = \{3, 5, \ldots, n+3\} \cup \{n+5, n+6, \ldots, 2n-1\}$ (n even) and $S = \{3, 5, \ldots, n+2\} \cup \{n+5, n+6, \ldots, 2n-1\}$ (n odd). In this case, the generating sets have cardinality $\lceil \frac{3}{2}n \rceil - 4$ and the 4-cycle $(1, 2, 3, 4, 1)$ is not extendable in the corresponding graphs. To see this, note that this is a chordless cycle and hence it is extendable if and only if at least one of the pairs of vertices $\{1, 2\}$, $\{1, 4\}$, $\{3, 2\}$, $\{3, 4\}$ have a common neighbour among $\{5, 6, \ldots, n\}$. This would imply two consecutive generating elements in $\{6, 7, \ldots, n + 4\}$.

3.4.6 G has a hamiltonian prism

The *prism* of a graph G is the graph obtained from two copies of G by connecting all the pairs of images of the same vertex by an edge. It was recently shown by Horák *et al.* [28] that the prism of the middle-levels graph is hamiltonian.

Theorem 3.4.19. *(Theorem 1, [30]) The property P: "G has a hamiltonian prism" satisfies $\frac{4}{3}n - 5 \leq s(P) \leq \frac{4}{3}(n - 1)$.* $\square$

Corollary 3.4.20. $|S| \geq \frac{4}{3}(n - 1)$ *implies that $G(n, S)$ has a hamiltonian prism.* $\square$

3.4.7 G contains a path P_k

Having considered thresholds for cycles of all lengths, it is natural to consider the threshold for Hamilton paths (and hence paths of all lengths). Recall the question posed by Larson [31] that asks when the additive graph of order n generated by the prime numbers has a Hamilton path. Corollary 3.4.22 implies that a Hamilton path is not inevitable given the proportion of primes in any interval $[3, 2n - 1]$.

Theorem 3.4.21. *(Theorem 9.4, [3]) The property P: "G contains a path P_k of length $k - 1$" satisfies $s(P) = n - 1$ for $4 \leq k \leq n$.* □

Corollary 3.4.22. *$|S| \geq n - 1$ implies that $G(n, S)$ has a Hamilton path, and this is best possible.* □

Limit Example 3.4.23. $G(n, S);\ S = \{3, 4, \ldots, n\}$

Corollary 3.4.22 is not an obvious consequence of the fact that $G(n, S)$, $|S| = n$, is hamiltonian, as the removal of a single generating set element typically results in the removal of many edges.

Theorem 3.4.24. *(Remark 4.10, [15]) The property P: "G contains a P_4" satisfies $\sqrt{8n+9} - 3 \leq cs(P) \leq \sqrt{8n+26} - 3$.* □

This property is one of the sole properties considered for which the complete stability and the threshold in $G(n, S)$ graphs differs markedly. For $n = 4$ the threshold is $|S| = 3$, and for $n = 5$ the threshold is $|S| = 4$. This follows from Corollary 3.4.22, and the limit examples $G(4, \{3, 7\})$ and $G(5, \{3, 4, 9\})$.

Theorem 3.4.25. *For $n \geq 6$, $|S| \geq 5$ implies that $G(n, S)$ has a P_4, and this is best possible.*

Proof. Suppose $n \geq 6$ and $|S| = 5$. Since $\deg(1) + \deg(n) \geq 5$, without loss of generality $\deg(1) \geq 3$. Let u, v, and w be three distinct neighbours of vertex 1, with $2 \leq u < v, w \leq n$. It follows that $\{u+1, v+1, w+1\} \subset S$. Without loss of generality, $v + 1 \neq 2u$. Hence $G(n, S)$ contains the P_4 $(1 + v - u, u, 1, v)$. □

Limit Example 3.4.26. $G(n, S);\ S = \{3, 4, 2n - 2, 2n - 1\}$

3.4.8 G is panconnected

Rather than generalize the notion of connectivity by demanding $k > 1$ disjoint paths between every pair of vertices, we can focus on the lengths of the (not necessarily disjoint) paths between every pair of vertices. A graph G is *panconnected* if every pair of vertices is connected by a path of length ℓ for $2 \leq \ell \leq n - 1$. The following is a simple corollary of Theorem 3.4.13.

Corollary 3.4.27. *(Corollary 31, [41]) The property P: "G is panconnected" satisfies $cs(P) = \lceil \frac{1}{2}(3n - 3) \rceil$.* □

Corollary 3.4.28. $|S| \geq \lceil \frac{1}{2}(3n - 3) \rceil$ *implies that $G(n, S)$ is panconnected, and this is best possible.* □

Limit Example 3.4.29. $G(n, S);\ S = \{3, 4, \ldots, n - 1\} \cup \{n, n + 2, \ldots, 2n - 1\}$ *(n odd)*
$G(n, S);\ S = \{3, 4, \ldots, n - 1\} \cup \{n, n + 2, \ldots, 2n - 2, 2n - 1\}$ *(n even)*

In the above limit examples, the vertices $n - 1$ and n are not connected by a path of length 2.

3.4.9 G is k-Hamilton-connected

We can generalize the concept of being *Hamilton-connected* in the same way as we generalized the concept of hamiltonian to k-hamiltonian. That is, a graph G is *k-Hamilton-connected* if the deletion of at most k vertices from G results in a Hamilton-connected graph.

Theorem 3.4.30. *[15] The property P: "G is k-Hamilton-connected" satisfies $cs(P) = n + k + 1$.* □

Corollary 3.4.31. *$|S| \geq n+k+1$ implies that $G(n, S)$ is k-Hamilton-connected, and this is best possible.* □

Limit Example 3.4.32. $G(n, S);\ S = \{n-k, n-k+1, \ldots, 2n-1\}$

To see that the above limit example is not k-Hamilton-connected, note the following proposition.

Proposition 3.4.33. *(Exercise 18.1.6, [4]) A graph is traceable from a vertex v if it has a Hamilton path with endpoint v. Let G be a graph and let H be the graph obtained from G by adding a new vertex x and joining it to every vertex of G.*

(i) H is Hamilton-connected if and only if G is traceable from every vertex.

(ii) H is traceable from every vertex if and only if G is traceable from some vertex.

Proof. $((i), \Rightarrow)$ Consider any vertex v in G. There exists a Hamilton path with endpoints v and x in H. Deleting x yields a Hamilton path in G with endpoint v.

$((i), \Leftarrow)$ Consider any pair of distinct vertices u, v in H. Suppose neither of these vertices is vertex x. The vertex v is in G, and hence there exists a Hamilton path P

with endpoints v and v'. If u' is the neighbour of u in P closest to the endpoint v, then $P + (u', x) + (x, v') - (u', u)$ is a Hamilton path in H with endpoints u and v. Otherwise, without loss of generality $u = x$. Then $P + (v', x)$ is a Hamilton path in H with endpoints u and v.

$((ii), \Rightarrow)$ There exists a Hamilton path with endpoint x in H. Deleting x yields a Hamilton path in G.

$((ii), \Leftarrow)$ Consider a Hamilton path P in G with endpoints u and v. It follows that $P + (u, x) + (x, v)$ is a Hamilton cycle in H. □

Thus given $G(n, \{n-k, n-k+1, \ldots, 2n-1\})$, delete vertices $\{n-k+1, n-k+2, \ldots, n\}$. The remaining graph is $G(N, \{N, N+1, \ldots, 2N-1\})$, where $N := n-k$. Note that vertices $N-1$ and N are connected to all other vertices. By Proposition 3.4.33, it follows that $G(N, \{N, N+1, \ldots, 2N-1\})$ is Hamilton-connected if and only if $G(N-2, \{N, N+1, \ldots, 2N-5\})$ is traceable from some vertex. But the latter graph has no Hamilton path because $\deg(1) = 0$.

3.4.10 G is k-leaf-connected

A Hamilton-connected graph has every pair of vertices as precisely the set of *leafs* of a *spanning tree*. In general, we say that a graph G is *k-leaf-connected* if $k < n$ and given any $L \subset V$ with $|L| = k$, G has a spanning tree F such that L is precisely the set of leafs of F.

Theorem 3.4.34. *(Theorem 4, [25]) The property P: "G is k-leaf-connected" satisfies $s(P) = n + k - 1$.* □

Corollary 3.4.35. *$|S| \geq n + k - 1$ ($k < n - 1$) implies that G is k-leaf-connected, and this is best possible. $|S| = 2n - 3$ implies that G is $(n-1)$-leaf-connected, and this is best possible.* □

Limit Example 3.4.36. $G(n, S);\ S = \{3, 4, \ldots, n + k\}$ ($k < n - 1$)
$G(n, S);\ S = \{3, 4, \ldots, 2n - 2\}$ ($k = n - 1$)

To see that the first limit example is not k-leaf-connected, suppose $k < n - 1$ and let $L = \{1, 2, \ldots, k\}$. Then the set of neighbours of vertex n is precisely the set L, and any spanning tree with set of leafs exactly L must be a spanning *star* with vertex n as its centre. This contradicts the assumption that $k \neq n-1$. If $|S| = 2n-3$, then $G(n, S) = K_n$, and hence this graph is $(n-1)$-leaf-connected. To see that the second limit example is not $(n-1)$-leaf-connected, let $L = \{1, 2, \ldots, n-1\}$. There is no spanning star with vertex n as its centre since n is not adjacent to $n-1$, and hence there is no spanning tree with set of leafs exactly L.

3.4.11 G contains a $K_{2,k}$

The notion of having a subgraph that is the *complete bipartite graph* $K_{2,k}$ has an interesting number-theoretic interpretation that we consider in Section 4.3.

Theorem 3.4.37. *(Theorem 4.11, [15]) The property P: "G contains a $K_{2,k}$" satisfies $\sqrt{8n+9} - 4 \leq cs(P) \leq \sqrt{8(k-1)n}$.* □

Corollary 3.4.38. *$|S| \geq \sqrt{8(k-1)n}$ implies that $G(n, S)$ contains a $K_{2,k}$.* □

3.4.12 G contains a matching of size k

Matchings in additive graphs generated by specific integer sequences were considered in the previous chapter. We now extend our consideration to arbitrary generating sets and matchings of any size.

Theorem 3.4.39. *[15] The property P: "G contains a matching of size k" satisfies $cs(P) = 2k - 1$.* □

Corollary 3.4.40. *$|S| \geq 2k-1$ ($2k \leq n$) implies that $G(n,S)$ contains a matching of size k, and this is best possible.* □

Limit Example 3.4.41. $G(n,S);\ S = \{3, 4, \ldots, 2k\}$

3.4.13 G is k-factor-critical

For some properties, the stability is known but the complete stability is not. For example, for any nonnegative integer k such that $n-k$ is even, a graph G is said to be *k-factor-critical* if, for any set $F \subset V$ with $|F| = k$, the graph $G - F$ has a perfect matching. Equivalently, every induced subgraph of order $n-k$ of G has a perfect matching.

Theorem 3.4.42. *(Theorem 2, [38]) The property P: "G is k-factor-critical" satisfies $s(P) \leq n + k - 1$.* □

Corollary 3.4.43. *$|S| \geq n + k - 1$ ($n-k$ even) implies that $G(n,S)$ is k-factor-critical, and this is best possible.* □

Limit Example 3.4.44. $G(n,S);\ S = \{3, 4, \ldots, n+k\}$

To see that the limit example is not k-factor-critical, let $F = \{1, 2, \ldots, k\}$. Since F is precisely the set of neighbours of vertex n, $G(n, S) - F$ has no perfect matching ($\deg(n) = 0$ in this subgraph).

Corollary 3.4.45. *The property P: "G is k-factor-critical" satisfies $cs(P) = n+k-1$.* □

3.4.14 G is k-matching-extendable

For an integer k, $0 \leq k \leq \frac{1}{2}|V|$, a graph G of even order is *k-matching-extendable* if G has a matching of size k and every matching of size k can be extended to a perfect matching of G.

Theorem 3.4.46. *(Theorem 3, [38]) The property P: "G is k-matching-extendable" satisfies $s(P) \leq n + 2k - 1$.* □

Corollary 3.4.47. *$|S| \geq n+2k-1$ (n even) implies that $G(n, S)$ is k-matching-extendable, and this is best possible.* □

Limit Example 3.4.48. $G(n, S)$; $S = \{3, 4, \ldots, n-2k\} \cup \{2n-4k, 2n-4k+1, \ldots, 2n-1\}$

To see that the limit example is not k-matching-extendable, consider the matching $M = \{(n-2k+1, n-2k+2), \ldots, (n-1, n)\}$ of size k on vertices $\{n-2k+1, n-2k+2, \ldots, n\}$. This matching is extendable only if there is a perfect matching on $\{1, 2, \ldots, n-2k\}$. There is no such matching, since vertex $n-2k$ has no neighbours among $\{1, 2, \ldots, n-2k-1\}$ in $G(n, S)$.

Corollary 3.4.49. *The property P: "G is k-matching-extendable" satisfies $cs(P) = n + 2k - 1$.* □

3.4.15 G contains a k-factor

Previously, we showed the existence of a 2-factor in $G(2m, P)$ (m even) by taking the union of two disjoint perfect matchings. As in the case of Hamilton cycles, Hamilton paths, and perfect matchings, the deterministic threshold for a 2-factor in general additive graphs is fairly high. Hence this result does not subsume what was previously proved regarding the additive graphs generated by the prime numbers.

Theorem 3.4.50. *[36] The property P: "G contains a k-factor ($1 \leq k \leq n-1$, kn even)" satisfies $cs(P) = n + k - 2$ for $1 \leq k \leq 3$ and $n + k - 2 \leq cs(P) \leq n + k - 1$ for $k \geq 4$.* □

Niessen [36] made the following conjecture, which appears to be true for our family of graphs.

Conjecture 3.4.51. *[36] $cs(P) = n + k - 2$ for $k \geq 4$.*

We note that some progress has since been made towards resolving this conjecture.

Theorem 3.4.52. *(Theorem 1, [29]) Suppose*

$$n > \max\left\{\frac{1}{8}(3k^2 + 2k + 3), 2k - 1 + \sqrt{3k^2 - 6k + 3}\right\}. \tag{3.1}$$

Then the property P: "G contains a k-factor" satisfies $cs(P) = n + k - 2$. □

Corollary 3.4.53. *$|S| \geq n + k - 2$ ($1 \leq k \leq 3$ or n satisfies inequality (3.1), kn even) implies that $G(n, S)$ has a k-factor, and this is best possible. Otherwise, $|S| \geq n + k - 1$ (kn even) implies that $G(n, S)$ has a k-factor.* □

Limit Example 3.4.54. $G(n, S);\ S = \{3, 4, \dots, n + k - 1\}$

3.4.16 G is k-connected or k-edge-connected

The notions of k-connectivity and k-edge-connectivity are fundamental to the study of graphs.

Theorem 3.4.55. *[15] The property P: "G is k-connected" and the property P': "G is k-edge-connected" satisfy $cs(P) = cs(P') = n + k - 2$.* □

Corollary 3.4.56. *$|S| \geq n + k - 2$ implies that $G(n, S)$ is k-connected and k-edge-connected, and this is best possible.* □

Limit Example 3.4.57. $G(n, S);\ S = \{3, 4, \dots, n + k - 1\}$

3.4.17 $\alpha(G) \leq k$

The *independence number* of an additive graph is another important graph-theoretic property. This is one of the only cases in which we are aware of a simple proof of the threshold for a property that does not depend on results from closure theory.

Theorem 3.4.58. *[15] The property P: "$\alpha(G) \leq k$" satisfies $cs(P) = 2n - 2k - 1$.* □

Corollary 3.4.59. *$|S| \geq 2n - 2k - 1$ implies that $\alpha(G(n, S)) \leq k$, and this is best possible.* □

Limit Example 3.4.60. $G(n, S);\ S = \{3, 4, \dots, 2n - 2k\}$

Remark 3.4.61. *An equivalent statement in additive number theory is that*

$$|\{a + b|\, a, b \in A,\ a \neq b\}| \geq 2|A| - 3$$

for any $A \subset [n]$, $|A| \geq 2$. This is easy to prove by induction.

3.4.18 $\mu(G) \leq k$

A perfect matching is a set of $\frac{1}{2}n$ pairwise disjoint paths that cover all the vertices of an even order graph, and a Hamilton path is a single path that covers all the vertices of a graph. Generalizing this idea, let $\mu(G)$ denote the smallest number of pairwise disjoint paths needed to cover all the vertices of a graph G.

Theorem 3.4.62. *[15] The property P: "$\mu(G) \leq k$" satisfies $cs(P) = n - k$.* □

Corollary 3.4.63. *$|S| \geq n - k$ implies that $\mu(G(n, S)) \leq k$, and this is best possible.* □

Limit Example 3.4.64. $G(n, S);\ S = \{3, 4, \ldots, n - k + 1\}$

3.4.19 G contains a clique K_t

We observe that the *complement* of an additive graph $G(n, S)$ is the additive graph generated by $\overline{S} = \{3, 4, \ldots, 2n-1\} - S$. Also, it is easy to see that *cliques* in a graph G correspond to independent sets in its complement $\overline{G}$. Hence if $G(n, S)$, $|S| = k$, has a clique of size t, then there exists a graph $G(n, S')$, $|S'| = 2n - 3 - k$, with an independent set of size t.

Theorem 3.4.65. *(Theorem 4.5, [15]) The property P: "G contains a clique K_t" satisfies $cs(P) = 2\lfloor \frac{t-2}{t-1} n \rfloor + 1$.* □

Corollary 3.4.66. *$|S| \geq 2\lfloor \frac{t-2}{t-1} n \rfloor + 1$ implies that $G(n, S)$ contains a clique K_t.* □

We show in the next chapter (Theorem 4.2.3) that this is not tight for $t = 3$.

3.4.20 G contains every tree on k vertices

We conclude with yet another result by Faudree *et al.* [15], which generalizes the property of having a path P_k.

Theorem 3.4.67. *(Theorem 4.9, [15]) The property P: "G contains every tree on k vertices" satisfies $2^{\frac{-1}{4}}\sqrt{(k-2)n} \leq cs(P) \leq 2\sqrt{2(k-2)n}$.* □

Corollary 3.4.68. $|S| \geq 2\sqrt{2(k-2)n}$ *implies that $G(n, S)$ contains every tree on k vertices.* □

Chapter 4

Other Results Concerning Additive Graphs

We begin this chapter with two alternate proofs of the threshold for connectivity; one of these proofs has an interesting corollary regarding the *diameter* of additive graphs. We also determine the tight threshold for triangles. In contrast to Section 3.4, these results demonstrate some ways in which additive graphs are not fully representative of general graphs with complete k-closure. As an example of the number-theoretic interpretations of these graphs and their properties, a theorem by Cramer [11] concerning prime numbers is generalized to a number of different sequences.

4.1 Alternate Proofs of the Threshold for Connectivity

In this section, we offer two other proofs of the fact that $|S| = n - 1$ implies that $G(n, S)$ is connected. The first is due to Vašek Chvátal [8] and the second is due to

Nithum Thain [49]. These proofs demonstrate the interesting combinatorial properties of these graphs, and the utility of the general results of the previous chapter (i.e., Theorem 3.2.2 and its many corollaries). As well, the proof by Thain implies a corollary regarding the diameter of $G(n, S)$ when $|S| \geq n-1$.

Theorem 4.1.1. $|S| = n-1$ *implies that* $G(n, S)$ *is connected.*

Proof. (due to Vašek Chvátal) Note that the statement "$|S| = n-1$ implies connectivity" is equivalent to the statement "For every partition of $\{1, 2, \ldots, n\}$ into two sets A and B, $|A + B| \geq n-1$." Here $A + B$ refers to the *sumset* of A and B. This equivalence follows from the following three observations:

- A graph is connected if and only if there is an edge coming out of every separating set.
- There is an edge coming out of every separating set in $G(n, S)$ if and only if for every partition of $[n]$ into two sets A and B, $(A + B) \cap S \neq \varnothing$.
- $(A + B) \cap S \neq \varnothing$ for all A and B if and only if $|A + B| > 2n - 3 - |S|$.

The partition statement follows from an easy induction argument: Remove n from the partition; Apply the induction hypothesis; Replace n and note that it adds at least one more item to the sumset. $\square$

Proof. (due to Nithum Thain) Fix n and consider a generating set S of cardinality $n-1$. Firstly, assume that there is no path between vertices 1 and n. This implies that vertices 1 and n are not adjacent and that they have no common neighbours. Recall that $S \subset R(1) \cup R(n) = R$, where $R(i) = \{i+1, i+2, \ldots, i+n\} - \{2i\}$ is the region of vertex i and $R = \{3, 4, \ldots, 2n-1\}$, and hence that every element in S

contributes to the degree of either vertex 1 or vertex n. Thus $\deg(1)+\deg(n) \geq n-1$. Since neither of these two vertices are counted in the left-hand side of the inequality and every other vertex is counted at most once, this implies that there are at least $(n-1)+2 = n+1$ vertices. This is a contradiction, and hence there exists a path (of length at most 2) between vertices 1 and n.

Secondly, we show that every other vertex i is connected to at least one of the vertices 1 and n. Suppose not. Then i is not adjacent to, and has no common neighbours with, either of these two vertices. Let $A := S \cap \{3, 4, \ldots, i\}$ and $B := S \cap \{n+i+1, n+i+2, \ldots, 2n-1\}$. Then $|A|$ is the number of neighbours of vertex 1 from among the vertices $\{2, 3, \ldots, i-1\}$ and $|B|$ is the number of neighbours of vertex n from among the vertices $\{i+1, i+2, \ldots, n-1\}$. It follows that

$$\begin{aligned} |A| + |B| + |R(i) \cap S| &= |S \cap (R - \{2i\})| \\ &\geq |S| - 1 \\ &= n - 2 \end{aligned}$$

as these three sets are disjoint by assumption. Every vertex of the graph is counted in the left-hand side of the inequality at most once, and clearly none of the vertices 1, i, and n are counted at all. This implies that there are at least $(n-2)+3 = n+1$ vertices. This is a contradiction, and hence there exists a path (of length at most 2) between vertex i and at least one of vertices 1 and n. It follows that $|S| = n-1$ implies connectivity. □

Corollary 4.1.2. $|S| \geq n-1$ *implies that* $G(n, S)$ *has diameter at most* 6. □

In fact, currently we have no example in which the diameter is greater than 4 for generating sets in this cardinality range. Clark *et al.* [9] determined that the maximum diameter $d(n)$ of all graphs of order n whose n-closure is complete is bounded by

$$3.2 \log n - 9 \leq d(n) \leq 8.3 \log n + 16.$$

This demonstrates an important distinction between our family of graphs and general graphs with complete k-closure.

4.2 A Tight Threshold for Odd Cycles

Recall that additive partitions of the natural numbers are related to bipartite additive graphs. In this section we answer the following question, which was not addressed in the previous chapter.

Question 4.2.1. *What is the minimum cardinality k such that $G(n, S)$ is not bipartite for all $|S| = k$?*

There are many generating sets of small size that induce odd cycles; two general examples are $\{2i+1, 2i+2, 2i+3\} \subseteq \{3, 4, \ldots, 2n-1\}$, which produces a triangle among vertices $i, i+1, i+2$, and $\{2i, 2i+2, 2i+4\} \subset \{3, 4, \ldots, 2n-1\}$, which produces a triangle among vertices $i-1, i+1, i+3$. Despite this, note that the set of odd generating set elements $S = \{3, 5, 7, ..., 2n-1\}$ has cardinality $n-1$ and produces a graph without an odd cycle (namely $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$). The following theorem states that for $n \geq 5$, $|S| = n$ implies that $G(n, S)$ has an odd cycle, thereby proving that the minimum cardinality k such that $G(n, S)$ is not bipartite for all $|S| = k$ is $k = n$. The statement in the theorem is trivially true for $n = 3$ and false for $n = 4$, as

$G(4, \{3, 4, 6, 7\}) = C_4$. To reduce the checking of base cases, we use the following lemma.

Lemma 4.2.2. *(Corollary 1, [32]) If G is a hamiltonian graph of order n with $\Delta > \frac{1}{2}n$, then G contains a triangle.* □

Theorem 4.2.3. *For any $n \geq 5$, $|S| = n$ implies that $G(n, S)$ has a triangle.*

Proof. We proceed by strong induction on n. As base cases, consider $n = 5$, $n = 6$, and $n = 7$. By Corollary 3.2.6, Lemma 4.2.2, and the fact that $\deg(1) + \deg(n) \geq n$, it suffices to check the cases when $n = |S| = 6$ and $\deg(1) = \deg(6) = 3$. In these cases, $7 \notin S$ and hence there are $\binom{4}{3}\binom{4}{3} = 16$ choices for S. It is easily confirmed that these generating sets induce triangles in their corresponding graphs.

For any $n > 7$, assume the induction hypothesis for $n - 1$, $n - 2$, and $n - 3$. Below is a simple case analysis.

- If $\{2n - 2, 2n - 1\} \not\subset S$, then delete vertex n and any generating set elements in S strictly greater than $2n - 3$ (there is at most 1). We arrive at a graph $G(n - 1, S')$, where $|S'| \geq n - 1$, and by induction this graph has a triangle.

- Thus assume henceforth that $\{2n-2, 2n-1\} \subset S$. If $2n-4 \notin S$ and $2n-3 \notin S$, then delete vertices $n - 1$ and n, as well as the two generating set elements in S strictly greater than $2n - 5$. We arrive at a graph $G(n - 2, S')$, where $|S'| = n - 2$, and by induction this graph has a triangle.

- If $2n-3 \in S$, then $G(n, S)$ contains a triangle among the vertices $n-2, n-1, n$.

- Thus assume henceforth that $2n - 4 \in S$ and $2n - 3 \notin S$. If $2n - 6 \notin S$ and $2n - 5 \notin S$, then delete vertices $n - 2$, $n - 1$, and n, as well as the three generating set elements in S strictly greater than $2n-7$. We arrive at a graph $G(n-3, S')$, where $|S'| = n-3$, and by induction this graph has a triangle.

- If $2n-5 \in S$, then $G(n,S)$ contains a triangle among the vertices $n-4, n-1, n$.

- If $2n-6 \in S$, then $G(n,S)$ contains a triangle among the vertices $n-4, n-2, n$.

It follows by induction that $|S| = n$ ($n \geq 5$) implies that $G(n,S)$ has a triangle.

□

4.3 A Number Theory Result

To demonstrate the possible number-theoretic interpretations of our results, we prove a generalization of the following theorem by Cramer. We use one of the many corollaries of the main result of the previous chapter.

Theorem 4.3.1. *[11] For arbitrarily large $M > 0$, there is an even integer K such that there are M pairs of primes that differ by K.* □

Theorem 4.3.2. *Let $\{a_n\}$ ($a_1 \geq 3$) be any strictly increasing integer sequence with enumerating function $E(x) \in \omega(\sqrt{x})$.[1] For arbitrarily large $M > 0$, there is an integer K such that there are M pairs of sequence elements (a_i, a_j) that differ by K.*

Proof. Recall that for a fixed n, $S_a = \{a_i < 2n\}$. Note that the statement in the theorem is equivalent to the statement that for any $M > 0$, $G(n, S_a)$ contains $K_{2,M}$ for sufficiently large n. To see this, suppose that $G(n, S_a)$ contains $K_{2,M}$. The integers in the first *partite set* differ by some number K, so that they are x and $x+K$. Let

[1] $f(n) \in \omega(g(n))$ if $\forall C > 0\ \exists N$ such that $|f(n)| > |Cg(n)|\ \forall\, n \geq N$.

the integers in the second partite set be $\{b_1, b_2, \ldots, b_M\}$, and let $a_{(i)} = b_i + x$. Then the $a_{(i)}$ are elements in the sequence $\{a_n\}$, and so are the $a_{(i)} + K$. It follows that there are M pairs of sequence elements that differ by K.

By Corollary 3.4.38, we have that $G(n, S_a)$ contains $K_{2,M}$ provided $|S_a| = E(2n-1) \geq \sqrt{8(M-1)n}$. By the assumption regarding the enumerating function of $\{a_n\}$, this is true for sufficiently large n. $\square$

We highlight in Table 4.1 some important integer sequences that satisfy the enumerating function condition of Theorem 4.3.2.

Table 4.1: Some sequences of natural numbers that satisfy the condition of Theorem 4.3.1.

Sequence	Definition	First ten terms greater than 3
Abundant numbers	A number n for which $\sigma(n) > 2n$.	12, 18, 20, 24, 30, 36, 40, 42, 48, 54
Deficient numbers	A number n for which $\sigma(n) < 2n$.	3, 4, 5, 7, 8, 9, 10, 11, 13, 14
Nonhypotenuse numbers	A number n whose square cannot be written as the sum of two nonzero squares.	3, 4, 6, 7, 8, 9, 11, 12, 14, 16
Practical numbers	A number n for which every integer $1 \leq k \leq n-1$ can be written as the sum of distinct proper divisors of n.	4, 6, 8, 12, 16, 18, 20, 24, 28, 30
Prime numbers	A number n that has no integer divisors except for 1 and itself.	3, 5, 7, 11, 13, 17, 19, 23, 29, 31
Refactorable numbers	A number n that is divisible by the count of its divisors.	8, 9, 12, 18, 24, 36, 40, 56, 60, 72
Square-free numbers	A number n that is not divisible by any perfect square greater than 1.	3, 5, 6, 7, 10, 11, 13, 14, 15, 17
Størmer numbers	A number n for which the greatest prime factor of n^2+1 meets or exceeds $2n$.	4, 5, 6, 9, 10, 11, 12, 14, 15, 16
Ulam numbers	$U_1 = 1$, $U_2 = 2$, and U_n is the smallest integer that is the sum of two distinct earlier terms in exactly one way.	3, 4, 6, 8, 11, 13, 16, 18, 26, 28

Chapter 5

Conclusion

In this thesis, we have reintroduced a family of graphs with a natural construction and obvious number-theoretic interpretations. These additive graphs generated by subsets of the natural numbers were shown to be the first known wide-ranging and representative subclass of graphs with complete k-closure. These graphs provide a new and simple construction of minimum graphs with complete k-closure, and they may be useful in investigating conjectures in the field of closure theory. We made the first detailed study of the Traversal by Prime Sum Problem, and we introduced a related problem of our devising termed the Traversal by Practical Sum Problem. In both cases we showed that the additive graphs in question satisfy several necessary conditions for Hamilton cycles.

Many open questions remain in relation to additive graphs generated by subsets of the natural numbers. First and foremost, both the Traversal by Prime Sum Problem and the Traversal by Practical Sum Problem remain unsolved. As well, some of the thresholds for monotone properties considered in Chapter 3 were not shown to be tight. Sharpening these thresholds would suggest the complete stability of these properties in the cases where this has not yet been precisely determined. For these and other properties, further interpreting our threshold results from the point of view of additive number theory may yield some interesting conclusions.

There has been no consideration of random generating sets. Given the pathological nature of the lower bound threshold examples, it seems likely that thresholds for the appearance of monotone properties in the case of random generating sets will be significantly lower than the corresponding deterministic thresholds. In particular, we may ask what is the probabilistic threshold for a Hamilton cycle when generating set elements are selected uniformly at random from the set of odd integers $\{3, 5, \ldots, 2n-1\}$.

A basic algorithmic question asks for the complexity of the problem of recognizing an unlabeled graph as $G(n, S)$ for some generating set S. Our results provide several necessary conditions for a graph G to be an additive graph generated by a set S of cardinality k; namely, that $cl_k(G) = K_n$, $cl_{2n-3-k}(\overline{G}) = K_n$, and $f(k) \leq |E| \leq g(n, k)$.

Finally, recall that if $|S| \geq n$, then a Hamilton cycle exists in $G(n, S)$ and can be found in polynomial time. The following question was posed by Vašek Chvátal [8].

Question 5.0.3. *Does there exist a graph $G(n, S)$ that is 1-tough but not hamiltonian?*

If the answer to this question is no, then there exists a polynomial time algorithm for determining whether a graph $G(n, S)$ with arbitrary generating set cardinality is hamiltonian. There exist examples of $G(n, S)$ graphs that are 2-connected but not hamiltonian (e.g., Figure 1.1), but these 2-connected graphs have failed to be 1-tough (removing vertices 2 and 8 in the previous example separates the graph into three components).

Glossary

All graph-theoretic definitions are consistent with those used in the text by Bondy and Murty [4], and all number-theoretic definitions are consistent with those used in the text by Rosen [42]. We have restricted the definition of a graph so that graphs are always undirected and simple.

Adjacent: Vertices that are the endpoints of an edge.

Bipartite graph: A graph whose vertices can be covered by two independent sets. It is well-known that a graph is bipartite if and only if it has no odd cycles.

Centre: Given a star $K_{1,n-1}$ with $n > 3$, the vertex in the partite set of cardinality 1.

Chordal graph: A graph that has no chordless cycles.

Chordless cycle: An induced cycle of length at least 4.

Chromatic number $\chi(G)$: The minimum number of colours in a proper colouring of G.

Clique K_t: A subgraph that is the complete graph K_t.

Complement $\overline{G}$: A simple graph with the same vertex set as G, and with $(u, v) \in E(\overline{G})$ if and only if $(u, v) \notin E(G)$.

Complete bipartite graph K_{n_1,n_2}: A bipartite graph in which every pair of vertices not belonging to the same partite set is adjacent, where the sizes of partite sets are

n_1 and n_2.

Complete graph K_n: A simple graph of order n in which every pair of vertices is adjacent.

Component: A maximal connected subgraph.

Connected: Having a $u,\ v-$path for every pair of vertices u, v.

k**-connected**: Having connectivity at least k.

Connectivity $\kappa(G)$: The minimum number of vertices whose deletion disconnects G or reduces it to one vertex.

Cycle: A simple graph whose vertices can be placed on a circle so that vertices are adjacent if and only if they appear consecutively on the circle.

k**-cycle** C_k: A cycle of length k.

Degree $\deg(v)$: The number of times a vertex v appears as an endpoint of an edge.

Diameter: The maximum of the distance $d(u, v)$ over all pairs of vertices u, v.

Disconnected: A graph with more than one component.

Disjoint: Edge-disjoint.

Distance $d(u, v)$: The minimum length of a $u,\ v-$path.

Divisor: Given an integer n, an integer a such that there exists another integer b with $ab = n$.

Edge: An unordered pair of vertices.

k**-edge-connected**: Having edge connectivity at least k.

Edge connectivity: The minimum number of edges whose deletion disconnects G.

Edge-disjoint: Two subgraphs whose edge sets are disjoint.

Edge set $E(G)$: The set of edges of G. If the graph G is clear from the context, it is simply denoted by E.

Empty graph: A graph having no edges.

Endpoint: Each member of an edge; the first or last vertex of a path.

Enumerating function $E(x)$: For a particular sequence of integers, the number of its terms not exceeding x.

k**-factor**: A spanning k-regular graph.

Fibonacci numbers: The terms of the Fibonacci sequence.

Fibonacci primes: Fibonacci numbers that are also prime numbers.

Fibonacci sequence $\{f_n\}$: A sequence defined recursively by $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$.

Finite graph: A graph with a finite number of vertices and edges.

Graph G: An ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of vertices and a set $E(G)$ of edges.

Graph density: For a simple undirected graph G, $\frac{2|E|}{|V|(|V|-1)}$.

$(m \times n)-$**grid graph**: A graph with vertex set $V = [m] \times [n]$, and vertex (u_1, v_1) adjacent to vertex (u_2, v_2) if and only if $u_1 = u_2$ or $v_1 = v_2$.

Hamilton-connected: Every pair of vertices is the set of endpoints of a Hamilton path.

Hamilton cycle: A spanning cycle.

Hamilton path: A spanning path.

Hamiltonian graph: A graph having a Hamilton cycle.

Independence number $\alpha(G)$: The maximum size of an independent set of vertices in G.

Independent set: A set of pairwise nonadjacent vertices.

Induced subgraph: The subgraph on a subset A of the vertex set obtained by taking A and all edges of G having both endpoints in A.

Integer graph: A finite graph whose vertex set is $[n]$; an infinite graph whose vertex set is the set of natural numbers.

Isomorphic: Two graphs that are the same up to a labeling of the vertices.

Leaf: A vertex of a tree of degree 1.

Length: The number of edges in a path or cycle.

Linear recurrence: A recurrence equation on a sequence of numbers $\{x_n\}$ expressing x_n as a first-degree polynomial in x_k with $k < n$.

Matched: A vertex that is an endpoint of an edge in a particular matching.

Matching: A set of edges sharing no endpoints.

Maximum degree $\Delta(G)$: The maximum of the vertex degrees. If the graph G is clear from the context, it is simply denoted by Δ.

Minimum degree $\delta(G)$: The minimum of the vertex degrees. If the graph G is clear from the context, it is simply denoted by δ.

Monotone property: A property that is preserved in a graph when edges are added.

Natural numbers $\mathbb{N}$: The set $\{1, 2, 3, \ldots\}$.

Neighbour: Given a vertex v, a vertex adjacent to v.

Nested sequence of graphs: A sequence $(G_o, G_1, \ldots, G_k)$ of graphs such that G_i is a subgraph of G_{i+1} for all $0 \leq i < k$.

Nonadjacent: Two vertices that are not adjacent.

Odd cycle: A cycle of odd length.

Order: The cardinality of the vertex set.

Partite set: One of the sets in a vertex partition into independent sets.

Partition: Given a set X, a collection of disjoint subsets of X whose union is X.

Path: A simple graph whose vertices can be listed so that vertices are adjacent if and only if they are consecutive in the list.

k**-path** P_k: A path of order k and length $k-1$.

$u,\ v-$**path**: A path with u and v as its endpoints.

Perfect matching: A matching in which every vertex in G is matched.

Planar graph: A graph that can be drawn in the plane in such a way that edges meet only at points corresponding to their common endpoints.

Prime counting function $\pi(x)$: The enumerating function for the sequence of prime numbers.

Prime number: An integer greater than 1 that is divisible by no positive integers other than 1 and itself.

Prime number theorem: $\lim_{x\to\infty} \frac{\pi(x)}{x/\ln x} = 1$.

Proper colouring: An assignment of colours to vertices such that no two adjacent vertices are assigned the same colour.

k**-regular**: Having all vertex degrees equal to k.

Separating set: A set of vertices whose deletion increases the number of components.

Spanning tree: A spanning subgraph that is a tree.

Spanning subgraph: Given a graph G, a subgraph containing each vertex of G.

Star graph: The complete bipartite graph $K_{1,n-1}$.

Strictly increasing: Given a sequence $\{a_n\}$ of integers, having the property that $a_i < a_{i+1}$ for all i.

Subgraph: Given a graph G, a graph whose vertices and edges all belong to G.

Sum of divisors function σ: For a given integer n, $\sigma(n)$ is defined to be the sum of all positive divisors of n.

Sumset $A+B$: Given two subsets A and B of an abelian group, the set $\{a+b|\ a \in A, b \in B\}$.

1**-tough**: A connected graph such that for every $S \subset V$, the number of components upon deletion of S is at most $|S|$.

Tree: A connected graph with no cycles.

Triangle: A cycle of length 3.

Unmatched: Not an endpoint of any edge in a particular matching.

Vertex: One of the points on which the graph is defined and which may be connected by graph edges.

Vertex set $V(G)$: The set of vertices of a graph G. If the graph G is clear from the context, it is simply denoted by V.

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